Instructor's and Solutions Manual

to accompany

Vector Mechanics for Engineers, Statics

Eleventh Edition

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TO THE INSTRUCTOR

As indicated in its preface, *Vector Mechanics for Engineers: Statics* is designed for the first course in statics offered in the sophomore year of college. New concepts have, therefore, been presented in simple terms and every step has been explained in detail. However, because of the large number of optional sections which have been included and the maturity of approach which has been achieved, this text can also be used to teach a course which will challenge the more advanced student.

The text has been divided into units, each corresponding to a well-defined topic and consisting of one or several theory sections, one or several Sample Problems, a section entitled *Solving Problems on Your Own*, and a large number of problems to be assigned. To assist instructors in making up a schedule of assignments that will best fit their classes, the various topics covered in the text have been listed in Table I and a suggested number of periods to be spent on each topic has been indicated. Both a minimum and a maximum number of periods have been suggested, and the topics which form the standard basic course in statics have been separated from those which are optional. The total number of periods required to teach the basic material varies from 26 to 39, while covering the entire text would require from 41 to 65 periods. If allowance is made for the time spent for review and exams, it is seen that this text is equally suitable for teaching a basic statics course to students with limited preparation (since this can be done in 39 periods or less) and for teaching a more complete statics course to advanced students (since 41 periods or more are necessary to cover the entire text). In most instances, of course, the instructor will want to include some, but not all, of the additional material presented in the text. In addition, it is noted that the text is suitable for teaching an abridged course in statics which can be used as an introduction to the study of dynamics (see Table I).

The problems have been grouped according to the portions of material they illustrate and have been arranged in order of increasing difficulty, with problems requiring special attention indicated by asterisks. We note that, in most cases, problems have been arranged in groups of six or more, all problems of the same group being closely related. This means that instructors will easily find additional problems to amplify a particular point which they may have brought up in discussing a problem assigned for homework. Accessible through Connect are problem sets for each chapter that are designed to be solved with computational software. Solutions for these problems, including analyses of the problems and problem solutions and output for the most widely used computational programs, are also available through Connect.

To assist in the preparation of homework assignments, Table II provides a brief description of all groups of problems and a classification of the problems in each group according to the units used. It should also be noted that the answers to all problems are given at the end of the text, except for those with a number in italic. Because of the large number of problems available in both systems of units, the instructor has the choice of assigning problems using SI units and problems using U.S. customary units in whatever proportion is found to be most desirable for a given class. To illustrate this point, sample lesson schedules are shown in Tables III, IV, and V, together with various alternative lists of assigned homework problems. Half of the problems in each of the six lists suggested in Table III and Table V are stated in SI units and half in U.S. customary units. On the other hand, 75% of the problems in the four lists suggested in Table IV are stated in SI units and 25% in U.S. customary units.

Since the approach used in this text differs in a number of respects from the approach used in other books, instructors will be well advised to read the preface to *Vector Mechanics for Engineers,* in which the authors have outlined their general philosophy. In addition, instructors will find in the following pages a description, chapter by chapter, of the more significant features of this text. It is hoped that this material will help instructors in organizing their courses to best fit the needs of their students. Acknowledgement and thanks are given to Amy Mazurek for her careful preparation of the solutions contained in this manual.

David Mazurek

DESCRIPTION OF THE MATERIAL CONTAINED IN *VECTOR MECHANICS FOR ENGINEERS: STATICS,* Eleventh Edition

Chapter 1
Introduction

The material in this chapter can be used as a first assignment or for later reference. The six fundamental principles listed in Sec. 1.2 are introduced separately and are discussed at greater length in the following chapters. Section 1.3 deals with the two systems of units used in the text. The SI metric units are discussed first. The base units are defined and the use of multiples and submultiples is explained. The various SI prefixes are presented in Table 1.1, while the principal SI units used in statics and dynamics are listed in Table 1.2. In the second part of Sec. 1.3, the base U.S. customary units used in mechanics are defined, and in Sec. l.4, it is shown how numerical data stated in U.S. customary units can be converted into SI units, and vice versa. The SI equivalents of the principal U.S. customary units used in statics and dynamics are listed in Table 1.3.

The instructor’s attention is called to the fact that the various rules relating to the use of SI units have been observed throughout the text. For instance, multiples and submultiples (such as kN and mm) are used whenever possible to avoid writing more than four digits to the left of the decimal point or zeros to the right of the decimal point. When 5-digit or larger numbers involving SI units are used, spaces rather than commas are utilized to separate digits into groups of three (for example, 20 000 km). Also, prefixes are never used in the denominator of derived units; for example, the constant of a spring which stretches 20 mm under a load of 100 N is expressed as 5 kN/m, not as 5 N/mm.

In order to achieve as much uniformity as possible between results expressed respectively in SI and U.S. customary units, a center point, rather than a hyphen, has been used to combine the symbols representing U.S. customary units (for example, 10 lb·ft); furthermore, the unit of time has been represented by the symbol s, rather than sec, whether SI or U.S. customary units are involved (for example, 5 s, 50 ft/s, 15 m/s).

However, the traditional use of commas to separate digits into groups of three has been maintained for 5-digit and larger numbers involving U.S. customary units.

Chapter 2
Statics of Particles

This is the first of two chapters dealing with the fundamental properties of force systems. A simple, intuitive classification of forces has been used: forces acting on a particle (Chap. 2) and forces acting on a rigid body (Chap. 3).

Chapter 2 begins with the parallelogram law of addition of forces and with the introduction of the fundamental properties of vectors. In the text, forces and other vector quantities are always shown in bold-face type. Thus, a force **F** (boldface), which is a vector quantity, is clearly distinguished from the magnitude *F* (italic) of the force, which is a scalar quantity. On the blackboard and in handwritten work, where bold-face lettering is not practical, vector quantities can be indicated by underlining. Both the magnitude and the direction of a vector quantity must be given to completely define that quantity. Thus, a force **F** of magnitude $F = 280$ lb, directed upward to the right at an angle of 25° with the horizontal, is indicated as $\mathbf{F} = 280$ lb ∠ 25° when printed or as $\underline{F} = 280$ lb ∠ 25° when handwritten. Unit vectors **i** and **j** are introduced in Sec. 2.7, where the rectangular components of forces are considered.

In the early sections of Chap. 2 the following basic topics are presented: the equilibrium of a particle, Newton's first law, and the concept of the free-body diagram. These first sections provide a review of the methods of plane trigonometry and familiarize the students with the proper use of a calculator. A general procedure for the solution of problems involving concurrent forces is given: when a problem involves only three forces, the use of a force triangle and a trigonometric solution is preferred; when a problem involves more than three forces, the forces should be resolved into rectangular components and the equations $\Sigma F_x = 0$, $\Sigma F_y = 0$ should be used.

The second part of Chap. 2 deals with forces in space and with the equilibrium of particles in space. Unit vectors are used and forces are expressed in the form $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} = F\boldsymbol{\lambda}$, where **i**, **j**, and **k** are the unit vectors directed respectively along the *x*, *y*, and *z* axes, and $\boldsymbol{\lambda}$ is the unit vector directed along the line of action of **F**.

Note that since this chapter deals only with particles or bodies which can be considered as particles, problems involving compression members have been postponed with only a few exceptions until Chap. 4, where students will learn to handle rigid-body problems in a uniform fashion and will not be tempted to erroneously assume that forces are concurrent or that reactions are directed along members.

It should be observed that when SI units are used a body is generally specified by its mass expressed in kilograms. The weight of the body, however, should be expressed in newtons. Therefore, in many equilibrium problems involving SI units, an additional calculation is required before a free-body diagram can be drawn (compare the example in Sec. 2.3C and Sample Probs. 2.5 and 2.9). This apparent disadvantage of the SI system of units, when compared to the U.S. customary units, will be offset in dynamics, where the mass of a body expressed in kilograms can be entered directly into the equation $\mathbf{F} = m\mathbf{a}$, whereas with U.S. customary units the mass of the body must first be determined in lb·s^2/ft (or slugs) from its weight in pounds.

Chapter 3
Rigid Bodies:
Equivalent Systems of Forces

The principle of transmissibility is presented as the basic assumption of the statics of rigid bodies. However, it is pointed out that this principle can be derived from Newton's three laws of motion (see Sec. 16.1D of *Dynamics*). The vector product is then introduced and used to define the moment of a force about a point. The convenience of using the determinant form (Eqs. 3.19 and 3.21) to express the moment of a force about a point should be noted. The scalar product and the mixed triple product are introduced and used to define the moment of a force about an axis. Again, the convenience of using the determinant form (Eqs. 3.41 and 3.44) should be noted. The amount of time which should be assigned to this part of the chapter will depend on the extent to which vector algebra has been considered and used in prerequisite mathematics and physics courses. It is felt that, even with no previous knowledge of vector algebra, a maximum of four periods is adequate (see Table I).

In Secs. 3.12 through 3.15 couples are introduced, and it is proved that couples are equivalent if they have the same moment. While this fundamental property of couples is often taken for granted, the authors believe that its rigorous and logical proof is necessary if rigor and logic are to be demanded of the students in the solution of their mechanics problems.

In Section 3.3, the concept of equivalent systems of forces is carefully presented. This concept is made more intuitive through the extensive use of *free-body-diagram equations* (see Figs. 3.34 through 3.41). Note that the *moment* of a force is either not shown or is represented by a green vector (Figs. 3.10 and 3.22). A red vector with the symbol is used only to represent a *couple*, that is, an actual system consisting of two forces (Figs. 3.33 through 3.41). Section 3.4D is optional; it introduces the concept of a *wrench* and shows how the most general system of forces in space can be reduced to this combination of a force and a couple with the same line of action.

Since one of the purposes of Chap. 3 is to familiarize students with the fundamental operations of vector algebra, students should be encouraged to solve all problems in this chapter (two-dimensional as well as three-dimensional) using the methods of vector algebra. However, many students may be expected to develop solutions of their own, particularly in the case of two-dimensional problems, based on the direct computation of the moment of a force about a given point as the product of the magnitude of the force and the perpendicular distance to the point considered. Such alternative solutions may occasionally be indicated by the instructor (as in Sample Prob. 3.9), who may then wish to compare the solutions of the sample problems of this chapter with the solutions of the same sample problems given in Chaps. 3 and 4 of the parallel text *Mechanics for*

Engineers. It should be pointed out that in later chapters the use of vector products will generally be reserved for the solution of three-dimensional problems.

Chapter 4
Equilibrium of Rigid Bodies

In the first part of this chapter, problems involving the equilibrium of rigid bodies in two dimensions are considered and solved using ordinary algebra, while problems involving three dimensions and requiring the full use of vector algebra are discussed in the second part of the chapter. Particular emphasis is placed on the correct drawing and use of free-body diagrams and on the types of reactions produced by various supports and connections (see Figs. 4.1 and 4.10). Note that a distinction is made between hinges used in pairs and hinges used alone; in the first case the reactions consist only of force components, while in the second case the reactions may, if necessary, include couples.

For a rigid body in two dimensions, it is shown (Sec. 4.1B) that no more than three independent equations can be written for a given free body, so that a problem involving the equilibrium of a single rigid body can be solved for no more than three unknowns. It is also shown that it is possible to choose equilibrium equations containing only one unknown to avoid the necessity of solving simultaneous equations. Section 4.1C introduces the concepts of statical indeterminacy and partial constraints. Section 4.2 is devoted to the equilibrium of two- and three-force bodies; it is shown how these concepts can be used to simplify the solution of certain problems. This topic is presented only after the general case of equilibrium of a rigid body to lessen the possibility of students misusing this particular method of solution.

The equilibrium of a rigid body in three dimensions is considered with full emphasis placed on the free-body diagram. While the tool of vector algebra is freely used to simplify the computations involved, vector algebra does not, and indeed cannot, replace the free-body diagram as the focal point of an equilibrium problem. Therefore, the solution of every sample problem in this section begins with a reference to the drawing of a free-body diagram. Emphasis is also placed on the fact that the number of unknowns and the number of equations must be equal if a structure is to be statically determinate and completely constrained.

Chapter 5
Distributed Forces:
Centroids and Centers of Gravity

Chapter 5 starts by defining the center of gravity of a body as the point of application of the resultant of the weights of the various particles forming the body. This definition is then used to establish the concept of the centroid of an area or line. Section 5.1C introduces the concept of the first moment of an area or line, a concept fundamental to the analysis of shearing stresses in beams in a later study of mechanics of materials. All problems assigned for the first period involve only areas and lines made of simple geometric shapes; thus, they can be solved without using calculus.

Section 5.2A explains the use of differential elements in the determination of centroids by integration. The theorems of Pappus-Guldinus are given in Sec. 5.2B. Sections 5.3A and 5.3B are optional; they show how the resultant of a distributed load can be determined by evaluating an area and by locating its centroid. Section 5.4 deals with centers of gravity and centroids of volumes. Here again the determination of the centroids of composite shapes precedes the calculation of centroids by integration.

It should be noted that when SI units are used, a given material is generally characterized by its density (mass per unit volume, expressed in kg/m^3), rather than by its specific weight (weight per unit volume, expressed in N/m^3). The specific weight of the material can then be obtained by multiplying its density by $g = 9.81$ m/s^2 (see footnote, page 234 of the text).

Chapter 6
Analysis of Structures

In this chapter students learn to determine the internal forces exerted on the members of pin-connected structures. The chapter starts with the statement of Newton's third law (action and reaction) and is divided into two parts: (*a*) trusses, that is, structures consisting of two-force members only, (*b*) frames and machines, that is, structures involving multiforce members.

After trusses and simple trusses have been defined in Sec. 6.1A, the method of joints and the method of sections are explained in detail in Sec. 6.1B and Sec. 6.2A, respectively. Since a discussion of Maxwell's diagram is not included in this text, the use of Bow's notation has been avoided, and a uniform notation has been used in presenting the method of joints and the method of sections.

In the method of joints, a free-body diagram should be drawn for each pin. Since all forces are of known direction, their magnitudes, rather than their components, should be used as unknowns. Following the general procedure outlined in Chap. 2, joints involving only three forces are solved using a force triangle, while joints involving more than three forces are solved by summing *x* and *y* components. Sections 6.1C and 6.1D are optional. It is shown in Sec. 6.1C how the analysis of certain trusses can be expedited by recognizing joints under special loading conditions, while in Sec. 6.1D the method of joints is applied to the solution of three-dimensional trusses.

It is pointed out in Sec. 6.1B that forces in a simple truss can be determined by analyzing the truss joint by joint and that joints can always be found that involve only two unknown forces. The method of sections (Sec. 6.2A) should be used (*a*) if only the forces in a few members are desired, or (*b*) if the truss is not a simple truss and if the solution of simultaneous equations is to be avoided (for example, Fink truss). Students should be urged to draw a separate free-body diagram for each section used. The free body obtained should be emphasized by shading and the intersected members should be removed and replaced by the forces they exerted on the free body. It is shown that, through a judicious choice of equilibrium equations, the force in any given member can be obtained in most cases by solving a single equation. Section 6.2B is optional; it deals with the trusses obtained by combining several simple trusses and discusses the statical determinacy of such structures as well as the completeness of their constraints.

Structures involving multiforce members are separated into frames and machines. Frames are designed to support loads, while machines are designed to transmit and modify forces. It is shown that while some frames remain rigid after they have been detached from their supports, others will collapse (Sec. 6.3B). In the latter case, the equations obtained by considering the entire frame as a free body provide necessary but not sufficient conditions for the equilibrium of the frame. It is then necessary to dismember the frame and to consider the equilibrium of its component parts in order to determine the reactions at the external supports. The same procedure is necessary with most machines in order to determine the output force **Q** from the input force **P** or inversely (Sec. 6.4).

Students should be urged to resolve a force of unknown magnitude and direction into two components but to represent a force of known direction by a single unknown, namely its magnitude. While this rule may sometimes result in slightly more complicated arithmetic, it has the advantage of matching the numbers of equations and unknowns and thus makes it possible for students to know at any time during the computations what is known and what is yet to be determined.

Chapter 7
Forces in Beams and Cables

This chapter consists of five groups of sections, all of which are optional. The first three groups deal with forces in beams and the last two groups with forces in cables. Most likely the instructor will not have time to cover the entire chapter and will have to choose between beams and cables.

Section 7.1 defines the internal forces in a member. While these forces are limited to tension or compression in a straight two-force member, they include a shearing force and a bending couple in the case of multiforce members or curved two-force members. Problems in this section do not make use of sign conventions for shear and bending moment and answers should specify which part of the member is used as the free body.

In Section 7.2 the usual sign conventions are introduced and shear and bending-moment diagrams are drawn. All problems in these sections should be solved by drawing the free-body diagrams of the various portions of the beams.

The relations among load, shear, and bending moment are introduced in Sec. 7.3. Problems in this section should be solved by evaluating areas under load and shear curves or by formal integration (as in Probs. 7.85 through 7.88). Some instructors may feel that the special methods used in this section detract from the unity achieved in the rest of the text through the use of the free-body diagram, and they may wish to omit Sec. 7.3. Others will feel that the study of shear and bending-moment diagrams is incomplete without this section, and they will want to include it. The latter view is particularly justified when the course in statics is immediately followed by a course in mechanics of materials.

Section 7.4 is devoted to cables, first with concentrated loads and then with distributed loads. In both cases, the analysis is based on free-body diagrams. The differential equation approach is considered in the last problems of this group (Probs. 7.124 through 7.126). Section 7.5 is devoted to catenaries and requires the use of hyperbolic functions.

Chapter 8
Friction

This chapter not only introduces the general topic of friction but also provides an opportunity for students to consolidate their knowledge of the methods of analysis presented in Chaps. 2, 3, 4, and 6. It is recommended that each course in statics include at least a portion of this chapter.

Section 8.1 is devoted to the presentation of the laws of dry friction and to their application to various problems. The different cases which can be encountered are illustrated by diagrams in Figs. 8.2, 8.3, and 8.4. Particular emphasis is placed on the fact that no relation exists between the friction force and the normal force except when motion is impending or when motion is actually taking place. Following the general procedure outlined in Chap. 2, problems involving only three forces are solved by a force triangle, while problems involving more than three forces are solved by summing x and y components. In the first case the reaction of the surface of contact should be represented by the resultant **R** of the friction force and normal force, while in the second case it should be resolved into its components **F** and **N**.

Special applications of friction are considered in Secs. 8.2 through 8.4. They are divided into the following groups: wedges and screws (Sec. 8.2); axle and disk friction, rolling resistance (Sec. 8.3); belt friction (Sec. 8.4). The sections on axle and disk friction and on rolling resistance are not essential to the understanding of the rest of the text and thus may be omitted.

Chapter 9
Distributed Forces
Moments of Inertia

The purpose of Sec. 9.1A is to give motivation to the study of moments of inertia of areas. Two examples are considered: one deals with the pure bending of a beam and the other with the hydrostatic forces exerted on a submerged circular gate. It is shown in each case that the solution of the problem reduces to the computation of the moment of inertia of an area. The other sections in the first assignment are devoted to the definition and the computation of rectangular moments of inertia, polar moments of inertia, and the corresponding radii of gyration. It is shown how the same differential element can be used to determine the moment of inertia of an area about each of the two coordinate axes.

Section 9.2introduces the parallel-axis theorem and its application to the determination of moments of inertia of composite areas. Particular emphasis is placed on the proper use of the parallel-axis theorem (see Sample Prob. 9.5). Sections 9.3 and 9.4 are optional; they are devoted to products of inertia and to the determination of principal axes of inertia.

Sections 9.5 and 9.6 deal with the moments of inertia of masses. Particular emphasis is placed on the moments of inertia of thin plates (Sec. 9.5C) and on the use of these plates as differential elements in the computation of moments of inertia of symmetrical three-dimensional bodies (Sec. 9.5D). Section 9.6 is optional but should be used whenever the following dynamics course includes the motion of rigid bodies in three dimensions. Sections 9.6A and 9.6B introduce the moment of inertia of a body with respect to an arbitrary axis as well as the concepts of mass products of inertia and principal axes of inertia. Section 9.6C discusses the determination of the principal axes and principal moments of inertia of a body of arbitrary shape.

When solving many of the problems of Chap. 5, information on the specific weight of a material was generally required. This information was readily available in problems stated in U.S. customary units, while it had to be obtained from the density of the material in problems stated in SI units (see the last paragraph of our discussion of Chap. 5). In Chap. 9, when SI units are used, the mass and mass moment of inertia of a given body are respectively obtained in kg and kg·m^2 directly from the dimensions of the body in meters and from its density in kg/m^3. However, if U.S. customary units are used, the density of the body must first be calculated from its specific weight or, alternatively, the weight of the body can be obtained from its dimensions and specific weight and then converted into the corresponding mass expressed in lb·s^2/ft (or slugs). The mass moment of inertia of the body is then obtained in lb·ft·s^2 (or slug·ft^2). Sample Problem 9.12 provides an example of such a computation. Attention is also called to the footnote on page of the 530 regarding the conversion of mass moments of inertia from U.S. customary units to SI units.

Chapter 10
Method of Virtual Work

While this chapter is optional, the instructor should give serious consideration to its inclusion in the basic statics course. Indeed, students who learn the method of virtual work in their first course in mechanics will remember it as a fundamental and natural principle. They may, on the other hand, consider it as an artificial device if its presentation is postponed to a more advanced course.

Section 10.1 is devoted to the derivation of the principle of virtual work and to its direct application to the solution of equilibrium problems. Section 10.2 introduces the concept of potential energy and shows that equilibrium requires that the derivative of the potential energy be zero. Section 10.1D defines the mechanical efficiency of a machine and Sec. 10.2D discusses the stability of equilibrium.

The first groups of problems in each assignment utilize the principle of virtual work as an alternative method for the computation of unknown forces. Subsequent problems call for the determination of positions of equilibrium, while other problems combine the conventional methods of statics with the method of virtual work to determine displacements (Probs. 10.55 through 10.58).

TABLE I: LIST OF THE TOPICS COVERED IN *VECTOR MECHANICS FOR ENGINEERS: STATICS*

		Suggested Number of Periods		
Sections	**Topics**	**Basic Course**	**Additional Topics**	**Abridged Course to be used as an introduction to dynamics**[+]
1.	INTRODUCTION			
1.1–6	This material may be used for the first assignment or for later reference			
2.	STATICS OF PARTICLES			
2.1	Addition and Resolution of Forces	0.5–1		0.5–1
2.2	Rectangular Components	0.5–1		0.5–1
2.3	Equilibrium of a Particle	1		1
2.4	Forces in Space	1		1
2.5	Equilibrium in Space	1		1
3.	RIGID BODIES: EQUIVALENT SYSTEMS OF FORCES			
3.1	Vector Product; Moment of a Force about a Point	1–2		1–2
3.2	Scalar Product; Moment of a Force about an Axis	1–2		1–2
3.3	Couples	1		1
3.4A–4C	Equivalent Systems of Forces	1–1.5		1–1.5
*3.4D	Reduction of a Wrench		0.5–1	
4.	EQUILIBRIUM OF RIGID BODIES			
4.1A–1B	Equilibrium in Two Dimensions	1.5–2		1.5–2
4.1C	Indeterminate Reactions; Partial Constraints	0.5–1		
4.2	Two- and Three-Force Bodies	1		
4.3	Equilibrium in Three Dimensions	2		2
5.	CENTROIDS AND CENTERS OF GRAVITY			
5.1	Centroids and First Moments of Areas and Lines	1–2		
5.2	Centroids by Integration	1–2		
5.3	Beams and Submerged Surfaces		1–1.5	
5.4	Centroids of Volumes	1–2		
6.	ANALYSIS OF STRUCTURES			
6.1A–1B	Trusses by Method of Joints	1–1.5		
*6.1C	Joints under Special Loading Conditions		0.25–0.5	
*6.1D	Space Trusses		0.5–1	
6.2A	Trusses by Method of Sections	1–2		
6.2B	Combined Trusses		0.25–0.5	
6.3	Frames	2–3		1–2
6.4	Machines	1–2		0.5–1.5
7.	INTERNAL FORCES AND MOMENTS			
7.1	Internal Forces in Members		1	
7.2	Shear and Moment Diagrams by FB Diagram		1–2	
7.3	Shear and Moment Diagrams by Integration		1–2	
*7.4	Cables with Concentrated Loads; Parabolic Cable		1–2	
*7.5	Catenary		1	
8.	FRICTION			
8.1	Laws of Friction and Applications	1–2		1–2
8.2	Wedges and Screws	1		
*8.3	Axle and Disk Friction; Rolling Resistance		1–2	
8.4	Belt Friction	1		
9.	MOMENTS OF INERTIA			
9.1	Moments of Inertial of Areas	1		
9.2	Composite Areas	1–2		
*9.3	Products of Inertia; Principal Axes		1–2	
*9.4	Mohr’s Circle		1	
9.5	Moments of Inertia of Masses[#]		1–2	
*9.6	Mass Products of Inertia; Principal Axes and Principal Moments of Inertia		1–2	
10.	METHOD OF VIRTUAL WORK			
10.1A–1C	Principle of Virtual Work		1–2	
10.1D	Mechanical Efficiency		0.5–1	
10.2	Potential Energy; Stability		1–1.5	
	Total Number of Periods	26–39	15–26	14–21

[+] A sample assignment schedule for a course in dynamics including this minimum amount of introductory material in statics is given Table V. It is recommended that a more complete statics course, such as the one outlined in Tables III and IV of this manual, be used in curricula which include the study of mechanics of materials.

[#] Mass moments of inertia have not been included in the basic statics course since this material is often taught in dynamics.

TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS

Problem Number*		
SI Units	U.S. Units	Problem Description
		CHAPTER 2: STATICS OF PARTICLES
		FORCES IN A PLANE
		Resultant of concurrent forces
2.1, *3*	2.2, 4	graphical method
2.5, 8	2.6, *7*	law of sines
2.9, 10	2.11, *12*	
2.14	2.13	special problems
2.16, *18*	2.15, 17	laws of cosines and sines
2.19, *20*		
		Rectangular components of force
2.*22*, 23	2.21, 24	simple problems
2.*25*, 27	2.26, 28	more advanced problems
2.29, *30*		
2.32, 34	2.31, *33*	Resultant by $\Sigma F_x = 0$, $\Sigma F_y = 0$
2.35, 36	2.37, *38*	
2.39, 40	2.*41*, 42	Select force so that resultant has a given direction
2.*F1*, *F4*	2.*F2*, *F3*	Free Body Practice Problems
		Equilibrium. Free-Body Diagram
2.44, *45*	2.43, 48	equilibrium of 3 forces
2.46, *47*		
2.49, 50	2.51, *52*	equilibrium of 4 forces
2.53, 54	2.55, *56*	
2.57, 58	2.59, *60*	find parameter to satisfy specified conditions
2.61, 62	2.63, *64*	
2.65, *66*	2.67, 68	special problems
2.69, *70*		
		FORCES IN SPACE
		Rectangular components of a force in space
2.71, 72	2.75, *76*	given F, θ, and ϕ, find components and direction angles
2.73, 74	2.77, *78*	
2.*80*, 82	2.81, 82	relations between components and direction angles
2.*83*, 84		
2.85, *86*	2.87, 88	direction of force defined by two points on its line of action
2.89, *90*		
2.91, 92	2.*93*, 94	resultant of two or three forces
2.95, 96	2.97, *98*	
2.*F5*, *F6*	2.*F7*, *F8*	Free Body Practice Problems
		Equilibrium of a particle in space
2.99, *100*	2.103, 104	load applied to three cables, introductory problems
2.101, *102*		

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	U.S. Units	Problem Description
2.107, 108	2.*105*, 106	intermediate problems
2.109, 110	2.*111*, 112	
2.115, 116	2.113, *114*	advanced problems
2.117, 118	2.119, *120*	
2.121, *122*	2.123, 124	problems involving cable through ring
2.125, *126*		special problems
2.127, 131	2.128, *129*	Review problems
2.*132*, *134*	2.130, 133	
2.135, 136	2.137, *138*	
2.C1, C3	2.C2	Computer problems
2.C4	2.C5	

CHAPTER 3: RIGID BODIES; EQUIVALENT SYSTEMS OF FORCES

SI Units	U.S. Units	Problem Description
		Moment of a force about a point: Two dimensions
3.1, 2	3.*3*, 6	introductory problems
3.4, 5	3.7, *8*	
3.9, *10*	3.11, 12	direction of a force defined by two points on its line of action
3.13, *14*		
3.15		derivation of a formula
3.17, 18	3.*16*	applications of the vector product
		Moment of a force about a point: Three dimensions
3.*19*	3.20	computing **M** = **r** × **F**, introductory problems
3.*21*, 23	3.22, 25	computing **M** = **r** × **F**, more involved problems
3.*24*, 26		
3.28, 29	3.27, 30	using **M** to find the perpendicular distance from a point to a line
3.32, 33	3.*31*, *34*	
3.35	3.36	Scalar Product
3.37, *38*	3.39, 40	Finding the angle between two lines
3.41, *42*	3.43, 44	
3.45	3.46	Mixed triple product
3.47, 48	3.49, *50*	Moment of a force about the coordinate axes
3.51, 52	3.53, *54*	
3.57, 58	3.55, *56*	Moment of a force about an oblique axis
3.59, 60	3.61, *62*	
	3.63	
*3.*66*, *67	*3.64, *65	Finding the perpendicular distance between two lines
*3.68, *69		
3.70, 71	3.*72*, 73	Couples in two dimensions
3.74		
3.78, 79	3.*75*, 76	Couples in three dimensions
	3.77, 80	

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Answers are not given to problems with a number set in italic type.

TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	U.S. Units	Problem Description
3._81_, 83	3.82, 84	Replacing a force by an equivalent force-couple system: two dimensions
3._85_, 86		
3.87, _88_	3.89, 91	Replacing a force-couple system by an equivalent force or forces
3.90, _92_		
3.93, 95	3._94_, 97	Replacing a force by an equivalent force-couple system: three dimensions
3.96, 98	3.99, _100_	
3.101, 102	3.104	Equivalent force-couple systems
3._103_		
3._107_	3.105, 106	Finding the resultant of parallel forces: two dimensions
3.110, 111	3.108, _109_	Finding the resultant and its line of action: two dimensions
3.116, 117	3._112_, 113	
3.118	3.114, 115	
3.119, 120	3._121_, 122	Reducing a three-dimensional system of forces to a single force-couple system
3.124, 125	3._123_, 126	
3.127, 128	3.129, 130	Finding the resultant of parallel forces: three dimensions
*3._131_, *_132_		
		Reducing three-dimensional systems of forces or forces and couples to a wrench
*3.133, *_135_	*3.134, *136	axis of wrench is parallel to a coordinate axis or passes through _O_
*3.137		force-couple system parallel to the coordinate axes
*3._139_, 140	*3._138_	general, three-dimensional case
*3.141	*3._142_	special cases where the wrench reduces to a single force
*3.143, *144	*3.145, *146	special, more advanced problems
3.147, 148	3._149_, 150	Review problems
3.151, 153	3._152_, 154	
3.156, _157_	3._155_, 158	
3.C1, C4	3.C2, C3	Computer problems
3.C5	3.C6	

CHAPTER 4: EQUILIBRIUM OF RIGID BODIES

EQUILIBRIUM IN TWO DIMENSIONS

_4.F1, F4_	_4.F2, F3_	Free Body Practice Problems
4.1, 2	4._3_, 4	Parallel forces
4.5, 6	4.7, _8_	
4.9, _10_	4.12, 14	Parallel forces, find range of values of loads to satisfy multiple criteria
4.11, _13_		
4.15, _16_	4.17, 18	Rigid bodies with one reaction of unknown direction and one of known direction
4.19, _20_		
4.22, 25	4._21_, 23	
4.26, 28	4.24, _27_	
4._29_, _30_	4.33, 34	
4.31, 32		
4.35, 36	4._37_, _38_	Rigid bodies with three reactions of known direction
4.39, 40	4.41, 42	

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Answers are not given to problems with a number set in italic type.

TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	U.S. Units	Problem Description
4.45, 46	4.43, *44*	Rigid bodies with a couple included in the reactions
4.47, 50	4.48, *49*	
4.51, 52	4.53, 54	Find position of rigid body in equilibrium
4.*56*, *58*	4.55, 57	
4.59	4.*60*	Partial constraints, statical indeterminacy
		Three-force bodies
4.62, 63	4.61, *64*	simple geometry, solution of a right triangle required
4.65, 67	4.66, *68*	simple geometry, frame includes a two-force member
4.69, *70*	4.71, 72	more involved geometry
4.73, *74*		
4.75, *76*	4.77, 79	
4.78, 81	4.80	
4.*82*		
4.83, 84	4.86, *87*	find position of equilibrium
4.85, 88	4.*89*, 90	

EQUILIBRIUM IN THREE DIMENSIONS

SI Units	U.S. Units	Problem Description
4.*F6*, *F7*	4.*F5*	Free Body Practice Problems
4.91, *92*	4.93, 94	Rigid bodies with two hinges along a coordinate axis and an additional reaction parallel to another coordinate axis
4.95, *96*		
4.99, 100	4.97, *98*	Rigid bodies supported by three vertical wires or by vertical reactions
4.101, 102	4.103, *104*	
4.106, 107	4.105, 110	Derrick and boom problems involving unknown tension in two cables
4.108, 109	4.*111*, 112	
4.113, *114*	4.115, 116	Rigid bodies with two hinges along a coordinate axis and an additional reaction not parallel to a coordinate axis
4.117, *118*		
4.119, 122	4.120, 121	Problems involving couples as part of the reaction at a hinge
4.123, 124	4.*125*, *126*	Advanced problems
4.129, *130*	4.127, 128	
4.131, *132*		
4.133, *134*	4.135, 136	Problems involving taking moments about an oblique line passing through two supports
4.138, *139*	4.137, 140	
	4.141	
4.*144*, 145	4.142, 143	Review problems
4.146, 148	4.*147*, 149	
4.*150*, 153	4.151, *152*	
4.C2, C5	4.C1, C3	Computer problems
4.C6	4.C4	

CHAPTER 5: DISTRIBUTED FORCES: CENTROIDS AND CENTERS OF GRAVITY

SI Units	U.S. Units	Problem Description
		Centroid of an area formed by combining
5.1, 3	5.2	rectangles and triangles
5.4		
5.6, 9	5.*5*, 7	rectangles, triangles, and portions of circular areas
	5.*8*	

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number* SI Units	 U.S. Units	 Problem Description
5.11, *12*	5.10, 13	triangles, portions of circular or elliptical areas, and areas of analytical functions
5.14, *15*		
5.16, 17	5.*18*, 19	Derive expression for location of centroid
5.20, *22*	5.21, 23	First moment of an area
5.24, *25*	5.26, *27*	Center of gravity of a wire figure
5.29, 30	5.*28*, 33	Equilibrium of wire figures
5.31, 32		Find dimension to maximize distance to centroid
		Use integration to find centroid of
5.*36*	5.34, 35	simple areas
5.37, *38*	5.39	areas obtained by combining shapes of Fig. 5.8A
5.40	5.41	
	5.*42*	parabolic area
5.43	5.44	areas defined by different functions over the interval of interest
5.45, 46	*5._47_	homogeneous wires
*5.48, *49		areas defined by exponential or cosine functions
	5.*50*, 51	areas defined by a hyperbola
		Find areas or volumes by Pappus - Guldinus
5.52, 54	5.53, *56*	rotate simple geometric figures
5.55, 57		
5.60, *61*	5.58, *59*	practical applications
5.64, *_65_	5.62, 63	
		Distributed load on beams
5.66, 67		resultant of loading
5.71	5.*68*, 69	reactions at supports
	5.70	
5.73	5.*72*	parabolic loadings
5.74, *75*	5.76, 77	special problems
5.78, *79*		
		Forces on submerged surfaces
5.81, 82	5.80, 84	reactions on dams or vertical gates
5.*83*, *86*	5.85	
5.87		
5.88, 89	5.90, *91*	reactions on non-vertical gates
5.92, 93	5.94, *95*	
		Centroids and centers of gravity of three-dimensional bodies
5.98, 99	5.96, 97	composite bodies formed from two common shapes
5.100, *101*	5.102, 103	composite bodies formed from three or more elements
5.104, *105*		
5.106, 107	5.*108*, 109	composite bodies formed from a material of uniform thickness
5.110, 113	5.111, *112*	
5.114, 116	5.*115*, 117	composite bodies formed from a wire or structural shape of uniform cross section
5.118, 121	5.119, *120*	composite bodies made of two different materials
		use integration to locate the centroid of
5.122	5.123, 124	standard shapes: single integration
5.125, *126*	*5.128, *129	bodies of revolution: single integration
5.*127*		

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Answers are not given to problems with a number set in italic type.

TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	U.S. Units	Problem Description
5.132	*5.130, _131_	special applications: single integration
5._133_, *134		bodies formed by cutting a standard shape with an oblique plane: single integration
	5.135, 136	special applications: double integration
5.138, 141	5.137, 139	Review problems
5._142_, 144	5._140_, 143	
5._145_, 148	5.146, _147_	
5.C2, C3	5.C1, C4	Computer problems
5.C5, C6	*5.C7	

CHAPTER 6: ANALYSIS OF STRUCTURES

TRUSSES

SI Units	U.S. Units	Problem Description
		Method of joints
6.2, 4	6.1, 3	simple problems
6.6, 8	6._5_, _7_	
6.9, _10_	6.11, 12	problems of average difficulty
6.13, _14_		
6.17, 18	6.15, _16_	more advanced problems
6.21, 24	6.19, _20_	
6._25_, _26_	6.22, 23	
	6.27, 28	
6.29, 30		designate simple trusses
6.32, 34	6._31_, _33_	find zero-force members
*6.36, *37	*6.35, *38	space trusses
*6.39, *40	*6._41_, *_42_	
		Method of sections
6.43, 44	6.45, 46	two of the members cut are parallel
6._47_, _48_		
6.49, 50	6.51, 52	none of the members cut are parallel
6.53, 54	6._57_, _58_	
6.55, 56	6.59, 60	
6.61, 62	6._63_, _64_	K-type trusses
6.65, 66	6.67, 68	trusses with counters
6.69, 70	6.71, 72	Classify trusses according to constraints
6._73_, _74_		

FRAMES AND MACHINES

SI Units	U.S. Units	Problem Description
6._F1_, _F4_	6._F2_, _F3_	Free Body Practice Problems – Frames
		Analysis of Frames
6.75, 77	6.76, _78_	easy problems
6.79, 80	6.81, _82_	

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	U.S. Units	Problem Description
6.83, *84*	6.87, 88	problems where internal forces are changed by repositioning a couple or by moving a force along its line of action
6.85, *86*	6.89	
6.90		replacement of pulleys by equivalent loadings
6.91, 92	6.93, 94	analysis of frames supporting pulleys or pipes
	6.95, 96	analysis of highway vehicles
6.*97*, *98*	6.99, 100	analysis of frames consisting of multiforce members
6.101, 102	6.103, 104	
6.*105*, *106*		
6.107, 108	6.109, 110	problems involving the solution of simultaneous equations
6.112, 113	6.*111*, *114*	
6.115	6.116	
6.117	6.118	unusual floor systems
6.*119*, *120*	6.121	rigid and non-rigid frames
6.*F6*, *F8*	6.*F5*, *F7*	Free Body Practice Problems – Machines
		Analysis of Machines
6.122, 125	6.123, 124	toggle-type machines
6.*126*, 127		
6.128		
6.131, 132	6.129, 130	machines involving cranks
6.133, 134		
6.137, 138	6.*135*, *136*	machines involving a crank with a collar
6.139, 140		robotic machines
6.143, 144	6.141, *142*	tongs
6.147, 148	6.145, 146	pliers, boltcutters, pruning shears
6.149, *150*	6.151, 152	wrench, find force to maintain position of toggle
6.155, *156*	6.153, 154	force in hydraulic cylinder
	6.*157*, *158*	
6.159, 160	*6.161, *162	gears and universal joints
*6.163		special tongs
6.164, 165	6.166, *167*	Review problems
6.*169*, 170	6.168, 171	
6.*173*, 174	6.172, *175*	
6.C2, C4	6.C1, C3	Computer problems
6.C6	6.C5	

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	U.S. Units	Problem Description
		CHAPTER 7: FORCES IN BEAMS AND CABLES
		Internal forces in members
7.3, 4	7.1, 2	simple frames
7.*5*, *6*		
7.9, 10	7.7, 8	curved members
7.11, 12	7.*13*, *14*	
7.15, 16	7.17, 18	frames with pulleys or pipes
7.19, 20		
	7.*21*, *22*	effect of supports
7.23, 24	7.25, 26	bending moment in circular rods due to their own weight
7.*27*, *28*		
		BEAMS
		Shear and bending-moment diagrams using portions of beam as free bodies
7.29, 30	7.31, 32	problems involving no numerical values
	7.*33*, *34*	
7.35, 36	7.*37*, *38*	beams with concentrated loads
7.39, 40	7.41, 42	beams with mixed loads
7.*43*, *44*	7.45, 46	beams resting on the ground
7.47, 48		
7.49, 50	7.51, 52	beams subjected to forces and couples
7.*53*, *54*		
7.55, 56	7.59, *60*	find value of parameter to minimize absolute value of bending moment
7.57, 58	7.*61*, *62	
		Shear and bending-moment diagrams using relations among ω, V, and M
7.63, 64	7.65, 66	problems involving no numerical values
7.*67*, *68*		
7.69, 70	7.73, 74	problems involving numerical values
7.71, 72	7.*75*, *76*	
7.77, 78	7.81, 82	find magnitude and location of maximum bending moment
7.79, 80	7.*83*, *84*	
7.*85*, *88*	7.86, 87	Determine V and M by integrating ω twice
7.89, 90	*7.91, *92	Find values of loads given bending moment at two points
		CABLES
		Cables with concentrated loads
7.93, 94	7.95, 96	vertical loads
7.97, 98	7.*99*, *100*	
7.101, 102	7.103, 104	horizontal and vertical loads
7.*105*, *106*		
		Parabolic cables
7.107, *108*	7.109, 110	supports at the same elevation
7.112, *113*	7.111, 114	
7.115, 116	7.117, 118	supports at different elevations

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	U.S. Units	Problem Description
*7.119		Derive analogy between a beam and a cable
7.120, 121	7._122_, _123_	use analogy to solve previous problems
*7.124	*7.125, *126	Derive or use $d^2y/dx^2 = w(x)\ T_0$
		Catenary
7.129, 130	7.127, 128	given length of cable and sag or T_m, find span of cable
7._131_, _132_		
7.133, 136	7.134, 135	given span and length of cable, find sag and/or weight
7.139, 140	7._137_, _138_	given span, T_m, and w, find sag
7._141_, _142_	7.143, 144	given T_0, w, and sag or slope, find span or sag
7.145, 146	*7.147, *148	
*7.151, *152	7._149_, *_150_	special problems
*7.153		
7.154, _155_	7.156, 158	Review problems
7.157, _159_	7._160_, 161	
7.163, 164	7._162_, 165	
7.C2, C4	7.C1, C3	Computer problems
7.C5, C6		

CHAPTER 8: FRICTION

SI Units	U.S. Units	Problem Description
8._F1_, _F4_	8._F2_, _F3_	Free Body Practice Problems – Frames
8.1, 2	8.3, 4	For given loading, determine whether block is in equilibrium and find friction force
8.7, 9	8.5, _6_	Find minimum force required to start, maintain, or prevent motion
8.10	8._8_	
8.13, 14	8.11, 12	Analyze motion of several blocks
8._15_, _16_	8.17, 18	Sliding and/or tipping of a rigid body
8.21, 22	8.19, 20	Problems involving wheels and cylinders
8.25, _26_	8.23, _24_	Problems involving rods
8.27, _28_	8.29, 30	Analysis of mechanisms with friction
8.32, _33_	8._31_, 34	
	8.35	
8.37, 39	8.36, _38_	Analysis of more advanced rod and beam problems
8.40	8.41, _42_	
8.43, 44	8.46, 47	Analysis of systems with possibility of slippage for various loadings
8.45		
8.48, 49	8.52, 53	Wedges, introductory problems
8._50_, _51_	8.57	
8.54, 55		
8.56		
8.59, 64	8._58_, 60	Wedges, more advanced problems
8.65	8._61_, 62	
*8.66, *67	8.63	

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	U.S. Units	Problem Description
8.68	8._69_, _70_	Square-threaded screws
8.72, 73	8.71, 74	
8._75_, _76_		
8.77, 79	8.78, _83_	Axle friction
8.80, 81	8.84, _87_	
8.82, _85_	8.88, 89	
8.86, _91_	8.90	
8.92, *94	8.93, _97_	Disk friction
*8._95_, *_96_		
8.100, 101	8.98, 99	Rolling resistance
8.102		
		Belt friction
8.105, 106	8.103, 104	belt passing over fixed drum
8._107_, _108_		
8.109, 112	8.110, 111	transmission belts and band brakes
8.113, 114	8._115_, _116_	
	8.117	
8.118, 119	8.120, 121	advanced problems
8.124, 125	8._122_, _123_	
8.126, _127_	8.128, 129	
8._132_, 133	8.130, 131	derivations, V belts
8.134, _135_	8.136, 137	Review problems
8.138, _139_	8.140, 141	
8.143, 144	8._142_, _145_	
8.C1, C3	8.C2, C5	Computer problems
8.C4, C6	8.C7	
8.C8		

CHAPTER 9: DISTRIBUTED FORCES: MOMENTS OF INERTIA

MOMENTS OF INERTIA OF AREAS

		Find by direct integration
9.1, 3	9.2, 4	moments of inertia of an area
9.6, 8	9._5_, _7_	
9.9, 12	9.10, 11	
9._13_, _14_		
9.15, 17	9.16, 18	moments of inertia and radii of gyration of an area
9._19_, _20_		
9.21, 22	9.23, _24_	polar moments of inertia and polar radii of gyration of an area
9.25, 26	9._27_, 28	
*9.29	*9.30	Special problems
		Parallel-axis theorem applied to composite areas to find
9.31, 33	9.32, 34	moment of inertia and radius of gyration
9._35_, _36_		

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	U.S. Units	Problem Description
9.39, 40	9.37, *38*	centroidal moment of inertia, given I or J
9.41, 42	9.43, 44	centroidal moments of inertia
9.47, 48	9.*45*, *46*	centroidal polar moment of inertia
		centroidal moments of inertia of composite areas consisting of rolled-steel shapes:
9.49, 52	9.50, 51	doubly-symmetrical composite areas
9.54, 55	9.*53*, *56*	singly-symmetrical composite areas (first locate centroid of area)
		Center of pressure
9.58, *60	9.57, 59	for end panel of a trough
9.*61*, *62*		for a submerged vertical gate or cover
*9.64	*9.63	used to locate centroid of a volume
*9.65	*9.66	special problems
		Products of inertia of areas found by
9.67, 68	9.69, *70*	direct integration
9.71, 72	9.*73*, 74	parallel-axis theorem
9.75, 78	9.76, *77*	
		Using the equations defining the moments and products of inertia with respect to rotated axes to find
9.79, 80	9.81, 83	$I_{x'}$, $I_{y'}$, $I_{x'y'}$ for a given angle of rotation
9.*82*, *84*		
9.85, 86	9.87, 89	principal axes and principal moments of inertia
9.*88*, *90*		
		Using Mohr's circle to find
9.91, 92	9.93, 95	$I_{x'}$, $I_{y'}$, $I_{x'y'}$ for a given angle of rotation
9.*94*, *96*		
9.97, 98	9.99, 100	principal axes and principal moments of inertia
	9.*101*, *102*	
9.104, 105	9.103, *106	
9.107, 108	9.*109*, *110*	Special problems
		MOMENTS OF INERTIA OF MASSES
		Mass moment of inertia
9.111, 112	9.113, 114	of thin plates: two-dimensions
9.117, 118	9.*115*, *116*	
9.119, 120	9.121, 122	of simple geometric shapes by direct single integration
9.*123*, *125*	9.124, *126	
9.128	9.127	and radius of gyration of composite bodies
9.*129*, 130	9.131, *134*	special problems using the parallel-axis theorem
9.132, 133		
9.135, 136	9.138, 139	of bodies formed of sheet metal or of thin plates: three-dimensions
9.*137*, *140		
9.141, 142	9.143, *144*	of machine elements and of bodies formed of homogeneous wire
9.145, 148	9.*146*, 147	
		Mass products of inertia
9.149, 150	9.151, 152	of machine elements
9.*153*, *154*		of bodies formed of sheet metal or of thin plates
9.155, 156		
9.157, 158	9.159, 160	of bodies formed of homogeneous wire
9.162	9.161	Derivation and special problem

* Problems which do not involve any specific system of units have been indicated by underlining their number. Answers are not given to problems with a number set in italic type.

TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number* SI Units	 U.S. Units	 Problem Description
9.165, 166	9.*163*, *164*	Mass moment of inertia of a body with respect to a skew axis
9.169, 170	9.167, 168	
	9.*171*, *172*	
9.175, 176	9.173, 174	Ellipsoid of inertia and special problems
9.*177*, *178*		
*9.179, *180	*9.182, *183	Principal mass moments of inertia and principal axes of inertia
*9._181_, _*184_		
9.*187*, 188	9.185, 186	Review problems
9.189, 193	9.*190*, 191	
9.*194*, 195	9.*192*, 196	
9.C3, C5	9.C1, C2	Computer problems
*9.C7, *C8	9.C4, C6	

CHAPTER 10: METHOD OF VIRTUAL WORK

		For linkages and simple machines, find
10.1, 3	10.2, 4	force or couple required for equilibrium (linear relations among displacements)
10.5, 6	10.*7*, *8*	
10.*11*, 12	10.9, 10	force required for equilibrium (trigonometric relations among displacements)
10.*13*, 14		
10.17, 18	10.15, 16	couple required for equilibrium (trigonometric relations among displacements)
10.*21*, *22*	10.19, 20	force or couple required for equilibrium
		Find position of equilibrium
10.23, 24	10.*25*, *26*	for numerical values of loads
10.27, 28		
10.*29*, 32	10.30, 31	linear springs included in mechanism
10.33, *34*	10.36, 38	
10.35, 37		
10.39, 40		torsional spring included in mechanism
10.43, 44	10.*41*, *42*	Problems requiring the use of the law of cosines
	10.45, 46	
10.48, 50	10.*47*, 49	Problems involving the effect of friction
10.*51*, 52		
		Use method of virtual work to find:
10.53, 54		reactions of a beam
10.*55*	10.*56*	internal forces in a mechanism
	10.57, 58	movement of a truss joint
		Potential energy method used to
10.62, *63*	10.*59*, 60	solve problems from Sec. 10.4
10.64, 65	10.61, 66	
10.67	10.68	establish that equilibrium is neutral
		find position of equilibrium and determine its stability for a system involving:
10.69, 72	10.70, 71	gears and drums
10.73, 74		torsional springs
10.77, 78	10.*75*, *76*	linear springs
10.*79*, *82*	10.80, 81	
10.*85*, 86	10.87, 88	

* Problems which do not involve any specific system of units have been indicated by underlining their number.
Answers are not given to problems with a number set in italic type.

TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	U.S. Units	Problem Description
10.83, *84*		two applied forces
		determine stability of a known position of equilibrium for a system with
10.89, 93	10.*90*, 91	one degree of freedom
10.94, 96	10.92, *95*	
*10.*97*, *98	*10.*99*, *100	two degrees of freedom
10.103, *104*	10.101, 102	Review problems
10.105, 107	10.*106*, 108	
10.*111*, 112	10.*109*, 110	
10.C2, C3	10.C1, C4	Computer problems
10.C5, C6		
10.C7		

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* Problems which do not involve any specific system of units have been indicated by underlining their number.
Answers are not given to problems with a number set in italic type.

TABLE III: SAMPLE ASSIGNMENT SCHEDULE FOR A COURSE IN STATICS

This schedule includes all of the material of *VECTOR MECHANICS FOR ENGINEERS: STATICS* with the exception of Sections 3.4D, 6.1D, 7.4–7.5, 8.3, and 9.6C.

			50% OF THE PROBLEMS IN EACH OF THE FOLLOWING LISTS USE SI UNITS AND 50% U.S. CUSTOMARY UNITS					
			ANSWERS TO ALL OF THESE PROBLEMS ARE GIVEN IN THE BACK OF THE BOOK				NO ANSWERS ARE GIVEN IN THE BOOK FOR ANY OF THESE PROBLEMS	
Period	Sections	Topics	List 1	List 2	List 3	List 4	List 5	List 6
1	1.1–6	Introduction						
2	2.1–2	Addition and Resolution of Forces	2.5, 15, 27, 31	2.10, 17, 29, 37	2.11, 16, 26, 36	2.6, 19, 28, 35	2.12, 18, 30, 33	2.7, 20, 25, 38
3	2.3	Equilibrium of a Particle	2.43, 50, 63, 65	2.48, 49, 59, 69	2.46, 55, 61, 67	2.44, 51, 62, 68	2.45, 56, 64, 66	2.47, 52, 60, 70
4	2.4	Forces in Space	2.71, 81, 85, 97	2.72, 82, 89, 94	2.75, 84, 88, 92	2.77, 79, 87, 91	2.76, 80, 86, 93	2.78, 83, 90, 98
5	2.5	Equilibrium in Space	2.103, 107, 119, 125	2.104, 108, 113, 121	2.101, 106, 116, 123	2.99, 112, 115, 124	2.102, 105, 120, 126	2.100, 111, 114, 122
6	3.1	Vector Product, Moment of a Force about a Point	3.4, 12, 23, 27	3.5, 11, 26, 30	3.7, 13, 25, 28	3.6, 9, 22, 29	3.3, 14, 21, 31	3.8, 10, 24, 34
7	3.2	Scalar Product, Moment of a Force about an Axis	3.43, 48, 61, *68	3.39, 47, 55, *67	3.41, 53, 57, *64	3.37, 49, 58, *65	3.42, 50, 56, *69	3.38, 54, 62, *66
8	3.3	Couples	3.71, 84, 87, 99	3.70, 82, 90, 97	3.73, 86, 89, 96	3.80, 83, 91, 95	3.75, 85, 88, 100	3.72, 81, 92, 94
9	3.4	Equivalent Systems of Forces	3.106, 111, 126, 127	3.105, 110, 122, 128	3.102, 108, 120, 129	3.101, 114, 119, 130	3.107, 112, 121, 131	3.103, 109, 123, 132
10		EXAM NUMBER ONE						
11	4.1	Equilibrium in Two Dimensions	4.1, 14, 19, 23	4.2, 12, 15, 24	4.4, 13, 17, 25	4.7, 9, 18, 22	4.3, 10, 16, 27	4.8, 11, 20, 21
12	4.1	Equilibrium in Two Dimensions	4.34, 36, 43, 52	4.33, 35, 48, 51	4.31, 42, 46, 57	4.32, 41, 47, 55	4.30, 37, 44, 58	4.29, 38, 49, 56
13	4.2	Two- and Three-Force Bodies	4.63, 72, 75, 86	4.62, 71, 81, 90	4.66, 73, 77, 88	4.61, 69, 79, 83	4.68, 74, 76, 87	4.64, 70, 82, 89
14	4.3	Equilibrium in Three Dimensions	4.93, 100, 110, 117	4.94, 99, 105, 113	4.91, 103, 107, 116	4.95, 97, 106, 115	4.92, 104, 111, 118	4.96, 98, 112, 114
15	4.3	Equilibrium in Three Dimensions	4.122, 127, 133	4.119, 128, 138	4.121, 131, 135	4.120, 129, 140	4.126, 132, 134	4.125, 130, 139
16	5.1	Centroids and First Moments	5.1, 10, 24, 33	5.3, 13, 20, 26	5.5, 14, 23, 30	5.2, 11, 21, 31	5.8, 15, 25, 28	5.7, 12, 22, 27
17	5.2	Centroids by Integration	5.34, 43, 58, 64	5.35, 40, 53, 60	5.37, 44, 54, 63	5.39, 41, 52, 62	5.38, 42, 59, *65	5.36, *47, 56, 61
18	5.3	Beams and Submerged Surfaces	5.66, 76, 81, 94	5.67, 77, 82, 90	5.70, 74, 80, 89	5.69, 78, 84, 88	5.72, 75, 83, 95	5.68, 79, 86, 91
19	5.4	Centroids of Volumes	5.102, 107, 119, 122	5.103, 106, 117, 125	5.100, 111, 114, 124	5.104, 109, 116, 123	5.101, 112, 115, 126	5.105, 108, 120, 127
20		EXAM NUMBER TWO						
21	6.1	Trusses: Method of Joints	6.2, 11, 21, 30	6.4, 12, 24, 27	6.1, 13, 23, 29	6.3, 9, 22, 28	6.7, 14, 25, 33	6.5, 10, 26, 31
22	6.2	Trusses: Method of Sections	6.45, 54, 68, 69	6.46, 53, 67, 70	6.43, 59, 62, 71	6.44, 60, 61, 72	6.47, 57, 64, 73	6.48, 58, 63, 74
23	6.3	Analysis of Frames	6.75, 88, 102, 109	6.77, 89, 101, 110	6.76, 83, 100, 107	6.81, 85, 99, 108	6.78, 84, 105, 111	6.82, 86, 106, 114
24	6.4	Analysis of Machines	6.123, 134, 145, 149	6.124, 133, 141, 155	6.127, 129, 144, 153	6.122, 130, 143, 154	6.126, 135, 142, 150	6.128, 136, 146, 156
25	6.1–4	Review of Chapter 6	6.15, 52, 91, 151	6.19, 51, 92, 152	6.17, 50, 94, 140	6.18, 49, 93, 137	6.16, 48, 98, 157	6.20, 47, 97, 158
26	7.1	Internal Forces in Members	7.1, 12, 18, 23	7.2, 11, 17, 24	7.3, 8, 19, 26	7.4, 7, 20, 25	7.5, 14, 22, 27	7.6, 13, 21, 28
27	7.2	Beams	7.36, 46, 50, 59	7.35, 45, 49, 59	7.42, 48, 52, 55	7.41, 47, 51, 58	7.38, 43, 54, 60	7.37, 44, 53, 61
28	7.3	Beams	7.65, 70, 81, *89	7.66, 69, 82, *90	7.64, 73, 78, 86	7.63, 74, 77, 87	7.68, 75, 83, 85	7.67, 76, 84, 88
29		EXAM NUMBER THREE						
30	8.1	Laws of Friction	8.1, 18, 32, 36	8.2, 17, 27, 41	8.3, 14, 30, 40	8.4, 13, 29, 39	8.8, 16, 33, 38	8.6, 15, 28, 42
31	8.2	Wedges and Screws	8.52, 54, 63, 73	8.53, 55, 62, 72	8.48, 57, 65, 74	8.49, 60, 64, 71	8.50, 58, 70, 76	8.51, 61, 69, 75
32	8.4	Belt Friction	8.104, 109, 121, 126	8.103, 112, 120, 133	8.105, 110, 119, 128	8.106, 111, 118, 129	8.108, 115, 123, 127	8.107, 116, 122, 132
33	9.1	Moments of Inertia of Areas	9.1, 10, 17, 23	9.3, 11, 15, 28	9.4, 12, 16, 21	9.2, 9, 18, 22	9.5, 14, 19, 24	9.7, 13, 20, 27
34	9.2	Composite Areas	9.34, 42, 51, 58	9.32, 41, 50, *60	9.31, 43, 52, 57	9.33, 44, 49, 59	9.35, 45, 56, 61	9.36, 46, 53, 62
35	9.3	Product of Inertia	9.72, 83, 86	9.71, 81, 85	9.76, 79, 87	9.74, 80, 89	9.77, 84, 88	9.73, 82, 90
36	9.4	Mohr’s Circle	9.93, 97, 103	9.95, 98, *106	9.92, 100, 107	9.91, 99, 108	9.94, 102, 110	9.96, 101, 109
37	9.5	Moments of Inertia of Masses	9.112, 121, 135, 147	9.111, 122, 136, 143	9.113, 119, 138, 142	9.114, 120, 139, 141	9.116, 123, *140, 146	9.115, 125, 137, 144
38	9.6	Principal Axes	9.152, 158, 167, 175	9.151, 155, 168, 176	9.149, 159, 165, 174	9.150, 160, 166, 173	9.154, 163, 171, 178	9.153, 164, 172, 177
39		EXAM NUMBER FOUR						
40	10.1	Virtual Work	10.1, 9, 18, 20	10.3, 10, 17, 19	10.2, 12, 16, 24	10.4, 14, 15, 23	10.8, 13, 21, 25	10.7, 11, 22, 26
41	10.1	Virtual Work	10.31, 35, 46, 53	10.30, 37, 45, 54	10.32, 36, 43, 57	10.33, 38, 44, 58	10.34, 42, 47, 55	10.29, 41, 51, 56
42	10.2	Potential Energy	10.69, 80, 86, 91	10.72, 81, 83, 92	10.71, 77, 88, 89	10.70, 78, 87, 93	10.76, 79, 84, 95	10.75, 82, 85, 90

TABLE IV: SAMPLE ASSIGNMENT SCHEDULE FOR A COURSE IN STATICS

This schedule includes all of the material of *VECTOR MECHANICS FOR ENGINEERS: STATICS* with the exception of Sections 3.4D, 6.1D, 7.4–7.5, 8.3, and 9.6C.

75% OF THE PROBLEMS IN EACH OF THE FOLLOWING LISTS USE SI UNITS AND 25% U.S. CUSTOMARY UNITS

ANSWERS TO ALL OF THESE PROBLEMS ARE GIVEN IN THE BACK OF THE BOOK

Period	Sections	Topics	List 1a	List 2a	List 3a	List 4a
1	1.1–6	Introduction				
2	2.1–2	Addition and Resolution of Forces	2.5, 15, 27, 34	2.10, 17, 29, 32	2.8, 16, 26, 36	2.9, 19, 28, 35
3	2.3	Equilibrium of a Particle	2.43, 50, 57, 65	2.48, 49, 58, 69	2.46, 54, 61, 67	2.44, 53, 62, 68
4	2.4	Forces in Space	2.71, 81, 85, 95	2.72, 82, 89, 96	2.74, 84, 88, 92	2.73, 79, 87, 91
5	2.5	Equilibrium in Space	2.103, 107, 118, 125	2.104, 108, 117, 121	2.101, 109, 116, 123	2.99, 110, 115, 124
6	3.1	Vector Product, Moment of a Force about a Point	3.4, 12, 23, 32	3.5, 11, 26, 33	3.1, 13, 25, 28	3.2, 9, 22, 29
7	3.2	Scalar Product, Moment of a Force about an Axis	3.43, 48, 59, *68	3.39, 47, 60, *67	3.41, 52, 57, *64	3.37, 51, 58, *65
8	3.3	Couples	3.71, 84, 87, 98	3.70, 82, 90, 93	3.74, 86, 89, 96	3.78, 83, 91, 95
9	3.4	Equivalent Systems of Forces	3.106, 111, 125, 127	3.105, 110, 124, 128	3.102, 116, 120, 129	3.101, 117, 119, 130
10		EXAM NUMBER ONE				
11	4.1	Equilibrium in Two Dimensions	4.1, 14, 19, 28	4.2, 12, 15, 26	4.5, 13, 17, 25	4.6, 9, 18, 22
12	4.1	Equilibrium in Two Dimensions	4.34, 36, 45, 52	4.33, 35, 50, 51	4.31, 40, 46, 57	4.32, 39, 47, 55
13	4.2	Two- and Three-Force Bodies	4.63, 72, 75, 84	4.62, 71, 81, 85	4.67, 73, 77, 88	4.65, 69, 79, 83
14	4.3	Equilibrium in Three Dimensions	4.93, 100, 108, 117	4.94, 99, 109, 113	4.91, 102, 107, 116	4.95, 101, 106, 115
15	4.3	Equilibrium in Three Dimensions	4.122, 127, 133	4.119, 128, 138	4.124, 131, 135	4.123, 129, 140
16	5.1	Centroids and First Moments	5.1, 10, 24, 29	5.3, 13, 20, 32	5.6, 14, 23, 30	5.9, 11, 21, 31
17	5.2	Centroids by Integration	5.34, 43, 57, 64	5.35, 40, 55, 60	5.37, 46, 54, 63	5.39, 45, 52, 62
18	5.3	Beams and Submerged Surfaces	5.66, 76, 81, 93	5.67, 77, 82, 92	5.73, 74, 80, 89	5.71, 78, 84, 88
19	5.4	Centroids of Volumes	5.102, 107, 118, 122	5.103, 106, 121, 125	5.100, 113, 114, 124	5.104, 110, 116, 123
20		EXAM NUMBER TWO				
21	6.1	Trusses: Method of Joints	6.2, 11, 21, 32	6.4, 12, 24, 34	6.6, 13, 23, 29	6.8, 9, 22, 28
22	6.2	Trusses: Method of Sections	6.45, 54, 66, 69	6.46, 53, 65, 70	6.43, 56, 62, 71	6.44, 55, 61, 72
23	6.3	Analysis of Frames	6.75, 88, 102, 113	6.77, 89, 101, 112	6.79, 83, 100, 107	6.80, 85, 99, 108
24	6.4	Analysis of Machines	6.123, 134, 147, 149	6.124, 133, 148, 155	6.127, 132, 144, 153	6.122, 131, 143, 154
25	6.1–4	Review of Chapter 6	6.15, 52, 91, 125	6.19, 51, 92, 139	6.50, 94, 140, 164	6.49, 93, 137, 165
26	7.1	Internal Forces in Members	7.1, 12, 15, 23	7.2, 11, 16, 24	7.3, 10, 19, 26	7.4, 9, 20, 25
27	7.2	Beams	7.36, 46, 50, 57	7.35, 45, 49, 56	7.40, 48, 52, 55	7.39, 47, 51, 58
28	7.3	Beams	7.65, 70, 80, *89	7.66, 69, 79, *90	7.64, 71, 78, 86	7.63, 72, 77, 87
29		EXAM NUMBER THREE				
30	8.1	Laws of Friction	8.1, 18, 32, 37	8.2, 17, 27, 43	8.7, 14, 30, 40	8.9, 13, 29, 39
31	8.2	Wedges and Screws	8.52, 54, *67, 73	8.53, 55, *66, 72	8.48, 56, 65, 74	8.49, 59, 64, 71
32	8.4	Belt Friction	8.104, 109, 125, 126	8.103, 112, 124, 133	8.105, 113, 119, 128	8.106, 114, 118, 129
33	9.1	Moments of Inertia of Areas	9.1, 10, 17, 26	9.3, 11, 15, 25	9.8, 12, 16, 21	9.6, 9, 18, 22
34	9.2	Composite Areas	9.34, 42, 55, 58	9.32, 41, 54, *60	9.31, 47, 52, 57	9.33, 48, 49, 59
35	9.3	Product of Inertia	9.72, 83, 86	9.71, 81, 85	9.75, 79, 87	9.78, 80, 89
36	9.4	Mohr’s Circle	9.93, 97, 103	9.95, 98, *106	9.92, 104, 107	9.91, 105, 108
37	9.5	Moments of Inertia of Masses	9.112, 121, 135, 148	9.111, 122, 136, 145	9.117, 119, 138, 142	9.118, 120, 139, 141
38	9.6	Principal Axes	9.152, 158, 170, 175	9.151, 155, 169, 176	9.149, 157, 165, 174	9.150, 156, 166, 173
39		EXAM NUMBER FOUR				
40	10.1	Virtual Work	10.1, 9, 18, 28	10.3, 10, 17, 27	10.5, 12, 16, 24	10.6, 14, 15, 23
41	10.1	Virtual Work	10.31, 35, 52, 53	10.30, 37, 48, 54	10.32, 40, 43, 57	10.33, 39, 44, 58
42	10.2	Potential Energy	10.69, 80, 86, 96	10.72, 81, 83, 94	10.74, 77, 88, 89	10.73, 78, 87, 93

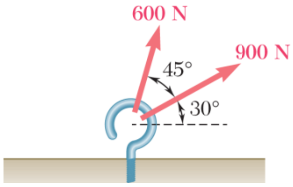
TABLE V: SAMPLE ASSIGNMENT SCHEDULE FOR A COMBINED COURSE IN STATICS AND DYNAMICS

This schedule is intended for a 4-semester-credit-hour course consisting of (1) the abridged Statics course defined in Table I of this manual and (2) the standard Dynamics course defined in Table I of the Dynamics Instructor's Manual. Note that the Statics portion of the course is designed to serve only as an introduction to Dynamics; a more complete coverage of Statics is recommended for curricula including courses in Mechanics of Materials (see Tables III and ff. of the Statics Manual).

50% OF THE PROBLEMS IN EACH OF THE FOLLOWING LISTS USE SI UNITS AND 50% U.S. CUSTOMARY UNITS

			ANSWERS TO ALL OF THESE PROBLEMS ARE GIVEN IN THE BACK OF THE BOOK				NO ANSWERS ARE GIVEN IN THE BOOK FOR ANY OF THESE PROBLEMS	
Period	Sections	Topics	List 1	List 2	List 3	List 4	List 5	List 6
1	1.1–6	Introduction						
2	2.1–2	Addition and Resolution of Forces	2.5, 15, 27, 31	2.10, 17, 29, 37	2.11, 16, 26, 36	2.6, 19, 28, 35	2.12, 18, 30, 33	2.7, 20, 25, 38
3	2.3	Equilibrium of a Particle	2.43, 50, 63, 65	2.48, 49, 59, 69	2.46, 55, 61, 67	2.44, 51, 62, 68	2.45, 56, 64, 66	2.47, 52, 60, 70
4	2.4	Forces in Space	2.71, 81, 85, 97	2.72, 82, 89, 94	2.75, 84, 88, 92	2.77, 79, 87, 91	2.76, 80, 86, 93	2.78, 83, 90, 98
5	2.5	Equilibrium in Space	2.103, 107, 119, 125	2.104, 108, 113, 121	2.101, 106, 116, 123	2.99, 112, 115, 124	2.102, 105, 120, 126	2.100, 111, 114, 122
6	3.1	Vector Product, Moment of a Force about a Point	3.4, 12, 23, 27	3.5, 11, 26, 30	3.7, 13, 25, 28	3.6, 9, 22, 29	3.3, 14, 21, 31	3.8, 10, 24, 34
7	3.2	Scalar Product, Moment of a Force about an Axis	3.43, 48, 61, *68	3.39, 47, 55, *67	3.41, 53, 57, *64	3.37, 49, 58, *65	3.42, 50, 56, *69	3.38, 54, 62, *66
8	3.3	Couples	3.71, 84, 87, 99	3.70, 82, 90, 97	3.73, 86, 89, 96	3.80, 83, 91, 95	3.75, 85, 88, 100	3.72, 81, 92, 94
9	3.4	Equivalent Systems of Forces	3.106, 111, 126, 127	3.105, 110, 122, 128	3.102, 108, 120, 129	3.101, 114, 119, 130	3.107, 112, 121, 131	3.103, 109, 123, 132
10		EXAM NUMBER ONE						
11	4.1	Equilibrium in Two Dimensions	4.1, 14, 19, 23	4.2, 12, 15, 24	4.4, 13, 17, 25	4.7, 9, 18, 22	4.3, 10, 16, 27	4.8, 11, 20, 21
12	4.1	Equilibrium in Two Dimensions	4.34, 36, 43, 52	4.33, 35, 48, 51	4.31, 42, 46, 57	4.32, 41, 47, 55	4.30, 37, 44, 58	4.29, 38, 49, 56
13	6.3	Analysis of Frames	6.75, 88, 102, 109	6.77, 89, 101, 110	6.76, 83, 100, 107	6.81, 85, 99, 108	6.78, 84, 105, 111	6.82, 86, 106, 114
14	6.4	Analysis of Machines	6.123, 134, 145, 149	6.124, 133, 141, 155	6.127, 129, 144, 153	6.122, 130, 143, 154	6.126, 135, 142, 150	6.128, 136, 146, 156
15	4.3	Equilibrium in Three Dimensions	4.93, 100, 110, 117	4.94, 99, 105, 113	4.91, 103, 107, 116	4.95, 97, 106, 115	4.92, 104, 111, 118	4.96, 98, 112, 114
16	4.3	Equilibrium in Three Dimensions	4.122, 127, 133	4.119, 128, 138	4.121, 131, 135	4.120, 129, 140	4.126, 132, 134	4.125, 130, 139
17	8.1	Laws of Friction	8.1, 18, 32, 36	8.2, 17, 27, 41	8.3, 14, 30, 40	8.4, 13, 29, 39	8.8, 16, 33, 38	8.6, 15, 28, 42
18	8.2	Wedges and Screws	8.52, 54, 63, 73	8.53, 55, 62, 72	8.48, 57, 65, 74	8.49, 60, 64, 71	8.50, 58, 70, 76	8.51, 61, 69, 75
19		EXAM NUMBER TWO						
20	11.1	Rectilinear Motion						
21	11.2	Uniformly Accelerated Motion						
22	11.4	Curvilinear Motion (Rect. Comps)						
23	11.5	Curvilinear Motion (Other Comp.)						
24	12.1	Equations of Motion						
25	12.2	Angular Momentum						
26		Review						
27		EXAM NUMBER THREE						
28	13.1	Work and Energy, Power						
29	13.2	Conservation of Energy						
30	13.2D	Application to Space Mechanics						
31	13.3	Impulse and Momentum						
32	13.4	Impact						
33	14.1	Systems of Particles						
34	14.2	Systems of Particles						
35		EXAM NUMBER FOUR						
36	15.1	Translation, Rotation						
37	15.2	General Plane Motion						
38	15.3	Instantaneous Center						
39	15.4	Acceleration in Plane Motion						
40	15.5	Coriolis Acceleration in Plane Motion						
41	9.5	Moments of Inertia of Masses	9.112, 121, 135, 147	9.111, 122, 136, 143	9.113, 119, 138, 142	9.114, 120, 139, 141	9.116, 123, *140, 146	9.115, 125, 137, 144
42		Review						
43		EXAM NUMBER FIVE						
44	16.1	Plane Motion of Rigid Bodies						
45	16.1	Plane Motion of Rigid Bodies						
46	16.2	Constrained Plane Motion						
47	16.2	Constrained Plane Motion						
48	17.1	Work and Energy						
49	17.2	Impulse and Momentum						
50	17.3	Eccentric Impact						
51		Review						
52		EXAM NUMBER SIX						
53	19.1	Free Vibrations of Particles						
54	19.2	Free Vibrations of Rigid Bodies						
55	19.3	Energy Methods						
56	19.4	Forced Vibrations						

CHAPTER 2

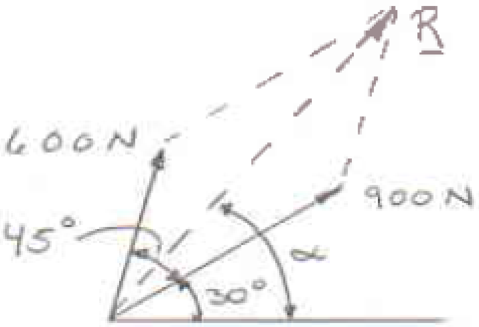

600 N
900 N
45°
30°

PROBLEM 2.1

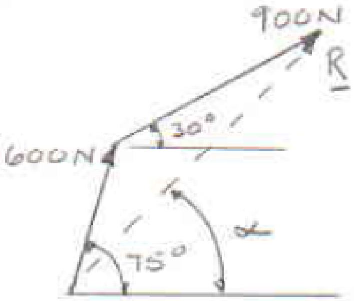
Two forces are applied as shown to a hook. Determine graphically the magnitude and direction of their resultant using (*a*) the parallelogram law, (*b*) the triangle rule.

SOLUTION

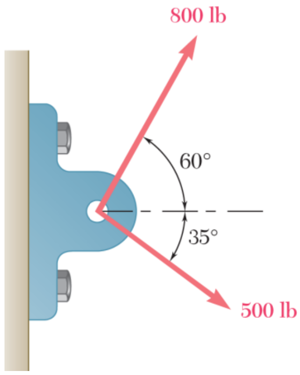
(*a*) Parallelogram law:


R
600 N
900 N
45°
30°
α

(*b*) Triangle rule:


900 N
R
600 N
30°
75°
α

We measure: $R = 1391 \text{ kN}, \quad \alpha = 47.8°$ $\mathbf{R} = 1391 \text{ N} \measuredangle 47.8°$ ◀

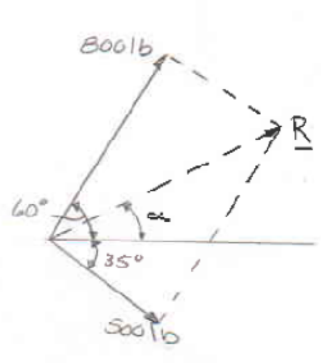

800 lb
60°
35°
500 lb

PROBLEM 2.2

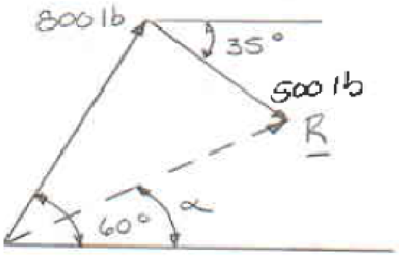
Two forces are applied as shown to a bracket support. Determine graphically the magnitude and direction of their resultant using (*a*) the parallelogram law, (*b*) the triangle rule.

SOLUTION

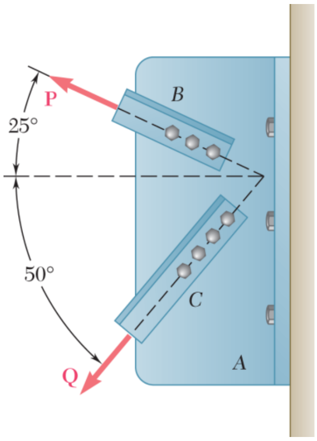
(*a*) Parallelogram law:


800lb
R
60°
α
35°
500 lb

(*b*) Triangle rule:


800 lb
35°
500 lb
R
60°
α

We measure: $R = 906$ lb, $\alpha = 26.6°$ $R = 906$ lb ∠ 26.6° ◀

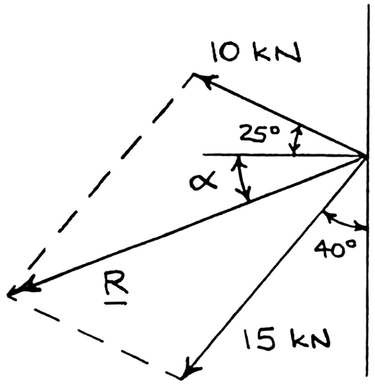


PROBLEM 2.3

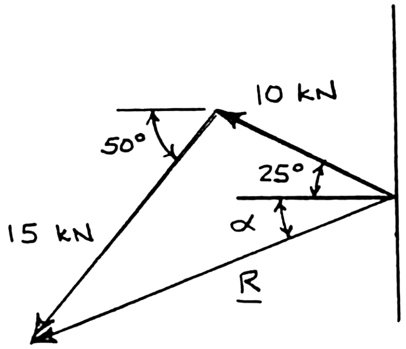
Two structural members *B* and *C* are bolted to bracket *A*. Knowing that both members are in tension and that $P = 10$ kN and $Q = 15$ kN, determine graphically the magnitude and direction of the resultant force exerted on the bracket using (*a*) the parallelogram law, (*b*) the triangle rule.

SOLUTION

(*a*) Parallelogram law:

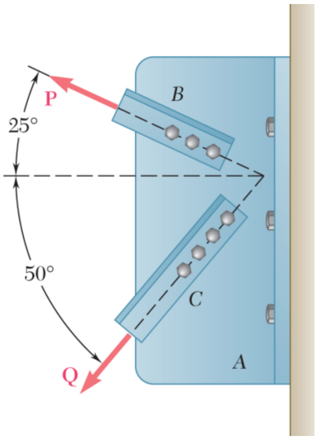

10 kN
25°
α
40°
R
15 kN

(*b*) Triangle rule:


10 kN
50°
25°
α
15 kN
R

We measure: $R = 20.1$ kN, $\alpha = 21.2°$

R = 20.1 kN ∡ 21.2° ◀

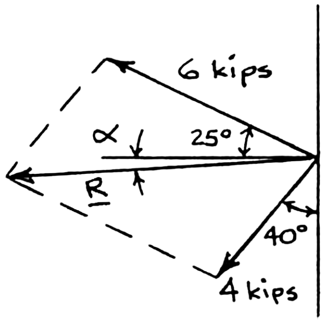


PROBLEM 2.4

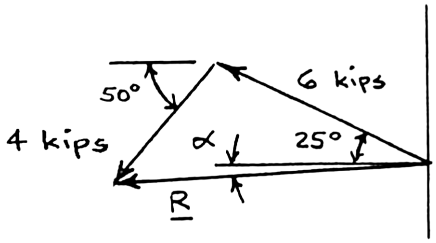
Two structural members *B* and *C* are bolted to bracket *A*. Knowing that both members are in tension and that $P = 6$ kips and $Q = 4$ kips, determine graphically the magnitude and direction of the resultant force exerted on the bracket using (*a*) the parallelogram law, (*b*) the triangle rule.

SOLUTION

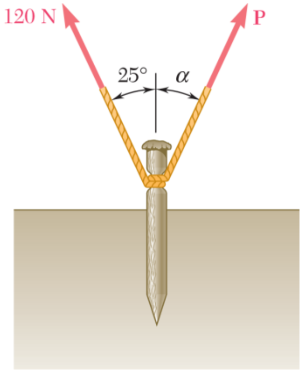
(*a*) Parallelogram law:


6 kips
α
25°
R
40°
4 kips

(*b*) Triangle rule:


50°
6 kips
4 kips
α
25°
R

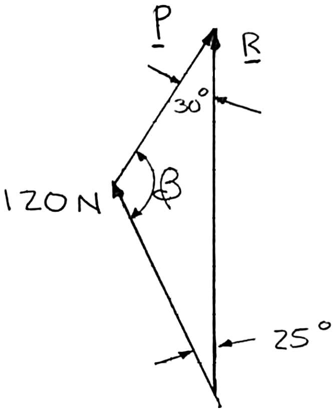
We measure: $R = 8.03$ kips, $\alpha = 3.8°$ $\mathbf{R} = 8.03$ kips ⦣ 3.8° ◀



PROBLEM 2.5

A stake is being pulled out of the ground by means of two ropes as shown. Knowing that $\alpha = 30°$, determine by trigonometry (*a*) the magnitude of the force **P** so that the resultant force exerted on the stake is vertical, (*b*) the corresponding magnitude of the resultant.

SOLUTION



Using the triangle rule and the law of sines:

(*a*)

$$\frac{120\text{ N}}{\sin 30°} = \frac{P}{\sin 25°}$$

$P = 101.4\text{ N}$ ◀

(*b*)

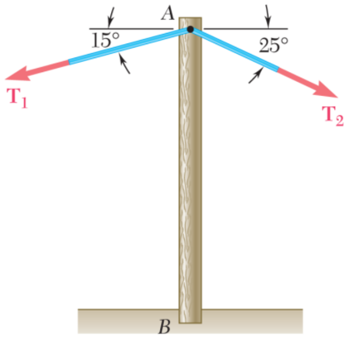
$$30° + \beta + 25° = 180°$$

$$\beta = 180° - 25° - 30°$$

$$= 125°$$

$$\frac{120\text{ N}}{\sin 30°} = \frac{R}{\sin 125°}$$

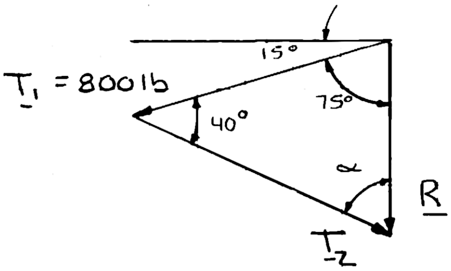
$R = 196.6\text{ N}$ ◀



PROBLEM 2.6

A telephone cable is clamped at *A* to the pole *AB*. Knowing that the tension in the left-hand portion of the cable is $T_1 = 800$ lb, determine by trigonometry (*a*) the required tension T_2 in the right-hand portion if the resultant **R** of the forces exerted by the cable at *A* is to be vertical, (*b*) the corresponding magnitude of **R**.

SOLUTION



Using the triangle rule and the law of sines:

(*a*) $$75° + 40° + \alpha = 180°$$

$$\alpha = 180° - 75° - 40°$$

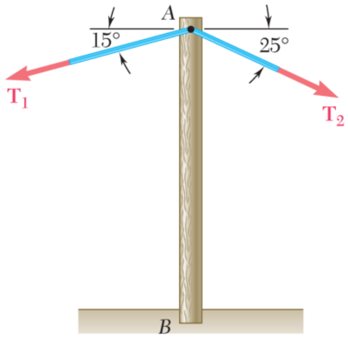
$$= 65°$$

$$\frac{800\text{ lb}}{\sin 65°} = \frac{T_2}{\sin 75°}$$

$T_2 = 853$ lb ◀

(*b*) $$\frac{800\text{ lb}}{\sin 65°} = \frac{R}{\sin 40°}$$

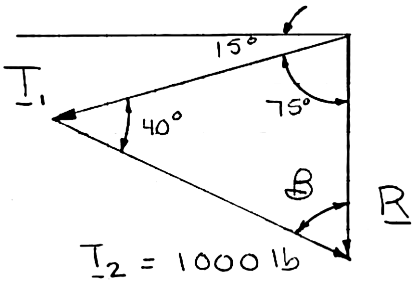
$R = 567$ lb ◀



PROBLEM 2.7

A telephone cable is clamped at A to the pole AB. Knowing that the tension in the right-hand portion of the cable is $T_2 = 1000$ lb, determine by trigonometry (*a*) the required tension T_1 in the left-hand portion if the resultant **R** of the forces exerted by the cable at A is to be vertical, (*b*) the corresponding magnitude of **R**.

SOLUTION



Using the triangle rule and the law of sines:

(*a*)

$$75° + 40° + \beta = 180°$$

$$\beta = 180° - 75° - 40° = 65°$$

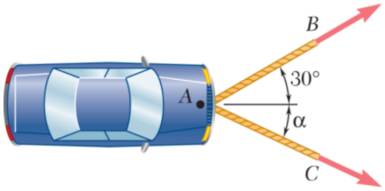
$$\frac{1000 \text{ lb}}{\sin 75°} = \frac{T_1}{\sin 65°}$$

$T_1 = 938$ lb ◀

(*b*)

$$\frac{1000 \text{ lb}}{\sin 75°} = \frac{R}{\sin 40°}$$

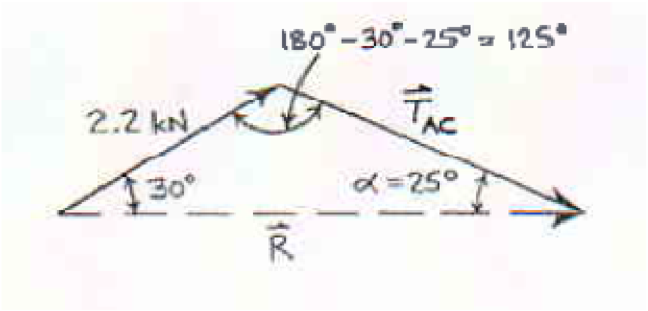
$R = 665$ lb ◀



PROBLEM 2.8

A disabled automobile is pulled by means of two ropes as shown. The tension in rope *AB* is 2.2 kN, and the angle *α* is 25°. Knowing that the resultant of the two forces applied at *A* is directed along the axis of the automobile, determine by trigonometry (*a*) the tension in rope *AC*, (*b*) the magnitude of the resultant of the two forces applied at *A*.

SOLUTION



Using the law of sines:

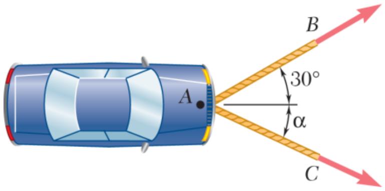
$$\frac{T_{AC}}{\sin 30^\circ} = \frac{R}{\sin 125^\circ} = \frac{2.2 \text{ kN}}{\sin 25^\circ}$$

$$T_{AC} = 2.603 \text{ kN}$$

$$R = 4.264 \text{ kN}$$

(*a*) $T_{AC} = 2.60 \text{ kN}$ ◀

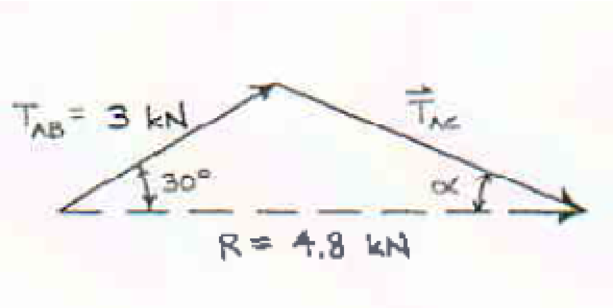
(*b*) $R = 4.26 \text{ kN}$ ◀



PROBLEM 2.9

A disabled automobile is pulled by means of two ropes as shown. Knowing that the tension in rope *AB* is 3 kN, determine by trigonometry the tension in rope *AC* and the value of α so that the resultant force exerted at *A* is a 4.8-kN force directed along the axis of the automobile.

SOLUTION



Using the law of cosines:

$$T_{AC}^2 = (3\text{ kN})^2 + (4.8\text{ kN})^2 - 2(3\text{ kN})(4.8\text{ kN})\cos 30°$$

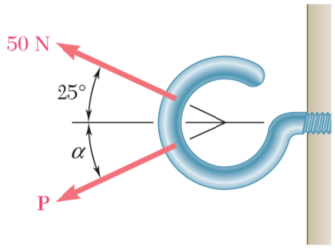
$$T_{AC} = 2.6643\text{ kN}$$

Using the law of sines:

$$\frac{\sin\alpha}{3\text{ kN}} = \frac{\sin 30°}{2.6643\text{ kN}}$$

$$\alpha = 34.3°$$

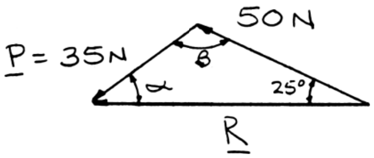
$$\mathbf{T}_{AC} = 2.66\text{ kN} \searrow 34.3° \blacktriangleleft$$



PROBLEM 2.10

Two forces are applied as shown to a hook support. Knowing that the magnitude of **P** is 35 N, determine by trigonometry (*a*) the required angle α if the resultant **R** of the two forces applied to the support is to be horizontal, (*b*) the corresponding magnitude of **R**.

SOLUTION



Using the triangle rule and law of sines:

(*a*)

$$\frac{\sin\alpha}{50\text{ N}} = \frac{\sin 25^\circ}{35\text{ N}}$$

$$\sin\alpha = 0.60374$$

$$\alpha = 37.138^\circ$$

$\alpha = 37.1^\circ$ ◀

(*b*)

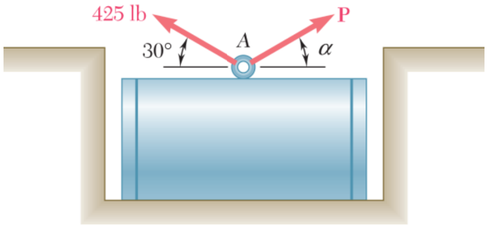
$$\alpha + \beta + 25^\circ = 180^\circ$$

$$\beta = 180^\circ - 25^\circ - 37.138^\circ$$

$$= 117.862^\circ$$

$$\frac{R}{\sin 117.862^\circ} = \frac{35\text{ N}}{\sin 25^\circ}$$

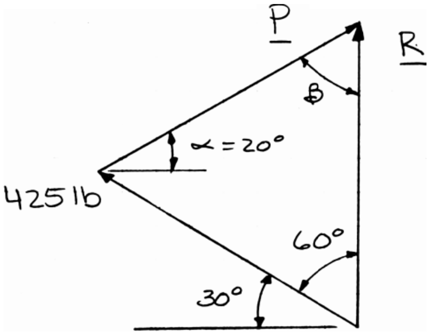
$R = 73.2$ N ◀



PROBLEM 2.11

A steel tank is to be positioned in an excavation. Knowing that $\alpha = 20°$, determine by trigonometry (*a*) the required magnitude of the force **P** if the resultant **R** of the two forces applied at *A* is to be vertical, (*b*) the corresponding magnitude of **R**.

SOLUTION

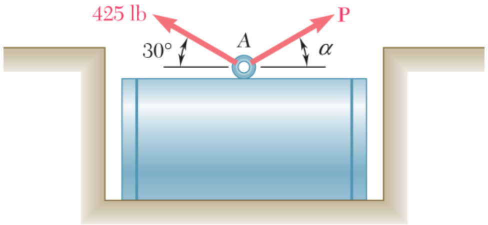


Using the triangle rule and the law of sines:

(*a*)
$$\beta + 50° + 60° = 180°$$
$$\beta = 180° - 50° - 60°$$
$$= 70°$$

$$\frac{425 \text{ lb}}{\sin 70°} = \frac{P}{\sin 60°} \qquad P = 392 \text{ lb} \blacktriangleleft$$

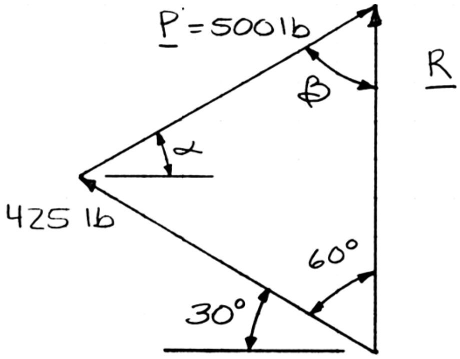
(*b*)
$$\frac{425 \text{ lb}}{\sin 70°} = \frac{R}{\sin 50°} \qquad R = 346 \text{ lb} \blacktriangleleft$$



PROBLEM 2.12

A steel tank is to be positioned in an excavation. Knowing that the magnitude of **P** is 500 lb, determine by trigonometry (*a*) the required angle α if the resultant **R** of the two forces applied at *A* is to be vertical, (*b*) the corresponding magnitude of **R**.

SOLUTION



Using the triangle rule and the law of sines:

(*a*)

$$(\alpha + 30°) + 60° + \beta = 180°$$

$$\beta = 180° - (\alpha + 30°) - 60°$$

$$\beta = 90° - \alpha$$

$$\frac{\sin(90° - \alpha)}{425 \text{ lb}} = \frac{\sin 60°}{500 \text{ lb}}$$

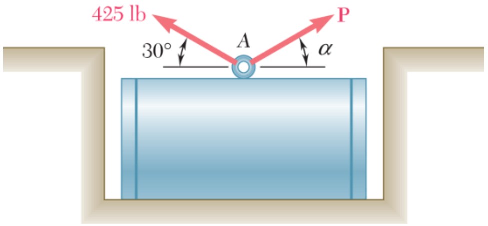
$$90° - \alpha = 47.402°$$

$\alpha = 42.6°$ ◀

(*b*)

$$\frac{R}{\sin(42.598° + 30°)} = \frac{500 \text{ lb}}{\sin 60°}$$

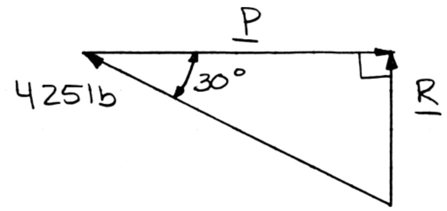
$R = 551$ lb ◀



PROBLEM 2.13

A steel tank is to be positioned in an excavation. Determine by trigonometry (*a*) the magnitude and direction of the smallest force **P** for which the resultant **R** of the two forces applied at *A* is vertical, (*b*) the corresponding magnitude of **R**.

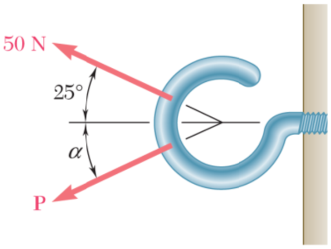
SOLUTION



The smallest force *P* will be perpendicular to *R*.

(*a*) $P = (425 \text{ lb})\cos 30°$ $\mathbf{P} = 368 \text{ lb}$ →

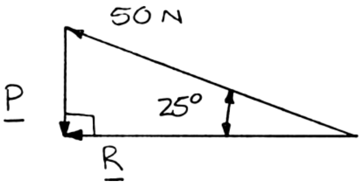
(*b*) $R = (425 \text{ lb})\sin 30°$ $R = 213 \text{ lb}$



PROBLEM 2.14

For the hook support of Prob. 2.10, determine by trigonometry (*a*) the magnitude and direction of the smallest force **P** for which the resultant **R** of the two forces applied to the support is horizontal, (*b*) the corresponding magnitude of **R**.

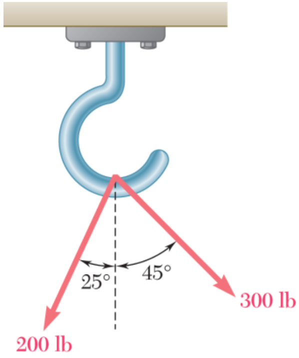
SOLUTION



The smallest force *P* will be perpendicular to *R*.

(*a*) $P = (50\text{ N})\sin 25°$ $\mathbf{P} = 21.1\text{ N}\downarrow$ ◀

(*b*) $R = (50\text{ N})\cos 25°$ $R = 45.3\text{ N}$ ◀

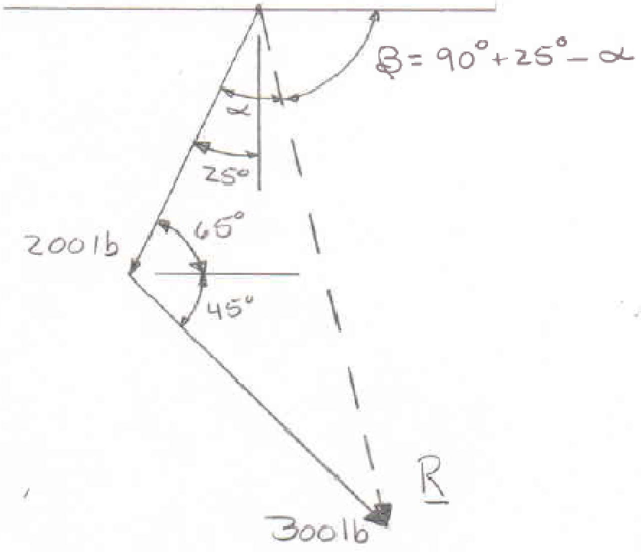


25°
45°
300 lb
200 lb

PROBLEM 2.15

For the hook support shown, determine by trigonometry the magnitude and direction of the resultant of the two forces applied to the support.

SOLUTION



ß = 90°+25°− α
α
25°
65°
200 lb
45°
R
300 lb

Using the law of cosines:

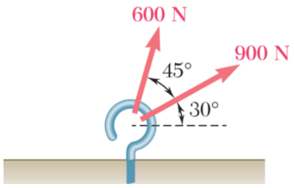
$$R^2 = (200 \text{ lb})^2 + (300 \text{ lb})^2$$
$$-2(200 \text{ lb})(300 \text{ lb})\cos(45^\circ + 65^\circ)$$
$$R = 413.57 \text{ lb}$$

Using the law of sines:

$$\frac{\sin\alpha}{300 \text{ lb}} = \frac{\sin(45^\circ + 65^\circ)}{413.57 \text{ lb}}$$
$$\alpha = 42.972^\circ$$

$$\beta = 90^\circ + 25^\circ - 42.972^\circ$$

R = 414 lb ⟍ 72.0° ◀



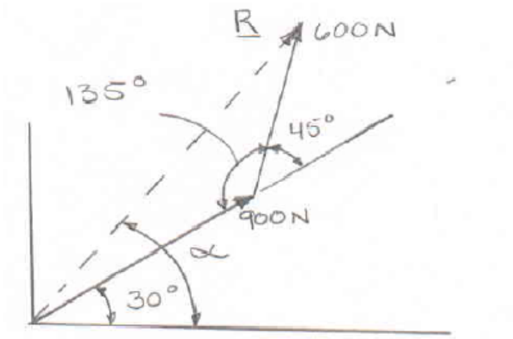
PROBLEM 2.16

Solve Prob. 2.1 by trigonometry.

PROBLEM 2.1

Two forces are applied as shown to a hook. Determine graphically the magnitude and direction of their resultant using (*a*) the parallelogram law, (*b*) the triangle rule.

SOLUTION



Using the law of cosines:

$$R^2 = (900\,\text{N})^2 + (600\,\text{N})^2 - 2(900\,\text{N})(600\,\text{N})\cos(135°)$$

$$R = 1390.57\text{N}$$

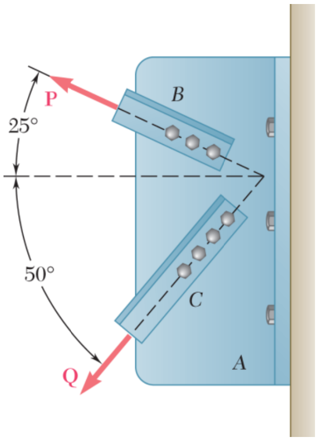
Using the law of sines:

$$\frac{\sin(\alpha - 30^\circ)}{600\text{N}} = \frac{\sin(135°)}{1390.57\text{N}}$$

$$\alpha - 30^\circ = 17.7642°$$

$$\alpha = 47.764^\circ$$

$\mathbf{R} = 1391\text{N}$ ∠ 47.8° ◀



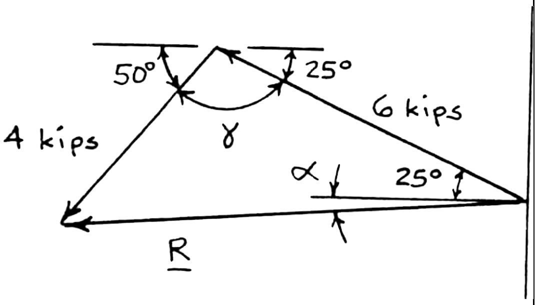
PROBLEM 2.17

Solve Problem 2.4 by trigonometry.

PROBLEM 2.4 Two structural members *B* and *C* are bolted to bracket *A*. Knowing that both members are in tension and that $P = 6$ kips and $Q = 4$ kips, determine graphically the magnitude and direction of the resultant force exerted on the bracket using (*a*) the parallelogram law, (*b*) the triangle rule.

SOLUTION

Using the force triangle and the laws of cosines and sines:



We have: $\gamma = 180° - (50° + 25°)$

$= 105°$

Then $R^2 = (4\text{ kips})^2 + (6\text{ kips})^2 - 2(4\text{ kips})(6\text{ kips})\cos 105°$

$= 64.423\text{ kips}^2$

$R = 8.0264\text{ kips}$

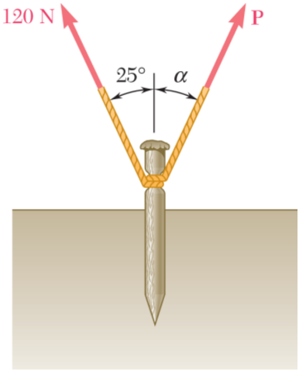
And $$\frac{4\text{ kips}}{\sin(25° + \alpha)} = \frac{8.0264\text{ kips}}{\sin 105°}$$

$\sin(25° + \alpha) = 0.48137$

$25° + \alpha = 28.775°$

$\alpha = 3.775°$

$\mathbf{R} = 8.03\text{ kips}$ ⦨ 3.8° ◀

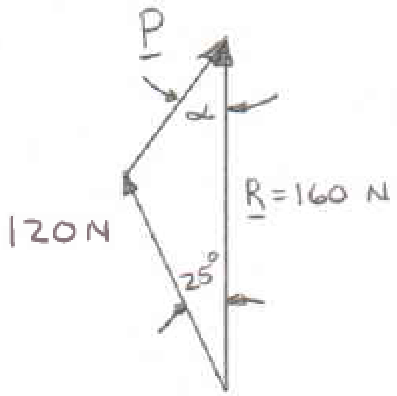


PROBLEM 2.18

For the stake of Prob. 2.5, knowing that the tension in one rope is 120 N, determine by trigonometry the magnitude and direction of the force **P** so that the resultant is a vertical force of 160 N.

PROBLEM 2.5 A stake is being pulled out of the ground by means of two ropes as shown. Knowing that $\alpha = 30°$, determine by trigonometry (*a*) the magnitude of the force **P** so that the resultant force exerted on the stake is vertical, (*b*) the corresponding magnitude of the resultant.

SOLUTION

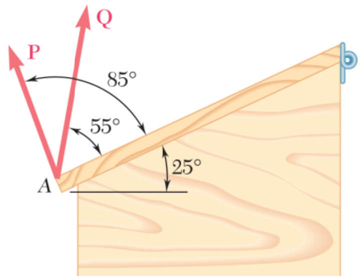


Using the laws of cosines and sines:

$$P^2 = (120\text{ N})^2 + (160\text{ N})^2 - 2(120\text{ N})(160\text{ N})\cos 25°$$
$$P = 72.096\text{ N}$$

And
$$\frac{\sin\alpha}{120\text{ N}} = \frac{\sin 25°}{72.096\text{ N}}$$
$$\sin\alpha = 0.70343$$
$$\alpha = 44.703°$$

$\mathbf{P} = 72.1\text{ N} \angle 44.7°$ ◀

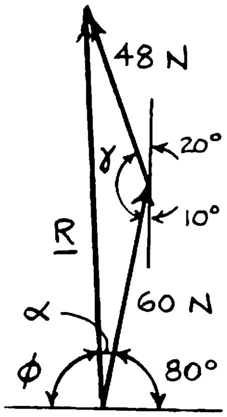


PROBLEM 2.19

Two forces **P** and **Q** are applied to the lid of a storage bin as shown. Knowing that P = 48 N and Q = 60 N, determine by trigonometry the magnitude and direction of the resultant of the two forces.

SOLUTION

Using the force triangle and the laws of cosines and sines:



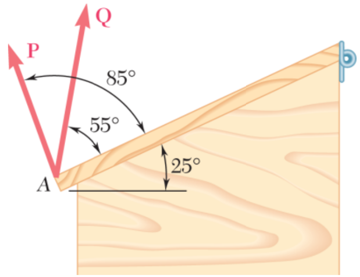
We have
$$\gamma = 180° - (20° + 10°)$$
$$= 150°$$

Then
$$R^2 = (48 \text{ N})^2 + (60 \text{ N})^2$$
$$-2(48 \text{ N})(60 \text{ N})\cos 150°$$
$$R = 104.366 \text{ N}$$

and
$$\frac{48 \text{ N}}{\sin\alpha} = \frac{104.366 \text{ N}}{\sin 150°}$$
$$\sin\alpha = 0.22996$$
$$\alpha = 13.2947°$$

Hence:
$$\phi = 180° - \alpha - 80°$$
$$= 180° - 13.2947° - 80°$$
$$= 86.705°$$

R = 104.4 N ⦣ 86.7° ◀

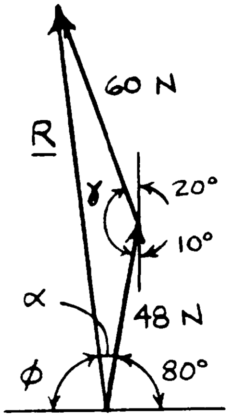


PROBLEM 2.20

Two forces **P** and **Q** are applied to the lid of a storage bin as shown. Knowing that $P = 60$ N and $Q = 48$ N, determine by trigonometry the magnitude and direction of the resultant of the two forces.

SOLUTION

Using the force triangle and the laws of cosines and sines:



We have
$$\gamma = 180° - (20° + 10°)$$
$$= 150°$$

Then
$$R^2 = (60\text{ N})^2 + (48\text{ N})^2 - 2(60\text{ N})(48\text{ N})\cos 150°$$
$$R = 104.366\text{ N}$$

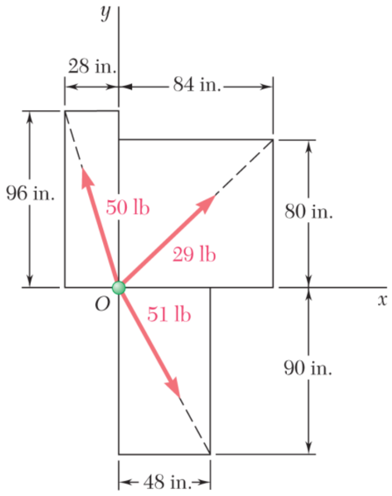
and
$$\frac{60\text{ N}}{\sin\alpha} = \frac{104.366\text{ N}}{\sin 150°}$$
$$\sin\alpha = 0.28745$$
$$\alpha = 16.7054°$$

Hence:
$$\phi = 180° - \alpha - 180°$$
$$= 180° - 16.7054° - 80°$$
$$= 83.295°$$

$\mathbf{R} = 104.4$ N ⦪ 83.3° ◀

PROBLEM 2.21

Determine the x and y components of each of the forces shown.


y
28 in.
84 in.
96 in.
50 lb
29 lb
80 in.
O
51 lb
x
90 in.
48 in.

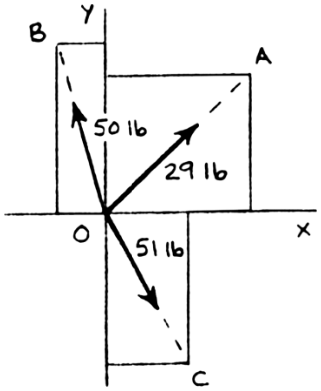
SOLUTION

Compute the following distances:

$$OA = \sqrt{(84)^2 + (80)^2}$$
$$= 116 \text{ in.}$$

$$OB = \sqrt{(28)^2 + (96)^2}$$
$$= 100 \text{ in.}$$

$$OC = \sqrt{(48)^2 + (90)^2}$$
$$= 102 \text{ in.}$$


B
Y
A
50 lb
29 lb
O
X
51 lb
C

29-lb Force: $F_x = +(29 \text{ lb})\dfrac{84}{116}$ $F_x = +21.0 \text{ lb}$ ◀

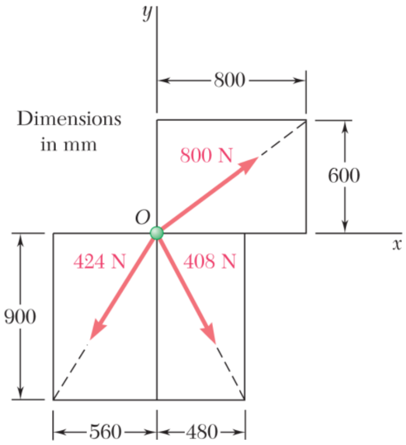
$F_y = +(29 \text{ lb})\dfrac{80}{116}$ $F_y = +20.0 \text{ lb}$ ◀

50-lb Force: $F_x = -(50 \text{ lb})\dfrac{28}{100}$ $F_x = -14.00 \text{ lb}$ ◀

$F_y = +(50 \text{ lb})\dfrac{96}{100}$ $F_y = +48.0 \text{ lb}$ ◀

51-lb Force: $F_x = +(51 \text{ lb})\dfrac{48}{102}$ $F_x = +24.0 \text{ lb}$ ◀

$F_y = -(51 \text{ lb})\dfrac{90}{102}$ $F_y = -45.0 \text{ lb}$ ◀

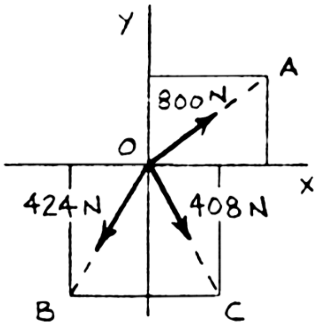

y
800
Dimensions in mm
800 N
600
O
x
424 N
408 N
900
560
480

PROBLEM 2.22

Determine the *x* and *y* components of each of the forces shown.

SOLUTION

Compute the following distances:


y
A
800 N
O
x
424 N
408 N
B
C

$$OA = \sqrt{(600)^2 + (800)^2}$$
$$= 1000 \text{ mm}$$

$$OB = \sqrt{(560)^2 + (900)^2}$$
$$= 1060 \text{ mm}$$

$$OC = \sqrt{(480)^2 + (900)^2}$$
$$= 1020 \text{ mm}$$

800-N Force:

$$F_x = +(800 \text{ N})\frac{800}{1000} \qquad F_x = +640 \text{ N} \blacktriangleleft$$

$$F_y = +(800 \text{ N})\frac{600}{1000} \qquad F_y = +480 \text{ N} \blacktriangleleft$$

424-N Force:

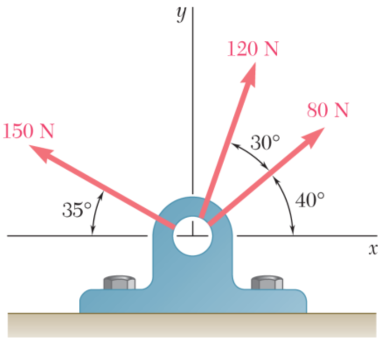
$$F_x = -(424 \text{ N})\frac{560}{1060} \qquad F_x = -224 \text{ N} \blacktriangleleft$$

$$F_y = -(424 \text{ N})\frac{900}{1060} \qquad F_y = -360 \text{ N} \blacktriangleleft$$

408-N Force:

$$F_x = +(408 \text{ N})\frac{480}{1020} \qquad F_x = +192.0 \text{ N} \blacktriangleleft$$

$$F_y = -(408 \text{ N})\frac{900}{1020} \qquad F_y = -360 \text{ N} \blacktriangleleft$$



PROBLEM 2.23

Determine the *x* and *y* components of each of the forces shown.

SOLUTION

80-N Force: $F_x = +(80\text{ N})\cos 40°$ $\quad F_x = 61.3\text{ N}$ ◀

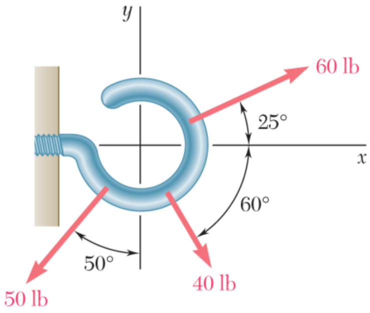
$F_y = +(80\text{ N})\sin 40°$ $\quad F_y = 51.4\text{ N}$ ◀

120-N Force: $F_x = +(120\text{ N})\cos 70°$ $\quad F_x = 41.0\text{ N}$ ◀

$F_y = +(120\text{ N})\sin 70°$ $\quad F_y = 112.8\text{ N}$ ◀

150-N Force: $F_x = -(150\text{ N})\cos 35°$ $\quad F_x = -122.9\text{ N}$ ◀

$F_y = +(150\text{ N})\sin 35°$ $\quad F_y = 86.0\text{ N}$ ◀



PROBLEM 2.24

Determine the x and y components of each of the forces shown.

SOLUTION

40-lb Force: $F_x = +(40\text{ lb})\cos 60°$ $\qquad F_x = 20.0\text{ lb}$ ◀

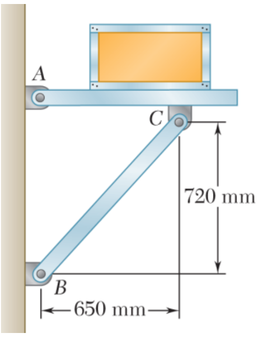
$F_y = -(40\text{ lb})\sin 60°$ $\qquad F_y = -34.6\text{ lb}$ ◀

50-lb Force: $F_x = -(50\text{ lb})\sin 50°$ $\qquad F_x = -38.3\text{ lb}$ ◀

$F_y = -(50\text{ lb})\cos 50°$ $\qquad F_y = -32.1\text{ lb}$ ◀

60-lb Force: $F_x = +(60\text{ lb})\cos 25°$ $\qquad F_x = 54.4\text{ lb}$ ◀

$F_y = +(60\text{ lb})\sin 25°$ $\qquad F_y = 25.4\text{ lb}$ ◀



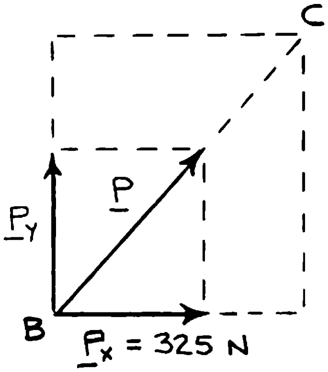
PROBLEM 2.25

Member *BC* exerts on member *AC* a force **P** directed along line *BC*. Knowing that **P** must have a 325-N horizontal component, determine (*a*) the magnitude of the force **P**, (*b*) its vertical component.

SOLUTION

$$BC = \sqrt{(650\text{ mm})^2 + (720\text{ mm})^2}$$

$$= 970\text{ mm}$$



(*a*)

$$P_x = P\left(\frac{650}{970}\right)$$

or

$$P = P_x\left(\frac{970}{650}\right)$$

$$= 325\text{ N}\left(\frac{970}{650}\right)$$

$$= 485\text{ N}$$

$P = 485$ N ◀

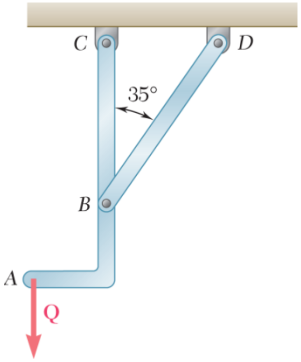
(*b*)

$$P_y = P\left(\frac{720}{970}\right)$$

$$= 485\text{ N}\left(\frac{720}{970}\right)$$

$$= 360\text{ N}$$

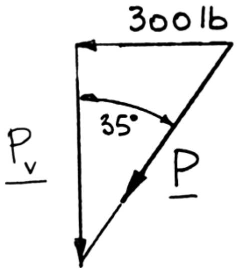
$P_y = 970$ N ◀



PROBLEM 2.26

Member *BD* exerts on member *ABC* a force **P** directed along line *BD*. Knowing that **P** must have a 300-lb horizontal component, determine (*a*) the magnitude of the force **P**, (*b*) its vertical component.

SOLUTION


300 lb
35°
$\underline{P}_v$
$\underline{P}$

(*a*) $P \sin 35° = 300 \text{ lb}$

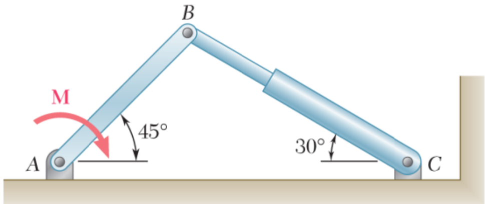
$$P = \frac{300 \text{ lb}}{\sin 35°}$$

$P = 523 \text{ lb}$ ◀

(*b*) Vertical component $P_v = P \cos 35°$

$= (523 \text{ lb}) \cos 35°$

$P_v = 428 \text{ lb}$ ◀



PROBLEM 2.27

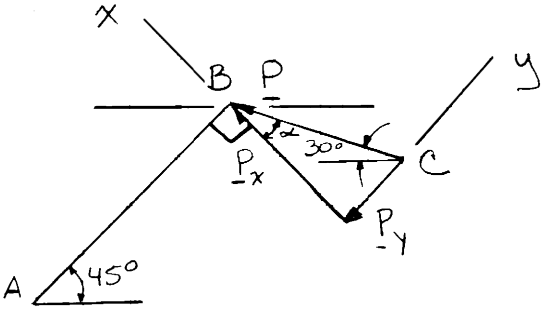
The hydraulic cylinder *BC* exerts on member *AB* a force **P** directed along line *BC*. Knowing that **P** must have a 600-N component perpendicular to member *AB*, determine (*a*) the magnitude of the force **P**, (*b*) its component along line *AB*.

SOLUTION

$$180° = 45° + \alpha + 90° + 30°$$

$$\alpha = 180° - 45° - 90° - 30°$$

$$= 15°$$



(*a*)

$$\cos\alpha = \frac{P_x}{P}$$

$$P = \frac{P_x}{\cos\alpha}$$

$$= \frac{600 \text{ N}}{\cos 15°}$$

$$= 621.17 \text{ N}$$

$$P = 621 \text{ N} \blacktriangleleft$$

(*b*)

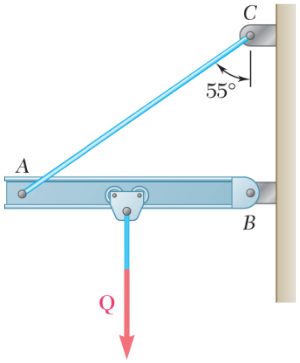
$$\tan\alpha = \frac{P_y}{P_x}$$

$$P_y = P_x \tan\alpha$$

$$= (600 \text{ N}) \tan 15°$$

$$= 160.770 \text{ N}$$

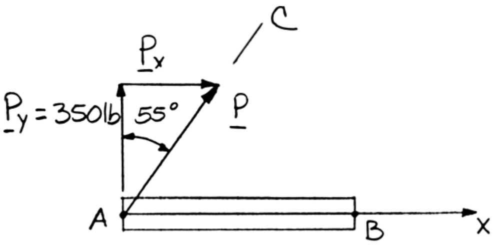
$$P_y = 160.8 \text{ N} \blacktriangleleft$$



PROBLEM 2.28

Cable *AC* exerts on beam *AB* a force **P** directed along line *AC*. Knowing that **P** must have a 350-lb vertical component, determine (*a*) the magnitude of the force **P**, (*b*) its horizontal component.

SOLUTION



(*a*)

$$P = \frac{P_y}{\cos 55°}$$

$$= \frac{350 \text{ lb}}{\cos 55°}$$

$$= 610.21 \text{ lb}$$

$P = 610$ lb ◀

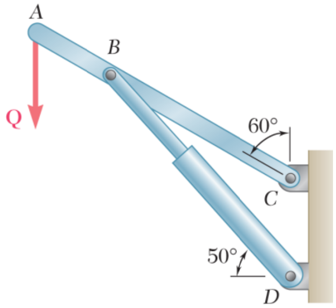
(*b*)

$$P_x = P \sin 55°$$

$$= (610.21 \text{ lb}) \sin 55°$$

$$= 499.85 \text{ lb}$$

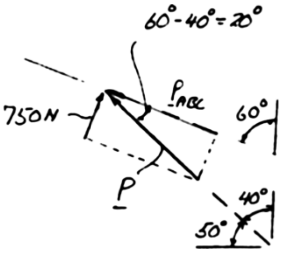
$P_x = 500$ lb ◀



PROBLEM 2.29

The hydraulic cylinder *BD* exerts on member *ABC* a force **P** directed along line *BD*. Knowing that **P** must have a 750-N component perpendicular to member *ABC*, determine (*a*) the magnitude of the force **P**, (*b*) its component parallel to *ABC*.

SOLUTION

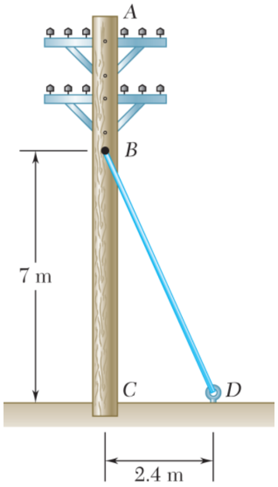


(*a*) $750 \text{ N} = P\sin 20°$

$P = 2192.9 \text{ N}$ $\qquad P = 2190 \text{ N}$ ◀

(*b*) $P_{ABC} = P\cos 20°$

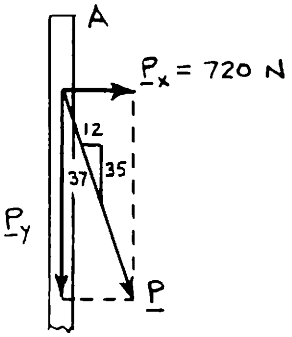
$= (2192.9 \text{ N})\cos 20°$ $\qquad P_{ABC} = 2060 \text{ N}$ ◀



PROBLEM 2.30

The guy wire *BD* exerts on the telephone pole *AC* a force **P** directed along *BD*. Knowing that **P** must have a 720-N component perpendicular to the pole *AC*, determine (*a*) the magnitude of the force **P**, (*b*) its component along line *AC*.

SOLUTION



A
$\underline{P}_x = 720$ N
12
37
35
$\underline{P}_y$
$\underline{P}$

(*a*)

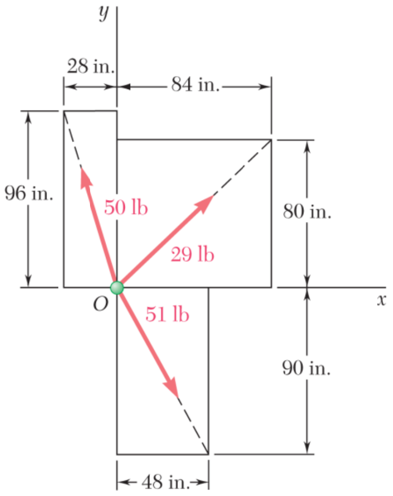
$$P = \frac{37}{12}P_x$$
$$= \frac{37}{12}(720\text{ N})$$
$$= 2220\text{ N}$$

$P = 2.22$ kN ◀

(*b*)

$$P_y = \frac{35}{12}P_x$$
$$= \frac{35}{12}(720\text{ N})$$
$$= 2100\text{ N}$$

$P_y = 2.10$ kN ◀


y
28 in.
84 in.
96 in.
50 lb
29 lb
80 in.
O
51 lb
x
90 in.
48 in.

PROBLEM 2.31

Determine the resultant of the three forces of Problem 2.21.

PROBLEM 2.21 Determine the *x* and *y* components of each of the forces shown.

SOLUTION

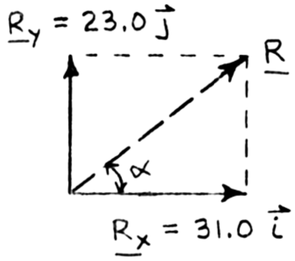
Components of the forces were determined in Problem 2.21:

Force	*x* Comp. (lb)	*y* Comp. (lb)
29 lb	+21.0	+20.0
50 lb	–14.00	+48.0
51 lb	+24.0	–45.0
	$R_x = +31.0$	$R_y = +23.0$

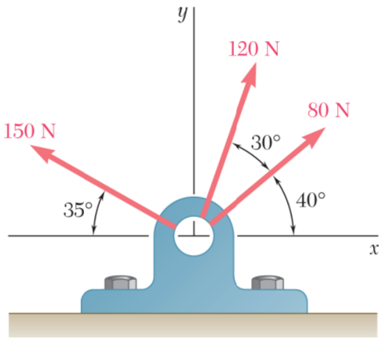
$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$$
$$= (31.0\text{ lb})\mathbf{i} + (23.0\text{ lb})\mathbf{j}$$

$$\tan\alpha = \frac{R_y}{R_x}$$
$$= \frac{23.0}{31.0}$$
$$\alpha = 36.573°$$

$$R = \frac{23.0\text{ lb}}{\sin(36.573°)}$$
$$= 38.601\text{ lb}$$


$\underline{R_y} = 23.0\,\vec{j}$
$\underline{R}$
α
$\underline{R_x} = 31.0\,\vec{i}$

R = 38.6 lb ⦣ 36.6° ◀


y
120 N
80 N
150 N
30°
40°
35°
x

PROBLEM 2.32

Determine the resultant of the three forces of Problem 2.23.

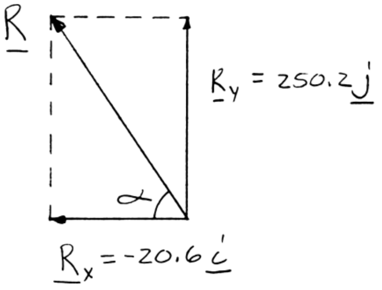
PROBLEM 2.23 Determine the *x* and *y* components of each of the forces shown.

SOLUTION

Components of the forces were determined in Problem 2.23:

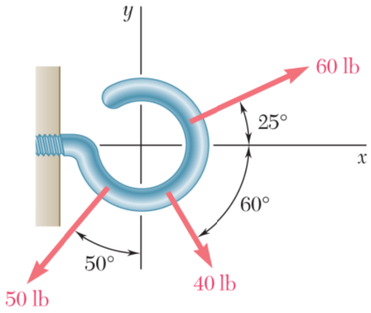
Force	*x* Comp. (N)	*y* Comp. (N)
80 N	+61.3	+51.4
120 N	+41.0	+112.8
150 N	−122.9	+86.0
	$R_x = -20.6$	$R_y = +250.2$

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$$
$$= (-20.6 \text{ N})\mathbf{i} + (250.2 \text{ N})\mathbf{j}$$


$\underline{R}$
$\underline{R}_y = 250.2\underline{j}$
α
$\underline{R}_x = -20.6\underline{i}$

$$\tan\alpha = \frac{R_y}{R_x}$$
$$\tan\alpha = \frac{250.2 \text{ N}}{20.6 \text{ N}}$$
$$\tan\alpha = 12.1456$$
$$\alpha = 85.293°$$
$$R = \frac{250.2 \text{ N}}{\sin 85.293°}$$

R = 251 N ⦣ 85.3° ◀


y
60 lb
25°
x
60°
50°
40 lb
50 lb

PROBLEM 2.33

Determine the resultant of the three forces of Problem 2.24.

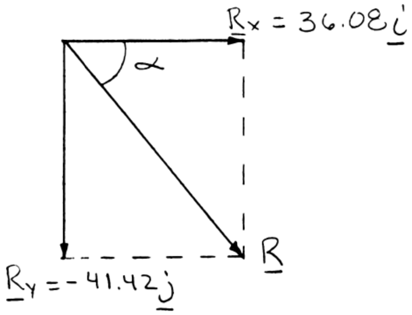
PROBLEM 2.24 Determine the x and y components of each of the forces shown.

SOLUTION

Force	x Comp. (lb)	y Comp. (lb)
40 lb	+20.00	–34.64
50 lb	–38.30	–32.14
60 lb	+54.38	+25.36
	$R_x = +36.08$	$R_y = -41.42$

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$$

$$= (+36.08 \text{ lb})\mathbf{i} + (-41.42 \text{ lb})\mathbf{j}$$


$\underline{R}_x = 36.08\underline{i}$
α
$\underline{R}$
$\underline{R}_y = -41.42\underline{j}$

$$\tan\alpha = \frac{R_y}{R_x}$$

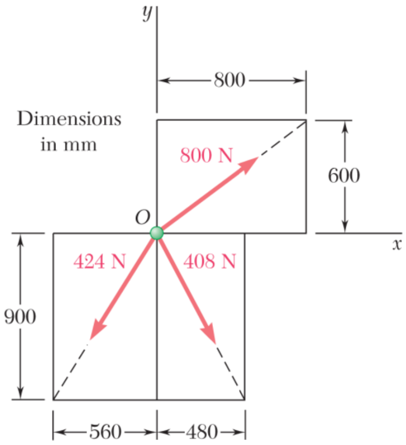
$$\tan\alpha = \frac{41.42 \text{ lb}}{36.08 \text{ lb}}$$

$$\tan\alpha = 1.14800$$

$$\alpha = 48.942°$$

$$R = \frac{41.42 \text{ lb}}{\sin 48.942°}$$

$\mathbf{R} = 54.9$ lb ⦪ 48.9° ◀


y
800
Dimensions in mm
800 N
600
O
x
424 N
408 N
900
560
480

PROBLEM 2.34

Determine the resultant of the three forces of Problem 2.22.

PROBLEM 2.22 Determine the *x* and *y* components of each of the forces shown.

SOLUTION

Components of the forces were determined in Problem 2.22:

Force	*x* Comp. (N)	*y* Comp. (N)
800 lb	+640	+480
424 lb	–224	–360
408 lb	+192	–360
	$R_x = +608$	$R_y = -240$

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$$

$$= (608 \text{ lb})\mathbf{i} + (-240 \text{ lb})\mathbf{j}$$

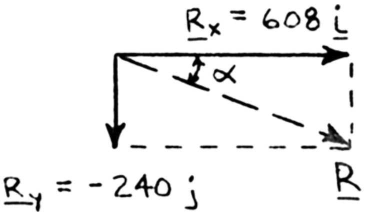
$$\tan\alpha = \frac{R_y}{R_x}$$

$$= \frac{240}{608}$$

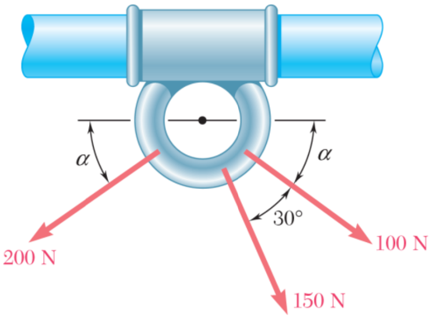
$$\alpha = 21.541°$$

$$R = \frac{240 \text{ N}}{\sin(21.541°)}$$

$$= 653.65 \text{ N}$$


Rx = 608 i
α
Ry = -240 j
R

R = 654 N ↘ 21.5° ◀


α
α
30°
200 N
100 N
150 N

PROBLEM 2.35

Knowing that $\alpha = 35°$, determine the resultant of the three forces shown.

SOLUTION

100-N Force:

$$F_x = +(100\text{ N})\cos 35° = +81.915\text{ N}$$
$$F_y = -(100\text{ N})\sin 35° = -57.358\text{ N}$$

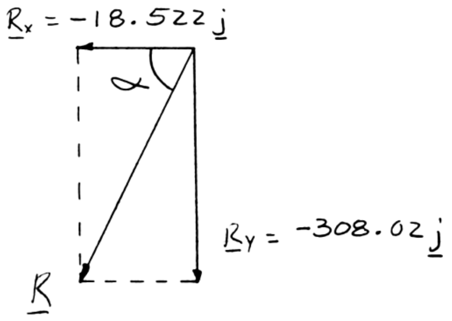
150-N Force:

$$F_x = +(150\text{ N})\cos 65° = +63.393\text{ N}$$
$$F_y = -(150\text{ N})\sin 65° = -135.946\text{ N}$$

200-N Force:

$$F_x = -(200\text{ N})\cos 35° = -163.830\text{ N}$$
$$F_y = -(200\text{ N})\sin 35° = -114.715\text{ N}$$

Force	x Comp. (N)	y Comp. (N)
100 N	+81.915	−57.358
150 N	+63.393	−135.946
200 N	−163.830	−114.715
	$R_x = -18.522$	$R_y = -308.02$

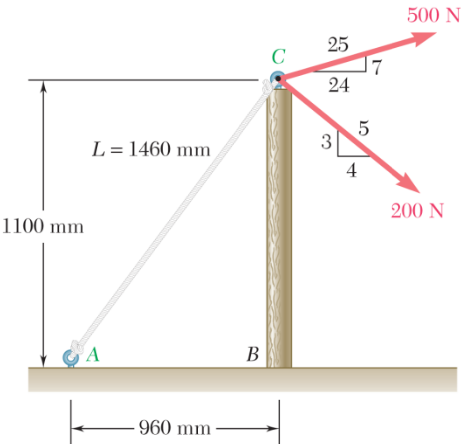

$\underline{R}_x = -18.522\underline{j}$
α
$\underline{R}_y = -308.02\underline{j}$
$\underline{R}$

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$$
$$= (-18.522\text{ N})\mathbf{i} + (-308.02\text{ N})\mathbf{j}$$

$$\tan\alpha = \frac{R_y}{R_x}$$
$$= \frac{308.02}{18.522}$$
$$\alpha = 86.559°$$

$$R = \frac{308.02\text{ N}}{\sin 86.559}$$

$\mathbf{R} = 309$ N ∠ 86.6° ◀

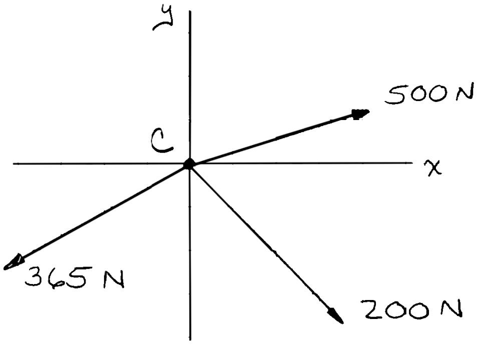

500 N
25
7
24
C
L = 1460 mm
3
5
4
200 N
1100 mm
A
B
960 mm

PROBLEM 2.36

Knowing that the tension in rope *AC* is 365 N, determine the resultant of the three forces exerted at point *C* of post *BC*.

SOLUTION

Determine force components:


y
500 N
C
x
365 N
200 N

Cable force *AC*:

$$F_x = -(365\text{ N})\frac{960}{1460} = -240\text{ N}$$

$$F_y = -(365\text{ N})\frac{1100}{1460} = -275\text{ N}$$

500-N Force:

$$F_x = (500\text{ N})\frac{24}{25} = 480\text{ N}$$

$$F_y = (500\text{ N})\frac{7}{25} = 140\text{ N}$$

200-N Force:

$$F_x = (200\text{ N})\frac{4}{5} = 160\text{ N}$$

$$F_y = -(200\text{ N})\frac{3}{5} = -120\text{ N}$$

and

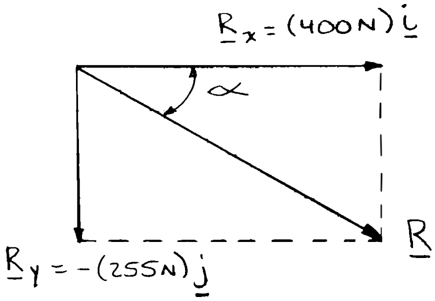
$$R_x = \Sigma F_x = -240\text{ N} + 480\text{ N} + 160\text{ N} = 400\text{ N}$$

$$R_y = \Sigma F_y = -275\text{ N} + 140\text{ N} - 120\text{ N} = -255\text{ N}$$

$$R = \sqrt{R_x^2 + R_y^2}$$

$$= \sqrt{(400\text{ N})^2 + (-255\text{ N})^2}$$

$$= 474.37\text{ N}$$

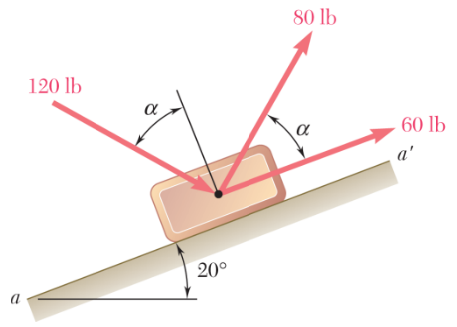

$R_x = (400\text{ N})\,\underline{i}$
α
R
$R_y = -(255\text{N})\,\underline{j}$

Further:

$$\tan\alpha = \frac{255}{400}$$

$$\alpha = 32.5°$$

$\mathbf{R} = 474$ N ⦪ 32.5° ◀


80 lb
120 lb
α
α
60 lb
a′
20°
a

PROBLEM 2.37

Knowing that $\alpha = 40°$, determine the resultant of the three forces shown.

SOLUTION

60-lb Force: $F_x = (60\text{ lb})\cos 20° = 56.382\text{ lb}$

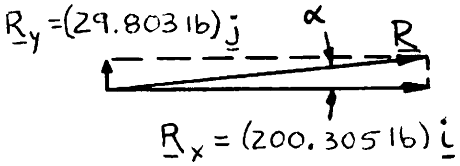
$F_y = (60\text{ lb})\sin 20° = 20.521\text{ lb}$

80-lb Force: $F_x = (80\text{ lb})\cos 60° = 40.000\text{ lb}$

$F_y = (80\text{ lb})\sin 60° = 69.282\text{ lb}$

120-lb Force: $F_x = (120\text{ lb})\cos 30° = 103.923\text{ lb}$

$F_y = -(120\text{ lb})\sin 30° = -60.000\text{ lb}$


$\underline{R}_y = (29.803\text{ lb})\underline{j}$
α
$\underline{R}$
$\underline{R}_x = (200.305\text{ lb})\underline{i}$

and $R_x = \Sigma F_x = 200.305\text{ lb}$

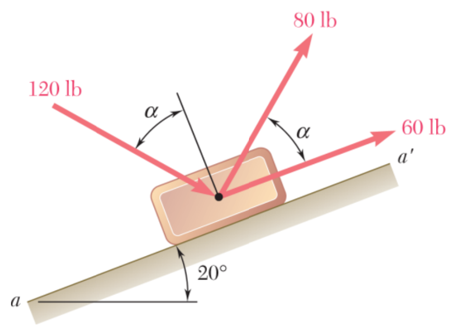
$R_y = \Sigma F_y = 29.803\text{ lb}$

$$R = \sqrt{(200.305\text{ lb})^2 + (29.803\text{ lb})^2}$$
$$= 202.510\text{ lb}$$

Further: $$\tan\alpha = \frac{29.803}{200.305}$$

$$\alpha = \tan^{-1}\frac{29.803}{200.305}$$
$$= 8.46°$$

$\mathbf{R} = 203\text{ lb} \measuredangle 8.46°$ ◀


80 lb
120 lb
α
α
60 lb
a′
20°
a

PROBLEM 2.38

Knowing that $\alpha = 75°$, determine the resultant of the three forces shown.

SOLUTION

60-lb Force:

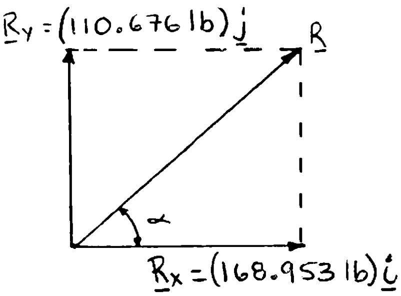
$$F_x = (60 \text{ lb}) \cos 20° = 56.382 \text{ lb}$$
$$F_y = (60 \text{ lb}) \sin 20° = 20.521 \text{ lb}$$

80-lb Force:

$$F_x = (80 \text{ lb}) \cos 95° = -6.9725 \text{ lb}$$
$$F_y = (80 \text{ lb}) \sin 95° = 79.696 \text{ lb}$$

120-lb Force:

$$F_x = (120 \text{ lb}) \cos 5° = 119.543 \text{ lb}$$
$$F_y = (120 \text{ lb}) \sin 5° = 10.459 \text{ lb}$$


$\underline{R}_y = (110.676 \text{ lb})\underline{j}$
$\underline{R}$
α
$\underline{R}_x = (168.953 \text{ lb})\underline{i}$

Then

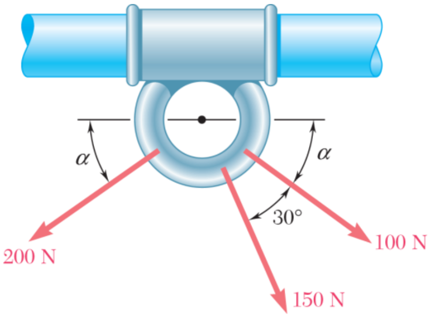
$$R_x = \Sigma F_x = 168.953 \text{ lb}$$
$$R_y = \Sigma F_y = 110.676 \text{ lb}$$

and

$$R = \sqrt{(168.953 \text{ lb})^2 + (110.676 \text{ lb})^2}$$
$$= 201.976 \text{ lb}$$

$$\tan\alpha = \frac{110.676}{168.953}$$
$$\tan\alpha = 0.65507$$
$$\alpha = 33.228°$$

$\mathbf{R} = 202 \text{ lb} \measuredangle 33.2°$ ◀



PROBLEM 2.39

For the collar of Problem 2.35, determine (*a*) the required value of α if the resultant of the three forces shown is to be vertical, (*b*) the corresponding magnitude of the resultant.

SOLUTION

$$R_x = \Sigma F_x$$

$$= (100\text{ N})\cos\alpha + (150\text{ N})\cos(\alpha + 30°) - (200\text{ N})\cos\alpha$$

$$R_x = -(100\text{ N})\cos\alpha + (150\text{ N})\cos(\alpha + 30°) \qquad (1)$$

$$R_y = \Sigma F_y$$

$$= -(100\text{ N})\sin\alpha - (150\text{ N})\sin(\alpha + 30°) - (200\text{ N})\sin\alpha$$

$$R_y = -(300\text{ N})\sin\alpha - (150\text{ N})\sin(\alpha + 30°) \qquad (2)$$

(*a*) For **R** to be vertical, we must have $R_x = 0$. We make $R_x = 0$ in Eq. (1):

$$-100\cos\alpha + 150\cos(\alpha + 30°) = 0$$

$$-100\cos\alpha + 150(\cos\alpha\cos 30° - \sin\alpha\sin 30°) = 0$$

$$29.904\cos\alpha = 75\sin\alpha$$

$$\tan\alpha = \frac{29.904}{75}$$

$$= 0.39872$$

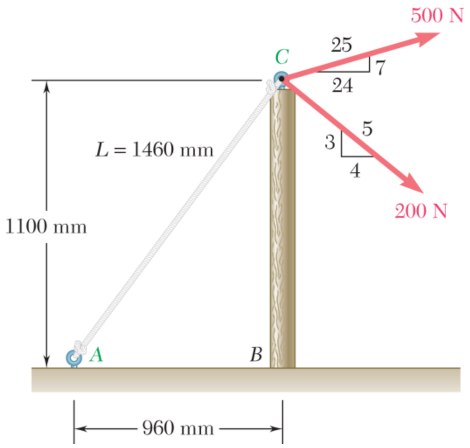
$$\alpha = 21.738° \qquad \alpha = 21.7° \blacktriangleleft$$

(*b*) Substituting for α in Eq. (2):

$$R_y = -300\sin 21.738° - 150\sin 51.738°$$

$$= -228.89\text{ N}$$

$$R = |R_y| = 228.89\text{ N} \qquad R = 229\text{ N} \blacktriangleleft$$



PROBLEM 2.40

For the post of Prob. 2.36, determine (*a*) the required tension in rope *AC* if the resultant of the three forces exerted at point *C* is to be horizontal, (*b*) the corresponding magnitude of the resultant.

SOLUTION

$$R_x = \Sigma F_x = -\frac{960}{1460}T_{AC} + \frac{24}{25}(500\text{ N}) + \frac{4}{5}(200\text{ N})$$

$$R_x = -\frac{48}{73}T_{AC} + 640\text{ N} \qquad (1)$$

$$R_y = \Sigma F_y = -\frac{1100}{1460}T_{AC} + \frac{7}{25}(500\text{ N}) - \frac{3}{5}(200\text{ N})$$

$$R_y = -\frac{55}{73}T_{AC} + 20\text{ N} \qquad (2)$$

(*a*) For **R** to be horizontal, we must have $R_y = 0$.

Set $R_y = 0$ in Eq. (2): $-\frac{55}{73}T_{AC} + 20\text{ N} = 0$

$$T_{AC} = 26.545\text{ N}$$

$T_{AC} = 26.5\text{ N}$ ◀

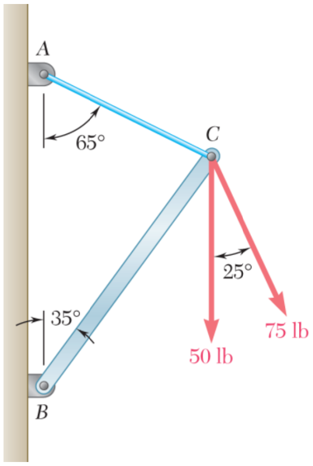
(*b*) Substituting for T_{AC} into Eq. (1) gives

$$R_x = -\frac{48}{73}(26.545\text{ N}) + 640\text{ N}$$

$$R_x = 622.55\text{ N}$$

$$R = R_x = 623\text{ N}$$

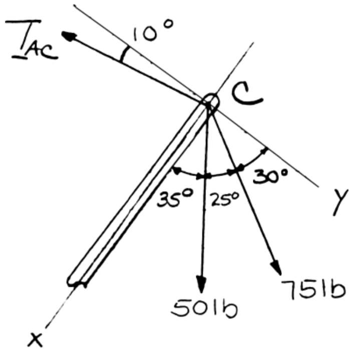
$R = 623\text{ N}$ ◀



PROBLEM 2.41

Determine (*a*) the required tension in cable *AC*, knowing that the resultant of the three forces exerted at Point *C* of boom *BC* must be directed along *BC*, (*b*) the corresponding magnitude of the resultant.

SOLUTION



Using the *x* and *y* axes shown:

$$R_x = \Sigma F_x = T_{AC}\sin 10° + (50\text{ lb})\cos 35° + (75\text{ lb})\cos 60°$$

$$= T_{AC}\sin 10° + 78.458\text{ lb} \qquad (1)$$

$$R_y = \Sigma F_y = (50\text{ lb})\sin 35° + (75\text{ lb})\sin 60° - T_{AC}\cos 10°$$

$$R_y = 93.631\text{ lb} - T_{AC}\cos 10° \qquad (2)$$

(*a*) Set $R_y = 0$ in Eq. (2):

$$93.631\text{ lb} - T_{AC}\cos 10° = 0$$

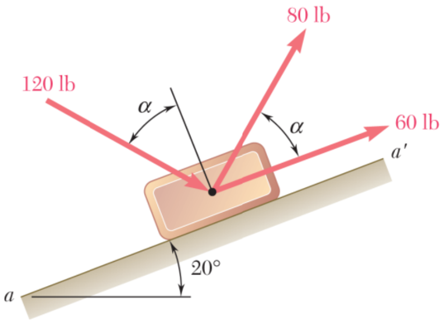
$$T_{AC} = 95.075\text{ lb} \qquad T_{AC} = 95.1\text{ lb} \blacktriangleleft$$

(*b*) Substituting for T_{AC} in Eq. (1):

$$R_x = (95.075\text{ lb})\sin 10° + 78.458\text{ lb}$$

$$= 94.968\text{ lb}$$

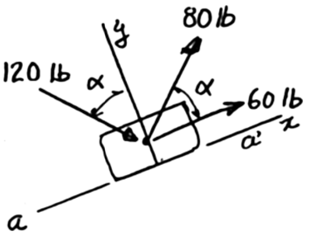
$$R = R_x \qquad R = 95.0\text{ lb} \blacktriangleleft$$



PROBLEM 2.42

For the block of Problems 2.37 and 2.38, determine (*a*) the required value of α if the resultant of the three forces shown is to be parallel to the incline, (*b*) the corresponding magnitude of the resultant.

SOLUTION



Select the x axis to be along $a\,a'$.

Then

$$R_x = \Sigma F_x = (60\text{ lb}) + (80\text{ lb})\cos\alpha + (120\text{ lb})\sin\alpha \qquad (1)$$

and

$$R_y = \Sigma F_y = (80\text{ lb})\sin\alpha - (120\text{ lb})\cos\alpha \qquad (2)$$

(*a*) Set $R_y = 0$ in Eq. (2).

$$(80\text{ lb})\sin\alpha - (120\text{ lb})\cos\alpha = 0$$

Dividing each term by $\cos\alpha$ gives:

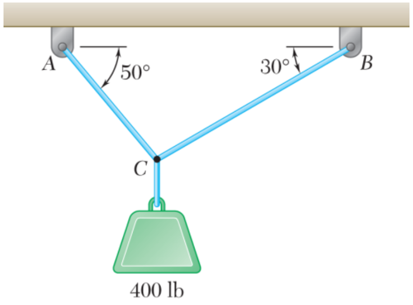
$$(80\text{ lb})\tan\alpha = 120\text{ lb}$$

$$\tan\alpha = \frac{120\text{ lb}}{80\text{ lb}}$$

$$\alpha = 56.310^\circ \qquad \alpha = 56.3^\circ \blacktriangleleft$$

(*b*) Substituting for α in Eq. (1) gives:

$$R_x = 60\text{ lb} + (80\text{ lb})\cos 56.31^\circ + (120\text{ lb})\sin 56.31^\circ = 204.22\text{ lb} \qquad R_x = 204\text{ lb} \blacktriangleleft$$

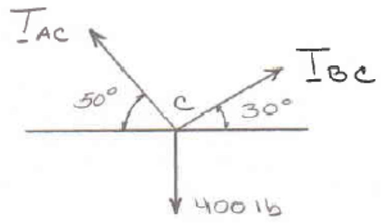

A
50°
30°
B
C
400 lb

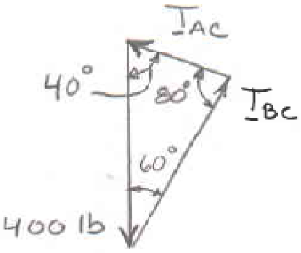
PROBLEM 2.43

Two cables are tied together at C and are loaded as shown. Determine the tension (a) in cable AC, (b) in cable BC.

SOLUTION

Free-Body Diagram **Force Triangle**


T_{AC}
T_{BC}
50°
C
30°
400 lb

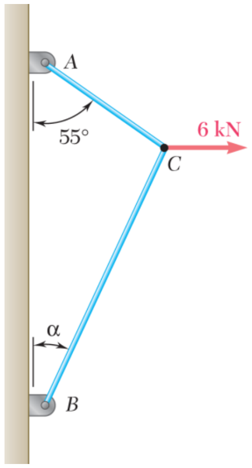

T_{AC}
40°
80°
T_{BC}
60°
400 lb

Law of sines:

$$\frac{T_{AC}}{\sin 60°} = \frac{T_{BC}}{\sin 40°} = \frac{400 \text{ lb}}{\sin 80°}$$

(a) $$T_{AC} = \frac{400 \text{ lb}}{\sin 80°}(\sin 60°)$$ $T_{AC} = 352 \text{ lb}$ ◀

(b) $$T_{BC} = \frac{400 \text{ lb}}{\sin 80°}(\sin 40°)$$ $T_{BC} = 261 \text{ lb}$ ◀

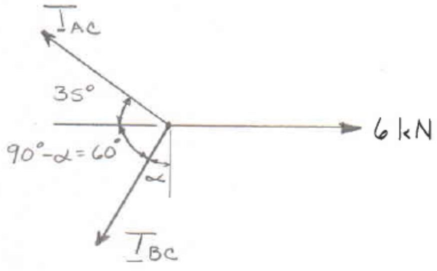


PROBLEM 2.44

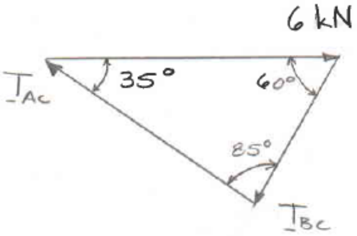
Two cables are tied together at C and are loaded as shown. Knowing that $\alpha = 30°$, determine the tension (a) in cable AC, (b) in cable BC.

SOLUTION

Free-Body Diagram


T_{AC}
35°
90°−α = 60°
α
6 kN
T_{BC}

Force Triangle

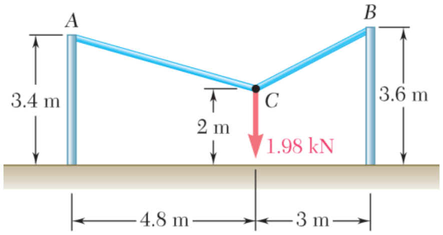

6 kN
T_{AC}
35°
60°
85°
T_{BC}

Law of sines:

$$\frac{T_{AC}}{\sin 60°} = \frac{T_{BC}}{\sin 35°} = \frac{6\text{ kN}}{\sin 85°}$$

(a) $$T_{AC} = \frac{6\text{ kN}}{\sin 85°}(\sin 60°)$$ $T_{AC} = 5.22$ kN ◀

(b) $$T_{BC} = \frac{6\text{ kN}}{\sin 85°}(\sin 35°)$$ $T_{BC} = 3.45$ kN ◀


A
B
3.4 m
2 m
C
1.98 kN
3.6 m
4.8 m
3 m

PROBLEM 2.45

Two cables are tied together at C and loaded as shown. Determine the tension (a) in cable AC, (b) in cable BC.

SOLUTION

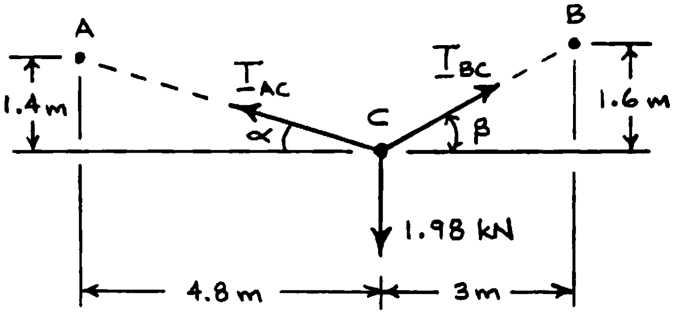
Free-Body Diagram

$$\tan\alpha = \frac{1.4}{4.8}$$

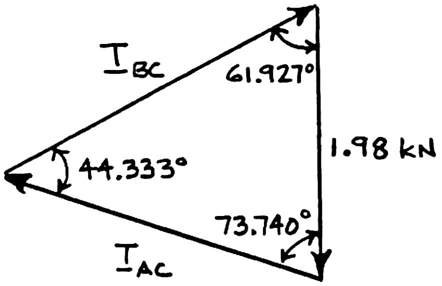
$$\alpha = 16.2602^\circ$$

$$\tan\beta = \frac{1.6}{3}$$

$$\beta = 28.073^\circ$$


A
B
T_{AC}
T_{BC}
1.4 m
1.6 m
C
α
β
1.98 kN
4.8 m
3 m

Force Triangle

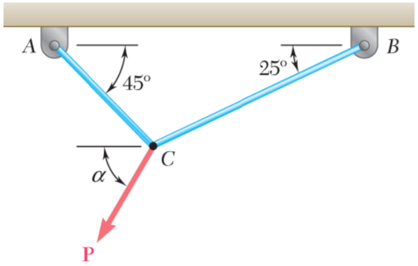

T_{BC}
61.927°
1.98 kN
44.333°
73.740°
T_{AC}

Law of sines:

$$\frac{T_{AC}}{\sin 61.927^\circ} = \frac{T_{BC}}{\sin 73.740^\circ} = \frac{1.98\text{ kN}}{\sin 44.333^\circ}$$

(a) $$T_{AC} = \frac{1.98\text{ kN}}{\sin 44.333^\circ}\sin 61.927^\circ$$ $T_{AC} = 2.50$ kN ◀

(b) $$T_{BC} = \frac{1.98\text{ kN}}{\sin 44.333^\circ}\sin 73.740^\circ$$ $T_{BC} = 2.72$ kN ◀

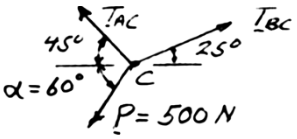


PROBLEM 2.46

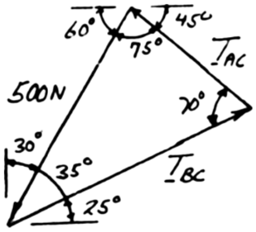
Two cables are tied together at C and are loaded as shown. Knowing that $\mathbf{P}$ = 500 N and $\alpha = 60°$, determine the tension in (*a*) in cable *AC*, (*b*) in cable *BC*.

SOLUTION

Free-Body Diagram


T_{AC}
T_{BC}
45°
25°
$\alpha = 60°$
C
P = 500 N

Force Triangle


60°
75°
45°
T_{AC}
500N
70°
30°
35°
T_{BC}
25°

Law of sines:

$$\frac{T_{AC}}{\sin 35°} = \frac{T_{BC}}{\sin 75°} = \frac{500\text{ N}}{\sin 70°}$$

(*a*)

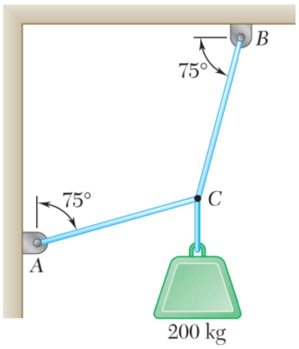
$$T_{AC} = \frac{500\text{ N}}{\sin 70°}\sin 35°$$

$T_{AC} = 305\text{ N}$ ◀

(*b*)

$$T_{BC} = \frac{500\text{ N}}{\sin 70°}\sin 75°$$

$T_{BC} = 514\text{ N}$ ◀



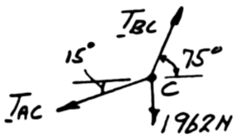
75°
B
75°
C
A
200 kg

PROBLEM 2.47

Two cables are tied together at *C* and are loaded as shown. Determine the tension (*a*) in cable *AC*, (*b*) in cable *BC*.

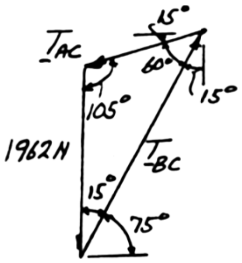
SOLUTION

Free-Body Diagram



T_{BC}
15°
75°
C
T_{AC}
1962N

Force Triangle



15°
T_{AC}
60°
105°
15°
1962N
T_{BC}
15°
75°

$$\begin{aligned} W &= \text{mg} \\ &= (200\text{ kg})(9.81\text{ m/s}^2) \\ &= 1962\text{ N} \end{aligned}$$

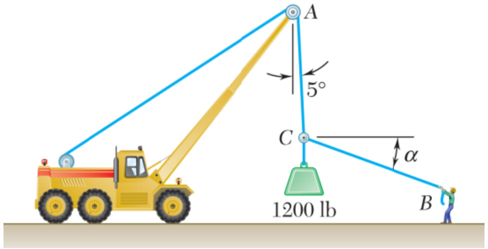
Law of sines: $$\frac{T_{AC}}{\sin 15°} = \frac{T_{BC}}{\sin 105°} = \frac{1962\text{ N}}{\sin 60°}$$

(*a*) $$T_{AC} = \frac{(1962\text{ N})\ \sin 15°}{\sin 60°}$$

$T_{AC} = 586\text{ N}$ ◀

(*b*) $$T_{BC} = \frac{(1962\text{ N})\sin 105°}{\sin 60°}$$

$T_{BC} = 2190\text{ N}$ ◀

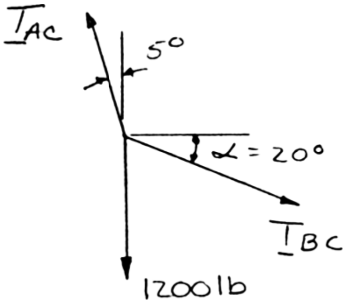

A
5°
C
α
1200 lb
B

PROBLEM 2.48

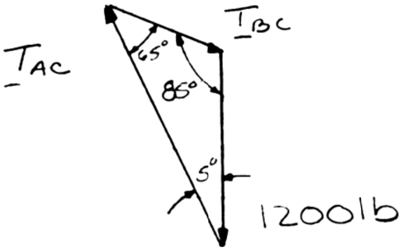
Knowing that $\alpha = 20°$, determine the tension (*a*) in cable *AC*, (*b*) in rope *BC*.

SOLUTION

Free-Body Diagram


T_{AC}
5°
α = 20°
T_{BC}
1200 lb

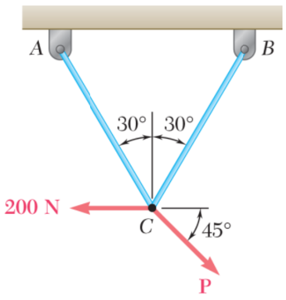
Force Triangle


T_{AC}
T_{BC}
65°
85°
5°
1200 lb

Law of sines: $$\frac{T_{AC}}{\sin 110°} = \frac{T_{BC}}{\sin 5°} = \frac{1200 \text{ lb}}{\sin 65°}$$

(*a*) $$T_{AC} = \frac{1200 \text{ lb}}{\sin 65°} \sin 110°$$ $T_{AC} = 1244 \text{ lb}$ ◀

(*b*) $$T_{BC} = \frac{1200 \text{ lb}}{\sin 65°} \sin 5°$$ $T_{BC} = 115.4 \text{ lb}$ ◀

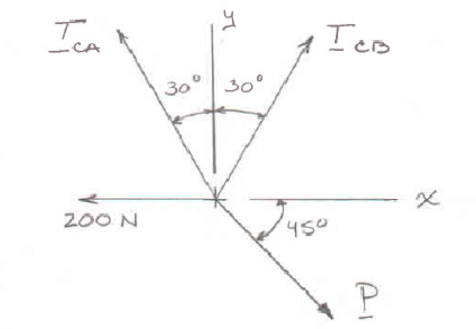


PROBLEM 2.49

Two cables are tied together at *C* and are loaded as shown. Knowing that *P* = 300 N, determine the tension in cables *AC* and *BC*.

SOLUTION

Free-Body Diagram



$\xrightarrow{+} \Sigma F_x = 0 \qquad -T_{CA}\sin 30^\circ + T_{CB}\sin 30^\circ - P\cos 45^\circ - 200\text{N} = 0$

For $P = 200\text{N}$ we have,

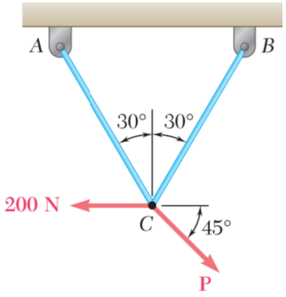
$$-0.5T_{CA} + 0.5T_{CB} + 212.13 - 200 = 0 \quad (1)$$

$+\uparrow \Sigma F_y = 0 \qquad T_{CA}\cos 30^\circ - T_{CB}\cos 30^\circ - P\sin 45^\circ = 0$

$$0.86603T_{CA} + 0.86603T_{CB} - 212.13 = 0 \quad (2)$$

Solving equations (1) and (2) simultaneously gives, $T_{CA} = 134.6\text{ N}$ ◀

$T_{CB} = 110.4\text{ N}$ ◀

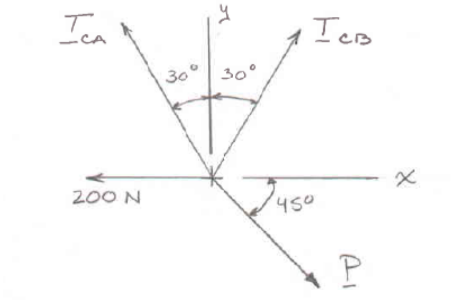


PROBLEM 2.50

Two cables are tied together at *C* and are loaded as shown. Determine the range of values of **P** for which both cables remain taut.

SOLUTION

Free-Body Diagram



$\xrightarrow{+} \Sigma F_x = 0 \qquad -T_{CA}\sin 30^\circ + T_{CB}\sin 30^\circ - P\cos 45^\circ - 200\text{N} = 0$

For $T_{CA} = 0$ we have,

$$0.5T_{CB} + 0.70711P - 200 = 0 \quad (1)$$

$+\uparrow \Sigma F_y = 0 \qquad T_{CA}\cos 30^\circ - T_{CB}\cos 30^\circ - P\sin 45^\circ = 0$; again setting $T_{CA} = 0$ yields,

$$0.86603T_{CB} - 0.70711P = 0 \quad (2)$$

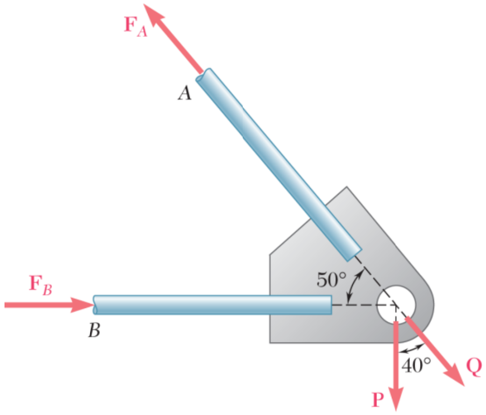
Adding equations (1) and (2) gives, $1.36603T_{CB} = 200$ hence $T_{CB} = 146.410\text{N}$ and $P = 179.315\text{N}$

Substituting for $T_{CB} = 0$ into the equilibrium equations and solving simultaneously gives,

$$-0.5T_{CA} + 0.70711P - 200 = 0$$
$$0.86603T_{CA} - 0.70711P = 0$$

And $T_{CA} = 546.40\text{N}$, $P = 669.20\text{N}$ Thus for both cables to remain taut, load *P* must be within the range of 179.315 N and 669.20 N.

179.3 N < *P* < 669 N ◀

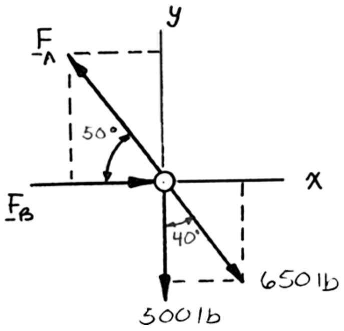


PROBLEM 2.51

Two forces **P** and **Q** are applied as shown to an aircraft connection. Knowing that the connection is in equilibrium and that $P = 500$ lb and $Q = 650$ lb, determine the magnitudes of the forces exerted on the rods *A* and *B*.

SOLUTION

Free-Body Diagram



Resolving the forces into *x*- and *y*-directions:

$$\mathbf{R} = \mathbf{P} + \mathbf{Q} + \mathbf{F}_A + \mathbf{F}_B = 0$$

Substituting components:

$$\begin{aligned}\mathbf{R} &= -(500\text{ lb})\mathbf{j} + [(650\text{ lb})\cos 50°]\mathbf{i}\\ &\quad -[(650\text{ lb})\sin 50°]\mathbf{j}\\ &\quad + F_B\mathbf{i} - (F_A\cos 50°)\mathbf{i} + (F_A\sin 50°)\mathbf{j} = 0\end{aligned}$$

In the *y*-direction (one unknown force):

$$-500\text{ lb} - (650\text{ lb})\sin 50° + F_A\sin 50° = 0$$

Thus,

$$F_A = \frac{500\text{ lb} + (650\text{ lb})\sin 50°}{\sin 50°}$$

$$= 1302.70\text{ lb}$$

$F_A = 1303$ lb ◀

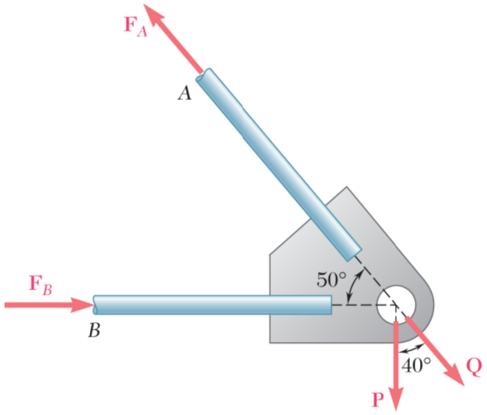
In the *x*-direction:

$$(650\text{ lb})\cos 50° + F_B - F_A\cos 50° = 0$$

Thus,

$$\begin{aligned}F_B &= F_A\cos 50° - (650\text{ lb})\cos 50°\\ &= (1302.70\text{ lb})\cos 50° - (650\text{ lb})\cos 50°\\ &= 419.55\text{ lb}\end{aligned}$$

$F_B = 420$ lb ◀

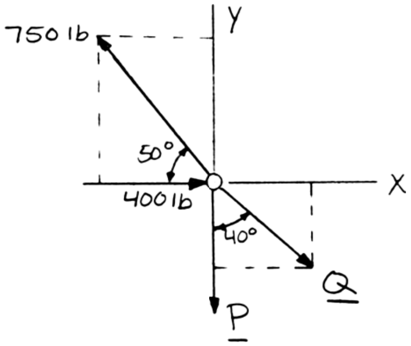


PROBLEM 2.52

Two forces **P** and **Q** are applied as shown to an aircraft connection. Knowing that the connection is in equilibrium and that the magnitudes of the forces exerted on rods *A* and *B* are $F_A = 750$ lb and $F_B = 400$ lb, determine the magnitudes of **P** and **Q**.

SOLUTION

Free-Body Diagram



Resolving the forces into *x*- and *y*-directions:

$$\mathbf{R} = \mathbf{P} + \mathbf{Q} + \mathbf{F}_A + \mathbf{F}_B = 0$$

Substituting components:
$$\mathbf{R} = -P\mathbf{j} + Q\cos 50°\mathbf{i} - Q\sin 50°\mathbf{j} - [(750\text{ lb})\cos 50°]\mathbf{i} + [(750\text{ lb})\sin 50°]\mathbf{j} + (400\text{ lb})\mathbf{i}$$

In the *x*-direction (one unknown force):

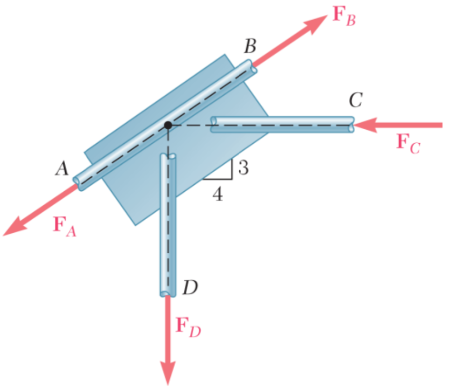
$$Q\cos 50° - [(750\text{ lb})\cos 50°] + 400\text{ lb} = 0$$

$$Q = \frac{(750\text{ lb})\cos 50° - 400\text{ lb}}{\cos 50°} = 127.710\text{ lb}$$

In the *y*-direction:
$$-P - Q\sin 50° + (750\text{ lb})\sin 50° = 0$$

$$P = -Q\sin 50° + (750\text{ lb})\sin 50° = -(127.710\text{ lb})\sin 50° + (750\text{ lb})\sin 50° = 476.70\text{ lb}$$

$P = 477$ lb; $Q = 127.7$ lb ◀

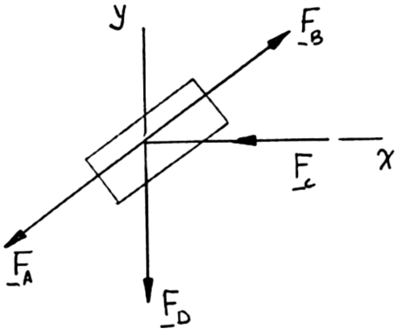


PROBLEM 2.53

A welded connection is in equilibrium under the action of the four forces shown. Knowing that $F_A = 8$ kN and $F_B = 16$ kN, determine the magnitudes of the other two forces.

SOLUTION

Free-Body Diagram of Connection



$$\Sigma F_x = 0: \quad \frac{3}{5}F_B - F_C - \frac{3}{5}F_A = 0$$

With $F_A = 8$ kN

$F_B = 16$ kN

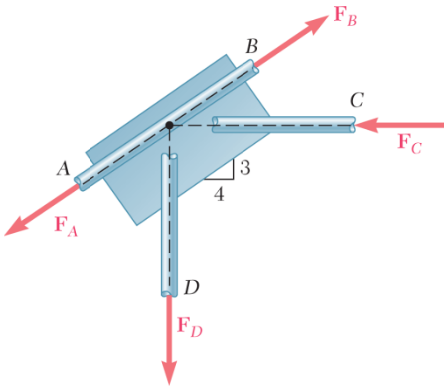
$$F_C = \frac{4}{5}(16 \text{ kN}) - \frac{4}{5}(8 \text{ kN})$$

$F_C = 6.40$ kN ◀

$$\Sigma F_y = 0: \quad -F_D + \frac{3}{5}F_B - \frac{3}{5}F_A = 0$$

With F_A and F_B as above: $F_D = \frac{3}{5}(16 \text{ kN}) - \frac{3}{5}(8 \text{ kN})$

$F_D = 4.80$ kN ◀

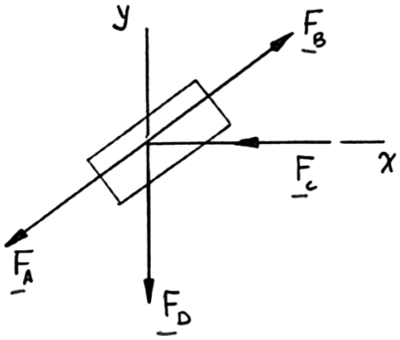

$\mathbf{F}_B$
B
C
$\mathbf{F}_C$
A
3
4
$\mathbf{F}_A$
D
$\mathbf{F}_D$

PROBLEM 2.54

A welded connection is in equilibrium under the action of the four forces shown. Knowing that $F_A = 5$ kN and $F_D = 6$ kN, determine the magnitudes of the other two forces.

SOLUTION

Free-Body Diagram of Connection


y
$\underline{F}_B$
$\underline{F}_C$
x
$\underline{F}_A$
$\underline{F}_D$

$$\Sigma F_y = 0: \quad -F_D - \frac{3}{5}F_A + \frac{3}{5}F_B = 0$$

or

$$F_B = F_D + \frac{3}{5}F_A$$

With

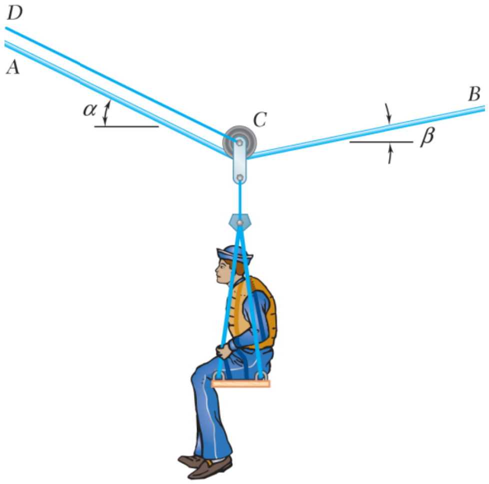
$$F_A = 5\text{ kN}, \quad F_D = 8\text{ kN}$$

$$F_B = \frac{5}{3}\left[6\text{ kN} + \frac{3}{5}(5\text{ kN})\right] \qquad F_B = 15.00\text{ kN} \blacktriangleleft$$

$$\Sigma F_x = 0: \quad -F_C + \frac{4}{5}F_B - \frac{4}{5}F_A = 0$$

$$F_C = \frac{4}{5}(F_B - F_A)$$

$$= \frac{4}{5}(15\text{ kN} - 5\text{ kN}) \qquad F_C = 8.00\text{ kN} \blacktriangleleft$$

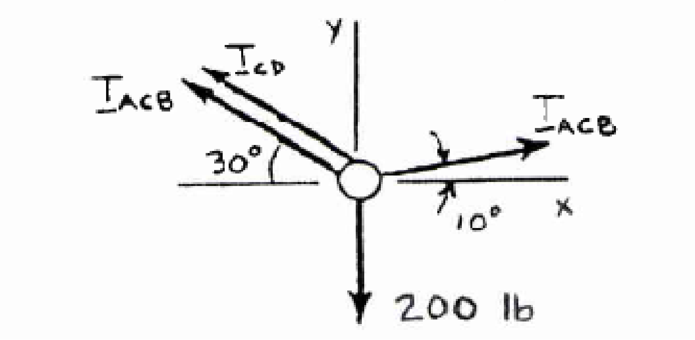


PROBLEM 2.55

A sailor is being rescued using a boatswain's chair that is suspended from a pulley that can roll freely on the support cable *ACB* and is pulled at a constant speed by cable *CD*. Knowing that $\alpha = 30°$ and $\beta = 10°$ and that the combined weight of the boatswain's chair and the sailor is 200 lb, determine the tension (*a*) in the support cable *ACB*, (*b*) in the traction cable *CD*.

SOLUTION

Free-Body Diagram



$$\overset{+}{\rightarrow} \Sigma F_x = 0: \quad T_{ACB} \cos 10° - T_{ACB} \cos 30° - T_{CD} \cos 30° = 0$$

$$T_{CD} = 0.137158 T_{ACB} \qquad (1)$$

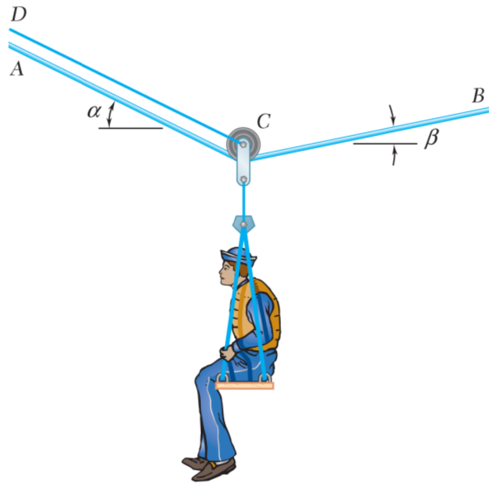
$$+\!\uparrow \Sigma F_y = 0: \quad T_{ACB} \sin 10° + T_{ACB} \sin 30° + T_{CD} \sin 30° - 200 = 0$$

$$0.67365 T_{ACB} + 0.5 T_{CD} = 200 \qquad (2)$$

(*a*) Substitute (1) into (2): $0.67365 T_{ACB} + 0.5(0.137158 T_{ACB}) = 200$

$T_{ACB} = 269.46 \text{ lb}$ $\qquad T_{ACB} = 269 \text{ lb}$ ◀

(*b*) From (1): $T_{CD} = 0.137158(269.46 \text{ lb})$ $\qquad T_{CD} = 37.0 \text{ lb}$ ◀

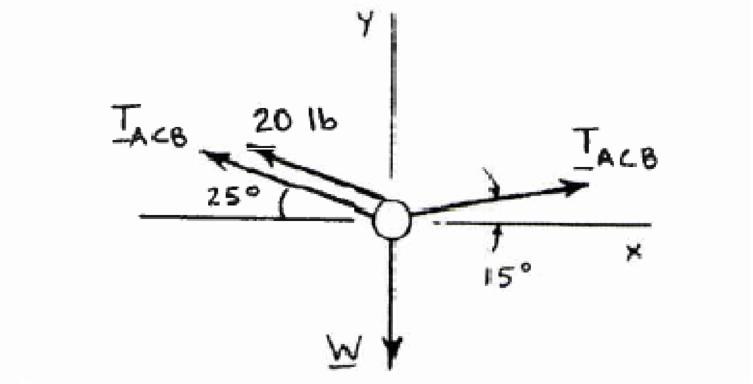


PROBLEM 2.56

A sailor is being rescued using a boatswain's chair that is suspended from a pulley that can roll freely on the support cable *ACB* and is pulled at a constant speed by cable *CD*. Knowing that $\alpha = 25°$ and $\beta = 15°$ and that the tension in cable *CD* is 20 lb, determine (*a*) the combined weight of the boatswain's chair and the sailor, (*b*) the tension in the support cable *ACB*.

SOLUTION

Free-Body Diagram



$$\xrightarrow{+} \Sigma F_x = 0: \quad T_{ACB}\cos 15° - T_{ACB}\cos 25° - (20\text{ lb})\cos 25° = 0$$

$$T_{ACB} = 304.04\text{ lb}$$

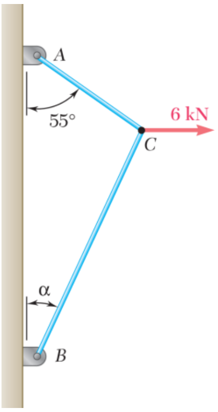
$$+\!\uparrow \Sigma F_y = 0: \quad (304.04\text{ lb})\sin 15° + (304.04\text{ lb})\sin 25°$$

$$+(20\text{ lb})\sin 25° - W = 0$$

$$W = 215.64\text{ lb}$$

(*a*) $W = 216\text{ lb}$ ◀

(*b*) $T_{ACB} = 304\text{ lb}$ ◀

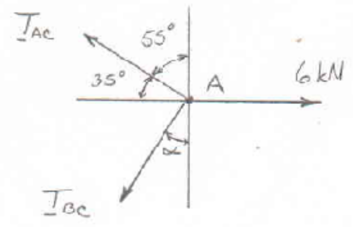


PROBLEM 2.57

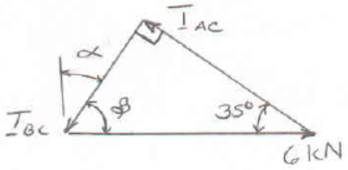
For the cables of prob. 2.44, find the value of α for which the tension is as small as possible (*a*) in cable *bc*, (*b*) in both cables simultaneously. In each case determine the tension in each cable.

SOLUTION

Free-Body Diagram



Force Triangle



(*a*) For a minimum tension in cable *BC,* set angle between cables to 90 degrees.

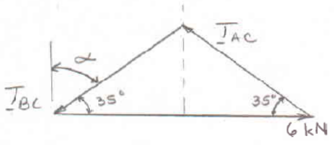
By inspection, $\alpha = 35.0^\circ$ ◀

$T_{AC} = (6\text{ kN})\cos 35^\circ$ $T_{AC} = 4.91\text{ kN}$ ◀

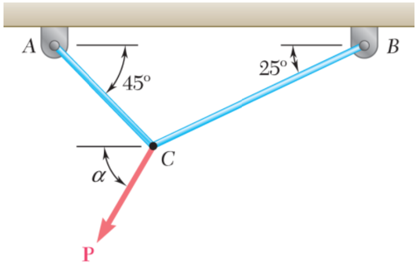
$T_{BC} = (6\text{ kN})\sin 35^\circ$ $T_{BC} = 3.44\text{ kN}$ ◀

(*b*) For equal tension in both cables, the force triangle will be an isosceles.

Therefore, by inspection, $\alpha = 55.0^\circ$ ◀



$$T_{AC} = T_{BC} = (1/2)\frac{6\text{ kN}}{\cos 35^\circ}$$ $T_{AC} = T_{BC} = 3.66\text{ kN}$ ◀

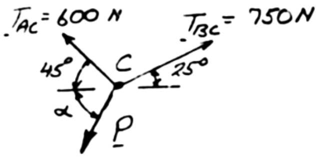


PROBLEM 2.58

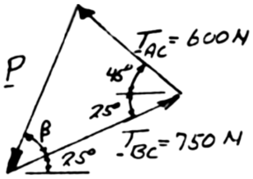
For the cables of Problem 2.46, it is known that the maximum allowable tension is 600 N in cable *AC* and 750 N in cable *BC*. Determine (*a*) the maximum force **P** that can be applied at *C*, (*b*) the corresponding value of *α*.

SOLUTION

Free-Body Diagram


T_{AC} = 600 N
T_{BC} = 750N
45°
C
25°
α
P

Force Triangle


P
T_{AC} = 600N
45°
25°
β
25°
T_{BC} = 750 N

(*a*) Law of cosines

$$P^2 = (600)^2 + (750)^2 - 2(600)(750)\cos(25° + 45°)$$

$$P = 784.02 \text{ N}$$

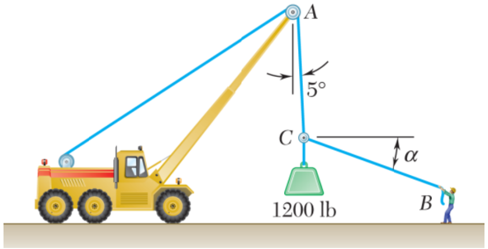
$$P = 784 \text{ N} \blacktriangleleft$$

(*b*) Law of sines

$$\frac{\sin\beta}{600 \text{ N}} = \frac{\sin(25° + 45°)}{784.02 \text{ N}}$$

$$\beta = 46.0° \qquad \therefore \quad \alpha = 46.0° + 25°$$

$$\alpha = 71.0° \blacktriangleleft$$

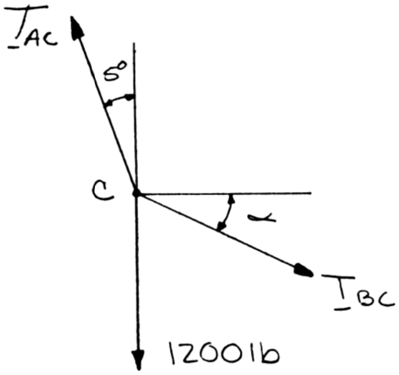

A
5°
C
α
1200 lb
B

PROBLEM 2.59

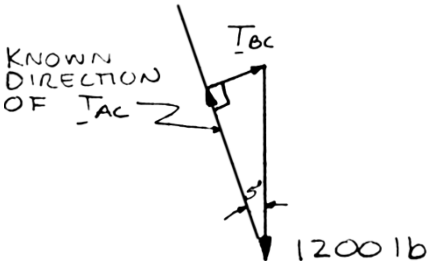
For the situation described in Figure P2.48, determine (*a*) the value of α for which the tension in rope *BC* is as small as possible, (*b*) the corresponding value of the tension.

SOLUTION

Free-Body Diagram


T_{AC}
5°
C
α
T_{BC}
1200 lb

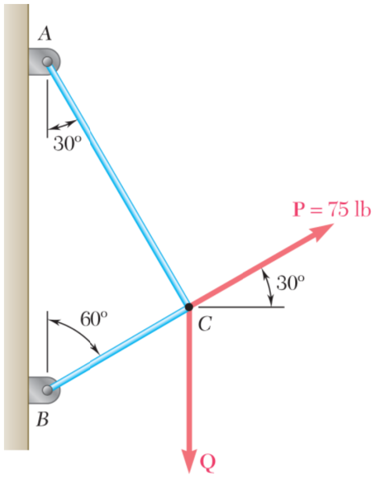
Force Triangle


T_{BC}
KNOWN DIRECTION OF T_{AC}
5°
1200 lb

To be smallest, T_{BC} must be perpendicular to the direction of T_{AC}.

(*a*) Thus, $\alpha = 5.00°$ $\alpha = 5.00°$ ◀

(*b*) $T_{BC} = (1200 \text{ lb})\sin 5°$ $T_{BC} = 104.6 \text{ lb}$ ◀

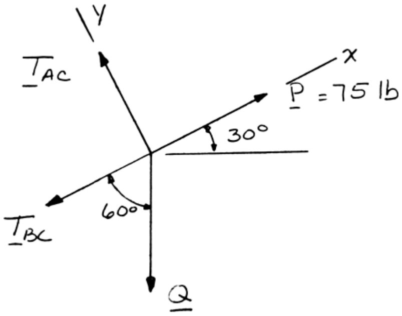


PROBLEM 2.60

Two cables tied together at C are loaded as shown. Determine the range of values of Q for which the tension will not exceed 60 lb in either cable.

SOLUTION

Free-Body Diagram



$$\Sigma F_x = 0: \quad -T_{BC} - Q\cos 60° + 75 \text{ lb} = 0$$

$$T_{BC} = 75 \text{ lb} - Q\cos 60° \qquad (1)$$

$$\Sigma F_y = 0: \quad T_{AC} - Q\sin 60° = 0$$

$$T_{AC} = Q\sin 60° \qquad (2)$$

Requirement: $T_{AC} = 60$ lb:

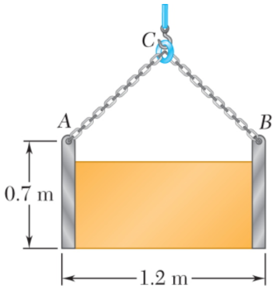
From Eq. (2): $Q\sin 60° = 60$ lb

$$Q = 69.3 \text{ lb}$$

Requirement: $T_{BC} = 60$ lb:

From Eq. (1): $75 \text{ lb} - Q\cos 60° = 60$ lb

$$Q = 30.0 \text{ lb} \quad 30.0 \text{ lb} \le Q \le 69.3 \text{ lb}$$ ◀

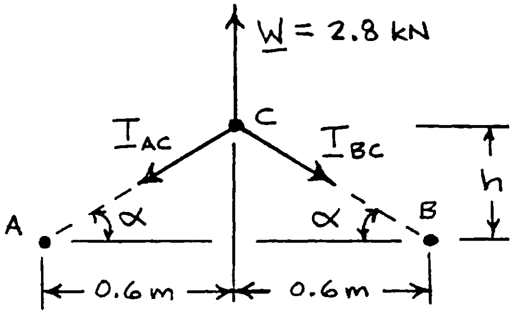


PROBLEM 2.61

A movable bin and its contents have a combined weight of 2.8 kN. Determine the shortest chain sling *ACB* that can be used to lift the loaded bin if the tension in the chain is not to exceed 5 kN.

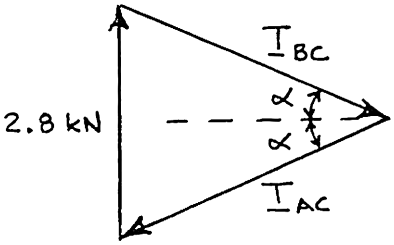
SOLUTION

Free-Body Diagram



$$\tan\alpha = \frac{h}{0.6\text{ m}} \qquad (1)$$

Isosceles Force Triangle



Law of sines: $\sin\alpha = \dfrac{\frac{1}{2}(2.8\text{ kN})}{T_{AC}}$

$$T_{AC} = 5\text{ kN}$$

$$\sin\alpha = \frac{\frac{1}{2}(2.8\text{ kN})}{5\text{ kN}}$$

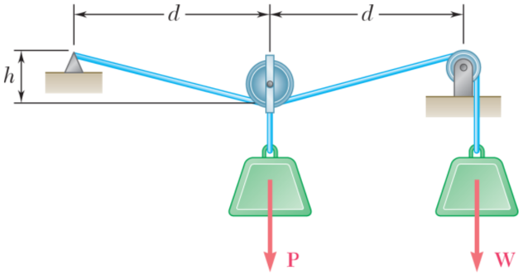
$$\alpha = 16.2602°$$

From Eq. (1): $\tan 16.2602° = \dfrac{h}{0.6\text{ m}} \qquad \therefore \quad h = 0.175000\text{ m}$

Half-length of chain $= AC = \sqrt{(0.6\text{ m})^2 + (0.175\text{ m})^2}$

$= 0.625\text{ m}$

Total length: $= 2\times 0.625\text{ m}$ $\qquad$ 1.250 m ◀

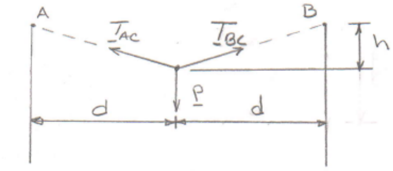


PROBLEM 2.62

For $W = 800$ N, $P = 200$ N, and $d = 600$ mm, determine the value of h consistent with equilibrium.

SOLUTION

Free-Body Diagram



$$T_{AC} = T_{BC} = 800 \text{ N}$$

$$AC = BC = \sqrt{(h^2 + d^2)}$$

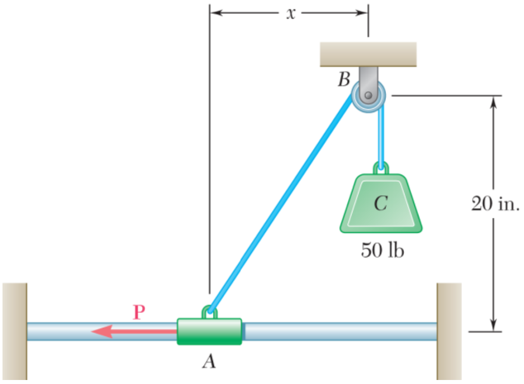
$$\Sigma F_y = 0: \quad 2(800 \text{ N})\frac{h}{\sqrt{h^2 + d^2}} - P = 0$$

$$800 = \frac{P}{2}\sqrt{1 + \left(\frac{d}{h}\right)^2}$$

Data: $P = 200$ N, $d = 600$ mm and solving for h

$$800 \text{ N} = \frac{200 \text{ N}}{2}\sqrt{1 + \left(\frac{600 \text{ mm}}{h}\right)^2}$$

$h = 75.6$ mm ◀

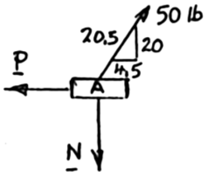

x
B
C
50 lb
20 in.
P
A

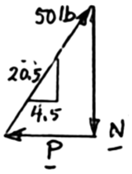
PROBLEM 2.63

Collar *A* is connected as shown to a 50-lb load and can slide on a frictionless horizontal rod. Determine the magnitude of the force **P** required to maintain the equilibrium of the collar when (*a*) $x = 4.5$ in., (*b*) $x = 15$ in.

SOLUTION

(*a*) **Free Body: Collar *A***


50 lb
20.5
20
4.5
P
A
N

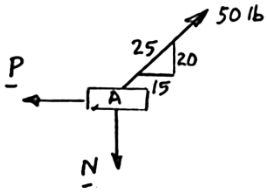

50 lb
20.5
4.5
P
N

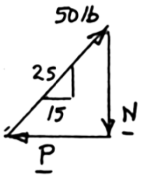
Force Triangle

$$\frac{P}{4.5} = \frac{50 \text{ lb}}{20.5}$$

$P = 10.98$ lb ◀

(*b*) **Free Body: Collar *A***


50 lb
25
20
15
P
A
N


50 lb
25
15
P
N

Force Triangle

$$\frac{P}{15} = \frac{50 \text{ lb}}{25}$$

$P = 30.0$ lb ◀

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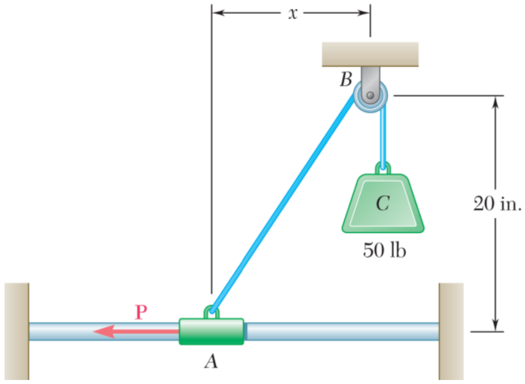
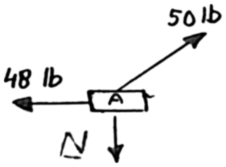

x
B
C
50 lb
20 in.
P
A

PROBLEM 2.64

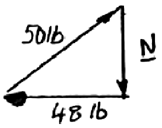
Collar *A* is connected as shown to a 50-lb load and can slide on a frictionless horizontal rod. Determine the distance *x* for which the collar is in equilibrium when $P = 48$ lb.

SOLUTION

Free Body: Collar *A*


50 lb
48 lb
A
N

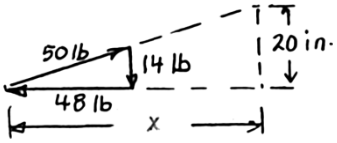
Force Triangle



$$N^2 = (50)^2 - (48)^2 = 196$$

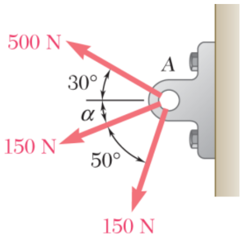
$$N = 14.00 \text{ lb}$$

Similar Triangles


50 lb
14 lb
20 in.
48 lb
x

$$\frac{x}{20 \text{ in.}} = \frac{48 \text{ lb}}{14 \text{ lb}}$$

$x = 68.6$ in. ◀

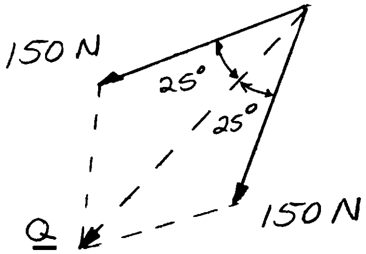


PROBLEM 2.65

Three forces are applied to a bracket as shown. The directions of the two 150-N forces may vary, but the angle between these forces is always 50°. Determine the range of values of α for which the magnitude of the resultant of the forces acting at A is less than 600 N.

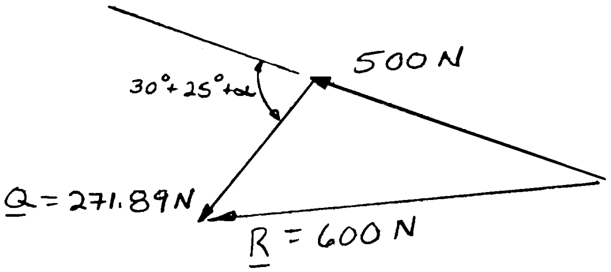
SOLUTION

Combine the two 150-N forces into a resultant force Q:



$$Q = 2(150\text{ N})\cos 25°$$
$$= 271.89\text{ N}$$

Equivalent loading at A:



Using the law of cosines:

$$(600\text{ N})^2 = (500\text{ N})^2 + (271.89\text{ N})^2 + 2(500\text{ N})(271.89\text{ N})\cos(55° + \alpha)$$
$$\cos(55° + \alpha) = 0.132685$$

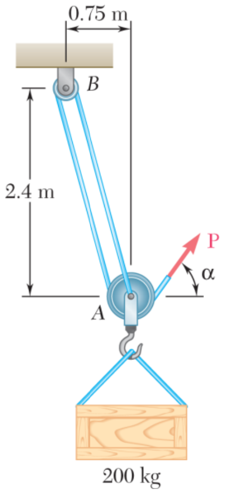
Two values for α: $55° + \alpha = 82.375$

$$\alpha = 27.4°$$

$$55° + \alpha = -82.375°$$
$$55° + \alpha = 360° - 82.375°$$

or $\alpha = 222.6°$

For $R < 600$ lb: $27.4° < \alpha < 222.6°$ ◀

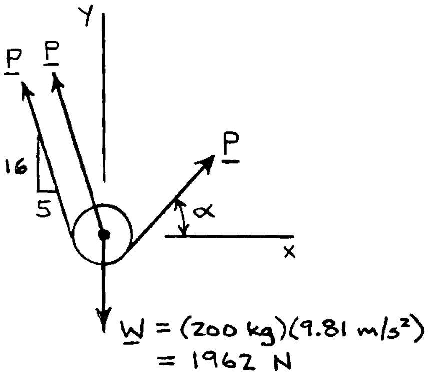


PROBLEM 2.66

A 200-kg crate is to be supported by the rope-and-pulley arrangement shown. Determine the magnitude and direction of the force **P** that must be exerted on the free end of the rope to maintain equilibrium. (*Hint:* The tension in the rope is the same on each side of a simple pulley. This can be proved by the methods of Ch. 4.)

SOLUTION

Free-Body Diagram: Pulley *A*



$$\xrightarrow{+} \Sigma F_x = 0: \quad -2P\left(\frac{5}{\sqrt{281}}\right) + P\cos\alpha = 0$$

$$\cos\alpha = 0.59655$$

$$\alpha = \pm 53.377^\circ$$

For $\alpha = +53.377^\circ$:

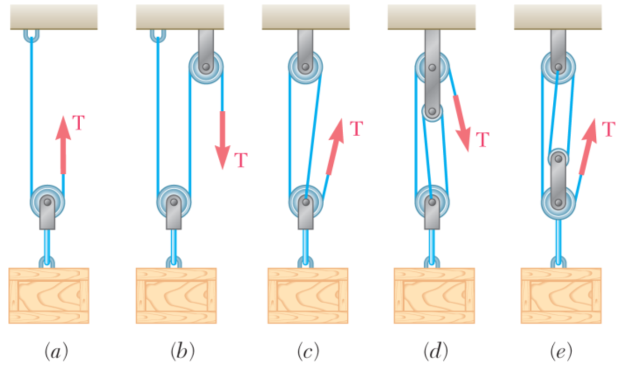
$$+\uparrow \Sigma F_y = 0: \quad 2P\left(\frac{16}{\sqrt{281}}\right) + P\sin 53.377^\circ - 1962\text{ N} = 0$$

$$\mathbf{P} = 724\text{ N} \measuredangle 53.4^\circ \blacktriangleleft$$

For $\alpha = -53.377^\circ$:

$$+\uparrow \Sigma F_y = 0: \quad 2P\left(\frac{16}{\sqrt{281}}\right) + P\sin(-53.377^\circ) - 1962\text{ N} = 0$$

$$\mathbf{P} = 1773 \searrow 53.4^\circ \blacktriangleleft$$



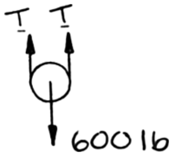
PROBLEM 2.67

A 600-lb crate is supported by several rope-and-pulley arrangements as shown. Determine for each arrangement the tension in the rope. (See the hint for Problem 2.66.)

SOLUTION

Free-Body Diagram of Pulley

(*a*)

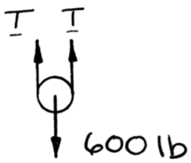


$$+\uparrow \Sigma F_y = 0: \quad 2T - (600 \text{ lb}) = 0$$

$$T = \frac{1}{2}(600 \text{ lb})$$

$$T = 300 \text{ lb} \blacktriangleleft$$

(*b*)

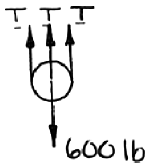


$$+\uparrow \Sigma F_y = 0: \quad 2T - (600 \text{ lb}) = 0$$

$$T = \frac{1}{2}(600 \text{ lb})$$

$$T = 300 \text{ lb} \blacktriangleleft$$

(*c*)

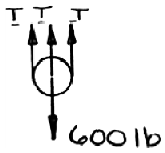


$$+\uparrow \Sigma F_y = 0: \quad 3T - (600 \text{ lb}) = 0$$

$$T = \frac{1}{3}(600 \text{ lb})$$

$$T = 200 \text{ lb} \blacktriangleleft$$

(*d*)

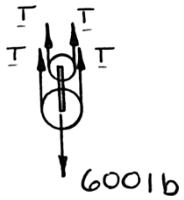


$$+\uparrow \Sigma F_y = 0: \quad 3T - (600 \text{ lb}) = 0$$

$$T = \frac{1}{3}(600 \text{ lb})$$

$$T = 200 \text{ lb} \blacktriangleleft$$

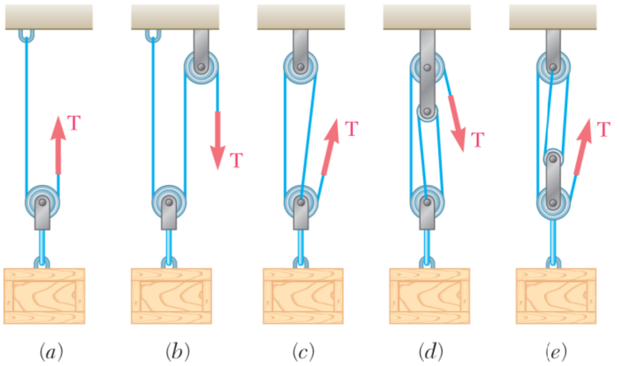
(*e*)



$$+\uparrow \Sigma F_y = 0: \quad 4T - (600 \text{ lb}) = 0$$

$$T = \frac{1}{4}(600 \text{ lb})$$

$$T = 150.0 \text{ lb} \blacktriangleleft$$



PROBLEM 2.68

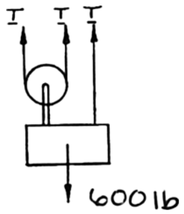
Solve Parts *b* and *d* of Problem 2.67, assuming that the free end of the rope is attached to the crate.

PROBLEM 2.67 A 600-lb crate is supported by several rope-and-pulley arrangements as shown. Determine for each arrangement the tension in the rope. (See the hint for Problem 2.66.)

SOLUTION

Free-Body Diagram of Pulley and Crate

(*b*)

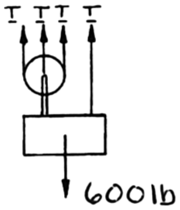


$$+\uparrow \Sigma F_y = 0: \quad 3T - (600\text{ lb}) = 0$$

$$T = \frac{1}{3}(600\text{ lb})$$

$$T = 200\text{ lb} \blacktriangleleft$$

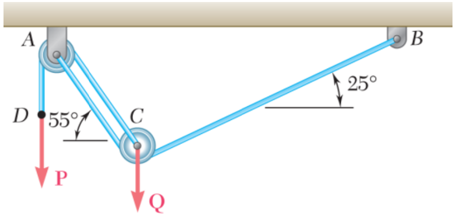
(*d*)



$$+\uparrow \Sigma F_y = 0: \quad 4T - (600\text{ lb}) = 0$$

$$T = \frac{1}{4}(600\text{ lb})$$

$$T = 150.0\text{ lb} \blacktriangleleft$$

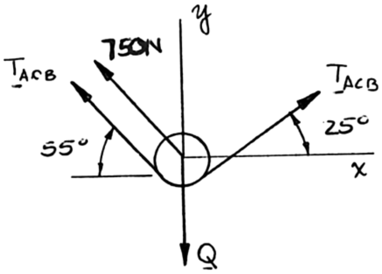


PROBLEM 2.69

A load **Q** is applied to the pulley C, which can roll on the cable ACB. The pulley is held in the position shown by a second cable CAD, which passes over the pulley A and supports a load **P**. Knowing that $P = 750$ N, determine (a) the tension in cable ACB, (b) the magnitude of load **Q**.

SOLUTION

Free-Body Diagram: Pulley *C*



(a) $\xrightarrow{+} \Sigma F_x = 0{:}\quad T_{ACB}(\cos 25° - \cos 55°) - (750\text{ N})\cos 55° = 0$

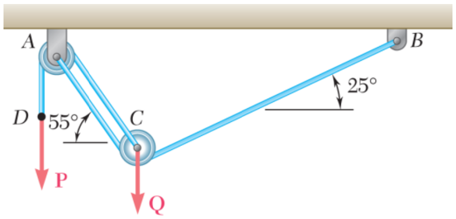
Hence: $T_{ACB} = 1292.88$ N

$T_{ACB} = 1293$ N ◀

(b) $+\!\uparrow \Sigma F_y = 0{:}\quad T_{ACB}(\sin 25° + \sin 55°) + (750\text{ N})\sin 55° - Q = 0$

$(1292.88\text{ N})(\sin 25° + \sin 55°) + (750\text{ N})\sin 55° - Q = 0$

or $Q = 2219.8$ N $\quad Q = 2220$ N ◀

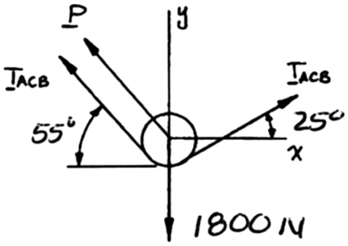


PROBLEM 2.70

An 1800-N load **Q** is applied to the pulley *C*, which can roll on the cable *ACB*. The pulley is held in the position shown by a second cable *CAD*, which passes over the pulley *A* and supports a load **P**. Determine (*a*) the tension in cable *ACB*, (*b*) the magnitude of load **P**.

SOLUTION

Free-Body Diagram: Pulley *C*



$$\xrightarrow{+} \Sigma F_x = 0: \quad T_{ACB}(\cos 25° - \cos 55°) - P\cos 55° = 0$$

or
$$P = 0.58010 T_{ACB} \quad (1)$$

$$+\!\uparrow \Sigma F_y = 0: \quad T_{ACB}(\sin 25° + \sin 55°) + P\sin 55° - 1800\text{ N} = 0$$

or
$$1.24177 T_{ACB} + 0.81915 P = 1800\text{ N} \quad (2)$$

(*a*) Substitute Equation (1) into Equation (2):

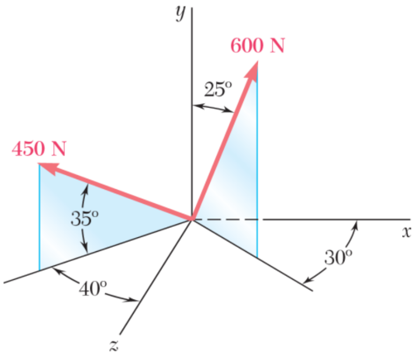
$$1.24177 T_{ACB} + 0.81915(0.58010 T_{ACB}) = 1800\text{ N}$$

Hence: $T_{ACB} = 1048.37\text{ N}$

$T_{ACB} = 1048\text{ N}$ ◀

(*b*) Using (1), $P = 0.58010(1048.37\text{ N}) = 608.16\text{ N}$

$P = 608\text{ N}$ ◀



PROBLEM 2.71

Determine (*a*) the *x*, *y*, and *z* components of the 600-N force, (*b*) the angles θ_x, θ_y, and θ_z that the force forms with the coordinate axes.

SOLUTION

(*a*)

$$F_x = (600\text{ N})\sin 25^\circ \cos 30^\circ$$

$$F_x = 219.60\text{ N} \qquad F_x = 220\text{ N} \blacktriangleleft$$

$$F_y = (600\text{ N})\cos 25^\circ$$

$$F_y = 543.78\text{ N} \qquad F_y = 544\text{ N} \blacktriangleleft$$

$$F_z = (380.36\text{ N})\sin 25^\circ \sin 30^\circ$$

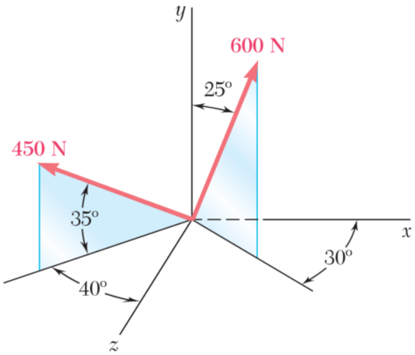
$$F_z = 126.785\text{ N} \qquad F_z = 126.8\text{ N} \blacktriangleleft$$

(*b*)

$$\cos\theta_x = \frac{F_x}{F} = \frac{219.60\text{ N}}{600\text{ N}} \qquad \theta_x = 68.5^\circ \blacktriangleleft$$

$$\cos\theta_y = \frac{F_y}{F} = \frac{543.78\text{ N}}{600\text{ N}} \qquad \theta_y = 25.0^\circ \blacktriangleleft$$

$$\cos\theta_z = \frac{F_z}{F} = \frac{126.785\text{ N}}{600\text{ N}} \qquad \theta_z = 77.8^\circ \blacktriangleleft$$



PROBLEM 2.72

Determine (*a*) the *x*, *y*, and *z* components of the 450-N force, (*b*) the angles θ_x, θ_y, and θ_z that the force forms with the coordinate axes.

SOLUTION

(*a*)

$$F_x = -(450\text{ N})\cos 35°\sin 40°$$

$$F_x = -236.94\text{ N} \qquad F_x = -237\text{ N} \blacktriangleleft$$

$$F_y = (450\text{ N})\sin\ 35°$$

$$F_y = 258.11\text{ N} \qquad F_y = 258\text{ N} \blacktriangleleft$$

$$F_z = (450\text{ N})\cos 35°\cos 40°$$

$$F_z = 282.38\text{ N} \qquad F_z = 282\text{ N} \blacktriangleleft$$

(*b*)

$$\cos\theta_x = \frac{F_x}{F} = \frac{-236.94\text{ N}}{450\text{ N}} \qquad \theta_x = 121.8° \blacktriangleleft$$

$$\cos\theta_y = \frac{F_y}{F} = \frac{258.11\text{ N}}{450\text{ N}} \qquad \theta_y = 55.0° \blacktriangleleft$$

$$\cos\theta_z = \frac{F_z}{F} = \frac{282.38\text{ N}}{450\text{ N}} \qquad \theta_z = 51.1° \blacktriangleleft$$

Note: From the given data, we could have computed directly

$\theta_y = 90° - 35° = 55°$, which checks with the answer obtained.

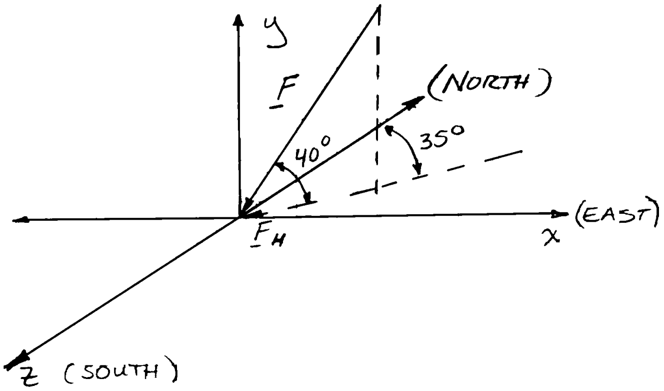
PROBLEM 2.73

A gun is aimed at a point *A* located 35° east of north. Knowing that the barrel of the gun forms an angle of 40° with the horizontal and that the maximum recoil force is 400 N, determine (*a*) the *x*, *y*, and *z* components of that force, (*b*) the values of the angles θ_x, θ_y, and θ_z defining the direction of the recoil force. (Assume that the *x*, *y*, and *z* axes are directed, respectively, east, up, and south.)

SOLUTION

Recoil force $F = 400 \text{ N}$

$$\therefore \quad F_H = (400 \text{ N})\cos 40° = 306.42 \text{ N}$$



(*a*)

$$F_x = -F_H \sin 35° = -(306.42 \text{ N})\sin 35° = -175.755 \text{ N}$$

$F_x = -175.8 \text{ N}$ ◀

$$F_y = -F \sin 40° = -(400 \text{ N})\sin 40° = -257.12 \text{ N}$$

$F_y = -257 \text{ N}$ ◀

$$F_z = +F_H \cos 35° = +(306.42 \text{ N})\cos 35° = +251.00 \text{ N}$$

$F_z = +251 \text{ N}$ ◀

(*b*)

$$\cos\theta_x = \frac{F_x}{F} = \frac{-175.755 \text{ N}}{400 \text{ N}}$$

$\theta_x = 116.1°$ ◀

$$\cos\theta_y = \frac{F_y}{F} = \frac{-257.12 \text{ N}}{400 \text{ N}}$$

$\theta_y = 130.0°$ ◀

$$\cos\theta_z = \frac{F_z}{F} = \frac{251.00 \text{ N}}{400 \text{ N}}$$

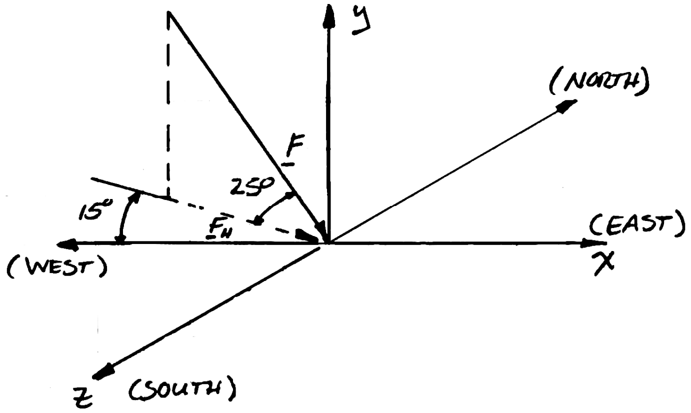
$\theta_z = 51.1°$ ◀

PROBLEM 2.74

Solve Problem 2.73, assuming that point *A* is located 15° north of west and that the barrel of the gun forms an angle of 25° with the horizontal.

PROBLEM 2.73 A gun is aimed at a point *A* located 35° east of north. Knowing that the barrel of the gun forms an angle of 40° with the horizontal and that the maximum recoil force is 400 N, determine (*a*) the *x*, *y*, and *z* components of that force, (*b*) the values of the angles θ_x, θ_y, and θ_z defining the direction of the recoil force. (Assume that the *x*, *y*, and *z* axes are directed, respectively, east, up, and south.)

SOLUTION



Recoil force $F = 400\text{ N}$

$$\therefore\quad F_H = (400\text{ N})\cos 25° = 362.52\text{ N}$$

(*a*)

$$F_x = +F_H\cos 15° = +(362.52\text{ N})\cos 15° = +350.17\text{ N}$$

$F_x = +350\text{ N}$ ◀

$$F_y = -F\sin 25° = -(400\text{ N})\sin 25° = -169.047\text{ N}$$

$F_y = -169.0\text{ N}$ ◀

$$F_z = +F_H\sin 15° = +(362.52\text{ N})\sin 15° = +93.827\text{ N}$$

$F_z = +93.8\text{ N}$ ◀

(*b*)

$$\cos\theta_x = \frac{F_x}{F} = \frac{+350.17\text{ N}}{400\text{ N}}$$

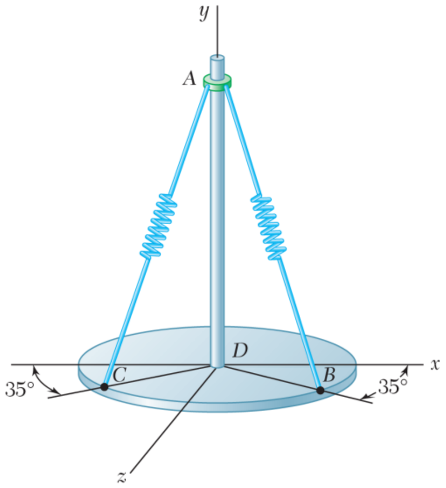
$\theta_x = 28.9°$ ◀

$$\cos\theta_y = \frac{F_y}{F} = \frac{-169.047\text{ N}}{400\text{ N}}$$

$\theta_y = 115.0°$ ◀

$$\cos\theta_z = \frac{F_z}{F} = \frac{+93.827\text{ N}}{400\text{ N}}$$

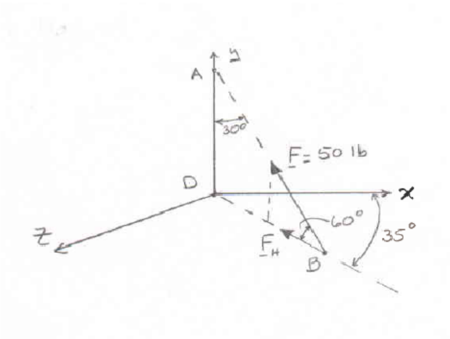
$\theta_z = 76.4°$ ◀



PROBLEM 2.75

The angle between spring *AB* and the post *DA* is 30°. Knowing that the tension in the spring is 50 lb, determine (*a*) the *x*, *y*, and *z* components of the force exerted on the circular plate at *B*, (*b*) the angles θ_x, θ_y, and θ_z defining the direction of the force at *B*.

SOLUTION



$$F_h = F\cos 60°$$
$$= (50 \text{ lb})\cos 60°$$
$$F_h = 25.0 \text{ lb}$$

$$F_x = -F_h \cos 35° \qquad F_y = F\sin 60° \qquad F_z = -F_h \sin 35°$$
$$F_x = (-25.0 \text{ lb})\cos 35° \qquad F_y = (50.0 \text{ lb})\sin 60° \qquad F_z = (-25.0 \text{ lb})\sin 35°$$
$$F_x = -20.479 \text{ lb} \qquad F_y = 43.301 \text{ lb} \qquad F_z = -14.3394 \text{ lb}$$

(*a*)

$F_x = -20.5 \text{ lb}$ ◀

$F_y = 43.3 \text{ lb}$ ◀

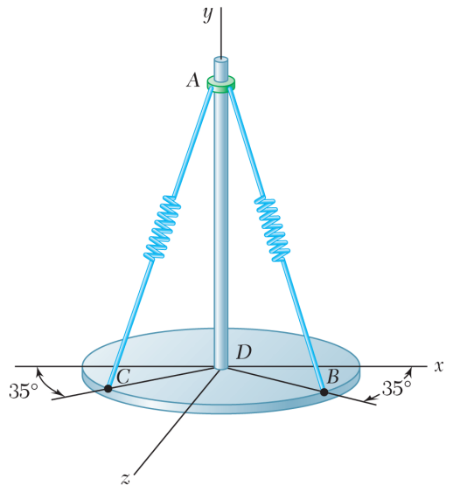
$F_z = -14.33 \text{ lb}$ ◀

(*b*)

$$\cos\theta_x = \frac{F_x}{F} = \frac{-20.479 \text{ lb}}{50 \text{ lb}}$$ $\theta_x = 114.2°$ ◀

$$\cos\theta_y = \frac{F_y}{F} = \frac{43.301 \text{ lb}}{50 \text{ lb}}$$ $\theta_y = 30.0°$ ◀

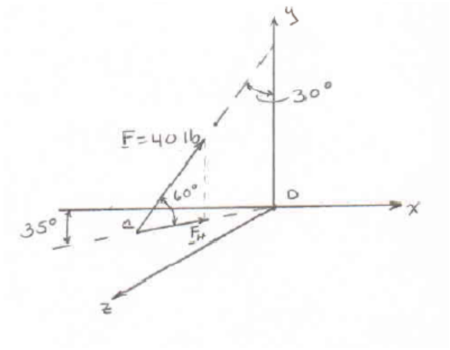
$$\cos\theta_z = \frac{F_z}{F} = \frac{-14.3394 \text{ lb}}{50 \text{ lb}}$$ $\theta_z = 106.7°$ ◀



PROBLEM 2.76

The angle between spring *AC* and the post *DA* is 30°. Knowing that the tension in the spring is 40 lb, determine (*a*) the *x*, *y*, and *z* components of the force exerted on the circular plate at *C*, (*b*) the angles θ_x, θ_y, and θ_z defining the direction of the force at *C*.

SOLUTION



$$F_h = F\cos 60°$$
$$= (40 \text{ lb})\cos 60°$$
$$F_h = 20.0 \text{ lb}$$

(*a*)

$$F_x = F_h \cos 35° \qquad F_y = F\sin 60° \qquad F_z = -F_h \sin 35°$$
$$= (20.0 \text{ lb})\cos 35° \qquad = (40 \text{ lb})\sin 60° \qquad = -(20.0 \text{ lb})\sin 35°$$
$$F_x = 16.3830 \text{ lb} \qquad F_y = 34.641 \text{ lb} \qquad F_z = -11.4715 \text{ lb}$$

$F_x = 16.38$ lb ◀

$F_y = 34.6$ lb ◀

$F_z = -11.47$ lb ◀

(*b*)

$$\cos\theta_x = \frac{F_x}{F} = \frac{16.3830 \text{ lb}}{40 \text{ lb}}$$

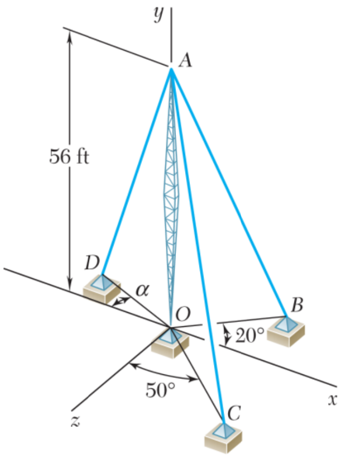
$\theta_x = 65.8°$ ◀

$$\cos\theta_y = \frac{F_y}{F} = \frac{34.641 \text{ lb}}{40 \text{ lb}}$$

$\theta_y = 30.0°$ ◀

$$\cos\theta_z = \frac{F_z}{F} = \frac{-11.4715 \text{ lb}}{40 \text{ lb}}$$

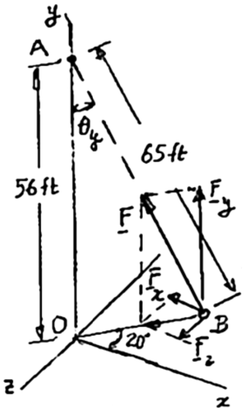
$\theta_z = 106.7°$ ◀



PROBLEM 2.77

Cable AB is 65 ft long, and the tension in that cable is 3900 lb. Determine (a) the x, y, and z components of the force exerted by the cable on the anchor B, (b) the angles θ_x, θ_y, and θ_z defining the direction of that force.

SOLUTION



From triangle AOB:

$$\cos\theta_y = \frac{56\text{ ft}}{65\text{ ft}}$$

$$= 0.86154$$

$$\theta_y = 30.51°$$

(a)

$$F_x = -F\sin\theta_y\cos 20°$$

$$= -(3900\text{ lb})\sin 30.51°\cos 20°$$

$F_x = -1861$ lb ◀

$$F_y = +F\cos\theta_y = (3900\text{ lb})(0.86154)$$

$F_y = +3360$ lb ◀

$$F_z = +(3900\text{ lb})\sin 30.51°\sin 20°$$

$F_z = +677$ lb ◀

(b)

$$\cos\theta_x = \frac{F_x}{F} = -\frac{1861\text{ lb}}{3900\text{ lb}} = -0.4771$$

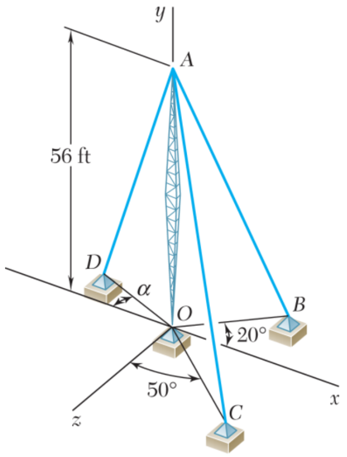
$\theta_x = 118.5°$ ◀

From above: $\theta_y = 30.51°$

$\theta_y = 30.5°$ ◀

$$\cos\theta_z = \frac{F_z}{F} = +\frac{677\text{ lb}}{3900\text{ lb}} = +0.1736$$

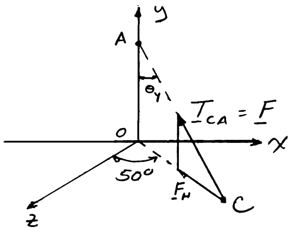
$\theta_z = 80.0°$ ◀



PROBLEM 2.78

Cable *AC* is 70 ft long, and the tension in that cable is 5250 lb. Determine (*a*) the *x*, *y*, and *z* components of the force exerted by the cable on the anchor *C*, (*b*) the angles θ_x, θ_y, and θ_z defining the direction of that force.

SOLUTION



In triangle *AOB*:

$$AC = 70\text{ ft}$$
$$OA = 56\text{ ft}$$
$$F = 5250\text{ lb}$$

$$\cos\theta_y = \frac{56\text{ ft}}{70\text{ ft}}$$
$$\theta_y = 36.870°$$
$$F_H = F\sin\theta_y$$
$$= (5250\text{ lb})\sin 36.870°$$
$$= 3150.0\text{ lb}$$

(*a*) $F_x = -F_H\sin 50° = -(3150.0\text{ lb})\sin 50° = -2413.0\text{ lb}$

$F_x = -2410\text{ lb}$ ◀

$F_y = +F\cos\theta_y = +(5250\text{ lb})\cos 36.870° = +4200.0\text{ lb}$

$F_y = +4200\text{ lb}$ ◀

$F_z = -F_H\cos 50° = -3150\cos 50° = -2024.8\text{ lb}$ $\quad F_z = -2025\text{ lb}$ ◀

(*b*) $\cos\theta_x = \dfrac{F_x}{F} = \dfrac{-2413.0\text{ lb}}{5250\text{ lb}}$ $\quad \theta_x = 117.4°$ ◀

From above: $\theta_y = 36.870°$ $\quad \theta_y = 36.9°$ ◀

$\theta_z = \dfrac{F_z}{F} = \dfrac{-2024.8\text{ lb}}{5250\text{ lb}}$ $\quad \theta_z = 112.7°$ ◀

PROBLEM 2.79

Determine the magnitude and direction of the force **F** = (240 N)**i** – (270 N)**j** + (680 N)**k**.

SOLUTION

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$F = \sqrt{(240\text{ N})^2 + (-270\text{ N})^2 + (-680\text{ N})^2}$$

$F = 770\text{ N}$ ◀

$$\cos\theta_x = \frac{F_x}{F} = \frac{240\text{ N}}{770\text{ N}}$$

$\theta_x = 71.8°$ ◀

$$\cos\theta_y = \frac{F_y}{F} = \frac{-270\text{ N}}{770\text{ N}}$$

$\theta_y = 110.5°$ ◀

$$\cos\theta_y = \frac{F_z}{F} = \frac{680\text{ N}}{770\text{ N}}$$

$\theta_z = 28.0°$ ◀

PROBLEM 2.80

Determine the magnitude and direction of the force **F** = (320 N)**i** + (400 N)**j** − (250 N)**k**.

SOLUTION

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$F = \sqrt{(320\text{ N})^2 + (400\text{ N})^2 + (-250\text{ N})^2}$$

$F = 570\text{ N}$ ◀

$$\cos\theta_x = \frac{F_x}{F} = \frac{320\text{ N}}{570\text{ N}}$$

$\theta_x = 55.8°$ ◀

$$\cos\theta_y = \frac{F_y}{F} = \frac{400\text{ N}}{570\text{ N}}$$

$\theta_y = 45.4°$ ◀

$$\cos\theta_y = \frac{F_z}{F} = \frac{-250\text{ N}}{570\text{ N}}$$

$\theta_z = 116.0°$ ◀

PROBLEM 2.81

A force acts at the origin of a coordinate system in a direction defined by the angles $\theta_x = 69.3°$ and $\theta_z = 57.9°$. Knowing that the y component of the force is –174.0 lb, determine (*a*) the angle θ_y, (*b*) the other components and the magnitude of the force.

SOLUTION

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

$$\cos^2(69.3°) + \cos^2\theta_y + \cos^2(57.9°) = 1$$

$$\cos\theta_y = \pm 0.7699$$

(*a*) Since $F_y < 0,$ we choose $\cos\theta_y = -0.7699$ $\therefore\ \theta_y = 140.3°$ ◀

(*b*)

$$F_y = F\cos\theta_y$$

$$-174.0\text{ lb} = F(-0.7699)$$

$$F = 226.0\text{ lb}$$ $F = 226\text{ lb}$ ◀

$$F_x = F\cos\theta_x = (226.0\text{ lb})\cos 69.3°$$ $F_x = 79.9\text{ lb}$ ◀

$$F_z = F\cos\theta_z = (226.0\text{ lb})\cos 57.9°$$ $F_z = 120.1\text{ lb}$ ◀

PROBLEM 2.82

A force acts at the origin of a coordinate system in a direction defined by the angles $\theta_x = 70.9°$ and $\theta_y = 144.9°$. Knowing that the z component of the force is -52.0 lb, determine (*a*) the angle θ_z, (*b*) the other components and the magnitude of the force.

SOLUTION

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

$$\cos^2 70.9^\circ + \cos^2 144.9^\circ + \cos^2\theta_z{}^\circ = 1$$

$$\cos\theta_z = \pm 0.47282$$

(*a*) Since $F_z < 0$, we choose $\cos\theta_z = -0.47282$ $\therefore\ \theta_z = 118.2^\circ$ ◀

(*b*)

$$F_z = F\cos\theta_z$$

$$-52.0\ lb = F(-0.47282)$$

$$F = 110.0 \text{ lb}$$

$F = 110.0$ lb ◀

$$F_x = F\cos\theta_x = (110.0 \text{ lb})\cos 70.9^\circ$$

$F_x = 36.0$ lb ◀

$$F_y = F\cos\theta_y = (110.0 \text{ lb})\cos 144.9^\circ$$

$F_y = -90.0$ lb ◀

PROBLEM 2.83

A force **F** of magnitude 210 N acts at the origin of a coordinate system. Knowing that $F_x = 80$ N, $\theta_z = 151.2°$, and $F_y < 0$, determine (*a*) the components F_y and F_z, (*b*) the angles θ_x and θ_y.

SOLUTION

(*a*)

$$F_z = F\cos\theta_z = (210\text{ N})\cos 151.2°$$

$$= -184.024\text{ N} \qquad F_z = -184.0\text{ N} \blacktriangleleft$$

Then: $$F^2 = F_x^2 + F_y^2 + F_z^2$$

So: $$(210\text{ N})^2 = (80\text{ N})^2 + (F_y)^2 + (184.024\text{ N})^2$$

Hence: $$F_y = -\sqrt{(210\text{ N})^2 - (80\text{ N})^2 - (184.024\text{ N})^2}$$

$$= -61.929\text{ N} \qquad F_y = -62.0\text{ lb} \blacktriangleleft$$

(*b*)

$$\cos\theta_x = \frac{F_x}{F} = \frac{80\text{ N}}{210\text{ N}} = 0.38095 \qquad \theta_x = 67.6° \blacktriangleleft$$

$$\cos\theta_y = \frac{F_y}{F} = \frac{61.929\text{ N}}{210\text{ N}} = -0.29490$$

$$\theta_y = 107.2° \blacktriangleleft$$

PROBLEM 2.84

A force **F** of magnitude 1200 N acts at the origin of a coordinate system. Knowing that $\theta_x = 65°$, $\theta_y = 40°$, and $F_z > 0$, determine (*a*) the components of the force, (*b*) the angle θ_z.

SOLUTION

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

$$\cos^2 65^\circ + \cos^2 40^\circ + \cos^2\theta_z{}^\circ = 1$$

$$\cos\theta_z = \pm 0.48432$$

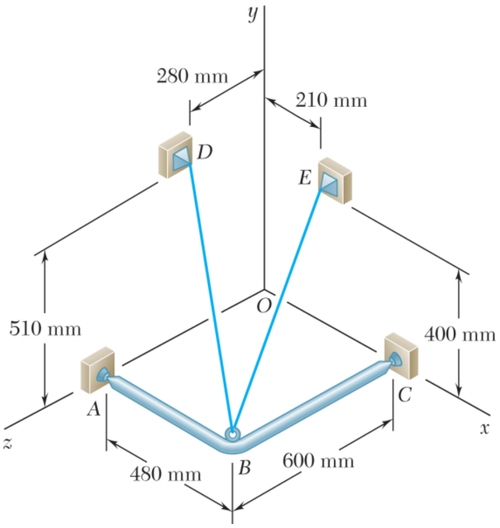
(*b*) Since $F_z > 0$, we choose $\cos\theta_z = 0.48432$, or $\theta_z = 61.032^\circ$ $\quad\therefore\ \theta_z = 61.0°$ ◀

(*a*) $F = 1200\text{ N}$

$$F_x = F\cos\theta_x = (1200\text{ N})\cos 65^\circ \qquad F_x = 507\text{ N}$$ ◀

$$F_y = F\cos\theta_y = (1200\text{ N})\cos 40^\circ \qquad F_y = 919\text{ N}$$ ◀

$$F_z = F\cos\theta_z = (1200\text{ N})\cos 61.032^\circ \qquad F_z = 582\text{ N}$$ ◀



PROBLEM 2.85

A frame *ABC* is supported in part by cable *DBE* that passes through a frictionless ring at *B*. Knowing that the tension in the cable is 385 N, determine the components of the force exerted by the cable on the support at *D*.

SOLUTION

$$\overrightarrow{DB} = (480 \text{ mm})\mathbf{i} - (510 \text{ mm})\mathbf{j} + (320 \text{ mm})\mathbf{k}$$

$$DB = \sqrt{(480 \text{ mm})^2 + (510 \text{ mm}^2) + (320 \text{ mm})^2}$$

$$= 770 \text{ mm}$$

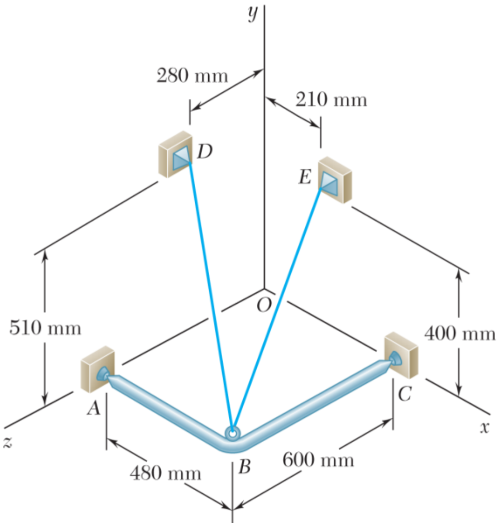
$$\mathbf{F} = F\boldsymbol{\lambda}_{DB}$$

$$= F\frac{\overrightarrow{DB}}{DB}$$

$$= \frac{385 \text{ N}}{770 \text{ mm}}[(480 \text{ mm})\mathbf{i} - (510 \text{ mm})\mathbf{j} + (320 \text{ mm})\mathbf{k}]$$

$$= (240 \text{ N})\mathbf{i} - (255 \text{ N})\mathbf{j} + (160 \text{ N})\mathbf{k}$$

$$F_x = +240 \text{ N}, \quad F_y = -255 \text{ N}, \quad F_z = +160.0 \text{ N} \blacktriangleleft$$



PROBLEM 2.86

For the frame and cable of Problem 2.85, determine the components of the force exerted by the cable on the support at *E*.

PROBLEM 2.85 A frame *ABC* is supported in part by cable *DBE* that passes through a frictionless ring at *B*. Knowing that the tension in the cable is 385 N, determine the components of the force exerted by the cable on the support at *D*.

SOLUTION

$$\overrightarrow{EB} = (270\text{ mm})\mathbf{i} - (400\text{ mm})\mathbf{j} + (600\text{ mm})\mathbf{k}$$

$$EB = \sqrt{(270\text{ mm})^2 + (400\text{ mm})^2 + (600\text{ mm})^2}$$

$$= 770\text{ mm}$$

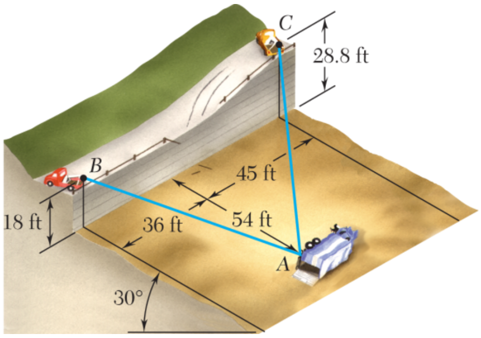
$$\mathbf{F} = F\boldsymbol{\lambda}_{EB}$$

$$= F\frac{\overrightarrow{EB}}{EB}$$

$$= \frac{385\text{ N}}{770\text{ mm}}[(270\text{ mm})\mathbf{i} - (400\text{ mm})\mathbf{j} + (600\text{ mm})\mathbf{k}]$$

$$\mathbf{F} = (135\text{ N})\mathbf{i} - (200\text{ N})\mathbf{j} + (300\text{ N})\mathbf{k}$$

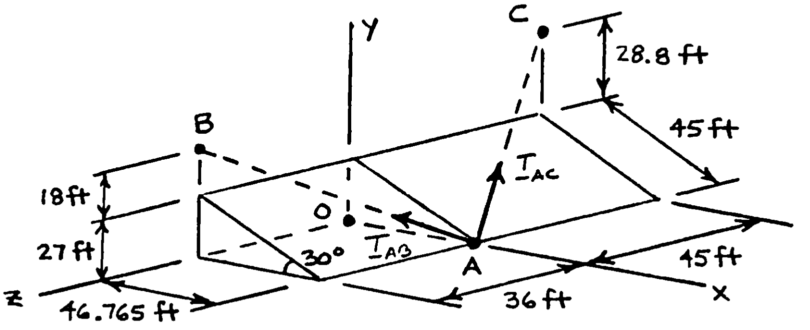
$F_x = +135.0\text{ N},\quad F_y = -200\text{ N},\quad F_z = +300\text{ N}$ ◀


C
28.8 ft
B
45 ft
54 ft
18 ft
36 ft
A
30°

PROBLEM 2.87

In order to move a wrecked truck, two cables are attached at *A* and pulled by winches *B* and *C* as shown. Knowing that the tension in cable *AB* is 2 kips, determine the components of the force exerted at *A* by the cable.

SOLUTION


y
C
28.8 ft
45 ft
B
T_{AC}
18 ft
O
T_{AB}
27 ft
30°
A
45 ft
z
46.765 ft
36 ft
x

AB = 74.216 ft　　AC = 85.590 ft

Cable *AB*:

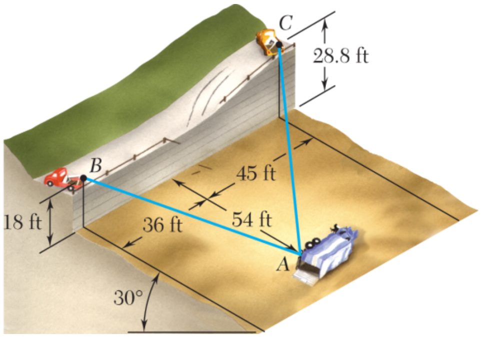
$$\lambda_{AB} = \frac{\overrightarrow{AB}}{AB} = \frac{(-46.765 \text{ ft})\mathbf{i} + (45 \text{ ft})\mathbf{j} + (36 \text{ ft})\mathbf{k}}{74.216 \text{ ft}}$$

$$\mathbf{T}_{AB} = T_{AB}\lambda_{AB} = \frac{-46.765\mathbf{i} + 45\mathbf{j} + 36\mathbf{k}}{74.216}$$

$(T_{AB})_x = -1.260$ kips ◀

$(T_{AB})_y = +1.213$ kips ◀

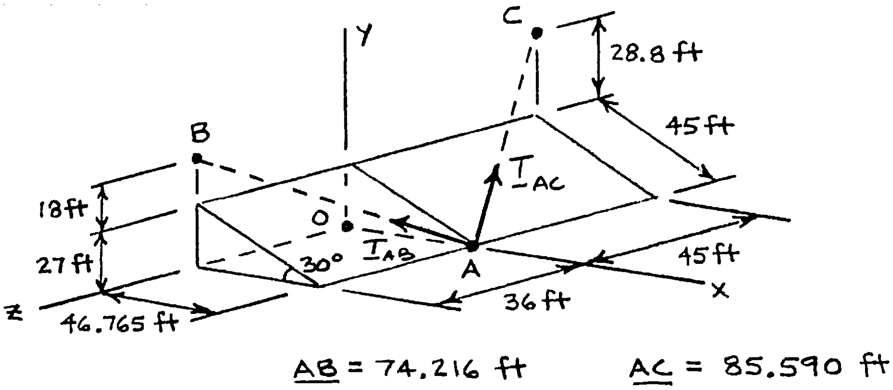
$(T_{AB})_z = +0.970$ kips ◀


C
28.8 ft
B
45 ft
18 ft
36 ft
54 ft
A
30°

PROBLEM 2.88

In order to move a wrecked truck, two cables are attached at *A* and pulled by winches *B* and *C* as shown. Knowing that the tension in cable *AC* is 1.5 kips, determine the components of the force exerted at *A* by the cable.

SOLUTION


Y
C
28.8 ft
45 ft
B
T_{AC}
18 ft
O
27 ft
30°
T_{AB}
A
45 ft
x
z
46.765 ft
36 ft
AB = 74.216 ft
AC = 85.590 ft

Cable *AB*:

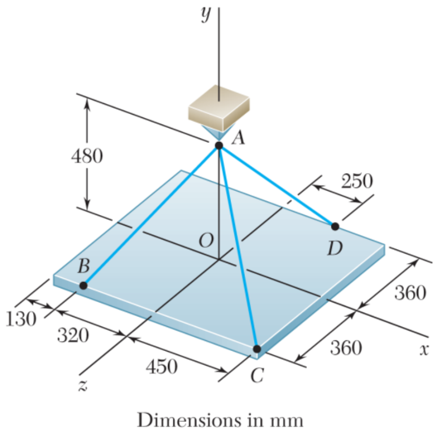
$$\boldsymbol{\lambda}_{AC} = \frac{\overrightarrow{AC}}{AC} = \frac{(-46.765\text{ ft})\mathbf{i} + (55.8\text{ ft})\mathbf{j} + (-45\text{ ft})\mathbf{k}}{85.590\text{ ft}}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = (1.5\text{ kips})\frac{-46.765\mathbf{i} + 55.8\mathbf{j} - 45\mathbf{k}}{85.590}$$

$(T_{AC})_x = -0.820\text{ kips}$ ◀

$(T_{AC})_y = +0.978\text{ kips}$ ◀

$(T_{AC})_z = -0.789\text{ kips}$ ◀



PROBLEM 2.89

A rectangular plate is supported by three cables as shown. Knowing that the tension in cable *AB* is 408 N, determine the components of the force exerted on the plate at *B*.

SOLUTION

We have:

$$\overrightarrow{BA} = +(320\text{ mm})\mathbf{i} + (480\text{ mm})\mathbf{j} - (360\text{ mm})\mathbf{k} \qquad BA = 680\text{ mm}$$

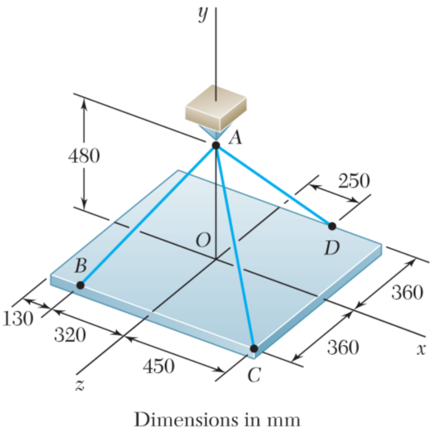
Thus:

$$\mathrm{F}_B = T_{BA}\boldsymbol{\lambda}_{BA} = T_{BA}\frac{\overrightarrow{BA}}{BA} = T_{BA}\left(\frac{8}{17}\mathbf{i} + \frac{12}{17}\mathbf{j} - \frac{9}{17}\mathbf{k}\right)$$

$$\left(\frac{8}{17}T_{BA}\right)\mathbf{i} + \left(\frac{12}{17}T_{BA}\right)\mathbf{j} - \left(\frac{9}{17}T_{BA}\right)\mathbf{k} = 0$$

Setting $T_{BA} = 408\text{ N}$ yields,

$$F_x = +192.0\text{ N},\quad F_y = +288\text{ N},\quad F_z = -216\text{ N} \blacktriangleleft$$



PROBLEM 2.90

A rectangular plate is supported by three cables as shown. Knowing that the tension in cable *AD* is 429 N, determine the components of the force exerted on the plate at *D*.

SOLUTION

We have:

$$\overrightarrow{DA} = -(250\text{ mm})\mathbf{i} + (480\text{ mm})\mathbf{j} + (360\text{ mm})\mathbf{k} \quad DA = 650\text{ mm}$$

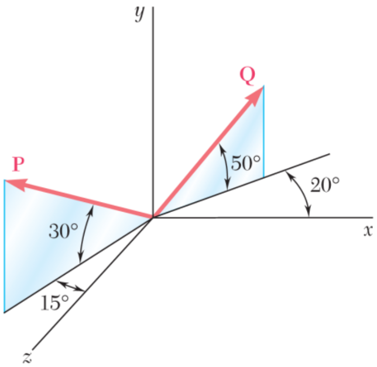
Thus:

$$\mathrm{F}_D = T_{DA}\boldsymbol{\lambda}_{DA} = T_{DA}\frac{\overrightarrow{DA}}{DA} = T_{DA}\left(-\frac{5}{13}\mathbf{i} + \frac{48}{65}\mathbf{j} + \frac{36}{65}\mathbf{k}\right)$$

$$-\left(\frac{5}{13}T_{DA}\right)\mathbf{i} + \left(\frac{48}{65}T_{DA}\right)\mathbf{j} + \left(\frac{36}{65}T_{DA}\right)\mathbf{k} = 0$$

Setting $T_{DA} = 429\text{ N}$ yields,

$$F_x = -165.0\text{ N}, \quad F_y = +317\text{ N}, \quad F_z = +238\text{ N} \blacktriangleleft$$



PROBLEM 2.91

Find the magnitude and direction of the resultant of the two forces shown knowing that $P = 300$ N and $Q = 400$ N.

SOLUTION

$$\begin{aligned}\mathbf{P} &= (300\text{ N})[-\cos 30^\circ \sin 15^\circ \mathbf{i} + \sin 30^\circ \mathbf{j} + \cos 30^\circ \cos 15^\circ \mathbf{k}] \\ &= -(67.243\text{ N})\mathbf{i} + (150\text{ N})\mathbf{j} + (250.95\text{ N})\mathbf{k} \\ \mathbf{Q} &= (400\text{ N})[\cos 50^\circ \cos 20^\circ \mathbf{i} + \sin 50^\circ \mathbf{j} - \cos 50^\circ \sin 20^\circ \mathbf{k}] \\ &= (400\text{ N})[0.60402\mathbf{i} + 0.76604\mathbf{j} - 0.21985] \\ &= (241.61\text{ N})\mathbf{i} + (306.42\text{ N})\mathbf{j} - (87.939\text{ N})\mathbf{k} \\ \mathbf{R} &= \mathbf{P} + \mathbf{Q} \\ &= (174.367\text{ N})\mathbf{i} + (456.42\text{ N})\mathbf{j} + (163.011\text{ N})\mathbf{k} \\ R &= \sqrt{(174.367\text{ N})^2 + (456.42\text{ N})^2 + (163.011\text{ N})^2} \\ &= 515.07\text{ N}\end{aligned}$$

$R = 515$ N ◀

$$\cos\theta_x = \frac{R_x}{R} = \frac{174.367\text{ N}}{515.07\text{ N}} = 0.33853$$

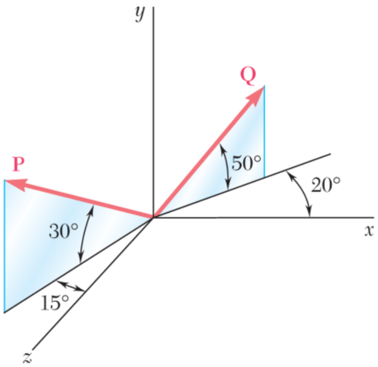
$\theta_x = 70.2°$ ◀

$$\cos\theta_y = \frac{R_y}{R} = \frac{456.42\text{ N}}{515.07\text{ N}} = 0.88613$$

$\theta_y = 27.6°$ ◀

$$\cos\theta_z = \frac{R_z}{R} = \frac{163.011\text{ N}}{515.07\text{ N}} = 0.31648$$

$\theta_z = 71.5°$ ◀



PROBLEM 2.92

Find the magnitude and direction of the resultant of the two forces shown knowing that $P = 400$ N and $Q = 300$ N.

SOLUTION

$$\mathbf{P} = (400\text{ N})[-\cos 30°\sin 15°\mathbf{i} + \sin 30°\mathbf{j} + \cos 30°\cos 15°\mathbf{k}]$$

$$= -(89.678\text{ N})\mathbf{i} + (200\text{ N})\mathbf{j} + (334.61\text{ N})\mathbf{k}$$

$$\mathbf{Q} = (300\text{ N})[\cos 50°\cos 20°\mathbf{i} + \sin 50°\mathbf{j} - \cos 50°\sin 20°\mathbf{k}]$$

$$= (181.21\text{ N})\mathbf{i} + (229.81\text{ N})\mathbf{j} - (65.954\text{ N})\mathbf{k}$$

$$\mathbf{R} = \mathbf{P} + \mathbf{Q}$$

$$= (91.532\text{ N})\mathbf{i} + (429.81\text{ N})\mathbf{j} + (268.66\text{ N})\mathbf{k}$$

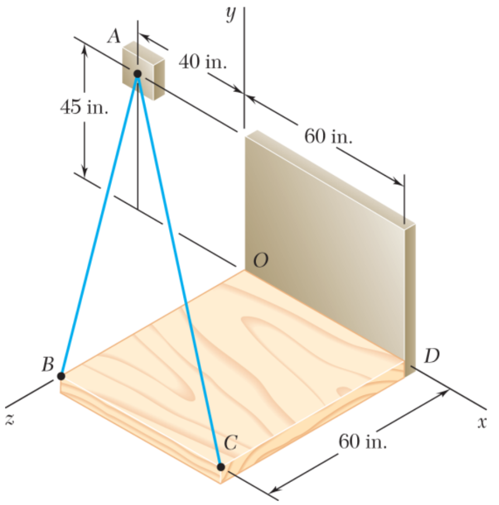
$$R = \sqrt{(91.532\text{ N})^2 + (429.81\text{ N})^2 + (268.66\text{ N})^2}$$

$$= 515.07\text{ N} \qquad R = 515\text{ N} ◀$$

$$\cos\theta_x = \frac{R_x}{R} = \frac{91.532\text{ N}}{515.07\text{ N}} = 0.177708 \qquad \theta_x = 79.8° ◀$$

$$\cos\theta_y = \frac{R_y}{R} = \frac{429.81\text{ N}}{515.07\text{ N}} = 0.83447 \qquad \theta_y = 33.4° ◀$$

$$\cos\theta_z = \frac{R_z}{R} = \frac{268.66\text{ N}}{515.07\text{ N}} = 0.52160 \qquad \theta_z = 58.6° ◀$$



PROBLEM 2.93

Knowing that the tension is 425 lb in cable *AB* and 510 lb in cable *AC*, determine the magnitude and direction of the resultant of the forces exerted at *A* by the two cables.

SOLUTION

$$\overrightarrow{AB} = (40\text{ in.})\mathbf{i} - (45\text{ in.})\mathbf{j} + (60\text{ in.})\mathbf{k}$$

$$AB = \sqrt{(40\text{ in.})^2 + (45\text{ in.})^2 + (60\text{ in.})^2} = 85\text{ in.}$$

$$\overrightarrow{AC} = (100\text{ in.})\mathbf{i} - (45\text{ in.})\mathbf{j} + (60\text{ in.})\mathbf{k}$$

$$AC = \sqrt{(100\text{ in.})^2 + (45\text{ in.})^2 + (60\text{ in.})^2} = 125\text{ in.}$$

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = (425\text{ lb})\left[\frac{(40\text{ in.})\mathbf{i} - (45\text{ in.})\mathbf{j} + (60\text{ in.})\mathbf{k}}{85\text{ in.}}\right]$$

$$\mathbf{T}_{AB} = (200\text{ lb})\mathbf{i} - (225\text{ lb})\mathbf{j} + (300\text{ lb})\mathbf{k}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = (510\text{ lb})\left[\frac{(100\text{ in.})\mathbf{i} - (45\text{ in.})\mathbf{j} + (60\text{ in.})\mathbf{k}}{125\text{ in.}}\right]$$

$$\mathbf{T}_{AC} = (408\text{ lb})\mathbf{i} - (183.6\text{ lb})\mathbf{j} + (244.8\text{ lb})\mathbf{k}$$

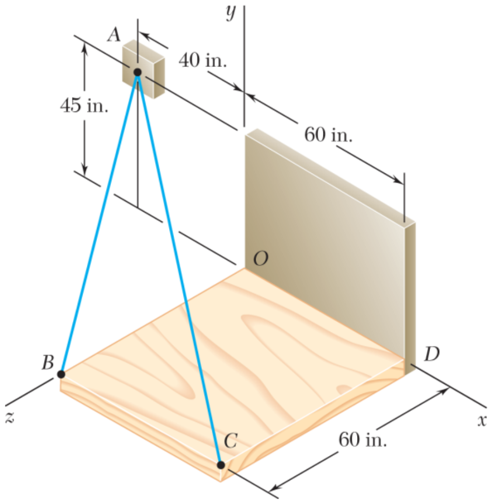
$$\mathbf{R} = \mathbf{T}_{AB} + \mathbf{T}_{AC} = (608)\mathbf{i} - (408.6\text{ lb})\mathbf{j} + (544.8\text{ lb})\mathbf{k}$$

Then: $R = 912.92\text{ lb}$ $\qquad R = 913\text{ lb}$ ◀

and $\cos\theta_x = \dfrac{608\text{ lb}}{912.92\text{ lb}} = 0.66599$ $\qquad \theta_x = 48.2°$ ◀

$\cos\theta_y = \dfrac{408.6\text{ lb}}{912.92\text{ lb}} = -0.44757$ $\qquad \theta_y = 116.6°$ ◀

$\cos\theta_z = \dfrac{544.8\text{ lb}}{912.92\text{ lb}} = 0.59677$ $\qquad \theta_z = 53.4°$ ◀



PROBLEM 2.94

Knowing that the tension is 510 lb in cable *AB* and 425 lb in cable *AC*, determine the magnitude and direction of the resultant of the forces exerted at *A* by the two cables.

SOLUTION

$$\overrightarrow{AB} = (40\text{ in.})\mathbf{i} - (45\text{ in.})\mathbf{j} + (60\text{ in.})\mathbf{k}$$

$$AB = \sqrt{(40\text{ in.})^2 + (45\text{ in.})^2 + (60\text{ in.})^2} = 85\text{ in.}$$

$$\overrightarrow{AC} = (100\text{ in.})\mathbf{i} - (45\text{ in.})\mathbf{j} + (60\text{ in.})\mathbf{k}$$

$$AC = \sqrt{(100\text{ in.})^2 + (45\text{ in.})^2 + (60\text{ in.})^2} = 125\text{ in.}$$

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = (510\text{ lb})\left[\frac{(40\text{ in.})\mathbf{i} - (45\text{ in.})\mathbf{j} + (60\text{ in.})\mathbf{k}}{85\text{ in.}}\right]$$

$$\mathbf{T}_{AB} = (240\text{ lb})\mathbf{i} - (270\text{ lb})\mathbf{j} + (360\text{ lb})\mathbf{k}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = (425\text{ lb})\left[\frac{(100\text{ in.})\mathbf{i} - (45\text{ in.})\mathbf{j} + (60\text{ in.})\mathbf{k}}{125\text{ in.}}\right]$$

$$\mathbf{T}_{AC} = (340\text{ lb})\mathbf{i} - (153\text{ lb})\mathbf{j} + (204\text{ lb})\mathbf{k}$$

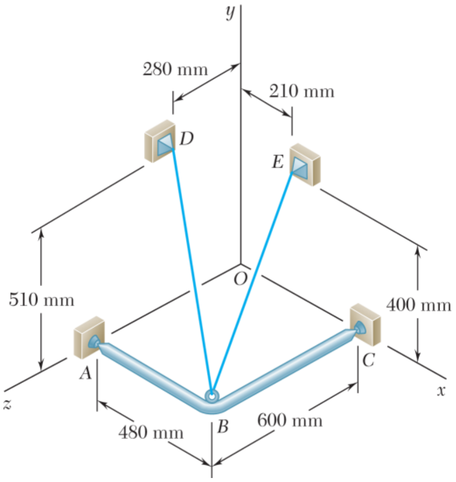
$$\mathbf{R} = \mathbf{T}_{AB} + \mathbf{T}_{AC} = (580\text{ lb})\mathbf{i} - (423\text{ lb})\mathbf{j} + (564\text{ lb})\mathbf{k}$$

Then: $R = 912.92\text{ lb}$ $\qquad R = 913\text{ lb}$ ◀

and $\cos\theta_x = \dfrac{580\text{ lb}}{912.92\text{ lb}} = 0.63532$ $\qquad \theta_x = 50.6°$ ◀

$\cos\theta_y = \dfrac{-423\text{ lb}}{912.92\text{ lb}} = -0.46335$ $\qquad \theta_y = 117.6°$ ◀

$\cos\theta_z = \dfrac{564\text{ lb}}{912.92\text{ lb}} = 0.61780$ $\qquad \theta_z = 51.8°$ ◀



PROBLEM 2.95

For the frame of Problem 2.85, determine the magnitude and direction of the resultant of the forces exerted by the cable at *B* knowing that the tension in the cable is 385 N.

PROBLEM 2.85 A frame *ABC* is supported in part by cable *DBE* that passes through a frictionless ring at *B*. Knowing that the tension in the cable is 385 N, determine the components of the force exerted by the cable on the support at *D*.

SOLUTION

$$\overrightarrow{BD} = -(480\text{ mm})\mathbf{i} + (510\text{ mm})\mathbf{j} - (320\text{ mm})\mathbf{k}$$

$$BD = \sqrt{(480\text{ mm})^2 + (510\text{ mm})^2 + (320\text{ mm})^2} = 770\text{ mm}$$

$$\mathbf{F}_{BD} = T_{BD}\boldsymbol{\lambda}_{BD} = T_{BD}\frac{\overrightarrow{BD}}{BD}$$

$$= \frac{(385\text{ N})}{(770\text{ mm})}[-(480\text{ mm})\mathbf{i} + (510\text{ mm})\mathbf{j} - (320\text{ mm})\mathbf{k}]$$

$$= -(240\text{ N})\mathbf{i} + (255\text{ N})\mathbf{j} - (160\text{ N})\mathbf{k}$$

$$\overrightarrow{BE} = -(270\text{ mm})\mathbf{i} + (400\text{ mm})\mathbf{j} - (600\text{ mm})\mathbf{k}$$

$$BE = \sqrt{(270\text{ mm})^2 + (400\text{ mm})^2 + (600\text{ mm})^2} = 770\text{ mm}$$

$$\mathbf{F}_{BE} = T_{BE}\boldsymbol{\lambda}_{BE} = T_{BE}\frac{\overrightarrow{BE}}{BE}$$

$$= \frac{(385\text{ N})}{(770\text{ mm})}[-(270\text{ mm})\mathbf{i} + (400\text{ mm})\mathbf{j} - (600\text{ mm})\mathbf{k}]$$

$$= -(135\text{ N})\mathbf{i} + (200\text{ N})\mathbf{j} - (300\text{ N})\mathbf{k}$$

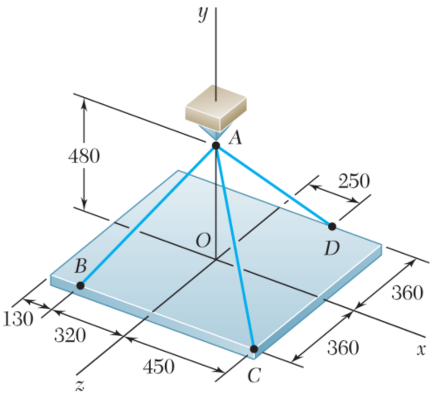
$$\mathbf{R} = \mathbf{F}_{BD} + \mathbf{F}_{BE} = -(375\text{ N})\mathbf{i} + (455\text{ N})\mathbf{j} - (460\text{ N})\mathbf{k}$$

$$R = \sqrt{(375\text{ N})^2 + (455\text{ N})^2 + (460\text{ N})^2} = 747.83\text{ N} \qquad R = 748\text{ N} \blacktriangleleft$$

$$\cos\theta_x = \frac{-375\text{ N}}{747.83\text{ N}} \qquad \theta_x = 120.1° \blacktriangleleft$$

$$\cos\theta_y = \frac{455\text{ N}}{747.83\text{ N}} \qquad \theta_y = 52.5° \blacktriangleleft$$

$$\cos\theta_z = \frac{-460\text{ N}}{747.83\text{ N}} \qquad \theta_z = 128.0° \blacktriangleleft$$



Dimensions in mm

PROBLEM 2.96

For the plate of Prob. 2.89, determine the tensions in cables *AB* and *AD* knowing that the tension in cable *AC* is 54 N and that the resultant of the forces exerted by the three cables at *A* must be vertical.

SOLUTION

We have:

$$\overrightarrow{AB} = -(320\text{ mm})\mathbf{i} - (480\text{ mm})\mathbf{j} + (360\text{ mm})\mathbf{k} \qquad AB = 680\text{ mm}$$

$$\overrightarrow{AC} = (450\text{ mm})\mathbf{i} - (480\text{ mm})\mathbf{j} + (360\text{ mm})\mathbf{k} \qquad AC = 750\text{ mm}$$

$$\overrightarrow{AD} = (250\text{ mm})\mathbf{i} - (480\text{ mm})\mathbf{j} - (360\text{ mm})\mathbf{k} \qquad AD = 650\text{ mm}$$

Thus:

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = \frac{T_{AB}}{680}(-320\mathbf{i} - 480\mathbf{j} + 360\mathbf{k})$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = \frac{54}{750}(450\mathbf{i} - 480\mathbf{j} + 360\mathbf{k})$$

$$\mathbf{T}_{AD} = T_{AD}\boldsymbol{\lambda}_{AD} = T_{AD}\frac{\overrightarrow{AD}}{AD} = \frac{T_{AD}}{650}(250\mathbf{i} - 480\mathbf{j} - 360\mathbf{k})$$

Substituting into the Eq. $\mathbf{R} = \Sigma\mathbf{F}$ and factoring $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\begin{aligned}\mathbf{R} = &\left(-\frac{320}{680}T_{AB} + 32.40 + \frac{250}{650}T_{AD}\right)\mathbf{i}\\ &+\left(-\frac{480}{680}T_{AB} - 34.560 - \frac{480}{650}T_{AD}\right)\mathbf{j}\\ &+\left(\frac{360}{680}T_{AB} + 25.920 - \frac{360}{650}T_{AD}\right)\mathbf{k}\end{aligned}$$

PROBLEM 2.96 (Continued)

Since **R** is vertical, the coefficients of **i** and **k** are zero:

$$\mathbf{i}: \qquad -\frac{320}{680}T_{AB} + 32.40 + \frac{250}{650}T_{AD} = 0 \qquad (1)$$

$$\mathbf{k}: \qquad \frac{360}{680}T_{AB} + 25.920 - \frac{360}{650}T_{AD} = 0 \qquad (2)$$

Multiply (1) by 3.6 and (2) by 2.5 then add:

$$-\frac{252}{680}T_{AB} + 181.440 = 0$$

$$T_{AB} = 489.60 \text{ N}$$

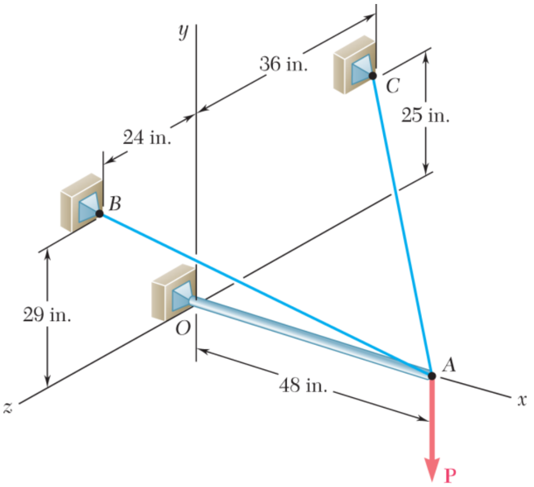
$T_{AB} = 490$ N ◀

Substitute into (2) and solve for T_{AD}:

$$\frac{360}{680}(489.60 \text{ N}) + 25.920 - \frac{360}{650}T_{AD} = 0$$

$$T_{AD} = 514.80 \text{ N}$$

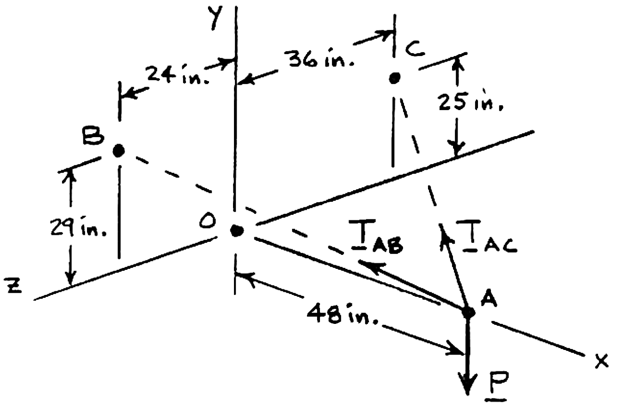
$T_{AD} = 515$ N ◀



PROBLEM 2.97

The boom *OA* carries a load **P** and is supported by two cables as shown. Knowing that the tension in cable *AB* is 183 lb and that the resultant of the load **P** and of the forces exerted at *A* by the two cables must be directed along *OA*, determine the tension in cable *AC*.

SOLUTION



Cable *AB*: $T_{AB} = 183\text{ lb}$

$$\mathbf{T}_{AB} = T_{AB}\lambda_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = (183\text{ lb})\frac{(-48\text{ in.})\mathbf{i} + (29\text{ in.})\mathbf{j} + (24\text{ in.})\mathbf{k}}{61\text{ in.}}$$

$$\mathbf{T}_{AB} = -(144\text{ lb})\mathbf{i} + (87\text{ lb})\mathbf{j} + (72\text{ lb})\mathbf{k}$$

Cable *AC*:

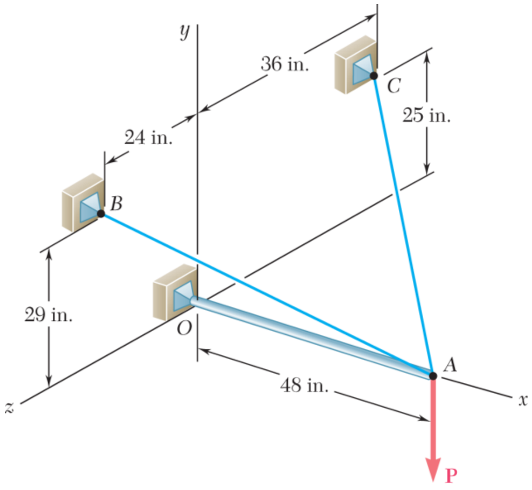
$$\mathbf{T}_{AC} = T_{AC}\lambda_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = T_{AC}\frac{(-48\text{ in.})\mathbf{i} + (25\text{ in.})\mathbf{j} + (-36\text{ in.})\mathbf{k}}{65\text{ in.}}$$

$$\mathbf{T}_{AC} = -\frac{48}{65}T_{AC}\mathbf{i} + \frac{25}{65}T_{AC}\mathbf{j} - \frac{36}{65}T_{AC}\mathbf{k}$$

Load *P*: $\mathbf{P} = P\,\mathbf{j}$

For resultant to be directed along *OA*, i.e., *x*-axis

$$R_z = 0: \quad \Sigma F_z = (72\text{ lb}) - \frac{36}{65}T'_{AC} = 0 \qquad T_{AC} = 130.0\text{ lb} \blacktriangleleft$$



PROBLEM 2.98

For the boom and loading of Problem. 2.97, determine the magnitude of the load **P**.

PROBLEM 2.97 The boom *OA* carries a load **P** and is supported by two cables as shown. Knowing that the tension in cable *AB* is 183 lb and that the resultant of the load **P** and of the forces exerted at *A* by the two cables must be directed along *OA*, determine the tension in cable *AC*.

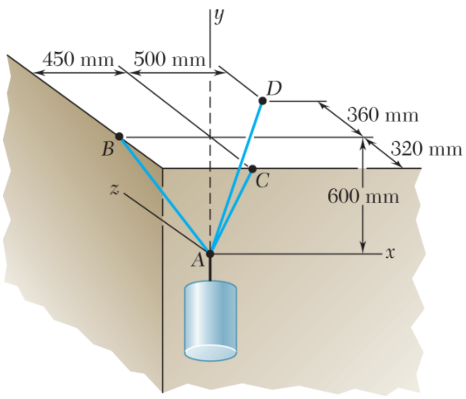
SOLUTION

See Problem 2.97. Since resultant must be directed along *OA*, i.e., the *x*-axis, we write

$$R_y = 0: \quad \Sigma F_y = (87\text{ lb}) + \frac{25}{65}T_{AC} - P = 0$$

$T_{AC} = 130.0\text{ lb}$ from Problem 2.97.

Then $$(87\text{ lb}) + \frac{25}{65}(130.0\text{ lb}) - P = 0$$ $P = 137.0\text{ lb}$ ◀

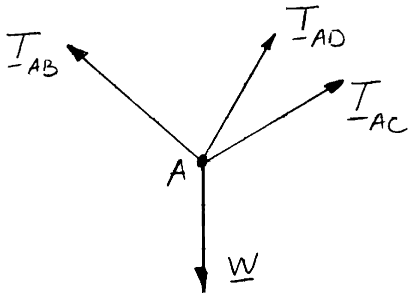


PROBLEM 2.99

A container is supported by three cables that are attached to a ceiling as shown. Determine the weight W of the container, knowing that the tension in cable AB is 6 kN.

SOLUTION

Free-Body Diagram at A:



The forces applied at A are: $\mathbf{T}_{AB}, \mathbf{T}_{AC}, \mathbf{T}_{AD}$, and $\mathbf{W}$

where $\mathbf{W} = W\mathbf{j}$. To express the other forces in terms of the unit vectors **i**, **j**, **k**, we write

$$\overrightarrow{AB} = -(450\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{j} \qquad AB = 750\text{ mm}$$

$$\overrightarrow{AC} = +(600\text{ mm})\mathbf{j} - (320\text{ mm})\mathbf{k} \qquad AC = 680\text{ mm}$$

$$\overrightarrow{AD} = +(500\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{j} + (360\text{ mm})\mathbf{k} \qquad AD = 860\text{ mm}$$

and

$$\mathbf{T}_{AB} = \boldsymbol{\lambda}_{AB}T_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = T_{AB}\frac{(-450\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{j}}{750\text{ mm}}$$

$$= \left(-\frac{45}{75}\mathbf{i} + \frac{60}{75}\mathbf{j}\right)T_{AB}$$

$$\mathbf{T}_{AC} = \boldsymbol{\lambda}_{AC}T_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = T_{AC}\frac{(600\text{ mm})\mathbf{i} - (320\text{ mm})\mathbf{j}}{680\text{ mm}}$$

$$= \left(\frac{60}{68}\mathbf{j} - \frac{32}{68}\mathbf{k}\right)T_{AC}$$

$$\mathbf{T}_{AD} = \boldsymbol{\lambda}_{AD}T_{AD} = T_{AD}\frac{\overrightarrow{AD}}{AD} = T_{AD}\frac{(500\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{j} + (360\text{ mm})\mathbf{k}}{860\text{ mm}}$$

$$= \left(\frac{50}{86}\mathbf{i} + \frac{60}{86}\mathbf{j} + \frac{36}{86}\mathbf{k}\right)T_{AD}$$

PROBLEM 2.99 (Continued)

Equilibrium condition: $\Sigma F = 0: \quad \therefore \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AD} + \mathbf{W} = 0$

Substituting the expressions obtained for $\mathbf{T}_{AB}, \mathbf{T}_{AC}$, and $\mathbf{T}_{AD}$; factoring **i**, **j**, and **k**; and equating each of the coefficients to zero gives the following equations:

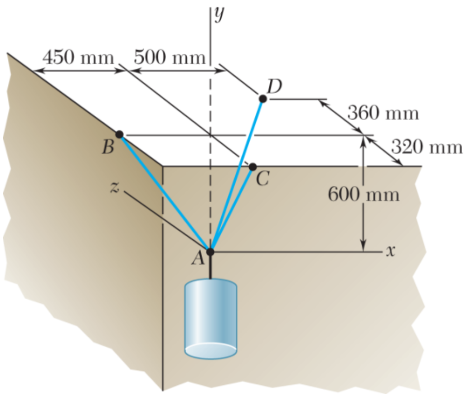
From **i**: $$-\frac{45}{75}T_{AB} + \frac{50}{86}T_{AD} = 0 \qquad (1)$$

From **j**: $$\frac{60}{75}T_{AB} + \frac{60}{68}T_{AC} + \frac{60}{86}T_{AD} - W = 0 \qquad (2)$$

From **k**: $$-\frac{32}{68}T_{AC} + \frac{36}{86}T_{AD} = 0 \qquad (3)$$

Setting $T_{AB} = 6\text{ kN}$ in (1) and (2), and solving the resulting set of equations gives

$$T_{AC} = 6.1920\text{ kN}$$
$$T_{AC} = 5.5080\text{ kN} \qquad W = 13.98\text{ kN} \blacktriangleleft$$



PROBLEM 2.100

A container is supported by three cables that are attached to a ceiling as shown. Determine the weight *W* of the container, knowing that the tension in cable *AD* is 4.3 kN.

SOLUTION

See Problem 2.99 for the figure and analysis leading to the following set of linear algebraic equations:

$$-\frac{45}{75}T_{AB}+\frac{50}{86}T_{AD}=0 \qquad (1)$$

$$\frac{60}{75}T_{AB}+\frac{60}{68}T_{AC}+\frac{60}{86}T_{AD}-W=0 \qquad (2)$$

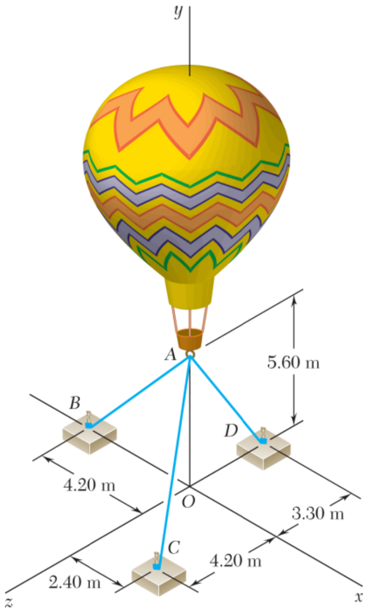
$$-\frac{32}{68}T_{AC}+\frac{36}{86}T_{AD}=0 \qquad (3)$$

Setting $T_{AD}=4.3\text{ kN}$ into the above equations gives

$$T_{AB}=4.1667\text{ kN}$$

$$T_{AC}=3.8250\text{ kN}$$

$$W=9.71\text{ kN} \blacktriangleleft$$

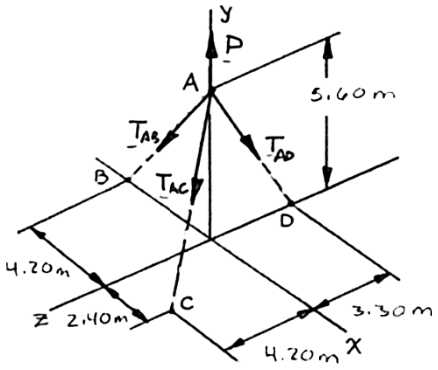


PROBLEM 2.101

Three cables are used to tether a balloon as shown. Determine the vertical force **P** exerted by the balloon at *A* knowing that the tension in cable *AD* is 481 N.

SOLUTION

FREE-BODY DIAGRAM AT *A*



The forces applied at *A* are: $\mathbf{T}_{AB}, \mathbf{T}_{AC}, \mathbf{T}_{AD},$ and $\mathbf{P}$

where $\mathbf{P} = P\mathbf{j}$. To express the other forces in terms of the unit vectors **i**, **j**, **k**, we write

$$\overrightarrow{AB} = -(4.20 \text{ m})\mathbf{i} - (5.60 \text{ m})\mathbf{j} \qquad AB = 7.00 \text{ m}$$

$$\overrightarrow{AC} = (2.40 \text{ m})\mathbf{i} - (5.60 \text{ m})\mathbf{j} + (4.20 \text{ m})\mathbf{k} \qquad AC = 7.40 \text{ m}$$

$$\overrightarrow{AD} = -(5.60 \text{ m})\mathbf{j} - (3.30 \text{ m})\mathbf{k} \qquad AD = 6.50 \text{ m}$$

and

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = (-0.6\mathbf{i} - 0.8\mathbf{j})T_{AB}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = (0.32432\mathbf{i} - 0.75676\mathbf{j} + 0.56757\mathbf{k})T_{AC}$$

$$\mathbf{T}_{AD} = T_{AD}\boldsymbol{\lambda}_{AD} = T_{AD}\frac{\overrightarrow{AD}}{AD} = (-0.86154\mathbf{j} - 0.50769\mathbf{k})T_{AD}$$

PROBLEM 2.101 (Continued)

Equilibrium condition: $\Sigma F = 0$: $\mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AD} + P\mathbf{j} = 0$

Substituting the expressions obtained for $\mathbf{T}_{AB}$, $\mathbf{T}_{AC}$, and $\mathbf{T}_{AD}$ and factoring **i**, **j**, and **k**:

$$(-0.6T_{AB} + 0.32432T_{AC})\mathbf{i} + (-0.8T_{AB} - 0.75676T_{AC} - 0.86154T_{AD} + P)\mathbf{j}$$
$$+ (0.56757T_{AC} - 0.50769T_{AD})\mathbf{k} = 0$$

Equating to zero the coefficients of **i**, **j**, **k**:

$$-0.6T_{AB} + 0.32432T_{AC} = 0 \tag{1}$$

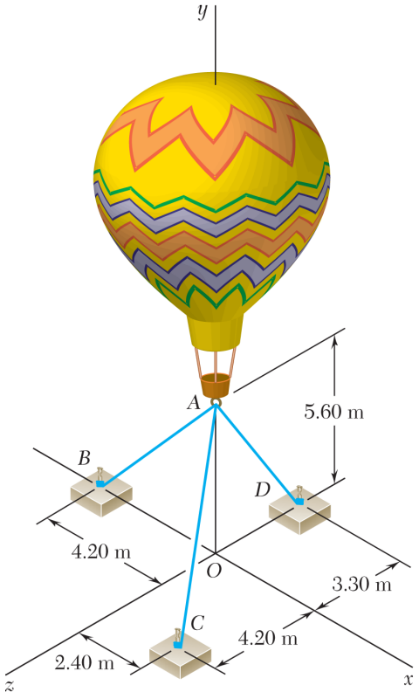
$$-0.8T_{AB} - 0.75676T_{AC} - 0.86154T_{AD} + P = 0 \tag{2}$$

$$0.56757T_{AC} - 0.50769T_{AD} = 0 \tag{3}$$

Setting $T_{AD} = 481\text{ N}$ in (2) and (3), and solving the resulting set of equations gives

$$T_{AC} = 430.26\text{ N}$$
$$T_{AD} = 232.57\text{ N}$$

$\mathbf{P} = 926\text{ N}\uparrow$ ◀



PROBLEM 2.102

Three cables are used to tether a balloon as shown. Knowing that the balloon exerts an 800-N vertical force at *A*, determine the tension in each cable.

SOLUTION

See Problem 2.101 for the figure and analysis leading to the linear algebraic Equations (1), (2), and (3).

$$-0.6T_{AB} + 0.32432T_{AC} = 0 \tag{1}$$

$$-0.8T_{AB} - 0.75676T_{AC} - 0.86154T_{AD} + P = 0 \tag{2}$$

$$0.56757T_{AC} - 0.50769T_{AD} = 0 \tag{3}$$

From Eq. (1): $T_{AB} = 0.54053T_{AC}$

From Eq. (3): $T_{AD} = 1.11795T_{AC}$

Substituting for T_{AB} and T_{AD} in terms of T_{AC} into Eq. (2) gives

$$-0.8(0.54053T_{AC}) - 0.75676T_{AC} - 0.86154(1.11795T_{AC}) + P = 0$$

$$2.1523T_{AC} = P; \quad P = 800 \text{ N}$$

$$T_{AC} = \frac{800 \text{ N}}{2.1523}$$

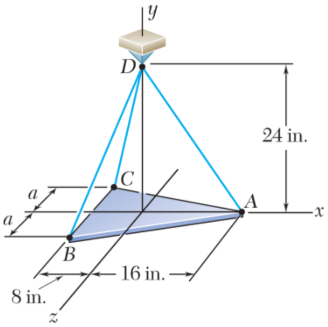
$$= 371.69 \text{ N}$$

Substituting into expressions for T_{AB} and T_{AD} gives

$$T_{AB} = 0.54053(371.69 \text{ N})$$

$$T_{AD} = 1.11795(371.69 \text{ N})$$

$$T_{AB} = 201 \text{ N}, \quad T_{AC} = 372 \text{ N}, \quad T_{AD} = 416 \text{ N} \blacktriangleleft$$



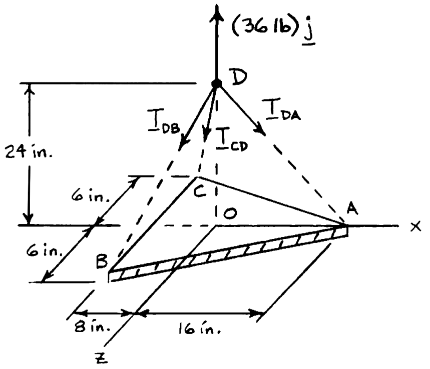
PROBLEM 2.103

A 36-lb triangular plate is supported by three wires as shown. Determine the tension in each wire, knowing that $a = 6$ in.

SOLUTION

By Symmetry $T_{DB} = T_{DC}$

Free-Body Diagram of Point *D*:



The forces applied at D are: $\mathbf{T}_{DB}, \mathbf{T}_{DC}, \mathbf{T}_{DA}$, and $\mathbf{P}$

where $\mathbf{P} = P\mathbf{j} = (36 \text{ lb})\mathbf{j}$. To express the other forces in terms of the unit vectors **i**, **j**, **k**, we write

$$\overrightarrow{DA} = (16 \text{ in.})\mathbf{i} - (24 \text{ in.})\mathbf{j} \qquad DA = 28.844 \text{ in.}$$

$$\overrightarrow{DB} = -(8 \text{ in.})\mathbf{i} - (24 \text{ in.})\mathbf{j} + (6 \text{ in.})\mathbf{k} \qquad DB = 26.0 \text{ in.}$$

$$\overrightarrow{DC} = -(8 \text{ in.})\mathbf{i} - (24 \text{ in.})\mathbf{j} - (6 \text{ in.})\mathbf{k} \qquad DC = 26.0 \text{ in.}$$

and

$$\mathbf{T}_{DA} = T_{DA}\boldsymbol{\lambda}_{DA} = T_{DA}\frac{\overrightarrow{DA}}{DA} = (0.55471\mathbf{i} - 0.83206\mathbf{j})T_{DA}$$

$$\mathbf{T}_{DB} = T_{DB}\boldsymbol{\lambda}_{DB} = T_{DB}\frac{\overrightarrow{DB}}{DB} = (-0.30769\mathbf{i} - 0.92308\mathbf{j} + 0.23077\mathbf{k})T_{DB}$$

$$\mathbf{T}_{DC} = T_{DC}\boldsymbol{\lambda}_{DC} = T_{DC}\frac{\overrightarrow{DC}}{DC} = (-0.30769\mathbf{i} - 0.92308\mathbf{j} - 0.23077\mathbf{k})T_{DC}$$

PROBLEM 2.103 (Continued)

Equilibrium condition: $\Sigma F = 0\colon\quad \mathbf{T}_{DA} + \mathbf{T}_{DB} + \mathbf{T}_{DC} + (36\text{ lb})\mathbf{j} = 0$

Substituting the expressions obtained for $\mathbf{T}_{DA}$, $\mathbf{T}_{DB}$, and $\mathbf{T}_{DC}$ and factoring **i**, **j**, and **k**:

$$(0.55471T_{DA} - 0.30769T_{DB} - 0.30769T_{DC})\mathbf{i} + (-0.83206T_{DA} - 0.92308T_{DB} - 0.92308T_{DC} + 36\text{ lb})\mathbf{j}$$
$$+ (0.23077T_{DB} - 0.23077T_{DC})\mathbf{k} = 0$$

Equating to zero the coefficients of **i**, **j**, **k**:

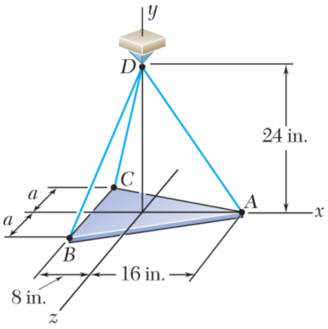
$$0.55471T_{DA} - 0.30769T_{DB} - 0.30769T_{DC} = 0 \qquad (1)$$

$$-0.83206T_{DA} - 0.92308T_{DB} - 0.92308T_{DC} + 36\text{ lb} = 0 \qquad (2)$$

$$0.23077T_{DB} - 0.23077T_{DC} = 0 \qquad (3)$$

Equation (3) confirms that $T_{DB} = T_{DC}$. Solving simultaneously gives,

$T_{DA} = 14.42\text{ lb};\qquad T_{DB} = T_{DC} = 13.00\text{ lb}$ ◀



PROBLEM 2.104

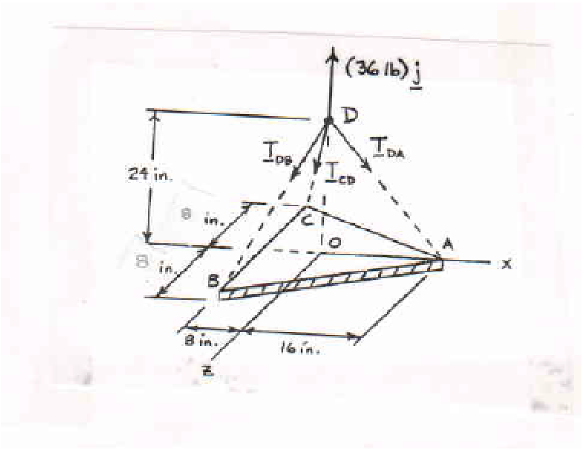
Solve Prob. 2.103, assuming that $a = 8$ in.

PROBLEM 2.103 A 36-lb triangular plate is supported by three wires as shown. Determine the tension in each wire, knowing that $a = 6$ in.

SOLUTION

By Symmetry $T_{DB} = T_{DC}$

Free-Body Diagram of Point *D*:



The forces applied at *D* are: $\mathbf{T}_{DB}, \mathbf{T}_{DC}, \mathbf{T}_{DA}$, and $\mathbf{P}$

where $\mathbf{P} = P\mathbf{j} = (36\text{ lb})\mathbf{j}$. To express the other forces in terms of the unit vectors **i**, **j**, **k**, we write

$$\overrightarrow{DA} = (16\text{ in.})\mathbf{i} - (24\text{ in.})\mathbf{j} \qquad DA = 28.844\text{ in.}$$

$$\overrightarrow{DB} = -(8\text{ in.})\mathbf{i} - (24\text{ in.})\mathbf{j} + (8\text{ in.})\mathbf{k} \qquad DB = 26.533\text{ in.}$$

$$\overrightarrow{DC} = -(8\text{ in.})\mathbf{i} - (24\text{ in.})\mathbf{j} - (8\text{ in.})\mathbf{k} \qquad DC = 26.533\text{ in.}$$

and

$$\mathbf{T}_{DA} = T_{DA}\boldsymbol{\lambda}_{DA} = T_{DA}\frac{\overrightarrow{DA}}{DA} = (0.55471\mathbf{i} - 0.83206\mathbf{j})T_{DA}$$

$$\mathbf{T}_{DB} = T_{DB}\boldsymbol{\lambda}_{DB} = T_{DB}\frac{\overrightarrow{DB}}{DB} = (-0.30151\mathbf{i} - 0.90453\mathbf{j} + 0.30151\mathbf{k})T_{DB}$$

$$\mathbf{T}_{DC} = T_{DC}\boldsymbol{\lambda}_{DC} = T_{DC}\frac{\overrightarrow{DC}}{DC} = (-0.30151\mathbf{i} - 0.90453\mathbf{j} - 0.30151\mathbf{k})T_{DC}$$

PROBLEM 2.104 (Continued)

Equilibrium condition: $\Sigma F = 0$: $\mathbf{T}_{DA} + \mathbf{T}_{DB} + \mathbf{T}_{DC} + (36 \text{ lb})\mathbf{j} = 0$

Substituting the expressions obtained for $\mathbf{T}_{DA}$, $\mathbf{T}_{DB}$, and $\mathbf{T}_{DC}$ and factoring **i**, **j**, and **k**:

$$(0.55471T_{DA} - 0.30151T_{DB} - 0.30151T_{DC})\mathbf{i} + (-0.83206T_{DA} - 0.90453T_{DB} - 0.90453T_{DC} + 36 \text{ lb})\mathbf{j}$$
$$+ (0.30151T_{DB} - 0.30151T_{DC})\mathbf{k} = 0$$

Equating to zero the coefficients of **i**, **j**, **k**:

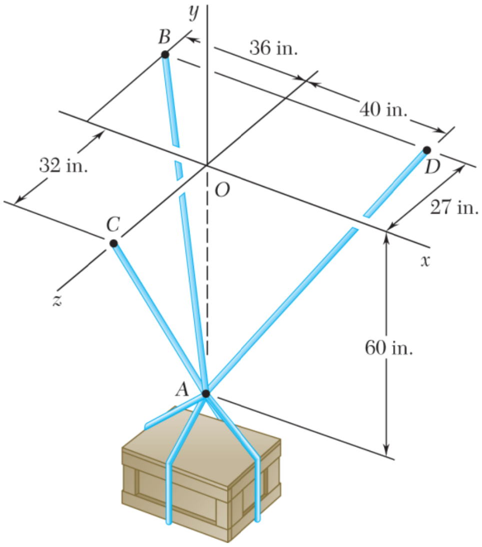
$$0.55471T_{DA} - 0.30151T_{DB} - 0.30151T_{DC} = 0 \quad (1)$$

$$-0.83206T_{DA} - 0.90453T_{DB} - 0.90453T_{DC} + 36 \text{ lb} = 0 \quad (2)$$

$$0.30151T_{DB} - 0.30151T_{DC} = 0 \quad (3)$$

Equation (3) confirms that $T_{DB} = T_{DC}$. Solving simultaneously gives,

$T_{DA} = 14.42$ lb; $T_{DB} = T_{DC} = 13.27$ lb ◀



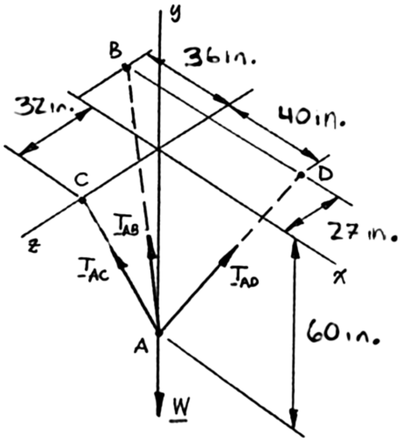
PROBLEM 2.105

A crate is supported by three cables as shown. Determine the weight of the crate knowing that the tension in cable AC is 544 lb.

Solution The forces applied at A are:

$$\mathbf{T}_{AB}, \mathbf{T}_{AC}, \mathbf{T}_{AD} \text{ and } \mathbf{W}$$

where $\mathbf{P} = P\mathbf{j}$. To express the other forces in terms of the unit vectors **i**, **j**, **k**, we write



$$\overrightarrow{AB} = -(36 \text{ in.})\mathbf{i} + (60 \text{ in.})\mathbf{j} - (27 \text{ in.})\mathbf{k}$$
$$AB = 75 \text{ in.}$$
$$\overrightarrow{AC} = (60 \text{ in.})\mathbf{j} + (32 \text{ in.})\mathbf{k}$$
$$AC = 68 \text{ in.}$$
$$\overrightarrow{AD} = (40 \text{ in.})\mathbf{i} + (60 \text{ in.})\mathbf{j} - (27 \text{ in.})\mathbf{k}$$
$$AD = 77 \text{ in.}$$

and

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB}$$
$$= (-0.48\mathbf{i} + 0.8\mathbf{j} - 0.36\mathbf{k})T_{AB}$$
$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC}$$
$$= (0.88235\mathbf{j} + 0.47059\mathbf{k})T_{AC}$$
$$\mathbf{T}_{AD} = T_{AD}\boldsymbol{\lambda}_{AD} = T_{AD}\frac{\overrightarrow{AD}}{AD}$$
$$= (0.51948\mathbf{i} + 0.77922\mathbf{j} - 0.35065\mathbf{k})T_{AD}$$

Equilibrium Condition with $\mathbf{W} = -W\mathbf{j}$

$$\Sigma F = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AD} - W\mathbf{j} = 0$$

PROBLEM 2.105 (Continued)

Substituting the expressions obtained for $\mathbf{T}_{AB}$, $\mathbf{T}_{AC}$, and $\mathbf{T}_{AD}$ and factoring **i**, **j**, and **k**:

$$(-0.48T_{AB} + 0.51948T_{AD})\mathbf{i} + (0.8T_{AB} + 0.88235T_{AC} + 0.77922T_{AD} - W)\mathbf{j}$$
$$+(-0.36T_{AB} + 0.47059T_{AC} - 0.35065T_{AD})\mathbf{k} = 0$$

Equating to zero the coefficients of **i**, **j**, **k**:

$$-0.48T_{AB} + 0.51948T_{AD} = 0 \qquad (1)$$

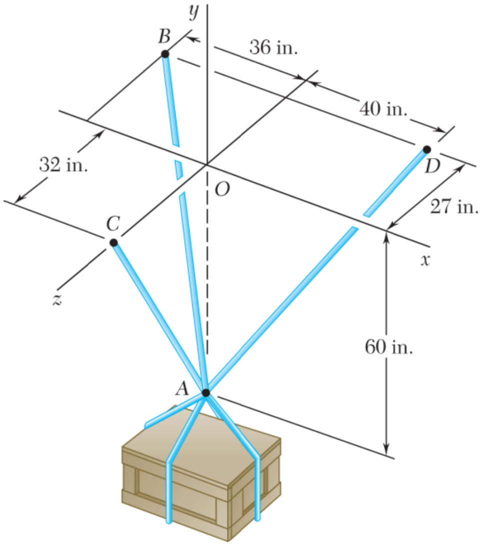
$$0.8T_{AB} + 0.88235T_{AC} + 0.77922T_{AD} - W = 0 \qquad (2)$$

$$-0.36T_{AB} + 0.47059T_{AC} - 0.35065T_{AD} = 0 \qquad (3)$$

Substituting $T_{AC} = 544\text{ lb}$ in Equations (1), (2), and (3) above, and solving the resulting set of equations using conventional algorithms, gives:

$$T_{AB} = 374.27\text{ lb}$$
$$T_{AD} = 345.82\text{ lb}$$

$W = 1049\text{ lb}$ ◀



PROBLEM 2.106

A 1600-lb crate is supported by three cables as shown. Determine the tension in each cable.

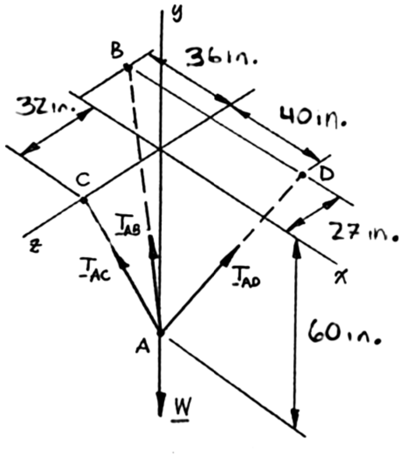
SOLUTION

The forces applied at A are:

$$\mathbf{T}_{AB}, \mathbf{T}_{AC}, \mathbf{T}_{AD} \text{ and } \mathbf{W}$$

where $\mathbf{P} = P\mathbf{j}$. To express the other forces in terms of the unit vectors **i**, **j**, **k**, we write

$$\overrightarrow{AB} = -(36 \text{ in.})\mathbf{i} + (60 \text{ in.})\mathbf{j} - (27 \text{ in.})\mathbf{k}$$
$$AB = 75 \text{ in.}$$
$$\overrightarrow{AC} = (60 \text{ in.})\mathbf{j} + (32 \text{ in.})\mathbf{k}$$
$$AC = 68 \text{ in.}$$
$$\overrightarrow{AD} = (40 \text{ in.})\mathbf{i} + (60 \text{ in.})\mathbf{j} - (27 \text{ in.})\mathbf{k}$$
$$AD = 77 \text{ in.}$$



and

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB}$$
$$= (-0.48\mathbf{i} + 0.8\mathbf{j} - 0.36\mathbf{k})T_{AB}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC}$$
$$= (0.88235\mathbf{j} + 0.47059\mathbf{k})T_{AC}$$

$$\mathbf{T}_{AD} = T_{AD}\boldsymbol{\lambda}_{AD} = T_{AD}\frac{\overrightarrow{AD}}{AD}$$
$$= (0.51948\mathbf{i} + 0.77922\mathbf{j} - 0.35065\mathbf{k})T_{AD}$$

Equilibrium Condition with $\mathbf{W} = -W\mathbf{j}$

$$\Sigma F = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AD} - W\mathbf{j} = 0$$

PROBLEM 2.106 (Continued)

Substituting the expressions obtained for $\mathbf{T}_{AB}$, $\mathbf{T}_{AC}$, and $\mathbf{T}_{AD}$ and factoring **i**, **j**, and **k**:

$$(-0.48T_{AB}+0.51948T_{AD})\mathbf{i}+(0.8T_{AB}+0.88235T_{AC}+0.77922T_{AD}-W)\mathbf{j}$$
$$+(-0.36T_{AB}+0.47059T_{AC}-0.35065T_{AD})\mathbf{k}=0$$

Equating to zero the coefficients of **i**, **j**, **k**:

$$-0.48T_{AB}+0.51948T_{AD}=0 \qquad (1)$$

$$0.8T_{AB}+0.88235T_{AC}+0.77922T_{AD}-W=0 \qquad (2)$$

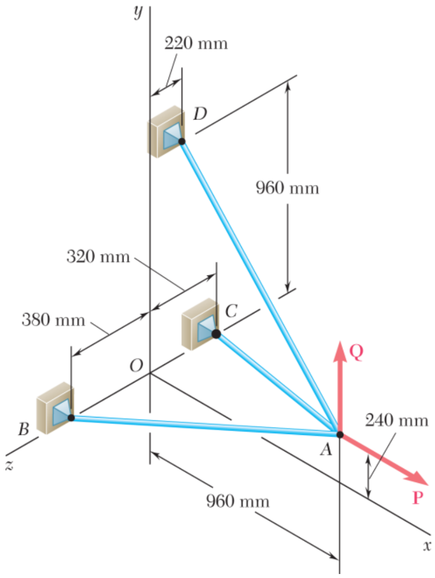
$$-0.36T_{AB}+0.47059T_{AC}-0.35065T_{AD}=0 \qquad (3)$$

Substituting $W=1600$ lb in Equations (1), (2), and (3) above, and solving the resulting set of equations using conventional algorithms gives,

$T_{AB} = 571$ lb ◀

$T_{AC} = 830$ lb ◀

$T_{AD} = 528$ lb ◀



PROBLEM 2.107

Three cables are connected at A, where the forces **P** and **Q** are applied as shown. Knowing that $Q = 0$, find the value of P for which the tension in cable AD is 305 N.

SOLUTION

$$\Sigma \mathbf{F}_A = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AD} + \mathbf{P} = 0 \quad \text{where} \quad \mathbf{P} = P\mathbf{i}$$

$$\overrightarrow{AB} = -(960 \text{ mm})\mathbf{i} - (240 \text{ mm})\mathbf{j} + (380 \text{ mm})\mathbf{k} \qquad AB = 1060 \text{ mm}$$

$$\overrightarrow{AC} = -(960 \text{ mm})\mathbf{i} - (240 \text{ mm})\mathbf{j} - (320 \text{ mm})\mathbf{k} \qquad AC = 1040 \text{ mm}$$

$$\overrightarrow{AD} = -(960 \text{ mm})\mathbf{i} + (720 \text{ mm})\mathbf{j} - (220 \text{ mm})\mathbf{k} \qquad AD = 1220 \text{ mm}$$

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = T_{AB}\left(-\frac{48}{53}\mathbf{i} - \frac{12}{53}\mathbf{j} + \frac{19}{53}\mathbf{k}\right)$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = T_{AC}\left(-\frac{12}{13}\mathbf{i} - \frac{3}{13}\mathbf{j} - \frac{4}{13}\mathbf{k}\right)$$

$$\mathbf{T}_{AD} = T_{AD}\boldsymbol{\lambda}_{AD} = \frac{305 \text{ N}}{1220 \text{ mm}}[(-960 \text{ mm})\mathbf{i} + (720 \text{ mm})\mathbf{j} - (220 \text{ mm})\mathbf{k}]$$

$$= -(240 \text{ N})\mathbf{i} + (180 \text{ N})\mathbf{j} - (55 \text{ N})\mathbf{k}$$

Substituting into $\Sigma \mathbf{F}_A = 0$, factoring $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and setting each coefficient equal to ϕ gives:

$$\mathbf{i}: \quad P = \frac{48}{53}T_{AB} + \frac{12}{13}T_{AC} + 240 \text{ N} \tag{1}$$

$$\mathbf{j}: \quad \frac{12}{53}T_{AB} + \frac{3}{13}T_{AC} = 180 \text{ N} \tag{2}$$

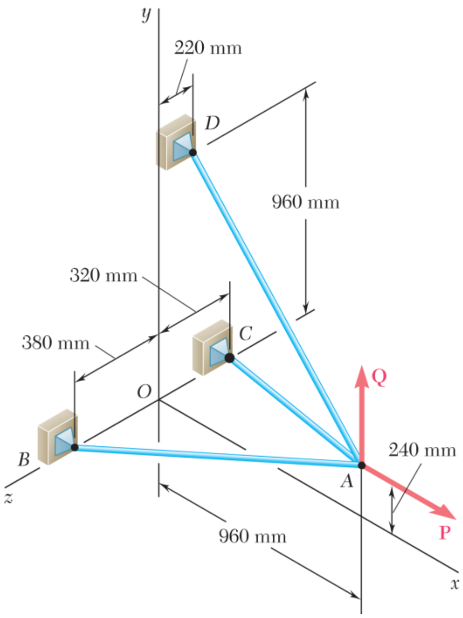
$$\mathbf{k}: \quad \frac{19}{53}T_{AB} - \frac{4}{13}T_{AC} = 55 \text{ N} \tag{3}$$

Solving the system of linear equations using conventional algorithms gives:

$$T_{AB} = 446.71 \text{ N}$$

$$T_{AC} = 341.71 \text{ N}$$

$P = 960$ N ◀



PROBLEM 2.108

Three cables are connected at A, where the forces **P** and **Q** are applied as shown. Knowing that $P = 1200$ N, determine the values of Q for which cable AD is taut.

SOLUTION

We assume that $T_{AD} = 0$ and write $\quad \Sigma \mathbf{F}_A = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + Q\mathbf{j} + (1200 \text{ N})\mathbf{i} = 0$

$$\overrightarrow{AB} = -(960 \text{ mm})\mathbf{i} - (240 \text{ mm})\mathbf{j} + (380 \text{ mm})\mathbf{k} \quad AB = 1060 \text{ mm}$$

$$\overrightarrow{AC} = -(960 \text{ mm})\mathbf{i} - (240 \text{ mm})\mathbf{j} - (320 \text{ mm})\mathbf{k} \quad AC = 1040 \text{ mm}$$

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = \left(-\frac{48}{53}\mathbf{i} - \frac{12}{53}\mathbf{j} + \frac{19}{53}\mathbf{k}\right)T_{AB}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = \left(-\frac{12}{13}\mathbf{i} - \frac{3}{13}\mathbf{j} - \frac{4}{13}\mathbf{k}\right)T_{AC}$$

Substituting into $\Sigma \mathbf{F}_A = 0$, factoring $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and setting each coefficient equal to ϕ gives:

$$\mathbf{i}: \quad -\frac{48}{53}T_{AB} - \frac{12}{13}T_{AC} + 1200 \text{ N} = 0 \qquad (1)$$

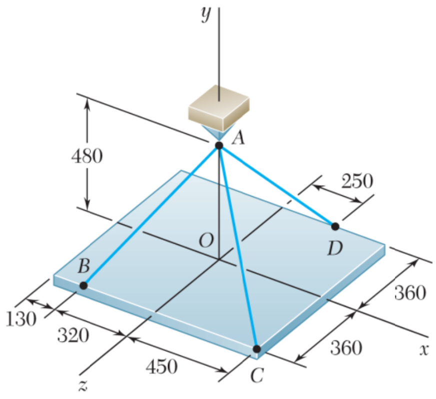
$$\mathbf{j}: \quad -\frac{12}{53}T_{AB} - \frac{3}{13}T_{AC} + Q = 0 \qquad (2)$$

$$\mathbf{k}: \quad \frac{19}{53}T_{AB} - \frac{4}{13}T_{AC} = 0 \qquad (3)$$

Solving the resulting system of linear equations using conventional algorithms gives:

$$T_{AB} = 605.71 \text{ N}$$
$$T_{AC} = 705.71 \text{ N}$$
$$Q = 300.00 \text{ N} \qquad 0 \le Q < 300 \text{ N} \blacktriangleleft$$

Note: This solution assumes that Q is directed upward as shown $(Q \ge 0)$, if negative values of Q are considered, cable AD remains taut, but AC becomes slack for $Q = -460$ N.



Dimensions in mm

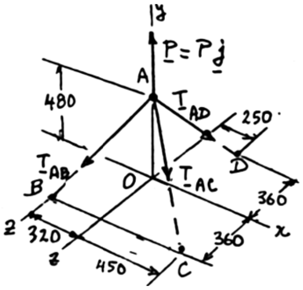
PROBLEM 2.109

A rectangular plate is supported by three cables as shown. Knowing that the tension in cable *AC* is 60 N, determine the weight of the plate.

SOLUTION

We note that the weight of the plate is equal in magnitude to the force **P** exerted by the support on Point *A*.

Free Body *A*:



Dimensions in mm

$$\Sigma F = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AD} + P\mathbf{j} = 0$$

We have:

$$\overrightarrow{AB} = -(320 \text{ mm})\mathbf{i} - (480 \text{ mm})\mathbf{j} + (360 \text{ mm})\mathbf{k} \qquad AB = 680 \text{ mm}$$

$$\overrightarrow{AC} = (450 \text{ mm})\mathbf{i} - (480 \text{ mm})\mathbf{j} + (360 \text{ mm})\mathbf{k} \qquad AC = 750 \text{ mm}$$

$$\overrightarrow{AD} = (250 \text{ mm})\mathbf{i} - (480 \text{ mm})\mathbf{j} - \left(360 \text{ mm}\right)\mathbf{k} \qquad AD = 650 \text{ mm}$$

Thus:

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = \left(-\frac{8}{17}\mathbf{i} - \frac{12}{17}\mathbf{j} + \frac{9}{17}\mathbf{k}\right)T_{AB}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = \left(0.6\mathbf{i} - 0.64\mathbf{j} + 0.48\mathbf{k}\right)T_{AC}$$

$$\mathbf{T}_{AD} = T_{AD}\boldsymbol{\lambda}_{AD} = T_{AD}\frac{\overrightarrow{AD}}{AD} = \left(\frac{5}{13}\mathbf{i} - \frac{9.6}{13}\mathbf{j} - \frac{7.2}{13}\mathbf{k}\right)T_{AD}$$

Substituting into the Eq. $\Sigma F = 0$ and factoring **i**, **j**, **k**:

$$\left(-\frac{8}{17}T_{AB} + 0.6T_{AC} + \frac{5}{13}T_{AD}\right)\mathbf{i}$$

$$+\left(-\frac{12}{17}T_{AB} - 0.64T_{AC} - \frac{9.6}{13}T_{AD} + P\right)\mathbf{j}$$

$$+\left(\frac{9}{17}T_{AB} + 0.48T_{AC} - \frac{7.2}{13}T_{AD}\right)\mathbf{k} = 0$$

PROBLEM 2.109 (Continued)

Setting the coefficient of **i**, **j**, **k** equal to zero:

$$\mathbf{i}: \quad -\frac{8}{17}T_{AB} + 0.6T_{AC} + \frac{5}{13}T_{AD} = 0 \tag{1}$$

$$\mathbf{j}: \quad -\frac{12}{7}T_{AB} - 0.64T_{AC} - \frac{9.6}{13}T_{AD} + P = 0 \tag{2}$$

$$\mathbf{k}: \quad \frac{9}{17}T_{AB} + 0.48T_{AC} - \frac{7.2}{13}T_{AD} = 0 \tag{3}$$

Making $T_{AC} = 60\text{ N}$ in (1) and (3):

$$-\frac{8}{17}T_{AB} + 36\text{ N} + \frac{5}{13}T_{AD} = 0 \tag{1$'$}$$

$$\frac{9}{17}T_{AB} + 28.8\text{ N} - \frac{7.2}{13}T_{AD} = 0 \tag{3$'$}$$

Multiply (1′) by 9, (3′) by 8, and add:

$$554.4\text{ N} - \frac{12.6}{13}T_{AD} = 0 \quad T_{AD} = 572.0\text{ N}$$

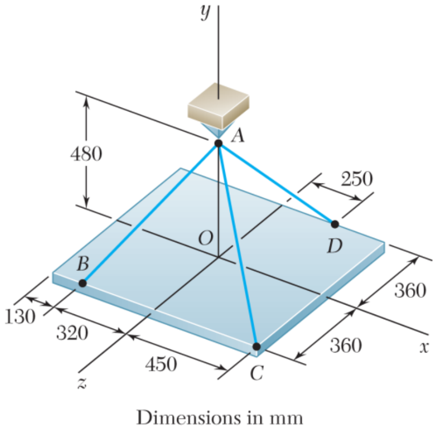
Substitute into (1′) and solve for T_{AB}:

$$T_{AB} = \frac{17}{8}\left(36 + \frac{5}{13} \times 572\right) \qquad T_{AB} = 544.0\text{ N}$$

Substitute for the tensions in Eq. (2) and solve for P:

$$P = \frac{12}{17}(544\text{ N}) + 0.64(60\text{ N}) + \frac{9.6}{13}(572\text{ N})$$
$$= 844.8\text{ N}$$

Weight of plate $= P = 845\text{ N}$ ◀



PROBLEM 2.110

A rectangular plate is supported by three cables as shown. Knowing that the tension in cable *AD* is 520 N, determine the weight of the plate.

SOLUTION

See Problem 2.109 for the figure and the analysis leading to the linear algebraic Equations (1), (2), and (3) below:

$$-\frac{8}{17}T_{AB}+0.6T_{AC}+\frac{5}{13}T_{AD}=0 \tag{1}$$

$$-\frac{12}{17}T_{AB}+0.64T_{AC}-\frac{9.6}{13}T_{AD}+P=0 \tag{2}$$

$$\frac{9}{17}T_{AB}+0.48T_{AC}-\frac{7.2}{13}T_{AD}=0 \tag{3}$$

Making $T_{AD}=520\text{ N}$ in Eqs. (1) and (3):

$$-\frac{8}{17}T_{AB}+0.6T_{AC}+200\text{ N}=0 \tag{1$'$}$$

$$\frac{9}{17}T_{AB}+0.48T_{AC}-288\text{ N}=0 \tag{3$'$}$$

Multiply (1′) by 9, (3′) by 8, and add:

$$9.24T_{AC}-504\text{ N}=0 \quad T_{AC}=54.5455\text{ N}$$

Substitute into (1′) and solve for T_{AB}:

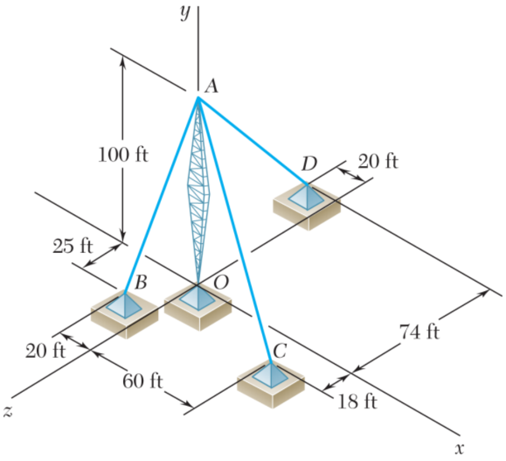
$$T_{AB}=\frac{17}{8}(0.6\times 54.5455+200) \quad T_{AB}=494.545\text{ N}$$

Substitute for the tensions in Eq. (2) and solve for *P*:

$$P=\frac{12}{17}(494.545\text{ N})+0.64(54.5455\text{ N})+\frac{9.6}{13}(520\text{ N})$$

$$=768.00\text{ N}$$

Weight of plate $=P=768\text{ N}$ ◀



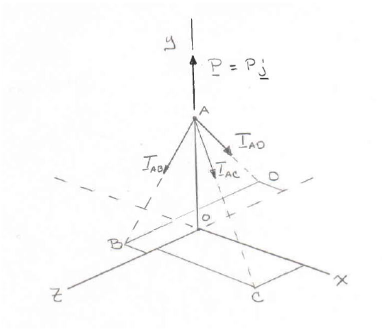
PROBLEM 2.111

A transmission tower is held by three guy wires attached to a pin at A and anchored by bolts at B, C, and D. If the tension in wire AB is 840 lb, determine the vertical force $\mathbf{P}$ exerted by the tower on the pin at A.

SOLUTION

$$\Sigma \mathbf{F} = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AD} + P\mathbf{j} = 0$$

Free-Body Diagram at *A*:



$$\overrightarrow{AB} = -20\mathbf{i} - 100\mathbf{j} + 25\mathbf{k} \qquad AB = 105 \text{ ft}$$

$$\overrightarrow{AC} = 60\mathbf{i} - 100\mathbf{j} + 18\mathbf{k} \qquad AC = 118 \text{ ft}$$

$$\overrightarrow{AD} = -20\mathbf{i} - 100\mathbf{j} - 74\mathbf{k} \qquad AD = 126 \text{ ft}$$

We write

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB}$$

$$= \left(-\frac{4}{21}\mathbf{i} - \frac{20}{21}\mathbf{j} + \frac{5}{21}\mathbf{k}\right)T_{AB}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC}$$

$$= \left(\frac{30}{59}\mathbf{i} - \frac{50}{59}\mathbf{j} + \frac{9}{59}\mathbf{k}\right)T_{AC}$$

$$\mathbf{T}_{AD} = T_{AD}\boldsymbol{\lambda}_{AD} = T_{AD}\frac{\overrightarrow{AD}}{AD}$$

$$= \left(-\frac{10}{63}\mathbf{i} - \frac{50}{63}\mathbf{j} - \frac{37}{63}\mathbf{k}\right)T_{AD}$$

Substituting into the Eq. $\Sigma \mathbf{F} = 0$ and factoring $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$:

PROBLEM 2.111 (Continued)

$$\left(-\frac{4}{21}T_{AB}+\frac{30}{59}T_{AC}-\frac{10}{63}T_{AD}\right)\mathbf{i}$$
$$+\left(-\frac{20}{21}T_{AB}-\frac{50}{59}T_{AC}-\frac{50}{63}T_{AD}+P\right)\mathbf{j}$$
$$+\left(\frac{5}{21}T_{AB}+\frac{9}{59}T_{AC}-\frac{37}{63}T_{AD}\right)\mathbf{k}=0$$

Setting the coefficients of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, equal to zero:

$$\mathbf{i}: \quad -\frac{4}{21}T_{AB}+\frac{30}{59}T_{AC}-\frac{10}{63}T_{AD}=0 \tag{1}$$

$$\mathbf{j}: \quad -\frac{20}{21}T_{AB}-\frac{50}{59}T_{AC}-\frac{50}{63}T_{AD}+P=0 \tag{2}$$

$$\mathbf{k}: \quad \frac{5}{21}T_{AB}+\frac{9}{59}T_{AC}-\frac{37}{63}T_{AD}=0 \tag{3}$$

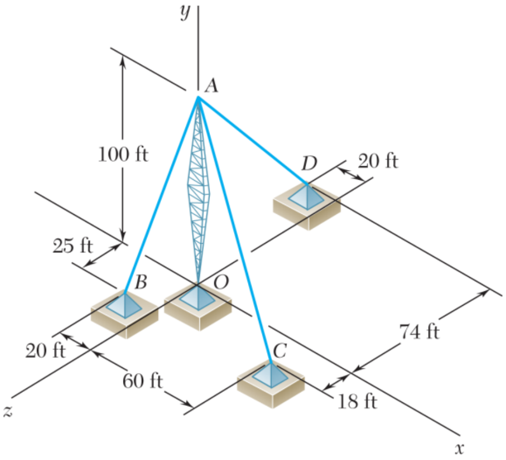
Set $T_{AB}=840$ lb in Eqs. (1) – (3):

$$-160\text{ lb}+\frac{30}{59}T_{AC}-\frac{10}{63}T_{AD}=0 \tag{1$'$}$$

$$-800\text{ lb}-\frac{50}{59}T_{AC}-\frac{50}{63}T_{AD}+P=0 \tag{2$'$}$$

$$200\text{ lb}+\frac{9}{59}T_{AC}-\frac{37}{63}T_{AD}=0 \tag{3$'$}$$

Solving, $T_{AC}=458.12$ lb $\quad T_{AD}=459.53$ lb $\quad P=1552.94$ lb $\qquad P=1553$ lb ◀



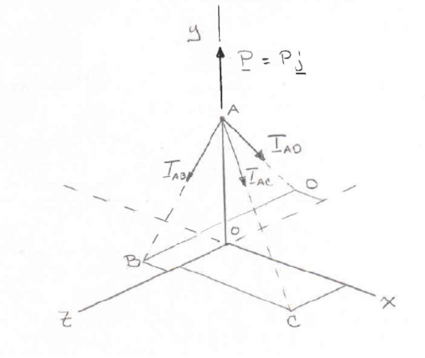
PROBLEM 2.112

A transmission tower is held by three guy wires attached to a pin at *A* and anchored by bolts at *B*, *C*, and *D*. If the tension in wire *AC* is 590 lb, determine the vertical force **P** exerted by the tower on the pin at *A*.

SOLUTION

$$\Sigma \mathbf{F} = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AD} + P\mathbf{j} = 0$$

Free-Body Diagram at *A*:



$$\overrightarrow{AB} = -20\mathbf{i} - 100\mathbf{j} + 25\mathbf{k} \qquad AB = 105 \text{ ft}$$

$$\overrightarrow{AC} = 60\mathbf{i} - 100\mathbf{j} + 18\mathbf{k} \qquad AC = 118 \text{ ft}$$

$$\overrightarrow{AD} = -20\mathbf{i} - 100\mathbf{j} - 74\mathbf{k} \qquad AD = 126 \text{ ft}$$

We write

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB}$$

$$= \left(-\frac{4}{21}\mathbf{i} - \frac{20}{21}\mathbf{j} + \frac{5}{21}\mathbf{k}\right)T_{AB}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC}$$

$$= \left(\frac{30}{59}\mathbf{i} - \frac{50}{59}\mathbf{j} + \frac{9}{59}\mathbf{k}\right)T_{AC}$$

$$\mathbf{T}_{AD} = T_{AD}\boldsymbol{\lambda}_{AD} = T_{AD}\frac{\overrightarrow{AD}}{AD}$$

$$= \left(-\frac{10}{63}\mathbf{i} - \frac{50}{63}\mathbf{j} - \frac{37}{63}\mathbf{k}\right)T_{AD}$$

Substituting into the Eq. $\Sigma \mathbf{F} = 0$ and factoring $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$:

PROBLEM 2.112 (Continued)

$$\left(-\frac{4}{21}T_{AB}+\frac{30}{59}T_{AC}-\frac{10}{63}T_{AD}\right)\mathbf{i}$$
$$+\left(-\frac{20}{21}T_{AB}-\frac{50}{59}T_{AC}-\frac{50}{63}T_{AD}+P\right)\mathbf{j}$$
$$+\left(\frac{5}{21}T_{AB}+\frac{9}{59}T_{AC}-\frac{37}{63}T_{AD}\right)\mathbf{k}=0$$

Setting the coefficients of **i**, **j**, **k**, equal to zero:

$$\mathbf{i}: \quad -\frac{4}{21}T_{AB}+\frac{30}{59}T_{AC}-\frac{10}{63}T_{AD}=0 \qquad (1)$$

$$\mathbf{j}: \quad -\frac{20}{21}T_{AB}-\frac{50}{59}T_{AC}-\frac{50}{63}T_{AD}+P=0 \qquad (2)$$

$$\mathbf{k}: \quad \frac{5}{21}T_{AB}+\frac{9}{59}T_{AC}-\frac{37}{63}T_{AD}=0 \qquad (3)$$

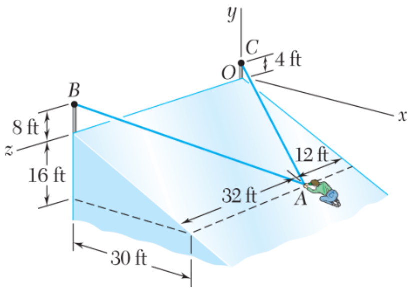
Set $T_{AC}=590$ lb in Eqs. (1) – (3):

$$-\frac{4}{21}T_{AB}+300\text{ lb}-\frac{10}{63}T_{AD}=0 \qquad (1')$$

$$-\frac{20}{21}T_{AB}-500\text{ lb}-\frac{50}{63}T_{AD}+P=0 \qquad (2')$$

$$\frac{5}{21}T_{AB}+90\text{ lb}-\frac{37}{63}T_{AD}=0 \qquad (3')$$

Solving, $T_{AB}=1081.82$ lb $\quad T_{AD}=591.82$ lb $\quad$ $P=2000$ lb ◀

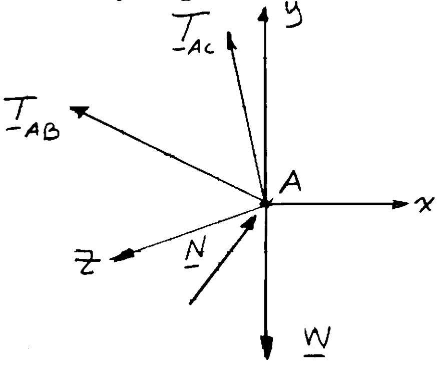


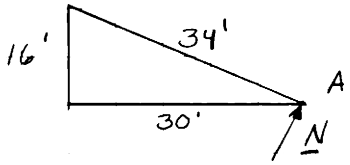
PROBLEM 2.113

In trying to move across a slippery icy surface, a 175-lb man uses two ropes *AB* and *AC*. Knowing that the force exerted on the man by the icy surface is perpendicular to that surface, determine the tension in each rope.

SOLUTION

Free-Body Diagram at *A*





$$\mathbf{N} = N\left(\frac{16}{34}\mathbf{i} + \frac{30}{34}\mathbf{j}\right)$$

and $\mathbf{W} = W\,\mathbf{j} = -(175\text{ lb})\mathbf{j}$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = T_{AC}\frac{(-30\text{ ft})\mathbf{i} + (20\text{ ft})\mathbf{j} - (12\text{ ft})\mathbf{k}}{38\text{ ft}}$$

$$= T_{AC}\left(-\frac{15}{19}\mathbf{i} + \frac{10}{19}\mathbf{j} - \frac{6}{19}\mathbf{k}\right)$$

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = T_{AB}\frac{(-30\text{ ft})\mathbf{i} + (24\text{ ft})\mathbf{j} + (32\text{ ft})\mathbf{k}}{50\text{ ft}}$$

$$= T_{AB}\left(-\frac{15}{25}\mathbf{i} + \frac{12}{25}\mathbf{j} + \frac{16}{25}\mathbf{k}\right)$$

Equilibrium condition: $\Sigma\mathbf{F} = 0$

$$\mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{N} + \mathbf{W} = 0$$

PROBLEM 2.113 (Continued)

Substituting the expressions obtained for $\mathbf{T}_{AB}, \mathbf{T}_{AC}, \mathbf{N},$ and **W**; factoring **i**, **j**, and **k**; and equating each of the coefficients to zero gives the following equations:

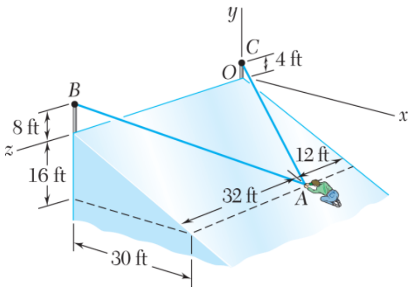
From **i**:
$$-\frac{15}{25}T_{AB} - \frac{15}{19}T_{AC} + \frac{16}{34}N = 0 \tag{1}$$

From **j**:
$$\frac{12}{25}T_{AB} + \frac{10}{19}T_{AC} + \frac{30}{34}N - (175\text{ lb}) = 0 \tag{2}$$

From **k**:
$$\frac{16}{25}T_{AB} - \frac{6}{19}T_{AC} = 0 \tag{3}$$

Solving the resulting set of equations gives:

$T_{AB} = 30.8$ lb; $T_{AC} = 62.5$ lb ◀



PROBLEM 2.114

Solve Problem 2.113, assuming that a friend is helping the man at *A* by pulling on him with a force $\mathbf{P} = -(45\text{ lb})\mathbf{k}$.

PROBLEM 2.113 In trying to move across a slippery icy surface, a 175-lb man uses two ropes *AB* and *AC*. Knowing that the force exerted on the man by the icy surface is perpendicular to that surface, determine the tension in each rope.

SOLUTION

Refer to Problem 2.113 for the figure and analysis leading to the following set of equations, Equation (3) being modified to include the additional force $\mathbf{P} = (-45\text{ lb})\mathbf{k}$.

$$-\frac{15}{25}T_{AB} - \frac{15}{19}T_{AC} + \frac{16}{34}N = 0 \qquad (1)$$

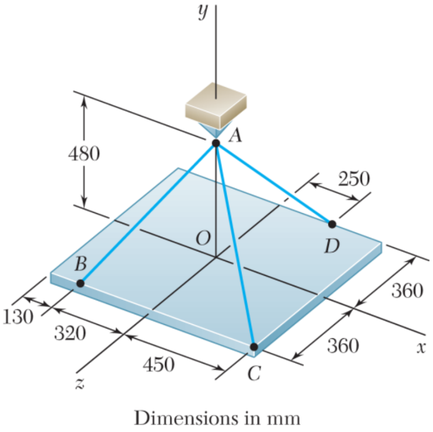
$$\frac{12}{25}T_{AB} + \frac{10}{19}T_{AC} + \frac{30}{34}N - (175\text{ lb}) = 0 \qquad (2)$$

$$\frac{16}{25}T_{AB} - \frac{6}{19}T_{AC} - (45\text{ lb}) = 0 \qquad (3)$$

Solving the resulting set of equations simultaneously gives:

$T_{AB} = 81.3$ lb ◀

$T_{AC} = 22.2$ lb ◀



PROBLEM 2.115

For the rectangular plate of Problems 2.109 and 2.110, determine the tension in each of the three cables knowing that the weight of the plate is 792 N.

SOLUTION

See Problem 2.109 for the figure and the analysis leading to the linear algebraic Equations (1), (2), and (3) below. Setting $P = 792\text{ N}$ gives:

$$-\frac{8}{17}T_{AB} + 0.6T_{AC} + \frac{5}{13}T_{AD} = 0 \qquad (1)$$

$$-\frac{12}{17}T_{AB} - 0.64T_{AC} - \frac{9.6}{13}T_{AD} + 792\text{ N} = 0 \qquad (2)$$

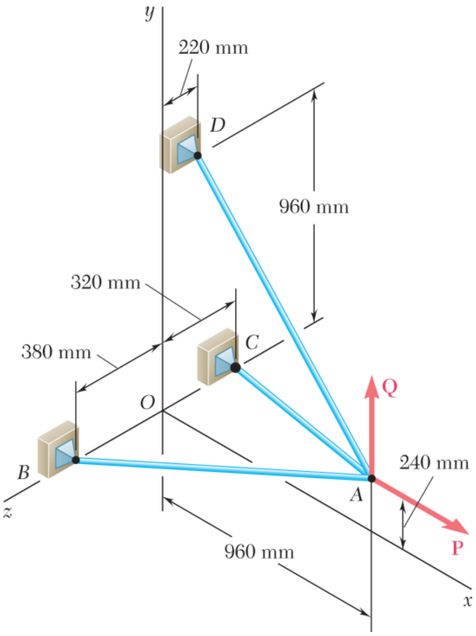
$$\frac{9}{17}T_{AB} + 0.48T_{AC} - \frac{7.2}{13}T_{AD} = 0 \qquad (3)$$

Solving Equations (1), (2), and (3) by conventional algorithms gives

$T_{AB} = 510.00\text{ N}$ $\qquad T_{AB} = 510\text{ N}$ ◀

$T_{AC} = 56.250\text{ N}$ $\qquad T_{AC} = 56.2\text{ N}$ ◀

$T_{AD} = 536.25\text{ N}$ $\qquad T_{AD} = 536\text{ N}$ ◀



PROBLEM 2.116

For the cable system of Problems 2.107 and 2.108, determine the tension in each cable knowing that $P = 2880$ N and $Q = 0$.

SOLUTION

$$\Sigma \mathbf{F}_A = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AD} + \mathbf{P} + \mathbf{Q} = 0$$

Where $\mathbf{P} = P\mathbf{i}$ and $\mathbf{Q} = Q\mathbf{j}$

$$\overrightarrow{AB} = -(960 \text{ mm})\mathbf{i} - (240 \text{ mm})\mathbf{j} + (380 \text{ mm})\mathbf{k} \quad AB = 1060 \text{ mm}$$

$$\overrightarrow{AC} = -(960 \text{ mm})\mathbf{i} - (240 \text{ mm})\mathbf{j} - (320 \text{ mm})\mathbf{k} \quad AC = 1040 \text{ mm}$$

$$\overrightarrow{AD} = -(960 \text{ mm})\mathbf{i} + (720 \text{ mm})\mathbf{j} - (220 \text{ mm})\mathbf{k} \quad AD = 1220 \text{ mm}$$

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = T_{AB}\left(-\frac{48}{53}\mathbf{i} - \frac{12}{53}\mathbf{j} + \frac{19}{53}\mathbf{k}\right)$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = T_{AC}\left(-\frac{12}{13}\mathbf{i} - \frac{3}{13}\mathbf{j} - \frac{4}{13}\mathbf{k}\right)$$

$$\mathbf{T}_{AD} = T_{AD}\boldsymbol{\lambda}_{AD} = T_{AD}\frac{\overrightarrow{AD}}{AD} = T_{AD}\left(-\frac{48}{61}\mathbf{i} + \frac{36}{61}\mathbf{j} - \frac{11}{61}\mathbf{k}\right)$$

Substituting into $\Sigma \mathbf{F}_A = 0$, setting $P = (2880 \text{ N})\mathbf{i}$ and $Q = 0$, and setting the coefficients of **i**, **j**, **k** equal to 0, we obtain the following three equilibrium equations:

$$\mathbf{i}: \quad -\frac{48}{53}T_{AB} - \frac{12}{13}T_{AC} - \frac{48}{61}T_{AD} + 2880 \text{ N} = 0 \qquad (1)$$

$$\mathbf{j}: \quad -\frac{12}{53}T_{AB} - \frac{3}{13}T_{AC} + \frac{36}{61}T_{AD} = 0 \qquad (2)$$

$$\mathbf{k}: \quad \frac{19}{53}T_{AB} - \frac{4}{13}T_{AC} - \frac{11}{61}T_{AD} = 0 \qquad (3)$$

PROBLEM 2.116 (Continued)

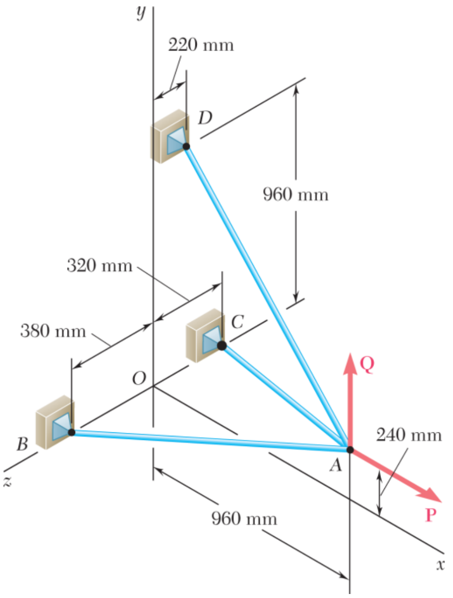
Solving the system of linear equations using conventional algorithms gives:

$$T_{AB} = 1340.14 \text{ N}$$
$$T_{AC} = 1025.12 \text{ N}$$
$$T_{AD} = 915.03 \text{ N}$$

$T_{AB} = 1340 \text{ N}$ ◀

$T_{AC} = 1025 \text{ N}$ ◀

$T_{AD} = 915 \text{ N}$ ◀



PROBLEM 2.117

For the cable system of Problems 2.107 and 2.108, determine the tension in each cable knowing that $P = 2880$ N and $Q = 576$ N.

SOLUTION

See Problem 2.116 for the analysis leading to the linear algebraic Equations (1), (2), and (3) below:

$$-\frac{48}{53}T_{AB} - \frac{12}{13}T_{AC} - \frac{48}{61}T_{AD} + P = 0 \tag{1}$$

$$-\frac{12}{53}T_{AB} - \frac{3}{13}T_{AC} + \frac{36}{61}T_{AD} + Q = 0 \tag{2}$$

$$\frac{19}{53}T_{AB} - \frac{4}{13}T_{AC} - \frac{11}{61}T_{AD} = 0 \tag{3}$$

Setting $P = 2880$ N and $Q = 576$ N gives:

$$-\frac{48}{53}T_{AB} - \frac{12}{13}T_{AC} - \frac{48}{61}T_{AD} + 2880\text{ N} = 0 \tag{1$'$}$$

$$-\frac{12}{53}T_{AB} - \frac{3}{13}T_{AC} + \frac{36}{61}T_{AD} + 576\text{ N} = 0 \tag{2$'$}$$

$$\frac{19}{53}T_{AB} - \frac{4}{13}T_{AC} - \frac{11}{61}T_{AD} = 0 \tag{3$'$}$$

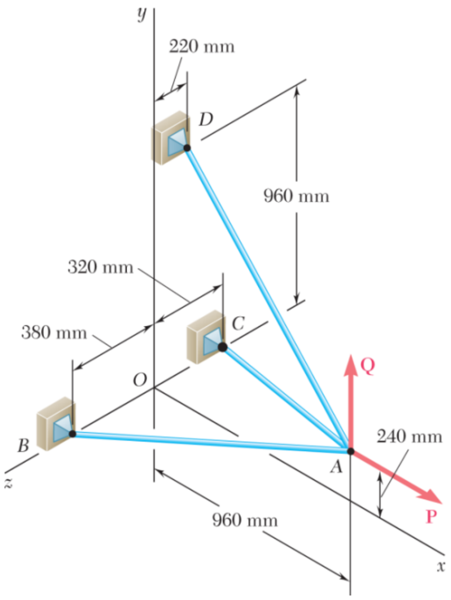
Solving the resulting set of equations using conventional algorithms gives:

$$T_{AB} = 1431.00\text{ N}$$
$$T_{AC} = 1560.00\text{ N}$$
$$T_{AD} = 183.010\text{ N}$$

$T_{AB} = 1431$ N ◀

$T_{AC} = 1560$ N ◀

$T_{AD} = 183.0$ N ◀



PROBLEM 2.118

For the cable system of Problems 2.107 and 2.108, determine the tension in each cable knowing that $P = 2880$ N and $Q = -576$ N. (**Q** is directed downward).

SOLUTION

See Problem 2.116 for the analysis leading to the linear algebraic Equations (1), (2), and (3) below:

$$-\frac{48}{53}T_{AB} - \frac{12}{13}T_{AC} - \frac{48}{61}T_{AD} + P = 0 \qquad (1)$$

$$-\frac{12}{53}T_{AB} - \frac{3}{13}T_{AC} + \frac{36}{61}T_{AD} + Q = 0 \qquad (2)$$

$$\frac{19}{53}T_{AB} - \frac{4}{13}T_{AC} - \frac{11}{61}T_{AD} = 0 \qquad (3)$$

Setting $P = 2880$ N and $Q = -576$ N gives:

$$-\frac{48}{53}T_{AB} - \frac{12}{13}T_{AC} - \frac{48}{61}T_{AD} + 2880\text{ N} = 0 \qquad (1')$$

$$-\frac{12}{53}T_{AB} - \frac{3}{13}T_{AC} + \frac{36}{61}T_{AD} - 576\text{ N} = 0 \qquad (2')$$

$$\frac{19}{53}T_{AB} - \frac{4}{13}T_{AC} - \frac{11}{61}T_{AD} = 0 \qquad (3')$$

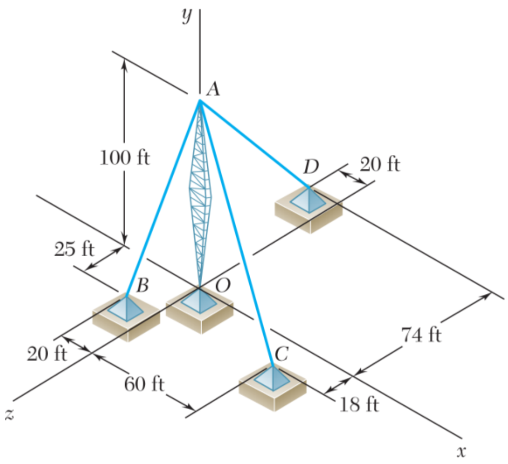
Solving the resulting set of equations using conventional algorithms gives:

$$T_{AB} = 1249.29\text{ N}$$
$$T_{AC} = 490.31\text{ N}$$
$$T_{AD} = 1646.97\text{ N}$$

$T_{AB} = 1249$ N ◀

$T_{AC} = 490$ N ◀

$T_{AD} = 1647$ N ◀



PROBLEM 2.119

For the transmission tower of Probs. 2.111 and 2.112, determine the tension in each guy wire knowing that the tower exerts on the pin at *A* an upward vertical force of 1800 lb.

PROBLEM 2.111 A transmission tower is held by three guy wires attached to a pin at *A* and anchored by bolts at *B*, *C*, and *D*. If the tension in wire *AB* is 840 lb, determine the vertical force **P** exerted by the tower on the pin at *A*.

SOLUTION

See Problem 2.111 for the figure and the analysis leading to the linear algebraic Equations (1), (2), and (3) below:

$$\mathbf{i}: \quad -\frac{4}{21}T_{AB}+\frac{30}{59}T_{AC}-\frac{10}{63}T_{AD}=0 \qquad (1)$$

$$\mathbf{j}: \quad -\frac{20}{21}T_{AB}-\frac{50}{59}T_{AC}-\frac{50}{63}T_{AD}+P=0 \qquad (2)$$

$$\mathbf{k}: \quad \frac{5}{21}T_{AB}+\frac{9}{59}T_{AC}-\frac{37}{63}T_{AD}=0 \qquad (3)$$

Substituting for $P = 1800$ lb in Equations (1), (2), and (3) above and solving the resulting set of equations using conventional algorithms gives:

$$-\frac{4}{21}T_{AB}+\frac{30}{59}T_{AC}-\frac{10}{63}T_{AD}=0 \qquad (1')$$

$$-\frac{20}{21}T_{AB}-\frac{50}{59}T_{AC}-\frac{50}{63}T_{AD}+1800\text{ lb}=0 \qquad (2')$$

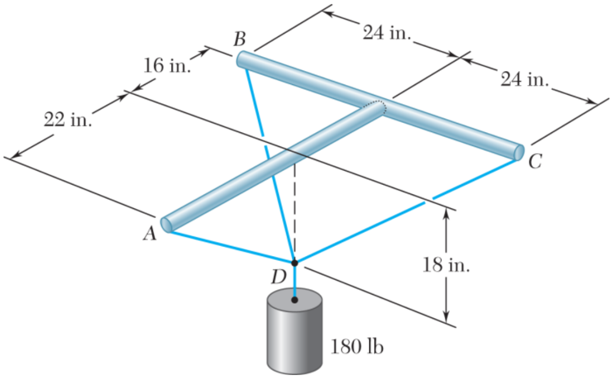
$$\frac{5}{21}T_{AB}+\frac{9}{59}T_{AC}-\frac{37}{63}T_{AD}=0 \qquad (3')$$

$$T_{AB} = 973.64\text{ lb}$$
$$T_{AC} = 531.00\text{ lb}$$
$$T_{AD} = 532.64\text{ lb}$$

$T_{AB} = 974$ lb ◀

$T_{AC} = 531$ lb ◀

$T_{AD} = 533$ lb ◀

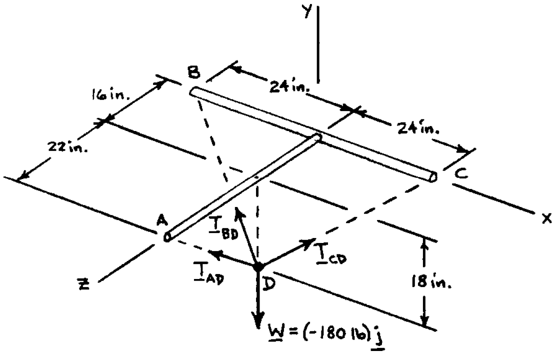

B
16 in.
24 in.
24 in.
22 in.
C
A
D
18 in.
180 lb

PROBLEM 2.120

Three wires are connected at point *D*, which is located 18 in. below the T-shaped pipe support *ABC*. Determine the tension in each wire when a 180-lb cylinder is suspended from point *D* as shown.

SOLUTION

Free-Body Diagram of Point *D*:


y
B
16 in.
24 in.
24 in.
22 in.
C
x
A
T_{BD}
T_{CD}
T_{AD}
z
D
18 in.
$W = (-180 \text{ lb})j$

The forces applied at *D* are:

$$\mathbf{T}_{DA}, \mathbf{T}_{DB}, \mathbf{T}_{DC} \text{ and } \mathbf{W}$$

where $\mathbf{W} = -180.0 \text{ lb}\mathbf{j}$. To express the other forces in terms of the unit vectors **i**, **j**, **k**, we write

$$\overrightarrow{DA} = (18 \text{ in.})\mathbf{j} + (22 \text{ in.})\mathbf{k}$$
$$DA = 28.425 \text{ in.}$$
$$\overrightarrow{DB} = -(24 \text{ in.})\mathbf{i} + (18 \text{ in.})\mathbf{j} - (16 \text{ in.})\mathbf{k}$$
$$DB = 34.0 \text{ in.}$$
$$\overrightarrow{DC} = (24 \text{ in.})\mathbf{i} + (18 \text{ in.})\mathbf{j} - (16 \text{ in.})\mathbf{k}$$
$$DC = 34.0 \text{ in.}$$

PROBLEM 2.120 (Continued)

and
$$\mathbf{T}_{DA} = T_{Da}\boldsymbol{\lambda}_{DA} = T_{Da}\frac{\overrightarrow{DA}}{DA}$$
$$= (0.63324\mathbf{j} + 0.77397\mathbf{k})T_{DA}$$
$$\mathbf{T}_{DB} = T_{DB}\boldsymbol{\lambda}_{DB} = T_{DB}\frac{\overrightarrow{DB}}{DB}$$
$$= (-0.70588\mathbf{i} + 0.52941\mathbf{j} - 0.47059\mathbf{k})T_{DB}$$
$$\mathbf{T}_{DC} = T_{DC}\boldsymbol{\lambda}_{DC} = T_{DC}\frac{\overrightarrow{DC}}{DC}$$
$$= (0.70588\mathbf{i} + 0.52941\mathbf{j} - 0.47059\mathbf{k})T_{DC}$$

Equilibrium Condition with $\mathbf{W} = -W\mathbf{j}$

$$\Sigma F = 0: \quad \mathbf{T}_{DA} + \mathbf{T}_{DB} + \mathbf{T}_{DC} - W\mathbf{j} = 0$$

Substituting the expressions obtained for $\mathbf{T}_{DA}, \mathbf{T}_{DB}$, and $\mathbf{T}_{DC}$ and factoring **i**, **j**, and **k**:

$$(-0.70588T_{DB} + 0.70588T_{DC})\mathbf{i}$$
$$(0.63324T_{DA} + 0.52941T_{DB} + 0.52941T_{DC} - W)\mathbf{j}$$
$$(0.77397T_{DA} - 0.47059T_{DB} - 0.47059T_{DC})\mathbf{k}$$

Equating to zero the coefficients of **i**, **j**, **k**:

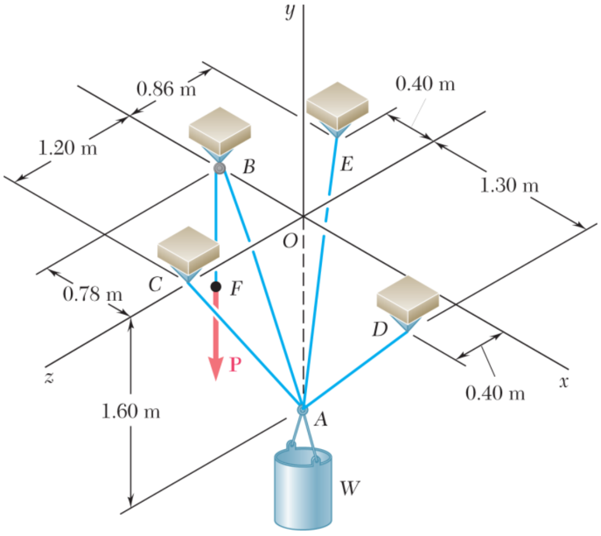
$$-0.70588T_{DB} + 0.70588T_{DC} = 0 \tag{1}$$
$$0.63324T_{DA} + 0.52941T_{DB} + 0.52941T_{DC} - W = 0 \tag{2}$$
$$0.77397T_{DA} - 0.47059T_{DB} - 0.47059T_{DC} = 0 \tag{3}$$

Substituting $W = 180$ lb in Equations (1), (2), and (3) above, and solving the resulting set of equations using conventional algorithms gives,

$T_{DA} = 119.7$ lb ◀

$T_{DB} = 98.4$ lb ◀

$T_{DC} = 98.4$ lb ◀



PROBLEM 2.121

A container of weight W is suspended from ring A, to which cables AC and AE are attached. A force **P** is applied to the end F of a third cable that passes over a pulley at B and through ring A and that is attached to a support at D. Knowing that $W = 1000$ N, determine the magnitude of P. (*Hint:* The tension is the same in all portions of cable $FBAD$.)

SOLUTION

The (vector) force in each cable can be written as the product of the (scalar) force and the unit vector along the cable. That is, with

$$\overrightarrow{AB} = -(0.78\text{ m})\mathbf{i} + (1.6\text{ m})\mathbf{j} + (0\text{ m})\mathbf{k}$$

$$AB = \sqrt{(-0.78\text{ m})^2 + (1.6\text{ m})^2 + (0)^2}$$

$$= 1.78\text{ m}$$

$$\mathbf{T}_{AB} = T\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB}$$

$$= \frac{T_{AB}}{1.78\text{ m}}[-(0.78\text{ m})\mathbf{i} + (1.6\text{ m})\mathbf{j} + (0\text{ m})\mathbf{k}]$$

$$\mathbf{T}_{AB} = T_{AB}(-0.4382\mathbf{i} + 0.8989\mathbf{j} + 0\mathbf{k})$$

and

$$\overrightarrow{AC} = (0)\mathbf{i} + (1.6\text{ m})\mathbf{j} + (1.2\text{ m})\mathbf{k}$$

$$AC = \sqrt{(0\text{ m})^2 + (1.6\text{ m})^2 + (1.2\text{ m})^2} = 2\text{ m}$$

$$\mathbf{T}_{AC} = T\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = \frac{T_{AC}}{2\text{ m}}[(0)\mathbf{i} + (1.6\text{ m})\mathbf{j} + (1.2\text{ m})\mathbf{k}]$$

$$\mathbf{T}_{AC} = T_{AC}(0.8\mathbf{j} + 0.6\mathbf{k})$$

and

$$\overrightarrow{AD} = (1.3\text{ m})\mathbf{i} + (1.6\text{ m})\mathbf{j} + (0.4\text{ m})\mathbf{k}$$

$$AD = \sqrt{(1.3\text{ m})^2 + (1.6\text{ m})^2 + (0.4\text{ m})^2} = 2.1\text{ m}$$

$$\mathbf{T}_{AD} = T\boldsymbol{\lambda}_{AD} = T_{AD}\frac{\overrightarrow{AD}}{AD} = \frac{T_{AD}}{2.1\text{ m}}[(1.3\text{ m})\mathbf{i} + (1.6\text{ m})\mathbf{j} + (0.4\text{ m})\mathbf{k}]$$

$$\mathbf{T}_{AD} = T_{AD}(0.6190\mathbf{i} + 0.7619\mathbf{j} + 0.1905\mathbf{k})$$

PROBLEM 2.121 (Continued)

Finally,

$$\overrightarrow{AE} = -(0.4 \text{ m})\mathbf{i} + (1.6 \text{ m})\mathbf{j} - (0.86 \text{ m})\mathbf{k}$$

$$AE = \sqrt{(-0.4 \text{ m})^2 + (1.6 \text{ m})^2 + (-0.86 \text{ m})^2} = 1.86 \text{ m}$$

$$\mathbf{T}_{AE} = T\boldsymbol{\lambda}_{AE} = T_{AE}\frac{\overrightarrow{AE}}{AE}$$

$$= \frac{T_{AE}}{1.86 \text{ m}}[-(0.4 \text{ m})\mathbf{i} + (1.6 \text{ m})\mathbf{j} - (0.86 \text{ m})\mathbf{k}]$$

$$\mathbf{T}_{AE} = T_{AE}(-0.2151\mathbf{i} + 0.8602\mathbf{j} - 0.4624\mathbf{k})$$

With the weight of the container $\mathbf{W} = -W\mathbf{j}$, at A we have:

$$\Sigma\mathbf{F} = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{T}_{AD} - W\mathbf{j} = 0$$

Equating the factors of **i**, **j**, and **k** to zero, we obtain the following linear algebraic equations:

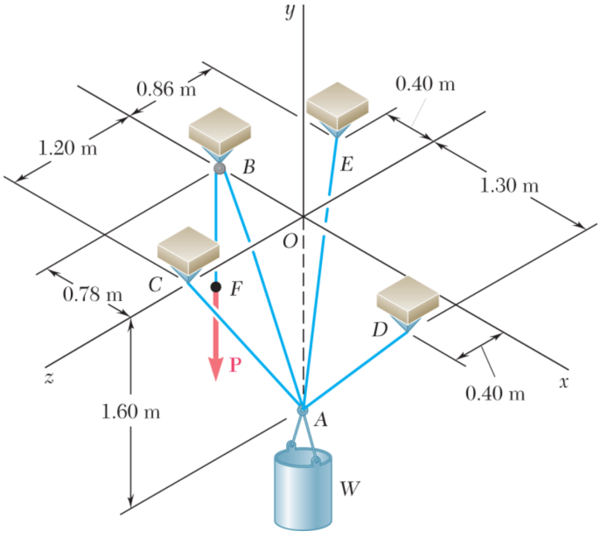
$$-0.4382T_{AB} + 0.6190T_{AD} - 0.2151T_{AE} = 0 \qquad (1)$$

$$0.8989T_{AB} + 0.8T_{AC} + 0.7619T_{AD} + 0.8602T_{AE} - W = 0 \qquad (2)$$

$$0.6T_{AC} + 0.1905T_{AD} - 0.4624T_{AE} = 0 \qquad (3)$$

Knowing that $W = 1000$ N and that because of the pulley system at $B\,T_{AB} = T_{AD} = P$, where P is the externally applied (unknown) force, we can solve the system of linear Equations (1), (2) and (3) uniquely for P.

$P = 378$ N ◀



PROBLEM 2.122

Knowing that the tension in cable *AC* of the system described in Problem 2.121 is 150 N, determine (*a*) the magnitude of the force **P**, (*b*) the weight *W* of the container.

PROBLEM 2.121 A container of weight *W* is suspended from ring *A*, to which cables *AC* and *AE* are attached. A force **P** is applied to the end *F* of a third cable that passes over a pulley at *B* and through ring *A* and that is attached to a support at *D*. Knowing that $W = 1000$ N, determine the magnitude of *P*. (*Hint:* The tension is the same in all portions of cable *FBAD*.)

SOLUTION

Here, as in Problem 2.121, the support of the container consists of the four cables *AE*, *AC*, *AD*, and *AB*, with the condition that the force in cables *AB* and *AD* is equal to the externally applied force *P*. Thus, with the condition

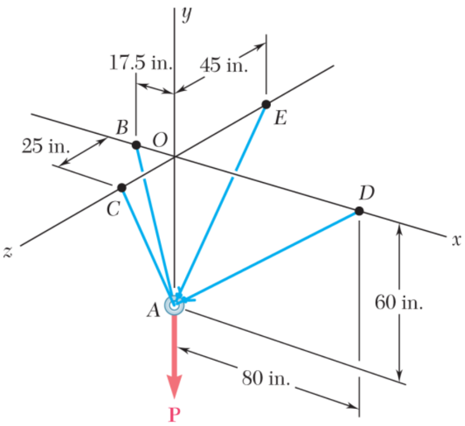
$$T_{AB} = T_{AD} = P$$

and using the linear algebraic equations of Problem 2.131 with $T_{AC} = 150$ N, we obtain

(*a*) $P = 454$ N ◀

(*b*) $W = 1202$ N ◀

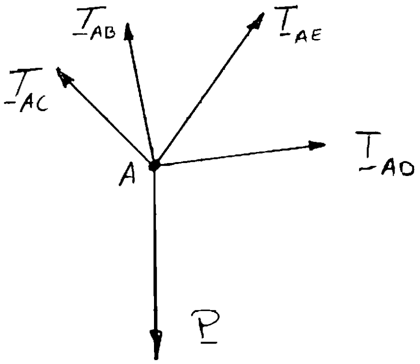
PROBLEM 2.123



Cable *BAC* passes through a frictionless ring *A* and is attached to fixed supports at *B* and *C*, while cables *AD* and *AE* are both tied to the ring and are attached, respectively, to supports at *D* and *E*. Knowing that a 200-lb vertical load **P** is applied to ring *A*, determine the tension in each of the three cables.

SOLUTION

Free Body Diagram at *A*:



Since T_{BAC} = tension in cable *BAC*, it follows that

$$T_{AB} = T_{AC} = T_{BAC}$$

$$\mathbf{T}_{AB} = T_{BAC}\boldsymbol{\lambda}_{AB} = T_{BAC}\frac{(-17.5\text{ in.})\mathbf{i} + (60\text{ in.})\mathbf{j}}{62.5\text{ in.}} = T_{BAC}\left(\frac{-17.5}{62.5}\mathbf{i} + \frac{60}{62.5}\mathbf{j}\right)$$

$$\mathbf{T}_{AC} = T_{BAC}\boldsymbol{\lambda}_{AC} = T_{BAC}\frac{(60\text{ in.})\mathbf{i} + (25\text{ in.})\mathbf{k}}{65\text{ in.}} = T_{BAC}\left(\frac{60}{65}\mathbf{j} + \frac{25}{65}\mathbf{k}\right)$$

$$\mathbf{T}_{AD} = T_{AD}\boldsymbol{\lambda}_{AD} = T_{AD}\frac{(80\text{ in.})\mathbf{i} + (60\text{ in.})\mathbf{j}}{100\text{ in.}} = T_{AD}\left(\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}\right)$$

$$\mathbf{T}_{AE} = T_{AE}\boldsymbol{\lambda}_{AE} = T_{AE}\frac{(60\text{ in.})\mathbf{j} - (45\text{ in.})\mathbf{k}}{75\text{ in.}} = T_{AE}\left(\frac{4}{5}\mathbf{j} - \frac{3}{5}\mathbf{k}\right)$$

PROBLEM 2.123 (Continued)

Substituting into $\Sigma \mathbf{F}_A = 0$, setting $\mathbf{P} = (-200 \text{ lb})\mathbf{j}$, and setting the coefficients of **i**, **j**, **k** equal to ϕ, we obtain the following three equilibrium equations:

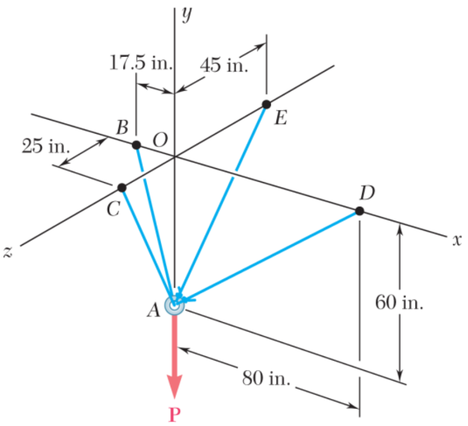
From **i**: $-\dfrac{17.5}{62.5}T_{BAC} + \dfrac{4}{5}T_{AD} = 0$ (1)

From **j**: $\left(\dfrac{60}{62.5} + \dfrac{60}{65}\right)T_{BAC} + \dfrac{3}{5}T_{AD} + \dfrac{4}{5}T_{AE} - 200 \text{ lb} = 0$ (2)

From **k**: $\dfrac{25}{65}T_{BAC} - \dfrac{3}{5}T_{AE} = 0$ (3)

Solving the system of linear equations using conventional algorithms gives:

$T_{BAC} = 76.7 \text{ lb};\ T_{AD} = 26.9 \text{ lb};\ T_{AE} = 49.2 \text{ lb}$ ◀



PROBLEM 2.124

Knowing that the tension in cable *AE* of Prob. 2.123 is 75 lb, determine (*a*) the magnitude of the load **P**, (*b*) the tension in cables *BAC* and *AD*.

PROBLEM 2.123 Cable *BAC* passes through a frictionless ring *A* and is attached to fixed supports at *B* and *C*, while cables *AD* and *AE* are both tied to the ring and are attached, respectively, to supports at *D* and *E*. Knowing that a 200-lb vertical load **P** is applied to ring *A*, determine the tension in each of the three cables.

SOLUTION

Refer to the solution to Problem 2.123 for the figure and analysis leading to the following set of equilibrium equations, Equation (2) being modified to include $P\mathbf{j}$ as an unknown quantity:

$$-\frac{17.5}{62.5}T_{BAC}+\frac{4}{5}T_{AD}=0 \tag{1}$$

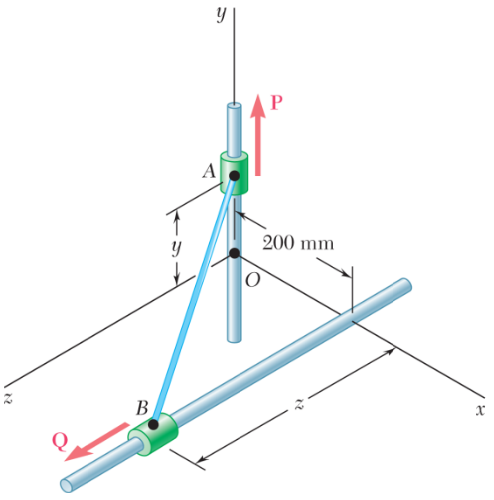
$$\left(\frac{60}{62.5}+\frac{60}{65}\right)T_{BAC}+\frac{3}{5}T_{AD}+\frac{4}{5}T_{AE}-P=0 \tag{2}$$

$$\frac{25}{65}T_{BAC}-\frac{3}{5}T_{AE}=0 \tag{3}$$

Substituting for $T_{AE}=75\text{ lb}$ and solving simultaneously gives:

(*a*) $P=305\text{ lb}$ ◀

(*b*) $T_{BAC}=117.0\text{ lb};\ T_{AD}=40.9\text{ lb}$ ◀



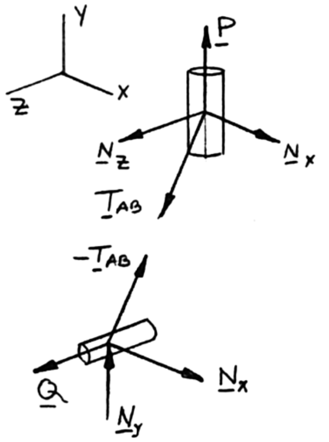
PROBLEM 2.125

Collars A and B are connected by a 525-mm-long wire and can slide freely on frictionless rods. If a force $\mathbf{P} = (341\ \text{N})\mathbf{j}$ is applied to collar A, determine (a) the tension in the wire when $y = 155$ mm, (b) the magnitude of the force $\mathbf{Q}$ required to maintain the equilibrium of the system.

SOLUTION

For both Problems 2.125 and 2.126:

Free-Body Diagrams of Collars:



$$(AB)^2 = x^2 + y^2 + z^2$$

Here $$(0.525\ \text{m})^2 = (0.20\ \text{m})^2 + y^2 + z^2$$

or $$y^2 + z^2 = 0.23563\ \text{m}^2$$

Thus, when y given, z is determined,

Now $$\boldsymbol{\lambda}_{AB} = \frac{\overrightarrow{AB}}{AB}$$
$$= \frac{1}{0.525\ \text{m}}(0.20\mathbf{i} - y\mathbf{j} + z\mathbf{k})\text{m}$$
$$= 0.38095\mathbf{i} - 1.90476y\mathbf{j} + 1.90476z\mathbf{k}$$

Where y and z are in units of meters, m.

From the F.B. Diagram of collar A: $\quad \Sigma\mathbf{F} = 0\text{:} \quad N_x\mathbf{i} + N_z\mathbf{k} + P\mathbf{j} + T_{AB}\lambda_{AB} = 0$

Setting the **j** coefficient to zero gives $\quad P - (1.90476y)T_{AB} = 0$

With $$P = 341\ \text{N}$$
$$T_{AB} = \frac{341\ \text{N}}{1.90476y}$$

Now, from the free body diagram of collar B: $\quad \Sigma\mathbf{F} = 0\text{:} \quad N_x\mathbf{i} + N_y\mathbf{j} + Q\mathbf{k} - T_{AB}\boldsymbol{\lambda}_{AB} = 0$

Setting the **k** coefficient to zero gives $\quad Q - T_{AB}(1.90476z) = 0$

And using the above result for T_{AB}, we have $\quad Q = T_{AB}z = \dfrac{341\ \text{N}}{(1.90476)y}(1.90476z) = \dfrac{(341\ \text{N})(z)}{y}$

PROBLEM 2.125 (Continued)

Then from the specifications of the problem, $y = 155\text{ mm} = 0.155\text{ m}$

$$z^2 = 0.23563\text{ m}^2 - (0.155\text{ m})^2$$

$$z = 0.46\text{ m}$$

and

(*a*)

$$T_{AB} = \frac{341\text{ N}}{0.155(1.90476)}$$

$$= 1155.00\text{ N}$$

or $T_{AB} = 1155\text{ N}$ ◀

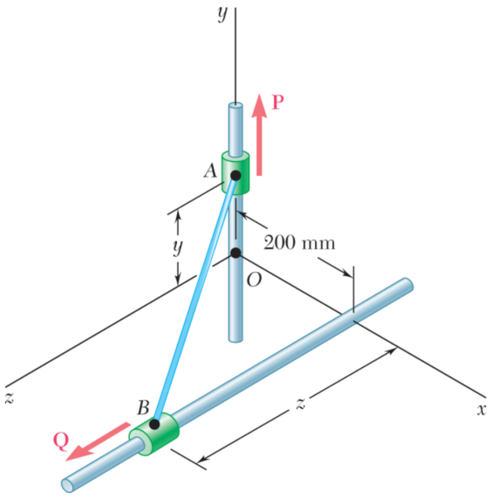
and

(*b*)

$$Q = \frac{341\text{ N}(0.46\text{ m})(0.866)}{(0.155\text{ m})}$$

$$= (1012.00\text{ N})$$

or $Q = 1012\text{ N}$ ◀



PROBLEM 2.126

Solve Problem 2.125 assuming that $y = 275$ mm.

PROBLEM 2.125 Collars *A* and *B* are connected by a 525-mm-long wire and can slide freely on frictionless rods. If a force $\mathbf{P} = (341\text{ N})\mathbf{j}$ is applied to collar *A*, determine (*a*) the tension in the wire when $y = 155$ mm, (*b*) the magnitude of the force **Q** required to maintain the equilibrium of the system.

SOLUTION

From the analysis of Problem 2.125, particularly the results:

$$y^2 + z^2 = 0.23563\text{ m}^2$$

$$T_{AB} = \frac{341\text{ N}}{1.90476y}$$

$$Q = \frac{341\text{ N}}{y}z$$

With $y = 275\text{ mm} = 0.275\text{ m}$, we obtain:

$$z^2 = 0.23563\text{ m}^2 - (0.275\text{ m})^2$$

$$z = 0.40\text{ m}$$

and

(*a*) $$T_{AB} = \frac{341\text{ N}}{(1.90476)(0.275\text{ m})} = 651.00$$

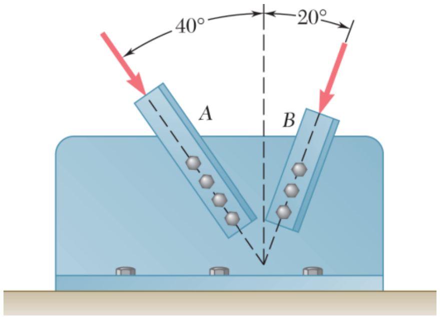
or $$T_{AB} = 651\text{ N} \blacktriangleleft$$

and

(*b*) $$Q = \frac{341\text{ N}(0.40\text{ m})}{(0.275\text{ m})}$$

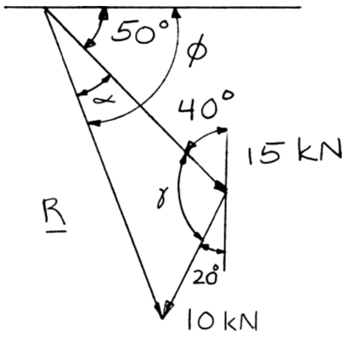
or $$Q = 496\text{ N} \blacktriangleleft$$

PROBLEM 2.127



Two structural members *A* and *B* are bolted to a bracket as shown. Knowing that both members are in compression and that the force is 15 kN in member *A* and 10 kN in member *B*, determine by trigonometry the magnitude and direction of the resultant of the forces applied to the bracket by members *A* and *B*.

SOLUTION



Using the force triangle and the laws of cosines and sines,

we have $\gamma = 180° - (40° + 20°)$

$= 120°$

Then $R^2 = (15\text{ kN})^2 + (10\text{ kN})^2 - 2(15\text{ kN})(10\text{ kN})\cos 120°$

$= 475\text{ kN}^2$

$R = 21.794\text{ kN}$

and $\dfrac{10\text{ kN}}{\sin\alpha} = \dfrac{21.794\text{ kN}}{\sin 120°}$

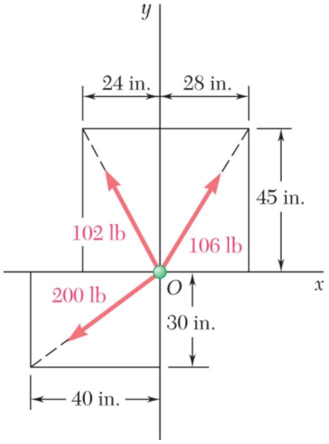
$\sin\alpha = \left(\dfrac{10\text{ kN}}{21.794\text{ kN}}\right)\sin 120°$

$= 0.39737$

$\alpha = 23.414$

Hence: $\phi = \alpha + 50° = 73.414$

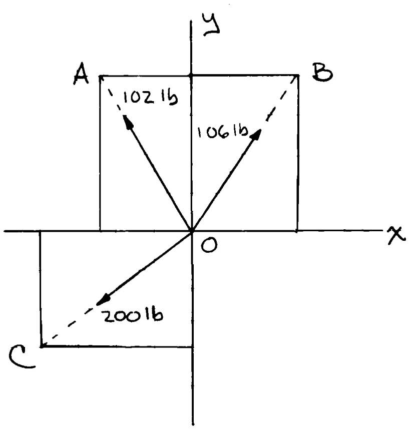
R = 21.8 kN ⦫ 73.4° ◀


y
24 in.
28 in.
45 in.
102 lb
106 lb
200 lb
O
x
30 in.
40 in.

PROBLEM 2.128

Determine the x and y components of each of the forces shown.

SOLUTION


y
A
B
102 lb
106 lb
x
O
200 lb
C

Compute the following distances:

$$OA = \sqrt{(24\text{ in.})^2 + (45\text{ in.})^2}$$
$$= 51.0\text{ in.}$$

$$OB = \sqrt{(28\text{ in.})^2 + (45\text{ in.})^2}$$
$$= 53.0\text{ in.}$$

$$OC = \sqrt{(40\text{ in.})^2 + (30\text{ in.})^2}$$
$$= 50.0\text{ in.}$$

102-lb Force: $F_x = -102\text{ lb}\dfrac{24\text{ in.}}{51.0\text{ in.}}$ $\quad F_x = -48.0\text{ lb}$ ◀

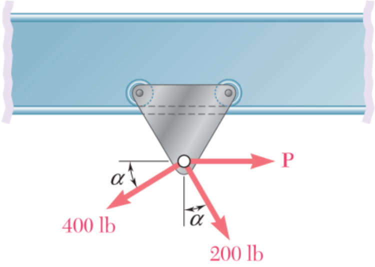
$F_y = +102\text{ lb}\dfrac{45\text{ in.}}{51.0\text{ in.}}$ $\quad F_y = +90.0\text{ lb}$ ◀

106-lb Force: $F_x = +106\text{ lb}\dfrac{28\text{ in.}}{53.0\text{ in.}}$ $\quad F_x = +56.0\text{ lb}$ ◀

$F_y = +106\text{ lb}\dfrac{45\text{ in.}}{53.0\text{ in.}}$ $\quad F_y = +90.0\text{ lb}$ ◀

200-lb Force: $F_x = -200\text{ lb}\dfrac{40\text{ in.}}{50.0\text{ in.}}$ $\quad F_x = -160.0\text{ lb}$ ◀

$F_y = -200\text{ lb}\dfrac{30\text{ in.}}{50.0\text{ in.}}$ $\quad F_y = -120.0\text{ lb}$ ◀



PROBLEM 2.129

A hoist trolley is subjected to the three forces shown. Knowing that $\alpha = 40°$, determine (*a*) the required magnitude of the force **P** if the resultant of the three forces is to be vertical, (*b*) the corresponding magnitude of the resultant.

SOLUTION

$$R_x = \xrightarrow{+} \Sigma F_x = P + (200 \text{ lb})\sin 40° - (400 \text{ lb})\cos 40°$$

$$R_x = P - 177.860 \text{ lb} \qquad (1)$$

$$R_y = +\!\downarrow \Sigma F_y = (200 \text{ lb})\cos 40° + (400 \text{ lb})\sin 40°$$

$$R_y = 410.32 \text{ lb} \qquad (2)$$

(*a*) For **R** to be vertical, we must have $R_x = 0$.

Set $R_x = 0$ in Eq. (1)

$$0 = P - 177.860 \text{ lb}$$

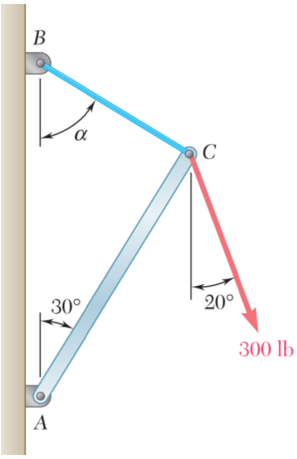
$$P = 177.860 \text{ lb}$$

$P = 177.9$ lb ◀

(*b*) Since **R** is to be vertical:

$$R = R_y = 410 \text{ lb}$$

$R = 410$ lb ◀

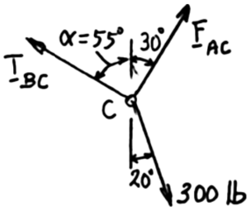

B
α
C
30°
20°
300 lb
A

PROBLEM 2.130

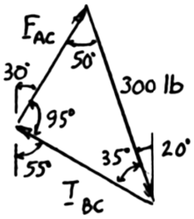
Knowing that $\alpha = 55°$ and that boom AC exerts on pin C a force directed along line AC, determine (a) the magnitude of that force, (b) the tension in cable BC.

SOLUTION

Free-Body Diagram


α = 55°
30°
F_{AC}
T_{BC}
C
20°
300 lb

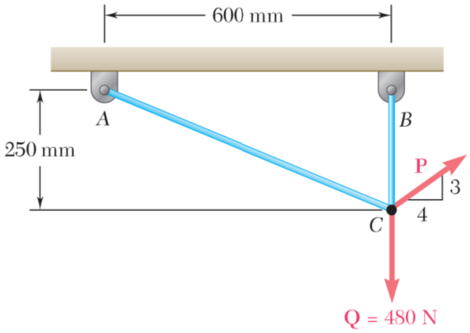
Force Triangle


F_{AC}
50°
30°
300 lb
95°
35°
20°
55°
T_{BC}

Law of sines:
$$\frac{F_{AC}}{\sin 35°} = \frac{T_{BC}}{\sin 50°} = \frac{300 \text{ lb}}{\sin 95°}$$

(a)
$$F_{AC} = \frac{300 \text{ lb}}{\sin 95°} \sin 35°$$
$F_{AC} = 172.7$ lb ◀

(b)
$$T_{BC} = \frac{300 \text{ lb}}{\sin 95°} \sin 50°$$
$T_{BC} = 231$ lb ◀

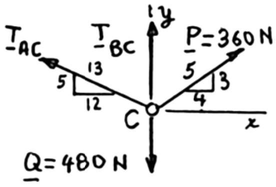

600 mm
A
B
250 mm
P
3
4
C
Q = 480 N

PROBLEM 2.131

Two cables are tied together at C and loaded as shown. Knowing that $P = 360$ N, determine the tension (a) in cable AC, (b) in cable BC.

SOLUTION

Free Body: *C*


T_{AC}
T_{BC}
y
P=360 N
13
5
12
5
3
4
C
x
Q=480 N

(a)

$$\Sigma F_x = 0: \quad -\frac{12}{13}T_{AC} + \frac{4}{5}(360 \text{ N}) = 0$$

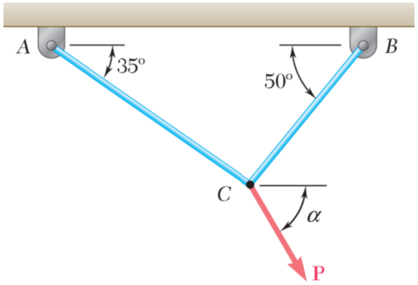
$$T_{AC} = 312 \text{ N} \blacktriangleleft$$

(b)

$$\Sigma F_y = 0: \quad \frac{5}{13}(312 \text{ N}) + T_{BC} + \frac{3}{5}(360 \text{ N}) - 480 \text{ N} = 0$$

$$T_{BC} = 480 \text{ N} - 120 \text{ N} - 216 \text{ N}$$

$$T_{BC} = 144.0 \text{ N} \blacktriangleleft$$

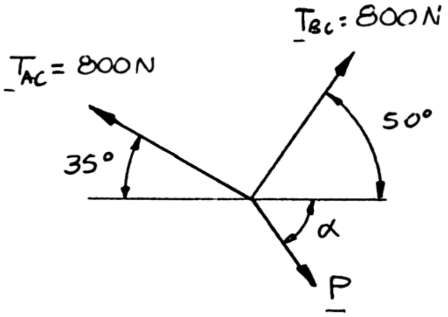


PROBLEM 2.132

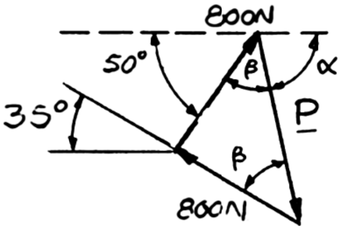
Two cables tied together at C are loaded as shown. Knowing that the maximum allowable tension in each cable is 800 N, determine (a) the magnitude of the largest force **P** that can be applied at C, (b) the corresponding value of α.

SOLUTION

Free-Body Diagram: *C*


T_{BC} = 800 N
T_{AC} = 800 N
35°
50°
α
P

Force Triangle


800N
50°
β
α
35°
P
β
800N

Force triangle is isosceles with

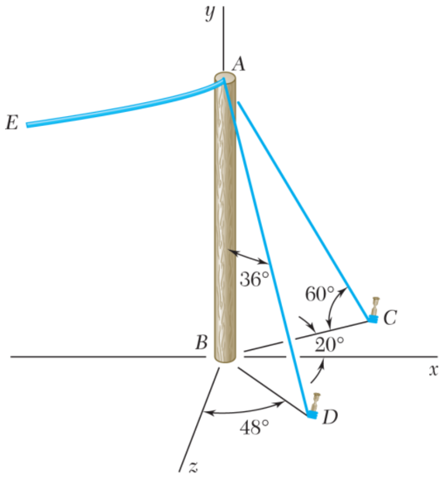
$$2\beta = 180° - 85°$$

$$\beta = 47.5°$$

(a) $$P = 2(800 \text{ N})\cos 47.5° = 1081 \text{ N}$$

Since $P > 0$, the solution is correct. $P = 1081$ N ◀

(b) $$\alpha = 180° - 50° - 47.5° = 82.5°$$ $\alpha = 82.5°$ ◀



PROBLEM 2.133

The end of the coaxial cable *AE* is attached to the pole *AB*, which is strengthened by the guy wires *AC* and *AD*. Knowing that the tension in wire *AC* is 120 lb, determine (*a*) the components of the force exerted by this wire on the pole, (*b*) the angles θ_x, θ_y, and θ_z that the force forms with the coordinate axes.

SOLUTION

(*a*)

$$F_x = (120\text{ lb})\cos 60°\cos 20°$$

$$F_x = 56.382\text{ lb} \qquad F_x = +56.4\text{ lb} ◀$$

$$F_y = -(120\text{ lb})\sin 60°$$

$$F_y = -103.923\text{ lb} \qquad F_y = -103.9\text{ lb} ◀$$

$$F_z = -(120\text{ lb})\cos 60°\sin 20°$$

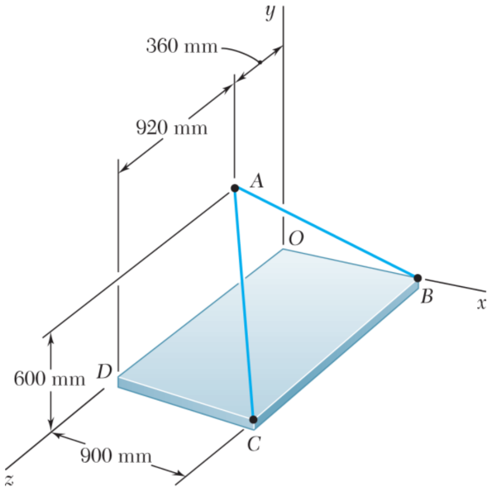
$$F_z = -20.521\text{ lb} \qquad F_z = -20.5\text{ lb} ◀$$

(*b*)

$$\cos\theta_x = \frac{F_x}{F} = \frac{56.382\text{ lb}}{120\text{ lb}} \qquad \theta_x = 62.0° ◀$$

$$\cos\theta_y = \frac{F_y}{F} = \frac{-103.923\text{ lb}}{120\text{ lb}} \qquad \theta_y = 150.0° ◀$$

$$\cos\theta_z = \frac{F_z}{F} = \frac{-20.52\text{ lb}}{120\text{ lb}} \qquad \theta_z = 99.8° ◀$$



PROBLEM 2.134

Knowing that the tension in cable *AC* is 2130 N, determine the components of the force exerted on the plate at *C*.

SOLUTION

$$\overrightarrow{CA} = -(900\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{j} - (920\text{ mm})\mathbf{k}$$

$$CA = \sqrt{(900\text{ mm})^2 + (600\text{ mm})^2 + (920\text{ mm})^2}$$

$$= 1420\text{ mm}$$

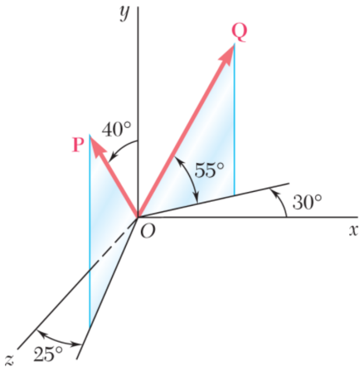
$$\mathbf{T}_{CA} = T_{CA}\boldsymbol{\lambda}_{CA}$$

$$= T_{CA}\frac{\overrightarrow{CA}}{CA}$$

$$\mathbf{T}_{CA} = \frac{2130\text{ N}}{1420\text{ mm}}[-(900\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{j} - (920\text{ mm})\mathbf{k}]$$

$$= -(1350\text{ N})\mathbf{i} + (900\text{ N})\mathbf{j} - (1380\text{ N})\mathbf{k}$$

$$(T_{CA})_x = -1350\text{ N},\quad (T_{CA})_y = 900\text{ N},\quad (T_{CA})_z = -1380\text{ N} \blacktriangleleft$$



PROBLEM 2.135

Find the magnitude and direction of the resultant of the two forces shown knowing that $P = 600$ N and $Q = 450$ N.

SOLUTION

$$\mathbf{P} = (600\text{ N})[\sin 40° \sin 25° \mathbf{i} + \cos 40° \mathbf{j} + \sin 40° \cos 25° \mathbf{k}]$$

$$= (162.992\text{ N})\mathbf{i} + (459.63\text{ N})\mathbf{j} + (349.54\text{ N})\mathbf{k}$$

$$\mathbf{Q} = (450\text{ N})[\cos 55° \cos 30° \mathbf{i} + \sin 55° \mathbf{j} - \cos 55° \sin 30° \mathbf{k}]$$

$$= (223.53\text{ N})\mathbf{i} + (368.62\text{ N})\mathbf{j} - (129.055\text{ N})\mathbf{k}$$

$$\mathbf{R} = \mathbf{P} + \mathbf{Q}$$

$$= (386.52\text{ N})\mathbf{i} + (828.25\text{ N})\mathbf{j} + (220.49\text{ N})\mathbf{k}$$

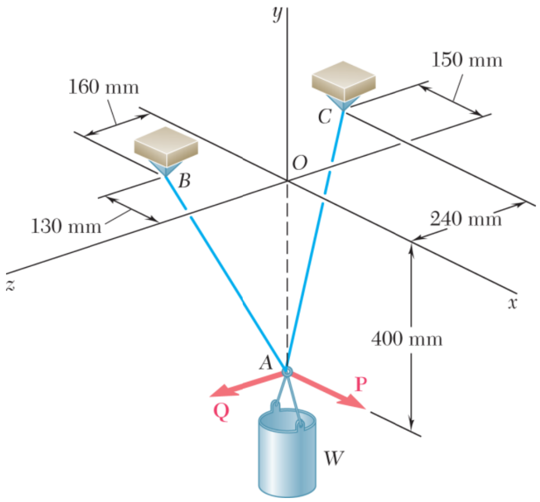
$$R = \sqrt{(386.52\text{ N})^2 + (828.25\text{ N})^2 + (220.49\text{ N})^2}$$

$$= 940.22\text{ N} \qquad R = 940\text{ N} ◀$$

$$\cos\theta_x = \frac{R_x}{R} = \frac{386.52\text{ N}}{940.22\text{ N}} \qquad \theta_x = 65.7° ◀$$

$$\cos\theta_y = \frac{R_y}{R} = \frac{828.25\text{ N}}{940.22\text{ N}} \qquad \theta_y = 28.2° ◀$$

$$\cos\theta_z = \frac{R_z}{R} = \frac{220.49\text{ N}}{940.22\text{ N}} \qquad \theta_z = 76.4° ◀$$

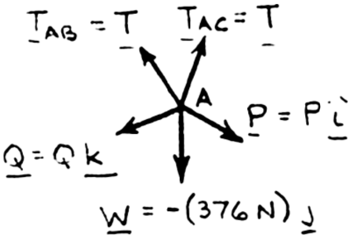


PROBLEM 2.136

A container of weight W is suspended from ring A. Cable BAC passes through the ring and is attached to fixed supports at B and C. Two forces $\mathbf{P} = P\mathbf{i}$ and $\mathbf{Q} = Q\mathbf{k}$ are applied to the ring to maintain the container in the position shown. Knowing that $W = 376$ N, determine P and Q. (*Hint:* The tension is the same in both portions of cable BAC.)

SOLUTION

Free-Body A:



$$\mathbf{T}_{AB} = T\boldsymbol{\lambda}_{AB}$$

$$= T\frac{\overline{AB}}{AB}$$

$$= T\frac{(-130\text{ mm})\mathbf{i} + (400\text{ mm})\mathbf{j} + (160\text{ mm})\mathbf{k}}{450\text{ mm}}$$

$$= T\left(-\frac{13}{45}\mathbf{i} + \frac{40}{45}\mathbf{j} + \frac{16}{45}\mathbf{k}\right)$$

$$\mathbf{T}_{AC} = T\boldsymbol{\lambda}_{AC}$$

$$= T\frac{\overline{AC}}{AC}$$

$$= T\frac{(-150\text{ mm})\mathbf{i} + (400\text{ mm})\mathbf{j} + (-240\text{ mm})\mathbf{k}}{490\text{ mm}}$$

$$= T\left(-\frac{15}{49}\mathbf{i} + \frac{40}{49}\mathbf{j} - \frac{24}{49}\mathbf{k}\right)$$

$$\Sigma F = 0: \quad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{Q} + \mathbf{P} + \mathbf{W} = 0$$

Setting coefficients of **i**, **j**, **k** equal to zero:

$$\mathbf{i}: \quad -\frac{13}{45}T - \frac{15}{49}T + P = 0 \qquad 0.59501T = P \qquad (1)$$

$$\mathbf{j}: \quad +\frac{40}{45}T + \frac{40}{49}T - W = 0 \qquad 1.70521T = W \qquad (2)$$

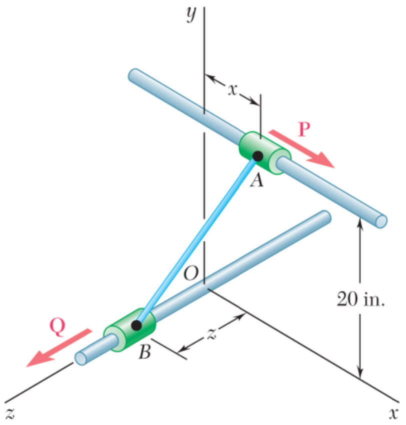
$$\mathbf{k}: \quad +\frac{16}{45}T - \frac{24}{49}T + Q = 0 \qquad 0.134240T = Q \qquad (3)$$

PROBLEM 2.136 (Continued)

Data: $W = 376\text{ N} \quad 1.70521T = 376\text{ N} \quad T = 220.50\text{ N}$

$0.59501(220.50\text{ N}) = P$ $\qquad P = 131.2\text{ N}$ ◀

$0.134240(220.50\text{ N}) = Q$ $\qquad Q = 29.6\text{ N}$ ◀


y
x
P
A
O
20 in.
Q
B
z
z
x

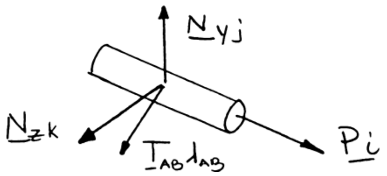
PROBLEM 2.137

Collars *A* and *B* are connected by a 25-in.-long wire and can slide freely on frictionless rods. If a 60-lb force **Q** is applied to collar *B* as shown, determine (*a*) the tension in the wire when $x = 9$ in., (*b*) the corresponding magnitude of the force **P** required to maintain the equilibrium of the system.

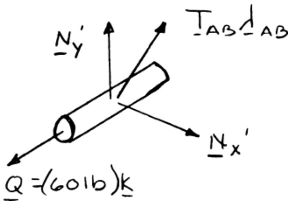
SOLUTION

Free-Body Diagrams of Collars:

***A*:**


$\underline{N}_y \underline{j}$
$\underline{N}_z k$
$\underline{T}_{AB} \underline{\lambda}_{AB}$
$P \underline{i}$

***B*:**


$\underline{N}_y'$
$\underline{T}_{AB} \underline{\lambda}_{AB}$
$\underline{N}_x'$
$\underline{Q} = (60 \text{ lb})\underline{k}$

$$\boldsymbol{\lambda}_{AB} = \frac{\overrightarrow{AB}}{AB} = \frac{-x\mathbf{i} - (20 \text{ in.})\mathbf{j} + z\mathbf{k}}{25 \text{ in.}}$$

Collar *A*: $\Sigma\mathbf{F} = 0\text{:} \quad P\mathbf{i} + N_y\mathbf{j} + N_z\mathbf{k} + T_{AB}\boldsymbol{\lambda}_{AB} = 0$

Substitute for $\boldsymbol{\lambda}_{AB}$ and set coefficient of **i** equal to zero:

$$P - \frac{T_{AB}x}{25 \text{ in.}} = 0 \qquad (1)$$

Collar *B*: $\Sigma\mathbf{F} = 0\text{:} \quad (60 \text{ lb})\mathbf{k} + N_x'\mathbf{i} + N_y'\mathbf{j} - T_{AB}\boldsymbol{\lambda}_{AB} = 0$

Substitute for $\boldsymbol{\lambda}_{AB}$ and set coefficient of **k** equal to zero:

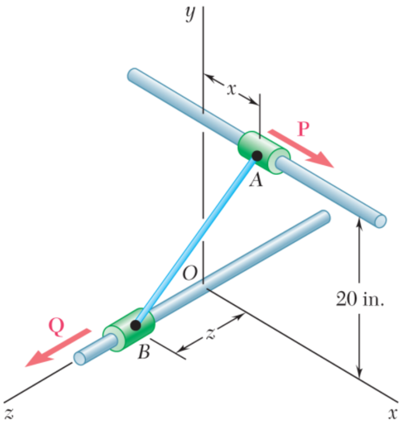
$$60 \text{ lb} - \frac{T_{AB}z}{25 \text{ in.}} = 0 \qquad (2)$$

(*a*) $x = 9$ in. $\quad (9 \text{ in.})^2 + (20 \text{ in.})^2 + z^2 = (25 \text{ in.})^2$

$$z = 12 \text{ in.}$$

From Eq. (2): $\dfrac{60 \text{ lb} - T_{AB}(12 \text{ in.})}{25 \text{ in.}}$ $\qquad T_{AB} = 125.0 \text{ lb}$ ◀

(*b*) From Eq. (1): $P = \dfrac{(125.0 \text{ lb})(9 \text{ in.})}{25 \text{ in.}}$ $\qquad P = 45.0 \text{ lb}$ ◀



PROBLEM 2.138

Collars A and B are connected by a 25-in.-long wire and can slide freely on frictionless rods. Determine the distances x and z for which the equilibrium of the system is maintained when $P = 120$ lb and $Q = 60$ lb.

SOLUTION

See Problem 2.137 for the diagrams and analysis leading to Equations (1) and (2) below:

$$P = \frac{T_{AB}x}{25 \text{ in.}} = 0 \tag{1}$$

$$60 \text{ lb} - \frac{T_{AB}z}{25 \text{ in.}} = 0 \tag{2}$$

For $P = 120$ lb, Eq. (1) yields $T_{AB}x = (25 \text{ in.})(20 \text{ lb})$ (1′)

From Eq. (2): $T_{AB}z = (25 \text{ in.})(60 \text{ lb})$ (2′)

Dividing Eq. (1′) by (2′), $\dfrac{x}{z} = 2$ (3)

Now write $x^2 + z^2 + (20 \text{ in.})^2 = (25 \text{ in.})^2$ (4)

Solving (3) and (4) simultaneously,

$$4z^2 + z^2 + 400 = 625$$

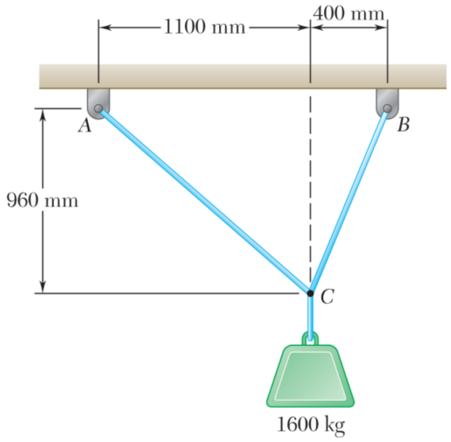
$$z^2 = 45$$

$$z = 6.7082 \text{ in.}$$

From Eq. (3): $x = 2z = 2(6.7082 \text{ in.})$

$$= 13.4164 \text{ in.}$$

$x = 13.42$ in., $z = 6.71$ in. ◀

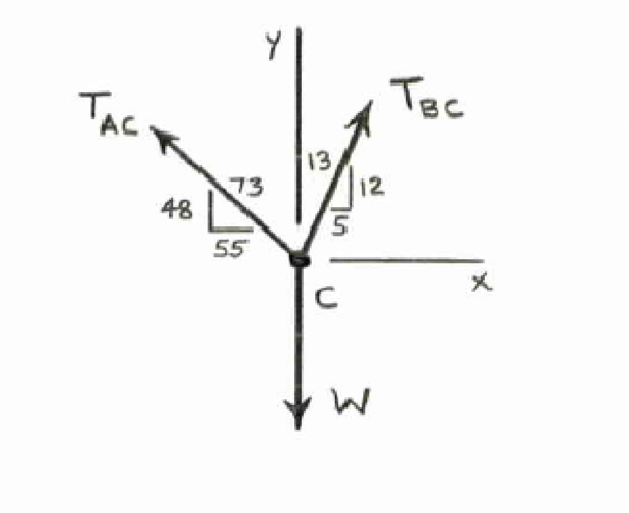

1100 mm
400 mm
A
B
960 mm
C
1600 kg

PROBLEM 2F1

Two cables are tied together at *C* and loaded as shown. Draw the free-body diagram needed to determine the tension in *AC* and *BC*.

SOLUTION

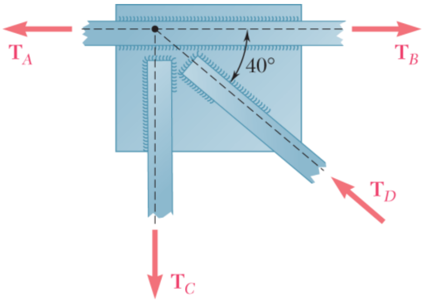
Free-Body Diagram of Point *C*:


y
T_AC
T_BC
13
12
5
73
48
55
C
x
W

$W = (1600\text{ kg})(9.81\text{ m/s}^2)$

$W = 15.6960(10^3)\text{ N}$

$W = 15.696\text{ kN}$

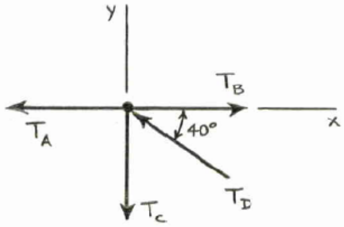


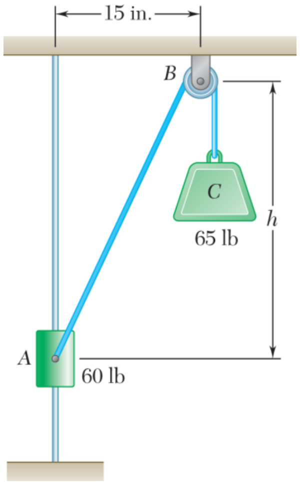
PROBLEM 2.F2

Two forces of magnitude T_A = 8 kips and T_B = 15 kips are applied as shown to a welded connection. Knowing that the connection is in equilibrium, draw the free-body diagram needed to determine the magnitudes of the forces T_C and T_D.

SOLUTION

Free-Body Diagram of Point *E*:



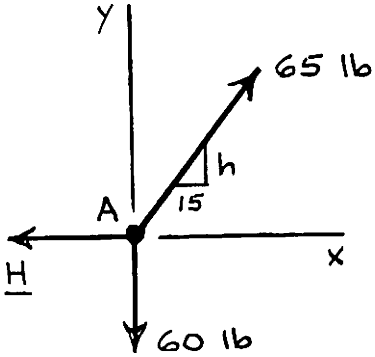

15 in.
B
C
65 lb
h
A
60 lb

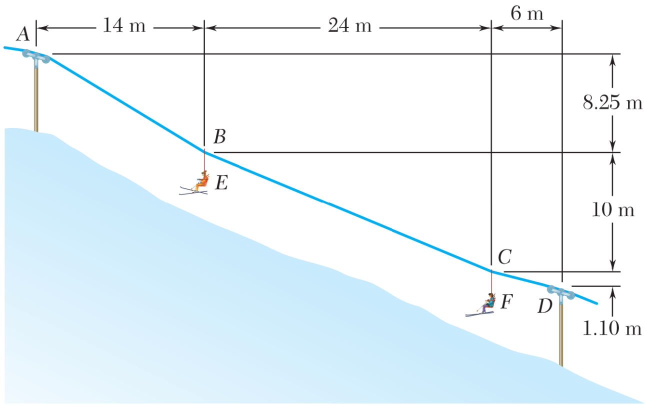
PROBLEM 2.F3

The 60-lb collar *A* can slide on a frictionless vertical rod and is connected as shown to a 65-lb counterweight *C*. Draw the free-body diagram needed to determine the value of *h* for which the system is in equilibrium.

SOLUTION

Free-Body Diagram of Point *A*:


y
65 lb
h
15
A
H
x
60 lb

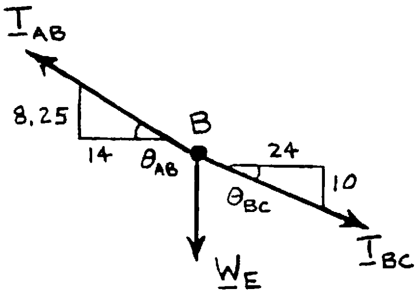

A
14 m
24 m
6 m
8.25 m
B
E
10 m
C
F
D
1.10 m

PROBLEM 2.F4

A chairlift has been stopped in the position shown. Knowing that each chair weighs 250 N and that the skier in chair *E* weighs 765 N, draw the free-body diagrams needed to determine the weight of the skier in chair *F*.

SOLUTION

Free-Body Diagram of Point *B*:


$\underline{T}_{AB}$
8.25
14
θ_{AB}
B
24
10
θ_{BC}
$\underline{W}_E$
$\underline{T}_{BC}$

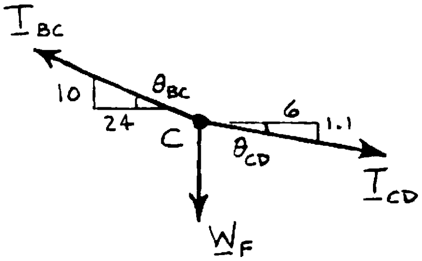
$$W_E = 250\text{ N} + 765\text{ N} = 1015\text{ N}$$

$$\theta_{AB} = \tan^{-1}\frac{8.25}{14} = 30.510°$$

$$\theta_{BC} = \tan^{-1}\frac{10}{24} = 22.620°$$

Use this free body to determine T_{AB} and T_{BC}.

Free-Body Diagram of Point *C*:


$\underline{T}_{BC}$
10
24
θ_{BC}
C
6
1.1
θ_{CD}
$\underline{T}_{CD}$
$\underline{W}_F$

$$\theta_{CD} = \tan^{-1}\frac{1.1}{6} = 10.3889°$$

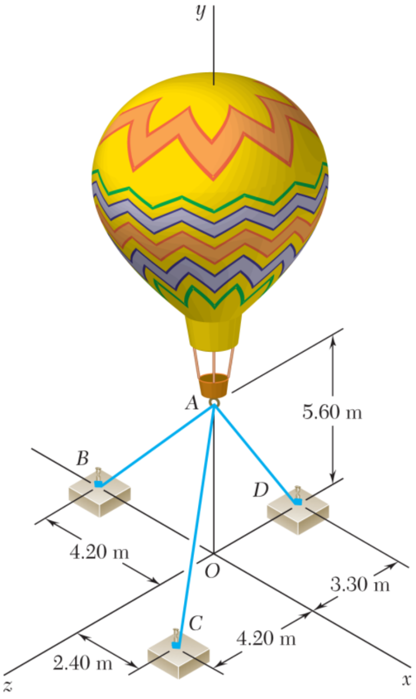
Use this free body to determine T_{CD} and W_F.

Then weight of skier W_S is found by

$$W_S = W_F - 250\text{ N} \blacktriangleleft$$

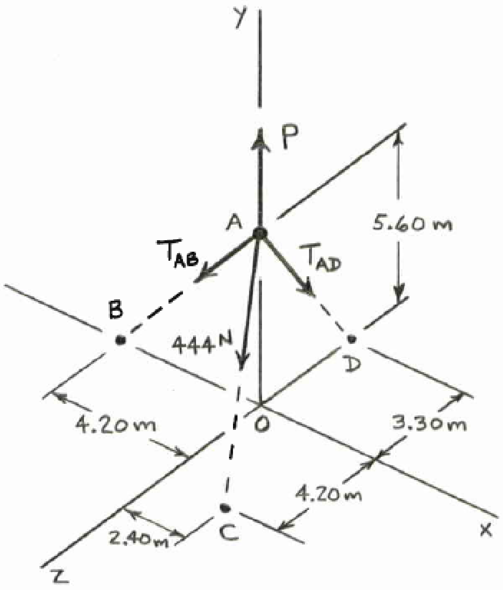
PROBLEM 2.F5

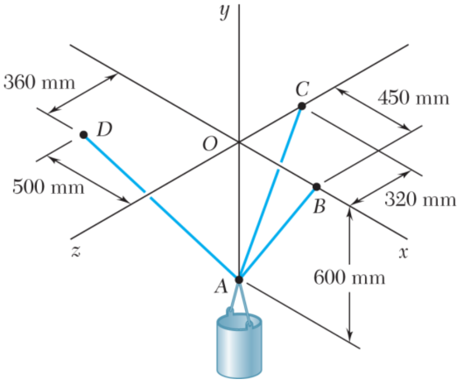
Three cables are used to tether a balloon as shown. Knowing that the tension in cable *AC* is 444 N, draw the free-body diagram needed to determine the vertical force **P** exerted by the balloon at *A*.


y
A
5.60 m
B
D
4.20 m
O
3.30 m
C
4.20 m
2.40 m
z
x

SOLUTION

Free-Body Diagram of Point *A*:


y
P
A
5.60 m
T_{AB}
T_{AD}
B
444 N
D
4.20 m
O
3.30 m
4.20 m
2.40 m
C
z
x

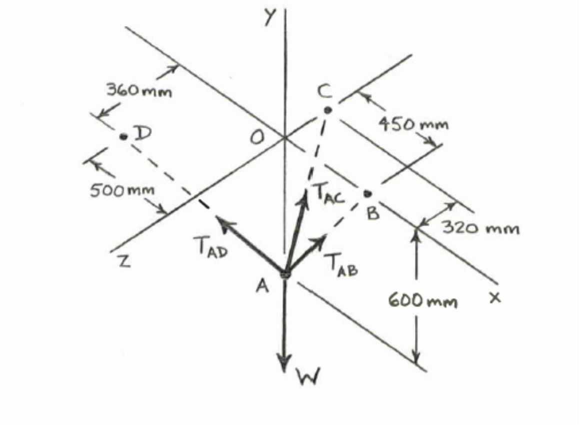

y
360 mm
C
450 mm
D
O
500 mm
B
320 mm
z
x
A
600 mm

PROBLEM 2.F6

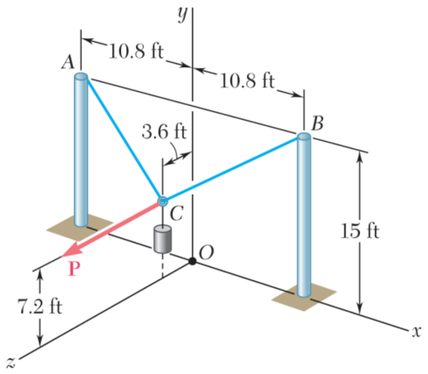
A container of mass $m = 120$ kg is supported by three cables as shown. Draw the free-body diagram needed to determine the tension in each cable

SOLUTION

Free-Body Diagram of Point *A*:


Y
360mm
C
450 mm
D
O
500mm
T_{AC}
B
320 mm
T_{AD}
T_{AB}
Z
A
600mm
X
W

$$W = (120\text{ kg})(9.81\text{ m/s}^2)$$
$$= 1177.2\text{ N}$$

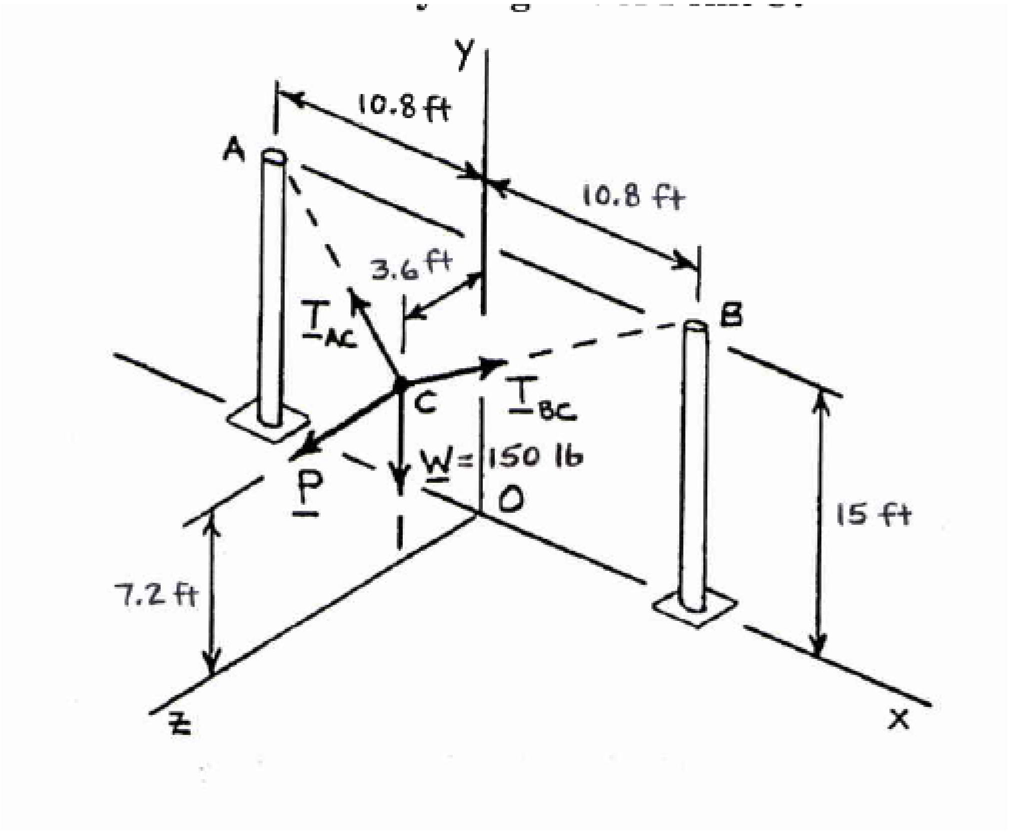

y
A
10.8 ft
10.8 ft
B
3.6 ft
C
15 ft
P
O
7.2 ft
x
z

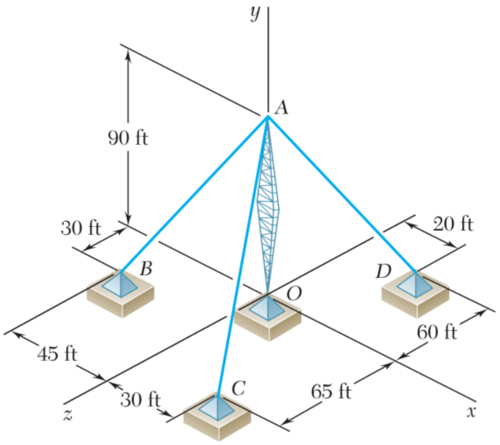
PROBLEM 2.F7

A 150-lb cylinder is supported by two cables *AC* and *BC* that are attached to the top of vertical posts. A horizontal force **P**, which is perpendicular to the plane containing the posts, holds the cylinder in the position shown. Draw the free-body diagram needed to determine the magnitude of **P** and the force in each cable.

SOLUTION

Free-Body Diagram of Point *C*:


y
10.8 ft
A
10.8 ft
3.6 ft
$\underline{T}_{AC}$
B
C
$\underline{T}_{BC}$
$\underline{W}$ = 150 lb
$\underline{P}$
O
15 ft
7.2 ft
z
x

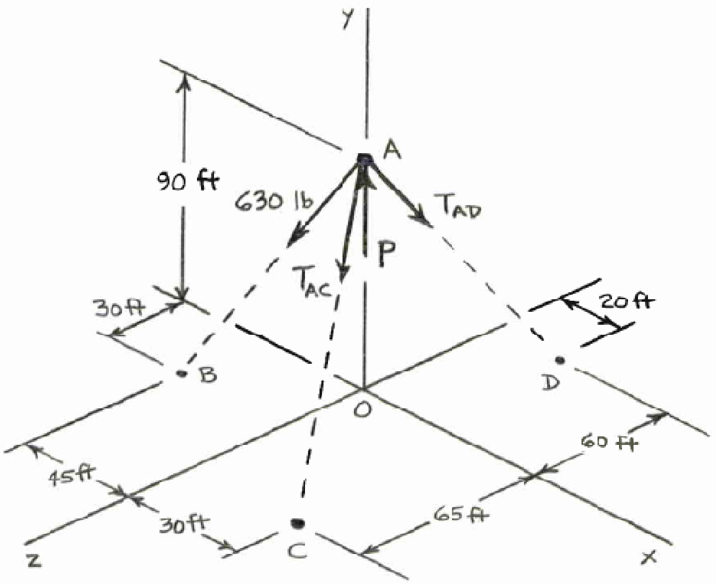

y
A
90 ft
30 ft
B
O
D
20 ft
60 ft
45 ft
C
65 ft
30 ft
z
x

PROBLEM 2.F8

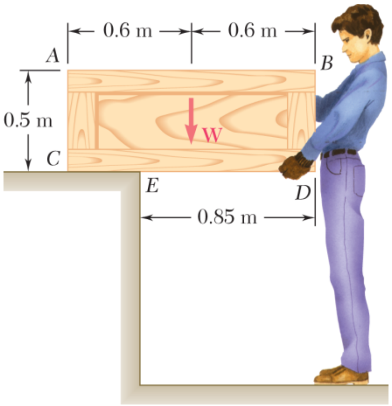
A transmission tower is held by three guy wires attached to a pin at *A* and anchored by bolts at *B*, *C*, and *D*. Knowing that the tension in wire *AB* is 630 lb, draw the free-body diagram needed to determine the vertical force **P** exerted by the tower on the pin at *A*.

SOLUTION

Free-Body Diagram of point *A*:


y
A
90 ft
630 lb
T_{AD}
T_{AC}
P
30 ft
20 ft
B
D
O
60 ft
45 ft
30 ft
C
65 ft
z
x

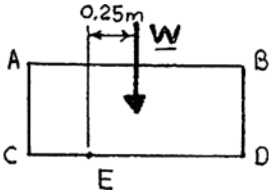
CHAPTER 3



PROBLEM 3.1

A crate of mass 80 kg is held in the position shown. Determine (*a*) the moment produced by the weight **W** of the crate about *E*, (*b*) the smallest force applied at *B* that creates a moment of equal magnitude and opposite sense about *E*.

SOLUTION

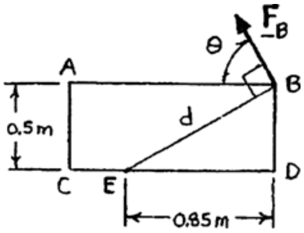


(*a*) By definition, $W = mg = 80\text{ kg}(9.81\text{ m/s}^2) = 784.8\text{ N}$

We have $\Sigma M_E\!: \quad M_E = (784.8\text{ N})(0.25\text{ m})$

$$\mathbf{M}_E = 196.2\text{ N}\cdot\text{m} \circlearrowright \blacktriangleleft$$

(*b*) For the force at *B* to be the smallest, resulting in a moment ($\mathbf{M}_E$) about *E*, the line of action of force $\mathbf{F}_B$ must be perpendicular to the line connecting *E* to *B*. The sense of $\mathbf{F}_B$ must be such that the force produces a counterclockwise moment about *E*.



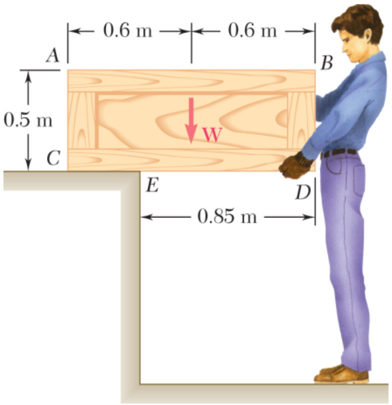
Note: $d = \sqrt{(0.85\text{ m})^2 + (0.5\text{ m})^2} = 0.98615\text{ m}$

We have $\Sigma M_E\!: \quad 196.2\text{ N}\cdot\text{m} = F_B(0.98615\text{ m})$

$$F_B = 198.954\text{ N}$$

and $\theta = \tan^{-1}\left(\dfrac{0.85\text{ m}}{0.5\text{ m}}\right) = 59.534^\circ$

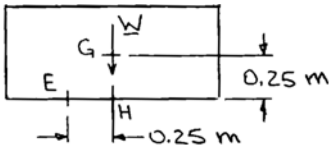
or $$\mathbf{F}_B = 199.0\text{ N} \measuredangle 59.5^\circ \blacktriangleleft$$



PROBLEM 3.2

A crate of mass 80 kg is held in the position shown. Determine (*a*) the moment produced by the weight **W** of the crate about *E*, (*b*) the smallest force applied at *A* that creates a moment of equal magnitude and opposite sense about *E*, (*c*) the magnitude, sense, and point of application on the bottom of the crate of the smallest vertical force that creates a moment of equal magnitude and opposite sense about *E*.

SOLUTION

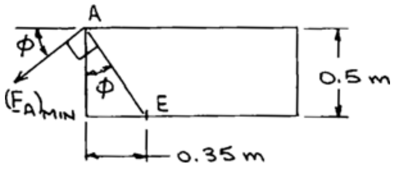


First note. . .

$$W = mg = (80\text{ kg})(9.81\text{ m/s}^2) = 784.8\text{ N}$$

(*a*) We have $M_E = r_{H/E}W = (0.25\text{ m})(784.8\text{ N}) = 196.2\text{ N}\cdot\text{m}$ or $\mathbf{M}_E = 196.2\text{ N}\cdot\text{m} \circlearrowright$ ◀

(*b*)



For $\mathbf{F}_A$ to be minimum, it must be perpendicular to the line joining Points *A* and *E*. Then with $\mathbf{F}_A$ directed as shown, we have $(-M_E) = r_{A/E}(F_A)_{\min}$.

Where $r_{A/E} = \sqrt{(0.35\text{ m})^2 + (0.5\text{ m})^2} = 0.61033\text{ m}$

then $196.2\text{ N}\cdot\text{m} = (0.61033\text{ m})(F_A)_{\min}$

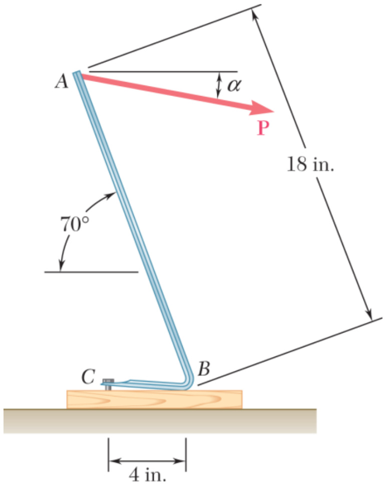
or $(F_A)_{\min} = 321\text{ N}$

Also $\tan\phi = \dfrac{0.35\text{ m}}{0.5\text{ m}}$ or $\phi = 35.0^\circ$ $(\mathbf{F}_A)_{\min} = 321\text{ N} \measuredangle 35.0^\circ$ ◀

(*c*) For $\mathbf{F}_{vertical}$ to be minimum, the perpendicular distance from its line of action to Point *E* must be maximum. Thus, apply $(\mathbf{F}_{vertical})_{\min}$ at Point *D*, and then

$$(-M_E) = r_{D/E}(F_{vertical})_{\min}$$

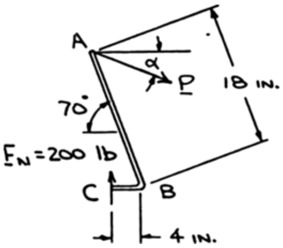
$$196.2\text{ N}\cdot\text{m} = (0.85\text{ m})(F_{vertical})_{\min}$$ or $(\mathbf{F}_{vertical})_{\min} = 231\text{ N}\uparrow$ at Point *D* ◀



PROBLEM 3.3

It is known that a vertical force of 200 lb is required to remove the nail at *C* from the board. As the nail first starts moving, determine (*a*) the moment about *B* of the force exerted on the nail, (*b*) the magnitude of the force **P** that creates the same moment about *B* if $\alpha = 10°$, (*c*) the smallest force **P** that creates the same moment about *B*.

SOLUTION

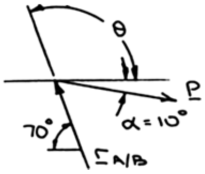


(*a*) We have $M_B = r_{C/B}F_N$

$$= (4\text{ in.})(200\text{ lb})$$

$$= 800\text{ lb}\cdot\text{in.}$$

or $M_B = 800\text{ lb}\cdot\text{in.}$ ↻ ◀



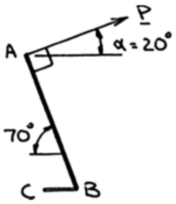
(*b*) By definition, $M_B = r_{A/B}P\sin\theta$

$$\theta = 10° + (180° - 70°)$$

$$= 120°$$

Then $800\text{ lb}\cdot\text{in.} = (18\text{ in.})\times P\sin 120°$

or $P = 51.3$ lb ◀



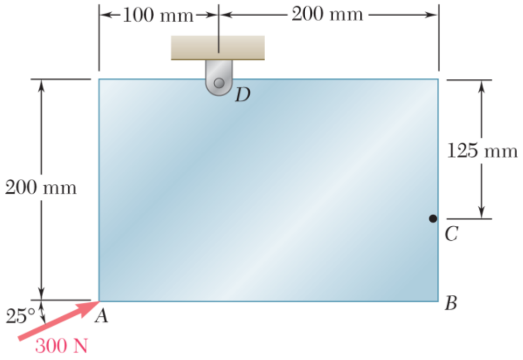
(*c*) For **P** to be minimum, it must be perpendicular to the line joining Points *A* and *B*. Thus, **P** must be directed as shown.

Thus $M_B = dP_{\min}$

$$d = r_{A/B}$$

or $800\text{ lb}\cdot\text{in.} = (18\text{ in.})P_{\min}$

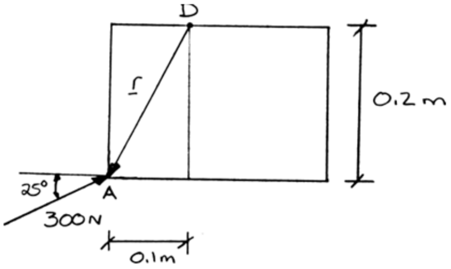
or $P_{\min} = 44.4$ lb $\mathbf{P}_{\min} = 44.4$ lb ∠ 20° ◀



PROBLEM 3.4

A 300-N force is applied at *A* as shown. Determine (*a*) the moment of the 300-N force about *D*, (*b*) the smallest force applied at *B* that creates the same moment about *D*.

SOLUTION



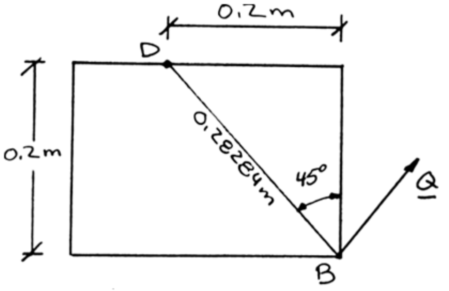
(*a*) $F_x = (300\text{ N})\cos 25°$

$= 271.89\text{ N}$

$F_y = (300\text{ N})\sin 25°$

$= 126.785\text{ N}$

$\mathbf{F} = (271.89\text{ N})\mathbf{i} + (126.785\text{ N})\mathbf{j}$



$$\mathbf{r} = \overrightarrow{DA} = -(0.1\text{ m})\mathbf{i} - (0.2\text{ m})\mathbf{j}$$

$$\mathbf{M}_D = \mathbf{r} \times \mathbf{F}$$

$$\mathbf{M}_D = [-(0.1\text{ m})\mathbf{i} - (0.2\text{ m})\mathbf{j}] \times [(271.89\text{ N})\mathbf{i} + (126.785\text{ N})\mathbf{j}]$$

$$= -(12.6785\text{ N}\cdot\text{m})\mathbf{k} + (54.378\text{ N}\cdot\text{m})\mathbf{k}$$

$$= (41.700\text{ N}\cdot\text{m})\mathbf{k}$$

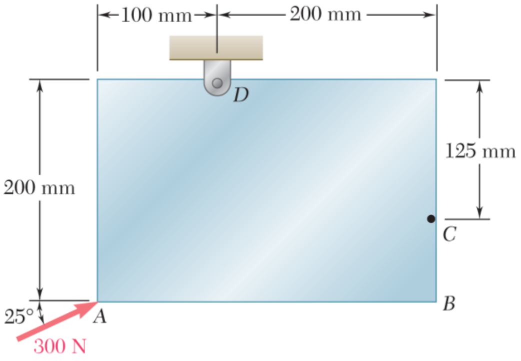
$\mathbf{M}_D = 41.7\text{ N}\cdot\text{m}$ ↺ ◀

(*b*) The smallest force *Q* at *B* must be perpendicular to $\overrightarrow{DB}$ at 45°

$$\mathbf{M}_D = Q(\overrightarrow{DB})$$

$$41.700\text{ N}\cdot\text{m} = Q(0.28284\text{ m})$$

$Q = 147.4\text{ N}$ 45.0° ◀



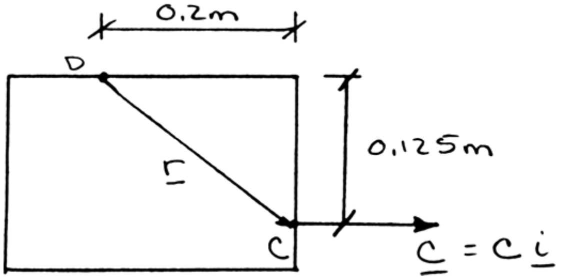
PROBLEM 3.5

A 300-N force is applied at *A* as shown. Determine (*a*) the moment of the 300-N force about *D*, (*b*) the magnitude and sense of the horizontal force applied at *C* that creates the same moment about *D*, (*c*) the smallest force applied at *C* that creates the same moment about *D*.

SOLUTION

(*a*) See Problem 3.3 for the figure and analysis leading to the determination of $\mathbf{M}_D$

$\mathbf{M}_D = 41.7\text{ N}\cdot\text{m} \circlearrowleft$ ◀



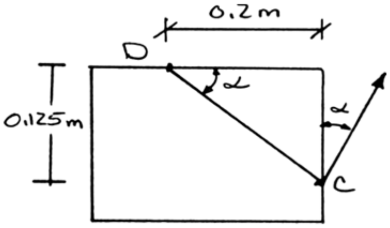
(*b*) Since **C** is horizontal $\mathbf{C} = C\mathbf{i}$

$$\mathbf{r} = \overrightarrow{DC} = (0.2\text{ m})\mathbf{i} - (0.125\text{ m})\mathbf{j}$$
$$\mathbf{M}_D = \mathbf{r} \times C\mathbf{i} = C(0.125\text{ m})\mathbf{k}$$
$$41.7\text{ N}\cdot\text{m} = (0.125\text{ m})(C)$$
$$C = 333.60\text{ N}$$

$C = 334\text{ N}$ ◀

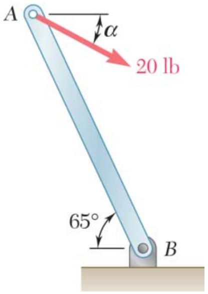
(*c*) The smallest force *C* must be perpendicular to *DC*; thus, it forms α with the vertical

$$\tan\alpha = \frac{0.125\text{ m}}{0.2\text{ m}}$$
$$\alpha = 32.0°$$
$$\mathbf{M}_D = C(DC); \quad DC = \sqrt{(0.2\text{ m})^2 + (0.125\text{ m})^2}$$
$$= 0.23585\text{ m}$$



$$41.70\text{ N}\cdot\text{m} = C(0.23585\text{ m})$$

$\mathbf{C} = 176.8\text{ N} \measuredangle 58.0°$ ◀

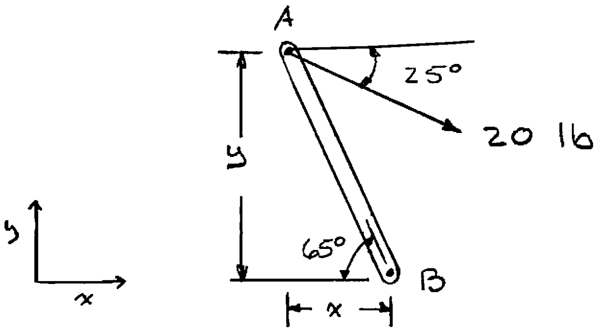


PROBLEM 3.6

A 20-lb force is applied to the control rod *AB* as shown. Knowing that the length of the rod is 9 in. and that $\alpha = 25°$, determine the moment of the force about Point *B* by resolving the force into horizontal and vertical components.

SOLUTION

Free-Body Diagram of Rod *AB*:



$$x = (9\text{ in.})\cos 65° = 3.8036\text{ in.}$$

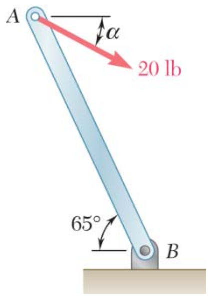
$$y = (9\text{ in.})\sin 65° = 8.1568\text{ in.}$$

$$\begin{aligned}\mathbf{F} &= F_x\mathbf{i} + F_y\mathbf{j}\\ &= (20\text{ lb}\cos 25°)\mathbf{i} + (-20\text{ lb}\sin 25°)\mathbf{j}\\ &= (18.1262\text{ lb})\mathbf{i} - (8.4524\text{ lb})\mathbf{j}\end{aligned}$$

$$\mathbf{r}_{A/B} = \overrightarrow{BA} = (-3.8036\text{ in.})\mathbf{i} + (8.1568\text{ in.})\mathbf{j}$$

$$\begin{aligned}\mathbf{M}_B &= \mathbf{r}_{A/B} \times \mathbf{F}\\ &= (-3.8036\mathbf{i} + 8.1568\mathbf{j}) \times (18.1262\mathbf{i} - 8.4524\mathbf{j})\\ &= 32.150\mathbf{k} - 147.852\mathbf{k}\\ &= -115.702\text{ lb-in.}\end{aligned}$$

$\mathbf{M}_B = 115.7$ lb-in. $\curvearrowright$ ◀

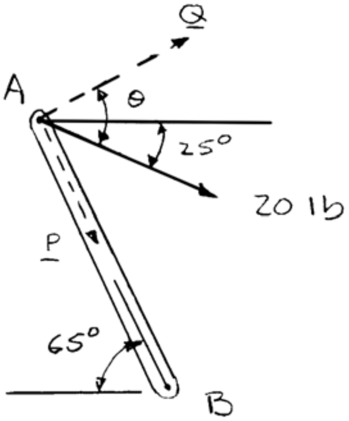


PROBLEM 3.7

A 20-lb force is applied to the control rod *AB* as shown. Knowing that the length of the rod is 9 in. and that $\alpha = 25°$, determine the moment of the force about Point *B* by resolving the force into components along *AB* and in a direction perpendicular to *AB*.

SOLUTION

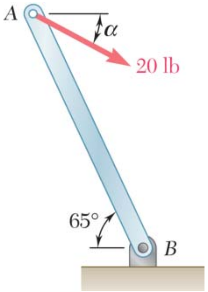
Free-Body Diagram of Rod *AB*:



$$\theta = 90° - (65° - 25°)$$
$$= 50°$$

$$Q = (20\text{ lb})\cos 50°$$
$$= 12.8558\text{ lb}$$
$$M_B = Q(9\text{ in.})$$
$$= (12.8558\text{ lb})(9\text{ in.})$$
$$= 115.702\text{ lb-in.}$$

$\mathbf{M}_B = 115.7\text{ lb-in.} \circlearrowright$ ◀

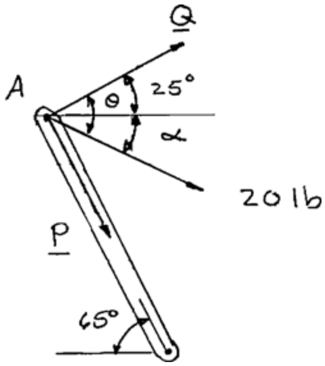


PROBLEM 3.8

A 20-lb force is applied to the control rod *AB* as shown. Knowing that the length of the rod is 9 in. and that the moment of the force about *B* is 120 lb · in. clockwise, determine the value of *α*.

SOLUTION

Free-Body Diagram of Rod *AB*:



$$\alpha = \theta - 25°$$

$$Q = (20 \text{ lb})\cos\theta$$

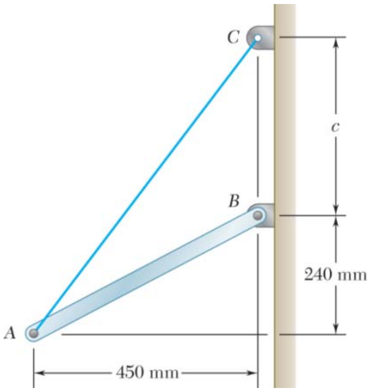
and $$M_B = (Q)(9 \text{ in.})$$

Therefore, $$120 \text{ lb-in.} = (20 \text{ lb})(\cos\theta)(9 \text{ in.})$$

$$\cos\theta = \frac{120 \text{ lb-in.}}{180 \text{ lb-in.}}$$

or $$\theta = 48.190°$$

Therefore, $$\alpha = 48.190° - 25°$$ $\alpha = 23.2°$ ◀



PROBLEM 3.9

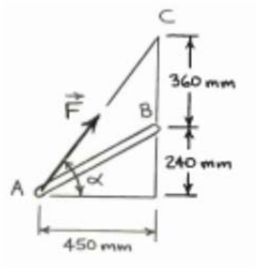
Rod *AB* is held in place by the cord *AC*. Knowing that the tension in the cord is 1350 N and that $c = 360$ mm, determine the moment about *B* of the force exerted by the cord at point *A* by resolving that force into horizontal and vertical components applied (*a*) at point *A*, (*b*) at point *C*.

SOLUTION

Free-Body Diagram of Rod *AB*:

$$(a)\quad F = 1350\ \text{N} \qquad AC = \sqrt{(450)^2 + (600)^2} = 750\ \text{mm}$$

$$\cos\alpha = \frac{450}{750} = 0.6 \qquad \sin\alpha = \frac{600}{750} = 0.8$$



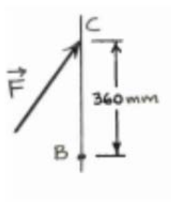
$$\begin{aligned}\vec{\mathbf{F}} &= F\cos\alpha\mathbf{i} + F\sin\alpha\mathbf{j}\\ &= (1350\ \text{N})0.6\mathbf{i} + (1350\ \text{N})0.8\mathbf{j}\\ &= (810\ \text{N})\mathbf{i} + (1080\text{N})\mathbf{j}\end{aligned}$$

$$\vec{\mathbf{r}}_{A/B} = -(0.45\ \text{m})\mathbf{i} - (0.24\ \text{m})\mathbf{j}$$

$$\begin{aligned}\vec{\mathbf{M}}_B &= \vec{\mathbf{r}}_{A/B} \times \vec{\mathbf{F}} = (-0.45\mathbf{i} - 0.24\mathbf{j}) \times (810\mathbf{i} + 1080\mathbf{j})\\ &= -486\mathbf{k} + 194.4\mathbf{k}\\ &= -(291.6\ \text{N}\cdot\text{m})\mathbf{k}\end{aligned}$$

$\mathbf{M}_B = 292\ \text{N}\cdot\text{m}$ ↻ ◀

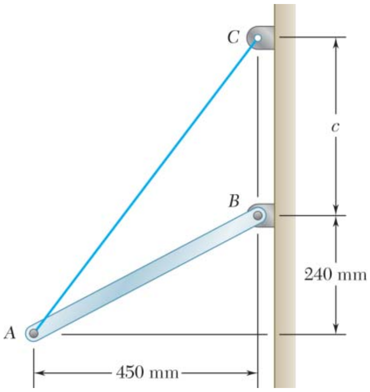
(*b*) $\mathbf{F} = (810\ \text{N})\mathbf{i} + (1080\ \text{N})\mathbf{j}$



$$\mathbf{r}_{C/B} = (0.36\ \text{m})\mathbf{j}$$

$$\begin{aligned}\mathbf{M}_B &= \mathbf{r}_{C/B} \times \mathbf{F} = 0.36\mathbf{j} \times (810\mathbf{i} + 1080\mathbf{j})\\ &= -(291.6\ \text{N}\cdot\text{m})\mathbf{k}\end{aligned}$$

$\mathbf{M}_B = 292\ \text{N}\cdot\text{m}$ ↻ ◀



PROBLEM 3.10

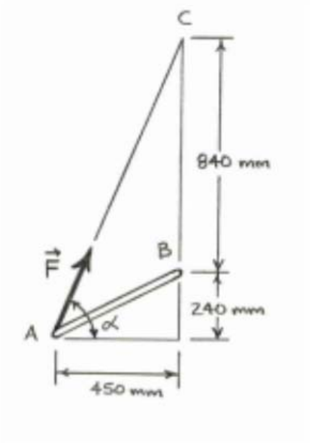
Rod *AB* is held in place by the cord *AC*. Knowing that $c = 840$ mm and that the moment about *B* of the force exerted by the cord at point *A* is 756 N·m, determine the tension in the cord.

SOLUTION

Free-Body Diagram of Rod *AB*:

$$\mathbf{M}_B = -(756\ \text{N}\cdot\text{m})\mathbf{k} \qquad AC = \sqrt{(450)^2 + (1080)^2} = 1170\ \text{mm}$$

$$\cos\alpha = \frac{450}{1170} \qquad \sin\alpha = \frac{1080}{1170}$$



$$\vec{\mathbf{F}} = F\cos\alpha\mathbf{i} + F\sin\alpha\mathbf{j}$$

$$= \frac{450}{1170}F\mathbf{i} + \frac{1080}{1170}F\mathbf{j}$$

$$\vec{\mathbf{r}}_{A/B} = -(0.45\ \text{m})\mathbf{i} - (0.24\ \text{m})\mathbf{j}$$

$$\overrightarrow{\mathbf{M}}_B = \vec{\mathbf{r}}_{A/B} \times \vec{\mathbf{F}} = (-0.45\mathbf{i} - 0.24\mathbf{j}) \times (450\mathbf{i} + 1080\mathbf{j})\frac{F}{1170}$$

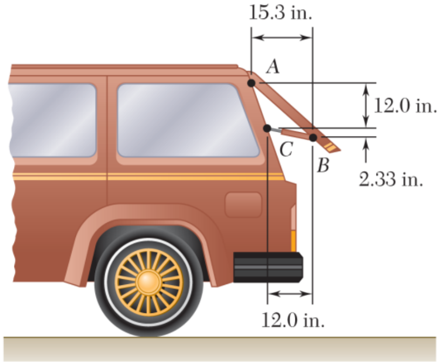
$$= (-486\mathbf{k} + 108\mathbf{k})\frac{F}{1170}$$

$$= \left(-\frac{378}{1170}F\right)\mathbf{k}$$

Substituting for M_B:

$$-756 = -\frac{378}{1170}F$$

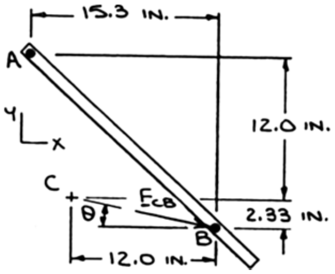
$F = 2340$ N ◀



PROBLEM 3.11

The tailgate of a car is supported by the hydraulic lift *BC*. If the lift exerts a 125-lb force directed along its centerline on the ball and socket at *B*, determine the moment of the force about *A*.

SOLUTION



First note
$$d_{CB} = \sqrt{(12.0\text{ in.})^2 + (2.33\text{ in.})^2}$$
$$= 12.2241\text{ in.}$$

Then
$$\cos\theta = \frac{12.0\text{ in.}}{12.2241\text{ in.}}$$
$$\sin\theta = \frac{2.33\text{ in.}}{12.2241\text{ in.}}$$

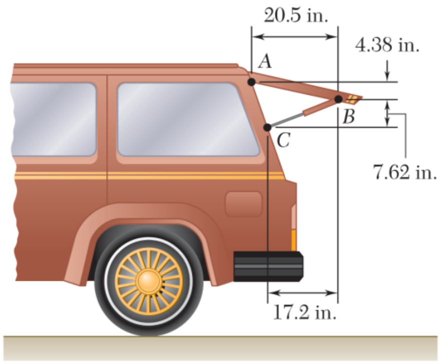
and
$$\mathbf{F}_{CB} = F_{CB}\cos\theta\,\mathbf{i} - F_{CB}\sin\theta\,\mathbf{j}$$
$$= \frac{125\text{ lb}}{12.2241\text{ in.}}[(12.0\text{ in.})\,\mathbf{i} - (2.33\text{ in.})\,\mathbf{j}]$$

Now
$$\mathbf{M}_A = \mathbf{r}_{B/A} \times \mathbf{F}_{CB}$$

where
$$\mathbf{r}_{B/A} = (15.3\text{ in.})\,\mathbf{i} - (12.0\text{ in.} + 2.33\text{ in.})\,\mathbf{j}$$
$$= (15.3\text{ in.})\,\mathbf{i} - (14.33\text{ in.})\,\mathbf{j}$$

Then
$$\mathbf{M}_A = [(15.3\text{ in.})\mathbf{i} - (14.33\text{ in.})\mathbf{j}] \times \frac{125\text{ lb}}{12.2241\text{ in.}}(12.0\mathbf{i} - 2.33\mathbf{j})$$
$$= (1393.87\text{ lb}\cdot\text{in.})\mathbf{k}$$
$$= (116.156\text{ lb}\cdot\text{ft})\mathbf{k}$$

or $\mathbf{M}_A = 116.2\text{ lb}\cdot\text{ft}$ ↻ ◀

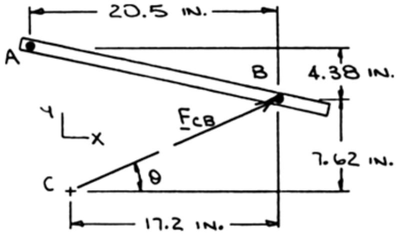

20.5 in.
4.38 in.
A
B
C
7.62 in.
17.2 in.

PROBLEM 3.12

The tailgate of a car is supported by the hydraulic lift *BC*. If the lift exerts a 125-lb force directed along its centerline on the ball and socket at *B*, determine the moment of the force about *A*.

SOLUTION

First note
$$d_{CB} = \sqrt{(17.2\text{ in.})^2 + (7.62\text{ in.})^2}$$
$$= 18.8123\text{ in.}$$



Then
$$\cos\theta = \frac{17.2\text{ in.}}{18.8123\text{ in.}}$$
$$\sin\theta = \frac{7.62\text{ in.}}{18.8123\text{ in.}}$$

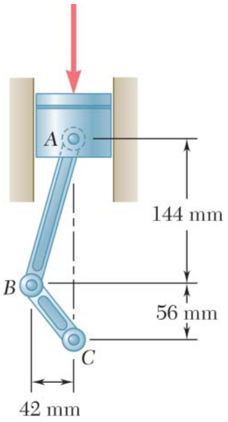
and
$$\mathbf{F}_{CB} = (F_{CB}\cos\theta)\mathbf{i} - (F_{CB}\sin\theta)\mathbf{j}$$
$$= \frac{125\text{ lb}}{18.8123\text{ in.}}[(17.2\text{ in.})\mathbf{i} + (7.62\text{ in.})\mathbf{j}]$$

Now
$$\mathbf{M}_A = \mathbf{r}_{B/A} \times \mathbf{F}_{CB}$$

where
$$\mathbf{r}_{B/A} = (20.5\text{ in.})\mathbf{i} - (4.38\text{ in.})\mathbf{j}$$

Then
$$\mathbf{M}_A = [(20.5\text{ in.})\mathbf{i} - (4.38\text{ in.})\mathbf{j}] \times \frac{125\text{ lb}}{18.8123\text{ in.}}(17.2\mathbf{i} - 7.62\mathbf{j})$$
$$= (1538.53\text{ lb}\cdot\text{in.})\mathbf{k}$$
$$= (128.2\text{ lb}\cdot\text{ft})\mathbf{k}$$

or $\mathbf{M}_A = 128.2\text{ lb}\cdot\text{ft}$ ↺ ◀



PROBLEM 3.13

It is known that the connecting rod *AB* exerts on the crank *BC* a 2.5-kN force directed down and to the left along the centerline of *AB*. Determine the moment of the force about *C*.

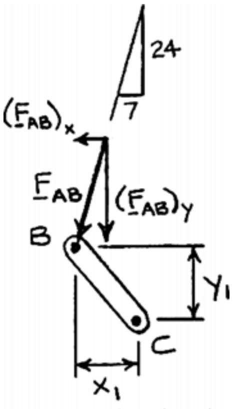
SOLUTION

Using (*a*):

$$M_C = y_1(F_{AB})_x + x_1(F_{AB})_y$$

$$= (0.056 \text{ m})\left(\frac{7}{25} \times 2500 \text{ N}\right) + (0.042 \text{ m})\left(\frac{24}{25} \times 2500 \text{ N}\right)$$

$$= 140.0 \text{ N} \cdot \text{m}$$



(*a*)

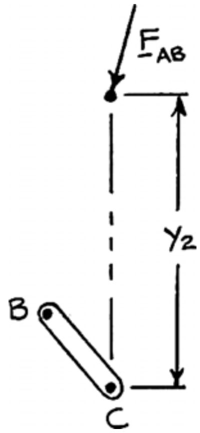
$\mathbf{M}_C = 140.0 \text{ N} \cdot \text{m}$ ↺ ◀

Using (*b*):

$$M_C = y_2(F_{AB})_x$$

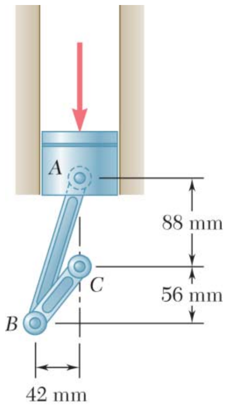
$$= (0.2 \text{ m})\left(\frac{7}{25} \times 2500 \text{ N}\right)$$

$$= 140.0 \text{ N} \cdot \text{m}$$



(*b*)

$\mathbf{M}_C = 140.0 \text{ N} \cdot \text{m}$ ↺ ◀



PROBLEM 3.14

It is known that the connecting rod *AB* exerts on the crank *BC* a 2.5-kN force directed down and to the left along the centerline of *AB*. Determine the moment of the force about *C*.

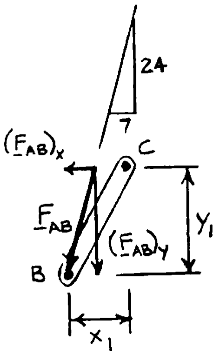
SOLUTION

Using (*a*):

$$M_C = -y_1(F_{AB})_x + x_1(F_{AB})_y$$

$$= -(0.056\text{ m})\left(\frac{7}{25}\times 2500\text{ N}\right) + (0.042\text{ m})\left(\frac{24}{25}\times 2500\text{ N}\right)$$

$$= +61.6\text{ N}\cdot\text{m}$$



(*a*)

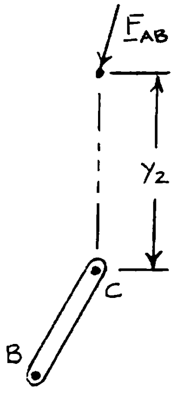
$\mathbf{M}_C = 61.6\text{ N}\cdot\text{m} \circlearrowleft$ ◀

Using (*b*):

$$M_C = y_2(F_{AB})_x$$

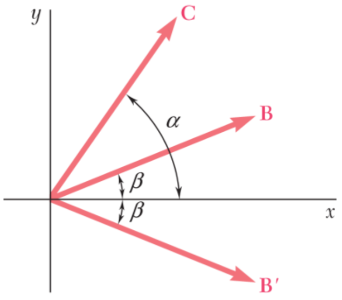
$$= (0.088\text{ m})\left(\frac{7}{25}\times 2500\text{ N}\right)$$

$$= +61.6\text{ N}\cdot\text{m}$$



(*b*)

$\mathbf{M}_C = 61.6\text{ N}\cdot\text{m} \circlearrowleft$ ◀



PROBLEM 3.15

Form the vector products **B** × **C** and **B′** × **C**, where $B = B'$, and use the results obtained to prove the identity

$$\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta).$$

SOLUTION

Note:

$$\mathbf{B} = B(\cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j})$$
$$\mathbf{B}' = B(\cos\beta\,\mathbf{i} - \sin\beta\,\mathbf{j})$$
$$\mathbf{C} = C(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j})$$

By definition,

$$|\mathbf{B}\times\mathbf{C}| = BC\sin(\alpha-\beta) \quad (1)$$

$$|\mathbf{B}'\times\mathbf{C}| = BC\sin(\alpha+\beta) \quad (2)$$

Now

$$\mathbf{B}\times\mathbf{C} = B(\cos\beta\mathbf{i} + \sin\beta\mathbf{j}) \times C(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$$
$$= BC(\cos\beta\sin\alpha - \sin\beta\cos\alpha)\mathbf{k} \quad (3)$$

and

$$\mathbf{B}'\times\mathbf{C} = B(\cos\beta\mathbf{i} - \sin\beta\mathbf{j}) \times C(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$$
$$= BC(\cos\beta\sin\alpha + \sin\beta\cos\alpha)\mathbf{k} \quad (4)$$

Equating the magnitudes of $\mathbf{B}\times\mathbf{C}$ from Equations (1) and (3) yields:

$$BC\sin(\alpha-\beta) = BC(\cos\beta\sin\alpha - \sin\beta\cos\alpha) \quad (5)$$

Similarly, equating the magnitudes of $\mathbf{B}'\times\mathbf{C}$ from Equations (2) and (4) yields:

$$BC\sin(\alpha+\beta) = BC(\cos\beta\sin\alpha + \sin\beta\cos\alpha) \quad (6)$$

Adding Equations (5) and (6) gives:

$$\sin(\alpha-\beta) + \sin(\alpha+\beta) = 2\cos\beta\sin\alpha$$

$$\text{or} \quad \sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta) \blacktriangleleft$$

PROBLEM 3.16

The vectors **P** and **Q** are two adjacent sides of a parallelogram. Determine the area of the parallelogram when (*a*) **P** = –8**i** + 4**j** – 4**k** and **Q** = 3**i** + 3**j** + 6**k**, (*b*) **P** = 7**i** – 6**j** – 3**k** and **Q** = –3**i** + 6**j** – 2**k**

SOLUTION

(*a*) We have $A = |\mathbf{P}\times\mathbf{Q}|$

where $\mathbf{P} = -8\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

$\mathbf{Q} = 3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

Then
$$\mathbf{P}\times\mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 4 & -4 \\ 3 & 3 & 6 \end{vmatrix}$$

$$= [(24+12)\mathbf{i} + (-12+48)\mathbf{j} + (-24-12)\mathbf{k}]$$
$$= (36)\mathbf{i} + (36)\mathbf{j} + (-36)\mathbf{k}$$

$$A = \sqrt{(36)^2 + (36)^2 + (-36)^2}$$

or $A = 62.4$ ◀

(*b*) We have $A = |\mathbf{P}\times\mathbf{Q}|$

where $\mathbf{P} = 7\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$

$\mathbf{Q} = -3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

Then
$$\mathbf{P}\times\mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -6 & -3 \\ -3 & 6 & -2 \end{vmatrix}$$

$$= [(12+18)\mathbf{i} + (9+14)\mathbf{j} + (42-18)\mathbf{k}]$$
$$= (30)\mathbf{i} + (23)\mathbf{j} + (24)\mathbf{k}$$

$$A = \sqrt{(30)^2 + (23)^2 + (24)^2}$$

or $A = 44.8$ ◀

PROBLEM 3.17

A plane contains the vectors **A** and **B**. Determine the unit vector normal to the plane when **A** and **B** are equal to, respectively, (*a*) 2**i** + 3**j** – 6**k** and 5**i** – 8**j** – 6**k**,

(*b*) 4**i** – 4**j** + 3**k** and –3**i** + 7**j** – 5**k**.

SOLUTION

(*a*) We have
$$\boldsymbol{\lambda} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|}$$

where
$$\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$$
$$\mathbf{B} = 5\mathbf{i} - 8\mathbf{j} - 6\mathbf{k}$$

Then
$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -6 \\ 5 & -8 & -6 \end{vmatrix}$$
$$= (-18-15)\mathbf{i} + (-30+12)\mathbf{j} + (-16-15)\mathbf{k}$$
$$= (-33\mathbf{i} - 18\mathbf{j} - 31\mathbf{k})$$

and
$$|\mathbf{A}\times\mathbf{B}| = \sqrt{(-33)^2 + (-18)^2 + (-31)^2} = \sqrt{2374}$$

$$\boldsymbol{\lambda} = \frac{(-33\mathbf{i} - 18\mathbf{j} - 31\mathbf{k})}{\sqrt{2374}} \qquad \text{or} \quad \boldsymbol{\lambda} = -0.677\mathbf{i} - 0.369\mathbf{j} - 0.636\mathbf{k} \ ◀$$

(*b*) We have
$$\boldsymbol{\lambda} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|}$$

where
$$\mathbf{A} = 4\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$$
$$\mathbf{B} = -3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$$

Then
$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 3 \\ -3 & 7 & -5 \end{vmatrix}$$
$$= (20-21)\mathbf{i} + (-9+20)\mathbf{j} + (28-12)\mathbf{k}$$
$$= (-\mathbf{i} + 11\mathbf{j} + 16\mathbf{k})$$

and
$$|\mathbf{A}\times\mathbf{B}| = \sqrt{(-1) + (11)^2 + (16)^2} = \sqrt{378}$$

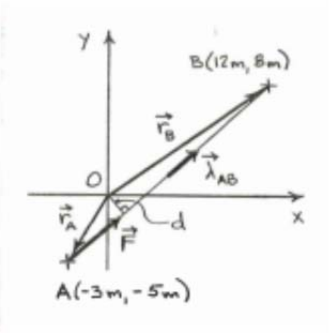
$$\boldsymbol{\lambda} = \frac{(-\mathbf{i} + 11\mathbf{j} + 16\mathbf{k})}{\sqrt{378}} \qquad \text{or} \quad \boldsymbol{\lambda} = -0.0514\mathbf{i} + 0.566\mathbf{j} + 0.823\mathbf{k} \ ◀$$

PROBLEM 3.18

A line passes through the points (12 m, 8 m) and (–3 m, –5 m). Determine the perpendicular distance *d* from the line to the origin *O* of the system of coordinates.

SOLUTION

$$d_{AB} = \sqrt{[12\text{ m} - (-3\text{ m})]^2 + [8\text{ m} - (-5\text{ m})]^2}$$
$$= \sqrt{394}\text{ m}$$



Assume that a force **F**, or magnitude *F*(N), acts at Point *A* and is directed from *A* to *B*.

Then
$$\mathbf{F} = F\boldsymbol{\lambda}_{AB}$$

where
$$\boldsymbol{\lambda}_{AB} = \frac{\mathbf{r}_B - \mathbf{r}_A}{d_{AB}}$$
$$= \frac{1}{\sqrt{394}}(15\mathbf{i} + 13\mathbf{j})$$

By definition,
$$\mathbf{M}_O = |\mathbf{r}_A \times \mathbf{F}| = dF$$

where
$$\mathbf{r}_A = (-3\text{ m})\mathbf{i} + (-5\text{ m})\mathbf{j}$$

Then
$$\mathbf{M}_O = [(-3\text{ m})\mathbf{i} + (-5\text{ m})\mathbf{j}] \times \frac{F}{\sqrt{394}\text{m}}[(15\text{ m})\mathbf{i} + (13\text{ m})\mathbf{j}]$$
$$= \frac{F}{\sqrt{394}}[-39\mathbf{k} + 75\mathbf{k}]\text{ N}\cdot\text{m}$$
$$= \left(\frac{36}{\sqrt{394}}F\right)\mathbf{k}\text{ N}\cdot\text{m}$$

Finally,
$$\left(\frac{36}{\sqrt{394}}F\right) = d(F)$$
$$d = \frac{36}{\sqrt{394}}\text{ m}$$

$d = 1.184$ m ◀

PROBLEM 3.19

Determine the moment about the origin O of the force $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ that acts at a Point A. Assume that the position vector of A is (*a*) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, (*b*) $\mathbf{r} = -8\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$, (c) $\mathbf{r} = 8\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$.

SOLUTION

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

(*a*)
$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 4 & -3 & 5 \end{vmatrix}$$

$$= (15-12)\mathbf{i} + (-16-10)\mathbf{j} + (-6-12)\mathbf{k} \qquad \mathbf{M}_O = 3\mathbf{i} - 26\mathbf{j} - 18\mathbf{k} \blacktriangleleft$$

(*b*)
$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 6 & -10 \\ 4 & -3 & 5 \end{vmatrix}$$

$$= (30-30)\mathbf{i} + (-40+40)\mathbf{j} + (24-24)\mathbf{k} \qquad \mathbf{M}_O = 0 \blacktriangleleft$$

(*c*)
$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -6 & 5 \\ 4 & -3 & 5 \end{vmatrix}$$

$$= (-30+15)\mathbf{i} + (20-40)\mathbf{j} + (-24+24)\mathbf{k} \qquad \mathbf{M}_O = -15\mathbf{i} - 20\mathbf{j} \blacktriangleleft$$

Note: The answer to Part (*b*) could have been anticipated since the elements of the last two rows of the determinant are proportional.

PROBLEM 3.20

Determine the moment about the origin *O* of the force **F** = 2**i** + 3**j** – 4**k** that acts at a Point *A*. Assume that the position vector of *A* is (*a*) **r** = 3**i** – 6**j** + 5**k**, (*b*) **r** = **i** – 4**j** – 2**k**, (*c*) **r** = 4**i** + 6**j** – 8**k**.

SOLUTION

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

(*a*)

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & 5 \\ 2 & 3 & -4 \end{vmatrix}$$

$$= (24 - 15)\mathbf{i} + (10 + 12)\mathbf{j} + (9 + 12)\mathbf{k} \qquad \mathbf{M}_O = 9\mathbf{i} + 22\mathbf{j} + 21\mathbf{k} \blacktriangleleft$$

(*b*)

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -2 \\ 2 & 3 & -4 \end{vmatrix}$$

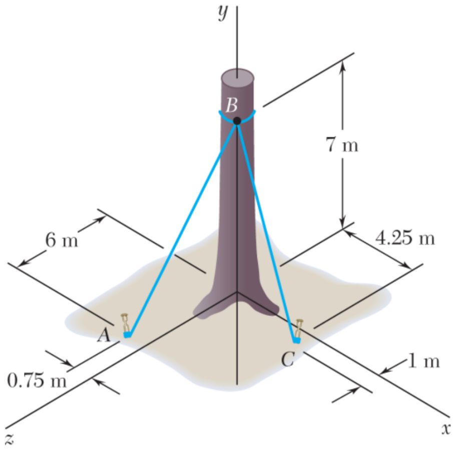
$$= (16 + 6)\mathbf{i} + (-4 + 4)\mathbf{j} + (3 + 8)\mathbf{k} \qquad \mathbf{M}_O = 22\mathbf{i} + 11\mathbf{k} \blacktriangleleft$$

(*c*)

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & -8 \\ 2 & 3 & -4 \end{vmatrix}$$

$$= (-24 + 24)\mathbf{i} + (-16 + 16)\mathbf{j} + (12 - 12)\mathbf{k} \qquad \mathbf{M}_O = 0 \blacktriangleleft$$

Note: The answer to Part (*c*) could have been anticipated since the elements of the last two rows of the determinant are proportional.



PROBLEM 3.21

Before the trunk of a large tree is felled, cables *AB* and *BC* are attached as shown. Knowing that the tensions in cables *AB* and *BC* are 555 N and 660 N, respectively, determine the moment about *O* of the resultant force exerted on the tree by the cables at *B*.

SOLUTION

$$d_{BA} = \sqrt{(-0.75\text{ m})^2 + (-7\text{ m})^2 + (6\text{ m})^2} = 9.25\text{ m}$$

$$d_{BC} = \sqrt{(4.25\text{ m})^2 + (-7\text{ m})^2 + (1\text{ m})^2} = 8.25\text{ m}$$

Have

$$\mathbf{T}_{BA} = T_{BA}\frac{\overrightarrow{BA}}{d_{BA}} = \frac{555\text{ N}}{9.25\text{ m}}(-0.75\text{ m}\,\mathbf{i} - 7\text{ m}\,\mathbf{j} + 6\text{ m}\,\mathbf{k})$$

$$\mathbf{T}_{BA} = -(45\text{ N})\mathbf{i} - (420\text{ N})\mathbf{j} + (360\text{ N})\mathbf{k}$$

$$\mathbf{T}_{BC} = T_{BC}\frac{\overrightarrow{BC}}{d_{BC}} = \frac{660\text{ N}}{8.25\text{ m}}(4.25\text{ m}\,\mathbf{i} - 7\text{ m}\,\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}_{BC} = (340\text{ N})\mathbf{i} - (540\text{ N})\mathbf{j} + (80\text{ N})\mathbf{k}$$

$$\mathbf{R} = \mathbf{T}_{BA} + \mathbf{T}_{BC}$$

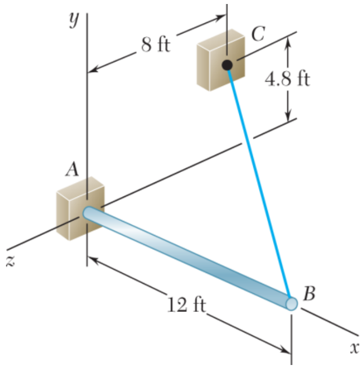
$$\therefore\ \mathbf{R} = (295\text{ N})\mathbf{i} - (980\text{ N})\mathbf{j} + (440\text{ N})\mathbf{k}$$

$$\mathbf{M}_O = \mathbf{r}_{B/O} \times \mathbf{R} \quad \text{where } \mathbf{r}_{B/O} = (7\text{ m})\mathbf{j}$$

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 7 & 0 \\ 295 & -980 & 440 \end{vmatrix} \text{N}\cdot\text{m}$$

$$= (3080\text{ N}\cdot\text{m})\mathbf{i} - (2065\text{ N}\cdot\text{m})\mathbf{k}$$

$$\mathbf{M}_O = (3080\text{ N}\cdot\text{m})\mathbf{i} - (2070\text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$$



PROBLEM 3.22

The 12-ft boom *AB* has a fixed end *A*. A steel cable is stretched from the free end *B* of the boom to a point *C* located on the vertical wall. If the tension in the cable is 380 lb, determine the moment about *A* of the force exerted by the cable at *B*.

SOLUTION

First note

$$d_{BC} = \sqrt{(-12)^2 + (4.8)^2 + (-8)^2}$$
$$= 15.2 \text{ ft}$$

Then

$$\mathbf{T}_{BC} = \frac{380 \text{ lb}}{15.2}(-12\mathbf{i} + 4.8\mathbf{j} - 8\mathbf{k})$$

We have

$$\mathbf{M}_A = \mathbf{r}_{B/A} \times \mathbf{T}_{BC}$$

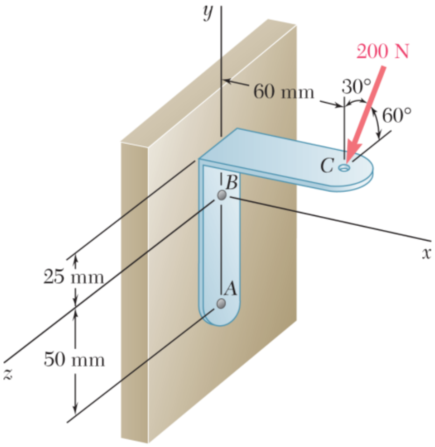
where

$$\mathbf{r}_{B/A} = (12 \text{ ft})\mathbf{i}$$

Then

$$\mathbf{M}_A = (12 \text{ ft})\mathbf{i} \times \frac{380 \text{ lb}}{15.2}(-12\mathbf{i} + 4.8\mathbf{j} - 8\mathbf{k})$$

or $\mathbf{M}_A = (2400 \text{ lb}\cdot\text{ft})\mathbf{j} + (1440 \text{ lb}\cdot\text{ft})\mathbf{k}$ ◀



PROBLEM 3.23

A 200-N force is applied as shown to the bracket *ABC*. Determine the moment of the force about *A*.

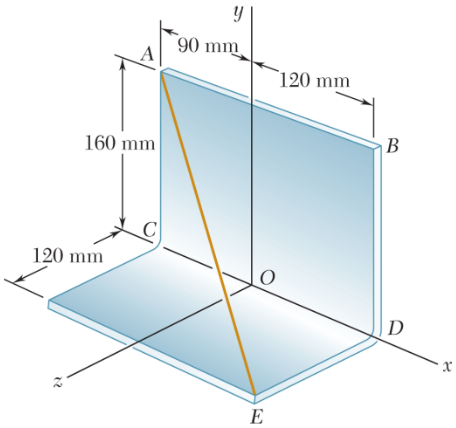
SOLUTION

We have
$$\mathbf{M}_A = \mathbf{r}_{C/A} \times \mathbf{F}_C$$

where
$$\mathbf{r}_{C/A} = (0.06\text{ m})\mathbf{i} + (0.075\text{ m})\mathbf{j}$$
$$\mathbf{F}_C = -(200\text{ N})\cos 30°\mathbf{j} + (200\text{ N})\sin 30°\mathbf{k}$$

Then
$$\mathbf{M}_A = 200\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.06 & 0.075 & 0 \\ 0 & -\cos 30° & \sin 30° \end{vmatrix}$$
$$= 200[(0.075\sin 30°)\mathbf{i} - (0.06\sin 30°)\mathbf{j} - (0.06\cos 30°)\mathbf{k}]$$

or $\mathbf{M}_A = (7.50\text{ N}\cdot\text{m})\mathbf{i} - (6.00\text{ N}\cdot\text{m})\mathbf{j} - (10.39\text{ N}\cdot\text{m})\mathbf{k}$ ◀



PROBLEM 3.24

The wire *AE* is stretched between the corners *A* and *E* of a bent plate. Knowing that the tension in the wire is 435 N, determine the moment about *O* of the force exerted by the wire (*a*) on corner *A*, (*b*) on corner *E*.

SOLUTION

$$\overrightarrow{AE} = (0.21\text{ m})\mathbf{i} - (0.16\text{ m})\mathbf{j} + (0.12\text{ m})\mathbf{k}$$

$$AE = \sqrt{(0.21\text{ m})^2 + (-0.16\text{ m})^2 + (0.12\text{ m})^2} = 0.29\text{ m}$$

(*a*)
$$\mathbf{F}_A = F_A \boldsymbol{\lambda}_{AE} = F\frac{\overrightarrow{AE}}{AE}$$

$$= (435\text{ N})\frac{0.21\mathbf{i} - 0.16\mathbf{j} + 0.12\mathbf{k}}{0.29}$$

$$= (315\text{ N})\mathbf{i} - (240\text{ N})\mathbf{j} + (180\text{ N})\mathbf{k}$$

$$\mathbf{r}_{A/O} = -(0.09\text{ m})\mathbf{i} + (0.16\text{ m})\mathbf{j}$$

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.09 & 0.16 & 0 \\ 315 & -240 & 180 \end{vmatrix}$$

$$= 28.8\mathbf{i} + 16.20\mathbf{j} + (21.6 - 50.4)\mathbf{k} \qquad \mathbf{M}_O = (28.8\text{ N}\cdot\text{m})\mathbf{i} + (16.20\text{ N}\cdot\text{m})\mathbf{j} - (28.8\text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$$

(*b*)
$$\mathbf{F}_E = -\mathbf{F}_A = -(315\text{ N})\mathbf{i} + (240\text{ N})\mathbf{j} - (180\text{ N})\mathbf{k}$$

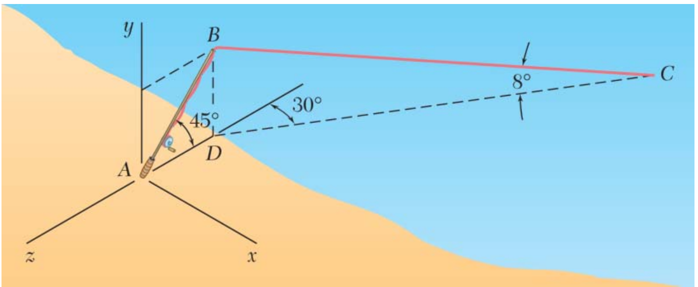
$$\mathbf{r}_{E/O} = (0.12\text{ m})\mathbf{i} + (0.12\text{ m})\mathbf{k}$$

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.12 & 0 & 0.12 \\ -315 & 240 & -180 \end{vmatrix}$$

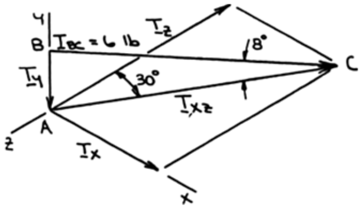
$$= -28.8\mathbf{i} + (-37.8 + 21.6)\mathbf{j} + 28.8\mathbf{k} \qquad \mathbf{M}_O = -(28.8\text{ N}\cdot\text{m})\mathbf{i} - (16.20\text{ N}\cdot\text{m})\mathbf{j} + (28.8\text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$$

PROBLEM 3.25

A 6-ft-long fishing rod *AB* is securely anchored in the sand of a beach. After a fish takes the bait, the resulting force in the line is 6 lb. Determine the moment about *A* of the force exerted by the line at *B*.



SOLUTION



We have
$$T_{xz} = (6\text{ lb})\cos 8° = 5.9416\text{ lb}$$

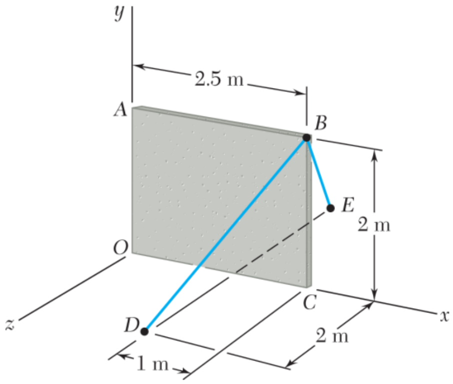
Then
$$T_x = T_{xz}\sin 30° = 2.9708\text{ lb}$$
$$T_y = T_{BC}\sin 8° = -0.83504\text{ lb}$$
$$T_z = T_{xz}\cos 30° = -5.1456\text{ lb}$$

Now
$$\mathbf{M}_A = \mathbf{r}_{B/A} \times \mathbf{T}_{BC}$$

where
$$\mathbf{r}_{B/A} = (6\sin 45°)\mathbf{j} - (6\cos 45°)\mathbf{k}$$
$$= \frac{6\text{ ft}}{\sqrt{2}}(\mathbf{j} - \mathbf{k})$$

Then
$$\mathbf{M}_A = \frac{6}{\sqrt{2}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 2.9708 & -0.83504 & -5.1456 \end{vmatrix}$$
$$= \frac{6}{\sqrt{2}}(-5.1456 - 0.83504)\mathbf{i} - \frac{6}{\sqrt{2}}(2.9708)\mathbf{j} - \frac{6}{\sqrt{2}}(2.9708)\mathbf{k}$$

or
$$\mathbf{M}_A = -(25.4\text{ lb}\cdot\text{ft})\mathbf{i} - (12.60\text{ lb}\cdot\text{ft})\mathbf{j} - (12.60\text{ lb}\cdot\text{ft})\mathbf{k}$$ ◀



PROBLEM 3.26

A precast concrete wall section is temporarily held by two cables as shown. Knowing that the tension in cable *BD* is 900 N, determine the moment about Point *O* of the force exerted by the cable at *B*.

SOLUTION

$$\mathbf{F} = F\frac{\overrightarrow{BD}}{BD} \quad \text{where} \quad F = 900\text{ N}$$

$$\overrightarrow{BD} = -(1\text{ m})\mathbf{i} - (2\text{ m})\mathbf{j} + (2\text{ m})\mathbf{k}$$

$$BD = \sqrt{(-1\text{ m})^2 + (-2\text{ m})^2 + (2\text{ m})^2}$$

$$= 3\text{ m}$$

$$\mathbf{F} = (900\text{ N})\frac{-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{3}$$

$$= -(300\text{ N})\mathbf{i} - (600\text{ N})\mathbf{j} + (600\text{ N})\mathbf{k}$$

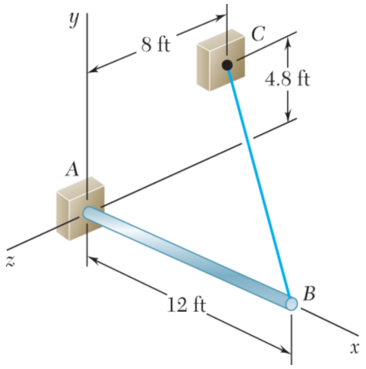
$$\mathbf{r}_{B/O} = (2.5\text{ m})\mathbf{i} + (2\text{ m})\mathbf{j}$$

$$\mathbf{M}_O = \mathbf{r}_{B/O} \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2.5 & 2 & 0 \\ -300 & -600 & 600 \end{vmatrix}$$

$$= 1200\mathbf{i} - 1500\mathbf{j} + (-1500 + 600)\mathbf{k}$$

$$\mathbf{M}_O = (1200\text{ N}\cdot\text{m})\mathbf{i} - (1500\text{ N}\cdot\text{m})\mathbf{j} - (900\text{ N}\cdot\text{m})\mathbf{k} \quad ◀$$



PROBLEM 3.27

In Prob. 3.22, determine the perpendicular distance from point *A* to cable *BC*.

PROBLEM 3.22 The 12-ft boom *AB* has a fixed end *A*. A steel cable is stretched from the free end *B* of the boom to a point *C* located on the vertical wall. If the tension in the cable is 380 lb, determine the moment about *A* of the force exerted by the cable at *B*.

SOLUTION

From the solution to problem 3.22

$$M_A = \sqrt{(2400)^2 + (1440)^2}$$
$$= 2798.9 \text{ lb} \cdot \text{ft}$$

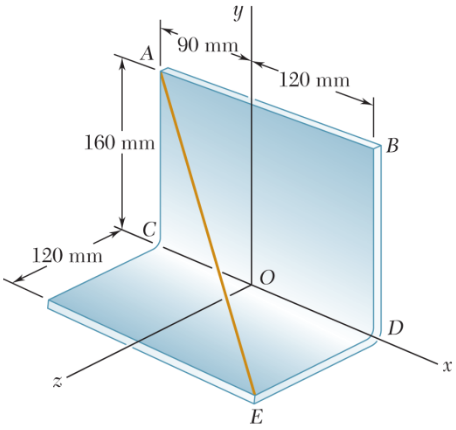
But

$$M_A = F_B \times d$$

$$d = \frac{M_A}{F_B}$$

$$d = \frac{2798.9 \text{ lb} \cdot \text{ft}}{380 \text{ lb}}$$

or $d = 7.37$ ft ◀



PROBLEM 3.28

In Prob. 3.24, determine the perpendicular distance from point *O* to wire *AE*.

PROBLEM 3.24 The wire *AE* is stretched between the corners *A* and *E* of a bent plate. Knowing that the tension in the wire is 435 N, determine the moment about *O* of the force exerted by the wire (*a*) on corner *A*, (*b*) on corner *E*.

SOLUTION

From the solution to Prob. 3.24

$$\mathbf{M}_O = (28.8\ \text{N}\cdot\text{m})\mathbf{i} + (16.20\ \text{N}\cdot\text{m})\mathbf{j} - (28.8\ \text{N}\cdot\text{m})\mathbf{k}$$

$$M_O = \sqrt{(28.8)^2 + (16.20)^2 + (28.8)^2}$$

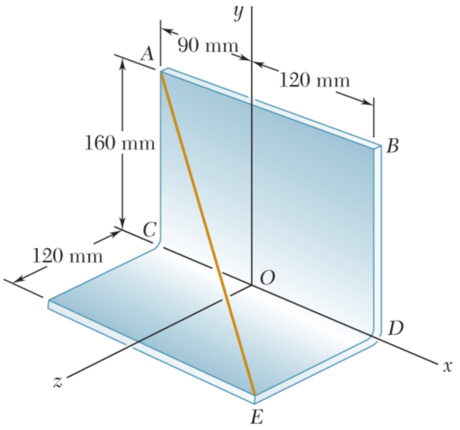
$$= 43.8329\ \text{N}\cdot\text{m}$$

But $M_O = F_A d$ or $d = \dfrac{M_O}{F_A}$

$$d = \frac{43.8329\ \text{N}\cdot\text{m}}{435\ \text{N}}$$

$$= 0.100765\ \text{m}$$

$$d = 100.8\ \text{mm}\ \blacktriangleleft$$



PROBLEM 3.29

In Prob. 3.24, determine the perpendicular distance from point *B* to wire *AE*.

PROBLEM 3.24 The wire *AE* is stretched between the corners *A* and *E* of a bent plate. Knowing that the tension in the wire is 435 N, determine the moment about *O* of the force exerted by the wire (*a*) on corner *A*, (*b*) on corner *E*.

SOLUTION

From the solution to Prob. 3.24

$$\mathbf{F}_A = (315\text{ N})\mathbf{i} - (240\text{ N})\mathbf{j} + (180\text{ N})\mathbf{k}$$

$$\mathbf{r}_{A/B} = -(0.210\text{ m})\mathbf{i}$$

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F}_A = -0.21\mathbf{i} \times (315\mathbf{i} - 240\mathbf{j} + 180\mathbf{k})$$

$$= 50.4\mathbf{k} + 37.8\mathbf{j}$$

$$M_B = \sqrt{(50.4)^2 + (37.8)^2}$$

$$= 63.0\text{ N}\cdot\text{m}$$

$$M_B = F_A d \qquad \text{or} \qquad d = \frac{M_B}{F_A}$$

$$d = \frac{63.0\text{ N}\cdot\text{m}}{435\text{ N}}$$

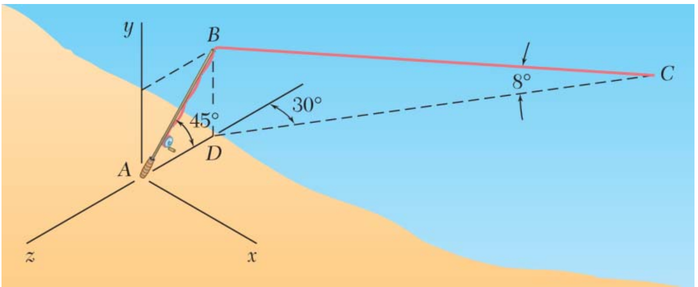
$$= 0.144829\text{ m}$$

$d = 144.8$ mm ◀

PROBLEM 3.30

In Prob. 3.25, determine the perpendicular distance from point *A* to a line drawn through points *B* and *C*.

PROBLEM 3.25 A 6-ft-long fishing rod *AB* is securely anchored in the sand of a beach. After a fish takes the bait, the resulting force in the line is 6 lb. Determine the moment about *A* of the force exerted by the line at *B*.



SOLUTION

From the solution to Prob. 3.25:

$$\mathbf{M}_A = -(25.4\ \text{lb}\cdot\text{ft})\mathbf{i} - (12.60\ \text{lb}\cdot\text{ft})\mathbf{j} - (12.60\ \text{lb}\cdot\text{ft})\mathbf{k}$$

$$M_A = \sqrt{(-25.4)^2 + (-12.60)^2 + (-12.60)^2}$$

$$= 31.027\ \text{lb}\cdot\text{ft}$$

$$M_A = T_{BC}d \qquad \text{or} \qquad d = \frac{M_A}{T_{BC}}$$

$$= \frac{31.027\ \text{lb}\cdot\text{ft}}{6\ \text{lb}}$$

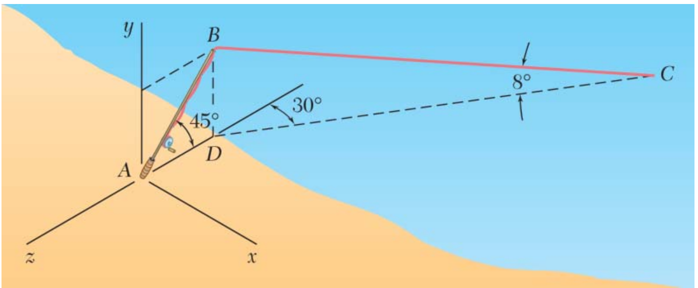
$$= 5.1712\ \text{ft}$$

$d = 5.17$ ft ◀

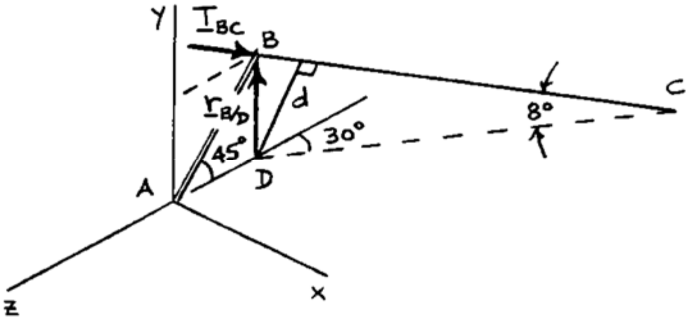
PROBLEM 3.31

In Prob. 3.25, determine the perpendicular distance from point *D* to a line drawn through points *B* and *C*.

PROBLEM 3.25 A 6-ft-long fishing rod *AB* is securely anchored in the sand of a beach. After a fish takes the bait, the resulting force in the line is 6 lb. Determine the moment about *A* of the force exerted by the line at *B*.



SOLUTION



$\overline{AB} = 6$ ft

$T_{BC} = 6$ lb

We have $|\mathbf{M}_D| = T_{BC}d$

where d = perpendicular distance from *D* to line *BC*.

$$\mathbf{M}_D = \mathbf{r}_{B/D} \times \mathbf{T}_{BC} \qquad \mathbf{r}_{B/D} = (6\sin 45°\text{ ft})\mathbf{j} = (4.2426\text{ ft})$$

$$\mathbf{T}_{BC}: \quad (T_{BC})_x = (6\text{ lb})\cos 8° \sin 30° = 2.9708\text{ lb}$$

$$(T_{BC})_y = -(6\text{ lb})\sin 8° = -0.83504\text{ lb}$$

$$(T_{BC})_z = -(6\text{ lb})\cos 8° \cos 30° = -5.1456\text{ lb}$$

$$\mathbf{T}_{BC} = (2.9708\text{ lb})\mathbf{i} - (0.83504\text{ lb})\mathbf{j} - (5.1456\text{ lb})\mathbf{k}$$

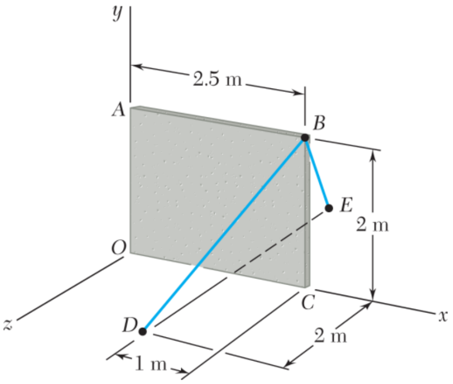
$$\mathbf{M}_D = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4.2426 & 0 \\ 2.9708 & -0.83504 & -5.1456 \end{vmatrix}$$

$$= -(21.831\text{ lb}\cdot\text{ft})\mathbf{i} - (12.6039\text{ lb}\cdot\text{ft})$$

$$|M_D| = \sqrt{(-21.831)^2 + (-12.6039)^2} = 25.208\text{ lb}\cdot\text{ft}$$

$$25.208\text{ lb}\cdot\text{ft} = (6\text{ lb})d$$

$d = 4.20$ ft ◀



PROBLEM 3.32

In Prob. 3.26, determine the perpendicular distance from point *O* to cable *BD*.

PROBLEM 3.26 A precast concrete wall section is temporarily held by two cables as shown. Knowing that the tension in cable *BD* is 900 N, determine the moment about Point *O* of the force exerted by the cable at *B*.

SOLUTION

From the solution to Prob. 3.26 we have

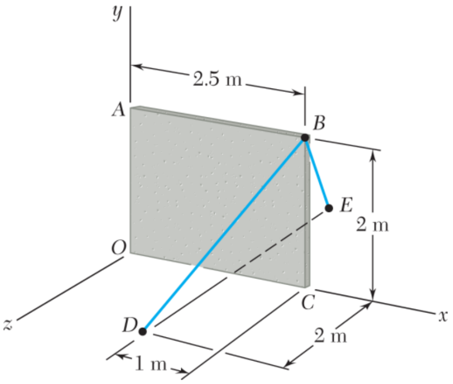
$$\mathbf{M}_O = (1200\ \text{N}\cdot\text{m})\mathbf{i} - (1500\ \text{N}\cdot\text{m})\mathbf{j} - (900\ \text{N}\cdot\text{m})\mathbf{k}$$

$$M_O = \sqrt{(1200)^2 + (-1500)^2 + (-900)^2} = 2121.3\ \text{N}\cdot\text{m}$$

$$M_O = Fd \qquad d = \frac{M_O}{F}$$

$$= \frac{2121.3\ \text{N}\cdot\text{m}}{900\ \text{N}}$$

$d = 2.36$ m ◀



PROBLEM 3.33

In Prob. 3.26, determine the perpendicular distance from point *C* to cable *BD*.

PROBLEM 3.26 A precast concrete wall section is temporarily held by two cables as shown. Knowing that the tension in cable *BD* is 900 N, determine the moment about Point *O* of the force exerted by the cable at *B*.

SOLUTION

From the solution to Prob. 3.26 we have

$$\mathbf{F} = -(300\text{ N})\mathbf{i} - (600\text{ N})\mathbf{j} + (600\text{ N})\mathbf{k}$$

$$\mathbf{r}_{B/C} = (2\text{ m})\mathbf{j}$$

$$\mathbf{M}_C = \mathbf{r}_{B/C} \times \mathbf{F} = (2\text{ m})\mathbf{j} \times (-300\text{ N}\mathbf{i} - 600\text{ N}\mathbf{j} + 600\text{ N}\mathbf{k})$$

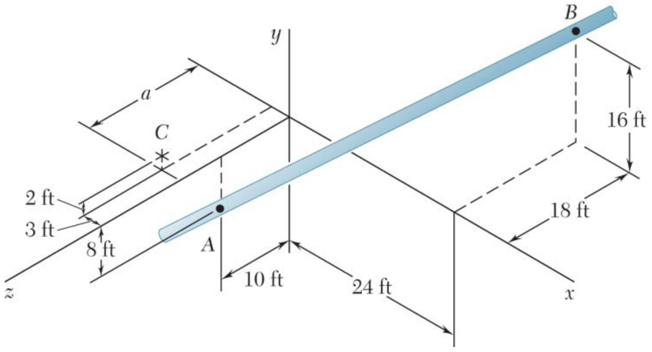
$$= (600\text{ N}\cdot\text{m})\mathbf{k} + (1200\text{ N}\cdot\text{m})\mathbf{i}$$

$$M_C = \sqrt{(600)^2 + (1200)^2} = 1341.64\text{ N}\cdot\text{m}$$

$$M_C = Fd \qquad d = \frac{M_C}{F}$$

$$= \frac{1341.64\text{ N}\cdot\text{m}}{900\text{ N}}$$

$d = 1.491$ m ◀



PROBLEM 3.34

Determine the value of a that minimizes the perpendicular distance from Point C to a section of pipeline that passes through Points A and B.

SOLUTION

Assuming a force **F** acts along AB,

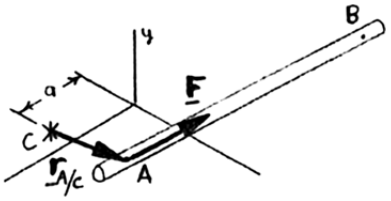
$$|\mathbf{M}_C| = |\mathbf{r}_{A/C} \times \mathbf{F}| = F(d)$$

where $d =$ perpendicular distance from C to line AB

$$\mathbf{F} = \boldsymbol{\lambda}_{AB} F$$

$$= \frac{(24\text{ ft})\,\mathbf{i} + (24\text{ ft})\,\mathbf{j} - (28\text{ ft})\,\mathbf{k}}{\sqrt{(24)^2 + (24)^2 + (28)^2}\text{ ft}} F$$

$$= \frac{F}{11}(6)\,\mathbf{i} + (6)\,\mathbf{j} - (7)\,\mathbf{k}$$



$$\mathbf{r}_{A/C} = (3\text{ ft})\mathbf{i} - (10\text{ ft})\mathbf{j} - (a - 10\text{ ft})\mathbf{k}$$

$$\mathbf{M}_C = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -10 & 10a \\ 6 & 6 & -7 \end{vmatrix} \frac{F}{11}$$

$$= [(10 + 6a)\mathbf{i} + (81 - 6a)\mathbf{j} + 78\,\mathbf{k}]\frac{F}{11}$$

Since $|\mathbf{M}_C| = \sqrt{|\mathbf{r}_{A/C} \times \mathbf{F}^2|}$ or $|\mathbf{r}_{A/C} \times \mathbf{F}^2| = (dF)^2$

$$\frac{1}{121}(10 + 6a)^2 + (81 - 6a)^2 + (78)^2 = d^2$$

Setting $\frac{d}{da}(d^2) = 0$ to find a to minimize d:

$$\frac{1}{121}[2(6)(10 + 6a) + 2(-6)(81 - 6a)] = 0$$

Solving $a = 5.92$ ft

or $a = 5.92$ ft ◀

PROBLEM 3.35

Given the vectors **P** = 2**i** + 3**j** – **k**, **Q** = 5**i** – 4**j** + 3**k**, and **S** = –3**i** + 2**j** – 5**k**, compute the scalar products **P** · **Q**, **P** · **S**, and **Q** · **S**.

SOLUTION

$$\begin{aligned}\mathbf{P}\cdot\mathbf{Q} &= (2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(5\mathbf{i}-4\mathbf{j}+3\mathbf{k})\\ &= (2)(5)+(3)(-4)+(-1)(3)\\ &= 10-12-3\end{aligned}$$

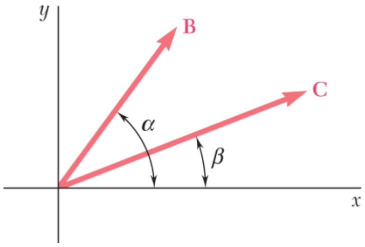
$$\mathbf{P}\cdot\mathbf{Q} = -5 \blacktriangleleft$$

$$\begin{aligned}\mathbf{P}\cdot\mathbf{S} &= (2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(-3\mathbf{i}+2\mathbf{j}-5\mathbf{k})\\ &= (2)(-3)+(3)(2)+(-1)(-5)\\ &= -6+6+5\end{aligned}$$

$$\mathbf{P}\cdot\mathbf{S} = +5 \blacktriangleleft$$

$$\begin{aligned}\mathbf{Q}\cdot\mathbf{S} &= (5\mathbf{i}-4\mathbf{j}+3\mathbf{k})\cdot(-3\mathbf{i}+2\mathbf{j}-5\mathbf{k})\\ &= (5)(-3)+(-4)(2)+(3)(-5)\\ &= -15-8-15\end{aligned}$$

$$\mathbf{Q}\cdot\mathbf{S} = -38 \blacktriangleleft$$

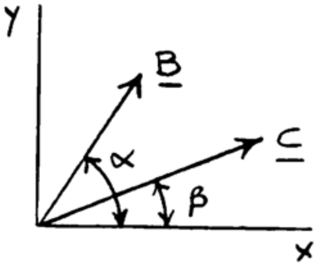


PROBLEM 3.36

Form the scalar product **B** · **C** and use the result obtained to prove the identity

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\,.$$

SOLUTION



$$\mathbf{B} = B\cos\alpha\,\mathbf{i} + B\sin\alpha\,\mathbf{j} \qquad (1)$$

$$\mathbf{C} = C\cos\beta\,\mathbf{i} + C\sin\beta\,\mathbf{j} \qquad (2)$$

By definition:

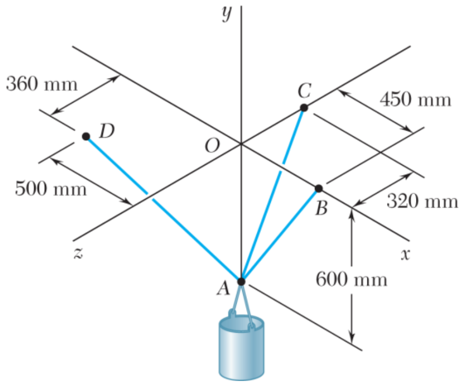
$$\mathbf{B}\cdot\mathbf{C} = BC\cos(\alpha-\beta) \qquad (3)$$

From (1) and (2):

$$\begin{aligned}\mathbf{B}\cdot\mathbf{C} &= (B\cos\alpha\,\mathbf{i} + B\sin\alpha\,\mathbf{j})\cdot(C\cos\beta\,\mathbf{i} + C\sin\beta\,\mathbf{j})\\ &= BC(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \qquad (4)\end{aligned}$$

Equating the right-hand members of (3) and (4),

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad ◀$$



PROBLEM 3.37

Three cables are used to support a container as shown. Determine the angle formed by cables *AB* and *AD*.

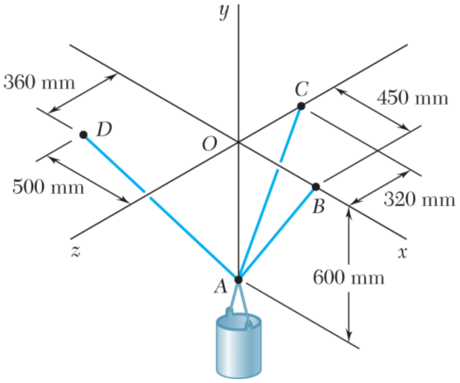
SOLUTION

First note:
$$AB = \sqrt{(450\text{ mm})^2 + (600\text{ mm})^2}$$
$$= 750\text{ mm}$$
$$AD = \sqrt{(-500\text{ mm})^2 + (600\text{ mm})^2 + (360\text{ mm})^2}$$
$$= 860\text{ mm}$$

and
$$\overrightarrow{AB} = (450\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{j}$$
$$\overrightarrow{AD} = (-500\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{j} + (360\text{ mm})\mathbf{k}$$

By definition,
$$\overrightarrow{AB} \cdot \overrightarrow{AD} = (AB)(AD)\cos\theta$$
$$(450\mathbf{i} + 600\mathbf{j}) \cdot (-500\mathbf{i} - 600\mathbf{j} + 360\mathbf{k}) = (750)(860)\cos\theta$$
$$(450)(-500) + (600)(600) + (0)(360) = (750)(860)\cos\theta$$

or
$$\cos\theta = 0.20930 \qquad \theta = 77.9° \blacktriangleleft$$



PROBLEM 3.38

Three cables are used to support a container as shown. Determine the angle formed by cables AC and AD.

SOLUTION

First note:

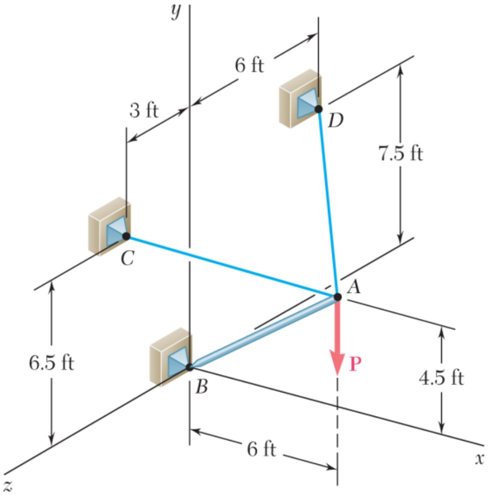
$$AC = \sqrt{(600 \text{ mm})^2 + (-320 \text{ mm})^2}$$
$$= 680 \text{ mm}$$
$$AD = \sqrt{(-500 \text{ mm})^2 + (600 \text{ mm})^2 + (360 \text{ mm})^2}$$
$$= 860 \text{ mm}$$

and

$$\overrightarrow{AC} = (600 \text{ mm})\mathbf{j} + (-320 \text{ mm})\mathbf{k}$$
$$\overrightarrow{AD} = (-500 \text{ mm})\mathbf{i} + (600 \text{ mm})\mathbf{j} + (360 \text{ mm})\mathbf{k}$$

By definition,

$$\overrightarrow{AC} \cdot \overrightarrow{AD} = (AC)(AD)\cos\theta$$
$$(600\mathbf{j} - 320\mathbf{k}) \cdot (-500\mathbf{i} + 600\mathbf{j} + 360\mathbf{k}) = (680)(860)\cos\theta$$
$$0(-500) + (600)(600) + (-320)(360) = (680)(860)\cos\theta$$
$$\cos\theta = 0.41860 \qquad \theta = 65.3° \blacktriangleleft$$



PROBLEM 3.39

Knowing that the tension in cable *AC* is 280 lb, determine (*a*) the angle between cable *AC* and the boom *AB*, (*b*) the projection on *AB* of the force exerted by cable *AC* at point *A*.

SOLUTION

(*a*) First note $AC = \sqrt{(-6)^2 + (2)^2 + (3)^2}$

$= 7.00$ ft

$AB = \sqrt{(-6)^2 + (-4.5)^2 + (0)^2}$

$= 7.50$ ft

and $\overrightarrow{AC} = -(6\text{ ft})\mathbf{i} + (2\text{ ft})\mathbf{j} + (3\text{ ft})\mathbf{k}$

$\overrightarrow{AB} = -(6\text{ ft})\mathbf{i} - (4.5\text{ ft})\mathbf{j}$

By definition $\overrightarrow{AC} \cdot \overrightarrow{AB} = (AC)(AB)\cos\theta$

or $(-6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (-6\mathbf{i} - 4.5\mathbf{j}) = (7.00)(7.50) \times \cos\theta$

or $(-6)(-6) + (2)(-4.5) + (3)(0) = 52.50\cos\theta$

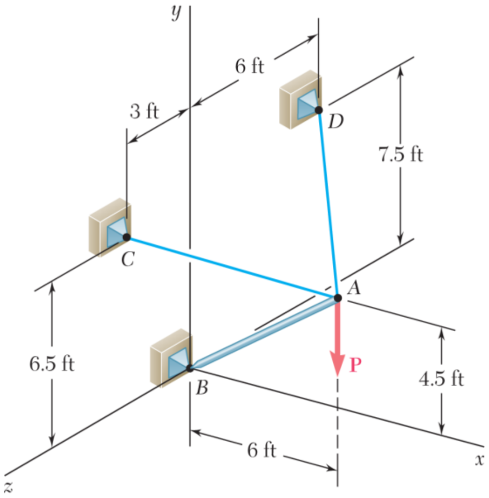
or $\cos\theta = 0.51429$ or $\theta = 59.0°$ ◀

(*b*) We have $(T_{AC})_{AB} = \mathbf{T}_{AC} \cdot \boldsymbol{\lambda}_{AB}$

$= T_{AC}\cos\theta$

$= (280\text{ lb})(0.51429)$

or $(T_{AC})_{AB} = 144.0$ lb ◀



PROBLEM 3.40

Knowing that the tension in cable *AD* is 180 lb, determine (*a*) the angle between cable *AD* and the boom *AB*, (*b*) the projection on *AB* of the force exerted by cable *AD* at point *A*.

SOLUTION

(*a*) First note

$$AD = \sqrt{(-6)^2 + (3)^2 + (-6)^2}$$
$$= 9.00 \text{ ft}$$
$$AB = \sqrt{(-6)^2 + (-4.5)^2 + (0)^2}$$
$$= 7.50 \text{ ft}$$

and

$$\overrightarrow{AD} = -(6 \text{ ft})\mathbf{i} + (3 \text{ ft})\mathbf{j} - (6 \text{ ft})\mathbf{k}$$
$$\overrightarrow{AB} = -(6 \text{ ft})\mathbf{i} - (4.5 \text{ ft})\mathbf{j}$$

By definition,

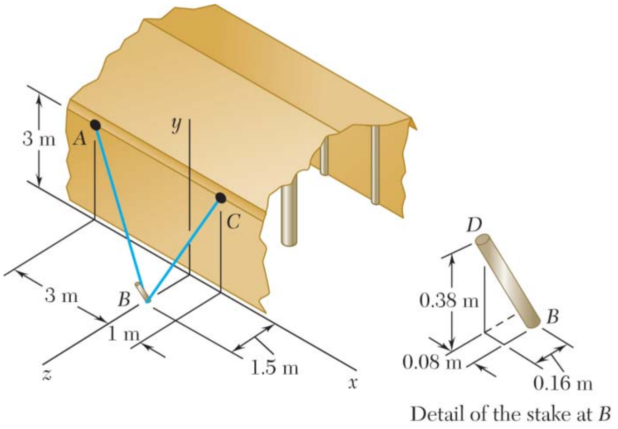
$$\overrightarrow{AD} \cdot \overrightarrow{AB} = (AD)(AB)\cos\theta$$
$$(-6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) \cdot (-6\mathbf{i} - 4.5\mathbf{j}) = (9.00)(7.50)\cos\theta$$
$$(-6)(-6) + (3)(-4.5) + (-6)(0) = 67.50\cos\theta$$
$$\cos\theta = \frac{1}{3}$$

$\theta = 70.5°$ ◀

(*b*)

$$(T_{AD})_{AB} = \mathbf{T}_{AD} \cdot \boldsymbol{\lambda}_{AB}$$
$$= T_{AD}\cos\theta$$
$$= (180 \text{ lb})\left(\frac{1}{3}\right)$$

$(T_{AD})_{AB} = 60.0 \text{ lb}$ ◀



PROBLEM 3.41

Ropes *AB* and *BC* are two of the ropes used to support a tent. The two ropes are attached to a stake at *B*. If the tension in rope *AB* is 540 N, determine (*a*) the angle between rope *AB* and the stake, (*b*) the projection on the stake of the force exerted by rope *AB* at Point *B*.

SOLUTION

First note:

$$BA = \sqrt{(-3)^2 + (3)^2 + (-1.5)^2} = 4.5 \text{ m}$$

$$BD = \sqrt{(-0.08)^2 + (0.38)^2 + (0.16)^2} = 0.42 \text{ m}$$

Then

$$\mathbf{T}_{BA} = \frac{T_{BA}}{4.5}(-3\mathbf{i} + 3\mathbf{j} - 1.5\mathbf{k})$$

$$= \frac{T_{BA}}{3}(-2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

$$\boldsymbol{\lambda}_{BD} = \frac{\overrightarrow{BD}}{BD} = \frac{1}{0.42}(-0.08\mathbf{i} + 0.38\mathbf{j} + 0.16\mathbf{k})$$

$$= \frac{1}{21}(-4\mathbf{i} + 19\mathbf{j} + 8\mathbf{k})$$

(*a*) We have

$$\mathbf{T}_{BA} \cdot \boldsymbol{\lambda}_{BD} = T_{BA} \cos\theta$$

or

$$\frac{T_{BA}}{3}(-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot \frac{1}{21}(-4\mathbf{i} + 19\mathbf{j} + 8\mathbf{k}) = T_{BA} \cos\theta$$

or

$$\cos\theta = \frac{1}{63}[(-2)(-4) + (2)(19) + (-1)(8)]$$

$$= 0.60317$$

or $\theta = 52.9°$ ◀

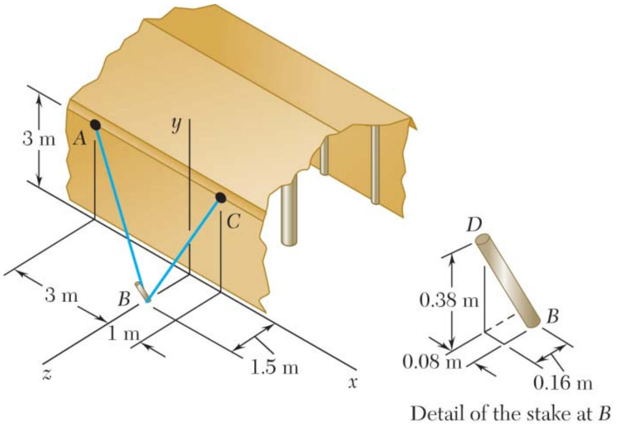
(*b*) We have

$$(T_{BA})_{BD} = \mathbf{T}_{BA} \cdot \boldsymbol{\lambda}_{BD}$$

$$= T_{BA} \cos\theta$$

$$= (540 \text{ N})(0.60317)$$

or $(T_{BA})_{BD} = 326 \text{ N}$ ◀



PROBLEM 3.42

Ropes *AB* and *BC* are two of the ropes used to support a tent. The two ropes are attached to a stake at *B*. If the tension in rope *BC* is 490 N, determine (*a*) the angle between rope *BC* and the stake, (*b*) the projection on the stake of the force exerted by rope *BC* at Point *B*.

SOLUTION

First note:

$$BC = \sqrt{(1)^2 + (3)^2 + (-1.5)^2} = 3.5 \text{ m}$$

$$BD = \sqrt{(-0.08)^2 + (0.38)^2 + (0.16)^2} = 0.42 \text{ m}$$

$$\mathbf{T}_{BC} = \frac{T_{BC}}{3.5}(\mathbf{i} + 3\mathbf{j} - 1.5\mathbf{k})$$

$$= \frac{T_{BC}}{7}(2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$$

$$\lambda_{BD} = \frac{\overrightarrow{BD}}{BD} = \frac{1}{0.42}(-0.08\mathbf{i} + 0.38\mathbf{j} + 0.16\mathbf{k})$$

$$= \frac{1}{21}(-4\mathbf{i} + 19\mathbf{j} + 8\mathbf{k})$$

(*a*)

$$\mathbf{T}_{BC} \cdot \lambda_{BD} = T_{BC} \cos\theta$$

$$\frac{T_{BC}}{7}(2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) \cdot \frac{1}{21}(-4\mathbf{i} + 19\mathbf{j} + 8\mathbf{k}) = T_{BC} \cos\theta$$

$$\cos\theta = \frac{1}{147}[(2)(-4) + (6)(19) + (-3)(8)]$$

$$= 0.55782$$

$$\theta = 56.1° \blacktriangleleft$$

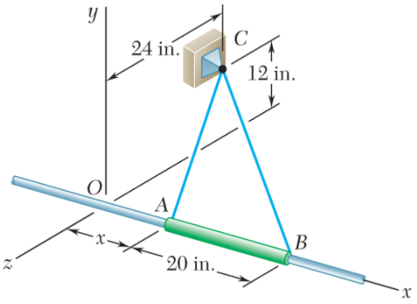
(*b*)

$$(T_{BC})_{BD} = \mathbf{T}_{BC} \cdot \lambda_{BD}$$

$$= T_{BC} \cos\theta$$

$$= (490 \text{ N})(0.55782)$$

$$(T_{BC})_{BD} = 273 \text{ N} \blacktriangleleft$$



PROBLEM 3.43

The 20-in. tube AB can slide along a horizontal rod. The ends A and B of the tube are connected by elastic cords to the fixed point C. For the position corresponding to $x = 11$ in., determine the angle formed by the two cords (*a*) using Eq. (3.32), (*b*) applying the law of cosines to triangle ABC.

SOLUTION

(*a*) Using Eq. (3.32):

$$\overline{CA} = 11\mathbf{i} - 12\mathbf{j} + 24\mathbf{k}$$

$$CA = \sqrt{(11)^2 + (-12)^2 + (24)^2} = 29 \text{ in.}$$

$$\overline{CB} = 31\mathbf{i} - 12\mathbf{j} + 24\mathbf{k}$$

$$CB = \sqrt{(31)^2 + (-12)^2 + (24)^2} = 41 \text{ in.}$$

$$\begin{aligned}\cos\theta &= \frac{\overline{CA}\cdot\overline{CB}}{(CA)(CB)} \\ &= \frac{(11\mathbf{i} - 12\mathbf{j} + 24\mathbf{k})\cdot(31\mathbf{i} - 12\mathbf{j} + 24\mathbf{k})}{(29)(41)} \\ &= \frac{(11)(31) + (-12)(-12) + (24)(24)}{(29)(41)} \\ &= 0.89235\end{aligned}$$

$\theta = 26.8°$ ◀

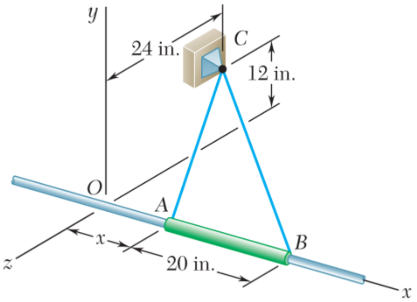
(*b*) Law of cosines:

$$(AB)^2 = (CA)^2 + (CB)^2 - 2(CA)(CB)\cos\theta$$

$$(20)^2 = (29)^2 + (41)^2 - 2(29)(41)\cos\theta$$

$$\cos\theta = 0.89235$$

$\theta = 26.8°$ ◀



PROBLEM 3.44

Solve Prob. 3.43 for the position corresponding to $x = 4$ in.

PROBLEM 3.43 The 20-in. tube *AB* can slide along a horizontal rod. The ends *A* and *B* of the tube are connected by elastic cords to the fixed point *C*. For the position corresponding to $x = 11$ in., determine the angle formed by the two cords (*a*) using Eq. (3.32), (*b*) applying the law of cosines to triangle *ABC*.

SOLUTION

(*a*) Using Eq. (3.32):

$$\overrightarrow{CA} = 4\mathbf{i} - 12\mathbf{j} + 24\mathbf{k}$$

$$CA = \sqrt{(4)^2 + (-12)^2 + (24)^2} = 27.129 \text{ in.}$$

$$\overrightarrow{CB} = 24\mathbf{i} - 12\mathbf{j} + 24\mathbf{k}$$

$$CB = \sqrt{(24)^2 + (-12)^2 + (24)^2} = 36 \text{ in.}$$

$$\cos\theta = \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{(CA)(CB)}$$

$$= \frac{(4\mathbf{i} - 12\mathbf{j} + 24\mathbf{k}) \cdot (24\mathbf{i} - 12\mathbf{j} + 24\mathbf{k})}{(27.129)(36)}$$

$$= 0.83551 \qquad \theta = 33.3° \blacktriangleleft$$

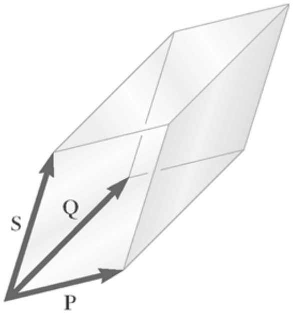
(*b*) Law of cosines:

$$(AB)^2 = (CA)^2 + (CB)^2 - 2(CA)(CB)\cos\theta$$

$$(20)^2 = (27.129)^2 + (36)^2 - 2(27.129)(36)\cos\theta$$

$$\cos\theta = 0.83551$$

$$\theta = 33.3° \blacktriangleleft$$



PROBLEM 3.45

Determine the volume of the parallelepiped of Fig. 3.25 when (*a*) **P** = 4**i** − 3**j** + 2**k**, **Q** = −2**i** − 5**j** + **k**, and **S** = 7**i** + **j** − **k**, (*b*) **P** = 5**i** − **j** + 6**k**, **Q** = 2**i** + 3**j** + **k**, and **S** = −3**i** − 2**j** + 4**k**.

SOLUTION

Volume of a parallelepiped is found using the mixed triple product.

(*a*)

$$\text{Vol.} = \mathbf{P}\cdot(\mathbf{Q}\times\mathbf{S})$$

$$= \begin{vmatrix} 4 & -3 & 2 \\ -2 & -5 & 1 \\ 7 & 1 & -1 \end{vmatrix} \text{in.}^3$$

$$= (20-21-4+70+6-4)$$

$$= 67$$

or Volume = 67.0 ◀

(*b*)

$$\text{Vol.} = \mathbf{P}\cdot(\mathbf{Q}\times\mathbf{S})$$

$$= \begin{vmatrix} 5 & -1 & 6 \\ 2 & 3 & 1 \\ -3 & -2 & 4 \end{vmatrix} \text{in.}^3$$

$$= (60+3-24+54+8+10)$$

$$= 111$$

or Volume = 111.0 ◀

PROBLEM 3.46

Given the vectors $\mathbf{P} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{Q} = 4\mathbf{i} + Q_y\mathbf{j} - 2\mathbf{k}$, and $\mathbf{S} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, determine the value of Q_y for which the three vectors are coplanar.

SOLUTION

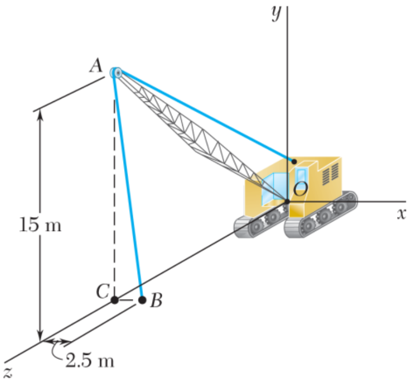
If **P**, **Q**, and **S** are coplanar, then **P** must be perpendicular to $(\mathbf{Q} \times \mathbf{S})$.

$$\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{S}) = 0$$

(or, the volume of a parallelepiped defined by **P**, **Q**, and **S** is zero).

Then
$$\begin{vmatrix} 3 & -1 & 1 \\ 4 & Q_y & -2 \\ 2 & -2 & 2 \end{vmatrix} = 0$$

or
$$6Q_y + 4 - 8 - 2Q_y + 8 - 12 = 0 \qquad Q_y = 2 \blacktriangleleft$$



PROBLEM 3.47

A crane is oriented so that the end of the 25-m boom *AO* lies in the *yz* plane. At the instant shown, the tension in cable *AB* is 4 kN. Determine the moment about each of the coordinate axes of the force exerted on *A* by cable *AB*.

SOLUTION

$$\begin{aligned} OC &= \sqrt{(OA)^2 - (AC)^2} \\ &= \sqrt{(25\text{ m})^2 - (15\text{ m})^2} \\ &= 20\text{ m} \end{aligned}$$

$$\mathbf{r}_A = (15\text{ m})\mathbf{j} + (20\text{ m})\mathbf{k}$$

$$AB = \sqrt{(2.5\text{ m})^2 + (15\text{ m})^2} = 15.2069\text{ m}$$

$$\begin{aligned} \mathbf{P} &= P\frac{\overline{AB}}{AB} \\ &= (4\text{ kN})\frac{2.5\mathbf{i} - 15\mathbf{j}}{15.2069} \\ &= (0.65760\text{ kN})\mathbf{i} - (3.9456\text{ kN})\mathbf{j} \end{aligned}$$

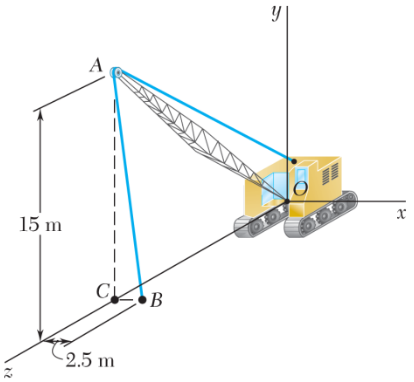
Using Eq. (3.11):

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{P} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 15 & 20 \\ 0.65760 & 3.9456 & 0 \end{vmatrix}$$

$$\mathbf{M}_O = (78.912\text{ kN}\cdot\text{m})\mathbf{i} + (13.1520\text{ kN}\cdot\text{m})\mathbf{j} - (9.8640\text{ kN}\cdot\text{m})\mathbf{k}$$

But $\mathbf{M}_O = M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}$

Therefore, $M_x = 78.9\text{ kN}\cdot\text{m},\ M_y = 13.15\text{ kN}\cdot\text{m},\ M_z = -9.86\text{ kN}\cdot\text{m}$ ◀



PROBLEM 3.48

The 25-m crane boom *AO* lies in the *yz* plane. Determine the maximum permissible tension in cable *AB* if the absolute value of moments about the coordinate axes of the force exerted on *A* by cable *AB* must be

$$|M_x| \le 60 \text{ kN·m}, \ |M_y| \le 12 \text{ kN·m}, \ |M_z| \le 8 \text{ kN·m}$$

$$OC = \sqrt{(OA)^2 - (AC)^2}$$
$$= \sqrt{(25 \text{ m})^2 - (15 \text{ m})^2}$$
$$= 20 \text{ m}$$

$$\mathbf{r}_A = (15 \text{ m})\mathbf{j} + (20 \text{ m})\mathbf{k}$$

$$AB = \sqrt{(2.5 \text{ m})^2 + (15 \text{ m})^2} = 15.2069 \text{ m}$$

$$\mathbf{P} = P\frac{\overrightarrow{AB}}{AB}$$
$$= (P)\frac{2.5\mathbf{i} - 15\mathbf{j}}{15.2069}$$
$$= (0.164399P \text{ kN})\mathbf{i} - (0.98639P \text{ kN})\mathbf{j}$$

Using Eq. (3.11):

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{P} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 15 & 20 \\ 0.163499P & -0.98639P & 0 \end{vmatrix}$$

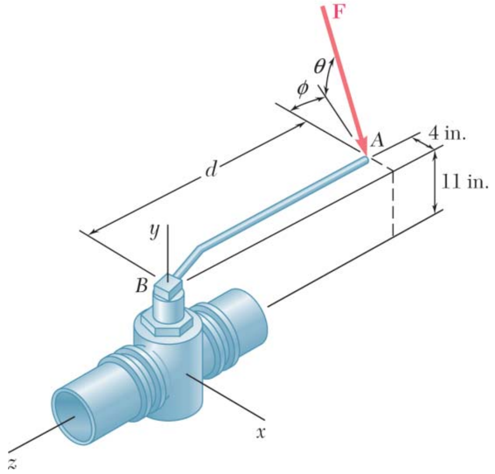
$$\mathbf{M}_O = (19.7278P)\mathbf{i} + (3.2700P)\mathbf{j} - (2.4525P)\mathbf{k}$$

$M_x < 60 \text{ kN} \cdot \text{m}$: $\quad 19.7278P < 60 \quad P \le 3.04 \text{ kN}$

But $\quad M_y < 12 \text{ kN} \cdot \text{m}$: $\quad 3.270P < 12 \quad P \le 3.67 \text{ kN}$

$M_z < 8 \text{ kN} \cdot \text{m}$: $\quad 2.4525P < 8 \quad P \le 3.15 \text{ kN}$

Therefore, the maximum permissible tension is $P = 3.04 \text{ kN}$ ◀



PROBLEM 3.49

To loosen a frozen valve, a force **F** of magnitude 70 lb is applied to the handle of the valve. Knowing that $\theta = 25°$, $M_x = -61\text{ lb}\cdot\text{ft}$, and $M_z = -43\text{ lb}\cdot\text{ft}$, determine ϕ and d.

SOLUTION

We have $\Sigma\mathbf{M}_O: \quad \mathbf{r}_{A/O} \times \mathbf{F} = \mathbf{M}_O$

where $\mathbf{r}_{A/O} = -(4\text{ in.})\mathbf{i} + (11\text{ in.})\mathbf{j} - (d)\mathbf{k}$

$$\mathbf{F} = F(\cos\theta\cos\phi\mathbf{i} - \sin\theta\mathbf{j} + \cos\theta\sin\phi\mathbf{k})$$

For $F = 70\text{ lb}, \quad \theta = 25°$

$$\mathbf{F} = (70\text{ lb})[(0.90631\cos\phi)\mathbf{i} - 0.42262\mathbf{j} + (0.90631\sin\phi)\mathbf{k}]$$

$$\mathbf{M}_O = (70\text{ lb})\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 11 & -d \\ -0.90631\cos\phi & -0.42262 & 0.90631\sin\phi \end{vmatrix}\text{ in.}$$

$$= (70\text{ lb})[(9.9694\sin\phi - 0.42262d)\mathbf{i} + (-0.90631d\cos\phi + 3.6252\sin\phi)\mathbf{j}$$
$$+ (1.69048 - 9.9694\cos\phi)\mathbf{k}]\text{ in.}$$

and

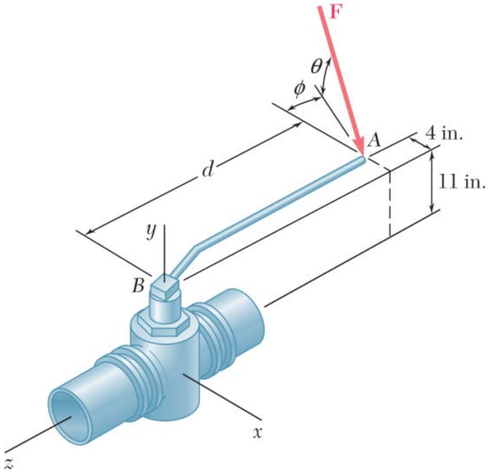
$$M_x = (70\text{ lb})(9.9694\sin\phi - 0.42262d)\text{ in.} = -(61\text{ lb}\cdot\text{ft})(12\text{ in./ft}) \qquad (1)$$

$$M_y = (70\text{ lb})(-0.90631d\cos\phi + 3.6252\sin\phi)\text{ in.} \qquad (2)$$

$$M_z = (70\text{ lb})(1.69048 - 9.9694\cos\phi)\text{ in.} = -43\text{ lb}\cdot\text{ft}(12\text{ in./ft}) \qquad (3)$$

From Equation (3): $\phi = \cos^{-1}\left(\dfrac{634.33}{697.86}\right) = 24.636°$ or $\phi = 24.6°$ ◀

From Equation (1): $d = \left(\dfrac{1022.90}{29.583}\right) = 34.577\text{ in.}$ or $d = 34.6\text{ in.}$ ◀



PROBLEM 3.50

When a force **F** is applied to the handle of the valve shown, its moments about the x and z axes are, respectively, $M_x = -77\ \text{lb}\cdot\text{ft}$ and $M_z = -81\ \text{lb}\cdot\text{ft}$. For $d = 27$ in., determine the moment M_y of **F** about the y axis.

SOLUTION

We have $$\Sigma \mathbf{M}_O:\quad \mathbf{r}_{A/O}\times\mathbf{F} = \mathbf{M}_O$$

Where $$\mathbf{r}_{A/O} = -(4\text{ in.})\mathbf{i} + (11\text{ in.})\mathbf{j} - (27\text{ in.})\mathbf{k}$$

$$\mathbf{F} = F(\cos\theta\cos\phi\mathbf{i} - \sin\theta\mathbf{j} + \cos\theta\sin\phi\mathbf{k})$$

$$\mathbf{M}_O = F\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 11 & -27 \\ \cos\theta\cos\phi & -\sin\theta & \cos\theta\sin\phi \end{vmatrix}\ \text{lb}\cdot\text{in.}$$

$$\begin{aligned}&= F[(11\cos\theta\sin\phi - 27\sin\theta)\mathbf{i}\\ &\quad + (-27\cos\theta\cos\phi + 4\cos\theta\sin\phi)\mathbf{j}\\ &\quad + (4\sin\theta - 11\cos\theta\cos\phi)\mathbf{k}](\text{lb}\cdot\text{in.})\end{aligned}$$

and $$M_x = F(11\cos\theta\sin\phi - 27\sin\theta)(\text{lb}\cdot\text{in.}) \qquad (1)$$

$$M_y = F(-27\cos\theta\cos\phi + 4\cos\theta\sin\phi)\ (\text{lb}\cdot\text{in.}) \qquad (2)$$

$$M_z = F(4\sin\theta - 11\cos\theta\cos\phi)\ (\text{lb}\cdot\text{in.}) \qquad (3)$$

Now, Equation (1) $$\cos\theta\sin\phi = \frac{1}{11}\left(\frac{M_x}{F} + 27\sin\theta\right) \qquad (4)$$

and Equation (3) $$\cos\theta\cos\phi = \frac{1}{11}\left(4\sin\theta - \frac{M_z}{F}\right) \qquad (5)$$

Substituting Equations (4) and (5) into Equation (2),

$$M_y = F\left\{-27\left[\frac{1}{11}\left(4\sin\theta - \frac{M_z}{F}\right)\right] + 4\left[\frac{1}{11}\left(\frac{M_x}{F} + 27\sin\theta\right)\right]\right\}$$

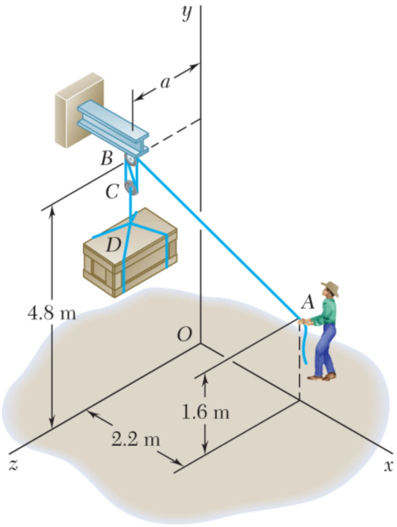
or $$M_y = \frac{1}{11}(27M_z + 4M_x)$$

PROBLEM 3.50 (Continued)

Noting that the ratios $\frac{27}{11}$ and $\frac{4}{11}$ are the ratios of lengths, have

$$M_y = \frac{27}{11}(-81\text{ lb}\cdot\text{ft}) + \frac{4}{11}(-77\text{ lb}\cdot\text{ft})$$
$$= 226.82\text{ lb}\cdot\text{ft}$$

or $M_y = -227\text{ lb}\cdot\text{ft}$ ◀



PROBLEM 3.51

To lift a heavy crate, a man uses a block and tackle attached to the bottom of an I-beam at hook *B*. Knowing that the moments about the *y* and the *z* axes of the force exerted at *B* by portion *AB* of the rope are, respectively, 120 N · m and –460 N · m, determine the distance *a*.

SOLUTION

First note

$$\overrightarrow{BA} = (2.2\text{ m})\mathbf{i} - (3.2\text{ m})\mathbf{j} - (a\text{ m})\mathbf{k}$$

Now

$$\mathbf{M}_D = \mathbf{r}_{A/D} \times \mathbf{T}_{BA}$$

where

$$\mathbf{r}_{A/D} = (2.2\text{ m})\mathbf{i} + (1.6\text{ m})\mathbf{j}$$

$$\mathbf{T}_{BA} = \frac{T_{BA}}{d_{BA}}(2.2\mathbf{i} - 3.2\mathbf{j} - a\mathbf{k})\ (\text{N})$$

Then

$$\mathbf{M}_D = \frac{T_{BA}}{d_{BA}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2.2 & 1.6 & 0 \\ 2.2 & -3.2 & -a \end{vmatrix}$$

$$= \frac{T_{BA}}{d_{BA}}\{-1.6a\,\mathbf{i} + 2.2a\,\mathbf{j} + [(2.2)(-3.2) - (1.6)(2.2)]\mathbf{k}\}$$

Thus

$$M_y = 2.2\frac{T_{BA}}{d_{BA}}a \qquad (\text{N}\cdot\text{m})$$

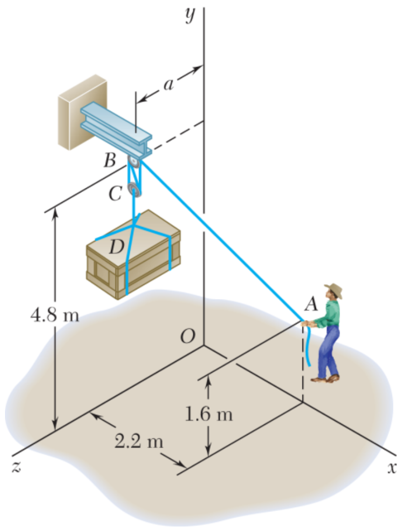
$$M_z = -10.56\frac{T_{BA}}{d_{BA}} \qquad (\text{N}\cdot\text{m})$$

Then forming the ratio

$$\frac{M_y}{M_z}$$

$$\frac{120\text{ N}\cdot\text{m}}{-460\text{ N}\cdot\text{m}} = \frac{2.2\frac{T_{BA}}{d_{BA}}\ (\text{N}\cdot\text{m})}{-10.56\frac{T_{BA}}{d_{BA}}\ (\text{N}\cdot\text{m})}$$

or $a = 1.252$ m ◀



PROBLEM 3.52

To lift a heavy crate, a man uses a block and tackle attached to the bottom of an I-beam at hook *B*. Knowing that the man applies a 195-N force to end *A* of the rope and that the moment of that force about the *y* axis is 132 N · m, determine the distance *a*.

SOLUTION

First note

$$d_{BA} = \sqrt{(2.2)^2 + (-3.2)^2 + (-a)^2}$$

$$= \sqrt{15.08 + a^2} \text{ m}$$

and

$$\mathbf{T}_{BA} = \frac{195 \text{ N}}{d_{BA}}(2.2\mathbf{i} - 3.2\mathbf{j} - a\,\mathbf{k})$$

Now

$$M_y = \mathbf{j} \cdot (\mathbf{r}_{A/D} \times \mathbf{T}_{BA})$$

where

$$\mathbf{r}_{A/0} = (2.2 \text{ m})\mathbf{i} + (1.6 \text{ m})\mathbf{j}$$

Then

$$M_y = \frac{195}{d_{BA}}\begin{vmatrix} 0 & 1 & 0 \\ 2.2 & 1.6 & 0 \\ 2.2 & -3.2 & -a \end{vmatrix}$$

$$= \frac{195}{d_{BA}}(2.2a) \text{ (N} \cdot \text{m)}$$

Substituting for M_y and d_{BA}

$$132 \text{ N} \cdot \text{m} = \frac{195}{\sqrt{15.08 + a^2}}(2.2a)$$

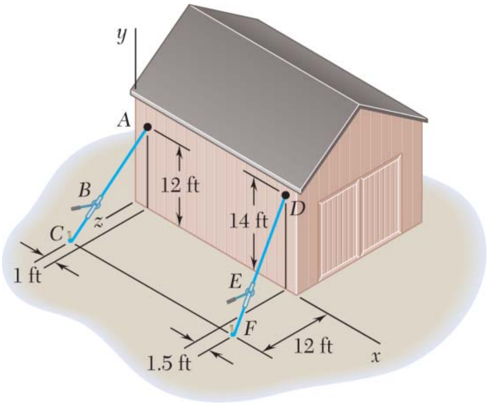
or

$$0.30769\sqrt{15.08 + a^2} = a$$

Squaring both sides of the equation

$$0.094675(15.08 + a^2) = a^2$$

or $a = 1.256$ m ◀

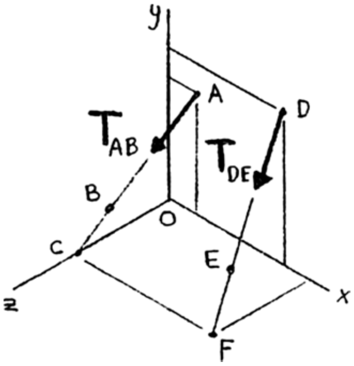


PROBLEM 3.53

A farmer uses cables and winch pullers B and E to plumb one side of a small barn. If it is known that the sum of the moments about the x-axis of the forces exerted by the cables on the barn at Points A and D is equal to 4728 lb · ft, determine the magnitude of $\mathbf{T}_{DE}$ when $T_{AB} = 255$ lb.

SOLUTION

The moment about the x-axis due to the two cable forces can be found using the z components of each force acting at their intersection with the xy plane (A and D). The x components of the forces are parallel to the x-axis, and the y components of the forces intersect the x-axis. Therefore, neither the x or y components produce a moment about the x-axis.



We have ΣM_x: $(T_{AB})_z(y_A) + (T_{DE})_z(y_D) = M_x$

where

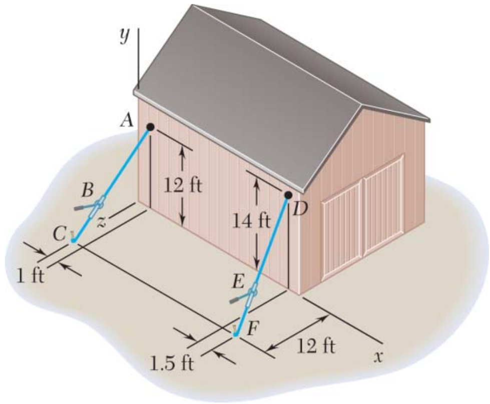
$$(T_{AB})_z = \mathbf{k}\cdot\mathbf{T}_{AB}$$
$$= \mathbf{k}\cdot(T_{AB}\lambda_{AB})$$
$$= \mathbf{k}\cdot\left[255\text{ lb}\left(\frac{-\mathbf{i}-12\mathbf{j}+12\mathbf{k}}{17}\right)\right]$$
$$= 180\text{ lb}$$

$$(T_{DE})_z = \mathbf{k}\cdot\mathbf{T}_{DE}$$
$$= \mathbf{k}\cdot(T_{DE}\lambda_{DE})$$
$$= \mathbf{k}\cdot\left[T_{DE}\left(\frac{1.5\mathbf{i}-14\mathbf{j}+12\mathbf{k}}{18.5}\right)\right]$$
$$= 0.64865T_{DE}$$

$$y_A = 12\text{ ft}$$
$$y_D = 14\text{ ft}$$
$$M_x = 4728\text{ lb}\cdot\text{ft}$$

$$(180\text{ lb})(12\text{ ft}) + (0.64865T_{DE})(14\text{ ft}) = 4728\text{ lb}\cdot\text{ft}$$

and $T_{DE} = 282.79$ lb or $T_{DE} = 283$ lb ◀



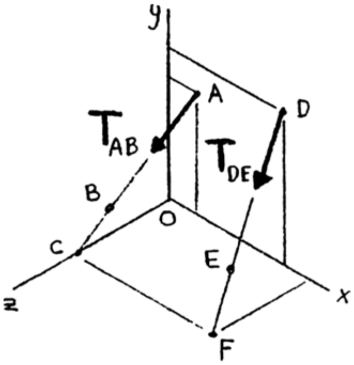
PROBLEM 3.54

Solve Problem 3.53 when the tension in cable *AB* is 306 lb.

PROBLEM 3.53 A farmer uses cables and winch pullers *B* and *E* to plumb one side of a small barn. If it is known that the sum of the moments about the *x*-axis of the forces exerted by the cables on the barn at Points *A* and *D* is equal to 4728 lb · ft, determine the magnitude of $\mathbf{T}_{DE}$ when $T_{AB} = 255$ lb.

SOLUTION

The moment about the *x*-axis due to the two cable forces can be found using the *z* components of each force acting at the intersection with the *xy* plane (*A* and *D*). The *x* components of the forces are parallel to the *x*-axis, and the *y* components of the forces intersect the *x*-axis. Therefore, neither the *x* or *y* components produce a moment about the *x*-axis.



We have ΣM_x: $(T_{AB})_z(y_A) + (T_{DE})_z(y_D) = M_x$

Where

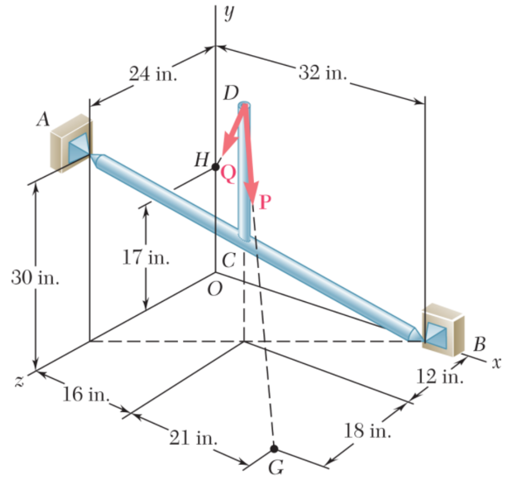
$$\begin{aligned}(T_{AB})_z &= \mathbf{k}\cdot\mathbf{T}_{AB}\\ &= \mathbf{k}\cdot(T_{AB}\boldsymbol{\lambda}_{AB})\\ &= \mathbf{k}\cdot\left[306\text{ lb}\left(\frac{-\mathbf{i}-12\mathbf{j}+12\mathbf{k}}{17}\right)\right]\\ &= 216\text{ lb}\end{aligned}$$

$$\begin{aligned}(T_{DE})_z &= \mathbf{k}\cdot\mathbf{T}_{DE}\\ &= \mathbf{k}\cdot(T_{DE}\boldsymbol{\lambda}_{DE})\\ &= \mathbf{k}\cdot\left[T_{DE}\left(\frac{1.5\mathbf{i}-14\mathbf{j}+12\mathbf{k}}{18.5}\right)\right]\\ &= 0.64865T_{DE}\end{aligned}$$

$$y_A = 12\text{ ft}$$
$$y_D = 14\text{ ft}$$
$$M_x = 4728\text{ lb}\cdot\text{ft}$$

$$(216\text{ lb})(12\text{ ft}) + (0.64865T_{DE})(14\text{ ft}) = 4728\text{ lb}\cdot\text{ft}$$

and $T_{DE} = 235.21$ lb or $T_{DE} = 235$ lb ◀



PROBLEM 3.55

The 23-in. vertical rod *CD* is welded to the midpoint *C* of the 50-in. rod *AB*. Determine the moment about *AB* of the 235-lb force **P**.

SOLUTION

$$\overline{AB} = (32\text{ in.})\mathbf{i} - (30\text{ in.})\mathbf{j} - (24\text{ in.})\mathbf{k}$$

$$AB = \sqrt{(32)^2 + (-30)^2 + (-24)^2} = 50\text{ in.}$$

$$\boldsymbol{\lambda}_{AB} = \frac{\overline{AB}}{AB} = 0.64\mathbf{i} - 0.60 - 0.48\mathbf{k}$$

We shall apply the force *P* at Point *G*:

$$\mathbf{r}_{G/B} = (5\text{ in.})\mathbf{i} + (30\text{ in.})\mathbf{k}$$

$$\overline{DG} = (21\text{ in.})\mathbf{i} - (38\text{ in.})\mathbf{j} + (18\text{ in.})\mathbf{k}$$

$$DG = \sqrt{(21)^2 + (-38)^2 + (18)^2} = 47\text{ in.}$$

$$\mathbf{P} = P\frac{\overline{DG}}{DG} = (235\text{ lb})\frac{21\mathbf{i} - 38\mathbf{j} + 18\mathbf{k}}{47}$$

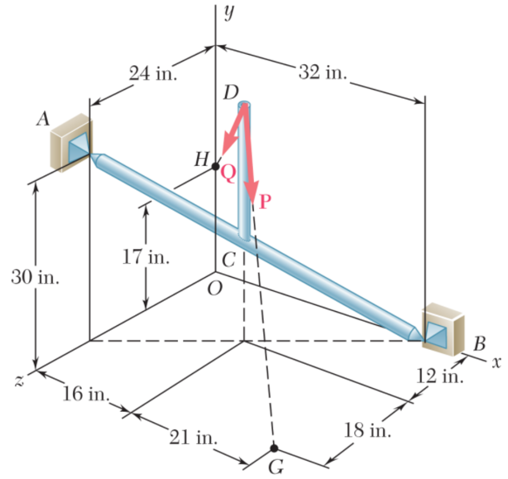
$$\mathbf{P} = (105\text{ lb})\mathbf{i} - (190\text{ lb})\mathbf{j} + (90\text{ lb})\mathbf{k}$$

The moment of **P** about *AB* is given by Eq. (3.46):

$$\mathbf{M}_{AB} = \boldsymbol{\lambda}_{AB} \cdot (\mathbf{r}_{G/B} \times P) = \begin{vmatrix} 0.64 & -0.60 & -0.48 \\ 5\text{ in.} & 0 & 30\text{ in.} \\ 105\text{ lb} & -190\text{ lb} & 90\text{ lb} \end{vmatrix}$$

$$\begin{aligned} \mathbf{M}_{AB} &= 0.64[0 - (30\text{ in.})(-190\text{ lb})] \\ &\quad - 0.60[(30\text{ in.})(105\text{ lb}) - (5\text{ in.})(90\text{ lb})] \\ &\quad - 0.48[(5\text{ in.})(-190\text{ lb}) - 0] \\ &= +2484\text{ lb}\cdot\text{in.} \end{aligned}$$

$$\mathbf{M}_{AB} = +207\text{ lb}\cdot\text{ft}$$ ◀



PROBLEM 3.56

The 23-in. vertical rod *CD* is welded to the midpoint *C* of the 50-in. rod *AB*. Determine the moment about *AB* of the 174-lb force **Q**.

SOLUTION

$$\overline{AB} = (32\text{ in.})\mathbf{i} - (30\text{ in.})\mathbf{j} - (24\text{ in.})\mathbf{k}$$

$$AB = \sqrt{(32)^2 + (-30)^2 + (-24)^2} = 50\text{ in.}$$

$$\boldsymbol{\lambda}_{AB} = \frac{\overline{AB}}{AB} = 0.64\mathbf{i} - 0.60\mathbf{j} - 0.48\mathbf{k}$$

We shall apply the force **Q** at Point *H*:

$$\mathbf{r}_{H/B} = -(32\text{ in.})\mathbf{i} + (17\text{ in.})\mathbf{j}$$

$$\overline{DH} = -(16\text{ in.})\mathbf{i} - (21\text{ in.})\mathbf{j} - (12\text{ in.})\mathbf{k}$$

$$DH = \sqrt{(16)^2 + (-21)^2 + (-12)^2} = 29\text{ in.}$$

$$\mathbf{Q} = \frac{\overline{DH}}{DH} = (174\text{ lb})\frac{-16\mathbf{i} - 21\mathbf{j} - 12\mathbf{k}}{29}$$

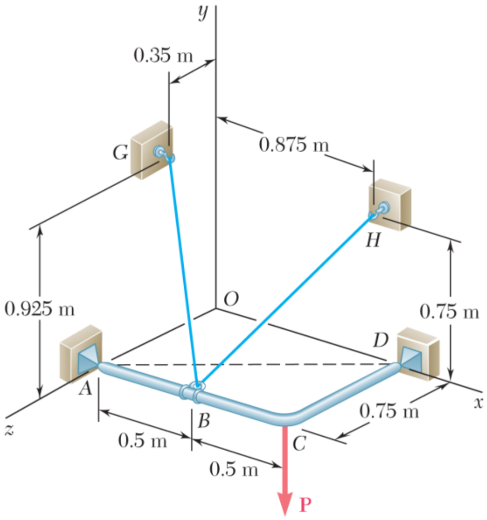
$$Q = -(96\text{ lb})\mathbf{i} - (126\text{ lb})\mathbf{j} - (72\text{ lb})\mathbf{k}$$

The moment of **Q** about *AB* is given by Eq. (3.46):

$$\mathbf{M}_{AB} = \boldsymbol{\lambda}_{AB} \cdot (\mathbf{r}_{H/B} \times \mathbf{Q}) = \begin{vmatrix} 0.64 & -0.60 & -0.48 \\ -32\text{ in.} & 17\text{ in.} & 0 \\ -96\text{ lb} & -126\text{ lb} & -72\text{ lb} \end{vmatrix}$$

$$\begin{aligned}\mathbf{M}_{AB} &= 0.64[(17\text{ in.})(-72\text{ lb}) - 0] \\ &\quad - 0.60[(0 - (-32\text{ in.})(-72\text{ lb})] \\ &\quad - 0.48[(-32\text{ in.})(-126\text{ lb}) - (17\text{ in.})(-96\text{ lb})] \\ &= -2119.7\text{ lb}\cdot\text{in.}\end{aligned}$$

$\mathbf{M}_{AB} = -176.6\text{ lb}\cdot\text{ft}$ ◀



PROBLEM 3.57

The frame *ACD* is hinged at *A* and *D* and is supported by a cable that passes through a ring at *B* and is attached to hooks at *G* and *H*. Knowing that the tension in the cable is 450 N, determine the moment about the diagonal *AD* of the force exerted on the frame by portion *BH* of the cable.

SOLUTION

$$M_{AD} = \boldsymbol{\lambda}_{AD} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}_{BH})$$

Where

$$\boldsymbol{\lambda}_{AD} = \frac{1}{5}(4\mathbf{i} - 3\mathbf{k})$$

$$\mathbf{r}_{B/A} = (0.5 \text{ m})\mathbf{i}$$

and

$$d_{BH} = \sqrt{(0.375)^2 + (0.75)^2 + (-0.75)^2}$$

$$= 1.125 \text{ m}$$

Then

$$\mathbf{T}_{BH} = \frac{450 \text{ N}}{1.125}(0.375\mathbf{i} + 0.75\mathbf{j} - 0.75\mathbf{k})$$

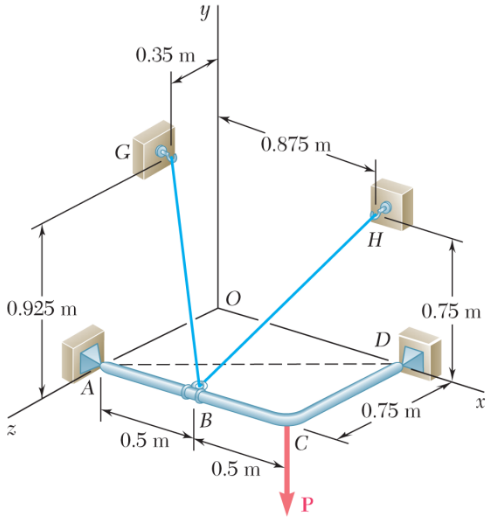
$$= (150 \text{ N})\mathbf{i} + (300 \text{ N})\mathbf{j} - (300 \text{ N})\mathbf{k}$$

Finally,

$$M_{AD} = \frac{1}{5}\begin{vmatrix} 4 & 0 & -3 \\ 0.5 & 0 & 0 \\ 150 & 300 & -300 \end{vmatrix}$$

$$= \frac{1}{5}[(-3)(0.5)(300)]$$

or $M_{AD} = -90.0 \text{ N}\cdot\text{m}$ ◀



PROBLEM 3.58

In Problem 3.57, determine the moment about the diagonal *AD* of the force exerted on the frame by portion *BG* of the cable.

PROBLEM 3.57 The frame *ACD* is hinged at *A* and *D* and is supported by a cable that passes through a ring at *B* and is attached to hooks at *G* and *H*. Knowing that the tension in the cable is 450 N, determine the moment about the diagonal *AD* of the force exerted on the frame by portion *BH* of the cable.

SOLUTION

$$M_{AD} = \boldsymbol{\lambda}_{AD} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}_{BG})$$

Where

$$\boldsymbol{\lambda}_{AD} = \frac{1}{5}(4\mathbf{i} - 3\mathbf{k})$$

$$\mathbf{r}_{B/A} = (0.5\text{ m})\mathbf{j}$$

and

$$BG = \sqrt{(-0.5)^2 + (0.925)^2 + (-0.4)^2}$$

$$= 1.125\text{ m}$$

Then

$$\mathbf{T}_{BG} = \frac{450\text{ N}}{1.125}(-0.5\mathbf{i} + 0.925\mathbf{j} - 0.4\mathbf{k})$$

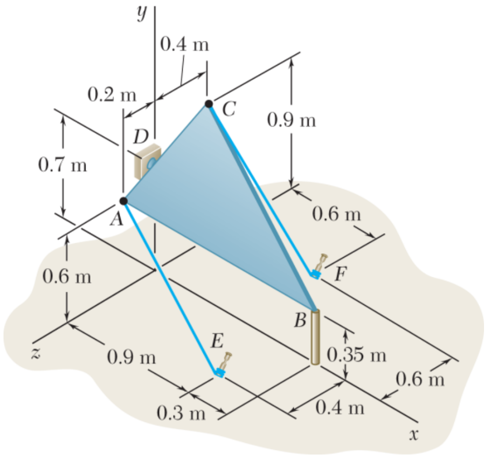
$$= -(200\text{ N})\mathbf{i} + (370\text{ N})\mathbf{j} - (160\text{ N})\mathbf{k}$$

Finally,

$$M_{AD} = \frac{1}{5}\begin{vmatrix} 4 & 0 & -3 \\ 0.5 & 0 & 0 \\ -200 & 370 & -160 \end{vmatrix}$$

$$= \frac{1}{5}[(-3)(0.5)(370)]$$

$M_{AD} = -111.0\text{ N}\cdot\text{m}$ ◀



PROBLEM 3.59

The triangular plate *ABC* is supported by ball-and-socket joints at *B* and *D* and is held in the position shown by cables *AE* and *CF*. If the force exerted by cable *AE* at *A* is 55 N, determine the moment of that force about the line joining Points *D* and *B*.

SOLUTION

First note:

$$\mathbf{T}_{AE} = T_{AE}\frac{\overrightarrow{AE}}{AE}$$

$$AE = \sqrt{(0.9)^2 + (-0.6)^2 + (0.2)^2} = 1.1 \text{ m}$$

Then

$$\mathbf{T}_{AE} = \frac{55 \text{ N}}{1.1}(0.9\mathbf{i} - 0.6\mathbf{j} + 0.2\mathbf{k})$$

$$= 5[(9 \text{ N})\mathbf{i} - (6 \text{ N})\mathbf{j} + (2 \text{ N})\mathbf{k}]$$

Also,

$$DB = \sqrt{(1.2)^2 + (-0.35)^2 + (0)^2}$$

$$= 1.25 \text{ m}$$

Then

$$\boldsymbol{\lambda}_{DB} = \frac{\overrightarrow{DB}}{DB}$$

$$= \frac{1}{1.25}(1.2\mathbf{i} - 0.35\mathbf{j})$$

$$= \frac{1}{25}(24\mathbf{i} - 7\mathbf{j})$$

Now

$$M_{DB} = \boldsymbol{\lambda}_{DB} \cdot (\mathbf{r}_{A/D} \times \mathbf{T}_{AE})$$

where

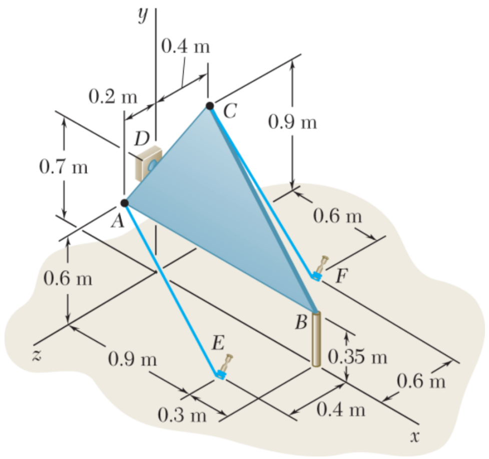
$$\mathbf{r}_{A/D} = -(0.1 \text{ m})\mathbf{j} + (0.2 \text{ m})\mathbf{k}$$

Then

$$M_{DB} = \frac{1}{25}(5)\begin{vmatrix} 24 & -7 & 0 \\ 0 & -0.1 & 0.2 \\ 9 & -6 & 2 \end{vmatrix}$$

$$= \frac{1}{5}(-4.8 - 12.6 + 28.8)$$

or $M_{DB} = 2.28 \text{ N}\cdot\text{m}$ ◀



PROBLEM 3.60

The triangular plate *ABC* is supported by ball-and-socket joints at *B* and *D* and is held in the position shown by cables *AE* and *CF*. If the force exerted by cable *CF* at *C* is 33 N, determine the moment of that force about the line joining Points *D* and *B*.

SOLUTION

First note:

$$\mathbf{T}_{CF} = T_{CF}\frac{\overrightarrow{CF}}{CF}$$

$$CF = \sqrt{(0.6)^2 + (-0.9)^2 + (-0.2)^2} = 1.1 \text{ m}$$

Then

$$\mathbf{T}_{CF} = \frac{33 \text{ N}}{1.1}(0.6\mathbf{i} - 0.9\mathbf{j} + 0.2\mathbf{k})$$

$$= 3[(6 \text{ N})\mathbf{i} - (9 \text{ N})\mathbf{j} - (2 \text{ N})\mathbf{k}]$$

Also,

$$DB = \sqrt{(1.2)^2 + (-0.35)^2 + (0)^2}$$

$$= 1.25 \text{ m}$$

Then

$$\boldsymbol{\lambda}_{DB} = \frac{\overrightarrow{DB}}{DB}$$

$$= \frac{1}{1.25}(1.2\mathbf{i} - 0.35\mathbf{j})$$

$$= \frac{1}{25}(24\mathbf{i} - 7\mathbf{j})$$

Now

$$M_{DB} = \boldsymbol{\lambda}_{DB} \cdot (\mathbf{r}_{C/D} \times \mathbf{T}_{CF})$$

where

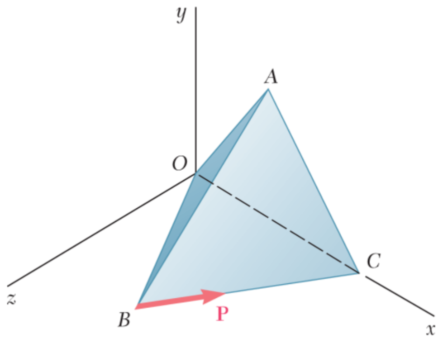
$$\mathbf{r}_{C/D} = (0.2 \text{ m})\mathbf{j} - (0.4 \text{ m})\mathbf{k}$$

Then

$$M_{DB} = \frac{1}{25}(3)\begin{vmatrix} 24 & -7 & 0 \\ 0 & 0.2 & -0.4 \\ 6 & -9 & -2 \end{vmatrix}$$

$$= \frac{3}{25}(-9.6 + 16.8 - 86.4)$$

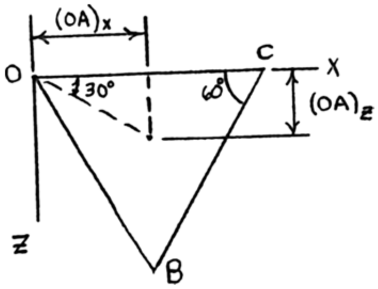
or $M_{DB} = -9.50 \text{ N} \cdot \text{m}$ ◀



PROBLEM 3.61

A regular tetrahedron has six edges of length a. A force **P** is directed as shown along edge BC. Determine the moment of **P** about edge OA.

SOLUTION



We have $$M_{OA} = \boldsymbol{\lambda}_{OA} \cdot (\mathbf{r}_{C/O} \times \mathbf{P})$$

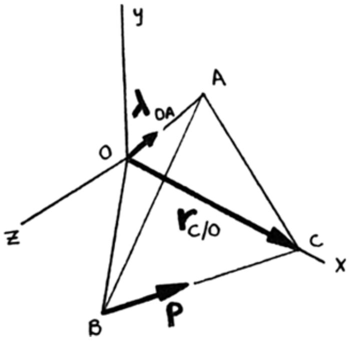
From triangle *OBC*: $$(OA)_x = \frac{a}{2}$$

$$(OA)_z = (OA)_x \tan 30° = \frac{a}{2}\left(\frac{1}{\sqrt{3}}\right) = \frac{a}{2\sqrt{3}}$$

Since $$(OA)^2 = (OA)_x^2 + (OA)_y^2 + (OA_z)^2$$

or $$a^2 = \left(\frac{a}{2}\right)^2 + (OA)_y^2 + \left(\frac{a}{2\sqrt{3}}\right)^2$$

$$(OA)_y = \sqrt{a^2 - \frac{a^2}{4} - \frac{a^2}{12}} = a\sqrt{\frac{2}{3}}$$



Then $$\mathbf{r}_{A/O} = \frac{a}{2}\mathbf{i} + a\sqrt{\frac{2}{3}}\,\mathbf{j} + \frac{a}{2\sqrt{3}}\mathbf{k}$$

and $$\boldsymbol{\lambda}_{OA} = \frac{1}{2}\mathbf{i} + \sqrt{\frac{2}{3}}\mathbf{j} + \frac{1}{2\sqrt{3}}\mathbf{k}$$

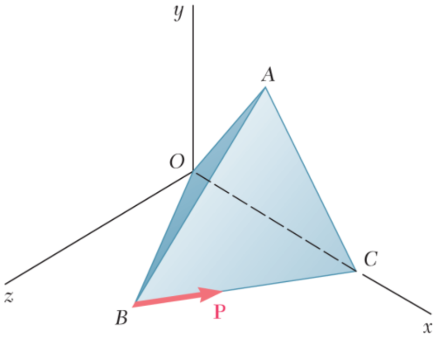
$$\mathbf{P} = \boldsymbol{\lambda}_{BC} P = \frac{(a \sin 30°)\mathbf{i} - (a \cos 30°)\mathbf{k}}{a}(P) = \frac{P}{2}(\mathbf{i} - \sqrt{3}\mathbf{k})$$

$$\mathbf{r}_{C/O} = a\mathbf{i}$$

$$M_{OA} = \begin{vmatrix} \frac{1}{2} & \sqrt{\frac{2}{3}} & \frac{1}{2\sqrt{3}} \\ 1 & 0 & 0 \\ 1 & 0 & -\sqrt{3} \end{vmatrix} (a)\left(\frac{P}{2}\right)$$

$$= \frac{aP}{2}\left(-\sqrt{\frac{2}{3}}\right)(1)(-\sqrt{3}) = \frac{aP}{\sqrt{2}}$$

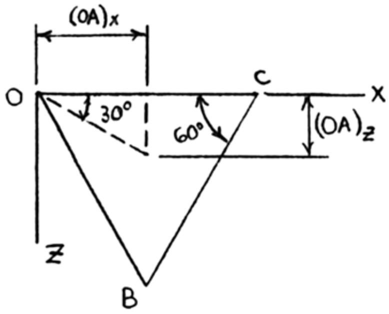
$$M_{OA} = \frac{aP}{\sqrt{2}} \blacktriangleleft$$



PROBLEM 3.62

A regular tetrahedron has six edges of length a. (a) Show that two opposite edges, such as OA and BC, are perpendicular to each other. (b) Use this property and the result obtained in Problem 3.61 to determine the perpendicular distance between edges OA and BC.

SOLUTION



(a) For edge OA to be perpendicular to edge BC,

$$\overrightarrow{OA} \cdot \overrightarrow{BC} = 0$$

From triangle OBC:

$$(OA)_x = \frac{a}{2}$$

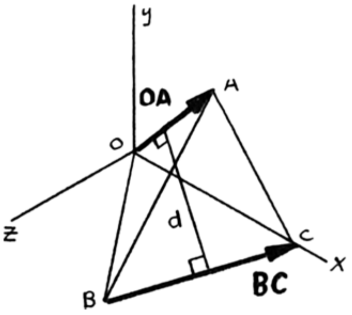
$$(OA)_z = (OA)_x \tan 30° = \frac{a}{2}\left(\frac{1}{\sqrt{3}}\right) = \frac{a}{2\sqrt{3}}$$

$$\overrightarrow{OA} = \left(\frac{a}{2}\right)\mathbf{i} + (OA)_y\,\mathbf{j} + \left(\frac{a}{2\sqrt{3}}\right)\mathbf{k}$$

and

$$\overrightarrow{BC} = (a\sin 30°)\mathbf{i} - (a\cos 30°)\mathbf{k}$$

$$= \frac{a}{2}\mathbf{i} - \frac{a\sqrt{3}}{2}\mathbf{k} = \frac{a}{2}(\mathbf{i} - \sqrt{3}\,\mathbf{k})$$



Then

$$\left[\frac{a}{2}\mathbf{i} + (OA)_y\,\mathbf{j} + \left(\frac{a}{2\sqrt{3}}\right)\mathbf{k}\right] \cdot (\mathbf{i} - \sqrt{3}\mathbf{k})\frac{a}{2} = 0$$

or

$$\frac{a^2}{4} + (OA)_y(0) - \frac{a^2}{4} = 0$$

$$\overrightarrow{OA} \cdot \overrightarrow{BC} = 0$$

so that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{BC}$. ◀

(b) We have $M_{OA} = Pd$, with P acting along BC and d the perpendicular distance from $\overrightarrow{OA}$ to $\overrightarrow{BC}$.

From the results of Problem 3.57,

$$M_{OA} = \frac{Pa}{\sqrt{2}}$$

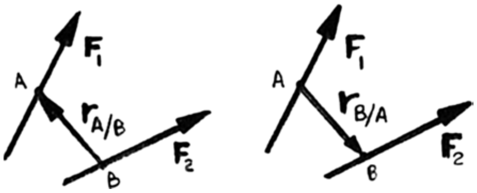
$$\frac{Pa}{\sqrt{2}} = Pd$$

or $d = \dfrac{a}{\sqrt{2}}$ ◀

PROBLEM 3.63

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in space have the same magnitude F. Prove that the moment of $\mathbf{F}_1$ about the line of action of $\mathbf{F}_2$ is equal to the moment of $\mathbf{F}_2$ about the line of action of $\mathbf{F}_1$.

SOLUTION



First note that $\mathbf{F}_1 = F_1\boldsymbol{\lambda}_1$ and $\mathbf{F}_2 = F_2\boldsymbol{\lambda}_2$

Let $M_1 =$ moment of $\mathbf{F}_2$ about the line of action of $\mathbf{F}_1$ and $M_2 =$ moment of $\mathbf{F}_1$ about the line of action of $\mathbf{F}_2$.

Now, by definition,

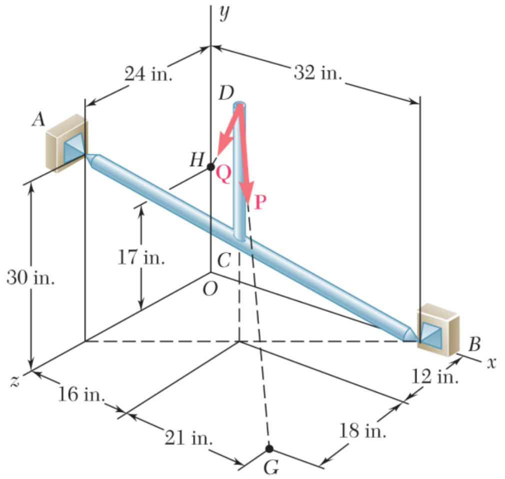
$$M_1 = \boldsymbol{\lambda}_1 \cdot (\mathbf{r}_{B/A} \times \mathbf{F}_2)$$
$$= \boldsymbol{\lambda}_1 \cdot (\mathbf{r}_{B/A} \times \boldsymbol{\lambda}_2)F_2$$
$$M_2 = \boldsymbol{\lambda}_2 \cdot (\mathbf{r}_{A/B} \times \mathbf{F}_1)$$
$$= \boldsymbol{\lambda}_2 \cdot (\mathbf{r}_{A/B} \times \boldsymbol{\lambda}_1)F_1$$

Since $F_1 = F_2 = F$ and $\mathbf{r}_{A/B} = -\mathbf{r}_{B/A}$

$$M_1 = \boldsymbol{\lambda}_1 \cdot (\mathbf{r}_{B/A} \times \boldsymbol{\lambda}_2)F$$
$$M_2 = \boldsymbol{\lambda}_2 \cdot (-\mathbf{r}_{B/A} \times \boldsymbol{\lambda}_1)F$$

Using Equation (3.39): $\boldsymbol{\lambda}_1 \cdot (\mathbf{r}_{B/A} \times \boldsymbol{\lambda}_2) = \boldsymbol{\lambda}_2 \cdot (-\mathbf{r}_{B/A} \times \boldsymbol{\lambda}_1)$

so that $M_2 = \boldsymbol{\lambda}_1 \cdot (\mathbf{r}_{B/A} \times \boldsymbol{\lambda}_2)F$ $\qquad M_{12} = M_{21}$ ◀



PROBLEM 3.64

In Prob. 3.55, determine the perpendicular distance between rod *AB* and the line of action of **P**.

PROBLEM 3.55 The 23-in. vertical rod *CD* is welded to the midpoint *C* of the 50-in. rod *AB*. Determine the moment about *AB* of the 235-lb force **P**.

SOLUTION

$$\overline{AB} = (32\text{ in.})\mathbf{i} - (30\text{ in.})\mathbf{j} - (24\text{ in.})\mathbf{k}$$

$$AB = \sqrt{(32)^2 + (-30)^2 + (-24)^2} = 50\text{ in.}$$

$$\boldsymbol{\lambda}_{AB} = \frac{\overline{AB}}{AB} = 0.64\mathbf{i} - 0.60\mathbf{j} - 0.48\mathbf{k}$$

$$\boldsymbol{\lambda}_P = \frac{\mathbf{P}}{P} = \frac{105\mathbf{i} - 190\mathbf{j} + 90\mathbf{k}}{235}$$

Angle θ between *AB* and **P**:

$$\begin{aligned}\cos\theta &= \boldsymbol{\lambda}_{AB} \cdot \boldsymbol{\lambda}_P \\ &= (0.64\mathbf{i} - 0.60\mathbf{j} - 0.48\mathbf{k}) \cdot \frac{105\mathbf{i} - 190\mathbf{j} + 90\mathbf{k}}{235} \\ &= 0.58723\end{aligned}$$

$$\therefore \quad \theta = 54.039°$$

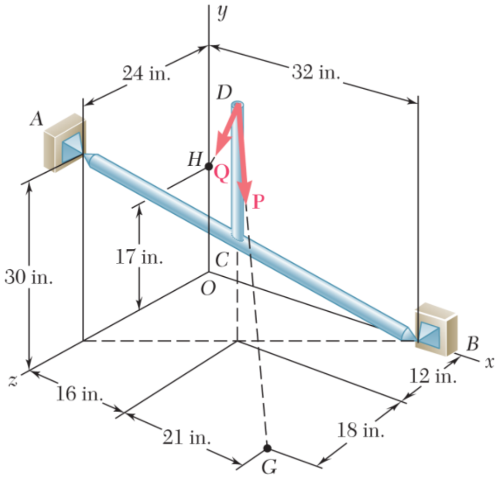
The moment of **P** about *AB* may be obtained by multiplying the projection of **P** on a plane perpendicular to *AB* by the perpendicular distance *d* from *AB* to **P**:

$$\mathbf{M}_{AB} = (P\sin\theta)d$$

From the solution to Prob. 3.55: $\mathbf{M}_{AB} = 207\text{ lb}\cdot\text{ft} = 2484\text{ lb}\cdot\text{in.}$

We have $\quad 2484\text{ lb}\cdot\text{in.} = (235\text{ lb})(\sin 54.039)d$

$$d = 13.06\text{ in.} \blacktriangleleft$$



PROBLEM 3.65

In Prob. 3.56, determine the perpendicular distance between rod *AB* and the line of action of **Q**.

PROBLEM 3.56 The 23-in. vertical rod *CD* is welded to the midpoint *C* of the 50-in. rod *AB*. Determine the moment about *AB* of the 174-lb force **Q**.

SOLUTION

$$\overline{AB} = (32\text{ in.})\mathbf{i} - (30\text{ in.})\mathbf{j} - (24\text{ in.})\mathbf{k}$$

$$AB = \sqrt{(32)^2 + (-30)^2 + (-24)^2} = 50\text{ in.}$$

$$\boldsymbol{\lambda}_{AB} = \frac{\overline{AB}}{AB} = 0.64\mathbf{i} - 0.60\mathbf{j} - 0.48\mathbf{k}$$

$$\boldsymbol{\lambda}_{Q} = \frac{\mathbf{Q}}{Q} = \frac{-96\mathbf{i} - 126\mathbf{j} - 72\mathbf{k}}{174}$$

Angle θ between *AB* and **Q**:

$$\begin{aligned}\cos\theta &= \boldsymbol{\lambda}_{AB} \cdot \boldsymbol{\lambda}_{Q} \\ &= (0.64\mathbf{i} - 0.60\mathbf{j} - 0.48\mathbf{k}) \cdot \frac{(-96\mathbf{i} - 126\mathbf{j} - 72\mathbf{k})}{174} \\ &= 0.28000 \\ \therefore\quad \theta &= 73.740^\circ\end{aligned}$$

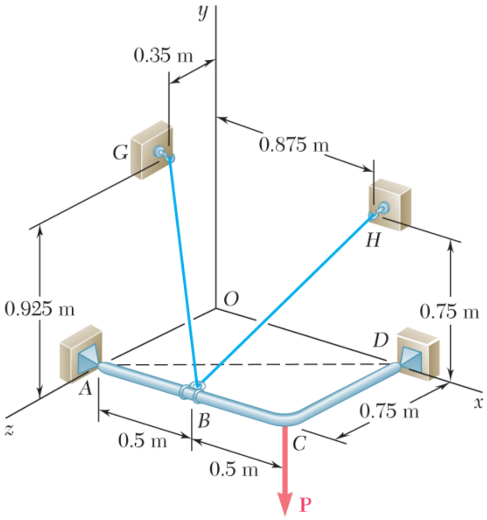
The moment of **Q** about *AB* may be obtained by multiplying the projection of **Q** on a plane perpendicular to *AB* by the perpendicular distance *d* from *AB* to **Q**:

$$\mathbf{M}_{AB} = (Q\sin\theta)d$$

From the solution to Prob. 3.56: $\mathbf{M}_{AB} = 176.6\text{ lb}\cdot\text{ft} = 2119.2\text{ lb}\cdot\text{in.}$

$$2119.2\text{ lb}\cdot\text{in.} = (174\text{ lb})(\sin 73.740^\circ)d$$

$d = 12.69$ in. ◀



PROBLEM 3.66

In Problem 3.57, determine the perpendicular distance between portion *BH* of the cable and the diagonal *AD*.

PROBLEM 3.57 The frame *ACD* is hinged at *A* and *D* and is supported by a cable that passes through a ring at *B* and is attached to hooks at *G* and *H*. Knowing that the tension in the cable is 450 N, determine the moment about the diagonal *AD* of the force exerted on the frame by portion *BH* of the cable.

SOLUTION

From the solution to Problem 3.57:

$$T_{BH} = 450 \text{ N}$$

$$\mathbf{T}_{BH} = (150 \text{ N})\mathbf{i} + (300 \text{ N})\mathbf{j} - (300 \text{ N})\mathbf{k}$$

$$|M_{AD}| = 90.0 \text{ N}\cdot\text{m}$$

$$\boldsymbol{\lambda}_{AD} = \frac{1}{5}(4\mathbf{i} - 3\mathbf{k})$$

Based on the discussion of Section 3.11, it follows that only the perpendicular component of $\mathbf{T}_{BH}$ will contribute to the moment of $\mathbf{T}_{BH}$ about line $\overrightarrow{AD}$.

Now
$$(T_{BH})_{\text{parallel}} = \mathbf{T}_{BH}\cdot\boldsymbol{\lambda}_{AD}$$

$$= (150\mathbf{i} + 300\mathbf{j} - 300\mathbf{k})\cdot\frac{1}{5}(4\mathbf{i} - 3\mathbf{k})$$

$$= \frac{1}{5}[(150)(4) + (-300)(-3)]$$

$$= 300 \text{ N}$$

Also,
$$\mathbf{T}_{BH} = (\mathbf{T}_{BH})_{\text{parallel}} + (\mathbf{T}_{BH})_{\text{perpendicular}}$$

so that
$$(T_{BH})_{\text{perpendicular}} = \sqrt{(450)^2 - (300)^2} = 335.41 \text{ N}$$

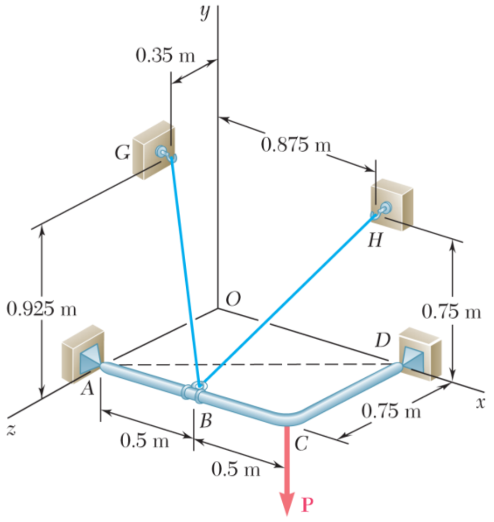
Since $\boldsymbol{\lambda}_{AD}$ and $(\mathbf{T}_{BH})_{\text{perpendicular}}$ are perpendicular, it follows that

$$M_{AD} = d(T_{BH})_{\text{perpendicular}}$$

or
$$90.0 \text{ N}\cdot\text{m} = d(335.41 \text{ N})$$

$$d = 0.26833 \text{ m}$$

$d = 0.268$ m ◀



PROBLEM 3.67

In Problem 3.58, determine the perpendicular distance between portion *BG* of the cable and the diagonal *AD*.

PROBLEM 3.58 In Problem 3.57, determine the moment about the diagonal *AD* of the force exerted on the frame by portion *BG* of the cable.

PROBLEM 3.57 The frame *ACD* is hinged at *A* and *D* and is supported by a cable that passes through a ring at *B* and is attached to hooks at *G* and *H*. Knowing that the tension in the cable is 450 N, determine the moment about the diagonal *AD* of the force exerted on the frame by portion *BH* of the cable.

SOLUTION

From the solution to Problem 3.58: $\mathbf{T}_{BG} = 450\text{ N}$

$$\mathbf{T}_{BG} = -(200\text{ N})\mathbf{i} + (370\text{ N})\mathbf{j} - (160\text{ N})\mathbf{k}$$

$$|M_{AD}| = 111\text{ N}\cdot\text{m}$$

$$\lambda_{AD} = \frac{1}{5}(4\mathbf{i} - 3\mathbf{k})$$

Based on the discussion of Section 3.11, it follows that only the perpendicular component of $\mathbf{T}_{BG}$ will contribute to the moment of $\mathbf{T}_{BG}$ about line $\overline{AD}$.

Now $$(T_{BG})_{\text{parallel}} = \mathbf{T}_{BG}\cdot\lambda_{AD}$$

$$= (-200\mathbf{i} + 370\mathbf{j} - 160\mathbf{k})\cdot\frac{1}{5}(4\mathbf{i} - 3\mathbf{k})$$

$$= \frac{1}{5}[(-200)(4) + (-160)(-3)]$$

$$= -64\text{ N}$$

Also, $$\bar{\mathbf{T}}_{BG} = (\mathbf{T}_{BG})_{\text{parallel}} + (\mathbf{T}_{BG})_{\text{perpendicular}}$$

so that $$(\mathbf{T}_{BG})_{\text{perpendicular}} = \sqrt{(450)^2 - (-64)^2} = 445.43\text{ N}$$

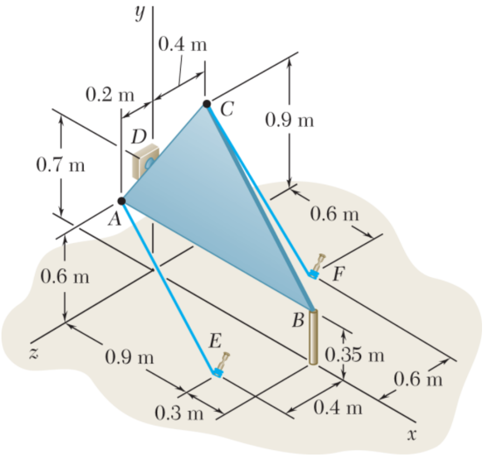
Since λ_{AD} and $(\mathbf{T}_{BG})_{\text{perpendicular}}$ are perpendicular, it follows that

$$M_{AD} = d(T_{BG})_{\text{perpendicular}}$$

or $$111\text{ N}\cdot\text{m} = d(445.43\text{ N})$$

$$d = 0.24920\text{ m}$$

$d = 0.249$ m ◀



PROBLEM 3.68

In Problem 3.59, determine the perpendicular distance between cable *AE* and the line joining Points *D* and *B*.

PROBLEM 3.59 The triangular plate *ABC* is supported by ball-and-socket joints at *B* and *D* and is held in the position shown by cables *AE* and *CF*. If the force exerted by cable *AE* at *A* is 55 N, determine the moment of that force about the line joining Points *D* and *B*.

SOLUTION

From the solution to Problem 3.59:

$$T_{AE} = 55\text{ N}$$

$$\mathbf{T}_{AE} = 5[(9\text{ N})\mathbf{i} - (6\text{ N})\mathbf{j} + (2\text{ N})\mathbf{k}]$$

$$|M_{DB}| = 2.28\text{ N}\cdot\text{m}$$

$$\boldsymbol{\lambda}_{DB} = \frac{1}{25}(24\mathbf{i} - 7\mathbf{j})$$

Based on the discussion of Section 3.11, it follows that only the perpendicular component of $\mathbf{T}_{AE}$ will contribute to the moment of $\mathbf{T}_{AE}$ about line $\overrightarrow{DB}$.

Now

$$\begin{aligned}(T_{AE})_{\text{parallel}} &= \mathbf{T}_{AE}\cdot\boldsymbol{\lambda}_{DB}\\ &= 5(9\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})\cdot\frac{1}{25}(24\mathbf{i} - 7\mathbf{j})\\ &= \frac{1}{5}[(9)(24) + (-6)(-7)]\\ &= 51.6\text{ N}\end{aligned}$$

Also,

$$\mathbf{T}_{AE} = (\mathbf{T}_{AE})_{\text{parallel}} + (\mathbf{T}_{AE})_{\text{perpendicular}}$$

so that

$$(\mathbf{T}_{AE})_{\text{perpendicular}} = \sqrt{(55)^2 + (51.6)^2} = 19.0379\text{ N}$$

Since $\boldsymbol{\lambda}_{DB}$ and $(\mathbf{T}_{AE})_{\text{perpendicular}}$ are perpendicular, it follows that

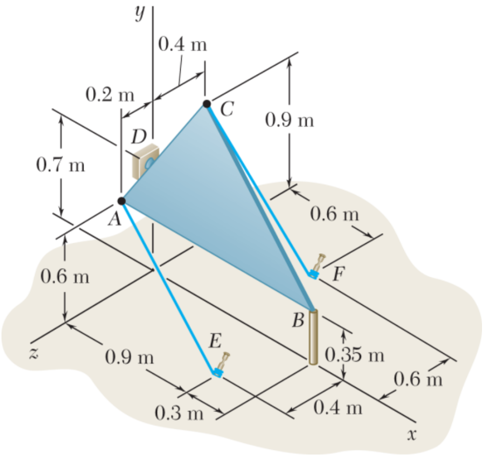
$$M_{DB} = d(T_{AE})_{\text{perpendicular}}$$

or

$$2.28\text{ N}\cdot\text{m} = d(19.0379\text{ N})$$

$$d = 0.119761$$

$d = 0.1198$ m ◀



PROBLEM 3.69

In Problem 3.60, determine the perpendicular distance between cable *CF* and the line joining Points *D* and *B*.

PROBLEM 3.60 The triangular plate *ABC* is supported by ball-and-socket joints at *B* and *D* and is held in the position shown by cables *AE* and *CF*. If the force exerted by cable *CF* at *C* is 33 N, determine the moment of that force about the line joining Points *D* and *B*.

SOLUTION

From the solution to Problem 3.60:

$$T_{CF} = 33 \text{ N}$$

$$\mathbf{T}_{CF} = 3[(6\text{ N})\mathbf{i} - (9\text{ N})\mathbf{j} - (2\text{ N})\mathbf{k}]$$

$$|M_{DB}| = 9.50 \text{ N}\cdot\text{m}$$

$$\boldsymbol{\lambda}_{DB} = \frac{1}{25}(24\mathbf{i} - 7\mathbf{j})$$

Based on the discussion of Section 3.11, it follows that only the perpendicular component of $\mathbf{T}_{CF}$ will contribute to the moment of $\mathbf{T}_{CF}$ about line $\overrightarrow{DB}$.

Now

$$\begin{aligned}(\mathbf{T}_{CF})_{\text{parallel}} &= \mathbf{T}_{CF}\cdot\boldsymbol{\lambda}_{DB}\\ &= 3(6\mathbf{i} - 9\mathbf{j} - 2\mathbf{k})\cdot\frac{1}{25}(24\mathbf{i} - 7\mathbf{j})\\ &= \frac{3}{25}[(6)(24) + (-9)(-7)]\\ &= 24.84 \text{ N}\end{aligned}$$

Also,

$$\mathbf{T}_{CF} = (\mathbf{T}_{CF})_{\text{parallel}} + (\mathbf{T}_{CF})_{\text{perpendicular}}$$

so that

$$\begin{aligned}(\mathbf{T}_{CF})_{\text{perpendicular}} &= \sqrt{(33)^2 - (24.84)^2}\\ &= 21.725 \text{ N}\end{aligned}$$

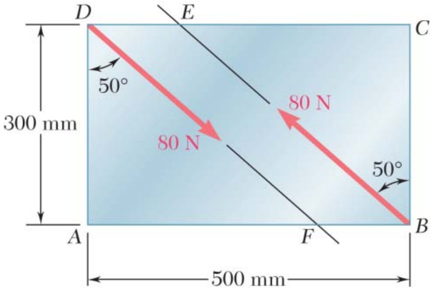
Since $\boldsymbol{\lambda}_{DB}$ and $(\mathbf{T}_{CF})_{\text{perpendicular}}$ are perpendicular, it follows that

$$|M_{DB}| = d(T_{CF})_{\text{perpendicular}}$$

or

$$9.50 \text{ N}\cdot\text{m} = d \times 21.725 \text{ N}$$

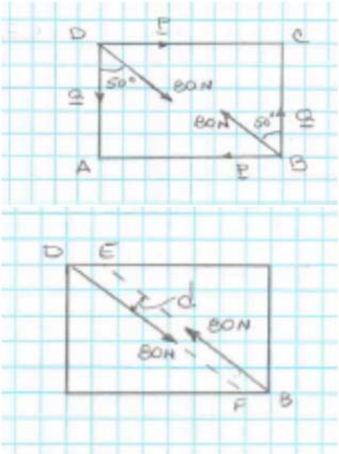
or $d = 0.437$ m ◀



PROBLEM 3.70

Two 80-N forces are applied as shown to the corners *B* and *D* of a rectangular plate. (*a*) Determine the moment of the couple formed by the two forces by resolving each force into horizontal and vertical components and adding the moments of the two resulting couples. (*b*) Use the result obtained to determine the perpendicular distance between lines *BE* and *DF*.

SOLUTION



(*a*) Resolving forces into components:

$$P = (80\text{ N})\sin 50^\circ = 61.284\text{ N}$$

$$Q = (80\text{ N})\cos 50^\circ = 51.423\text{ N}$$

$$M = (51.423\text{ N})(0.5\text{ m}) - (61.284\text{ N})(0.3\text{ m})$$
$$= 7.3263\text{ N}\cdot\text{m}$$

$$\mathbf{M} = 7.33\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

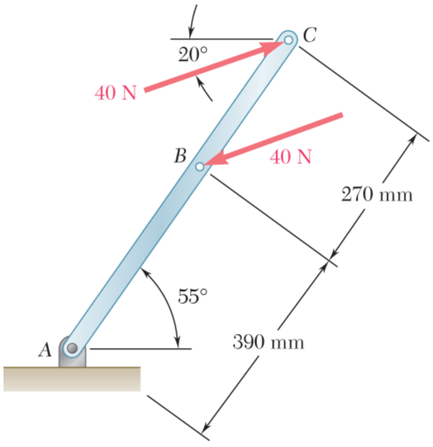
(*b*) Distance between lines *BE* and *DF*

$\mathbf{M} = Fd$

or $7.3263\text{ N}\cdot\text{m} = (80\text{ N})d$

$d = 0.091579\text{ m}$

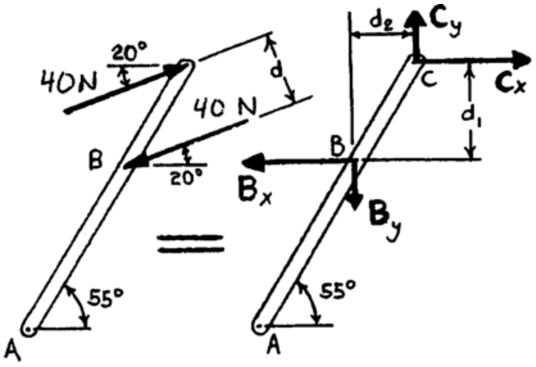
$$d = 91.6\text{ mm} \blacktriangleleft$$



PROBLEM 3.71

Two parallel 40-N forces are applied to a lever as shown. Determine the moment of the couple formed by the two forces (*a*) by resolving each force into horizontal and vertical components and adding the moments of the two resulting couples, (*b*) by using the perpendicular distance between the two forces, (*c*) by summing the moments of the two forces about Point *A*.

SOLUTION



(*a*) We have $\Sigma\mathbf{M}_B:\quad -d_1C_x + d_2C_y = \mathbf{M}$

where

$$d_1 = (0.270\text{ m})\sin 55° = 0.22117\text{ m}$$

$$d_2 = (0.270\text{ m})\cos 55° = 0.154866\text{ m}$$

$$C_x = (40\text{ N})\cos 20° = 37.588\text{ N}$$

$$C_y = (40\text{ N})\sin 20° = 13.6808\text{ N}$$

$$\mathbf{M} = -(0.22117\text{ m})(37.588\text{ N})\mathbf{k} + (0.154866\text{ m})(13.6808\text{ N})\mathbf{k}$$

$$= -(6.1946\text{ N}\cdot\text{m})\mathbf{k}$$

or $\mathbf{M} = 6.19\text{ N}\cdot\text{m}$ ↻ ◀

(*b*) We have

$$\mathbf{M} = Fd(-\mathbf{k})$$

$$= 40\text{ N}[(0.270\text{ m})\sin(55° - 20°)](-\mathbf{k})$$

$$= -(6.1946\text{ N}\cdot\text{m})\mathbf{k}$$

or $\mathbf{M} = 6.19\text{ N}\cdot\text{m}$ ↻ ◀

(*c*) We have $\Sigma\mathbf{M}_A:\quad \Sigma(\mathbf{r}_A \times \mathbf{F}) = \mathbf{r}_{B/A} \times \mathbf{F}_B + \mathbf{r}_{C/A} \times \mathbf{F}_C = \mathbf{M}$

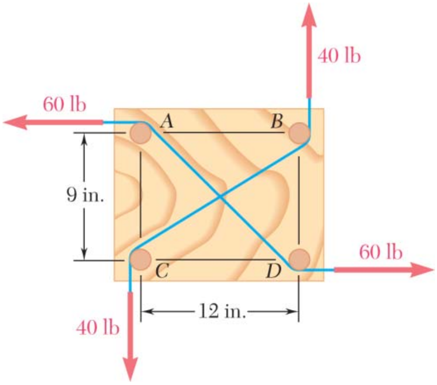
$$M = (0.390\text{ m})(40\text{ N})\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos 55° & \sin 55° & 0 \\ -\cos 20° & -\sin 20° & 0 \end{vmatrix}$$

$$+ (0.660\text{ m})(40\text{ N})\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos 55° & \sin 55° & 0 \\ \cos 20° & \sin 20° & 0 \end{vmatrix}$$

$$= (8.9478\text{ N}\cdot\text{m} - 15.1424\text{ N}\cdot\text{m})\mathbf{k}$$

$$= -(6.1946\text{ N}\cdot\text{m})\mathbf{k}$$

or $\mathbf{M} = 6.19\text{ N}\cdot\text{m}$ ↻ ◀



PROBLEM 3.72

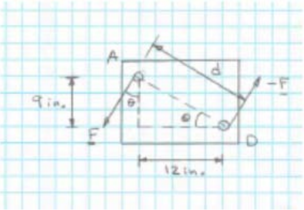
Four $1\frac{1}{2}$-in.-diameter pegs are attached to a board as shown. Two strings are passed around the pegs and pulled with the forces indicated. (*a*) Determine the resultant couple acting on the board. (*b*) If only one string is used, around which pegs should it pass and in what directions should it be pulled to create the same couple with the minimum tension in the string? (*c*) What is the value of that minimum tension?

SOLUTION

(*a*)

$$+\circlearrowright M = (60\text{ lb})(10.5\text{ in.}) + (40\text{ lb})(13.5\text{ in.})$$
$$= 630\text{ lb}\cdot\text{in.} + 540\text{ lb}\cdot\text{in.}$$

$$M = 1170\text{ lb}\cdot\text{in.}\ \circlearrowright$$ ◀



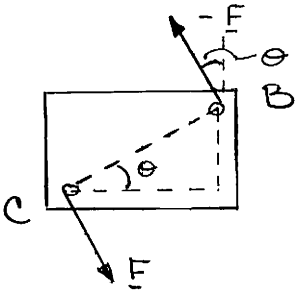
(*b*) With only one string, pegs *A* and *D*, or *B* and *C* should be used. We have

$$\tan\theta = \frac{9}{12} \qquad \theta = 36.9^\circ \qquad 90^\circ - \theta = 53.1^\circ$$

Direction of forces:

With pegs *A* and *D*: $\theta = 53.1^\circ$ ↗ ◀

With pegs *B* and *C*: $\theta = 53.1^\circ$ ↖ ◀



(*c*) The distance between the centers of the two pegs is

$$\sqrt{12^2 + 9^2} = 15\text{ in.}$$

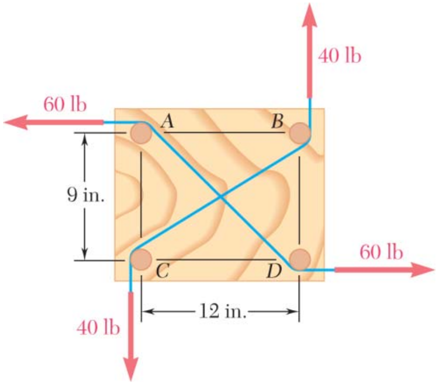
Therefore, the perpendicular distance *d* between the forces is

$$d = 15\text{ in.} + 2\left(\frac{3}{4}\text{ in.}\right)$$
$$= 16.5\text{ in.}$$

We must have $M = Fd \quad 1170\text{ lb}\cdot\text{in.} = F(16.5\text{ in.})$

$$F = 70.9\text{ lb}$$ ◀

PROBLEM 3.73

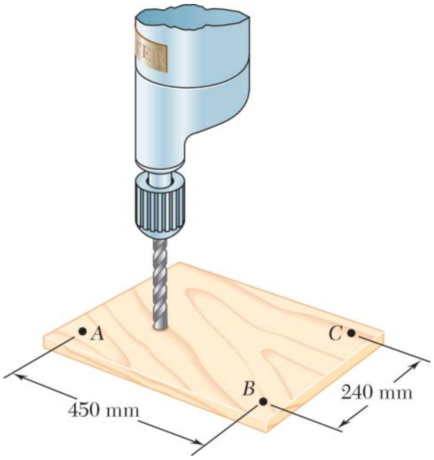


Four pegs of the same diameter are attached to a board as shown. Two strings are passed around the pegs and pulled with the forces indicated. Determine the diameter of the pegs knowing that the resultant couple applied to the board is 1132.5 lb·in. counterclockwise.

SOLUTION

$$M = d_{AD}F_{AD} + d_{BC}F_{BC}$$

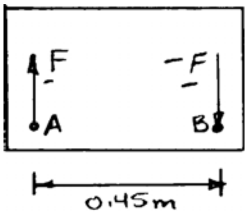
$$1132.5 \text{ lb}\cdot\text{in.} = [(9+d)\text{ in.}](60\text{ lb}) + [(12+d)\text{ in.}](40\text{ lb}) \qquad d = 1.125\text{ in.} \blacktriangleleft$$



PROBLEM 3.74

A piece of plywood in which several holes are being drilled successively has been secured to a workbench by means of two nails. Knowing that the drill exerts a 12-N·m couple on the piece of plywood, determine the magnitude of the resulting forces applied to the nails if they are located (*a*) at *A* and *B*, (*b*) at *B* and *C*, (*c*) at *A* and *C*.

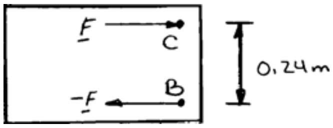
SOLUTION



(*a*) $M = Fd$

$12\ \text{N}\cdot\text{m} = F(0.45\ \text{m})$

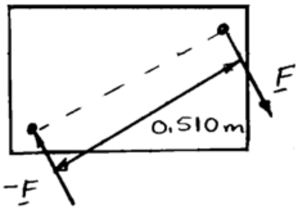
$F = 26.7\ \text{N}$ ◀



(*b*) $M = Fd$

$12\ \text{N}\cdot\text{m} = F(0.24\ \text{m})$

$F = 50.0\ \text{N}$ ◀

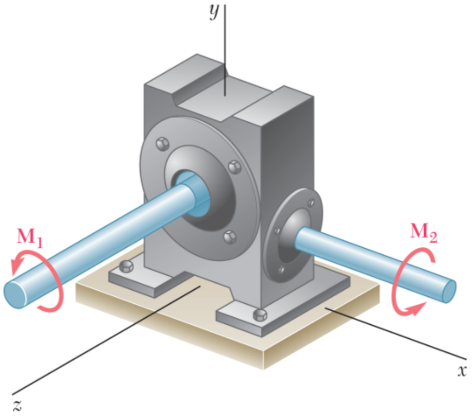


(*c*) $M = Fd \qquad d = \sqrt{(0.45\ \text{m})^2 + (0.24\ \text{m})^2}$

$= 0.510\ \text{m}$

$12\ \text{N}\cdot\text{m} = F(0.510\ \text{m})$

$F = 23.5\ \text{N}$ ◀



PROBLEM 3.75

The two shafts of a speed-reducer unit are subjected to couples of magnitude $M_1 = 15$ lb·ft and $M_2 = 3$ lb·ft, respectively. Replace the two couples with a single equivalent couple, specifying its magnitude and the direction of its axis.

SOLUTION

$$M_1 = (15 \text{ lb}\cdot\text{ft})\mathbf{k}$$

$$M_2 = (3 \text{ lb}\cdot\text{ft})\mathbf{i}$$

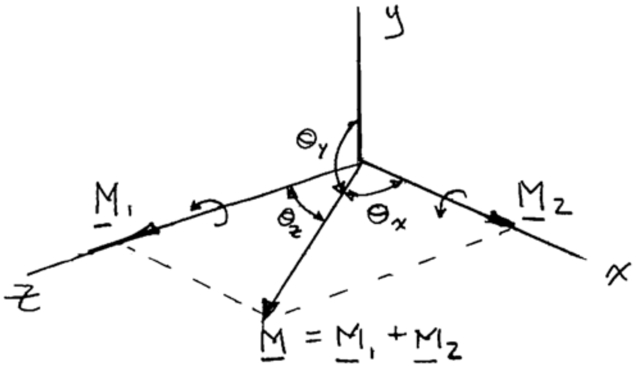
$$\begin{aligned} M &= \sqrt{M_1^2 + M_2^2} \\ &= \sqrt{(15)^2 + (3)^2} \\ &= 15.30 \text{ lb}\cdot\text{ft} \end{aligned}$$

$$\tan\theta_x = \frac{15}{3} = 5$$

$$\theta_x = 78.7°$$

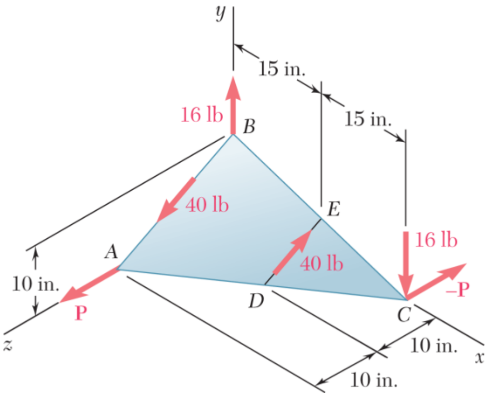
$$\theta_y = 90°$$

$$\begin{aligned} \theta_z &= 90° - 78.7° \\ &= 11.30° \end{aligned}$$



y
θ_y
$\underline{M}_1$
θ_z
θ_x
$\underline{M}_2$
x
z
$\underline{M} = \underline{M}_1 + \underline{M}_2$

$M = 15.30 \text{ lb}\cdot\text{ft};\ \theta_x = 78.7°,\ \theta_y = 90.0°,\ \theta_z = 11.30°$ ◀



PROBLEM 3.76

If $P = 0$, replace the two remaining couples with a single equivalent couple, specifying its magnitude and the direction of its axis.

SOLUTION

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2; \quad F_1 = 16 \text{ lb}, \quad F_2 = 40 \text{ lb}$$

$$\mathbf{M_1} = \mathbf{r}_C \times \mathbf{F}_1 = (30 \text{ in.})\mathbf{i} \times [-(16 \text{ lb})\mathbf{j}] = -(480 \text{ lb} \cdot \text{in.})\mathbf{k}$$

$$\mathbf{M_2} = \mathbf{r}_{E/B} \times \mathbf{F}_2; \quad \mathbf{r}_{E/B} = (15 \text{ in.})\mathbf{i} - (5 \text{ in.})\mathbf{j}$$

$$d_{DE} = \sqrt{(0)^2 + (5)^2 + (10)^2} = 5\sqrt{5} \text{ in.}$$

$$F_2 = \frac{40 \text{ lb}}{5\sqrt{5}}(5\mathbf{j} - 10\mathbf{k})$$

$$= 8\sqrt{5}[(1 \text{ lb})\mathbf{j} - (2 \text{ lb})\mathbf{k}]$$

$$\mathbf{M}_2 = 8\sqrt{5}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & -5 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 8\sqrt{5}[(10 \text{ lb} \cdot \text{in.})\mathbf{i} + (30 \text{ lb} \cdot \text{in.})\mathbf{j} + (15 \text{ lb} \cdot \text{in.})\mathbf{k}]$$

$$\mathbf{M} = -(480 \text{ lb} \cdot \text{in.})\mathbf{k} + 8\sqrt{5}[(10 \text{ lb} \cdot \text{in.})\mathbf{i} + (30 \text{ lb} \cdot \text{in.})\mathbf{j} + (15 \text{ lb} \cdot \text{in.})\mathbf{k}]$$

$$= (178.885 \text{ lb} \cdot \text{in.})\mathbf{i} + (536.66 \text{ lb} \cdot \text{in.})\mathbf{j} - (211.67 \text{ lb} \cdot \text{in.})\mathbf{k}$$

$$M = \sqrt{(178.885)^2 + (536.66)^2 + (-211.67)^2}$$

$$= 603.99 \text{ lb} \cdot \text{in}$$

$M = 604 \text{ lb} \cdot \text{in.}$ ◀

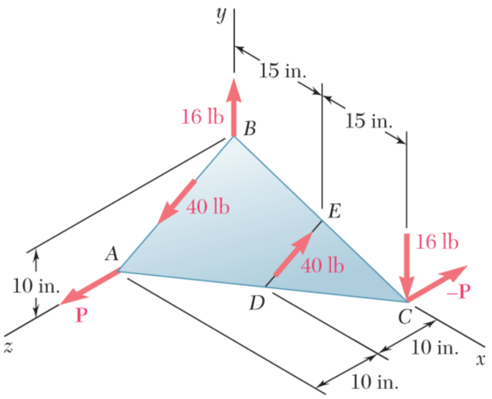
$$\lambda_{\text{axis}} = \frac{\mathbf{M}}{M} = 0.29617\mathbf{i} + 0.88852\mathbf{j} - 0.35045\mathbf{k}$$

$$\cos\theta_x = 0.29617$$

$$\cos\theta_y = 0.88852$$

$$\cos\theta_z = -0.35045$$

$\theta_x = 72.8°$ $\theta_y = 27.3°$ $\theta_z = 110.5°$ ◀



PROBLEM 3.77

If $P = 20$ lb, replace the three couples with a single equivalent couple, specifying its magnitude and the direction of its axis.

SOLUTION

From the solution to Problem. 3.78:

16-lb force: $M_1 = -(480\text{ lb}\cdot\text{in.})\mathbf{k}$

40-lb force: $M_2 = 8\sqrt{5}[(10\text{ lb}\cdot\text{in.})\mathbf{i} + (30\text{ lb}\cdot\text{in.})\mathbf{j} + (15\text{ lb}\cdot\text{in.})\mathbf{k}]$

$P = 20$ lb

$$\begin{aligned} M_3 &= \mathbf{r}_C \times P \\ &= (30\text{ in.})\mathbf{i} \times (20\text{ lb})\mathbf{k} \\ &= (600\text{ lb}\cdot\text{in.})\mathbf{j} \end{aligned}$$

$$\begin{aligned} \mathbf{M} &= \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 \\ &= -(480)\mathbf{k} + 8\sqrt{5}\,(10\mathbf{i} + 30\mathbf{j} + 15\mathbf{k}) + 600\mathbf{j} \\ &= (178.885\text{ lb}\cdot\text{in.})\mathbf{i} + (1136.66\text{ lb}\cdot\text{in.})\mathbf{j} - (211.67\text{ lb}\cdot\text{in.})\mathbf{k} \end{aligned}$$

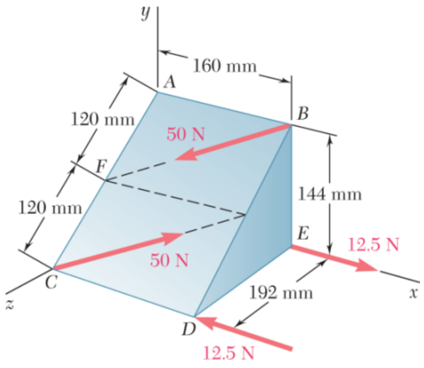
$$\begin{aligned} M &= \sqrt{(178.885)^2 + (113.66)^2 + (211.67)^2} \\ &= 1169.96\text{ lb}\cdot\text{in.} \end{aligned}$$

$M = 1170\text{ lb}\cdot\text{in.}$ ◀

$$\lambda_{\text{axis}} = \frac{\mathbf{M}}{M} = 0.152898\mathbf{i} + 0.97154\mathbf{j} - 0.180921\mathbf{k}$$

$$\begin{aligned} \cos\theta_x &= 0.152898 \\ \cos\theta_y &= 0.97154 \\ \cos\theta_z &= -0.180921 \end{aligned}$$

$\theta_x = 81.2°$ $\theta_y = 13.70°$ $\theta_z = 100.4°$ ◀

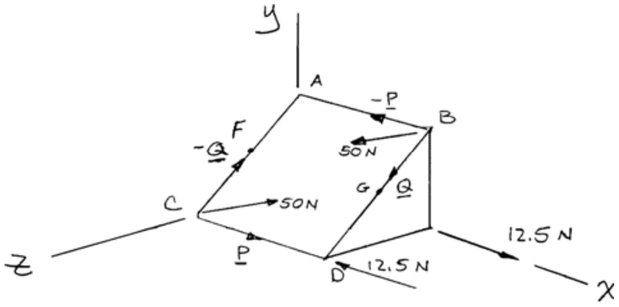


y
160 mm
A
B
120 mm
50 N
F
144 mm
120 mm
E
12.5 N
50 N
C
192 mm
x
z
D
12.5 N

PROBLEM 3.78

Replace the two couples shown with a single equivalent couple, specifying its magnitude and the direction of its axis.

SOLUTION



y
A
-P
B
F
-Q
50 N
G
Q
C
50 N
12.5 N
z
P
D
12.5 N
x

Replace the couple in the *ABCD* plane with two couples *P* and *Q* shown:

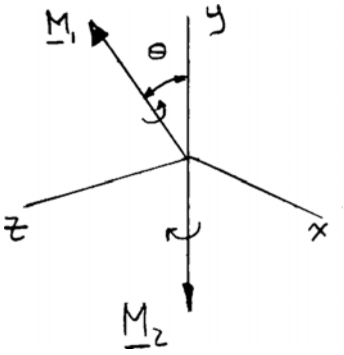
$$P = (50\text{ N})\frac{CD}{CG} = (50\text{ N})\left(\frac{160\text{ mm}}{200\text{ mm}}\right) = 40\text{ N}$$

$$Q = (50\text{ N})\frac{CF}{CG} = (50\text{ N})\left(\frac{120\text{ mm}}{200\text{ mm}}\right) = 30\text{ N}$$

Couple vector $\mathbf{M}_1$ perpendicular to plane *ABCD*:

$$+\circlearrowleft M_1 = (40\text{ N})(0.24\text{ m}) - (30\text{ N})(0.16\text{ m}) = 4.80\text{ N}\cdot\text{m}$$

Couple vector $\mathbf{M}_2$ in the *xy* plane:



$$+\circlearrowleft M_2 = -(12.5\text{ N})(0.192\text{ m}) = -2.40\text{ N}\cdot\text{m}$$

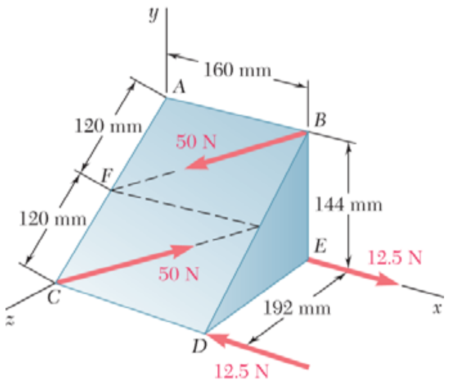
$$\tan\theta = \frac{144\text{ mm}}{192\text{ mm}} \qquad \theta = 36.870°$$

$$\begin{aligned}\mathbf{M}_1 &= (4.80\cos 36.870°)\mathbf{j} + (4.80\sin 36.870°)\mathbf{k}\\ &= 3.84\mathbf{j} + 2.88\mathbf{k}\end{aligned}$$

$$\mathbf{M}_2 = -2.40\mathbf{j}$$

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 = 1.44\mathbf{j} + 2.88\mathbf{k}$$

$$\mathbf{M} = 3.22\text{ N}\cdot\text{m};\ \theta_x = 90.0°,\ \theta_y = 53.1°,\ \theta_z = 36.9° \blacktriangleleft$$

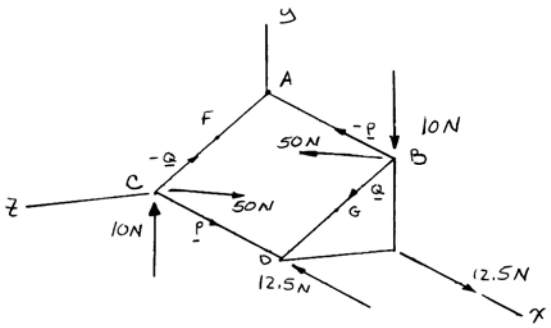


PROBLEM 3.79

Solve Prob. 3.78, assuming that two 10-N vertical forces have been added, one acting upward at *C* and the other downward at *B*.

PROBLEM 3.78 Replace the two couples shown with a single equivalent couple, specifying its magnitude and the direction of its axis.

SOLUTION



Replace the couple in the *ABCD* plane with two couples *P* and *Q* shown.

$$P = (50\text{ N})\frac{CD}{CG} = (50\text{ N})\left(\frac{160\text{ mm}}{200\text{ mm}}\right) = 40\text{ N}$$

$$Q = (50\text{ N})\frac{CF}{CG} = (50\text{ N})\left(\frac{120\text{ mm}}{200\text{ mm}}\right) = 30\text{ N}$$

Couple vector $\mathbf{M}_1$ perpendicular to plane *ABCD*.

$$+\circlearrowleft M_1 = (40\text{ N})(0.24\text{ m}) - (30\text{ N})(0.16\text{ m}) = 4.80\text{ N}\cdot\text{m}$$

$$\tan\theta = \frac{144\text{ mm}}{192\text{ mm}} \qquad \theta = 36.870°$$

$$\begin{aligned}\mathbf{M}_1 &= (4.80\cos 36.870°)\mathbf{j} + (4.80\sin 36.870°)\mathbf{k}\\ &= 3.84\mathbf{j} + 2.88\mathbf{k}\end{aligned}$$

$$\begin{aligned}+\circlearrowleft M_2 &= -(12.5\text{ N})(0.192\text{ m}) = -2.40\text{ N}\cdot\text{m}\\ &= -2.40\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{M}_3 &= \mathbf{r}_{B/C} \times M_3;\ \mathbf{r}_{B/C} = (0.16\text{ m})\mathbf{i} + (0.144\text{ m})\mathbf{j} - (0.192\text{ m})\mathbf{k}\\ &= (0.16\text{ m})\mathbf{i} + (0.144\text{ m})\mathbf{j} - (0.192\text{ m})\mathbf{k} \times (-10\text{ N})\mathbf{j}\\ &= -1.92\mathbf{i} - 1.6\mathbf{k}\end{aligned}$$

PROBLEM 3.79 (Continued)

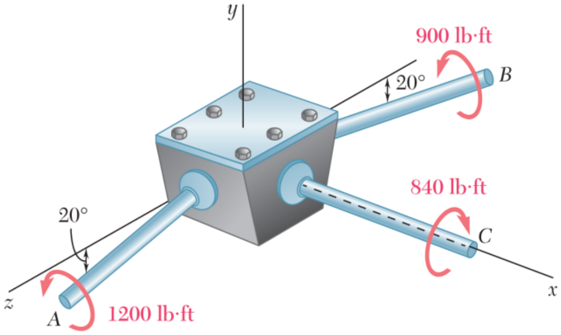
$$M = M_1 + M_2 + M_3 = (3.84\mathbf{j} + 2.88\mathbf{k}) - 2.40\mathbf{j} + (-1.92\mathbf{i} - 1.6\mathbf{k})$$
$$= -(1.92\text{ N}\cdot\text{m})\mathbf{i} + (1.44\text{ N}\cdot\text{m})\mathbf{j} + (1.28\text{ N}\cdot\text{m})\mathbf{k}$$

$$M = \sqrt{(-1.92)^2 + (1.44)^2 + (1.28)^2} = 2.72\text{ N}\cdot\text{m}$$

$M = 2.72\text{ N}\cdot\text{m}$ ◀

$$\cos\theta_x = -1.92/2.72$$
$$\cos\theta_y = 1.44/2.72$$
$$\cos\theta_z = 1.28/2.72$$

$\theta_x = 134.9°$ $\theta_y = 58.0°$ $\theta_z = 61.9°$ ◀



PROBLEM 3.80

Shafts *A* and *B* connect the gear box to the wheel assemblies of a tractor, and shaft *C* connects it to the engine. Shafts *A* and *B* lie in the vertical *yz* plane, while shaft *C* is directed along the *x* axis. Replace the couples applied to the shafts by a single equivalent couple, specifying its magnitude and the direction of its axis.

SOLUTION

$$\mathbf{M}_A = -1200\sin 20^\circ\,\mathbf{j} + 1200\cos 20^\circ\,\mathbf{k} = -410.42\mathbf{j} + 1127.63\mathbf{k}$$

$$\mathbf{M}_B = 900\sin 20^\circ\,\mathbf{j} + 900\cos 20^\circ\,\mathbf{k} = 307.82\mathbf{j} + 845.72\mathbf{k}$$

$$\mathbf{M}_C = -840\mathbf{i}$$

The single equivalent couple is the sum of the three moments

$$\mathbf{M} = -(840\text{ lb}\cdot\text{ft})\mathbf{i} - (102.60\text{ lb}\cdot\text{ft})\mathbf{j} + (1973.35\text{ lb}\cdot\text{ft})\mathbf{k}$$

$$M = \sqrt{(-840)^2 + (-102.60)^2 + (1973.35)^2}$$

$$= 2147.3\text{ lb}\cdot\text{ft} \qquad M = 2150\text{ lb}\cdot\text{ft} \blacktriangleleft$$

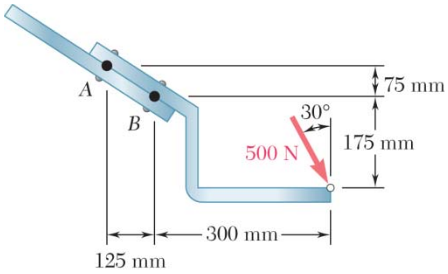
$$\lambda_{\text{axis}} = \frac{\mathbf{M}}{M} = -0.39119\mathbf{i} - 0.047921\mathbf{j} + 0.91899\mathbf{k}$$

$$\cos\theta_x = -0.39119$$

$$\cos\theta_y = -0.047921$$

$$\cos\theta_z = 0.91899$$

$$\theta_x = 113.0^\circ \quad \theta_y = 92.7^\circ \quad \theta_z = 23.2^\circ \blacktriangleleft$$



PROBLEM 3.81

A 500-N force is applied to a bent plate as shown. Determine (*a*) an equivalent force-couple system at *B*, (*b*) an equivalent system formed by a vertical force at *A* and a force at *B*.

SOLUTION

(*a*) Force-couple system at *B*

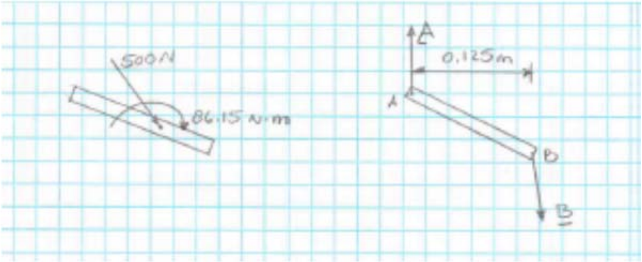
$$\mathbf{F} = (500\text{ N})\sin 30^\circ\,\mathbf{i} - (500\text{ N})\cos 30^\circ\,\mathbf{k}$$
$$\mathbf{F} = (250.0\text{ N})\mathbf{i} - (433.01\text{ N})\mathbf{k}$$

$$\mathbf{M}_B = (0.3\mathbf{i} - 0.175\mathbf{j}) \times (250\mathbf{i} - 433.01\mathbf{j})$$
$$= -86.153\mathbf{k}\ \text{N}\cdot\text{m}$$

The equivalent force-couple system at *B* is

$\mathbf{F}_B = 500\text{ N}$ ↘ 60.0° $\qquad\qquad$ $\mathbf{M}_B = 86.2\ \text{N}\cdot\text{m}$ ↻ ◀

(*b*) Require

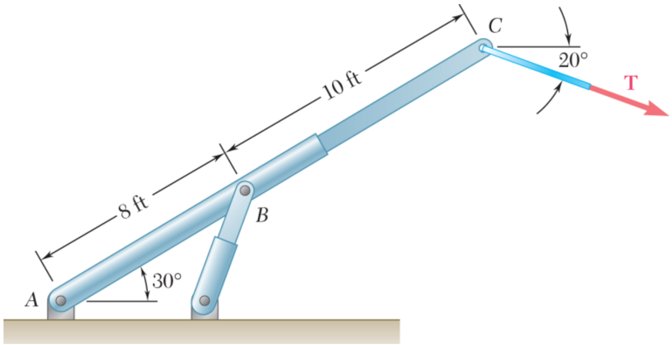


Equivalence requires

$$\Sigma M_B = \Sigma M_B$$
$$86.153\ \text{N}\cdot\text{m} = \mathbf{A}(0.125\text{ m});\quad \mathbf{A} = (689.22\text{ N})\mathbf{j}$$
$$\mathbf{F} = \mathbf{A} + \mathbf{B} \text{ or } \mathbf{B} = \mathbf{F} - \mathbf{A};\quad \mathbf{B} = (250\text{ N})\mathbf{i} - (433.01\text{ N})\mathbf{j} - (689.22\text{ N})\mathbf{j}$$

$\mathbf{B} = (250\text{ N})\mathbf{i} - (1122.23\text{ N})\mathbf{j} = 1149.74\text{ N}$ ↘ 77.4° $\qquad\qquad$ $\mathbf{B} = 1150\text{ N}$ ↘ 77.4° ◀

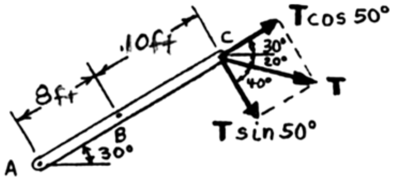
$\mathbf{A} = 689\text{ N}$ ↑ ◀


C
20°
T
10 ft
8 ft
B
30°
A

PROBLEM 3.82

The tension in the cable attached to the end C of an adjustable boom ABC is 560 lb. Replace the force exerted by the cable at C with an equivalent force-couple system (*a*) at A, (*b*) at B.

SOLUTION


T cos 50°
10 ft
8 ft
C
30°
20°
40°
T
T sin 50°
B
30°
A

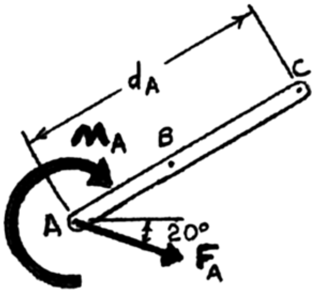
(*a*) Based on ΣF: $F_A = T = 560 \text{ lb}$

or $\mathbf{F}_A = 560 \text{ lb}$ $\searrow$ 20.0° ◀

ΣM_A: $M_A = (T \sin 50°)(d_A)$

$= (560 \text{ lb}) \sin 50° (18 \text{ ft})$
$= 7721.7 \text{ lb} \cdot \text{ft}$

or $\mathbf{M}_A = 7720 \text{ lb} \cdot \text{ft}$ $\circlearrowright$ ◀


d_A
C
M_A
B
A
20°
F_A

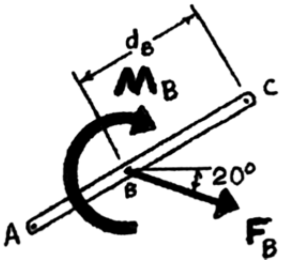
(*b*) Based on ΣF: $F_B = T = 560 \text{ lb}$

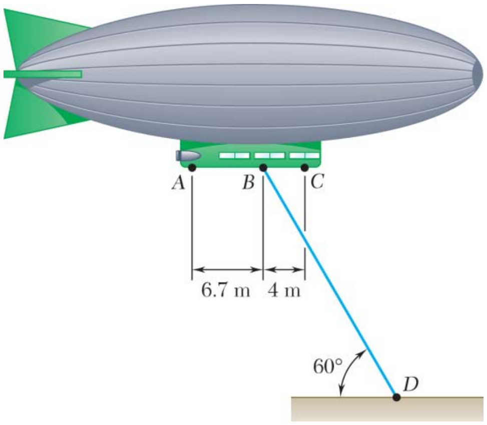
or $\mathbf{F}_B = 560 \text{ lb}$ $\searrow$ 20.0° ◀

ΣM_B: $M_B = (T \sin 50°)(d_B)$

$= (560 \text{ lb}) \sin 50° (10 \text{ ft})$
$= 4289.8 \text{ lb} \cdot \text{ft}$

or $\mathbf{M}_B = 4290 \text{ lb} \cdot \text{ft}$ $\circlearrowright$ ◀

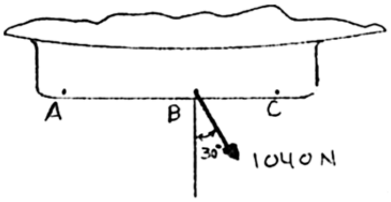

d_B
M_B
C
20°
B
A
F_B

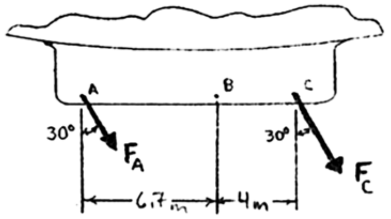


PROBLEM 3.83

A dirigible is tethered by a cable attached to its cabin at *B*. If the tension in the cable is 1040 N, replace the force exerted by the cable at *B* with an equivalent system formed by two parallel forces applied at *A* and *C*.

SOLUTION





Require the equivalent forces acting at *A* and *C* be parallel and at an angle of α with the vertical.

Then for equivalence,

$$\Sigma F_x: \quad (1040\text{ N})\sin 30° = F_A \sin\alpha + F_B \sin\alpha \qquad (1)$$

$$\Sigma F_y: \quad -(1040\text{ N})\cos 30° = -F_A \cos\alpha - F_B \cos\alpha \qquad (2)$$

Dividing Equation (1) by Equation (2),

$$\frac{(1040\text{ N})\sin 30°}{-(1040\text{ N})\cos 30°} = \frac{(F_A + F_B)\sin\alpha}{-(F_A + F_B)\cos\alpha}$$

Simplifying yields $\alpha = 30°$.

Based on

$$\Sigma M_C: \quad [(1040\text{ N})\cos 30°](4\text{ m}) = (F_A \cos 30°)(10.7\text{ m})$$

$$F_A = 388.79\text{ N}$$

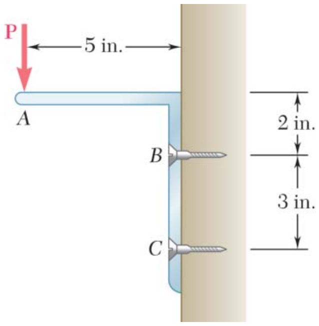
or $\mathbf{F}_A = 389\text{ N}$ ↘ 60.0° ◀

Based on

$$\Sigma M_A: \quad -[(1040\text{ N})\cos 30°](6.7\text{ m}) = (F_C \cos 30°)(10.7\text{ m})$$

$$F_C = 651.21\text{ N}$$

or $\mathbf{F}_C = 651\text{ N}$ ↘ 60.0° ◀



PROBLEM 3.84

A 30-lb vertical force **P** is applied at *A* to the bracket shown, which is held by screws at *B* and *C*. (*a*) Replace **P** with an equivalent force-couple system at *B*. (*b*) Find the two horizontal forces at *B* and *C* that are equivalent to the couple obtained in part *a*.

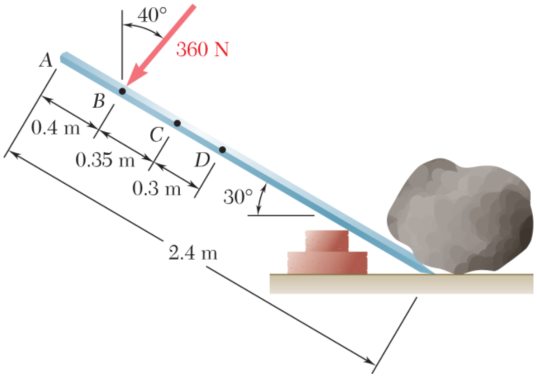
SOLUTION

(*a*)
$$M_B = (30 \text{ lb})(5 \text{ in.})$$
$$= 150.0 \text{ lb} \cdot \text{in.}$$

$$\mathbf{F} = 30.0 \text{ lb} \downarrow, \quad \mathbf{M}_B = 150.0 \text{ lb} \cdot \text{in.} \circlearrowright \blacktriangleleft$$

(*b*)
$$B = C = \frac{150 \text{ lb} \cdot \text{in.}}{3.0 \text{ in.}} = 50.0 \text{ lb}$$

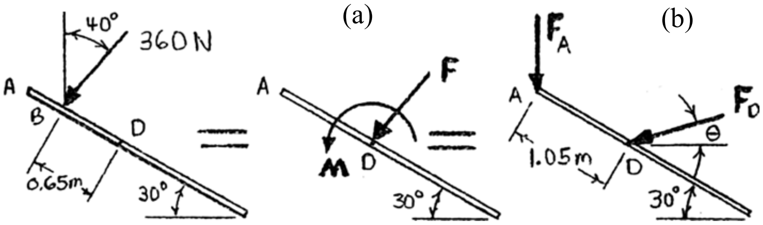
$$\mathbf{B} = 50.0 \text{ lb} \longleftarrow; \quad \mathbf{C} = 50.0 \text{ lb} \longrightarrow \blacktriangleleft$$



PROBLEM 3.85

A worker tries to move a rock by applying a 360-N force to a steel bar as shown. (*a*) Replace that force with an equivalent force-couple system at *D*. (b) Two workers attempt to move the same rock by applying a vertical force at *A* and another force at *D*. Determine these two forces if they are to be equivalent to the single force of Part *a*.

SOLUTION



(*a*) We have $\Sigma\mathbf{F}: \quad 360\text{ N}(-\sin 40°\mathbf{i} - \cos 40°\mathbf{j}) = -(231.40\text{ N})\mathbf{i} - (275.78\text{ N})\mathbf{j} = \mathbf{F}$

or $\mathbf{F} = 360\text{ N} \swarrow 50°$ ◀

We have $\Sigma\mathbf{M}_D: \quad \mathbf{r}_{B/D} \times \mathbf{R} = \mathbf{M}$

where

$$\begin{aligned}\mathbf{r}_{B/D} &= -[(0.65\text{ m})\cos 30°]\mathbf{i} + [(0.65\text{ m})\sin 30°]\mathbf{j}\\ &= -(0.56292\text{ m})\mathbf{i} + (0.32500\text{ m})\mathbf{j}\end{aligned}$$

$$\mathbf{M} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.56292 & 0.32500 & 0 \\ -231.40 & -275.78 & 0 \end{vmatrix} \text{N}\cdot\text{m}$$

$$= [155.240 + 75.206)\text{N}\cdot\text{m}]\mathbf{k}$$

$$= (230.45\text{ N}\cdot\text{m})\mathbf{k}$$

or $\mathbf{M} = 230\text{ N}\cdot\text{m} \circlearrowleft$ ◀

(*b*) We have $\Sigma\mathbf{M}_D: \quad \mathbf{M} = \mathbf{r}_{A/D} \times \mathbf{F}_A$

where

$$\begin{aligned}\mathbf{r}_{B/D} &= -[(1.05\text{ m})\cos 30°]\mathbf{i} + [(1.05\text{ m})\sin 30°]\mathbf{j}\\ &= -(0.90933\text{ m})\mathbf{i} + (0.52500\text{ m})\mathbf{j}\end{aligned}$$

$$F_A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.90933 & 0.52500 & 0 \\ 0 & -1 & 0 \end{vmatrix} \text{N}\cdot\text{m}$$

$$= [230.45\text{ N}\cdot\text{m}]\mathbf{k}$$

PROBLEM 3.85 (Continued)

or $(0.90933F_A)\mathbf{k} = 230.45\mathbf{k}$

$F_A = 253.42 \text{ N}$ or $\mathbf{F}_A = 253 \text{ N}\downarrow$ ◀

We have $\Sigma\mathbf{F}: \quad \mathbf{F} = \mathbf{F}_A + \mathbf{F}_D$

$$-(231.40 \text{ N})\mathbf{i} - (275.78 \text{ N})\mathbf{j} = -(253.42 \text{ N})\mathbf{j} + F_D(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j})$$

From $\mathbf{i}: \quad 231.40 \text{ N} = F_D \cos\theta \qquad (1)$

$\mathbf{j}: \quad 22.36 \text{ N} = F_D \sin\theta \qquad (2)$

Equation (2) divided by Equation (1)

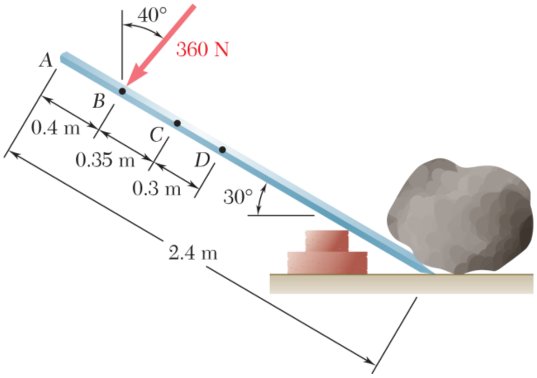
$\tan\theta = 0.096629$

$\theta = 5.5193°$ or $\theta = 5.52°$

Substitution into Equation (1)

$$F_D = \frac{231.40}{\cos 5.5193°} = 232.48 \text{ N}$$

or $\mathbf{F}_D = 232 \text{ N} \swarrow 5.52°$ ◀



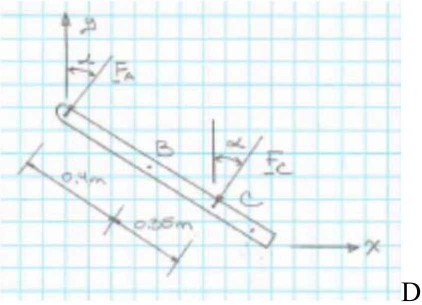
PROBLEM 3.86

A worker tries to move a rock by applying a 360-N force to a steel bar as shown. If two workers attempt to move the same rock by applying a force at *A* and a parallel force at *C*, determine these two forces so that they will be equivalent to the single 360-N force shown in the figure.

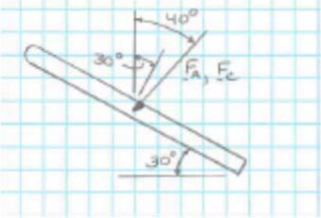
SOLUTION

$$\Sigma F_X:\ \ -360\sin 40^\circ = -F_A\sin\alpha - F_C\sin\alpha \quad (1)$$

$$\Sigma F_Y:\ \ -360\cos 40^\circ = F_A\cos\alpha - \mathrm{F}_C\cos\alpha \quad (2)$$



Dividing (1) by (2) yields $\alpha = 40^\circ$ and then $F_A + F_C = 360\text{ N} \quad (3)$



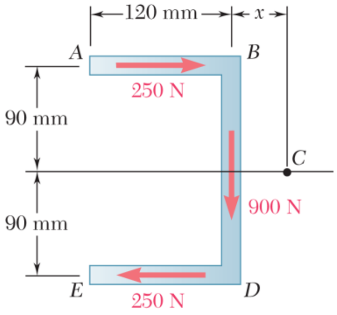
$$\Sigma\mathbf{M}_B:\ \ 0 = (0.4\text{ m})F_A\cos 10^\circ - (0.35\text{ m})F_C\cos 10^\circ$$

$$F_A = \frac{7}{8}F_C \quad (4)$$

Substituting into (3) ....

$$\frac{7}{8}F_C + F_C = 360$$

Finally $\mathbf{F}_A = 168.0\text{ N}$ ↙ 50.0°; $\mathbf{F}_C = 192.0\text{ N}$ ↙ 50.0°; ◀

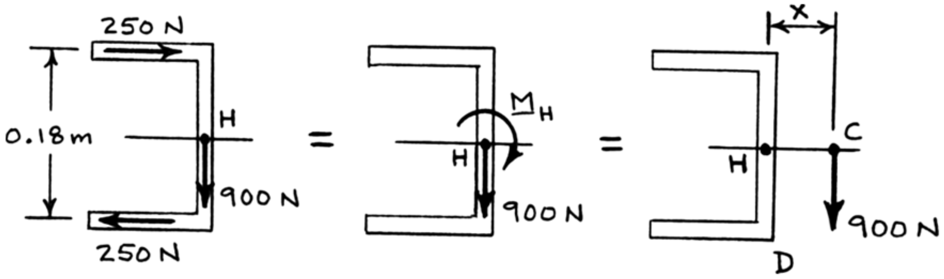

120 mm
x
A
B
250 N
90 mm
C
900 N
90 mm
E
250 N
D

PROBLEM 3.87

The shearing forces exerted on the cross section of a steel channel can be represented by a 900-N vertical force and two 250-N horizontal forces as shown. Replace this force and couple with a single force **F** applied at Point *C*, and determine the distance *x* from *C* to line *BD*. (Point *C* is defined as the *shear center* of the section.)

SOLUTION

Replace the 250-N forces with a couple and move the 900-N force to Point *C* such that its moment about *H* is equal to the moment of the couple

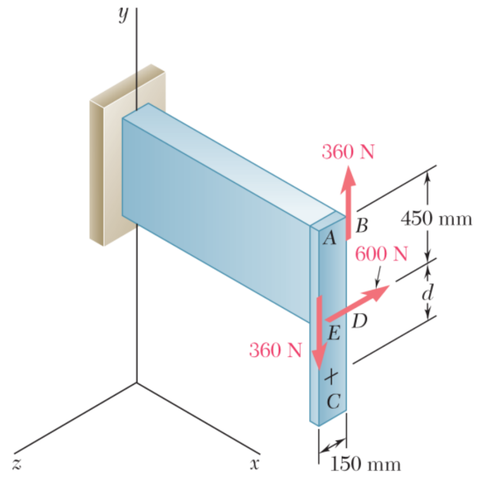

250 N
0.18 m
H
900 N
250 N
=
M_H
H
900 N
=
x
C
H
900 N
D

$$M_H = (0.18)(250\text{ N})$$
$$= 45\text{ N}\cdot\text{m}$$

Then $$M_H = x(900\text{ N})$$

or $$45\text{ N}\cdot\text{m} = x(900\text{ N})$$
$$x = 0.05\text{ m}$$

$\mathbf{F} = 900\text{ N}\downarrow \quad x = 50.0\text{ mm}$ ◀

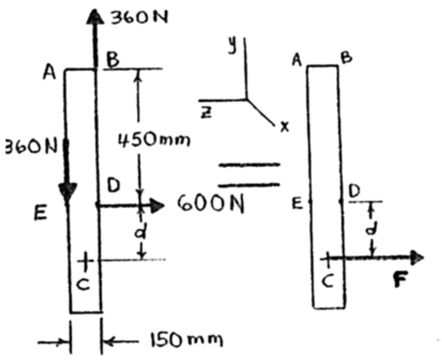


y
360 N
B
A
450 mm
600 N
d
D
E
360 N
C
z
x
150 mm

PROBLEM 3.88

A force and a couple are applied as shown to the end of a cantilever beam. (*a*) Replace this system with a single force **F** applied at Point *C*, and determine the distance *d* from *C* to a line drawn through Points *D* and *E*. (*b*) Solve part *a* if the directions of the two 360-N forces are reversed.

SOLUTION



360 N
A
B
360 N
450 mm
E
D
600 N
d
C
150 mm
y
z
x
=
A
B
E
D
d
C
F

(*a*) We have $\Sigma \mathbf{F}$: $\mathbf{F} = (360\text{ N})\mathbf{j} - (360\text{ N})\mathbf{j} - (600\text{ N})\mathbf{k}$

or $\mathbf{F} = -(600\text{ N})\mathbf{k}$ ◀

and ΣM_D: $(360\text{ N})(0.15\text{ m}) = (600\text{ N})(d)$

$d = 0.09\text{ m}$

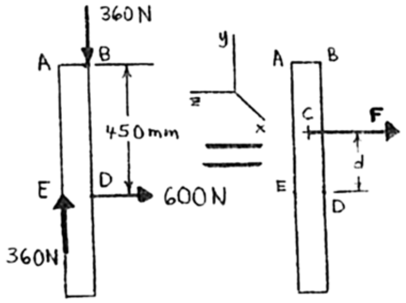
or $d = 90.0$ mm below *ED* ◀

(*b*) We have from part *a*: $\mathbf{F} = -(600\text{ N})\mathbf{k}$ ◀

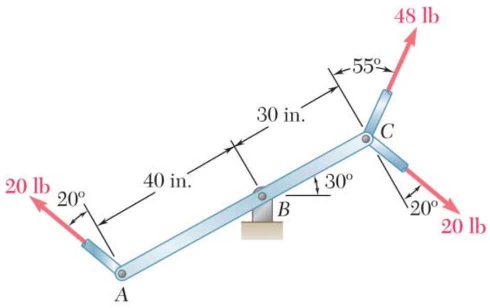
and ΣM_D: $-(360\text{ N})(0.15\text{ m}) = -(600\text{ N})(d)$

$d = 0.09\text{ m}$

or $d = 90.0$ mm above *ED* ◀



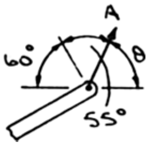
360 N
A
B
450 mm
E
D
600 N
360 N
y
z
x
=
A
B
C
F
d
E
D



PROBLEM 3.89

Three control rods attached to a lever *ABC* exert on it the forces shown. (*a*) Replace the three forces with an equivalent force-couple system at *B*. (*b*) Determine the single force that is equivalent to the force-couple system obtained in Part *a*, and specify its point of application on the lever.

SOLUTION



(*a*) First note that the two 20-lb forces form *A* couple. Then

$$\mathbf{F} = 48 \text{ lb} \measuredangle \theta$$

where $\theta = 180° - (60° + 55°) = 65°$

and

$$\begin{aligned} M &= \Sigma M_B \\ &= (30 \text{ in.})(48 \text{ lb})\cos 55° - (70 \text{ in.})(20 \text{ lb})\cos 20° \\ &= -489.62 \text{ lb}\cdot\text{in} \end{aligned}$$

The equivalent force-couple system at *B* is

$$\mathbf{F} = 48.0 \text{ lb} \measuredangle 65.0°; \qquad \mathbf{M} = 490 \text{ lb}\cdot\text{in.} \circlearrowright$$ ◀

(*b*) The single equivalent force $\mathbf{F}'$ is equal to **F**. Further, since the sense of **M** is clockwise, $\mathbf{F}'$ must be applied between *A* and *B*. For equivalence.

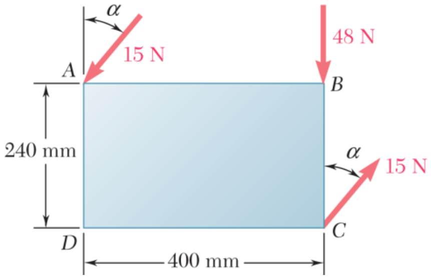
$$\Sigma M_B: \quad M = -aF' \cos 55°$$

where *a* is the distance from *B* to the point of application of $\mathbf{F}'$. Then

$$-489.62 \text{ lb}\cdot\text{in.} = -a(48.0 \text{ lb})\cos 55°$$

or $a = 17.78$ in. $\qquad \mathbf{F}' = 48.0 \text{ lb} \measuredangle 65.0°$ ◀

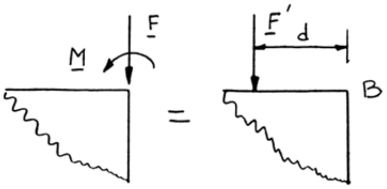
and is applied to the lever 17.78 in. to the left of pin *B* ◀



PROBLEM 3.90

A rectangular plate is acted upon by the force and couple shown. This system is to be replaced with a single equivalent force. (*a*) For $\alpha = 40°$, specify the magnitude and the line of action of the equivalent force. (*b*) Specify the value of α if the line of action of the equivalent force is to intersect line *CD* 300 mm to the right of *D*.

SOLUTION



(*a*) The given force-couple system (*F*, *M*) at *B* is

$$\mathbf{F} = 48 \text{ N}\downarrow$$

and $M = \Sigma M_B$

$$= (0.4 \text{ m})(15 \text{ N})\cos 40° + (0.24 \text{ m})(15 \text{ N})\sin 40°$$

or $\mathbf{M} = 6.9103 \text{ N}\cdot\text{m} \circlearrowright$

The single equivalent force F' is equal to **F**. Further for equivalence

$$\Sigma M_B:\quad M = dF'$$

or $6.9103 \text{ N}\cdot\text{m} = d \times 48 \text{ N}$

or $d = 0.14396 \text{ m}$ $\quad F' = 48.0 \text{ N}$ ◀

and the line of action of F' intersects line *AB* 144.0 mm to the right of *A*. ◀

(*b*) Following the solution to Part *a* but with $d = 0.1$ m and α unknown, have

$$\Sigma M_B:\quad (0.4 \text{ m})(15 \text{ N})\cos\alpha + (0.24 \text{ m})(15 \text{ N})\sin\alpha$$

$$= (0.1 \text{ m})(48 \text{ N})$$

or $5\cos\alpha + 3\sin\alpha = 4$

Rearranging and squaring $\quad 25\cos^2\alpha = (4 - 3\sin\alpha)^2$

Using $\cos^2\alpha = 1 - \sin^2\alpha$ and expanding

$$25(1 - \sin^2\alpha) = 16 - 24\sin\alpha + 9\sin^2\alpha$$

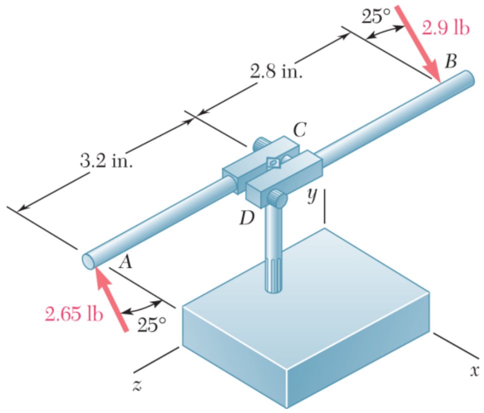
or $34\sin^2\alpha - 24\sin\alpha - 9 = 0$

Then

$$\sin\alpha = \frac{24 \pm \sqrt{(-24)^2 - 4(34)(-9)}}{2(34)}$$

$$\sin\alpha = 0.97686 \quad\text{or}\quad \sin\alpha = -0.27098$$

$$\alpha = 77.7° \quad\text{or}\quad \alpha = -15.72°$$ ◀



PROBLEM 3.91

While tapping a hole, a machinist applies the horizontal forces shown to the handle of the tap wrench. Show that these forces are equivalent to a single force, and specify, if possible, the point of application of the single force on the handle.

SOLUTION

Since the forces at *A* and *B* are parallel, the force at *B* can be replaced with the sum of two forces with one of the forces equal in magnitude to the force at *A* except with an opposite sense, resulting in a force-couple.

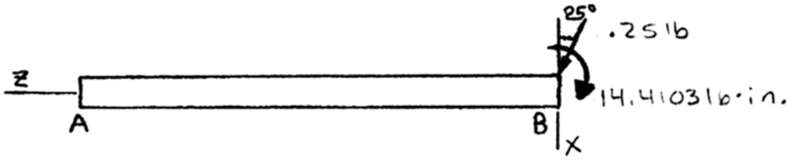
We have $F_B = 2.9\text{ lb} - 2.65\text{ lb} = 0.25\text{ lb}$, where the 2.65-lb force is part of the couple. Combining the two parallel forces,

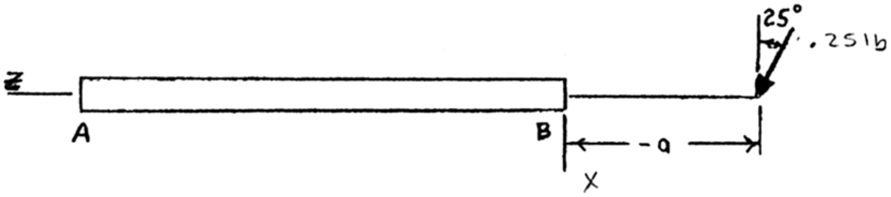
$$M_{\text{couple}} = (2.65\text{ lb})[(3.2\text{ in.} + 2.8\text{ in.})\cos 25°]$$

$$= 14.4103\text{ lb}\cdot\text{in.}$$

and

$$\mathbf{M}_{\text{couple}} = 14.4103\text{ lb}\cdot\text{in.}$$





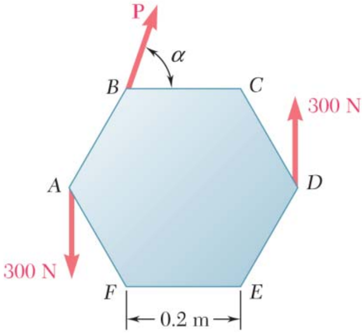
A single equivalent force will be located in the negative *z* direction.

Based on $\Sigma M_B: \quad -14.4103\text{ lb}\cdot\text{in.} = [(0.25\text{ lb})\cos 25°](a)$

$$a = 63.600\text{ in.}$$

$$\mathbf{F}' = (0.25\text{ lb})(\cos 25°\mathbf{i} + \sin 25°\mathbf{k})$$

$\mathbf{F}' = (0.227\text{ lb})\mathbf{i} + (0.1057\text{ lb})\mathbf{k}$ and is applied on an extension of handle *BD* at a distance of 63.6 in. to the right of *B*. ◀

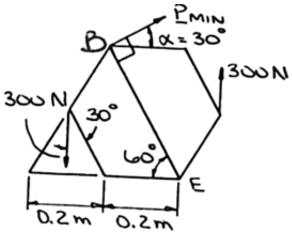


PROBLEM 3.92

A hexagonal plate is acted upon by the force **P** and the couple shown. Determine the magnitude and the direction of the smallest force **P** for which this system can be replaced with a single force at *E*.

SOLUTION

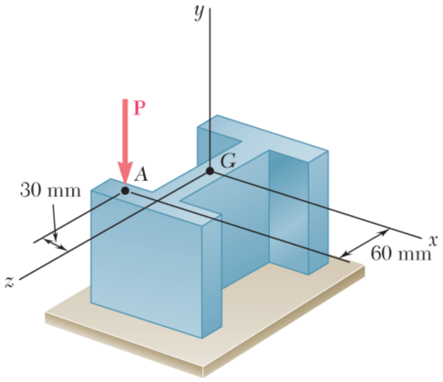
From the statement of the problem, it follows that $\Sigma M_E = 0$ for the given force-couple system. Further, for $\mathbf{P}_{\text{min}}$, we must require that **P** be perpendicular to $\mathbf{r}_{B/E}$. Then



$$\Sigma M_E: \quad (0.2 \sin 30° + 0.2)\text{m} \times 300\text{ N}$$
$$+ (0.2\text{ m}) \sin 30° \times 300\text{ N}$$
$$- (0.4\text{ m}) P_{\text{min}} = 0$$

or
$$P_{\text{min}} = 300\text{ N}$$

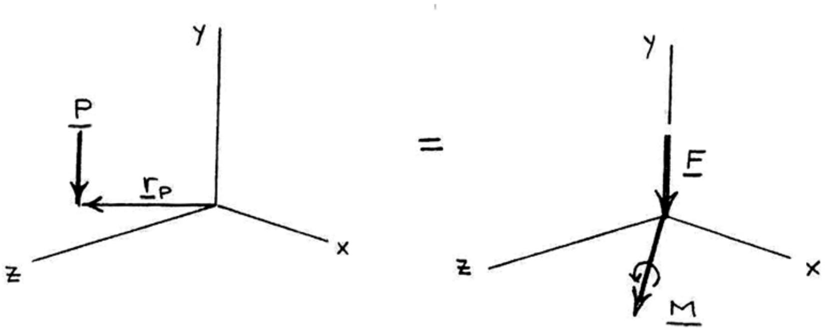
$\mathbf{P}_{\text{min}} = 300\text{ N} \measuredangle 30.0°$ ◀



PROBLEM 3.93

An eccentric, compressive 250-kN force **P** is applied to the end of a column. Replace **P** with an equivalent force-couple system at *G*.

SOLUTION



Have $\Sigma\mathbf{F}$: $-(250 \text{ kN})\mathbf{j} = \mathbf{F}$

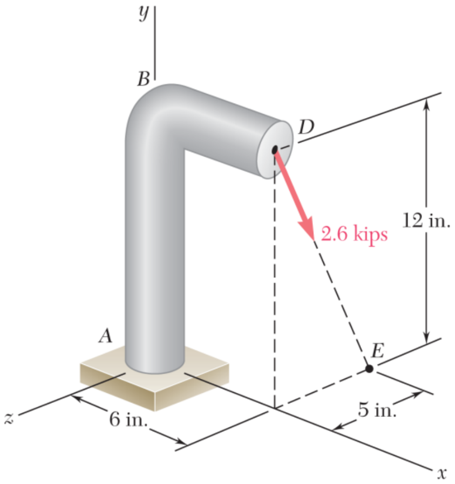
or $\mathbf{F} = -(250 \text{ kN})\mathbf{j}$ ◀

Also have $\Sigma\mathbf{M}_G$: $\mathbf{r}_P \times \mathbf{P} = \mathbf{M}$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.030 & 0 & 0.060 \\ 0 & -250 & 0 \end{vmatrix} \text{kN}\cdot\text{m} = \mathbf{M}$$

$$\therefore\ \mathbf{M} = (15 \text{ kN}\cdot\text{m})\mathbf{i} + (7.5 \text{ kN}\cdot\text{m})\mathbf{k}$$

or $\mathbf{M} = (15.00 \text{ kN}\cdot\text{m})\mathbf{i} + (7.50 \text{ kN}\cdot\text{m})\mathbf{k}$ ◀



PROBLEM 3.94

A 2.6-kip force is applied at Point D of the cast iron post shown. Replace that force with an equivalent force-couple system at the center A of the base section.

SOLUTION

$$\overrightarrow{DE} = -(12\text{ in.})\mathbf{j} - (5\text{ in.})\mathbf{k}; \quad DE = 13.00\text{ in.}$$

$$\mathbf{F} = (2.6\text{ kips})\frac{\overrightarrow{DE}}{DE}$$

$$\mathbf{F} = (2.6\text{ kips})\frac{-12\mathbf{j} - 5\mathbf{k}}{13}$$

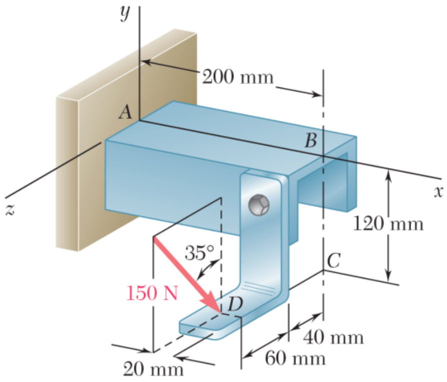
$$\mathbf{F} = -(2.40\text{ kips})\mathbf{j} - (1.000\text{ kip})\mathbf{k} \blacktriangleleft$$

$$\mathbf{M}_A = \mathbf{r}_{D/A} \times \mathbf{F}$$

where $\mathbf{r}_{D/A} = (6\text{ in.})\mathbf{i} + (12\text{ in.})\mathbf{j}$

$$\mathbf{M}_A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6\text{ in.} & 12\text{ in.} & 0 \\ 0 & -2.4\text{ kips} & -1.0\text{ kips} \end{vmatrix}$$

$$\mathbf{M}_A = -(12.00\text{ kip}\cdot\text{in.})\mathbf{i} + (6.00\text{ kip}\cdot\text{in.})\mathbf{j} - (14.40\text{ kip}\cdot\text{in.})\mathbf{k} \blacktriangleleft$$



PROBLEM 3.95

Replace the 150-N force with an equivalent force-couple system at *A*.

SOLUTION

Equivalence requires $\Sigma\mathbf{F}:\quad \mathbf{F} = (150\text{ N})(-\cos 35°\mathbf{j} - \sin 35°\mathbf{k})$

$$= -(122.873\text{ N})\mathbf{j} - (86.036\text{ N})\mathbf{k}$$

$$\Sigma\mathbf{M}_A:\quad \mathbf{M} = \mathbf{r}_{D/A} \times \mathbf{F}$$

where $\mathbf{r}_{D/A} = (0.18\text{ m})\mathbf{i} - (0.12\text{ m})\mathbf{j} + (0.1\text{ m})\mathbf{k}$

Then
$$\mathbf{M} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.18 & -0.12 & 0.1 \\ 0 & -122.873 & -86.036 \end{vmatrix} \text{N}\cdot\text{m}$$

$$= [(-0.12)(-86.036) - (0.1)(-122.873)]\mathbf{i}$$

$$+ [-(0.18)(-86.036)]\mathbf{j}$$

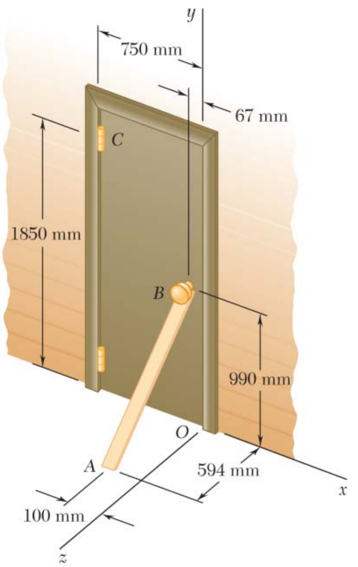
$$+ [(0.18)(-122.873)]\mathbf{k}$$

$$= (22.6\text{ N}\cdot\text{m})\mathbf{i} + (15.49\text{ N}\cdot\text{m})\mathbf{j} - (22.1\text{ N}\cdot\text{m})\mathbf{k}$$

The equivalent force-couple system at *A* is

$\mathbf{F} = -(122.9\text{ N})\mathbf{j} - (86.0\text{ N})\mathbf{k}$ ◀

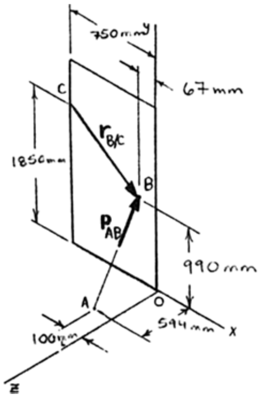
$\mathbf{M} = (22.6\text{ N}\cdot\text{m})\mathbf{i} + (15.49\text{ N}\cdot\text{m})\mathbf{j} - (22.1\text{ N}\cdot\text{m})\mathbf{k}$ ◀


y
750 mm
67 mm
C
1850 mm
B
990 mm
O
A
594 mm
x
100 mm
z

PROBLEM 3.96

To keep a door closed, a wooden stick is wedged between the floor and the doorknob. The stick exerts at *B* a 175-N force directed along line *AB*. Replace that force with an equivalent force-couple system at *C*.

SOLUTION


750 mm
y
67 mm
C
$\mathbf{r}_{B/C}$
1850 mm
B
$\mathbf{P}_{AB}$
990 mm
A
O
594 mm
x
100 mm
z

We have

$$\Sigma\mathbf{F}: \quad \mathbf{P}_{AB} = \mathbf{F}_C$$

where

$$\mathbf{P}_{AB} = \boldsymbol{\lambda}_{AB} P_{AB}$$

$$= \frac{(33\text{ mm})\mathbf{i} + (990\text{ mm})\mathbf{j} - (594\text{ mm})\mathbf{k}}{1155.00\text{ mm}}(175\text{ N})$$

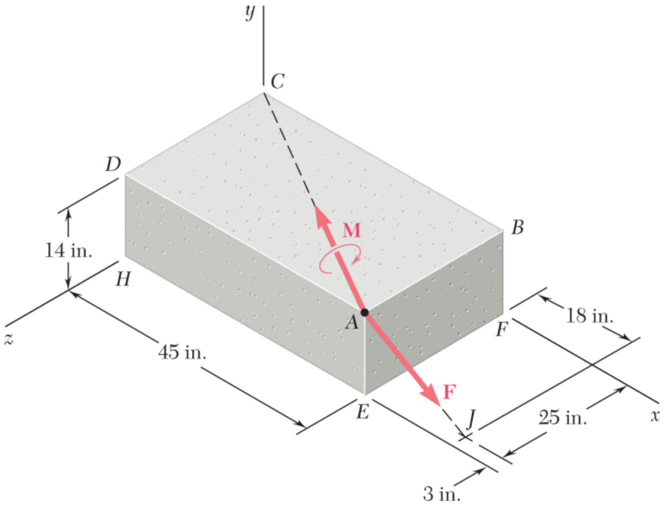
or $\mathbf{F}_C = (5.00\text{ N})\mathbf{i} + (150.0\text{ N})\mathbf{j} - (90.0\text{ N})\mathbf{k}$ ◀

We have $\Sigma\mathbf{M}_C: \quad \mathbf{r}_{B/C} \times \mathbf{P}_{AB} = \mathbf{M}_C$

$$\mathbf{M}_C = 5\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.683 & -0.860 & 0 \\ 1 & 30 & -18 \end{vmatrix} \text{N}\cdot\text{m}$$

$$= (5)\{(-0.860)(-18)\mathbf{i} - (0.683)(-18)\mathbf{j}$$
$$+ [(0.683)(30) - (0.860)(1)]\mathbf{k}\}$$

or $\mathbf{M}_C = (77.4\text{ N}\cdot\text{m})\mathbf{i} + (61.5\text{ N}\cdot\text{m})\mathbf{j} + (106.8\text{ N}\cdot\text{m})\mathbf{k}$ ◀



PROBLEM 3.97

A 46-lb force **F** and a 2120-lb·in. couple **M** are applied to corner *A* of the block shown. Replace the given force-couple system with an equivalent force-couple system at corner *H*.

SOLUTION

We have $$d_{AJ} = \sqrt{(18)^2 + (-14)^2 + (-3)^2} = 23 \text{ in.}$$

Then $$\mathbf{F} = \frac{46 \text{ lb}}{23}(18\mathbf{i} - 14\mathbf{j} - 3\mathbf{k})$$
$$= (36 \text{ lb})\mathbf{i} - (28 \text{ lb})\mathbf{j} - (6 \text{ lb})\mathbf{k}$$

Also $$d_{AC} = \sqrt{(-45)^2 + (0)^2 + (-28)^2} = 53 \text{ in.}$$

Then $$\mathbf{M} = \frac{2120 \text{ lb}\cdot\text{in.}}{53}(-45\mathbf{i} - 28\mathbf{k})$$
$$= -(1800 \text{ lb}\cdot\text{in.})\mathbf{i} - (1120 \text{ lb}\cdot\text{in.})\mathbf{k}$$

Now $$\mathbf{M}' = \mathbf{M} + \mathbf{r}_{A/H} \times \mathbf{F}$$

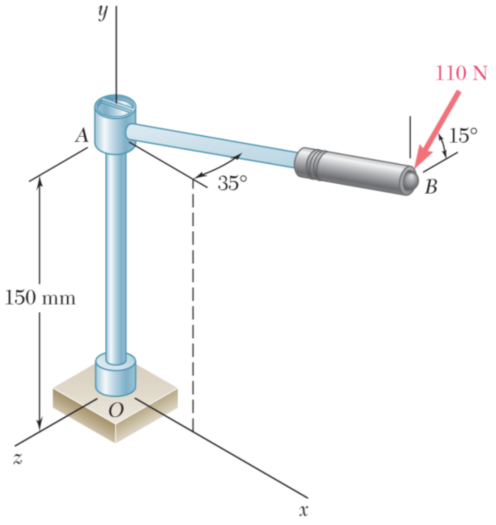
where $$\mathbf{r}_{A/H} = (45 \text{ in.})\mathbf{i} + (14 \text{ in.})\mathbf{j}$$

Then $$\mathbf{M}' = (-1800\mathbf{i} - 1120\mathbf{k}) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 45 & 14 & 0 \\ 36 & -28 & -6 \end{vmatrix}$$

$$= (-1800\mathbf{i} - 1120\mathbf{k}) + \{[(14)(-6)]\mathbf{i} + [-(45)(-6)]\mathbf{j} + [(45)(-28) - (14)(36)]\mathbf{k}\}$$
$$= (-1800 - 84)\mathbf{i} + (270)\mathbf{j} + (-1120 - 1764)\mathbf{k}$$
$$= -(1884 \text{ lb}\cdot\text{in.})\mathbf{i} + (270 \text{ lb}\cdot\text{in.})\mathbf{j} - (2884 \text{ lb}\cdot\text{in.})\mathbf{k}$$
$$= -(157 \text{ lb}\cdot\text{ft})\mathbf{i} + (22.5 \text{ lb}\cdot\text{ft})\mathbf{j} - (240 \text{ lb}\cdot\text{ft})\mathbf{k}$$

The equivalent force-couple system at *H* is $$\mathbf{F}' = (36.0 \text{ lb})\mathbf{i} - (28.0 \text{ lb})\mathbf{j} - (6.00 \text{ lb})\mathbf{k} \blacktriangleleft$$

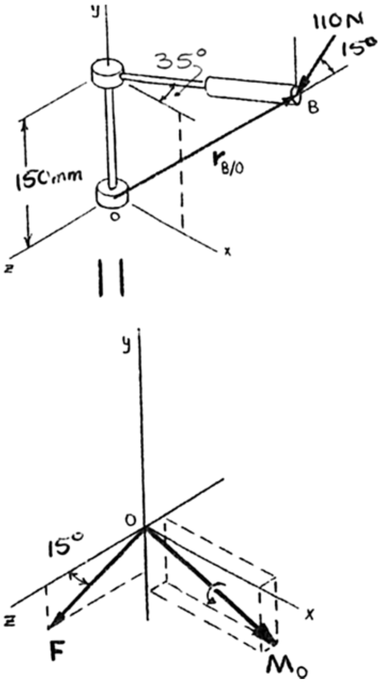
$$\mathbf{M}' = -(157.0 \text{ lb}\cdot\text{ft})\mathbf{i} + (22.5 \text{ lb}\cdot\text{ft})\mathbf{j} - (240 \text{ lb}\cdot\text{ft})\mathbf{k} \blacktriangleleft$$



PROBLEM 3.98

A 110-N force acting in a vertical plane parallel to the *yz*-plane is applied to the 220-mm-long horizontal handle *AB* of a socket wrench. Replace the force with an equivalent force-couple system at the origin *O* of the coordinate system.

SOLUTION



We have $\Sigma\mathbf{F}: \quad \mathbf{P}_B = \mathbf{F}$

where $\mathbf{P}_B = 110\text{ N}[-(\sin 15°)\mathbf{j} + (\cos 15°)\mathbf{k}]$

$$= -(28.470\text{ N})\mathbf{j} + (106.252\text{ N})\mathbf{k}$$

or $\mathbf{F} = -(28.5\text{ N})\mathbf{j} + (106.3\text{ N})\mathbf{k}$ ◀

We have $\Sigma M_O: \quad \mathbf{r}_{B/O} \times \mathbf{P}_B = \mathbf{M}_O$

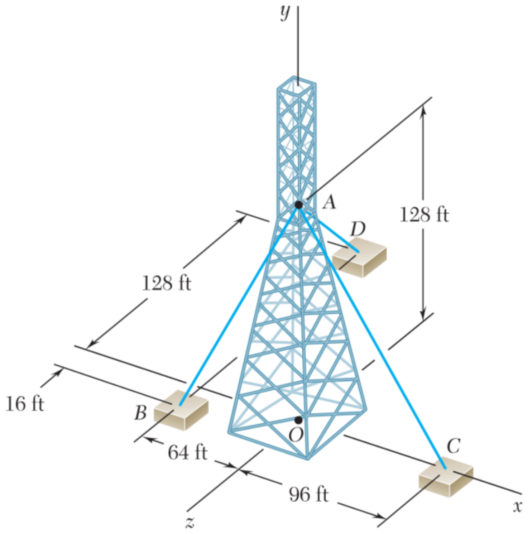
where $\mathbf{r}_{B/O} = [(0.22\cos 35°)\mathbf{i} + (0.15)\mathbf{j} - (0.22\sin 35°)\mathbf{k}]\text{ m}$

$$= (0.180213\text{ m})\mathbf{i} + (0.15\text{ m})\mathbf{j} - (0.126187\text{ m})\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.180213 & 0.15 & 0.126187 \\ 0 & -28.5 & 106.3 \end{vmatrix} \text{N} \cdot \text{m} = \mathbf{M}_O$$

$$\mathbf{M}_O = [(12.3487)\mathbf{i} - (19.1566)\mathbf{j} - (5.1361)\mathbf{k}]\text{ N} \cdot \text{m}$$

or $\mathbf{M}_O = (12.35\text{ N} \cdot \text{m})\mathbf{i} - (19.16\text{ N} \cdot \text{m})\mathbf{j} - (5.13\text{ N} \cdot \text{m})\mathbf{k}$ ◀



PROBLEM 3.99

An antenna is guyed by three cables as shown. Knowing that the tension in cable *AB* is 288 lb, replace the force exerted at *A* by cable *AB* with an equivalent force-couple system at the center *O* of the base of the antenna.

SOLUTION

We have $$d_{AB} = \sqrt{(-64)^2 + (-128)^2 + (16)^2} = 144 \text{ ft}$$

Then $$\mathbf{T}_{AB} = \frac{288 \text{ lb}}{144}(-64\mathbf{i} - 128\mathbf{j} + 16\mathbf{k})$$

$$= (32 \text{ lb})(-4\mathbf{i} - 8\mathbf{j} + \mathbf{k})$$

Now $$\mathbf{M} = \mathbf{M}_O = \mathbf{r}_{A/O} \times \mathbf{T}_{AB}$$

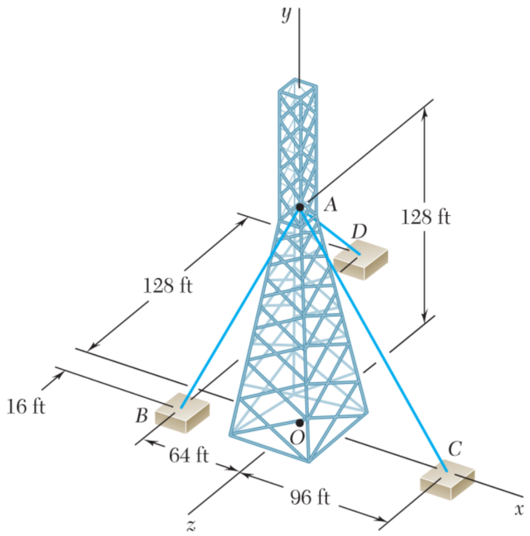
$$= 128\mathbf{j} \times 32(-4\mathbf{i} - 8\mathbf{j} + \mathbf{k})$$

$$= (4096 \text{ lb} \cdot \text{ft})\mathbf{i} + (16{,}384 \text{ lb} \cdot \text{ft})\mathbf{k}$$

The equivalent force-couple system at *O* is

$$\mathbf{F} = -(128.0 \text{ lb})\mathbf{i} - (256 \text{ lb})\mathbf{j} + (32.0 \text{ lb})\mathbf{k} \blacktriangleleft$$

$$\mathbf{M} = (4.10 \text{ kip} \cdot \text{ft})\mathbf{i} + (16.38 \text{ kip} \cdot \text{ft})\mathbf{k} \blacktriangleleft$$



PROBLEM 3.100

An antenna is guyed by three cables as shown. Knowing that the tension in cable *AD* is 270 lb, replace the force exerted at *A* by cable *AD* with an equivalent force-couple system at the center *O* of the base of the antenna.

SOLUTION

We have
$$d_{AD} = \sqrt{(-64)^2 + (-128)^2 + (-128)^2}$$
$$= 192 \text{ ft}$$

Then
$$\mathbf{T}_{AD} = \frac{270 \text{ lb}}{192}(-64\mathbf{i} - 128\mathbf{j} + 128\mathbf{k})$$
$$= (90 \text{ lb})(-\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$

Now
$$\mathbf{M} = \mathbf{M}_O = \mathbf{r}_{A/O} \times \mathbf{T}_{AD}$$
$$= 128\mathbf{j} \times 90(-\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$
$$= -(23{,}040 \text{ lb}\cdot\text{ft})\mathbf{i} + (11{,}520 \text{ lb}\cdot\text{ft})\mathbf{k}$$

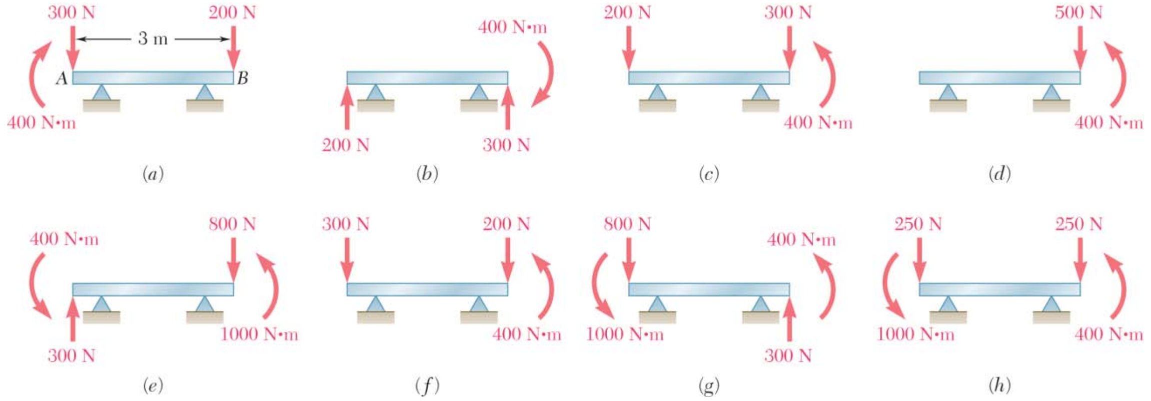
The equivalent force-couple system at *O* is

$$\mathbf{F} = -(90.0 \text{ lb})\mathbf{i} - (180.0 \text{ lb})\mathbf{j} - (180.0 \text{ lb})\mathbf{k} \blacktriangleleft$$

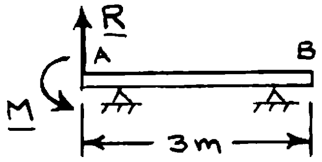
$$\mathbf{M} = -(23.0 \text{ kip}\cdot\text{ft})\mathbf{i} + (11.52 \text{ kip}\cdot\text{ft})\mathbf{k} \blacktriangleleft$$

PROBLEM 3.101

A 3-m-long beam is subjected to a variety of loadings. (*a*) Replace each loading with an equivalent force-couple system at end *A* of the beam. (*b*) Which of the loadings are equivalent?


300 N
200 N
3 m
A
B
400 N·m
(a)
400 N·m
200 N
300 N
(b)
200 N
300 N
400 N·m
(c)
500 N
400 N·m
(d)
400 N·m
800 N
300 N
1000 N·m
(e)
300 N
200 N
400 N·m
(f)
800 N
400 N·m
1000 N·m
300 N
(g)
250 N
250 N
1000 N·m
400 N·m
(h)

SOLUTION



(*a*) (*a*) We have $\Sigma F_Y: \quad -300\text{ N} - 200\text{ N} = R_a$

or $\mathbf{R}_a = 500\text{ N}\downarrow$ ◀

and $\Sigma M_A: \quad -400\text{ N}\cdot\text{m} - (200\text{ N})(3\text{ m}) = M_a$

or $\mathbf{M}_a = 1000\text{ N}\cdot\text{m}\ \circlearrowright$ ◀

(*b*) We have $\Sigma F_Y: \quad 200\text{ N} + 300\text{ N} = R_b$

or $\mathbf{R}_b = 500\text{ N}\uparrow$ ◀

and $\Sigma M_A: \quad -400\text{ N}\cdot\text{m} + (300\text{ N})(3\text{ m}) = M_b$

or $\mathbf{M}_b = 500\text{ N}\cdot\text{m}\ \circlearrowleft$ ◀

(*c*) We have $\Sigma F_Y: \quad -200\text{ N} - 300\text{ N} = R_c$

or $\mathbf{R}_c = 500\text{ N}\downarrow$ ◀

and $\Sigma M_A: \quad 400\text{ N}\cdot\text{m} - (300\text{ N})(3\text{ m}) = M_c$

or $\mathbf{M}_c = 500\text{ N}\cdot\text{m}\ \circlearrowright$ ◀

PROBLEM 3.101 (Continued)

(*d*) We have ΣF_Y: $-500\text{ N} = R_d$

or $\mathbf{R}_d = 500\text{ N}$ ↓ ◀

and ΣM_A: $400\text{ N}\cdot\text{m} - (500\text{ N})(3\text{ m}) = M_d$

or $\mathbf{M}_d = 1100\text{ N}\cdot\text{m}$ ↻ ◀

(*e*) We have ΣF_Y: $300\text{ N} - 800\text{ N} = R_e$

or $\mathbf{R}_e = 500\text{ N}$ ↓ ◀

and ΣM_A: $400\text{ N}\cdot\text{m} + 1000\text{ N}\cdot\text{m} - (800\text{ N})(3\text{ m}) = M_e$

or $\mathbf{M}_e = 1000\text{ N}\cdot\text{m}$ ↻ ◀

(*f*) We have ΣF_Y: $-300\text{ N} - 200\text{ N} = R_f$

or $\mathbf{R}_f = 500\text{ N}$ ↓ ◀

and ΣM_A: $400\text{ N}\cdot\text{m} - (200\text{ N})(3\text{ m}) = M_f$

or $\mathbf{M}_f = 200\text{ N}\cdot\text{m}$ ↻ ◀

(*g*) We have ΣF_Y: $-800\text{ N} + 300\text{ N} = R_g$

or $\mathbf{R}_g = 500\text{ N}$ ↓ ◀

and ΣM_A: $1000\text{ N}\cdot\text{m} + 400\text{ N}\cdot\text{m} + (300\text{ N})(3\text{ m}) = M_g$

or $\mathbf{M}_g = 2300\text{ N}\cdot\text{m}$ ↺ ◀

(*h*) We have ΣF_Y: $-250\text{ N} - 250\text{ N} = R_h$

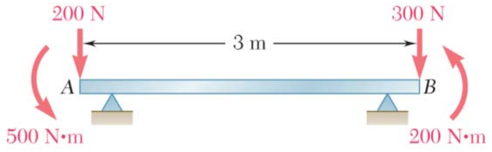
or $\mathbf{R}_h = 500\text{ N}$ ↓ ◀

and ΣM_A: $1000\text{ N}\cdot\text{m} + 400\text{ N}\cdot\text{m} - (250\text{ N})(3\text{ m}) = M_h$

or $\mathbf{M}_h = 650\text{ N}\cdot\text{m}$ ↺ ◀

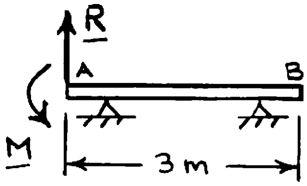
(*b*) Therefore, loadings (*a*) and (*e*) are equivalent.

PROBLEM 3.102



A 3-m-long beam is loaded as shown. Determine the loading of Prob. 3.101 that is equivalent to this loading.

SOLUTION



We have ΣF_Y: $-200\text{ N} - 300\text{ N} = R$

or $\mathbf{R} = 500\text{ N}\downarrow$

and ΣM_A: $500\text{ N}\cdot\text{m} + 200\text{ N}\cdot\text{m} - (300\text{ N})(3\text{ m}) = M$

or $\mathbf{M} = 200\text{ N}\cdot\text{m}$ ↻

Problem 3.101 equivalent force-couples at *A*:

Case	$\overline{\mathbf{R}}$	$\overline{\mathbf{M}}$	
(*a*)	500 N↓	1000 N·m ↻	
(*b*)	500 N↑	500 N·m ↺	
(*c*)	500 N↓	500 N·m ↻	
(*d*)	500 N↓	1100 N·m ↻	
(*e*)	500 N↓	1000 N·m ↻	
(*f*)	500 N↓	200 N·m ↻	←
(*g*)	500 N↓	2300 N·m ↺	
(*h*)	500 N↓	650 N·m ↺	

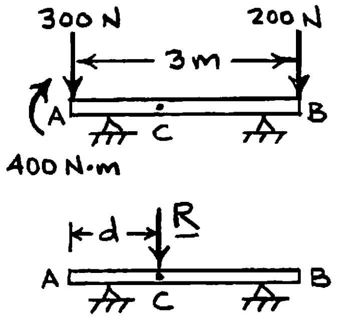
Equivalent to case (*f*) of Problem 3.101 ◀

PROBLEM 3.103

Determine the single equivalent force and the distance from Point A to its line of action for the beam and loading of (*a*) Prob. 3.101*a*, (*b*) Prob. 3.101*b*, (*c*) Prob. 3.102.

SOLUTION

For equivalent single force at distance d from A:

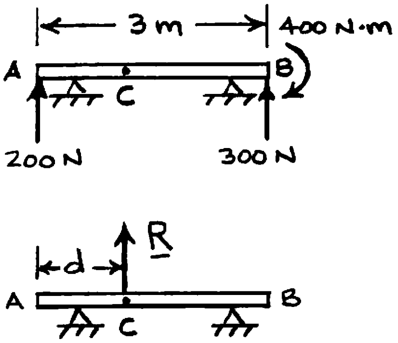

300 N
200 N
3 m
A
B
C
400 N·m
d
R
A
C
B

(*a*) We have ΣF_Y: $-300\text{ N} - 200\text{ N} = R$

or $\mathbf{R} = 500\text{ N}\downarrow$ ◀

and ΣM_C: $-400\text{ N}\cdot\text{m} + (300\text{ N})(d)$
$-(200\text{ N})(3 - d) = 0$

or $d = 2.00\text{ m}$ ◀

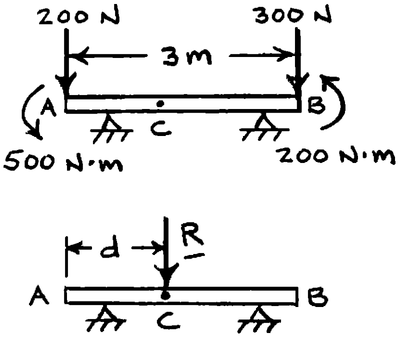

3 m
400 N·m
A
B
C
200 N
300 N
d
R
A
C
B

(*b*) We have ΣF_Y: $200\text{ N} + 300\text{ N} = R$

or $\mathbf{R} = 500\text{ N}\uparrow$ ◀

and ΣM_C: $-400\text{ N}\cdot\text{m} - (200\text{ N})(d)$
$+(300\text{ N})(3 - d) = 0$

or $d = 1.000\text{ m}$ ◀


200 N
300 N
3 m
A
B
C
500 N·m
200 N·m
d
R
A
C
B

(*c*) We have ΣF_Y: $-200\text{ N} - 300\text{ N} = R$

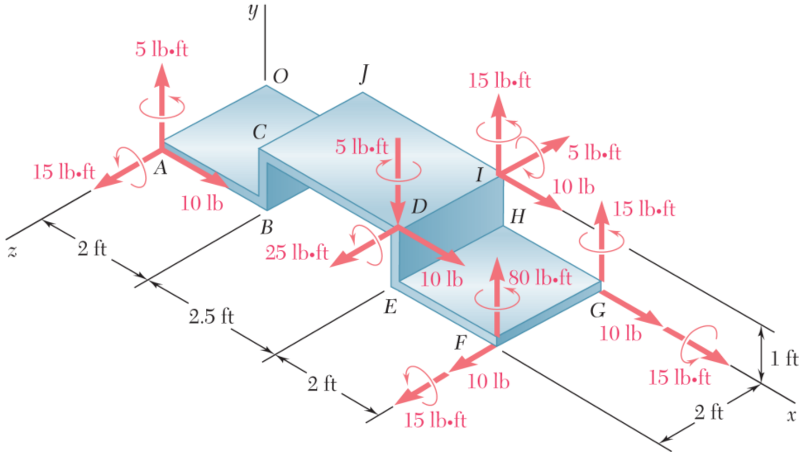
or $\mathbf{R} = 500\text{ N}\downarrow$ ◀

and ΣM_C: $500\text{ N}\cdot\text{m} + 200\text{ N}\cdot\text{m}$
$+(200\text{ N})(d) - (300\text{ N})(3 - d) = 0$

or $d = 0.400\text{ m}$ ◀

PROBLEM 3.104

Five separate force-couple systems act at the corners of a piece of sheet metal, which has been bent into the shape shown. Determine which of these systems is equivalent to a force $\mathbf{F} = (10\text{ lb})\mathbf{i}$ and a couple of moment $\mathbf{M} = (15\text{ lb}\cdot\text{ft})\mathbf{j} + (15\text{ lb}\cdot\text{ft})\mathbf{k}$ located at the origin.



SOLUTION

First note that the force-couple system at F cannot be equivalent because of the direction of the force [The force of the other four systems is $(10\text{ lb})\mathbf{i}$]. Next, move each of the systems to the origin O; the forces remain unchanged.

$$A:\quad \mathbf{M}_A = \Sigma\mathbf{M}_O = (5\text{ lb}\cdot\text{ft})\mathbf{j} + (15\text{ lb}\cdot\text{ft})\mathbf{k} + (2\text{ ft})\mathbf{k}\times(10\text{ lb})\mathbf{i}$$

$$= (25\text{ lb}\cdot\text{ft})\mathbf{j} + (15\text{ lb}\cdot\text{ft})\mathbf{k}$$

$$D:\quad \mathbf{M}_D = \Sigma\mathbf{M}_O = -(5\text{ lb}\cdot\text{ft})\mathbf{j} + (25\text{ lb}\cdot\text{ft})\mathbf{k}$$

$$+[(4.5\text{ ft})\mathbf{i} + (1\text{ ft})\mathbf{j} + (2\text{ ft})\mathbf{k}]\times 10\text{ lb})\mathbf{i}$$

$$= (15\text{ lb}\cdot\text{ft})\mathbf{i} + (15\text{ lb}\cdot\text{ft})\mathbf{k}$$

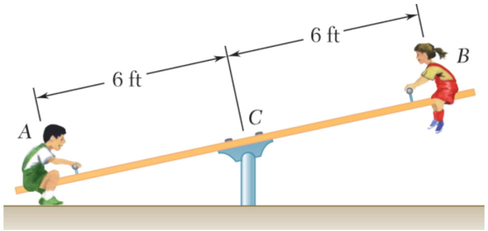
$$G:\quad \mathbf{M}_G = \Sigma\mathbf{M}_O = (15\text{ lb}\cdot\text{ft})\mathbf{i} + (15\text{ lb}\cdot\text{ft})\mathbf{j}$$

$$I:\quad \mathbf{M}_I = \Sigma\mathbf{M}_I = (15\text{ lb}\cdot\text{ft})\mathbf{j} - (5\text{ lb}\cdot\text{ft})\mathbf{k}$$

$$+[(4.5\text{ ft})\mathbf{i} + (1\text{ ft})\mathbf{j}]\times(10\text{ lb})\mathbf{j}$$

$$= (15\text{ lb}\cdot\text{ft})\mathbf{j} - (15\text{ lb}\cdot\text{ft})\mathbf{k}$$

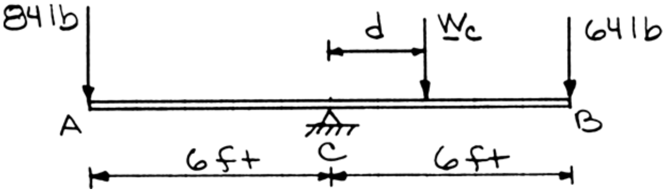
The equivalent force-couple system is the system at corner D. ◀



PROBLEM 3.105

The weights of two children sitting at ends *A* and *B* of a seesaw are 84 lb and 64 lb, respectively. Where should a third child sit so that the resultant of the weights of the three children will pass through *C* if she weighs (*a*) 60 lb, (*b*) 52 lb.

SOLUTION



(*a*) For the resultant weight to act at *C*, $\Sigma M_C = 0 \quad W_C = 60 \text{ lb}$

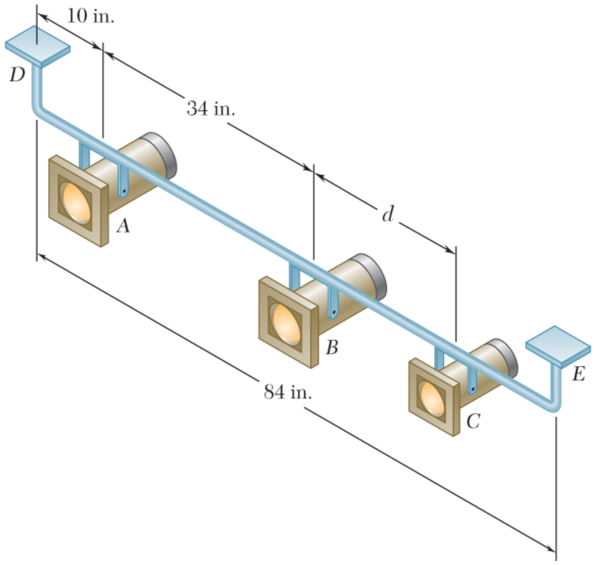
Then $(84 \text{ lb})(6 \text{ ft}) - 60 \text{ lb}(d) - 64 \text{ lb}(6 \text{ ft}) = 0$

$d = 2.00$ ft to the right of C ◀

(*b*) For the resultant weight to act at *C*, $\Sigma M_C = 0 \quad W_C = 52 \text{ lb}$

Then $(84 \text{ lb})(6 \text{ ft}) - 52 \text{ lb}(d) - 64 \text{ lb}(6 \text{ ft}) = 0$

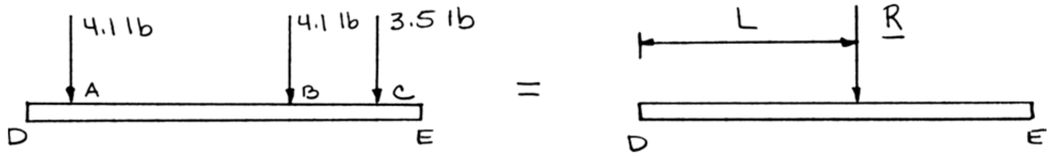
$d = 2.31$ ft to the right of C ◀



PROBLEM 3.106

Three stage lights are mounted on a pipe as shown. The lights at *A* and *B* each weigh 4.1 lb, while the one at *C* weighs 3.5 lb. (*a*) If $d = 25$ in., determine the distance from *D* to the line of action of the resultant of the weights of the three lights. (*b*) Determine the value of *d* so that the resultant of the weights passes through the midpoint of the pipe.

SOLUTION



For equivalence

$$\Sigma F_y: \quad -4.1 - 4.1 - 3.5 = -R \quad \text{or} \quad \mathbf{R} = 11.7 \text{ lb}\downarrow$$

$$\Sigma F_D: \quad -(10 \text{ in.})(4.1 \text{ lb}) - (44 \text{ in.})(4.1 \text{ lb})$$
$$-[(4.4 + d)\text{in.}](3.5 \text{ lb}) = -(L \text{ in.})(11.7 \text{ lb})$$

or $$375.4 + 3.5d = 11.7L \quad (d, L \text{ in in.})$$

(*a*) $$d = 25 \text{ in.}$$

We have $$375.4 + 3.5(25) = 11.7L \quad \text{or} \quad L = 39.6 \text{ in.}$$

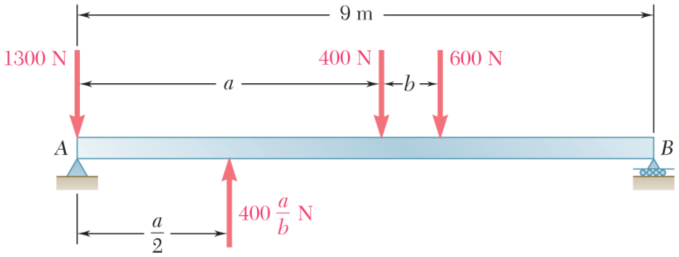
The resultant passes through a Point 39.6 in. to the right of *D*. ◀

(*b*) $$L = 42 \text{ in.}$$

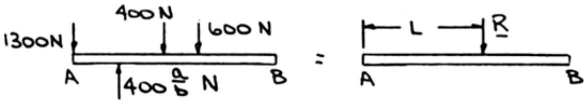
We have $$375.4 + 3.5d = 11.7(42)$$ or $d = 33.1$ in. ◀

PROBLEM 3.107

A beam supports three loads of given magnitude and a fourth load whose magnitude is a function of position. If $b = 1.5$ m and the loads are to be replaced with a single equivalent force, determine (*a*) the value of *a* so that the distance from support *A* to the line of action of the equivalent force is maximum, (*b*) the magnitude of the equivalent force and its point of application on the beam.



SOLUTION



For equivalence,

$$\Sigma F_y:\quad -1300 + 400\frac{a}{b} - 400 - 600 = -R$$

or

$$R = \left(2300 - 400\frac{a}{b}\right)\text{ N} \qquad (1)$$

$$\Sigma M_A:\quad \frac{a}{2}\left(400\frac{a}{b}\right) - a(400) - (a+b)(600) = -LR$$

or

$$L = \frac{1000a + 600b - 200\frac{a^2}{b}}{2300 - 400\frac{a}{b}}$$

Then with

$$b = 1.5\text{ m} \qquad L = \frac{10a + 9 - \frac{4}{3}a^2}{23 - \frac{8}{3}a} \qquad (2)$$

where *a*, *L* are in m.

(*a*) Find value of *a* to maximize *L*.

$$\frac{dL}{da} = \frac{\left(10 - \frac{8}{3}a\right)\left(23 - \frac{8}{3}a\right) - \left(10a + 9 - \frac{4}{3}a^2\right)\left(-\frac{8}{3}\right)}{\left(23 - \frac{8}{3}a\right)^2}$$

PROBLEM 3.107 (Continued)

or $$230 - \frac{184}{3}a - \frac{80}{3}a + \frac{64}{9}a^2 + \frac{80}{3}a + 24 - \frac{32}{9}a^2 = 0$$

or $$16a^2 - 276a + 1143 = 0$$

Then $$a = \frac{276 \pm \sqrt{(-276)^2 - 4(16)(1143)}}{2(16)}$$

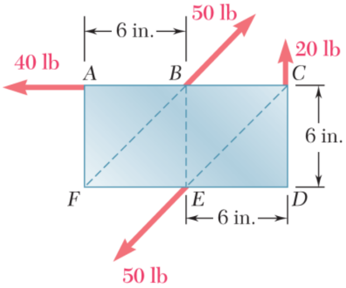
or $a = 10.3435$ m and $a = 6.9065$ m

Since $AB = 9$ m, a must be less than 9 m $a = 6.91$ m ◀

(*b*) Using Eq. (1), $$R = 2300 - 400\frac{6.9065}{1.5}$$ or $R = 458$ N ◀

and using Eq. (2), $$L = \frac{10(6.9065) + 9 - \frac{4}{3}(6.9065)^2}{23 - \frac{8}{3}(6.9065)} = 3.16 \text{ m}$$

R is applied 3.16 m to the right of *A*. ◀

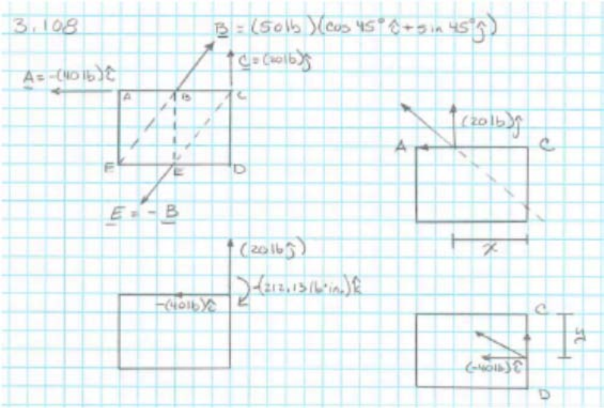

50 lb
6 in.
40 lb
20 lb
A
B
C
6 in.
F
E
D
6 in.
50 lb

PROBLEM 3.108

A 6 × 12-in. plate is subjected to four loads as shown. Find the resultant of the four loads and the two points at which the line of action of the resultant intersects the edge of the plate.

SOLUTION

We have


3.108
B = (50 lb)(cos 45° i + sin 45° j)
C = (20 lb) j
A = −(40 lb) i
A
B
C
F
E
D
E = − B
(20 lb) j
A
C
x
(20 lb j)
−(212.13 lb·in.) k
−(40 lb) i
C
y
(−40 lb) i
D

For equivalence, $\mathbf{R} = -(40 \text{ lb})\mathbf{i} + (20 \text{ lb})\mathbf{j}$

$\mathbf{R} = 44.721$ lb

R = 44.7 lb $\measuredangle$ 26.6° ◀

$$M_C = \mathbf{r}_{B/C} \times \mathbf{B}$$

$$= -(6 \text{ in.})\mathbf{i} \times (50 \text{ lb})(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j})$$

$$= -(212.13 \text{ lb} \cdot \text{in.})\mathbf{k}$$

Intersection with edge *AC*

$$M_C = -(x\mathbf{i}) \times (20)\mathbf{j}$$

$$-212.13\mathbf{k} = -(20x)\mathbf{k}$$

$$x = 10.61 \text{ in.}$$

10.61 in. to the left of *C*. ◀

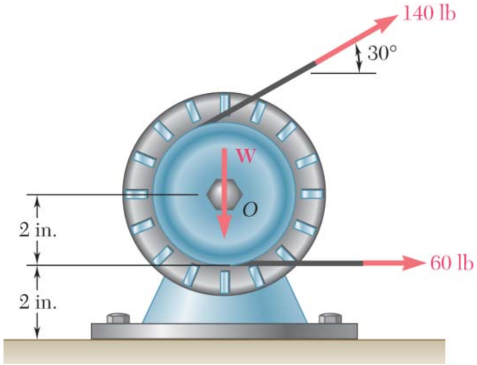
Intersection with edge *CD*

$$M_C = -(y\mathbf{j}) \times (-40\mathbf{i})$$

$$-213.13\mathbf{k} = -(40y)\mathbf{k}$$

$$y = 5.30 \text{ in.}$$

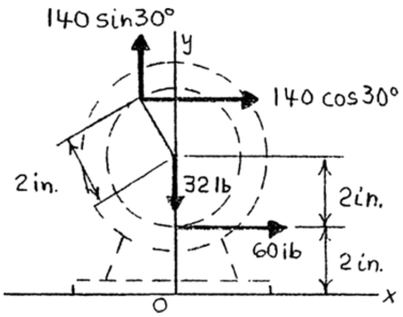
5.30 in. below *C*. ◀



PROBLEM 3.109

A 32-lb motor is mounted on the floor. Find the resultant of the weight and the forces exerted on the belt, and determine where the line of action of the resultant intersects the floor.

SOLUTION



We have

$$\Sigma\mathbf{F}: \quad (60\text{ lb})\mathbf{i} - (32\text{ lb})\mathbf{j} + (140\text{ lb})(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) = \mathbf{R}$$

$$\mathbf{R} = (181.244\text{ lb})\mathbf{i} + (38.0\text{ lb})\mathbf{j}$$

or $\mathbf{R} = 185.2\text{ lb} \measuredangle\, 11.84°$ ◀

We have $\Sigma M_O: \quad \Sigma M_O = xR_y$

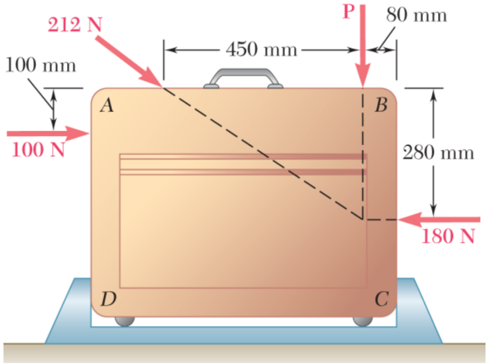
$$-[(140\text{ lb})\cos 30°][(4 + 2\cos 30°)\text{in.}] - [(140\text{ lb})\sin 30°][(2\text{ in.})\sin 30°]$$

$$-(60\text{ lb})(2\text{ in.}) = x(38.0\text{ lb})$$

$$x = \frac{1}{38.0}(-694.97 - 70.0 - 120)\text{ in.}$$

and $x = -23.289$ in.

Or resultant intersects the base (x-axis) 23.3 in. to the left of the vertical centerline (y-axis) of the motor. ◀

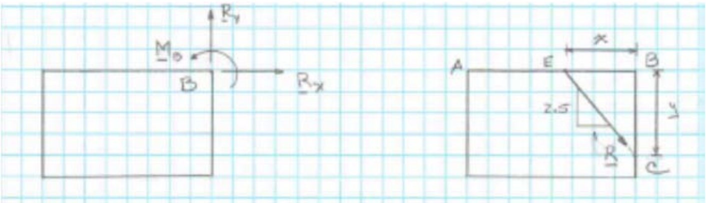


PROBLEM 3.110

To test the strength of a 625 × 500-mm suitcase, forces are applied as shown. If $P = 88$ N, (*a*) determine the resultant of the applied forces, (*b*) locate the two points where the line of action of the resultant intersects the edge of the suitcase.

SOLUTION

We have



First replace the applied forces and couples with an equivalent force-couple system at *B*.

(*a*)Thus, $\quad \Sigma F_x\!: \quad 100 + \dfrac{45}{53}(212) - 180 = R_x; \qquad R_x = 100\text{ N}$

$$\Sigma F_y\!: \quad -\frac{28}{53}(212) - 88 = R_y; \qquad R_y = -200\text{ N}$$

$$\mathbf{R} = (100\text{ N})\mathbf{i} - (200\text{ N})\mathbf{j}$$

$\mathbf{R} = 224\text{ N} \searrow 63.4°$ ◀

(*b*)

$$\Sigma M_B\!: \quad (0.1\text{ m})(100\text{ N}) + (0.53\text{ m})\left(\frac{28}{53}\right)(212\text{ N})$$

$$+ (0.08\text{ m})(88\text{ N}) - (0.28\text{ m})(180\text{ N}) = M_B$$

$$M_B = 26.0\text{ N}\cdot\text{m} \qquad (1)$$

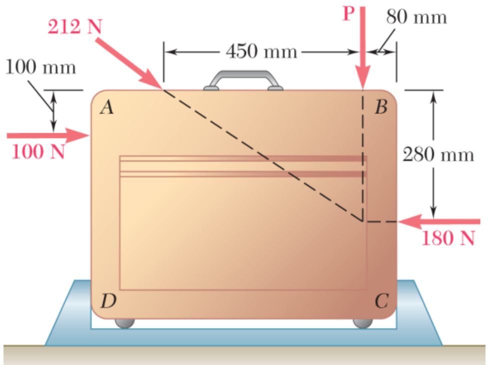
The single equivalent force **R** must then act as indicated. Then with **R** at *E*:

$$\Sigma M_B\!: \quad 26.0\text{ N}\cdot\text{m} = -x(200\text{ N})$$

$$x = 130\text{ mm}$$

$$\frac{y}{x} = \frac{2}{1} \quad \text{or } y = 260\text{ mm}$$

Therefore, the line of action of **R** intersects top *AB* 130.0 mm to the left of *B* and intersects side *BC* 260 mm below *B*. ◀



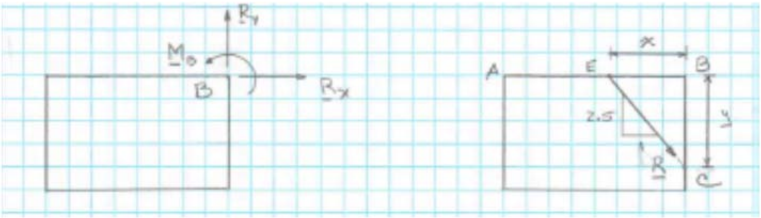
PROBLEM 3.111

Solve Prob. 3.110, assuming that $P = 138$ N.

Problem 3.110 To test the strength of a 625 × 500-mm suitcase, forces are applied as shown. If $P = 88$ N, (*a*) determine the resultant of the applied forces, (*b*) locate the two points where the line of action of the resultant intersects the edge of the suitcase.

SOLUTION

We have



First replace the applied forces and couples with an equivalent force-couple system at *B*.

(*a*) Thus,

$$\Sigma F_x:\quad 100 + \frac{45}{53}(212) - 180 = R_x; \quad R_x = 100\text{ N}$$

$$\Sigma F_y:\quad -\frac{28}{53}(212) - 138 = R_y; \qquad R_y = -250\text{ N}$$

$\mathbf{R} = 269\text{ N} \measuredangle 68.2^\circ$ ◀

(*b*)

$$\Sigma M_B:\quad (0.1\text{ m})(100\text{ N}) + (0.53\text{ m})\left(\frac{28}{53}\right)(212\text{ N})$$

$$+ (0.08\text{ m})(138\text{ N}) - (0.28\text{ m})(180\text{ N}) = M_B$$

$$M_B = 30.0\text{ N}\cdot\text{m}$$

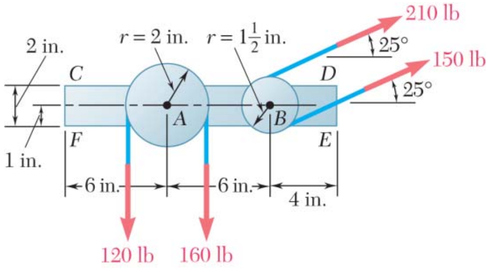
The single equivalent force **R** must then act as indicated. Then with **R** at *E*:

$$\Sigma M_B:\quad 30\text{ N}\cdot\text{m} = -x(250\text{ N})$$

$$x = 120\text{ mm}$$

$$\frac{y}{x} = \frac{5}{2} \quad \text{or } y = 300\text{ mm}$$

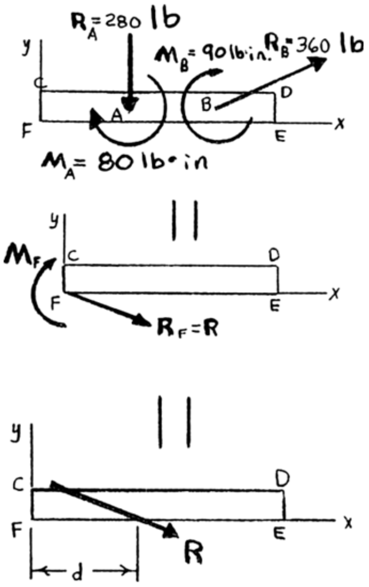
Therefore, the line of action of **R** intersects top *AB* 120 mm to the left of *B* and intersects side *BC* 300 mm below *B*. ◀


2 in.
$r = 2$ in.
$r = 1\frac{1}{2}$ in.
210 lb
25°
150 lb
25°
C
D
A
B
F
E
1 in.
6 in.
6 in.
4 in.
120 lb
160 lb

PROBLEM 3.112

Pulleys *A* and *B* are mounted on bracket *CDEF*. The tension on each side of the two belts is as shown. Replace the four forces with a single equivalent force, and determine where its line of action intersects the bottom edge of the bracket.

SOLUTION


$R_A = 280$ lb
$M_B = 90$ lb·in.
$R_B = 360$ lb
y
C
A
B
D
F
E
x
$M_A = 80$ lb·in
||
M_F
y
C
D
F
E
x
$R_F = R$
||
y
C
D
F
E
x
R
d

Equivalent force-couple at *A* due to belts on pulley *A*

We have $\Sigma\mathbf{F}: \quad -120 \text{ lb} - 160 \text{ lb} = R_A$

$$\mathbf{R}_A = 280 \text{ lb}\downarrow$$

We have $\Sigma\mathbf{M}_A: \quad -40 \text{ lb}(2 \text{ in.}) = M_A$

$$\mathbf{M}_A = 80 \text{ lb}\cdot\text{in.}\circlearrowright$$

Equivalent force-couple at *B* due to belts on pulley *B*

We have $\Sigma\mathbf{F}: \quad (210 \text{ lb} + 150 \text{ lb}) \measuredangle\ 25° = \mathbf{R}_B$

$$\mathbf{R}_B = 360 \text{ lb} \measuredangle\ 25°$$

We have $\Sigma\mathbf{M}_B: \quad -60 \text{ lb}(1.5 \text{ in.}) = M_B$

$$\mathbf{M}_B = 90 \text{ lb}\cdot\text{in.}\circlearrowright$$

Equivalent force-couple at *F*

We have $\Sigma\mathbf{F}: \quad \mathbf{R}_F = (-280 \text{ lb})\mathbf{j} + (360 \text{ lb})(\cos 25°\mathbf{i} + \sin 25°\mathbf{j})$

$$= (326.27 \text{ lb})\mathbf{i} - (127.857 \text{ lb})\mathbf{j}$$

$$R = R_F$$

$$= \sqrt{R_{Fx}^2 + R_{Fy}^2}$$

$$= \sqrt{(326.27)^2 + (127.857)^2}$$

$$= 350.43 \text{ lb}$$

$$\theta = \tan^{-1}\left(\frac{R_{Fy}}{R_{Fx}}\right)$$

$$= \tan^{-1}\left(\frac{-127.857}{326.27}\right)$$

$$= -21.399°$$

or $\mathbf{R}_F = \mathbf{R} = 350 \text{ lb} \measuredangle\ 21.4°$ ◀

PROBLEM 3.112 (Continued)

We have $\Sigma\mathbf{M}_F$: $M_F = -(280\text{ lb})(6\text{ in.}) - 80\text{ lb}\cdot\text{in.}$

$$-[(360\text{ lb})\cos 25°](1.0\text{ in.})$$
$$+[(360\text{ lb})\sin 25°](12\text{ in.}) - 90\text{ lb}\cdot\text{in.}$$

$$\mathbf{M}_F = -(350.56\text{ lb}\cdot\text{in.})\mathbf{k}$$

To determine where a single resultant force will intersect line *FE*,

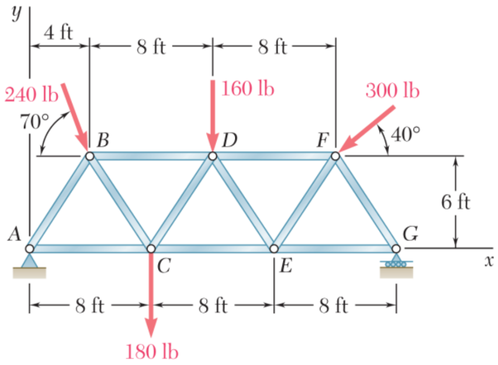
$$M_F = dR_y$$

$$d = \frac{M_F}{R_y}$$

$$= \frac{-350.56\text{ lb}\cdot\text{in.}}{-127\cdot857\text{ lb}}$$

$$= 2.7418\text{ in.}$$

or $d = 2.74$ in. ◀


y
4 ft
8 ft
8 ft
240 lb
70°
160 lb
300 lb
40°
B
D
F
6 ft
A
C
E
G
x
8 ft
8 ft
8 ft
180 lb

PROBLEM 3.113

A truss supports the loading shown. Determine the equivalent force acting on the truss and the point of intersection of its line of action with a line drawn through Points A and G.

SOLUTION

We have

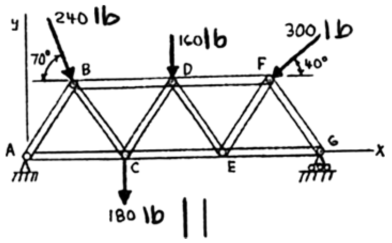
$$\mathbf{R} = \Sigma\mathbf{F}$$

$$\mathbf{R} = (240\text{ lb})(\cos 70°\mathbf{i} - \sin 70°\mathbf{j}) - (160\text{ lb})\mathbf{j}$$
$$+ (300\text{ lb})(-\cos 40°\mathbf{i} - \sin 40°\mathbf{j}) - (180\text{ lb})\mathbf{j}$$

$$\mathbf{R} = -(147.728\text{ lb})\mathbf{i} - (758.36\text{ lb})\mathbf{j}$$

$$R = \sqrt{R_x^2 + R_y^2}$$
$$= \sqrt{(147.728)^2 + (758.36)^2}$$
$$= 772.62\text{ lb}$$

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$
$$= \tan^{-1}\left(\frac{-758.36}{-147.728}\right)$$
$$= 78.977°$$


y
240 lb
160 lb
300 lb
70°
40°
B
D
F
A
C
E
G
x
180 lb

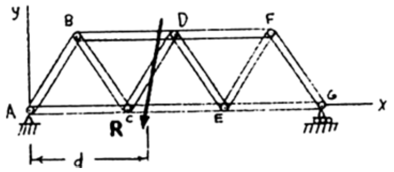
or $\mathbf{R} = 773\text{ lb}$ ↙ 79.0° ◀

We have $\Sigma M_A = dR_y$

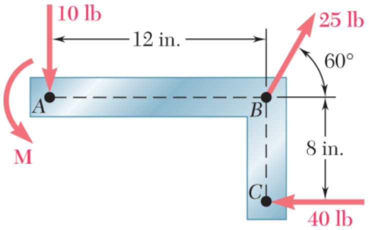
where

$$\Sigma M_A = -[240\text{ lb}\cos 70°](6\text{ ft}) - [240\text{ lb}\sin 70°](4\text{ ft})$$
$$-(160\text{ lb})(12\text{ ft}) + [300\text{ lb}\cos 40°](6\text{ ft})$$
$$-[300\text{ lb}\sin 40°](20\text{ ft}) - (180\text{ lb})(8\text{ ft})$$
$$= -7232.5\text{ lb}\cdot\text{ft}$$

$$d = \frac{-7232.5\text{ lb}\cdot\text{ft}}{-758.36\text{ lb}}$$
$$= 9.5370\text{ ft}$$


y
B
D
F
A
R
C
E
G
x
d

or $d = 9.54\text{ ft}$ to the right of A ◀



PROBLEM 3.114

A couple of magnitude $M = 80$ lb·in. and the three forces shown are applied to an angle bracket. (*a*) Find the resultant of this system of forces. (*b*) Locate the points where the line of action of the resultant intersects line *AB* and line *BC*.

SOLUTION

(*a*) We have

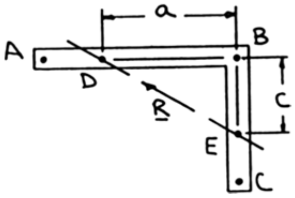
$$\Sigma\mathbf{F}: \quad \mathbf{R} = (-10\mathbf{j}) + (25\cos 60°)\mathbf{i} + (25\sin 60°)\mathbf{j} + (-40\mathbf{i})$$
$$= -(27.50\text{ lb})\mathbf{i} + (11.6506\text{ lb})\mathbf{j}$$

or $\mathbf{R} = 29.9\text{ lb} \measuredangle 23.0°$ ◀

(*b*) First reduce the given forces and couple to an equivalent force-couple system $(\mathbf{R},\ \mathbf{M}_B)$ at *B*.

We have

$$\Sigma M_B: \quad M_B = (80\text{ lb}\cdot\text{in}) + (12\text{ in.})(10\text{ lb}) - (8\text{ in.})(40\text{ lb})$$
$$= -120.0\text{ lb}\cdot\text{in.}$$



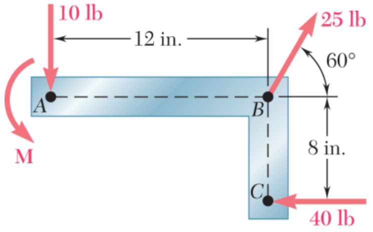
Then with **R** at *D*, $\Sigma M_B: \quad -120.0\text{ lb}\cdot\text{in} = a(11.6506\text{ lb})$

or $a = 10.30$ in.

and with **R** at *E*, $\Sigma M_B: \quad -120.0\text{ lb}\cdot\text{in} = c(27.5\text{ lb})$

or $c = 4.36$ in.

The line of action of **R** intersects line *AB* 10.30 in. to the left of *B* and intersects line *BC* 4.36 in. below *B*. ◀



PROBLEM 3.115

A couple **M** and the three forces shown are applied to an angle bracket. Find the moment of the couple if the line of action of the resultant of the force system is to pass through (*a*) point *A*, (*b*) point *B*, (*c*) point *C*.

SOLUTION

In each case, we must have $\mathbf{M}_1^R = 0$

(*a*) $+\circlearrowleft M_A^B = \Sigma M_A = M + (12\text{ in.})[(25\text{ lb})\sin 60°] - (8\text{ in.})(40\text{ lb}) = 0$

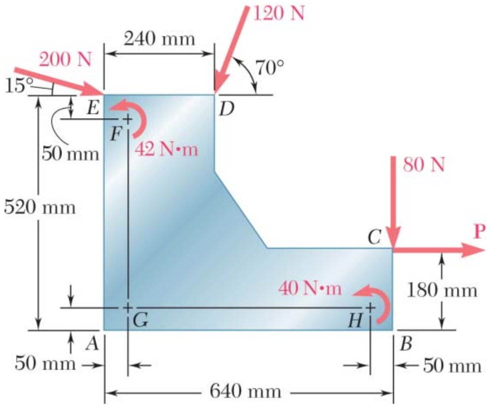
$M = +60.192\text{ lb}\cdot\text{in.}$ $\qquad \mathbf{M} = 60.2\text{ lb}\cdot\text{in.} \circlearrowleft$ ◀

(*b*) $+\circlearrowleft M_B^R = \Sigma M_B = M + (12\text{ in.})(10\text{ lb}) - (8\text{ in.})(40\text{ lb}) = 0$

$M = +200\text{ lb}\cdot\text{in.}$ $\qquad \mathbf{M} = 200\text{ lb}\cdot\text{in.} \circlearrowleft$ ◀

(*c*) $+\circlearrowleft M_C^R = \Sigma M_C = M + (12\text{ in.})(10\text{ lb}) - (8\text{ in.})[(25\text{ lb})\cos 60°] = 0$

$M = -20.0\text{ lb}\cdot\text{in.}$ $\qquad \mathbf{M} = 20.0\text{ lb}\cdot\text{in.} \circlearrowright$ ◀

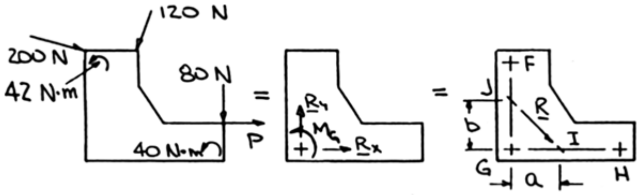

120 N
240 mm
200 N
70°
15°
E
D
F
50 mm
42 N·m
80 N
520 mm
C
P
40 N·m
180 mm
G
H
A
B
50 mm
50 mm
640 mm

PROBLEM 3.116

A machine component is subjected to the forces and couples shown. The component is to be held in place by a single rivet that can resist a force but not a couple. For $P = 0$, determine the location of the rivet hole if it is to be located (*a*) on line *FG*, (*b*) on line *GH*.

SOLUTION

We have


120 N
200 N
42 N·m
80 N
40 N·m
P
=
R_y
M_G
R_x
=
F
J
b
R
I
G
a
H

First replace the applied forces and couples with an equivalent force-couple system at *G*.

Thus, $\Sigma F_x\colon\quad 200\cos 15° - 120\cos 70° + P = R_x$

or $R_x = (152.142 + P)\text{ N}$

$\Sigma F_y\colon\quad -200\sin 15° - 120\sin 70° - 80 = R_y$

or $R_y = -244.53\text{ N}$

$$\begin{aligned}\Sigma M_G\colon\quad & -(0.47\text{ m})(200\text{ N})\cos 15° + (0.05\text{ m})(200\text{ N})\sin 15° \\ & + (0.47\text{ m})(120\text{ N})\cos 70° - (0.19\text{ m})(120\text{ N})\sin 70° \\ & - (0.13\text{ m})(P\text{ N}) - (0.59\text{ m})(80\text{ N}) + 42\text{ N}\cdot\text{m} \\ & + 40\text{ N}\cdot\text{m} = M_G\end{aligned}$$

or $M_G = -(55.544 + 0.13P)\text{ N}\cdot\text{m}$ (1)

Setting $P = 0$ in Eq. (1):

Now with **R** at *I*, $\Sigma M_G\colon\quad -55.544\text{ N}\cdot\text{m} = -a(244.53\text{ N})$

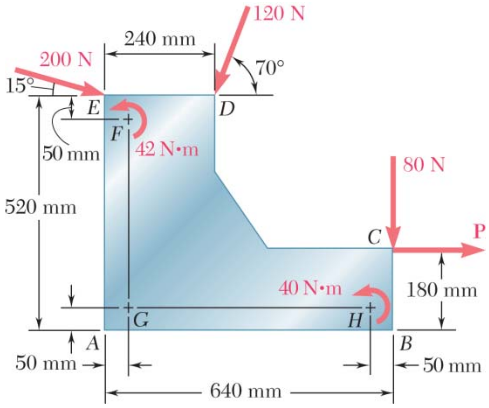
or $a = 0.227\text{ m}$

and with **R** at *J*, $\Sigma M_G\colon\quad -55.544\text{ N}\cdot\text{m} = -b(152.142\text{ N})$

or $b = 0.365\text{ m}$

(*a*) The rivet hole is 0.365 m above *G*. ◀

(*b*) The rivet hole is 0.227 m to the right of *G*. ◀



PROBLEM 3.117

Solve Problem 3.116, assuming that $P = 60$ N.

PROBLEM 3.116 A machine component is subjected to the forces and couples shown. The component is to be held in place by a single rivet that can resist a force but not a couple. For $P = 0$, determine the location of the rivet hole if it is to be located (*a*) on line *FG*, (*b*) on line *GH*.

SOLUTION

See the solution to Problem 3.116 leading to the development of Equation (1):

$$M_G = -(55.544 + 0.13P)\ \text{N} \cdot \text{m}$$

and $$R_x = (152.142 + P)\ \text{N}$$

For $$P = 60\ \text{N}$$

we have
$$\begin{aligned} R_x &= (152.142 + 60) \\ &= 212.14\ \text{N} \\ M_G &= -[55.544 + 0.13(60)] \\ &= -63.344\ \text{N} \cdot \text{m} \end{aligned}$$

Then with **R** at *I*, $$\Sigma M_G\!: \quad -63.344\ \ \text{N} \cdot \text{m} = -a(244.53\ \text{N})$$

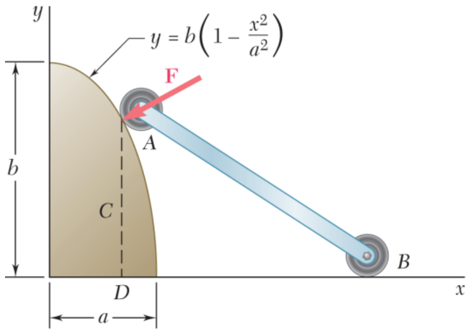
or $$a = 0.259\ \text{m}$$

and with **R** at *J*, $$\Sigma M_G\!: \quad -63.344\ \text{N} \cdot \text{m} = -b(212.14\ \text{N})$$

or $$b = 0.299\ \text{m}$$

(*a*) The rivet hole is 0.299 m above *G*. ◀

(*b*) The rivet hole is 0.259 m to the right of *G*. ◀



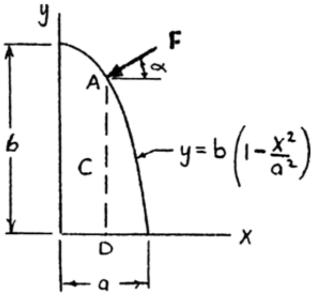
PROBLEM 3.118

As follower *AB* rolls along the surface of member *C*, it exerts a constant force **F** perpendicular to the surface. (*a*) Replace **F** with an equivalent force-couple system at Point *D* obtained by drawing the perpendicular from the point of contact to the *x*-axis. (*b*) For $a = 1$ m and $b = 2$ m, determine the value of *x* for which the moment of the equivalent force-couple system at *D* is maximum.

SOLUTION

(*a*) The slope of any tangent to the surface of member *C* is

$$\frac{dy}{dx} = \frac{d}{dx}\left[b\left(1 - \frac{x^2}{a^2}\right)\right] = \frac{-2b}{a^2}x$$



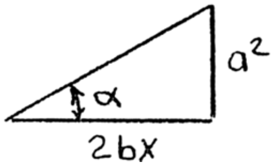
Since the force **F** is perpendicular to the surface,

$$\tan\alpha = -\left(\frac{dy}{dx}\right)^{-1} = \frac{a^2}{2b}\left(\frac{1}{x}\right)$$

For equivalence,

$$\Sigma F: \quad \mathbf{F} = \mathbf{R}$$

$$\Sigma M_D: \quad (F\cos\alpha)(y_A) = M_D$$



where

$$\cos\alpha = \frac{2bx}{\sqrt{(a^2)^2 + (2bx)^2}}$$

$$y_A = b\left(1 - \frac{x^2}{a^2}\right)$$

$$M_D = \frac{2Fb^2\left(x - \frac{x^3}{a^2}\right)}{\sqrt{a^4 + 4b^2x^2}}$$

Therefore, the equivalent force-couple system at *D* is

$$\mathbf{R} = F \swarrow \tan^{-1}\left(\frac{a^2}{2bx}\right) \blacktriangleleft$$

$$\mathbf{M} = \frac{2Fb^2\left(x - \frac{x^3}{a^2}\right)}{\sqrt{a^4 + 4b^2x^2}} \blacktriangleleft$$

PROBLEM 3.118 (Continued)

(*b*) To maximize *M*, the value of *x* must satisfy $\dfrac{dM}{dx} = 0$

where for $a = 1\text{ m}, \quad b = 2\text{ m}$

$$M = \frac{8F(x - x^3)}{\sqrt{1 + 16x^2}}$$

$$\frac{dM}{dx} = 8F\frac{\sqrt{1+16x^2}(1-3x^2) - (x - x^3)\left[\frac{1}{2}(32x)(1+16x^2)^{-1/2}\right]}{(1+16x^2)} = 0$$

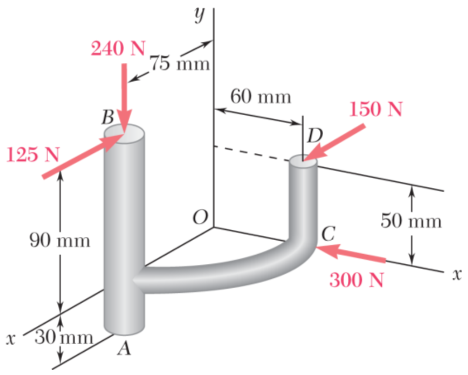
$$(1 + 16x^2)(1 - 3x^2) - 16x(x - x^3) = 0$$

or

$$32x^4 + 3x^2 - 1 = 0$$

$$x^2 = \frac{-3 \pm \sqrt{9 - 4(32)(-1)}}{2(32)} = 0.136011\text{ m}^2 \quad \text{and} \quad -0.22976\text{ m}^2$$

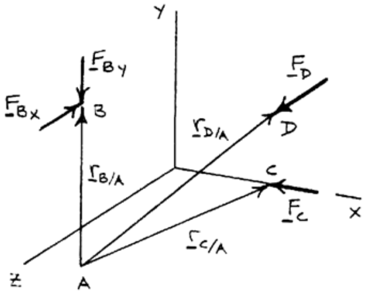
Using the positive value of x^2: $x = 0.36880\text{ m}$ or $x = 0.369\text{ m}$ ◀



PROBLEM 3.119

A machine component is subjected to the forces shown, each of which is parallel to one of the coordinate axes. Replace these forces with an equivalent force-couple system at *A*.

SOLUTION



For equivalence

$$\Sigma \mathbf{F}: \quad \mathbf{F}_B + \mathbf{F}_C + \mathbf{F}_D = \mathbf{R}_A$$

$$\mathbf{R}_A = -(240\text{ N})\mathbf{j} - (125\text{ N})\mathbf{k} - (300\text{ N})\mathbf{i} + (150\text{ N})\mathbf{k}$$

$$\therefore \quad \mathbf{R}_A = -(300\text{ N})\mathbf{i} - (240\text{ N})\mathbf{j} + (25.0\text{ N})\mathbf{k} \blacktriangleleft$$

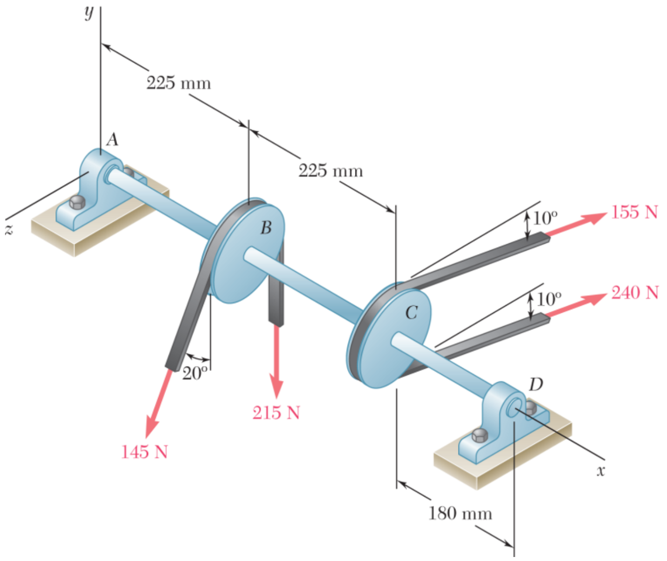
Also for equivalence

$$\Sigma \mathbf{M}_A: \quad \mathbf{r}_{B/A} \times \mathbf{F}_B + \mathbf{r}_{C/A} \times \mathbf{F}_C + \mathbf{r}_{D/A} \times \mathbf{F}_D = \mathbf{M}_A$$

$$\text{or } M_A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.12\text{ m} & 0 \\ 0 & -240\text{ N} & -125\text{ N} \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.06\text{ m} & 0.03\text{ m} & -0.075\text{ m} \\ -300\text{ N} & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.06\text{ m} & 0.08\text{ m} & -0.75\text{ m} \\ 0 & 0 & 150\text{ N} \end{vmatrix}$$

$$= \left[-(15\text{ N}\cdot\text{m})\mathbf{i}\right] + \left[(22.5\text{ N}\cdot\text{m})\mathbf{j} + (9\text{ N}\cdot\text{m})\mathbf{k}\right] + \left[(12\text{ N}\cdot\text{m})\mathbf{i} - (9\text{ N}\cdot\text{m})\mathbf{j}\right]$$

or $\quad \mathbf{M}_A = -(3.00\text{ N}\cdot\text{m})\mathbf{i} + (13.50\text{ N}\cdot\text{m})\mathbf{j} + (9.00\text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$

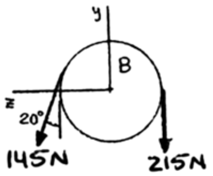


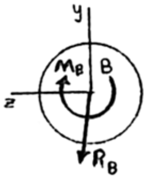
PROBLEM 3.120

Two 150-mm-diameter pulleys are mounted on line shaft *AD*. The belts at *B* and *C* lie in vertical planes parallel to the *yz*-plane. Replace the belt forces shown with an equivalent force-couple system at *A*.

SOLUTION

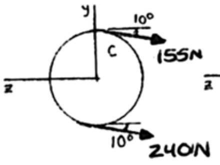
Equivalent force-couple at each pulley:

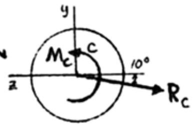




Pulley *B*:

$$\mathbf{R}_B = (145\text{ N})(-\cos 20°\mathbf{j} + \sin 20°\mathbf{k}) - 215\text{ N}\mathbf{j}$$
$$= -(351.26\text{ N})\mathbf{j} + (49.593\text{ N})\mathbf{k}$$
$$\mathbf{M}_B = -(215\text{ N} - 145\text{ N})(0.075\text{ m})\mathbf{i}$$
$$= -(5.25\text{ N}\cdot\text{m})\mathbf{i}$$





Pulley *C*:

$$\mathbf{R}_C = (155\text{ N} + 240\text{ N})(-\sin 10°\mathbf{j} - \cos 10°\mathbf{k})$$
$$= -(68.591\text{ N})\mathbf{j} - (389.00\text{ N})\mathbf{k}$$
$$\mathbf{M}_C = (240\text{ N} - 155\text{ N})(0.075\text{ m})\mathbf{i}$$
$$= (6.3750\text{ N}\cdot\text{m})\mathbf{i}$$

Then

$$\mathbf{R} = \mathbf{R}_B + \mathbf{R}_C = -(419.85\text{ N})\mathbf{j} - (339.41)\mathbf{k} \qquad \text{or} \quad \mathbf{R} = (420\text{ N})\mathbf{j} - (339\text{ N})\mathbf{k} \blacktriangleleft$$

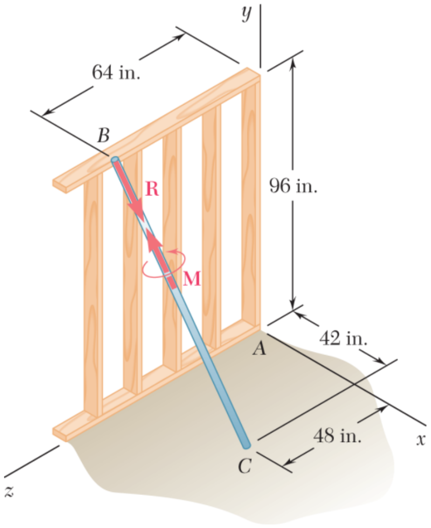
$$\mathbf{M}_A = \mathbf{M}_B + \mathbf{M}_C + \mathbf{r}_{B/A} \times \mathbf{R}_B + \mathbf{r}_{C/A} \times \mathbf{R}_C$$

$$= -(5.25\text{ N}\cdot\text{m})\mathbf{i} + (6.3750\text{ N}\cdot\text{m})\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.225 & 0 & 0 \\ 0 & -351.26 & 49.593 \end{vmatrix} \text{N}\cdot\text{m}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.45 & 0 & 0 \\ 0 & -68.591 & -389.00 \end{vmatrix} \text{N}\cdot\text{m}$$

$$= (1.12500\text{ N}\cdot\text{m})\mathbf{i} + (163.892\text{ N}\cdot\text{m})\mathbf{j} - (109.899\text{ N}\cdot\text{m})\mathbf{k}$$

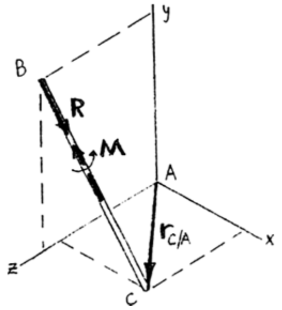
$$\text{or} \quad \mathbf{M}_A = (1.125\text{ N}\cdot\text{m})\mathbf{i} + (163.9\text{ N}\cdot\text{m})\mathbf{j} - (109.9\text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$$



PROBLEM 3.121

As an adjustable brace *BC* is used to bring a wall into plumb, the force-couple system shown is exerted on the wall. Replace this force-couple system with an equivalent force-couple system at *A* if $R = 21.2$ lb and $M = 13.25$ lb · ft.

SOLUTION



We have $\quad \Sigma\mathbf{F}: \quad \mathbf{R} = \mathbf{R}_A = \mathbf{R}\lambda_{BC}$

where $\boldsymbol{\lambda}_{BC} = \dfrac{(42 \text{ in.})\mathbf{i} - (96 \text{ in.})\mathbf{j} - (16 \text{ in.})\mathbf{k}}{106 \text{ in.}}$

$$\mathbf{R}_A = \frac{21.2 \text{ lb}}{106}(42\mathbf{i} - 96\mathbf{j} - 16\mathbf{k})$$

or $\quad \mathbf{R}_A = (8.40 \text{ lb})\mathbf{i} - (19.20 \text{ lb})\mathbf{j} - (3.20 \text{ lb})\mathbf{k}$ ◀

We have $\quad \Sigma\mathbf{M}_A: \quad \mathbf{r}_{C/A} \times \mathbf{R} + \mathbf{M} = \mathbf{M}_A$

where $\mathbf{r}_{C/A} = (42 \text{ in.})\mathbf{i} + (48 \text{ in.})\mathbf{k} = \dfrac{1}{12}(42\mathbf{i} + 48\mathbf{k})\text{ft}$

$$= (3.5 \text{ ft})\mathbf{i} + (4.0 \text{ ft})\mathbf{k}$$

$$\mathbf{R} = (8.40 \text{ lb})\mathbf{i} - (19.50 \text{ lb})\mathbf{j} - (3.20 \text{ lb})\mathbf{k}$$

$$\mathbf{M} = -\lambda_{BC}M$$

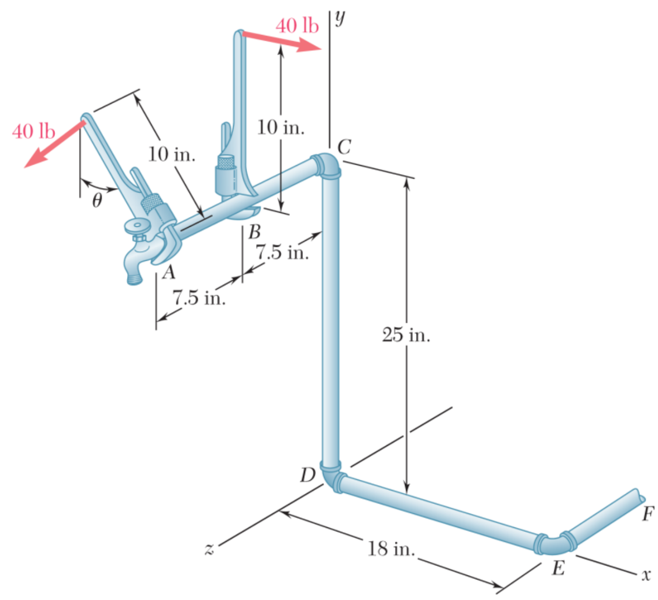
$$= \frac{-42\mathbf{i} + 96\mathbf{j} + 16\mathbf{k}}{106}(13.25 \text{ lb} \cdot \text{ft})$$

$$= -(5.25 \text{ lb} \cdot \text{ft})\mathbf{i} + (12 \text{ lb} \cdot \text{ft})\mathbf{j} + (2 \text{ lb} \cdot \text{ft})\mathbf{k}$$

Then $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3.5 & 0 & 4.0 \\ 8.40 & -19.20 & -3.20 \end{vmatrix} \text{lb} \cdot \text{ft} + (-5.25\mathbf{i} + 12\mathbf{j} + 2\mathbf{k}) \text{ lb} \cdot \text{ft} = \mathbf{M}_A$

$$\mathbf{M}_A = (71.55 \text{ lb} \cdot \text{ft})\mathbf{i} + (56.80 \text{ lb} \cdot \text{ft})\mathbf{j} - (65.20 \text{ lb} \cdot \text{ft})\mathbf{k}$$

or $\quad \mathbf{M}_A = (71.6 \text{ lb} \cdot \text{ft})\mathbf{i} + (56.8 \text{ lb} \cdot \text{ft})\mathbf{j} - (65.2 \text{ lb} \cdot \text{ft})\mathbf{k}$ ◀



PROBLEM 3.122

In order to unscrew the tapped faucet *A*, a plumber uses two pipe wrenches as shown. By exerting a 40-lb force on each wrench, at a distance of 10 in. from the axis of the pipe and in a direction perpendicular to the pipe and to the wrench, he prevents the pipe from rotating, and thus avoids loosening or further tightening the joint between the pipe and the tapped elbow *C*. Determine (*a*) the angle *θ* that the wrench at *A* should form with the vertical if elbow *C* is not to rotate about the vertical, (*b*) the force-couple system at *C* equivalent to the two 40-lb forces when this condition is satisfied.

SOLUTION

We first reduce the given forces to force-couple systems at *A* and *B*, noting that

$$|\mathbf{M}_A| = |\mathbf{M}_B| = (40\text{ lb})(10\text{ in.})$$
$$= 400\text{ lb}\cdot\text{in.}$$

We now determine the equivalent force-couple system at *C*.

$$\mathbf{R} = (40\text{ lb})(1-\cos\theta)\mathbf{i} - (40\text{ lb})\sin\theta\mathbf{j} \qquad (1)$$

$$\mathbf{M}_C^R = \mathbf{M}_A + \mathbf{M}_B + (15\text{ in.})\mathbf{k}\times[-(40\text{ lb})\cos\theta\mathbf{i} - (40\text{ lb})\sin\theta\mathbf{j}]$$
$$+ (7.5\text{ in.})\mathbf{k}\times(40\text{ lb})\mathbf{i}$$
$$= +400 - 400 - 600\cos\theta\mathbf{j} + 600\sin\theta\mathbf{i} + 300\mathbf{j}$$
$$= (600\text{ lb}\cdot\text{in.})\sin\theta\mathbf{i} + (300\text{ lb}\cdot\text{in.})(1-2\cos\theta)\mathbf{j} \qquad (2)$$

(*a*) For no rotation about vertical, *y* component of $\mathbf{M}_C^R$ must be zero.

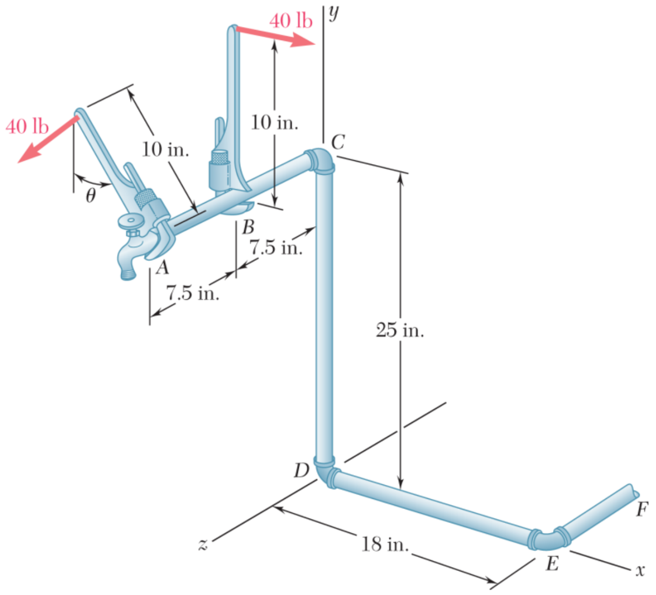
$$1 - 2\cos\theta = 0$$
$$\cos\theta = 1/2$$

$\theta = 60.0°$ ◀

(*b*) For $\theta = 60.0°$ in Eqs. (1) and (2),

$$\mathbf{R} = (20.0\text{ lb})\mathbf{i} - (34.641\text{ lb})\mathbf{j};\ \mathbf{M}_C^R = (519.62\text{ lb}\cdot\text{in.})\mathbf{i}$$

$\mathbf{R} = (20.0\text{ lb})\mathbf{i} - (34.6\text{ lb})\mathbf{j};\ \mathbf{M}_C^R = (520\text{ lb}\cdot\text{in.})\mathbf{i}$ ◀



PROBLEM 3.123

Assuming $\theta = 60°$ in Prob. 3.122, replace the two 40-lb forces with an equivalent force-couple system at *D* and determine whether the plumber's action tends to tighten or loosen the joint between (*a*) pipe *CD* and elbow *D*, (*b*) elbow *D* and pipe *DE*. Assume all threads to be right-handed.

PROBLEM 3.122 In order to unscrew the tapped faucet *A*, a plumber uses two pipe wrenches as shown. By exerting a 40-lb force on each wrench, at a distance of 10 in. from the axis of the pipe and in a direction perpendicular to the pipe and to the wrench, he prevents the pipe from rotating, and thus avoids loosening or further tightening the joint between the pipe and the tapped elbow *C*. Determine (*a*) the angle θ that the wrench at *A* should form with the vertical if elbow *C* is not to rotate about the vertical, (*b*) the force-couple system at *C* equivalent to the two 40-lb forces when this condition is satisfied.

SOLUTION

The equivalent force-couple system at *C* for $\theta = 60°$ was obtained in the solution to Prob. 3.122:

$$\mathbf{R} = (20.0 \text{ lb})\mathbf{i} - (34.641 \text{ lb})\mathbf{j}$$

$$\mathbf{M}_C^R = (519.62 \text{ lb}\cdot\text{in.})\mathbf{i}$$

The equivalent force-couple system at *D* is made of **R** and $\mathbf{M}_D^R$ where

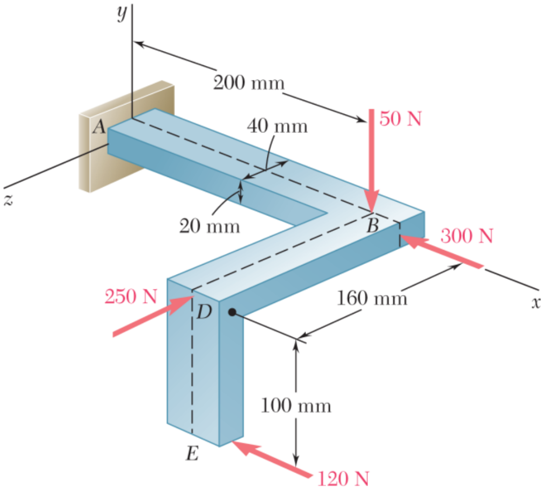
$$\begin{aligned}\mathbf{M}_D^R &= \mathbf{M}_C^R + \mathbf{r}_{C/D} \times \mathbf{R}\\ &= (519.62 \text{ lb}\cdot\text{in.})\mathbf{i} + (25.0 \text{ in.})\mathbf{j} \times [(20.0 \text{ lb})\mathbf{i} - (34.641 \text{ lb})\mathbf{j}]\\ &= (519.62 \text{ lb}\cdot\text{in.})\mathbf{i} - (500 \text{ lb}\cdot\text{in.})\mathbf{k}\end{aligned}$$

Equivalent force-couple at *D*:

$$\mathbf{R} = (20.0 \text{ lb})\mathbf{i} - (34.6 \text{ lb})\mathbf{j};\ \ \mathbf{M}_C^R = (520 \text{ lb}\cdot\text{in.})\mathbf{i} - (500 \text{ lb}\cdot\text{in.})\mathbf{k} \ \blacktriangleleft$$

(*a*) Since $\mathbf{M}_D^R$ has no component along the *y*-axis, the plumber's action will neither loosen nor tighten the joint between pipe *CD* and elbow. ◀

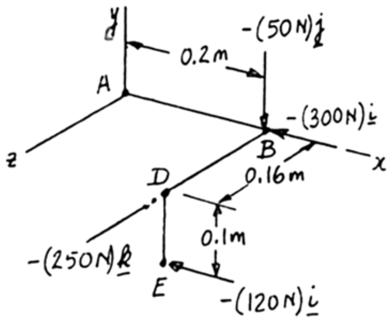
(*b*) Since the *x* component of $\mathbf{M}_D^R$ is ↶, the plumber's action will tend to tighten the joint between elbow and pipe *DE*. ◀



PROBLEM 3.124

Four forces are applied to the machine component *ABDE* as shown. Replace these forces with an equivalent force-couple system at *A*.

SOLUTION



$$\mathbf{R} = -(50\text{ N})\mathbf{j} - (300\text{ N})\mathbf{i} - (120\text{ N})\mathbf{i} - (250\text{ N})\mathbf{k}$$

$$\mathbf{R} = -(420\text{ N})\mathbf{i} - (50\text{ N})\mathbf{j} - (250\text{ N})\mathbf{k}$$

$$\mathbf{r}_B = (0.2\text{ m})\mathbf{i}$$

$$\mathbf{r}_D = (0.2\text{ m})\mathbf{i} + (0.16\text{ m})\mathbf{k}$$

$$\mathbf{r}_E = (0.2\text{ m})\mathbf{i} - (0.1\text{ m})\mathbf{j} + (0.16\text{ m})\mathbf{k}$$

$$\mathbf{M}_A^R = \mathbf{r}_B \times [-(300\text{ N})\mathbf{i} - (50\text{ N})\mathbf{j}]$$

$$+\mathbf{r}_D \times (-250\text{ N})\mathbf{k} + \mathbf{r} \times (-120\text{ N})\mathbf{i}$$

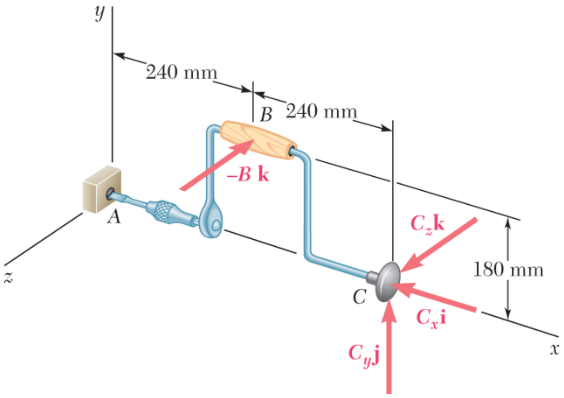
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.2\text{ m} & 0 & 0 \\ -300\text{ N} & -50\text{ N} & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.2\text{ m} & 0 & 0.16\text{ m} \\ 0 & 0 & -250\text{ N} \end{vmatrix}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.2\text{ m} & -0.1\text{ m} & 0.16\text{ m} \\ -120\text{ N} & 0 & 0 \end{vmatrix}$$

$$= -(10\text{ N}\cdot\text{m})\mathbf{k} + (50\text{ N}\cdot\text{m})\mathbf{j} - (19.2\text{ N}\cdot\text{m})\mathbf{j} - (12\text{ N}\cdot\text{m})\mathbf{k}$$

Force-couple system at *A* is

$$\mathbf{R} = -(420\text{ N})\mathbf{i} - (50\text{ N})\mathbf{j} - (250\text{ N})\mathbf{k} \quad \mathbf{M}_A^R = (30.8\text{ N}\cdot\text{m})\mathbf{j} - (22.0\text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$$



PROBLEM 3.125

A blade held in a brace is used to tighten a screw at A. (*a*) Determine the forces exerted at B and C, knowing that these forces are equivalent to a force-couple system at A consisting of $\mathbf{R} = -(25\text{ N})\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$ and $\mathbf{M}_A^R = -(13.5\text{ N}\cdot\text{m})\mathbf{i}$. (*b*) Find the corresponding values of R_y and R_z. (*c*) What is the orientation of the slot in the head of the screw for which the blade is least likely to slip when the brace is in the position shown?

SOLUTION

(*a*) Equivalence requires $\Sigma\mathbf{F}$: $\mathbf{R} = \mathbf{B} + \mathbf{C}$

or

$$-(25\text{ N})\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k} = -B\mathbf{k} + (-C_x\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k})$$

Equating the **i** coefficients: $\mathbf{i}:\ -25\text{ N} = -C_x$ or $C_x = 25\text{ N}$

Also, $\Sigma\mathbf{M}_A$: $\mathbf{M}_A^R = \mathbf{r}_{B/A}\times\mathbf{B} + \mathbf{r}_{C/A}\times\mathbf{C}$

or

$$-(13.5\text{ N}\cdot\text{m})\mathbf{i} = [(0.2\text{ m})\mathbf{i} + (0.15\text{ m})\mathbf{j}]\times(-B)\mathbf{k}$$
$$+(0.4\text{ m})\mathbf{i}\times[-(25\text{ N})\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k}]$$

Equating coefficients:

$\mathbf{i}:\ -13.5\text{ N}\cdot\text{m} = -(0.15\text{ m})B$ or $B = 75\text{ N}$

$\mathbf{k}:\ 0 = (0.4\text{ m})C_y$ or $C_y = 0$

$\mathbf{j}:\ 0 = (0.2\text{ m})(75\text{ N}) - (0.4\text{ m})C_z$ or $C_z = 37.5\text{ N}$

$$\mathbf{B} = -(75.0\text{ N})\mathbf{k}\quad \mathbf{C} = -(25.0\text{ N})\mathbf{i} + (37.5\text{ N})\mathbf{k}\ ◀$$

(*b*) Now we have for the equivalence of forces

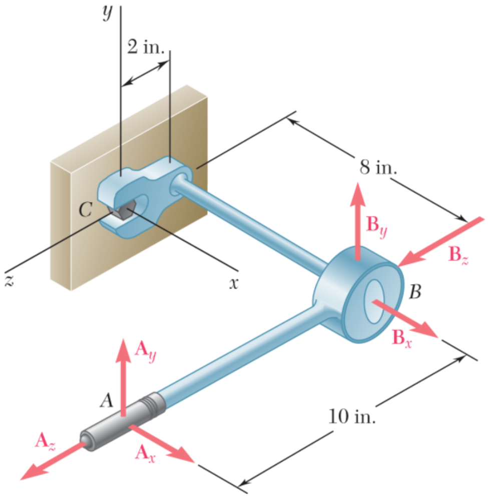
$$-(25\text{ N})\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k} = -(75\text{ N})\mathbf{k} + [(-25\text{ N})\mathbf{i} + (37.5\text{ N})\mathbf{k}]$$

Equating coefficients:

$\mathbf{j}:\ R_y = 0$ $\qquad R_y = 0$ ◀

$\mathbf{k}:\ R_z = -75 + 37.5$ or $R_z = -37.5\text{ N}$ ◀

(*c*) First note that $\mathbf{R} = -(25\text{ N})\mathbf{i} - (37.5\text{ N})\mathbf{k}$. Thus, the screw is best able to resist the lateral force R_z when the slot in the head of the screw is vertical. ◀



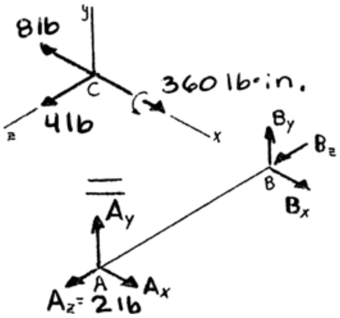
PROBLEM 3.126

A mechanic uses a crowfoot wrench to loosen a bolt at *C*. The mechanic holds the socket wrench handle at Points *A* and *B* and applies forces at these points. Knowing that these forces are equivalent to a force-couple system at *C* consisting of the force $\mathbf{C} = (8\text{ lb})\mathbf{i} + (4\text{ lb})\mathbf{k}$ and the couple $\mathbf{M}_C = (360\text{ lb}\cdot\text{in.})\mathbf{i}$, determine the forces applied at *A* and at *B* when $A_z = 2$ lb.

SOLUTION

We have $\Sigma\mathbf{F}$: $\mathbf{A} + \mathbf{B} = \mathbf{C}$

or F_x: $A_x + B_x = 8\text{ lb}$



(1)

$A_y = -B_y$ (2)

ΣF_z: $2\text{ lb} + B_z = 4\text{ lb}$

or $B_z = 2\text{ lb}$ (3)

We have $\Sigma\mathbf{M}_C$: $\mathbf{r}_{B/C} \times \mathbf{B} + \mathbf{r}_{A/C} \times \mathbf{A} = \mathbf{M}_C$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 0 & 2 \\ B_x & B_y & 2 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 0 & 8 \\ A_x & A_y & 2 \end{vmatrix} \text{lb}\cdot\text{in.} = (360\text{ lb}\cdot\text{in.})\mathbf{i}$$

or $(2B_y - 8A_y)\mathbf{i} + (2B_x - 16 + 8A_x - 16)\mathbf{j}$

$+(8B_y + 8A_y)\mathbf{k} = (360\text{ lb}\cdot\text{in.})\mathbf{i}$

From **i**-coefficient: $2B_y - 8A_y = 360\text{ lb}\cdot\text{in.}$ (4)

j-coefficient: $-2B_x + 8A_x = 32\text{ lb}\cdot\text{in.}$ (5)

k-coefficient: $8B_y + 8A_y = 0$ (6)

PROBLEM 3.126 (Continued)

From Equations (2) and (4): $2B_y - 8(-B_y) = 360$

$B_y = 36 \text{ lb} \quad A_y = 36 \text{ lb}$

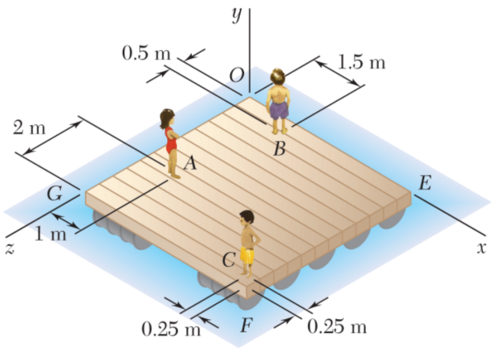
From Equations (1) and (5): $2(-A_x - 8) + 8A_x = 32$

$A_x = 1.6 \text{ lb}$

From Equation (1): $B_x = -(1.6 + 8) = -9.6 \text{ lb}$

$\mathbf{A} = (1.600 \text{ lb})\mathbf{i} - (36.0 \text{ lb})\mathbf{j} + (2.00 \text{ lb})\mathbf{k}$ ◀

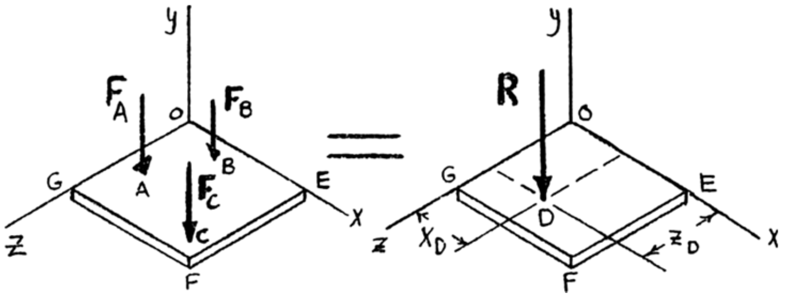
$\mathbf{B} = -(9.60 \text{ lb})\mathbf{i} + (36.0 \text{ lb})\mathbf{j} + (2.00 \text{ lb})\mathbf{k}$ ◀



PROBLEM 3.127

Three children are standing on a 5×5-m raft. If the weights of the children at Points *A*, *B*, and *C* are 375 N, 260 N, and 400 N, respectively, determine the magnitude and the point of application of the resultant of the three weights.

SOLUTION



We have $\Sigma\mathbf{F}:\quad \mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_C = \mathbf{R}$

$$-(375\text{ N})\mathbf{j} - (260\text{ N})\mathbf{j} - (400\text{ N})\mathbf{j} = \mathbf{R}$$

$$-(1035\text{ N})\mathbf{j} = \mathbf{R}$$

or $R = 1035\text{ N}$ ◀

We have $\Sigma M_x:\quad F_A(z_A) + F_B(z_B) + F_C(z_C) = R(z_D)$

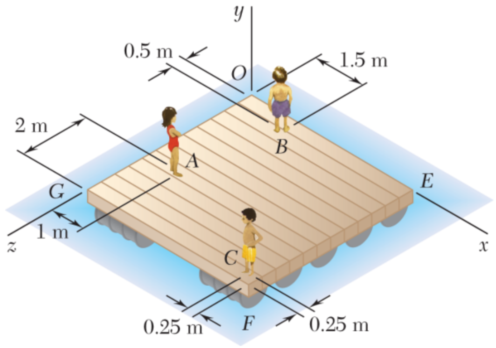
$$(375\text{ N})(3\text{ m}) + (260\text{ N})(0.5\text{ m}) + (400\text{ N})(4.75\text{ m}) = (1035\text{ N})(z_D)$$

$z_D = 3.0483\text{ m}$ or $z_D = 3.05\text{ m}$ ◀

We have $\Sigma M_z:\quad F_A(x_A) + F_B(x_B) + F_C(x_C) = R(x_D)$

$$375\text{ N}(1\text{ m}) + (260\text{ N})(1.5\text{ m}) + (400\text{ N})(4.75\text{ m}) = (1035\text{ N})(x_D)$$

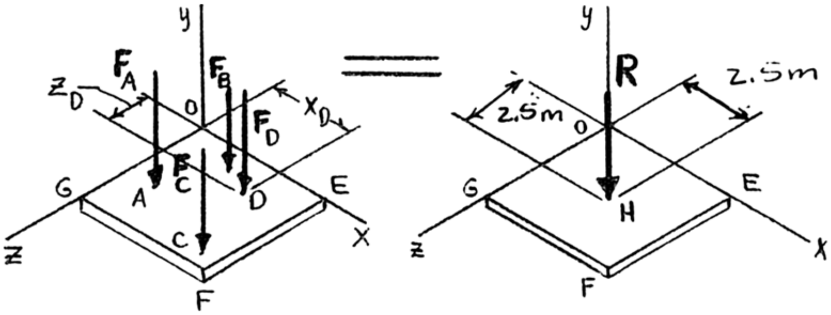
$x_D = 2.5749\text{ m}$ or $x_D = 2.57\text{ m}$ ◀



PROBLEM 3.128

Three children are standing on a 5×5-m raft. The weights of the children at Points *A*, *B*, and *C* are 375 N, 260 N, and 400 N, respectively. If a fourth child of weight 425 N climbs onto the raft, determine where she should stand if the other children remain in the positions shown and the line of action of the resultant of the four weights is to pass through the center of the raft.

SOLUTION



We have $\qquad \Sigma \mathbf{F}: \quad \mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_C = \mathbf{R}$

$$-(375\text{ N})\mathbf{j} - (260\text{ N})\mathbf{j} - (400\text{ N})\mathbf{j} - (425\text{ N})\mathbf{j} = \mathbf{R}$$

$$\mathbf{R} = -(1460\text{ N})\mathbf{j}$$

We have $\qquad \Sigma M_x: \quad F_A(z_A) + F_B(z_B) + F_C(z_C) + F_D(z_D) = R(z_H)$

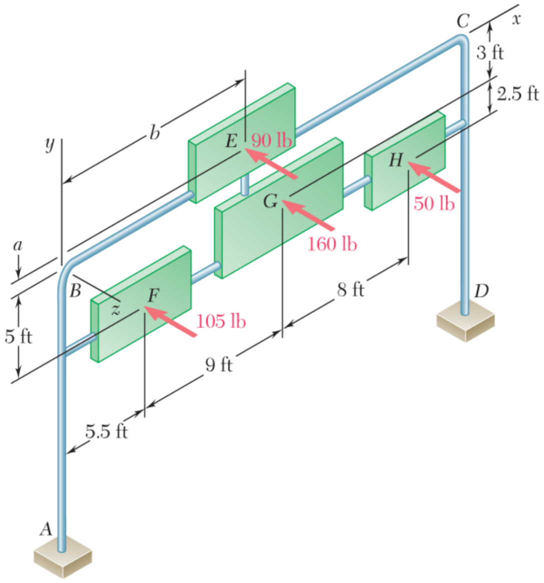
$$(375\text{ N})(3\text{ m}) + (260\text{ N})(0.5\text{ m}) + (400\text{ N})(4.75\text{ m})$$
$$+ (425\text{ N})(z_D) = (1460\text{ N})(2.5\text{ m})$$

$z_D = 1.16471\text{ m}$ $\qquad$ or $\quad z_D = 1.165\text{ m}$ ◀

We have $\qquad \Sigma M_z: \quad F_A(x_A) + F_B(x_B) + F_C(x_C) + F_D(x_D) = R(x_H)$

$$(375\text{ N})(1\text{ m}) + (260\text{ N})(1.5\text{ m}) + (400\text{ N})(4.75\text{ m})$$
$$+ (425\text{ N})(x_D) = (1460\text{ N})(2.5\text{ m})$$

$x_D = 2.3235\text{ m}$ $\qquad$ or $\quad x_D = 2.32\text{ m}$ ◀

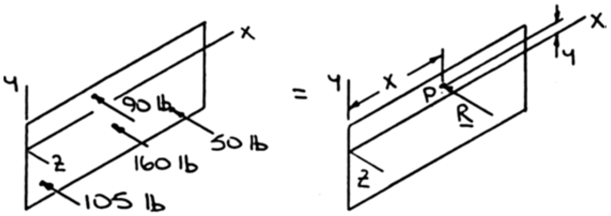

C
x
3 ft
2.5 ft
y
b
E
90 lb
H
G
50 lb
160 lb
a
B
F
z
105 lb
8 ft
D
5 ft
9 ft
5.5 ft
A

PROBLEM 3.129

Four signs are mounted on a frame spanning a highway, and the magnitudes of the horizontal wind forces acting on the signs are as shown. Determine the magnitude and the point of application of the resultant of the four wind forces when $a = 1$ ft and $b = 12$ ft.

SOLUTION

We have


y
x
90 lb
50 lb
z
160 lb
105 lb
=
y
x
x
y
P
R
z

Assume that the resultant **R** is applied at Point *P* whose coordinates are $(x, y, 0)$.

Equivalence then requires

$$\Sigma F_z: \quad -105 - 90 - 160 - 50 = -R$$

or $R = 405$ lb ◀

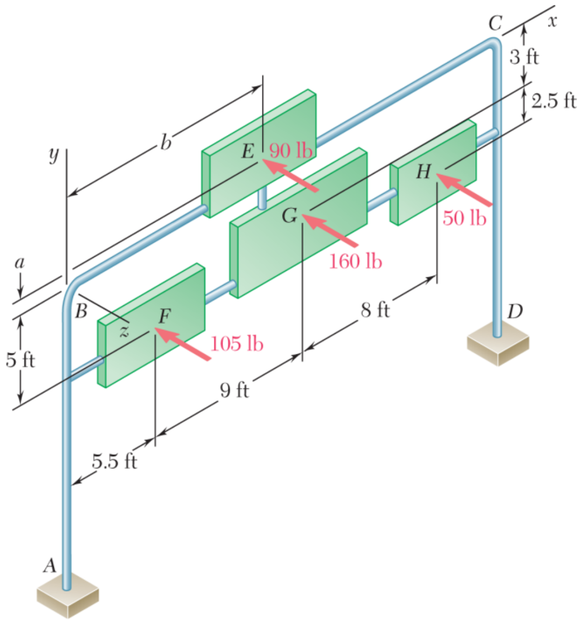
$$\Sigma M_x: \quad (5 \text{ ft})(105 \text{ lb}) - (1 \text{ ft})(90 \text{ lb}) + (3 \text{ ft})(160 \text{ lb}) + (5.5 \text{ ft})(50 \text{ lb}) = -y(405 \text{ lb})$$

or $\quad y = -2.94$ ft

$$\Sigma M_y: \quad (5.5 \text{ ft})(105 \text{ lb}) + (12 \text{ ft})(90 \text{ lb}) + (14.5 \text{ ft})(160 \text{ lb}) + (22.5 \text{ ft})(50 \text{ lb}) = -x(405 \text{ lb})$$

or $\quad x = 12.60$ ft

R acts 12.60 ft to the right of member *AB* and 2.94 ft below member *BC*. ◀



C
x
3 ft
2.5 ft
y
b
E
90 lb
H
G
50 lb
160 lb
a
B
F
z
8 ft
D
5 ft
105 lb
9 ft
5.5 ft
A

PROBLEM 3.130

Four signs are mounted on a frame spanning a highway, and the magnitudes of the horizontal wind forces acting on the signs are as shown. Determine *a* and *b* so that the point of application of the resultant of the four forces is at *G*.

SOLUTION

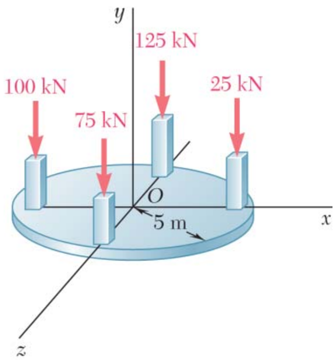
Since **R** acts at *G*, equivalence then requires that $\Sigma \mathbf{M}_G$ of the applied system of forces also be zero. Then at

$$G:\Sigma M_x: \quad -(a+3)\text{ ft}\times(90\text{ lb})+(2\text{ ft})(105\text{ lb}) + (2.5\text{ ft})(50\text{ lb})=0$$

or $a = 0.722$ ft ◀

$$\Sigma M_y: \quad -(9\text{ ft})(105\text{ ft})-(14.5-b)\text{ ft}\times(90\text{ lb}) + (8\text{ ft})(50\text{ lb})=0$$

or $b = 20.6$ ft ◀



PROBLEM 3.131

A concrete foundation mat of 5-m radius supports four equally spaced columns, each of which is located 4 m from the center of the mat. Determine the magnitude and the point of application of the resultant of the four loads.

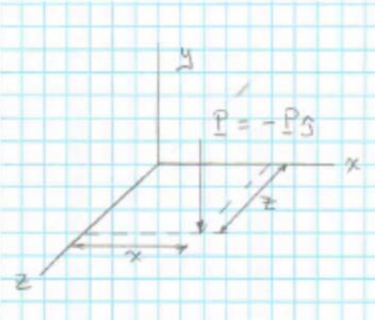
SOLUTION

Have: $\Sigma\mathbf{F}$: $\quad \mathbf{F}_A + \mathbf{F}_C + \mathbf{F}_D + \mathbf{F}_E = \mathbf{R}$

$$\mathbf{R} = -(100\text{ kN})\mathbf{j} - (125\text{ kN})\mathbf{j} - (25\text{kN})\mathbf{j} - (75\text{ kN})\mathbf{j}$$

$$= -(325\text{ kN})\mathbf{j}$$

$\mathbf{R} = \mathbf{P} = -(325\text{ kN})\mathbf{j}$ or $R = 325\text{ kN}$ ◀



Have: ΣM_x: $\quad F_C(z_C) + F_E(z_E) = R(z_G)$

$$-(125\text{ kN})(4\text{ m}) + (75\text{ kN})(4\text{ m})$$

$$= (325\text{ kN})z_G$$

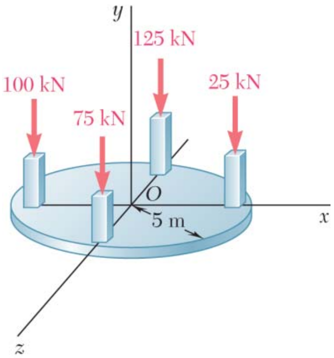
$\therefore \quad z_G = -0.61539\text{ m}$

or $z_G = -0.615\text{ m}$ ◀

ΣM_z: $\quad F_A(x_A) + F_D(x_D) = R(x_G)$

$$(100\text{ kN})(4\text{ m}) - (25\text{ kN})(4\text{ m}) = -(325\text{ kN})x_G$$

or $x_G = -0.923\text{ m}$ ◀

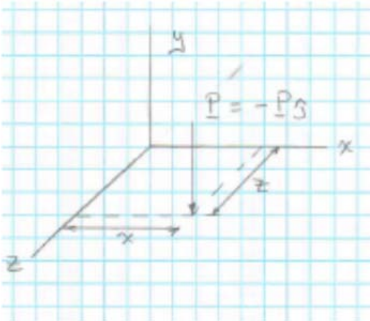


PROBLEM 3.132

Determine the magnitude and the point of application of the smallest additional load that must be applied to the foundation mat of Prob. 3.131 if the resultant of the five loads is to pass through the center of the mat.

SOLUTION

The smallest load P will be placed on the edge of the mat at a radius of 5 m. Therefore $x_P{}^2 + z_P{}^2 = (5\text{ m})^2$ (1)



Have: ΣM_x: $F_C(z_C) + F_E(z_E) + P(z_P) = R(z_G)$

$$-(125\text{ kN})(4\text{ m}) + (75\text{ kN})(4\text{ m}) + P(z_P) = R(0)$$

$$P(z_G) = 200\text{ kN}\cdot\text{m} \qquad (2)$$

Also ΣM_z: $F_A(x_A) + F_D(x_D) + P(x_P) = R(x_G)$

$$(100\text{ kN})(4\text{ m}) - (25\text{ kN})(4\text{ m}) - P(x_P) = R(0)$$

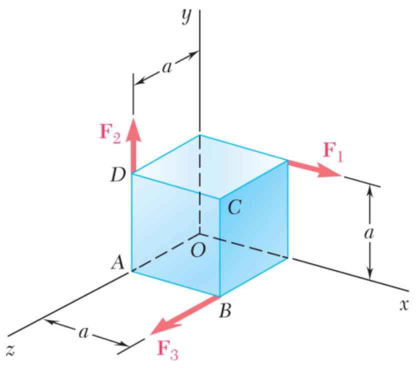
$$P(x_P) = 300\text{ kN}\cdot\text{m}$$

(3)

Substituting for x and z from equations (2) and (3) into (1), $\left(\dfrac{300}{P}\right)^2 + \left(\dfrac{200}{P}\right)^2 = (5)^2$

$$P^2 = \frac{(300)^2 + (200)^2}{(5)^2} = 5200 \quad \text{or } P = 72.111\text{ kN} \qquad P = 72.1\text{ kN} ◀$$

Finally, from (2) and (3) $x_P = 4.16\text{ m}$; $z_P = 2.77\text{ m}$◀

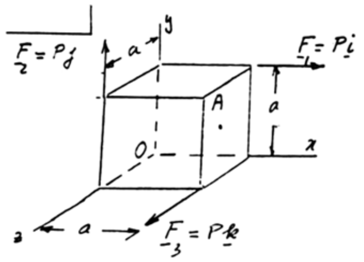

y
a
F_2
D
F_1
C
a
O
A
B
x
z
a
F_3

PROBLEM 3.133*

Three forces of the same magnitude P act on a cube of side a as shown. Replace the three forces by an equivalent wrench and determine (a) the magnitude and direction of the resultant force **R**, (b) the pitch of the wrench, (c) the axis of the wrench.

SOLUTION

Force-couple system at O:


$\underline{F}_2 = P\underline{j}$
a
y
$\underline{F}_1 = P\underline{i}$
A
a
O
x
z
a
$\underline{F}_3 = P\underline{k}$

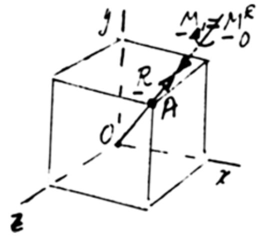
$$\mathbf{R} = P\mathbf{i} + P\mathbf{j} + P\mathbf{k} = P(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\mathbf{M}_O^R = a\mathbf{j} \times P\mathbf{i} + a\mathbf{k} \times P\mathbf{j} + a\mathbf{i} \times P\mathbf{k}$$

$$= -Pa\mathbf{k} - Pa\mathbf{i} - Pa\mathbf{j}$$

$$\mathbf{M}_O^R = -Pa(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Since **R** and $\mathbf{M}_O^R$ have the same direction, they form a wrench with $\mathbf{M}_1 = \mathbf{M}_O^R$. Thus, the axis of the wrench is the diagonal OA. We note that


y
$\underline{M}_1 = \underline{M}^R_O$
R
A
O
x
z

$$\cos\theta_x = \cos\theta_y = \cos\theta_z = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$R = P\sqrt{3} \quad \theta_x = \theta_y = \theta_z = 54.7°$$

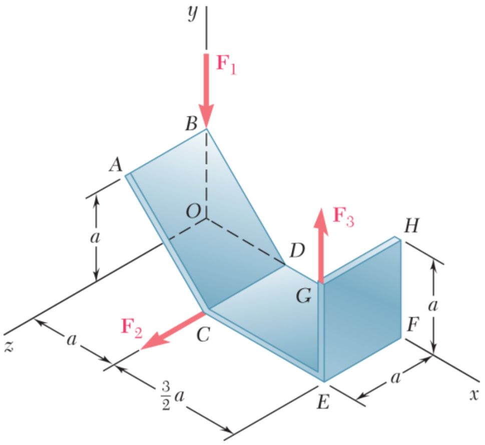
$$M_1 = M_O^R = -Pa\sqrt{3}$$

$$\text{Pitch} = p = \frac{M_1}{R} = \frac{-Pa\sqrt{3}}{P\sqrt{3}} = -a$$

(a) $R = P\sqrt{3} \quad \theta_x = \theta_y = \theta_z = 54.7°$ ◀

(b) $-a$ ◀

(c) Axis of the wrench is diagonal OA. ◀



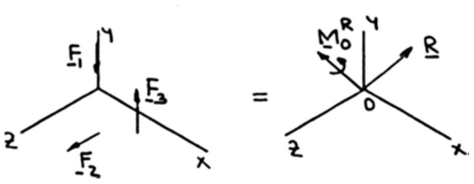
PROBLEM 3.134*

A piece of sheet metal is bent into the shape shown and is acted upon by three forces. If the forces have the same magnitude P, replace them with an equivalent wrench and determine (*a*) the magnitude and the direction of the resultant force **R**, (*b*) the pitch of the wrench, (*c*) the axis of the wrench.

SOLUTION

First reduce the given forces to an equivalent force-couple system $(\mathbf{R},\ \mathbf{M}_O^R)$ at the origin.

We have



$$\Sigma\mathbf{F}:\quad -P\mathbf{j} + P\mathbf{j} + P\mathbf{k} = R$$

or $\quad \mathbf{R} = P\mathbf{k}$

$$\Sigma\mathbf{M}_O:\quad -(aP)\mathbf{j} + \left[-(aP)\mathbf{i} + \left(\frac{5}{2}aP\right)\mathbf{k}\right] = M_O^R$$

or $\quad \mathbf{M}_O^R = aP\left(-\mathbf{i} - \mathbf{j} + \frac{5}{2}\mathbf{k}\right)$

(*a*) Then for the wrench,

$$R = P \blacktriangleleft$$

and $\quad \boldsymbol{\lambda}_{\text{axis}} = \frac{\mathbf{R}}{R} = \mathbf{k}$

$$\cos\theta_x = 0 \quad \cos\theta_y = 0 \quad \cos\theta_z = 1$$

or $\quad \theta_x = 90° \quad \theta_y = 90° \quad \theta_z = 0° \quad \blacktriangleleft$

(*b*) Now

$$M_1 = \boldsymbol{\lambda}_{\text{axis}} \cdot \mathbf{M}_O^R$$

$$= \mathbf{k} \cdot aP\left(-\mathbf{i} - \mathbf{j} + \frac{5}{2}\mathbf{k}\right)$$

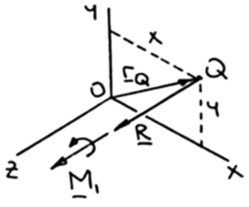
$$= \frac{5}{2}aP$$

Then $\quad P = \frac{\mathbf{M}_1}{R} = \frac{\frac{5}{2}aP}{P} \quad$ or $\quad P = \frac{5}{2}a \blacktriangleleft$

PROBLEM 3.134* (Continued)

(*c*) The components of the wrench are $(\mathbf{R},\ \mathbf{M}_1)$, where $\mathbf{M}_1 = M_1\boldsymbol{\lambda}_{\text{axis}}$, and the axis of the wrench is assumed to intersect the *xy*-plane at Point *Q*, whose coordinates are (*x*, *y*, 0). Thus, we require

$$\mathbf{M}_z = \mathbf{r}_Q \times \mathbf{R}_R$$



where $$\mathbf{M}_z = \mathbf{M}_O \times \mathbf{M}_1$$

Then

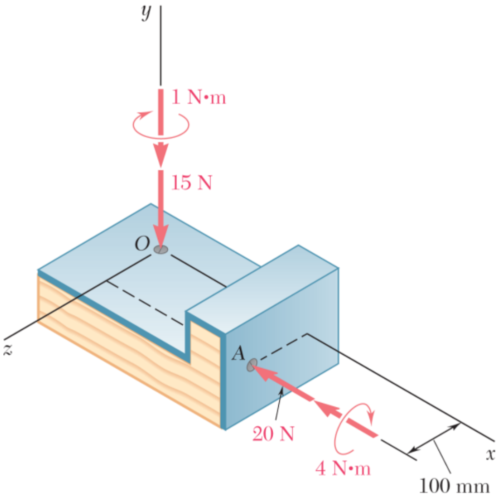
$$aP\left(-\mathbf{i}-\mathbf{j}+\frac{5}{2}\mathbf{k}\right)-\frac{5}{2}aP\mathbf{k} = (x\mathbf{i}+y\mathbf{j})+P\mathbf{k}$$

Equating coefficients:

$\mathbf{i}$: $\ -aP = yP \quad$ or $\quad y = -a$

$\mathbf{j}$: $\ -aP = -xP \quad$ or $\quad x = a$

The axis of the wrench is parallel to the *z*-axis and intersects the *xy*-plane at $x = a,\ y = -a.$ ◀



PROBLEM 3.135*

The forces and couples shown are applied to two screws as a piece of sheet metal is fastened to a block of wood. Reduce the forces and the couples to an equivalent wrench and determine (*a*) the resultant force **R**, (*b*) the pitch of the wrench, (*c*) the point where the axis of the wrench intersects the *xz*-plane.

SOLUTION

First, reduce the given force system to a force-couple system.

We have $\Sigma\mathbf{F}: \quad -(20\text{ N})\mathbf{i} - (15\text{ N})\mathbf{j} = \mathbf{R} \quad R = 25\text{ N}$

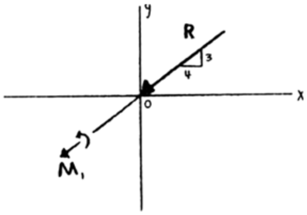
We have $\Sigma\mathbf{M}_O: \quad \Sigma(\mathbf{r}_O \times \mathbf{F}) + \Sigma\mathbf{M}_C = \mathbf{M}_O^R$

$$\mathbf{M}_O^R = -20\text{ N}(0.1\text{ m})\mathbf{j} - (4\text{ N}\cdot\text{m})\mathbf{i} - (1\text{ N}\cdot\text{m})\mathbf{j}$$
$$= -(4\text{ N}\cdot\text{m})\mathbf{i} - (3\text{ N}\cdot\text{m})\mathbf{j}$$

(*a*) $\mathbf{R} = -(20.0\text{ N})\mathbf{i} - (15.00\text{ N})\mathbf{j}$ ◀

(*b*) We have $M_1 = \boldsymbol{\lambda}_R \cdot \mathbf{M}_O^R \quad \boldsymbol{\lambda} = \dfrac{\mathbf{R}}{R}$

$$= (-0.8\mathbf{i} - 0.6\mathbf{j}) \cdot [-(4\text{ N}\cdot\text{m})\mathbf{i} - (3\text{ N}\cdot\text{m})\mathbf{j}]$$
$$= 5\text{ N}\cdot\text{m}$$



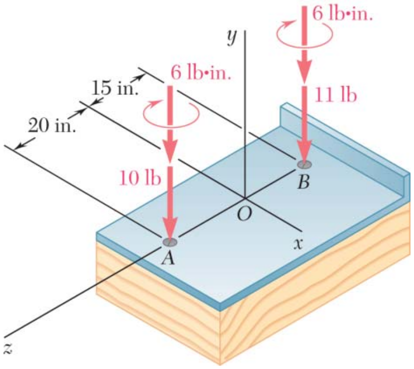
Pitch: $p = \dfrac{M_1}{R} = \dfrac{5\text{ N}\cdot\text{m}}{25\text{ N}} = 0.200\text{ m}$

or $p = 0.200\text{ m}$ ◀

(*c*) From above, note that

$\mathbf{M}_1 = \mathbf{M}_O^R$

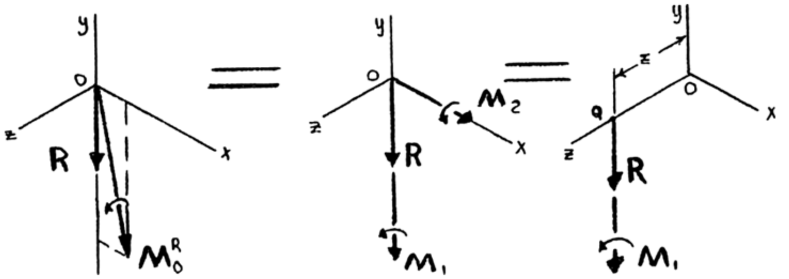
Therefore, the axis of the wrench goes through the origin. The line of action of the wrench lies in the *xy*-plane with a slope of $y = \dfrac{3}{4}x$ ◀



PROBLEM 3.136*

The forces and couples shown are applied to two screws as a piece of sheet metal is fastened to a block of wood. Reduce the forces and the couples to an equivalent wrench and determine (*a*) the resultant force **R**, (*b*) the pitch of the wrench, (*c*) the point where the axis of the wrench intersects the *xz*-plane.

SOLUTION



First, reduce the given force system to a force-couple at the origin.

We have $\Sigma \mathbf{F}: \quad -(10\text{ lb})\mathbf{j} - (11\text{ lb})\mathbf{j} = \mathbf{R}$

$$\mathbf{R} = -(21\text{ lb})\mathbf{j}$$

We have $\Sigma \mathbf{M}_O: \quad \Sigma(\mathbf{r}_O \times \mathbf{F}) + \Sigma \mathbf{M}_C = \mathbf{M}_O^R$

$$\mathbf{M}_O^R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 20 \\ 0 & -10 & 0 \end{vmatrix} \text{lb}\cdot\text{in.} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -15 \\ 0 & -11 & 0 \end{vmatrix} \text{lb}\cdot\text{in.} - (12\text{ lb}\cdot\text{in})\mathbf{j}$$

$$= (35\text{ lb}\cdot\text{in.})\mathbf{i} - (12\text{ lb}\cdot\text{in.})\mathbf{j}$$

(*a*) $\mathbf{R} = -(21\text{ lb})\mathbf{j}$ or $\mathbf{R} = -(21.0\text{ lb})\mathbf{j}$ ◀

(*b*) We have $M_1 = \boldsymbol{\lambda}_R \cdot \mathbf{M}_O^R \quad \boldsymbol{\lambda}_R = \dfrac{\mathbf{R}}{R}$

$$= (-\mathbf{j}) \cdot [(35\text{ lb}\cdot\text{in.})\mathbf{i} - (12\text{ lb}\cdot\text{in.})\mathbf{j}]$$

$$= 12\text{ lb}\cdot\text{in.} \quad \text{and} \quad \mathbf{M}_1 = -(12\text{ lb}\cdot\text{in.})\mathbf{j}$$

and pitch $p = \dfrac{M_1}{R} = \dfrac{12\text{ lb}\cdot\text{in.}}{21\text{ lb}} = 0.57143\text{ in.}$ or $p = 0.571\text{ in.}$ ◀

(*c*) We have

$$\mathbf{M}_O^R = \mathbf{M}_1 + \mathbf{M}_2$$

$$\mathbf{M}_2 = \mathbf{M}_O^R - \mathbf{M}_1 = (35\text{ lb}\cdot\text{in.})\mathbf{i}$$

We require

$$\mathbf{M}_2 = \mathbf{r}_{Q/O} \times \mathbf{R}$$

$$(35\text{ lb}\cdot\text{in.})\mathbf{i} = (x\mathbf{i} + z\mathbf{k}) \times [-(21\text{ lb})\mathbf{j}]$$

$$35\mathbf{i} = -(21x)\mathbf{k} + (21z)\mathbf{i}$$

From **i**:

$$35 = 21z$$

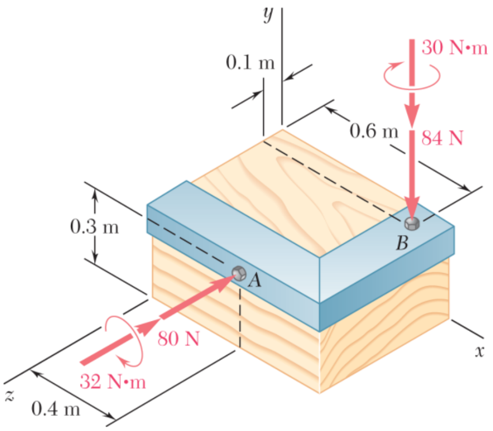
$$z = 1.66667\text{ in.}$$

From **k**:

$$0 = -21x$$

$$z = 0$$

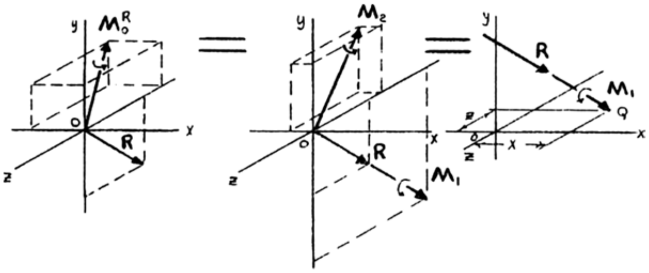
The axis of the wrench is parallel to the *y*-axis and intersects the *xz*-plane at $x = 0$, $z = 1.667$ in. ◀



PROBLEM 3.137*

Two bolts at *A* and *B* are tightened by applying the forces and couples shown. Replace the two wrenches with a single equivalent wrench and determine (*a*) the resultant **R**, (*b*) the pitch of the single equivalent wrench, (*c*) the point where the axis of the wrench intersects the *xz*-plane.

SOLUTION



First, reduce the given force system to a force-couple at the origin.

We have $\quad \Sigma\mathbf{F}: \quad -(84\text{ N})\mathbf{j} - (80\text{ N})\mathbf{k} = \mathbf{R} \quad R = 116\text{ N}$

and $\quad \Sigma\mathbf{M}_O: \quad \Sigma(\mathbf{r}_O \times \mathbf{F}) + \Sigma\mathbf{M}_C = \mathbf{M}_O^R$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.6 & 0 & 0.1 \\ 0 & 84 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.4 & 0.3 & 0 \\ 0 & 0 & 80 \end{vmatrix} + (-30\mathbf{j} - 32\mathbf{k})\text{ N}\cdot\text{m} = \mathbf{M}_O^R$$

$$\mathbf{M}_O^R = -(15.6\text{ N}\cdot\text{m})\mathbf{i} + (2\text{ N}\cdot\text{m})\mathbf{j} - (82.4\text{ N}\cdot\text{m})\mathbf{k}$$

(*a*) $\quad \mathbf{R} = -(84.0\text{ N})\mathbf{j} - (80.0\text{ N})\mathbf{k}$ ◀

(*b*) We have $\quad M_1 = \boldsymbol{\lambda}_R \cdot \mathbf{M}_O^R \quad \boldsymbol{\lambda}_R = \dfrac{\mathbf{R}}{R}$

$$= -\frac{-84\mathbf{j} - 80\mathbf{k}}{116} \cdot [-(15.6\text{ N}\cdot\text{m})\mathbf{i} + (2\text{ N}\cdot\text{m})\mathbf{j} - (82.4\text{ N}\cdot\text{m})\mathbf{k}]$$

$$= 55.379\text{ N}\cdot\text{m}$$

and $\quad \mathbf{M}_1 = M_1\lambda_R = -(40.102\text{ N}\cdot\text{m})\mathbf{j} - (38.192\text{ N}\cdot\text{m})\mathbf{k}$

Then pitch $\quad p = \dfrac{M_1}{R} = \dfrac{55.379\text{ N}\cdot\text{m}}{116\text{ N}} = 0.47741\text{ m}$ $\quad$ or $\quad p = 0.477\text{ m}$ ◀

PROBLEM 3.137* (Continued)

(*c*) We have $\mathbf{M}_O^R = \mathbf{M}_1 + \mathbf{M}_2$

$$\mathbf{M}_2 = \mathbf{M}_O^R - \mathbf{M}_1 = [(-15.6\mathbf{i} + 2\mathbf{j} - 82.4\mathbf{k}) - (40.102\mathbf{j} - 38.192\mathbf{k})]\ \text{N}\cdot\text{m}$$
$$= -(15.6\ \text{N}\cdot\text{m})\mathbf{i} + (42.102\ \text{N}\cdot\text{m})\mathbf{j} - (44.208\ \text{N}\cdot\text{m})\mathbf{k}$$

We require $\mathbf{M}_2 = \mathbf{r}_{Q/O} \times \mathbf{R}$

$$(-15.6\mathbf{i} + 42.102\mathbf{j} - 44.208\mathbf{k}) = (x\mathbf{i} + z\mathbf{k}) \times (84\mathbf{j} - 80\mathbf{k})$$
$$= (84z)\mathbf{i} + (80x)\mathbf{j} - (84x)\mathbf{k}$$

From **i**: $-15.6 = 84z$

$$z = -0.185714\ \text{m}$$

or $z = -0.1857\ \text{m}$

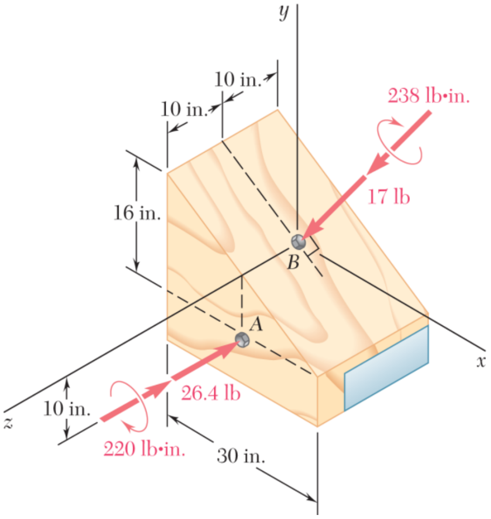
From **k**: $-44.208 = -84x$

$$x = 0.52629\ \text{m}$$

or $x = 0.526\ \text{m}$

The axis of the wrench intersects the *xz*-plane at

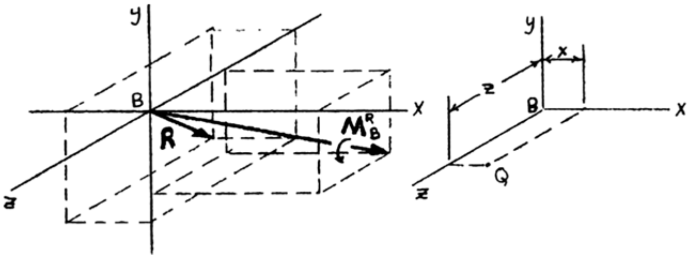
$x = 0.526\ \text{m} \quad y = 0 \quad z = -0.1857\ \text{m}$ ◀



PROBLEM 3.138*

Two bolts at *A* and *B* are tightened by applying the forces and couples shown. Replace the two wrenches with a single equivalent wrench and determine (*a*) the resultant **R**, (*b*) the pitch of the single equivalent wrench, (*c*) the point where the axis of the wrench intersects the *xz*-plane.

SOLUTION



First, reduce the given force system to a force-couple at the origin at *B*.

(*a*) We have $\Sigma\mathbf{F}:\quad -(26.4\text{ lb})\mathbf{k}-(17\text{ lb})\left(\frac{8}{17}\mathbf{i}+\frac{15}{17}\mathbf{j}\right)=\mathbf{R}$

$$\mathbf{R}=-(8.00\text{ lb})\mathbf{i}-(15.00\text{ lb})\mathbf{j}-(26.4\text{ lb})\mathbf{k}\ ◀$$

and $R=31.4\text{ lb}$

We have $\Sigma\mathbf{M}_B:\quad \mathbf{r}_{A/B}\times\mathbf{F}_A+\mathbf{M}_A+\mathbf{M}_B=\mathbf{M}_B^R$

$$\mathbf{M}_B^R=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & -10 & 0\\ 0 & 0 & -26.4\end{vmatrix}-220\mathbf{k}-238\left(\frac{8}{17}\mathbf{i}+\frac{15}{17}\mathbf{j}\right)=264\mathbf{i}-220\mathbf{k}-14(8\mathbf{i}+15\mathbf{j})$$

$$\mathbf{M}_B^R=(152\text{ lb}\cdot\text{in.})\mathbf{i}-(210\text{ lb}\cdot\text{in.})\mathbf{j}-(220\text{ lb}\cdot\text{in.})\mathbf{k}$$

(*b*) We have $M_1=\boldsymbol{\lambda}_R\cdot\mathbf{M}_O^R\quad \boldsymbol{\lambda}_R=\frac{\mathbf{R}}{R}$

$$=\frac{-8.00\mathbf{i}-15.00\mathbf{j}-26.4\mathbf{k}}{31.4}\cdot[(152\text{ lb}\cdot\text{in.})\mathbf{i}-(210\text{ lb}\cdot\text{in.})\mathbf{j}-(220\text{ lb}\cdot\text{in.})\mathbf{k}]$$

$$=246.56\text{ lb}\cdot\text{in.}$$

PROBLEM 3.138* (Continued)

and $$\mathbf{M}_1 = M_1\lambda_R = -(62.818\ \text{lb}\cdot\text{in.})\mathbf{i} - (117.783\ \text{lb}\cdot\text{in.})\mathbf{j} - (207.30\ \text{lb}\cdot\text{in.})\mathbf{k}$$

Then pitch $$p = \frac{M_1}{R} = \frac{246.56\ \text{lb}\cdot\text{in.}}{31.4\ \text{lb}} = 7.8522\ \text{in.}$$ or $p = 7.85$ in. ◀

(*c*) We have $$\mathbf{M}_B^R = \mathbf{M}_1 + \mathbf{M}_2$$

$$\mathbf{M}_2 = \mathbf{M}_B^R - \mathbf{M}_1 = (152\mathbf{i} - 210\mathbf{j} - 220\mathbf{k}) - (-62.818\mathbf{i} - 117.783\mathbf{j} - 207.30\mathbf{k})$$
$$= (214.82\ \text{lb}\cdot\text{in.})\mathbf{i} - (92.217\ \text{lb}\cdot\text{in.})\mathbf{j} - (12.7000\ \text{lb}\cdot\text{in.})\mathbf{k}$$

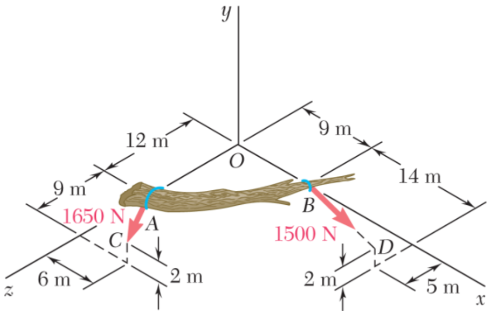
We require $$\mathbf{M}_2 = \mathbf{r}_{Q/B} \times \mathbf{R}$$

$$214.82\mathbf{i} - 92.217\mathbf{j} - 12.7000\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & z \\ -8 & -15 & -26.4 \end{vmatrix}$$
$$= (15z)\mathbf{i} - (8z)\mathbf{j} + (26.4x)\mathbf{j} - (15x)\mathbf{k}$$

From **i**: $214.82 = 15z \qquad z = 14.3213$ in.

From **k**: $-12.7000 = -15x \qquad x = 0.84667$ in.

The axis of the wrench intersects the *xz*-plane at $x = 0.847$ in. $\quad y = 0 \quad z = 14.32$ in. ◀



PROBLEM 3.139*

Two ropes attached at *A* and *B* are used to move the trunk of a fallen tree. Replace the forces exerted by the ropes with an equivalent wrench and determine (*a*) the resultant force **R**, (*b*) the pitch of the wrench, (*c*) the point where the axis of the wrench intersects the *yz*-plane.

SOLUTION

(*a*) First replace the given forces with an equivalent force-couple system $\left(\mathbf{R},\ \mathbf{M}_O^R\right)$ at the origin.

We have

$$d_{AC} = \sqrt{(6)^2 + (2)^2 + (9)^2} = 11 \text{ m}$$

$$d_{BD} = \sqrt{(14)^2 + (2)^2 + (5)^2} = 15 \text{ m}$$

Then

$$\begin{aligned} T_{AC} &= \frac{1650 \text{ N}}{11} = (6\mathbf{i} + 2\mathbf{j} + 9\mathbf{k}) \\ &= (900 \text{ N})\mathbf{i} + (300 \text{ N})\mathbf{j} + (1350 \text{ N})\mathbf{k} \end{aligned}$$

and

$$\begin{aligned} T_{BD} &= \frac{1500 \text{ N}}{15} = (14\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}) \\ &= (1400 \text{ N})\mathbf{i} + (200 \text{ N})\mathbf{j} + (500 \text{ N})\mathbf{k} \end{aligned}$$

Equivalence then requires

$$\begin{aligned} \Sigma\mathbf{F}: \quad \mathbf{R} &= \mathbf{T}_{AC} + \mathbf{T}_{BD} \\ &= (900\mathbf{i} + 300\mathbf{j} + 1350\mathbf{k}) \\ &\quad + (1400\mathbf{i} + 200\mathbf{j} + 500\mathbf{k}) \\ &= (2300 \text{ N})\mathbf{i} + (500 \text{ N})\mathbf{j} + (1850 \text{ N})\mathbf{k} \end{aligned}$$

$$\begin{aligned} \Sigma\mathbf{M}_O: \quad \mathbf{M}_O^R &= \mathbf{r}_A \times \mathbf{T}_{AC} + \mathbf{r}_B \times \mathbf{T}_{BD} \\ &= (12 \text{ m})\mathbf{k} \times [(900 \text{ N})\mathbf{i} + (300 \text{ N})\mathbf{j} + (1350 \text{ N})\mathbf{k}] \\ &\quad + (9 \text{ m})\mathbf{i} \times [(1400 \text{ N})\mathbf{i} + (200 \text{ N})\mathbf{j} + (500 \text{ N})\mathbf{k}] \\ &= -(3600)\mathbf{i} + (10{,}800 - 4500)\mathbf{j} + (1800)\mathbf{k} \\ &= -(3600 \text{ N}\cdot\text{m})\mathbf{i} + (6300 \text{ N}\cdot\text{m})\mathbf{j} + (1800 \text{ N}\cdot\text{m})\mathbf{k} \end{aligned}$$

The components of the wrench are $(\mathbf{R},\ \mathbf{M}_1)$, where

$$\mathbf{R} = (2300 \text{ N})\mathbf{i} + (500 \text{ N})\mathbf{j} + (1850 \text{ N})\mathbf{k}$$ ◀

PROBLEM 3.139* (Continued)

(*b*) We have

$$R = 100\sqrt{(23)^2 + (5)^2 + (18.5)^2} = 2993.7 \text{ N}$$

Let

$$\boldsymbol{\lambda}_{\text{axis}} = \frac{\mathbf{R}}{R} = \frac{1}{29.937}(23\mathbf{i} + 5\mathbf{j} + 18.5\mathbf{k})$$

Then

$$\begin{aligned} M_1 &= \boldsymbol{\lambda}_{\text{axis}} \cdot \mathbf{M}_O^R \\ &= \frac{1}{29.937}(23\mathbf{i} + 5\mathbf{j} + 18.5\mathbf{k}) \cdot (-3600\mathbf{i} + 6300\mathbf{j} + 1800\mathbf{k}) \\ &= \frac{1}{0.29937}[(23)(-36) + (5)(63) + (18.5)(18)] \\ &= -601.26 \text{ N} \cdot \text{m} \end{aligned}$$

Finally,

$$P = \frac{M_1}{R} = \frac{-601.26 \text{ N} \cdot \text{m}}{2993.7 \text{ N}}$$

or $P = -0.201$ m ◀

(*c*) We have

$$\begin{aligned} M_1 &= M_1\boldsymbol{\lambda}_{\text{axis}} \\ &= (-601.26 \text{ N} \cdot \text{m}) \times \frac{1}{29.937}(23\mathbf{i} + 5\mathbf{j} + 18.5\mathbf{k}) \end{aligned}$$

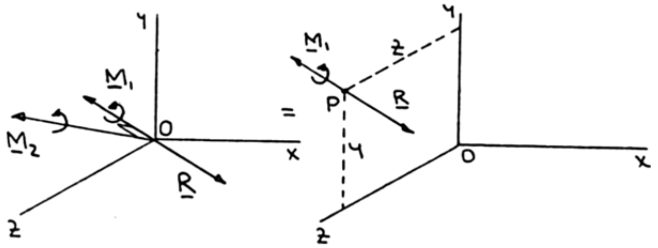
or

$$\mathbf{M}_1 = -(461.93 \text{ N} \cdot \text{m})\mathbf{i} - (100.421 \text{ N} \cdot \text{m})\mathbf{j} - (371.56 \text{ N} \cdot \text{m})\mathbf{k}$$

Now

$$\begin{aligned} \mathbf{M}_2 &= \mathbf{M}_O^R - \mathbf{M}_1 \\ &= (-3600\mathbf{i} + 6300\mathbf{j} + 1800\mathbf{k}) \\ &\quad -(-461.93\mathbf{i} - 100.421\mathbf{j} - 371.56\mathbf{k}) \\ &= -(3138.1 \text{ N} \cdot \text{m})\mathbf{i} + (6400.4 \text{ N} \cdot \text{m})\mathbf{j} + (2171.6 \text{ N} \cdot \text{m})\mathbf{k} \end{aligned}$$

For equivalence:



PROBLEM 3.139* (Continued)

Thus, we require $\mathbf{M}_2 = \mathbf{r}_P \times \mathbf{R} \qquad \mathbf{r} = (y\mathbf{j} + z\mathbf{k})$

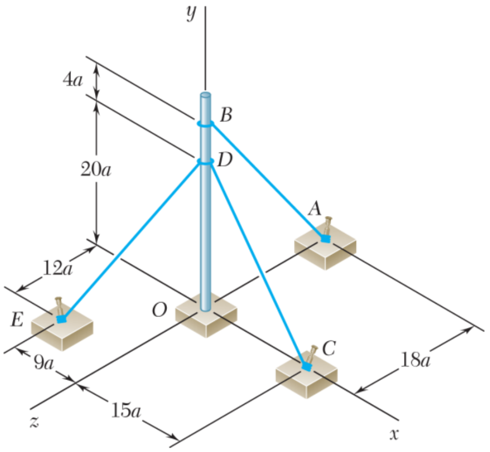
Substituting:

$$-3138.1\mathbf{i} + 6400.4\mathbf{j} + 2171.6\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y & z \\ 2300 & 500 & 1850 \end{vmatrix}$$

Equating coefficients:

$$\mathbf{j}: \quad 6400.4 = 2300z \qquad \text{or} \qquad z = 2.78 \text{ m}$$
$$\mathbf{k}: \quad 2171.6 = -2300y \qquad \text{or} \qquad y = -0.944 \text{ m}$$

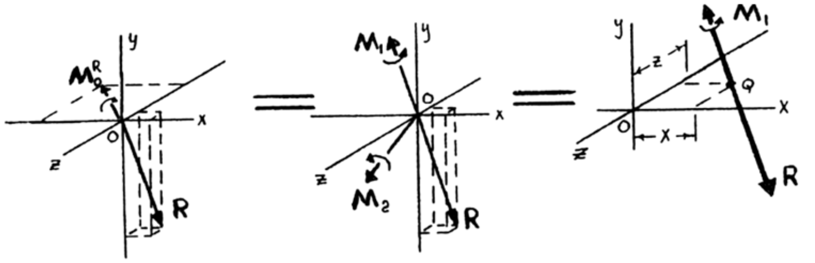
The axis of the wrench intersects the *yz*-plane at $y = -0.944$ m $\quad z = 2.78$ m ◀



PROBLEM 3.140*

A flagpole is guyed by three cables. If the tensions in the cables have the same magnitude P, replace the forces exerted on the pole with an equivalent wrench and determine (*a*) the resultant force **R**, (*b*) the pitch of the wrench, (*c*) the point where the axis of the wrench intersects the *xz*-plane.

SOLUTION



(*a*) First reduce the given force system to a force-couple at the origin.

We have $\Sigma\mathbf{F}$: $P\boldsymbol{\lambda}_{BA} + P\boldsymbol{\lambda}_{DC} + P\boldsymbol{\lambda}_{DE} = \mathbf{R}$

$$\mathbf{R} = P\left[\left(\frac{4}{5}\mathbf{j} - \frac{3}{5}\mathbf{k}\right) + \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) + \left(\frac{-9}{25}\mathbf{i} - \frac{4}{5}\mathbf{j} + \frac{12}{25}\mathbf{k}\right)\right]$$

$$\mathbf{R} = \frac{3P}{25}(2\mathbf{i} - 20\mathbf{j} - \mathbf{k}) \blacktriangleleft$$

$$R = \frac{3P}{25}\sqrt{(2)^2 + (20)^2 + (1)^2} = \frac{27\sqrt{5}}{25}P$$

We have $\Sigma\mathbf{M}$: $\Sigma(\mathbf{r}_O \times P) = \mathbf{M}_O^R$

$$(24a)\mathbf{j} \times \left(\frac{-4P}{5}\mathbf{j} - \frac{3P}{5}\mathbf{k}\right) + (20a)\mathbf{j} \times \left(\frac{3P}{5}\mathbf{i} - \frac{4P}{5}\mathbf{j}\right) + (20a)\mathbf{j} \times \left(\frac{-9P}{25}\mathbf{i} - \frac{4P}{5}\mathbf{j} + \frac{12P}{25}\mathbf{k}\right) = \mathbf{M}_O^R$$

$$\mathbf{M}_O^R = \frac{24Pa}{5}(-\mathbf{i} - \mathbf{k})$$

PROBLEM 3.140* (Continued)

(*b*) We have $M_1 = \boldsymbol{\lambda}_R \cdot \mathbf{M}_O^R$

where $\boldsymbol{\lambda}_R = \dfrac{\mathbf{R}}{R} = \dfrac{3P}{25}(2\mathbf{i} - 20\mathbf{j} - \mathbf{k})\dfrac{25}{27\sqrt{5}P} = \dfrac{1}{9\sqrt{5}}(2\mathbf{i} - 20\mathbf{j} - \mathbf{k})$

Then $M_1 = \dfrac{1}{9\sqrt{5}}(2\mathbf{i} - 20\mathbf{j} - \mathbf{k}) \cdot \dfrac{24Pa}{5}(-\mathbf{i} - \mathbf{k}) = \dfrac{-8Pa}{15\sqrt{5}}$

and pitch $p = \dfrac{M_1}{R} = \dfrac{-8Pa}{15\sqrt{5}}\left(\dfrac{25}{27\sqrt{5}P}\right) = \dfrac{-8a}{81}$ or $p = -0.0988a$ ◀

(*c*)

$$\mathbf{M}_1 = M_1\boldsymbol{\lambda}_R = \frac{-8Pa}{15\sqrt{5}}\left(\frac{1}{9\sqrt{5}}\right)(2\mathbf{i} - 20\mathbf{j} - \mathbf{k}) = \frac{8Pa}{675}(-2\mathbf{i} + 20\mathbf{j} + \mathbf{k})$$

Then $\mathbf{M}_2 = \mathbf{M}_O^R - \mathbf{M}_1 = \dfrac{24Pa}{5}(-\mathbf{i} - \mathbf{k}) - \dfrac{8Pa}{675}(-2\mathbf{i} + 20\mathbf{j} + \mathbf{k}) = \dfrac{8Pa}{675}(-430\mathbf{i} - 20\mathbf{j} - 406\mathbf{k})$

We require $\mathbf{M}_2 = \mathbf{r}_{Q/O} \times \mathbf{R}$

$$\left(\frac{8Pa}{675}\right)(-403\mathbf{i} - 20\mathbf{j} - 406\mathbf{k}) = (x\mathbf{i} + z\mathbf{k}) \times \left(\frac{3P}{25}\right)(2\mathbf{i} - 20\mathbf{j} - \mathbf{k})$$

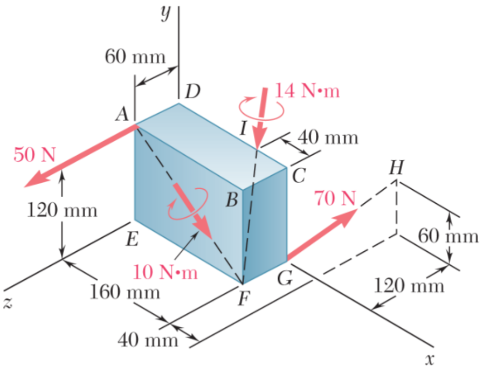
$$= \left(\frac{3P}{25}\right)[20z\mathbf{i} + (x + 2z)\mathbf{j} - 20x\mathbf{k}]$$

From **i**: $8(-403)\dfrac{Pa}{675} = 20z\left(\dfrac{3P}{25}\right)$ $z = -1.99012a$

From **k**: $8(-406)\dfrac{Pa}{675} = -20x\left(\dfrac{3P}{25}\right)$ $x = 2.0049a$

The axis of the wrench intersects the *xz*-plane at

$x = 2.00a,\ z = -1.990a$ ◀



PROBLEM 3.141*

Determine whether the force-and-couple system shown can be reduced to a single equivalent force **R**. If it can, determine **R** and the point where the line of action of **R** intersects the *yz*-plane. If it cannot be so reduced, replace the given system with an equivalent wrench and determine its resultant, its pitch, and the point where its axis intersects the *yz*-plane.

SOLUTION

First, reduce the given force system to a force-couple at the origin.

We have $\Sigma\mathbf{F}: \quad \mathbf{F}_A + \mathbf{F}_G = \mathbf{R}$

$$\mathbf{R} = (50\text{ N})\mathbf{k} + 70\text{ N}\left[\frac{(40\text{ mm})\mathbf{i} + (60\text{ mm})\mathbf{j} - (120\text{ mm})\mathbf{k}}{140\text{ mm}}\right]$$

$$= (20\text{ N})\mathbf{i} + (30\text{ N})\mathbf{j} - (10\text{ N})\mathbf{k}$$

and $R = 37.417\text{ N}$

We have $\Sigma\mathbf{M}_O: \quad \Sigma(\mathbf{r}_O \times \mathbf{F}) + \Sigma\mathbf{M}_C = \mathbf{M}_O^R$

$$\mathbf{M}_O^R = [(0.12\text{ m})\mathbf{j} \times (50\text{ N})\mathbf{k}] + \{(0.16\text{ m})\mathbf{i} \times [(20\text{ N})\mathbf{i} + (30\text{ N})\mathbf{j} - (60\text{ N})\mathbf{k}]\}$$

$$+ (10\text{ N}\cdot\text{m})\left[\frac{(160\text{ mm})\mathbf{i} - (120\text{ mm})\mathbf{j}}{200\text{ mm}}\right]$$

$$+ (14\text{ N}\cdot\text{m})\left[\frac{(40\text{ mm})\mathbf{i} - (120\text{ mm})\mathbf{j} + (60\text{ mm})\mathbf{k}}{140\text{ mm}}\right]$$

$$\mathbf{M}_0^R = (18\text{ N}\cdot\text{m})\mathbf{i} - (8.4\text{ N}\cdot\text{m})\mathbf{j} + (10.8\text{ N}\cdot\text{m})\mathbf{k}$$

To be able to reduce the original forces and couples to a single equivalent force, **R** and **M** must be perpendicular. Thus, $\mathbf{R}\cdot\mathbf{M} = 0$.

Substituting

$$(20\mathbf{i} + 30\mathbf{j} - 10\mathbf{k})\cdot(18\mathbf{i} - 8.4\mathbf{j} + 10.8\mathbf{k}) \overset{?}{=} 0$$

or

$$(20)(18) + (30)(-8.4) + (-10)(10.8) \overset{?}{=} 0$$

or

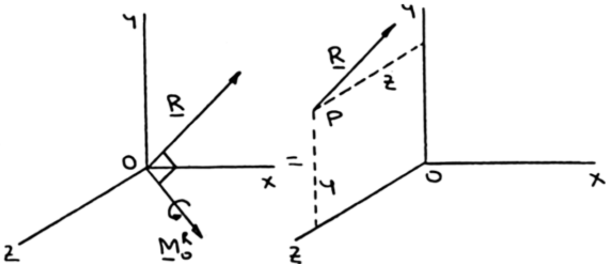
$$0 \overset{\checkmark}{=} 0$$

R and **M** are perpendicular so that the given system can be reduced to the single equivalent force.

$\mathbf{R} = (20.0\text{ N})\mathbf{i} + (30.0\text{ N})\mathbf{j} - (10.00\text{ N})\mathbf{k}$ ◀

PROBLEM 3.141* (Continued)

Then for equivalence,



Thus, we require $\mathbf{M}_O^R = \mathbf{r}_p \times \mathbf{R} \quad \mathbf{r}_p = y\mathbf{j} + z\mathbf{k}$

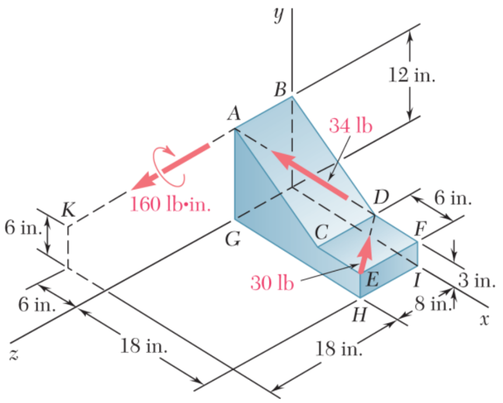
Substituting:

$$18\mathbf{i} - 8.4\mathbf{j} + 10.8\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y & z \\ 20 & 30 & -10 \end{vmatrix}$$

Equating coefficients:

$$\mathbf{j}: \quad -8.4 = 20z \qquad \text{or} \qquad z = -0.42 \text{ m}$$
$$\mathbf{k}: \quad 10.8 = -20y \qquad \text{or} \qquad y = -0.54 \text{ m}$$

The line of action of **R** intersects the *yz*-plane at $x = 0 \quad y = -0.540 \text{ m} \quad z = -0.420 \text{ m}$ ◀



PROBLEM 3.142*

Determine whether the force-and-couple system shown can be reduced to a single equivalent force **R**. If it can, determine **R** and the point where the line of action of **R** intersects the *yz*-plane. If it cannot be so reduced, replace the given system with an equivalent wrench and determine its resultant, its pitch, and the point where its axis intersects the *yz*-plane.

SOLUTION

First determine the resultant of the forces at *D*. We have

$$d_{DA} = \sqrt{(-12)^2 + (9)^2 + (8)^2} = 17 \text{ in.}$$

$$d_{ED} = \sqrt{(-6)^2 + (0)^2 + (-8)^2} = 10 \text{ in.}$$

Then

$$\mathbf{F}_{DA} = \frac{34 \text{ lb}}{17} = (-12\mathbf{i} + 9\mathbf{j} + 8\mathbf{k})$$

$$= -(24 \text{ lb})\mathbf{i} + (18 \text{ lb})\mathbf{j} + (16 \text{ lb})\mathbf{k}$$

and

$$\mathbf{F}_{ED} = \frac{30 \text{ lb}}{10} = (-6\mathbf{i} - 8\mathbf{k})$$

$$= -(18 \text{ lb})\mathbf{i} - (24 \text{ lb})\mathbf{k}$$

Then

$$\Sigma\mathbf{F}: \quad \mathbf{R} = \mathbf{F}_{DA} + \mathbf{F}_{ED}$$

$$= (-24\mathbf{i} + 18\mathbf{j} + 16\mathbf{k} + (-18\mathbf{i} - 24\mathbf{k})$$

$$= -(42 \text{ lb})\mathbf{i} + (18 \text{ lb})\mathbf{j} - (8 \text{ lb})\mathbf{k}$$

For the applied couple

$$d_{AK} = \sqrt{(-6)^2 + (-6)^2 + (18)^2} = 6\sqrt{11} \text{ in.}$$

Then

$$\mathbf{M} = \frac{160 \text{ lb} \cdot \text{in.}}{6\sqrt{11}}(-6\mathbf{i} - 6\mathbf{j} + 18\mathbf{k})$$

$$= \frac{160}{\sqrt{11}}[-(1 \text{ lb} \cdot \text{in.})\mathbf{i} - (1 \text{ lb} \cdot \text{in.})\mathbf{j} + (3 \text{ lb} \cdot \text{in.})\mathbf{k}]$$

To be able to reduce the original forces and couple to a single equivalent force, **R** and **M** must be perpendicular. Thus

$$\mathbf{R} \cdot \mathbf{M} \overset{?}{=} 0$$

PROBLEM 3.142* (Continued)

Substituting

$$(-42\mathbf{i}+18\mathbf{j}-8\mathbf{k})\cdot\frac{160}{\sqrt{11}}(-\mathbf{i}-\mathbf{j}+3\mathbf{k}) \overset{?}{=} 0$$

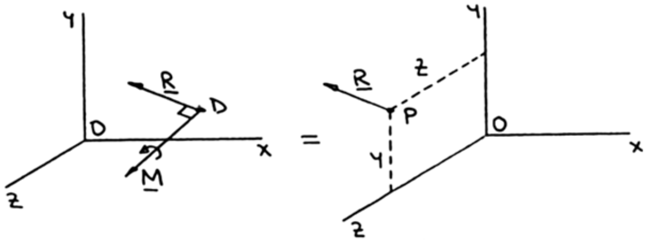
or $$\frac{160}{\sqrt{11}}[(-42)(-1)+(18)(-1)+(-8)(3)] \overset{?}{=} 0$$

or $$0 \overset{\checkmark}{=} 0$$

R and **M** are perpendicular so that the given system can be reduced to the single equivalent force.

$$\mathbf{R} = -(42.0\text{ lb})\mathbf{i}+(18.00\text{ lb})\mathbf{j}-(8.00\text{ lb})\mathbf{k} \blacktriangleleft$$

Then for equivalence,



Thus, we require $\mathbf{M} = \mathbf{r}_{P/D}\times\mathbf{R}$

where $\mathbf{r}_{P/D} = -(12\text{ in.})\mathbf{i}+[(y-3)\text{in.}]\mathbf{j}+(z\text{ in.})\mathbf{k}$

Substituting:

$$\frac{160}{\sqrt{11}}(-\mathbf{i}-\mathbf{j}+3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -12 & (y-3) & z \\ -42 & 18 & -8 \end{vmatrix}$$

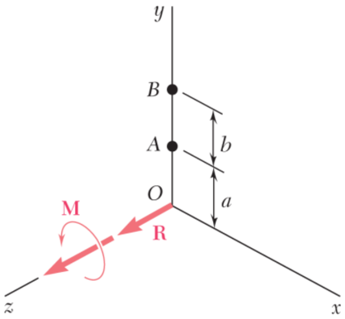
$$\begin{aligned} &= [(y-3)(-8)-(z)(18)]\mathbf{i} \\ &\quad +[(z)(-42)-(-12)(-8)]\mathbf{j} \\ &\quad +[(-12)(18)-(y-3)(-42)]\mathbf{k} \end{aligned}$$

Equating coefficients:

$$\mathbf{j}:\quad -\frac{160}{\sqrt{11}} = -42z-96 \qquad \text{or} \quad z = -1.137\text{ in.}$$

$$\mathbf{k}:\quad \frac{480}{\sqrt{11}} = -216+42(y-3) \qquad \text{or} \quad y = 11.59\text{ in.}$$

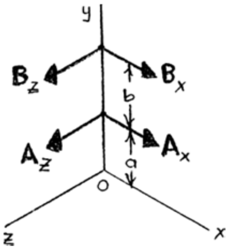
The line of action of **R** intersects the *yz*-plane at $x = 0 \quad y = 11.59\text{ in.} \quad z = -1.137\text{ in.}$ ◀



PROBLEM 3.143*

Replace the wrench shown with an equivalent system consisting of two forces perpendicular to the *y*-axis and applied respectively at *A* and *B*.

SOLUTION



Express the forces at *A* and *B* as

$$\mathbf{A} = A_x\mathbf{i} + A_z\mathbf{k}$$

$$\mathbf{B} = B_x\mathbf{i} + B_z\mathbf{k}$$

Then, for equivalence to the given force system,

$$\Sigma F_x: \quad A_x + B_x = 0 \qquad (1)$$

$$\Sigma F_z: \quad A_z + B_z = R \qquad (2)$$

$$\Sigma M_x: \quad A_z(a) + B_z(a+b) = 0 \qquad (3)$$

$$\Sigma M_z: \quad -A_x(a) - B_x(a+b) = M \qquad (4)$$

From Equation (1), $B_x = -A_x$

Substitute into Equation (4):

$$-A_x(a) + A_x(a+b) = M$$

$$A_x = \frac{M}{b} \quad \text{and} \quad B_x = -\frac{M}{b}$$

From Equation (2), $B_z = R - A_z$

and Equation (3), $A_z a + (R - A_z)(a+b) = 0$

$$A_z = R\left(1 + \frac{a}{b}\right)$$

and $B_z = R - R\left(1 + \frac{a}{b}\right)$

$$B_z = -\frac{a}{b}R$$

Then

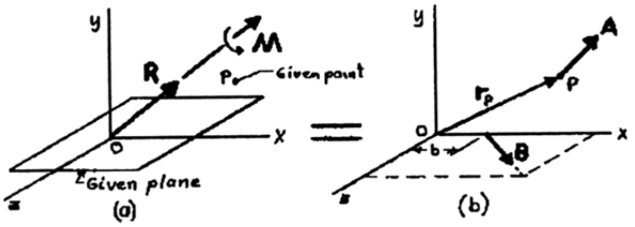
$$\mathbf{A} = \left(\frac{M}{b}\right)\mathbf{i} + R\left(1 + \frac{a}{b}\right)\mathbf{k} \blacktriangleleft$$

$$\mathbf{B} = -\left(\frac{M}{b}\right)\mathbf{i} - \left(\frac{a}{b}R\right)\mathbf{k} \blacktriangleleft$$

PROBLEM 3.144*

Show that, in general, a wrench can be replaced with two forces chosen in such a way that one force passes through a given point while the other force lies in a given plane.

SOLUTION



First, choose a coordinate system so that the *xy*-plane coincides with the given plane. Also, position the coordinate system so that the line of action of the wrench passes through the origin as shown in Figure *a*. Since the orientation of the plane and the components (**R**, **M**) of the wrench are known, it follows that the scalar components of **R** and **M** are known relative to the shown coordinate system.

A force system to be shown as equivalent is illustrated in Figure *b*. Let **A** be the force passing through the given Point *P* and **B** be the force that lies in the given plane. Let *b* be the *x*-axis intercept of **B**.

The known components of the wrench can be expressed as

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k} \quad \text{and} \quad M = M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}$$

while the unknown forces **A** and **B** can be expressed as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \quad \text{and} \quad \mathbf{B} = B_x\mathbf{i} + B_z\mathbf{k}$$

Since the position vector of Point *P* is given, it follows that the scalar components (*x*, *y*, *z*) of the position vector $\mathbf{r}_P$ are also known.

Then, for equivalence of the two systems,

$$\Sigma F_x: \quad R_x = A_x + B_x \tag{1}$$

$$\Sigma F_y: \quad R_y = A_y \tag{2}$$

$$\Sigma F_z: \quad R_z = A_z + B_z \tag{3}$$

$$\Sigma M_x: \quad M_x = yA_z - zA_y \tag{4}$$

$$\Sigma M_y: \quad M_y = zA_x - xA_z - bB_z \tag{5}$$

$$\Sigma M_z: \quad M_z = xA_y - yA_x \tag{6}$$

Based on the above six independent equations for the six unknowns $(A_x, A_y, A_z, B_x, B_z, b)$, there exists a unique solution for **A** and **B**.

From Equation (2), $A_y = R_y$ ◀

PROBLEM 3.144* (Continued)

Equation (6): $$A_x = \left(\frac{1}{y}\right)(xR_y - M_z) \blacktriangleleft$$

Equation (1): $$B_x = R_x - \left(\frac{1}{y}\right)(xR_y - M_z) \blacktriangleleft$$

Equation (4): $$A_z = \left(\frac{1}{y}\right)(M_x + zR_y) \blacktriangleleft$$

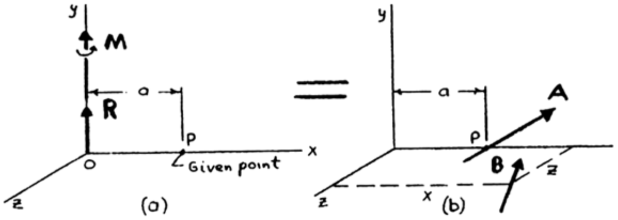
Equation (3): $$B_z = R_z - \left(\frac{1}{y}\right)(M_x + zR_y) \blacktriangleleft$$

Equation (5): $$b = \frac{(xM_x + yM_y + zM_z)}{(M_x - yR_z + zR_y)} \blacktriangleleft$$

PROBLEM 3.145*

Show that a wrench can be replaced with two perpendicular forces, one of which is applied at a given point.

SOLUTION



First, observe that it is always possible to construct a line perpendicular to a given line so that the constructed line also passes through a given point. Thus, it is possible to align one of the coordinate axes of a rectangular coordinate system with the axis of the wrench while one of the other axes passes through the given point.

See Figures *a* and *b*.

We have $\mathbf{R} = R\mathbf{j}$ and $\mathbf{M} = M\mathbf{j}$ and are known.

The unknown forces **A** and **B** can be expressed as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \quad \text{and} \quad \mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

The distance *a* is known. It is assumed that force **B** intersects the *xz*-plane at (*x*, 0, *z*). Then for equivalence,

$$\Sigma F_x: \quad 0 = A_x + B_x \tag{1}$$

$$\Sigma F_y: \quad R = A_y + B_y \tag{2}$$

$$\Sigma F_z: \quad 0 = A_z + B_z \tag{3}$$

$$\Sigma M_x: \quad 0 = -zB_y \tag{4}$$

$$\Sigma M_y: \quad M = -aA_z - xB_z + zB_x \tag{5}$$

$$\Sigma M_z: \quad 0 = aA_y + xB_y \tag{6}$$

Since **A** and **B** are made perpendicular,

$$\mathbf{A} \cdot \mathbf{B} = 0 \quad \text{or} \quad A_xB_x + A_yB_y + A_zB_z = 0 \tag{7}$$

There are eight unknowns: $A_x, A_y, A_z, B_x, B_y, B_z, x, z$

But only seven independent equations. Therefore, *there exists an infinite number of solutions.*

Next, consider Equation (4): $0 = -zB_y$

If $B_y = 0$, Equation (7) becomes $A_xB_x + A_zB_z = 0$

Using Equations (1) and (3), this equation becomes $A_x^2 + A_z^2 = 0$

Since the components of **A** must be real, a nontrivial solution is not possible. Thus, it is required that $B_y \neq 0,$ so that from Equation (4), $z = 0.$

To obtain one possible solution, arbitrarily let $A_x = 0.$

(*Note:* Setting $A_y, A_z,$ or B_z equal to zero results in unacceptable solutions.)

The defining equations then become

$$0 = B_x \tag{1)$'$ (}$$

$$R = A_y + B_y \qquad (2)$$

$$0 = A_z + B_z \qquad (3)$$

$$M = -aA_z - xB_z \qquad (5)'$$

$$0 = aA_y + xB_y \qquad (6)$$

$$A_yB_y + A_zB_z = 0 \qquad (7)'$$

Then Equation (2) can be written $A_y = R - B_y$

Equation (3) can be written $B_z = -A_z$

Equation (6) can be written $x = -\dfrac{aA_y}{B_y}$

Substituting into Equation (5)′,

$$M = -aA_z - \left(-a\frac{R-B_y}{B_y}\right)(-A_z)$$

or

$$A_z = -\frac{M}{aR}B_y \qquad (8)$$

Substituting into Equation (7)′,

$$(R-B_y)B_y + \left(-\frac{M}{aR}B_y\right)\left(\frac{M}{aR}B_y\right) = 0$$

or

$$B_y = \frac{a^2R^3}{a^2R^2 + M^2}$$

Then from Equations (2), (8), and (3),

$$A_y = R - \frac{a^2R^2}{a^2R^2 + M^2} = \frac{RM^2}{a^2R^2 + M^2}$$

$$A_z = -\frac{M}{aR}\left(\frac{a^2R^3}{a^2R^2 + M^2}\right) = -\frac{aR^2M}{a^2R^2 + M^2}$$

$$B_z = \frac{aR^2M}{a^2R^2 + M^2}$$

In summary,

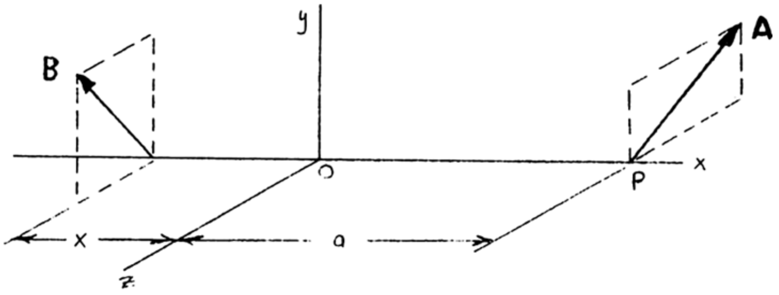
$$\mathbf{A} = \frac{RM}{a^2R^2 + M^2}(M\mathbf{j} - aR\mathbf{k}) \quad \blacktriangleleft$$

$$\mathbf{B} = \frac{aR^2}{a^2R^2 + M^2}(aR\mathbf{j} + M\mathbf{k}) \quad \blacktriangleleft$$

Which shows that it is possible to replace a wrench with two perpendicular forces, one of which is applied at a given point.

Lastly, if $R > 0$ and $M > 0$, it follows from the equations found for **A** and **B** that $A_y > 0$ and $B_y > 0$.

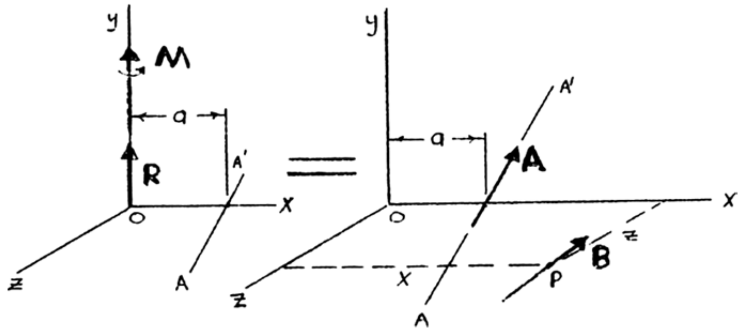
From Equation (6), $x < 0$ (assuming $a > 0$). Then, as a consequence of letting $A_x = 0$, force **A** lies in a plane parallel to the *yz*-plane and to the right of the origin, while force **B** lies in a plane parallel to the *yz*-plane but to the left to the origin, as shown in the figure below.



PROBLEM 3.146*

Show that a wrench can be replaced with two forces, one of which has a prescribed line of action.

SOLUTION



First, choose a rectangular coordinate system where one axis coincides with the axis of the wrench and another axis intersects the prescribed line of action (AA'). Note that it has been assumed that the line of action of force **B** intersects the xz-plane at Point $P(x, 0, z)$. Denoting the known direction of line AA' by

$$\lambda_A = \lambda_x \mathbf{i} + \lambda_y \mathbf{j} + \lambda_z \mathbf{k}$$

it follows that force **A** can be expressed as

$$\mathbf{A} = A\lambda_A = A(\lambda_x \mathbf{i} + \lambda_y \mathbf{j} + \lambda_z \mathbf{k})$$

Force **B** can be expressed as

$$\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$$

Next, observe that since the axis of the wrench and the prescribed line of action AA' are known, it follows that the distance a can be determined. In the following solution, it is assumed that a is known.

Then for equivalence,

$$\Sigma F_x: \quad 0 = A\lambda_x + B_x \tag{1}$$

$$\Sigma F_y: \quad R = A\lambda_y + B_y \tag{2}$$

$$\Sigma F_z: \quad 0 = A\lambda_z + B_z \tag{3}$$

$$\Sigma M_x: \quad 0 = -zB_y \tag{4}$$

$$\Sigma M_y: \quad M = -aA\lambda_z + zB_x - xB_z \tag{5}$$

$$\Sigma M_x: \quad 0 = -aA\lambda_y + xB_y \tag{6}$$

Since there are six unknowns (A, B_x, B_y, B_z, x, z) and six independent equations, it will be possible to obtain a solution.

PROBLEM 3.146* (Continued)

Case 1: Let $z = 0$ to satisfy Equation (4).

Now Equation (2): $$A\lambda_y = R - B_y$$

Equation (3): $$B_z = -A\lambda_z$$

Equation (6): $$x = -\frac{aA\lambda_y}{B_y} = -\left(\frac{a}{B_y}\right)(R - B_y)$$

Substitution into Equation (5):

$$M = -aA\lambda_z - \left[-\left(\frac{a}{B_y}\right)(R - B_y)(-A\lambda_z)\right]$$

$$A = -\frac{1}{\lambda_z}\left(\frac{M}{aR}\right)B_y$$

Substitution into Equation (2):

$$R = -\frac{1}{\lambda_z}\left(\frac{M}{aR}\right)B_y\lambda_y + B_y$$

$$B_y = \frac{\lambda_z aR^2}{\lambda_z aR - \lambda_y M}$$

Then $$A = -\frac{MR}{\lambda_z aR - \lambda_y M} = \frac{R}{\lambda_y - \frac{aR}{M}\lambda_z}$$

$$B_x = -A\lambda_x = \frac{\lambda_x MR}{\lambda_z aR - \lambda_y M}$$

$$B_z = -A\lambda_z = \frac{\lambda_z MR}{\lambda_z aR - \lambda_y M}$$

In summary, $$\mathbf{A} = \frac{P}{\lambda_y - \frac{aR}{M}\lambda_z}\lambda_A \blacktriangleleft$$

$$\mathbf{B} = \frac{R}{\lambda_z aR - \lambda_y M}(\lambda_x M\,\mathbf{i} + \lambda_z aR\mathbf{j} + \lambda_z M\,\mathbf{k}) \blacktriangleleft$$

and $$x = a\left(1 - \frac{R}{B_y}\right)$$

$$= a\left[1 - R\left(\frac{\lambda_z aR - \lambda_y M}{\lambda_z aR^2}\right)\right]$$

or $$x = \frac{\lambda_y M}{\lambda_z R} \blacktriangleleft$$

Note that for this case, the lines of action of both **A** and **B** intersect the *x*-axis.

PROBLEM 3.146* (Continued)

Case 2: Let $B_y = 0$ to satisfy Equation (4).

Now Equation (2): $A = \frac{R}{\lambda_y}$

Equation (1): $B_x = -R\left(\frac{\lambda_x}{\lambda_y}\right)$

Equation (3): $B_z = -R\left(\frac{\lambda_z}{\lambda_y}\right)$

Equation (6): $aA\lambda_y = 0$ which requires $a = 0$

Substitution into Equation (5):

$$M = z\left[-R\left(\frac{\lambda_x}{\lambda_y}\right)\right] - x\left[-R\left(\frac{\lambda_z}{\lambda_y}\right)\right] \quad \text{or} \quad \lambda_z x - \lambda_x z = \left(\frac{M}{R}\right)\lambda_y$$

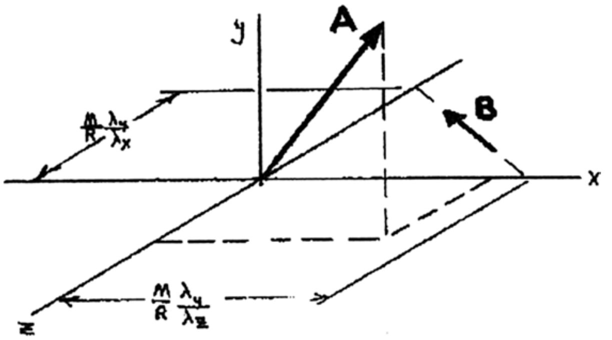
This last expression is the equation for the line of action of force **B**.

In summary,

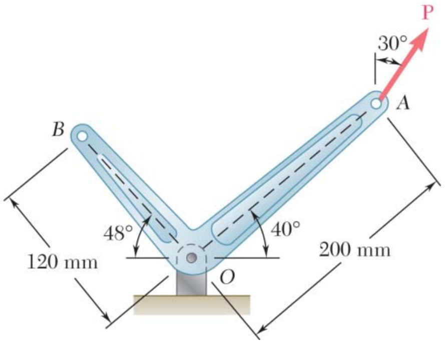
$$\mathbf{A} = \left(\frac{R}{\lambda_y}\right)\lambda_A \quad ◀$$

$$\mathbf{B} = \left(\frac{R}{\lambda_y}\right)(-\lambda_x\mathbf{i} - \lambda_x\mathbf{k}) \quad ◀$$

Assuming that $\lambda_x, \lambda_y, \lambda_z$. 0, the equivalent force system is as shown below.


y
A
B
x
z
$\frac{M}{R}\frac{\lambda_y}{\lambda_x}$
$\frac{M}{R}\frac{\lambda_y}{\lambda_z}$

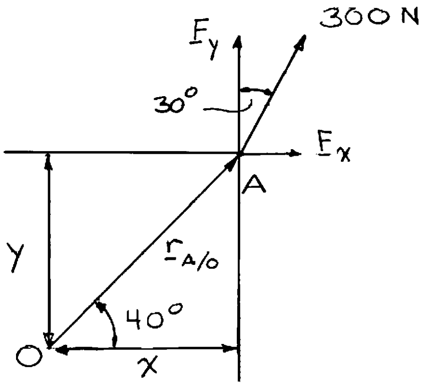
Note that the component of **A** in the *xz*-plane is parallel to **B**.



PROBLEM 3.147

A 300-N force **P** is applied at Point *A* of the bell crank shown. (*a*) Compute the moment of the force **P** about *O* by resolving it into horizontal and vertical components. (*b*) Using the result of part (*a*), determine the perpendicular distance from *O* to the line of action of **P**.

SOLUTION



$$x = (0.2\ \text{m})\cos 40°$$
$$= 0.153209\ \text{m}$$
$$y = (0.2\ \text{m})\sin 40°$$
$$= 0.128558\ \text{m}$$
$$\therefore\quad \mathbf{r}_{A/O} = (0.153209\ \text{m})\mathbf{i} + (0.128558\ \text{m})\mathbf{j}$$

(*a*) $F_x = (300\ \text{N})\sin 30°$

$= 150\ \text{N}$

$F_y = (300\ \text{N})\cos 30°$

$= 259.81\ \text{N}$

$\mathbf{F} = (150\ \text{N})\mathbf{i} + (259.81\ \text{N})\mathbf{j}$

$\mathbf{M}_O = \mathbf{r}_{A/O} \times \mathbf{F}$

$= (0.153209\mathbf{i} + 0.128558\mathbf{j})\ \text{m} \times (150\mathbf{i} + 259.81\mathbf{j})\ \text{N}$

$= (39.805\mathbf{k} - 19.2837\mathbf{k})\ \text{N}\cdot\text{m}$

$= (20.521\ \text{N}\cdot\text{m})\mathbf{k}$

$\mathbf{M}_O = 20.5\ \text{N}\cdot\text{m} \circlearrowleft$ ◀

(*b*) $M_O = Fd$

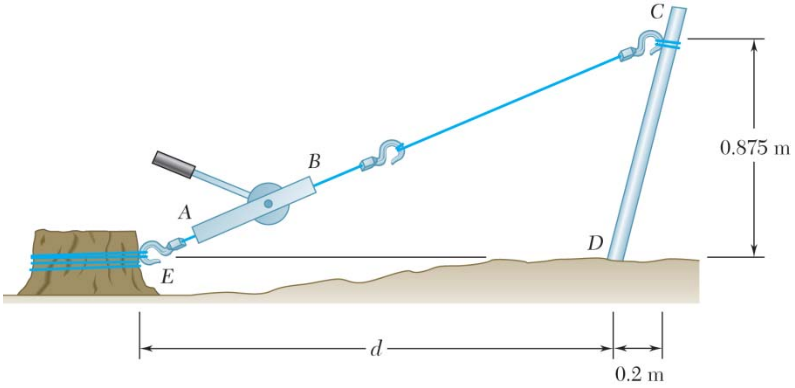
$20.521\ \text{N}\cdot\text{m} = (300\ \text{N})(d)$

$d = 0.068403\ \text{m}$

$d = 68.4\ \text{mm}$ ◀

PROBLEM 3.148

A winch puller *AB* is used to straighten a fence post. Knowing that the tension in cable *BC* is 1040 N and length *d* is 1.90 m, determine the moment about *D* of the force exerted by the cable at *C* by resolving that force into horizontal and vertical components applied (*a*) at Point *C*, (*b*) at Point *E*.

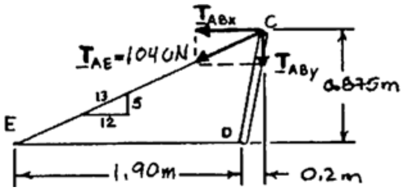

C
B
A
D
E
0.875 m
d
0.2 m

SOLUTION

(*a*) Slope of line: $EC = \dfrac{0.875\text{ m}}{1.90\text{ m} + 0.2\text{ m}} = \dfrac{5}{12}$

Then

$$T_{ABx} = \frac{12}{13}(T_{AB})$$
$$= \frac{12}{13}(1040\text{ N})$$
$$= 960\text{ N}$$


T_ABx
C
T_AE=1040N
T_ABy
0.875m
13
5
12
E
D
1.90m
0.2m

(*a*)

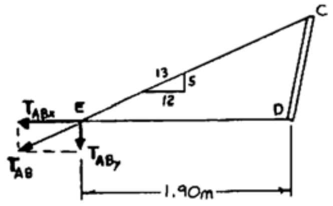
and

$$T_{ABy} = \frac{5}{13}(1040\text{ N})$$
$$= 400\text{ N}$$

Then

$$M_D = T_{ABx}(0.875\text{ m}) - T_{ABy}(0.2\text{ m})$$
$$= (960\text{ N})(0.875\text{ m}) - (400\text{ N})(0.2\text{ m})$$
$$= 760\text{ N}\cdot\text{m}$$

or $\mathbf{M}_D = 760\text{ N}\cdot\text{m}$ ↻ ◀

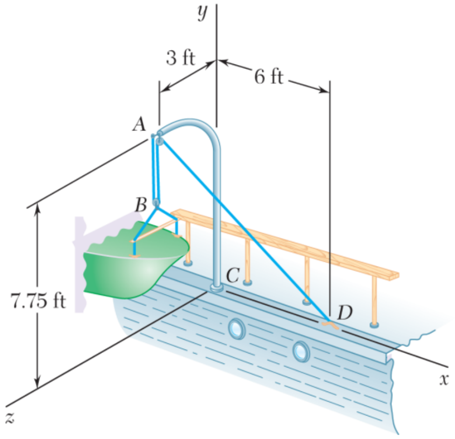

C
13
5
12
T_ABx
E
D
T_AB
T_ABy
1.90m

(*b*)

(*b*) We have

$$M_D = T_{ABx}(y) + T_{ABx}(x)$$
$$= (960\text{ N})(0) + (400\text{ N})(1.90\text{ m})$$
$$= 760\text{ N}\cdot\text{m}$$

or $\mathbf{M}_D = 760\text{ N}\cdot\text{m}$ ↻ ◀



PROBLEM 3.149

A small boat hangs from two davits, one of which is shown in the figure. The tension in line *ABAD* is 82 lb. Determine the moment about *C* of the resultant force $\mathbf{R}_A$ exerted on the davit at *A*.

SOLUTION

We have $$\mathbf{R}_A = 2\mathbf{F}_{AB} + \mathbf{F}_{AD}$$

where $$\mathbf{F}_{AB} = -(82\text{ lb})\mathbf{j}$$

and $$\mathbf{F}_{AD} = F_{AD}\frac{\overrightarrow{AD}}{AD} = (82\text{ lb})\frac{6\mathbf{i} - 7.75\mathbf{j} - 3\mathbf{k}}{10.25}$$

$$\mathbf{F}_{AD} = (48\text{ lb})\mathbf{i} - (62\text{ lb})\mathbf{j} - (24\text{ lb})\mathbf{k}$$

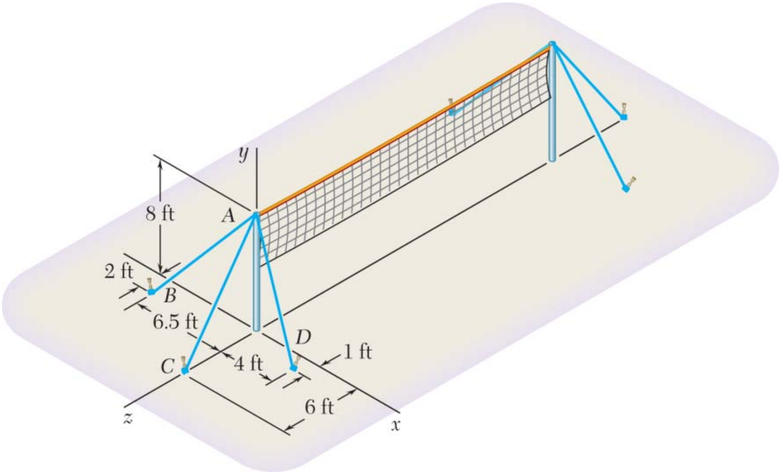
Thus $$\mathbf{R}_A = 2\mathbf{F}_{AB} + \mathbf{F}_{AD} = (48\text{ lb})\mathbf{i} - (226\text{ lb})\mathbf{j} - (24\text{ lb})\mathbf{k}$$

Also $$\mathbf{r}_{A/C} = (7.75\text{ ft})\mathbf{j} + (3\text{ ft})\mathbf{k}$$

Using Eq. (3.21): $$\mathbf{M}_C = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 7.75 & 3 \\ 48 & -226 & -24 \end{vmatrix}$$

$$= (492\text{ lb}\cdot\text{ft})\mathbf{i} + (144.0\text{ lb}\cdot\text{ft})\mathbf{j} - (372\text{ lb}\cdot\text{ft})\mathbf{k}$$

$$\mathbf{M}_C = (492\text{ lb}\cdot\text{ft})\mathbf{i} + (144.0\text{ lb}\cdot\text{ft})\mathbf{j} - (372\text{ lb}\cdot\text{ft})\mathbf{k}$$ ◀



PROBLEM 3.150

Consider the volleyball net shown. Determine the angle formed by guy wires *AB* and *AC*.

SOLUTION

First note:

$$AB = \sqrt{(-6.5)^2 + (-8)^2 + (2)^2} = 10.5 \text{ ft}$$

$$AC = \sqrt{(0)^2 + (-8)^2 + (6)^2} = 10 \text{ ft}$$

and

$$\overrightarrow{AB} = -(6.5 \text{ ft})\mathbf{i} - (8 \text{ ft})\mathbf{j} + (2 \text{ ft})\mathbf{k}$$

$$\overrightarrow{AC} = -(8 \text{ ft})\mathbf{j} + (6 \text{ ft})\mathbf{k}$$

By definition,

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = (AB)(AC)\cos\theta$$

or

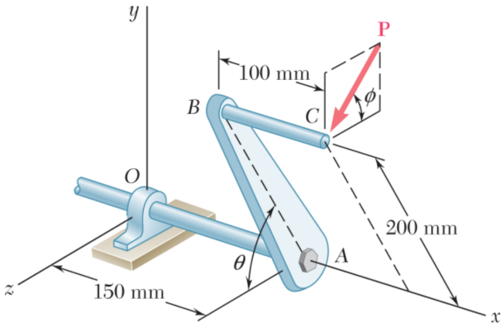
$$(-6.5\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}) \cdot (-8\mathbf{j} + 6\mathbf{k}) = (10.5)(10)\cos\theta$$

$$(-6.5)(0) + (-8)(-8) + (2)(6) = 105\cos\theta$$

or

$$\cos\theta = 0.72381$$

or $\theta = 43.6°$ ◀



PROBLEM 3.151

A single force **P** acts at *C* in a direction perpendicular to the handle *BC* of the crank shown. Determine the moment M_x of **P** about the *x*-axis when $\theta = 65°$, knowing that $M_y = -15\ \text{N}\cdot\text{m}$ and $M_z = -36\ \text{N}\cdot\text{m}$.

SOLUTION

See the solution to Prob. 3.53 for the derivation of the following equations:

$$M_x = (0.2)P\sin(\theta+\phi) \qquad (1)$$

$$\tan\phi = \frac{M_z}{M_y} \qquad (4)$$

$$P = 4\sqrt{M_y^2 + M_z^2} \qquad (5)$$

Substituting for known data gives:

$$\tan\phi = \frac{-36\ \text{N}\cdot\text{m}}{-15\ \text{N}\cdot\text{m}}$$

$$\phi = 67.380°$$

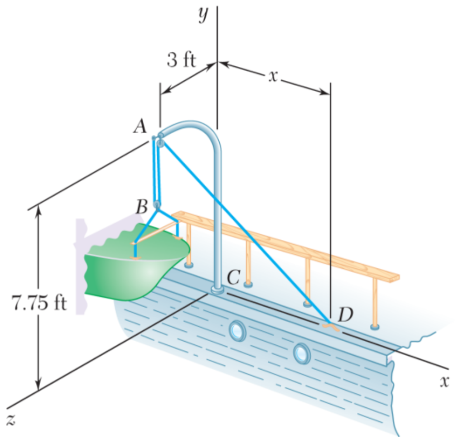
$$P = 4\sqrt{(-15)^2 + (-36)^2}$$

$$P = 156.0\ \text{N}$$

$$M_x = 0.2\ \text{m}(156.0\ \text{N})\sin(65° + 67.380°)$$

$$= 23.047\ \text{N}\cdot\text{m}$$

$$M_x = 23.0\ \text{N}\cdot\text{m} \blacktriangleleft$$



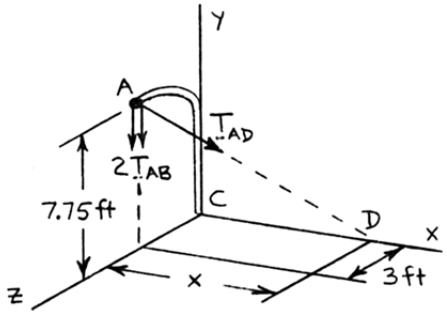
PROBLEM 3.152

A small boat hangs from two davits, one of which is shown in the figure. It is known that the moment about the *z*-axis of the resultant force $\mathbf{R}_A$ exerted on the davit at *A* must not exceed 279 lb·ft in absolute value. Determine the largest allowable tension in line *ABAD* when $x = 6$ ft.

SOLUTION

First note: $$\mathbf{R}_A = 2\mathbf{T}_{AB} + \mathbf{T}_{AD}$$

Also note that only $\mathbf{T}_{AD}$ will contribute to the moment about the *z*-axis.



Now $$AD = \sqrt{(6)^2 + (-7.75)^2 + (-3)^2}$$
$$= 10.25 \text{ ft}$$

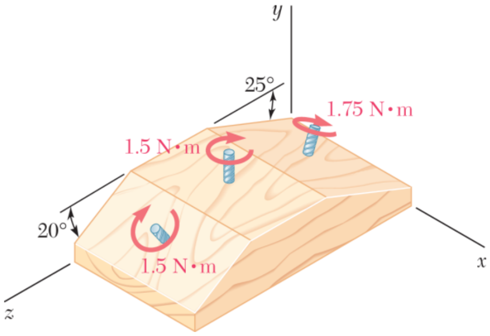
Then $$\mathbf{T}_{AD} = T\frac{\overrightarrow{AD}}{AD}$$
$$= \frac{T}{10.25}(6\mathbf{i} - 7.75\mathbf{j} - 3\mathbf{k})$$

Now $$M_z = \mathbf{k} \cdot (\mathbf{r}_{A/C} \times \mathbf{T}_{AD})$$

where $$\mathbf{r}_{A/C} = (7.75 \text{ ft})\mathbf{j} + (3 \text{ ft})\mathbf{k}$$

Then for $T_{\max}$, $$279 = \left|\frac{T_{\max}}{10.25}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 7.75 & 3 \\ 6 & -7.75 & -3 \end{vmatrix}\right|$$
$$= \frac{T_{\max}}{10.25}\left|-(1)(7.75)(6)\right|$$

or $T_{\max} = 61.5$ lb ◀



PROBLEM 3.153

In a manufacturing operation, three holes are drilled simultaneously in a workpiece. If the holes are perpendicular to the surfaces of the workpiece, replace the couples applied to the drills with a single equivalent couple, specifying its magnitude and the direction of its axis.

SOLUTION

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3$$

$$= (1.5\text{ N}\cdot\text{m})(-\cos 20°\mathbf{j} + \sin 20°\mathbf{k}) - (1.5\text{ N}\cdot\text{m})\mathbf{j}$$

$$+ (1.75\text{ N}\cdot\text{m})(-\cos 25°\mathbf{j} + \sin 25°\mathbf{k})$$

$$= -(4.4956\text{ N}\cdot\text{m})\mathbf{j} + (0.22655\text{ N}\cdot\text{m})\mathbf{k}$$

$$M = \sqrt{(0)^2 + (-4.4956)^2 + (0.22655)^2}$$

$$= 4.5013\text{ N}\cdot\text{m}$$

$M = 4.50\text{ N}\cdot\text{m}$ ◀

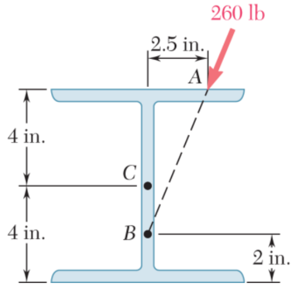
$$\lambda_{\text{axis}} = \frac{\mathbf{M}}{M} = -(0.99873\mathbf{j} + 0.050330\mathbf{k})$$

$$\cos\theta_x = 0$$

$$\cos\theta_y = -0.99873$$

$$\cos\theta_z = 0.050330$$

$\theta_x = 90.0°, \quad \theta_y = 177.1°, \quad \theta_z = 87.1°$ ◀



PROBLEM 3.154

A 260-lb force is applied at A to the rolled-steel section shown. Replace that force with an equivalent force-couple system at the center C of the section.

SOLUTION

$$AB = \sqrt{(2.5\text{ in.})^2 + (6.0\text{ in.})^2} = 6.50\text{ in.}$$

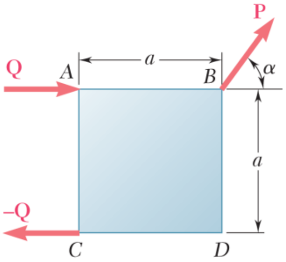
$$\sin\alpha = \frac{2.5\text{ in.}}{6.5\text{ in.}} = \frac{5}{13}$$

$$\cos\alpha = \frac{6.0\text{ in.}}{6.5\text{ in.}} = \frac{12}{13} \qquad \alpha = 22.6^\circ$$

$$\begin{aligned}\mathbf{F} &= -F\sin\alpha\mathbf{i} - F\cos\alpha\mathbf{j}\\ &= -(260\text{ lb})\frac{5}{13}\mathbf{i} - (260\text{ lb})\frac{12}{13}\mathbf{j}\\ &= -(100.0\text{ lb})\mathbf{i} - (240\text{ lb})\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{M}_C &= \mathbf{r}_{A/C} \times \mathbf{F}\\ &= (2.5\mathbf{i} + 4.0\mathbf{j}) \times (-100.0\mathbf{i} - 240\mathbf{j})\\ &= 400\mathbf{k} - 600\mathbf{k}\\ &= -(200\text{ lb}\cdot\text{in.})\mathbf{k}\end{aligned}$$

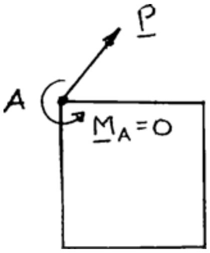
$\mathbf{F} = 260\text{ lb} \measuredangle 67.4^\circ$; $\mathbf{M}_C = 200\text{ lb}\cdot\text{in.} \circlearrowright$ ◀



PROBLEM 3.155

The force and couple shown are to be replaced by an equivalent single force. Knowing that $P = 2Q$, determine the required value of α if the line of action of the single equivalent force is to pass through (*a*) Point A, (*b*) Point C.

SOLUTION

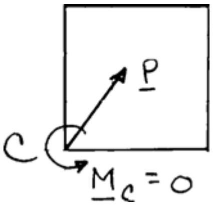


(*a*) We must have $M_A = 0$

$$(P\sin\alpha)a - Q(a) = 0$$

$$\sin\alpha = \frac{Q}{P} = \frac{Q}{2Q} = \frac{1}{2}$$

$\alpha = 30.0°$ ◀



(*b*) We must have $M_C = 0$

$$(P\sin\alpha)a - (P\cos\alpha)a - Q(a) = 0$$

$$\sin\alpha - \cos\alpha = \frac{Q}{P} = \frac{Q}{2Q} = \frac{1}{2}$$

$$\sin\alpha = \cos\alpha + \frac{1}{2} \qquad (1)$$

$$\sin^2\alpha = \cos^2\alpha + \cos\alpha + \frac{1}{4}$$

$$1 - \cos^2\alpha = \cos^2\alpha + \cos\alpha + \frac{1}{4}$$

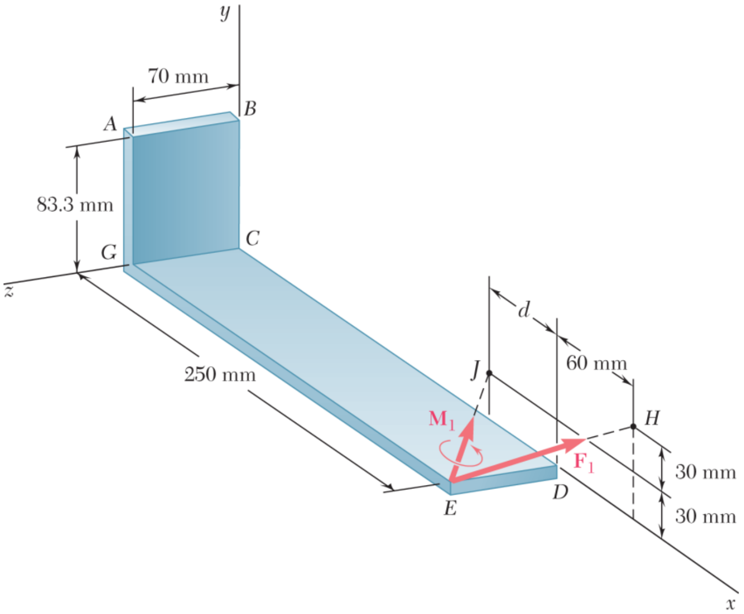
$$2\cos^2\alpha + \cos\alpha - 0.75 = 0 \qquad (2)$$

Solving the quadratic in $\cos\alpha$:

$$\cos\alpha = \frac{-1 \pm \sqrt{7}}{4} \qquad \alpha = 65.7° \text{ or } 155.7°$$

Only the first value of α satisfies Eq. (1),

therefore $\alpha = 65.7°$ ◀



PROBLEM 3.156

A 77-N force $\mathbf{F}_1$ and a 31-N · m couple $\mathbf{M}_1$ are applied to corner *E* of the bent plate shown. If $\mathbf{F}_1$ and $\mathbf{M}_1$ are to be replaced with an equivalent force-couple system $(\mathbf{F}_2, \mathbf{M}_2)$ at corner *B* and if $(M_2)_z = 0$, determine (*a*) the distance *d*, (*b*) $\mathbf{F}_2$ and $\mathbf{M}_2$.

SOLUTION

(*a*) We have

$$\Sigma M_{Bz}: \quad M_{2z} = 0$$

$$\mathbf{k}\cdot(\mathbf{r}_{H/B}\times \mathbf{F}_1) + M_{1z} = 0 \qquad (1)$$

where

$$\mathbf{r}_{H/B} = (0.31\text{ m})\mathbf{i} - (0.0233)\mathbf{j}$$

$$\mathbf{F}_1 = \boldsymbol{\lambda}_{EH} F_1$$

$$= \frac{(0.06\text{ m})\mathbf{i} + (0.06\text{ m})\mathbf{j} - (0.07\text{ m})\mathbf{k}}{0.11\text{ m}}(77\text{ N})$$

$$= (42\text{ N})\mathbf{i} + (42\text{ N})\mathbf{j} - (49\text{ N})\mathbf{k}$$

$$M_{1z} = \mathbf{k}\cdot\mathbf{M}_1$$

$$\mathbf{M}_1 = \boldsymbol{\lambda}_{EJ} M_1$$

$$= \frac{-d\mathbf{i} + (0.03\text{ m})\mathbf{j} - (0.07\text{ m})\mathbf{k}}{\sqrt{d^2 + 0.0058}\text{ m}}(31\text{ N}\cdot\text{m})$$

Then from Equation (1),

$$\begin{vmatrix} 0 & 0 & 1 \\ 0.31 & -0.0233 & 0 \\ 42 & 42 & -49 \end{vmatrix} + \frac{(-0.07\text{ m})(31\text{ N}\cdot\text{m})}{\sqrt{d^2 + 0.0058}} = 0$$

Solving for *d*, Equation (1) reduces to

$$(13.0200 + 0.9786) - \frac{2.17\text{ N}\cdot\text{m}}{\sqrt{d^2 + 0.0058}} = 0$$

from which $d = 0.1350\text{ m}$ or $d = 135.0\text{ mm}$ ◀

PROBLEM 3.156 (Continued)

(*b*) $\mathbf{F}_2 = \mathbf{F}_1 = (42\mathbf{i} + 42\mathbf{j} - 49\mathbf{k})\ \text{N}$ or $\mathbf{F}_2 = (42.0\ \text{N})\mathbf{i} + (42.0\ \text{N})\mathbf{j} - (49.0\ \text{N})\mathbf{k}$ ◀

$$\mathbf{M}_2 = \mathbf{r}_{H/B} \times \mathbf{F}_1 + \mathbf{M}_1$$

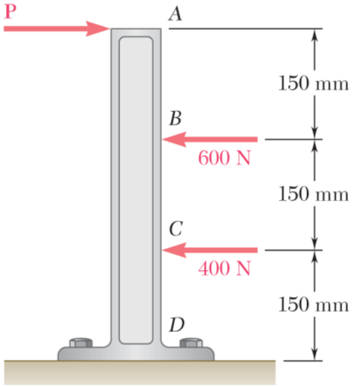
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.31 & -0.0233 & 0 \\ 42 & 42 & -49 \end{vmatrix} + \frac{(0.1350)\mathbf{i} + 0.03\mathbf{j} - 0.07\mathbf{k}}{0.155000}(31\ \text{N}\cdot\text{m})$$

$$= (1.14170\mathbf{i} + 15.1900\mathbf{j} + 13.9986\mathbf{k})\ \text{N}\cdot\text{m}$$

$$+ (-27.000\mathbf{i} + 6.0000\mathbf{j} - 14.0000\mathbf{k})\ \text{N}\cdot\text{m}$$

$$\mathbf{M}_2 = -(25.858\ \text{N}\cdot\text{m})\mathbf{i} + (21.190\ \text{N}\cdot\text{m})\mathbf{j}$$

or $\mathbf{M}_2 = -(25.9\ \text{N}\cdot\text{m})\mathbf{i} + (21.2\ \text{N}\cdot\text{m})\mathbf{j}$ ◀

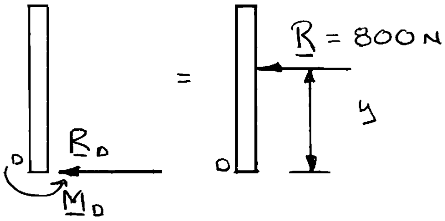


PROBLEM 3.157

Three horizontal forces are applied as shown to a vertical cast iron arm. Determine the resultant of the forces and the distance from the ground to its line of action when (*a*) P = 200 N, (*b*) P = 2400 N, (*c*) P = 1000 N.

SOLUTION

(*a*)



$$+\longrightarrow R_D = +200\text{ N} - 600\text{ N} - 400\text{ N} = -800\text{ N}$$

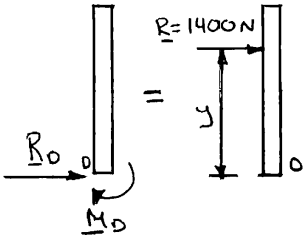
$$+\circlearrowleft M_D = -(200\text{ N})(0.450\text{ m}) + (600\text{ N})(0.300\text{ m}) + (400\text{ N})(0.1500\text{ m})$$

$$= +150.0\text{ N}\cdot\text{m}$$

$$y = \frac{M_D}{R} = \frac{150\text{ N}\cdot\text{m}}{800\text{ N}} = 0.1875\text{ m}$$

$\mathbf{R} = 800\text{ N} \longleftarrow;\ y = 187.5\text{ mm}$ ◀

(*b*)



$$+\longrightarrow R_D = +2400\text{ N} - 600\text{ N} - 400\text{ N} = +1400\text{ N}$$

$$+\circlearrowleft M_D = -(2400\text{ N})(0.450\text{ m}) + (600\text{ N})(0.300\text{ m}) + (400\text{ N})(0.1500\text{ m})$$

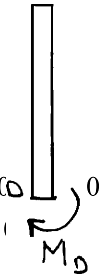
$$= -840\text{ N}\cdot\text{m}$$

$$y = \frac{M_D}{R} = \frac{840\text{ N}\cdot\text{m}}{1400\text{ N}} = 0.600\text{ m}$$

$\mathbf{R} = 1400\text{ N} \longrightarrow;\ y = 600\text{ mm}$ ◀

PROBLEM 3.157 (Continued)

(*c*)



$$+\longrightarrow R_D = +1000 - 600 - 400 = 0$$

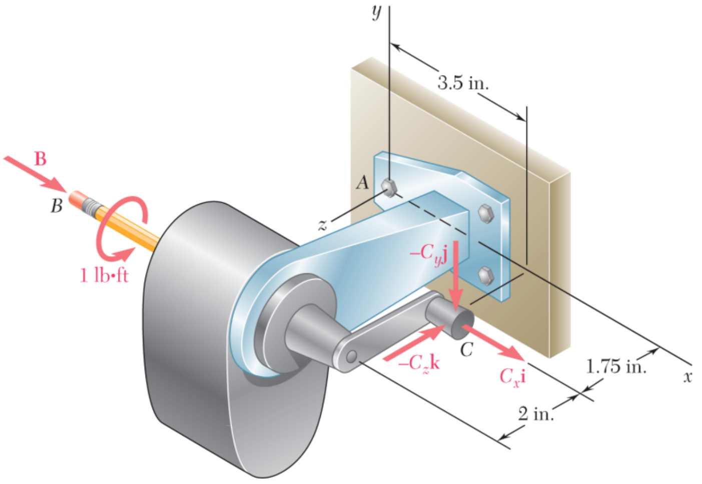
$$+\circlearrowleft$$

$$M_D = -(1 \quad 0.450\text{ m}) + (600\text{ N})(0.300\text{ m}) + (400\text{ N})(0.1500\text{ m})$$

$$= -21$$

$\therefore \quad y = \infty$ System reduces to a couple.

$\mathbf{M}_D = 210\text{ N}\cdot\text{m} \circlearrowright$ ◀



PROBLEM 3.158

While using a pencil sharpener, a student applies the forces and couple shown. (*a*) Determine the forces exerted at *B* and *C* knowing that these forces and the couple are equivalent to a force-couple system at *A* consisting of the force $\mathbf{R} = (2.6\text{ lb})\mathbf{i} + R_y\mathbf{j} - (0.7\text{ lb})\mathbf{k}$ and the couple $\mathbf{M}_A^R = M_x\mathbf{i} + (1.0\text{ lb}\cdot\text{ft})\mathbf{j} - (0.72\text{ lb}\cdot\text{ft})\mathbf{k}$. (*b*) Find the corresponding values of R_y and M_x.

SOLUTION

(*a*) From the statement of the problem, equivalence requires

$$\Sigma\mathbf{F}: \quad \mathbf{B} + \mathbf{C} = \mathbf{R}$$

or $$\Sigma F_x: \quad B_x + C_x = 2.6\text{ lb} \qquad (1)$$

$$\Sigma F_y: \quad -C_y = R_y \qquad (2)$$

$$\Sigma F_z: \quad -C_z = -0.7\text{ lb} \quad \text{or} \quad C_z = 0.7\text{ lb}$$

and $$\Sigma\mathbf{M}_A: \quad (\mathbf{r}_{B/A} \times \mathbf{B} + \mathbf{M}_B) + \mathbf{r}_{C/A} \times \mathbf{C} = M_A^R$$

or $$\Sigma\mathbf{M}_x: \quad (1\text{ lb}\cdot\text{ft}) + \left(\frac{1.75}{12}\text{ ft}\right)(C_y) = M_x \qquad (3)$$

$$\Sigma M_y: \quad \left(\frac{3.75}{12}\text{ ft}\right)(B_x) + \left(\frac{1.75}{12}\text{ ft}\right)(C_x) + \left(\frac{3.5}{12}\text{ ft}\right)(0.7\text{ lb}) = 1\text{ lb}\cdot\text{ft}$$

or $$3.75B_x + 1.75C_x = 9.55$$

Using Eq. (1): $$3.75B_x + 1.75(2.6B_x) = 9.55$$

or $$B_x = 2.5\text{ lb}$$

and $$C_x = 0.1\text{ lb}$$

$$\Sigma M_z: \quad -\left(\frac{3.5}{12}\text{ ft}\right)(C_y) = -0.72\text{ lb}\cdot\text{ft}$$

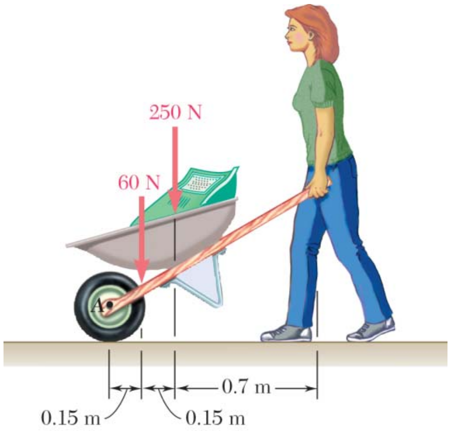
or $$C_y = 2.4686\text{ lb}$$

$$\mathbf{B} = (2.50\text{ lb})\mathbf{i} \quad \mathbf{C} = (0.1000\text{ lb})\mathbf{i} - (2.47\text{ lb})\mathbf{j} - (0.700\text{ lb})\mathbf{k} \blacktriangleleft$$

(*b*) Eq. (2) ⇒ $$R_y = -2.47\text{ lb} \blacktriangleleft$$

Using Eq. (3): $$1 + \left(\frac{1.75}{12}\right)(2.4686) = M_x \qquad \text{or} \quad M_x = 1.360\text{ lb}\cdot\text{ft} \blacktriangleleft$$

CHAPTER 4

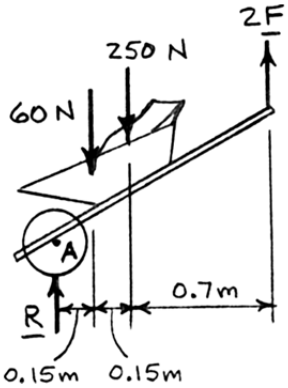

250 N
60 N
A
0.7 m
0.15 m
0.15 m

PROBLEM 4.1

A gardener uses a 60-N wheelbarrow to transport a 250-N bag of fertilizer. What force must she exert on each handle?

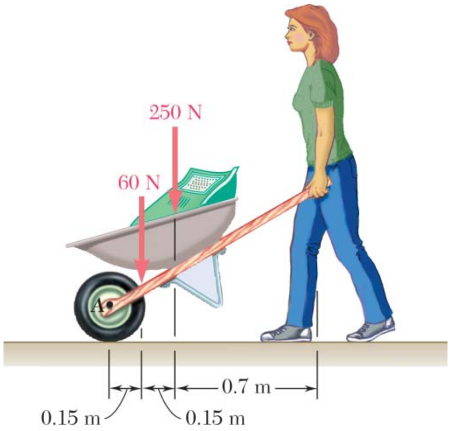
SOLUTION

Free-Body Diagram:


2F
250 N
60 N
A
R
0.7m
0.15m
0.15m

$$+\circlearrowleft \Sigma M_A = 0: \quad (2F)(1\text{ m}) - (60\text{ N})(0.15\text{ m}) - (250\text{ N})(0.3\text{ m}) = 0$$

$$\mathbf{F} = 42.0\text{ N}\uparrow \blacktriangleleft$$



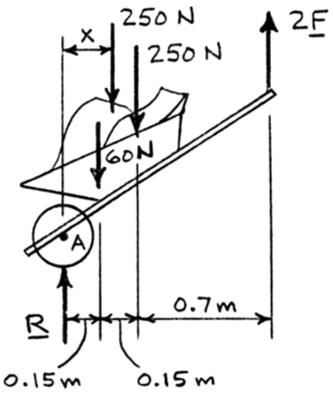
PROBLEM 4.2

The gardener of Problem 4.1 wishes to transport a second 250-N bag of fertilizer at the same time as the first one. Determine the maximum allowable horizontal distance from the axle *A* of the wheelbarrow to the center of gravity of the second bag if she can hold only 75 N with each arm.

PROBLEM 4.1 A gardener uses a 60-N wheelbarrow to transport a 250-N bag of fertilizer. What force must she exert on each handle?

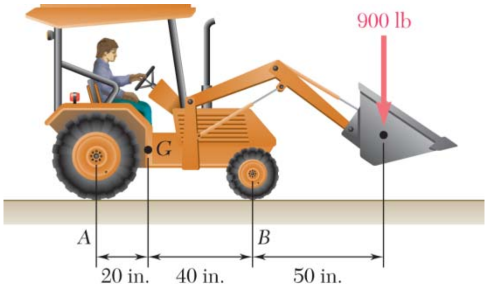
SOLUTION

Free-Body Diagram:



$$+\circlearrowleft \Sigma M_A = 0: \quad 2(75\text{ N})(1\text{ m}) - (60\text{ N})(0.15\text{ m}) - (250\text{ N})(0.3\text{ m}) - (250\text{ N})x = 0$$

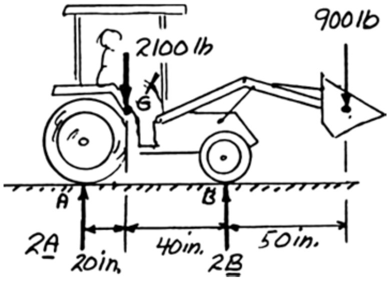
$$x = 0.264\text{ m} \blacktriangleleft$$


900 lb
G
A
B
20 in.
40 in.
50 in.

PROBLEM 4.3

A 2100-lb tractor is used to lift 900 lb of gravel. Determine the reaction at each of the two (*a*) rear wheels *A*, (*b*) front wheels *B*.

SOLUTION


900 lb
2100 lb
G
A
B
2A
20 in.
40 in.
50 in.
2B

(*a*) Rear wheels $+\circlearrowleft \Sigma M_B = 0: \quad +(2100\text{ lb})(40\text{ in.}) - (900\text{ lb})(50\text{ in.}) + 2A(60\text{ in.}) = 0$

$A = +325\text{ lb}$ $\qquad \mathbf{A} = 325\text{ lb}\uparrow$ ◀

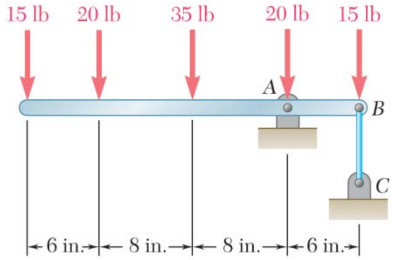
(*b*) Front wheels $+\circlearrowleft \Sigma M_A: \quad -(2100\text{ lb})(20\text{ in.}) - (900\text{ lb})(110\text{ in.}) - 2B(60\text{ in.}) = 0$

$B = +1175\text{ lb}$ $\qquad \mathbf{B} = 1175\text{ lb}\uparrow$ ◀

Check: $+\Sigma F_y = 0: \quad 2A + 2B - 2100\text{ lb} - 900\text{ lb} = 0$

$2(325\text{ lb}) + 2(1175\text{ lb}) - 2100\text{ lb} - 900 = 0$

$0 = 0$ (Checks)

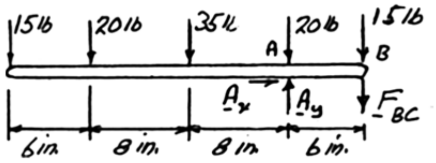

15 lb
20 lb
35 lb
20 lb
15 lb
A
B
C
6 in.
8 in.
8 in.
6 in.

PROBLEM 4.4

For the beam and loading shown, determine (*a*) the reaction at *A*, (*b*) the tension in cable *BC*.

SOLUTION

Free-Body Diagram:


15lb
20lb
35lb
20lb
15lb
A
B
A_x
A_y
F_{BC}
6in
8in
8in
6in

(*a*) Reaction at *A*: $\Sigma F_x = 0\colon \quad A_x = 0$

$$+\circlearrowleft \Sigma M_B = 0\colon \quad (15\text{ lb})(28\text{ in.}) + (20\text{ lb})(22\text{ in.}) + (35\text{ lb})(14\text{ in.})$$
$$+ (20\text{ lb})(6\text{ in.}) - A_y(6\text{ in.}) = 0$$

$$A_y = +245\text{ lb} \qquad \mathbf{A} = 245\text{ lb}\uparrow \blacktriangleleft$$

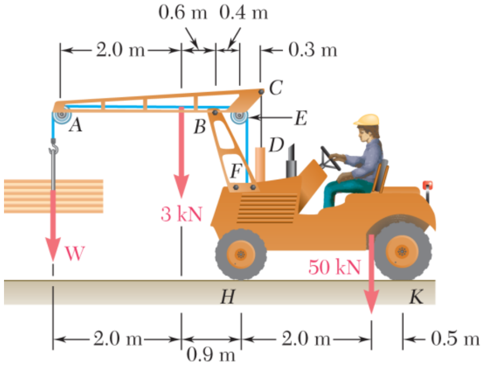
(*b*) Tension in *BC*:

$$+\circlearrowleft \Sigma M_A = 0\colon \quad (15\text{ lb})(22\text{ in.}) + (20\text{ lb})(16\text{ in.}) + (35\text{ lb})(8\text{ in.})$$
$$- (15\text{ lb})(6\text{ in.}) - F_{BC}(6\text{ in.}) = 0$$

$$F_{BC} = +140.0\text{ lb} \qquad F_{BC} = 140.0\text{ lb} \blacktriangleleft$$

Check:

$$+\uparrow \Sigma F_y = 0\colon \quad -15\text{ lb} - 20\text{ lb} = 35\text{ lb} - 20\text{ lb} + A - F_{BC} = 0$$
$$-105\text{ lb} + 245\text{ lb} - 140.0 = 0$$
$$0 = 0 \quad \text{(Checks)}$$

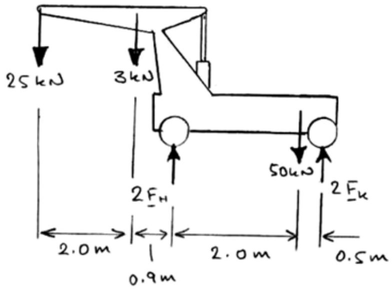

0.6 m
0.4 m
2.0 m
0.3 m
C
E
A
B
D
F
3 kN
W
50 kN
H
K
2.0 m
0.9 m
2.0 m
0.5 m

PROBLEM 4.5

A load of lumber weighing $W = 25$ kN is being raised by a mobile crane. The weight of the boom *ABC* and the combined weight of the truck and driver are as shown. Determine the reaction at each of the two (*a*) front wheels *H*, (*b*) rear wheels *K*.

SOLUTION

Free-Body Diagram:

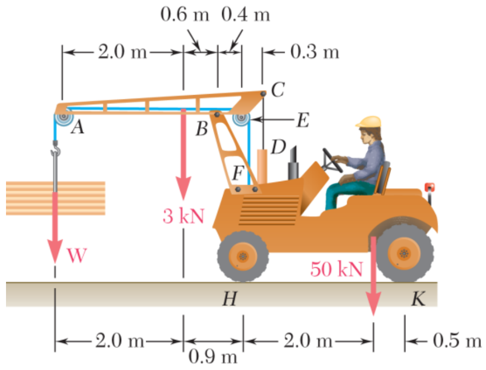

25 kN
3 kN
2 F_H
50 kN
2 F_K
2.0 m
0.9 m
2.0 m
0.5 m

(*a*) $+\circlearrowleft \Sigma M_K = 0$: $$(25\text{ kN})(5.4\text{ m}) + (3\text{ kN})(3.4\text{ m}) - 2F_H(2.5\text{ m}) + (50\text{ kN})(0.5\text{ m}) = 0$$

$$2F_H = 68.080\text{ kN}$$ or $\mathbf{F}_H = 34.0\text{ kN}\uparrow$ ◀

(*b*) $+\circlearrowleft \Sigma M_H = 0$: $$(25\text{ kN})(2.9\text{ m}) + (3\text{ kN})(0.9\text{ m}) - (50\text{ kN})(2.0\text{ m}) + 2F_K(2.5\text{ m}) = 0$$

$$2F_K = 9.9200\text{ kN}$$ or $\mathbf{F}_K = 4.96\text{ kN}\uparrow$ ◀

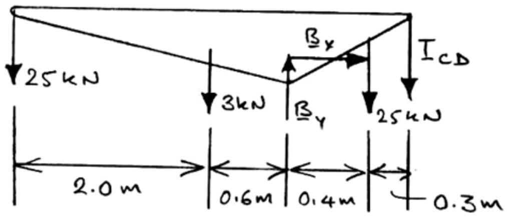

0.6 m
0.4 m
2.0 m
0.3 m
C
E
A
B
D
F
3 kN
W
50 kN
H
K
2.0 m
0.9 m
2.0 m
0.5 m

PROBLEM 4.6

A load of lumber weighing $W = 25$ kN is being raised as shown by a mobile crane. Knowing that the tension is 25 kN in all portions of cable *AEF* and that the weight of boom *ABC* is 3 kN, determine (*a*) the tension in rod *CD*, (*b*) the reaction at pin *B*.

SOLUTION

Free-Body Diagram: (boom)


Bx
TCD
25kN
3kN
By
25kN
2.0 m
0.6m
0.4m
0.3m

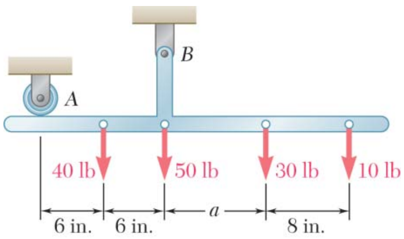
(*a*) $+\circlearrowleft\ \Sigma M_B = 0$: $\quad (25\text{ kN})(2.6\text{ m}) + (3\text{ kN})(0.6\text{ m}) - (25\text{ kN})(0.4\text{ m}) - T_{CD}(0.7\text{ m}) = 0$

$T_{CD} = 81.143$ kN $\qquad$ or $T_{CD} = 81.1$ kN ◀

(*b*) $\xrightarrow{+}\ \Sigma F_x = 0$: $\quad B_x = 0$ so that $B = B_y$

$+\!\uparrow\ \Sigma F_y = 0$: $\quad (-25 - 3 - 25 - 81.143)\text{kN} + B = 0$

$B = 134.143$ kN $\qquad$ or $\mathbf{B} = 134.1\text{ kN}\uparrow$ ◀

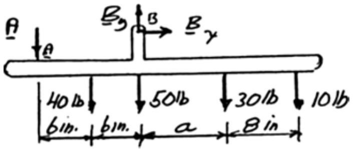


PROBLEM 4.7

A T-shaped bracket supports the four loads shown. Determine the reactions at *A* and *B* (*a*) if $a = 10$ in., (*b*) if $a = 7$ in.

SOLUTION

Free-Body Diagram:



$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (40 \text{ lb})(6 \text{ in.}) - (30 \text{ lb})a - (10 \text{ lb})(a + 8 \text{ in.}) + (12 \text{ in.})A = 0$$

$$A = \frac{(40a - 160)}{12} \qquad (1)$$

$$+\circlearrowleft \Sigma M_A = 0: \quad -(40 \text{ lb})(6 \text{ in.}) - (50 \text{ lb})(12 \text{ in.}) - (30 \text{ lb})(a + 12 \text{ in.})$$
$$- (10 \text{ lb})(a + 20 \text{ in.}) + (12 \text{ in.})B_y = 0$$

$$B_y = \frac{(1400 + 40a)}{12}$$

Since $$B_x = 0, \quad B = \frac{(1400 + 40a)}{12} \qquad (2)$$

(*a*) For $a = 10$ in.,

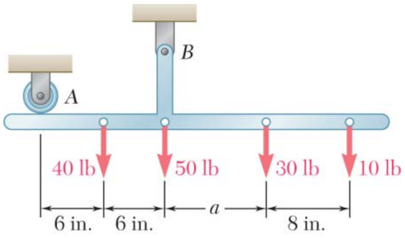
Eq. (1): $$A = \frac{(40 \times 10 - 160)}{12} = +20.0 \text{ lb} \qquad \mathbf{A} = 20.0 \text{ lb}\downarrow \blacktriangleleft$$

Eq. (2): $$B = \frac{(1400 + 40 \times 10)}{12} = +150.0 \text{ lb} \qquad \mathbf{B} = 150.0 \text{ lb}\uparrow \blacktriangleleft$$

(*b*) For $a = 7$ in.,

Eq. (1): $$A = \frac{(40 \times 7 - 160)}{12} = +10.00 \text{ lb} \qquad \mathbf{A} = 10.00 \text{ lb}\downarrow \blacktriangleleft$$

Eq. (2): $$B = \frac{(1400 + 40 \times 7)}{12} = +140.0 \text{ lb} \qquad \mathbf{B} = 140.0 \text{ lb}\uparrow \blacktriangleleft$$


B
A
40 lb
50 lb
30 lb
10 lb
6 in.
6 in.
a
8 in.

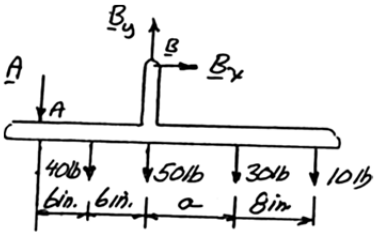
PROBLEM 4.8

For the bracket and loading of Problem 4.7, determine the smallest distance *a* if the bracket is not to move.

PROBLEM 4.7 A T-shaped bracket supports the four loads shown. Determine the reactions at *A* and *B* (*a*) if $a = 10$ in., (*b*) if $a = 7$ in.

SOLUTION

Free-Body Diagram:

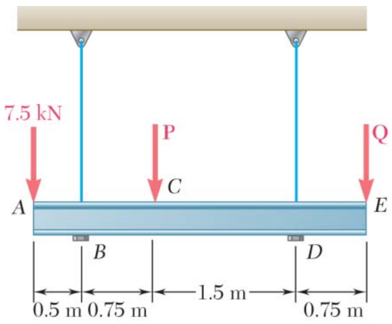

B_y
B
B_x
A
A
40lb
50lb
30lb
10lb
6in.
6in.
a
8in

For no motion, reaction at *A* must be downward or zero; smallest distance *a* for no motion corresponds to $\mathbf{A} = 0$.

$$+\circlearrowleft \Sigma M_B = 0: \quad (40 \text{ lb})(6 \text{ in.}) - (30 \text{ lb})a - (10 \text{ lb})(a + 8 \text{ in.}) + (12 \text{ in.})A = 0$$

$$A = \frac{(40a - 160)}{12}$$

$$A = 0: \quad (40a - 160) = 0 \qquad a = 4.00 \text{ in.} \blacktriangleleft$$

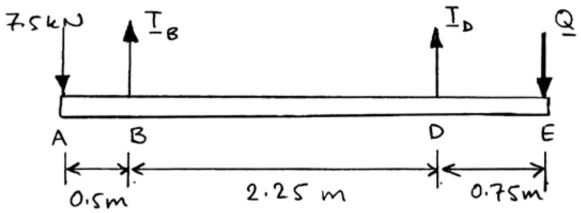

7.5 kN
P
Q
C
A
E
B
D
0.5 m
0.75 m
1.5 m
0.75 m

PROBLEM 4.9

Three loads are applied as shown to a light beam supported by cables attached at *B* and *D*. Neglecting the weight of the beam, determine the range of values of *Q* for which neither cable becomes slack when $P = 0$.

SOLUTION

Free-Body Diagram:


7.5 kN
T_B
T_D
Q
A
B
D
E
0.5m
2.25 m
0.75m

For $Q_{\min}$, $T_D = 0$

$+\circlearrowleft \Sigma M_B = 0$: $\quad (7.5 \text{ kN})(0.5 \text{ m}) - Q_{\min}(3 \text{ m}) = 0$

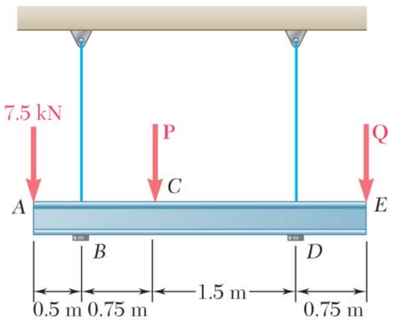
$$Q_{\min} = 1.250 \text{ kN}$$

For $Q_{\max}$, $T_B = 0$

$+\circlearrowleft \Sigma M_D = 0$: $\quad (7.5 \text{ kN})(2.75 \text{ m}) - Q_{\max}(0.75 \text{ m}) = 0$

$$Q_{\max} = 27.5 \text{ kN}$$

Therefore: $1.250 \text{ kN} \leq Q \leq 27.5 \text{ kN}$ ◀

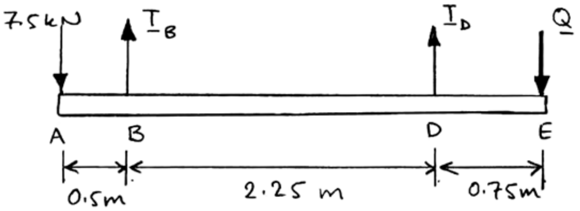


PROBLEM 4.10

Three loads are applied as shown to a light beam supported by cables attached at *B* and *D*. Knowing that the maximum allowable tension in each cable is 12 kN and neglecting the weight of the beam, determine the range of values of *Q* for which the loading is safe when $P = 0$.

SOLUTION

Free-Body Diagram:



$$+\circlearrowleft \Sigma M_D = 0: \qquad (7.5\text{ kN})(2.75\text{ m}) - T_B(2.25\text{ m}) - Q(0.75\text{ m}) = 0$$

$$Q = 27.5 - 3T_B \qquad (1)$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad (7.5\text{ kN})(0.5\text{ m}) + T_D(2.25\text{ m}) - Q(3\text{ m}) = 0$$

$$Q = 1.25 + (0.75\, T_D) \qquad (2)$$

For the loading to be safe, cables must not be slack and tension must not exceed 12 kN.

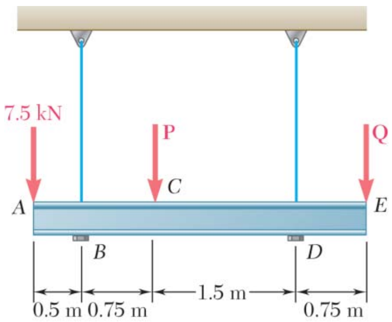
Thus, making $0 \le T_B \le 12$ kN in (1), we have

$$-8.50\text{ kN} \le Q \le 27.5\text{ kN} \qquad (3)$$

And making $0 \le T_D \le 12$ kN in (2), we have

$$1.25 \le Q \le 10.25\text{ kN} \qquad (4)$$

(3) and (4) now give: $1.250\text{ kN} \le Q \le 10.25\text{ kN}$ ◀

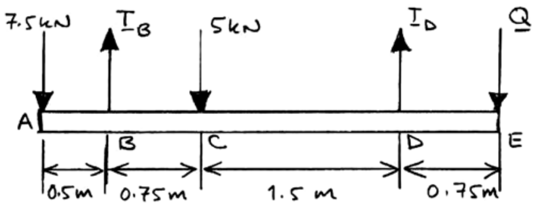


PROBLEM 4.11

For the beam of Prob. 4.10, determine the range of values of Q for which the loading is safe when $P = 5$ kN.

SOLUTION

Free-Body Diagram:



$$+\circlearrowleft \Sigma M_D = 0: \quad (7.5\text{ kN})(2.75\text{ m}) - T_B(2.25\text{ m}) + (5\text{ kN})(1.5\text{ m}) - Q(0.75\text{ m}) = 0$$

$$Q = (37.5 - 3T_B)\text{ kN} \qquad (1)$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (7.5\text{ kN})(0.5\text{ m}) - (5\text{ kN})(0.75\text{ m}) + T_D(2.25\text{ m}) - Q(3\text{ m}) = 0$$

$$Q = (0.75\,T_D)\text{ kN} \qquad (2)$$

For the loading to be safe, cables must not be slack and tension must not exceed 12 kN.

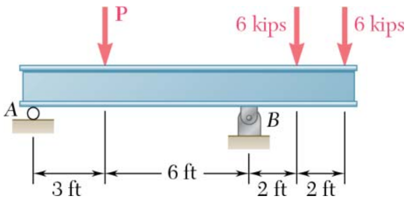
Thus, making $0 \le T_B \le 12$ kN in (1), we have

$$1.500\text{ kN} \le Q \le 37.5\text{ kN} \qquad (3)$$

And making $0 \le T_D \le 12$ kN in (2), we have

$$0 \le Q \le 9.00\text{ kN} \qquad (4)$$

(3) and (4) now give: $1.500\text{ kN} \le Q \le 9.00\text{ kN}$ ◀

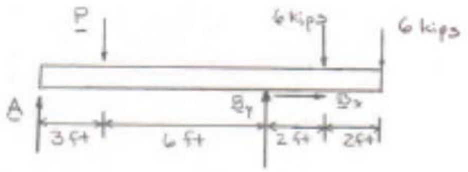


PROBLEM 4.12

For the beam of Sample Prob. 4.2, determine the range of values of *P* for which the beam will be safe, knowing that the maximum allowable value of each of the reactions is 25 kips and that the reaction at *A* must be directed upward.

SOLUTION

Free-Body Diagram:



$$+\circlearrowleft \Sigma M_A = 0: \quad -P(3\text{ ft}) + B_y(9\text{ ft}) - (6\text{ kips})(11\text{ft}) - (6\text{ kips})(13\text{ ft}) = 0$$

$$Q = (3\text{B}_y - 48)\text{ kips} \qquad (1)$$

$$+\circlearrowleft \Sigma M_B = 0: \quad -A_y(9\text{ ft}) + P(6\text{ ft}) - (6\text{ kips})(2\text{ ft}) - (6\text{ kips})(4\text{ ft}) = 0$$

$$P = (1.5A_y + 6)\text{ kips} \qquad (2)$$

For the loading to meet the design criteria, the reactions must not exceed 25 kips.

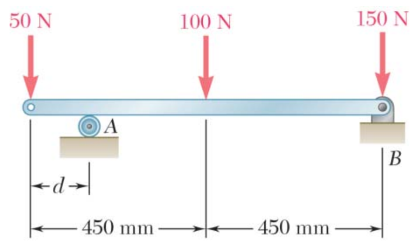
Thus, making $B_y \leq 25$ kips in (1), we have

$$P \leq 27.0\text{ kips} \qquad (3)$$

And making $0 \leq A_y \leq 25$ kips in (2), we have

$$6.00 \leq P \leq 43.5\text{ kips} \qquad (4)$$

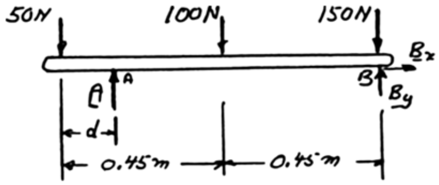
(3) and (4) now give: $6.00\text{ kips} \leq P \leq 27.0\text{ kips}$ ◀



PROBLEM 4.13

The maximum allowable value of each of the reactions is 180 N. Neglecting the weight of the beam, determine the range of the distance *d* for which the beam is safe.

SOLUTION



$$\Sigma F_x = 0: \quad B_x = 0$$

$$B = B_y$$

$$+\circlearrowleft \Sigma M_A = 0: \quad (50\text{ N})d - (100\text{ N})(0.45\text{ m} - d) - (150\text{ N})(0.9\text{ m} - d) + B(0.9\text{ m} - d) = 0$$

$$50d - 45 + 100d - 135 + 150d + 0.9B - Bd = 0$$

$$d = \frac{180\text{ N}\cdot\text{m} - (0.9\text{ m})B}{300A - B} \qquad (1)$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (50\text{ N})(0.9\text{ m}) - A(0.9\text{ m} - d) + (100\text{ N})(0.45\text{ m}) = 0$$

$$45 - 0.9A + Ad + 45 = 0$$

$$d = \frac{(0.9\text{ m})A - 90\text{ N}\cdot\text{m}}{A} \qquad (2)$$

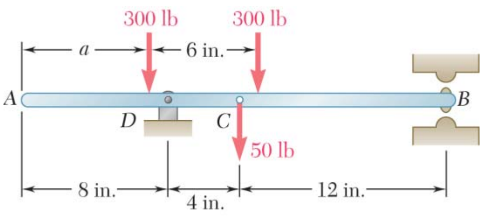
Since $B \le 180$ N, Eq. (1) yields

$$d \ge \frac{180 - (0.9)180}{300 - 180} = \frac{18}{120} = 0.15\text{ m} \qquad d \ge 150.0\text{ mm} \ \triangleleft$$

Since $A \le 180$ N, Eq. (2) yields

$$d \le \frac{(0.9)180 - 90}{180} = \frac{72}{180} = 0.40\text{ m} \qquad d \le 400\text{ mm} \ \triangleleft$$

Range: $150.0\text{ mm} \le d \le 400\text{ mm}$ ◀

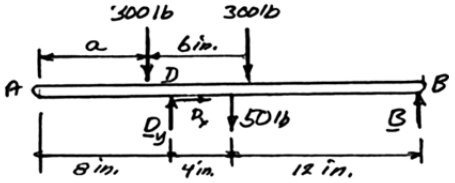

300 lb
300 lb
a
6 in.
A
B
D
C
50 lb
8 in.
4 in.
12 in.

PROBLEM 4.14

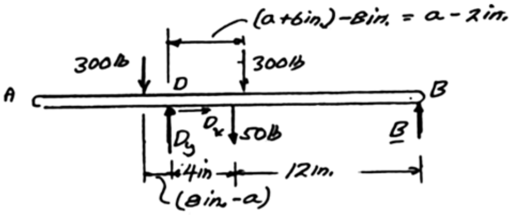
For the beam and loading shown, determine the range of the distance *a* for which the reaction at *B* does not exceed 100 lb downward or 200 lb upward.

SOLUTION

Assume *B* is positive when directed ↑.


300 lb
300 lb
a
6 in.
D
A
B
D_y
D_x
50 lb
B
8 in.
4 in.
12 in.

Sketch showing distance from *D* to forces.


(a + 6 in.) − 8 in. = a − 2 in.
300 lb
300 lb
A
D
B
D_y
D_x
50 lb
B
4 in
12 in.
(8 in. − a)

$$+\circlearrowleft \Sigma M_D = 0: \quad (300\text{ lb})(8\text{ in.} - a) - (300\text{ lb})(a - 2\text{ in.}) - (50\text{ lb})(4\text{ in.}) + 16B = 0$$

$$-600a + 2800 + 16B = 0$$

$$a = \frac{(2800 + 16B)}{600} \qquad (1)$$

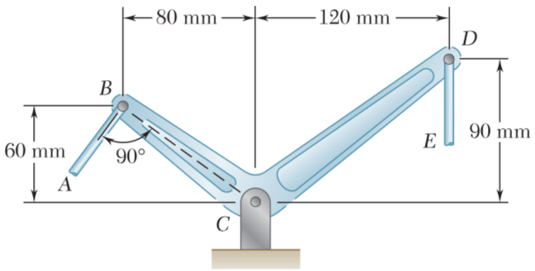
For $B = 100\text{ lb}\downarrow = -100\text{ lb}$, Eq. (1) yields:

$$a \geq \frac{[2800 + 16(-100)]}{600} = \frac{1200}{600} = 2\text{ in.} \qquad a \geq 2.00\text{ in.} \;\triangleleft$$

For $B = 200\uparrow = +200\text{ lb}$, Eq. (1) yields:

$$a \leq \frac{[2800 + 16(200)]}{600} = \frac{6000}{600} = 10\text{ in.} \qquad a \leq 10.00\text{ in.} \;\triangleleft$$

Required range: $2.00\text{ in.} \leq a \leq 10.00\text{ in.}$ ◀

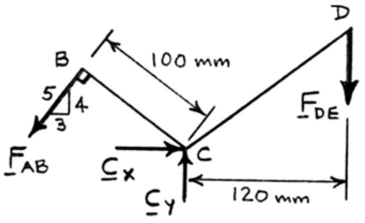

80 mm
120 mm
D
B
90 mm
E
60 mm
90°
A
C

PROBLEM 4.15

Two links *AB* and *DE* are connected by a bell crank as shown. Knowing that the tension in link *AB* is 720 N, determine (*a*) the tension in link *DE*, (*b*) the reaction at *C*.

SOLUTION

Free-Body Diagram:


D
B
100 mm
5
4
3
F_{AB}
C_x
C
C_y
F_{DE}
120 mm

$$+\circlearrowleft \Sigma M_C = 0: \quad F_{AB}(100\text{ mm}) - F_{DE}(120\text{ mm}) = 0$$

$$F_{DE} = \frac{5}{6} F_{AB} \qquad (1)$$

(*a*) For

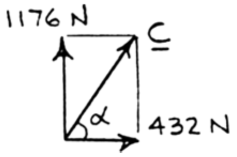
$$F_{AB} = 720\text{ N}$$

$$F_{DE} = \frac{5}{6}(720\text{ N}) \qquad F_{DE} = 600\text{ N} \blacktriangleleft$$

(*b*)

$$\xrightarrow{+} \Sigma F_x = 0: \quad -\frac{3}{5}(720\text{ N}) + C_x = 0$$

$$C_x = +432\text{ N}$$


1176 N
C
α
432 N

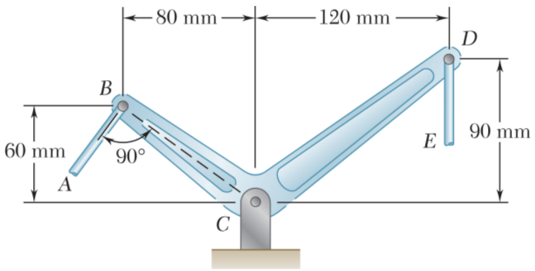
$$+\uparrow \Sigma F_y = 0: \quad -\frac{4}{5}(720\text{ N}) + C_y - 600\text{ N} = 0$$

$$C_y = +1176\text{ N}$$

$$C = 1252.84\text{ N}$$

$$\alpha = 69.829°$$

$$\mathbf{C} = 1253\text{ N} \measuredangle 69.8° \blacktriangleleft$$



PROBLEM 4.16

Two links *AB* and *DE* are connected by a bell crank as shown. Determine the maximum force that may be safely exerted by link *AB* on the bell crank if the maximum allowable value for the reaction at *C* is 1600 N.

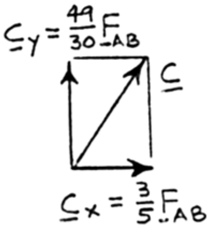
SOLUTION

See solution to Problem 4.15 for F.B.D. and derivation of Eq. (1).

$$F_{DE} = \frac{5}{6}F_{AB} \qquad (1)$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -\frac{3}{5}F_{AB} + C_x = 0 \qquad C_x = \frac{3}{5}F_{AB}$$

$$+\!\uparrow \Sigma F_y = 0: \quad -\frac{4}{5}F_{AB} + C_y - F_{DE} = 0$$



$$-\frac{4}{5}F_{AB} + C_y - \frac{5}{6}F_{AB} = 0$$

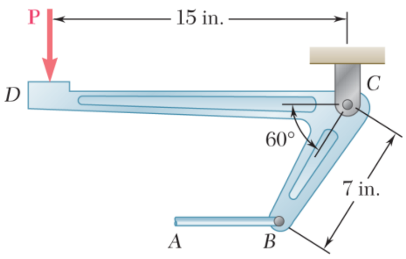
$$C_y = \frac{49}{30}F_{AB}$$

$$C = \sqrt{C_x^2 + C_y^2}$$

$$= \frac{1}{30}\sqrt{(49)^2 + (18)^2}\,F_{AB}$$

$$C = 1.74005F_{AB}$$

For $C = 1600$ N, $\quad 1600 \text{ N} = 1.74005F_{AB}$ $\qquad F_{AB} = 920$ N ◀

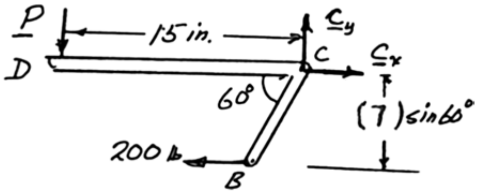

P
15 in.
D
C
60°
7 in.
A
B

PROBLEM 4.17

The required tension in cable *AB* is 200 lb. Determine (*a*) the vertical force **P** that must be applied to the pedal, (*b*) the corresponding reaction at *C*.

SOLUTION

Free-Body Diagram:

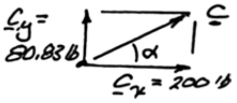

P
15 in.
C_y
C
C_x
D
60°
(7) sin 60°
200 lb
B

$BC = 7$ in.

(*a*) $+\circlearrowleft \Sigma M_C = 0\!: \quad P(15\text{ in.}) - (200\text{ lb})(6.062\text{ in.}) = 0$

$P = 80.83$ lb $\quad$ $\mathbf{P} = 80.8$ lb ↓ ◀

(*b*) $\xrightarrow{+} \Sigma F_y = 0\!: \quad C_x - 200\text{ lb} = 0$ $\quad$ $\mathbf{C}_x = 200$ lb ⟶

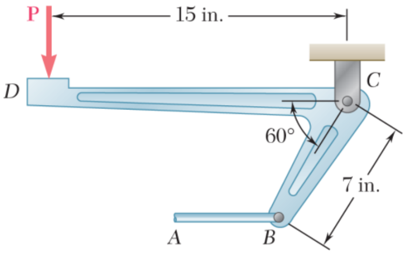

C_y = 80.83 lb
C
α
C_x = 200 lb

$+\uparrow \Sigma F_y = 0\!: \quad C_y - P = 0 \quad C_y - 80.83\text{ lb} = 0$ $\quad$ $\mathbf{C}_y = 80.83$ lb ↑

$\alpha = 22.0°$

$C = 215.7$ lb

$\mathbf{C} = 216$ lb ⦣ 22.0° ◀

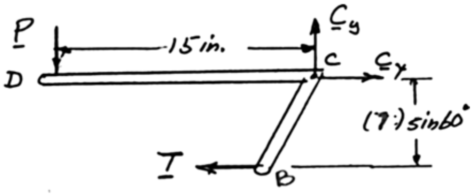

P
15 in.
D
C
60°
7 in.
A
B

PROBLEM 4.18

Determine the maximum tension that can be developed in cable *AB* if the maximum allowable value of the reaction at *C* is 250 lb.

SOLUTION

Free-Body Diagram:


P
15 in.
C_y
D
C
C_x
(7) sin60°
T
B

$BC = 7$ in.

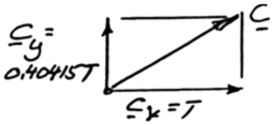
$$+\circlearrowleft \Sigma M_C = 0: \quad P(15\text{ in.}) - T(6.062\text{ in.}) = 0 \qquad P = 0.40415T$$

$$+\uparrow \Sigma F_y = 0: \quad -P + C_y = 0 \qquad -0.40415P + C_y = 0$$

$$C_y = 0.40415T$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -T + C_x = 0 \qquad C_x = T$$

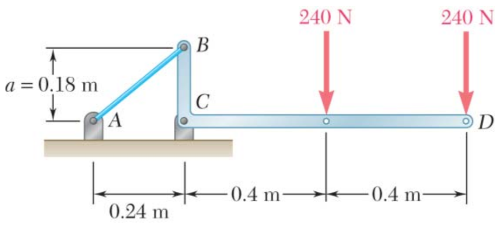
$$C = \sqrt{C_x^2 + C_y^2} = \sqrt{T^2 + (0.40415T)^2}$$


$C_y =$
0.40415T
C
$C_x = T$

$$C = 1.0786T$$

For $\quad C = 250$ lb

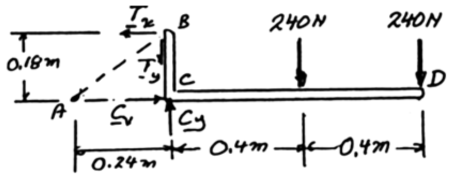
$$250\text{ lb} = 1.0786T \qquad T = 232\text{ lb} \blacktriangleleft$$


240 N
240 N
B
a = 0.18 m
C
A
D
0.24 m
0.4 m
0.4 m

PROBLEM 4.19

The bracket *BCD* is hinged at *C* and attached to a control cable at *B*. For the loading shown, determine (*a*) the tension in the cable, (*b*) the reaction at *C*.

SOLUTION


T_x
B
240N
240N
0.18m
T_y
C
D
A
C_x
C_y
0.24m
0.4m
0.4m

At *B*:

$$\frac{T_y}{T_x} = \frac{0.18\text{ m}}{0.24\text{ m}}$$

$$T_y = \frac{3}{4}T_x \qquad (1)$$

(*a*) $+\circlearrowleft \Sigma M_C = 0\colon \quad T_x(0.18\text{ m}) - (240\text{ N})(0.4\text{ m}) - (240\text{ N})(0.8\text{ m}) = 0$

$$T_x = +1600\text{ N}$$

From Eq. (1):

$$T_y = \frac{3}{4}(1600\text{ N}) = 1200\text{ N}$$

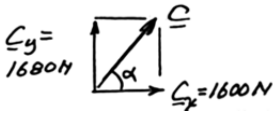
$$T = \sqrt{T_x^2 + T_y^2} = \sqrt{1600^2 + 1200^2} = 2000\text{ N} \qquad T = 2.00\text{ kN}$$ ◀

(*b*) $\xrightarrow{+} \Sigma F_x = 0\colon \quad C_x - T_x = 0$

$$C_x - 1600\text{ N} = 0 \quad C_x = +1600\text{ N} \qquad \mathbf{C}_x = 1600\text{ N} \longrightarrow$$

$$+\uparrow \Sigma F_y = 0\colon \quad C_y - T_y - 240\text{ N} - 240\text{ N} = 0$$

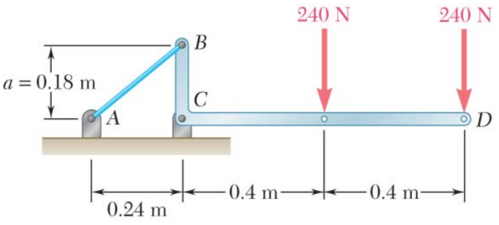
$$C_y - 1200\text{ N} - 480\text{ N} = 0$$


C_y = 1680N
C
α
C_x=1600N

$$C_y = +1680\text{ N} \qquad \mathbf{C}_y = 1680\text{ N} \uparrow$$

$$\alpha = 46.4°$$

$$C = 2320\text{ N} \qquad \mathbf{C} = 2.32\text{ kN} \measuredangle 46.4°$$ ◀

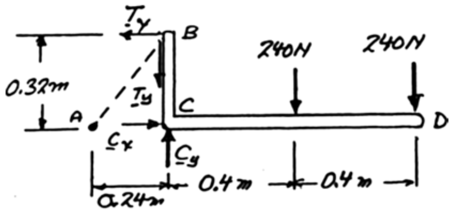

240 N
240 N
B
a = 0.18 m
C
A
D
0.24 m
0.4 m
0.4 m

PROBLEM 4.20

Solve Problem 4.19, assuming that $a = 0.32$ m.

PROBLEM 4.19 The bracket *BCD* is hinged at *C* and attached to a control cable at *B*. For the loading shown, determine (*a*) the tension in the cable, (*b*) the reaction at *C*.

SOLUTION


Ty
B
240N
240N
0.32m
Ty
C
A
D
Cx
Cy
0.4m
0.4m
0.24m

At *B*:

$$\frac{T_y}{T_x} = \frac{0.32\text{ m}}{0.24\text{ m}}$$

$$T_y = \frac{4}{3}T_x$$

$$+\circlearrowleft \Sigma M_C = 0: \quad T_x(0.32\text{ m}) - (240\text{ N})(0.4\text{ m}) - (240\text{ N})(0.8\text{ m}) = 0$$

$$T_x = 900\text{ N}$$

From Eq. (1):

$$T_y = \frac{4}{3}(900\text{ N}) = 1200\text{ N}$$

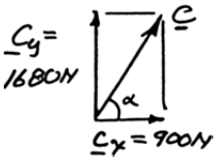
$$T = \sqrt{T_x^2 + T_y^2} = \sqrt{900^2 + 1200^2} = 1500\text{ N} \qquad T = 1.500\text{ kN} \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x - T_x = 0$$

$$C_x - 900\text{ N} = 0 \quad C_x = +900\text{ N} \qquad \mathbf{C}_x = 900\text{ N} \longrightarrow$$

$$+\uparrow \Sigma F_y = 0: \quad C_y - T_y - 240\text{ N} - 240\text{ N} = 0$$

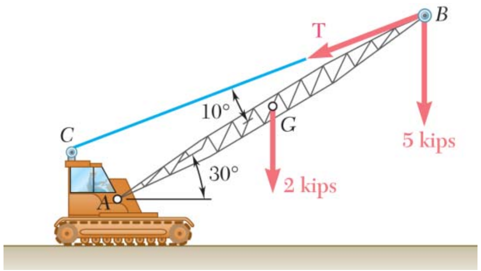
$$C_y - 1200\text{ N} - 480\text{ N} = 0$$


Cy =
1680N
C
α
Cx = 900N

$$C_y = +1680\text{ N} \qquad \mathbf{C}_y = 1680\text{ N} \uparrow$$

$$\alpha = 61.8°$$

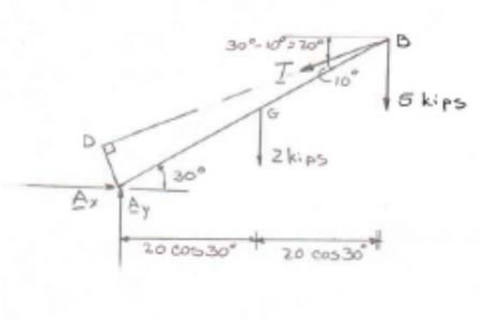
$$C = 1906\text{ N} \qquad \mathbf{C} = 1.906\text{ kN} \measuredangle 61.8° \blacktriangleleft$$


B
T
10°
G
C
5 kips
30°
2 kips
A

PROBLEM 4.21

The 40-ft boom *AB* weighs 2 kips; the distance from the axle *A* to the center of gravity *G* of the boom is 20 ft. For the position shown, determine (*a*) the tension *T* in the cable, (*b*) the reaction at *A*.

SOLUTION


30° - 10° = 20°
B
T
10°
5 kips
G
D
2 kips
30°
A_x
A_y
20 cos 30°
20 cos 30°

$$AD = (AB)\sin 10^\circ = (40\text{ ft})\sin 10^\circ$$

$$AD = 6.9459\text{ ft}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad T(AD) - 2(20\cos 30^\circ) - 5(40\cos 30^\circ) = 0$$

$$T = 29.924\text{ kips}$$

$T = 29.9$ kips ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - (29.924)\cos 20^\circ = 0$$

$$A_x = +28.119\text{ kips}$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - (29.924)\sin 20^\circ - 2 - 5 = 0$$

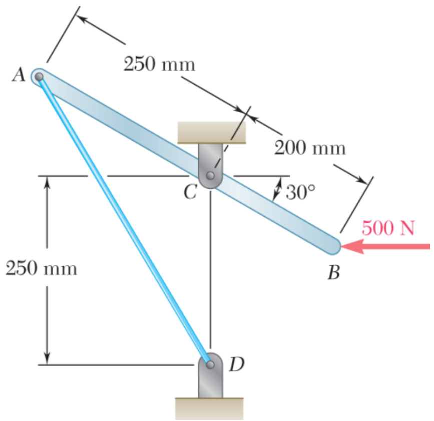
$$A_y = 17.2346\text{ kips}$$

$$\mathbf{A} = \sqrt{28.119^2 + 17.2346^2} = 32.980\text{ kips}$$

$$\alpha = \tan^{-1}\left(\frac{17.2346}{28.119}\right)$$

$$= 31.5^\circ$$

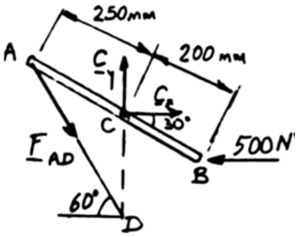
$\mathbf{A} = 33.0$ kips $\measuredangle$ 31.5° ◀


A
250 mm
200 mm
C
30°
500 N
B
250 mm
D

PROBLEM 4.22

A lever *AB* is hinged at *C* and attached to a control cable at *A*. If the lever is subjected to a 500-N horizontal force at *B*, determine (*a*) the tension in the cable, (*b*) the reaction at *C*.

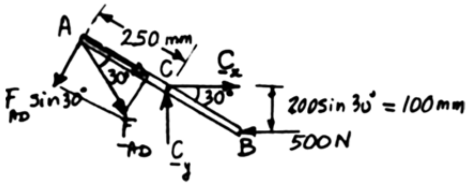
SOLUTION


250mm
200 mm
A
C_y
C_x
C
30°
500N
F_{AD}
B
60°
D

Triangle *ACD* is isosceles with $\measuredangle C = 90° + 30° = 120°$ $\measuredangle A = \measuredangle D = \frac{1}{2}(180° - 120°) = 30°.$

Thus, *DA* forms angle of 60° with the horizontal axis.

(*a*) We resolve $\mathbf{F}_{AD}$ into components along *AB* and perpendicular to *AB*.


A
250 mm
C
C_x
30°
30°
F_{AD} sin30°
200sin 30° = 100mm
F_{AD}
C_y
B
500N

$$+\circlearrowleft \Sigma M_C = 0: \quad (F_{AD} \sin 30°)(250 \text{ mm}) - (500 \text{ N})(100 \text{ mm}) = 0 \qquad F_{AD} = 400 \text{ N} \blacktriangleleft$$

(*b*)

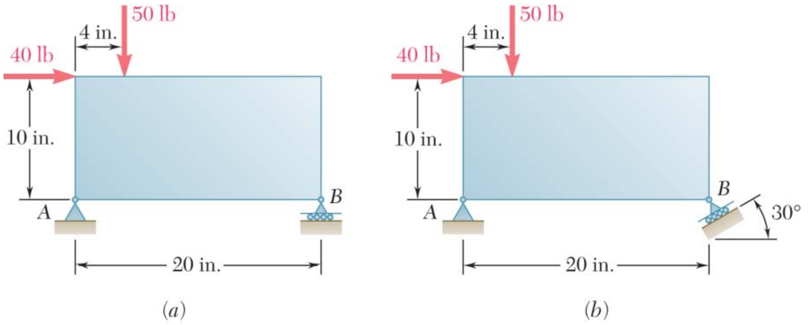
$$\xrightarrow{+} \Sigma F_x = 0: \quad -(400 \text{ N})\cos 60° + C_x - 500 \text{ N} = 0 \qquad C_x = +300 \text{ N}$$

$$+\!\uparrow \Sigma F_y = 0: \quad -(400 \text{ N})\sin 60° + C_y = 0 \qquad C_y = +346.4 \text{ N}$$

$$\mathbf{C} = 458 \text{ N} \measuredangle 49.1° \blacktriangleleft$$

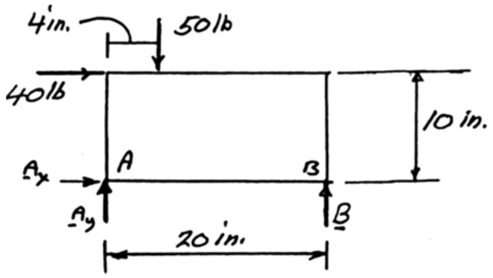
PROBLEM 4.23

For each of the plates and loadings shown, determine the reactions at *A* and *B*.


50 lb
4 in.
40 lb
10 in.
A
B
20 in.
(a)
50 lb
4 in.
40 lb
10 in.
A
B
30°
20 in.
(b)

SOLUTION

(*a*) **Free-Body Diagram:**


4 in.
50 lb
40 lb
10 in.
A_x
A
B
A_y
B
20 in.

$$+\circlearrowleft \Sigma M_A = 0: \quad B(20 \text{ in.}) - (50 \text{ lb})(4 \text{ in.}) - (40 \text{ lb})(10 \text{ in.}) = 0$$

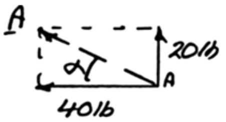
$$B = +30 \text{ lb} \qquad \mathbf{B} = 30.0 \text{ lb}\uparrow \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + 40 \text{ lb} = 0$$

$$A_x = -40 \text{ lb} \qquad \mathbf{A}_x = 40.0 \text{ lb} \longleftarrow$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + B - 50 \text{ lb} = 0$$

$$A_y + 30 \text{ lb} - 50 \text{ lb} = 0$$


A
20 lb
α
A
40 lb

$$A_y = +20 \text{ lb} \qquad \mathbf{A}_y = 20.0 \text{ lb}\uparrow$$

$$\alpha = 26.56^\circ$$

$$A = 44.72 \text{ lb} \qquad \mathbf{A} = 44.7 \text{ lb} \;\nwarrow\; 26.6^\circ \blacktriangleleft$$

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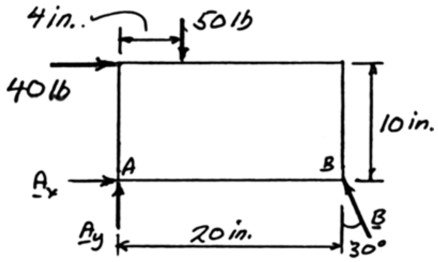
PROBLEM 4.23 (Continued)

(*b*) **Free-Body Diagram:**



$+\circlearrowleft \Sigma M_A = 0$: $(B\cos 30°)(20\text{ in.}) - (40\text{ lb})(10\text{ in.}) - (50\text{ lb})(4\text{ in.}) = 0$

$B = 34.64\text{ lb}$ $\qquad$ $\mathbf{B} = 34.6\text{ lb} \measuredangle 60.0°$ ◀

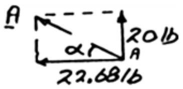
$\xrightarrow{+} \Sigma F_x = 0$: $A_x - B\sin 30° + 40\text{ lb}$

$A_x - (34.64\text{ lb})\sin 30° + 40\text{ lb} = 0$

$A_x = -22.68\text{ lb}$ $\qquad$ $\mathbf{A}_x = 22.68\text{ lb} \longleftarrow$

$+\uparrow \Sigma F_y = 0$: $A_y + B\cos 30° - 50\text{ lb} = 0$

$A_y + (34.64\text{ lb})\cos 30° - 50\text{ lb} = 0$



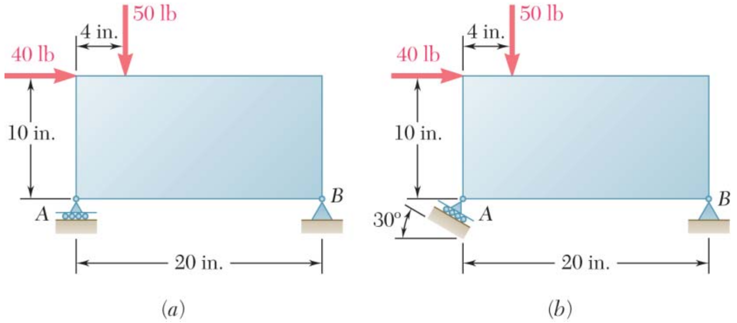
$A_y = +20\text{ lb}$ $\qquad$ $\mathbf{A}_y = 20.0\text{ lb} \uparrow$

$\alpha = 41.4°$

$A = 30.24\text{ lb}$ $\qquad$ $\mathbf{A} = 30.2\text{ lb} \measuredangle 41.4°$ ◀

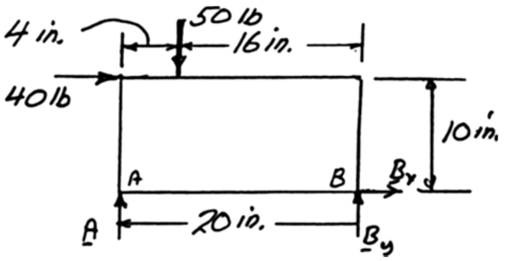
PROBLEM 4.24

For each of the plates and loadings shown, determine the reactions at *A* and *B*.


50 lb
4 in.
40 lb
10 in.
A
B
20 in.
(a)
50 lb
4 in.
40 lb
10 in.
30°
A
B
20 in.
(b)

SOLUTION

(*a*) **Free-Body Diagram:**


4 in.
50 lb
16 in.
40 lb
10 in.
A
B
B_x
A
20 in.
B_y

$$+\circlearrowleft \Sigma M_B = 0: \quad A(20\text{ in.}) + (50\text{ lb})(16\text{ in.}) - (40\text{ lb})(10\text{ in.}) = 0$$

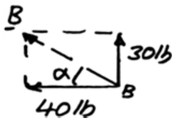
$A = +20$ lb **A** = 20.0 lb ↑ ◀

$\xrightarrow{+} \Sigma F_x = 0: \quad 40\text{ lb} + B_x = 0$

$B_x = -40$ lb $\mathbf{B}_x = 40$ lb ←

$+\uparrow \Sigma F_y = 0: \quad A + B_y - 50\text{ lb} = 0$

$20\text{ lb} + B_y - 50\text{ lb} = 0$

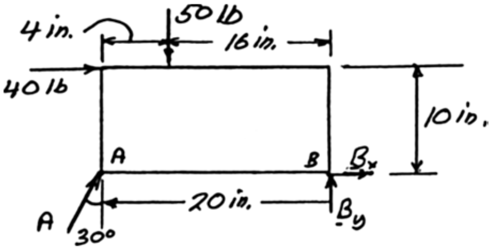

B
30 lb
α
B
40 lb

$B_y = +30$ lb $\mathbf{B}_y = 30$ lb ↑

$\alpha = 36.87°$ $B = 50$ lb **B** = 50.0 lb $\measuredangle$ 36.9° ◀

PROBLEM 4.24 (Continued)

(*b*)


4 in.
50 lb
16 in.
40 lb
10 in.
A
B
B_x
20 in.
A
30°
B_y

$$+\circlearrowleft \Sigma M_A = 0: \quad -(A\cos 30°)(20\text{ in.}) - (40\text{ lb})(10\text{ in.}) + (50\text{ lb})(16\text{ in.}) = 0$$

$A = 23.09$ lb **A** = 23.1 lb ∠ 60.0° ◀

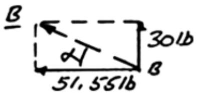
$$\xrightarrow{+} \Sigma F_x = 0: \quad A\sin 30° + 40\text{ lb} + B_x = 0$$

$$(23.09\text{ lb})\sin 30° + 40\text{ lb} + 8_x = 0$$

$B_x = -51.55$ lb $\mathbf{B}_x = 51.55$ lb ←

$$+\!\uparrow \Sigma F_y = 0: \quad A\cos 30° + B_y - 50\text{ lb} = 0$$

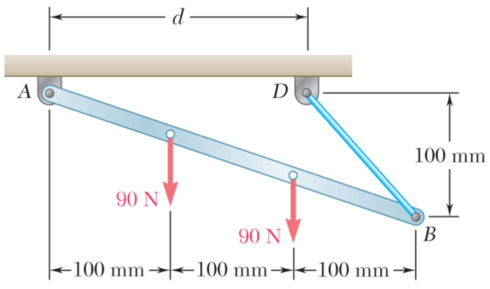
$$(23.09\text{ lb})\cos 30° + B_y - 50\text{ lb} = 0$$


B
30 lb
α
B
51.55 lb

$B_y = +30$ lb $\mathbf{B}_y = 30$ lb ↑

$\alpha = 30.2°$

$B = 59.64$ lb **B** = 59.6 lb ∠ 30.2° ◀

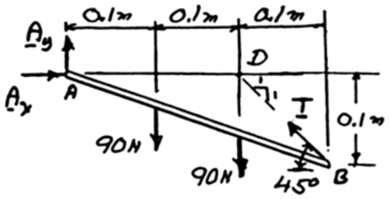

d
A
D
100 mm
90 N
90 N
B
100 mm
100 mm
100 mm

PROBLEM 4.25

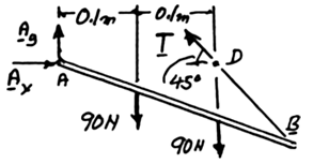
A rod *AB* hinged at *A* and attached at *B* to cable *BD* supports the loads shown. Knowing that $d = 200$ mm, determine (*a*) the tension in cable *BD*, (*b*) the reaction at *A*.

SOLUTION

Free-Body Diagram:


0.1m
0.1m
0.1m
A_y
A_x
A
D
T
0.1m
90N
90N
45°
B

(*a*) Move **T** along *BD* until it acts at Point *D*.


0.1m
0.1m
A_y
T
45°
D
A_x
A
90N
90N
B

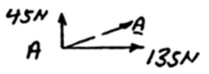
$$+\circlearrowleft \Sigma M_A = 0: \quad (T \sin 45^\circ)(0.2\text{ m}) + (90\text{ N})(0.1\text{ m}) + (90\text{ N})(0.2\text{ m}) = 0$$

$$T = 190.919\text{ N} \qquad T = 190.9\text{ N}$$ ◀

(*b*)

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - (190.919\text{ N})\cos 45^\circ = 0$$

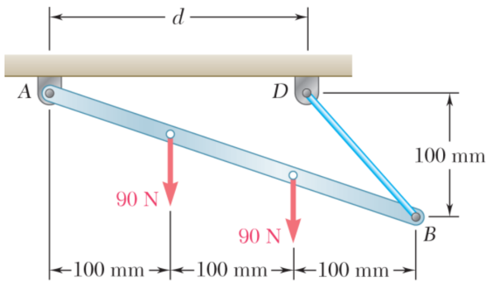
$$A_x = +135.0\text{ N} \qquad \mathbf{A}_x = 135.0\text{ N} \longrightarrow$$



$$+\uparrow \Sigma F_y = 0: \quad A_y - 90\text{ N} - 90\text{ N} + (190.919\text{ N})\sin 45^\circ = 0$$

$$A_y = +45.0\text{ N} \qquad \mathbf{A}_y = 45.0\text{ N} \uparrow$$

$$\mathbf{A} = 142.3\text{ N} \measuredangle 18.43^\circ$$ ◀

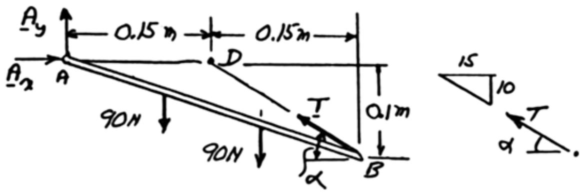

d
A
D
100 mm
90 N
90 N
B
100 mm
100 mm
100 mm

PROBLEM 4.26

A rod *AB*, hinged at *A* and attached at *B* to cable *BD*, supports the loads shown. Knowing that $d = 150$ mm, determine (*a*) the tension in cable *BD*, (*b*) the reaction at *A*.

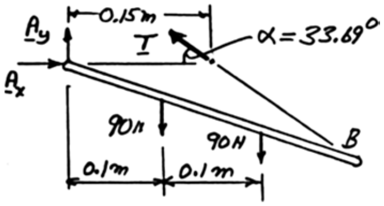
SOLUTION

Free-Body Diagram:


A_y
0.15 m
0.15 m
A_x
A
D
T
0.1 m
90 N
90 N
α
B
15
10
T
α

$$\tan\alpha = \frac{10}{15}; \quad \alpha = 33.690°$$

(*a*) Move **T** along *BD* until it acts at Point *D*.


A_y
0.15 m
T
α = 33.69°
A_x
90 N
90 N
B
0.1 m
0.1 m

$+\circlearrowleft \Sigma M_A = 0$: $(T\sin 33.690°)(0.15\text{ m}) - (90\text{ N})(0.1\text{ m}) - (90\text{ N})(0.2\text{ m}) = 0$

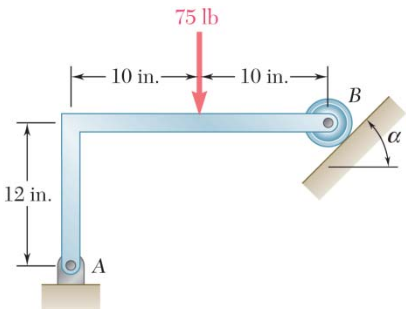
$T = 324.50$ N $\qquad T = 324$ N ◀

(*b*) $\xrightarrow{+} \Sigma F_x = 0$: $A_x - (324.50\text{ N})\cos 33.690° = 0$

$A_x = +270$ N $\qquad \mathbf{A}_x = 270$ N →

$+\uparrow \Sigma F_y = 0$: $A_y - 90\text{ N} - 90\text{ N} + (324.50\text{ N})\sin 33.690° = 0$

$A_y = 0$ $\qquad \mathbf{A} = 270$ N → ◀

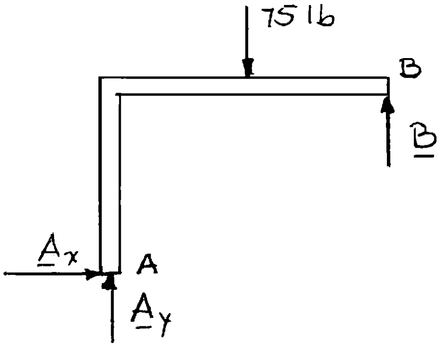


PROBLEM 4.27

Determine the reactions at *A* and *B* when (*a*) $\alpha = 0$, (*b*) $\alpha = 90°$, (*c*) $\alpha = 30°$.

SOLUTION

(*a*) $\alpha = 0$



$$+\circlearrowleft \Sigma M_A = 0: \quad B(20 \text{ in.}) - 75 \text{ lb}(10 \text{ in.}) = 0$$
$$B = 37.5 \text{ lb}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 75 \text{ lb} + 37.5 \text{ lb} = 0$$
$$A_y = 37.5 \text{ lb}$$

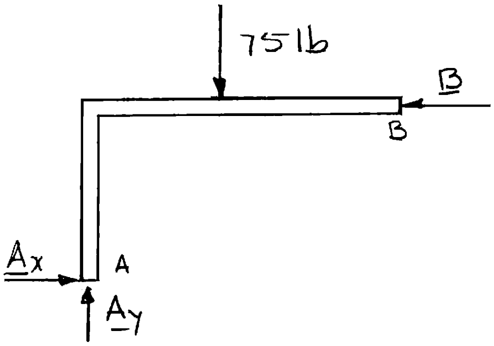
$$\mathbf{A} = \mathbf{B} = 37.5 \text{ lb} \uparrow \blacktriangleleft$$

(*b*) $\alpha = 90°$

$$+\circlearrowleft \Sigma M_A = 0: \quad B(12 \text{ in.}) - 75 \text{ lb}(10 \text{ in.}) = 0$$
$$B = 62.5 \text{ lb}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - B = 0$$
$$A_x = 62.5 \text{ lb}$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 75 \text{ lb} = 0$$
$$A_y = 75 \text{ lb}$$



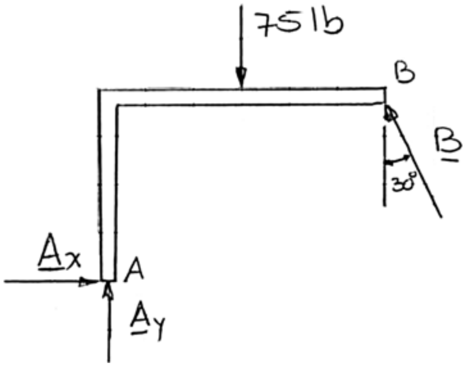
$$A = \sqrt{A_x^2 + A_y^2}$$
$$= \sqrt{(62.5 \text{ lb})^2 + (75 \text{ lb})^2}$$
$$= 97.6 \text{ lb}$$

$$\tan\theta = \frac{75}{62.5}$$
$$\theta = 50.2°$$

$$\mathbf{A} = 97.6 \text{ lb} \measuredangle 50.2°; \ \mathbf{B} = 62.51 \text{ lb} \longleftarrow \blacktriangleleft$$

PROBLEM 4.27 (Continued)

(*c*) $\alpha = 30°$



$$+\circlearrowleft \Sigma M_A = 0: \quad (B\cos 30°)(20\text{ in.}) + (B\sin 30°)(12\text{ in.}) - (75\text{ lb})(10\text{ in.}) = 0$$

$$B = 32.161\text{ lb}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - (32.161)\sin 30° = 0$$

$$A_x = 16.0805\text{ lb}$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + (32.161)\cos 30° - 75 = 0$$

$$A_y = 47.148\text{ lb}$$

$$A = \sqrt{A_x^2 + A_y^2}$$

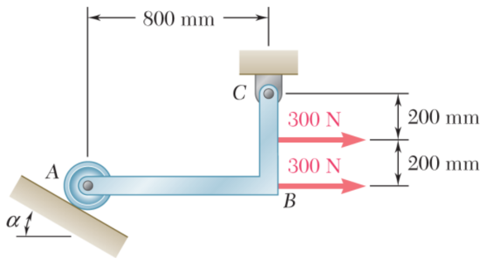
$$= \sqrt{(16.0805)^2 + (47.148)^2}$$

$$= 49.8\text{ lb}$$

$$\tan\theta = \frac{47.148}{16.0805}$$

$$\theta = 71.2°$$

$\mathbf{A} = 49.8\text{ lb} \measuredangle 71.2°$; $\mathbf{B} = 32.2\text{ lb} \searrow 60.0°$ ◀



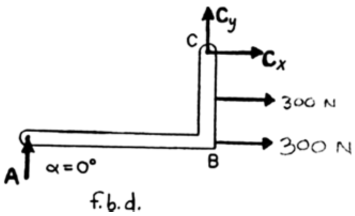
PROBLEM 4.28

Determine the reactions at A and C when (a) $\alpha = 0$, (b) $\alpha = 30°$.

SOLUTION

(a) $\alpha = 0$

From F.B.D. of member ABC:



$$+\circlearrowleft \Sigma M_C = 0: \quad (300\text{ N})(0.2\text{ m}) + (300\text{ N})(0.4\text{ m}) - A(0.8\text{ m}) = 0$$

$$A = 225\text{ N} \qquad \text{or} \qquad \mathbf{A} = 225\text{ N}\uparrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad C_y + 225\text{ N} = 0$$

$$C_y = -225\text{ N} \quad \text{or} \quad \mathbf{C}_y = 225\text{ N}\downarrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad 300\text{ N} + 300\text{ N} + C_x = 0$$

$$C_x = -600\text{ N} \quad \text{or} \quad \mathbf{C}_x = 600\text{ N} \longleftarrow$$

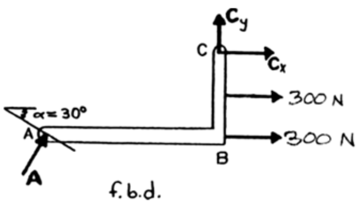
Then
$$C = \sqrt{C_x^2 + C_y^2} = \sqrt{(600)^2 + (225)^2} = 640.80\text{ N}$$

and
$$\theta = \tan^{-1}\left(\frac{C_y}{C_x}\right) = \tan^{-1}\left(\frac{-225}{-600}\right) = 20.556°$$

$$\text{or} \quad \mathbf{C} = 641\text{ N} \measuredangle 20.6° \blacktriangleleft$$

(b) $\alpha = 30°$

From F.B.D. of member ABC:



$$+\circlearrowleft \Sigma M_C = 0: \quad (300\text{ N})(0.2\text{ m}) + (300\text{ N})(0.4\text{ m}) - (A\cos 30°)(0.8\text{ m})$$
$$+ (A\sin 30°)(20\text{ in.}) = 0$$

$$A = 365.24\text{ N} \qquad \text{or} \qquad \mathbf{A} = 365\text{ N} \measuredangle 60.0° \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad 300\text{ N} + 300\text{ N} + (365.24\text{ N})\sin 30° + C_x = 0$$

$$C_x = -782.62$$

PROBLEM 4.28 (Continued)

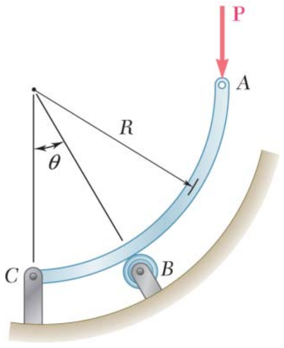
$$+\uparrow \Sigma F_y = 0: \quad C_y + (365.24\text{ N})\cos 30° = 0$$

$$C_y = -316.31\text{ N} \quad \text{or} \quad \mathbf{C}_y = 316\text{ N} \downarrow$$

Then $$C = \sqrt{C_x^2 + C_y^2} = \sqrt{(782.62)^2 + (316.31)^2} = 884.12\text{ N}$$

and $$\theta = \tan^{-1}\left(\frac{C_y}{C_x}\right) = \tan^{-1}\left(\frac{-316.31}{-782.62}\right) = 22.007°$$

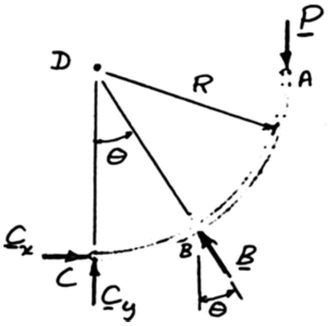
or $$\mathbf{C} = 884\text{ N} \swarrow 22.0° \;◀$$



PROBLEM 4.29

Rod *ABC* is bent in the shape of an arc of circle of radius *R*. Knowing the $\theta = 30°$, determine the reaction (*a*) at *B*, (*b*) at *C*.

SOLUTION



Free-Body Diagram:

$$+\circlearrowleft \Sigma M_D = 0: \quad C_x(R) - P(R) = 0$$

$$C_x = +P$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x - B\sin\theta = 0$$

$$P - B\sin\theta = 0$$

$$B = P/\sin\theta$$

$$\mathbf{B} = \frac{P}{\sin\theta} \nwarrow \theta$$

$$+\uparrow \Sigma F_y = 0: \quad C_y + B\cos\theta - P = 0$$

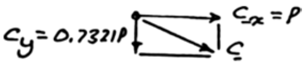
$$C_y + (P/\sin\theta)\cos\theta - P = 0$$

$$C_y = P\left(1 - \frac{1}{\tan\theta}\right)$$

For $\theta = 30°$,

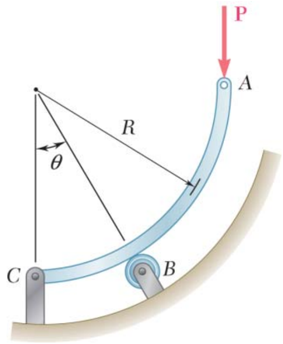
(*a*) $B = P/\sin 30° = 2P$ $\quad \mathbf{B} = 2P \nwarrow 60.0°$ ◀

(*b*) $C_x = +P \qquad C_x = P \longrightarrow$



$C_y = P(1 - 1/\tan 30°) = -0.732/P$ $\quad \mathbf{C}_y = 0.7321P \downarrow$

$\mathbf{C} = 1.239P \searrow 36.2°$ ◀



PROBLEM 4.30

Rod *ABC* is bent in the shape of an arc of circle of radius *R*. Knowing that $\theta = 60°$, determine the reaction (*a*) at *B*, (*b*) at *C*.

SOLUTION

See the solution to Problem 4.33 for the free-body diagram and analysis leading to the following expressions:

$$C_x = +P$$

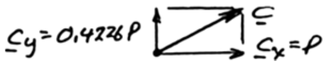
$$C_y = P\left(1 - \frac{1}{\tan\theta}\right)$$

$$B = \frac{P}{\sin\theta}$$

For $\theta = 60°$,

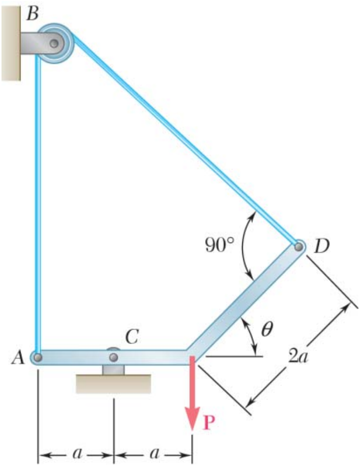
(*a*) $B = P/\sin 60° = 1.1547P$ $\mathbf{B} = 1.155P \searrow 30.0°$ ◀

(*b*) $C_x = +P \qquad C_x = P \longrightarrow$



$C_y = P(1 - 1/\tan 60°) = +0.4226P$ $\mathbf{C}_y = 0.4226P \downarrow$

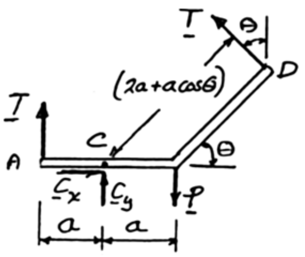
$\mathbf{C} = 1.086P \nearrow 22.9°$ ◀



PROBLEM 4.31

Neglecting friction, determine the tension in cable *ABD* and the reaction at *C* when $\theta = 60°$.

SOLUTION



$$+\circlearrowleft \Sigma M_C = 0: \quad T(2a + a\cos\theta) - Ta + Pa = 0$$

$$T = \frac{P}{1+\cos\theta} \qquad (1)$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x - T\sin\theta = 0$$

$$C_x = T\sin\theta = \frac{P\sin\theta}{1+\cos\theta}$$

$$+\circlearrowleft \Sigma F_y = 0: \quad C_y + T + T\cos\theta - P = 0$$

$$C_y = P - T(1+\cos\theta) = P - P\frac{1+\cos\theta}{1+\cos\theta}$$

$$C_y = 0$$

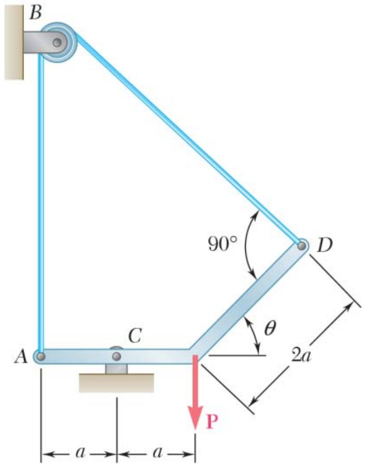
Since $C_y = 0, \quad C = C_x$ $\qquad \mathbf{C} = P\dfrac{\sin\theta}{1+\cos\theta}$

$\longrightarrow$ (2)

For $\theta = 60°$:

Eq. (1): $T = \dfrac{P}{1+\cos 60°} = \dfrac{P}{1+\frac{1}{2}}$ $\qquad T = \dfrac{2}{3}P$ ◀

Eq. (2): $C = P\dfrac{\sin 60°}{1+\cos 60°} = P\dfrac{0.866}{1+\frac{1}{2}}$ $\qquad \mathbf{C} = 0.577P \longrightarrow$ ◀

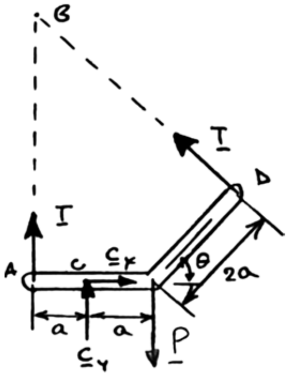


PROBLEM 4.32

Neglecting friction, determine the tension in cable *ABD* and the reaction at *C* when $\theta = 45°$.

SOLUTION

Free-Body Diagram:



Equilibrium for bracket:

$$+\circlearrowleft \Sigma M_C = 0: \quad -T(a) - P(a) + (T\sin 45°)(2a\sin 45°) + (T\cos 45°)(a + 2a\cos 45°) = 0$$

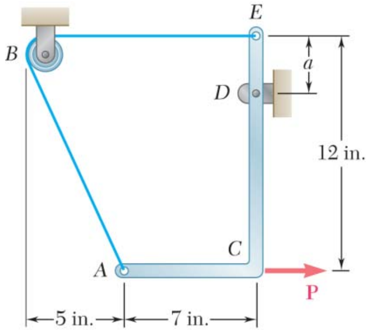
$$T = 0.58579 \qquad \text{or } T = 0.586P \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x + (0.58579P)\sin 45° = 0$$

$$C_x = 0.41422P$$

$$+\uparrow \Sigma F_y = 0: \quad C_y + 0.58579P - P + (0.58579P)\cos 45° = 0$$

$$C_y = 0 \qquad \text{or } \mathbf{C} = 0.414P \longrightarrow \blacktriangleleft$$

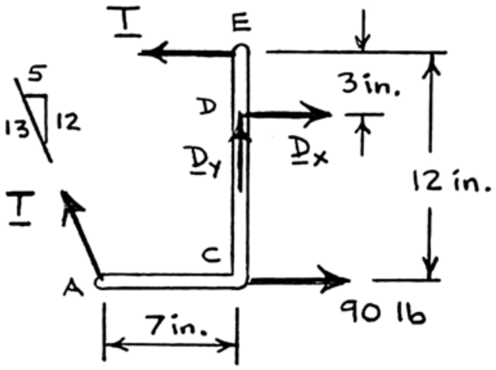

E
B
a
D
12 in.
C
A
P
5 in.
7 in.

PROBLEM 4.33

A force **P** of magnitude 90 lb is applied to member *ACDE*, which is supported by a frictionless pin at *D* and by the cable *ABE*. Since the cable passes over a small pulley at *B*, the tension may be assumed to be the same in portions *AB* and *BE* of the cable. For the case when $a = 3$ in., determine (*a*) the tension in the cable, (*b*) the reaction at *D*.

SOLUTION

Free-Body Diagram:

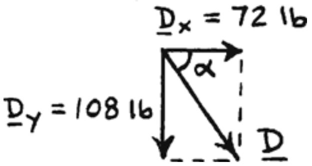

T
E
5
13
12
D
3 in.
D_y
D_x
12 in.
T
C
A
90 lb
7 in.

(*a*) $+\circlearrowleft \Sigma M_D = 0$: $(90\text{ lb})(9\text{ in.}) - \frac{5}{13}T(9\text{ in.}) - \frac{12}{13}T(7\text{ in.}) + T(3\text{ in.}) = 0$

$T = 117\text{ lb}$ $\qquad T = 117.0\text{ lb}$ ◀

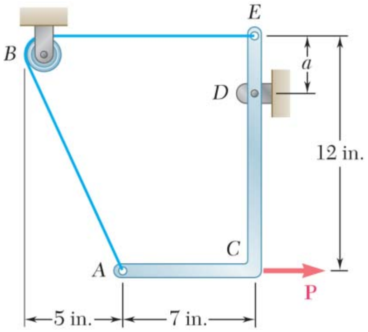
(*b*) $\xrightarrow{+} \Sigma F_x = 0$: $D_x - 117\text{ lb} - \frac{5}{13}(117\text{ lb}) + 90 = 0$

$D_x = +72\text{ lb}$


D_x = 72 lb
α
D_y = 108 lb
D

$+\uparrow \Sigma F_y = 0$: $D_y + \frac{12}{13}(117\text{ lb}) = 0$

$D_y = -108\text{ lb}$ $\qquad \mathbf{D} = 129.8\text{ lb} \measuredangle 56.3°$ ◀


E
B
a
D
12 in.
C
A
P
5 in.
7 in.

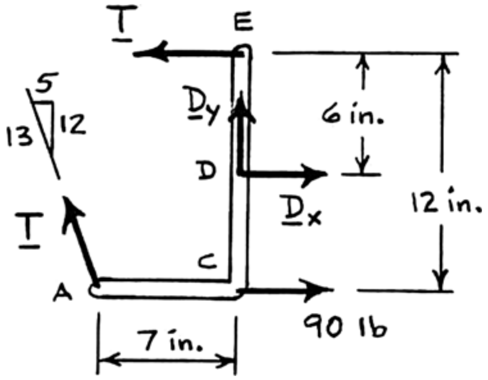
PROBLEM 4.34

Solve Problem 4.33 for $a = 6$ in.

PROBLEM 4.33 A force **P** of magnitude 90 lb is applied to member *ACDE*, which is supported by a frictionless pin at *D* and by the cable *ABE*. Since the cable passes over a small pulley at *B*, the tension may be assumed to be the same in portions *AB* and *BE* of the cable. For the case when $a = 3$ in., determine (*a*) the tension in the cable, (*b*) the reaction at *D*.

SOLUTION

Free-Body Diagram:


T
E
5
13
12
Dy
6 in.
D
Dx
12 in.
T
C
A
90 lb
7 in.

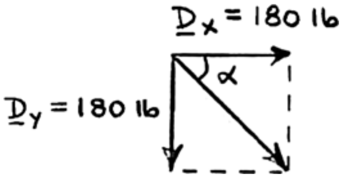
(*a*) $+\circlearrowleft \Sigma M_D = 0$: $(90 \text{ lb})(6 \text{ in.}) - \frac{5}{13}T(6 \text{ in.}) - \frac{12}{13}T(7 \text{ in.}) + T(6 \text{ in.}) = 0$

$T = 195$ lb

$T = 195.0$ lb ◀

(*b*) $\xrightarrow{+} \Sigma F_x = 0$: $D_x - 195 \text{ lb} - \frac{5}{13}(195 \text{ lb}) + 90 = 0$

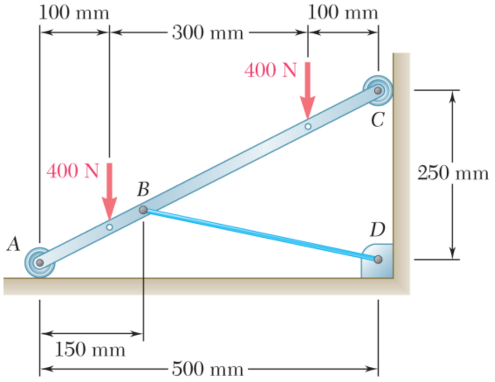
$D_x = +180$ lb


Dx = 180 lb
α
Dy = 180 lb

$+\uparrow \Sigma F_y = 0$: $D_y + \frac{12}{13}(195 \text{ lb}) = 0$

$D_y = -180$ lb

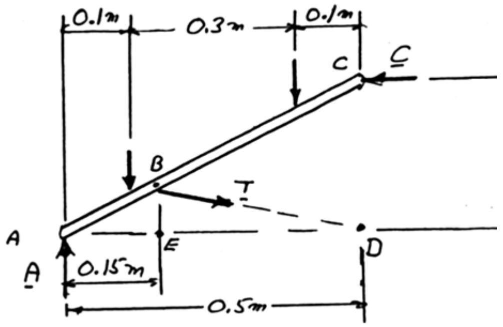
D = 255 lb ↘ 45.0° ◀


100 mm
300 mm
100 mm
400 N
C
250 mm
400 N
B
D
A
150 mm
500 mm

PROBLEM 4.35

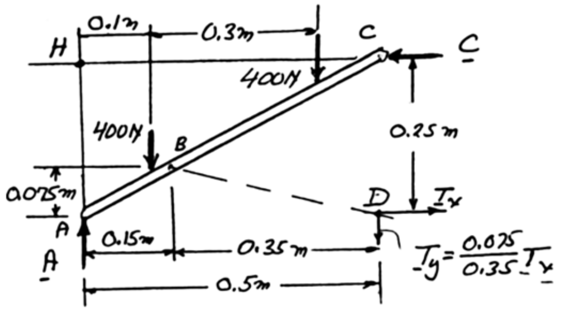
Bar *AC* supports two 400-N loads as shown. Rollers at *A* and *C* rest against frictionless surfaces and a cable *BD* is attached at *B*. Determine (*a*) the tension in cable *BD*, (*b*) the reaction at *A*, (*c*) the reaction at *C*.

SOLUTION


0.1m
0.3m
0.1m
C
C
B
T
A
E
D
A
0.15m
0.5m

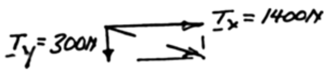
Similar triangles: *ABE* and *ACD*

$$\frac{AE}{AD}=\frac{BE}{CD};\quad \frac{0.15\text{ m}}{0.5\text{ m}}=\frac{BE}{0.25\text{ m}};\quad BE=0.075\text{ m}$$


H
0.1m
0.3m
C
C
400N
400N
B
0.25 m
0.075m
A
D
T_x
A
0.15m
0.35m
$T_y=\frac{0.075}{0.35}T_x$
0.5m

(*a*) $\quad +\circlearrowleft\Sigma M_A=0{:}\quad T_x(0.25\text{ m})-\left(\dfrac{0.075}{0.35}T_x\right)(0.5\text{ m})-(400\text{ N})(0.1\text{ m})-(400\text{ N})(0.4\text{ m})=0$

$$T_x=1400\text{ N}$$


$T_y=300N$
$T_x=1400N$

$$T_y=\frac{0.075}{0.35}(1400\text{ N})$$
$$=300\text{ N}$$

$T=1432\text{ N}$ ◀

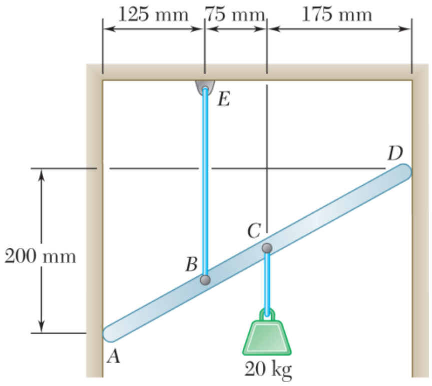
PROBLEM 4.35 (Continued)

(*b*) $+\uparrow \Sigma F_y = 0: \quad A - 300\text{ N} - 400\text{ N} - 400\text{ N} = 0$

$$A = +1100\text{ N} \qquad \mathbf{A} = 1100\text{ N}\uparrow \blacktriangleleft$$

(*c*) $\xrightarrow{+} \Sigma F_x = 0: \quad -C + 1400\text{ N} = 0$

$$C = +1400\text{ N} \qquad \mathbf{C} = 1400\text{ N} \longleftarrow \blacktriangleleft$$

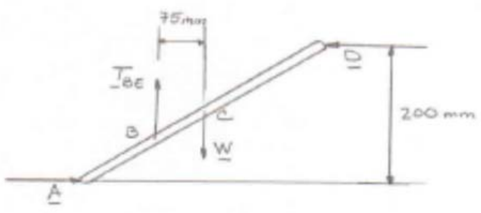

125 mm
75 mm
175 mm
E
D
C
B
200 mm
A
20 kg

PROBLEM 4.36

A light bar *AD* is suspended from a cable *BE* and supports a 20-kg block at *C*. The ends *A* and *D* of the bar are in contact with frictionless vertical walls. Determine the tension in cable *BE* and the reactions at *A* and *D*.

SOLUTION

Free-Body Diagram:


75 mm
$\underline{T}_{BE}$
B
C
W
D
200 mm
A

$$W = (20\text{ kg})(9.81\text{ m/s}^2) = 196.20\text{ N}$$

$$\Sigma F_x = 0: \quad A = D$$

$$\Sigma F_y = 0: \quad T_{BE} = W$$

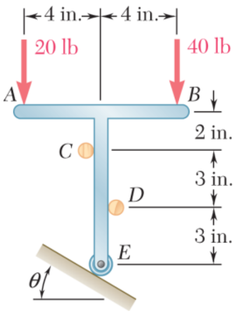
$T_{BE} = 196.2\text{ N}$ ◀

We note that the forces shown form two couples.

$$+\circlearrowleft \Sigma M_B = 0: \quad A(200\text{ mm}) - (196.20\text{ N})(75\text{ mm}) = 0$$

$$A = 73.575\text{ N}$$

$\mathbf{A} = 73.6\text{ N} \longrightarrow, \quad \mathbf{D} = 73.6\text{ N} \longleftarrow$ ◀

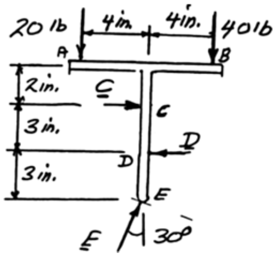

4 in.
4 in.
20 lb
40 lb
A
B
2 in.
C
3 in.
D
3 in.
E
θ

PROBLEM 4.37

The T-shaped bracket shown is supported by a small wheel at E and pegs at C and D. Neglecting the effect of friction, determine the reactions at C, D, and E when $\theta = 30°$.

SOLUTION

Free-Body Diagram:


20 lb
4in.
4in.
40lb
A
B
2in.
C
C
3in.
D
3in.
D
E
E
30°

$$+\uparrow \Sigma F_y = 0: \quad E\cos 30° - 20 - 40 = 0$$

$$E = \frac{60\text{ lb}}{\cos 30°} = 69.282\text{ lb} \qquad \mathbf{E} = 69.3\text{ lb} \measuredangle 60.0° \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_D = 0: \quad (20\text{ lb})(4\text{ in.}) - (40\text{ lb})(4\text{ in.})$$

$$- C(3\text{ in.}) + E\sin 30°(3\text{ in.}) = 0$$

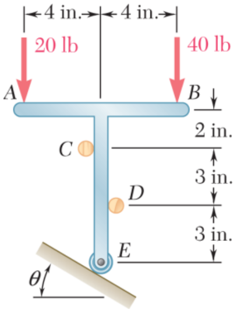
$$-80 - 3C + 69.282(0.5)(3) = 0$$

$$C = 7.9743\text{ lb} \qquad \mathbf{C} = 7.97\text{ lb} \longrightarrow \blacktriangleleft$$

$$\Sigma F_x = 0: \quad E\sin 30° + C - D = 0$$

$$(69.282\text{ lb})(0.5) + 7.9743\text{ lb} - D = 0$$

$$D = 42.615\text{ lb} \qquad \mathbf{D} = 42.6\text{ lb} \longleftarrow \blacktriangleleft$$

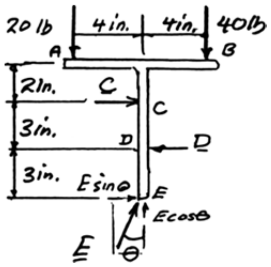

4 in.
4 in.
20 lb
40 lb
A
B
2 in.
C
3 in.
D
3 in.
E
θ

PROBLEM 4.38

The T-shaped bracket shown is supported by a small wheel at E and pegs at C and D. Neglecting the effect of friction, determine (a) the smallest value of θ for which the equilibrium of the bracket is maintained, (b) the corresponding reactions at C, D, and E.

SOLUTION

Free-Body Diagram:


20 lb
4 in.
4 in.
40 lb
A
B
2 in.
C
C
3 in.
D
D
3 in.
E sinθ
E
E cosθ
E
θ

$$+\uparrow \Sigma F_y = 0: \quad E\cos\theta - 20 - 40 = 0$$

$$E = \frac{60}{\cos\theta} \qquad (1)$$

$$+\circlearrowleft \Sigma M_D = 0: \quad (20 \text{ lb})(4 \text{ in.}) - (40 \text{ lb})(4 \text{ in.}) - C(3 \text{ in.})$$

$$+\left(\frac{60}{\cos\theta}\sin\theta\right)3 \text{ in.} = 0$$

$$C = \frac{1}{3}(180\tan\theta - 80)$$

(a) For $C = 0$, $\quad 180\tan\theta = 80$

$$\tan\theta = \frac{4}{9} \qquad \theta = 23.962°$$

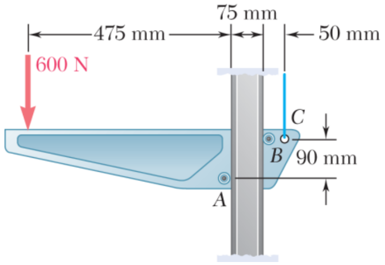
$\theta = 24.0°$ ◀

From Eq. (1): $\quad E = \dfrac{60}{\cos 23.962°} = 65.659$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -D + C + E\sin\theta = 0$$

$$D = (65.659)\sin 23.962 = 26.666 \text{ lb}$$

(b) $\quad \mathbf{C} = 0, \quad \mathbf{D} = 26.7 \text{ lb}$, ← $\quad \mathbf{E} = 65.7 \text{ lb}$ ∠ 66.0° ◀

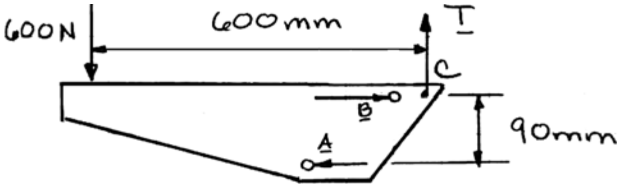


PROBLEM 4.39

A movable bracket is held at rest by a cable attached at *C* and by frictionless rollers at *A* and *B*. For the loading shown, determine (*a*) the tension in the cable, (*b*) the reactions at *A* and *B*.

SOLUTION

Free-Body Diagram:



(*a*) $+\uparrow \Sigma F_y = 0: \quad T - 600 \text{ N} = 0$

$T = 600 \text{ N}$ ◀

(*b*) $\xrightarrow{+} \Sigma F_x = 0: \quad B - A = 0 \qquad \therefore \quad B = A$

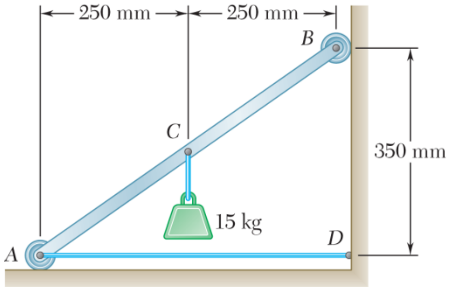
Note that the forces shown form two couples.

$+\circlearrowleft \Sigma M = 0: \quad (600 \text{ N})(600 \text{ mm}) - A(90 \text{ mm}) = 0$

$A = 4000 \text{ N}$

$\therefore \quad B = 4000 \text{ N}$

$\mathbf{A} = 4.00 \text{ kN} \longleftarrow; \quad \mathbf{B} = 4.00 \text{ kN} \longrightarrow$ ◀

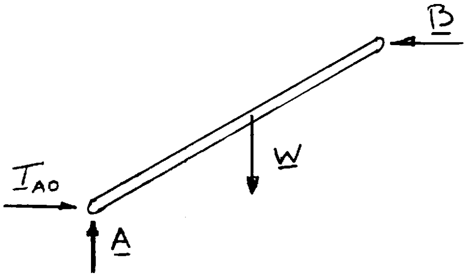


PROBLEM 4.40

A light bar *AB* supports a 15-kg block at its midpoint *C*. Rollers at *A* and *B* rest against frictionless surfaces, and a horizontal cable *AD* is attached at *A*. Determine (*a*) the tension in cable *AD*, (*b*) the reactions at *A* and *B*.

SOLUTION

Free-Body Diagram:



$$W = (15\text{ kg})(9.81\text{ m/s}^2)$$
$$= 147.150\text{ N}$$

(*a*) $\xrightarrow{+} \Sigma F_x = 0{:}\quad T_{AD} - 105.107\text{ N} = 0$

$T_{AD} = 105.1\text{ N}$ ◀

(*b*) $+\!\uparrow \Sigma F_y = 0{:}\quad A - W = 0$

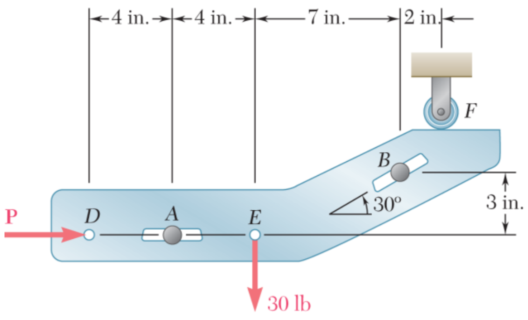
$A - 147.150\text{ N} = 0$

$\mathbf{A} = 147.2\text{ N}\uparrow$ ◀

$+\circlearrowleft \Sigma M_A = 0{:}\quad B(350\text{ mm}) - (147.150\text{ N})(250\text{ mm}) = 0$

$B = 105.107\text{ N}$

$\mathbf{B} = 105.1\text{ N} \longleftarrow$ ◀

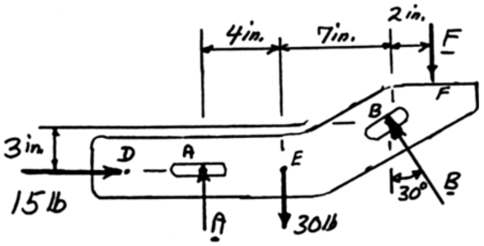


PROBLEM 4.41

Two slots have been cut in plate *DEF*, and the plate has been placed so that the slots fit two fixed, frictionless pins *A* and *B*. Knowing that $P = 15$ lb, determine (*a*) the force each pin exerts on the plate, (*b*) the reaction at *F*.

SOLUTION

Free-Body Diagram:



(*a*) $\overset{+}{\longrightarrow}\Sigma F_x = 0\text{:}\quad 15\text{ lb} - B\sin 30° = 0$ $\qquad\mathbf{B} = 30.0\text{ lb} \searrow 60.0°$ ◀

(*b*) $+\circlearrowleft\Sigma M_A = 0\text{:}\quad -(30\text{ lb})(4\text{ in.}) + B\sin 30°(3\text{ in.}) + B\cos 30°(11\text{ in.}) - F(13\text{ in.}) = 0$

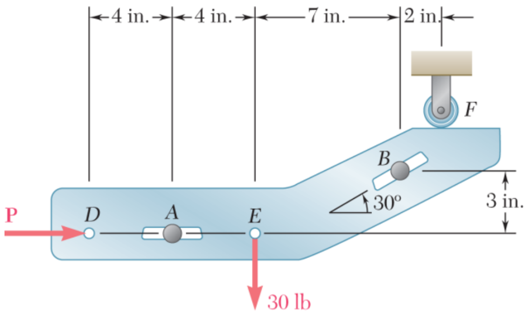
$$-120\text{ lb}\cdot\text{in.} + (30\text{ lb})\sin 30°(3\text{ in.}) + (30\text{ lb})\cos 30°(11\text{ in.}) - F(13\text{ in.}) = 0$$

$$F = +16.2145\text{ lb}$$ $\mathbf{F} = 16.21\text{ lb}\downarrow$ ◀

(*a*) $+\uparrow\Sigma F_y = 0\text{:}\quad A - 30\text{ lb} + B\cos 30° - F = 0$

$$A - 30\text{ lb} + (30\text{ lb})\cos 30° - 16.2145\text{ lb} = 0$$

$$A = +20.23\text{ lb}$$ $\mathbf{A} = 20.2\text{ lb}\uparrow$ ◀



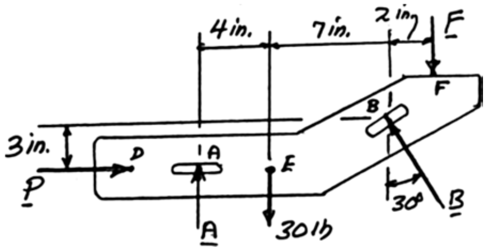
PROBLEM 4.42

For the plate of Problem 4.41 the reaction at *F* must be directed downward, and its maximum allowable value is 20 lb. Neglecting friction at the pins, determine the required range of values of *P*.

PROBLEM 4.41 Two slots have been cut in plate *DEF*, and the plate has been placed so that the slots fit two fixed, frictionless pins *A* and *B*. Knowing that *P* = 15 lb, determine (*a*) the force each pin exerts on the plate, (*b*) the reaction at *F*.

SOLUTION

Free-Body Diagram:



$\xrightarrow{+} \Sigma F_x = 0\text{:} \quad P - B\sin 30^\circ = 0$ $\qquad \mathbf{B} = 2P \searrow 60^\circ$

$+\circlearrowleft \Sigma M_A = 0\text{:} \quad -(30\text{ lb})(4\text{ in.}) + B\sin 30^\circ(3\text{ in.}) + B\cos 30^\circ(11\text{ in.}) - F(13\text{ in.}) = 0$

$$-120\text{ lb}\cdot\text{in.} + 2P\sin 30^\circ(3\text{ in.}) + 2P\cos 30^\circ(11\text{ in.}) - F(13\text{ in.}) = 0$$

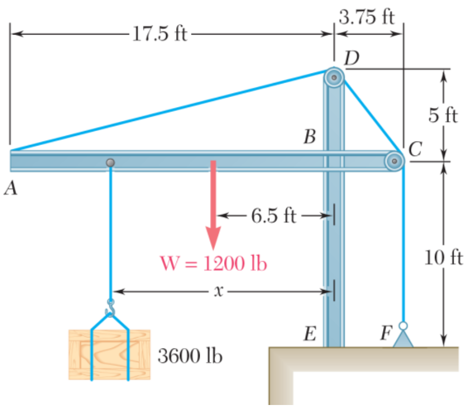
$$-120 + 3P + 19.0525P - 13F = 0$$

$$P = \frac{13F + 120}{22.0525} \qquad (1)$$

For $F = 0$: $\quad P = \dfrac{13(0) + 120}{22.0525} = 5.442\text{ lb}$

For $P = 20$ lb: $\quad P = \dfrac{13(20) + 120}{22.0525} = 17.232\text{ lb}$

For $\quad 0 \le F \le 20$ lb: $\qquad 5.44\text{ lb} \le P \le 17.23\text{ lb}$ ◀

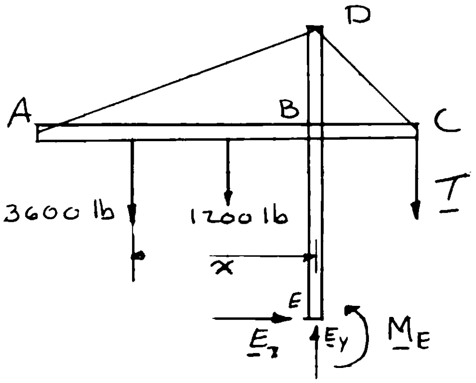


PROBLEM 4.43

The rig shown consists of a 1200-lb horizontal member *ABC* and a vertical member *DBE* welded together at *B*. The rig is being used to raise a 3600-lb crate at a distance $x = 12$ ft from the vertical member *DBE*. If the tension in the cable is 4 kips, determine the reaction at *E*, assuming that the cable is (*a*) anchored at *F* as shown in the figure, (*b*) attached to the vertical member at a point located 1 ft above *E*.

SOLUTION

Free-Body Diagram:



$$+\circlearrowleft M_E = 0: \quad M_E + (3600\text{ lb})x + (1200\text{ lb})(6.5\text{ ft}) - T(3.75\text{ ft}) = 0$$

$$M_E = 3.75T - 3600x - 7800 \qquad (1)$$

(*a*) For $x = 12$ ft and $T = 4000$ lbs,

$$M_E = 3.75(4000) - 3600(12) - 7800$$
$$= 36{,}000\text{ lb}\cdot\text{ft}$$

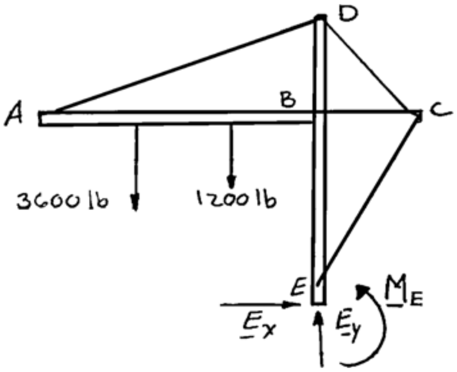
$$\xrightarrow{+} \Sigma F_x = 0 \quad \therefore \quad E_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad E_y - 3600\text{ lb} - 1200\text{ lb} - 4000 = 0$$

$$E_y = 8800\text{ lb}$$

$\mathbf{E} = 8.80\text{ kips}\uparrow;\ \mathbf{M}_E = 36.0\text{ kip}\cdot\text{ft}\circlearrowright$ ◀

PROBLEM 4.43 (Continued)


D
A
B
C
3600 lb
1200 lb
E
$\underline{M}_E$
$\underline{E}_x$
$\underline{E}_y$

(*b*)

$$\overset{\curvearrowleft}{+}\Sigma M_E = 0: \quad M_E + (3600 \text{ lb})(12 \text{ ft}) + (1200 \text{ lb})(6.5 \text{ ft}) = 0$$

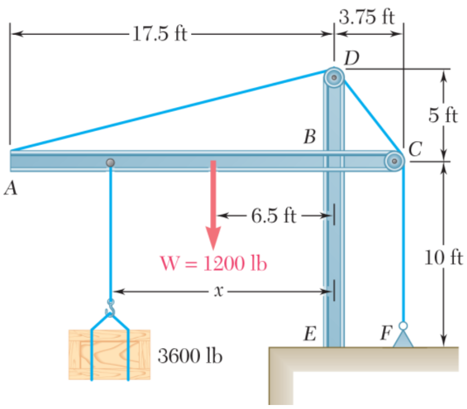
$$M_E = -51{,}000 \text{ lb}\cdot\text{ft}$$

$$\xrightarrow{+} \Sigma F_x = 0 \quad \therefore \quad E_x = 0$$

$$+\!\uparrow \Sigma F_y = 0: \qquad E_y - 3600 \text{ lb} - 1200 \text{ lb} = 0$$

$$E_y = 4800 \text{ lb}$$

$$\mathbf{E} = 4.80 \text{ kips} \uparrow; \ \mathbf{M}_E = 51.0 \text{ kip}\cdot\text{ft} \curvearrowright \blacktriangleleft$$

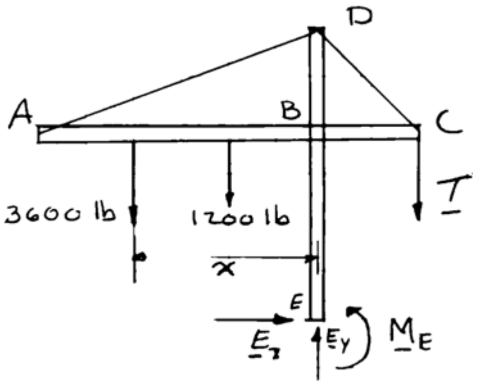

17.5 ft
3.75 ft
D
5 ft
B
C
A
6.5 ft
W = 1200 lb
10 ft
x
E
F
3600 lb

PROBLEM 4.44

For the rig and crate of Prob. 4.43, and assuming that cable is anchored at *F* as shown, determine (*a*) the required tension in cable *ADCF* if the maximum value of the couple at *E* as *x* varies from 1.5 to 17.5 ft is to be as small as possible, (*b*) the corresponding maximum value of the couple.

SOLUTION

Free-Body Diagram:


D
A
B
C
T
3600 lb
1200 lb
x
E
E_x
E_y
M_E

$$+\circlearrowleft M_E = 0: \quad M_E + (3600\text{ lb})x + (1200\text{ lb})(6.5\text{ ft}) - T(3.75\text{ ft}) = 0$$

$$M_E = 3.75T - 3600x - 7800 \qquad (1)$$

For $x = 1.5$ ft, Eq. (1) becomes

$$(M_E)_1 = 3.75T - 3600(1.5) - 7800 \qquad (2)$$

For $x = 17.5$ ft, Eq. (1) becomes

$$(M_E)_2 = 3.75T - 3600(17.5) - 7800$$

(*a*) For smallest max value of $|M_E|$, we set

$$(M_E)_1 =^{-} (M_E)_2$$

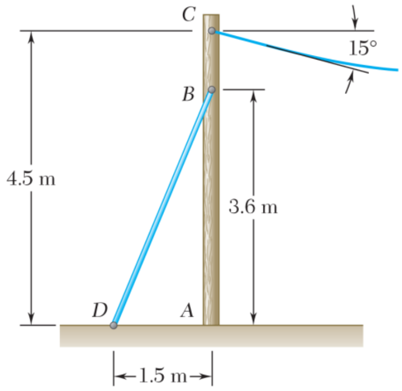
$$3.75T - 13{,}200 = -3.75T + 70{,}800$$

$\mathbf{T} = 11.20$ kips ◀

(*b*) From Equation (2), then

$$M_E = 3.75(11.20) - 13.20$$

$|M_E| = 28.8$ kip · ft ◀

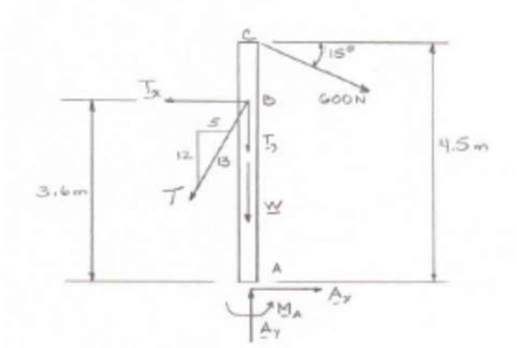


PROBLEM 4.45

A 175-kg utility pole is used to support at *C* the end of an electric wire. The tension in the wire is 600 N, and the wire forms an angle of 15° with the horizontal at *C*. Determine the largest and smallest allowable tensions in the guy cable *BD* if the magnitude of the couple at *A* may not exceed 500 N·m.

SOLUTION

Free-Body Diagram:



Geometry:

Distance $BD = \sqrt{(1.5)^2 + (3.6)^2} = 3.90$ m

Note also that: $W = mg = (175 \text{ kg})(9.81 \text{ m/s}^2) = 1716.75$ N

With $\mathbf{M}_A = 500 \text{ N}\cdot\text{m}$ clockwise: (i.e. corresponding to $T_{\max}$)

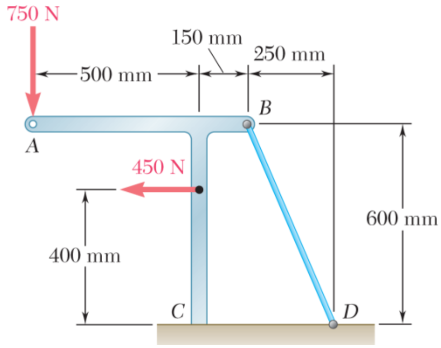
$$+\circlearrowleft \Sigma M_A = 0: \qquad -500 \text{ N}\cdot\text{m} - [(600 \text{ N})\cos 15^{\circ}](4.5 \text{ m}) + \left[\left(\frac{1.5}{3.90}\right)T_{\max}\right](3.6 \text{ m}) = 0$$

$T_{\max} = 2244.7$ N or $T_{\max} = 2240$ N ◀

With $\mathbf{M}_A = 500 \text{ N}\cdot\text{m}$ counter-clockwise: (i.e. corresponding to $T_{\min}$)

$$+\circlearrowleft \Sigma M_A = 0: \qquad 500 \text{ N}\cdot\text{m} - [(600 \text{ N})\cos 15^{\circ}](4.5 \text{ m}) + \left[\left(\frac{1.5}{3.90}\right)T_{\min}\right](3.6 \text{ m}) = 0$$

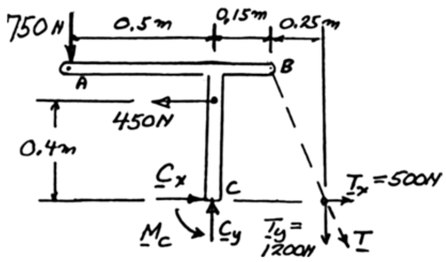
$T_{\min} = 1522.44$ N or $T_{\min} = 1522$ N ◀


750 N
150 mm
250 mm
500 mm
B
A
450 N
600 mm
400 mm
C
D

PROBLEM 4.46

Knowing that the tension in wire *BD* is 1300 N, determine the reaction at the fixed support *C* of the frame shown.

SOLUTION


750 N
0.5 m
0.15 m
0.25 m
A
B
450N
0.4 m
C_x
C
$T_x = 500N$
M_C
C_y
$T_y = 1200N$
T

$$T = 1300 \text{ N}$$

$$T_x = \frac{5}{13}T$$

$$= 500 \text{ N}$$

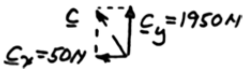
$$T_y = \frac{12}{13}T$$

$$= 1200 \text{ N}$$

$\overset{+}{\rightarrow}\Sigma M_x = 0\text{:} \quad C_x - 450 \text{ N} + 500 \text{ N} = 0 \qquad C_x = -50 \text{ N}$ $\qquad \mathbf{C}_x = 50 \text{ N} \leftarrow$

$+\!\uparrow \Sigma F_y = 0\text{:} \quad C_y - 750 \text{ N} - 1200 \text{ N} = 0 \qquad C_y = +1950 \text{ N}$

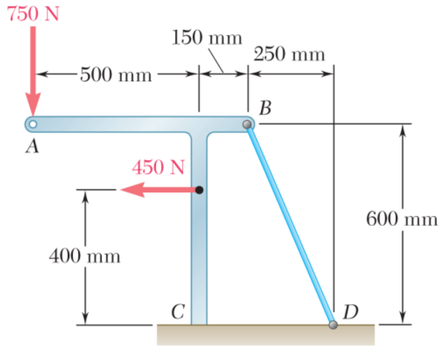
$\mathbf{C}_y = 1950 \text{ N} \uparrow$


C
$C_y = 1950N$
$C_x = 50N$

$\mathbf{C} = 1951 \text{ N} \measuredangle 88.5^\circ$ ◀

$+\circlearrowright \Sigma M_C = 0\text{:} \quad M_C + (750 \text{ N})(0.5 \text{ m}) + (4.50 \text{ N})(0.4 \text{ m})$
$\qquad - (1200 \text{ N})(0.4 \text{ m}) = 0$

$M_C = -75.0 \text{ N} \cdot \text{m}$ $\qquad \mathbf{M}_C = 75.0 \text{ N} \cdot \text{m} \circlearrowright$ ◀

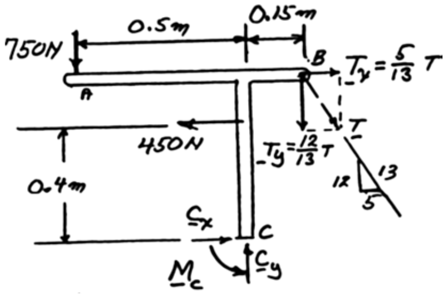


750 N
150 mm
250 mm
500 mm
B
A
450 N
600 mm
400 mm
C
D

PROBLEM 4.47

Determine the range of allowable values of the tension in wire *BD* if the magnitude of the couple at the fixed support *C* is not to exceed 100 N · m.

SOLUTION



0.5 m
0.15 m
750 N
B
$T_x = \frac{5}{13}T$
A
T
450 N
$T_y = \frac{12}{13}T$
13
12
5
0.4 m
C_x
C
M_C
C_y

$$+\circlearrowright \Sigma M_C = 0: \quad (750\text{ N})(0.5\text{ m}) + (450\text{ N})(0.4\text{ m}) - \left(\frac{5}{13}T\right)(0.6\text{ m})$$

$$-\left(\frac{12}{13}T\right)(0.15\text{ m}) + M_C = 0$$

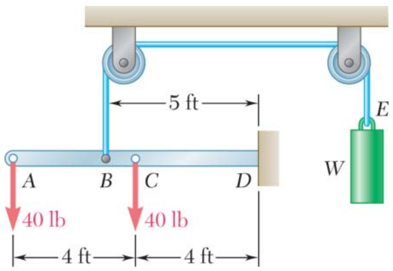
$$375\text{ N}\cdot\text{m} + 180\text{ N}\cdot\text{m} - \left(\frac{4.8}{13}\text{m}\right)T + M_C = 0$$

$$T = \frac{13}{4.8}(555 + M_C)$$

For $\quad M_C = -100\text{ N}\cdot\text{m}: \quad T = \frac{13}{4.8}(555 - 100) = 1232\text{ N}$

For $\quad M_C = +100\text{ N}\cdot\text{m}: \quad T = \frac{13}{4.8}(555 + 100) = 1774\text{ N}$

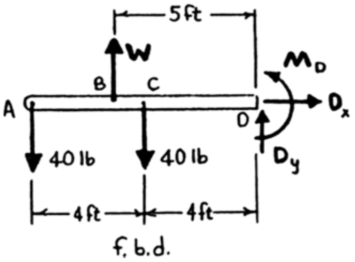
For $\quad |M_C| \leq 100\text{ N}\cdot\text{m}: \quad 1.232\text{ kN} \leq T \leq 1.774\text{ kN}$ ◀



PROBLEM 4.48

Beam AD carries the two 40-lb loads shown. The beam is held by a fixed support at D and by the cable BE that is attached to the counterweight W. Determine the reaction at D when (*a*) $W = 100$ lb, (*b*) $W = 90$ lb.

SOLUTION



(*a*) $W = 100$ lb

From F.B.D. of beam AD:

$$\xrightarrow{+} \Sigma F_x = 0{:} \quad D_x = 0$$

$$+\uparrow \Sigma F_y = 0{:} \quad D_y - 40\text{ lb} - 40\text{ lb} + 100\text{ lb} = 0$$

$$D_y = -20.0\text{ lb} \qquad \text{or} \qquad \mathbf{D} = 20.0\text{ lb}\downarrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_D = 0{:} \quad M_D - (100\text{ lb})(5\text{ ft}) + (40\text{ lb})(8\text{ ft}) + (40\text{ lb})(4\text{ ft}) = 0$$

$$M_D = 20.0\text{ lb}\cdot\text{ft} \qquad \text{or} \qquad \mathbf{M}_D = 20.0\text{ lb}\cdot\text{ft} \circlearrowleft \blacktriangleleft$$

(*b*) $W = 90$ lb

From F.B.D. of beam AD:

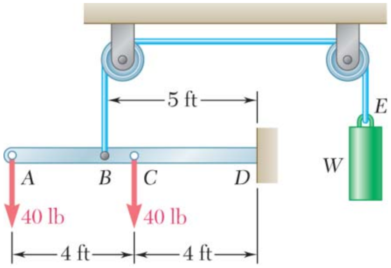
$$\xrightarrow{+} \Sigma F_x = 0{:} \quad D_x = 0$$

$$+\uparrow \Sigma F_y = 0{:} \quad D_y + 90\text{ lb} - 40\text{ lb} - 40\text{ lb} = 0$$

$$D_y = -10.00\text{ lb} \qquad \text{or} \qquad \mathbf{D} = 10.00\text{ lb}\downarrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_D = 0{:} \quad M_D - (90\text{ lb})(5\text{ ft}) + (40\text{ lb})(8\text{ ft}) + (40\text{ lb})(4\text{ ft}) = 0$$

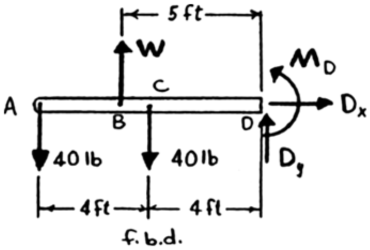
$$M_D = -30.0\text{ lb}\cdot\text{ft} \qquad \text{or} \qquad \mathbf{M}_D = 30.0\text{ lb}\cdot\text{ft} \circlearrowright \blacktriangleleft$$


5 ft
E
W
A
B
C
D
40 lb
40 lb
4 ft
4 ft

PROBLEM 4.49

For the beam and loading shown, determine the range of values of W for which the magnitude of the couple at D does not exceed 40 lb · ft.

SOLUTION


5 ft
W
M_D
C
A
B
D
D_x
40 lb
40 lb
D_y
4 ft
4 ft
f.b.d.

For W_{min}, $M_D = -40 \text{ lb} \cdot \text{ft}$

From F.B.D. of beam AD: $+\circlearrowleft \Sigma M_D = 0$: $(40 \text{ lb})(8 \text{ ft}) - W_{\text{min}}(5 \text{ ft})$

$$+(40 \text{ lb})(4 \text{ ft}) - 40 \text{ lb} \cdot \text{ft} = 0$$

$$W_{\text{min}} = 88.0 \text{ lb}$$

For W_{max}, $M_D = 40 \text{ lb} \cdot \text{ft}$

From F.B.D. of beam AD: $+\circlearrowleft \Sigma M_D = 0$: $(40 \text{ lb})(8 \text{ ft}) - W_{\text{max}}(5 \text{ ft})$

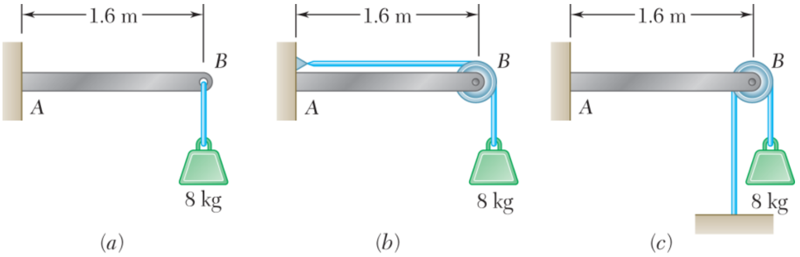
$$+(40 \text{ lb})(4 \text{ ft}) + 40 \text{ lb} \cdot \text{ft} = 0$$

$$W_{\text{max}} = 104.0 \text{ lb}$$

or $88.0 \text{ lb} \le W \le 104.0 \text{ lb}$ ◀

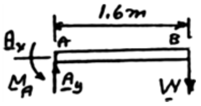
PROBLEM 4.50

An 8-kg mass can be supported in the three different ways shown. Knowing that the pulleys have a 100-mm radius, determine the reaction at *A* in each case.



SOLUTION

$$W = mg = (8\text{ kg})(9.81\text{ m/s}^2) = 78.480\text{ N}$$



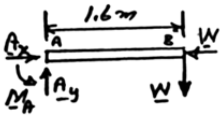
(*a*) $\Sigma F_x = 0: \quad A_x = 0$

$+\uparrow \Sigma F_y = 0: \quad A_y - W = 0$ $\qquad \mathbf{A}_y = 78.480\text{ N}\uparrow$

$+\circlearrowleft \Sigma M_A = 0: \quad M_A - W(1.6\text{ m}) = 0$

$M_A = +(78.480\text{ N})(1.6\text{ m})$ $\qquad \mathbf{M}_A = 125.568\text{ N}\cdot\text{m}\circlearrowleft$

$\mathbf{A} = 78.5\text{ N}\uparrow, \quad \mathbf{M}_A = 125.6\text{ N}\cdot\text{m}\circlearrowleft$ ◀



(*b*) $\overset{+}{\rightarrow} \Sigma F_x = 0: \quad A_x - W = 0$ $\qquad \mathbf{A}_x = 78.480\uparrow$

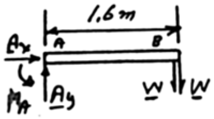
$+\uparrow \Sigma F_y = 0: \quad A_y - W = 0$ $\qquad \mathbf{A}_y = 78.480 \longleftarrow$

$$\mathbf{A} = (78.480\text{ N})\sqrt{2} = 110.987\text{ N} \measuredangle 45°$$

$+\circlearrowleft \Sigma M_A = 0: \quad M_A - W(1.6\text{ m}) = 0$

$M_A = +(78.480\text{ N})(1.6\text{ m})$ $\qquad \mathbf{M}_A = 125.568\text{ N}\cdot\text{m}\circlearrowleft$

$\mathbf{A} = 111.0\text{ N} \measuredangle 45°, \quad \mathbf{M}_A = 125.6\text{ N}\cdot\text{m}\circlearrowleft$ ◀



(*c*) $\Sigma F_x = 0: \quad A_x = 0$

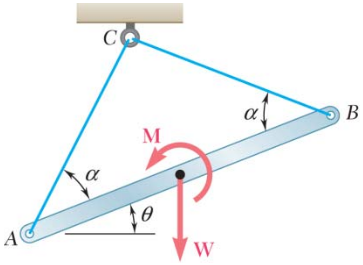
$+\uparrow \Sigma F_y = 0: \quad A_y - 2W = 0$

$$A_y = 2W = 2(78.480\text{ N}) = 156.960\text{ N}\uparrow$$

$+\circlearrowleft \Sigma M_A = 0: \quad M_A - 2W(1.6\text{ m}) = 0$

$M_A = +2(78.480\text{ N})(1.6\text{ m})$ $\qquad \mathbf{M}_A = 251.14\text{ N}\cdot\text{m}\circlearrowleft$

$\mathbf{A} = 157.0\text{ N}\uparrow, \quad \mathbf{M}_A = 251\text{ N}\cdot\text{m}\circlearrowleft$ ◀

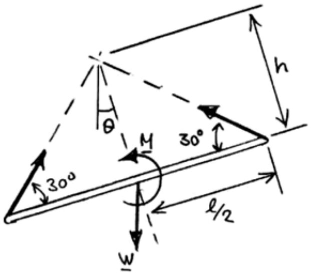


PROBLEM 4.51

A uniform rod *AB* with a length of *l* and weight of *W* is suspended from two cords *AC* and *BC* of equal length. Determine the angle θ corresponding to the equilibrium position when a couple **M** is applied to the rod.

SOLUTION

Free-Body Diagram:



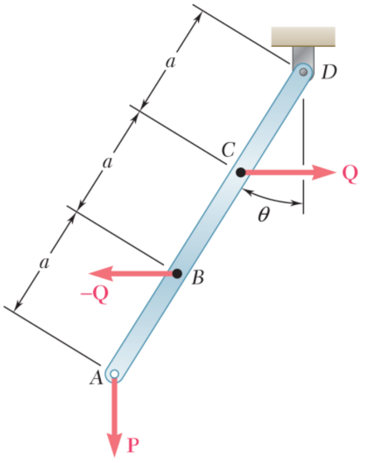
Using $h = \dfrac{l}{2}\tan\alpha$

$$\overset{+}{\circlearrowleft}\ \Sigma M_C = 0: \qquad M - (W)(h\sin\theta) = 0$$

$$\text{or } \sin\theta = \frac{M}{Wh}$$

$$= \frac{M}{W}\left(\frac{2}{l}\cot\alpha\right)$$

$$\text{or } \theta = \sin^{-1}\left(2M\frac{\cot\alpha}{Wl}\right) \blacktriangleleft$$

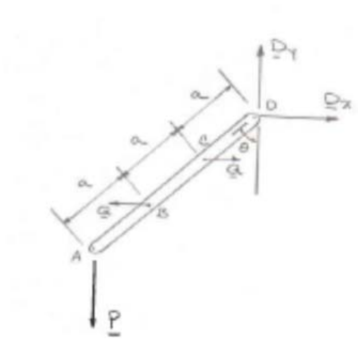


PROBLEM 4.52

Rod AD is acted upon by a vertical force **P** at end A and by two equal and opposite horizontal forces of magnitude Q at points B and C. Neglecting the weight of the rod, express the angle θ corresponding to the equilibrium position in terms of P and Q.

SOLUTION

Free-Body Diagram:

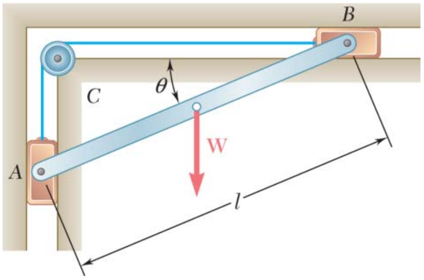


$$+\circlearrowleft \Sigma M_D = 0: \qquad P(3a\sin\theta) + Q(a\cos\theta) - Q(2a\cos\theta) = 0$$

$$3P\sin\theta = Q\cos\theta$$

$$\tan\theta = \frac{Q}{3P}$$

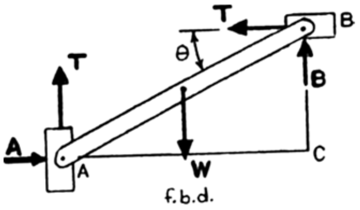
$$\text{or } \theta = \tan^{-1}\left(\frac{Q}{3P}\right) \blacktriangleleft$$



PROBLEM 4.53

A slender rod *AB*, of weight *W*, is attached to blocks *A* and *B*, which move freely in the guides shown. The blocks are connected by an elastic cord that passes over a pulley at *C*. (*a*) Express the tension in the cord in terms of *W* and *θ*. (*b*) Determine the value of *θ* for which the tension in the cord is equal to 3*W*.

SOLUTION



(*a*) From F.B.D. of rod *AB*:

$$\circlearrowleft + \Sigma M_C = 0: \quad T(l\sin\theta) + W\left[\left(\frac{1}{2}\right)\cos\theta\right] - T(l\cos\theta) = 0$$

$$T = \frac{W\cos\theta}{2(\cos\theta - \sin\theta)}$$

Dividing both numerator and denominator by cos *θ*,

$$T = \frac{W}{2}\left(\frac{1}{1-\tan\theta}\right) \qquad \text{or} \quad T = \frac{\left(\frac{W}{2}\right)}{(1-\tan\theta)} \blacktriangleleft$$

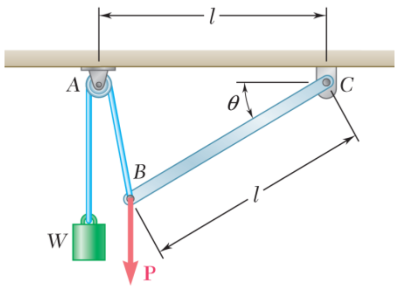
(*b*) For $T = 3W$,

$$3W = \frac{\left(\frac{W}{2}\right)}{(1-\tan\theta)}$$

$$1 - \tan\theta = \frac{1}{6}$$

or

$$\theta = \tan^{-1}\left(\frac{5}{6}\right) = 39.806° \qquad \text{or} \quad \theta = 39.8° \blacktriangleleft$$

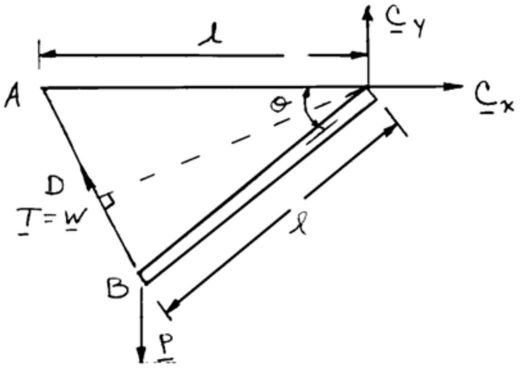


PROBLEM 4.54

A vertical load **P** is applied at end B of rod BC. (*a*) Neglecting the weight of the rod, express the angle θ corresponding to the equilibrium position in terms of P, l, and the counterweight W. (*b*) Determine the value of θ corresponding to equilibrium if $P = 2W$.

SOLUTION

Free-Body Diagram:



(*a*) Triangle ABC is isosceles. We have

$$CD = (BC)\cos\left(\frac{\theta}{2}\right) = l\cos\left(\frac{\theta}{2}\right)$$

$$+\circlearrowleft \Sigma M_C = 0: \quad P(l\cos\theta) - W\left(l\cos\frac{\theta}{2}\right) = 0$$

Setting $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$:

$$Pl\left(2\cos^2\frac{\theta}{2} - 1\right) - Wl\cos\frac{\theta}{2} = 0$$

$$\cos^2\frac{\theta}{2} - \left(\frac{W}{2P}\right)\cos\frac{\theta}{2} - \frac{1}{2} = 0$$

$$\cos\frac{\theta}{2} = \frac{1}{4}\left(\frac{W}{P} \pm \sqrt{\frac{W^2}{P^2} + 8}\right)$$

$$\theta = 2\cos^{-1}\left[\frac{1}{4}\left(\frac{W}{P} \pm \sqrt{\frac{W^2}{P^2} + 8}\right)\right] \blacktriangleleft$$

PROBLEM 4.54 (Continued)

(*b*) For $P = 2W$,

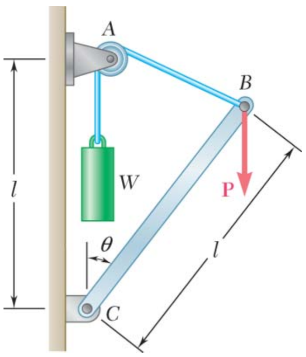
$$\cos\frac{\theta}{2} = \frac{1}{4}\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + 8}\right) = \frac{1}{8}\left(1 \pm \sqrt{33}\right)$$

$$\cos\frac{\theta}{2} = 0.84307 \quad \text{and} \quad \cos\frac{\theta}{2} = -0.59307$$

$$\frac{\theta}{2} = 32.534° \qquad \frac{\theta}{2} = 126.375°$$

$$\theta = 65.1° \qquad \theta = 252.75° \quad \text{(discard)}$$

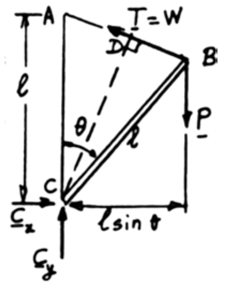
$\theta = 65.1°$ ◀



PROBLEM 4.55

A vertical load **P** is applied at end B of rod BC. (a) Neglecting the weight of the rod, express the angle θ corresponding to the equilibrium position in terms of P, l, and the counterweight W. (b) Determine the value of θ corresponding to equilibrium if $P = 2W$.

SOLUTION



(a) Triangle ABC is isosceles. We have

$$CD = (BC)\cos\frac{\theta}{2} = l\cos\frac{\theta}{2}$$

$$+\circlearrowleft \Sigma M_C = 0: \quad W\left(l\cos\frac{\theta}{2}\right) - P(l\sin\theta) = 0$$

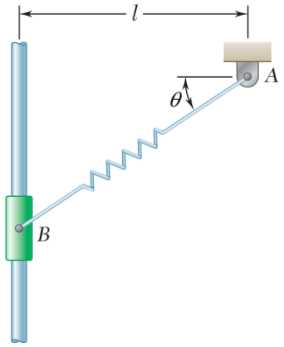
Setting $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$: $\quad Wl\cos\frac{\theta}{2} - 2Pl\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0$

$$W - 2P\sin\frac{\theta}{2} = 0 \qquad \theta = 2\sin^{-1}\left(\frac{W}{2P}\right) ◀$$

(b) For $P = 2W$, $\quad \sin\frac{\theta}{2} = \frac{W}{2P} = \frac{W}{4W} = 0.25$

$$\frac{\theta}{2} = 14.5° \qquad \theta = 29.0° ◀$$

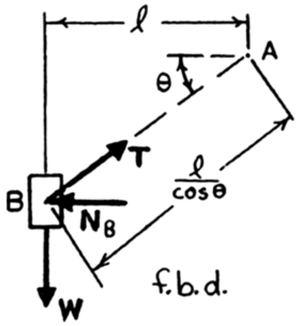
or $\quad \frac{\theta}{2} = 165.5° \quad \theta = 331° \quad$ (discard)



PROBLEM 4.56

A collar B of weight W can move freely along the vertical rod shown. The constant of the spring is k, and the spring is unstretched when $\theta = 0$. (a) Derive an equation in θ, W, k, and l that must be satisfied when the collar is in equilibrium. (b) Knowing that $W = 300\text{ N}$, $l = 500\text{ mm}$, and $k = 800\text{ N/m}$, determine the value of θ corresponding to equilibrium.

SOLUTION



First note: $T = ks$

where $k =$ spring constant

$s =$ elongation of spring

$$= \frac{l}{\cos\theta} - l$$

$$= \frac{l}{\cos\theta}(1-\cos\theta)$$

$$T = \frac{kl}{\cos\theta}(1-\cos\theta)$$

(a) From F.B.D. of collar B: $+\!\uparrow \Sigma F_y = 0\text{:} \quad T\sin\theta - W = 0$

or $$\frac{kl}{\cos\theta}(1-\cos\theta)\sin\theta - W = 0$$

or $\tan\theta - \sin\theta = \dfrac{W}{kl}$ ◀

(b) For $W = 3\text{ lb}$

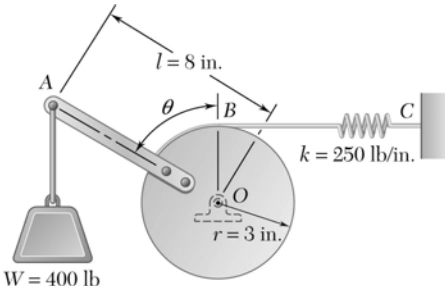
$l = 6\text{ in.}$

$k = 8\text{ lb/ft}$

$$l = \frac{6\text{ in.}}{12\text{ in./ft}} = 0.5\text{ ft}$$

$$\tan\theta - \sin\theta = \frac{3\text{ lb}}{(8\text{ lb/ft})(0.5\text{ ft})} = 0.75$$

Solving numerically, $\theta = 57.957°$ or $\theta = 58.0°$ ◀

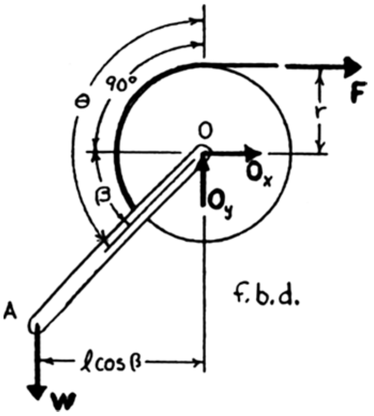


$l = 8$ in.
A
θ
B
C
$k = 250$ lb/in.
O
$r = 3$ in.
$W = 400$ lb

PROBLEM 4.57

Solve Sample Problem 4.5, assuming that the spring is unstretched when $\theta = 90°$.

SOLUTION



90°
θ
F
r
O
O_x
β
O_y
f.b.d.
A
$\ell \cos \beta$
W

First note: $T = \text{tension in spring} = ks$

where
$$s = \text{deformation of spring}$$
$$= r\beta$$
$$F = kr\beta$$

From F.B.D. of assembly: $+\circlearrowleft \Sigma M_0 = 0\text{:}\quad W(l\cos\beta) - F(r) = 0$

or
$$Wl\cos\beta - kr^2\beta = 0$$
$$\cos\beta = \frac{kr^2}{Wl}\beta$$

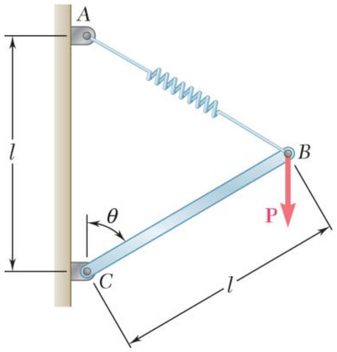
For
$$k = 250 \text{ lb/in.}$$
$$r = 3 \text{ in.}$$
$$l = 8 \text{ in.}$$
$$W = 400 \text{ lb}$$
$$\cos\beta = \frac{(250 \text{ lb/in.})(3 \text{ in.})^2}{(400 \text{ lb})(8 \text{ in.})}\beta$$

or $\cos\beta = 0.703125\beta$

Solving numerically, $\beta = 0.89245 \text{ rad}$

or $\beta = 51.134°$

Then $\theta = 90° + 51.134° = 141.134°$ or $\theta = 141.1°$ ◀

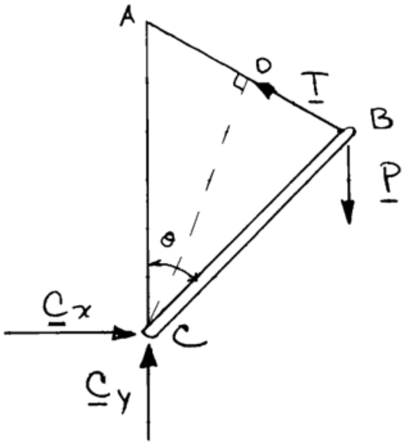


PROBLEM 4.58

A vertical load **P** is applied at end *B* of rod *BC*. The constant of the spring is *k*, and the spring is unstretched when $\theta = 60°$. (*a*) Neglecting the weight of the rod, express the angle θ corresponding to the equilibrium position terms of *P*, *k*, and *l*. (*b*) Determine the value of θ corresponding to equilibrium if $P = \frac{1}{4}kl$.

SOLUTION

Free-Body Diagram:



(*a*) Triangle *ABC* is isosceles. We have

$$AB = 2(AD) = 2l\sin\left(\frac{\theta}{2}\right);\ CD = l\cos\left(\frac{\theta}{2}\right)$$

Elongation of spring: $x = (AB)_\theta - (AB)_{\theta} = 60°$

$$= 2l\sin\left(\frac{\theta}{2}\right) - 2l\sin 30°$$

$$T = kx = 2kl\left(\sin\frac{\theta}{2} - \frac{1}{2}\right)$$

$$\circlearrowleft \Sigma M_C = 0:\quad T\left(l\cos\ \frac{\theta}{2}\right) - P(l\sin\theta) = 0$$

PROBLEM 4.58 (Continued)

$$2kl\left(\sin\frac{\theta}{2}-\frac{1}{2}\right)l\cos\frac{\theta}{2}-Pl\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)=0$$

$$\cos\frac{\theta}{2}=0 \quad \text{or} \quad 2(kl-P)\sin\frac{\theta}{2}-kl=0$$

$$\theta=180° \text{ (trivial)} \qquad \sin\frac{\theta}{2}=\frac{\frac{1}{2}kl}{kl-P}$$

$$\theta=2\sin^{-1}\left[\frac{1}{2}kl/(kl-P)\right] \blacktriangleleft$$

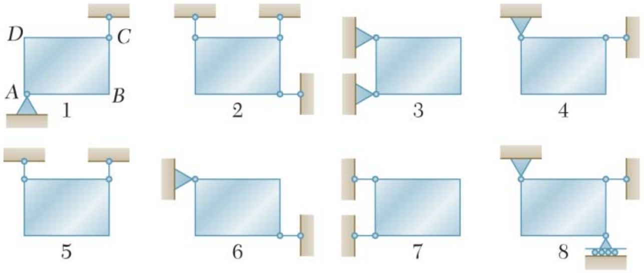
(*b*) For $P=\frac{1}{4}kl$,

$$\sin\frac{\theta}{2}=\frac{\frac{1}{2}kl}{\frac{3}{4}kl}=\frac{2}{3}$$

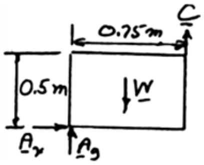
$$\frac{\theta}{2}=41.8° \qquad \theta=83.6° \blacktriangleleft$$

PROBLEM 4.59

Eight identical 500×750-mm rectangular plates, each of mass $m = 40$ kg, are held in a vertical plane as shown. All connections consist of frictionless pins, rollers, or short links. In each case, determine whether (*a*) the plate is completely, partially, or improperly constrained, (*b*) the reactions are statically determinate or indeterminate, (*c*) the equilibrium of the plate is maintained in the position shown. Also, wherever possible, compute the reactions.



SOLUTION



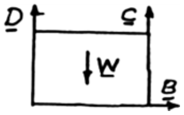
1. Three non-concurrent, non-parallel reactions:

   (*a*) Plate: completely constrained

   (*b*) Reactions: determinate

   (*c*) Equilibrium maintained

$$\mathbf{A} = \mathbf{C} = 196.2\text{ N}\uparrow$$



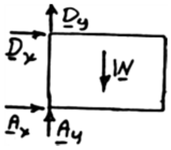
2. Three non-concurrent, non-parallel reactions:

   (*a*) Plate: completely constrained

   (*b*) Reactions: determinate

   (*c*) Equilibrium maintained

$$\mathbf{B} = 0, \quad \mathbf{C} = \mathbf{D} = 196.2\text{ N}\uparrow$$



3. Four non-concurrent, non-parallel reactions:

   (*a*) Plate: completely constrained

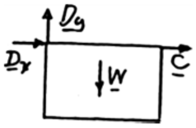
   (*b*) Reactions: indeterminate

   (*c*) Equilibrium maintained

$$\mathbf{A}_x = 294\text{ N}\longrightarrow, \quad \mathbf{D}_x = 294\text{ N}\longleftarrow$$

$$(\mathbf{A}_y + \mathbf{D}_y = 392\text{ N}\uparrow)$$

4. Three concurrent reactions (through *D*):



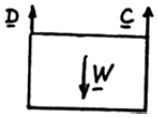
   (*a*) Plate: improperly constrained

   (*b*) Reactions: indeterminate

   (*c*) No equilibrium $(\Sigma M_D \neq 0)$

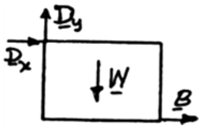
PROBLEM 4.59 (Continued)

5. Two reactions:
   - (*a*) Plate: partial constraint
   - (*b*) Reactions: determinate
   - (*c*) Equilibrium maintained



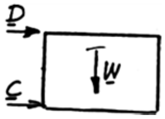
$$\mathbf{C} = \mathbf{D} = 196.2\text{ N}\uparrow$$

6. Three non-concurrent, non-parallel reactions:
   - (*a*) Plate: completely constrained
   - (*b*) Reactions: determinate
   - (*c*) Equilibrium maintained

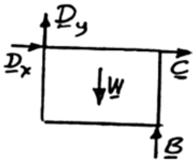


$$\mathbf{B} = 294\text{ N}\longrightarrow, \quad \mathbf{D} = 491\text{ N} \searrow 53.1^\circ$$

7. Two reactions:
   - (*a*) Plate: improperly constrained
   - (*b*) Reactions determined by dynamics
   - (*c*) No equilibrium $(\Sigma F_y \neq 0)$



8. Four non-concurrent, non-parallel reactions:
   - (*a*) Plate: completely constrained
   - (*b*) Reactions: indeterminate
   - (*c*) Equilibrium maintained

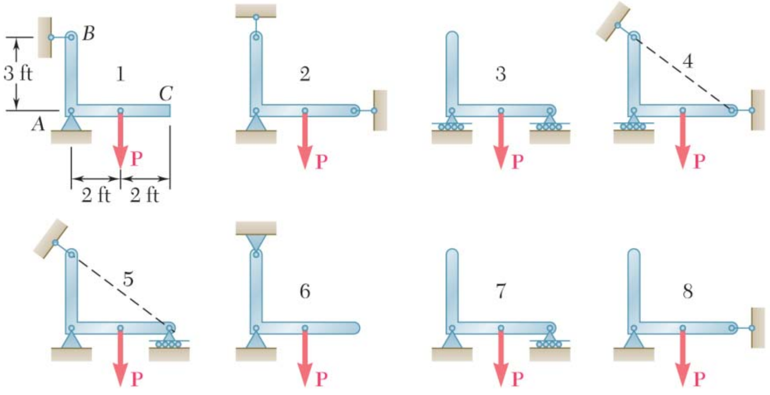


$$\mathbf{B} = \mathbf{D}_y = 196.2\text{ N}\uparrow$$

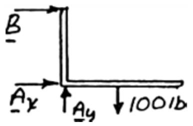
$$(\mathbf{C} + \mathbf{D}_x = 0)$$

PROBLEM 4.60

The bracket *ABC* can be supported in the eight different ways shown. All connections consist of smooth pins, rollers, or short links. For each case, answer the questions listed in Problem 4.59, and, wherever possible, compute the reactions, assuming that the magnitude of the force **P** is 100 lb.

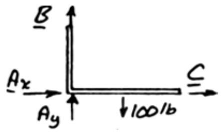


SOLUTION

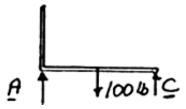


1. Three non-concurrent, non-parallel reactions:
    - (*a*) Bracket: complete constraint
    - (*b*) Reactions: determinate
    - (*c*) Equilibrium maintained

$$\mathbf{A} = 120.2\text{ lb} \measuredangle 56.3^\circ, \quad \mathbf{B} = 66.7\text{ lb} \longleftarrow$$

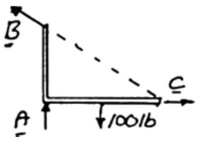


2. Four concurrent, reactions (through *A*):
    - (*a*) Bracket: improper constraint
    - (*b*) Reactions: indeterminate
    - (*c*) No equilibrium $(\Sigma M_A \neq 0)$



3. Two reactions:
    - (*a*) Bracket: partial constraint
    - (*b*) Reactions: indeterminate
    - (*c*) Equilibrium maintained

$$\mathbf{A} = 50\text{ lb}\uparrow, \quad \mathbf{C} = 50\text{ lb}\uparrow$$

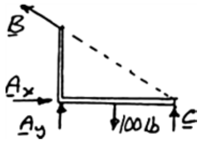


4. Three non-concurrent, non-parallel reactions:
    - (*a*) Bracket: complete constraint
    - (*b*) Reactions: determinate
    - (*c*) Equilibrium maintained

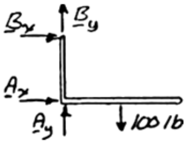
$$\mathbf{A} = 50\text{ lb}\uparrow, \quad \mathbf{B} = 83.3\text{ lb} \measuredangle 36.9^\circ, \quad \mathbf{C} = 66.7\text{ lb} \longrightarrow$$

PROBLEM 4.60 (Continued)

5. Four non-concurrent, non-parallel reactions:
   - (*a*) Bracket: complete constraint
   - (*b*) Reactions: indeterminate
   - (*c*) Equilibrium maintained $(\Sigma M_C = 0)\ \mathbf{A}_y = 50\text{ lb}\uparrow$



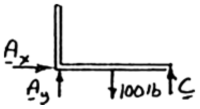
6. Four non-concurrent, non-parallel reactions:
   - (*a*) Bracket: complete constraint
   - (*b*) Reactions: indeterminate
   - (*c*) Equilibrium maintained



$$\mathbf{A}_x = 66.7\text{ lb} \longrightarrow \quad \mathbf{B}_x = 66.7\text{ lb} \longleftarrow$$

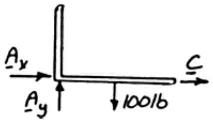
$$(\mathbf{A}_y + \mathbf{B}_y = 100\text{ lb}\uparrow)$$

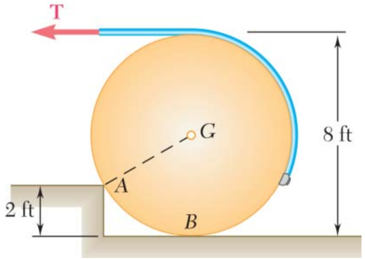
7. Three non-concurrent, non-parallel reactions:
   - (*a*) Bracket: complete constraint
   - (*b*) Reactions: determinate
   - (*c*) Equilibrium maintained



$$\mathbf{A} = \mathbf{C} = 50\text{ lb}\uparrow$$

8. Three concurrent, reactions (through *A*)
   - (*a*) Bracket: improper constraint
   - (*b*) Reactions: indeterminate
   - (*c*) No equilibrium $(\Sigma M_A \neq 0)$



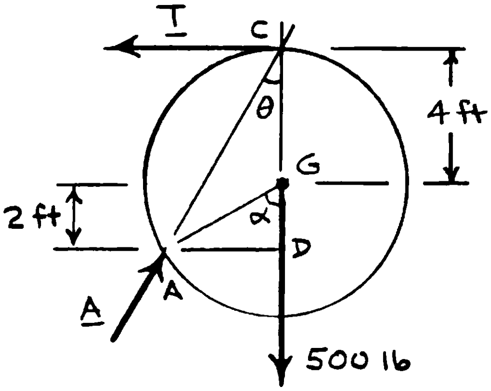


PROBLEM 4.61

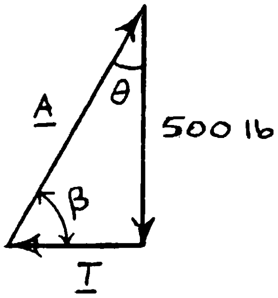
A 500-lb cylindrical tank, 8 ft in diameter, is to be raised over a 2-ft obstruction. A cable is wrapped around the tank and pulled horizontally as shown. Knowing that the corner of the obstruction at *A* is rough, find the required tension in the cable and the reaction at *A*.

SOLUTION

Free-Body Diagram:



Force triangle



$$\cos\alpha = \frac{GD}{AG} = \frac{2\text{ ft}}{4\text{ ft}} = 0.5 \qquad \alpha = 60°$$

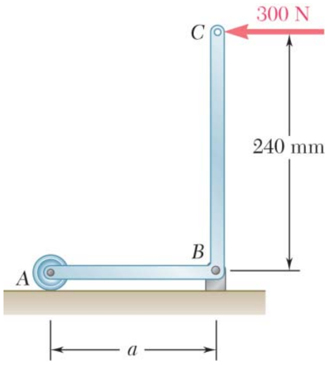
$$\theta = \frac{1}{2}\alpha = 30° \qquad (\beta = 60°)$$

$$T = (500\text{ lb})\tan 30°$$

T=289 lb ◀

$$A = \frac{500\text{ lb}}{\cos 30°}$$

$\mathbf{A} = 577\text{ lb} \measuredangle 60.0°$ ◀


300 N
C
240 mm
B
A
a

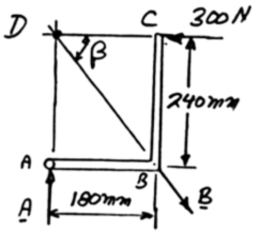
PROBLEM 4.62

Determine the reactions at *A* and *B* when $a = 180$ mm.

SOLUTION

Reaction at *B* must pass through *D* where *A* and 300-N load intersect.

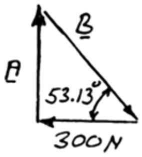
Free-Body Diagram:
(Three-force member)


D
C
300N
β
240mm
A
B
A
180mm
B

ΔBCD:

$$\tan\beta = \frac{240}{180}$$
$$\beta = 53.13°$$

Force triangle

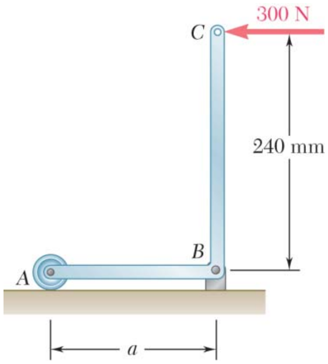

B
A
53.13°
300N

$$A = (300\text{ N})\tan 53.13°$$
$$= 400\text{ N}$$

$\mathbf{A} = 400\text{ N}\uparrow$ ◀

$$B = \frac{300\text{ N}}{\cos 53.13°}$$
$$= 500\text{ N}$$

$\mathbf{B} = 500\text{ N} \searrow 53.1°$ ◀


300 N
C
240 mm
B
A
a

PROBLEM 4.63

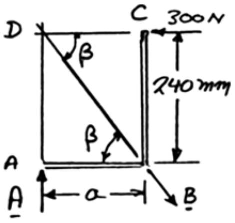
For the bracket and loading shown, determine the range of values of the distance a for which the magnitude of the reaction at B does not exceed 600 N.

SOLUTION

Reaction at B must pass through D where A and 300-N load intersect.

Free-Body Diagram:

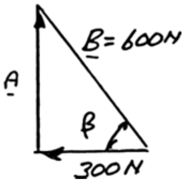
(Three-force member)


D
C
300 N
β
240 mm
β
A
A
a
B

$$a = \frac{240 \text{ mm}}{\tan \beta} \tag{1}$$

Force Triangle

(with $B = 600$ N)


B = 600 N
A
β
300 N

$$\cos \beta = \frac{300 \text{ N}}{600 \text{ N}} = 0.5$$

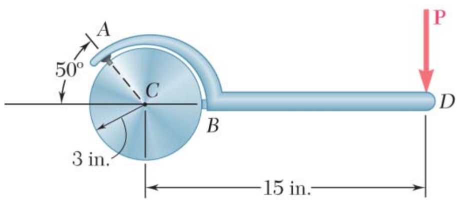
$$\beta = 60.0°$$

Eq. (1)

$$a = \frac{240 \text{ mm}}{\tan 60.0°}$$

$$= 138.56 \text{ mm}$$

For $B \leq 600$ N $\quad a \geq 138.6$ mm ◀


P
A
50°
C
D
B
3 in.
15 in.

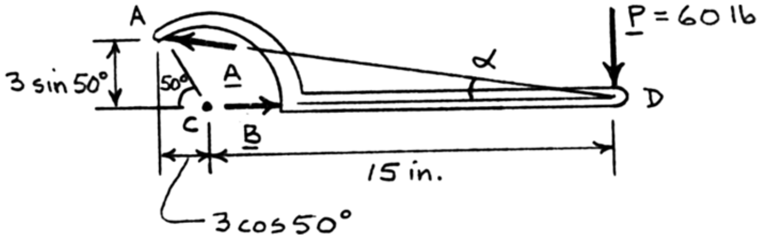
PROBLEM 4.64

The spanner shown is used to rotate a shaft. A pin fits in a hole at *A*, while a flat, frictionless surface rests against the shaft at *B*. If a 60-lb force **P** is exerted on the spanner at *D*, find the reactions at *A* and *B*.

SOLUTION

Free-Body Diagram:

(Three-force body)


A
P = 60 lb
α
3 sin 50°
50°
A
D
C
B
15 in.
3 cos 50°

The line of action of **A** must pass through *D*, where **B** and **P** intersect.

$$\tan\alpha = \frac{3\sin 50°}{3\cos 50° + 15}$$

$$= 0.135756$$

$$\alpha = 7.7310°$$

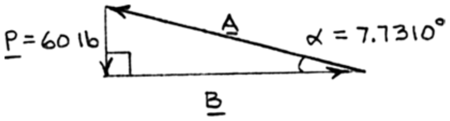
$$A = \frac{60 \text{ lb}}{\sin 7.7310°}$$

$$= 446.02 \text{ lb}$$

$$B = \frac{60 \text{ lb}}{\tan 7.7310°}$$

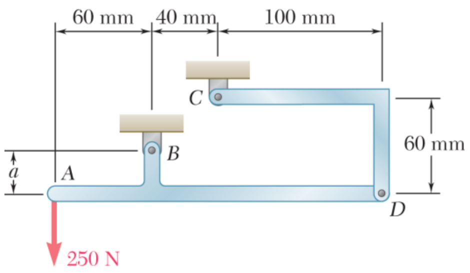
$$= 441.97 \text{ lb}$$

Force triangle


P = 60 lb
A
α = 7.7310°
B

A = 446 lb ↘ 7.73° ◀

B = 442 lb ⟶ ◀


60 mm
40 mm
100 mm
C
60 mm
B
a
A
D
250 N

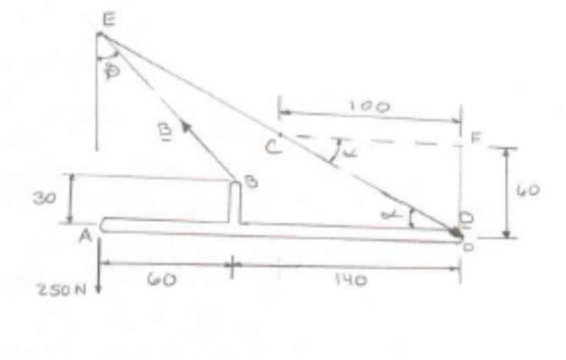
PROBLEM 4.65

Determine the reactions at *B* and *C* when *a* = 30 mm.

SOLUTION

Since *CD* is a two-force member, the force it exerts on member *ABD* is directed along *DC*.

Free-Body Diagram of *ABD*: (Three-Force member)


E
β
B
100
C
α
F
B
30
α
D
60
A
250 N
60
140
D

The reaction at *B* must pass through *E*, where **D** and the 250-N load intersect.

Triangle *CFD*:

$$\tan\alpha = \frac{60}{100} = 0.6$$

$$\alpha = 30.964°$$

Triangle *EAD*:

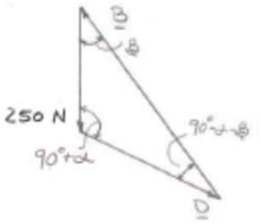
$$AE = 200\tan\alpha = 120 \text{ mm}$$

$$GE = AE - AG = 120 - 30 = 90 \text{ mm}$$

Triangle *EGB*:

$$\tan\beta = \frac{GB}{GE} = \frac{60}{90}$$

$$\beta = 33.690°$$


B
β
250 N
90°+α
90°-α-β
D

Force triangle

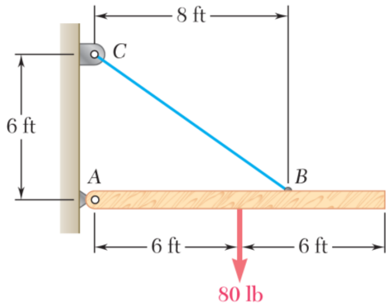
$$\frac{B}{\sin 120.964°} = \frac{D}{\sin 33.690°} = \frac{250 \text{ N}}{\sin 25.346°}$$

$$B = 500.7 \text{ N}$$

$$D = 323.9 \text{ N}$$

$\mathbf{B} = 501 \text{ N} \searrow 56.3°$ ◀

$\mathbf{C} = \mathbf{D} = 324 \text{ N} \swarrow 31.0°$ ◀



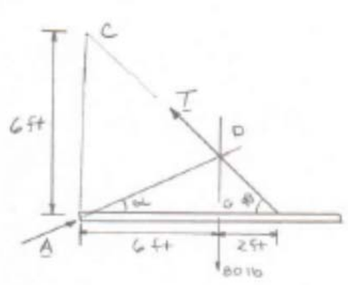
PROBLEM 4.66

A 12-ft wooden beam weighing 80 lb is supported by a pin and bracket at *A* and by cable *BC*. Find the reaction at *A* and the tension in the cable.

SOLUTION

Since *CB* is a two-force member, the force it exerts on member *AB* is directed along *CB*.

Free-Body Diagram of *AB*: (Three-Force member)



The reaction at *B* must pass through *E*, where **T** and the 80-lb load intersect.

Triangle *CFD*:

$$DG = \frac{BG}{AB}(AC)$$

$$DG = \frac{2}{8}(6\text{ ft}) = 1.50\text{ ft}$$

Triangle *EAD*:

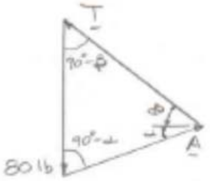
$$\tan\alpha = \frac{DG}{AE} = \frac{1.50}{6}$$

$$\alpha = 14.0362^\circ$$

Triangle *EGB*:

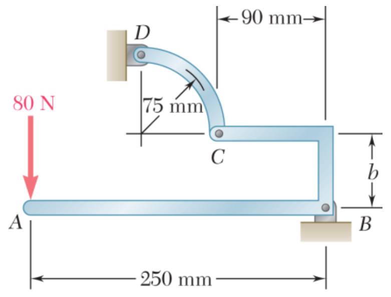
Force triangle

$$\frac{A}{\sin 53.13^\circ} = \frac{T}{\sin 75.964^\circ} = \frac{80\text{ lb}}{\sin 50.906^\circ}$$



$\mathbf{A} = 82.5\text{ lb} \measuredangle 14.04^\circ$ ◀

$T = 100.0\text{ lb}$ ◀


90 mm
D
80 N
75 mm
C
b
A
B
250 mm

PROBLEM 4.67

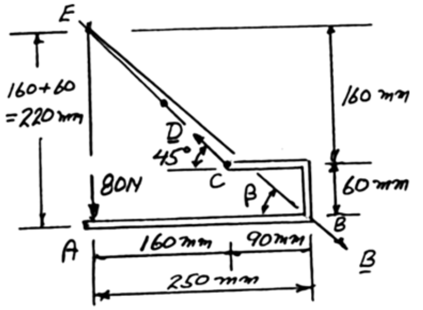
Determine the reactions at *B* and *D* when $b = 60$ mm.

SOLUTION

Since *CD* is a two-force member, the line of action of reaction at *D* must pass through Points *C* and *D*. 45° ⤡ D

Free-Body Diagram:

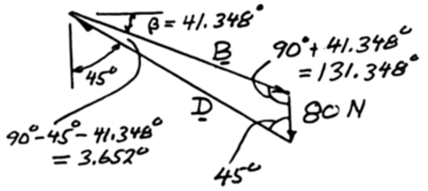
(Three-force body)


E
160+60
=220 mm
160 mm
D
45°
80N
C
β
60 mm
B
A
160 mm
90 mm
B
250 mm

Reaction at *B* must pass through *E*, where the reaction at *D* and the 80-N force intersect.

$$\tan\beta = \frac{220 \text{ mm}}{250 \text{ mm}}$$
$$\beta = 41.348°$$

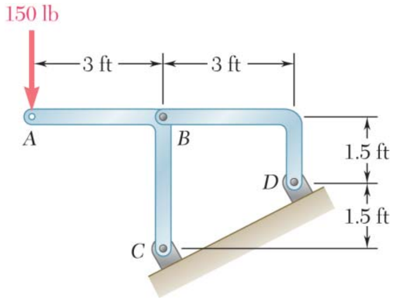
Force triangle


β = 41.348°
B
90°+ 41.348°
= 131.348°
45°
80 N
D
90°−45°−41.348°
= 3.652°
45°

Law of sines:

$$\frac{80 \text{ N}}{\sin 3.652°} = \frac{B}{\sin 45°} = \frac{D}{\sin 131.348°}$$
$$B = 888.0 \text{ N}$$
$$D = 942.8 \text{ N}$$

B = 888 N ↘ 41.3° **D** = 943 N ↖ 45.0° ◀



PROBLEM 4.68

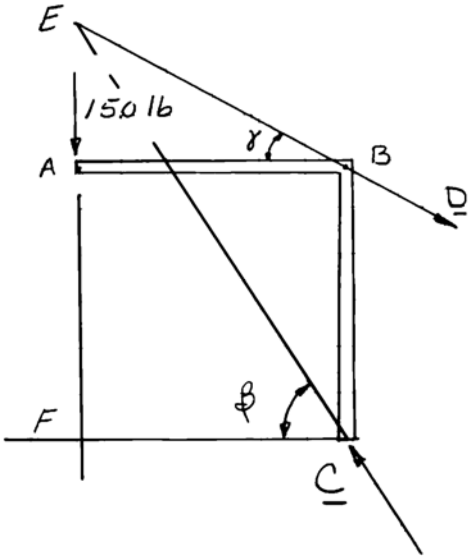
For the frame and loading shown, determine the reactions at *C* and *D*.

SOLUTION

Since *BD* is a two-force member, the reaction at *D* must pass through Points *B* and *D*.

Free-Body Diagram:

(Three-force body)



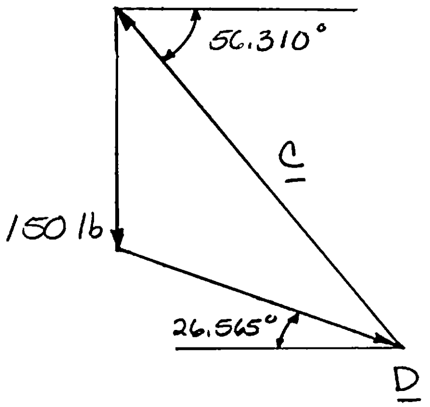
Reaction at *C* must pass through *E*, where the reaction at *D* and the 150-lb load intersect.

Triangle *CEF*: $\tan\beta = \dfrac{4.5\text{ ft}}{3\text{ ft}} \qquad \beta = 56.310°$

Triangle *ABE*: $\tan\gamma = \dfrac{1}{2} \qquad \gamma = 26.565°$

PROBLEM 4.68 (Continued)

Force Triangle



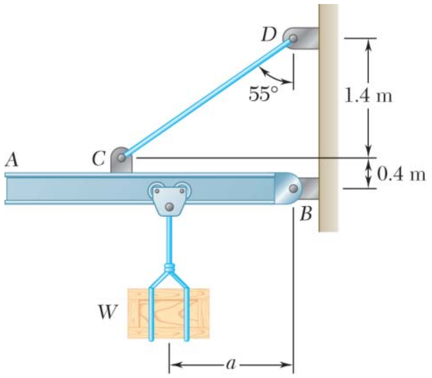
Law of sines:

$$\frac{150\text{ lb}}{\sin 29.745°} = \frac{C}{\sin 116.565°} = \frac{D}{\sin 33.690°}$$

$$C = 270.42\text{ lb},$$

$$D = 167.704\text{ lb}$$

$\mathbf{C} = 270\text{ lb} \searrow 56.3°$; $\mathbf{D} = 167.7\text{ lb} \searrow 26.6°$ ◀


D
55°
1.4 m
A
C
0.4 m
B
W
a

PROBLEM 4.69

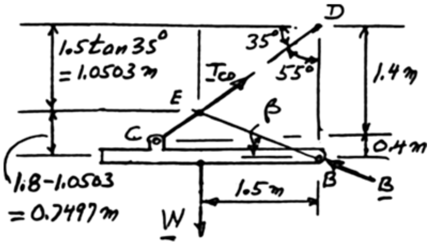
A 50-kg crate is attached to the trolley-beam system shown. Knowing that $a = 1.5$ m, determine (*a*) the tension in cable *CD*, (*b*) the reaction at *B*.

SOLUTION

Three-force body: **W** and $\mathbf{T}_{CD}$ intersect at *E*.

$$\tan\beta = \frac{0.7497 \text{ m}}{1.5 \text{ m}}$$

$$\beta = 26.556^\circ$$

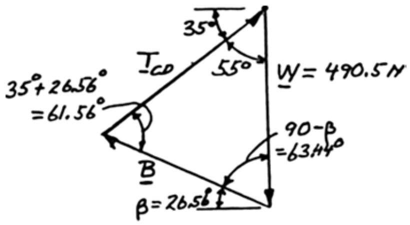

1.5 tan 35°
= 1.0503 m
35°
D
T_{CD}
55°
1.4 m
E
β
C
0.4 m
1.8−1.0503
= 0.7497 m
1.5 m
B
B
W

Three forces intersect at *E*.

$$W = (50 \text{ kg})\, 9.81 \text{ m/s}^2$$

$$= 490.50 \text{ N}$$

Force triangle


35°
T_{CD}
55°
W = 490.5 N
35°+26.56°
= 61.56°
90−β
= 63.44°
B
β = 26.56°

Law of sines:

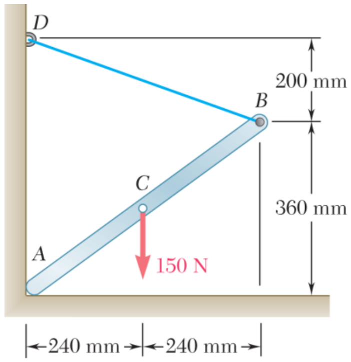
$$\frac{490.50 \text{ N}}{\sin 61.556^\circ} = \frac{T_{CD}}{\sin 63.444^\circ} = \frac{B}{\sin 55^\circ}$$

$$T_{CD} = 498.99 \text{ N}$$

$$B = 456.96 \text{ N}$$

(*a*) $T_{CD} = 499 \text{ N}$ ◀

(*b*) $\mathbf{B} = 457 \text{ N} \searrow 26.6^\circ$ ◀

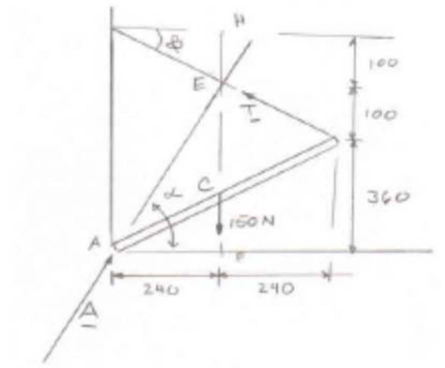

D
200 mm
B
C
360 mm
A
150 N
240 mm
240 mm

PROBLEM 4.70

One end of rod *AB* rests in the corner *A* and the other end is attached to cord *BD*. If the rod supports a 150-N load at its midpoint *C*, find the reaction at *A* and the tension in the cord.

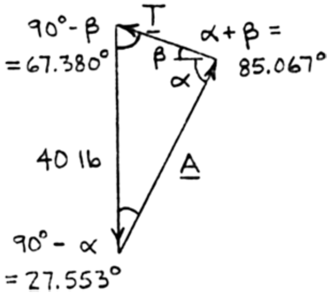
SOLUTION

Free-Body Diagram: (Three-force body) Dimensions in mm


β
H
E
T
100
100
360
α
C
150 N
A
F
240
240
A

The line of action of reaction at *A* must pass through *E*, where **T** and the 150-N load intersect.

Force triangle


90° - β
= 67.380°
T
α + β =
85.067°
β
α
40 lb
A
90° - α
= 27.553°

$$\tan\alpha = \frac{EF}{AF} = \frac{460}{240}$$

$$\alpha = 62.447^\circ$$

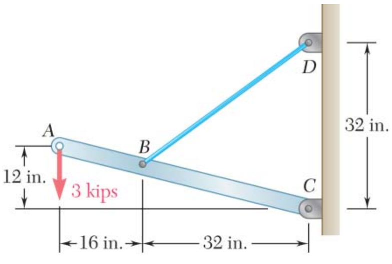
$$\tan\beta = \frac{EH}{DH} = \frac{100}{240}$$

$$\beta = 22.620^\circ$$

$$\frac{A}{\sin 67.380^\circ} = \frac{T}{\sin 27.553^\circ} = \frac{150\text{ N}}{\sin 85.067^\circ}$$

$\mathbf{A} = 139.0\text{ N} \measuredangle\ 62.4^\circ$ ◀

$T = 69.6\text{ N}$ ◀


D
32 in.
A
B
C
12 in.
3 kips
16 in.
32 in.

PROBLEM 4.71

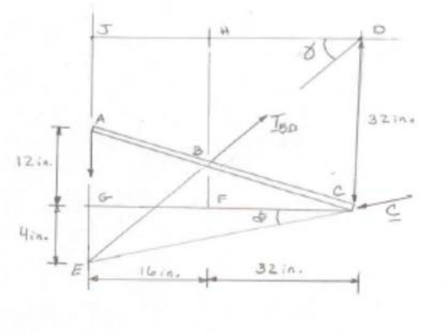
For the boom and loading shown, determine (*a*) the tension in cord *BD*, (*b*) the reaction at *C*.

SOLUTION

Free-Body Diagram:

Three-force body: 3-kip load and *T* intersect at *E*.

Geometry:


J
H
D
γ
A
T_{BD}
B
32 in.
12 in.
G
F
C
C
α
4 in.
E
16 in.
32 in.

$$\frac{BF}{AG}=\frac{CF}{CG};\quad \frac{BF}{12}=\frac{32}{48}$$

$$BF = 8 \text{ in.}$$

$$BH = 32 - BF = 32 - 8 = 24 \text{ in.}$$

$$\frac{JE}{BH}=\frac{DJ}{DH};\quad \frac{JE}{24}=\frac{48}{32};\quad JE = 36 \text{ in.}$$

$$EG = JE - JG = 36 - 32 = 4 \text{ in.}$$

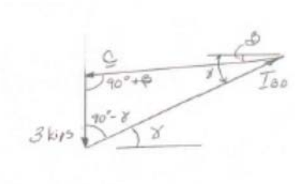
$$\tan\alpha = \frac{4 \text{ in.}}{48 \text{ in.}}$$

$$\alpha = 4.7636^\circ$$

$$\tan\beta = \frac{24 \text{ in.}}{32 \text{ in.}}$$

$$\beta = 36.870^\circ$$

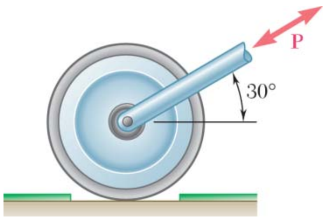
Force Triangle


C
90°+α
γ
α
T_{BD}
90°−γ
3 kips
γ

Law of sines: $$\frac{T_{BD}}{\sin 94.764^\circ}=\frac{3 \text{ kips}}{\sin 32.106^\circ}=\frac{C}{\sin 53.13^\circ}$$

(*a*) $T_{BD} = 5.63$ kips ◀

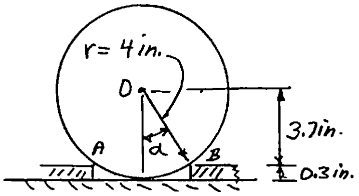
(*b*) **C** = 4.52 kips ↙ 4.76° ◀



PROBLEM 4.72

A 40-lb roller, of diameter 8 in., which is to be used on a tile floor, is resting directly on the subflooring as shown. Knowing that the thickness of each tile is 0.3 in., determine the force **P** required to move the roller onto the tiles if the roller is (*a*) pushed to the left, (*b*) pulled to the right.

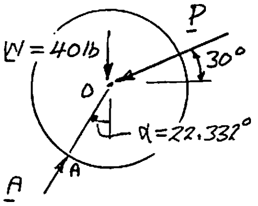
SOLUTION


r = 4 in.
O
3.7in.
A
α
B
0.3 in.

Geometry: For each case as roller comes into contact with tile,

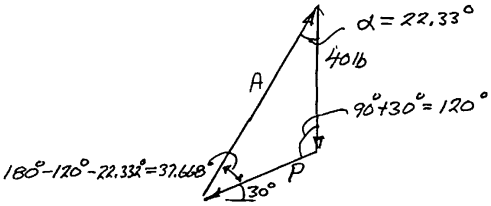
$$\alpha = \cos^{-1}\frac{3.7\text{ in.}}{4\text{ in.}}$$

$$\alpha = 22.332°$$


P
W = 40lb
30°
O
α = 22.332°
A
A

(*a*) Roller pushed to left (three-force body): Forces must pass through *O*.

Force Triangle


α = 22.33°
40lb
A
90°+30° = 120°
180°−120°−22.332° = 37.668°
P
30°

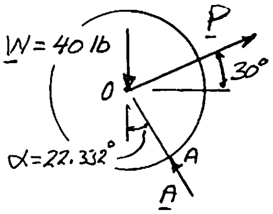
Law of sines:
$$\frac{40\text{ lb}}{\sin 37.668°} = \frac{P}{\sin 22.332°};\ P = 24.87\text{ lb}$$

$$\mathbf{P} = 24.9\text{ lb} \measuredangle 30.0°$$ ◀

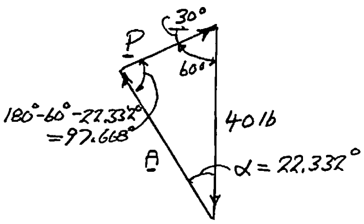
(*b*) Roller pulled to right (three-force body): Forces must pass through *O*.

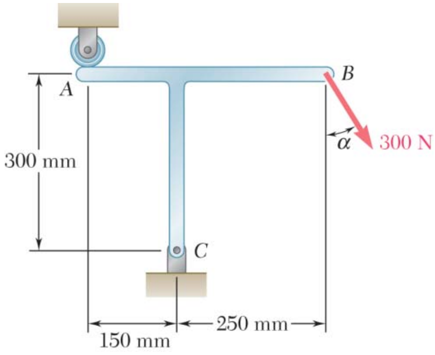
Law of sines:
$$\frac{40\text{ lb}}{\sin 97.668°} = \frac{P}{\sin 22.332°};\ P = 15.3361\text{ lb}$$

$$\mathbf{P} = 15.34\text{ lb} \measuredangle 30.0°$$ ◀


P
W = 40 lb
30°
O
α = 22.332°
A
A

Force Triangle


30°
P
60°
180°−60°−22.332°
= 97.668°
40lb
A
α = 22.332°


A
B
300 mm
α
300 N
C
150 mm
250 mm

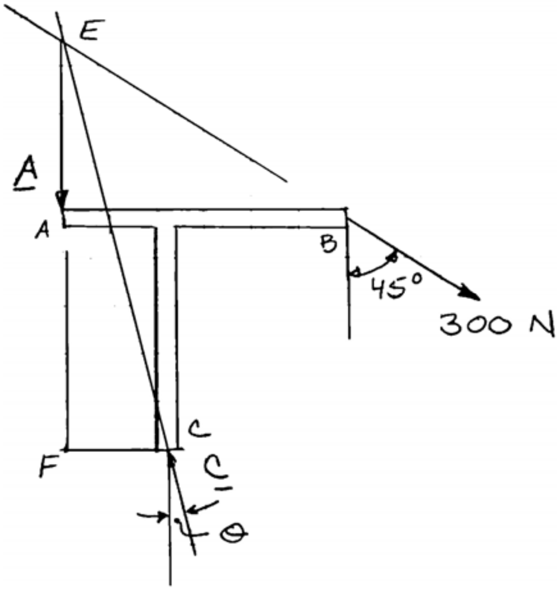
PROBLEM 4.73

A T-shaped bracket supports a 300-N load as shown. Determine the reactions at *A* and *C* when $\alpha = 45°$.

SOLUTION

Free-Body Diagram:

(Three-force body)


E
A
A
B
45°
300 N
C
F
C
θ

The line of action of **C** must pass through *E*, where **A** and the 300-N force intersect.

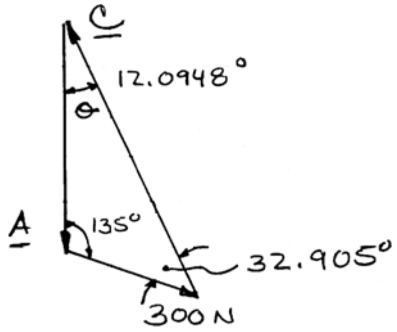
Triangle *ABE* is isosceles: $EA = AB = 400 \text{ mm}$

In triangle *CEF*:

$$\tan\theta = \frac{CF}{EF} = \frac{CF}{EA + AF} = \frac{150 \text{ mm}}{700 \text{ mm}} \qquad \theta = 12.0948°$$

PROBLEM 4.73 (Continued)

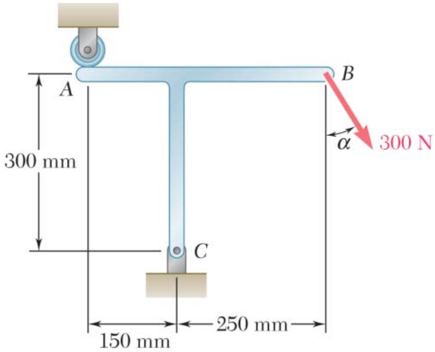
Force Triangle


C
12.0948°
θ
A
135°
32.905°
300 N

Law of sines:

$$\frac{A}{\sin 32.905^\circ} = \frac{C}{\sin 135^\circ} = \frac{300\text{ N}}{\sin 12.0948^\circ}$$

$\mathbf{A} = 778\text{ N}\downarrow$; $\mathbf{C} = 1012\text{ N}$ ⦫ 77.9° ◀



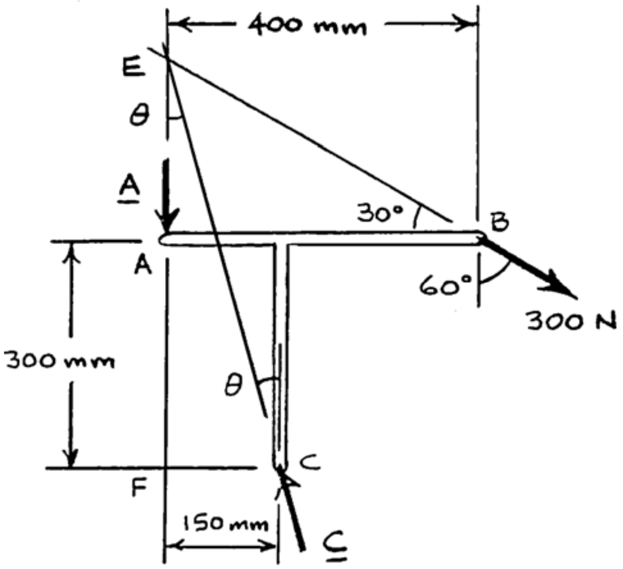
A
B
C
α
300 N
300 mm
150 mm
250 mm

PROBLEM 4.74

A T-shaped bracket supports a 300-N load as shown. Determine the reactions at *A* and *C* when $\alpha = 60°$.

SOLUTION

Free-Body Diagram:



400 mm
E
θ
A
30°
B
A
60°
300 N
300 mm
θ
C
F
150 mm
C

$$EA = (400 \text{ mm}) \tan 30°$$
$$= 230.94 \text{ mm}$$

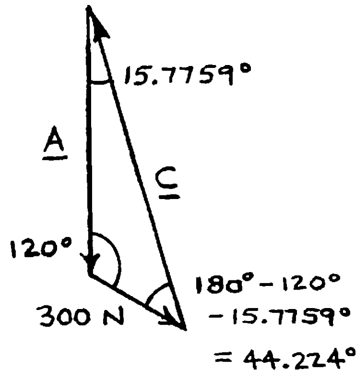
In triangle *CEF*:

$$\tan\theta = \frac{CF}{EF} = \frac{CF}{EA + AF}$$

$$\tan\theta = \frac{150}{230.94 + 300}$$
$$\theta = 15.7759°$$

PROBLEM 4.74 (Continued)

Force Triangle


15.7759°
A
C
120°
300 N
180° − 120°
− 15.7759°
= 44.224°

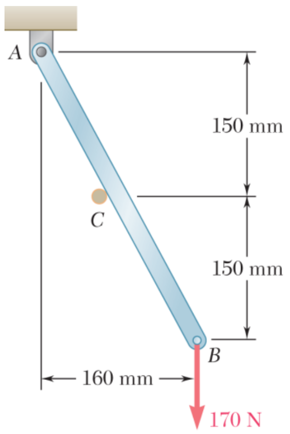
Law of sines:

$$\frac{A}{\sin 44.224^\circ} = \frac{C}{\sin 120^\circ} = \frac{300\text{ N}}{\sin 15.7759^\circ}$$

$$A = 770\text{ N}$$

$$C = 956\text{ N}$$

$\mathbf{A} = 770\text{ N}\downarrow$; $\mathbf{C} = 956\text{ N}\searrow 74.2^\circ$ ◀

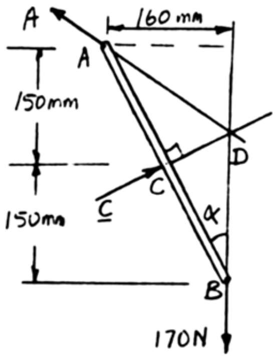


PROBLEM 4.75

Rod *AB* is supported by a pin and bracket at *A* and rests against a frictionless peg at *C*. Determine the reactions at *A* and *C* when a 170-N vertical force is applied at *B*.

SOLUTION

Free-Body Diagram:
(Three-force body)



The reaction at *A* must pass through *D* where **C** and the 170-N force intersect.

$$\tan\alpha = \frac{160\text{ mm}}{300\text{ mm}}$$

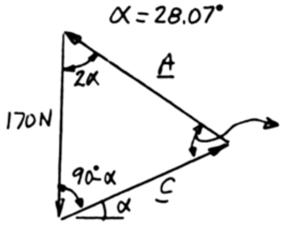
$$\alpha = 28.07°$$

We note that triangle *ABD* is isosceles (since $AC = BC$) and, therefore,

$$\sphericalangle CAD = \alpha = 28.07°$$

Also, since $CD \perp CB$, reaction **C** forms angle $\alpha = 28.07°$ with the horizontal axis.

Force triangle



We note that **A** forms angle 2α with the vertical axis. Thus, **A** and **C** form angle

$$180° - (90° - \alpha) - 2\alpha = 90° - \alpha$$

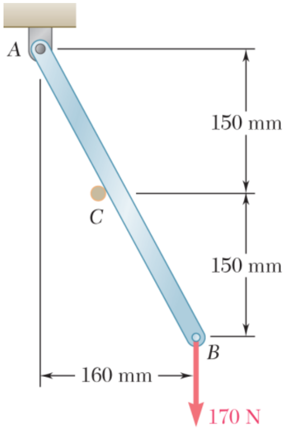
Force triangle is isosceles, and we have

$$A = 170\text{ N}$$

$$C = 2(170\text{ N})\sin\alpha$$

$$= 160.0\text{ N}$$

$\mathbf{A} = 170.0\text{ N}$ ⦩ 33.9°; $\mathbf{C} = 160.0\text{ N}$ ∠ 28.1° ◀



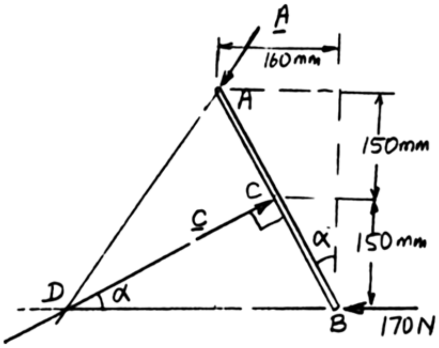
PROBLEM 4.76

Solve Problem 4.75, assuming that the 170-N force applied at *B* is horizontal and directed to the left.

PROBLEM 4.75 Rod *AB* is supported by a pin and bracket at *A* and rests against a frictionless peg at *C*. Determine the reactions at *A* and *C* when a 170-N vertical force is applied at *B*.

SOLUTION

Free-Body Diagram: (Three-Force body)



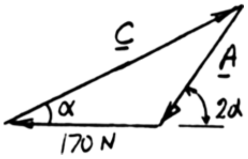
The reaction at *A* must pass through *D*, where **C** and the 170-N force intersect.

$$\tan\alpha = \frac{160\text{ mm}}{300\text{ mm}}$$

$$\alpha = 28.07^\circ$$

We note that triangle *ADB* is isosceles (since $AC = BC$). Therefore $\measuredangle A = \measuredangle B = 90^\circ - \alpha$.

Also $\measuredangle ADB = 2\alpha$



Force triangle

The angle between **A** and **C** must be $2\alpha - \alpha = \alpha$

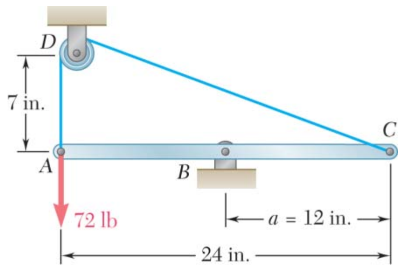
$\alpha = 28.07^\circ$

Thus, force triangle is isosceles and

$$A = 170.0\text{ N}$$

$$C = 2(170\text{ N})\cos\alpha = 300\text{ N}$$

$\mathbf{A} = 170.0\text{ N} \measuredangle 56.1^\circ \quad \mathbf{C} = 300\text{ N} \measuredangle 28.1^\circ$ ◀

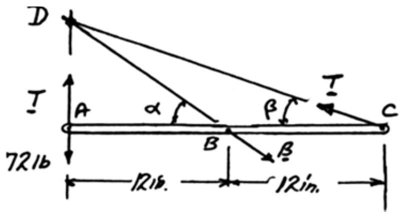

D
7 in.
C
A
B
72 lb
a = 12 in.
24 in.

PROBLEM 4.77

Member *ABC* is supported by a pin and bracket at *B* and by an inextensible cord attached at *A* and *C* and passing over a frictionless pulley at *D*. The tension may be assumed to be the same in portions *AD* and *CD* of the cord. For the loading shown and neglecting the size of the pulley, determine the tension in the cord and the reaction at *B*.

SOLUTION

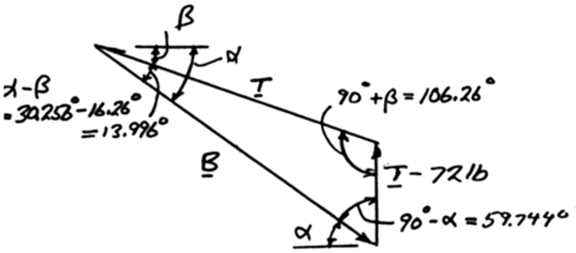
Free-Body Diagram:


D
T
A
α
β
T
C
B
B
72 lb
12 in.
12 in.

Reaction at *B* must pass through *D*.

$$\tan\alpha = \frac{7 \text{ in.}}{12 \text{ in.}}$$
$$\alpha = 30.256°$$
$$\tan\beta = \frac{7 \text{ in.}}{24 \text{ in.}}$$
$$\beta = 16.26°$$

Force Triangle


β
α
T
90° + β = 106.26°
α − β
= 30.256° − 16.26°
= 13.996°
B
T − 72 lb
90° − α = 59.744°
α

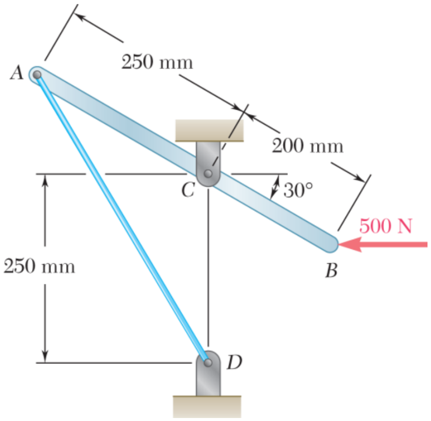
Law of sines:

$$\frac{T}{\sin 59.744°} = \frac{T - 72 \text{ lb}}{\sin 13.996°} = \frac{B}{\sin 106.26}$$
$$T(\sin 13.996°) = (T - 72 \text{ lb})(\sin 59.744°)$$
$$T(0.24185) = (T - 72)(0.86378)$$
$$T = 100.00 \text{ lb}$$

$T = 100.0$ lb ◀

$$B = (100 \text{ lb})\frac{\sin 106.26°}{\sin 59.744°}$$
$$= 111.14 \text{ lb}$$

$\mathbf{B} = 111.1$ lb ↘ 30.3° ◀


A
250 mm
200 mm
C
30°
500 N
B
250 mm
D

PROBLEM 4.78

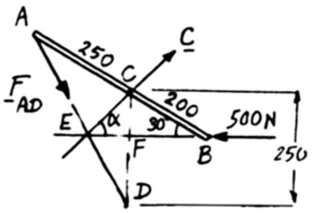
Using the method of Section 4.7, solve Problem 4.22.

PROBLEM 4.22 A lever *AB* is hinged at *C* and attached to a control cable at *A*. If the lever is subjected to a 500-N horizontal force at *B*, determine (*a*) the tension in the cable, (*b*) the reaction at *C*.

SOLUTION

Free-Body Diagram:

(Three-Force body)


A
250
C
$\underline{C}$
$\underline{F}_{AD}$
200
E
α
90°
500N
F
B
250
D

Dimensions in mm

Reaction at *C* must pass through *E*, where $\mathbf{F}_{AD}$ and 500-N force intersect.

Since $AC = CD = 250$ mm, triangle *ACD* is isosceles.

We have $$\sphericalangle C = 90° + 30° = 120°$$

and $$\sphericalangle A = \sphericalangle D = \frac{1}{2}(180° - 120°) = 30°$$

On the other hand, from triangle *BCF*:

$$CF = (BC)\sin 30° = 200 \sin 30° = 100 \text{ mm}$$
$$FD = CD - CF = 250 - 100 = 150 \text{ mm}$$

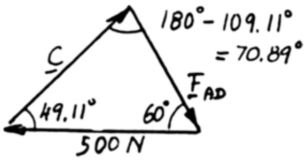
From triangle *EFD*, and since $\sphericalangle D = 30°$:

$$EF = (FD)\tan 30° = 150 \tan 30° = 86.60 \text{ mm}$$

From triangle *EFC*: $$\tan\alpha = \frac{CF}{EF} = \frac{100 \text{ mm}}{86.60 \text{ mm}}$$

$$\alpha = 49.11°$$

Force triangle

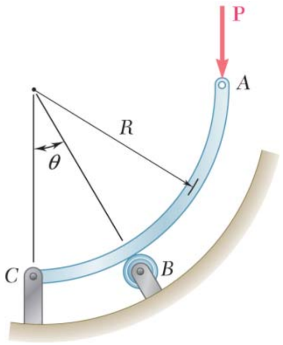

$180° - 109.11°$
$= 70.89°$
$\underline{C}$
$\underline{F}_{AD}$
49.11°
60°
500 N

Law of sines $$\frac{F_{AD}}{\sin 49.11°} = \frac{C}{\sin 60°} = \frac{500 \text{ N}}{\sin 70.89°}$$

$$F_{AD} = 400 \text{ N}, \quad C = 458 \text{ N}$$

(*a*) $F_{AD} = 400$ N ◀

(*b*) $\mathbf{C} = 458$ N ∡ 49.1° ◀

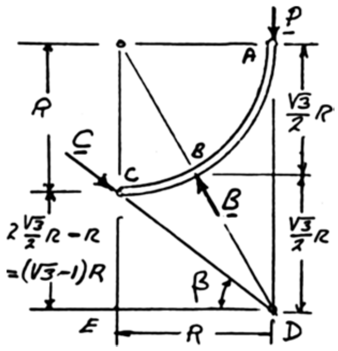


PROBLEM 4.79

Knowing that $\theta = 30°$, determine the reaction (*a*) at *B*, (*b*) at *C*.

SOLUTION

Free-Body Diagram:
(Three-force body)


P
A
R
$\frac{\sqrt{3}}{2}R$
C
C
B
B
β
$2\frac{\sqrt{3}}{2}R - R$
$=(\sqrt{3}-1)R$
$\frac{\sqrt{3}}{2}R$
E
R
D

Reaction at *C* must pass through *D* where force **P** and reaction at *B* intersect.

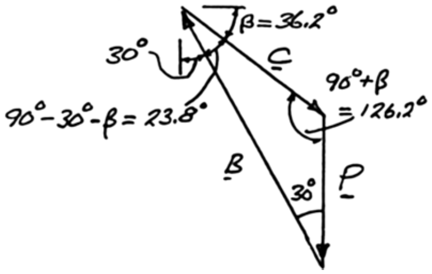
In Δ *CDE*:

$$\tan\beta = \frac{(\sqrt{3}-1)R}{R}$$

$$= \sqrt{3} - 1$$

$$\beta = 36.2°$$

Force Triangle


β=36.2°
30°
C
90°+β
= 126.2°
90°−30°−β = 23.8°
B
30°
P

Law of sines:

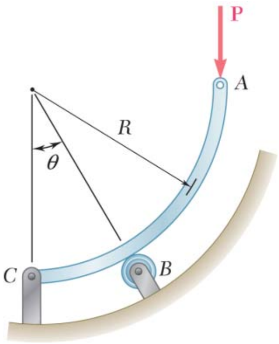
$$\frac{P}{\sin 23.8°} = \frac{B}{\sin 126.2°} = \frac{C}{\sin 30°}$$

$$B = 2.00P$$

$$C = 1.239P$$

(*a*) $\mathbf{B} = 2P$ ⦪ 60.0° ◀

(*b*) $\mathbf{C} = 1.239P$ ⦫ 36.2° ◀



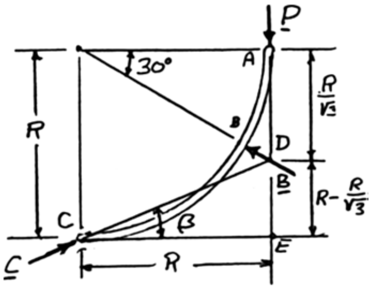
PROBLEM 4.80

Knowing that $\theta = 60°$, determine the reaction (*a*) at *B*, (*b*) at *C*.

SOLUTION

Reaction at *C* must pass through *D* where force **P** and reaction at *B* intersect.

Free-Body Diagram:
(Three-force body)


P
30°
A
R
B
D
$\frac{R}{\sqrt{3}}$
B
$R-\frac{R}{\sqrt{3}}$
C
β
C
E
R

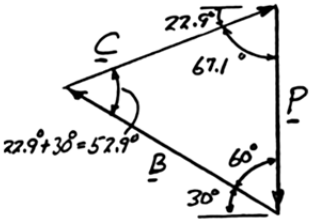
In ΔCDE:

$$\tan\beta = \frac{R - \frac{R}{\sqrt{3}}}{R}$$

$$= 1 - \frac{1}{\sqrt{3}}$$

$$\beta = 22.9°$$

Force Triangle


22.9°
C
67.1°
P
22.9°+30°=52.9°
B
60°
30°

Law of sines:

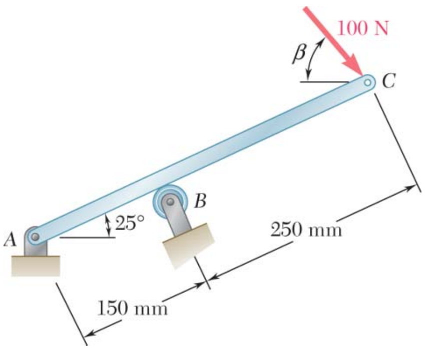
$$\frac{P}{\sin 52.9°} = \frac{B}{\sin 67.1°} = \frac{C}{\sin 60°}$$

$$B = 1.155P$$

$$C = 1.086P$$

(*a*) $\mathbf{B} = 1.155P \measuredangle 30.0°$ ◀

(*b*) $\mathbf{C} = 1.086P \measuredangle 22.9°$ ◀


100 N
β
C
B
A
25°
250 mm
150 mm

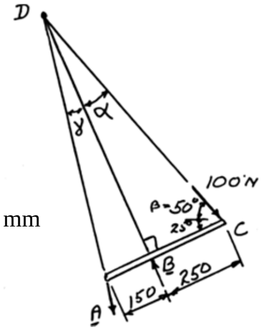
PROBLEM 4.81

Determine the reactions at A and B when $\beta = 50°$.

SOLUTION

Free-Body Diagram: (Three-force body)

Reaction **A** must pass through Point D where the 100-N force and **B** intersect.


D
γ
α
100 N
β = 50°
25°
C
B
250
150
A

Dimensions in mm

In right Δ BCD:

$$\alpha = 90° - 75° = 15°$$

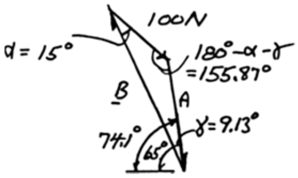
$$BD = 250 \tan 75° = 933.01\text{ mm}$$

In right Δ ABD:

$$\tan\gamma = \frac{AB}{BD} = \frac{150\text{ mm}}{933.01\text{ mm}}$$

$$\gamma = 9.1333°$$

Force Triangle

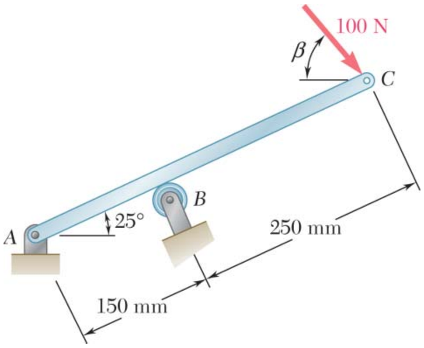

100N
α = 15°
180° − α − γ
= 155.87°
B
A
74.1°
65°
γ = 9.13°

Law of sines:

$$\frac{100\text{ N}}{\sin 9.1333°} = \frac{A}{\sin 15°} = \frac{B}{\sin 155.867°}$$

$$A = 163.1\text{ N}; \quad B = 257.6\text{ N}$$

$\mathbf{A} = 163.1\text{ N}$ ↘ 74.1° $\quad \mathbf{B} = 258\text{ N}$ ↖ 65.0° ◀



100 N
β
C
B
A
25°
250 mm
150 mm

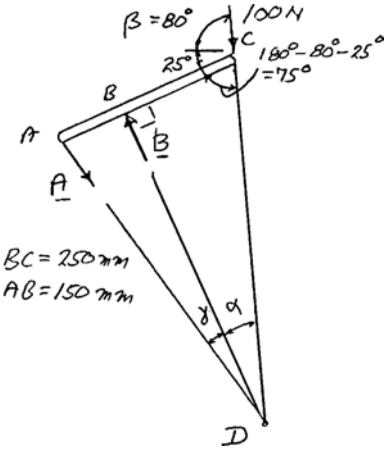
PROBLEM 4.82

Determine the reactions at *A* and *B* when $\beta = 80°$.

SOLUTION

Free-Body Diagram:

(Three-force body)



β = 80°
100 N
C
180° − 80° − 25°
= 75°
25°
B
A
B
A
BC = 250 mm
AB = 150 mm
γ
α
D

Reaction **A** must pass through *D* where the 100-N force and **B** intersect.

In right triangle *BCD*: $\alpha = 90° - 75° = 15°$

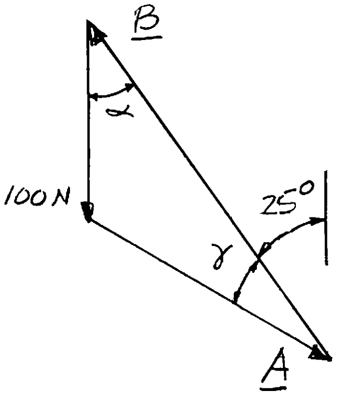
$$BD = BC \tan 75° = 250 \tan 75°$$

$$BD = 933.01 \text{ mm}$$

In right triangle *ABD*: $\tan \gamma = \dfrac{AB}{BD} = \dfrac{150 \text{ mm}}{933.01 \text{ mm}}$ $\qquad \gamma = 9.1333°$

PROBLEM 4.82 (Continued)

Force Triangle

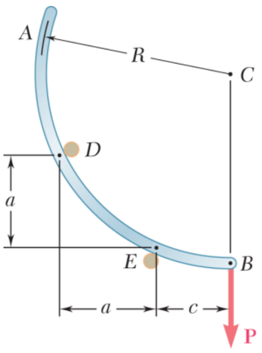


Law of sines:

$$\frac{100\text{ N}}{\sin 9.1333°} = \frac{A}{\sin 15°} = \frac{B}{\sin 155.867°}$$

A = 163.1 N ⦩ 55.9° ◀

B = 258 N ⦪ 65.0° ◀

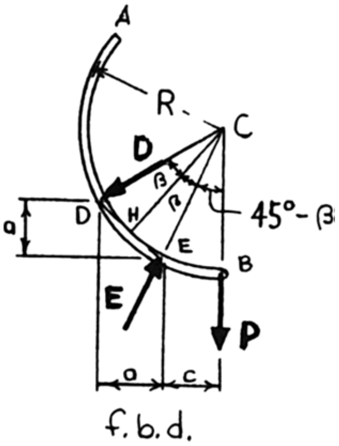


PROBLEM 4.83

Rod AB is bent into the shape of an arc of circle and is lodged between two pegs D and E. It supports a load **P** at end B. Neglecting friction and the weight of the rod, determine the distance c corresponding to equilibrium when $a = 20$ mm and $R = 100$ mm.

SOLUTION

Free-Body Diagram:



Since $y_{ED} = x_{ED} = a,$

slope of ED is ↘ 45°;

slope of HC is ↗ 45°.

Also $$DE = \sqrt{2}a$$

and $$DH = HE = \left(\frac{1}{2}\right)DE = \frac{a}{\sqrt{2}}$$

For triangles DHC and EHC, $$\sin\beta = \frac{\frac{a}{\sqrt{2}}}{R} = \frac{a}{\sqrt{2}R}$$

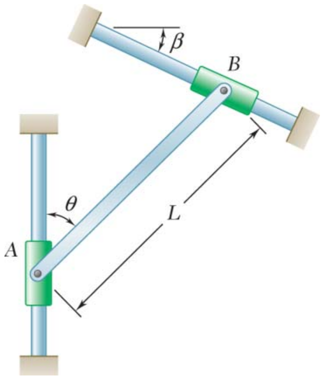
Now $$c = R\sin(45° - \beta)$$

For $$a = 20 \text{ mm} \quad \text{and} \quad R = 100 \text{ mm}$$

$$\sin\beta = \frac{20 \text{ mm}}{\sqrt{2}(100 \text{ mm})}$$
$$= 0.141421$$
$$\beta = 8.1301°$$

and $$c = (100 \text{ mm})\sin(45° - 8.1301°)$$
$$= 60.00 \text{ mm}$$

or $c = 60.0$ mm ◀



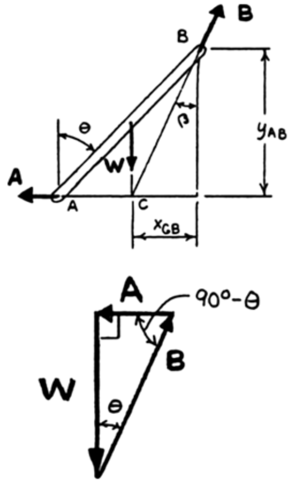
PROBLEM 4.84

A slender rod of length L is attached to collars that can slide freely along the guides shown. Knowing that the rod is in equilibrium, derive an expression for the angle θ in terms of the angle β.

SOLUTION

As shown in the free-body diagram of the slender rod AB, the three forces intersect at C. From the force geometry:

Free-Body Diagram:



$$\tan\beta = \frac{x_{GB}}{y_{AB}}$$

where

$$y_{AB} = L\cos\theta$$

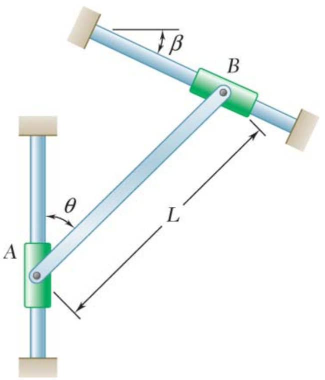
and

$$x_{GB} = \frac{1}{2}L\sin\theta$$

$$\tan\beta = \frac{\frac{1}{2}L\sin\theta}{L\cos\theta}$$

$$= \frac{1}{2}\tan\theta$$

or $\tan\theta = 2\tan\beta$ ◀



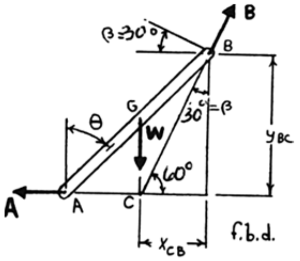
PROBLEM 4.85

An 8-kg slender rod of length L is attached to collars that can slide freely along the guides shown. Knowing that the rod is in equilibrium and that $\beta = 30°$, determine (*a*) the angle θ that the rod forms with the vertical, (*b*) the reactions at A and B.

SOLUTION

(*a*) As shown in the free-body diagram of the slender rod AB, the three forces intersect at C. From the geometry of the forces:

Free-Body Diagram:



$$\tan\beta = \frac{x_{CB}}{y_{BC}}$$

where

$$x_{CB} = \frac{1}{2}L\sin\theta$$

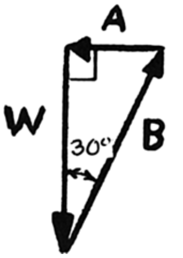
and

$$y_{BC} = L\cos\theta$$

$$\tan\beta = \frac{1}{2}\tan\theta$$

or $$\tan\theta = 2\tan\beta$$

For $$\beta = 30°$$



$$\tan\theta = 2\tan 30°$$
$$= 1.15470$$
$$\theta = 49.107°$$

or $\theta = 49.1°$ ◀

(*b*) $$W = mg = (8 \text{ kg})(9.81 \text{ m/s}^2) = 78.480 \text{ N}$$

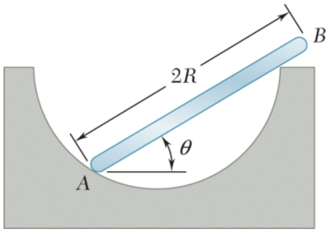
From force triangle:

$$A = W\tan\beta$$
$$= (78.480 \text{ N})\tan 30°$$
$$= 45.310 \text{ N}$$

or **A** = 45.3 N ← ◀

and $$B = \frac{W}{\cos\beta} = \frac{78.480 \text{ N}}{\cos 30°} = 90.621 \text{ N}$$

or **B** = 90.6 N ∠ 60.0° ◀



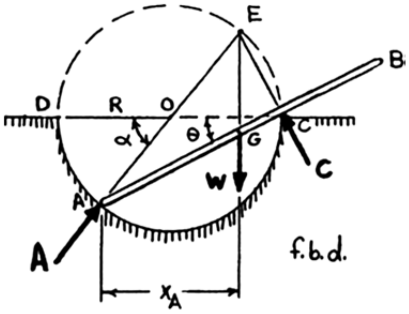
PROBLEM 4.86

A uniform rod *AB* of length 2*R* rests inside a hemispherical bowl of radius *R* as shown. Neglecting friction, determine the angle θ corresponding to equilibrium.

SOLUTION

Based on the F.B.D., the uniform rod *AB* is a three-force body. Point *E* is the point of intersection of the three forces. Since force *A* passes through *O*, the center of the circle, and since force **C** is perpendicular to the rod, triangle *ACE* is a right triangle inscribed in the circle. Thus, *E* is a point on the circle.

Note that the angle α of triangle *DOA* is the central angle corresponding to the inscribed angle θ of triangle *DCA*.



$$\alpha = 2\theta$$

The horizontal projections of $AE, (x_{AE}),$ and $AG, (x_{AG}),$ are equal.

$$x_{AE} = x_{AG} = x_A$$

or $$(AE)\cos 2\theta = (AG)\cos\theta$$

and $$(2R)\cos 2\theta = R\cos\theta$$

Now $$\cos 2\theta = 2\cos^2\theta - 1$$

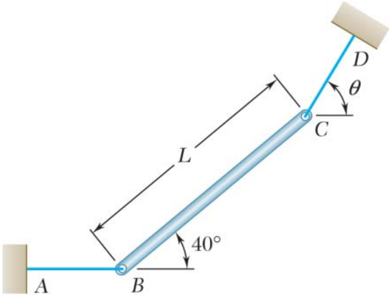
then $$4\cos^2\theta - 2 = \cos\theta$$

or $$4\cos^2\theta - \cos\theta - 2 = 0$$

Applying the quadratic equation,

$$\cos\theta = 0.84307 \quad \text{and} \quad \cos\theta = -0.59307$$

$$\theta = 32.534° \quad \text{and} \quad \theta = 126.375° \quad \text{(Discard)} \qquad \text{or} \quad \theta = 32.5° \blacktriangleleft$$



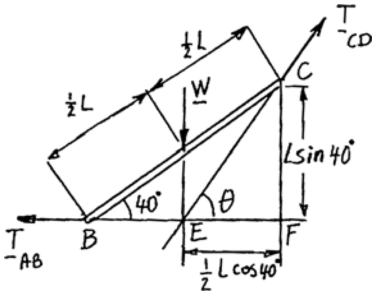
PROBLEM 4.87

A slender rod *BC* of length *L* and weight *W* is held by two cables as shown. Knowing that cable *AB* is horizontal and that the rod forms an angle of 40° with the horizontal, determine (*a*) the angle *θ* that cable *CD* forms with the horizontal, (*b*) the tension in each cable.

SOLUTION

Free-Body Diagram:

(Three-force body)



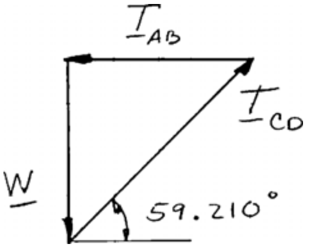
(*a*) The line of action of $\mathbf{T}_{CD}$ must pass through *E*, where $\mathbf{T}_{AB}$ and **W** intersect.

$$\begin{aligned}\tan\theta &= \frac{CF}{EF}\\ &= \frac{L\sin 40^\circ}{\frac{1}{2}L\cos 40^\circ}\\ &= 2\tan 40^\circ\\ &= 59.210^\circ\end{aligned}$$

$\theta = 59.2^\circ$ ◀

(b) Force Triangle

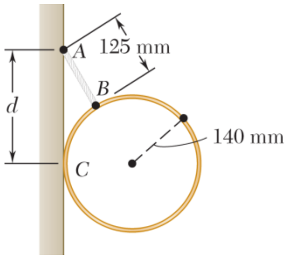
$$\begin{aligned}T_{AB} &= W\tan 30.790^\circ\\ &= 0.59588W\end{aligned}$$



$T_{AB} = 0.596W$ ◀

$$\begin{aligned}T_{CD} &= \frac{W}{\cos 30.790^\circ}\\ &= 1.16408W\end{aligned}$$

$T_{CD} = 1.164W$ ◀



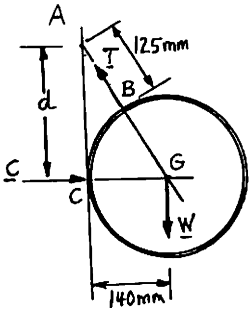
PROBLEM 4.88

A thin ring of mass 2 kg and radius $r = 140$ mm is held against a frictionless wall by a 125-mm string *AB*. Determine (*a*) the distance *d*, (*b*) the tension in the string, (*c*) the reaction at *C*.

SOLUTION

Free-Body Diagram:

(Three-force body)



The force *T* exerted at *B* must pass through the center *G* of the ring, since **C** and **W** intersect at that point.

Thus, points *A*, *B*, and *G* are in a straight line.

(*a*) From triangle *ACG*:

$$d = \sqrt{(AG)^2 - (CG)^2}$$

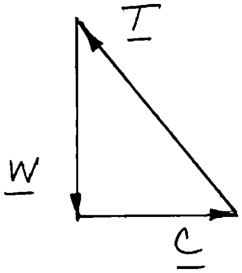
$$= \sqrt{(265 \text{ mm})^2 - (140 \text{ mm})^2}$$

$$= 225.00 \text{ mm}$$

$d = 225$ mm ◀

Force Triangle

$$W = (2 \text{ kg})(9.81 \text{ m/s}^2) = 19.6200 \text{ N}$$

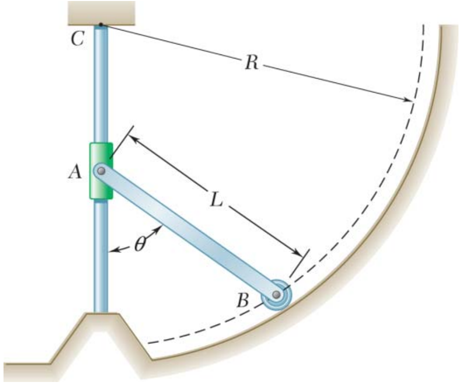


Law of sines:

$$\frac{T}{265 \text{ mm}} = \frac{C}{140 \text{ mm}} = \frac{19.6200 \text{ N}}{225.00 \text{ mm}}$$

(*b*) $T = 23.1$ N ◀

(*c*) $\mathbf{C} = 12.21 \text{ N} \longrightarrow$ ◀

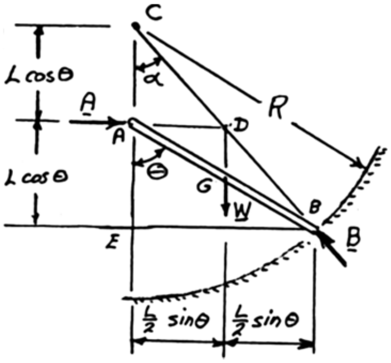


PROBLEM 4.89

A slender rod of length L and weight W is attached to a collar at A and is fitted with a small wheel at B. Knowing that the wheel rolls freely along a cylindrical surface of radius R, and neglecting friction, derive an equation in θ, L, and R that must be satisfied when the rod is in equilibrium.

SOLUTION

Free-Body Diagram (Three-force body)



Reaction **B** must pass through D where **B** and **W** intersect.

Note that ΔABC and ΔBGD are similar.

$$AC = AE = L\cos\theta$$

In ΔABC:

$$(CE)^2 + (BE)^2 = (BC)^2$$

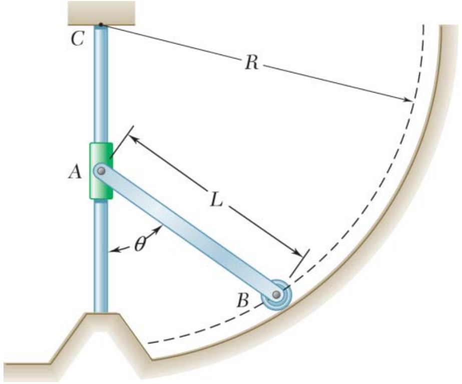
$$(2L\cos\theta)^2 + (L\sin\theta)^2 = R^2$$

$$\left(\frac{R}{L}\right)^2 = 4\cos^2\theta + \sin^2\theta$$

$$\left(\frac{R}{L}\right)^2 = 4\cos^2\theta + 1 - \cos^2\theta$$

$$\left(\frac{R}{L}\right)^2 = 3\cos^2\theta + 1$$

$$\cos^2\theta = \frac{1}{3}\left[\left(\frac{R}{L}\right)^2 - 1\right] \blacktriangleleft$$



PROBLEM 4.90

Knowing that for the rod of Problem 4.89, L = 15 in., R = 20 in., and W = 10 lb, determine (*a*) the angle θ corresponding to equilibrium, (*b*) the reactions at A and B.

SOLUTION

See the solution to Problem 4.86 for the free-body diagram and analysis leading to the following equation:

$$\cos^2\theta = \frac{1}{3}\left[\left(\frac{R}{L}\right)^2 - 1\right]$$

For $L = 15$ in., $R = 20$ in., and $W = 10$ lb,

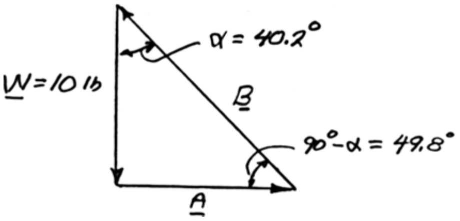
(*a*) $$\cos^2\theta = \frac{1}{3}\left[\left(\frac{20\text{ in.}}{15\text{ in.}}\right)^2 - 1\right]; \quad \theta = 59.39^\circ \qquad \theta = 59.4^\circ \blacktriangleleft$$

In ΔABC: $$\tan\alpha = \frac{BE}{CE} = \frac{L\sin\theta}{2L\cos\theta} = \frac{1}{2}\tan\theta$$

$$\tan\alpha = \frac{1}{2}\tan 59.39^\circ = 0.8452$$

$$\alpha = 40.2^\circ$$

Force Triangle

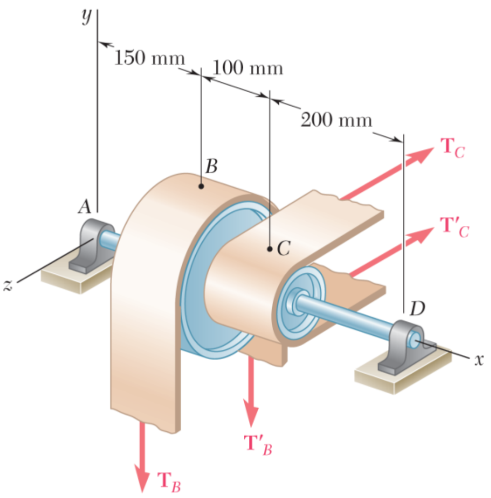


$$A = W\tan\alpha = (10\text{ lb})\tan 40.2^\circ = 8.45\text{ lb}$$

$$B = \frac{W}{\cos\alpha} = \frac{(10\text{ lb})}{\cos 40.2^\circ} = 13.09\text{ lb}$$

(*b*) $\mathbf{A} = 8.45$ lb → ◀

$\mathbf{B} = 13.09$ lb ↖ 49.8° ◀

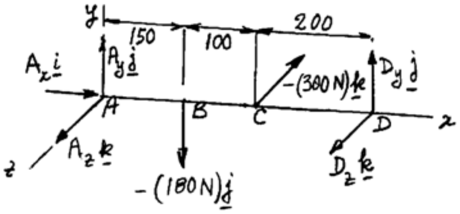

y
150 mm
100 mm
200 mm
B
A
C
D
z
x
$\mathbf{T}_C$
$\mathbf{T}'_C$
$\mathbf{T}'_B$
$\mathbf{T}_B$

PROBLEM 4.91

Two transmission belts pass over a double-sheaved pulley that is attached to an axle supported by bearings at *A* and *D*. The radius of the inner sheave is 125 mm and the radius of the outer sheave is 250 mm. Knowing that when the system is at rest, the tension is 90 N in both portions of belt *B* and 150 N in both portions of belt *C*, determine the reactions at *A* and *D*. Assume that the bearing at *D* does not exert any axial thrust.

SOLUTION

We replace $\mathbf{T}_B$ and $\mathbf{T}'_B$ by their resultant $(-180\text{ N})\mathbf{j}$ and $\mathbf{T}_C$ and $\mathbf{T}'_C$ by their resultant $(-300\text{ N})\mathbf{k}$.


y
150
100
200
$A_x\mathbf{i}$
$A_y\mathbf{j}$
$-(300\text{ N})\mathbf{k}$
$D_y\mathbf{j}$
A
B
C
D
x
z
$A_z\mathbf{k}$
$D_z\mathbf{k}$
$-(180\text{ N})\mathbf{j}$

Dimensions in mm

We have five unknowns and six equations of equilibrium. Axle *AD* is free to rotate about the *x*-axis, but equilibrium is maintained $(\Sigma M_x = 0)$.

$$\Sigma \mathbf{M}_A = 0: \quad (150\mathbf{i}) \times (-180\mathbf{j}) + (250\mathbf{i}) \times (-300\mathbf{k}) + (450\mathbf{i}) \times (D_y\mathbf{j} + D_z\mathbf{k}) = 0$$

$$-27 \times 10^3\mathbf{k} + 75 \times 10^3\mathbf{j} + 450D_y\mathbf{k} - 450D_z\mathbf{j} = 0$$

Equating coefficients of **j** and **k** to zero,

$$\mathbf{j}: \quad 75 \times 10^3 - 450D_z = 0 \qquad D_z = 166.7\text{ N}$$

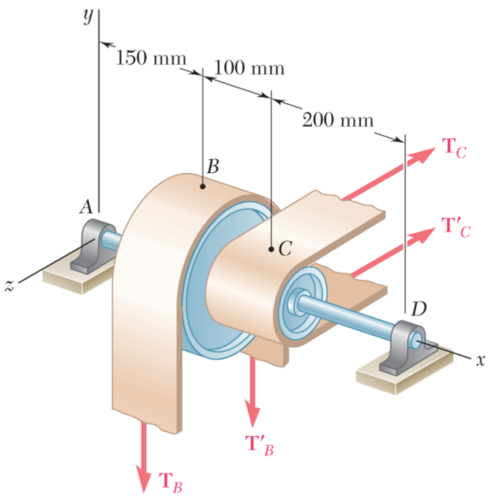
$$\mathbf{k}: \quad -27 \times 10^3 + 450D_y = 0 \qquad D_y = 60.0\text{ N}$$

$$\Sigma F_x = 0: \quad A_x = 0$$

$$\Sigma F_y = 0: \quad A_y + D_y - 180\text{ N} = 0 \qquad A_y = 180 - 60 = 120.0\text{ N}$$

$$\Sigma F_z = 0: \quad A_z + D_z - 300\text{ N} = 0 \qquad A_z = 300 - 166.7 = 133.3\text{ N}$$

$$\mathbf{A} = (120.0\text{ N})\mathbf{j} + (133.3\text{ N})\mathbf{k}; \quad \mathbf{D} = (60.0\text{ N})\mathbf{j} + (166.7\text{ N})\mathbf{k} \blacktriangleleft$$

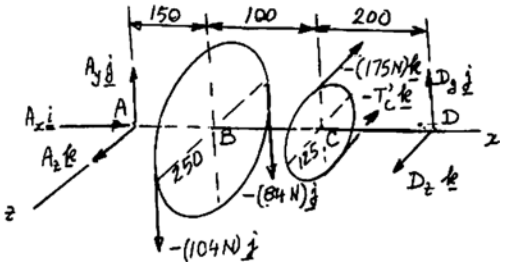

y
150 mm
100 mm
200 mm
$\mathbf{T}_C$
B
A
$\mathbf{T}'_C$
C
z
D
x
$\mathbf{T}'_B$
$\mathbf{T}_B$

PROBLEM 4.92

Solve Problem 4.91, assuming that the pulley rotates at a constant rate and that $T_B = 104$ N, $T'_B = 84$ N, $T_C = 175$ N.

PROBLEM 4.91 Two transmission belts pass over a double-sheaved pulley that is attached to an axle supported by bearings at *A* and *D*. The radius of the inner sheave is 125 mm and the radius of the outer sheave is 250 mm. Knowing that when the system is at rest, the tension is 90 N in both portions of belt *B* and 150 N in both portions of belt *C*, determine the reactions at *A* and *D*. Assume that the bearing at *D* does not exert any axial thrust.

SOLUTION


150
100
200
$A_y\mathbf{j}$
$A_x\mathbf{i}$
A
$A_z\mathbf{k}$
z
250
B
$-(84\text{ N})\mathbf{j}$
$-(104\text{ N})\mathbf{j}$
$-(175\text{ N})\mathbf{k}$
$-T'_C\mathbf{k}$
125
C
$D_y\mathbf{j}$
D
x
$D_z\mathbf{k}$

Dimensions in mm

We have six unknowns and six equations of equilibrium. —OK

$$\Sigma\mathbf{M}_A = 0: \quad (150\mathbf{i} + 250\mathbf{k}) \times (-104\mathbf{j}) + (150\mathbf{i} - 250\mathbf{k}) \times (-84\mathbf{j})$$
$$+ (250\mathbf{i} + 125\mathbf{j}) \times (-175\mathbf{k}) + (250\mathbf{i} - 125\mathbf{j}) \times (-T_C)$$
$$+ 450\mathbf{i} \times (D_y\mathbf{j} + D_z\mathbf{k}) = 0$$

$$-150(104 + 84)\mathbf{k} + 250(104 - 84)\mathbf{i} + 250(175 + T'_C)\,\mathbf{j} \;- 125(175 - T'_C)$$
$$+ 450D_y\mathbf{k} - 450D_z\mathbf{j} = 0$$

Equating the coefficients of the unit vectors to zero,

$$\mathbf{i}: \quad 250(104 - 84) - 125(175 - T'_C) = 0 \qquad 175 = T'_C = 40 \qquad T'_C = 135;$$

$$\mathbf{j}: \quad 250(175 + 135) - 450D_z = 0 \qquad D_z = 172.2 \text{ N}$$

$$\mathbf{k}: \quad -150(104 + 84) + 450\,D_y = 0 \qquad D_y = 62.7 \text{ N}$$

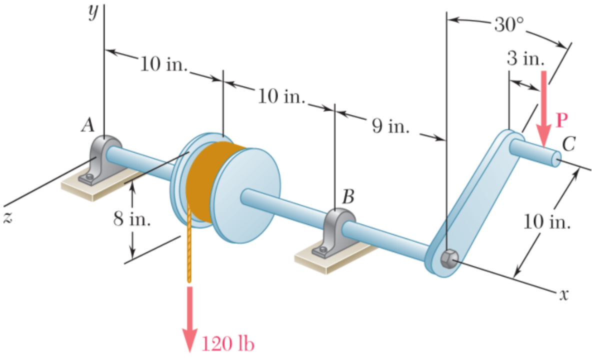
PROBLEM 4.92 (Continued)

$\Sigma F_x = 0$: $A_x = 0$

$\Sigma F_y = 0$: $A_y - 104 - 84 + 62.7 = 0$ $A_y = 125.3\text{ N}$

$\Sigma F_z = 0$: $A_z - 175 - 135 + 172.2 = 0$ $A_z = 137.8\text{ N}$

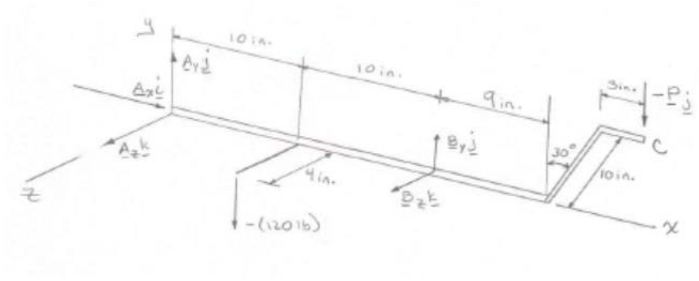
$\mathbf{A} = (125.3\text{ N})\mathbf{j} + (137.8\text{ N})\mathbf{k};\quad \mathbf{D} = (62.7\text{ N})\mathbf{j} + (172.2\text{ N})\mathbf{k}$ ◀


y
10 in.
10 in.
9 in.
30°
3 in.
P
C
A
B
z
8 in.
10 in.
x
120 lb

PROBLEM 4.93

A small winch is used to raise a 120-lb load. Find (*a*) the magnitude of the vertical force **P** that should be applied at *C* to maintain equilibrium in the position shown, (*b*) the reactions at *A* and *B*, assuming that the bearing at *B* does not exert any axial thrust.

SOLUTION


y
10 in.
10 in.
9 in.
3 in.
$-P\mathbf{j}$
$A_y\mathbf{j}$
$A_x\mathbf{i}$
$A_z\mathbf{k}$
$B_y\mathbf{j}$
$B_z\mathbf{k}$
30°
C
10 in.
4 in.
$-(120\text{ lb})$
z
x

Dimensions in in.

We have six unknowns and six equations of equilibrium.

$$\mathbf{r}_C = (32\text{ in.})\mathbf{i} + (10\text{ in.})\cos 30^\circ\,\mathbf{j} - (10\text{ in.})\sin 30^\circ\,\mathbf{k}$$
$$= 32\mathbf{i} + 8.6603\mathbf{j} - 5\mathbf{k}$$

$$\Sigma\mathbf{M}_A = 0:\quad (10\mathbf{i}+4\mathbf{k})\times(-120\mathbf{j}) + (20\mathbf{i})\times(B_y\mathbf{j}+B_z\mathbf{k}) + (32\mathbf{i}+8.6603\mathbf{j}-5\mathbf{k})\times(-P\mathbf{j}) = 0$$
$$-1200\mathbf{k} + 480\mathbf{i} + 20B_y\mathbf{k} - 20B_z\mathbf{j} - 32P\mathbf{k} - 5P\mathbf{i} = 0$$

Equating the coefficients of the unit vectors to zero,

$\mathbf{i}:\quad 480 - 5P) = 0 \qquad P = 96.0\text{ lb};$ $\qquad$ (*a*) $P = 96.0\text{ lb}$ ◀

$\mathbf{j}:\quad 20B_z = 0 \qquad B_z = 0$

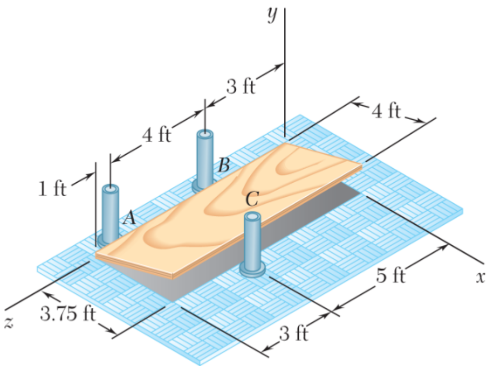
$\mathbf{k}:\quad -1200 + 20B_y - 32(96.0) = 0 \qquad B_y = 213.6\text{ lb}$

$\Sigma F_x = 0: \qquad A_x = 0$

$\Sigma F_y = 0: \qquad A_y - 120 + 213.6 - 96.0 = 0 \qquad A_y = 2.40\text{ lb}$

$\Sigma F_z = 0: \qquad A_z + B_z = 0 \qquad A_z = -B_z = 0$

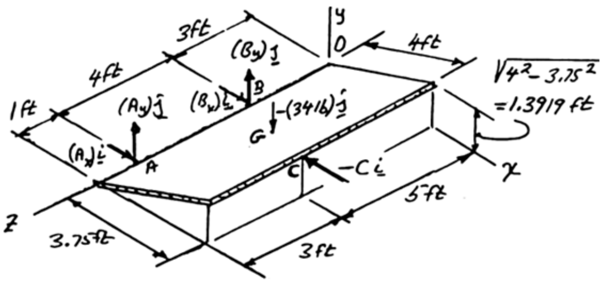
$\mathbf{A} = (2.40\text{ lb})\mathbf{j};\quad \mathbf{B} = (214\text{ lb})\mathbf{j}$ ◀


y
3 ft
4 ft
4 ft
1 ft
B
C
A
5 ft
x
z
3.75 ft
3 ft

PROBLEM 4.94

A 4×8-ft sheet of plywood weighing 34 lb has been temporarily placed among three pipe supports. The lower edge of the sheet rests on small collars at *A* and *B* and its upper edge leans against pipe *C*. Neglecting friction at all surfaces, determine the reactions at *A*, *B*, and *C*.

SOLUTION


3ft
$(B_y)\mathbf{j}$
y
O
4ft
$\sqrt{4^2-3.75^2}$
$=1.3919$ ft
4ft
1ft
$(A_y)\mathbf{j}$
$(B_x)\mathbf{i}$
B
$-(34\text{ lb})\mathbf{j}$
G
$(A_x)\mathbf{i}$
A
C
$-C\mathbf{i}$
x
5ft
z
3.75ft
3ft

$$\mathbf{r}_{G/B} = \frac{3.75}{2}\mathbf{i} + \frac{1.3919}{2}\mathbf{j} + \mathbf{k}$$

We have 5 unknowns and 6 Eqs. of equilibrium.

Plywood sheet is free to move in z direction, but equilibrium is maintained $(\Sigma F_z = 0)$.

$$\Sigma M_B = 0: \quad r_{A/B} \times (A_x\mathbf{i} + A_y\mathbf{j}) + r_{C/B} \times (-C\mathbf{i}) + r_{G/B} \times (-w\mathbf{j}) = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4 \\ A_x & A_y & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3.75 & 1.3919 & 2 \\ -C & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.875 & 0.696 & 1 \\ 0 & -34 & 0 \end{vmatrix} = 0$$

$$-4A_y\mathbf{i} + 4A_x\mathbf{j} - 2C\mathbf{j} + 1.3919C\mathbf{k} + 34\mathbf{i} - 63.75\mathbf{k} = 0$$

Equating coefficients of unit vectors to zero:

$$\mathbf{i}: \quad -4A_y + 34 = 0 \qquad A_y = 8.5\text{ lb}$$

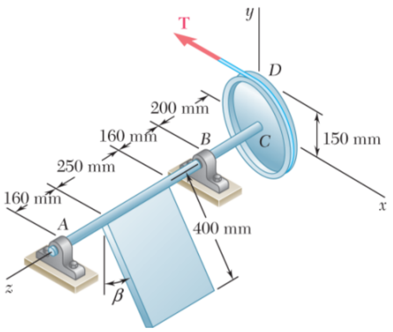
$$\mathbf{j}: \quad -2C + 4A_x = 0 \qquad A_x = \frac{1}{2}C = \frac{1}{2}(45.80) = 22.9\text{ lb}$$

$$\mathbf{k}: \quad 1.3919C - 63.75 = 0 \qquad C = 45.80\text{ lb} \qquad C = 45.8\text{ lb}$$

$$\Sigma F_x = 0: \quad A_x + B_x - C = 0: \qquad B_x = 45.8 - 22.9 = 22.9\text{ lb}$$

$$\Sigma F_y = 0: \quad A_y + B_y - W = 0: \qquad B_y = 34.0 - 8.50 = 25.5\text{ lb}$$

$$\mathbf{A} = (22.9\text{ lb})\mathbf{i} + (8.50\text{ lb})\mathbf{j} \quad \mathbf{B} = (22.9\text{ lb})\mathbf{i} + (25.5\text{ lb})\mathbf{j} \quad \mathbf{C} = -(45.8\text{ lb})\mathbf{i} \blacktriangleleft$$

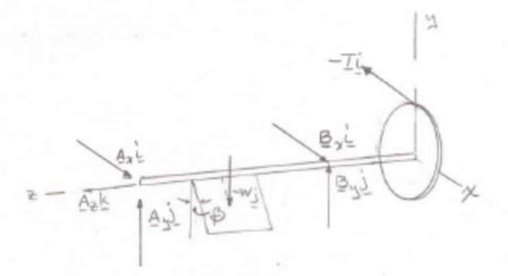


PROBLEM 4.95

A 250 × 400-mm plate of mass 12 kg and a 300-mm-diameter pulley are welded to axle *AC* that is supported by bearings at *A* and *B*. For $\beta = 30°$, determine (*a*) the tension in the cable, (*b*) the reactions at *A* and *B*. Assume that the bearing at *B* does not exert any axial thrust.

SOLUTION

Free-Body Diagram:



$$W = -mg = -(12\text{ kg})(9.81\text{ m/s}^2) = -117.720\mathbf{j}\text{ N}$$

$$\mathbf{r}_{B/A} = -570\mathbf{k};\quad \mathbf{r}_{D/\text{A}} = 150\mathbf{j} - 770\mathbf{k};$$

$$\mathbf{r}_{G/A} = (200\sin\beta)\mathbf{i} - (200\cos\beta)\mathbf{j} - 285\mathbf{k}$$

$$\Sigma M_A = 0:\quad \mathbf{r}_{B/A} \times (B_x\mathbf{i} + B_y\mathbf{j}) + \mathbf{r}_{\text{D/A}} \times -T\,\mathbf{i} + \mathbf{r}_{G/A} \times (-W)\mathbf{j} = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -570 \\ B_x & B_y & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 150 & -770 \\ -T & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 200\sin\beta & -200\cos\beta & -285 \\ 0 & -W & 0 \end{vmatrix} = 0$$

$$570B_y\,\mathbf{i} - 570B_x\mathbf{j} + 770T\mathbf{j} + 150T\mathbf{k} - 285W\mathbf{i} - (200\sin\beta)W\mathbf{k} = 0$$

Equating the coefficients of the unit vectors to zero,

$\mathbf{i}$: $\quad 570B_y - 285(117.72) = 0 \quad B_y = 58.860\text{ N}$

$\mathbf{j}$: $\quad -570B_x - 770T = 0 \quad B_x = \dfrac{77}{57}T;\quad B_x = 106.017\text{ N}$

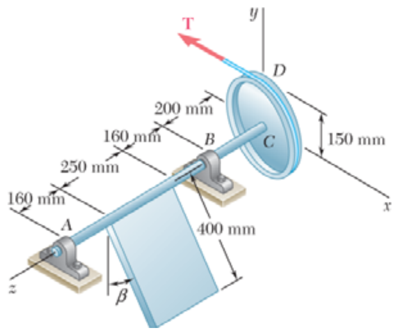
$\mathbf{k}$: $\quad 150T - (200\sin 30^\circ)117.720 = 0 \quad T = 78.480\text{ N} \qquad (a)\ T = 78.5\text{ N}$ ◀

$\Sigma F_y = 0$: $\quad A_y + B_y - W = 0 \quad A_y = 117.720 - 58.860 = 58.860\text{ N}$

$\Sigma F_z = 0$: $\quad A_z = 0$

$\Sigma F_x = 0$: $\quad A_x + B_x - T = 0;\quad A_x = -27.537\text{ N}$

(*b*) $\quad \mathbf{A} = -(27.5\text{ N})\mathbf{i} + (58.9\text{ N})\mathbf{j};\quad \mathbf{B} = (106.0\text{ N})\mathbf{i} + (58.9\text{ N})\mathbf{j}$ ◀


T
y
D
200 mm
160 mm
B
C
150 mm
250 mm
160 mm
A
400 mm
x
z
β

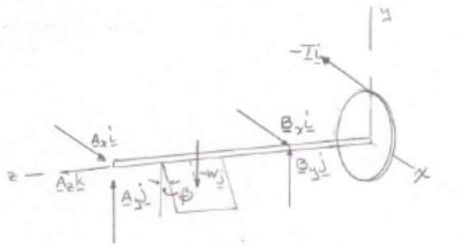
PROBLEM 4.96

Solve Prob. 4.95 for $\beta = 60°$.

PROBLEM 4.95 A 250 × 400-mm plate of mass 12 kg and a 300-mm-diameter pulley are welded to axle *AC* that is supported by bearings at *A* and *B*. For $\beta = 30°$, determine (*a*) the tension in the cable, (*b*) the reactions at *A* and *B*. Assume that the bearing at *B* does not exert any axial thrust.

SOLUTION

Free-Body Diagram:


y
$-T\underline{i}$
$B_x\underline{i}$
$A_x\underline{i}$
$B_y\underline{j}$
$-W\underline{j}$
z
$A_z\underline{k}$
$A_y\underline{j}$
β
x

$$W = -mg = -(12 \text{ kg})(9.81 \text{ m/s}^2) = -117.720\mathbf{j} \text{ N}$$

$$\mathbf{r}_{B/A} = -570\mathbf{k}; \quad \mathbf{r}_{D/\text{A}} = 150\mathbf{j} - 770\mathbf{k};$$

$$\mathbf{r}_{G/A} = (200\sin\beta)\mathbf{i} - (200\cos\beta)\mathbf{j} - 285\mathbf{k}$$

$$\Sigma M_A = 0: \quad \mathbf{r}_{B/A} \times (B_x\mathbf{i} + B_y\mathbf{j}) + \mathbf{r}_{\text{D}/A} \times -T\,\mathbf{i} + \mathbf{r}_{G/A} \times (-W)\mathbf{j} = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -570 \\ B_x & B_y & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 150 & -770 \\ -T & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 200\sin\beta & -200\cos\beta & -285 \\ 0 & -W & 0 \end{vmatrix} = 0$$

$$570B_y\,\mathbf{i} - 570B_x\mathbf{j} + 770T\mathbf{j} + 150T\mathbf{k} - 285W\mathbf{i} - (200\sin\beta)W\mathbf{k} = 0$$

$$\mathbf{i}: \qquad 570B_y - 285(117.72) = 0 \qquad B_y = 58.860 \text{ N}$$

$$\mathbf{j}: \qquad -570B_x - 770T = 0 \qquad B_x = \frac{77}{57}T; \quad B_x = 183.625 \text{ N}$$

$$\mathbf{k}: \qquad 150T - (200\sin 60^\circ)117.720 = 0 \qquad T = 135.931 \text{ N}$$

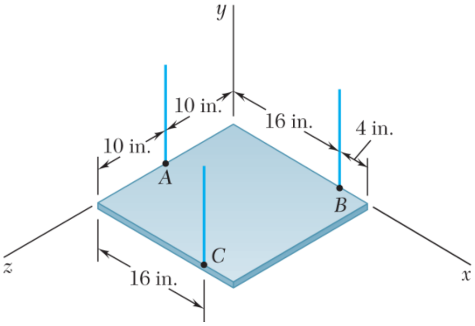
(*a*) $T = 135.9 \text{ N}$ ◀

$$\Sigma F_y = 0: \qquad A_y + B_y - W = 0 \qquad A_y = 117.720 - 58.860 = 58.860 \text{ N}$$

$$\Sigma F_z = 0: \qquad A_z = 0$$

$$\Sigma F_x = 0: \qquad A_x + B_x - T = 0; \qquad A_x = -47.694 \text{ N}$$

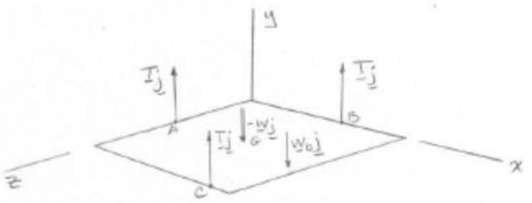
(*b*) $\mathbf{A} = -(47.7 \text{ N})\mathbf{i} + (58.9 \text{ N})\mathbf{j}; \quad \mathbf{B} = (183.6 \text{ N})\mathbf{i} + (58.9 \text{ N})\mathbf{j}$ ◀



PROBLEM 4.97

The 20 × 20-in. square plate shown weighs 56 lb and is supported by three vertical wires. Determine the tension in each wire.

SOLUTION



$W_b\mathbf{j} = 0$ and $W = -(56.0 \text{ lb})\mathbf{j}$

$$\Sigma M_A = 0: \quad \mathbf{r}_{B/A} \times T_B\mathbf{j} + \mathbf{r}_{C/A} \times T_C\mathbf{j} + \mathbf{r}_{G/A} \times (-W\mathbf{j}) = 0$$

$$(16\mathbf{i} - 10\mathbf{k}) \times T_B\mathbf{j} + (16\mathbf{i} + 10\mathbf{k}) \times \mathrm{T}_C\,\mathbf{j} + 10\mathbf{i} \times (-W\mathbf{j}) = 0$$

$$16T_B\mathbf{k} + 10T_B\mathbf{i} + 16T_C\mathbf{k} - 10T_C\mathbf{i} - 10W\mathbf{k} = 0$$

Equate coefficients of unit vectors to zero:

$$\mathbf{i}: \quad 10T_B - 10T_C = 0; \quad T_B = T_C$$

$$\mathbf{k}: \quad 16T_B + 16T_C - 10W = 0$$

$$16T_B + 16T_B - 10(56.0 \text{ lb}) = 0$$

$$T_B = T_C = 17.50 \text{ lb}$$

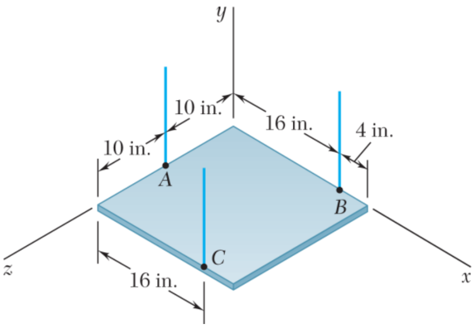
$$\Sigma F_y = 0: \quad T_A + T_B + T_C - W = 0$$

$$T_A + 17.50 + 17.50 - 56.0 = 0$$

$$T_A = 21.0 \text{ lb}$$

$T_A = 21.0$ lb ◀

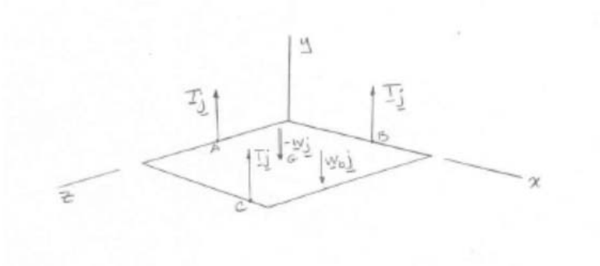
$T_B = T_C = 17.50$ lb ◀



PROBLEM 4.98

The 20 × 20-in. square plate shown weighs 56 lb and is supported by three vertical wires. Determine the weight and location of the lightest block that should be placed on the plate if the tensions in the three wires are to be equal.

SOLUTION



Let $-W_b\mathbf{j}$ be the weight of block; x and z its coordinates.

Since the tensions in the wires are equal, let $T_A = T_B = T_C = T$

$$\Sigma M_O = 0: \quad \mathbf{r}_A \times T\mathbf{j} + \mathbf{r}_B \times T\mathbf{j} + \mathbf{r}_C \times T\mathbf{j} + \mathbf{r}_G \times (-W\mathbf{j}) + (x\mathbf{i} + z\mathbf{k}) \times (-W_b\mathbf{j}) = 0$$

$$(10\mathbf{k}) \times T\mathbf{j} + (16\mathbf{i}) \times T\mathbf{j} + (16\mathbf{i} + 20\mathbf{k}) \times T\mathbf{j} \times (10\mathbf{i} + 10\mathbf{k}) \times (-W\mathbf{j}) + (x\mathbf{i} + z\mathbf{k}) \times (-W_b\mathbf{j}) = 0$$

$$-10T\mathbf{i} + 16T\mathbf{k} + 16T\mathbf{k} - 20T\mathbf{i} - 10W\mathbf{k} + 10W\mathbf{i} - W_b x\mathbf{k} + W_b z\mathbf{i} = 0$$

Equate coefficients of unit vectors to zero:

$$\mathbf{i}: \quad -30T + 10W + W_b z = 0; \quad (1)$$

$$\mathbf{k}: \quad 32T - 10W - W_b x = 0 \quad (2)$$

Also,

$$\Sigma F_y = 0: \quad 3T - W - W_b = 0 \quad (3)$$

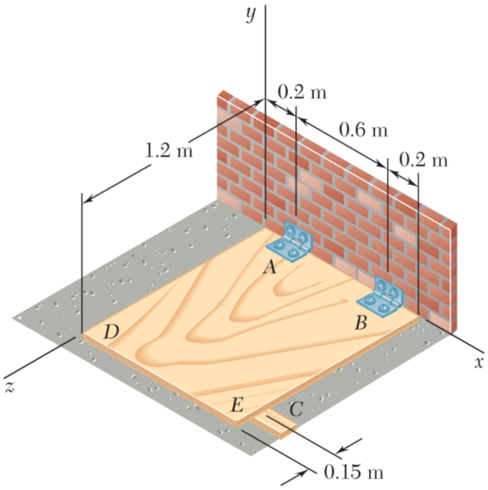
$$(1)+10(3): \quad (z-10)W_b = 0 \quad \text{since } x \text{ and } z \text{ must be} \le 20 \text{ in.} \;\; z = 10 \text{ in.}$$

$$3(2) - 32(3): \quad 2W + (-3x + 32)W_b = 0$$

$$\frac{W_b}{W} = \frac{2}{3x - 32}; \quad \text{since } x \le 20 \text{ in.} \quad \frac{W_b}{W} \ge \frac{2}{3(20) - 32}$$

Finally, with $W = 56$ lb; $W_b = (56 \text{ lb})\dfrac{1}{14} = 4.00$ lb

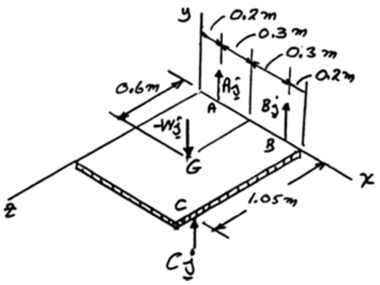
$W_b = 4.00$ lb at $x = 20.0$ in., $z = 10.00$ in. ◀



PROBLEM 4.99

An opening in a floor is covered by a $1 \times 1.2\text{-m}$ sheet of plywood of mass 18 kg. The sheet is hinged at *A* and *B* and is maintained in a position slightly above the floor by a small block *C*. Determine the vertical component of the reaction (*a*) at *A*, (*b*) at *B*, (*c*) at *C*.

SOLUTION



$$\mathbf{r}_{B/A} = 0.6\mathbf{i}$$

$$\mathbf{r}_{C/A} = 0.8\mathbf{i} + 1.05\mathbf{k}$$

$$\mathbf{r}_{G/A} = 0.3\mathbf{i} + 0.6\mathbf{k}$$

$$W = mg = (18\text{ kg})9.81$$

$$W = 176.58\text{ N}$$

$$\Sigma M_A = 0: \quad \mathbf{r}_{B/A} \times B\mathbf{j} + \mathbf{r}_{C/A} \times C\mathbf{j} + \mathbf{r}_{G/A} \times (-W\mathbf{j}) = 0$$

$$(0.6\mathbf{i}) \times B\mathbf{j} + (0.8\mathbf{i} + 1.05\mathbf{k}) \times C\mathbf{j} + (0.3\mathbf{i} + 0.6\mathbf{k}) \times (-W\mathbf{j}) = 0$$

$$0.6B\mathbf{k} + 0.8C\mathbf{k} - 1.05C\mathbf{i} - 0.3W\mathbf{k} + 0.6W\mathbf{i} = 0$$

Equate coefficients of unit vectors to zero:

$$\mathbf{i}: \quad 1.05C + 0.6W = 0 \quad C = \left(\frac{0.6}{1.05}\right)176.58\text{ N} = 100.90\text{ N}$$

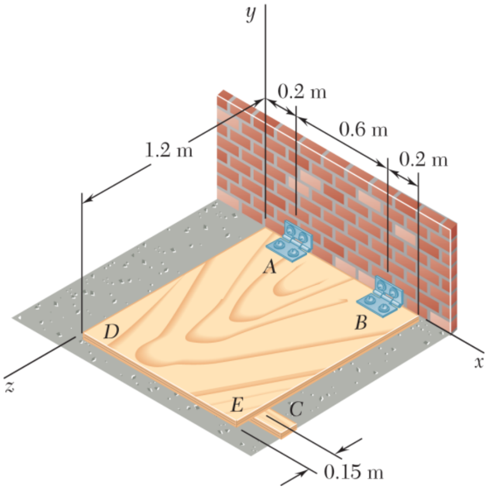
$$\mathbf{k}: \quad 0.6B + 0.8C - 0.3W = 0$$

$$0.6B + 0.8(100.90\text{ N}) - 0.3(176.58\text{ N}) = 0 \quad B = -46.24\text{ N}$$

$$\Sigma F_y = 0: \quad A + B + C - W = 0$$

$$A - 46.24\text{ N} + 100.90\text{ N} + 176.58\text{ N} = 0 \quad A = 121.92\text{ N}$$

$$(a) \quad A = 121.9\text{ N} \quad (b) \quad B = -46.2\text{ N} \quad (c) \quad C = 100.9\text{ N} \blacktriangleleft$$

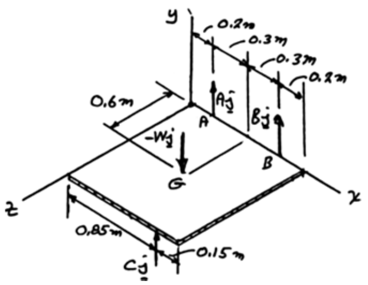


PROBLEM 4.100

Solve Problem 4.99, assuming that the small block *C* is moved and placed under edge *DE* at a point 0.15 m from corner *E*.

PROBLEM 4.99 An opening in a floor is covered by a 1×1.2-m sheet of plywood of mass 18 kg. The sheet is hinged at *A* and *B* and is maintained in a position slightly above the floor by a small block *C*. Determine the vertical component of the reaction (*a*) at *A*, (*b*) at *B*, (*c*) at *C*.

SOLUTION



$$\mathbf{r}_{B/A} = 0.6\mathbf{i}$$
$$\mathbf{r}_{C/A} = 0.65\mathbf{i} + 1.2\mathbf{k}$$
$$\mathbf{r}_{G/A} = 0.3\mathbf{i} + 0.6\mathbf{k}$$

$$W = mg = (18\text{ kg})\,9.81\text{ m/s}^2$$
$$W = 176.58\text{ N}$$

$$\Sigma M_A = 0\colon\quad \mathbf{r}_{B/A} \times B\mathbf{j} + \mathbf{r}_{C/A} \times C\mathbf{j} + \mathbf{r}_{G/A} \times (-W\mathbf{j}) = 0$$

$$0.6\mathbf{i} \times B\mathbf{j} + (0.65\mathbf{i} + 1.2\mathbf{k}) \times C\mathbf{j} + (0.3\mathbf{i} + 0.6\mathbf{k}) \times (-W\mathbf{j}) = 0$$

$$0.6B\mathbf{k} + 0.65C\mathbf{k} - 1.2C\mathbf{i} - 0.3W\mathbf{k} + 0.6W\mathbf{i} = 0$$

Equate coefficients of unit vectors to zero:

$$\mathbf{i}\colon\quad -1.2C + 0.6W = 0 \qquad C = \left(\frac{0.6}{1.2}\right)176.58\text{ N} = 88.29\text{ N}$$

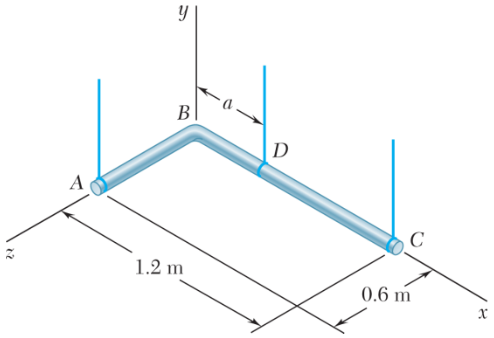
$$\mathbf{k}\colon\quad 0.6B + 0.65C - 0.3W = 0$$

$$0.6B + 0.65(88.29\text{ N}) - 0.3(176.58\text{ N}) = 0 \quad B = -7.36\text{ N}$$

$$\Sigma F_y = 0\colon\quad A + B + C - W = 0$$

$$A - 7.36\text{ N} + 88.29\text{ N} - 176.58\text{ N} = 0 \qquad A = 95.648\text{ N}$$

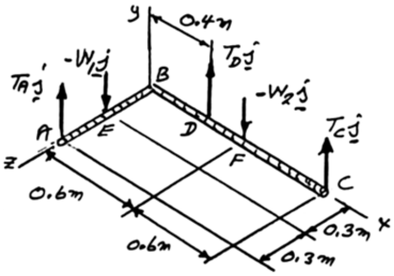
(*a*) $A = 95.6$ N (*b*) $B = -7.36$ N (*c*) $C = 88.3$ N ◀



PROBLEM 4.101

Two steel pipes *AB* and *BC*, each having a mass per unit length of 8 kg/m, are welded together at *B* and supported by three wires. Knowing that $a = 0.4$ m, determine the tension in each wire.

SOLUTION



$$W_1 = 0.6m'g$$
$$W_2 = 1.2m'g$$

$$\Sigma M_D = 0: \quad \mathbf{r}_{A/D} \times T_A\mathbf{j} + \mathbf{r}_{E/D} \times (-W_1\mathbf{j}) + \mathbf{r}_{F/D} \times (-W_2\mathbf{j}) + \mathbf{r}_{C/D} \times T_C\mathbf{j} = 0$$

$$(-0.4\mathbf{i} + 0.6\mathbf{k}) \times T_A\mathbf{j} + (-0.4\mathbf{i} + 0.3\mathbf{k}) \times (-W_1\mathbf{j}) + 0.2\mathbf{i} \times (-W_2\mathbf{j}) + 0.8\mathbf{i} \times T_C\mathbf{j} = 0$$

$$-0.4T_A\mathbf{k} - 0.6T_A\mathbf{i} + 0.4W_1\mathbf{k} + 0.3W_1\mathbf{i} - 0.2W_2\mathbf{k} + 0.8T_C\mathbf{k} = 0$$

Equate coefficients of unit vectors to zero:

$$\mathbf{i}: \quad -0.6T_A + 0.3W_1 = 0; \quad T_A = \frac{1}{2}W_1 = \frac{1}{2}0.6m'g = 0.3m'g$$

$$\mathbf{k}: \quad -0.4T_A + 0.4W_1 - 0.2W_2 + 0.8T_C = 0$$

$$-0.4(0.3m'g) + 0.4(0.6m'g) - 0.2(1.2m'g) + 0.8T_C = 0$$

$$T_C = \frac{(0.12 - 0.24 - 0.24)m'g}{0.8} = 0.15m'g$$

$$\Sigma F_y = 0: \quad T_A + T_C + T_D - W_1 - W_2 = 0$$

$$0.3m'g + 0.15m'g + T_D - 0.6m'g - 1.2m'g = 0$$

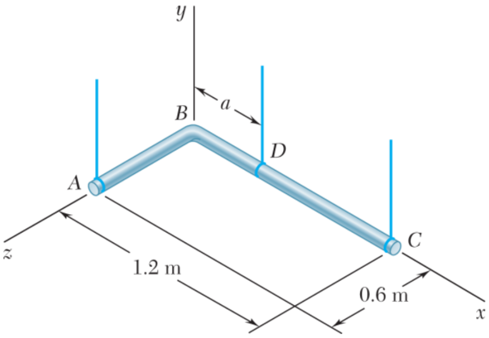
$$T_D = 1.35m'g$$

$$m'g = (8 \text{ kg/m})(9.81 \text{m/s}^2) = 78.480 \text{ N/m}$$

$T_A = 0.3m'g = 0.3 \times 78.480$ $\qquad$ $T_A = 23.5$ N ◀

$T_C = 0.15m'g = 0.15 \times 78.480$ $\qquad$ $T_C = 11.77$ N ◀

$T_D = 1.35m'g = 1.35 \times 78.480$ $\qquad$ $T_D = 105.9$ N ◀

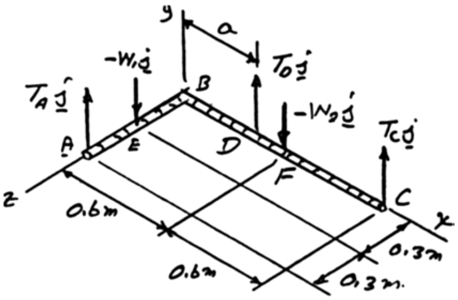


y
a
B
D
A
C
z
1.2 m
0.6 m
x

PROBLEM 4.102

For the pipe assembly of Problem 4.101, determine (*a*) the largest permissible value of *a* if the assembly is not to tip, (*b*) the corresponding tension in each wire.

SOLUTION



y
a
$-W_1\mathbf{j}$
$T_D\mathbf{j}$
$T_A\mathbf{j}$
B
$-W_2\mathbf{j}$
$T_C\mathbf{j}$
A
E
D
F
C
z
x
0.6m
0.6m
0.3m
0.3m

$$W_1 = 0.6m'g$$
$$W_2 = 1.2m'g$$

$$\Sigma M_D = 0: \quad \mathbf{r}_{A/D} \times T_A\mathbf{j} + \mathbf{r}_{E/D} \times (-W_1\mathbf{j}) + \mathbf{r}_{F/D} \times (-W_2\mathbf{j}) + \mathbf{r}_{C/D} \times T_C\mathbf{j} = 0$$

$$(-a\mathbf{i} + 0.6\mathbf{k}) \times T_A\mathbf{j} + (-a\mathbf{i} + 0.3\mathbf{k}) \times (-W_1\mathbf{j}) + (0.6 - a)\mathbf{i} \times (-W_2\mathbf{j}) + (1.2 - a)\mathbf{i} \times T_C\mathbf{j} = 0$$

$$-T_A a\mathbf{k} - 0.6T_A\mathbf{i} + W_1 a\mathbf{k} + 0.3W_1\mathbf{i} - W_2(0.6 - a)\mathbf{k} + T_C(1.2 - a)\mathbf{k} = 0$$

Equate coefficients of unit vectors to zero:

$$\mathbf{i}: \quad -0.6T_A + 0.3W_1 = 0; \quad T_A = \frac{1}{2}W_1 = \frac{1}{2}0.6m'g = 0.3m'g$$

$$\mathbf{k}: \quad -T_A a + W_1 a - W_2(0.6 - a) + T_C(1.2 - a) = 0$$

$$-0.3m'ga + 0.6m'ga - 1.2m'g(0.6 - a) + T_C(1.2 - a) = 0$$

$$T_C = \frac{0.3a - 0.6a + 1.2(0.6 - a)}{1.2 - a}$$

For maximum *a* and no tipping, $T_C = 0$.

(*a*)

$$-0.3a + 1.2(0.6 - a) = 0$$
$$-0.3a + 0.72 - 1.2a = 0$$
$$1.5a = 0.72$$

$a = 0.480$ m ◀

PROBLEM 4.102 (Continued)

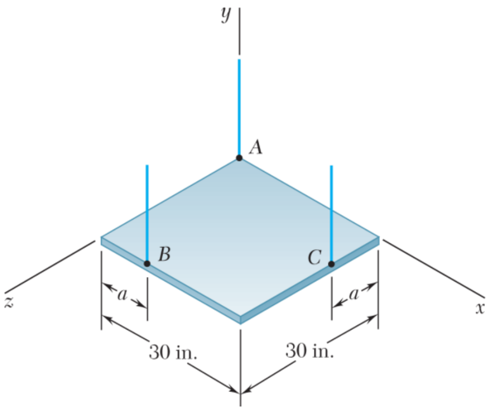
(*b*) Reactions: $m'g = (8 \text{ kg/m})\, 9.81 \text{ m/s}^2 = 78.48 \text{ N/m}$

$$T_A = 0.3m'g = 0.3 \times 78.48 = 23.544 \text{ N} \qquad T_A = 23.5 \text{ N} \blacktriangleleft$$

$$\Sigma F_y = 0: \quad T_A + T_C + T_D - W_1 - W_2 = 0$$

$$T_A + 0 + T_D - 0.6m'g - 1.2m'g = 0$$

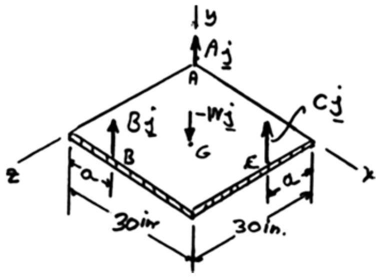
$$T_D = 1.8m'g - T_A = 1.8 \times 78.48 - 23.544 = 117.72 \qquad T_D = 117.7 \text{ N} \blacktriangleleft$$



PROBLEM 4.103

The 24-lb square plate shown is supported by three vertical wires. Determine (*a*) the tension in each wire when $a = 10$ in., (*b*) the value of a for which the tension in each wire is 8 lb.

SOLUTION



$$\mathbf{r}_{B/A} = a\mathbf{i} + 30\mathbf{k}$$
$$\mathbf{r}_{C/A} = 30\mathbf{i} + a\mathbf{k}$$
$$\mathbf{r}_{G/A} = 15\mathbf{i} + 15\mathbf{k}$$

By symmetry, $B = C$.

$$\Sigma M_A = 0: \quad \mathbf{r}_{B/A} \times B\mathbf{j} + \mathbf{r}_C \times C\mathbf{j} + \mathbf{r}_{G/A} \times (-W\mathbf{j}) = 0$$

$$(a\mathbf{i} + 30\mathbf{k}) \times B\mathbf{j} + (30\mathbf{i} + a\mathbf{k}) \times B\mathbf{j} + (15\mathbf{i} + 15\mathbf{k}) \times (-W\mathbf{j}) = 0$$

$$Ba\mathbf{k} - 30B\mathbf{i} + 30B\mathbf{k} - Ba\mathbf{i} - 15W\mathbf{k} + 15W\mathbf{i} = 0$$

Equate coefficient of unit vector **i** to zero:

$$\mathbf{i}: \quad -30B - Ba + 15W = 0$$

$$B = \frac{15W}{30 + a} \qquad C = B = \frac{15W}{30 + a} \tag{1}$$

$$\Sigma F_y = 0: \quad A + B + C - W = 0$$

$$A + 2\left[\frac{15W}{30 + a}\right] - W = 0; \quad A = \frac{aW}{30 + a} \tag{2}$$

(*a*) For $a = 10$ in.

From Eq. (1): $C = B = \dfrac{15(24\text{ lb})}{30 + 10} = 9.00\text{ lb}$

From Eq. (2): $A = \dfrac{10(24\text{ lb})}{30 + 10} = 6.00\text{ lb}$

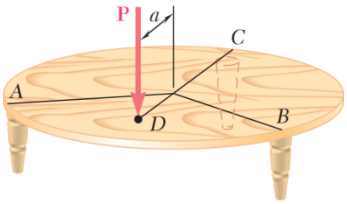
$A = 6.00\text{ lb}; \quad B = C = 9.00\text{ lb}$ ◀

PROBLEM 4.103 (Continued)

(*b*) For tension in each wire = 8 lb,

From Eq. (1): $8 \text{ lb} = \dfrac{15(24 \text{ lb})}{30 + a}$

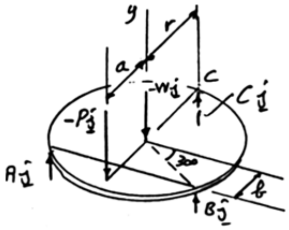
$30 \text{ in.} + a = 45$ $a = 15.00 \text{ in.}$ ◀



PROBLEM 4.104

The table shown weighs 30 lb and has a diameter of 4 ft. It is supported by three legs equally spaced around the edge. A vertical load **P** of magnitude 100 lb is applied to the top of the table at *D*. Determine the maximum value of *a* if the table is not to tip over. Show, on a sketch, the area of the table over which **P** can act without tipping the table.

SOLUTION



$$r = 2\text{ ft} \quad b = r\sin 30° = 1\text{ ft}$$

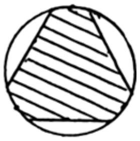
We shall sum moments about *AB*.

$$(b + r)C + (a - b)P - bW = 0$$

$$(1 + 2)C + (a - 1)100 - (1)30 = 0$$

$$C = \frac{1}{3}[30 - (a - 1)100]$$

If table is not to tip, $C \geq 0$.

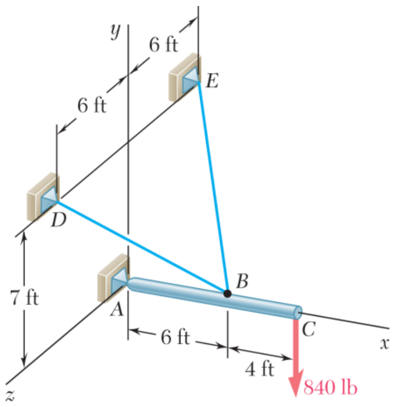


$$[30 - (a - 1)100] \geq 0$$

$$30 \geq (a - 1)100$$

$$a - 1 \leq 0.3 \quad a \leq 1.3\text{ ft} \qquad a = 1.300\text{ ft} ◀$$

Only ⊥ distance from *P* to *AB* matters. Same condition must be satisfied for each leg. *P* must be located in shaded area for no tipping.



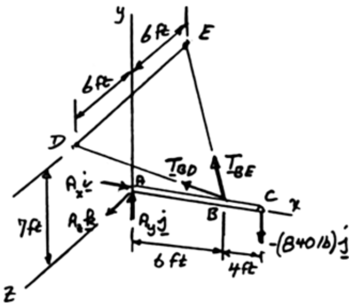
PROBLEM 4.105

A 10-ft boom is acted upon by the 840-lb force shown. Determine the tension in each cable and the reaction at the ball-and-socket joint at *A*.

SOLUTION

We have five unknowns and six equations of equilibrium, but equilibrium is maintained $(\Sigma M_x = 0)$.

Free-Body Diagram:



$$\overrightarrow{BD} = (-6\text{ ft})\mathbf{i} + (7\text{ ft})\mathbf{j} + (6\text{ ft})\mathbf{k} \qquad BD = 11\text{ ft}$$

$$\overrightarrow{BE} = (-6\text{ ft})\mathbf{i} + (7\text{ ft})\mathbf{j} - (6\text{ ft})\mathbf{k} \qquad BE = 11\text{ ft}$$

$$\mathbf{T}_{BD} = T_{BD}\frac{\overrightarrow{BD}}{BD} = \frac{T_{BD}}{11}(-6\mathbf{i} + 7\mathbf{j} + 6\mathbf{k})$$

$$\mathbf{T}_{BE} = T_{BE}\frac{\overrightarrow{BE}}{BE} = \frac{T_{BE}}{11}(-6\mathbf{i} + 7\mathbf{j} - 6\mathbf{k})$$

$$\Sigma M_A = 0: \quad \mathbf{r}_B \times \mathbf{T}_{BD} + \mathbf{r}_B \times \mathbf{T}_{BE} + \mathbf{r}_C \times (-840\mathbf{j}) = 0$$

$$6\mathbf{i} \times \frac{T_{BD}}{11}(-6\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}) + 6\mathbf{i} \times \frac{T_{BE}}{11}(-6\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}) + 10\mathbf{i} \times (-840\mathbf{j}) = 0$$

$$\frac{42}{11}T_{BD}\mathbf{k} - \frac{36}{11}T_{BD}\mathbf{j} + \frac{42}{11}T_{BE}\mathbf{k} + \frac{36}{11}T_{BE}\mathbf{j} - 8400\mathbf{k} = 0$$

Equate coefficients of unit vectors to zero:

$$\mathbf{i}: \quad -\frac{36}{11}T_{BD} + \frac{36}{11}T_{BE} = 0 \qquad T_{BE} = T_{BD}$$

$$\mathbf{k}: \quad \frac{42}{11}T_{BD} + \frac{42}{11}T_{BE} - 8400 = 0$$

$$2\left(\frac{42}{11}T_{BD}\right) = 8400$$

$T_{BD} = 1100\text{ lb}$ ◀

$T_{BE} = 1100\text{ lb}$ ◀

PROBLEM 4.105 (Continued)

$$\Sigma F_x = 0: \quad A_x - \frac{6}{11}(1100\text{ lb}) - \frac{6}{11}(1100\text{ lb}) = 0$$

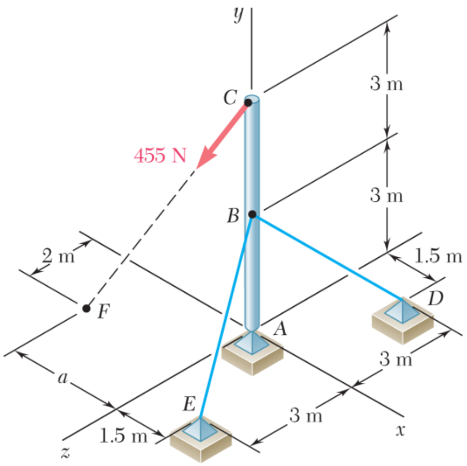
$$A_x = 1200\text{ lb}$$

$$\Sigma F_y = 0: \quad A_y + \frac{7}{11}(1100\text{ lb}) + \frac{7}{11}(1100\text{ lb}) - 840\text{ lb} = 0$$

$$A_y = -560\text{ lb}$$

$$\Sigma F_z = 0: \quad A_z + \frac{6}{11}(1100\text{ lb}) - \frac{6}{11}(1100\text{ lb}) = 0$$

$$A_z = 0 \qquad \mathbf{A} = (1200\text{ lb})\mathbf{i} - (560\text{ lb})\mathbf{j} \blacktriangleleft$$

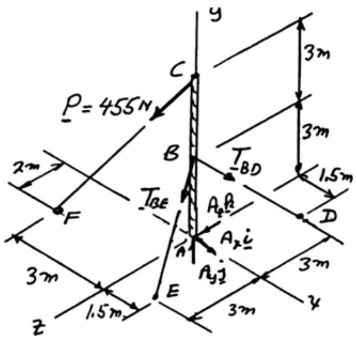


PROBLEM 4.106

The 6-m pole *ABC* is acted upon by a 455-N force as shown. The pole is held by a ball-and-socket joint at *A* and by two cables *BD* and *BE*. For $a = 3$ m, determine the tension in each cable and the reaction at *A*.

SOLUTION

Free-Body Diagram:



Five unknowns and six Eqs. of equilibrium, but equilibrium is maintained

$$(\Sigma M_{AC} = 0)$$

$$\mathbf{r}_B = 3\mathbf{j}$$
$$\mathbf{r}_C = 6\mathbf{j}$$

$$\overrightarrow{CF} = -3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \quad CF = 7 \text{ m}$$
$$\overrightarrow{BD} = 1.5\mathbf{i} - 3\mathbf{j} - 3\mathbf{k} \quad BD = 4.5 \text{ m}$$
$$\overrightarrow{BE} = 1.5\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} \quad BE = 4.5 \text{ m}$$

$$\mathbf{P} = P\frac{\overrightarrow{CF}}{CE} = \frac{P}{7}(-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$$

$$\mathbf{T}_{BD} = T_{BD}\frac{\overrightarrow{BD}}{BD} = \frac{T_{BD}}{4.5}(1.5\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}) = \frac{T_{BD}}{3}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{T}_{BE} = T_{BE} = \frac{\overrightarrow{BE}}{BE} = \frac{T_{BD}}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

$$\Sigma M_A = 0: \quad \mathbf{r}_B \times \mathbf{T}_{BD} + \mathbf{r}_B \times \mathbf{T}_{BE} + \mathbf{r}_C \times \mathbf{P} = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 0 \\ 1 & -2 & -2 \end{vmatrix} \frac{T_{BD}}{3} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 0 \\ 1 & -2 & 2 \end{vmatrix} \frac{T_{BE}}{3} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ -3 & -6 & 2 \end{vmatrix} \frac{P}{7} = 0$$

Coefficient of **i**: $$-2T_{BD} + 2T_{BE} + \frac{12}{7}P = 0 \qquad (1)$$

Coefficient of **k**: $$-T_{BD} - T_{BF} + \frac{18}{7}P = 0 \qquad (2)$$

PROBLEM 4.106 (Continued)

Eq. (1) + 2 Eq. (2): $-4T_{BD}+\dfrac{48}{7}P=0 \quad T_{BD}=\dfrac{12}{7}P$

Eq. (2): $-\dfrac{12}{7}P-T_{BE}+\dfrac{18}{7}P=0 \quad T_{BE}=\dfrac{6}{7}P$

Since $P=445\text{ N} \quad T_{BD}=\dfrac{12}{7}(455)$ $\qquad T_{BD}=780\text{ N}$ ◀

$$T_{BE}=\frac{6}{7}(455) \qquad T_{BE}=390\text{ N} \blacktriangleleft$$

$$\Sigma\mathbf{F}=0: \quad \mathbf{T}_{BD}+\mathbf{T}_{BE}+\mathbf{P}+\mathbf{A}=0$$

Coefficient of **i**: $\dfrac{780}{3}+\dfrac{390}{3}-\dfrac{455}{7}(3)+A_x=0$

$$260+130-195+A_x=0 \quad A_x=195.0\text{ N}$$

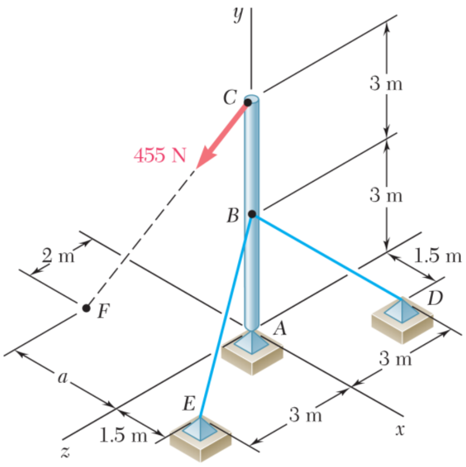
Coefficient of **j**: $-\dfrac{780}{3}(2)-\dfrac{390}{3}(2)-\dfrac{455}{7}(6)+A_y=0$

$$-520-260-390+A_y=0 \quad A_y=1170\text{ N}$$

Coefficient of **k**: $-\dfrac{780}{3}(2)+\dfrac{390}{3}(2)+\dfrac{455}{7}(2)+A_z=0$

$$-520+260+130+A_z=0 \quad A_z=+130.0\text{ N}$$

$$\mathbf{A}=-(195.0\text{ N})\mathbf{i}+(1170\text{ N})\mathbf{j}+(130.0\text{ N})\mathbf{k} \blacktriangleleft$$



PROBLEM 4.107

Solve Problem 4.106 for $a = 1.5$ m.

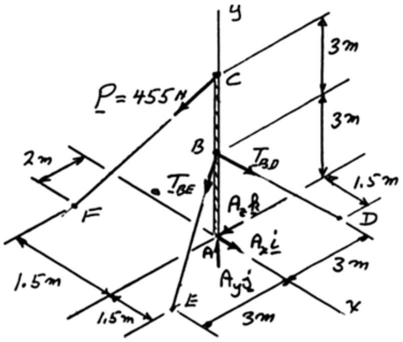
PROBLEM 4.106 The 6-m pole *ABC* is acted upon by a 455-N force as shown. The pole is held by a ball-and-socket joint at *A* and by two cables *BD* and *BE*. For $a = 3$ m, determine the tension in each cable and the reaction at *A*.

SOLUTION

Free-Body Diagram:

Five unknowns and six Eqs. of equilibrium but equilibrium is maintained

$$(\Sigma M_{AC} = 0)$$



$$\mathbf{r}_B = 3\mathbf{j}$$

$$\mathbf{r}_C = 6\mathbf{j}$$

$$\overrightarrow{CF} = -1.5\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \quad CF = 6.5 \text{ m}$$

$$\overrightarrow{BD} = 1.5\mathbf{i} - 3\mathbf{j} - 3\mathbf{k} \quad BD = 4.5 \text{ m}$$

$$\overrightarrow{BE} = 1.5\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} \quad BE = 4.5 \text{ m}$$

$$\mathbf{P} = P\frac{\overrightarrow{CF}}{CE} = \frac{P}{6.5}(-1.5\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) = \frac{P}{13}(-3\mathbf{i} - 12\mathbf{j} + 4\mathbf{k})$$

$$\mathbf{T}_{BD} = T_{BD}\frac{\overrightarrow{BD}}{BD} = \frac{T_{BD}}{4.5}(1.5\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}) = \frac{T_{BD}}{3}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{T}_{BE} = T_{BE}\frac{\overrightarrow{BE}}{BE} = \frac{T_{BD}}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

$$\Sigma M_A = 0: \quad \mathbf{r}_B \times \mathbf{T}_{BD} + \mathbf{r}_B \times \mathbf{T}_{BE} + \mathbf{r}_C \times \mathbf{P} = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 0 \\ 1 & -2 & -2 \end{vmatrix} \frac{T_{BD}}{3} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 0 \\ 1 & -2 & 2 \end{vmatrix} \frac{T_{BE}}{3} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ -3 & -12 & +4 \end{vmatrix} \frac{P}{13} = 0$$

Coefficient of **i**:

$$-2T_{BD} + 2T_{BE} + \frac{24}{13}P = 0 \tag{1}$$

Coefficient of **k**:

$$-T_{BD} - T_{BE} + \frac{18}{13}P = 0 \tag{2}$$

PROBLEM 4.107 (Continued)

Eq. (1) + 2 Eq. (2): $$-4T_{BD} + \frac{60}{13}P = 0 \quad T_{BD} = \frac{15}{13}P$$

Eq (2): $$-\frac{15}{13}P - T_{BE} + \frac{18}{13}P = 0 \quad T_{BE} = \frac{3}{13}P$$

Since $$P = 445\text{ N} \quad T_{BD} = \frac{15}{13}(455) \qquad T_{BD} = 525\text{ N} \blacktriangleleft$$

$$T_{BE} = \frac{3}{13}(455) \qquad T_{BE} = 105.0\text{ N} \blacktriangleleft$$

$$\Sigma\mathbf{F} = 0: \quad \mathbf{T}_{BD} + \mathbf{T}_{BE} + \mathbf{P} + \mathbf{A} = 0$$

Coefficient of **i**: $$\frac{525}{3} + \frac{105}{3} - \frac{455}{13}(3) + A_x = 0$$

$$175 + 35 - 105 + A_x = 0 \quad A_x = 105.0\text{ N}$$

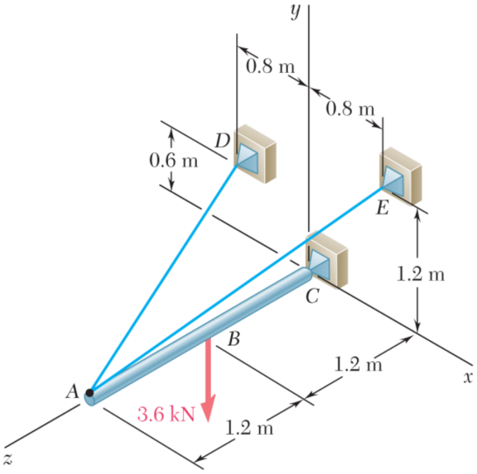
Coefficient of **j**: $$-\frac{525}{3}(2) - \frac{105}{3}(2) - \frac{455}{13}(12) + A_y = 0$$

$$-350 - 70 - 420 + A_y = 0 \quad A_y = 840\text{ N}$$

Coefficient of **k**: $$-\frac{525}{3}(2) + \frac{105}{3}(2) + \frac{455}{13}(4) + A_z = 0$$

$$-350 + 70 + 140 + A_z = 0 \quad A_z = 140.0\text{ N}$$

$$\mathbf{A} = -(105.0\text{ N})\mathbf{i} + (840\text{ N})\mathbf{j} + (140.0\text{ N})\mathbf{k} \blacktriangleleft$$



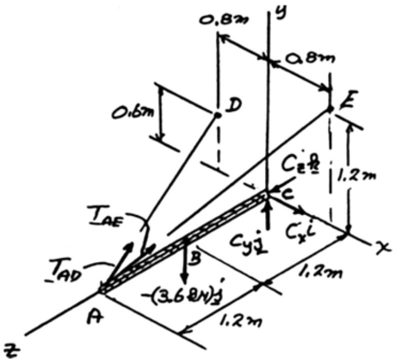
y
0.8 m
0.8 m
D
0.6 m
E
C
1.2 m
B
1.2 m
x
A
3.6 kN
1.2 m
z

PROBLEM 4.108

A 2.4-m boom is held by a ball-and-socket joint at *C* and by two cables *AD* and *AE*. Determine the tension in each cable and the reaction at *C*.

SOLUTION

Free-Body Diagram: Five unknowns and six equations of equilibrium, but equilibrium is maintained $(\Sigma M_{AC} = 0)$.



0.8m
y
0.8m
0.6m
D
E
$C_z\mathbf{k}$
1.2m
T_{AE}
$C_y\mathbf{j}$
$C_x\mathbf{i}$
C
x
T_{AD}
B
1.2m
$-(3.6\,\text{kN})\mathbf{j}$
A
1.2m
z

$$\mathbf{r}_B = 1.2\mathbf{k}$$

$$\mathbf{r}_A = 2.4\mathbf{k}$$

$$\overrightarrow{AD} = -0.8\mathbf{i} + 0.6\mathbf{j} - 2.4\mathbf{k} \qquad AD = 2.6\text{ m}$$

$$\overrightarrow{AE} = 0.8\mathbf{i} + 1.2\mathbf{j} - 2.4\mathbf{k} \qquad AE = 2.8\text{ m}$$

$$T_{AD} = \frac{\overrightarrow{AD}}{AD} = \frac{T_{AD}}{2.6}(-0.8\mathbf{i} + 0.6\mathbf{j} - 2.4\mathbf{k})$$

$$T_{AE} = \frac{\overrightarrow{AE}}{AE} = \frac{T_{AE}}{2.8}(0.8\mathbf{i} + 1.2\mathbf{j} - 2.4\mathbf{k})$$

$$\Sigma M_C = 0\text{:}\quad \mathbf{r}_A \times \mathbf{T}_{AD} + \mathbf{r}_A \times \mathbf{T}_{AE} + \mathbf{r}_B \times (-3\text{ kN})\mathbf{j} = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2.4 \\ -0.8 & 0.6 & -2.4 \end{vmatrix} \frac{T_{AD}}{2.6} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2.4 \\ 0.8 & 1.2 & -2.4 \end{vmatrix} \frac{T_{AE}}{2.8} + 1.2\mathbf{k} \times (-3.6\text{ kN})\mathbf{j} = 0$$

Equate coefficients of unit vectors to zero:

$$\mathbf{i}\text{:}\quad -0.55385T_{AD} - 1.02857T_{AE} + 4.32 = 0 \qquad (1)$$

$$\mathbf{j}\text{:}\quad -0.73846T_{AD} + 0.68671T_{AE} = 0$$

$$T_{AD} = 0.92857T_{AE} \qquad (2)$$

From Eq. (1):

$$-0.55385(0.92857)T_{AE} - 1.02857T_{AE} + 4.32 = 0$$

$$1.54286T_{AE} = 4.32$$

$$T_{AE} = 2.800\text{ kN}$$

$T_{AE} = 2.80\text{ kN}$ ◀

PROBLEM 4.108 (Continued)

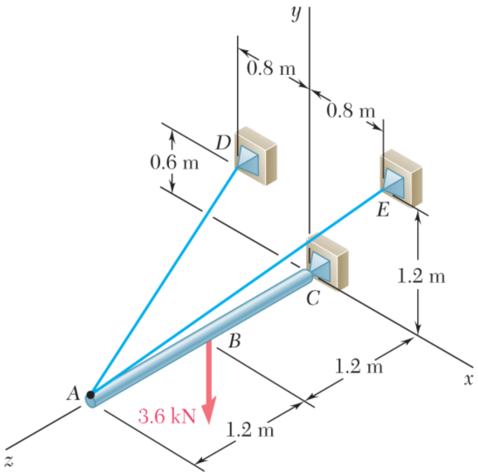
From Eq. (2): $T_{AD} = 0.92857(2.80) = 2.600 \text{ kN}$ $\qquad T_{AD} = 2.60 \text{ kN}$ ◀

$$\Sigma F_x = 0: \quad C_x - \frac{0.8}{2.6}(2.6 \text{ kN}) + \frac{0.8}{2.8}(2.8 \text{ kN}) = 0 \qquad C_x = 0$$

$$\Sigma F_y = 0: \quad C_y + \frac{0.6}{2.6}(2.6 \text{ kN}) + \frac{1.2}{2.8}(2.8 \text{ kN}) - (3.6 \text{ kN}) = 0 \qquad C_y = 1.800 \text{ kN}$$

$$\Sigma F_z = 0: \quad C_z - \frac{2.4}{2.6}(2.6 \text{ kN}) - \frac{2.4}{2.8}(2.8 \text{ kN}) = 0 \qquad C_z = 4.80 \text{ kN}$$

$\mathbf{C} = (1.800 \text{ kN})\mathbf{j} + (4.80 \text{ kN})\mathbf{k}$ ◀



y
0.8 m
0.8 m
D
0.6 m
E
C
1.2 m
B
1.2 m
x
A
3.6 kN
1.2 m
z

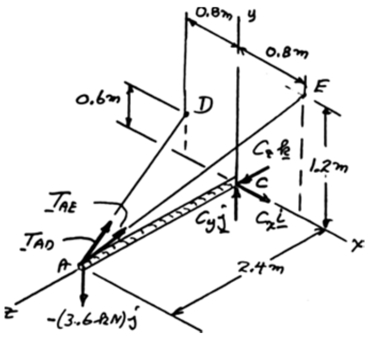
PROBLEM 4.109

Solve Problem 4.108, assuming that the 3.6-kN load is applied at Point *A*.

PROBLEM 4.108 A 2.4-m boom is held by a ball-and-socket joint at *C* and by two cables *AD* and *AE*. Determine the tension in each cable and the reaction at *C*.

SOLUTION

Free-Body Diagram: Five unknowns and six equations of equilibrium, but equilibrium is maintained $(\Sigma M_{AC} = 0)$.



0.8m
y
0.8m
E
0.6m
D
$C_z\mathbf{k}$
1.2m
$\mathbf{T}_{AE}$
C
$\mathbf{T}_{AD}$
$C_y\mathbf{j}$
$C_x\mathbf{i}$
x
A
2.4m
z
$-(3.6\,\text{kN})\mathbf{j}$

$$\overrightarrow{AD} = -0.8\mathbf{i} + 0.6\mathbf{j} - 2.4\mathbf{k} \qquad AD = 2.6\text{ m}$$

$$\overrightarrow{AE} = 0.8\mathbf{i} + 1.2\mathbf{j} - 2.4\mathbf{k} \qquad AE = 2.8\text{ m}$$

$$T_{AD}\frac{\overrightarrow{AD}}{AD} = \frac{T_{AD}}{2.6}(-0.8\mathbf{i} + 0.6\mathbf{j} - 2.4\mathbf{k})$$

$$T_{AE}\frac{\overrightarrow{AE}}{AE} = \frac{T_{AE}}{2.8}(0.8\mathbf{i} + 1.2\mathbf{j} - 2.4\mathbf{k})$$

$$\Sigma M_C = 0: \quad \mathbf{r}_A \times \mathbf{T}_{AD} + \mathbf{r}_A \times \mathbf{T}_{AE} + \mathbf{r}_A \times (-3.6\text{ kN})\mathbf{j}$$

Factor r_A:

$$\mathbf{r}_A \times (\mathbf{T}_{AD} + \mathbf{T}_{AE} - (3.6\text{ kN})\mathbf{j})$$

or

$$\mathbf{T}_{AD} + \mathbf{T}_{AE} - (3\text{ kN})\mathbf{j} = 0 \qquad \text{(Forces concurrent at } A\text{)}$$

Coefficient of **i**:

$$-\frac{T_{AD}}{2.6}(0.8) + \frac{T_{AE}}{2.8}(0.8) = 0$$

$$T_{AD} = \frac{2.6}{2.8}T_{AE} \qquad (1)$$

Coefficient of **j**:

$$\frac{T_{AD}}{2.6}(0.6) + \frac{T_{AE}}{2.8}(1.2) - 3.6\text{ kN} = 0$$

$$\frac{2.6}{2.8}T_{AE}\left(\frac{0.6}{2.6}\right) + \frac{1.2}{2.8}T_{AE} - 3.6\text{ kN} = 0$$

$$T_{AE}\left(\frac{0.6+1.2}{2.8}\right) = 3.6\text{ kN}$$

$$T_{AE} = 5.600\text{ kN} \qquad T_{AE} = 5.60\text{ kN} \blacktriangleleft$$

PROBLEM 4.109 (Continued)

From Eq. (1): $T_{AD} = \dfrac{2.6}{2.8}(5.6) = 5.200 \text{ kN}$ $\qquad T_{AD} = 5.20 \text{ kN}$ ◀

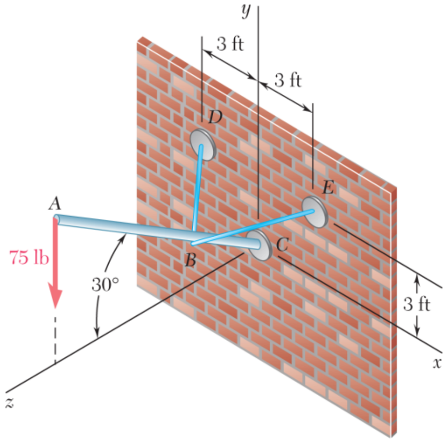
$$\Sigma F_x = 0: \quad C_x - \frac{0.8}{2.6}(5.2 \text{ kN}) + \frac{0.8}{2.8}(5.6 \text{ kN}) = 0 \qquad C_x = 0$$

$$\Sigma F_y = 0: \quad C_y + \frac{0.6}{2.6}(5.2 \text{ kN}) + \frac{1.2}{2.8}(5.6 \text{ kN}) - 3.6 \text{ kN} = 0 \qquad C_y = 0$$

$$\Sigma F_z = 0: \quad C_z - \frac{2.4}{2.6}(5.2 \text{ kN}) - \frac{2.4}{2.8}(5.6 \text{ kN}) = 0 \qquad C_z = 9.60 \text{ kN}$$

$\mathbf{C} = (9.60 \text{ kN})\mathbf{k}$ ◀

Note: Since the forces and reaction are concurrent at *A*, we could have used the methods of Chapter 2.

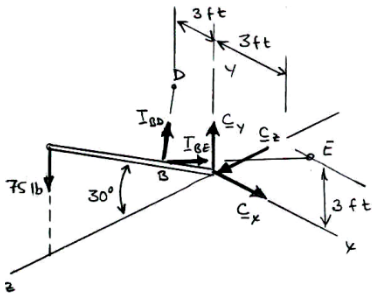


PROBLEM 4.110

The 10-ft flagpole AC forms an angle of $30°$ with the z axis. It is held by a ball-and-socket joint at C and by two thin braces BD and BE. Knowing that the distance BC is 3 ft, determine the tension in each brace and the reaction at C.

SOLUTION

Free-Body Diagram:



Express all forces in terms of rectangular components:

$$\mathbf{r}_E = (3\text{ ft})\mathbf{i} + (3\text{ ft})\mathbf{j}$$

$$\mathbf{r}_B = (3\text{ ft})\sin 30°\mathbf{j} + (3\text{ ft})\cos 30°\mathbf{k} = (1.5\text{ ft})\mathbf{j} + (2.598\text{ ft})\mathbf{k}$$

$$\mathbf{r}_D = -(3\text{ ft})\mathbf{i} + (3\text{ ft})\mathbf{j}$$

$$\mathbf{r}_A = (10\text{ ft})\sin 30°\mathbf{j} - (10\text{ ft})\cos 30°\mathbf{k} = (5\text{ ft})\mathbf{j} + (8.66\text{ ft})\mathbf{k}$$

$$\overrightarrow{BE} = \mathbf{r}_E - \mathbf{r}_B = (3\text{ ft})\mathbf{i} + (3\text{ ft})\mathbf{j} - (1.5\text{ ft})\mathbf{j} - (2.598\text{ ft})\mathbf{k}$$

or $\overrightarrow{BE} = (3\text{ ft})\mathbf{i} + (1.5\text{ ft})\mathbf{j} - (2.598\text{ ft})\mathbf{k}$, and $BE = 4.243\text{ ft}$

$$\overrightarrow{BD} = \mathbf{r}_D - \mathbf{r}_B = -(3\text{ ft})\mathbf{i} + (3\text{ ft})\mathbf{j} - (1.5\text{ ft})\mathbf{j} - (2.598\text{ ft})\mathbf{k}$$

or $\overrightarrow{BD} = -(3\text{ ft})\mathbf{i} + (1.5\text{ ft})\mathbf{j} - (2.598\text{ ft})\mathbf{k}$, and $BD = 4.243\text{ ft}$

Then

$$\vec{T}_{BD} = T_{BD}\frac{\overrightarrow{BD}}{BD} = T_{BD}(-0.707\mathbf{i} + 0.3535\mathbf{j} - 0.6123\mathbf{k})$$

$$\vec{T}_{BE} = T_{BE}\frac{\overrightarrow{BE}}{BE} = T_{BE}(0.707\mathbf{i} + 0.3535\mathbf{j} - 0.6123\mathbf{k})$$

PROBLEM 4.110 (Continued)

$$\Sigma \mathbf{M}_C = 0: \qquad \mathbf{r}_B \times \mathbf{T}_{BD} + \mathbf{r}_B \times \mathbf{T}_{BE} + \left[(5\text{ ft})\mathbf{j} + (8.66\text{ ft})\mathbf{k}\right] \times (-75\text{ lb})\mathbf{j}$$

or

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1.5 & 2.598 \\ -0.707 & 0.3535 & -0.6123 \end{vmatrix} T_{BD} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1.5 & 2.598 \\ 0.707 & 0.3535 & -0.6123 \end{vmatrix} + 649.5\mathbf{i} = 0$$

Equating the coefficients of the unit vectors to zero:

$\mathbf{j}$: $\quad -1.837T_{BD} + 1.837T_{BE} = 0$

$\mathbf{i}$: $\quad -1.837T_{BD} + 1.837T_{BE} + 649.5\text{ lb} = 0$

$T_{BD} = 176.8\text{ lb}$ ◀

$T_{BE} = 176.8\text{ lb}$ ◀

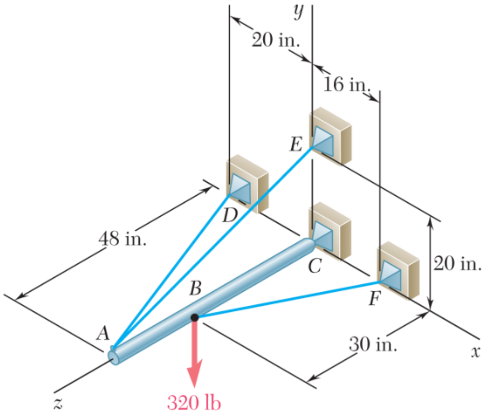
Force equations:

$C_x + (176.8)(-0.707) + (176.8)(0.707) = 0, \quad \text{or} \quad C_x = 0$

$C_y + (176.8)(0.3535) + (176.8)(0.3535) - 75\text{ lb} = 0, \quad \text{or} \quad C_y = -50\text{ lb}$

$C_z + (176.8)(-0.6123) + (176.8)(-0.6123) = 0, \quad \text{or} \quad C_z = 216.5\text{ lb}$

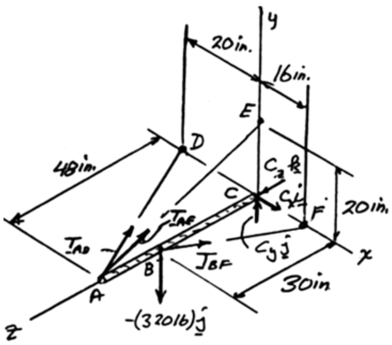
$\mathbf{C} = -(50\text{ lb})\mathbf{j} + (216.5\text{ lb})\mathbf{k}$ ◀



PROBLEM 4.111

A 48-in. boom is held by a ball-and-socket joint at *C* and by two cables *BF* and *DAE*; cable *DAE* passes around a frictionless pulley at *A*. For the loading shown, determine the tension in each cable and the reaction at *C*.

SOLUTION



Free-Body Diagram:

Five unknowns and six equations of equilibrium, but equilibrium is maintained $(\Sigma M_{AC} = 0)$.

T = Tension in both parts of cable *DAE*.

$$\mathbf{r}_B = 30\mathbf{k}$$
$$\mathbf{r}_A = 48\mathbf{k}$$

$$\overrightarrow{AD} = -20\mathbf{i} - 48\mathbf{k} \qquad AD = 52 \text{ in.}$$
$$\overrightarrow{AE} = 20\mathbf{j} - 48\mathbf{k} \qquad AE = 52 \text{ in.}$$
$$\overrightarrow{BF} = 16\mathbf{i} - 30\mathbf{k} \qquad BF = 34 \text{ in.}$$

$$\mathbf{T}_{AD} = T\frac{\overrightarrow{AD}}{AD} = \frac{T}{52}(-20\mathbf{i} - 48\mathbf{k}) = \frac{T}{13}(-5\mathbf{i} - 12\mathbf{k})$$
$$\mathbf{T}_{AE} = T\frac{\overrightarrow{AE}}{AE} = \frac{T}{52}(20\mathbf{j} - 48\mathbf{k}) = \frac{T}{13}(5\mathbf{j} - 12\mathbf{k})$$
$$\mathbf{T}_{BF} = T_{BF}\frac{\overrightarrow{BF}}{BF} = \frac{T_{BF}}{34}(16\mathbf{i} - 30\mathbf{k}) = \frac{T_{BF}}{17}(8\mathbf{i} - 15\mathbf{k})$$

$$\Sigma\mathbf{M}_C = 0: \quad \mathbf{r}_A \times \mathbf{T}_{AD} + \mathbf{r}_A \times \mathbf{T}_{AE} + \mathbf{r}_B \times \mathbf{T}_{BF} + \mathbf{r}_B \times (-320 \text{ lb})\mathbf{j} = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 48 \\ -5 & 0 & -12 \end{vmatrix}\frac{T}{13} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 48 \\ 0 & 5 & -12 \end{vmatrix}\frac{T}{13} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 30 \\ 8 & 0 & -15 \end{vmatrix}\frac{T_{BF}}{17} + (30\mathbf{k}) \times (-320\mathbf{j}) = 0$$

Coefficient of **i**: $$-\frac{240}{13}T + 9600 = 0 \qquad T = 520 \text{ lb}$$

PROBLEM 4.111 (Continued)

Coefficient of **j**: $$-\frac{240}{13}T + \frac{240}{17}T_{BD} = 0$$

$$T_{BD} = \frac{17}{13}T = \frac{17}{13}(520) \quad T_{BD} = 680 \text{ lb}$$

$$\Sigma \mathbf{F} = 0: \quad \mathbf{T}_{AD} + \mathbf{T}_{AE} + \mathbf{T}_{BF} - 320\mathbf{j} + C = 0$$

Coefficient of **i**: $$-\frac{20}{52}(520) + \frac{8}{17}(680) + C_x = 0$$

$$-200 + 320 + C_x = 0 \qquad C_x = -120 \text{ lb}$$

Coefficient of **j**: $$\frac{20}{52}(520) - 320 + C_y = 0$$

$$200 - 320 + C_y = 0 \qquad C_y = 120 \text{ lb}$$

Coefficient of **k**: $$-\frac{48}{52}(520) - \frac{48}{52}(520) - \frac{30}{34}(680) + C_z = 0$$

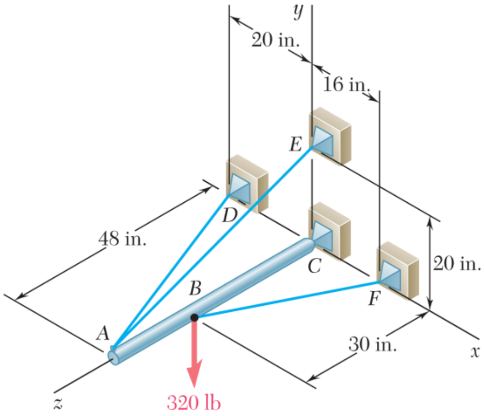
$$-480 - 480 - 600 + C_z = 0$$

$$C_z = 1560 \text{ lb}$$

Answers: $T_{DAE} = T$ $\quad T_{DAE} = 520 \text{ lb}$ ◀

$T_{BD} = 680 \text{ lb}$ ◀

$\mathbf{C} = -(120.0 \text{ lb})\mathbf{i} + (120.0 \text{ lb})\mathbf{j} + (1560 \text{ lb})\mathbf{k}$ ◀



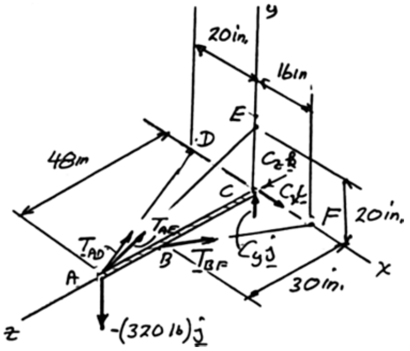
PROBLEM 4.112

Solve Problem 4.111, assuming that the 320-lb load is applied at *A*.

PROBLEM 4.111 A 48-in. boom is held by a ball-and-socket joint at *C* and by two cables *BF* and *DAE*; cable *DAE* passes around a frictionless pulley at *A*. For the loading shown, determine the tension in each cable and the reaction at *C*.

SOLUTION

Free-Body Diagram:



Five unknowns and six equations of equilibrium, but equilibrium is maintained $(\Sigma M_{AC} = 0)$.

T = tension in both parts of cable *DAE*.

$$\mathbf{r}_B = 30\mathbf{k}$$
$$\mathbf{r}_A = 48\mathbf{k}$$

$$\overrightarrow{AD} = -20\mathbf{i} - 48\mathbf{k} \qquad AD = 52\text{ in.}$$
$$\overrightarrow{AE} = 20\mathbf{j} - 48\mathbf{k} \qquad AE = 52\text{ in.}$$
$$\overrightarrow{BF} = 16\mathbf{i} - 30\mathbf{k} \qquad BF = 34\text{ in.}$$

$$\mathbf{T}_{AD} = T\frac{\overrightarrow{AD}}{AD} = \frac{T}{52}(-20\mathbf{i} - 48\mathbf{k}) = \frac{T}{13}(-5\mathbf{i} - 12\mathbf{k})$$
$$\mathbf{T}_{AE} = T\frac{\overrightarrow{AE}}{AE} = \frac{T}{52}(20\mathbf{j} - 48\mathbf{k}) = \frac{T}{13}(5\mathbf{j} - 12\mathbf{k})$$
$$\mathbf{T}_{BF} = T_{BF}\frac{\overrightarrow{BF}}{BF} = \frac{T_{BF}}{34}(16\mathbf{i} - 30\mathbf{k}) = \frac{T_{BF}}{17}(8\mathbf{i} - 15\mathbf{k})$$

$$\Sigma M_C = 0\!: \quad \mathbf{r}_A \times \mathbf{T}_{AD} + \mathbf{r}_A \times \mathbf{T}_{AE} + \mathbf{r}_B \times \mathbf{T}_{BF} + \mathbf{r}_A \times (-320\text{ lb})\mathbf{j} = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 48 \\ -5 & 0 & -12 \end{vmatrix}\frac{T}{13} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 48 \\ 0 & 5 & -12 \end{vmatrix}\frac{T}{13} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 30 \\ 8 & 0 & -15 \end{vmatrix}\frac{T_{BF}}{17} + 48\mathbf{k} \times (-320\mathbf{j}) = 0$$

Coefficient of **i**:

$$-\frac{240}{13}T + 15{,}360 = 0 \qquad T = 832\text{ lb}$$

PROBLEM 4.112 (Continued)

Coefficient of **j**: $$-\frac{240}{13}T+\frac{240}{17}T_{BD}=0$$

$$T_{BD}=\frac{17}{13}T=\frac{17}{13}(832) \qquad T_{BD}=1088\text{ lb}$$

$$\Sigma\mathbf{F}=0: \quad \mathbf{T}_{AD}+\mathbf{T}_{AE}+\mathbf{T}_{BF}-320\mathbf{j}+\mathbf{C}=0$$

Coefficient of **i**: $$-\frac{20}{52}(832)+\frac{8}{17}(1088)+C_x=0$$

$$-320+512+C_x=0 \qquad C_x=-192\text{ lb}$$

Coefficient of **j**: $$\frac{20}{52}(832)-320+C_y=0$$

$$320-320+C_y=0 \qquad C_y=0$$

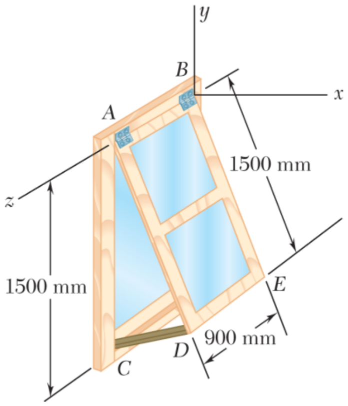
Coefficient of **k**: $$-\frac{48}{52}(832)-\frac{48}{52}(852)-\frac{30}{34}(1088)+C_z=0$$

$$-768-768-960+C_z=0 \qquad C_z=2496\text{ lb}$$

Answers: $T_{DAE}=T$ $\qquad$ $T_{DAE}=832\text{ lb}$ ◀

$T_{BD}=1088\text{ lb}$ ◀

$\mathbf{C}=-(192.0\text{ lb})\mathbf{i}+(2500\text{ lb})\mathbf{k}$ ◀



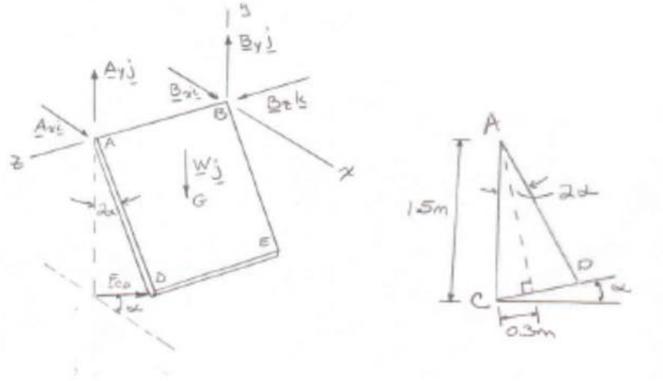
PROBLEM 4.113

A 10-kg storm window measuring 900 × 1500 mm is held by hinges at *A* and *B*. In the position shown, it is held away from the side of the house by a 600-mm stick *CD*. Assuming that the hinge at *A* does not exert any axial thrust, determine the magnitude of the force exerted by the stick and the components of the reactions at *A* and *B*.

SOLUTION

Free-Body Diagram: Since *CD* is a two-force member, F_{CD} is directed along *CD* and triangle *ACD* is isosceles. We have

$$\sin\alpha = \frac{0.3\text{ m}}{1.5\text{ m}} = 0.2; \quad \alpha = 11.5370^\circ \text{ and } 2\alpha = 23.074^\circ$$



$$\mathbf{W} = -(10\text{ kg})(9.81\text{ m/s}^2) = -(98.10\text{ N})\mathbf{j}$$

$$\mathbf{r}_G = (0.75\text{ m})\sin 23.074^\circ\,\mathbf{i} - (0.75\text{ m})\cos 23.074^\circ\,\mathbf{j} + (0.45\text{ m})\mathbf{k}$$
$$\mathbf{r}_G = (0.29394\text{ m})\mathbf{i} - (0.690\text{ m})\mathbf{j} + (0.45\text{ m})\mathbf{k}$$
$$\mathbf{F}_{CD} = F_{CD}(\cos 11.5370^\circ\,\mathbf{i} + \sin 11.5370^\circ\,\mathbf{j})$$
$$\mathbf{F}_{CD} = F_{CD}(0.97980\mathbf{i} + 0.20\mathbf{j})$$

$$\Sigma M_B = 0\text{:} \quad \mathbf{r}_A \times \mathbf{A} + \mathbf{r}_G \times \mathbf{W} + \mathbf{r}_C \times \mathbf{F}_{CD} = 0$$

$$0.9\mathbf{k} \times (A_x\mathbf{i} + A_y\mathbf{j}) + (0.29394\mathbf{i} - 0.690\mathbf{j} + 0.45\mathbf{k}) \times (-98.10\mathbf{j})$$
$$+ (1.5\mathbf{j} + 0.9\mathbf{k}) \times F_{CD}(0.97980\mathbf{i} + 0.20\mathbf{j}) = 0$$

$$0.90A_x\mathbf{j} - 0.90A_y\mathbf{i} - 28.836\mathbf{k} + 44.145\mathbf{i} + 1.46970F_{CD}\,\mathbf{k} + 0.88182F_{CD}\mathbf{j} - 0.180F_{CD}\mathbf{i} = 0$$

Equating the coefficients of the unit vectors to zero,

$$\mathbf{k}: \quad -28.836 + 1.46970F_{CD} = 0; \quad F_{CD} = 19.6203\text{ N}$$

PROBLEM 4.113 (Continued)

$$\mathbf{j}: \quad -0.90A_x + 0.88182(19.6203\text{ N}) = 0; \qquad A_x = 19.2240\text{ N}$$

$$\mathbf{i}: \quad -0.90A_y + 44.145 - 0.180(19.6203\text{ N}) = 0; \qquad A_y = 45.123\text{ N}$$

$F_{CD} = 19.62\text{ N}$ ◀

$\mathbf{A} = -(19.22\text{N})\mathbf{i} + (45.1\text{ N})\mathbf{j}$ ◀

$$\Sigma F_x = 0: \quad A_x + B_x + F_{CD}\cos 11.5370° = 0$$

$$-19.2240 + B_x + 19.6203\cos 11.5370^\circ = 0$$

$$B_x = 0$$

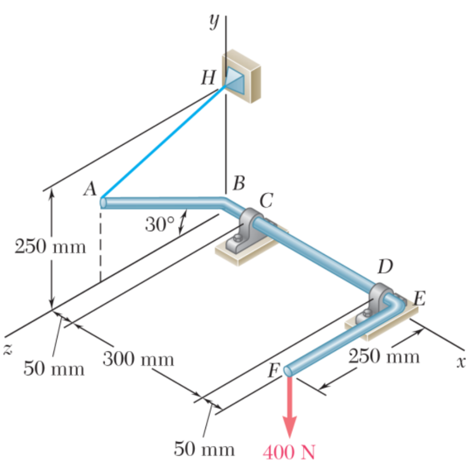
$$\Sigma F_y = 0: \quad A_y + B_y + F_{CD}\sin 11.5370^\circ - W = 0$$

$$45.122 + B_y + 19.6230\sin 11.5370^\circ - 98.1 = 0$$

$$B_y = 49.053\text{ N}$$

$$\Sigma F_z = 0: \quad B_z = 0$$

$\mathbf{B} = +(49.1\text{ N})\mathbf{j}$ ◀


y
H
A
B
C
30°
250 mm
D
E
z
50 mm
300 mm
F
250 mm
x
50 mm
400 N

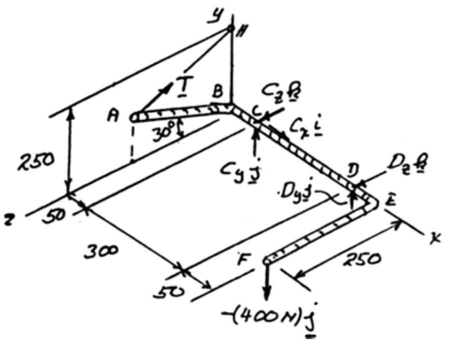
PROBLEM 4.114

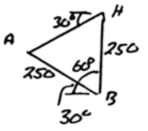
The bent rod *ABEF* is supported by bearings at *C* and *D* and by wire *AH*. Knowing that portion *AB* of the rod is 250 mm long, determine (*a*) the tension in wire *AH*, (*b*) the reactions at *C* and *D*. Assume that the bearing at *D* does not exert any axial thrust.

SOLUTION

Free-Body Diagram: ΔABH is equilateral.

Dimensions in mm


y
H
T
A
B
30°
$C_z\mathbf{k}$
$C_x\mathbf{i}$
$C_y\mathbf{j}$
D
$D_z\mathbf{k}$
$D_y\mathbf{j}$
E
250
z
50
300
F
250
x
50
$-(400\text{ N})\mathbf{j}$


A
30°
T
H
250
60°
250
B
30°

$$\mathbf{r}_{H/C} = -50\mathbf{i} + 250\mathbf{j}$$

$$\mathbf{r}_{D/C} = 300\mathbf{i}$$

$$\mathbf{r}_{F/C} = 350\mathbf{i} + 250\mathbf{k}$$

$$\mathbf{T} = T(\sin 30°)\mathbf{j} - T(\cos 30°)\mathbf{k} = T(0.5\mathbf{j} - 0.866\mathbf{k})$$

$$\Sigma\mathbf{M}_C = 0: \quad \mathbf{r}_{H/C} \times \mathbf{T} + \mathbf{r}_D \times \mathbf{D} + \mathbf{r}_{F/C} \times (-400\mathbf{j}) = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -50 & 250 & 0 \\ 0 & 0.5 & -0.866 \end{vmatrix} T + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 300 & 0 & 0 \\ 0 & D_y & D_z \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 350 & 0 & 250 \\ 0 & -400 & 0 \end{vmatrix} = 0$$

Coefficient **i**: $-216.5T + 100 \times 10^3 = 0$

$$T = 461.9\text{ N}$$

$T = 462\text{ N}$ ◀

Coefficient of **j**: $-43.3T - 300D_z = 0$

$$-43.3(461.9) - 300D_z = 0 \qquad D_z = -66.67\text{ N}$$

PROBLEM 4.114 (Continued)

Coefficient of **k**: $-25T + 300D_y - 140\times 10^3 = 0$

$-25(461.9) + 300D_y - 140\times 10^3 = 0 \qquad D_y = 505.1\text{ N}$

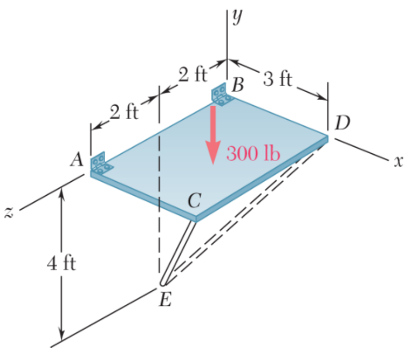
$\mathbf{D} = (505\text{ N})\mathbf{j} - (66.7\text{ N})\mathbf{k}$ ◀

$\Sigma\mathbf{F} = 0\text{:} \quad \mathbf{C} + \mathbf{D} + \mathbf{T} - 400\mathbf{j} = 0$

Coefficient **i**: $C_x = 0 \qquad C_x = 0$

Coefficient **j**: $C_y + (461.9)0.5 + 505.1 - 400 = 0 \quad C_y = -336\text{ N}$

Coefficient **k**: $C_z - (461.9)0.866 - 66.67 = 0 \qquad C_z = 467\text{ N} \qquad \mathbf{C} = -(336\text{ N})\mathbf{j} + (467\text{ N})\mathbf{k}$ ◀

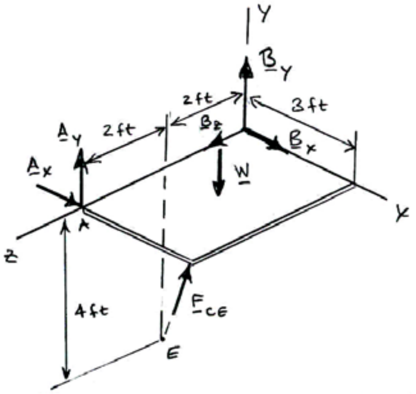


PROBLEM 4.115

The horizontal platform *ABCD* weighs 60 lb and supports a 240-lb load at its center. The platform is normally held in position by hinges at *A* and *B* and by braces *CE* and *DE*. If brace *DE* is removed, determine the reactions at the hinges and the force exerted by the remaining brace *CE*. The hinge at *A* does not exert any axial thrust.

SOLUTION

Free-Body Diagram:



Express forces, weight in terms of rectangular components:

$$\overrightarrow{EC} = (3\text{ ft})\mathbf{i} + (4\text{ ft})\mathbf{j} + (2\text{ ft})\mathbf{k}$$

$$\mathbf{F}_{CE} = F_{CE}\frac{\overrightarrow{EC}}{EC} = F_{CE}\frac{3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{(3)^2 + (4)^2 + (2)^2}}$$

$$= 0.55709F_{CE}\,\mathbf{i} + 0.74278F_{CE}\,\mathbf{j} + 0.37139F_{CE}\,\mathbf{k}$$

$$\mathbf{W} = -(mg)\mathbf{j} = -(300\text{ lb})\mathbf{j}$$

$$\Sigma\mathbf{M}_B = 0: \quad (4\text{ ft})\mathbf{k}\times\mathbf{A} + \left[(1.5\text{ ft})\mathbf{i} + (2\text{ ft})\mathbf{k}\right]\times(-300\text{ lb})\mathbf{j}$$

$$+\left[(3\text{ ft})\mathbf{i} + (4\text{ ft})\mathbf{k}\right]\times\mathbf{F}_{CE} = 0$$

or

$$-(4\text{ ft})A_y\mathbf{i} + (4\text{ ft})A_z\mathbf{j} - (1.5\text{ ft})(300\text{ lb})\mathbf{k} + (2\text{ ft})(300\text{ lb})\mathbf{i}$$

$$+\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 4 \\ 0.55709 & 0.74278 & 0.37139 \end{vmatrix} F_{CE}(\text{ft}) = 0$$

PROBLEM 4.115 (Continued)

Setting the coefficients of the unit vectors equal to zero:

k: $(0.74278F_{CE})(3\text{ ft}) - (300\text{ lb})(1.5\text{ ft}) = 0$

$$F_{CE} = 201.94\text{ lb}$$

or $F_{CE} = 202\text{ lb}$ ◀

j: $A_x(4\text{ ft}) + [0.55709(201.94\text{ lb})](4\text{ ft}) - [0.37139(201.94\text{ lb})](3\text{ ft}) = 0$

$$A_x = -56.250\text{ lb}$$

i: $-A_y(4\text{ ft}) - [0.74278(201.94\text{ lb})](4\text{ ft}) + (300\text{ lb})(2\text{ ft}) = 0$

$$A_y = 0$$

$\Sigma F_x = 0$: $-56.250\text{ lb} + B_x + 0.55709(201.94\text{ lb}) = 0$

$$B_x = -56.249\text{ lb}$$

$\Sigma F_y = 0$: $0 + B_y - 300\text{ lb} + 0.74278(201.94\text{ lb}) = 0$

$$B_y = 150.003\text{ lb}$$

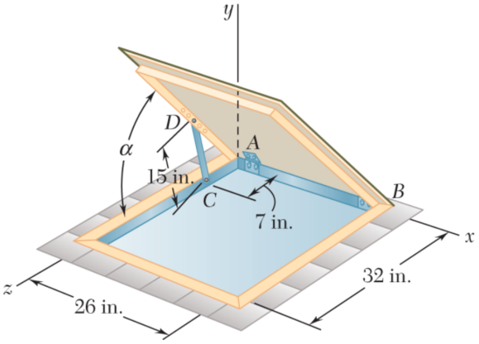
$\Sigma F_z = 0$: $B_z + 0.371391(201.94\text{ lb}) = 0$

$$B_z = -74.999\text{ lb}$$

Therefore:

$\mathbf{A} = -(56.3\text{ lb})\mathbf{i}$ ◀

$\mathbf{B} = -(56.2\text{ lb})\mathbf{i} + (150.0\text{ lb})\mathbf{j} - (75.0\text{ lb})\mathbf{k}$ ◀

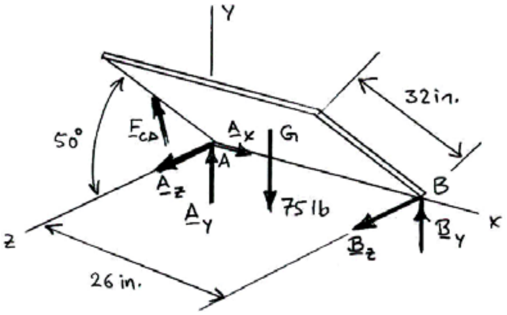


PROBLEM 4.116

The lid of a roof scuttle weighs 75 lb. It is hinged at corners *A* and *B* and maintained in the desired position by a rod *CD* pivoted at *C*; a pin at end *D* of the rod fits into one of several holes drilled in the edge of the lid. For the position shown, determine (*a*) the magnitude of the force exerted by rod *CD*, (*b*) the reactions at the hinges. Assume that the hinge at *B* does not exert any axial thrust.

SOLUTION

Free-Body Diagram:



Geometry:

Using triangle *ACD* and the law of sines

$$\frac{\sin\alpha}{7\text{ in.}} = \frac{\sin 50^\circ}{15\text{ in.}} \quad \text{or } \alpha = 20.946^\circ$$

$$\beta = 50^\circ + 20.946^\circ = 70.946^\circ$$

Expressing $\mathbf{F}_{CD}$ in terms of its rectangular coordinates:

$$\mathbf{F}_{CD} = F_{CD}\sin\beta\mathbf{j} + F_{CD}\cos\beta\mathbf{k}$$

$$= F_{CD}\sin 70.946^\circ\mathbf{j} + F_{CD}\cos 70.946^\circ\mathbf{k}$$

$$\mathbf{F}_{CD} = 0.94521F_{CD}\mathbf{j} + 0.32646F_{CD}\mathbf{k}$$

$$\Sigma\mathbf{M}_B = 0: \quad (-26\text{ in.})\mathbf{i}\times\mathbf{A} + \left[(-13\text{ in.})\mathbf{i} + (16\text{ in.})\sin 50^\circ\mathbf{j} + (16\text{ in.})\cos 50^\circ\mathbf{k}\right]\times(-75\text{ lb})\mathbf{j}$$

$$+\left[(-26\text{ in.})\mathbf{i} + (7\text{ in.})\mathbf{k}\right]\times\mathbf{F}_{CD} = 0$$

or

$$-(26\text{ in.})A_y\mathbf{k} + (26\text{ in.})A_z\mathbf{j} + (13\text{ in.})(75\text{ lb})\mathbf{k} + (16\text{ in.})(75\text{ lb})\cos 50^\circ\mathbf{i}$$

$$-(26\text{ in.})(0.94521F_{CD})\mathbf{k} + (26\text{ in.})(0.32646F_{CD})\mathbf{j} - (7\text{ in.})(0.94521F_{CD})\mathbf{i} = 0$$

PROBLEM 4.116 (Continued)

(*a*) Setting the coefficients of the unit vectors to zero:

$$\mathbf{i}: \quad (75\text{ lb})\left[(16\text{ in.})\cos 50°\right] - (0.94521F_{CD})(7\text{ in.}) = 0$$

$$F_{CD} = 116.6\text{ lb} \blacktriangleleft$$

(*b*) $\Sigma F_x = 0: \quad A_x = 0$

$$\mathbf{k}: \quad -\left[0.94521(116.580\text{ lb})\right](26\text{ in.}) + (75\text{ lb})(13\text{ in.}) - A_y(26\text{ in.}) = 0$$

$$A_y = -72.693\text{ lb}$$

$$\mathbf{j}: \quad \left[0.32646(116.580\text{ lb})\right](26\text{ in.}) + A_z(26\text{ in.}) = 0$$

$$A_z = -38.059\text{ lb}$$

$$\Sigma F_y = 0: \quad -72.693\text{ lb} + 0.94521(116.580\text{ lb}) - 75\text{ lb} + B_y = 0$$

$$B_y = 37.500\text{ lb}$$

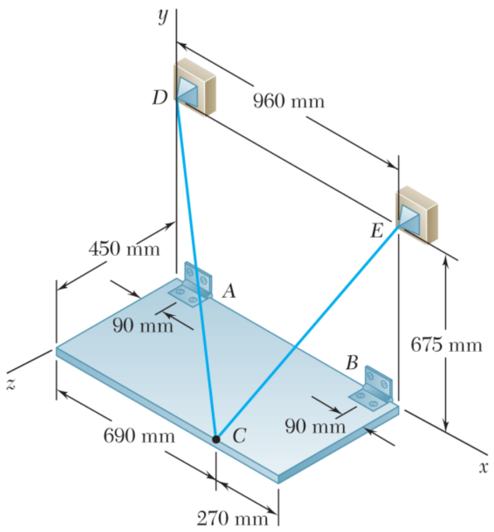
$$\Sigma F_z = 0: \quad -38.059\text{ lb} + 0.32646(116.580\text{ lb}) + B_z = 0$$

$$B_z = 0$$

Therefore:

$$\mathbf{A} = -(72.7\text{ lb})\mathbf{j} - (38.1\text{ lb})\mathbf{k} \blacktriangleleft$$

$$\mathbf{B} = (37.5\text{ lb})\mathbf{j} \blacktriangleleft$$



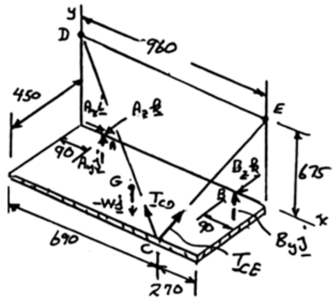
y
D
960 mm
450 mm
E
A
90 mm
675 mm
B
z
90 mm
690 mm
C
x
270 mm

PROBLEM 4.117

A 100-kg uniform rectangular plate is supported in the position shown by hinges *A* and *B* and by cable *DCE* that passes over a frictionless hook at *C*. Assuming that the tension is the same in both parts of the cable, determine (*a*) the tension in the cable, (*b*) the reactions at *A* and *B*. Assume that the hinge at *B* does not exert any axial thrust.

SOLUTION

Dimensions in mm



y
D
960
450
$A_y\mathbf{j}$
$A_z\mathbf{k}$
A
E
90
$A_x\mathbf{i}$
G
$-W\mathbf{j}$
T_{CD}
90
$B_z\mathbf{k}$
B
675
x
690
C
$B_y\mathbf{j}$
T_{CE}
270

$$\mathbf{r}_{B/A}(960-180)\mathbf{i} = 780\mathbf{i}$$

$$\mathbf{r}_{G/A} = \left(\frac{960}{2}-90\right)\mathbf{i} + \frac{450}{2}\mathbf{k}$$

$$= 390\mathbf{i} + 225\mathbf{k}$$

$$\mathbf{r}_{C/A} = 600\mathbf{i} + 450\mathbf{k}$$

T = Tension in cable DCE

$$\overrightarrow{CD} = -690\mathbf{i} + 675\mathbf{j} - 450\mathbf{k} \qquad CD = 1065 \text{ mm}$$

$$\overrightarrow{CE} = 270\mathbf{i} + 675\mathbf{j} - 450\mathbf{k} \qquad CE = 855 \text{ mm}$$

$$\mathbf{T}_{CD} = \frac{T}{1065}(-690\mathbf{i} + 675\mathbf{j} - 450\mathbf{k})$$

$$\mathbf{T}_{CE} = \frac{T}{855}(270\mathbf{i} + 675\mathbf{j} - 450\mathbf{k})$$

$$\mathbf{W} = -mg\mathbf{i} = -(100 \text{ kg})(9.81 \text{ m/s}^2)\mathbf{j} = -(981 \text{ N})\mathbf{j}$$

$$\Sigma\mathbf{M}_A = 0: \quad \mathbf{r}_{C/A}\times\mathbf{T}_{CD} + \mathbf{r}_{C/A}\times\mathbf{T}_{CE} + \mathbf{r}_{G/A}\times(-W\mathbf{j}) + \mathbf{r}_{B/A}\times\mathbf{B} = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 600 & 0 & 450 \\ -690 & 675 & -450 \end{vmatrix}\frac{T}{1065} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 600 & 0 & 450 \\ 270 & 675 & -450 \end{vmatrix}\frac{T}{855}$$

$$+\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 390 & 0 & 225 \\ 0 & -981 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 780 & 0 & 0 \\ 0 & B_y & B_z \end{vmatrix} = 0$$

PROBLEM 4.117 (Continued)

Coefficient of **i**: $$-(450)(675)\frac{T}{1065}-(450)(675)\frac{T}{855}+220.73\times 10^3=0$$

(a) $T=344.64\text{ N}\quad T=345\text{ N}$ ◀

Coefficient of **j**: $$(-690\times 450+600\times 450)\frac{344.64}{1065}+(270\times 450+600\times 450)\frac{344.64}{855}-780B_z=0$$

$$B_z=185.516\text{ N}$$

Coefficient of **k**: $$(600)(675)\frac{344.64}{1065}+(600)(675)\frac{344.64}{855}-382.59\times 10^3+780B_y=0\quad B_y=113.178\text{ N}$$

(*b*) $\mathbf{B}=(113.2\text{ N})\mathbf{j}+(185.5\text{ N})\mathbf{k}$ ◀

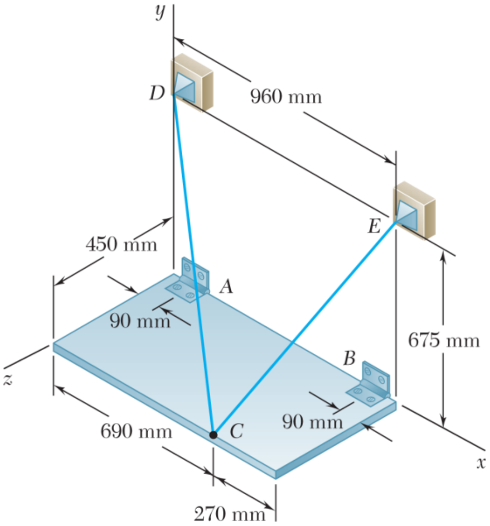
$$\Sigma\mathbf{F}=0:\quad \mathbf{A}+\mathbf{B}+\mathbf{T}_{CD}+\mathbf{T}_{CE}+\mathbf{W}=0$$

Coefficient of **i**: $$A_x-\frac{690}{1065}(344.64)+\frac{270}{855}(344.64)=0\qquad A_x=114.454\text{ N}$$

Coefficient of **j**: $$A_y+113.178+\frac{675}{1065}(344.64)+\frac{675}{855}(344.64)-981=0\qquad A_y=377.30\text{ N}$$

Coefficient of **k**: $$A_z+185.516-\frac{450}{1065}(344.64)-\frac{450}{855}(344.64)=0\qquad A_z=141.496\text{ N}$$

$\mathbf{A}=(114.4\text{ N})\mathbf{i}+(377\text{ N})\mathbf{j}+(141.5\text{ N})\mathbf{k}$ ◀



PROBLEM 4.118

Solve Problem 4.117, assuming that cable *DCE* is replaced by a cable attached to Point *E* and hook *C*.

PROBLEM 4.117 A 100-kg uniform rectangular plate is supported in the position shown by hinges *A* and *B* and by cable *DCE* that passes over a frictionless hook at *C*. Assuming that the tension is the same in both parts of the cable, determine (*a*) the tension in the cable, (*b*) the reactions at *A* and *B*. Assume that the hinge at *B* does not exert any axial thrust.

SOLUTION

See solution to Problem 4.113 for free-body diagram and analysis leading to the following:

$$CD = 1065 \text{ mm}$$

$$CE = 855 \text{ mm}$$

Now,

$$\mathbf{T}_{CD} = \frac{T}{1065}(-690\mathbf{i} + 675\mathbf{j} - 450\mathbf{k})$$

$$\mathbf{T}_{CE} = \frac{T}{855}(270\mathbf{i} + 675\mathbf{j} - 450\mathbf{k})$$

$$\mathbf{W} = -mg\mathbf{i} = -(100 \text{ kg})(9.81 \text{ m/s}^2)\mathbf{j} = -(981 \text{ N})\mathbf{j}$$

$$\Sigma\mathbf{M}_A = 0: \quad \mathbf{r}_{C/A} \times T_{CE} + \mathbf{r}_{G/A} \times (-W\mathbf{j}) + \mathbf{r}_{B/A} \times B = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 600 & 0 & 450 \\ 270 & 675 & -450 \end{vmatrix} \frac{T}{855} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 390 & 0 & 225 \\ 0 & -981 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 780 & 0 & 0 \\ 0 & B_y & B_z \end{vmatrix} = 0$$

Coefficient of **i**:

$$-(450)(675)\frac{T}{855} + 220.73 \times 10^3 = 0$$

$$T = 621.31 \text{ N} \qquad\qquad T = 621 \text{ N} \blacktriangleleft$$

Coefficient of **j**:

$$(270 \times 450 + 600 \times 450)\frac{621.31}{855} - 780B_z = 0 \quad B_z = 364.74 \text{ N}$$

Coefficient of **k**:

$$(600)(675)\frac{621.31}{855} - 382.59 \times 10^3 + 780B_y = 0 \quad B_y = 113.186 \text{ N}$$

$$\mathbf{B} = (113.2 \text{ N})\mathbf{j} + (365 \text{ N})\mathbf{k} \blacktriangleleft$$

PROBLEM 4.118 (Continued)

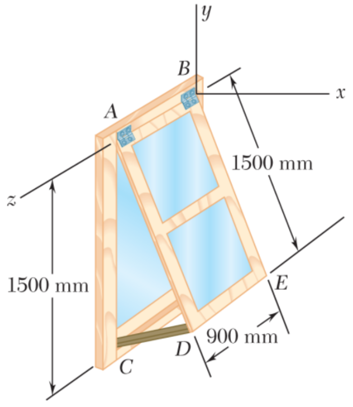
$$\Sigma \mathbf{F} = 0: \quad \mathbf{A} + \mathbf{B} + \mathbf{T}_{CE} + \mathbf{W} = 0$$

Coefficient of **i**: $A_x + \frac{270}{855}(621.31) = 0$ $\qquad A_x = -196.2 \text{ N}$

Coefficient of **j**: $A_y + 113.186 + \frac{675}{855}(621.31) - 981 = 0$ $\qquad A_y = 377.3 \text{ N}$

Coefficient of **k**: $A_z + 364.74 - \frac{450}{855}(621.31) = 0$ $\qquad A_z = -37.7 \text{ N}$

$$\mathbf{A} = -(196.2 \text{ N})\mathbf{i} + (377 \text{ N})\mathbf{j} - (37.7 \text{ N})\mathbf{k} \blacktriangleleft$$



PROBLEM 4.119

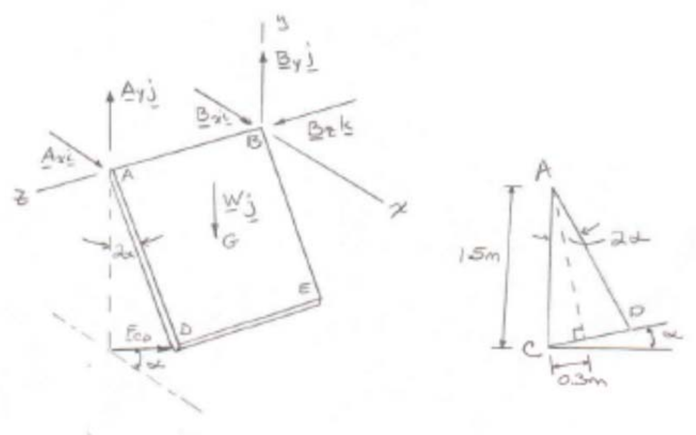
Solve Prob. 4.113, assuming that the hinge at *A* has been removed and that the hinge at *B* can exert couples about axes parallel to the *x* and *y* axes.

PROBLEM 4.113A 10-kg storm window measuring 900 × 1500 mm is held by hinges at *A* and *B*. In the position shown, it is held away from the side of the house by a 600-mm stick *CD*. Assuming that the hinge at *A* does not exert any axial thrust, determine the magnitude of the force exerted by the stick and the components of the reactions at *A* and *B*.

SOLUTION

Free-Body Diagram: Since *CD* is a two-force member, F_{CD}

is directed along *CD* and triangle *ACD* is isosceles. We have



$$\mathbf{W} = -(10\text{ kg})(9.81\text{ m/s}^2) = -(98.10\text{ N})\mathbf{j}$$

$$\mathbf{r}_G = (0.75\text{ m})\sin 23.074^\circ\,\mathbf{i} - (0.75\text{ m})\cos 23.074^\circ\,\mathbf{j} + (0.45\text{ m})\mathbf{k}$$
$$\mathbf{r}_G = (0.29394\text{ m})\mathbf{i} - (0.690\text{ m})\mathbf{j} + (0.45\text{ m})\mathbf{k}$$

$$\mathbf{F}_{CD} = F_{CD}(\cos 11.5370^\circ\,\mathbf{i} + \sin 11.5370^\circ\,\mathbf{j})$$
$$\mathbf{F}_{CD} = F_{CD}(0.97980\mathbf{i} + 0.20\mathbf{j})$$

With hinge at *A* removed, hinge at *B* will develop two couples to prevent rotation about the *x* and *z* axes.

$$\Sigma M_B = 0: \quad (M_B)_x\mathbf{i} + (M_B)_y\mathbf{j} + \mathbf{r}_G \times \mathbf{W} + \mathbf{r}_C \times \mathbf{F}_{CD} = 0$$

$$(M_B)_x\mathbf{i} + (M_B)_y\mathbf{j} + (0.29394\mathbf{i} - 0.690\mathbf{j} + 0.45\mathbf{k}) \times (-98.10)\mathbf{j}$$
$$+ (1.5\mathbf{j} + 0.9\mathbf{k}) \times F_{CD}(0.97980\mathbf{i} + 0.20\mathbf{j}) = 0$$

$$(M_B)_x\mathbf{i} + (M_B)_y\mathbf{j} - 28.836\mathbf{k} + 44.145\mathbf{i} + 1.46970F_{CD}\,\mathbf{k} + 0.88182F_{CD}\mathbf{j} - 0.180F_{CD}\mathbf{i} = 0$$

PROBLEM 4.119 (Continued)

Equating the coefficients of the unit vectors to zero,

$$\mathbf{k}: \quad -28.836 + 1.46970F_{CD} = 0; \quad F_{CD} = 19.6203 \text{ N}$$

$$\mathbf{j}: \quad (M_B)_y + 0.88182(19.6203) = 0; \qquad (M_B)_y = -17.3016 \text{ N}\cdot\text{m}$$

$$\mathbf{i}: \quad (M_B)_x + 44.145 - 0.180(19.6203) = 0; \quad (M_B)_x = -40.613 \text{ N}\cdot\text{m}$$

$$F_{CD} = 19.62 \text{ N} ◀$$

$$\mathbf{M}_B = -(40.6 \text{ N}\cdot\text{m})\mathbf{i} - (17.30 \text{ N}\cdot\text{m})\mathbf{j} ◀$$

$$\Sigma F_x = 0: \quad B_x + F_{CD}\cos 11.5370° = 0$$

$$B_x + 19.6203\cos 11.5370^\circ = 0$$

$$B_x = -19.22 \text{ N}$$

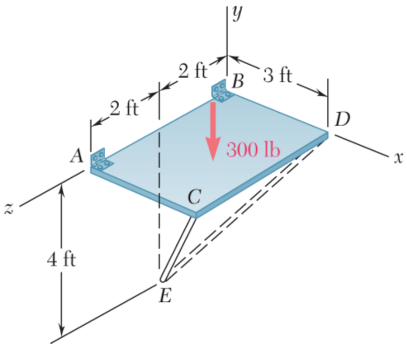
$$\Sigma F_y = 0: \quad B_y + F_{CD}\sin 11.5370^\circ - W = 0$$

$$B_y + 19.6230\sin 11.5370^\circ - 98.1 = 0$$

$$B_y = 94.2 \text{ N}$$

$$\Sigma F_z = 0: \quad B_z = 0$$

$$\mathbf{B} = -(19.22 \text{ N})\mathbf{i} + (94.2 \text{ N})\mathbf{j} ◀$$



PROBLEM 4.120

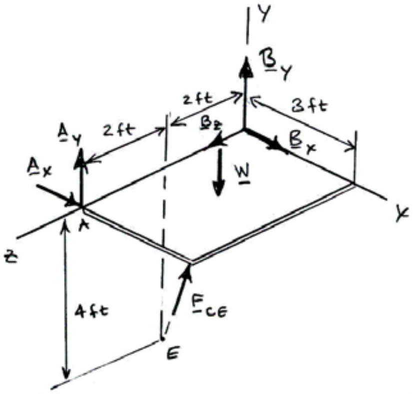
Solve Prob. 4.115, assuming that the hinge at *B* has been removed and that the hinge at *A* can exert an axial thrust, as well as couples about axes parallel to the *x* and *y* axes.

PROBLEM 4.115 The horizontal platform *ABCD* weighs 60 lb and supports a 240-lb load at its center. The platform is normally held in position by hinges at *A* and *B* and by braces *CE* and *DE*. If brace *DE* is removed, determine the reactions at the hinges and the force exerted by the remaining brace *CE*. The hinge at *A* does not exert any axial thrust.

SOLUTION

Free-Body Diagram:

$B_x = B_y = B_z = 0$



Express forces, weight in terms of rectangular components:

$$\overrightarrow{EC} = (3\text{ ft})\mathbf{i} + (4\text{ ft})\mathbf{j} + (2\text{ ft})\mathbf{k}$$

$$\mathbf{F}_{CE} = F_{CE}\frac{\overrightarrow{EC}}{EC} = F_{CE}\frac{3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{(3)^2 + (4)^2 + (2)^2}}$$

$$= 0.55709F_{CE}\,\mathbf{i} + 0.74278F_{CE}\,\mathbf{j} + 0.37139F_{CE}\,\mathbf{k}$$

$$\mathbf{W} = -(mg)\mathbf{j} = -(300\text{ lb})\mathbf{j}$$

$$\Sigma\mathbf{M}_A = 0\text{:}\ \ \mathbf{r}_{G/A} \times (-W)\mathbf{j} + \mathbf{r}_{C/A} \times F_{CE} + (M_{A_x})\mathbf{i} + (M_{A_y})\mathbf{j} = 0$$

$$\Sigma\mathbf{M}_A = 0\text{:}\qquad (1.5\mathbf{i} - 2\mathbf{k}) \times (-300\mathbf{j}) + 3\mathbf{i} \times F_{CE}(0.55709\mathbf{i} + 0.74278\mathbf{j} + 0.37139\mathbf{k}) + (M_{A_x})\mathbf{i} + (M_{A_y})\mathbf{j} = 0$$

or

$$-450\mathbf{k} - 600\mathbf{i} + 2.2283F_{CE}\mathbf{k} - 1.11417F_{CE}\mathbf{j} + M_{A_x}\mathbf{i} + M_{A_y}\mathbf{j} = 0$$

PROBLEM 4.120 (Continued)

Setting the coefficients of the unit vectors equal to zero:

$\mathbf{k}$: $-450 + \left(2.2283F_{CE}\right) = 0$

$$F_{CE} = 201.95 \text{ lb}$$

or $F_{CE} = 202 \text{ lb}$ ◀

$\mathbf{j}$: $-1.11417(201.95) + (M_{A_y}) = 0$

$$M_{A_y} = 225.00 \text{ lb} \cdot \text{ft}$$

$\mathbf{i}$: $-600 + M_{A_x} = 0$

$$M_{A_x} = 600 \text{ lb} \cdot \text{ft}$$

$\Sigma F_x = 0$: $A_x + 0.55709\left(201.94 \text{ lb}\right) = 0$

$$A_x = -112.499 \text{ lb}$$

$\Sigma F_y = 0$: $A_y - 300 \text{ lb} + 0.74278\left(201.94 \text{ lb}\right) = 0$

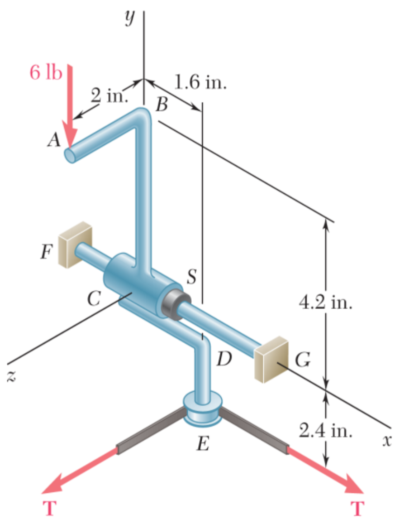
$$A_y = 150.003 \text{ lb}$$

$\Sigma F_z = 0$: $A_z + 0.371391\left(201.94 \text{ lb}\right) = 0$

$$A_z = -74.999 \text{ lb}$$

Therefore: $\mathbf{M}_A = \left(600 \text{ lb} \cdot \text{ft}\right)\mathbf{i} + (225 \text{ lb} \cdot \text{ft})\mathbf{j}$ ◀

$\mathbf{A} = -\left(112.5 \text{ lb}\right)\mathbf{i} + \left(150.0 \text{ lb}\right)\mathbf{j} - \left(75.0 \text{ lb}\right)\mathbf{k}$ ◀

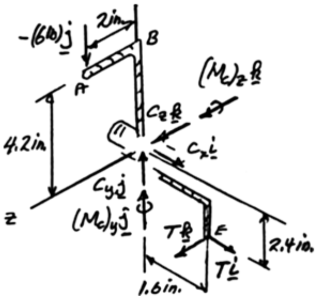


PROBLEM 4.121

The assembly shown is used to control the tension *T* in a tape that passes around a frictionless spool at *E*. Collar *C* is welded to rods *ABC* and *CDE*. It can rotate about shaft *FG* but its motion along the shaft is prevented by a washer *S*. For the loading shown, determine (*a*) the tension *T* in the tape, (*b*) the reaction at *C*.

SOLUTION

Free-Body Diagram:



$$\mathbf{r}_{A/C} = 4.2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}_{E/C} = 1.6\mathbf{i} - 2.4\mathbf{j}$$

$$\Sigma M_C = 0: \quad \mathbf{r}_{A/C} \times (-6\mathbf{j}) + \mathbf{r}_{E/C} \times T(\mathbf{i} + \mathbf{k}) + (M_C)_y\,\mathbf{j} + (M_C)_z\,\mathbf{k} = 0$$

$$(4.2\,\mathbf{j} + 2\mathbf{k}) \times (-6\mathbf{j}) + (1.6\mathbf{i} - 2.4\mathbf{j}) \times T(\mathbf{i} + \mathbf{k}) + (M_C)_y\,\mathbf{j} + (M_C)_z\,\mathbf{k} = 0$$

Coefficient of **i**: $12 - 2.4T = 0$ $T = 5.00 \text{ lb}$ ◀

Coefficient of **j**: $-1.6(5\text{ lb}) + (M_C)_y = 0 \quad (M_C)_y = 8 \text{ lb}\cdot\text{in.}$

Coefficient of **k**: $2.4(5\text{ lb}) + (M_C)_z = 0 \quad (M_C)_z = -12 \text{ lb}\cdot\text{in.}$

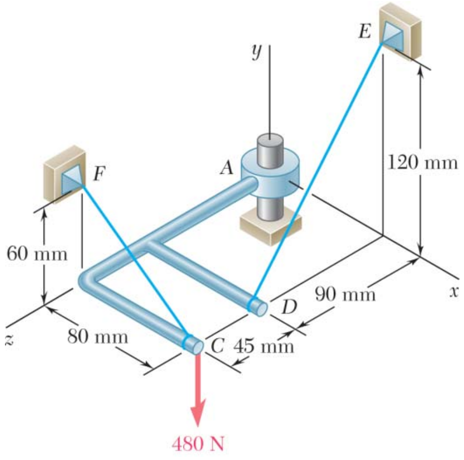
$$\mathbf{M}_C = (8.00 \text{ lb}\cdot\text{in.})\mathbf{j} - (12.00 \text{ lb}\cdot\text{in.})\mathbf{k}$$ ◀

$$\Sigma F = 0: \quad C_x\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k} - (6\text{ lb})\mathbf{j} + (5\text{ lb})\mathbf{i} + (5\text{ lb})\mathbf{k} = 0$$

Equate coefficients of unit vectors to zero.

$C_x = -5 \text{ lb} \quad C_y = 6 \text{ lb} \quad C_z = -5 \text{ lb}$ ◀

$$\mathbf{C} = -(5.00 \text{ lb})\mathbf{i} + (6.00 \text{ lb})\mathbf{j} - (5.00 \text{ lb})\mathbf{k}$$ ◀

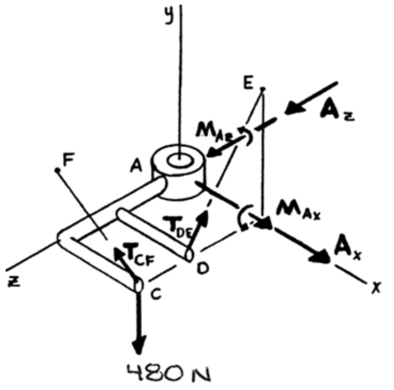


PROBLEM 4.122

The assembly shown is welded to collar *A* that fits on the vertical pin shown. The pin can exert couples about the *x* and *z* axes but does not prevent motion about or along the *y*-axis. For the loading shown, determine the tension in each cable and the reaction at *A*.

SOLUTION

Free-Body Diagram:



First note:

$$\mathbf{T}_{CF} = \boldsymbol{\lambda}_{CF} T_{CF} = \frac{-(0.08\text{ m})\mathbf{i} + (0.06\text{ m})\mathbf{j}}{\sqrt{(0.08)^2 + (0.06)^2}\text{ m}} T_{CF}$$

$$= T_{CF}(-0.8\mathbf{i} + 0.6\mathbf{j})$$

$$\mathbf{T}_{DE} = \boldsymbol{\lambda}_{DE} T_{DE} = \frac{(0.12\text{ m})\mathbf{j} - (0.09\text{ m})\mathbf{k}}{\sqrt{(0.12)^2 + (0.09)^2}\text{ m}} T_{DE}$$

$$= T_{DE}(0.8\mathbf{j} - 0.6\mathbf{k})$$

(*a*) From F.B.D. of assembly:

$$\Sigma F_y = 0\text{:} \quad 0.6T_{CF} + 0.8T_{DE} - 480\text{ N} = 0$$

or

$$0.6T_{CF} + 0.8T_{DE} = 480\text{ N} \tag{1}$$

$$\Sigma M_y = 0\text{:} \quad -(0.8T_{CF})(0.135\text{ m}) + (0.6T_{DE})(0.08\text{ m}) = 0$$

or

$$T_{DE} = 2.25T_{CF} \tag{2}$$

Substituting Equation (2) into Equation (1),

$$0.6T_{CF} + 0.8[(2.25)T_{CF}] = 480\text{ N}$$

$$T_{CF} = 200.00\text{ N}$$

or $T_{CF} = 200\text{ N}$ ◀

and from Equation (2): $T_{DE} = 2.25(200.00\text{ N}) = 450.00$

or $T_{DE} = 450\text{ N}$ ◀

(*b*) From F.B.D. of assembly:

$$\Sigma F_z = 0: \quad A_z - (0.6)(450.00 \text{ N}) = 0 \quad A_z = 270.00 \text{ N}$$

$$\Sigma F_x = 0: \quad A_x - (0.8)(200.00 \text{ N}) = 0 \quad A_x = 160.000 \text{ N}$$

or $\mathbf{A} = (160.0 \text{ N})\mathbf{i} + (270 \text{ N})\mathbf{k}$ ◀

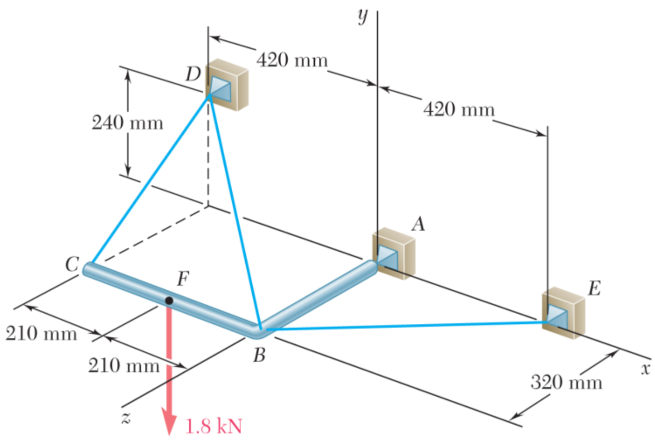
$$\Sigma M_x = 0: \quad M_{A_x} + (480 \text{ N})(0.135 \text{ m}) - [(200.00 \text{ N})(0.6)](0.135 \text{ m}) - [(450 \text{ N})(0.8)](0.09 \text{ m}) = 0$$

$$M_{A_x} = -16.2000 \text{ N}\cdot\text{m}$$

$$\Sigma M_z = 0: \quad M_{A_z} - (480 \text{ N})(0.08 \text{ m}) + [(200.00 \text{ N})(0.6)](0.08 \text{ m}) + [(450 \text{ N})(0.8)](0.08 \text{ m}) = 0$$

$$M_{A_z} = 0$$

or $\mathbf{M}_A = -(16.20 \text{ N}\cdot\text{m})\mathbf{i}$ ◀



y
420 mm
D
420 mm
240 mm
A
C
F
E
210 mm
210 mm
B
320 mm
x
z
1.8 kN

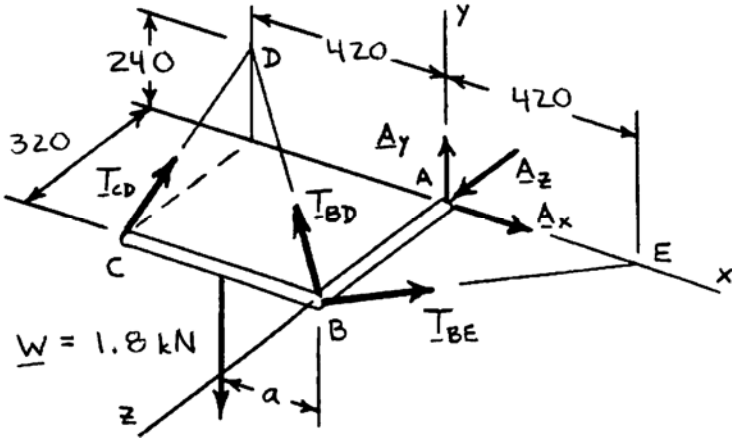
PROBLEM 4.123

The rigid L-shaped member *ABC* is supported by a ball-and-socket joint at *A* and by three cables. If a 1.8-kN load is applied at *F*, determine the tension in each cable.

SOLUTION

Free-Body Diagram:

Dimensions in mm



y
240
D
420
420
320
A_y
T_{CD}
A
A_z
T_{BD}
A_x
E
C
x
B
T_{BE}
W = 1.8 kN
a
z

In this problem: $a = 210$ mm

We have

$$\overrightarrow{CD} = (240 \text{ mm})\mathbf{j} - (320 \text{ mm})\mathbf{k} \quad CD = 400 \text{ mm}$$

$$\overrightarrow{BD} = -(420 \text{ mm})\mathbf{i} + (240 \text{ mm})\mathbf{j} - (320 \text{ mm})\mathbf{k} \quad BD = 580 \text{ mm}$$

$$\overrightarrow{BE} = (420 \text{ mm})\mathbf{i} - (320 \text{ mm})\mathbf{k} \quad BE = 528.02 \text{ mm}$$

Thus,

$$T_{CD} = T_{CD}\frac{\overrightarrow{CD}}{CD} = T_{CD}(0.6\mathbf{j} - 0.8\mathbf{k})$$

$$T_{BD} = T_{BD}\frac{\overrightarrow{BD}}{BD} = T_{BD}(-0.72414\mathbf{i} + 0.41379\mathbf{j} - 0.55172\mathbf{k})$$

$$T_{BE} = T_{BE}\frac{\overrightarrow{BE}}{BE} = T_{BE}(0.79542\mathbf{i} - 0.60604\mathbf{k})$$

$$\Sigma\mathbf{M}_A = 0: \quad (\mathbf{r}_C \times \mathbf{T}_{CD}) + (\mathbf{r}_B \times \mathbf{T}_{BD}) + (\mathbf{r}_B \times \mathbf{T}_{BE}) + (\mathbf{r}_W \times \mathbf{W}) = 0$$

Noting that

$$\mathbf{r}_C = -(420\text{ mm})\mathbf{i} + (320\text{ mm})\mathbf{k}$$
$$\mathbf{r}_B = (320\text{ mm})\mathbf{k}$$
$$\mathbf{r}_W = -a\mathbf{i} + (320\text{ mm})\mathbf{k}$$

and using determinants, we write

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -420 & 0 & 320 \\ 0 & 0.6 & -0.8 \end{vmatrix} T_{CD} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 320 \\ -0.72414 & 0.41379 & -0.55172 \end{vmatrix} T_{BD}$$
$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 320 \\ 0.79542 & 0 & -0.60604 \end{vmatrix} T_{BE} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & 0 & 320 \\ 0 & -1.8 & 0 \end{vmatrix} = 0$$

Equating to zero the coefficients of the unit vectors,

$$\mathbf{i}: \quad -192T_{CD} - 132.413T_{BD} + 576 = 0 \qquad (1)$$

$$\mathbf{j}: \quad -336T_{CD} - 231.72T_{BD} + 254.53T_{BE} = 0 \qquad (2)$$

$$\mathbf{k}: \quad -252T_{CD} + 1.8a = 0 \qquad (3)$$

Recalling that $a = 210$ mm, Eq. (3) yields

$$T_{CD} = \frac{1.8(210)}{252} = 1.500\text{ kN} \qquad T_{CD} = 1.500\text{ kN} \blacktriangleleft$$

From Eq. (1):

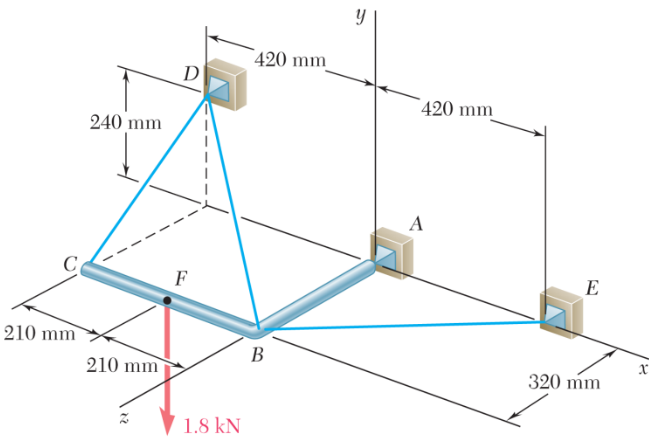
$$-192(1.5) - 132.413T_{BD} + 576 = 0$$

$$T_{BD} = 2.1751\text{ kN} \qquad T_{BD} = 2.18\text{ kN} \blacktriangleleft$$

From Eq. (2):

$$-336(1.5) - 231.72(2.1751) + 254.53T_{BE} = 0$$

$$T_{BE} = 3.9603\text{ kN} \qquad T_{BE} = 3.96\text{ kN} \blacktriangleleft$$



PROBLEM 4.124

Solve Problem 4.123, assuming that the 1.8-kN load is applied at *C*.

PROBLEM 4.123 The rigid L-shaped member *ABC* is supported by a ball-and-socket joint at *A* and by three cables. If a 1.8-kN load is applied at *F*, determine the tension in each cable.

SOLUTION

See solution of Problem 4.123 for free-body diagram and derivation of Eqs. (1), (2), and (3):

$$-192T_{CD} - 132.413T_{BD} + 576 = 0 \tag{1}$$

$$-336T_{CD} - 231.72T_{BD} + 254.53T_{BE} = 0 \tag{2}$$

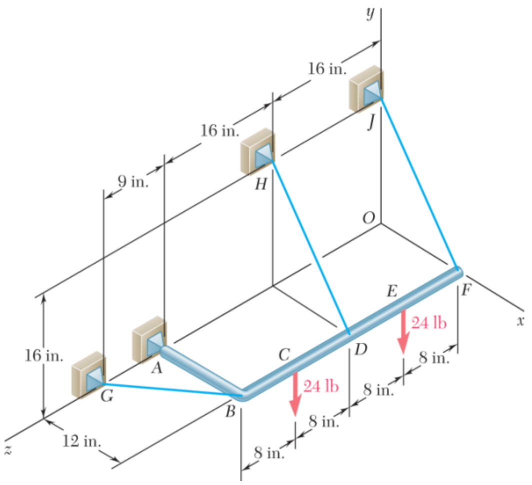
$$-252T_{CD} + 1.8a = 0 \tag{3}$$

In this problem, the 1.8-kN load is applied at *C* and we have $a = 420$ mm. Carrying into Eq. (3) and solving for T_{CD},

$$T_{CD} = 3.00$$ $T_{CD} = 3.00$ kN ◀

From Eq. (1): $-(192)(3) - 132.413T_{BD} + 576 = 0$ $T_{BD} = 0$ ◀

From Eq. (2): $-336(3) - 0 + 254.53T_{BE} = 0$ $T_{BE} = 3.96$ kN ◀

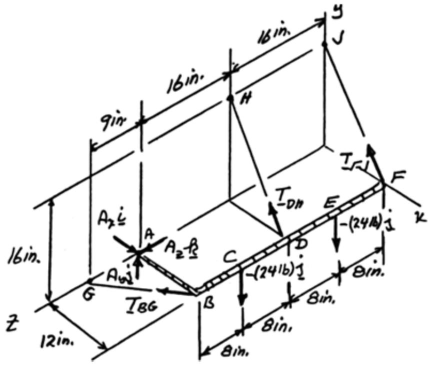


PROBLEM 4.125

The rigid L-shaped member *ABF* is supported by a ball-and-socket joint at *A* and by three cables. For the loading shown, determine the tension in each cable and the reaction at *A*.

SOLUTION

Free-Body Diagram:



$$\mathbf{r}_{B/A} = 12\mathbf{i}$$
$$\mathbf{r}_{F/A} = 12\mathbf{j} - 8\mathbf{k}$$
$$\mathbf{r}_{D/A} = 12\mathbf{i} - 16\mathbf{k}$$
$$\mathbf{r}_{E/A} = 12\mathbf{i} - 24\mathbf{k}$$
$$\mathbf{r}_{F/A} = 12\mathbf{i} - 32\mathbf{k}$$

$$\overrightarrow{BG} = -12\mathbf{i} + 9\mathbf{k}$$
$$BG = 15 \text{ in.}$$
$$\boldsymbol{\lambda}_{BG} = -0.8\mathbf{i} + 0.6\mathbf{k}$$

$$\overrightarrow{DH} = -12\mathbf{i} + 16\mathbf{j}; \quad DH = 20 \text{ in.}; \quad \lambda_{DH} = -0.6\mathbf{i} + 0.8\mathbf{j}$$
$$\overrightarrow{FJ} = -12\mathbf{i} + 16\mathbf{j}; \quad FJ = 20 \text{ in.}; \quad \lambda_{FJ} = -0.6\mathbf{i} + 0.8\mathbf{j}$$

$$\Sigma \mathbf{M}_A = 0: \quad \mathbf{r}_{B/A} \times \mathbf{T}_{BG}\lambda_{BG} + \mathbf{r}_{DH} \times \mathbf{T}_{DH}\lambda_{DH} + \mathbf{r}_{F/A} \times \mathbf{T}_{FJ}\lambda_{FJ}$$
$$+\mathbf{r}_{F/A} \times (-24\mathbf{j}) + \mathbf{r}_{E/A} \times (-24\mathbf{j}) = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & 0 \\ -0.8 & 0 & 0.6 \end{vmatrix} T_{BG} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & -16 \\ -0.6 & 0.8 & 0 \end{vmatrix} T_{DH} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & -32 \\ -0.6 & 0.8 & 0 \end{vmatrix} T_{FJ}$$
$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & -8 \\ 0 & -24 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & -24 \\ 0 & -24 & 0 \end{vmatrix} = 0$$

Coefficient of **i**: $+12.8T_{DH} + 25.6T_{FJ} - 192 - 576 = 0$ (1)

Coefficient of **k**: $+9.6T_{DH} + 9.6T_{FJ} - 288 - 288 = 0$ (2)

$\frac{3}{4}$ Eq. (1) – Eq. (2): $9.6T_{FJ} = 0$ $\quad T_{FJ} = 0$ ◀

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PROBLEM 4.125 (Continued)

From Eq. (1): $12.8T_{DH} - 268 = 0$ $\quad T_{DH} = 60 \text{ lb}$ ◀

Coefficient of **j:** $-7.2T_{BG} + (16 \times 0.6)(60.0 \text{ lb}) = 0$ $\quad T_{BG} = 80.0 \text{ lb}$ ◀

$$\Sigma\mathbf{F} = 0: \quad \mathbf{A} + T_{BG}\boldsymbol{\lambda}_{BG} + T_{DH}\boldsymbol{\lambda}_{DH} + T_{FJ} - 24\mathbf{j} - 24\mathbf{j} = 0$$

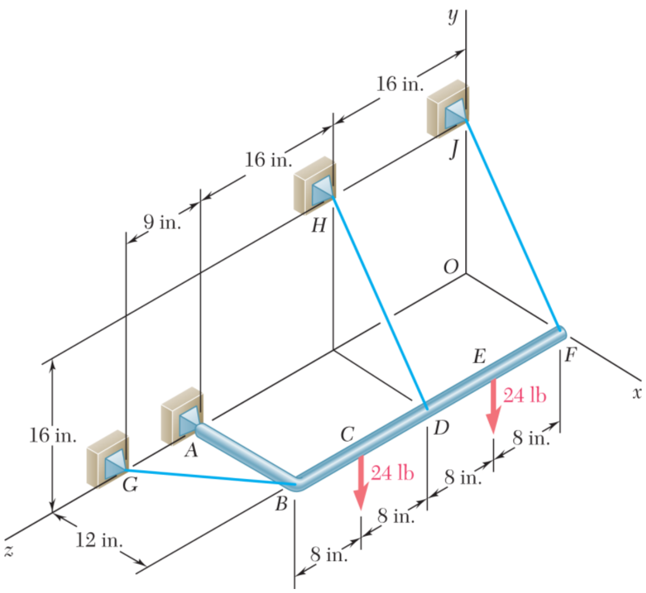
Coefficient of **i**: $A_x + (80)(-0.8) + (60.0)(-0.6) = 0 \quad A_x = 100.0 \text{ lb}$

Coefficient of **j**: $A_y + (60.0)(0.8) - 24 - 24 = 0 \quad A_y = 0$

Coefficient of **k:** $A_z + (80.0)(+0.6) = 0 \quad A_z = -48.0 \text{ lb}$

$\mathbf{A} = (100.0 \text{ lb})\mathbf{i} - (48.0 \text{ lb})\mathbf{k}$ ◀

Note: The value $A_y = 0$ can be confirmed by considering $\Sigma M_{BF} = 0$.


y
16 in.
16 in.
9 in.
J
H
O
E
F
x
24 lb
8 in.
D
C
8 in.
24 lb
8 in.
16 in.
A
G
B
8 in.
z
12 in.

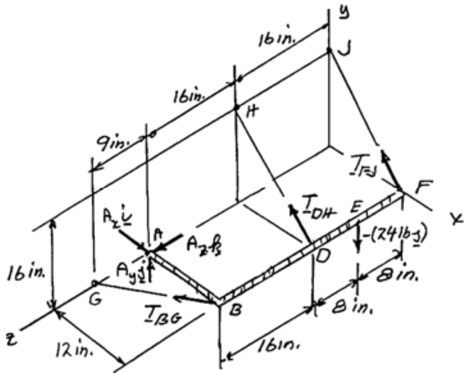
PROBLEM 4.126

Solve Problem 4.125, assuming that the load at *C* has been removed.

PROBLEM 4.125 The rigid L-shaped member *ABF* is supported by a ball-and-socket joint at *A* and by three cables. For the loading shown, determine the tension in each cable and the reaction at *A*.

SOLUTION

Free-Body Diagram:


y
16in.
J
16in.
H
9in.
T_{FJ}
F
T_{DH}
E
x
$-(24\text{ lb }\mathbf{j})$
$A_z\mathbf{k}$
A
$A_x\mathbf{i}$
16in.
$A_y\mathbf{j}$
D
8in.
G
T_{BG}
B
8in.
z
12in.
16in.

$$\mathbf{r}_{B/A} = 12\mathbf{i}$$

$$\mathbf{r}_{B/A} = 12\mathbf{i} - 16\mathbf{k}$$

$$\mathbf{r}_{E/A} = 12\mathbf{i} - 24\mathbf{k}$$

$$\mathbf{r}_{F/A} = 12\mathbf{i} - 32\mathbf{k}$$

$$\overrightarrow{BG} = -12\mathbf{i} + 9\mathbf{k}; \quad BG = 15\text{ in.}; \quad \boldsymbol{\lambda}_{BG} = -0.8\mathbf{i} + 0.6\mathbf{k}$$

$$\overrightarrow{DH} = -12\mathbf{i} + 16\mathbf{j}; \quad DH = 20\text{ in.}; \quad \boldsymbol{\lambda}_{DH} = -0.6\mathbf{i} + 0.8\mathbf{j}$$

$$\overrightarrow{FJ} = -12\mathbf{i} + 16\mathbf{j}; \quad FJ = 20\text{ in.}; \quad \boldsymbol{\lambda}_{FJ} = -0.6\mathbf{i} + 0.8\mathbf{j}$$

$$\Sigma\mathbf{M}_A = 0\text{:} \quad \mathbf{r}_{B/A} \times T_{BG}\,\boldsymbol{\lambda}_{BG} + \mathbf{r}_{D/A} \times T_{DH}\,\boldsymbol{\lambda}_{DH} + \mathbf{r}_{F/A} \times T_{FJ}\boldsymbol{\lambda}_{FJ} + \mathbf{r}_{E/A} \times (-24\mathbf{j}) = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & 0 \\ -0.8 & 0 & 0.6 \end{vmatrix} T_{BG} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & -16 \\ -0.6 & 0.8 & 0 \end{vmatrix} T_{DH} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & -32 \\ -0.6 & 0.8 & 0 \end{vmatrix} T_{FJ} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & -24 \\ 0 & -24 & 0 \end{vmatrix} = 0$$

$$\mathbf{i}\text{:} \quad +12.8T_{DH} + 25.6T_{FJ} - 576 = 0 \tag{1}$$

$$\mathbf{k}\text{:} \quad +9.6T_{DH} + 9.6T_{FJ} - 288 = 0 \tag{2}$$

PROBLEM 4.126 (Continued)

Multiply Eq. (1) by $\frac{3}{4}$ and subtract Eq. (2):

$$9.6T_{FJ} - 144 = 0 \qquad T_{FJ} = 15.00 \text{ lb} \blacktriangleleft$$

From Eq. (1):
$$12.8T_{DH} + 25.6(15.00) - 576 = 0 \qquad T_{DH} = 15.00 \text{ lb} \blacktriangleleft$$

$$\mathbf{j}: \quad -7.2T_{BG} + (16)(0.6)(15) + (32)(0.6)(15) = 0$$

$$-7.2T_{BG} + 432 = 0 \qquad T_{BG} = 60.0 \text{ lb} \blacktriangleleft$$

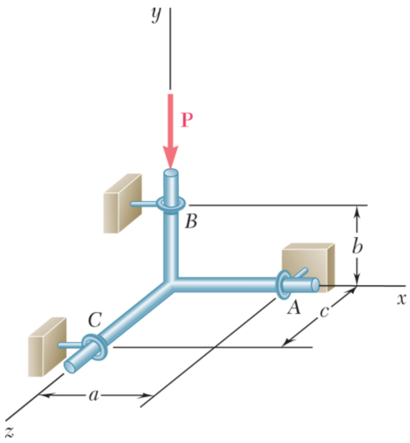
$$\Sigma F = 0: \quad \mathbf{A} + T_{BG}\boldsymbol{\lambda}_{BG} + T_{DA}\boldsymbol{\lambda}_{DH} + T_{FJ}\boldsymbol{\lambda}_{FJ} - 24\mathbf{j} = 0$$

$$\mathbf{i}: \quad A_x + (60)(-0.8) + (15)(-0.6) + (15)(-0.6) = 0 \qquad A_x = 66.0 \text{ lb}$$

$$\mathbf{j}: \quad A_y + (15)(0.8) + (15)(0.8) - 24 = 0 \qquad A_y = 0$$

$$\mathbf{k}: \quad A_z + (60)(0.6) = 0 \qquad A_z = -36.0 \text{ lb}$$

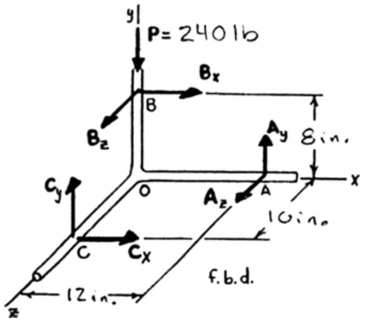
$$\mathbf{A} = (66.0 \text{ lb})\mathbf{i} - (36.0 \text{ lb})\mathbf{k} \blacktriangleleft$$



PROBLEM 4.127

Three rods are welded together to form a “corner” that is supported by three eyebolts. Neglecting friction, determine the reactions at A, B, and C when $P = 240$ lb, $a = 12$ in., $b = 8$ in., and $c = 10$ in.

SOLUTION



From F.B.D. of weldment:

$$\Sigma \mathbf{M}_O = 0: \quad \mathbf{r}_{A/O} \times \mathbf{A} + \mathbf{r}_{B/O} \times \mathbf{B} + \mathbf{r}_{C/O} \times \mathbf{C} = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & 0 \\ 0 & A_y & A_z \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & 0 \\ B_x & 0 & B_z \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 10 \\ C_x & C_y & 0 \end{vmatrix} = 0$$

$$(-12A_z\mathbf{j} + 12A_y\mathbf{k}) + (8B_z\mathbf{i} - 8B_x\mathbf{k}) + (-10C_y\mathbf{i} + 10C_x\mathbf{j}) = 0$$

From **i**-coefficient: $\quad 8B_z - 10C_y = 0$

or $\quad B_z = 1.25C_y \quad (1)$

j-coefficient: $\quad -12A_z + 10C_x = 0$

or $\quad C_x = 1.2A_z \quad (2)$

k-coefficient: $\quad 12A_y - 8B_x = 0$

or $\quad B_x = 1.5A_y \quad (3)$

$$\Sigma \mathbf{F} = 0: \quad \mathbf{A} + \mathbf{B} + \mathbf{C} - \mathbf{P} = 0$$

or $\quad (B_x + C_x)\mathbf{i} + (A_y + C_y - 240\text{ lb})\mathbf{j} + (A_z + B_z)\mathbf{k} = 0$

From **i**-coefficient: $\quad B_x + C_x = 0$

or $\quad C_x = -B_x \quad (4)$

j-coefficient: $\quad A_y + C_y - 240\text{ lb} = 0$

or $\quad A_y + C_y = 240\text{ lb} \quad (5)$

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PROBLEM 4.127 (Continued)

k-coefficient: $A_z + B_z = 0$

or $$A_z = -B_z \tag{6}$$

Substituting C_x from Equation (4) into Equation (2),

$$-B_z = 1.2A_z \tag{7}$$

Using Equations (1), (6), and (7),

$$C_y = \frac{B_z}{1.25} = \frac{-A_z}{1.25} = \frac{1}{1.25}\left(\frac{B_x}{1.2}\right) = \frac{B_x}{1.5} \tag{8}$$

From Equations (3) and (8):

$$C_y = \frac{1.5A_y}{1.5} \quad \text{or} \quad C_y = A_y$$

and substituting into Equation (5),

$$2A_y = 240 \text{ lb}$$

$$A_y = C_y = 120 \text{ lb} \tag{9}$$

Using Equation (1) and Equation (9),

$$B_z = 1.25(120 \text{ lb}) = 150.0 \text{ lb}$$

Using Equation (3) and Equation (9),

$$B_x = 1.5(120 \text{ lb}) = 180.0 \text{ lb}$$

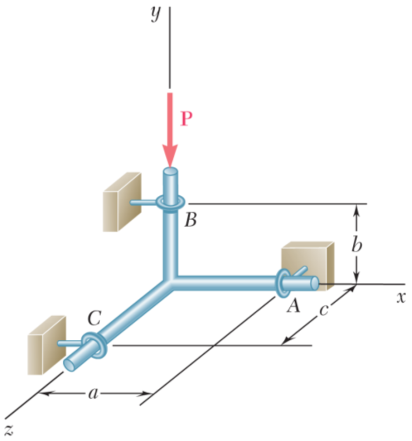
From Equation (4): $C_x = -180.0 \text{ lb}$

From Equation (6): $A_z = -150.0 \text{ lb}$

Therefore, $\mathbf{A} = (120.0 \text{ lb})\mathbf{j} - (150.0 \text{ lb})\mathbf{k}$ ◀

$\mathbf{B} = (180.0 \text{ lb})\mathbf{i} + (150.0 \text{ lb})\mathbf{k}$ ◀

$\mathbf{C} = -(180.0 \text{ lb})\mathbf{i} + (120.0 \text{ lb})\mathbf{j}$ ◀

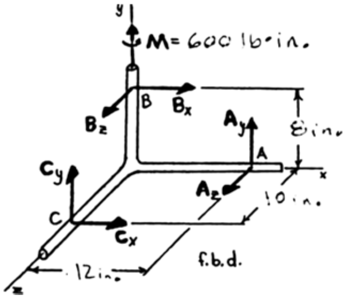


PROBLEM 4.128

Solve Problem 4.127, assuming that the force **P** is removed and is replaced by a couple $\mathbf{M} = +(600\text{ lb}\cdot\text{in.})\mathbf{j}$ acting at B.

PROBLEM 4.127 Three rods are welded together to form a "corner" that is supported by three eyebolts. Neglecting friction, determine the reactions at A, B, and C when $P = 240$ lb, $a = 12$ in., $b = 8$ in., and $c = 10$ in.

SOLUTION



From F.B.D. of weldment:

$$\Sigma\mathbf{M}_O = 0: \quad \mathbf{r}_{A/O} \times \mathbf{A} + \mathbf{r}_{B/O} \times \mathbf{B} + \mathbf{r}_{C/O} \times \mathbf{C} + \mathbf{M} = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 0 & 0 \\ 0 & A_y & A_z \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & 0 \\ B_x & 0 & B_z \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 10 \\ C_x & C_y & 0 \end{vmatrix} + (600\text{ lb}\cdot\text{in.})\mathbf{j} = 0$$

$$(-12A_z\mathbf{j} + 12A_y\mathbf{k}) + (8B_z\mathbf{j} - 8B_x\mathbf{k}) + (-10C_y\mathbf{i} + 10C_x\mathbf{j}) + (600\text{ lb}\cdot\text{in.})\mathbf{j} = 0$$

From **i**-coefficient: $8B_z - 10C_y = 0$

or $C_y = 0.8B_z$ (1)

j-coefficient: $-12A_z + 10C_x + 600 = 0$

or $C_x = 1.2A_z - 60$ (2)

k-coefficient: $12A_y - 8B_x = 0$

or $B_x = 1.5A_y$ (3)

$$\Sigma\mathbf{F} = 0: \quad \mathbf{A} + \mathbf{B} + \mathbf{C} = 0$$

$$(B_x + C_x)\mathbf{i} + (A_y + C_y)\mathbf{j} + (A_z + B_z)\mathbf{k} = 0$$

From **i**-coefficient: $C_x = -B_x$ (4)

j-coefficient: $C_y = -A_y$ (5)

k-coefficient: $A_z = -B_z$ (6)

PROBLEM 4.128 (Continued)

Substituting C_x from Equation (4) into Equation (2),

$$A_z = 50 - \left(\frac{B_x}{1.2}\right) \tag{7}$$

Using Equations (1), (6), and (7),

$$C_y = 0.8B_z = -0.8A_z = \left(\frac{2}{3}\right)B_x - 40 \tag{8}$$

From Equations (3) and (8):

$$C_y = A_y - 40$$

Substituting into Equation (5), $2A_y = 40$

$$A_y = 20.0 \text{ lb}$$

From Equation (5): $C_y = -20.0 \text{ lb}$

Equation (1): $B_z = -25.0 \text{ lb}$

Equation (3): $B_x = 30.0 \text{ lb}$

Equation (4): $C_x = -30.0 \text{ lb}$

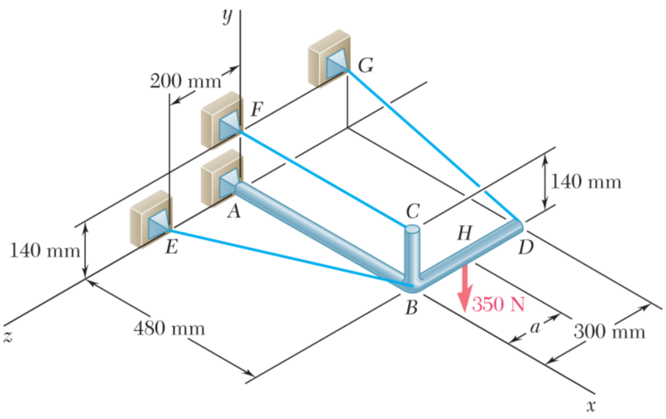
Equation (6): $A_z = 25.0 \text{ lb}$

Therefore,

$\mathbf{A} = (20.0 \text{ lb})\mathbf{j} + (25.0 \text{ lb})\mathbf{k}$ ◀

$\mathbf{B} = (30.0 \text{ lb})\mathbf{i} - (25.0 \text{ lb})\mathbf{k}$ ◀

$\mathbf{C} = -(30.0 \text{ lb})\mathbf{i} - (20.0 \text{ lb})\mathbf{j}$ ◀


y
200 mm
G
F
140 mm
A
C
H
D
140 mm
E
350 N
B
a
300 mm
480 mm
z
x

PROBLEM 4.129

Frame *ABCD* is supported by a ball-and-socket joint at *A* and by three cables. For $a = 150$ mm, determine the tension in each cable and the reaction at *A*.

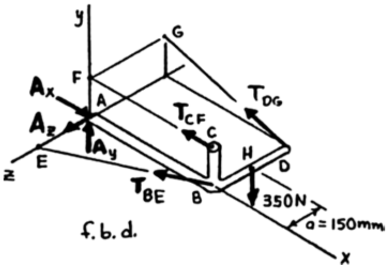
SOLUTION

First note:

$$\mathbf{T}_{DG} = \boldsymbol{\lambda}_{DG} T_{DG} = \frac{-(0.48 \text{ m})\mathbf{i} + (0.14 \text{ m})\mathbf{j}}{\sqrt{(0.48)^2 + (0.14)^2} \text{ m}} T_{DG}$$

$$= \frac{-0.48\mathbf{i} + 0.14\mathbf{j}}{0.50} T_{DG}$$

$$= \frac{T_{DG}}{25}(24\mathbf{i} + 7\mathbf{j})$$


y
G
F
T_DG
A_x
A
T_CF
A_z
C
A_y
H
z
E
D
T_BE
B
350N
a = 150mm
f.b.d.
x

$$\mathbf{T}_{BE} = \boldsymbol{\lambda}_{BE} T_{BE} = \frac{-(0.48 \text{ m})\mathbf{i} + (0.2 \text{ m})\mathbf{k}}{\sqrt{(0.48)^2 + (0.2)^2} \text{ m}} T_{BE}$$

$$= \frac{-0.48\mathbf{i} + 0.2\mathbf{k}}{0.52} T_{BE}$$

$$= \frac{T_{BE}}{13}(-12\mathbf{j} + 5\mathbf{k})$$

From F.B.D. of frame *ABCD*:

$$\Sigma M_x = 0: \quad \left(\frac{7}{25} T_{DG}\right)(0.3 \text{ m}) - (350 \text{ N})(0.15 \text{ m}) = 0$$

or $T_{DG} = 625 \text{ N}$ ◀

$$\Sigma M_y = 0: \quad \left(\frac{24}{25} \times 625 \text{ N}\right)(0.3 \text{ m}) - \left(\frac{5}{13} T_{BE}\right)(0.48 \text{ m}) = 0$$

or $T_{BE} = 975 \text{ N}$ ◀

$$\Sigma M_z = 0: \quad T_{CF}(0.14 \text{ m}) + \left(\frac{7}{25} \times 625 \text{ N}\right)(0.48 \text{ m}) - (350 \text{ N})(0.48 \text{ m}) = 0$$

or $T_{CF} = 600 \text{ N}$ ◀

PROBLEM 4.129 (Continued)

$$\Sigma F_x = 0: \quad A_x + T_{CF} + (T_{BE})_x + (T_{DG})_x = 0$$

$$A_x - 600\text{ N} - \left(\frac{12}{13}\times 975\text{ N}\right) - \left(\frac{24}{25}\times 625\text{ N}\right) = 0$$

$$A_x = 2100\text{ N}$$

$$\Sigma F_y = 0: \quad A_y + (T_{DG})_y - 350\text{ N} = 0$$

$$A_y + \left(\frac{7}{25}\times 625\text{ N}\right) - 350\text{ N} = 0$$

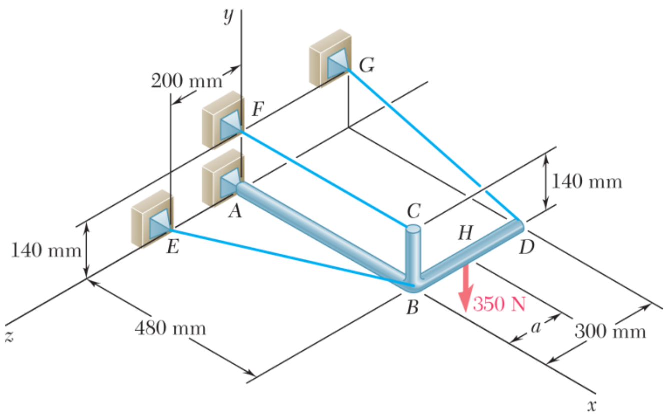
$$A_y = 175.0\text{ N}$$

$$\Sigma F_z = 0: \quad A_z + (T_{BE})_z = 0$$

$$A_z + \left(\frac{5}{13}\times 975\text{ N}\right) = 0$$

$$A_z = -375\text{ N}$$

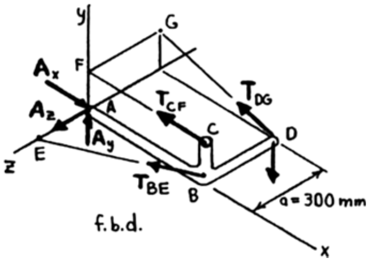
Therefore, $\mathbf{A} = (2100\text{ N})\mathbf{i} + (175.0\text{ N})\mathbf{j} - (375\text{ N})\mathbf{k}$ ◀



PROBLEM 4.130

Frame *ABCD* is supported by a ball-and-socket joint at *A* and by three cables. Knowing that the 350-N load is applied at *D* ($a = 300$ mm), determine the tension in each cable and the reaction at *A*.

SOLUTION



First note
$$\mathbf{T}_{DG} = \boldsymbol{\lambda}_{DG}T_{DG} = \frac{-(0.48\text{ m})\mathbf{i} + (0.14\text{ m})\mathbf{j}}{\sqrt{(0.48)^2 + (0.14)^2}\text{ m}}T_{DG}$$

$$= \frac{-0.48\mathbf{i} + 0.14\mathbf{j}}{0.50}T_{DG}$$

$$= \frac{T_{DG}}{25}(24\mathbf{i} + 7\mathbf{j})$$

$$\mathbf{T}_{BE} = \boldsymbol{\lambda}_{BE}T_{BE} = \frac{-(0.48\text{ m})\mathbf{i} + (0.2\text{ m})\mathbf{k}}{\sqrt{(0.48)^2 + (0.2)^2}\text{ m}}T_{BE}$$

$$= \frac{-0.48\mathbf{i} + 0.2\mathbf{k}}{0.52}T_{BE}$$

$$= \frac{T_{BE}}{13}(-12\mathbf{i} + 5\mathbf{k})$$

From f.b.d. of frame *ABCD*

$$\Sigma M_x = 0\text{:}\quad \left(\frac{7}{25}T_{DG}\right)(0.3\text{ m}) - (350\text{ N})(0.3\text{ m}) = 0$$

or $T_{DG} = 1250\text{ N}$ ◀

$$\Sigma M_y = 0\text{:}\quad \left(\frac{24}{25}\times 1250\text{ N}\right)(0.3\text{ m}) - \left(\frac{5}{13}T_{BE}\right)(0.48\text{ m}) = 0$$

or $T_{BE} = 1950\text{ N}$ ◀

$$\Sigma M_z = 0\text{:}\quad T_{CF}(0.14\text{ m}) + \left(\frac{7}{25}\times 1250\text{ N}\right)(0.48\text{ m}) - (350\text{ N})(0.48\text{ m}) = 0$$

or $T_{CF} = 0$ ◀

PROBLEM 4.130 (Continued)

$$\Sigma F_x = 0: \quad A_x + T_{CF} + (T_{BE})_x + (T_{DG})_x = 0$$

$$A_x + 0 - \left(\frac{12}{13} \times 1950 \text{ N}\right) - \left(\frac{24}{25} \times 1250 \text{ N}\right) = 0$$

$$A_x = 3000 \text{ N}$$

$$\Sigma F_y = 0: \quad A_y + (T_{DG})_y - 350 \text{ N} = 0$$

$$A_y + \left(\frac{7}{25} \times 1250 \text{ N}\right) - 350 \text{ N} = 0$$

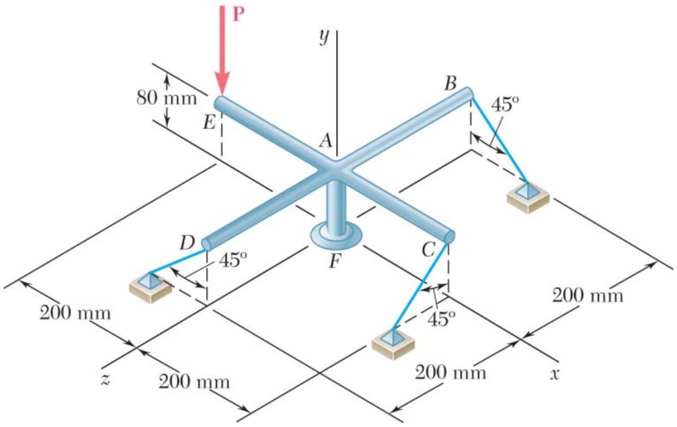
$$A_y = 0$$

$$\Sigma F_z = 0: \quad A_z + (T_{BE})_z = 0$$

$$A_z + \left(\frac{5}{13} \times 1950 \text{ N}\right) = 0$$

$$A_z = -750 \text{ N}$$

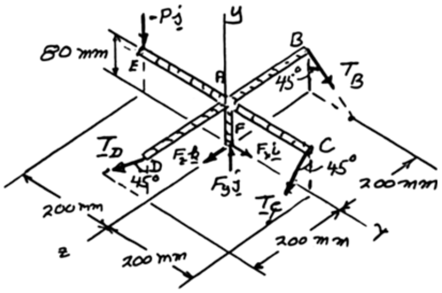
Therefore **A** = (3000 N)**i** – (750 N)**k** ◀



PROBLEM 4.131

The assembly shown consists of an 80-mm rod *AF* that is welded to a cross consisting of four 200-mm arms. The assembly is supported by a ball-and-socket joint at *F* and by three short links, each of which forms an angle of 45° with the vertical. For the loading shown, determine (*a*) the tension in each link, (*b*) the reaction at *F*.

SOLUTION



$$\mathbf{r}_{E/F} = -200\,\mathbf{i} + 80\mathbf{j}$$

$$\mathbf{T}_B = T_B(\mathbf{i} - \mathbf{j})/\sqrt{2} \qquad \mathbf{r}_{B/F} = 80\mathbf{j} - 200\mathbf{k}$$

$$\mathbf{T}_C = T_C(-\mathbf{j} + \mathbf{k})/\sqrt{2} \qquad \mathbf{r}_{C/F} = 200\mathbf{i} + 80\mathbf{j}$$

$$\mathbf{T}_D = T_D(-\mathbf{i} + \mathbf{j})/\sqrt{2} \qquad \mathbf{r}_{D/E} = 80\mathbf{j} + 200\mathbf{k}$$

$$\Sigma M_F = 0: \quad \mathbf{r}_{B/F} \times \mathbf{T}_B + \mathbf{r}_{C/F} \times \mathbf{T}_C + \mathbf{r}_{D/F} \times T_D + \mathbf{r}_{E/F} \times (-P\mathbf{j}) = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 80 & -200 \\ 1 & -1 & 0 \end{vmatrix} \frac{T_B}{\sqrt{2}} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 200 & 80 & 0 \\ 0 & -1 & 1 \end{vmatrix} \frac{T_C}{\sqrt{2}} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 80 & 200 \\ -1 & -1 & 0 \end{vmatrix} \frac{T_D}{\sqrt{2}} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -200 & 80 & 0 \\ 0 & -P & 0 \end{vmatrix} = 0$$

Equate coefficients of unit vectors to zero and multiply each equation by $\sqrt{2}$.

$$\mathbf{i}: \qquad -200T_B + 80T_C + 200\,T_D = 0 \qquad (1)$$

$$\mathbf{j}: \qquad -200T_B - 200T_C - 200T_D = 0 \qquad (2)$$

$$\mathbf{k}: \qquad -80T_B - 200T_C + 80T_D + 200\sqrt{2}P = 0 \qquad (3)$$

$$\frac{80}{200}(2): \qquad -80T_B - 80T_C - 80T_D = 0 \qquad (4)$$

Eqs. $(3)+(4)$: $\qquad -160T_B - 280T_C + 200\sqrt{2}P = 0 \qquad (5)$

Eqs. $(1)+(2)$: $\qquad -400T_B - 120T_C = 0$

$$T_B = -\frac{120}{400}T_C - 0.3T_C \qquad (6)$$

PROBLEM 4.131 (Continued)

Eqs. (6) $\longrightarrow$ (5): $-160(-0.3T_C) - 280T_C + 200\sqrt{2}P = 0$

$-232T_C + 200\sqrt{2}P = 0$

$T_C = 1.2191P$ $\quad$ $T_C = 1.219P$ ◀

From Eq. (6): $T_B = -0.3(1.2191P) = -0.36574 = P$ $\quad$ $T_B = -0.366P$ ◀

From Eq. (2): $-200(-0.3657P) - 200(1.2191P) - 200T_{\theta D} = 0$

$T_D = -0.8534P$ $\quad$ $T_D = -0.853P$ ◀

$\Sigma\mathbf{F} = 0$: $\mathbf{F} + \mathbf{T}_B + \mathbf{T}_C + \mathbf{T}_D - P\mathbf{j} = 0$

$\mathbf{i}$: $F_x + \dfrac{(-0.36574P)}{\sqrt{2}} - \dfrac{(-0.8534P)}{\sqrt{2}} = 0$

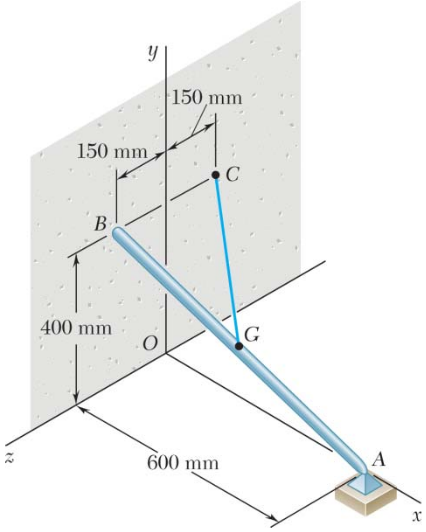
$F_x = -0.3448P$ $\quad$ $F_x = -0.345P$

$\mathbf{j}$: $F_y - \dfrac{(-0.36574P)}{\sqrt{2}} - \dfrac{(1.2191P)}{\sqrt{2}} - \dfrac{(-0.8534P)}{\sqrt{2}} - 200 = 0$

$F_y = P$ $\quad$ $F_y = P$

$\mathbf{k}$: $F_z + \dfrac{(1.2191P)}{\sqrt{2}} = 0$

$F_z = -0.8620P$ $\quad$ $F_z = -0.862P$ $\quad$ $\mathbf{F} = -0.345P\mathbf{i} + P\mathbf{j} - 0.862P\mathbf{k}$ ◀

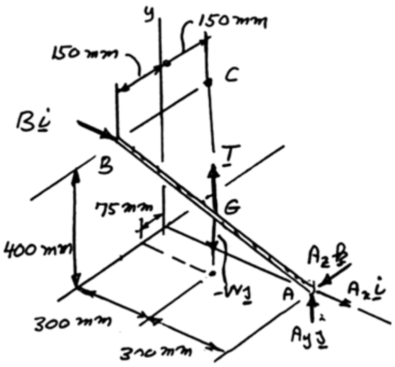


PROBLEM 4.132

The uniform 10-kg rod *AB* is supported by a ball-and-socket joint at *A* and by the cord *CG* that is attached to the midpoint *G* of the rod. Knowing that the rod leans against a frictionless vertical wall at *B*, determine (*a*) the tension in the cord, (*b*) the reactions at *A* and *B*.

SOLUTION

Free-Body Diagram:



Five unknowns and six equations of equilibrium, but equilibrium is maintained $(\Sigma M_{AB} = 0)$.

(*a*)

$$W = mg$$

$$= (10\text{ kg})\,9.81\text{ m/s}^2$$

$$W = 98.1\text{ N}$$

$$\overrightarrow{GC} = -300\mathbf{i} + 200\mathbf{j} - 225\mathbf{k} \qquad GC = 425\text{ mm}$$

$$\mathbf{T} = T\frac{\overrightarrow{GC}}{GC} = \frac{T}{425}(-300\mathbf{i} + 200\mathbf{j} - 225\mathbf{k})$$

$$\mathbf{r}_{B/A} = -600\mathbf{i} + 400\mathbf{j} + 150\text{ mm}$$

$$\mathbf{r}_{G/A} = -300\mathbf{i} + 200\mathbf{j} + 75\text{ mm}$$

$$\Sigma M_A = 0: \quad \mathbf{r}_{B/A} \times \mathbf{B} + \mathbf{r}_{G/A} \times \mathbf{T} + \mathbf{r}_{G/A} \times (-W\mathbf{j}) = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -600 & 400 & 150 \\ B & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -300 & 200 & 75 \\ -300 & 200 & -225 \end{vmatrix} \frac{T}{425} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -300 & 200 & 75 \\ 0 & -98.1 & 0 \end{vmatrix}$$

Coefficient of **i**: $(-105.88 - 35.29)T + 7357.5 = 0$

$$T = 52.12\text{ N}$$

$T = 52.1\text{ N}$ ◀

PROBLEM 4.132 (Continued)

(*b*)

Coefficient of **j**: $150B-(300\times 75+300\times 225)\dfrac{52.12}{425}=0$

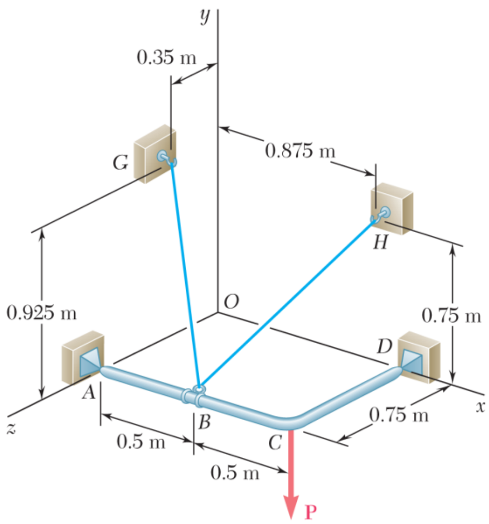
$B=73.58\text{ N}$ $\quad$ $\mathbf{B}=(73.6\text{ N})\mathbf{i}$ ◀

$\Sigma\mathbf{F}=0$: $\mathbf{A}+\mathbf{B}+\mathbf{T}-W\mathbf{j}=0$

Coefficient of **i**: $A_x+73.58-52.15\dfrac{300}{425}=0$ $\quad$ $A_x=-36.8\text{ N}$ ◀

Coefficient of **j**: $A_y+52.15\dfrac{200}{425}-98.1=0$ $\quad$ $A_y=73.6\text{ N}$ ◀

Coefficient of **k**: $A_z-52.15\dfrac{225}{425}=0$ $\quad$ $A_z=27.6\text{ N}$ ◀

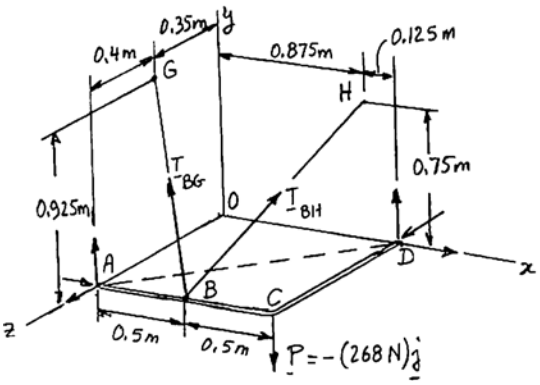

y
0.35 m
0.875 m
G
H
0.925 m
O
0.75 m
D
A
B
C
0.75 m
x
z
0.5 m
0.5 m
P

PROBLEM 4.133

The frame *ACD* is supported by ball-and-socket joints at *A* and *D* and by a cable that passes through a ring at *B* and is attached to hooks at *G* and *H*. Knowing that the frame supports at Point *C* a load of magnitude *P* = 268 N, determine the tension in the cable.

SOLUTION

Free-Body Diagram:


0.35m
y
0.4m
G
0.875m
0.125 m
H
0.75m
0.925m
T_BG
O
T_BH
A
D
x
B
C
z
0.5m
0.5m
P = −(268 N)j

$$\boldsymbol{\lambda}_{AD} = \frac{\overrightarrow{AD}}{AD} = \frac{(1\text{ m})\mathbf{i} - (0.75\text{ m})\mathbf{k}}{1.25\text{ m}}$$

$$\boldsymbol{\lambda}_{AD} = 0.8\mathbf{i} - 0.6\mathbf{k}$$

$$\mathbf{T}_{BG} = T_{BG}\frac{\overrightarrow{BG}}{BG}$$

$$= T_{BG}\frac{-0.5\mathbf{i} + 0.925\mathbf{j} - 0.4\mathbf{k}}{1.125}$$

$$T_{BH} = T_{BH}\frac{\overrightarrow{BH}}{BH}$$

$$= T_{BH}\frac{0.375\mathbf{i} + 0.75\mathbf{j} - 0.75\mathbf{k}}{1.125}$$

PROBLEM 4.133 (Continued)

$$\mathbf{r}_{B/A} = (0.5\text{ m})\mathbf{i}; \quad \mathbf{r}_{C/A} = (1\text{ m})\mathbf{i}; \quad \mathbf{P} = -(268\text{ N})\mathbf{j}$$

To eliminate the reactions at A and D, we shall write

$$\Sigma \mathbf{M}_{AD} = 0: \quad \boldsymbol{\lambda}_{AD} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}_{BG}) + \boldsymbol{\lambda}_{AD} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}_{BH}) + \boldsymbol{\lambda}_{AD} \cdot (\mathbf{r}_{C/A} \times \mathbf{P}) = 0 \tag{1}$$

Substituting for terms in Eq. (1) and using determinants,

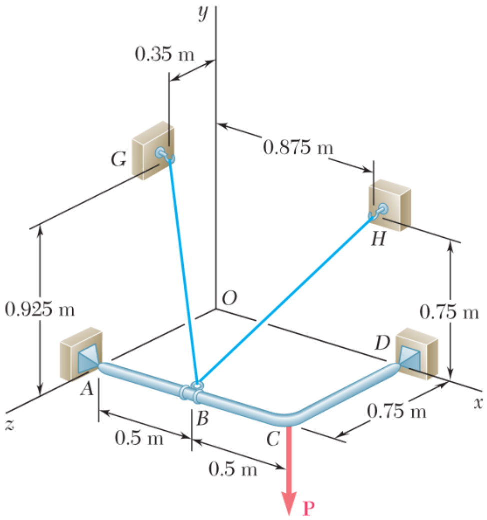
$$\begin{vmatrix} 0.8 & 0 & -0.6 \\ 0.5 & 0 & 0 \\ -0.5 & 0.925 & -0.4 \end{vmatrix} \frac{T_{BG}}{1.125} + \begin{vmatrix} 0.8 & 0 & -0.6 \\ 0.5 & 0 & 0 \\ 0.375 & 0.75 & -0.75 \end{vmatrix} \frac{T_{BH}}{1.125} + \begin{vmatrix} 0.8 & 0 & -0.6 \\ 1 & 0 & 0 \\ 0 & -268 & 0 \end{vmatrix} = 0$$

Multiplying all terms by (–1.125),

$$0.27750T_{BG} + 0.22500T_{BH} = 180.900 \tag{2}$$

For this problem, $T_{BG} = T_{BH} = T$

$$(0.27750 + 0.22500)T = 180.900 \qquad T = 360\text{ N} \blacktriangleleft$$


y
0.35 m
G
0.875 m
H
0.925 m
O
0.75 m
D
A
B
0.75 m
x
z
0.5 m
C
0.5 m
P

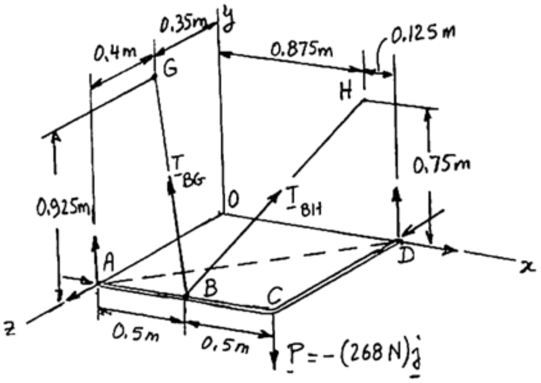
PROBLEM 4.134

Solve Prob. 4.133, assuming that cable *GBH* is replaced by a cable *GB* attached at *G* and *B*.

PROBLEM 4.133 The frame *ACD* is supported by ball-and-socket joints at *A* and *D* and by a cable that passes through a ring at *B* and is attached to hooks at *G* and *H*. Knowing that the frame supports at Point *C* a load of magnitude $P = 268$ N, determine the tension in the cable.

SOLUTION

Free-Body Diagram:


0.35m
y
0.4m
G
0.875m
0.125 m
H
0.75m
$\underline{T}_{BG}$
0.925m
O
$\underline{T}_{BH}$
A
B
C
D
x
z
0.5m
0.5m
$\underline{P} = -(268\text{ N})\underline{j}$

$$\lambda_{AD} = \frac{\overrightarrow{AD}}{AD} = \frac{(1\text{ m})\mathbf{i} - (0.75\text{ m})\mathbf{k}}{1.25\text{ m}}$$

$$\lambda_{AD} = 0.8\mathbf{i} - 0.6\mathbf{k}$$

$$T_{BG} = T_{BG}\frac{\overrightarrow{BG}}{BG}$$

$$= T_{BG}\frac{-0.5\mathbf{i} + 0.925\mathbf{j} - 0.4\mathbf{k}}{1.125}$$

$$T_{BH} = T_{BH}\frac{\overrightarrow{BH}}{BH}$$

$$= T_{BH}\frac{0.375\mathbf{i} + 0.75\mathbf{j} - 0.75\mathbf{k}}{1.125}$$

PROBLEM 4.134 (Continued)

$$\mathbf{r}_{B/A} = (0.5\text{ m})\mathbf{i}; \quad \mathbf{r}_{C/A} = (1\text{ m})\mathbf{i}; \quad \mathbf{P} = -(268\text{ N})\mathbf{j}$$

To eliminate the reactions at *A* and *D*, we shall write

$$\Sigma\mathbf{M}_{AD} = 0: \quad \boldsymbol{\lambda}_{AD}\cdot(\mathbf{r}_{B/A}\times\mathbf{T}_{BG}) + \boldsymbol{\lambda}_{AD}\cdot(\mathbf{r}_{B/A}\times\mathbf{T}_{BH}) + \boldsymbol{\lambda}_{AD}\cdot(\mathbf{r}_{C/A}\times\mathbf{P}) = 0 \qquad (1)$$

Substituting for terms in Eq. (1) and using determinants,

$$\begin{vmatrix} 0.8 & 0 & -0.6 \\ 0.5 & 0 & 0 \\ -0.5 & 0.925 & -0.4 \end{vmatrix}\frac{T_{BG}}{1.125} + \begin{vmatrix} 0.8 & 0 & -0.6 \\ 0.5 & 0 & 0 \\ 0.375 & 0.75 & -0.75 \end{vmatrix}\frac{T_{BH}}{1.125} + \begin{vmatrix} 0.8 & 0 & -0.6 \\ 1 & 0 & 0 \\ 0 & -268 & 0 \end{vmatrix} = 0$$

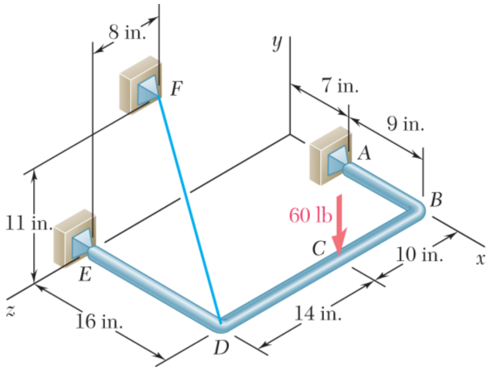
Multiplying all terms by (–1.125),

$$0.27750T_{BG} + 0.22500T_{BH} = 180.900 \qquad (2)$$

For this problem, $T_{BH} = 0$.

Thus, Eq. (2) reduces to

$$0.27750T_{BG} = 180.900 \qquad T_{BG} = 652\text{ N} \blacktriangleleft$$

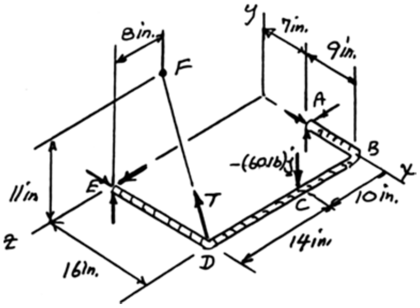


PROBLEM 4.135

The bent rod *ABDE* is supported by ball-and-socket joints at *A* and *E* and by the cable *DF*. If a 60-lb load is applied at *C* as shown, determine the tension in the cable.

SOLUTION

Free-Body Diagram:



$$\overrightarrow{DF} = -16\mathbf{i} + 11\mathbf{j} - 8\mathbf{k} \qquad DF = 21\text{ in.}$$

$$\mathbf{T} = T\frac{\overrightarrow{DE}}{DF} = \frac{T}{21}(-16\mathbf{i} + 11\mathbf{j} - 8\mathbf{k})$$

$$\mathbf{r}_{D/E} = 16\mathbf{i}$$

$$\mathbf{r}_{C/E} = 16\mathbf{i} - 14\mathbf{k}$$

$$\boldsymbol{\lambda}_{EA} = \frac{\overrightarrow{EA}}{EA} = \frac{7\mathbf{i} - 24\mathbf{k}}{25}$$

$$\Sigma M_{EA} = 0: \quad \boldsymbol{\lambda}_{EA} \cdot (\mathbf{r}_{B/E} \times \mathbf{T}) + \boldsymbol{\lambda}_{EA} \cdot (\mathbf{r}_{C/E} \cdot (-60\mathbf{j})) = 0$$

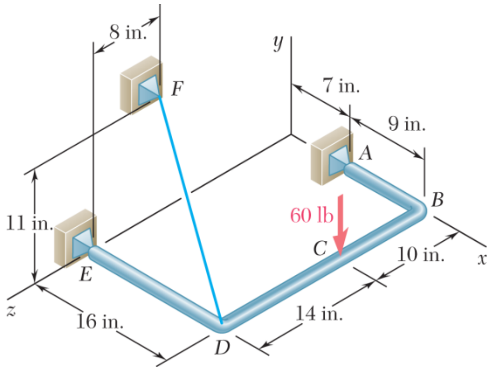
$$\begin{vmatrix} 7 & 0 & -24 \\ 16 & 0 & 0 \\ -16 & 11 & -8 \end{vmatrix} \frac{T}{21 \times 25} + \begin{vmatrix} 7 & 0 & -24 \\ 16 & 0 & -14 \\ 0 & -60 & 0 \end{vmatrix} \frac{1}{25} = 0$$

$$-\frac{24 \times 16 \times 11}{21 \times 25} T + \frac{-7 \times 14 \times 60 + 24 \times 16 \times 60}{25} = 0$$

$$201.14T + 17{,}160 = 0$$

$$T = 85.314\text{ lb}$$

$T = 85.3\text{ lb}$ ◀



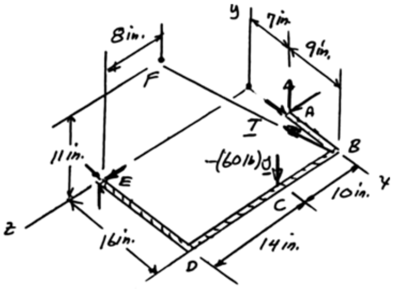
PROBLEM 4.136

Solve Problem 4.135, assuming that cable *DF* is replaced by a cable connecting *B* and *F*.

PROBLEM 4.135 The bent rod *ABDE* is supported by ball-and-socket joints at *A* and *E* and by the cable *DF*. If a 60-lb load is applied at *C* as shown, determine the tension in the cable.

SOLUTION

Free-Body Diagram:



$$\mathbf{r}_{B/A} = 9\mathbf{i}$$

$$\mathbf{r}_{C/A} = 9\mathbf{i} + 10\mathbf{k}$$

$$\overrightarrow{BF} = -16\mathbf{i} + 11\mathbf{j} + 16\mathbf{k} \qquad BF = 25.16 \text{ in.}$$

$$\mathbf{T} = T\frac{\overrightarrow{BF}}{BF} = \frac{T}{25.16}(-16\mathbf{i} + 11\mathbf{j} + 16\mathbf{k})$$

$$\boldsymbol{\lambda}_{AE} = \frac{\overrightarrow{AE}}{AE} = \frac{7\mathbf{i} - 24\mathbf{k}}{\mathbf{25}}$$

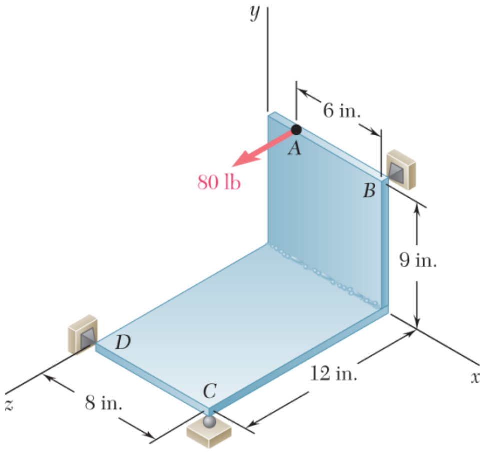
$$\Sigma M_{AE} = 0: \quad \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}) + \boldsymbol{\lambda}_{AE} \cdot (\mathbf{r}_{C/A} \cdot (-60\mathbf{j})) = 0$$

$$\begin{vmatrix} 7 & 0 & -24 \\ 9 & 0 & 0 \\ -16 & 11 & 16 \end{vmatrix} \frac{T}{25 \times 25.16} + \begin{vmatrix} 7 & 0 & -24 \\ 9 & 0 & 10 \\ 0 & -60 & 0 \end{vmatrix} \frac{1}{25} = 0$$

$$-\frac{24 \times 9 \times 11}{25 \times 25.16}T + \frac{24 \times 9 \times 60 + 7 \times 10 \times 60}{25} = 0$$

$$94.436T - 17{,}160 = 0$$

$T = 181.7$ lb ◀



PROBLEM 4.137

Two rectangular plates are welded together to form the assembly shown. The assembly is supported by ball-and-socket joints at *B* and *D* and by a ball on a horizontal surface at *C*. For the loading shown, determine the reaction at *C*.

SOLUTION

First note:

$$\lambda_{BD} = \frac{-(6\text{ in.})\mathbf{i} - (9\text{ in.})\mathbf{j} + (12\text{ in.})\mathbf{k}}{\sqrt{(6)^2 + (9)^2 + (12)^2}\text{ in.}}$$

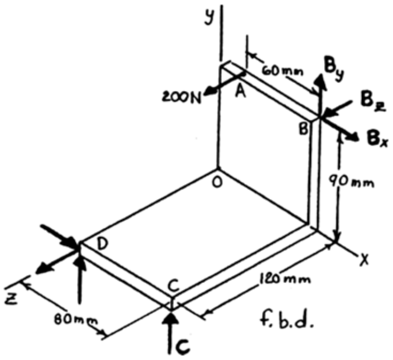
$$= \frac{1}{16.1555}(-6\mathbf{i} - 9\mathbf{j} + 12\mathbf{k})$$

$$\mathbf{r}_{A/B} = -(6\text{ in.})\mathbf{i}$$

$$\mathbf{P} = (80\text{ lb})\mathbf{k}$$

$$\mathbf{r}_{C/D} = (8\text{ in.})\mathbf{i}$$

$$\mathbf{C} = (C)\mathbf{j}$$



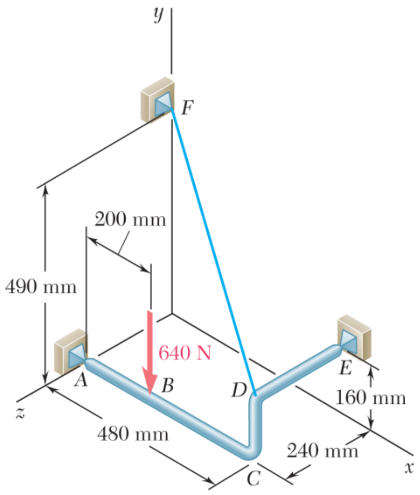
From the F.B.D. of the plates:

$$\Sigma M_{BD} = 0: \quad \lambda_{BD} \cdot (\mathbf{r}_{A/B} \times \mathrm{P}) + \lambda_{BD} \cdot (\mathbf{r}_{C/D} \times \mathrm{C}) = 0$$

$$\begin{vmatrix} -6 & -9 & 12 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \left[\frac{6(80)}{16.1555}\right] + \begin{vmatrix} -6 & -9 & 12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \left[\frac{C(8)}{16.1555}\right] = 0$$

$$(-9)(6)(80) + (12)(8)C = 0$$

$C = 45.0$ lb or $\mathbf{C} = (45.0\text{ lb})\mathbf{j}$ ◀



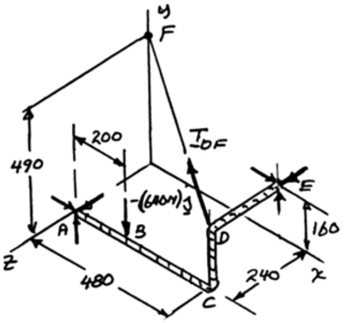
PROBLEM 4.138

The pipe *ACDE* is supported by ball-and-socket joints at *A* and *E* and by the wire *DF*. Determine the tension in the wire when a 640-N load is applied at *B* as shown.

SOLUTION

Free-Body Diagram:

Dimensions in mm



$$\overrightarrow{AE} = 480\mathbf{i} + 160\mathbf{j} - 240\mathbf{k}$$

$$AE = 560 \text{ mm}$$

$$\boldsymbol{\lambda}_{AE} = \frac{\overrightarrow{AE}}{AE} = \frac{480\mathbf{i} + 160\mathbf{j} - 240\mathbf{k}}{560}$$

$$\boldsymbol{\lambda}_{AE} = \frac{6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{7}$$

$$\mathbf{r}_{B/A} = 200\mathbf{i}$$

$$\mathbf{r}_{D/A} = 480\mathbf{i} + 160\mathbf{j}$$

$$\overrightarrow{DF} = -480\mathbf{i} + 330\mathbf{j} - 240\mathbf{k}; \qquad DF = 630 \text{ mm}$$

$$\mathbf{T}_{DF} = T_{DF}\frac{\overrightarrow{DF}}{DF} = T_{DF}\frac{-480\mathbf{i} + 330\mathbf{j} - 240\mathbf{k}}{630} = T_{DF}\frac{-16\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}}{21}$$

$$\Sigma M_{AE} = \boldsymbol{\lambda}_{AE} \cdot (\mathbf{r}_{D/A} \times \mathbf{T}_{DF}) + \boldsymbol{\lambda}_{AE} \cdot (\mathbf{r}_{B/A} \times (-600\mathbf{j})) = 0$$

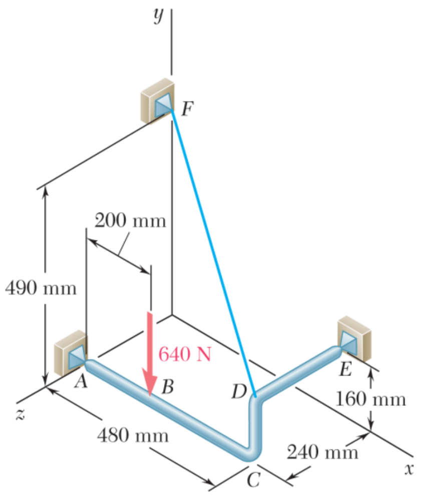
$$\begin{vmatrix} 6 & 2 & -3 \\ 480 & 160 & 0 \\ -16 & 11 & -8 \end{vmatrix} \frac{T_{DF}}{21 \times 7} + \begin{vmatrix} 6 & 2 & -3 \\ 200 & 0 & 0 \\ 0 & -640 & 0 \end{vmatrix} \frac{1}{7} = 0$$

$$\frac{-6 \times 160 \times 8 + 2 \times 480 \times 8 - 3 \times 480 \times 11 - 3 \times 160 \times 16}{21 \times 7} T_{DF} + \frac{3 \times 200 \times 640}{7} = 0$$

$$-1120 T_{DF} + 384 \times 10^3 = 0$$

$$T_{DF} = 342.86 \text{ N}$$

$$T_{DF} = 343 \text{ N} \blacktriangleleft$$



PROBLEM 4.139

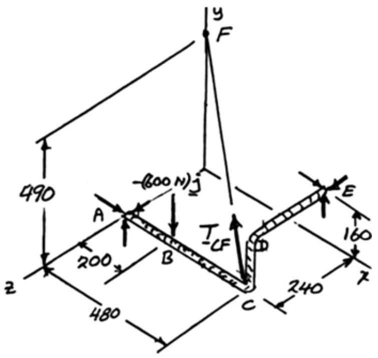
Solve Problem 4.138, assuming that wire *DF* is replaced by a wire connecting *C* and *F*.

PROBLEM 4.138 The pipe *ACDE* is supported by ball-and-socket joints at *A* and *E* and by the wire *DF*. Determine the tension in the wire when a 640-N load is applied at *B* as shown.

SOLUTION

Free-Body Diagram:

Dimensions in mm



$$\overrightarrow{AE} = 480\mathbf{i} + 160\mathbf{j} - 240\mathbf{k}$$

$$AE = 560 \text{ mm}$$

$$\boldsymbol{\lambda}_{AE} = \frac{\overrightarrow{AE}}{AE} = \frac{480\mathbf{i} + 160\mathbf{j} - 240\mathbf{k}}{560}$$

$$\boldsymbol{\lambda}_{AE} = \frac{6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{7}$$

$$\mathbf{r}_{B/A} = 200\mathbf{i}$$

$$\mathbf{r}_{C/A} = 480\mathbf{i}$$

$$\overrightarrow{CF} = -480\mathbf{i} + 490\mathbf{j} - 240\mathbf{k}; \quad CF = 726.70 \text{ mm}$$

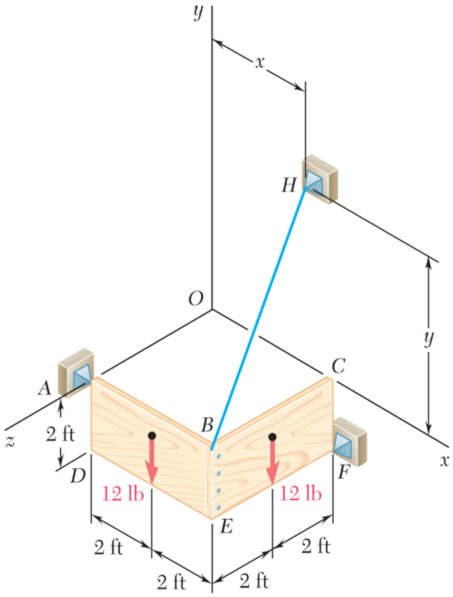
$$\mathbf{T}_{CF} = T_{CF}\frac{\overrightarrow{CE}}{CF} = \frac{-480\mathbf{i} + 490\mathbf{j} - 240\mathbf{k}}{726.70}$$

$$\Sigma M_{AE} = 0: \quad \boldsymbol{\lambda}_{AE} \cdot (\mathbf{r}_{C/A} \times \mathbf{T}_{CF}) + \boldsymbol{\lambda}_{AE} \cdot (\mathbf{r}_{B/A} \times (-600\mathbf{j})) = 0$$

$$\begin{vmatrix} 6 & 2 & -3 \\ 480 & 0 & 0 \\ -480 & +490 & -240 \end{vmatrix} \frac{T_{CF}}{726.7 \times 7} + \begin{vmatrix} 6 & 2 & -3 \\ 200 & 0 & 0 \\ 0 & -640 & 0 \end{vmatrix} \frac{1}{7} = 0$$

$$\frac{2 \times 480 \times 240 - 3 \times 480 \times 490}{726.7 \times 7} T_{CF} + \frac{3 \times 200 \times 640}{7} = 0$$

$$-653.91 T_{CF} + 384 \times 10^3 = 0 \qquad T_{CF} = 587 \text{ N} \blacktriangleleft$$

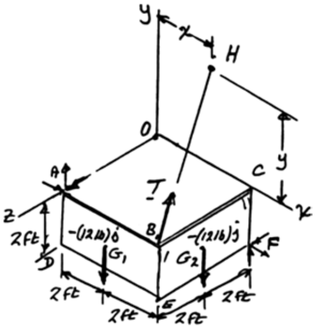


PROBLEM 4.140

Two 2×4-ft plywood panels, each of weight 12 lb, are nailed together as shown. The panels are supported by ball-and-socket joints at *A* and *F* and by the wire *BH*. Determine (*a*) the location of *H* in the *xy* plane if the tension in the wire is to be minimum, (*b*) the corresponding minimum tension.

SOLUTION

Free-Body Diagram:



$$\overrightarrow{AF} = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k} \qquad AF = 6\,\text{ft}$$

$$\boldsymbol{\lambda}_{AF} = \frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

$$\mathbf{r}_{G_1/A} = 2\mathbf{i} - \mathbf{j}$$

$$\mathbf{r}_{G_2/A} = 4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$$\mathbf{r}_{B/A} = 4\mathbf{i}$$

$$\Sigma M_{AF} = 0: \quad \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{G_1/A} \times (-12\mathbf{j}) + \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{G_2/A} \times (-12\mathbf{j})) + \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{B/A} \times T) = 0$$

$$\begin{vmatrix} 2 & -1 & -2 \\ 2 & -1 & 0 \\ 0 & -12 & 0 \end{vmatrix} \frac{1}{3} + \begin{vmatrix} 2 & -1 & -2 \\ 4 & -1 & -2 \\ 0 & -12 & 0 \end{vmatrix} \frac{1}{3} + \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}) = 0$$

$$(2 \times 2 \times 12)\frac{1}{3} + (-2 \times 2 \times 12 + 2 \times 4 \times 12)\frac{1}{3} + \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}) = 0$$

$$\boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}) = -32 \quad \text{or} \quad \mathbf{T} \cdot (\boldsymbol{\lambda}_{A/F} \times \mathbf{r}_{B/A}) = -32 \qquad (1)$$

PROBLEM 4.140 (Continued)

Projection of **T** on $(\boldsymbol{\lambda}_{AF} \times \mathbf{r}_{B/A})$ is constant. Thus, $T_{\min}$ is parallel to

$$\boldsymbol{\lambda}_{AF} \times \mathbf{r}_{B/A} = \frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \times 4\mathbf{i} = \frac{1}{3}(-8\mathbf{j} + 4\mathbf{k})$$

Corresponding unit vector is $\frac{1}{\sqrt{5}}(-2\mathbf{j} + \mathbf{k})$.

$$T_{\min} = T(-2\mathbf{j} + \mathbf{k})\frac{1}{\sqrt{5}} \qquad (2)$$

From Eq. (1):

$$\frac{T}{\sqrt{5}}(-2\mathbf{j} + \mathbf{k}) \cdot \left[\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \times 4\mathbf{i}\right] = -32$$

$$\frac{T}{\sqrt{5}}(-2\mathbf{j} + \mathbf{k}) \cdot \frac{1}{3}(-8\mathbf{j} + 4\mathbf{k}) = -32$$

$$\frac{T}{3\sqrt{5}}(16 + 4) = -32 \qquad T = -\frac{3\sqrt{5}(32)}{20} = 4.8\sqrt{5}$$

$$T = 10.7331 \text{ lb}$$

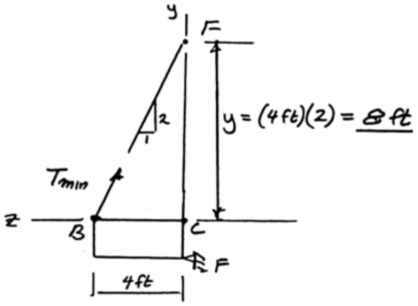
From Eq. (2):

$$T_{\min} = T(-2\mathbf{j} + \mathbf{k})\frac{1}{\sqrt{5}}$$

$$= 4.8\sqrt{5}(-2\mathbf{j} + \mathbf{k})\frac{1}{\sqrt{5}}$$

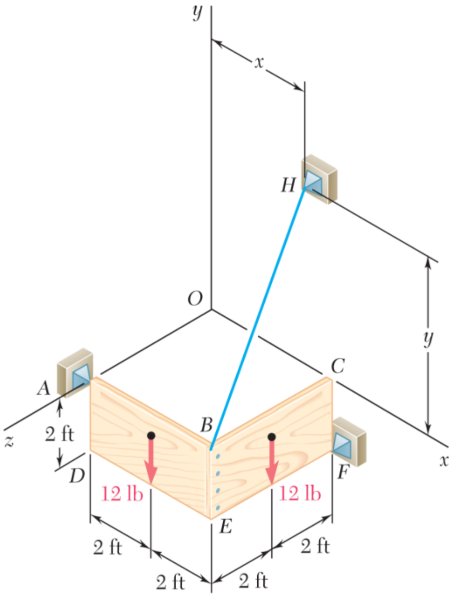
$$\mathbf{T}_{\min} = -(9.6 \text{ lb})\mathbf{j} + (4.8 \text{ lb } \mathbf{k})$$

Since $T_{\min}$ has no **i** component, wire *BH* is parallel to the *yz* plane, and $x = 4$ ft.



(*a*) $x = 4.00$ ft; $y = 8.00$ ft ◀

(*b*) $T_{\min} = 10.73$ lb ◀



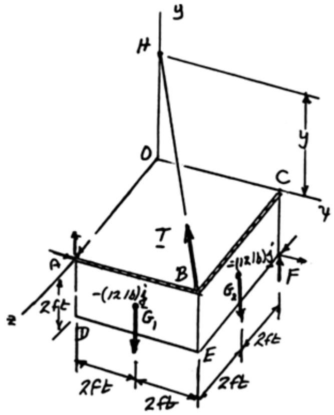
PROBLEM 4.141

Solve Problem 4.140, subject to the restriction that *H* must lie on the *y*-axis.

PROBLEM 4.140 Two 2×4-ft plywood panels, each of weight 12 lb, are nailed together as shown. The panels are supported by ball-and-socket joints at *A* and *F* and by the wire *BH*. Determine (*a*) the location of *H* in the *xy* plane if the tension in the wire is to be minimum, (*b*) the corresponding minimum tension.

SOLUTION

Free-Body Diagram:



$$\overrightarrow{AF} = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$$

$$\boldsymbol{\lambda}_{AF} = \frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

$$\mathbf{r}_{G_1/A} = 2\mathbf{i} - \mathbf{j}$$

$$\mathbf{r}_{G_2/A} = 4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$$\mathbf{r}_{B/A} = 4\mathbf{i}$$

$$\Sigma M_{AF} = 0: \quad \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{G/A} \times (-12\mathbf{j}) + \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{G_2/A} \times (-12\mathbf{j})) + \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{B/A} \times T) = 0$$

$$\begin{vmatrix} 2 & -1 & 2 \\ 2 & -1 & 0 \\ 0 & -12 & 0 \end{vmatrix} \frac{1}{3} + \begin{vmatrix} 2 & -1 & -2 \\ 4 & -1 & -2 \\ 0 & -12 & 0 \end{vmatrix} \frac{1}{3} + \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}) = 0$$

$$(2 \times 2 \times 12)\frac{1}{3} + (-2 \times 2 \times 12 + 2 \times 4 \times 12)\frac{1}{3} + \boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}) = 0$$

$$\boldsymbol{\lambda}_{AF} \cdot (\mathbf{r}_{B/A} \times \mathbf{T}) = -32 \qquad (1)$$

$$\overrightarrow{BH} = -4\mathbf{i} + y\mathbf{j} - 4\mathbf{k} \qquad BH = (32 + y^2)^{1/2}$$

$$\mathbf{T} = T\frac{\overrightarrow{BH}}{BH} = T\frac{-4\mathbf{i} + y\mathbf{j} - 4\mathbf{k}}{(32 + y^2)^{1/2}}$$

PROBLEM 4.141 (Continued)

From Eq. (1):

$$\boldsymbol{\lambda}_{AF}\cdot(\mathbf{r}_{B/A}\times T)=\begin{vmatrix}2 & -1 & -2\\ 4 & 0 & 0\\ -4 & y & -4\end{vmatrix}\frac{T}{3(32+y^2)^{1/2}}=-32$$

$$(-16-8y)T=-3\times 32(32+y^2)^{1/2}\qquad T=96\frac{(32+y^2)^{1/2}}{8y+16}\tag{2}$$

$$\frac{dT}{dy}=0:\quad 96\frac{(8y+16)\frac{1}{2}(32+y^2)^{-1/2}(2y)+(32+y^2)^{1/2}(8)}{(8y+16)^2}$$

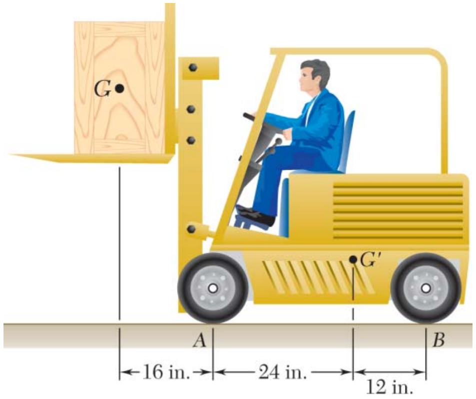
Numerator = 0: $(8y+16)y=(32+y^2)8$

$$8y^2+16y=32\times 8+8y^2$$

$x=0$ ft; $y=16.00$ ft ◀

From Eq. (2): $T=96\dfrac{(32+16^2)^{1/2}}{8\times 16+16}=11.3137$ lb

$T_{\min}=11.31$ lb ◀

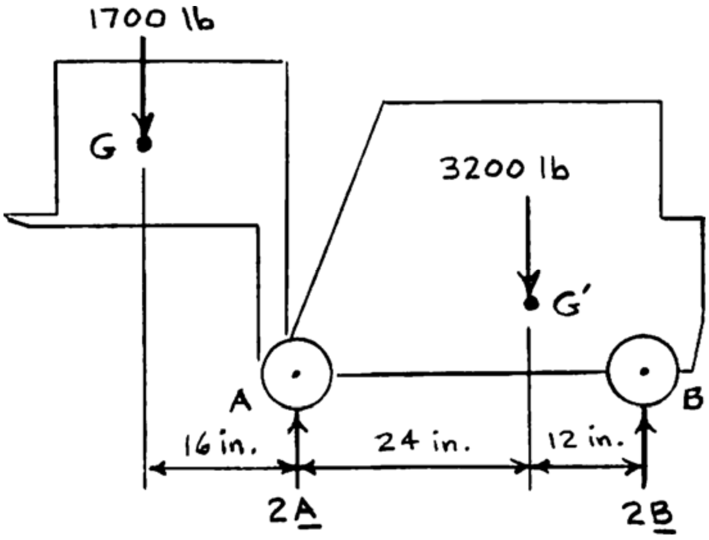

G
G′
A
B
16 in.
24 in.
12 in.

PROBLEM 4.142

A 3200-lb forklift truck is used to lift a 1700-lb crate. Determine the reaction at each of the two (*a*) front wheels *A*, (*b*) rear wheels *B*.

SOLUTION

Free-Body Diagram:

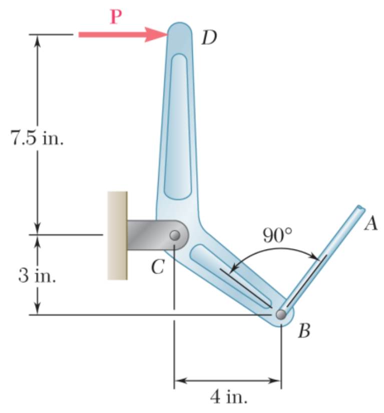

1700 lb
G
3200 lb
G′
A
B
16 in.
24 in.
12 in.
2A
2B

(*a*) Front wheels: $\circlearrowleft + \Sigma M_B = 0$: $(1700 \text{ lb})(52 \text{ in.}) + (3200 \text{ lb})(12 \text{ in.}) - 2A(36 \text{ in.}) = 0$

$A = +1761.11 \text{ lb}$ $\qquad \mathbf{A} = 1761 \text{ lb}\uparrow$ ◀

(*b*) Rear wheels: $+\uparrow \Sigma F_y = 0$: $-1700 \text{ lb} - 3200 \text{ lb} + 2(1761.11 \text{ lb}) + 2B = 0$

$B = +688.89 \text{ lb}$ $\qquad \mathbf{B} = 689 \text{ lb}\uparrow$ ◀

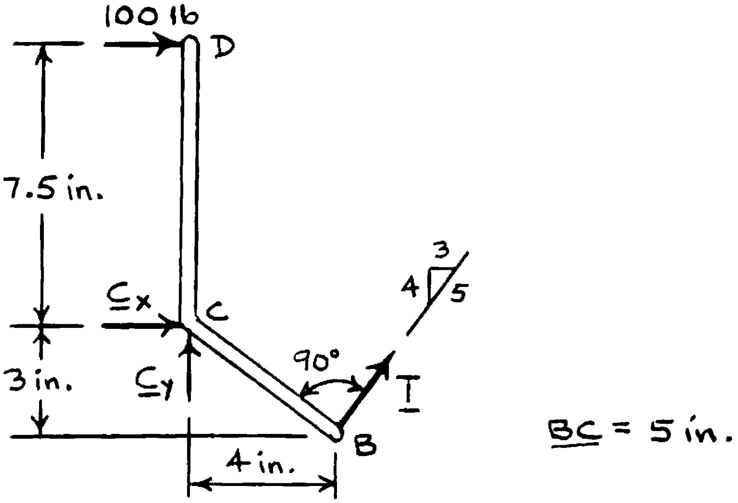

P
D
7.5 in.
90°
A
3 in.
C
B
4 in.

PROBLEM 4.143

The lever *BCD* is hinged at *C* and attached to a control rod at *B*. If *P* = 100 lb, determine (*a*) the tension in rod *AB*, (*b*) the reaction at *C*.

SOLUTION

Free-Body Diagram:

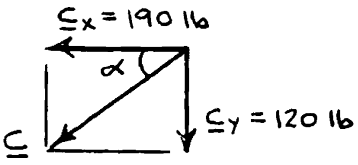

100 lb
D
7.5 in.
C_x
C
3 in.
C_y
90°
3
4
5
T
B
4 in.
BC = 5 in.

(*a*) $+\circlearrowleft \Sigma M_C = 0: \quad T(5 \text{ in.}) - (100 \text{ lb})(7.5 \text{ in.}) = 0$

$T = 150.0$ lb ◀

(*b*) $\xrightarrow{+} \Sigma F_x = 0: \quad C_x + 100 \text{ lb} + \frac{3}{5}(150.0 \text{ lb}) = 0$

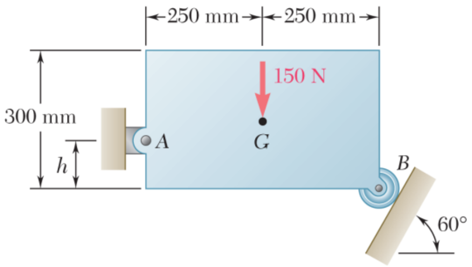
$C_x = -190$ lb $\qquad \mathbf{C}_x = 190$ lb ←


C_x = 190 lb
α
C_y = 120 lb
C

$+\uparrow \Sigma F_y = 0: \quad C_y + \frac{4}{5}(150.0 \text{ lb}) = 0$

$C_y = -120$ lb $\qquad \mathbf{C}_y = 120$ lb ↓

$\alpha = 32.3°$ $\qquad C = 225$ lb $\qquad \mathbf{C} = 225$ lb ↙ 32.3° ◀

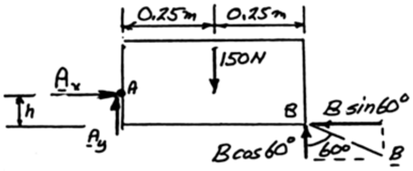

250 mm
250 mm
150 N
300 mm
A
G
B
h
60°

PROBLEM 4.144

Determine the reactions at A and B when (*a*) $h = 0$, (*b*) $h = 200$ mm.

SOLUTION

Free-Body Diagram:


0.25m
0.25m
150N
A_x
A
h
A_y
B
B sin60°
B cos60°
60°
B

$$+\circlearrowleft \Sigma M_A = 0: \quad (B\cos 60°)(0.5\text{ m}) - (B\sin 60°)h - (150\text{ N})(0.25\text{ m}) = 0$$

$$B = \frac{37.5}{0.25 - 0.866h} \qquad (1)$$

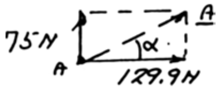
(*a*) When $h = 0$,

From Eq. (1): $B = \dfrac{37.5}{0.25} = 150$ N $\qquad \mathbf{B} = 150.0$ N ∡ 30.0° ◀

$$\xrightarrow{+} \Sigma F_y = 0: \quad A_x - B\sin 60° = 0$$

$A_x = (150)\sin 60° = 129.9$ N $\qquad \mathbf{A}_x = 129.9$ N →

$$+\uparrow \Sigma F_y = 0: \quad A_y - 150 + B\cos 60° = 0$$


75N
A
α
A
129.9N

$A_y = 150 - (150)\cos 60° = 75$ N $\qquad \mathbf{A}_y = 75$ N ↑

$\alpha = 30°$

$A = 150.0$ N $\qquad \mathbf{A} = 150.0$ N ∡ 30.0° ◀

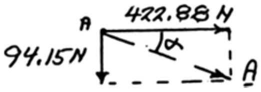
(*b*) When $h = 200$ mm $= 0.2$ m,

From Eq. (1): $B = \dfrac{37.5}{0.25 - 0.866(0.2)} = 488.3$ N $\qquad \mathbf{B} = 488$ N ∡ 30.0° ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - B\sin 60° = 0$$

$A_x = (488.3)\sin 60° = 422.88$ N $\qquad \mathbf{A}_x = 422.88$ N →

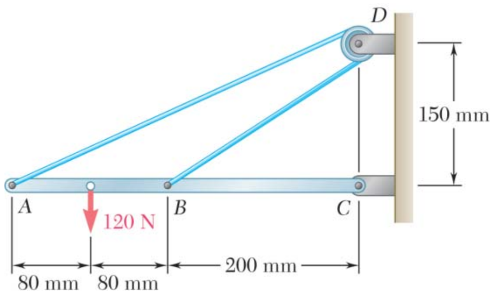
$$+\uparrow \Sigma F_y = 0: \quad A_y - 150 + B\cos 60° = 0$$


422.88 N
A
94.15N
α
A

$A_y = 150 - (488.3)\cos 60° = -94.15$ N $\qquad \mathbf{A}_y = 94.15$ N ↓

$\alpha = 12.55°$

$A = 433.2$ N $\qquad \mathbf{A} = 433$ N ∡ 12.55° ◀



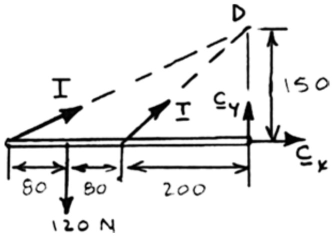
PROBLEM 4.145

Neglecting friction and the radius of the pulley, determine (*a*) the tension in cable *ADB*, (*b*) the reaction at *C*.

SOLUTION

Free-Body Diagram:

Dimensions in mm



Geometry:

Distance: $$AD = \sqrt{(0.36)^2 + (0.150)^2} = 0.39 \text{ m}$$

Distance: $$BD = \sqrt{(0.2)^2 + (0.15)^2} = 0.25 \text{ m}$$

Equilibrium for beam:

(*a*) $$+\circlearrowleft \Sigma M_C = 0: \quad (120 \text{ N})(0.28 \text{ m}) - \left(\frac{0.15}{0.39}T\right)(0.36 \text{ m}) - \left(\frac{0.15}{0.25}T\right)(0.2 \text{ m}) = 0$$

$$T = 130.000 \text{ N}$$

or $T = 130.0$ N ◀

(*b*) $$\xrightarrow{+} \Sigma F_x = 0: \quad C_x + \left(\frac{0.36}{0.39}\right)(130.000 \text{ N}) + \left(\frac{0.2}{0.25}\right)(130.000 \text{ N}) = 0$$

$$C_x = -224.00 \text{ N}$$

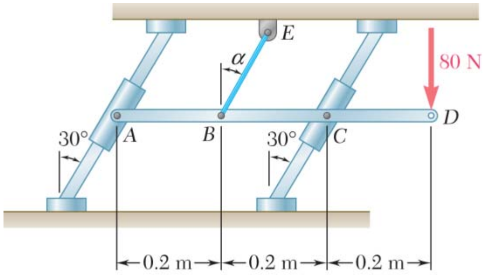
$$+\uparrow \Sigma F_y = 0: \quad C_y + \left(\frac{0.15}{0.39}\right)(130.00 \text{ N}) + \left(\frac{0.15}{0.25}\right)(130.00 \text{ N}) - 120 \text{ N} = 0$$

$$C_y = -8.0000 \text{ N}$$

Thus, $$C = \sqrt{C_x^2 + C_y^2} = \sqrt{(-224)^2 + (-8)^2} = 224.14 \text{ N}$$

and $$\theta = \tan^{-1}\frac{C_y}{C_x} = \tan^{-1}\frac{8}{224} = 2.0454^\circ$$

$\mathbf{C} = 224 \text{ N} \measuredangle 2.05^\circ$ ◀

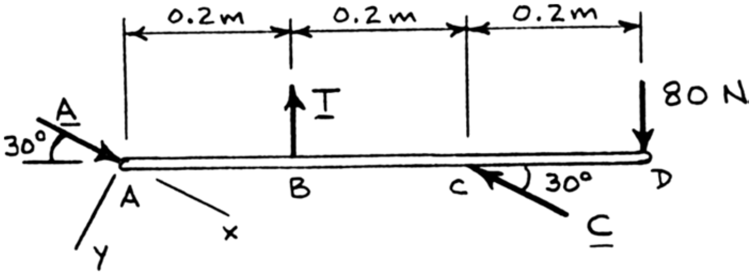


PROBLEM 4.146

Bar *AD* is attached at *A* and *C* to collars that can move freely on the rods shown. If the cord *BE* is vertical ($\alpha = 0$), determine the tension in the cord and the reactions at *A* and *C*.

SOLUTION

Free-Body Diagram:



$$+\swarrow \Sigma F_y = 0: \quad -T\cos 30° + (80\text{ N})\cos 30° = 0$$

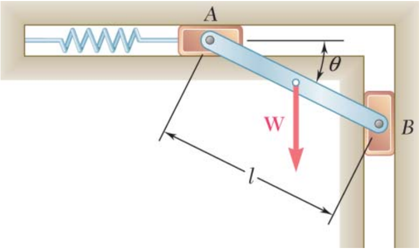
$$T = 80\text{ N} \qquad\qquad T = 80.0\text{ N}$$ ◀

$$+\circlearrowleft \Sigma M_C = 0: \quad (A\sin 30°)(0.4\text{ m}) - (80\text{ N})(0.2\text{ m}) - (80\text{ N})(0.2\text{ m}) = 0$$

$$A = +160\text{ N} \qquad \mathbf{A} = 160.0\text{ N} \nwarrow 30.0°$$ ◀

$$+\circlearrowleft \Sigma M_A = 0: \quad (80\text{ N})(0.2\text{ m}) - (80\text{ N})(0.6\text{ m}) + (C\sin 30°)(0.4\text{ m}) = 0$$

$$C = +160\text{ N} \qquad \mathbf{C} = 160.0\text{ N} \searrow 30.0°$$ ◀

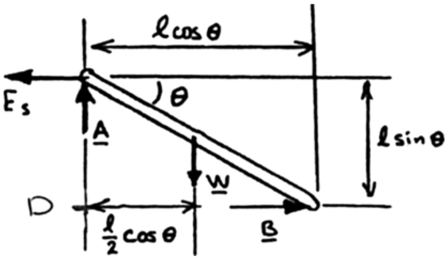


PROBLEM 4.147

A slender rod *AB*, of weight *W*, is attached to blocks *A* and *B* that move freely in the guides shown. The constant of the spring is *k*, and the spring is unstretched when $\theta = 0$. (*a*) Neglecting the weight of the blocks, derive an equation in *W*, *k*, *l*, and θ that must be satisfied when the rod is in equilibrium. (*b*) Determine the value of θ when $W = 75$ lb, $l = 30$ in., and $k = 3$ lb/in.

SOLUTION

Free-Body Diagram:



Spring force: $F_s = ks = k(l - l\cos\theta) = kl(1-\cos\theta)$

(*a*) $+\circlearrowleft \Sigma M_D = 0\colon \quad F_s(l\sin\theta) - W\left(\dfrac{l}{2}\cos\theta\right) = 0$

$$kl(1-\cos\theta)l\sin\theta - \frac{W}{2}l\cos\theta = 0$$

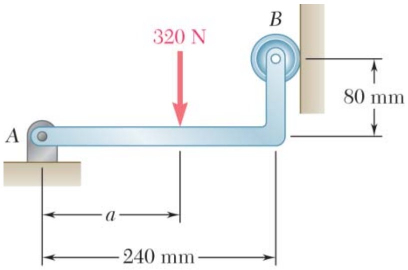
$$kl(1-\cos\theta)\tan\theta - \frac{W}{2} = 0 \quad \text{or} \quad (1-\cos\theta)\tan\theta = \frac{W}{2kl} \blacktriangleleft$$

(*b*) For given values of

$$W = 75 \text{ lb}$$
$$l = 30 \text{ in.}$$
$$k = 3 \text{ lb/in.}$$

$$(1-\cos\theta)\tan\theta = \tan\theta - \sin\theta$$
$$= \frac{75 \text{ lb}}{2(3 \text{ lb/in.})(30 \text{ in.})}$$
$$= 0.41667$$

Solving numerically, $\theta = 49.710°$ or $\theta = 49.7° \blacktriangleleft$

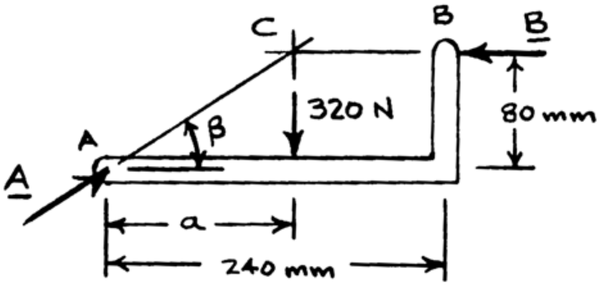

320 N
B
80 mm
A
a
240 mm

PROBLEM 4.148

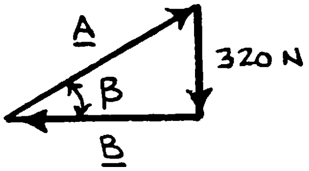
Determine the reactions at *A* and *B* when *a* = 150 mm.

SOLUTION

Free-Body Diagram:


C
B
B
320 N
80 mm
β
A
A
a
240 mm

Force triangle



$$\tan\beta = \frac{80\text{ mm}}{a} = \frac{80\text{ mm}}{150\text{ mm}}$$

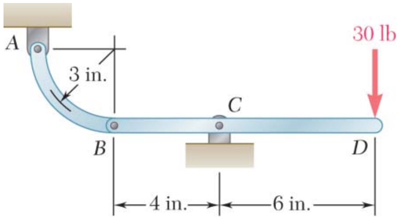
$$\beta = 28.072°$$

$$A = \frac{320\text{ N}}{\sin 28.072°}$$

$\mathbf{A} = 680\text{ N} \measuredangle 28.1°$ ◀

$$B = \frac{320\text{ N}}{\tan 28.072°}$$

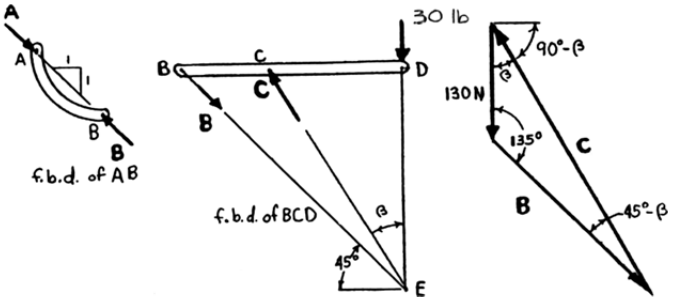
$\mathbf{B} = 600\text{ N} \leftarrow$ ◀



PROBLEM 4.149

For the frame and loading shown, determine the reactions at *A* and *C*.

SOLUTION



Since member *AB* is acted upon by two forces, **A** and **B**, they must be colinear, have the same magnitude, and be opposite in direction for *AB* to be in equilibrium. The force **B** acting at *B* of member *BCD* will be equal in magnitude but opposite in direction to force **B** acting on member *AB*. Member *BCD* is a three-force body with member forces intersecting at *E*. The F.B.D.'s of members *AB* and *BCD* illustrate the above conditions. The force triangle for member *BCD* is also shown. The angle β is found from the member dimensions:

$$\beta = \tan^{-1}\left(\frac{6 \text{ in.}}{10 \text{ in.}}\right) = 30.964°$$

Applying the law of sines to the force triangle for member *BCD*,

$$\frac{30 \text{ lb}}{\sin(45° - \beta)} = \frac{B}{\sin\beta} = \frac{C}{\sin 135°}$$

or

$$\frac{30 \text{ lb}}{\sin 14.036°} = \frac{B}{\sin 30.964°} = \frac{C}{\sin 135°}$$

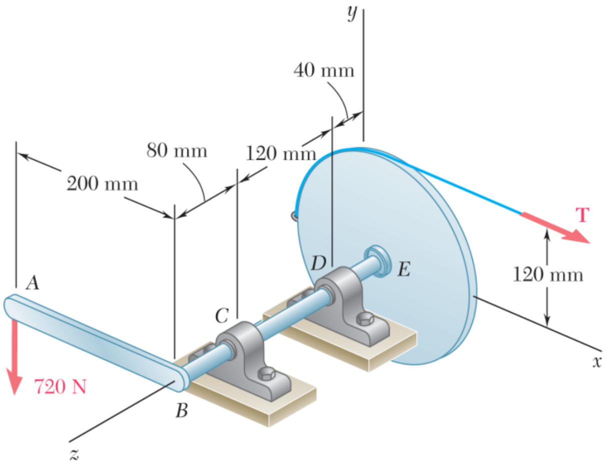
$$A = B = \frac{(30 \text{ lb})\sin 30.964°}{\sin 14.036°} = 63.641 \text{ lb}$$

or **A** = 63.6 lb ↘ 45.0° ◀

and

$$C = \frac{(30 \text{ lb})\sin 135°}{\sin 14.036°} = 87.466 \text{ lb}$$

or **C** = 87.5 lb ↖ 59.0° ◀

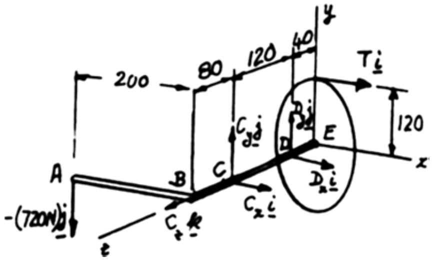

y
40 mm
80 mm
120 mm
200 mm
A
C
D
E
T
120 mm
720 N
B
z
x

PROBLEM 4.150

A 200-mm lever and a 240-mm-diameter pulley are welded to the axle *BE* that is supported by bearings at *C* and *D*. If a 720-N vertical load is applied at *A* when the lever is horizontal, determine (*a*) the tension in the cord, (*b*) the reactions at *C* and *D*. Assume that the bearing at *D* does not exert any axial thrust.

SOLUTION

Dimensions in mm


y
200
80
120
40
$T\mathbf{i}$
120
A
B
C
D
E
x
$C_y\mathbf{j}$
$D_y\mathbf{j}$
$D_x\mathbf{i}$
$C_x\mathbf{i}$
$-(720\text{N})\mathbf{j}$
$C_z\mathbf{k}$
z

We have six unknowns and six equations of equilibrium. —OK

$$\Sigma \mathbf{M}_C = 0: \quad (-120\mathbf{k}) \times (D_x\mathbf{i} + D_y\mathbf{j}) + (120\mathbf{j} - 160\mathbf{k}) \times T\mathbf{i} + (80\mathbf{k} - 200\mathbf{i}) \times (-720\mathbf{j}) = 0$$

$$-120D_x\mathbf{j} + 120D_y\mathbf{i} - 120T\mathbf{k} - 160T\mathbf{j} + 57.6 \times 10^3\mathbf{i} + 144 \times 10^3\mathbf{k} = 0$$

Equating to zero the coefficients of the unit vectors:

$\mathbf{k}$: $\quad -120T + 144 \times 10^3 = 0 \qquad$ (*a*) $T = 1200\text{ N}$ ◀

$\mathbf{i}$: $\quad 120D_y + 57.6 \times 10^3 = 0 \qquad D_y = -480\text{ N}$

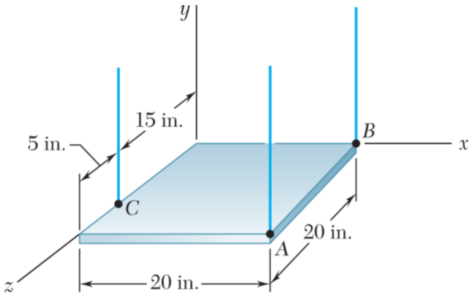
$\mathbf{j}$: $\quad -120D_x - 160(1200\text{ N}) = 0 \qquad D_x = -1600\text{ N}$

$\Sigma F_x = 0$: $\quad C_x + D_x + T = 0 \qquad C_x = 1600 - 1200 = 400\text{ N}$

$\Sigma F_y = 0$: $\quad C_y + D_y - 720 = 0 \qquad C_y = 480 + 720 = 1200\text{ N}$

$\Sigma F_z = 0$: $\quad C_z = 0$

(*b*) $\quad \mathbf{C} = (400\text{ N})\mathbf{i} + (1200\text{ N})\mathbf{j}; \quad \mathbf{D} = -(1600\text{ N})\mathbf{i} - (480\text{ N})\mathbf{j}$ ◀

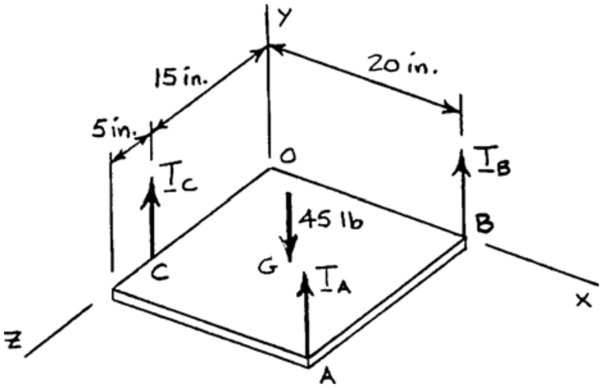


PROBLEM 4.151

The 45-lb square plate shown is supported by three vertical wires. Determine the tension in each wire.

SOLUTION

Free-Body Diagram:



$$\Sigma M_B = 0: \quad \mathbf{r}_{C/B} \times T_C\mathbf{j} + \mathbf{r}_{A/B} \times T_A\mathbf{j} + \mathbf{r}_{G/B} \times (-45\text{ lb})\mathbf{j} = 0$$

$$[-(20\text{ in.})\mathbf{i} + (15\text{ in.})\vec{\mathbf{k}}] \times T_C\mathbf{j} + (20\text{ in.})\mathbf{k} \times T_A\mathbf{j}$$

$$+[-(10\text{ in.})\mathbf{i} + (10\text{ in.})\mathbf{k}] \times [-(45\text{ lb})\mathbf{j}] = 0$$

$$-20T_C\mathbf{k} - 15T_C\mathbf{i} - 20T_A\mathbf{i} + 450\mathbf{k} + 450\mathbf{i} = 0$$

Equating to zero the coefficients of the unit vectors,

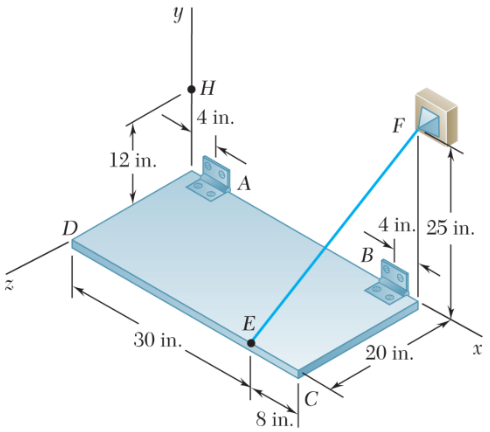
$$\mathbf{k}: \quad -20T_C + 450 = 0; \quad T_C = 22.5\text{ lb}$$

$$\mathbf{i}: \quad -15(22.5) - 20T_A + 450 = 0; \quad T_A = 5.625\text{ lb}$$

$$\Sigma F_y = 0: \quad T_A + T_B + T_C - 45\text{ lb} = 0$$

$$5.625\text{ lb} + T_B + 22.5\text{ lb} - 45\text{ lb} = 0; \quad T_B = 16.875\text{ lb}$$

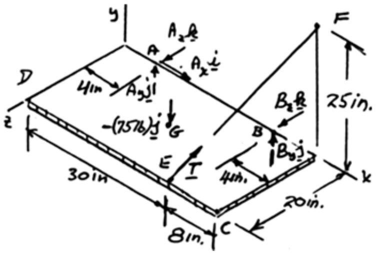
$$T_A = 5.63\text{ lb}; \; T_B = 16.88\text{ lb}; \; T_C = 22.5\text{ lb} \blacktriangleleft$$


y
H
4 in.
12 in.
A
F
D
4 in.
25 in.
B
z
E
30 in.
20 in.
x
C
8 in.

PROBLEM 4.152

The rectangular plate shown weighs 75 lb and is held in the position shown by hinges at *A* and *B* and by cable *EF*. Assuming that the hinge at *B* does not exert any axial thrust, determine (*a*) the tension in the cable, (*b*) the reactions at *A* and *B*.

SOLUTION


y
$A_z\mathbf{k}$
A
$A_x\mathbf{i}$
F
D
4in
$A_y\mathbf{j}$
$-(75\text{lb})\mathbf{j}$
G
z
$B_z\mathbf{k}$
25in.
B
$B_y\mathbf{j}$
E
T
4in.
30in
x
20in.
8in.
C

$$\mathbf{r}_{B/A} = (38-8)\mathbf{i} = 30\mathbf{i}$$

$$\mathbf{r}_{E/A} = (30-4)\mathbf{i} + 20\mathbf{k}$$

$$= 26\mathbf{i} + 20\mathbf{k}$$

$$\mathbf{r}_{G/A} = \frac{38}{2}\mathbf{i} + 10\mathbf{k}$$

$$= 19\mathbf{i} + 10\mathbf{k}$$

$$\overline{EF} = 8\mathbf{i} + 25\mathbf{j} - 20\mathbf{k}$$

$$EF = 33 \text{ in.}$$

$$\mathbf{T} = T\frac{\overrightarrow{AE}}{AE} = \frac{T}{33}(8\mathbf{i} + 25\mathbf{j} - 20\mathbf{k})$$

$$\Sigma\mathbf{M}_A = 0: \quad \mathbf{r}_{E/A} \times T + \mathbf{r}_{G/A} \times (-75\mathbf{j}) + \mathbf{r}_{B/A} \times B = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 26 & 0 & 20 \\ 8 & 25 & -20 \end{vmatrix}\frac{T}{33} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 19 & 0 & 10 \\ 0 & -75 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 30 & 0 & 0 \\ 0 & B_y & B_z \end{vmatrix} = 0$$

Coefficient of **i**: $-(25)(20)\dfrac{T}{33} + 750 = 0:$ $T = 49.5 \text{ lb}$ ◀

Coefficient of **j**: $(160+520)\dfrac{49.5}{33} - 30B_z = 0: \quad B_z = 34 \text{ lb}$

Coefficient of **k**: $(26)(25)\dfrac{49.5}{33} - 1425 + 30B_y = 0: \quad B_y = 15 \text{ lb}$ $\mathbf{B} = (15.00 \text{ lb})\mathbf{j} + (34.0 \text{ lb})\mathbf{k}$ ◀

PROBLEM 4.152 (Continued)

$$\Sigma \mathbf{F} = 0: \quad \mathbf{A} + \mathbf{B} + \mathbf{T} - (75 \text{ lb})\mathbf{j} = 0$$

Coefficient of **i**: $A_x + \dfrac{8}{33}(49.5) = 0 \quad A_x = -12.00 \text{ lb}$

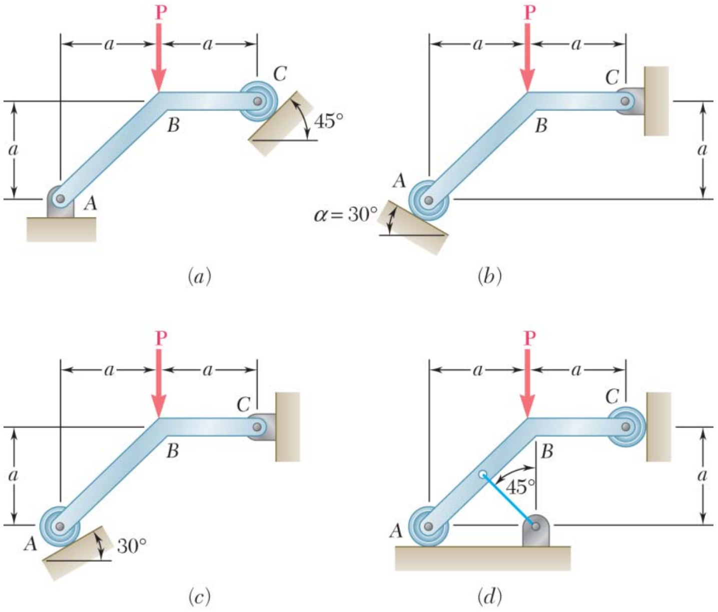
Coefficient of **j**: $A_y + 15 + \dfrac{25}{33}(49.5) - 75 = 0 \quad A_y = 22.5 \text{ lb}$

Coefficient of **k**: $A_z + 34 - \dfrac{20}{33}(49.5) = 0 \quad A_z = -4.00 \text{ lb}$

$\mathbf{A} = -(12.00 \text{ lb})\mathbf{i} + (22.5 \text{ lb})\mathbf{j} - (4.00 \text{ lb})\mathbf{k}$ ◀

PROBLEM 4.153

A force **P** is applied to a bent rod *ABC*, which may be supported in four different ways as shown. In each case, if possible, determine the reactions at the supports.

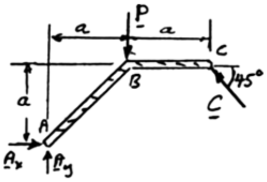


(*a*) (*b*)

(*c*) (*d*)

SOLUTION

(*a*)



$$+\circlearrowleft \Sigma M_A = 0: \quad -P_a + (C\sin 45°)2a + (\cos 45°)a = 0$$

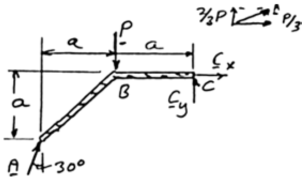
$$3\frac{C}{\sqrt{2}} = P \qquad C = \frac{\sqrt{2}}{3}P \qquad \mathbf{C} = 0.471P \measuredangle 45° \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - \left(\frac{\sqrt{2}}{3}P\right)\frac{1}{\sqrt{2}} \qquad A_x = \frac{P}{3} \longrightarrow$$

$$+\!\uparrow \Sigma F_y = 0: \quad A_y - P + \left(\frac{\sqrt{2}}{3}P\right)\frac{1}{\sqrt{2}} \qquad A_y = \frac{2P}{3}\uparrow$$

$$\mathbf{A} = 0.745P \measuredangle 63.4° \blacktriangleleft$$

(*b*)



$$+\circlearrowleft \Sigma M_C = 0: \quad +Pa - (A\cos 30°)2a + (A\sin 30°)a = 0$$

$$A(1.732 - 0.5) = P \quad A = 0.812P$$

$$\mathbf{A} = 0.812P \measuredangle 60.0° \blacktriangleleft$$

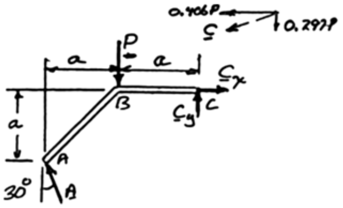
$$\xrightarrow{+} \Sigma F_x = 0: \quad (0.812P)\sin 30° + C_x = 0 \qquad C_x = -0.406P$$

$$+\!\uparrow \Sigma F_y = 0: \quad (0.812P)\cos 30° - P + C_y = 0 \quad C_y = -0.297P$$

$$\mathbf{C} = 0.503P \measuredangle 36.2° \blacktriangleleft$$

PROBLEM 4.153 (Continued)

(*c*)



$$+\circlearrowleft \Sigma M_C = 0: \quad +Pa - (A\cos 30°)2a + (A\sin 30°)a = 0$$

$$A(1.732 + 0.5) = P \quad A = 0.448P$$

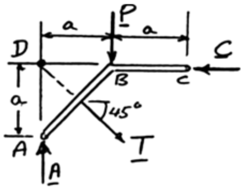
$$\mathbf{A} = 0.448P \ \measuredangle\ 60.0° \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -(0.448P)\sin 30° + C_x = 0 \qquad C_x = 0.224P \longrightarrow$$

$$+\uparrow \Sigma F_y = 0: \quad (0.448P)\cos 30° - P + C_y = 0 \quad C_y = 0.612P \uparrow$$

$$\mathbf{C} = 0.652P \ \measuredangle\ 69.9° \blacktriangleleft$$

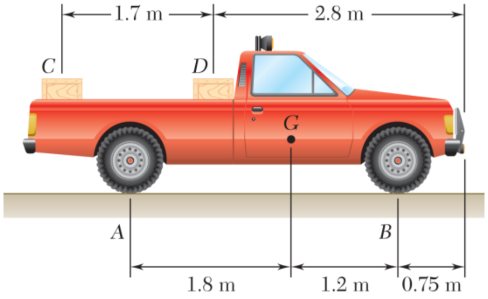
(*d*) Force **T** exerted by wire and reactions **A** and **C** all intersect at Point *D*.



$$+\circlearrowleft \Sigma M_D = 0: \quad P_a = 0$$

Equilibrium is not maintained.

Rod is improperly constrained. ◀

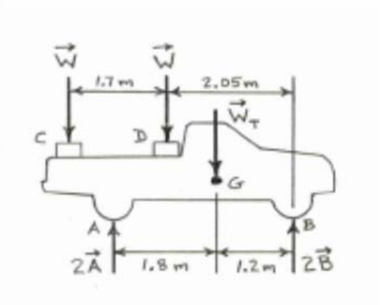

1.7 m
2.8 m
C
D
G
A
B
1.8 m
1.2 m
0.75 m

PROBLEM 4.F1

Two crates, each of mass 350 kg, are placed as shown in the bed of a 1400-kg pick-up truck. Draw the free-body diagram needed to determine the reactions at each of the two rear wheels *A* and front wheels *B*.

SOLUTION

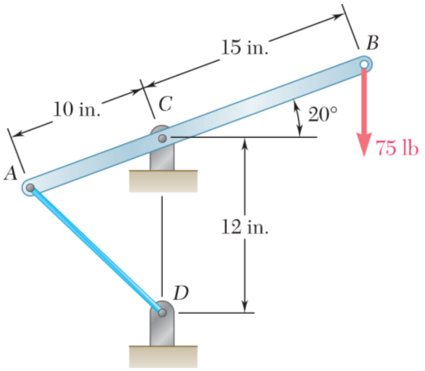
Free-Body Diagram of Truck:


$\vec{W}$
$\vec{W}$
1.7 m
2.05 m
$\vec{W}_T$
C
D
G
A
B
$2\vec{A}$
1.8 m
1.2 m
$2\vec{B}$

Weights:

$W = (350 \text{ kg})(9.81 \text{ m/s}^2) = 3.4335 \text{ N}$

$W_T = (1400 \text{ kg})(9.81 \text{ m/s}^2) = 13.734 \text{ N}$

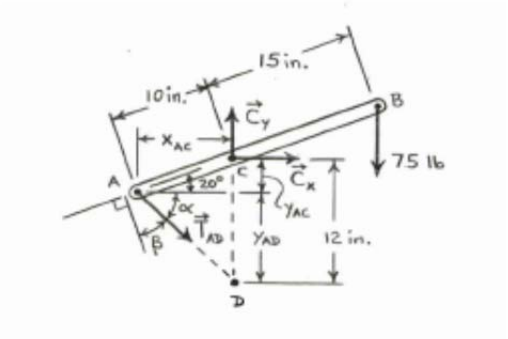

15 in.
B
10 in.
C
20°
75 lb
A
12 in.
D

PROBLEM 4.F2

A lever *AB* is hinged at *C* and attached to a control cable at *A*. If the lever is subjected to a 75-lb vertical force at *B*, draw the free-body diagram needed to determine the tension in the cable and the reaction at *C*.

SOLUTION

Free-Body Diagram of Lever *AB*:


15 in.
10 in.
B
$\vec{C}_y$
x_{AC}
75 lb
A
20°
C
$\vec{C}_x$
y_{AC}
α
$\vec{T}_{AD}$
β
y_{AD}
12 in.
D

Geometry:

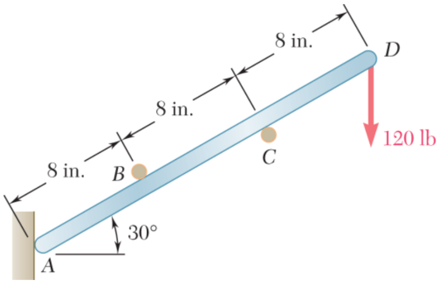
$$x_{AC} = (10\text{ in.})\cos 20^\circ = 9.3969\text{ in.}$$

$$y_{AC} = (10\text{ in.})\sin 20^\circ = 3.4202\text{ in.}$$

$$y_{AD} = 12\text{ in.} - 3.4202\text{ in.} = 8.5798\text{ in.}$$

$$\alpha = \tan^{-1}\left(\frac{y_{AD}}{x_{AC}}\right) = \tan^{-1}\left(\frac{8.5798\text{ in.}}{9.3969\text{ in.}}\right) = 42.398^\circ$$

$$\beta = 90^\circ - 20^\circ - 42.398^\circ = 27.602^\circ$$

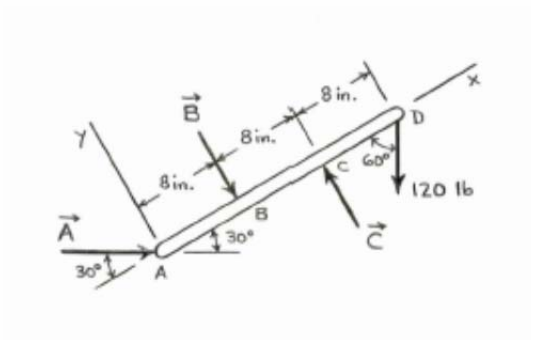

8 in.
8 in.
8 in.
D
B
C
120 lb
30°
A

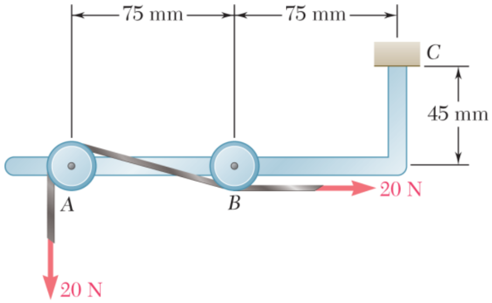
PROBLEM 4.F3

A light rod *AD* is supported by frictionless pegs at *B* and *C* and rests against a frictionless wall at *A*. A vertical 120-lb force is applied at D. Draw the free-body diagram needed to determine the reactions at *A*, *B*, and *C*.

SOLUTION

Free-Body Diagram of Rod *AD*:


y
x
$\vec{B}$
8 in.
8 in.
8 in.
D
C
60°
120 lb
B
30°
$\vec{A}$
30°
A
$\vec{C}$

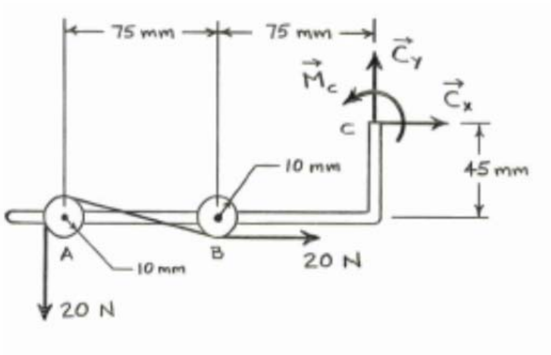

75 mm
75 mm
C
45 mm
20 N
A
B
20 N

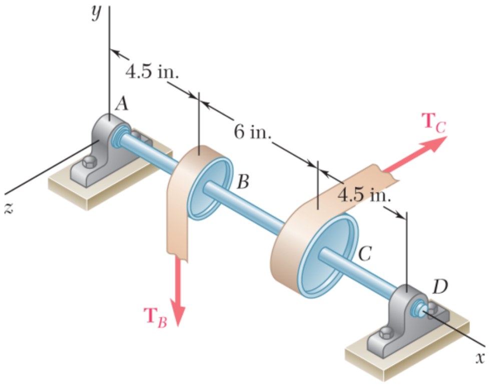
PROBLEM 4.F4

A tension of 20 N is maintained in a tape as it passes through the support system shown. Knowing that the radius of each pulley is 10 mm, draw the free-body diagram needed to determine the reaction at *C*.

SOLUTION

Free-Body Diagram of Support System:


75 mm
75 mm
$\vec{C}_y$
$\vec{M}_c$
$\vec{C}_x$
C
10 mm
45 mm
A
B
10 mm
20 N
20 N

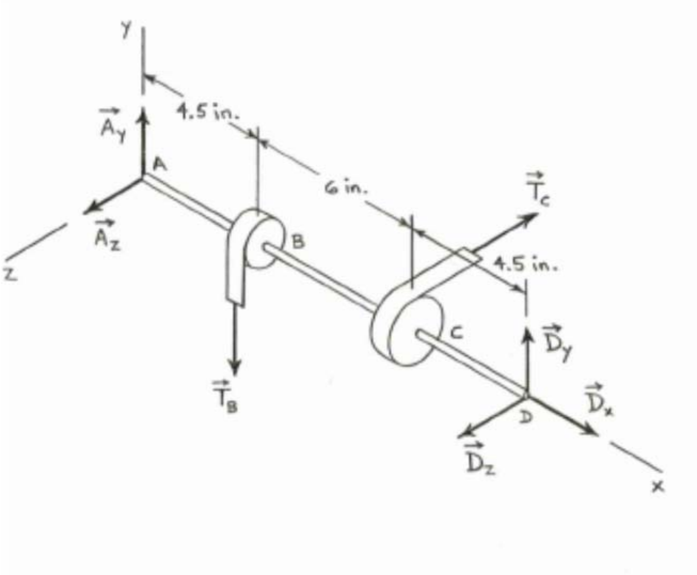

y
4.5 in.
A
6 in.
T_C
B
4.5 in.
C
D
z
T_B
x

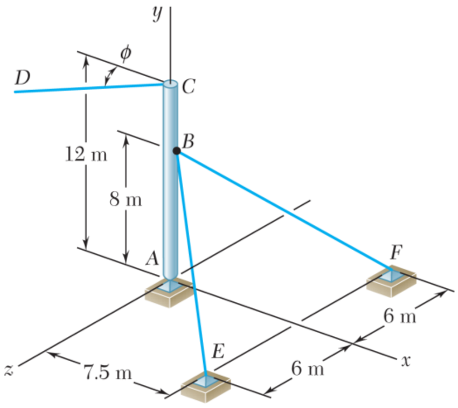
PROBLEM 4.F5

Two tape spools are attached to an axle supported by bearings at *A* and *D*. The radius of spool *B* is 1.5 in. and the radius of spool *C* is 2 in. Knowing that $T_B = 20$ lb and that the system rotates at a constant rate, draw the free-body diagram needed to determine the reactions at *A* and *D*. Assume that the bearing at *A* does not exert any axial thrust and neglect the weights of the spools and axle.

SOLUTION

Free-Body Diagram of Axle-Spool System:


y
4.5 in.
$\vec{A}_y$
A
$\vec{A}_z$
z
6 in.
B
$\vec{T}_c$
4.5 in.
C
$\vec{D}_y$
$\vec{T}_B$
D
$\vec{D}_x$
$\vec{D}_z$
x

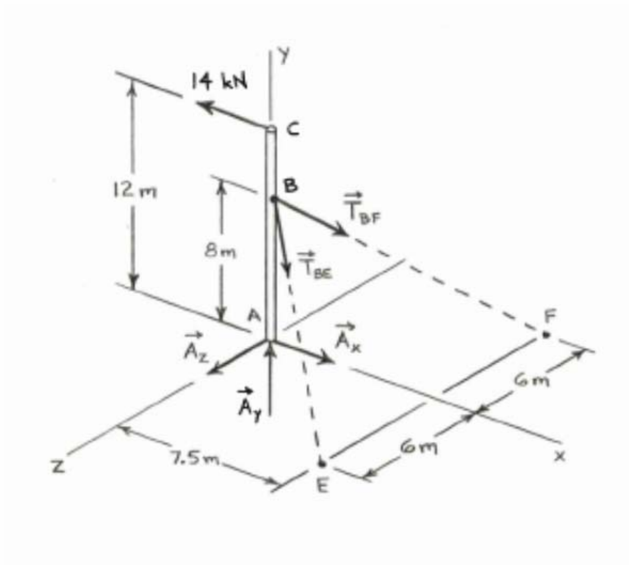

y
ϕ
D
C
B
12 m
8 m
A
F
6 m
E
6 m
x
z
7.5 m

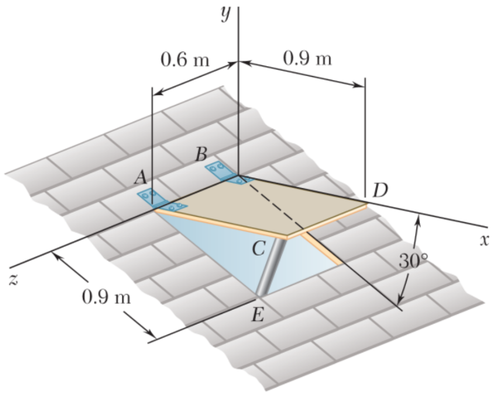
PROBLEM 4.F6

A 12-m pole supports a horizontal cable *CD* and is held by a ball and socket at *A* and two cables *BE* and *BF*. Knowing that the tension in cable *CD* is 14 kN and assuming that *CD* is parallel to the *x* axis ($\phi = 0$), draw the free-body diagram needed to determine the tension in cables *BE* and *BF* and the reaction at *A*.

SOLUTION

Free-Body Diagram of Pole:


Y
14 kN
C
B
12 m
$\vec{T}_{BF}$
8m
$\vec{T}_{BE}$
A
F
$\vec{A}_z$
$\vec{A}_x$
6m
$\vec{A}_y$
6m
Z
7.5m
E
x

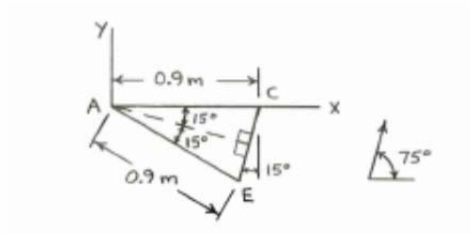

y
0.6 m
0.9 m
B
A
D
C
x
30°
z
0.9 m
E

PROBLEM 4.F7

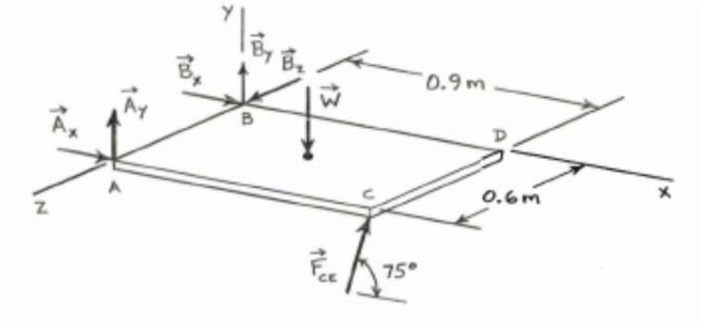
A 20-kg cover for a roof opening is hinged at corners *A* and *B*. The roof forms an angle of 30° with the horizontal, and the cover is maintained in a horizontal position by the brace *CE*. Draw the free-body diagram needed to determine the magnitude of the force exerted by the brace and the reactions at the hinges. Assume that the hinge at *A* does not exert any axial thrust.

SOLUTION

Geometry of Brace *CE*:

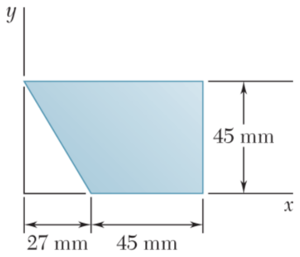

Y
0.9 m
C
A
X
15°
15°
0.9 m
15°
E
75°

Free-Body Diagram of Brace *CE*:


Y
$\vec{B}_y$
$\vec{B}_z$
$\vec{B}_x$
$\vec{A}_y$
$\vec{A}_x$
$\vec{W}$
0.9 m
B
D
A
C
Z
0.6 m
X
$\vec{F}_{CE}$
75°

Cover Weight: $W = (20\text{ kg})(9.81\text{ m/s}^2) = 196.2\text{ N}$

CHAPTER 5



PROBLEM 5.1

Locate the centroid of the plane area shown.

SOLUTION

Area 1: Rectangle 72 mm by 45 mm.

Area 2: Triangle $b = 27$ mm, $h = 45$ mm.

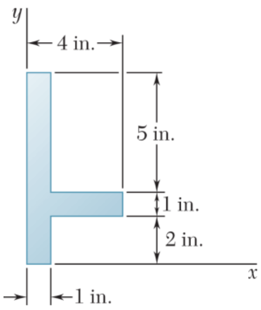
	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	3240	36	22.5	116,640	72,900
2	-607.5	9	215	-5467.5	-9112.5
Σ	2632.5			11,172.5	63,787.5

$$\bar{X}\,\Sigma A = \bar{x}A$$

$$\bar{X}(2632.5\text{ mm}^2) = 11{,}172.5\text{ mm}^3 \qquad \bar{X} = 42.2\text{ mm} \blacktriangleleft$$

$$\bar{Y}\,\Sigma A = \Sigma\bar{y}A$$

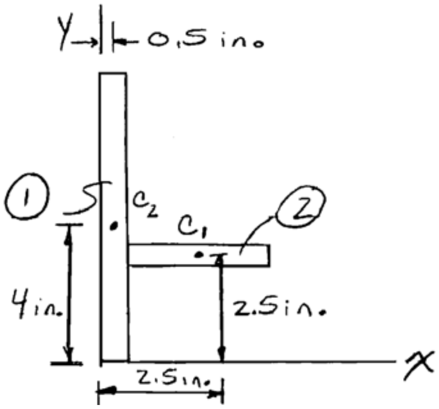
$$\bar{Y}(2632.5\text{ mm}^2) = 63787.5\text{ mm}^3 \qquad \bar{Y} = 24.2\text{ mm} \blacktriangleleft$$


y
4 in.
5 in.
1 in.
2 in.
x
1 in.

PROBLEM 5.2

Locate the centroid of the plane area shown.

SOLUTION


Y
0.5 in.
①
C_2
C_1
②
4 in.
2.5 in.
2.5 in.
x

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	8	0.5	4	4	32
2	3	2.5	2.5	7.5	7.5
Σ	11			11.5	39.5

$$\bar{X}\Sigma A = \bar{x}A$$

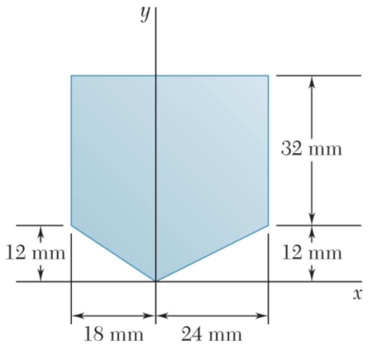
$$\bar{X}(11\text{ in}^2) = 11.5\text{ in}^3$$

$\bar{X} = 1.045$ in. ◀

$$\bar{Y}\Sigma A = \Sigma\bar{y}A$$

$$\bar{Y}(11\text{ in}^2) = 39.5\text{ in}^3$$

$\bar{Y} = 3.59$ in. ◀



PROBLEM 5.3

Locate the centroid of the plane area shown.

SOLUTION

Area 1: Rectangle 42 mm by 44 mm.

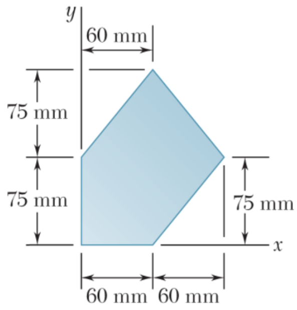
Area 2: Triangle $b = 18$ mm, $h = 12$ mm.

Area 3: Triangle $b = 24$ mm, $h = 12$ mm.

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$42 \times 44 = 1848$	3	22	5544	40,656
2	$-\frac{1}{2} \times 18 \times 12 = -108$	-12	4	1296	-432
3	$-\frac{1}{2} \times 24 \times 12 = -144$	16	4	-2304	-576
Σ	1596			4536	39,648

Then
$$\bar{X}\Sigma A = \Sigma \bar{x}A$$
$$\bar{X}(1596 \text{ mm}^2) = 4536 \text{ mm}^3$$
or $\bar{X} = 2.84$ mm ◀

and
$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$
$$\bar{Y}(1596 \text{ mm}^2) = 39{,}648 \text{ mm}^3$$
or $\bar{Y} = 24.8$ mm ◀



PROBLEM 5.4

Locate the centroid of the plane area shown.

SOLUTION

Area 1: Triangle $b = 60$ mm, $h = 75$ mm.

Area 2: Four triangles $b = 60$ mm, $h = 75$ mm forming the diamond.

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{1}{2} \times 60 \times 75 = 2250$	20	25	45,000	56,250
2	$4 \times \frac{1}{2} \times 60 \times 75 = 9000$	60	75	540,000	675,000
Σ	11,250			585,000	731,250

Then

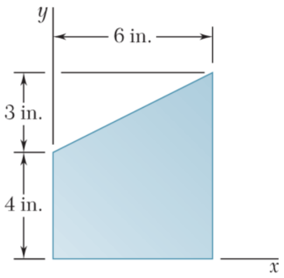
$$\bar{X}\Sigma A = \Sigma \bar{x} A$$

$$\bar{X}(11{,}250 \text{ mm}^2) = 585{,}000 \text{ mm}^3 \qquad \text{or} \quad \bar{X} = 52.0 \text{ mm} \blacktriangleleft$$

and

$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$

$$\bar{Y}(11{,}250 \text{ m}^2) = 731{,}250 \text{ mm}^3 \qquad \text{or} \quad \bar{Y} = 65.0 \text{ mm} \blacktriangleleft$$



PROBLEM 5.5

Locate the centroid of the plane area shown.

SOLUTION

Area 1: Rectangle 6 in. by 4 in.

Area 2: Triangle $b = 6$ in., $h = 3$ in.

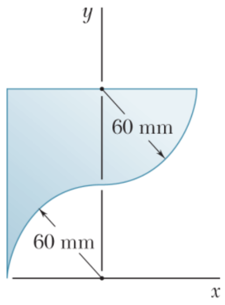
	A, in^2	$\bar{x}$, in	$\bar{y}$, in	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	24	3	2	72	48
2	9	4	5	36	45
Σ	33			108	93

$$\bar{X}\,\Sigma A = \bar{x}A$$

$$\bar{X}(33\text{ in}^2) = 108\text{ in}^3 \qquad \bar{X} = 3.27\text{ in.} \blacktriangleleft$$

$$\bar{Y}\,\Sigma A = \Sigma\bar{y}A$$

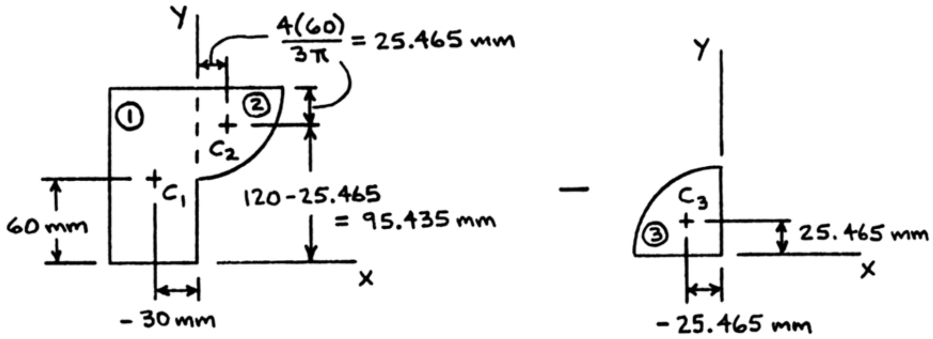
$$\bar{Y}(33\text{ in}^2) = 93\text{ in}^3 \qquad \bar{Y} = 2.82\text{ in.} \blacktriangleleft$$


y
60 mm
60 mm
x

PROBLEM 5.6

Locate the centroid of the plane area shown.

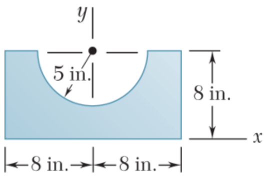
SOLUTION


Y
4(60)/3π = 25.465 mm
①
②
C₂
C₁
60 mm
120 − 25.465 = 95.435 mm
X
− 30 mm
−
Y
C₃
③
25.465 mm
X
− 25.465 mm

	A, mm^2	$\overline{x}$, mm	$\overline{y}$, mm	$\overline{x}A$, mm^3	$\overline{y}A$, mm^3
1	$(60)(120) = 7200$	-30	60	-216×10^3	432×10^3
2	$\frac{\pi}{4}(60)^2 = 2827.4$	25.465	95.435	72.000×10^3	269.83×10^3
3	$-\frac{\pi}{4}(60)^2 = -2827.4$	-25.465	25.465	72.000×10^3	-72.000×10^3
Σ	7200			-72.000×10^3	629.83×10^3

Then $\overline{X}A = \Sigma \overline{x} A$ $\quad \overline{X}(7200 \text{ mm}^2) = -72.000 \times 10^3 \text{ mm}^3$ $\quad \overline{X} = -10.00 \text{ mm}$ ◀

$\overline{Y}A = \Sigma \overline{y} A$ $\quad \overline{Y}(7200 \text{ mm}^2) = 629.83 \times 10^3 \text{ mm}^3$ $\quad \overline{Y} = 87.5 \text{ mm}$ ◀



PROBLEM 5.7

Locate the centroid of the plane area shown.

SOLUTION

Area 1: Rectangle 16 in. by 8 in.

Area 2: Semicircle radius of 5 in.

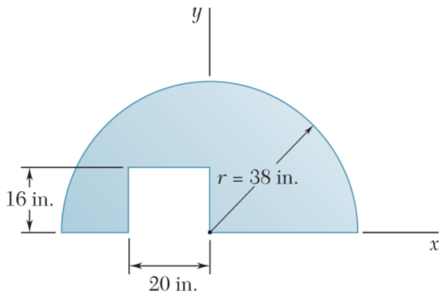
	A, in^2	$\bar{x}$, in	$\bar{y}$, in	$\bar{x}A, \text{in}^3$	$\bar{y}A, \text{in}^3$
1	128	0	4	0	512
2	-39.27	0	5.878	0	-230.83
Σ	88.73			0	281.17

$$\bar{X}\Sigma A = \bar{x}A$$

$$\bar{X} = 0 \text{ in.} \blacktriangleleft$$

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

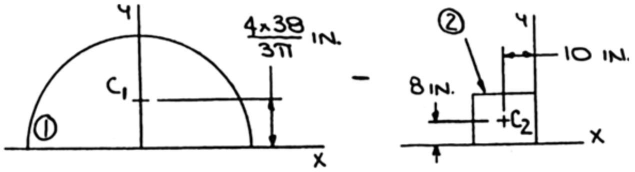
$$\bar{Y}(88.73 \text{ in}^2) = 281.17 \text{ in}^3 \qquad \bar{Y} = 3.17 \text{ in.} \blacktriangleleft$$


y
r = 38 in.
16 in.
20 in.
x

PROBLEM 5.8

Locate the centroid of the plane area shown.

SOLUTION


y
$\frac{4\times 38}{3\pi}$ IN.
C_1
①
x
②
y
10 IN.
8 IN.
C_2
x

	A, in^2	$\overline{x}$, in.	$\overline{y}$, in.	$\overline{x}A$, in^3	$\overline{y}A$, in^3
1	$\frac{\pi}{2}(38)^2 = 2268.2$	0	16.1277	0	36,581
2	$-20\times 16 = 320$	-10	8	3200	-2560
Σ	1948.23			3200	34,021

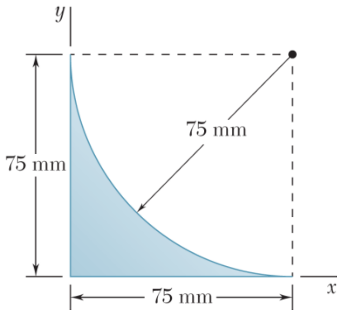
Then

$$\overline{X} = \frac{\Sigma\overline{x}A}{\Sigma A} = \frac{3200\text{ in}^3}{1948.23\text{ in}^2}$$

$\overline{X} = 1.643$ in. ◀

$$\overline{Y} = \frac{\Sigma\overline{y}A}{\Sigma A} = \frac{34{,}021\text{ in}^3}{1948.23\text{ in}^2}$$

$\overline{Y} = 17.46$ in. ◀



PROBLEM 5.9

Locate the centroid of the plane area shown.

SOLUTION

Area 1: Square 75 mm by 75 mm.

Area 2: Quarter circle radius of 75 mm.

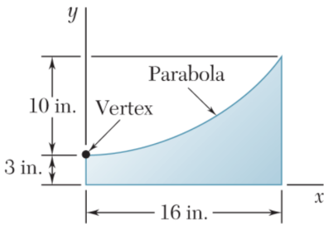
	A, mm^2	$\overline{x}$, mm	$\overline{y}$, mm	$\overline{x}A$, mm^3	$\overline{y}A$, mm^3
1	$75 \times 75 = 5625$	37.5	37.5	210,938	210,938
2	$-\frac{\pi}{4} \times 75^2 = -4417.9$	43.169	43.169	-190,715	−190,715
Σ	1207.10			20,223	20,223

Then by symmetry

$$\overline{X} = \overline{Y} = \frac{\Sigma \overline{x} A}{\Sigma A}$$

$$\overline{X}(1207.14 \text{ mm}^2) = 20,223 \text{ mm}^3$$

$\overline{X} = \overline{Y} = 16.75$ mm ◀



PROBLEM 5.10

Locate the centroid of the plane area shown.

SOLUTION

Area 1: Parabolic Spandrel, refer to fig 5.8A for centroid location.

Area 2: Rectangle 16 in. by 3 in.

	A, in.^{2}	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$\frac{1}{3}(10)(16) = 53.333$	12	6	640	320
2	$(3)(16) = 48.0$	8	1.5	384	72
Σ	101.333			1024	392

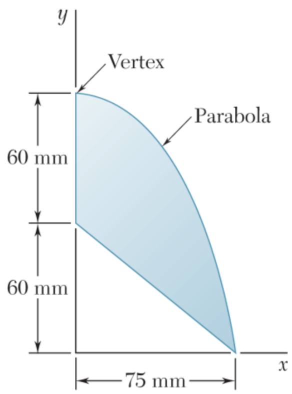
Then

$$\bar{X}A = \Sigma \bar{x}A$$

$$\bar{X}\,(101.333 \text{ in}^2) = 1024 \text{ in}^3 \qquad \bar{X} = 10.11 \text{ in.} \blacktriangleleft$$

$$\bar{Y}A = \Sigma\,\bar{y}A$$

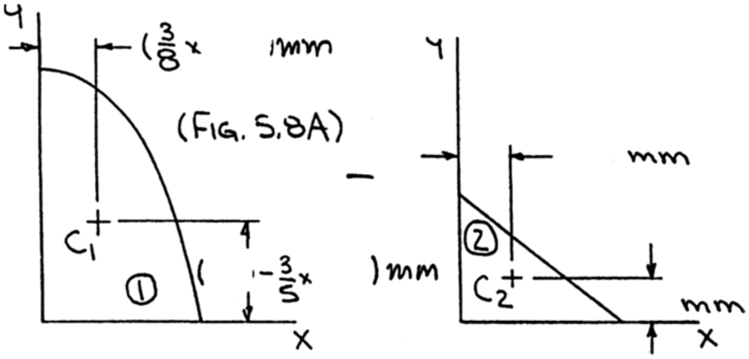
$$\bar{Y}(101.333 \text{ in}^2) = 393 \text{ in}^3 \qquad \bar{Y} = 3.88 \text{ in.} \blacktriangleleft$$


y
Vertex
Parabola
60 mm
60 mm
75 mm
x

PROBLEM 5.11

Locate the centroid of the plane area shown.

SOLUTION


y
$(\frac{3}{8}x$
)mm
(FIG. 5.8A)
C_1
①
(
$-\frac{3}{5}x$
)mm
x
−
y
mm
②
C_2
mm
x

	A, mm^2	$\overline{x}$, mm	$\overline{y}$, mm	$\overline{x}A$, mm^3	$\overline{y}A$, mm^3
1	$\frac{2}{3}(75)(120) = 6000$	28.125	48	168,750	288,000
2	$-\frac{1}{2}(75)(60) = -2250$	25	20	–56,250	–45,000
Σ	3750			112,500	243,000

Then

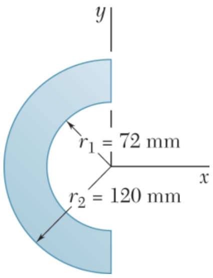
$$\overline{X}\Sigma A = \Sigma \overline{x}A$$

$$\overline{X}(3750\text{ mm}^2) = 112{,}500\text{ mm}^3 \qquad \text{or} \quad \overline{X} = 30.0\text{ mm} \blacktriangleleft$$

and

$$\overline{Y}\Sigma A = \Sigma \overline{y}A$$

$$\overline{Y}(3750\text{ mm}^2) = 243{,}000\text{ mm}^3 \qquad \text{or} \quad \overline{Y} = 64.8\text{ mm} \blacktriangleleft$$



PROBLEM 5.12

Locate the centroid of the plane area shown.

SOLUTION

Area 1: Outer semicircle diameter 120 mm.

Area 2: Inner semicircle diameter 72 mm.

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{\pi}{2}(120)^2 = 22,620$	-50.93	0	-1152.04×10^3	0
2	$-\frac{\pi}{2}(72)^2 = -8143.0$	-30.558	0	248.80×10^3	0
Σ	14,477			-903.20×10^3	0

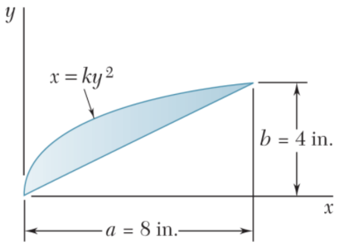
Then

$$\bar{X}A = \Sigma\bar{x}A$$

$$\bar{X}\,(14,477 \text{ mm}^2) = -903.20\times10^3 \text{ mm}^3 \qquad \bar{X} = -62.4 \text{ mm} ◀$$

$$\bar{Y}A = \Sigma\bar{y}A \qquad \bar{Y} = 0 ◀$$

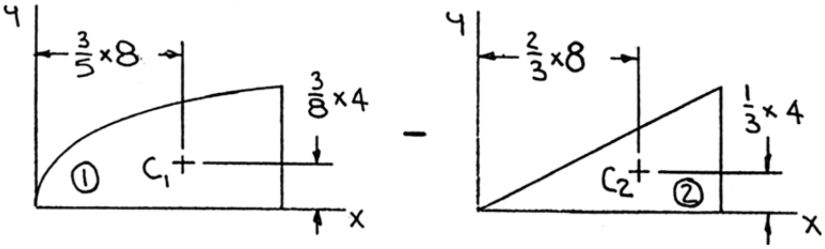
PROBLEM 5.13



y
$x = ky^2$
b = 4 in.
x
a = 8 in.

Locate the centroid of the plane area shown.

SOLUTION



y
$\frac{3}{5} \times 8$
$\frac{3}{8} \times 4$
①
C_1
x
−
y
$\frac{2}{3} \times 8$
$\frac{1}{3} \times 4$
C_2
②
x

Dimensions in in.

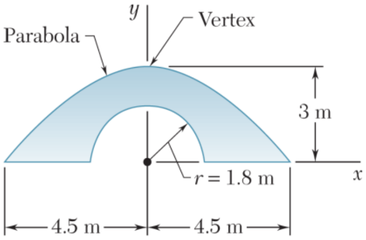
	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$\frac{2}{3}(4)(8) = 21.333$	4.8	1.5	102.398	32.000
2	$-\frac{1}{2}(4)(8) = -16.0000$	5.3333	1.33333	85.333	–21.333
Σ	5.3333			17.0650	10.6670

Then $$\bar{X}\Sigma A = \Sigma \bar{x}A$$

$$\bar{X}(5.3333\text{ in}^2) = 17.0650\text{ in}^3$$ or $\bar{X} = 3.20$ in. ◀

and $$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

$$\bar{Y}(5.3333\text{ in}^2) = 10.6670\text{ in}^3$$ or $\bar{Y} = 2.00$ in. ◀

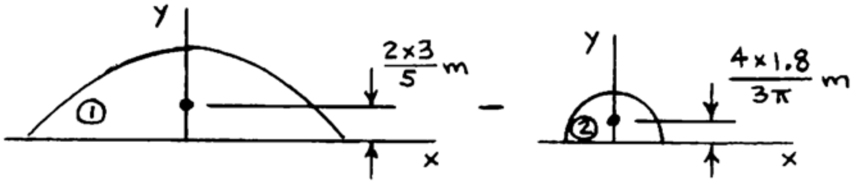

y
Vertex
Parabola
3 m
$r = 1.8$ m
x
4.5 m
4.5 m

PROBLEM 5.14

Locate the centroid of the plane area shown.

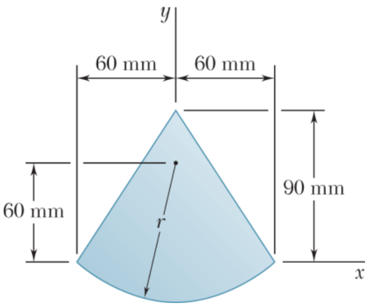
SOLUTION

First note that symmetry implies $\bar{X} = 0$ ◀


y
$\frac{2\times 3}{5}$ m
①
x
–
y
$\frac{4\times 1.8}{3\pi}$ m
②
x

	A, m^2	$\bar{y}$, m	$\bar{y}A$, m^3
1	$\frac{4}{3}\times 4.5\times 3 = 18$	1.2	21.6
2	$-\frac{\pi}{2}(1.8)^2 = -5.0894$	0.76394	–3.8880
Σ	12.9106		17.7120

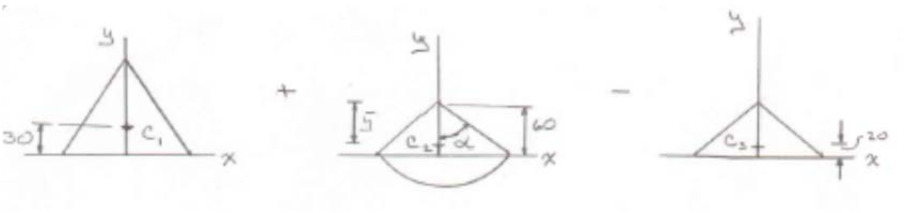
Then $$\bar{Y} = \frac{\Sigma\bar{y}A}{\Sigma A} = \frac{17.7120\text{ m}^3}{12.9106\text{ m}^2}$$ or $\bar{Y} = 1.372$ m ◀


y
60 mm
60 mm
90 mm
60 mm
r
x

PROBLEM 5.15

Locate the centroid of the plane area shown.

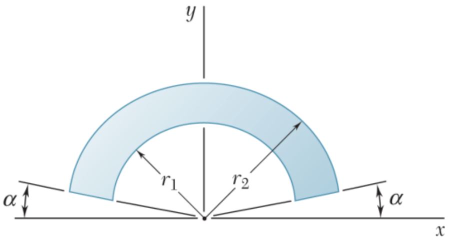
SOLUTION


y
30
C₁
x
+
y
15
C₂
α
60
x
−
y
C₃
20
x

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{1}{2}(120)(90)=5400$	0	30	0	162,000
2	$\frac{\pi}{4}(60\sqrt{2})^2=5654.9$	0	9.07	0	51,290
3	$-\frac{1}{2}(120)(60)=-3600$	0	20	0	-72,000
Σ	7454.9			0	141,290

Then $\bar{X}A=\Sigma\bar{x}A$ $\bar{X}=0$ mm ◀

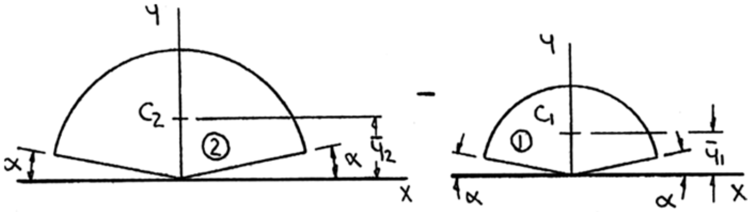
$\bar{Y}A=\Sigma\bar{y}A$ $\bar{Y}(7454.9\text{ mm}^2)=141{,}290\text{ mm}^3$ $\bar{Y}=18.95$ mm ◀



PROBLEM 5.16

Determine the y coordinate of the centroid of the shaded area in terms of r_1, r_2, and α.

SOLUTION



First, determine the location of the centroid.

From Figure 5.8A:
$$\bar{y}_2 = \frac{2}{3}r_2\frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\left(\frac{\pi}{2}-\alpha\right)} \qquad A_2 = \left(\frac{\pi}{2}-\alpha\right)r_2^2$$
$$= \frac{2}{3}r_2\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)}$$

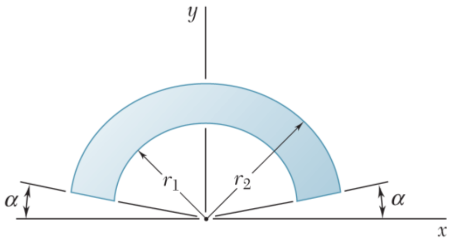
Similarly,
$$\bar{y}_1 = \frac{2}{3}r_1\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)} \qquad A_1 = \left(\frac{\pi}{2}-\alpha\right)r_1^2$$

Then
$$\Sigma\bar{y}A = \frac{2}{3}r_2\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)}\left[\left(\frac{\pi}{2}-\alpha\right)r_2^2\right] - \frac{2}{3}r_1\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)}\left[\left(\frac{\pi}{2}-\alpha\right)r_1^2\right]$$
$$= \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha$$

and
$$\Sigma A = \left(\frac{\pi}{2}-\alpha\right)r_2^2 - \left(\frac{\pi}{2}-\alpha\right)r_1^2$$
$$= \left(\frac{\pi}{2}-\alpha\right)\left(r_2^2 - r_1^2\right)$$

Now
$$\bar{Y}\Sigma A = \Sigma\bar{y}A$$

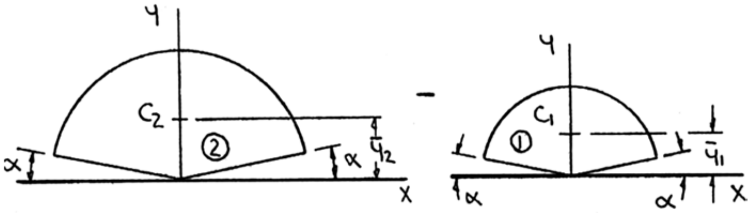
$$\bar{Y}\left[\left(\frac{\pi}{2}-\alpha\right)\left(r_2^2-r_1^2\right)\right] = \frac{2}{3}\left(r_2^3-r_1^3\right)\cos\alpha \qquad \bar{Y} = \frac{2}{3}\left(\frac{r_2^3-r_1^3}{r_2^2-r_1^2}\right)\left(\frac{2\cos\alpha}{\pi-2\alpha}\right) \blacktriangleleft$$



PROBLEM 5.17

Show that as r_1 approaches r_2, the location of the centroid approaches that for an arc of circle of radius $(r_1 + r_2)/2$.

SOLUTION



First, determine the location of the centroid.

From Figure 5.8A:

$$\bar{y}_2 = \frac{2}{3}r_2 \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\left(\frac{\pi}{2} - \alpha\right)} \qquad A_2 = \left(\frac{\pi}{2} - \alpha\right)r_2^2$$

$$= \frac{2}{3}r_2 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)}$$

Similarly,

$$\bar{y}_1 = \frac{2}{3}r_1 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)} \qquad A_1 = \left(\frac{\pi}{2} - \alpha\right)r_1^2$$

Then

$$\Sigma\bar{y}A = \frac{2}{3}r_2 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)}\left[\left(\frac{\pi}{2} - \alpha\right)r_2^2\right] - \frac{2}{3}r_1 \frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)}\left[\left(\frac{\pi}{2} - \alpha\right)r_1^2\right]$$

$$= \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha$$

and

$$\Sigma A = \left(\frac{\pi}{2} - \alpha\right)r_2^2 - \left(\frac{\pi}{2} - \alpha\right)r_1^2$$

$$= \left(\frac{\pi}{2} - \alpha\right)\left(r_2^2 - r_1^2\right)$$

Now

$$\bar{Y}\Sigma A = \Sigma\,\bar{y}A$$

$$\bar{Y}\left[\left(\frac{\pi}{2} - \alpha\right)\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha$$

$$\bar{Y} = \frac{2}{3}\left(\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\right)\left(\frac{2\cos\alpha}{\pi - 2\alpha}\right)$$

PROBLEM 5.17 (Continued)

Using Figure 5.8B, $\overline{Y}$ of an arc of radius $\frac{1}{2}(r_1 + r_2)$ is

$$\overline{Y} = \frac{1}{2}(r_1 + r_2)\frac{\sin(\frac{\pi}{2} - \alpha)}{(\frac{\pi}{2} - \alpha)}$$

$$= \frac{1}{2}(r_1 + r_2)\frac{\cos\alpha}{(\frac{\pi}{2} - \alpha)} \qquad (1)$$

Now
$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{(r_2 - r_1)\left(r_2^2 + r_1 r_2 + r_1^2\right)}{(r_2 - r_1)(r_2 + r_1)}$$

$$= \frac{r_2^2 + r_1 r_2 + r_1^2}{r_2 + r_1}$$

Let
$$r_2 = r + \Delta$$
$$r_1 = r - \Delta$$

Then
$$r = \frac{1}{2}(r_1 + r_2)$$

and
$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{(r + \Delta)^2 + (r + \Delta)(r - \Delta)(r - \Delta)^2}{(r + \Delta) + (r - \Delta)}$$

$$= \frac{3r^2 + \Delta^2}{2r}$$

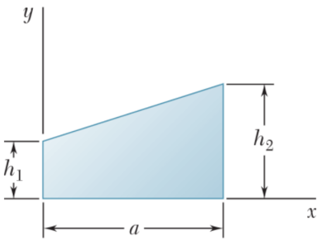
In the limit as $\Delta \longrightarrow 0$ (i.e., $r_1 = r_2$), then

$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{3}{2}r$$

$$= \frac{3}{2} \times \frac{1}{2}(r_1 + r_2)$$

So that
$$\overline{Y} = \frac{2}{3} \times \frac{3}{4}(r_1 + r_2)\frac{\cos\alpha}{\frac{\pi}{2} - \alpha}$$
or $\overline{Y} = (r_1 + r_2)\dfrac{\cos\alpha}{\pi - 2\alpha}$ ◀

which agrees with Equation (1).



PROBLEM 5.18

Determine the *x* coordinate of the centroid of the trapezoid shown in terms of h_1, h_2, and *a*.

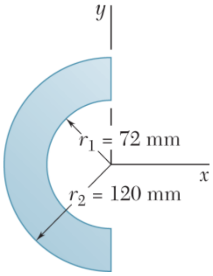
SOLUTION

Area 1:

	A	$\overline{x}$	$\overline{x}A$
1	$\frac{1}{2}h_1a$	$\frac{1}{3}a$	$\frac{1}{6}h_1a^2$
2	$\frac{1}{2}h_2a$	$\frac{2}{3}a$	$\frac{2}{6}h_2a^2$
Σ	$\frac{1}{2}a(h_1+h_2)$		$\frac{1}{6}a^2(h_1+2h_2)$

$$\overline{X} = \frac{\Sigma\overline{x}A}{\Sigma A} = \frac{\frac{1}{6}a^2(h_1+2h_2)}{\frac{1}{2}a(h_1+h_2)}$$

$$\overline{X} = \left(\frac{a}{3}\right)\frac{h_1+2h_2}{h_1+h_2} \blacktriangleleft$$



PROBLEM 5.19

For the semiannular area of Prob. 5.12, determine the ratio r_1 to r_2 so that the centroid of the area is located at $x = -\frac{1}{2}r_2$ and $y = 0$.

SOLUTION

First, determine the location of the centroid.

$$\bar{x}_2 = -\frac{4}{3\pi}r_2$$

Similarly, $$\bar{x}_1 = -\frac{4}{3\pi}r_1$$

$$\Sigma A = \frac{\pi}{2}\left(r_2^{\,2} - r_1^{\,2}\right)$$

Then $$\Sigma\bar{x}A = \frac{\pi}{2}r_2^{\,2}\left(\frac{-4r_2}{3\pi}\right) - \frac{\pi}{2}r_1^{\,2}\left(\frac{-4r_1}{3\pi}\right)$$

$$= -\frac{2}{3}\left(r_2^3 - r_1^3\right)$$

Now $$\bar{X}\Sigma A = \Sigma\bar{x}A$$

$$\bar{X}\left[\frac{\pi}{2}\left(r_2^2 - r_1^2\right)\right] = -\frac{2}{3}\left(r_2^3 - r_1^3\right) \text{ ; Dividing by } \left(r_2 - r_1\right)$$

$$\bar{X}\left(\frac{\pi}{2}\right)\left(r_2 + r_1\right) = -\frac{2}{3}\left(r_2^{\,2} + r_2r_1 + r_1^{\,2}\right) \quad (1)$$

$$\bar{X} = -\left(\frac{4}{3\pi}\right)\left(\frac{r_2^{\,2} + r_2r_1 + r_1^{\,2}}{r_2 + r_1}\right)$$

By symmetry, $\bar{Y} = 0$

PROBLEM 5.19 (Continued)

Now substituting for $\overline{X} = -\frac{1}{2}r_2$ in equation (1) gives.

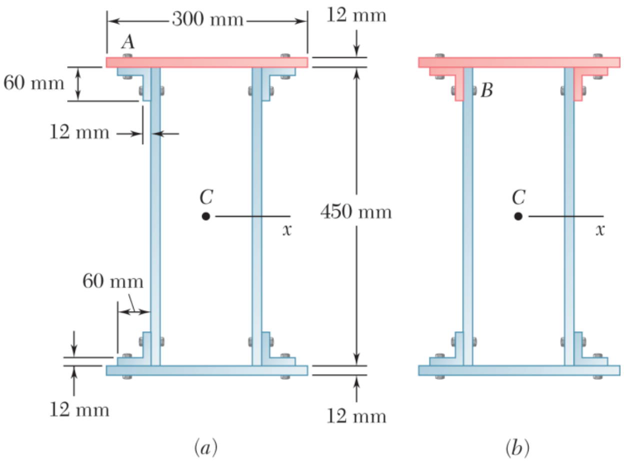
$$\left(-\frac{1}{2}r_2\right)\left(\frac{\pi}{2}\right)(r_2 + r_1) = -\frac{2}{3}(r_2{}^2 + r_1 r_2 + \mathrm{r}_1{}^2) \qquad (1)$$

Dividing through by $-r_2$ and 2/3,

$$\left(\frac{3}{2}\right)\left(\frac{\pi}{4}\right)\left(1 + \frac{r_1}{r_2}\right) = 1 + \frac{r_1}{r_2} + \left(\frac{r_1}{r_2}\right)^2$$

Then $$\left(\frac{r_1}{r_2}\right)^2 - 0.1781\left(\frac{r_1}{r_2}\right) - 0.1781 = 0$$

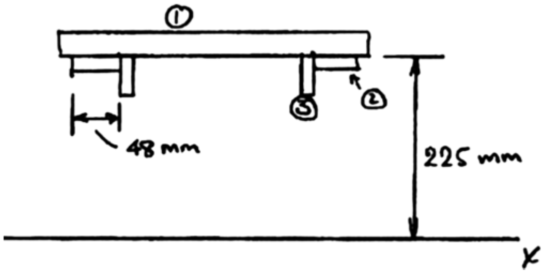
Solving the quadratic yields, $$\left(\frac{r_1}{r_2}\right) = 0.520 \blacktriangleleft$$



PROBLEM 5.20

A composite beam is constructed by bolting four plates to four 60 × 60 × 12-mm angles as shown. The bolts are equally spaced along the beam, and the beam supports a vertical load. As proved in mechanics of materials, the shearing forces exerted on the bolts at *A* and *B* are proportional to the first moments with respect to the centroidal *x*-axis of the red-shaded areas shown, respectively, in parts *a* and *b* of the figure. Knowing that the force exerted on the bolt at *A* is 280 N, determine the force exerted on the bolt at *B*.

SOLUTION



From the problem statement, *F* is proportional to Q_x.

Therefore, $$\frac{F_A}{(Q_x)_A} = \frac{F_B}{(Q_x)_B}, \quad \text{or} \quad F_B = \frac{(Q_x)_B}{(Q_x)_A} F_A$$

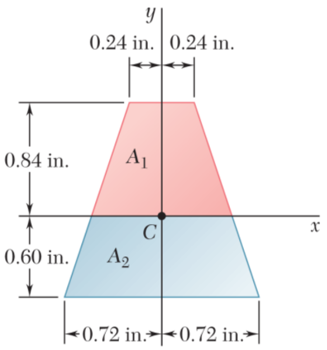
For the first moments, $$(Q_x)_A = \left(225 + \frac{12}{2}\right)(300 \times 12)$$

$$= 831{,}600 \text{ mm}^3$$

$$(Q_x)_B = (Q_x)A + 2\left(225 - \frac{12}{2}\right)(48 \times 12) + 2(225 - 30)(12 \times 60)$$

$$= 1{,}364{,}688 \text{ mm}^3$$

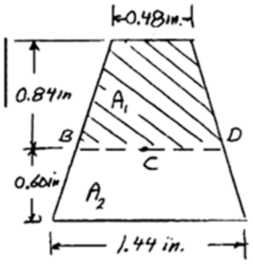
Then $$F_B = \frac{1{,}364{,}688}{831{,}600}(280 \text{ N})$$ or $F_B = 459$ N ◀

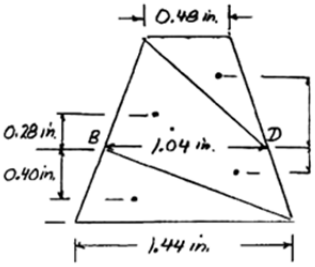

y
0.24 in.
0.24 in.
0.84 in.
A_1
C
x
0.60 in.
A_2
0.72 in.
0.72 in.

PROBLEM 5.21

The horizontal *x*-axis is drawn through the centroid *C* of the area shown, and it divides the area into two component areas A_1 and A_2. Determine the first moment of each component area with respect to the *x*-axis, and explain the results obtained.

SOLUTION


0.48in.
0.84in
A_1
B
D
C
0.60in
A_2
1.44 in.


0.48 in.
0.28 in.
B
1.04 in.
D
0.40in.
1.44 in.

Length of *BD*:

$$BD = 0.48\text{ in.} + (1.44\text{ in.} - 0.48\text{ in.})\frac{0.84\text{in.}}{0.84\text{ in.} \times 0.60\text{ in.}} = 0.48 + 0.56 = 1.04\text{ in.}$$

Area above *x*-axis (consider two triangular areas):

$$Q_1 = \Sigma \bar{y}_A = (0.28\text{ in.})\left[\frac{1}{2}(0.84\text{ in.})(1.04\text{ in.})\right] + (0.56\text{ in.})\left[\frac{1}{2}(0.84\text{ in.})(0.48\text{ in.})\right]$$
$$= 0.122304\text{ in}^3 + 0.112896\text{ in}^3$$

$$Q_1 = 0.235\text{ in}^3 \blacktriangleleft$$

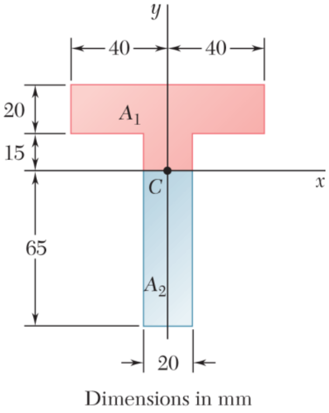
Area below *x*-axis:

$$Q_2 = \Sigma \bar{y}A = -(0.40\text{ in.})\left[\frac{1}{2}(0.60\text{ in.})(1.44\text{ in.})\right] - (0.20\text{ in.})\left[\frac{1}{2}(0.60\text{ in.})\right]$$
$$= -0.1728\text{ in}^3 - 0.0624\text{ in}^3$$

$$Q_2 = -0.235\text{ in}^3 \blacktriangleleft$$

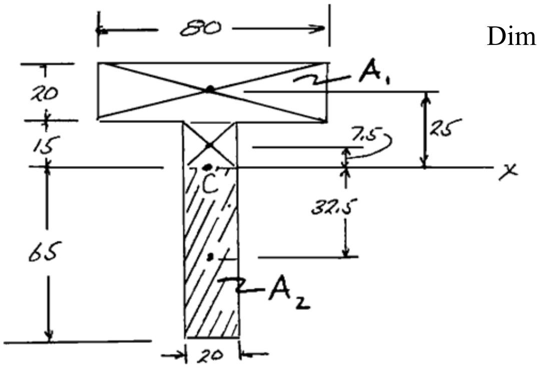
$|Q| = |Q_2|$, since C is centroid and thus, $Q = \Sigma \bar{y} A = 0$

PROBLEM 5.22


y
40
40
20
A_1
15
C
x
65
A_2
20
Dimensions in mm

The horizontal *x*-axis is drawn through the centroid *C* of the area shown, and it divides the area into two component areas A_1 and A_2. Determine the first moment of each component area with respect to the *x*-axis, and explain the results obtained.

SOLUTION


80
A_1
20
15
7.5
25
x
C
32.5
65
A_2
20

Dimensions in mm

Area above *x*-axis (Area A_1):

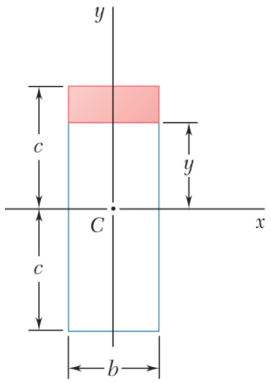
$$Q_1 = \Sigma \bar{y} A = (25)(20 \times 80) + (7.5)(15 \times 20)$$
$$= 40 \times 10^3 + 2.25 \times 10^3$$

$$Q_1 = 42.3 \times 10^3 \text{ mm}^3 \blacktriangleleft$$

Area below *x*-axis (Area A_2):

$$Q_2 = \Sigma \bar{y} A = (-32.5)(65 \times 20) \qquad Q_2 = -42.3 \times 10^3 \text{ mm}^3 \blacktriangleleft$$

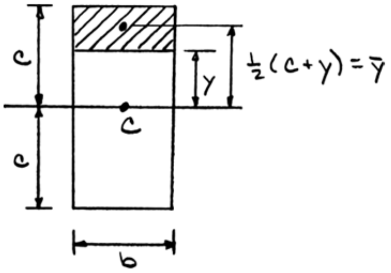
$|Q_1| = |Q_2|$, since *C* is centroid and thus, $Q = \Sigma \bar{y} A = 0$



PROBLEM 5.23

The first moment of the shaded area with respect to the x-axis is denoted by Q_x. (a) Express Q_x in terms of b, c, and the distance y from the base of the shaded area to the x-axis. (b) For what value of y is Q_x maximum, and what is that maximum value?

SOLUTION



Shaded area:

$$A = b(c-y)$$

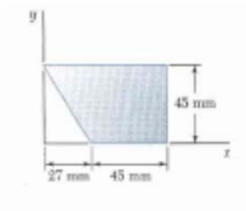
$$Q_x = \bar{y}A$$

$$= \frac{1}{2}(c+y)[b(c-y)]$$

(a) $$Q_x = \frac{1}{2}b(c^2 - y^2)$$ ◀

(b) For $Q_{\max}$, $$\frac{dQ}{dy} = 0 \quad \text{or} \quad \frac{1}{2}b(-2y) = 0$$ $y = 0$ ◀

For $y = 0$, $$(Q_x) = \frac{1}{2}bc^2$$ $(Q_x) = \frac{1}{2}bc^2$ ◀



PROBLEM 5.24

A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION

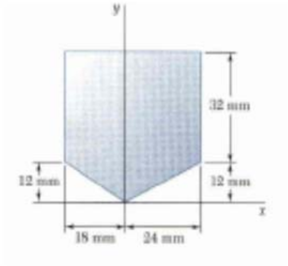
	L, mm	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}L$, mm^2	$\bar{y}L$, mm^2
1	45	49.5	0	2227.5	0
2	45	72	22.5	3240	1012.5
3	72	36	45	2592	3240
4	52.479	13.5	22.5	708.47	1180.78
Σ	2632.5			8768.0	5433.3

$$\bar{X}\,\Sigma L = \bar{x}L$$

$$\bar{X}(214.479\text{ mm}) = 8768.0\text{ mm}^2 \qquad \bar{X} = 40.9\text{ mm} \blacktriangleleft$$

$$\bar{Y}\,\Sigma L = \Sigma\,\bar{y}L$$

$$\bar{Y}(214.479\text{ mm}) = 5433.3\text{ mm}^2 \qquad \bar{Y} = 25.3\text{ mm} \blacktriangleleft$$



PROBLEM 5.25

A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION

	L, mm	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}L$, mm^2	$\bar{y}L$, mm^2
1	$\sqrt{24^2+12^2}=26.833$	12	6	322	160.998
2	32	24	28	768	896
3	42	3	44	126	1848
4	32	−18	28	−576	896
5	$\sqrt{18^2+12^2}=21.633$	−9	6	−194.697	129.798
Σ	154.466			445.30	3930.8

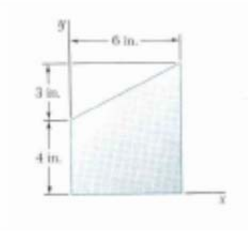
Then

$$\bar{X}\,\Sigma L = \Sigma \bar{x} L$$

$$\bar{X}(154.466) = 445.30 \qquad \text{or } \bar{X} = 2.88 \text{ mm} \blacktriangleleft$$

$$\bar{Y}\,\Sigma L = \Sigma \bar{y} L$$

$$\bar{Y}(154.466) = 3930.8 \qquad \text{or } \bar{Y} = 25.4 \text{ mm} \blacktriangleleft$$



PROBLEM 5.26

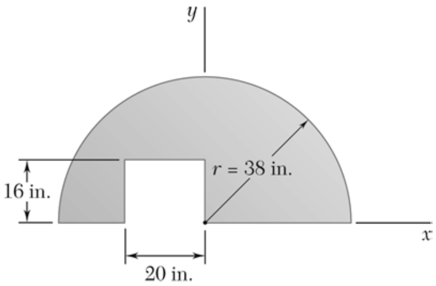
A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION

	L, in.	$\overline{x}$, in.	$\overline{y}$, in.	$\overline{x}L$, in^2	$\overline{y}L$, in^2
1	4	0	2	0	8
2	$\sqrt{3^2+6^2}=6.7082$	3	5.5	20.125	36.894
3	7	6	3.5	42	24.5
4	6	3	0	18	0
Σ	23.708			80.125	69.394

Then $\overline{X}\,\Sigma L = \Sigma \overline{x}\,L \qquad \overline{X}(23.708) = 80.125 \qquad \overline{X} = 3.38\text{ in.}$ ◀

$\overline{Y}\,\Sigma L = \Sigma \overline{y}\,L \qquad \overline{Y}(23.708) = 69.394 \qquad \overline{Y} = 2.93\text{ in.}$ ◀



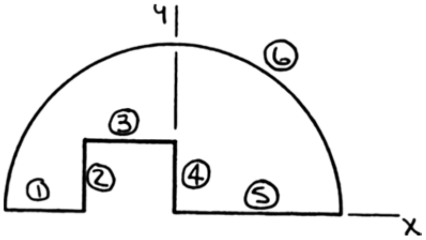
PROBLEM 5.27

A thin, homogeneous wire is bent to form the perimeter of the figure indicated. Locate the center of gravity of the wire figure thus formed.

SOLUTION

First note that because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line.

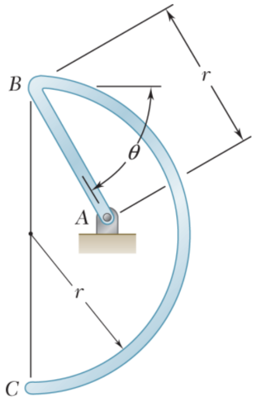
$$\bar{Y}_6 = \frac{2}{\pi}(38 \text{ in.})$$



	L, in.	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}L$, in.^{2}	$\bar{y}L$, in.^{2}
1	18	–29	0	–522	0
2	16	–20	8	–320	128
3	20	–10	16	–200	320
4	16	0	8	0	128
5	38	19	0	722	0
6	$\pi(38) = 119.381$	0	24.192	0	2888.1
Σ	227.38			–320	3464.1

Then $\bar{X} = \frac{\Sigma \bar{x} L}{\Sigma L} = \frac{-320}{227.38}$ $\bar{X} = -1.407$ in. ◀

$\bar{Y} = \frac{\Sigma \bar{y} L}{\Sigma L} = \frac{3464.1}{227.38}$ $\bar{Y} = 15.23$ in. ◀

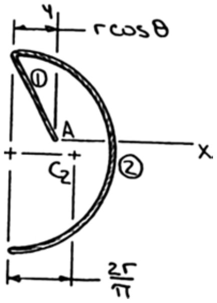


PROBLEM 5.28

The homogeneous wire *ABC* is bent into a semicircular arc and a straight section as shown and is attached to a hinge at *A*. Determine the value of θ for which the wire is in equilibrium for the indicated position.

SOLUTION

First note that for equilibrium, the center of gravity of the wire must lie on a vertical line through *A*. Further, because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line. Thus,



$$\overline{X} = 0$$

so that

$$\Sigma \overline{x} L = 0$$

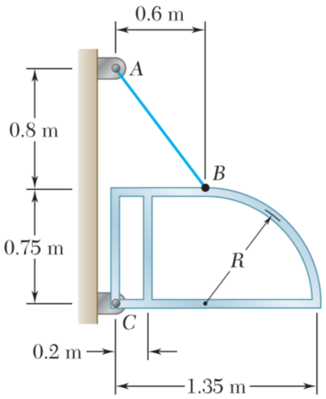
Then

$$\left(-\frac{1}{2} r\cos\theta\right)(r) + \left(\frac{2r}{\pi} - r\cos\theta\right)(\pi r) = 0$$

or

$$\cos\theta = \frac{4}{1+2\pi}$$
$$= 0.54921$$

or $\theta = 56.7°$ ◀



PROBLEM 5.29

The frame for a sign is fabricated from thin, flat steel bar stock of mass per unit length 4.73 kg/m. The frame is supported by a pin at *C* and by a cable *AB*. Determine (*a*) the tension in the cable, (*b*) the reaction at *C*.

SOLUTION

First note that because the frame is fabricated from uniform bar stock, its center of gravity will coincide with the centroid of the corresponding line.

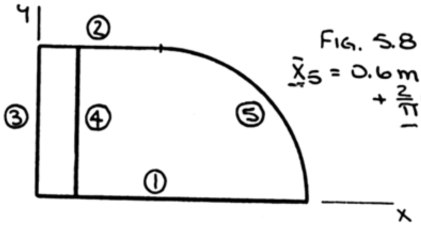


Fig. 5.8B

$\bar{x}_5 = 0.6\text{ m} + \frac{2}{\pi}(0.75\text{ m})$

	L, m	$\bar{x}$, m	$\bar{x}L$, m^2
1	1.35	0.675	0.91125
2	0.6	0.3	0.18
3	0.75	0	0
4	0.75	0.2	0.15
5	$\frac{\pi}{2}(0.75) = 1.17810$	1.07746	1.26936
Σ	4.62810		2.5106

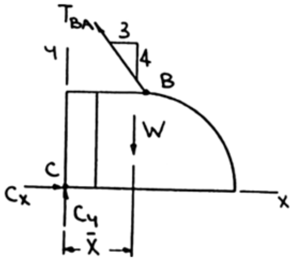
Then

$$\bar{X}\Sigma L = \Sigma \bar{x} L$$

$$\bar{X}(4.62810) = 2.5106$$

or

$$\bar{X} = 0.54247 \text{ m}$$



The free-body diagram of the frame is then

where

$$W = (m'\Sigma L)g$$

$$= 4.73 \text{ kg/m} \times 4.62810 \text{ m} \times 9.81 \text{ m/s}^2$$

$$= 214.75 \text{ N}$$

PROBLEM 5.29 (Continued)

Equilibrium then requires

(*a*) $$\Sigma M_C = 0: \quad (1.55\text{ m})\left(\frac{3}{5}T_{BA}\right) - (0.54247\text{ m})(214.75\text{ N}) = 0$$

or $$T_{BA} = 125.264\text{ N} \text{ or } T_{BA} = 125.3\text{ N} \blacktriangleleft$$

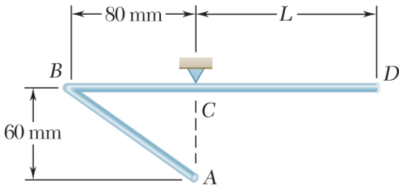
(*b*) $$\Sigma F_x = 0: \quad C_x - \frac{3}{5}(125.264\text{ N}) = 0$$

or $$\mathbf{C}_x = 75.158\text{ N} \longrightarrow$$

$$\Sigma F_y = 0: \quad C_y + \frac{4}{5}(125.264\text{ N}) - (214.75\text{ N}) = 0$$

or $$\mathbf{C}_y = 114.539\text{ N} \uparrow$$

Then $$\mathbf{C} = 137.0\text{ N} \measuredangle 56.7° \blacktriangleleft$$



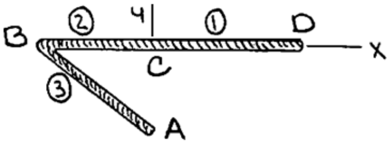
PROBLEM 5.30

The homogeneous wire *ABCD* is bent as shown and is attached to a hinge at *C*. Determine the length *L* for which portion *BCD* of the wire is horizontal.

SOLUTION

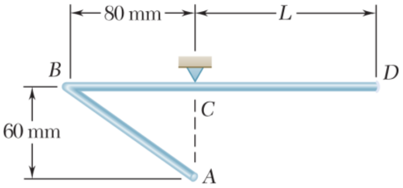
First note that for equilibrium, the center of gravity of the wire must lie on a vertical line through *C*. Further, because the wire is homogeneous, the center of gravity of the wire will coincide with the centroid of the corresponding line. Thus,

$$\bar{X} = 0 \text{ so that } \Sigma \bar{X} L = 0$$



Then
$$\frac{L}{2} + (-40\text{ mm})(80\text{ mm}) + (-40\text{ mm})(100\text{ mm}) = 0$$

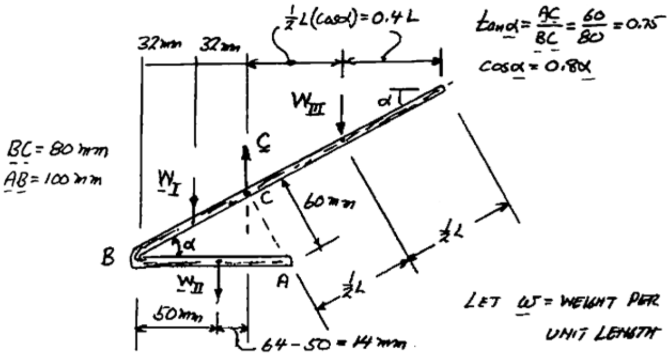
$$L^2 = 14,400\text{ mm}^2 \qquad L = 120.0\text{ mm} \blacktriangleleft$$


80 mm
L
B
D
C
60 mm
A

PROBLEM 5.31

The homogeneous wire *ABCD* is bent as shown and is attached to a hinge at *C*. Determine the length *L* for which portion *AB* of the wire is horizontal.

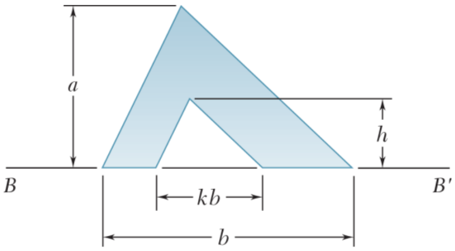
SOLUTION


$\frac{1}{2}L(\cos\alpha)=0.4L$
32 mm
32 mm
$\tan\alpha=\frac{AC}{BC}=\frac{60}{80}=0.75$
$\cos\alpha=0.80$
W_{III}
α
C
BC = 80 mm
AB = 100 mm
W_I
C
60 mm
B
α
$\frac{1}{2}L$
W_{II}
A
$\frac{1}{2}L$
50 mm
64 − 50 = 14 mm
LET w = WEIGHT PER UNIT LENGTH

$$W_{\mathrm{I}} = 80w \qquad W_{\mathrm{II}} = 100w \qquad W_{\mathrm{III}} = Lw$$

$$+\circlearrowleft \Sigma M_C = 0: \quad (80w)(32) + (100w)(14) - (Lw)(0.4L) = 0$$

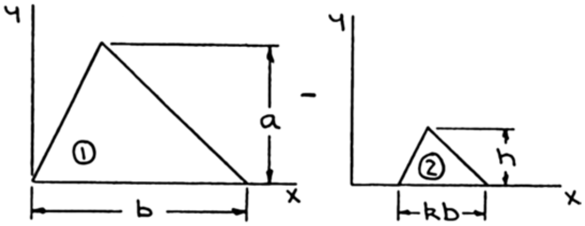
$$L^2 = 9900 \qquad L = 99.5 \text{ mm} \blacktriangleleft$$



PROBLEM 5.32

Determine the distance h for which the centroid of the shaded area is as far above line BB' as possible when (*a*) $k = 0.10$, (*b*) $k = 0.80$.

SOLUTION



	A	$\bar{y}$	$\bar{y}A$
1	$\frac{1}{2}ba$	$\frac{1}{3}a$	$\frac{1}{6}a^2b$
2	$-\frac{1}{2}(kb)h$	$\frac{1}{3}h$	$-\frac{1}{6}kbh^2$
Σ	$\frac{b}{2}(a-kh)$		$\frac{b}{6}(a^2-kh^2)$

Then $\bar{Y}\Sigma A = \Sigma\bar{y}A$

$$\bar{Y}\left[\frac{b}{2}(a-kh)\right] = \frac{b}{6}(a^2-kh^2)$$

or $\bar{Y} = \dfrac{a^2-kh^2}{3(a-kh)}$ (1)

and $\dfrac{d\bar{Y}}{dh} = \dfrac{1}{3}\dfrac{-2kh(a-kh)-(a^2-kh^2)(-k)}{(a-kh)^2} = 0$

or $2h(a-kh) - a^2 + kh^2 = 0$ (2)

Simplifying Eq. (2) yields $kh^2 - 2ah + a^2 = 0$

PROBLEM 5.32 (Continued)

Then $h = \dfrac{2a \pm \sqrt{(-2a)^2 - 4(k)(a^2)}}{2k}$

$$= \frac{a}{k}\left[1 \pm \sqrt{1-k}\right]$$

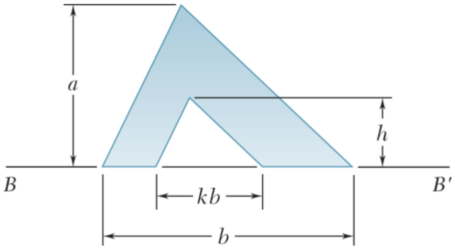
Note that only the negative root is acceptable since $h < a$. Then

(*a*) $k = 0.10$

$$h = \frac{a}{0.10}\left[1 - \sqrt{1-0.10}\right] \qquad \text{or } h = 0.513a \blacktriangleleft$$

(*b*) $k = 0.80$

$$h = \frac{a}{0.80}\left[1 - \sqrt{1-0.80}\right] \qquad \text{or } h = 0.691a \blacktriangleleft$$



PROBLEM 5.33

Knowing that the distance h has been selected to maximize the distance $\overline{y}$ from line BB' to the centroid of the shaded area, show that $\overline{y} = 2h/3$.

SOLUTION

See solution to Problem 5.32 for analysis leading to the following equations:

$$\overline{Y} = \frac{a^2 - kh^2}{3(a - kh)} \qquad (1)$$

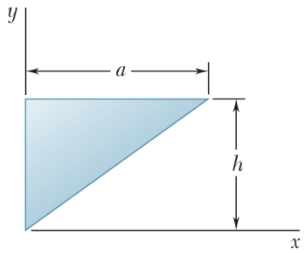
$$2h(a - kh) - a^2 + kh^2 = 0 \qquad (2)$$

Rearranging Eq. (2) (which defines the value of h which maximizes $\overline{Y}$) yields

$$a^2 - kh^2 = 2h(a - kh)$$

Then substituting into Eq. (1) (which defines $\overline{Y}$),

$$\overline{Y} = \frac{1}{3(a - kh)} \times 2h(a - kh) \qquad \text{or } \overline{Y} = \frac{2}{3}h \blacktriangleleft$$

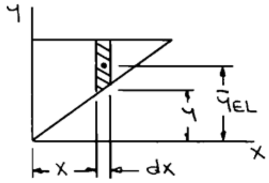


PROBLEM 5.34

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *h*.

SOLUTION

We have $y = \frac{h}{a}x$



and $dA = (h-y)dx$

$$= h\left(1-\frac{x}{a}\right)dx$$

$$\bar{x}_{EL} = x$$

$$\bar{y}_{EL} = \frac{1}{2}(h+y)$$

$$= \frac{h}{2}\left(1+\frac{x}{a}\right)$$

Then $A = \int dA = \int_0^a h\left(1-\frac{x}{a}\right)dx = h\left[x-\frac{x^2}{2a}\right]_0^a = \frac{1}{2}ah$

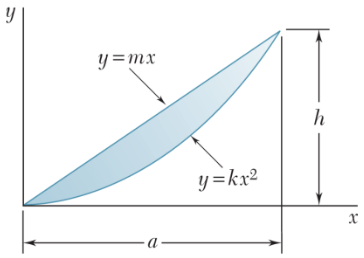
and $\int \bar{x}_{EL}\,dA = \int_0^a x\left[h\left(1-\frac{x}{a}\right)dx\right] = h\left[\frac{x^2}{2}-\frac{x^3}{3a}\right]_0^a = \frac{1}{6}a^2h$

$$\int \bar{y}_{EL}\,dA = \int_0^a \frac{h}{2}\left(1+\frac{x}{a}\right)\left[h\left(1-\frac{x}{a}\right)dx\right]$$

$$= \frac{h^2}{2}\int_0^a\left(1-\frac{x^2}{a^2}\right)dx = \frac{h^2}{2}\left[x-\frac{x^3}{3a^2}\right]_0^a = \frac{1}{3}ah^2$$

$\bar{x}A = \int \bar{x}_{EL}\,dA$: $\quad \bar{x}\left(\frac{1}{2}ah\right) = \frac{1}{6}a^2h$ $\qquad \bar{x} = \frac{2}{3}a$ ◀

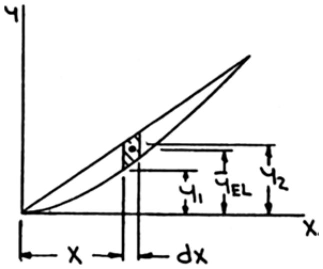
$\bar{y}A = \int \bar{y}_{EL}\,dA$: $\quad \bar{y}\left(\frac{1}{2}ah\right) = \frac{1}{3}ah^2$ $\qquad \bar{y} = \frac{2}{3}h$ ◀



PROBLEM 5.35

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *h*.

SOLUTION



At (a, h), $\quad y_1: \quad h = ka^2$

or $\quad k = \dfrac{h}{a^2}$

$y_2: \quad h = ma$

or $\quad m = \dfrac{h}{a}$

Now $\quad \overline{x}_{EL} = x$

$$\overline{y}_{EL} = \frac{1}{2}(y_1 + y_2)$$

and $$dA = (y_2 - y_1)dx = \left[\frac{h}{a}x - \frac{h}{a^2}x^2\right]dx$$

$$= \frac{h}{a^2}(ax - x^2)dx$$

Then $$A = \int dA = \int_0^a \frac{h}{a^2}(ax - x^2)dx = \frac{h}{a^2}\left[\frac{a}{2}x^2 - \frac{1}{3}x^3\right]_0^a = \frac{1}{6}ah$$

and $$\int \overline{x}_{EL}\, dA = \int_0^a x\left[\frac{h}{a^2}(ax - x^2)dx = \frac{h}{a^2}\left[\frac{a}{3}x^3 - \frac{1}{4}x^4\right]_0^a = \frac{1}{12}a^2h\right.$$

$$\int \overline{y}_{EL}\, dA = \int \frac{1}{2}(y_1 + y_2)[(y_2 - y_1)dx] = \int \frac{1}{2}\left(y_2^2 - y_1^2\right)dx$$

PROBLEM 5.35 (Continued)

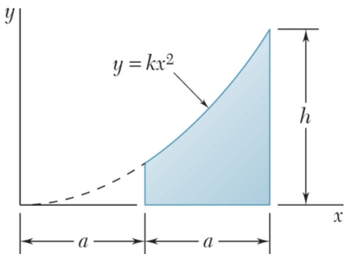
$$= \frac{1}{2}\int_0^a \left(\frac{h^2}{a^2}x^2 - \frac{h^2}{a^4}x^4\right)dx$$

$$= \frac{1}{2}\frac{h^2}{a^4}\left[\frac{a^2}{3}x^3 - \frac{1}{5}x^5\right]_0^a$$

$$= \frac{1}{15}ah^2$$

$\bar{x}A = \int \bar{x}_{EL}\,dA$: $\quad \bar{x}\left(\frac{1}{6}ah\right) = \frac{1}{12}a^2h$ $\qquad \bar{x} = \frac{1}{2}a$ ◀

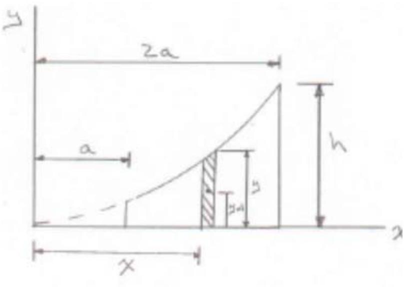
$\bar{y}A = \int \bar{y}_{EL}\,dA$: $\quad \bar{y}\left(\frac{1}{6}ah\right) = \frac{1}{15}ah^2$ $\qquad \bar{y} = \frac{2}{5}h$ ◀



PROBLEM 5.36

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *h*.

SOLUTION



For the element (el) shown at

$$x = 2a,\ y = h \qquad \text{or} \qquad h = k(2a)^2$$

$$\therefore\ k = \frac{h}{4a^2}$$

$$x_{el} = x$$

$$y_{el} = \frac{1}{2}y$$

$$dA = y\,dx$$

Then $$A = \int dA = \frac{h}{4a^2}\int_0^{2a} x^2\,dx = \frac{h}{4a^2}\left(\frac{x^3}{3}\right)\Bigg|_0^{2a}$$

$$= \frac{7}{12}ah$$

and $$\int \bar{x}_{el}dA = \int_a^{2a} x\left(\frac{h}{4a^2}\right)x^2\,dx = \frac{h}{4a^2}\left(\frac{x^4}{4}\right)\Bigg|_a^{2a}$$

$$= \frac{15}{16}ha^2$$

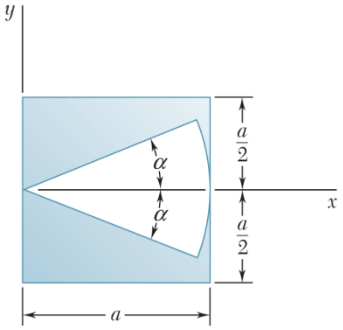
$$\int \bar{y}_{el}dA = \int_a^{2a}\frac{1}{2}\left(y^2\right)dx$$

$$= \frac{1}{2}\int\left(\frac{hx^2}{4a^2}\right)^2 dx = \frac{h^2}{32a^4}\int_a^{2a} x^4dx\Bigg|$$

$$= \frac{h^2}{32a^4}\left[\frac{x^5}{5}\right]_a^{2a} = \frac{31}{160}ah^2$$

Then $\bar{x}A = \int \bar{x}_{el}dA$: $\bar{x}\left(\frac{7}{12}ah\right) = \frac{15}{16}a^2h$ or $\bar{x} = 1.607a$ ◀

$\bar{y}A = \int \bar{y}_{EL}dA$: $\bar{y}\left(\frac{7}{12}ah\right) = \frac{31}{160}ah^2$ or $\bar{y} = 0.332h$ ◀

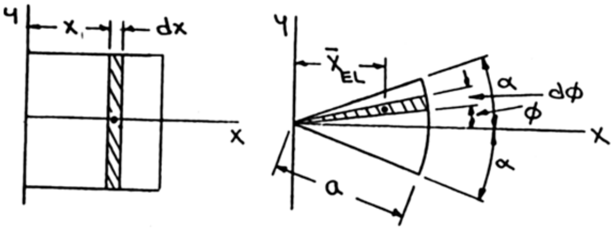


PROBLEM 5.37

Determine by direct integration the centroid of the area shown.

SOLUTION

First note that symmetry implies $\bar{y} = 0$ ◀



$$dA = \frac{1}{2}a(a\,d\phi)$$

$$dA = a\,dx \qquad \bar{x}_{EL} = \frac{2}{3}a\cos\phi$$

$$\bar{x}_{EL} = x$$

$$A = \int dA = \int_0^a a\,dx - \int_{-\alpha}^{\alpha} \frac{1}{2}a^2\,d\phi$$

$$= a[x]_0^a - \frac{a^2}{2}[\phi]_{\alpha}^{\alpha} = a^2(1-\alpha)$$

Then

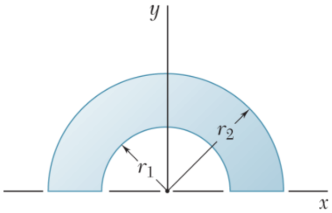
$$\int \bar{x}_{EL}\,dA = \int_0^a x(a\,dx) - \int_{-\alpha}^{\alpha} \frac{2}{3}a\cos\phi\left(\frac{1}{2}a^2\,d\phi\right)$$

$$= a\left[\frac{x^2}{2}\right]_0^a - \frac{1}{3}a^3[\sin\phi]_{-\alpha}^{\alpha}$$

$$= a^3\left(\frac{1}{2} - \frac{2}{3}\sin\alpha\right)$$

and

$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}[a^2(1-\alpha)] = a^3\left(\frac{1}{2} - \frac{2}{3}\sin\alpha\right) \qquad \text{or} \qquad \bar{x} = \frac{3-4\sin\alpha}{6(1-\alpha)}a$$ ◀



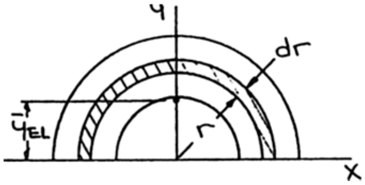
PROBLEM 5.38

Determine by direct integration the centroid of the area shown.

SOLUTION

First note that symmetry implies $\bar{x} = 0$ ◀

For the element (EL) shown



$$\bar{y}_{EL} = \frac{2r}{\pi} \quad \text{(Figure 5.8B)}$$

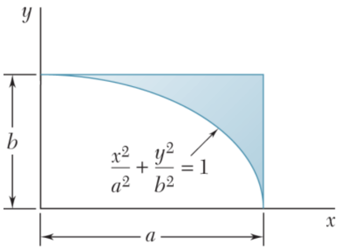
$$dA = \pi r dr$$

Then $A = \int dA = \int_{r_1}^{r_2} \pi r\, d r = \pi \left(\frac{r^2}{2} \right) \Bigg|_{r_1}^{r_2} = \frac{\pi}{2}\left(r_2^2 - r_1^2\right)$

and $\int \bar{y}_{EL} dA = \int_{r_1}^{r_2} \frac{2r}{\pi} (\pi r dr) = 2\left(\frac{1}{3} r^3 \right) \Bigg|_{r_1}^{r_2} = \frac{2}{3}\left(r_2^3 - r_1^3\right)$

So $\bar{y}A = \int \bar{y}_{EL} dA$: $\bar{y}\left[\frac{\pi}{2}\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right)$

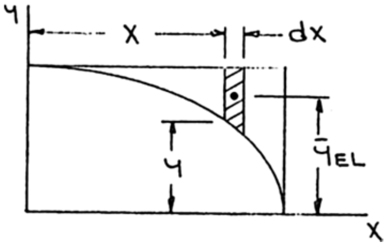
or $\bar{y} = \frac{4}{3\pi} \frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}$ ◀



PROBLEM 5.39

Determine by direct integration the centroid of the area shown.

SOLUTION



For the element (EL) shown, $y=\frac{b}{a}\sqrt{a^2-x^2}$

and $dA=(b-y)dx$

$$=\frac{b}{a}\left(a-\sqrt{a^2-x^2}\right)dx$$

$$\bar{x}_{EL}=x$$

$$\bar{y}_{EL}=\frac{1}{2}(y+b)$$

$$=\frac{b}{2a}\left(a+\sqrt{a^2-x^2}\right)$$

Then $A=\int dA=\int_0^a\frac{b}{a}\left(a-\sqrt{a^2-x^2}\right)dx$

To integrate, let $x=a\sin\theta$: $\sqrt{a^2-x^2}=a\cos\theta,\ dx=a\cos\theta\, d\theta$

Then $A=\int_0^{\pi/2}\frac{b}{a}(a-a\cos\theta)(a\cos\theta d\theta)$

$$=\frac{b}{a}\left[a^2\sin\theta-a^2\left(\frac{\theta}{2}+\sin\frac{2\theta}{4}\right)\right]\Bigg|_0^{\pi/2}$$

$$=ab\left(1-\frac{\pi}{4}\right)$$

and $\int\bar{x}_{EL}dA=\int_0^a x\left[\frac{b}{a}\left(a-\sqrt{a^2-x^2}\right)dx\right]$

$$=\frac{b}{a}\left[\left(\frac{a}{2}x^2+\frac{1}{3}(a^2-x^2)^{3/2}\right)\right]\Bigg|_0^{\pi/2}$$

$$=\frac{1}{6}a^3b$$

PROBLEM 5.39 (Continued)

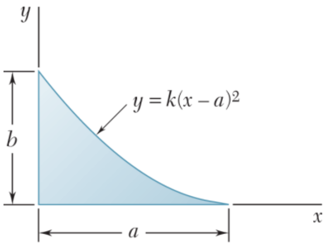
$$\int \overline{y}_{EL}\,dA = \int_0^a \frac{b}{2a}\left(a+\sqrt{a^2-x^2}\right)\left[\frac{b}{a}\left(a-\sqrt{a^2-x^2}\right)dx\right]$$

$$= \frac{b^2}{2a^2}\int_0^a (x^2)\;dx = \frac{b^2}{2a^2}\left(\frac{x^3}{3}\right)\Bigg|_0^a$$

$$= \frac{1}{6}ab^2$$

$\overline{x}A = \int \overline{x}_{EL}\,dA\text{:}\quad \overline{x}\left[ab\left(1-\frac{\pi}{4}\right)\right] = \frac{1}{6}a^2b$ or $\overline{x} = \frac{2a}{3(4-\pi)}$ ◀

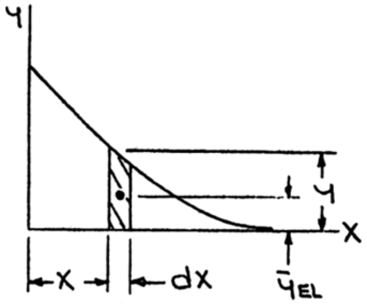
$\overline{y}A = \int \overline{y}_{EL}\,dA\text{:}\quad \overline{y}\left[ab\left(1-\frac{\pi}{4}\right)\right] = \frac{1}{6}ab^2$ or $\overline{y} = \frac{2b}{3(4-\pi)}$ ◀



PROBLEM 5.40

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

SOLUTION



At $x = 0, \quad y = b$

$$b = k(0-a)^2 \quad \text{or} \quad k = \frac{b}{a^2}$$

Then $y = \frac{b}{a^2}(x-a)^2$

Now $\bar{x}_{EL} = x$

$$\bar{y}_{EL} = \frac{y}{2} = \frac{b}{2a^2}(x-a)^2$$

and $dA = ydx = \frac{b}{a^2}(x-a)^2\,dx$

Then $A = \int dA = \int_0^a \frac{b}{a^2}(x-a)^2\,dx = \frac{b}{3a^2}\left[(x-a)^3\right]_0^a = \frac{1}{3}ab$

and $\int \bar{x}_{EL}\,dA = \int_0^a x\left[\frac{b}{a^2}(x-a)^2\,dx\right] = \frac{b}{a^2}\int_0^a \left(x^3 - 2ax^2 + a^2x\right)dx$

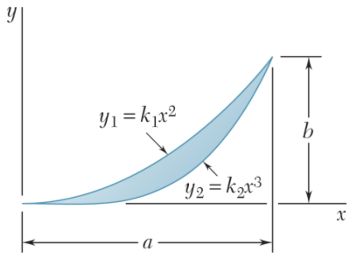
$$= \frac{b}{a^2}\left(\frac{x^4}{4} - \frac{2}{3}ax^3 + \frac{a^2}{2}x^2\right) = \frac{1}{12}a^2b$$

$$\int \bar{y}_{EL}\,dA = \int_0^a \frac{b}{2a^2}(x-a)^2\left[\frac{b}{a^2}(x-a)^2\,dx\right] = \frac{b^2}{2a^4}\left[\frac{1}{5}(x-a)^5\right]_0^a$$

$$= \frac{1}{10}ab^2$$

Hence $\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{1}{3}ab\right) = \frac{1}{12}a^2b$ $\qquad \bar{x} = \frac{1}{4}a$ ◀

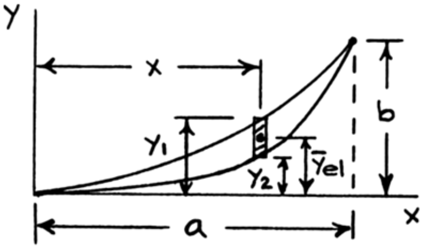
$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{1}{3}ab\right) = \frac{1}{10}ab^2$ $\qquad \bar{y} = \frac{3}{10}b$ ◀



PROBLEM 5.41

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

SOLUTION



$$y_1 = k_1x^2 \quad \text{but} \quad b = k_1a^2 \qquad y_1 = \frac{b}{a^2}x^2$$

$$y_2 = k_2x^3 \quad \text{but} \quad b = k_2a^3 \qquad y_2 = \frac{b}{a^3}x^3$$

$$dA = (y_2 - y_1)dx = \frac{b}{a^2}\left(x^2 - \frac{x^3}{a}\right)dx$$

$$\overline{x}_{EL} = x$$

$$\overline{y}_{EL} = \frac{1}{2}(y_1 + y_2)$$

$$= \frac{b}{2a^2}\left(x^2 + \frac{x^3}{a}\right)$$

$$A = \int dA = \frac{b}{a^2}\int_0^a\left(x^2 - \frac{x^3}{a}\right)dx$$

$$= \frac{b}{a^2}\left[\frac{x^3}{3} - \frac{x^4}{4a}\right]_0^a$$

$$= \frac{1}{12}ba$$

$$\int \overline{x}_{EL}\,dA = \int_0^a x\frac{b}{a^2}\left(x^2 - \frac{x^3}{a}\right)dx$$

$$= \frac{b}{a^2}\int_0^a\left(x^3 - \frac{x^4}{a}\right)dx$$

$$= \frac{b}{a^2}\left[\frac{x^4}{4} - \frac{x^5}{5a}\right]_0^a$$

$$= \frac{1}{20}a^2b$$

PROBLEM 5.41 (Continued)

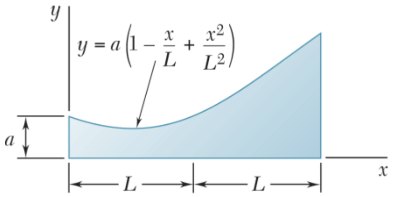
$$\int \bar{y}_{EL}\, dA = \int_0^a \frac{b}{2a^2}\left(x^2 + \frac{x^3}{a}\right)\frac{b}{a^2}\left(x^2 - \frac{x^3}{a}\right)dx$$

$$= \frac{b^2}{2a^4}\int_0^a \left(x^4 - \frac{x^6}{a^2}\right)dx$$

$$= \frac{b^2}{2a^4}\left[\frac{x^5}{5} - \frac{x^7}{7a^2}\right]_0^a = \frac{1}{35}ab^2$$

$\bar{x}A = \int \bar{x}_{EL}\, dA\text{:} \quad \bar{x}\left(\frac{1}{12}ba\right) = \frac{1}{20}a^2b$ $\qquad \bar{x} = \frac{3}{5}a$ ◀

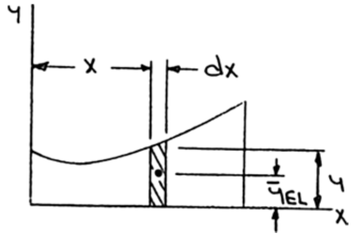
$\bar{y}A = \int \bar{y}_{EL}\, dA\text{:} \quad \bar{y}\left(\frac{1}{12}ba\right) = \frac{1}{35}ab^2$ $\qquad \bar{y} = \frac{12}{35}b$ ◀



PROBLEM 5.42

Determine by direct integration the centroid of the area shown.

SOLUTION



We have

$$\bar{x}_{EL} = x$$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{a}{2}\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)$$

$$dA = y\,dx = a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx$$

Then

$$A = \int dA = \int_0^{2L} a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx = a\left[x - \frac{x^2}{2L} + \frac{x^3}{3L^2}\right]_0^{2L}$$

$$= \frac{8}{3}aL$$

and

$$\int \bar{x}_{EL}\,dA = \int_0^{2L} x\left[a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx\right] = a\left[\frac{x^2}{2} - \frac{x^3}{3L} + \frac{x^4}{4L^2}\right]_0^{2L}$$

$$= \frac{10}{3}aL^2$$

$$\int \bar{y}_{EL}\,dA = \int_0^{2L} \frac{a}{2}\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)\left[a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx\right]$$

$$= \frac{a^2}{2}\int_0^{EL}\left(1 - 2\frac{x}{L} + 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} + \frac{x^4}{L^4}\right)dx$$

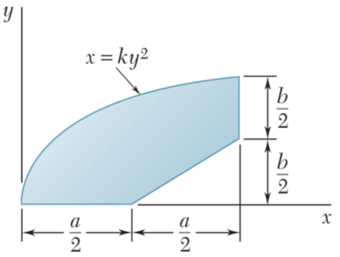
$$= \frac{a^2}{2}\left[x - \frac{x^2}{L} + \frac{x^3}{L^2} - \frac{x^4}{2L^3} + \frac{x^5}{5L^4}\right]_0^{2L}$$

$$= \frac{11}{5}a^2L$$

Hence,

$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{8}{3}aL\right) = \frac{10}{3}aL^2 \qquad \bar{x} = \frac{5}{4}L \blacktriangleleft$$

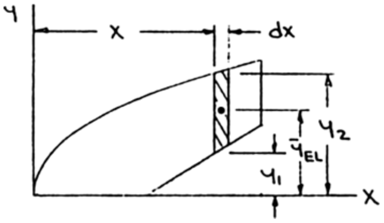
$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{1}{8}a\right) = \frac{11}{5}a^2 \qquad \bar{y} = \frac{33}{40}a \blacktriangleleft$$



PROBLEM 5.43

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

SOLUTION



For y_2 at $x = a$, $y = b$, $a = kb^2$, or $k = \frac{a}{b^2}$

Then $y_2 = \frac{b}{\sqrt{a}} x^{1/2}$

Now $\overline{x}_{EL} = x$

and for $0 \leq x \leq \frac{a}{2}$, $\overline{y}_{EL} = \frac{y_2}{2} = \frac{b}{2} \frac{x^{1/2}}{\sqrt{a}}$

$$dA = y_2\,dx = b\frac{x^{1/2}}{\sqrt{a}}dx$$

For $\frac{a}{2} \leq x \leq a$, $\overline{y}_{EL} = \frac{1}{2}(y_1 + y_2) = \frac{b}{2}\left(\frac{x}{a} - \frac{1}{2} + \frac{x^{1/2}}{\sqrt{a}}\right)$

$$dA = (y_2 - y_1)dx = b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx$$

Then
$$\begin{aligned} A = \int dA = \int_0^{a/2} b\frac{x^{1/2}}{\sqrt{a}}dx + \int_{a/2}^{a} b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx &= \frac{b}{\sqrt{a}}\left[\frac{2}{3}x^{3/2}\right]_0^{a/2} + b\left[\frac{2}{3}\frac{x^{3/2}}{\sqrt{a}} - \frac{x^2}{2a} + \frac{1}{2}x\right]_{a/2}^{a} \\ &= \frac{2}{3}\frac{b}{\sqrt{a}}\left[\left(\frac{a}{2}\right)^{3/2} + (a)^{3/2} - \left(\frac{a}{2}\right)^{3/2}\right] \\ &\quad + b\left\{-\frac{1}{2a}\left[(a^2) - \left(\frac{a}{2}\right)^2\right] + \frac{1}{2}\left[(a) - \left(\frac{a}{2}\right)\right]\right\} \\ &= \frac{13}{24}ab \end{aligned}$$

PROBLEM 5.43 (Continued)

and $$\int \bar{x}_{EL}\,dA = \int_0^{a/2} x\left(b\frac{x^{1/2}}{\sqrt{a}}dx\right) + \int_{a/2}^{a} x\left[b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)\right]dx$$

$$= \frac{b}{\sqrt{a}}\left[\frac{2}{5}x^{5/2}\right]_0^{a/2} + b\left[\frac{2}{5}\frac{x^{5/2}}{\sqrt{a}} - \frac{x^3}{3a} + \frac{x^4}{4}\right]_{a/2}^{a}$$

$$= \frac{2}{5}\frac{b}{\sqrt{a}}\left[\left(\frac{a}{2}\right)^{5/2} + (a)^{5/2} - \left(\frac{a}{2}\right)^{5/2}\right]$$

$$+ b\left\{-\frac{1}{3a}\left[(a)^3 - \left(\frac{a}{2}\right)^3\right] + \frac{1}{4}\left[(a)^2 - \left(\frac{a}{2}\right)^2\right]\right\}$$

$$= \frac{71}{240}a^2b$$

$$\int \bar{y}_{EL}\,dA = \int_0^{a/2} \frac{b}{2}\frac{x^{1/2}}{\sqrt{a}}\left[b\frac{x^{1/2}}{\sqrt{a}}dx\right]$$

$$+ \int_{a/2}^{a} \frac{b}{2}\left(\frac{x}{a} - \frac{1}{2} + \frac{x^{1/2}}{\sqrt{a}}\right)\left[b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx\right]$$

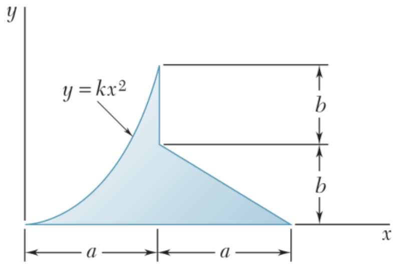
$$= \frac{b^2}{2a}\left[\frac{1}{2}x^2\right]_0^{a/2} + \frac{b^2}{2}\left[\left(\frac{x^2}{2a} - \frac{1}{3a}\left(\frac{x}{a} - \frac{1}{2}\right)^3\right)\right]_{a/2}^{a}$$

$$= \frac{b}{4a}\left[\left(\frac{a}{2}\right)^2 + (a)^2 - \left(\frac{a}{2}\right)^2\right] - \frac{b^2}{6a}\left(\frac{a}{2} - \frac{1}{2}\right)^3$$

$$= \frac{11}{48}ab^2$$

Hence, $\bar{x}A = \int \bar{x}_{EL}\,dA$: $\bar{x}\left(\frac{13}{24}ab\right) = \frac{71}{240}a^2b$ $\qquad \bar{x} = \frac{17}{130}a$ ◀

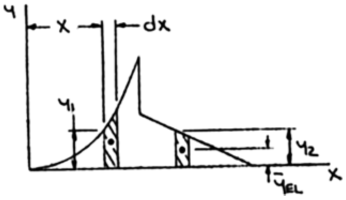
$\bar{y}A = \int \bar{y}_{EL}\,dA$: $\bar{y}\left(\frac{13}{24}ab\right) = \frac{11}{48}ab^2$ $\qquad \bar{y} = \frac{11}{26}b$ ◀



PROBLEM 5.44

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

SOLUTION



For y_1 at $x = a$, $\quad y = 2b, \quad 2b = ka^2, \quad$ or $\quad k = \dfrac{2b}{a^2}$

Then $\quad y_1 = \dfrac{2b}{a^2}x^2$

By observation, $\quad y_2 = -\dfrac{b}{a}(x + 2b) = b\left(2 - \dfrac{x}{a}\right)$

Now $\quad \overline{x}_{EL} = x$

and for $0 \le x \le a$, $\quad \overline{y}_{EL} = \dfrac{1}{2}y_1 = \dfrac{b}{a^2}x^2 \quad$ and $\quad dA = y_1\,dx = \dfrac{2b}{a^2}x^2dx$

For $a \le x \le 2a$, $\quad \overline{y}_{EL} = \dfrac{1}{2}y_2 = \dfrac{b}{2}\left(2 - \dfrac{x}{a}\right) \quad$ and $\quad dA = y_2\,dx = b\left(2 - \dfrac{x}{a}\right)dx$

Then $\quad A = \displaystyle\int dA = \int_0^a \frac{2b}{a^2}x^2dx + \int_a^{2a} b\left(2 - \frac{x}{a}\right)dx$

$$= \frac{2b}{a^2}\left[\frac{x^3}{3}\right]_0^a + b\left[-\frac{a}{2}\left(2 - \frac{x}{a}\right)^2\right]_0^{2a} = \frac{7}{6}ab$$

and $\quad \displaystyle\int \overline{x}_{EL}\,dA = \int_0^a x\left(\frac{2b}{a^2}x^2dx\right) + \int_a^{2a} x\left[b\left(2 - \frac{x}{a}\right)dx\right]$

$$= \frac{2b}{a^2}\left[\frac{x^4}{4}\right]_0^a + b\left[x^2 - \frac{x^3}{3a}\right]_0^{2a}$$

$$= \frac{1}{2}a^2b + b\left\{\left[(2a)^2 - (a)^2\right] + \frac{1}{3a}\left[(2a^2) - (a)^3\right]\right\}$$

$$= \frac{7}{6}a^2b$$

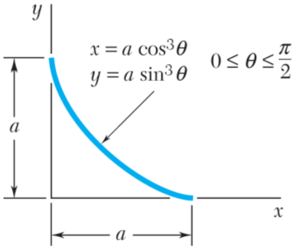
$$\int \bar{y}_{EL}\,dA = \int_0^a \frac{b}{a^2}x^2\left[\frac{2b}{a^2}x^2dx\right] + \int_0^{2a} \frac{b}{2}\left(2-\frac{x}{a}\right)\left[b\left(2-\frac{x}{a}\right)dx\right]$$

$$= \frac{2b^2}{a^4}\left[\frac{x^5}{5}\right]_0^a + \frac{b^2}{2}\left[-\frac{a}{3}\left(2-\frac{x}{a}\right)^3\right]_a^{2a}$$

$$= \frac{17}{30}ab^2$$

Hence, $\bar{x}A = \int \bar{x}_{EL}\,dA$: $\bar{x}\left(\frac{7}{6}ab\right) = \frac{7}{6}a^2b$ $\bar{x} = a$ ◀

$\bar{y}A = \int \bar{y}_{EL}\,dA$: $\bar{y}\left(\frac{7}{6}ab\right) = \frac{17}{30}ab^2$ $\bar{y} = \frac{17}{35}b$ ◀

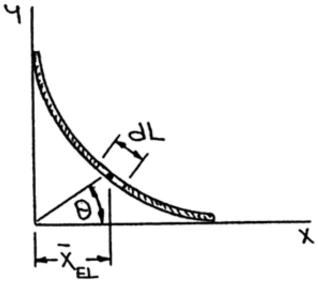


PROBLEM 5.45

A homogeneous wire is bent into the shape shown. Determine by direct integration the *x* coordinate of its centroid.

SOLUTION

First note that because the wire is homogeneous, its center of gravity coincides with the centroid of the corresponding line.



Now $\bar{x}_{EL} = a\cos^3\theta$ and $dL = \sqrt{dx^2 + dy^2}$

where $x = a\cos^3\theta: \quad dx = -3a\cos^2\theta\sin\theta d\theta$

$y = a\sin^3\theta: \quad dy = 3a\sin^2\theta\cos\theta d\theta$

Then
$$\begin{aligned} dL &= [(-3a\cos^2\theta\sin\theta d\theta)^2 + (3a\sin^2\theta\cos\theta d\theta)^2]^{1/2} \\ &= 3a\cos\theta\sin\theta(\cos^2\theta + \sin^2\theta)^{1/2} d\theta \\ &= 3a\cos\theta\sin\theta d\theta \end{aligned}$$

$$\begin{aligned} L &= \int dL = \int_0^{\pi/2} 3a\cos\theta\sin\theta d\theta = 3a\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} \\ &= \frac{3}{2}a \end{aligned}$$

and
$$\begin{aligned} \int \bar{x}_{EL}\, dL &= \int_0^{\pi/2} a\cos^3\theta(3a\cos\theta\sin\theta d\theta) \\ &= 3a^2\left[-\frac{1}{5}\cos^5\theta\right]_0^{\pi/2} = \frac{3}{5}a^2 \end{aligned}$$

Hence, $\bar{x}L = \int \bar{x}_{EL}\, dL: \quad \bar{x}\left(\frac{3}{2}a\right) = \frac{3}{5}a^2$ $\qquad \bar{x} = \frac{2}{5}a$ ◀

Alternative Solution:

$$x = a\cos^3\theta \Rightarrow \cos^2\theta = \left(\frac{x}{a}\right)^{2/3}$$

$$y = a\sin^3\theta \Rightarrow \sin^2\theta = \left(\frac{y}{a}\right)^{2/3}$$

PROBLEM 5.45 (Continued)

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1 \quad \text{or} \quad y = (a^{2/3} - x^{2/3})^{3/2}$$

$$\text{Then} \quad \frac{dy}{dx} = (a^{2/3} - x^{2/3})^{1/2}(-x^{-1/3})$$

$$\text{Now} \quad \overline{x}_{EL} = x$$

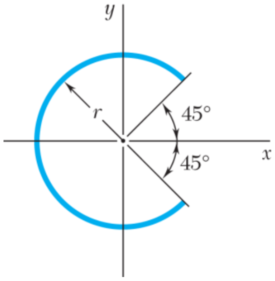
$$\text{and} \quad dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$dx = \left\{1 + \left[(a^{2/3} - x^{2/3})^{1/2}(-x^{-1/3})\right]^2\right\}^{1/2} dx$$

$$\text{Then} \quad L = \int dL = \int_0^a \frac{a^{1/3}}{x^{1/3}}\,dx = a^{1/3}\left[\frac{3}{2}x^{2/3}\right]_0^a = \frac{3}{2}a$$

$$\text{and} \quad \int \overline{x}_{EL}\,dL = \int_0^a x\left(\frac{a^{1/3}}{x^{1/3}}\,dx\right) = a^{1/3}\left[\frac{3}{5}x^{5/3}\right]_0^a = \frac{3}{5}a^2$$

Hence $\quad \overline{x}L = \int \overline{x}_{EL}\,dL: \quad \overline{x}\left(\frac{3}{2}a\right) = \frac{3}{5}a^2$ $\qquad \overline{x} = \frac{2}{5}a$ ◀



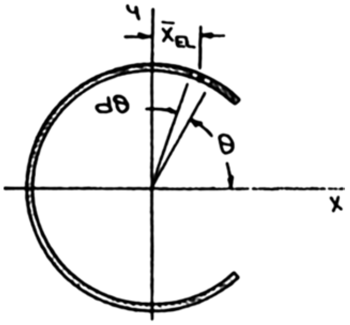
PROBLEM 5.46

A homogeneous wire is bent into the shape shown. Determine by direct integration the *x* coordinate of its centroid.

SOLUTION

First note that because the wire is homogeneous, its center of gravity coincides with the centroid of the corresponding line.

Now $\bar{x}_{EL} = r\cos\theta$ and $dL = r\,d\theta$



Then $L = \int dL = \int_{\pi/4}^{7\pi/4} r\,d\theta = r[\theta]_{\pi/4}^{7\pi/4} = \frac{3}{2}\pi r$

and $\int \bar{x}_{EL}\,dL = \int_{\pi/4}^{7\pi/4} r\cos\theta(r\,d\theta)$

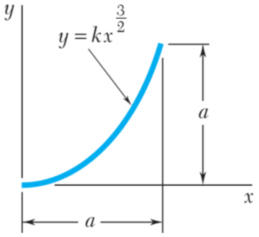
$$= r^2[\sin\theta]_{\pi/4}^{7\pi/4}$$

$$= r^2\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= -r^2\sqrt{2}$$

Thus $\bar{x}L = \int \bar{x}\,dL: \quad \bar{x}\left(\frac{3}{2}\pi r\right) = -r^2\sqrt{2}$

$\bar{x} = -\frac{2\sqrt{2}}{3\pi}r$ ◀

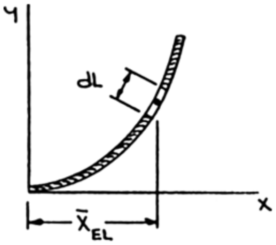


PROBLEM 5.47*

A homogeneous wire is bent into the shape shown. Determine by direct integration the *x* coordinate of its centroid. Express your answer in terms of *a*.

SOLUTION

First note that because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line.



We have at $x = a$, $y = a$, $a = ka^{3/2}$, or $k = \dfrac{1}{\sqrt{a}}$

Then $y = \dfrac{1}{\sqrt{a}}x^{3/2}$

and $\dfrac{dy}{dx} = \dfrac{3}{2\sqrt{a}}x^{1/2}$

Now $\bar{x}_{EL} = x$

and
$$dL = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \left[1+\left(\frac{3}{2\sqrt{a}}x^{1/2}\right)^2\right]^{1/2} dx$$
$$= \frac{1}{2\sqrt{a}}\sqrt{4a+9x}\,dx$$

Then
$$L = \int dL = \int_0^a \frac{1}{2\sqrt{a}}\sqrt{4a+9x}\,dx$$
$$= \frac{1}{2\sqrt{a}}\left[\frac{2}{3}\times\frac{1}{9}(4a+9x)^{3/2}\right]_0^a$$
$$= \frac{a}{27}[(13)^{3/2}-8]$$
$$= 1.43971a$$

and
$$\int \bar{x}_{EL}\,dL = \int_0^a x\left[\frac{1}{2\sqrt{a}}\sqrt{4a+9x}\,dx\right]$$

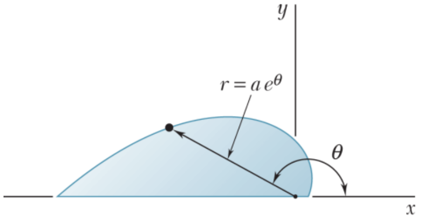
Use integration by parts with

$$u = x \qquad dv = \sqrt{4a+9x}\,dx$$
$$du = dx \qquad v = \frac{2}{27}(4a+9x)^{3/2}$$

Then $\displaystyle\int \overline{x}_{EL}\,dL = \frac{1}{2\sqrt{a}}\left\{\left[x\times\frac{2}{27}(4a+9x)^{3/2}\right]_0^a - \int_0^a \frac{2}{27}(4a+9x)^{3/2}\,dx\right\}$

$$= \frac{(13)^{3/2}}{27}a^2 - \frac{1}{27\sqrt{a}}\left[\frac{2}{45}(4a+9x)^{5/2}\right]_0^a$$
$$= \frac{a^2}{27}\left\{(13)^{3/2} - \frac{2}{45}[(13)^{5/2}-32]\right\}$$
$$= 0.78566a^2$$

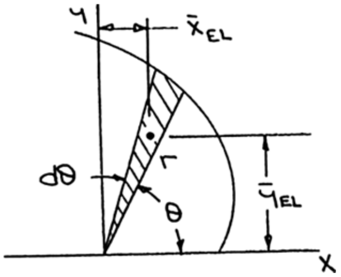
$\overline{x}L = \int x_{EL}\,dL$: $\quad \overline{x}(1.43971a) = 0.78566a^2$ or $\overline{x} = 0.546a$ ◀



PROBLEM 5.48*

Determine by direct integration the centroid of the area shown.

SOLUTION



We have $\bar{x}_{EL} = \frac{2}{3}r\cos\theta = \frac{2}{3}ae^{\theta}\cos\theta$

$$\bar{y}_{EL} = \frac{2}{3}r\sin\theta = \frac{2}{3}ae^{\theta}\sin\theta$$

and $dA = \frac{1}{2}(r)(rd\theta) = \frac{1}{2}a^2e^{2\theta}d\theta$

Then $A = \int dA = \int_0^{\pi}\frac{1}{2}a^2e^{2\theta}d\theta = \frac{1}{2}a^2\left[\frac{1}{2}e^{2\theta}\right]_0^{\pi}$

$$= \frac{1}{4}a^2(e^{2\pi} - 1)$$

$$= 133.623a^2$$

and $\int \bar{x}_{EL}dA = \int_0^{\pi}\frac{2}{3}ae^{\theta}\cos\theta\left(\frac{1}{2}a^2e^{2\theta}d\theta\right)$

$$= \frac{1}{3}a^3\int_0^{\pi}e^{3\theta}\cos\theta d\theta$$

To proceed, use integration by parts, with

$u = e^{3\theta}$ and $du = 3e^{3\theta}d\theta$

$dv = \cos\theta\, d\theta$ and $v = \sin\theta$

Then $\int e^{3\theta}\cos\theta d\theta = e^{3\theta}\sin\theta - \int \sin\theta(3e^{3\theta}d\theta)$

Now let $u = e^{3\theta}$ then $du = 3e^{3\theta}d\theta$

$dv = \sin\theta d\theta$, then $v = -\cos\theta$

Then $\int e^{3\theta}\cos\theta d\theta = e^{3\theta}\sin\theta - 3\left[-e^{3\theta}\cos\theta - \int(-\cos\theta)(3e^{3\theta}d\theta)\right]$

PROBLEM 5.48* (Continued)

so that $$\int e^{3\theta}\cos\theta d\theta = \frac{e^{3\theta}}{10}(\sin\theta + 3\cos\theta)$$

$$\int x_{EL} dA = \frac{1}{3}a^3\left[\frac{e^{3\theta}}{10}(\sin\theta + 3\cos\theta)\right]_0^{\pi}$$

$$= \frac{a^3}{30}(-3e^{3\pi} - 3) = -1239.26a^3$$

Also, $$\int \overline{y}_{EL} dA = \int_0^{\pi}\frac{2}{3}ae^{\theta}\sin\theta\left(\frac{1}{2}a^2e^{2\theta}d\theta\right)$$

$$= \frac{1}{3}a^3\int_0^{\pi}e^{3\theta}\sin\theta d\theta$$

Use integration by parts, as above, with

$$u = e^{3\theta} \quad \text{and} \quad du = 3e^{3\theta}d\theta$$

$$dv = \int\sin\theta d\theta \quad \text{and} \quad v = -\cos\theta$$

Then $$\int e^{3\theta}\sin\theta d\theta = -e^{3\theta}\cos\theta - \int(-\cos\theta)(3e^{3\theta}d\theta)$$

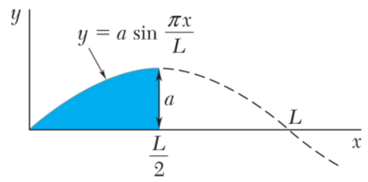
so that $$\int e^{3\theta}\sin\theta d\theta = \frac{e^{3\theta}}{10}(-\cos\theta + 3\sin\theta)$$

$$\int \overline{y}_{EL} dA = \frac{1}{3}a^3\left[\frac{e^{3\theta}}{10}(-\cos\theta + 3\sin\theta)\right]_0^{\pi}$$

$$= \frac{a^3}{30}(e^{3\pi} + 1) = 413.09a^3$$

Hence, $\overline{x}A = \int x_{EL} dA$: $\overline{x}(133.623a^2) = -1239.26a^3$ or $\overline{x} = -9.27a$ ◀

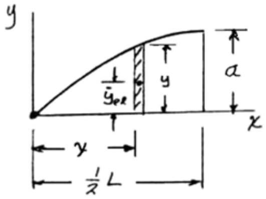
$\overline{y}A = \int \overline{y}_{EL} dA$: $\overline{y}(133.623a^2) = 413.09a^3$ or $\overline{y} = 3.09a$ ◀



PROBLEM 5.49*

Determine by direct integration the centroid of the area shown.

SOLUTION



$$y = a\sin\frac{\pi x}{L}$$

$$\bar{x}_{EL} = x, \quad \bar{y}_{EL} = \frac{1}{2}y, \quad dA = y\,dx$$

$$A = \int_0^{L/2} y\,dx = a\int_0^{L/2} \sin\frac{\pi x}{L}dx$$

$$A = a\left[\frac{L}{\pi}\left(-\cos\frac{\pi x}{L}\right)\right]_0^{L/2} = \frac{aL}{\pi}$$

$$\int \bar{x}_{EL}\,dA = \int_0^{L/2} xydx = \int_0^{L/2} xa\sin\frac{\pi x}{L}dx$$

Setting $u = \frac{\pi x}{L}$, we have $x = \frac{L}{\pi}u$, $dx = \frac{L}{\pi}du$,

$$\int \bar{x}_{EL}\,dA = \int_0^{\pi/2}\left(\frac{L}{\pi}u\right)a\sin u\left(\frac{L}{\pi}du\right) = a\left(\frac{L}{\pi}\right)^2\int_0^{\pi/2} u\sin x\,du$$

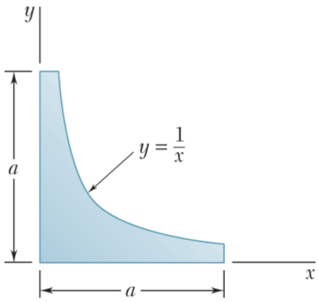
Integrating by parts, $\int \bar{x}_{EL}\,dA = a\left(\frac{L}{\pi}\right)^2\left\{[-u\cos u]_0^{\pi/2} + \int_0^{\pi/2}\cos u\,du\right\} = \frac{aL^2}{\pi^2}$

$$\int \bar{y}_{EL}\,dA = \int_0^{L/2}\frac{1}{2}y^2dx = \frac{1}{2}a^2\int_0^{L/2}\sin^2\frac{\pi x}{L}dx = \frac{a^2L}{2\pi}\int_0^{\pi/2}\sin^2 u\,du$$

$$= \frac{a^2L}{2\pi^2}\int_0^{\pi/2}\frac{1}{2}(1-\cos 2u)du = \frac{a^2L}{4\pi}\left[u - \frac{1}{2}\sin 2u\right]_0^{\pi/2} = \frac{1}{8}a^2L$$

$\bar{x}A = \int \bar{x}_{EL}\,dA$: $\quad \bar{x}\left(\frac{aL}{\pi}\right) = \frac{aL^2}{\pi^2}$ $\qquad \bar{x} = \frac{L}{\pi}$ ◀

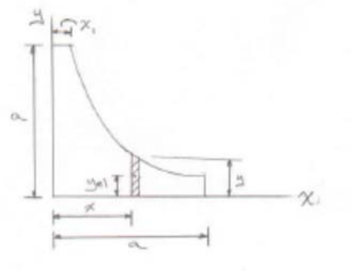
$\bar{y}A = \int \bar{y}_{EL}\,dA$: $\quad \bar{y}\left(\frac{aL}{\pi}\right) = \frac{1}{8}a^2L$ $\qquad \bar{y} = \frac{\pi}{8}a$ ◀



PROBLEM 5.50

Determine the centroid of the area shown in terms of *a*.

SOLUTION



We have $\bar{x}_{EL} = x$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{1}{2}\left(1 - \frac{1}{x}\right)$$

and $dA = y\,dx = \left(1 - \frac{1}{x}\right)dx$

Then $A = \int dA = \int_0^{\frac{1}{a}} a\,dx + \int_{\frac{1}{a}}^{a} \frac{dx}{x} = \left(ax\right)\Big|_0^{1/a} + \left(\ln x\right)\Big|_{1/a}^{a} = (1 + 2\ln a)$

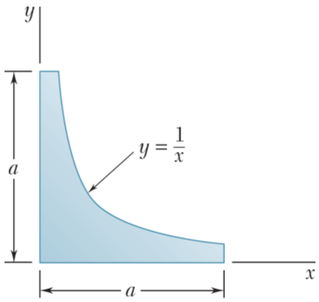
and $\int \bar{x}_{EL}\,dA = \int_0^{\frac{1}{a}} x\,dx + \int_{\frac{1}{a}}^{a} x\frac{dx}{x} = \left[\frac{ax^2}{2}\right]_0^{\frac{1}{a}} + \left[x\right]_{\frac{1}{a}}^{a} = \frac{1}{2a} + a - \frac{1}{a} = \frac{2a^2 - 1}{2a}$

$$\int \bar{y}_{EL}\,dA = \int_0^a \frac{1}{2}y^2\,dx = \frac{1}{2}\int_0^{\frac{1}{a}} a^2\,dx + \int_{\frac{1}{a}}^{a} \frac{dx}{2x^2} = \frac{a^2}{2}\left[x\right]_0^{1/a} - \frac{1}{2}\left(\frac{1}{x}\right)_{1/a}^{a} = \frac{a}{2} - \frac{1}{2a} + \frac{a}{2} = \frac{2a^2 - 1}{2a}$$

Because of symmetry, computation of only one coordinate was necessary.

$\bar{x}A = \int \bar{x}_{EL}\,dA$: $\bar{x}\left(1 + 2\ln a\right) = \frac{2a^2 - 1}{2a}$

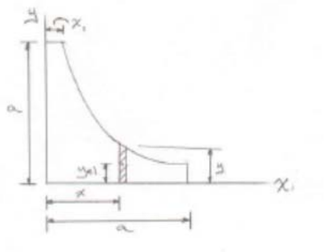
$$\bar{x} = \bar{y} = \frac{2a^2 - 1}{2a\left(1 + 2\ln a\right)} \blacktriangleleft$$



PROBLEM 5.51

Determine the centroid of the area shown when *a* = 4 in.

SOLUTION



We have $\bar{x}_{EL} = x$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{1}{2}\left(1 - \frac{1}{x}\right)$$

and $dA = y\,dx = \left(1 - \frac{1}{x}\right)dx$

Then $A = \int dA = \int_0^{\frac{1}{a}} a\,dx + \int_{\frac{1}{a}}^{a} \frac{dx}{x} = (ax)\Big|_0^{1/a} + (\ln x)\Big|_{1/a}^{a} = (1 + 2\ln a)$

and $\int \bar{x}_{EL}\,dA = \int_0^{\frac{1}{a}} x\,dx + \int_{\frac{1}{a}}^{a} x\frac{dx}{x} = \left[\frac{ax^2}{2}\right]_0^{\frac{1}{a}} + [x]_{\frac{1}{a}}^{a} = \frac{1}{2a} + a - \frac{1}{a} = \frac{2a^2 - 1}{2a}$

$$\int \bar{y}_{EL}\,dA = \int_0^a \frac{1}{2}y^2\,dx = \frac{1}{2}\int_0^{\frac{1}{a}} a^2\,dx + \int_{\frac{1}{a}}^{a} \frac{dx}{2x^2} = \frac{a^2}{2}[x]_0^{1/a} - \frac{1}{2}\left(\frac{1}{x}\right)_{1/a}^{a} = \frac{a}{2} - \frac{1}{2a} + \frac{a}{2} = \frac{2a^2 - 1}{2a}$$

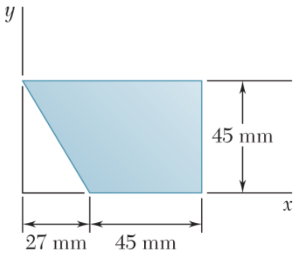
Because of symmetry, computation of only one coordinate was necessary.

$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}(1 + 2\ln a) = \frac{2a^2 - 1}{2a}$$

$$\bar{x} = \bar{y} = \frac{2a^2 - 1}{2a(1 + 2\ln a)}$$

Find $\bar{x}$ and $\bar{y}$ when $a = 4$ in.

We have $\bar{x} = \bar{y} = \frac{2(4)^2 - 1}{2(4)(1 + 2\ln 4)}$ or $\bar{x} = \bar{y} = 1.027$ in. ◀



PROBLEM 5.52

Determine the volume and the surface area of the solid obtained by rotating the area of Prob. 5.1 about (*a*) the *x* axis, (*b*) the line $x = 72$ mm.

SOLUTION

From the solution of Problem 5.1, we have

$$A = 2632.5 \text{ mm}^2$$
$$\Sigma \bar{x}A = 111{,}172.5 \text{ mm}^3$$
$$\Sigma \bar{y}A = 63{,}787.5 \text{ mm}^3$$

Applying the theorems of Pappus-Guldinus, we have

(*a*) Rotation about the *x*-axis:

$$\text{Volume} = 2\pi \bar{y}_{\text{area}} A = 2\pi \Sigma \bar{y}A$$
$$= 2\pi(63{,}787.5 \text{ mm}^3) \qquad \text{or Volume} = 401\times 10^3 \text{ mm}^3 \blacktriangleleft$$

$$\text{Area} = 2\pi\, \bar{y}_{\text{line}} L = 2\pi\, \Sigma(\bar{y}_{\text{line}})L$$
$$= 2\pi(\bar{y}_2 L_2 + \bar{y}_3 L_3 + \bar{y}_4 L_4)$$
$$= 2\pi[(22.5)(45) + (45)(72) + (22.5)\sqrt{27^2 + 45^2}\,]$$

$$\text{or Area} = 34.1\times 10^3 \text{ mm}^2 \blacktriangleleft$$

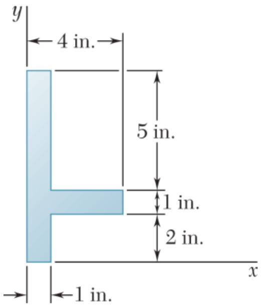
(*b*) Rotation about the line $x=72$ mm:

$$\text{Volume} = 2\pi(72 - \bar{x}_{\text{area}})A = 2\pi(72A - \Sigma\bar{x}A)$$
$$= 2\pi\left[(72 \text{ mm})(2632.5 \text{ mm}^2) - (111{,}172.5 \text{ mm}^3)\right] \qquad \text{or Volume} = 492\times 10^3 \text{ mm}^3 \blacktriangleleft$$

$\text{Area} = 2\pi\, \bar{x}_{\text{line}} L = 2\pi\, \Sigma(\bar{x}_{\text{line}})L$ where $\bar{x}_1$, $\bar{x}_3$ and $\bar{x}_4$ are measured wrt $x = 72$ mm

$$= 2\pi(\bar{x}_1 L_1 + \bar{x}_3 L_3 + \bar{x}_4 L_4)$$
$$= 2\pi[(22.5)(45) + (36)(72) + \left(\frac{45+72}{2}\right)\left(\sqrt{27^2 + 45^2}\right)]$$

$$\text{or Area} = 41.9\times 10^3 \text{ mm}^2 \blacktriangleleft$$



PROBLEM 5.53

Determine the volume and the surface area of the solid obtained by rotating the area of Problem 5.2 about (*a*) the *x*-axis, (*b*) the *y*-axis.

SOLUTION

From the solution of Problem 5.2, we have

$$A = 11\text{ in}^2$$
$$\Sigma \bar{x}A = 11.5\text{ in}^3$$
$$\Sigma \bar{y}A = 39.5\text{ in}^3$$

Applying the theorems of Pappus-Guldinus, we have

(*a*) Rotation about the *x*-axis:

$$\text{Volume} = 2\pi \bar{y}_{\text{area}} A = 2\pi \Sigma \bar{y}A$$
$$= 2\pi(39.5\text{ in}^3)$$

or $\text{Volume} = 248\text{ in}^3$ ◀

$$\text{Area} = 2\pi\,\bar{y}_{\text{line}}L = 2\pi\,\Sigma(\bar{y}_{\text{line}})L$$
$$= 2\pi(\bar{y}_2L_2 + \bar{y}_3L_3 + \bar{y}_4L_4 + \bar{y}_5L_5 + \bar{y}_6L_6 + \bar{y}_7L_7 + \bar{y}_8L_8)$$
$$= 2\pi[(1)(2) + (2)(3) + (2.5)(1) + (3)(3) + (5.5)(5) + (8)(1) + (4)(8)]$$

or $\text{Area} = 547\text{ in}^2$ ◀

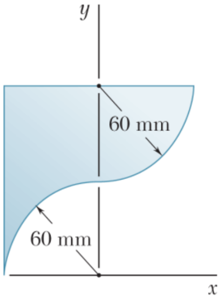
(*b*) Rotation about the *y*-axis:

$$\text{Volume} = 2\pi \bar{x}_{\text{area}} A = 2\pi \Sigma \bar{x}A$$
$$= 2\pi(11.5\text{ in}^3)$$

or $\text{Volume} = 72.3\text{ in}^3$ ◀

$$\text{Area} = 2\pi\,\bar{x}_{\text{line}}L = 2\pi\,\Sigma(\bar{x}_{\text{line}})L$$
$$= 2\pi(\bar{x}_1L_1 + \bar{x}_2L_2 + \bar{x}_3L_3 + \bar{x}_4L_4 + \bar{x}_5L_5 + \bar{x}_6L_6 + \bar{x}_7L_7)$$
$$= 2\pi[(0.5)(1) + (1)(2) + (2.5)(3) + (4)(1) + (2.5)(3) + (1)(5) + (0.5)(1)]$$

or $\text{Area} = 169.6\text{ in}^2$ ◀



PROBLEM 5.54

Determine the volume and the surface area of the solid obtained by rotating the area of Problem 5.6 about (*a*) the line $x = -60$ mm, (*b*) the line $y = 120$ mm.

SOLUTION

From the solution of Problem 5.6, we have

$$A = 7200 \text{ mm}^2$$
$$\Sigma \bar{x} A = -72 \times 10^3 \text{ mm}^3$$
$$\Sigma \bar{y} A = 629.83 \times 10^3 \text{ mm}^3$$

Applying the theorems of Pappus-Guldinus, we have

(*a*) Rotation about line $x = -60$ mm:

$$\text{Volume} = 2\pi(\bar{x} + 60)A = 2\pi(\Sigma \bar{x} A + 60A)$$
$$= 2\pi[-72 \times 10^3 + 60(7200)]$$

$\text{Volume} = 2.26 \times 10^6 \text{ mm}^3$ ◀

$$\text{Area} = 2\pi \bar{x}_{\text{line}} L = 2\pi \Sigma(\bar{x}_{\text{line}})L$$
$$= 2\pi(\bar{x}_1 L_1 + \bar{x}_2 L_2 + \bar{x}_3 L_3)$$
$$= 2\pi\left[\left(60 - \frac{2(60)}{\pi}\right)\left(\frac{\pi(60)}{2}\right) + \left(60 + \frac{2(60)}{\pi}\right)\left(\frac{\pi(60)}{2}\right) + (60)(120)\right]$$

where $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are measured with respect to line $x = -60$ mm.

$\text{Area} = 116.3 \times 10^3 \text{ mm}^2$ ◀

(*b*) Rotation about line $y = 120$ mm:

$$\text{Volume} = 2\pi(120 - \bar{y})A = 2\pi(120A - \Sigma \bar{y} A)$$
$$= 2\pi[120(7200) - 629.83 \times 10^3]$$

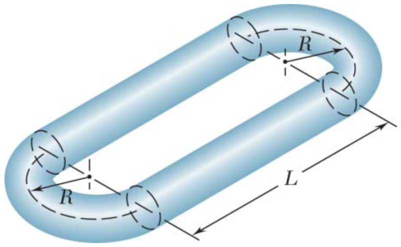
$\text{Volume} = 1.471 \times 10^6 \text{ mm}^3$ ◀

$$\text{Area} = 2\pi \bar{y}_{\text{line}} L = 2\pi \Sigma(\bar{y}_{\text{line}})L$$
$$= 2\pi(\bar{y}_1 L_1 + \bar{y}_2 L_2 + \bar{y}_4 L_4)$$

where $\bar{y}_1, \bar{y}_2, \bar{y}_4$ are measured with respect to line $y = 120$ mm.

$$\text{Area} = 2\pi\left[\left(120 - \frac{2(60)}{\pi}\right)\left(\frac{\pi(60)}{2}\right) + \left(\frac{2(60)}{\pi}\right)\left(\frac{\pi(60)}{2}\right) + (60)(120)\right]$$

$\text{Area} = 116.3 \times 10^3 \text{ mm}^2$ ◀



PROBLEM 5.55

Determine the volume and the surface area of the chain link shown, which is made from a 6-mm-diameter bar, if $R = 10$ mm and $L = 30$ mm.

SOLUTION

The area A and circumference C of the cross section of the bar are

$$A = \frac{\pi}{4}d^2 \quad \text{and} \quad C = \pi d.$$

Also, the semicircular ends of the link can be obtained by rotating the cross section through a horizontal semicircular arc of radius R. Now, applying the theorems of Pappus-Guldinus, we have for the volume V,

$$\begin{aligned} V &= 2(V_{\text{side}}) + 2(V_{\text{end}}) \\ &= 2(AL) + 2(\pi RA) \\ &= 2(L + \pi R)A \end{aligned}$$

or $$V = 2[30 \text{ mm} + \pi(10 \text{ mm})]\left[\frac{\pi}{4}(6 \text{ mm})^2\right]$$

$$= 3470 \text{ mm}^3$$

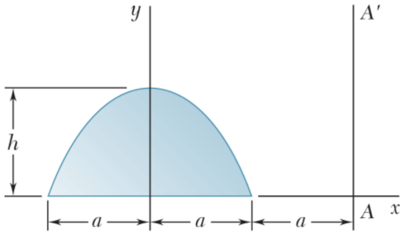
or $V = 3470 \text{ mm}^3$ ◀

For the area A, $$\begin{aligned} A &= 2(A_{\text{side}}) + 2(A_{\text{end}}) \\ &= 2(CL) + 2(\pi RC) \\ &= 2(L + \pi R)C \end{aligned}$$

or $$A = 2[30 \text{ mm} + \pi(10 \text{ mm})][\pi(6 \text{ mm})]$$

$$= 2320 \text{ mm}^2$$

or $A = 2320 \text{ mm}^2$ ◀



PROBLEM 5.56

Determine the volume of the solid generated by rotating the parabolic area shown about (*a*) the *x*-axis, (*b*) the axis *AA*′.

SOLUTION

First, from Figure 5.8a, we have $A = \frac{4}{3}ah$

$$\bar{y} = \frac{2}{5}h$$

Applying the second theorem of Pappus-Guldinus, we have

(*a*) Rotation about the *x*-axis:

$$\text{Volume} = 2\pi\bar{y}A$$

$$= 2\pi\left(\frac{2}{5}h\right)\left(\frac{4}{3}ah\right)$$

or $\text{Volume} = \frac{16}{15}\pi ah^2$ ◀

(*b*) Rotation about the line *AA*′:

$$\text{Volume} = 2\pi(2a)A$$

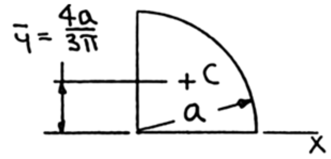
$$= 2\pi(2a)\left(\frac{4}{3}ah\right)$$

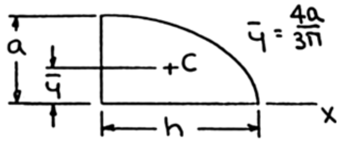
or $\text{Volume} = \frac{16}{3}\pi a^2 h$ ◀

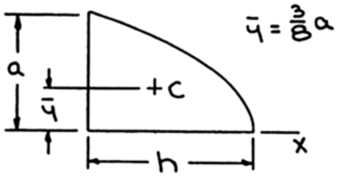
PROBLEM 5.57

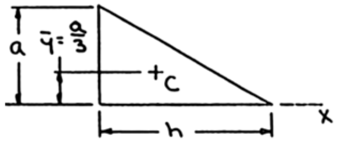
Verify that the expressions for the volumes of the first four shapes in Fig. 5.21 are correct.

SOLUTION









Following the second theorem of Pappus-Guldinus, in each case, a specific generating area *A* will be rotated about the *x*-axis to produce the given shape. Values of $\overline{y}$ are from Figure 5.8a.

(1) Hemisphere: the generating area is a quarter circle.

We have $$V = 2\pi \overline{y} A = 2\pi\left(\frac{4a}{3\pi}\right)\left(\frac{\pi}{4}a^2\right) \quad \text{or} \quad V = \frac{2}{3}\pi a^3 \blacktriangleleft$$

(2) Semiellipsoid of revolution: the generating area is a quarter ellipse.

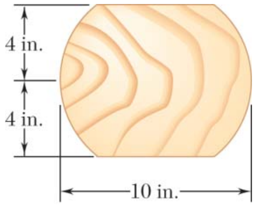
We have $$V = 2\pi \overline{y} A = 2\pi\left(\frac{4a}{3\pi}\right)\left(\frac{\pi}{4}ha\right) \quad \text{or} \quad V = \frac{2}{3}\pi a^2 h \blacktriangleleft$$

(3) Paraboloid of revolution: the generating area is a quarter parabola.

We have $$V = 2\pi \overline{y} A = 2\pi\left(\frac{3}{8}a\right)\left(\frac{2}{3}ah\right) \quad \text{or} \quad V = \frac{1}{2}\pi a^2 h \blacktriangleleft$$

(4) Cone: the generating area is a triangle.

We have $$V = 2\pi \overline{y} A = 2\pi\left(\frac{a}{3}\right)\left(\frac{1}{2}ha\right) \quad \text{or} \quad V = \frac{1}{3}\pi a^2 h \blacktriangleleft$$

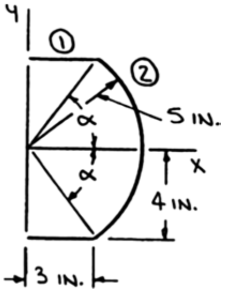

4 in.
4 in.
10 in.

PROBLEM 5.58

Knowing that two equal caps have been removed from a 10-in.-diameter wooden sphere, determine the total surface area of the remaining portion.

SOLUTION

The surface area can be generated by rotating the line shown about the *y*-axis. Applying the first theorem of Pappus-Guldinus, we have



$$A = 2\pi \overline{X}L = 2\pi \Sigma \overline{x}\, L$$
$$= 2\pi(2\overline{x}_1 L_1 + \overline{x}_2 L_2)$$

Now $\tan\alpha = \dfrac{4}{3}$

or $\alpha = 53.130°$

Then $\overline{x}_2 = \dfrac{5\text{ in.} \times \sin 53.130°}{53.130° \times \frac{\pi}{180°}}$

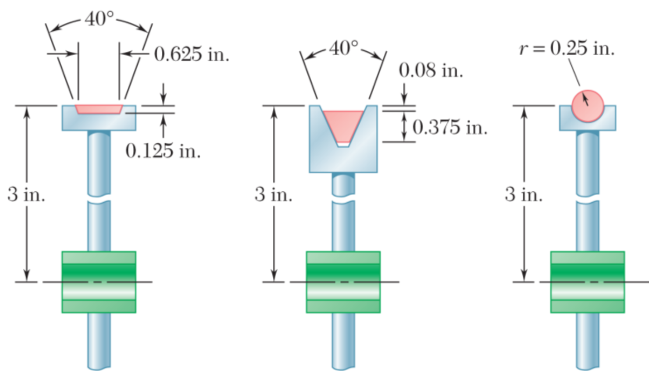
$= 4.3136$ in.

and $L_2 = 2\left(53.130° \times \dfrac{\pi}{180°}\right)(5\text{ in.})$

$= 9.2729$ in.

$$A = 2\pi\left[2\left(\frac{3}{2}\text{ in.}\right)(3\text{ in.}) + (4.3136\text{ in.})(9.2729\text{ in.})\right]$$

or $A = 308\text{ in}^2$ ◀


40°
0.625 in.
0.125 in.
3 in.
40°
0.08 in.
0.375 in.
3 in.
r = 0.25 in.
3 in.

(a) (b) (c)

PROBLEM 5.59

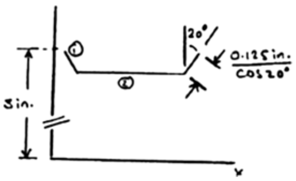
Three different drive belt profiles are to be studied. If at any given time each belt makes contact with one-half of the circumference of its pulley, determine the *contact area* between the belt and the pulley for each design.

SOLUTION

Applying the first theorem of Pappus-Guldinus, the contact area A_C of a belt is given by

$$A_C = \pi \bar{y} L = \pi \Sigma \bar{y} L$$

where the individual lengths are the lengths of the belt cross section that are in contact with the pulley.

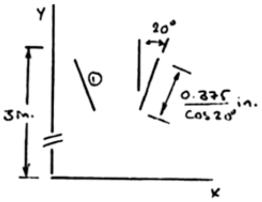

3 in.
①
②
20°
0.125 in. / cos 20°
x

(*a*)

$$A_C = \pi[2(\bar{y}_1 L_1) + \bar{y}_2 L_2]$$

$$= \pi \left\{ 2\left[\left(3 - \frac{0.125}{2}\right)\text{in.}\right]\left[\frac{0.125\text{ in.}}{\cos 20°}\right] + [(3 - 0.125)\text{ in.}](0.625\text{ in.}) \right\}$$

or $$A_C = 8.10\text{ in}^2 \blacktriangleleft$$

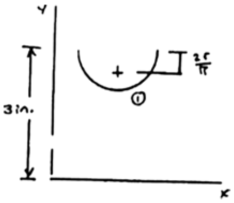

y
3 in.
①
20°
0.375 in. / cos 20°
x

(*b*)

$$A_C = \pi[2(\bar{y}_1 L_1)]$$

$$= 2\pi\left[\left(3 - 0.08 - \frac{0.375}{2}\right)\text{in.}\right]\left(\frac{0.375\text{ in.}}{\cos 20°}\right)$$

or $$A_C = 6.85\text{ in}^2 \blacktriangleleft$$

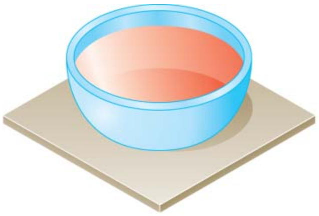


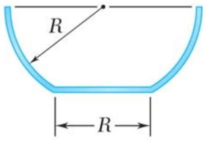
(*c*)

$$A_C = \pi[2(\bar{y}_1 L_1)]$$

$$= \pi\left[\left(3 - \frac{2(0.25)}{\pi}\right)\text{in.}\right][\pi(0.25\text{ in.})]$$

or $$A_C = 7.01\text{ in}^2 \blacktriangleleft$$





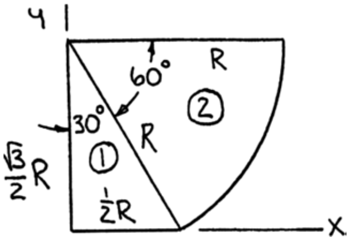
PROBLEM 5.60

Determine the capacity, in liters, of the punch bowl shown if $R = 250$ mm.

SOLUTION

The volume can be generated by rotating the triangle and circular sector shown about the *y*-axis. Applying the second theorem of Pappus-Guldinus and using Figure 5.8a, we have

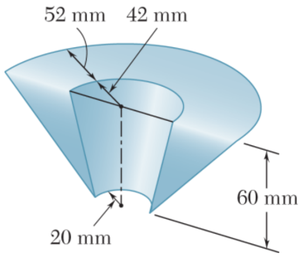
$$
\begin{aligned}
V &= 2\pi \bar{x} A = 2\pi \Sigma \bar{x} A \\
&= 2\pi(\bar{x}_1 A_1 + \bar{x}_2 A_2) \\
&= 2\pi\left[\left(\frac{1}{3}\times\frac{1}{2}R\right)\left(\frac{1}{2}\times\frac{1}{2}R\times\frac{\sqrt{3}}{2}R\right)+\left(\frac{2R\sin 30^\circ}{3\times\frac{\pi}{6}}\right)\left(\frac{\pi}{6}R^2\right)\right] \\
&= 2\pi\left(\frac{R^3}{16\sqrt{3}}+\frac{R^3}{2\sqrt{3}}\right) \\
&= \frac{3\sqrt{3}}{8}\pi R^3 \\
&= \frac{3\sqrt{3}}{8}\pi(0.25\text{ m})^3 \\
&= 0.031883\text{ m}^3
\end{aligned}
$$



Since $10^3\text{ l} = 1\text{ m}^3$

$$V = 0.031883\text{ m}^3 \times \frac{10^3\text{ l}}{1\text{ m}^3}$$

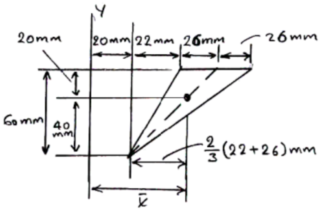
$V = 31.9$ liters ◀


52 mm
42 mm
60 mm
20 mm

PROBLEM 5.61

Determine the volume and total surface area of the bushing shown.

SOLUTION


Y
20mm
20mm
22mm
26mm
26mm
60mm
40 mm
$\frac{2}{3}(22+26)$mm
$\bar{x}$

Volume:

The volume can be obtained by rotating the triangular area shown through π radians about the y axis.

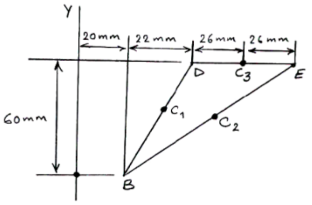
The area of the triangle is:

$$A = \frac{1}{2}(52)(60) = 1560 \text{ mm}^2$$

Applying the theorems of Pappus-Guldinus, we have

$$V = \pi \overline{x} A = \pi(52 \text{ mm})(1560 \text{ mm}^2) \qquad \text{or} \qquad V = 255 \times 10^3 \text{ mm}^3 \blacktriangleleft$$

The surface area can be obtained by rotating the triangle shown through an angle of π radians about the y axis.


Y
20mm
22mm
26mm
26mm
D
C_3
E
C_1
C_2
60mm
B

Considering each line BD, DE, and BE separately:

Line BD: $L_1 = \sqrt{22^2 + 60^2} = 63.906$ mm $\qquad \overline{x}_1 = 20 + \frac{22}{2} = 31$ mm

Line DE: $L_2 = 52$ mm $\qquad \overline{x}_2 = 20 + 22 + 26 = 68$ mm

Line BE: $L_3 = \sqrt{74^2 + 60^2} = 95.268$ mm $\qquad \overline{x}_1 = 20 + \frac{74}{2} = 57$ mm

PROBLEM 5.61 (Continued)

Then applying the theorems of Pappus-Guldinus for the part of the surface area generated by the lines:

$$A_L = \pi\Sigma\bar{x}A = \pi\left[(31)(63.906) + (68)(52) + (57)(95.268)\right] = \pi[10947.6] = 34.392 \times 10^3 \text{mm}^2$$

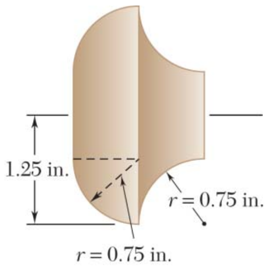
The area of the "end triangles":

$$A_E = 2\left[\frac{1}{2}(52)(60)\right] = 3.12 \times 10^3 \text{ mm}^2$$

Total surface area is therefore:

$$A = A_L + A_E = (34.392 + 3.12) \times 10^3 \text{ mm}^2$$

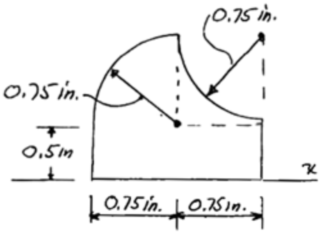
or $A = 37.5 \times 10^3 \text{ mm}^2$ ◀


1.25 in.
r = 0.75 in.
r = 0.75 in.

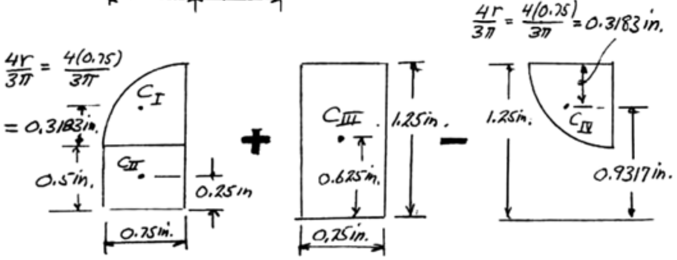
PROBLEM 5.62

Determine the volume and weight of the solid brass knob shown, knowing that the specific weight of brass is 0.306 lb/in^3.

SOLUTION


0.75 in.
0.75 in.
0.5 in
x
0.75 in.
0.75 in.

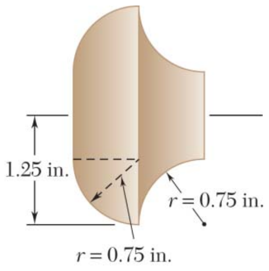
Volume of knob is obtained by rotating area at left about the *x*-axis. Consider area as made of components shown below.


4r/3π = 4(0.75)/3π
= 0.3183 in.
C_I
C_II
0.5 in.
0.25 in
0.75 in.
+
C_III
1.25 in.
0.625 in.
0.75 in.
−
4r/3π = 4(0.75)/3π = 0.3183 in.
1.25 in.
C_IV
0.9317 in.

	Area, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
1	$\frac{\pi}{4}(0.75)^2 = 0.4418$	0.8183	0.3615
2	$(0.5)(0.75) = 0.375$	0.25	0.0938
3	$(1.25)(0.75) = 0.9375$	0.625	0.5859
4	$\frac{-\pi}{4}(0.75)^2 = -0.4418$	0.9317	−0.4116
Σ			0.6296

$$V = 2\pi\Sigma\bar{y}A = 2\pi(0.6296\text{ in}^3) = 3.9559\text{ in}^3 \qquad V = 3.96\text{ in}^3 \blacktriangleleft$$

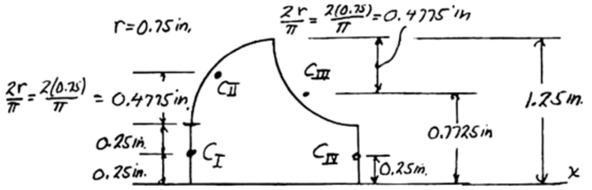
$$W = \gamma V = (0.306\text{ lb/in}^3)(3.9559\text{ in}^3) \qquad W = 1.211\text{ lb} \blacktriangleleft$$


1.25 in.
r = 0.75 in.
r = 0.75 in.

PROBLEM 5.63

Determine the total surface area of the solid brass knob shown.

SOLUTION


r=0.75 in.
$\frac{2r}{\pi}=\frac{2(0.75)}{\pi}=0.4775$ in
$\frac{2r}{\pi}=\frac{2(0.75)}{\pi}=0.4775$ in.
C_{II}
C_{III}
C_I
C_{IV}
0.25 in.
0.25 in.
0.7725 in
0.25 in.
1.25 in.
x

Area is obtained by rotating lines shown about the *x*-axis.

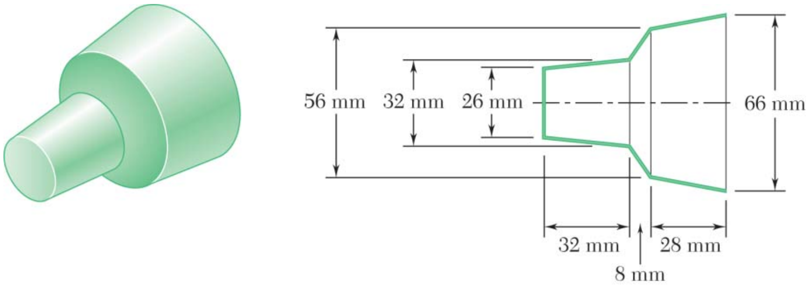
	L, in.	$\bar{y}$, in.	$\bar{y}L$, in^2
1	0.5	0.25	0.1250
2	$\frac{\pi}{2}(0.75)=1.1781$	0.9775	1.1516
3	$\frac{\pi}{2}(0.75)=1.1781$	0.7725	0.9101
4	0.5	0.25	0.1250
Σ			2.3117

$$A = 2\pi\Sigma\bar{y}L = 2\pi(2.3117\text{ in}^2)$$

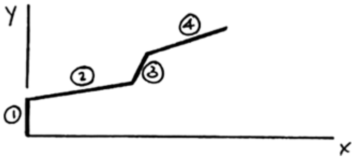
$A = 14.52\text{ in}^2$ ◀

PROBLEM 5.64

The aluminum shade for the small high-intensity lamp shown has a uniform thickness of 1 mm. Knowing that the density of aluminum is 2800 kg/m^3, determine the mass of the shade.



SOLUTION



The mass of the lamp shade is given by

$$m = \rho V = \rho A t$$

where A is the surface area and t is the thickness of the shade. The area can be generated by rotating the line shown about the x-axis. Applying the first theorem of Pappus Guldinus, we have

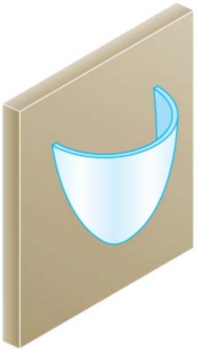
$$\begin{aligned} A &= 2\pi \bar{y} L = 2\pi \Sigma \bar{y} L \\ &= 2\pi(\bar{y}_1 L_1 + \bar{y}_2 L_2 + \bar{y}_3 L_3 + \bar{y}_4 L_4) \end{aligned}$$

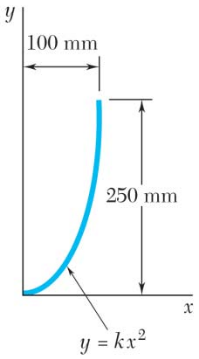
or
$$\begin{aligned} A = 2\pi\Bigg[&\frac{13 \text{ mm}}{2}(13 \text{ mm}) + \left(\frac{13+16}{2}\right)\text{mm} \times \sqrt{(32 \text{ mm})^2 + (3 \text{ mm})^2} \\ &+ \left(\frac{16+28}{2}\right)\text{mm} \times \sqrt{(8 \text{ mm})^2 + (12 \text{ mm})^2} \\ &+ \left(\frac{28+33}{2}\right)\text{mm} \times \sqrt{(28 \text{ mm})^2 + (5 \text{ mm})^2}\Bigg] \\ &= 2\pi(84.5 + 466.03 + 317.29 + 867.51) \\ &= 10{,}903.4 \text{ mm}^2 \end{aligned}$$

Then $m = \rho A t$ or

$$= (2800 \text{ kg/m}^3)(10.9034 \times 10^{-3} \text{ m}^2)(0.001 \text{ m})$$

$m = 0.0305$ kg ◀





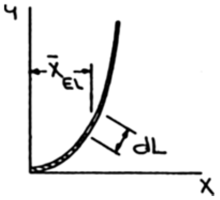
PROBLEM 5.65*

The shade for a wall-mounted light is formed from a thin sheet of translucent plastic. Determine the surface area of the outside of the shade, knowing that it has the parabolic cross section shown.

SOLUTION

First note that the required surface area A can be generated by rotating the parabolic cross section through π radians about the y-axis. Applying the first theorem of Pappus-Guldinus, we have

$$A = \pi \bar{x} L$$



Now at $x = 100\text{ mm}, \quad y = 250\text{ mm}$

$$250 = k(100)^2 \quad \text{or} \quad k = 0.025\text{ mm}^{-1}$$

and $x_{EL} = x$

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where $\dfrac{dy}{dx} = 2kx$

Then $dL = \sqrt{1 + 4k^2x^2}\, dx$

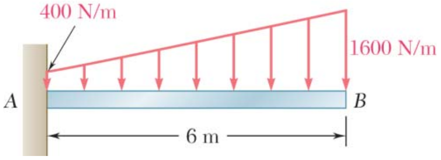
We have $\quad \bar{x}L = \int x_{EL}\, dL = \int_0^{100} x\left(\sqrt{1 + 4k^2x^2}\, dx\right)$

$$\bar{x}L = \left[\frac{1}{3}\frac{1}{4k^2}(1 + 4k^2x^2)^{3/2}\right]_0^{100}$$

$$= \frac{1}{12}\frac{1}{(0.025)^2}\left\{[1 + 4(0.025)^2(100)^2]^{3/2} - (1)^{3/2}\right\}$$

$$= 17{,}543.3\text{ mm}^2$$

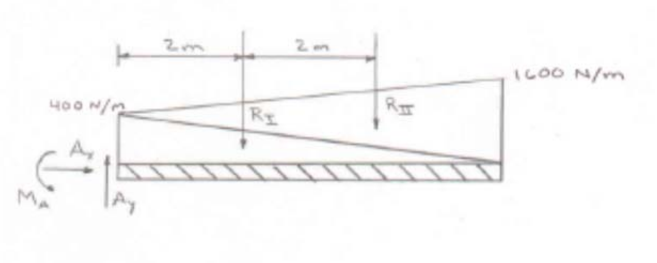
Finally, $\quad A = \pi(17{,}543.3\text{ mm}^2) \qquad$ or $\quad A = 55.1 \times 10^3\text{ mm}^2$ ◀



PROBLEM 5.66

For the beam and loading shown, determine (*a*) the magnitude and location of the resultant of the distributed load, (*b*) the reactions at the beam supports.

SOLUTION



(*a*) $R_{\text{I}} = \frac{1}{2}(400\text{ N/m})(6\text{ m})$

$= 1200\text{ N}$

$R_{\text{II}} = (1600\text{ N/m})(6\text{ m})$

$= 4800\text{ N}$

$+\uparrow \Sigma F_y = 0: \quad -R = -R_{\text{I}} - R_{\text{II}}$

$R = 1200\text{ N} + 4800\text{ N}$

$R = 6000\text{ N}$

$\Sigma M_A = 0: \quad -\overline{X}(6000) = -2(1200) - 4(4800)$

$\overline{X} = 3.60\text{ m}$

$\mathbf{R} = 6000\text{ N}\downarrow, \ \overline{X} = 3.60\text{ m}$ ◀

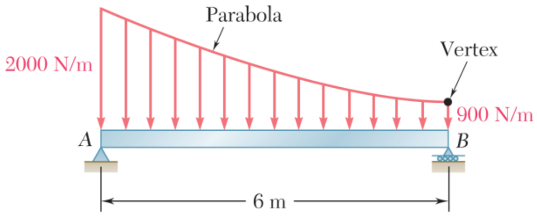
(*b*) Reactions $\Sigma F_x = 0: \quad A_x = 0$

$\Sigma F_y = 0: \quad A_y - 6000 = 0 \quad A_y = 6000\text{ N}$

$\mathbf{A} = 6000\text{ N}\uparrow$ ◀

$+\circlearrowleft \Sigma M_A = 0: \quad M_A - (6000\text{ N})(3.60\text{ m}) = 0$

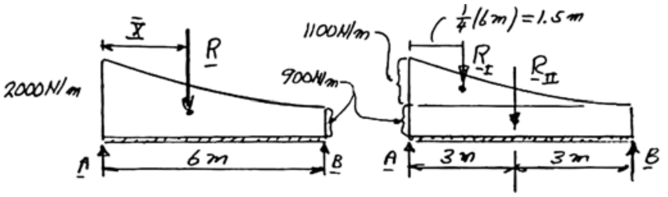
$\mathbf{M}_A = 21.6\text{ kN}\cdot\text{m} \circlearrowleft$ ◀


Parabola
Vertex
2000 N/m
900 N/m
A
B
6 m

PROBLEM 5.67

For the beam and loading shown, determine (*a*) the magnitude and location of the resultant of the distributed load, (*b*) the reactions at the beam supports.

SOLUTION


X̄
R
2000 N/m
1100 N/m
900 N/m
(1/4(6 m) = 1.5 m
R_I
R_II
A
6 m
B
A
3 m
3 m
B

(*a*) $$R_{\text{I}} = \frac{1}{3}(1100\text{ N/m})(6\text{ m}) = 2200\text{ N}$$

$$R_{\text{II}} = (900\text{ N/m})(6\text{ m}) = 5400\text{ N}$$

$$R = R_{\text{I}} + R_{\text{II}} = 2200 + 5400 = 7600\text{ N}$$

$$XR = \Sigma xR: \quad X(7600) = (2200)(1.5) + (5400)(3)$$

$$X = 2.5658\text{ m}$$

$\mathbf{R} = 7.60\text{ kN}\downarrow, \quad X = 2.57\text{ m}$ ◀

(*b*) $$+\circlearrowleft \Sigma M_A = 0: \quad B(6\text{ m}) - (7600\text{ N})(2.5658\text{ m}) = 0$$

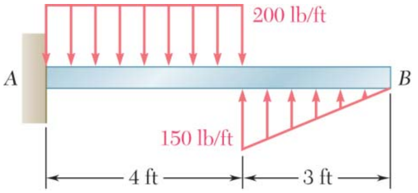
$$B = 3250.0\text{ N}$$

$\mathbf{B} = 3.25\text{ kN}\uparrow$ ◀

$$+\uparrow F_y = 0: \quad A + 3250.0\text{ N} - 7600\text{ N} = 0$$

$$A = 4350.0\text{ N}$$

$\mathbf{A} = 4.35\text{ kN}\uparrow$ ◀


200 lb/ft
A
B
150 lb/ft
4 ft
3 ft

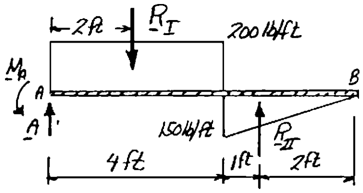
PROBLEM 5.68

Determine the reactions at the beam supports for the given loading.

SOLUTION

$$R_{\text{I}} = (200\text{ lb/ft})(4\text{ ft}) = 800\text{ lb}$$

$$R_{\text{II}} = \frac{1}{2}(150\text{ lb/ft})(3\text{ ft}) = 225\text{ lb}$$

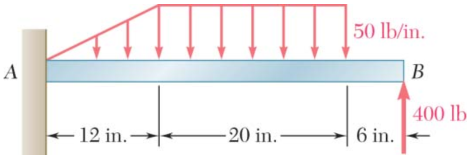

2ft
R_I
200 lb/ft
M_A
A
B
A
150 lb/ft
R_{II}
4ft
1ft
2ft

$$+\uparrow \Sigma F_y = 0: \quad A - 800\text{ lb} + 225\text{ lb} = 0$$

$\mathbf{A} = 575\text{ lb}\uparrow$ ◀

$$+\circlearrowleft \Sigma M_A = 0: \quad M_A - (800\text{ lb})(2\text{ ft}) + (225\text{ lb})(5\text{ ft}) = 0$$

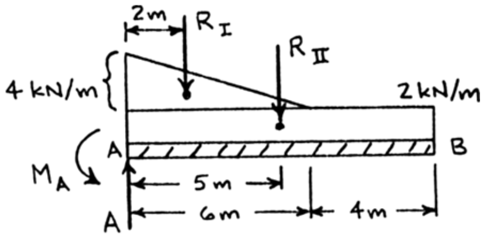
$\mathbf{M}_A = 475\text{ lb}\cdot\text{ft} +\circlearrowleft$ ◀


50 lb/in.
A
B
400 lb
12 in.
20 in.
6 in.

PROBLEM 5.69

Determine the reactions at the beam supports for the given loading.

SOLUTION


2m
R_I
R_{II}
4 kN/m
2 kN/m
A
B
M_A
5m
6m
4m
A

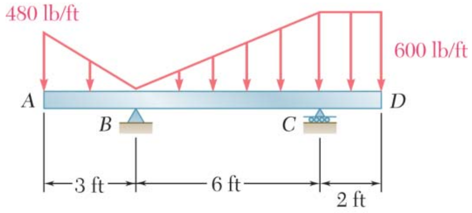
$$R_{\text{I}} = \frac{1}{2}(50\text{ lb/in.})(12\text{ in.})$$
$$= 300\text{ lb}$$
$$R_{\text{II}} = (50\text{ lb/in.})(20\text{ in.})$$
$$= 1000\text{ lb}$$

$$+\uparrow\ \Sigma F_y = 0:\quad A_y - 300\text{ lb} - 1000\text{ lb} + 400\text{ lb} = 0$$

$$\mathbf{A} = 900\text{ lb}\uparrow\ \blacktriangleleft$$

$$+\circlearrowleft\ \Sigma M_A = 0:\quad M_A - (300\text{ lb})(8\text{ in.}) - (1000\text{ lb})(22\text{ in.}) + (400\text{ lb})(38\text{ in.}) = 0$$

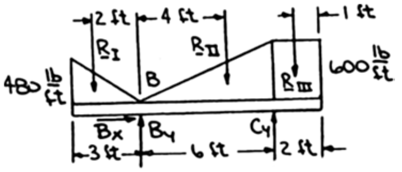
$$\mathbf{M}_A = 9200\text{ lb}\cdot\text{in.}\ \circlearrowleft\ \blacktriangleleft$$


480 lb/ft
600 lb/ft
A
B
C
D
3 ft
6 ft
2 ft

PROBLEM 5.70

Determine the reactions at the beam supports for the given loading.

SOLUTION


2 ft
4 ft
1 ft
R_I
R_II
B
R_III
480 lb/ft
600 lb/ft
Bx
By
Cy
3 ft
6 ft
2 ft

We have

$$R_{\text{I}} = \frac{1}{2}(3\,\text{ft})(480\text{ lb/ft}) = 720\text{ lb}$$

$$R_{\text{II}} = \frac{1}{2}(6\text{ ft})(600\text{ lb/ft}) = 1800\text{ lb}$$

$$R_{\text{III}} = (2\,\text{ft})(600\text{ lb/ft}) = 1200\text{ lb}$$

Then

$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (2\text{ ft})(720\text{ lb}) - (4\text{ ft})(1800\text{ lb}) + (6\text{ ft})C_y - (7\text{ ft})(1200\text{ lb}) = 0$$

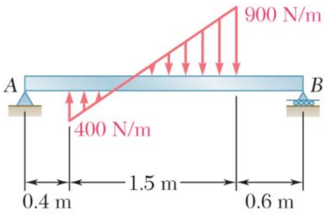
or

$$C_y = 2360\text{ lb} \qquad \mathbf{C} = 2360\text{ lb}\uparrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -720\text{ lb} + B_y - 1800\text{ lb} + 2360\text{ lb} - 1200\text{ lb} = 0$$

or

$$B_y = 1360\text{ lb} \qquad \mathbf{B} = 1360\text{ lb}\uparrow \blacktriangleleft$$

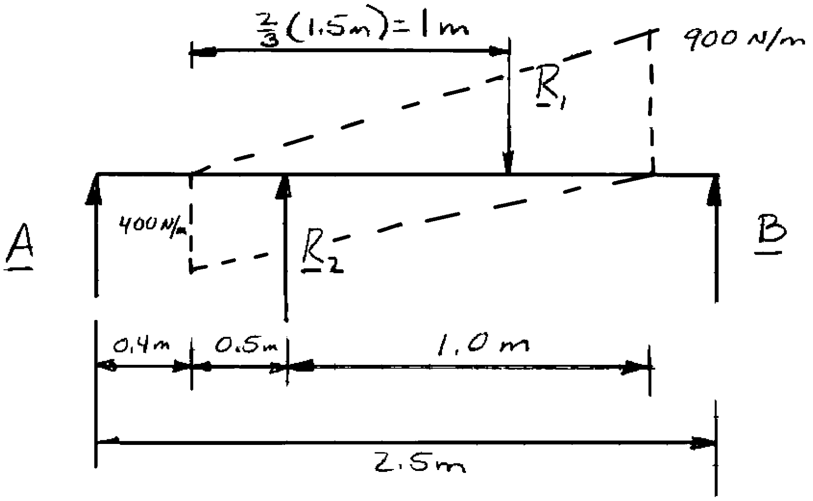

900 N/m
A
B
400 N/m
0.4 m
1.5 m
0.6 m

PROBLEM 5.71

Determine the reactions at the beam supports for the given loading.

SOLUTION

First replace the given loading by the loadings shown below. Both loadings are equivalent since they are both defined by a linear relation between load and distance and have the same values at the end points.


$\frac{2}{3}(1.5m)=1m$
900 N/m
R_1
400 N/m
A
R_2
B
0.4m
0.5m
1.0 m
2.5m

$$R_1 = \frac{1}{2}(900\text{ N/m})(1.5\text{ m}) = 675\text{ N}$$

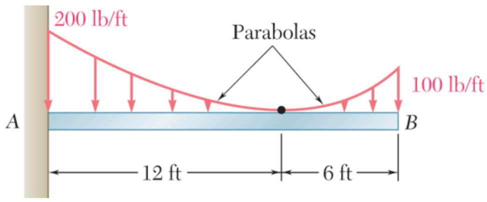
$$R_2 = \frac{1}{2}(400\text{ N/m})(1.5\text{ m}) = 300\text{ N}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad -(675\text{ N})(1.4\text{ m}) + (300\text{ N})(0.9\text{ m}) + B(2.5\text{ m}) = C$$

$B = 270\text{ N}$ $\mathbf{B} = 270\text{ N}\uparrow$ ◀

$$+\uparrow F_y = 0: \quad A - 675\text{ N} + 300\text{ N} + 270\text{ N} = 0$$

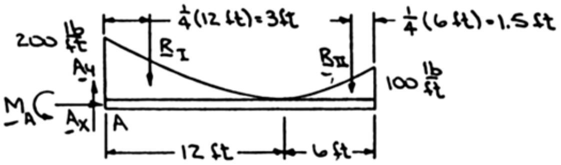
$A = 105.0\text{ N}$ $\mathbf{A} = 105.0\text{ N}\uparrow$ ◀


200 lb/ft
Parabolas
100 lb/ft
A
B
12 ft
6 ft

PROBLEM 5.72

Determine the reactions at the beam supports for the given loading.

SOLUTION


200 lb/ft
1/4(12 ft)=3ft
R_I
R_II
1/4(6 ft)=1.5 ft
100 lb/ft
A_y
M_A
A_x
A
12 ft
6 ft

We have $$R_{\text{I}} = \frac{1}{3}(12\text{ ft})(200\text{ lb/ft}) = 800\text{ lb}$$

$$R_{\text{II}} = \frac{1}{3}(6\text{ ft})(100\text{ lb/ft}) = 200\text{ lb}$$

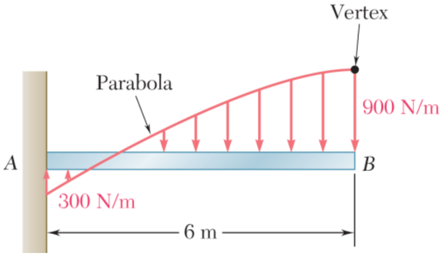
Then $$\xrightarrow{+}\ \Sigma F_x = 0:\quad A_x = 0$$

$$+\uparrow\ \Sigma F_y = 0:\quad A_y - 800\text{ lb} - 200\text{ lb} = 0$$

or $A_y = 1000\text{ lb}$ $\qquad\qquad$ $\mathbf{A} = 1000\text{ lb}\uparrow$ ◀

$$+\circlearrowleft \Sigma M_A = 0:\quad M_A - (3\text{ ft})(800\text{ lb}) - (16.5\text{ ft})(200\text{ lb}) = 0$$

or $M_A = 5700\text{ lb}\cdot\text{ft}$ $\qquad\qquad$ $\mathbf{M}_A = 5700\text{ lb}\cdot\text{ft}\circlearrowleft$ ◀

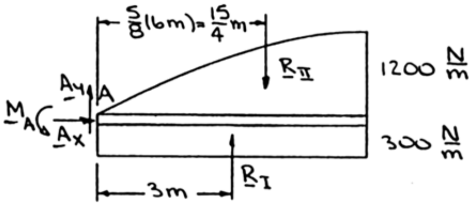

Vertex
Parabola
900 N/m
A
B
300 N/m
6 m

PROBLEM 5.73

Determine the reactions at the beam supports for the given loading.

SOLUTION

First replace the given loading with the loading shown below. The two loadings are equivalent because both are defined by a parabolic relation between load and distance and the values at the end points are the same.


$\frac{5}{8}(6\text{m}) = \frac{15}{4}\text{m}$
R_{II}
1200 N/m
A_y
A
M_A
A_x
300 N/m
3m
R_I

We have $R_\text{I} = (6\text{ m})(300\text{ N/m}) = 1800\text{ N}$

$$R_\text{II} = \frac{2}{3}(6\text{ m})(1200\text{ N/m}) = 4800\text{ N}$$

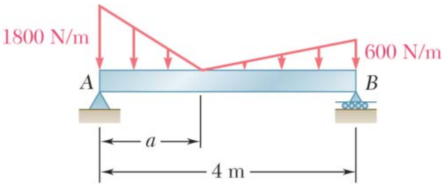
Then $\xrightarrow{+} \Sigma F_x = 0\text{:} \quad A_x = 0$

$$+\!\uparrow \Sigma F_y = 0\text{:} \quad A_y + 1800\text{ N} - 4800\text{ N} = 0$$

or $A_y = 3000\text{ N}$ **A** $= 3.00\text{ kN}\uparrow$ ◀

$$\overset{+}{\circlearrowleft} \Sigma M_A = 0\text{:} \quad M_A + (3\text{ m})(1800\text{ N}) - \left(\frac{15}{4}\text{ m}\right)(4800\text{ N}) = 0$$

or $M_A = 12.6\text{ kN}\cdot\text{m}$ $\mathbf{M}_A = 12.60\text{ kN}\cdot\text{m}\ \circlearrowleft$ ◀

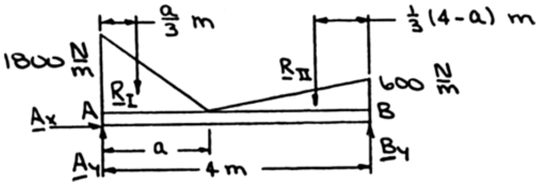


PROBLEM 5.74

Determine (*a*) the distance *a* so that the vertical reactions at supports *A* and *B* are equal, (*b*) the corresponding reactions at the supports.

SOLUTION

(*a*)



We have $R_{\text{I}} = \frac{1}{2}(a \text{ m})(1800 \text{ N/m}) = 900a \text{ N}$

$R_{\text{II}} = \frac{1}{2}[(4-a) \text{ m}](600 \text{ N/m}) = 300(4-a) \text{ N}$

Then $\Sigma F_y = 0\text{:} \quad A_y - 900a \overset{+\uparrow}{} - 300(4-a) + B_y = 0$

or $A_y + B_y = 1200 + 600a$

Now $A_y = B_y \Rightarrow A_y = B_y = 600 + 300a \text{ (N)} \qquad (1)$

Also, $+\circlearrowleft \Sigma M_B = 0\text{:} \quad -(4 \text{ m})A_y + \left[\left(4 - \frac{a}{3}\right) \text{m}\right][(900a) \text{ N}]$

$+\left[\frac{1}{3}(4-a) \text{ m}\right][300(4-a) \text{ N}] = 0$

or $A_y = 400 + 700a - 50a^2 \quad (2)$

Equating Eqs. (1) and (2), $600 + 300a = 400 + 700a - 50a^2$

or $a^2 - 8a + 4 = 0$

Then $a = \dfrac{8 \pm \sqrt{(-8)^2 - 4(1)(4)}}{2}$

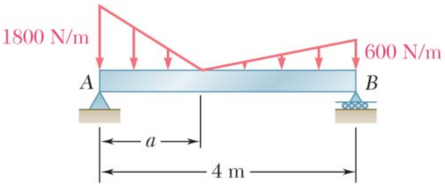
or $a = 0.53590 \text{ m} \qquad a = 7.4641 \text{ m} \qquad$ Now $a \le 4 \text{ m} \Rightarrow \qquad a = 0.536 \text{ m}$ ◀

PROBLEM 5.74 (Continued)

(*b*) We have $\xrightarrow{+} \Sigma F_x = 0\text{:} \quad A_x = 0$

From Eq. (1):
$$\begin{aligned} A_y &= B_y \\ &= 600 + 300(0.53590) \\ &= 761\ \text{N} \end{aligned}$$

$\mathbf{A} = \mathbf{B} = 761\ \text{N}\uparrow$ ◀

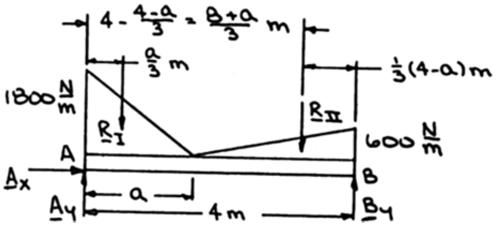

1800 N/m
600 N/m
A
B
a
4 m

PROBLEM 5.75

Determine (*a*) the distance *a* so that the reaction at support *B* is minimum, (*b*) the corresponding reactions at the supports.

SOLUTION

(*a*)


4 - (4-a)/3 = (8+a)/3 m
a/3 m
1/3 (4-a) m
1800 N/m
R_I
R_II
600 N/m
A
A_x
A_y
a
4 m
B
B_y

We have $R_{\text{I}} = \frac{1}{2}(a\text{ m})(1800\text{ N/m}) = 900a\text{ N}$

$$R_{\text{II}} = \frac{1}{2}[(4-a)\text{m}](600\text{ N/m}) = 300(4-a)\text{ N}$$

Then $+\circlearrowleft \Sigma M_A = 0$: $-\left(\frac{a}{3}\text{m}\right)(900a\text{ N}) - \left(\frac{8+a}{3}\text{m}\right)[300(4-a)\text{N}] + (4\text{ m})B_y = 0$

or $B_y = 50a^2 - 100a + 800 \quad (1)$

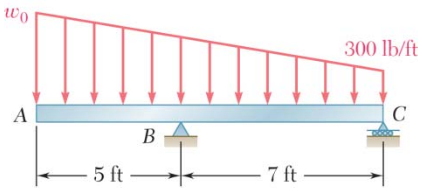
Then $\frac{dB_y}{da} = 100a - 100 = 0$ or $a = 1.000\text{ m}$ ◀

(*b*) From Eq. (1): $B_y = 50(1)^2 - 100(1) + 800 = 750\text{ N}$ $\mathbf{B} = 750\text{ N}\uparrow$ ◀

and $\xrightarrow{+} \Sigma F_x = 0$: $A_x = 0$

$+\uparrow \Sigma F_y = 0$: $A_y - 900(1)\text{ N} - 300(4-1)\text{ N} + 750\text{ N} = 0$

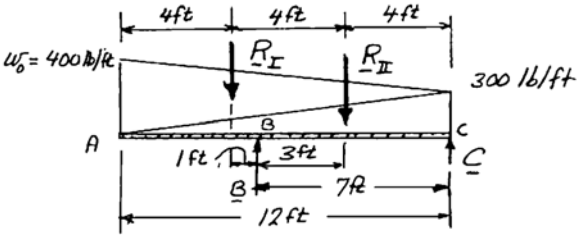
or $A_y = 1050\text{ N}$ $\mathbf{A} = 1050\text{ N}\uparrow$ ◀


w_0
300 lb/ft
A
B
C
5 ft
7 ft

PROBLEM 5.76

Determine the reactions at the beam supports for the given loading when $w_O = 400$ lb/ft.

SOLUTION


4ft
4ft
4ft
$w_0 = 400$ lb/ft
R_I
R_{II}
300 lb/ft
B
A
C
1ft
3ft
B
C
7ft
12ft

$$R_{\text{I}} = \frac{1}{2}w_O(12\text{ ft}) = \frac{1}{2}(400\text{ lb/ft})(12\text{ ft}) = 2400\text{ lb}$$

$$R_{\text{II}} = \frac{1}{2}(300\text{ lb/ft})(12\text{ ft}) = 1800\text{ lb}$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (2400\text{ lb})(1\text{ ft}) - (1800\text{ lb})(3\text{ ft}) + C(7\text{ ft}) = 0$$

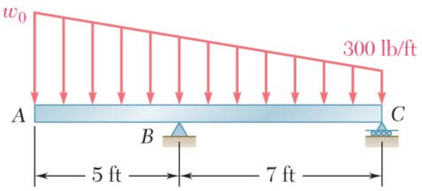
$$C = 428.57\text{ lb}$$

$\mathbf{C} = 429\text{ lb}\uparrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad B + 428.57\text{ lb} - 2400\text{ lb} - 1800\text{ lb} = 0$$

$$B = 3771\text{ lb}$$

$\mathbf{B} = 3770\text{ lb}\uparrow$ ◀

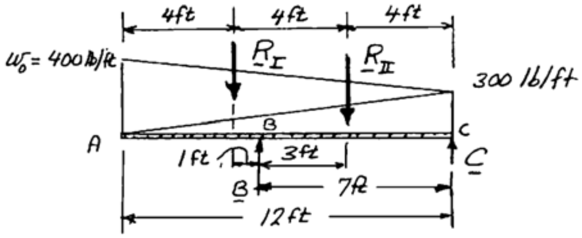


w_0
300 lb/ft
A
B
C
5 ft
7 ft

PROBLEM 5.77

Determine (*a*) the distributed load w_O at the end *A* of the beam *ABC* for which the reaction at *C* is zero, (*b*) the corresponding reaction at *B*.

SOLUTION



4ft
4ft
4ft
$w_0 = 400$ lb/ft
R_I
R_{II}
300 lb/ft
A
B
C
1ft
3ft
B
7ft
C
12ft

$$\text{For } w_O,\ R_\text{I} = \frac{1}{2} w_O (12 \text{ ft}) = 6\, w_O$$

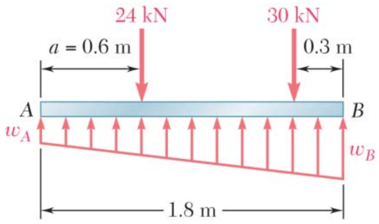
$$R_\text{II} = \frac{1}{2}(300 \text{ lb/ft})(12 \text{ ft}) = 1800 \text{ lb}$$

(*a*) For $C = 0$, $+\circlearrowleft \Sigma M_B = 0$: $(6\, w_O)(1 \text{ ft}) - (1800 \text{ lb})(3 \text{ ft}) = 0$ $w_O = 900 \text{ lb/ft}$ ◀

(*b*) Corresponding value of R_I:

$$R_\text{I} = 6(900) = 5400 \text{ lb}$$

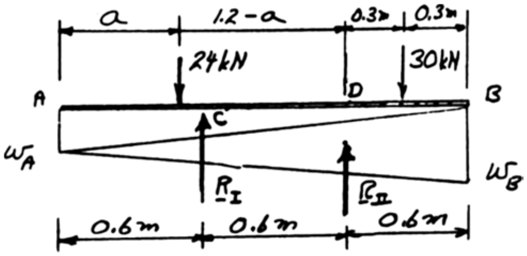
$+\uparrow \Sigma F_y = 0$: $B - 5400 \text{ lb} - 1800 \text{ lb} = 0$ $\mathbf{B} = 7200 \text{ lb} \uparrow$ ◀


24 kN
30 kN
a = 0.6 m
0.3 m
A
B
w_A
w_B
1.8 m

PROBLEM 5.78

The beam *AB* supports two concentrated loads and rests on soil that exerts a linearly distributed upward load as shown. Determine the values of ω_A and ω_B corresponding to equilibrium.

SOLUTION


a
1.2 − a
0.3m
0.3m
24kN
30kN
A
B
D
C
w_A
w_B
R_I
R_{II}
0.6m
0.6m
0.6m

$$R_{\mathrm{I}} = \frac{1}{2}\omega_A(1.8 \text{ m}) = 0.9\omega_A$$

$$R_{\mathrm{II}} = \frac{1}{2}\omega_B(1.8 \text{ m}) = 0.9\omega_B$$

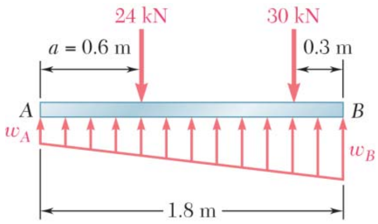
$$+\circlearrowleft \Sigma M_D = 0: \quad (24 \text{ kN})(1.2 - a) - (30 \text{ kN})(0.3 \text{ m}) - (0.9\omega_A)(0.6 \text{ m}) = 0 \qquad (1)$$

For $a = 0.6$ m,

$$24(1.2 - 0.6) - (30)(0.3) - 0.54\omega_a = 0$$

$$14.4 - 9 - 0.54\omega_A = 0 \qquad \omega_A = 10.00 \text{ kN/m} ◀$$

$$+\uparrow \Sigma F_y = 0: \quad -24 \text{ kN} - 30 \text{ kN} + 0.9(10 \text{ kN/m}) + 0.9\omega_B = 0 \qquad \omega_B = 50.0 \text{ kN/m} ◀$$

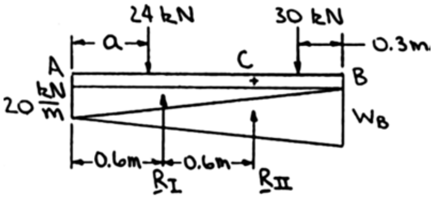


PROBLEM 5.79

For the beam and loading of Problem 5.78, determine (*a*) the distance *a* for which $\omega_A = 20$ kN/m, (*b*) the corresponding value of ω_B.

PROBLEM 5.78 The beam *AB* supports two concentrated loads and rests on soil that exerts a linearly distributed upward load as shown. Determine the values of ω_A and ω_B corresponding to equilibrium.

SOLUTION



We have
$$R_{\text{I}} = \frac{1}{2}(1.8 \text{ m})(20 \text{ kN/m}) = 18 \text{ kN}$$

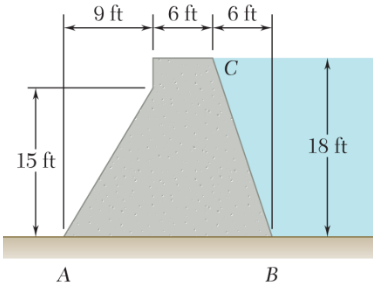
$$R_{\text{II}} = \frac{1}{2}(1.8 \text{ m})(\omega_B \text{ kN/m}) = 0.9\omega_B \text{ kN}$$

(*a*)
$$+\circlearrowleft \Sigma M_C = 0: \quad (1.2 - a)\text{m} \times 24 \text{ kN} - 0.6 \text{ m} \times 18 \text{ kN} - 0.3 \text{ m} \times 30 \text{ kN} = 0$$

or $a = 0.375$ m ◀

(*b*)
$$+\uparrow \Sigma F_y = 0: \quad -24 \text{ kN} + 18 \text{ kN} + (0.9\omega_B) \text{ kN} - 30 \text{ kN} = 0$$

or $\omega_B = 40.0$ kN/m ◀

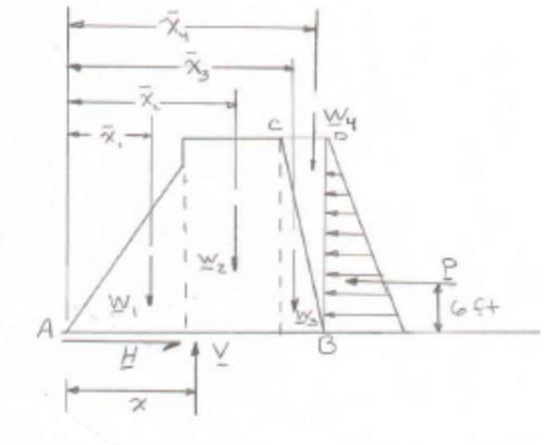


PROBLEM 5.80

The cross section of a concrete dam is as shown. For a 1-ft-wide dam section determine (*a*) the resultant of the reaction forces exerted by the ground on the base *AB* of the dam, (*b*) the point of application of the resultant of part *a*, (*c*) the resultant of the pressure forces exerted by the water on the face *BC* of the dam.

SOLUTION

The free body shown consists of a 1-ft thick section of the dam and the triangular section of water above the dam.



Note:

$$\bar{x}_1 = 6 \text{ ft}$$

$$\bar{x}_2 = (9+3)\text{ ft} = 12 \text{ ft}$$

$$\bar{x}_3 = (15+2)\text{ ft} = 17 \text{ ft}$$

$$\bar{x}_4 = (15+4)\text{ ft} = 19 \text{ ft}$$

(*a*) Now $W = \gamma V$

So that

$$W_1 = (150 \text{ lb/ft}^3)\left[\frac{1}{2}(9 \text{ ft})(15 \text{ ft})(1 \text{ ft})\right] = 10{,}125 \text{ lb}$$

$$W_2 = (150 \text{ lb/ft}^3)[(6 \text{ ft})(18 \text{ ft})(1 \text{ ft})] = 16{,}200 \text{ lb}$$

$$W_3 = (150 \text{ lb/ft}^3)\left[\frac{1}{2}(6 \text{ ft})(18 \text{ ft})(1 \text{ ft})\right] = 8100 \text{ lb}$$

$$W_4 = (62.4 \text{ lb/ft}^3)\left[\frac{1}{2}(6 \text{ ft})(18 \text{ ft})(1 \text{ ft})\right] = 3369.6 \text{ lb}$$

Also

$$P = \frac{1}{2}Ap = \frac{1}{2}[(18 \text{ ft})(1 \text{ ft})][(62.4 \text{ lb/ft}^3)(18 \text{ ft})] = 10{,}108.8 \text{ lb}$$

Then $\xrightarrow{+} \Sigma F_x = 0\text{:} \quad H - 10{,}108.8 \text{ lb} = 0$ or $\mathbf{H} = 10.11 \text{ kips} \longrightarrow$ ◀

PROBLEM 5.80 (Continued)

$+\uparrow \Sigma F_y = 0\text{:}\quad V - 10{,}125\text{ lb} - 16{,}200\text{ lb} - 8100\text{ lb} - 3369.6\text{ lb} = 0$

or $\quad V = 37{,}795\text{ lb}$ $\quad \mathbf{V} = 37.8\text{ kips}\uparrow$ ◀

(*b*) We have $\quad +\circlearrowleft \Sigma M_A = 0\text{:}\quad x(37{,}794.6\text{ lb}) - (6\text{ ft})(10{,}125\text{ lb}) - (12\text{ ft})(16{,}200\text{ lb})$

$$-(17\text{ ft})(8100\text{ lb}) - (19\text{ ft})(3369.6\text{ lb})$$

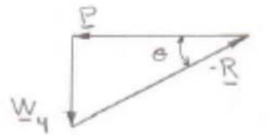
$$+(6\text{ ft})(10{,}108.8\text{ lb}) = 0$$

or $\quad 37{,}794.6x - 60{,}750 - 194{,}400 - 137{,}700 - 64{,}022.4 + 60{,}652.8 = 0$

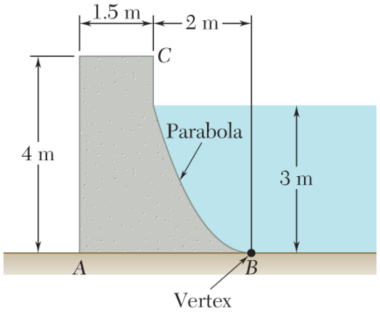
or $\quad x = 10.48\text{ ft}$ ◀

(*c*) Consider water section *BCD* as the free body

We have $\quad \Sigma\mathbf{F} = 0,\ W_4 = 3369.6\text{ lb},\ P = 10{,}108.8\text{ lb}$



Then $\quad -\mathbf{R} = 10.66\text{ kips} \measuredangle\ 18.43°$ or $\mathbf{R} = 10.66\text{ kips} \measuredangle\ 18.43°$ ◀

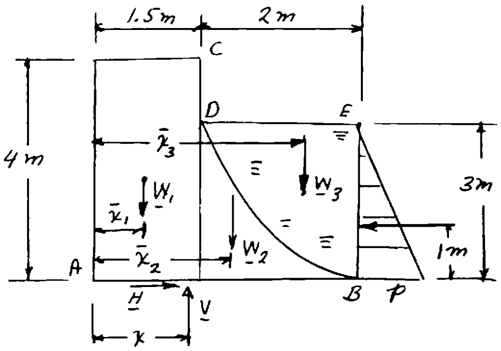


PROBLEM 5.81

The cross section of a concrete dam is as shown. For a 1-m-wide dam section, determine (*a*) the resultant of the reaction forces exerted by the ground on the base *AB* of the dam, (*b*) the point of application of the resultant of part *a*, (*c*) the resultant of the pressure forces exerted by the water on the face *BC* of the dam.

SOLUTION

(*a*) Consider free body made of dam and section *BDE* of water. (Thickness = 1 m)



$$p = (3\text{ m})(10\text{ kg/m}^3)(9.81\text{ m/s}^2)$$

$$W_1 = (1.5\text{ m})(4\text{ m})(1\text{ m})(2.4\times10^3\text{ kg/m}^3)(9.81\text{ m/s}^2) = 144.26\text{ kN}$$

$$W_2 = \frac{1}{3}(2\text{ m})(3\text{ m})(1\text{ m})(2.4\times10^3\text{ kg/m}^3)(9.81\text{ m/s}^2) = 47.09\text{ kN}$$

$$W_3 = \frac{2}{3}(2\text{ m})(3\text{ m})(1\text{ m})(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2) = 39.24\text{ kN}$$

$$P = \frac{1}{2}Ap = \frac{1}{2}(3\text{ m})(1\text{ m})(3\text{ m})(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2) = 44.145\text{ kN}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad H - 44.145\text{ kN} = 0$$

$H = 44.145\text{ kN}$ $\qquad$ $\mathbf{H} = 44.1\text{ kN} \longrightarrow$ ◀

$$+\!\uparrow \Sigma F_y = 0: \quad V - 141.26 - 47.09 - 39.24 = 0$$

$V = 227.6\text{ kN}$ $\qquad$ $\mathbf{V} = 228\text{ kN} \uparrow$ ◀

PROBLEM 5.81 (Continued)

$$\bar{x}_1 = \frac{1}{2}(1.5\text{ m}) = 0.75\text{ m}$$

$$\bar{x}_2 = 1.5\text{ m} + \frac{1}{4}(2\text{ m}) = 2\text{ m}$$

$$\bar{x}_3 = 1.5\text{ m} + \frac{5}{8}(2\text{ m}) = 2.75\text{ m}$$

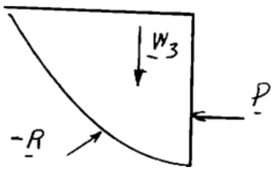
$$+\circlearrowleft \Sigma M_A = 0\text{:}\quad xV - \Sigma \bar{x}W + P(1\text{ m}) = 0$$

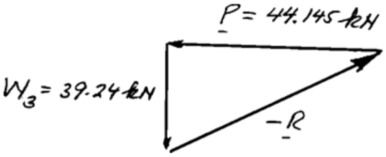
$$\begin{aligned} x(227.6\text{ kN}) &- (141.26\text{ kN})(0.75\text{ m}) - (47.09\text{ kN})(2\text{ m}) \\ &- (39.24\text{ kN})(2.75\text{ m}) + (44.145\text{ kN})(1\text{ m}) = 0 \\ x(227.6\text{ kN}) &- 105.9 - 94.2 - 107.9 + 44.145 = 0 \\ x(227.6) &- 263.9 = 0 \end{aligned}$$

$x = 1.159$ m (to right of A) ◀

(*b*) Resultant of face *BC*:

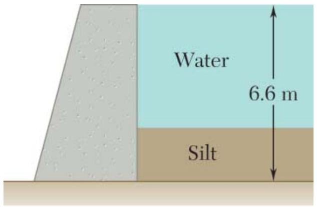
Consider free body of section *BDE* of water.





$$-\mathbf{R} = 59.1\text{ kN} \measuredangle 41.6^\circ$$

$\mathbf{R} = 59.1\text{ kN} \measuredangle 41.6^\circ$ ◀



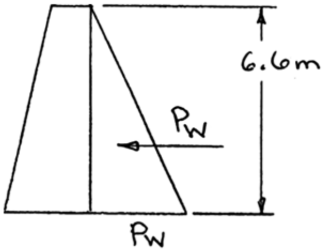
PROBLEM 5.82

The dam for a lake is designed to withstand the additional force caused by silt that has settled on the lake bottom. Assuming that silt is equivalent to a liquid of density $\rho_s = 1.76 \times 10^3\ \text{kg/m}^3$ and considering a 1-m-wide section of dam, determine the percentage increase in the force acting on the dam face for a silt accumulation of depth 2 m.

SOLUTION

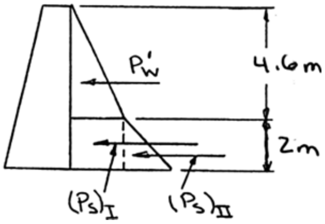
First, determine the force on the dam face without the silt.

We have
$$P_w = \frac{1}{2}Ap_w = \frac{1}{2}A(\rho g h)$$
$$= \frac{1}{2}[(6.6\ \text{m})(1\ \text{m})][(10^3\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(6.6\ \text{m})]$$
$$= 213.66\ \text{kN}$$



Next, determine the force on the dam face with silt

We have
$$P'_w = \frac{1}{2}[(4.6\ \text{m})(1\ \text{m})][(10^3\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(4.6\ \text{m})]$$
$$= 103.790\ \text{kN}$$
$$(P_s)_\text{I} = [(2.0\ \text{m})(1\ \text{m})][(10^3\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(4.6\ \text{m})]$$
$$= 90.252\ \text{kN}$$
$$(P_s)_\text{II} = \frac{1}{2}[(2.0\ \text{m})(1\ \text{m})][(1.76 \times 10^3\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(2.0\ \text{m})]$$
$$= 34.531\ \text{kN}$$

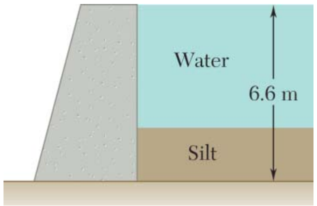


Then
$$P' = P'_w + (P_s)_\text{I} + (P_s)_\text{II} = 228.57\ \text{kN}$$

The percentage increase, % inc., is then given by

$$\%\ \text{inc.} = \frac{P' - P_w}{P_w} \times 100\%$$
$$= \frac{(228.57 - 213.66)}{213.66} \times 100\%$$
$$= 6.9874\%$$

$\%\ \text{inc.} = 6.98\%$ ◀

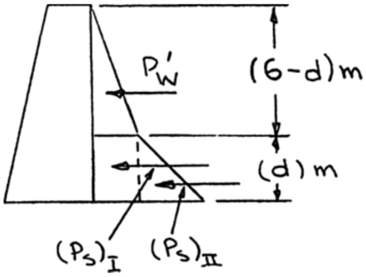


PROBLEM 5.83

The base of a dam for a lake is designed to resist up to 120 percent of the horizontal force of the water. After construction, it is found that silt (that is equivalent to a liquid of density $\rho_s = 1.76\times10^3\ \text{kg/m}^3$) is settling on the lake bottom at the rate of 12 mm/year. Considering a 1-m-wide section of dam, determine the number of years until the dam becomes unsafe.

SOLUTION

First determine force on dam without the silt,



$$\begin{aligned} P_w &= \frac{1}{2}A_{p_w} = \frac{1}{2}A(\rho g h) \\ &= \frac{1}{2}[(6.6\text{ m})(1\text{ m})][(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(6.6\text{ m})] \\ &= 213.66\text{ kN} \\ P_{\text{allow}} &= 1.2P_w = (1.5)(213.66\text{ kN}) = 256.39\text{ kN} \end{aligned}$$

Next determine the force P' on the dam face after a depth d of silt has settled.

We have
$$\begin{aligned} P_w' &= \frac{1}{2}[(6.6-d)\text{ m}\times(1\text{ m})][(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(6.6-d)\text{ m}] \\ &= 4.905(6.6-d)^2\text{ kN} \\ (P_s)_{\text{I}} &= [d(1\text{ m})][(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(6.6-d)\text{ m}] \\ &= 9.81(6.6d-d^2)\text{ kN} \\ (P_s)_{\text{II}} &= \frac{1}{2}[d(1\text{ m})][(1.76\times10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(d)\text{ m}] \\ &= 8.6328d^2\text{ kN} \end{aligned}$$

$$\begin{aligned} P' = P_w' + (P_s)_{\text{I}} + (P_s)_{\text{II}} &= [4.905(43.560 - 13.2000d + d^2) \\ &\quad + 9.81(6.6d - d^2) + 8.6328d^2]\text{ kN} \\ &= [3.7278d^2 + 213.66]\text{ kN} \end{aligned}$$

Now it's required that $P' = P_{\text{allow}}$ to determine the maximum value of d.

$$(3.7278d^2 + 213.66)\text{ kN} = 256.39\text{ kN}$$

or
$$d = 3.3856\text{ m}$$

Finally,
$$3.3856\text{ m} = 12\times10^{-3}\,\frac{\text{m}}{\text{year}}\times N$$
or $N = 282$ years ◀

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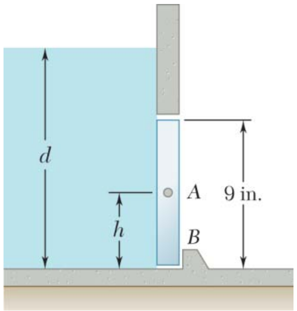


PROBLEM 5.84

An automatic valve consists of a 9 × 9-in. square plate that is pivoted about a horizontal axis through *A* located at a distance *h* = 3.6 in. above the lower edge. Determine the depth of water *d* for which the valve will open.

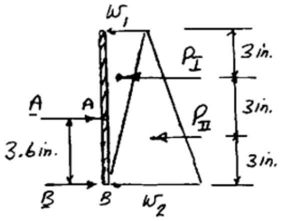
SOLUTION

Since valve is 9 in. wide, $w = 9p = 9\gamma h$, where all dimensions are in inches.

$$w_1 = 9\gamma(d-9), \quad w_2 = 9\gamma d$$

$$P_{\text{I}} = \frac{1}{2}(9\text{ in.})w_1 = \frac{1}{2}(9)(9\gamma)(d-9)$$

$$P_{\text{II}} = \frac{1}{2}(9\text{ in.})w_2 = \frac{1}{2}(9)(9\gamma d)$$



Valve opens when $B = 0$.

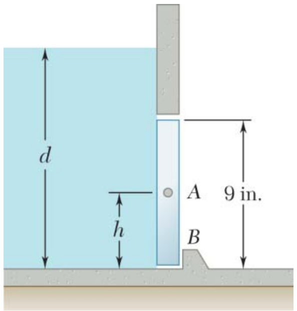
$$+\circlearrowleft \Sigma M_A = 0: \quad P_{\text{I}}(6\text{ in.} - 3.6\text{ in.}) - P_{\text{II}}(3.6\text{ in.} - 3\text{ in.}) = 0$$

$$\left[\frac{1}{2}(9)(9\gamma)(d-9)\right](2.4) - \left[\frac{1}{2}(9)(9\gamma d)\right](0.6) = 0$$

$$(d-9)(2.4) - d(0.6) = 0$$

$$1.8d - 21.6 = 0$$

$d = 12.00$ in. ◀



PROBLEM 5.85

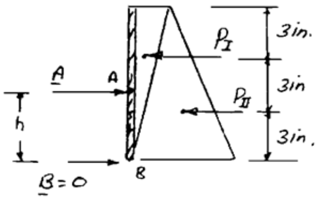
An automatic valve consists of a 9 × 9-in. square plate that is pivoted about a horizontal axis through *A*. If the valve is to open when the depth of water is $d = 18$ in., determine the distance *h* from the bottom of the valve to the pivot *A*.

SOLUTION

Since valve is 9 in. wide, $w = 9p = 9\gamma h$, where all dimensions are in inches.

$$w_1 = 9\gamma(d-9)$$
$$w_2 = 9\gamma d$$

For $d = 18$ in.,



$$w_1 = 9\gamma(18-9) = 81\gamma$$
$$w_2 = 9\gamma(18) = 162\gamma$$
$$P_I = \frac{1}{2}(9)(9\gamma)(18-9) = \frac{1}{2}(729\gamma)$$
$$P_{II} = \frac{1}{2}(9)(9\gamma)(18) = 729\gamma$$

Valve opens when $B = 0$.

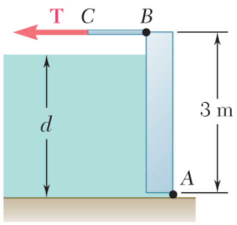
$$\circlearrowleft + \Sigma M_A = 0: \quad P_I(6-h) - P_{II}(h-3) = 0$$

$$\frac{1}{2}729\gamma(6-h) - 729(h-3) = 0$$

$$3 - \frac{1}{2}h - h + 3 = 0$$

$$6 - 1.5h = 0$$

$h = 4.00$ in. ◀

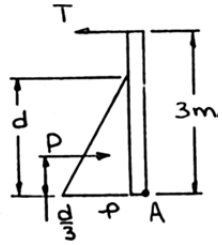


PROBLEM 5.86

The 3 × 4-m side *AB* of a tank is hinged at its bottom *A* and is held in place by a thin rod *BC*. The maximum tensile force the rod can withstand without breaking is 200 kN, and the design specifications require the force in the rod not to exceed 20 percent of this value. If the tank is slowly filled with water, determine the maximum allowable depth of water *d* in the tank.

SOLUTION

Consider the free-body diagram of the side.



We have $P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g d)$

Now $+\circlearrowleft \Sigma M_A = 0$: $hT - \frac{d}{3}P = 0$

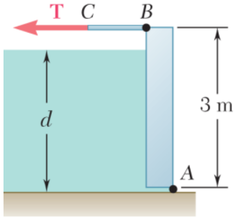
where $h = 3\text{ m}$

Then for d_{max},

$$(3\text{ m})(0.2 \times 200\times 10^3\text{N}) - \frac{d_{max}}{3}\left[\frac{1}{2}(4\text{ m}\times d_{max})\times(10^3\text{ kg/m}^3\times 9.81\text{ m/s}^2\times d_{max})\right] = 0$$

or $120\text{ N}\cdot\text{m} - 6.54 d_{max}^3\text{N/m}^2 = 0$

or $d_{max} = 2.64\text{ m}$ ◀

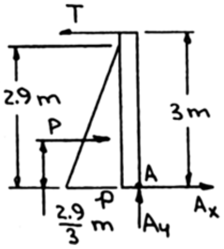


PROBLEM 5.87

The 3 × 4-m side of an open tank is hinged at its bottom *A* and is held in place by a thin rod *BC*. The tank is to be filled with glycerine, whose density is 1263 kg/m^3. Determine the force **T** in the rod and the reactions at the hinge after the tank is filled to a depth of 2.9 m.

SOLUTION

Consider the free-body diagram of the side.



We have $$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g d)$$

$$= \frac{1}{2}[(2.9\text{ m})(4\text{ m})]\,[(1263\text{ kg/m}^3)(9.81\text{ m/s}^2)(2.9\text{ m})]$$

$$= 208.40\text{ kN}$$

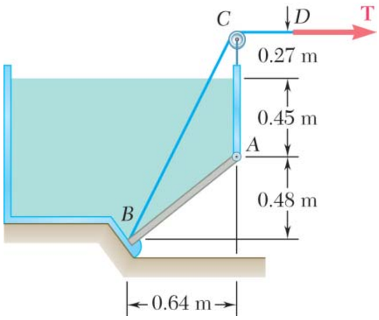
Then $$+\!\uparrow \Sigma F_y = 0: \quad A_y = 0$$

$$+\!\circlearrowleft \Sigma M_A = 0: \quad (3\text{ m})T - \left(\frac{2.9}{3}\text{m}\right)(208.4\text{ kN}) = 0$$

or $T = 67.151\text{ kN}$ $\quad$ $\mathbf{T} = 67.2\text{ kN} \longleftarrow$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + 208.40\text{ kN} - 67.151\text{ kN} = 0$$

or $A_x = -141.249\text{ kN}$ $\quad$ $\mathbf{A} = 141.2\text{ kN} \longleftarrow$ ◀



PROBLEM 5.88

A 0.5 × 0.8-m gate *AB* is located at the bottom of a tank filled with water. The gate is hinged along its top edge *A* and rests on a frictionless stop at *B*. Determine the reactions at *A* and *B* when cable *BCD* is slack.

SOLUTION

First consider the force of the water on the gate.

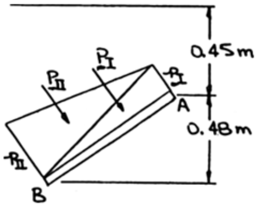
We have $P = \frac{1}{2}Ap = \frac{1}{2}A(\rho gh)$

so that $P_{\text{I}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})] \times [(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.45\text{ m})]$

$= 882.9\text{ N}$

$P_{\text{II}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})] \times [(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.93\text{ m})]$

$= 1824.66\text{ N}$

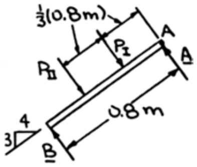


Reactions at *A* and *B* when *T* = 0:

We have

$$+\circlearrowright \Sigma M_A = 0: \quad \frac{1}{3}(0.8\text{ m})(882.9\text{ N}) + \frac{2}{3}(0.8\text{ m})(1824.66\text{ N}) - (0.8\text{ m})B = 0$$

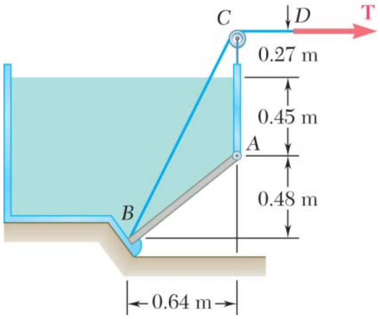
or $B = 1510.74\text{ N}$



or **B** $= 1511\text{ N} \measuredangle 53.1°$ ◀

$$+\nwarrow \Sigma F = 0: \quad A + 1510.74\text{ N} - 882.9\text{ N} - 1824.66\text{ N} = 0$$

or **A** $= 1197\text{ N} \measuredangle 53.1°$ ◀

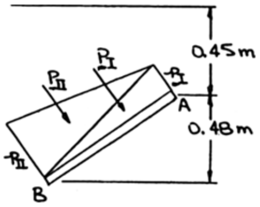


PROBLEM 5.89

A 0.5 × 0.8-m gate *AB* is located at the bottom of a tank filled with water. The gate is hinged along its top edge *A* and rests on a frictionless stop at *B*. Determine the minimum tension required in cable *BCD* to open the gate.

SOLUTION

First consider the force of the water on the gate.



We have $P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g h)$

so that $P_{\mathrm{I}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})] \times [(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.45\text{ m})]$

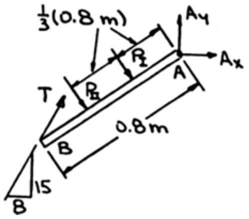
$= 882.9\text{ N}$

$P_{\mathrm{II}} = \frac{1}{2}[(0.5\text{ m})(0.8\text{ m})] \times [(10^3\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.93\text{ m})]$

$= 1824.66\text{ N}$

T to open gate:

First note that when the gate begins to open, the reaction at $B \longrightarrow 0$.

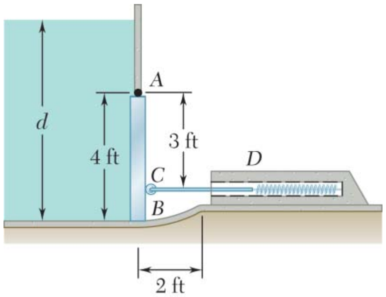


Then $+\circlearrowleft \Sigma M_A = 0$: $\frac{1}{3}(0.8\text{ m})(882.9\text{ N}) + \frac{2}{3}(0.8\text{ m})(1824.66\text{ N})$

$-(0.45 + 0.27)\text{m} \times \left(\frac{8}{17}T\right) = 0$

or $235.44 + 973.152 - 0.33882T = 0$

or $\mathbf{T} = 3570\text{ N}$ ◀

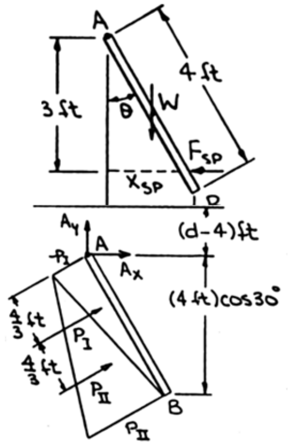


PROBLEM 5.90

A 4×2-ft gate is hinged at *A* and is held in position by rod *CD*. End *D* rests against a spring whose constant is 828 lb/ft. The spring is undeformed when the gate is vertical. Assuming that the force exerted by rod *CD* on the gate remains horizontal, determine the minimum depth of water *d* for which the bottom *B* of the gate will move to the end of the cylindrical portion of the floor.

SOLUTION

First determine the forces exerted on the gate by the spring and the water when *B* is at the end of the cylindrical portion of the floor.



We have $\sin\theta = \dfrac{2}{4} \quad \theta = 30°$

Then $x_{SP} = (3\text{ ft})\tan 30°$

and $F_{SP} = kx_{SP}$

Assume $d \geq 4$ ft

$= 828\text{ lb/ft} \times 3\text{ ft} \times \tan 30°$

$= 1434.14\text{ lb}$

We have $P = \dfrac{1}{2}Ap = \dfrac{1}{2}A(\gamma h)$

Then $P_{\text{I}} = \dfrac{1}{2}[(4\text{ ft})(2\text{ ft})] \times [(62.4\text{ lb/ft}^3)(d - 4)\text{ ft}]$

$= 249.6(d - 4)\text{ lb}$

$P_{\text{II}} = \dfrac{1}{2}[(4\text{ ft})(2\text{ ft})] \times [(62.4\text{ lb/ft}^3)(d - 4 + 4\cos 30°)]$

$= 249.6(d - 0.53590°)\text{ lb}$

For $d_{\min}$ so that the gate opens, $W = 0$

Using the above free-body diagrams of the gate, we have

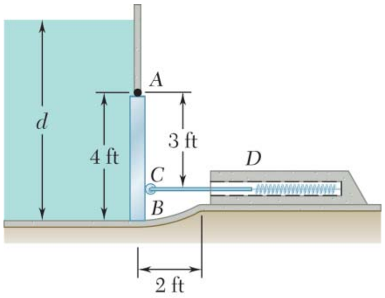
$$+\circlearrowleft \Sigma M_A = 0: \quad \left(\frac{4}{3}\text{ ft}\right)[249.6(d - 4)\text{ lb}] + \left(\frac{8}{3}\text{ ft}\right)[249.6(d - 0.53590)\text{ lb}]$$

$$-(3\text{ ft})(1434.14\text{ lb}) = 0$$

or $(332.8d - 1331.2) + (665.6d - 356.70) - 4302.4 = 0$

or $d = 6.00\text{ ft}$

$d \geq 4\text{ ft} \Rightarrow$ assumption correct $\qquad d = 6.00\text{ ft}$ ◀



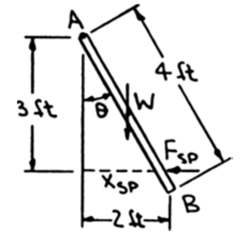
PROBLEM 5.91

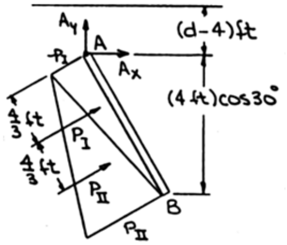
Solve Problem 5.90 if the gate weighs 1000 lb.

PROBLEM 5.90 A 4×2-ft gate is hinged at *A* and is held in position by rod *CD*. End *D* rests against a spring whose constant is 828 lb/ft. The spring is undeformed when the gate is vertical. Assuming that the force exerted by rod *CD* on the gate remains horizontal, determine the minimum depth of water *d* for which the bottom *B* of the gate will move to the end of the cylindrical portion of the floor.

SOLUTION

First determine the forces exerted on the gate by the spring and the water when *B* is at the end of the cylindrical portion of the floor.





We have $\sin\theta = \frac{2}{4} \quad \theta = 30°$

Then $x_{SP} = (3 \text{ ft})\tan 30°$

and $F_{SP} = kx_{SP} = 828 \text{ lb/ft} \times 3 \text{ ft} \times \tan 30°$

$= 1434.14 \text{ lb}$

Assume $d \geq 4 \text{ ft}$

We have $P = \frac{1}{2}Ap = \frac{1}{2}A(\gamma h)$

Then $P_{\text{I}} = \frac{1}{2}[(4 \text{ ft})(2 \text{ ft})] \times [(62.4 \text{ lb/ft}^3)(d-4) \text{ ft}]$

$= 249.6(d-4) \text{ lb}$

$P_{\text{II}} = \frac{1}{2}[(4 \text{ ft})(2 \text{ ft})] \times [(62.4 \text{ lb/ft}^3)(d - 4 + 4\cos 30°)]$

$= 249.6(d - 0.53590°) \text{ lb}$

For $d_{\min}$ so that the gate opens, $W = 1000 \text{ lb}$

Using the above free-body diagrams of the gate, we have

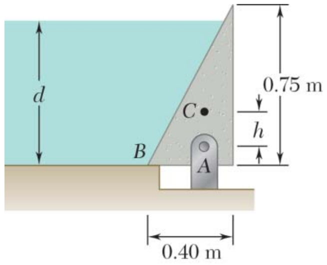
$$\circlearrowright + \Sigma M_A = 0: \left(\frac{4}{3} \text{ ft}\right)[249.6(d-4) \text{ lb}] + \left(\frac{8}{3} \text{ ft}\right)[249.6(d - 0.53590) \text{ lb}]$$

$$-(3 \text{ ft})(1434.14 \text{ lb}) - (1 \text{ ft})(1000 \text{ lb}) = 0$$

or $(332.8d - 1331.2) + (665.6d - 356.70) - 4302.4 - 1000 = 0$

or $d = 7.00 \text{ ft}$

$d \geq 4 \text{ ft} \Rightarrow$ assumption correct $d = 7.00 \text{ ft}$ ◀

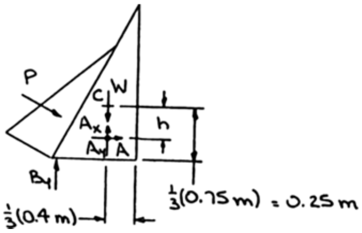

d
0.75 m
C
h
B
A
0.40 m

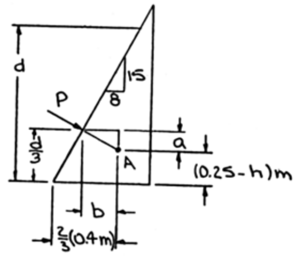
PROBLEM 5.92

A prismatically shaped gate placed at the end of a freshwater channel is supported by a pin and bracket at *A* and rests on a frictionless support at *B*. The pin is located at a distance $h = 0.10$ m below the center of gravity *C* of the gate. Determine the depth of water *d* for which the gate will open.

SOLUTION

First note that when the gate is about to open (clockwise rotation is impending), $B_y \longrightarrow 0$ and the line of action of the resultant **P** of the pressure forces passes through the pin at *A*. In addition, if it is assumed that the gate is homogeneous, then its center of gravity *C* coincides with the centroid of the triangular area. Then


P
C
W
A_x
A_y
A
h
B_y
$\frac{1}{3}$(0.4 m)
$\frac{1}{3}$(0.75 m) = 0.25 m


d
15
8
P
$\frac{d}{3}$
a
A
(0.25 − h)m
b
$\frac{2}{3}$(0.4 m)

$$a = \frac{d}{3} - (0.25 - h)$$

and $$b = \frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)$$

Now $$\frac{a}{b} = \frac{8}{15}$$

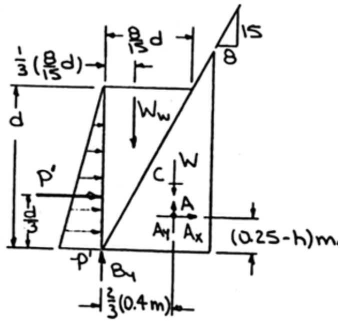
so that $$\frac{\frac{d}{3} - (0.25 - h)}{\frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)} = \frac{8}{15}$$

Simplifying yields

$$\frac{289}{45}d + 15h = \frac{70.6}{12} \qquad (1)$$

Alternative solution:

Consider a free body consisting of a 1-m thick section of the gate and the triangular section *BDE* of water above the gate.


$\frac{8}{15}$d
$\frac{1}{3}(\frac{8}{15}$d)
15
8
d
W_w
C
W
P'
A
$\frac{d}{3}$
A_y
A_x
(0.25 − h)m
−P'
B_y
$\frac{2}{3}$(0.4 m)

Now $$P' = \frac{1}{2}Ap' = \frac{1}{2}(d \times 1\text{ m})(\rho g d)$$

$$= \frac{1}{2}\rho g d^2 \text{ (N)}$$

$$W' = \rho g V = \rho g\left(\frac{1}{2} \times \frac{8}{15}d \times d \times 1\text{ m}\right)$$

$$= \frac{4}{15}\rho g d^2 \text{ (N)}$$

PROBLEM 5.92 (Continued)

Then with $B_y = 0$ (as explained above), we have

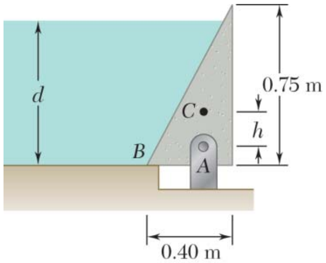
$$\overset{+}{\circlearrowleft}\Sigma M_A = 0:\quad \left[\frac{2}{3}(0.4) - \frac{1}{3}\left(\frac{8}{15}d\right)\right]\left(\frac{4}{15}\rho g d^2\right) - \left[\frac{d}{3} - (0.25 - h)\right]\left(\frac{1}{2}\rho g d^2\right) = 0$$

Simplifying yields $\dfrac{289}{45}d + 15h = \dfrac{70.6}{12}$

as above.

Find *d*: $h = 0.10\text{ m}$

Substituting into Eq. (1), $\dfrac{289}{45}d + 15(0.10) = \dfrac{70.6}{12}$ or $d = 0.683\text{ m}$ ◀

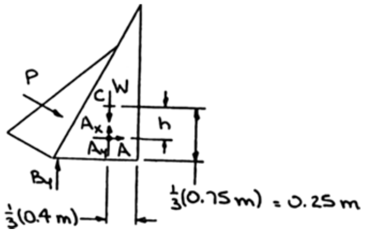


PROBLEM 5.93

A prismatically shaped gate placed at the end of a freshwater channel is supported by a pin and bracket at *A* and rests on a frictionless support at *B*. Determine the distance *h* if the gate is to open when $d = 0.75$ m.

SOLUTION

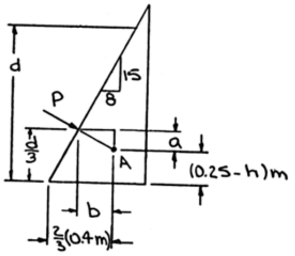
First note that when the gate is about to open (clockwise rotation is impending), $B_y \longrightarrow 0$ and the line of action of the resultant **P** of the pressure forces passes through the pin at *A*. In addition, if it is assumed that the gate is homogeneous, then its center of gravity *C* coincides with the centroid of the triangular area. Then



$$a = \frac{d}{3} - (0.25 - h)$$

and $$b = \frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)$$

Now $$\frac{a}{b} = \frac{8}{15}$$



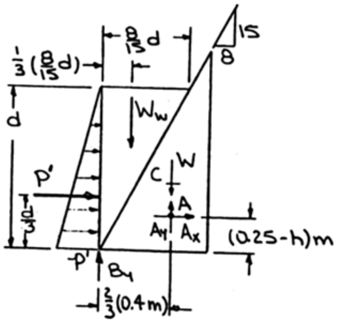
so that $$\frac{\frac{d}{3} - (0.25 - h)}{\frac{2}{3}(0.4) - \frac{8}{15}\left(\frac{d}{3}\right)} = \frac{8}{15}$$

Simplifying yields

$$\frac{289}{45}d + 15h = \frac{70.6}{12} \tag{1}$$

Alternative solution:

Consider a free body consisting of a 1-m thick section of the gate and the triangular section *BDE* of water above the gate.



Now $$P' = \frac{1}{2}Ap' = \frac{1}{2}(d \times 1\text{ m})(\rho g d)$$
$$= \frac{1}{2}\rho g d^2 \text{ (N)}$$

$$W' = \rho g V = \rho g\left(\frac{1}{2} \times \frac{8}{15}d \times d \times 1\text{ m}\right)$$
$$= \frac{4}{15}\rho g d^2 \text{ (N)}$$

PROBLEM 5.93 (Continued)

Then with $B_y = 0$ (as explained above), we have

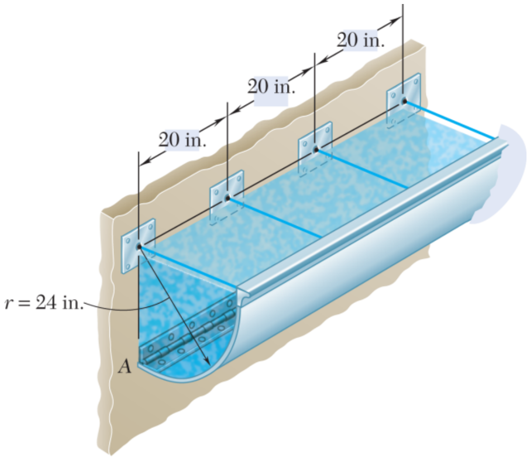
$$\overset{+}{\curvearrowleft}\Sigma M_A = 0: \quad \left[\frac{2}{3}(0.4) - \frac{1}{3}\left(\frac{8}{15}d\right)\right]\left(\frac{4}{15}\rho g d^2\right) - \left[\frac{d}{3} - (0.25 - h)\right]\left(\frac{1}{2}\rho g d^2\right) = 0$$

Simplifying yields $\quad \dfrac{289}{45}d + 15h = \dfrac{70.6}{12}$

as above.

Find *h*: $\quad d = 0.75 \text{ m}$

Substituting into Eq. (1), $\quad \dfrac{289}{45}(0.75) + 15h = \dfrac{70.6}{12} \quad$ or $\quad h = 0.0711 \text{ m}$ ◀


20 in.
20 in.
20 in.
$r = 24$ in.
A

PROBLEM 5.94

A long trough is supported by a continuous hinge along its lower edge and by a series of horizontal cables attached to its upper edge. Determine the tension in each of the cables, at a time when the trough is completely full of water.

SOLUTION

Consider free body consisting of 20-in. length of the trough and water.

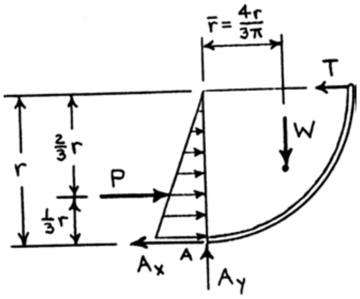
$$W = \gamma V = \gamma\left[\frac{\pi}{4}r^2 l\right]$$

l = 20-in. length of free body

$$P_A = \gamma r$$

$$P = \frac{1}{2}P_A r l = \frac{1}{2}(\gamma r) r l = \frac{1}{2}\gamma r^2 l$$

$$+\circlearrowleft \Sigma M_A = 0: \quad Tr - Wr - P\left(\frac{1}{3}r\right) = 0$$



$$Tr - \left(\gamma\frac{\pi}{4}r^2 l\right)\left(\frac{4r}{3\pi}\right) - \left(\frac{1}{2}\gamma r^2 l\right)\left(\frac{1}{3}r\right) = 0$$

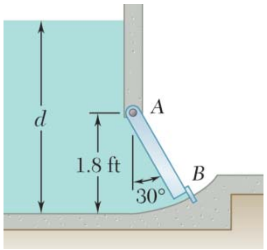
$$T = \frac{1}{3}\gamma r^2 l + \frac{1}{6}\gamma r^2 l = \frac{1}{2}\gamma r^2 l$$

Data: $\gamma = 62.4 \text{ lb/ft}^3 \quad r = \frac{24}{12} \text{ ft} = 2 \text{ ft} \quad l = \frac{20}{12} \text{ ft}$

Then $T = \frac{1}{2}(62.4 \text{ lb/ft}^3)(2 \text{ ft})^2\left(\frac{20}{12} \text{ ft}\right)$

$= 208.00$ lb

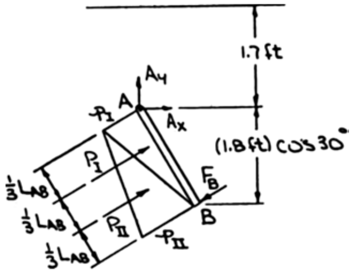
$T = 208$ lb ◀



PROBLEM 5.95

The square gate *AB* is held in the position shown by hinges along its top edge *A* and by a shear pin at *B*. For a depth of water $d = 3.5$ ft, determine the force exerted on the gate by the shear pin.

SOLUTION


1.7 ft
A_y
A
A_x
P_I
(1.8 ft) cos 30°
F_B
B
P_I
P_{II}
P_{II}
$\frac{1}{3}L_{AB}$
$\frac{1}{3}L_{AB}$
$\frac{1}{3}L_{AB}$

First consider the force of the water on the gate. We have

$$P = \frac{1}{2}Ap$$

$$= \frac{1}{2}A(\gamma h)$$

Then $P_{\text{I}} = \frac{1}{2}(1.8 \text{ ft})^2(62.4 \text{ lb/ft}^3)(1.7 \text{ ft})$

$$= 171.850 \text{ lb}$$

$$P_{\text{II}} = \frac{1}{2}(1.8 \text{ ft})^2(62.4 \text{ lb/ft}^3) \times (1.7 + 1.8\cos 30°) \text{ ft}$$

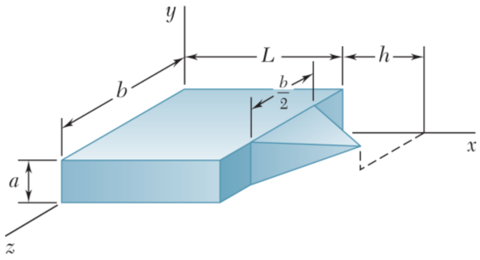
$$= 329.43 \text{ lb}$$

Now $\Sigma M_A = 0\text{:} \quad \left(\frac{1}{3}L_{AB}\right)P_{\text{I}} + \left(\frac{2}{3}L_{AB}\right)P_{\text{II}} - L_{AB}F_B = 0$

or $\frac{1}{3}(171.850 \text{ lb}) + \frac{2}{3}(329.43 \text{ lb}) - F_B = 0$

or $F_B = 276.90 \text{ lb}$ $\qquad$ $\mathbf{F}_B = 277 \text{ lb}$ ↗ 30.0° ◀

PROBLEM 5.96



Consider the composite body shown. Determine (*a*) the value of $\bar{x}$ when $h = L/2$, (*b*) the ratio h/L for which $\bar{x} = L$.

SOLUTION

	V	$\bar{x}$	$\bar{x}V$
Rectangular prism	Lab	$\frac{1}{2}L$	$\frac{1}{2}L^2ab$
Pyramid	$\frac{1}{3}a\left(\frac{b}{2}\right)h$	$L+\frac{1}{4}h$	$\frac{1}{6}abh\left(L+\frac{1}{4}h\right)$

Then $\quad \Sigma V = ab\left(L+\frac{1}{6}h\right)$

$$\Sigma\bar{x}V = \frac{1}{6}ab\left[3L^2 + h\left(L+\frac{1}{4}h\right)\right]$$

Now $$\bar{X}\Sigma V = \Sigma\bar{x}V$$

so that $$\bar{X}\left[ab\left(L+\frac{1}{6}h\right)\right] = \frac{1}{6}ab\left(3L^2 + hL + \frac{1}{4}h^2\right)$$

or $$\bar{X}\left(1+\frac{1}{6}\frac{h}{L}\right) = \frac{1}{6}L\left(3+\frac{h}{L}+\frac{1}{4}\frac{h^2}{L^2}\right) \qquad (1)$$

(*a*) $\quad \bar{X} = ?$ when $h = \frac{1}{2}L.$

Substituting $\frac{h}{L} = \frac{1}{2}$ into Eq. (1),

$$\bar{X}\left[1+\frac{1}{6}\left(\frac{1}{2}\right)\right] = \frac{1}{6}L\left[3+\left(\frac{1}{2}\right)+\frac{1}{4}\left(\frac{1}{2}\right)^2\right]$$

or $$\bar{X} = \frac{57}{104}L$$

$\bar{X} = 0.548L$ ◀

PROBLEM 5.96 (Continued)

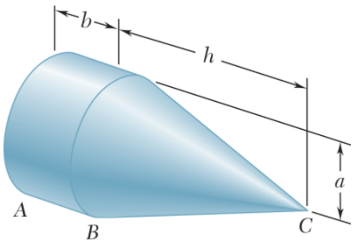
(b) $\frac{h}{L} = ?$ when $\bar{X} = L.$

Substituting into Eq. (1), $L\left(1+\frac{1}{6}\frac{h}{L}\right)=\frac{1}{6}L\left(3+\frac{h}{L}+\frac{1}{4}\frac{h^2}{L^2}\right)$

or $$1+\frac{1}{6}\frac{h}{L}=\frac{1}{2}+\frac{1}{6}\frac{h}{L}+\frac{1}{24}\frac{h^2}{L^2}$$

or $$\frac{h^2}{L^2}=12$$

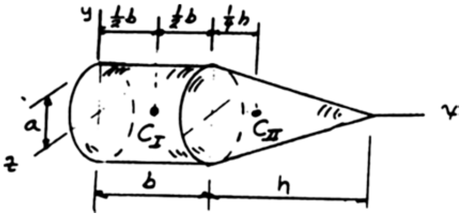
$\frac{h}{L}=2\sqrt{3}$ ◀



PROBLEM 5.97

Determine the location of the centroid of the composite body shown when (*a*) $h = 2b$, (*b*) $h = 2.5b$.

SOLUTION



	V	$\bar{x}$	$\bar{x}V$
Cylinder I	$\pi a^2 b$	$\frac{1}{2}b$	$\frac{1}{2}\pi a^2 b^2$
Cone II	$\frac{1}{3}\pi a^2 h$	$b+\frac{1}{4}h$	$\frac{1}{3}\pi a^2 h\left(b+\frac{1}{4}h\right)$

$$V = \pi a^2\left(b + \frac{1}{3}h\right)$$

$$\Sigma\bar{x}V = \pi a^2\left(\frac{1}{2}b^2 + \frac{1}{3}hb + \frac{1}{12}h^2\right)$$

(*a*) For $h = 2b$,

$$V = \pi a^2\left[b + \frac{1}{3}(2b)\right] = \frac{5}{3}\pi a^2 b$$

$$\Sigma\bar{x}V = \pi a^2\left[\frac{1}{2}b^2 + \frac{1}{3}(2b)b + \frac{1}{12}(2b)^2\right]$$

$$= \pi a^2 b^2\left[\frac{1}{2} + \frac{2}{3} + \frac{1}{3}\right] = \frac{3}{2}\pi a^2 b^2$$

$$\bar{X}V = \Sigma\bar{x}V: \quad \bar{X}\left(\frac{5}{3}\pi a^2 b\right) = \frac{3}{2}\pi a^2 b^2 \quad \bar{X} = \frac{9}{10}b$$

Centroid is $\frac{1}{10}b$ to the left of base of cone. ◀

PROBLEM 5.97 (Continued)

(*b*) For $h = 2.5b,$

$$V = \pi a^2\left[b + \frac{1}{3}(2.5b)\right] = 1.8333\pi a^2 b$$

$$\Sigma\bar{x}V = \pi a^2\left[\frac{1}{2}b^2 + \frac{1}{3}(2.5b)b + \frac{1}{12}(2.5b)^2\right]$$

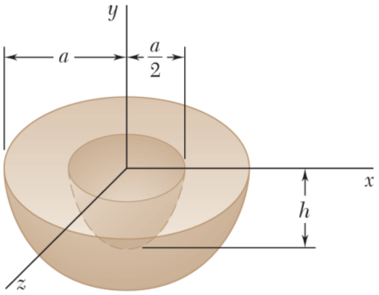
$$= \pi a^2 b^2[0.5 + 0.8333 + 0.52083]$$

$$= 1.85416\pi a^2 b^2$$

$$\bar{X}V = \Sigma\bar{x}V: \quad \bar{X}(1.8333\pi a^2 b) = 1.85416\pi a^2 b^2 \quad \bar{X} = 1.01136b$$

Centroid is 0.01136b to the right of base of cone. ◀

Note: Centroid is at base of cone for $h = \sqrt{6}b = 2.449b.$



PROBLEM 5.98

The composite body shown is formed by removing a semiellipsoid of revolution of semimajor axis h and semiminor axis $a/2$ from a hemisphere of radius a. Determine (*a*) the y coordinate of the centroid when $h = a/2$, (*b*) the ratio h/a for which $\bar{y} = -0.4a$.

SOLUTION

	V	$\bar{y}$	$\bar{y}V$
Hemisphere	$\frac{2}{3}\pi a^3$	$-\frac{3}{8}a$	$-\frac{1}{4}\pi a^4$
Semiellipsoid	$-\frac{2}{3}\pi\left(\frac{a}{2}\right)^2 h = -\frac{1}{6}\pi a^2 h$	$-\frac{3}{8}h$	$+\frac{1}{16}\pi a^2 h^2$

Then
$$\Sigma V = \frac{\pi}{6}a^2(4a - h)$$
$$\Sigma \bar{y}V = -\frac{\pi}{16}a^2(4a^2 - h^2)$$

Now
$$\bar{Y}\Sigma V = \Sigma \bar{y}V$$

So that
$$\bar{Y}\left[\frac{\pi}{6}a^2(4a-h)\right] = -\frac{\pi}{16}a^2(4a^2 - h^2)$$

or
$$\bar{Y}\left(4 - \frac{h}{a}\right) = -\frac{3}{8}a\left[4 - \left(\frac{h}{a}\right)^2\right] \qquad (1)$$

(*a*)
$$\bar{Y} = ? \quad \text{when} \quad h = \frac{a}{2}$$

Substituting $\frac{h}{a} = \frac{1}{2}$ into Eq. (1)

$$\bar{Y}\left(4 - \frac{1}{2}\right) = -\frac{3}{8}a\left[4 - \left(\frac{1}{2}\right)^2\right]$$

or
$$\bar{Y} = -\frac{45}{112}a$$

$\bar{Y} = -0.402a$ ◀

(*b*) $$\frac{h}{a} = ? \quad \text{when} \quad \overline{Y} = -0.4a$$

Substituting into Eq. (1)

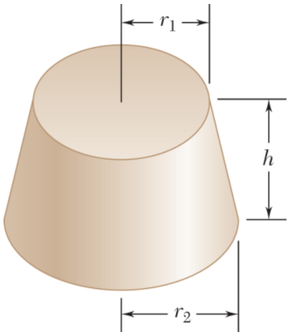
$$(-0.4a)\left(4 - \frac{h}{a}\right) = -\frac{3}{8}a\left[4 - \left(\frac{h}{a}\right)^2\right]$$

or $$3\left(\frac{h}{a}\right)^2 - 3.2\left(\frac{h}{a}\right) + 0.8 = 0$$

Then $$\frac{h}{a} = \frac{3.2 \pm \sqrt{(-3.2)^2 - 4(3)(0.8)}}{2(3)}$$

$$= \frac{3.2 \pm 0.8}{6}$$

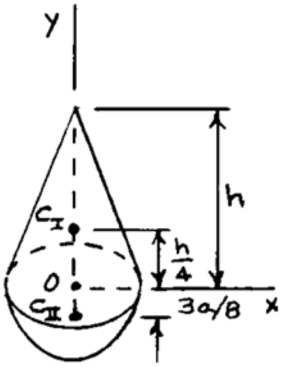
or $\frac{h}{a} = \frac{2}{5}$ and $\frac{h}{a} = \frac{2}{3}$ ◀



PROBLEM 5.99

Locate the centroid of the frustum of a right circular cone when $r_1 = 40$ mm, $r_2 = 50$ mm, and $h = 60$ mm.

SOLUTION



By similar triangles:

$$\frac{h_1}{40} = \frac{h_1 + 60}{50} \qquad h_1 = 240 \text{ mm, therefore } h_2 = 240 + 60 = 300 \text{ mm}$$

$$\bar{y}_1 = \frac{1}{4}h_2 = \frac{300}{4} = 75 \text{ mm} \qquad \bar{y}_2 = 60 + \frac{1}{4}h_1 = 60 + \frac{240}{4} = 120 \text{ mm}$$

$$V_1 = \frac{\pi}{3}(50)^2(300) = 250 \times 10^3 \pi \qquad V_2 = \frac{\pi}{3}(40)^2(240) = 128 \times 10^3 \pi$$

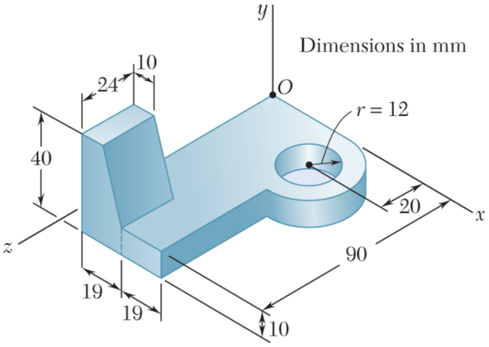
	V mm^3	$\bar{y}$ mm	$\bar{y}V$ mm^4
Triangle 1	$250 \times 10^3 \pi$	75	$18.75 \times 10^6 \pi$
Triangle 2	$-128 \times 10^3 \pi$	120	$-15.36 \times 10^6 \pi$
Σ	$122 \times 10^3 \pi$		$3.39 \times 10^6 \pi$

$$\bar{Y}\Sigma V = \Sigma \bar{y} V$$

$$\bar{Y}\left(122 \times 10^3 \pi\right) = 3.39 \times 10^6 \pi$$

$$\bar{Y} = 27.787 \text{ mm}$$

Centroid is 27.8 mm above base of cone. ◀

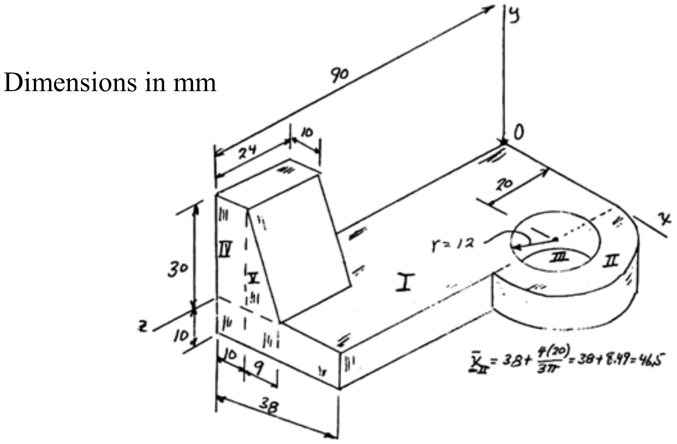


y
Dimensions in mm
10
24
O
r = 12
40
20
x
z
90
19
19
10

PROBLEM 5.100

For the machine element shown, locate the *x* coordinate of the center of gravity.

SOLUTION



Dimensions in mm
90
y
24
10
O
20
x
r = 12
III
II
I
IV
V
30
z
10
10
9
38

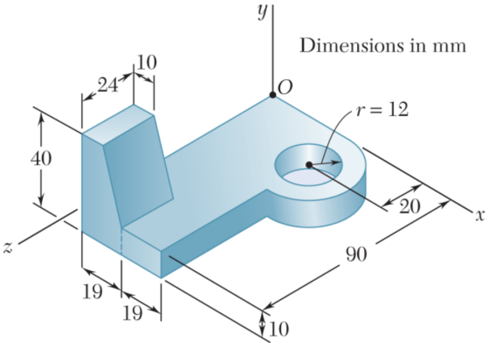
$\bar{x}_{II} = 38 + \frac{4(20)}{3\pi} = 38 + 8.49 = 46.5$

		V, mm^3	$\bar{x}$, mm	$\bar{z}$, mm	$\bar{x}V$, mm^4	$\bar{z}V$, mm^4
I	Rectangular plate	$(10)(90)(38) = 34.2 \times 10^3$	19	45	649.8×10^3	1539×10^3
II	Half cylinder	$\frac{\pi}{2}(20)^2(10) = 6.2832 \times 10^3$	46.5	20	292.17×10^3	125.664×10^3
III	–(Cylinder)	$-\pi(12)^2(10) = -4.5239 \times 10^3$	38	20	-171.908×10^3	-90.478×10^3
IV	Rectangular prism	$(30)(10)(24) = 7.2 \times 10^3$	5	78	36×10^3	561.6×10^3
V	Triangular prism	$\frac{1}{2}(30)(9)(24) = 3.24 \times 10^3$	13	78	42.12×10^3	252.72×10^3
	Σ	46.399×10^3			848.18×10^3	2388.5×10^3

PROBLEM 5.100 (Continued)

$$\bar{X}\Sigma V = \Sigma \bar{x} V$$

$$\bar{X} = \frac{\Sigma \bar{x} V}{\Sigma V} = \frac{848.18 \times 10^3 \text{ mm}^4}{46.399 \times 10^3 \text{ mm}^3} \qquad \bar{X} = 18.28 \text{ mm} \blacktriangleleft$$

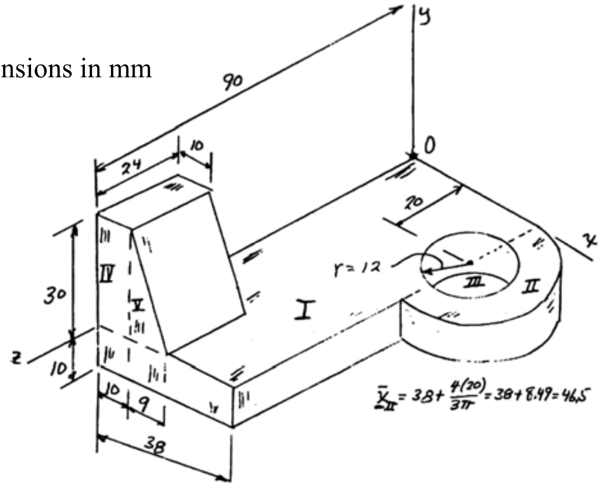

y
Dimensions in mm
O
r = 12
24
10
40
20
x
z
90
19
19
10

PROBLEM 5.101

For the machine element shown, locate the *z* coordinate of the center of gravity.

SOLUTION

Dimensions in mm


y
90
24
10
O
20
x
r = 12
30
10
z
10
9
38
I
II
III
IV
V

$\bar{x}_{II} = 38 + \frac{4(20)}{3\pi} = 38 + 8.49 = 46.5$

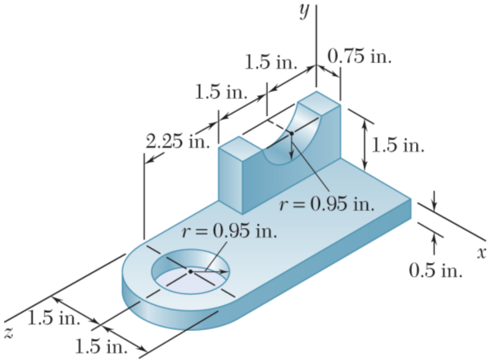
		V, mm^3	$\bar{x}$, mm	$\bar{z}$, mm	$\bar{x}V$, mm^4	$\bar{z}V$, mm^4
I	Rectangular plate	$(10)(90)(38) = 34.2 \times 10^3$	19	45	649.8×10^3	1539×10^3
II	Half cylinder	$\frac{\pi}{2}(20)^2(10) = 6.2832 \times 10^3$	46.5	20	292.17×10^3	125.664×10^3
III	–(Cylinder)	$-\pi(12)^2(10) = -4.5239 \times 10^3$	38	20	-171.908×10^3	-90.478×10^3
IV	Rectangular prism	$(30)(10)(24) = 7.2 \times 10^3$	5	78	36×10^3	561.6×10^3
V	Triangular prism	$\frac{1}{2}(30)(9)(24) = 3.24 \times 10^3$	13	78	42.12×10^3	252.72×10^3
	Σ	46.399×10^3			848.18×10^3	2388.5×10^3

PROBLEM 5.101 (Continued)

$$\bar{Z}\Sigma V = \Sigma \bar{z} V$$

$$\bar{Z} = \frac{\Sigma \bar{z} V}{\Sigma V} = \frac{2388.5 \times 10^3 \text{ mm}^4}{46.399 \times 10^3 \text{ mm}^3}$$

$\bar{Z} = 51.5 \text{ mm}$ ◀



PROBLEM 5.102

For the machine element shown, locate the y coordinate of the center of gravity.

SOLUTION

For half-cylindrical hole,

$$r = 0.95 \text{ in.}$$

$$\bar{y}_{\text{III}} = 1.5 - \frac{4(0.95)}{3\pi}$$

$$= 1.097 \text{ in.}$$

For half-cylindrical plate, $r = 1.5$ in.

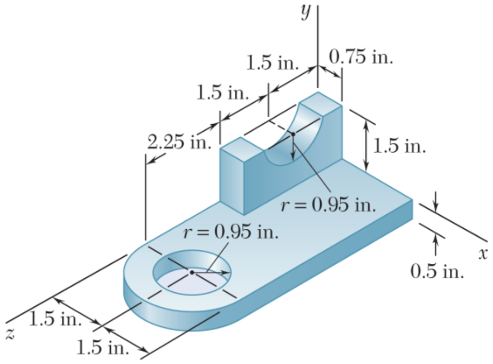
$$\bar{z}_{\text{IV}} = 5.25 + \frac{4(1.5)}{3\pi} = 5.887 \text{ in.}$$

		V, in^3	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{y}V$, in^4	$\bar{z}V$, in^4
I	Rectangular plate	$(5.25)(3)(0.5) = 7.875$	–0.25	2.625	–1.9688	20.672
II	Rectangular plate	$(3)(1.5)(0.75) = 3.375$	0.75	1.5	2.5313	5.0625
III	–(Half cylinder)	$-\frac{\pi}{2}(0.95)^2(0.75) = -1.063$	1.097	1.5	–1.1664	–1.595
IV	Half cylinder	$\frac{\pi}{2}(1.5)^2(0.5) = 1.767$	–0.25	5.887	–0.4418	10.403
V	–(Cylinder)	$-\pi(0.95)^2(0.5) = -1.418$	–0.25	5.25	0.3544	–7.443
	Σ	10.536			–0.6910	27.10

$$\bar{Y}\Sigma V = \Sigma \bar{y} V$$

$$\bar{Y}(10.536 \text{ in}^3) = -0.6910 \text{ in}^4$$

$\bar{Y} = -0.0656$ in. ◀



PROBLEM 5.103

For the machine element shown, locate the z coordinate of the center of gravity.

SOLUTION

For half-cylindrical hole,

$$r = 0.95 \text{ in.}$$

$$\bar{y}_{\text{III}} = 1.5 - \frac{4(0.95)}{3\pi}$$

$$= 1.097 \text{ in.}$$

For half-cylindrical plate, $r = 1.5$ in.

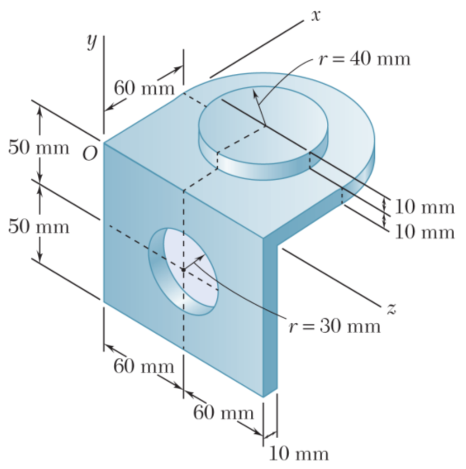
$$\bar{z}_{\text{IV}} = 5.25 + \frac{4(1.5)}{3\pi} = 5.887 \text{ in.}$$

		V, in^3	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{y}V$, in^4	$\bar{z}V$, in^4
I	Rectangular plate	$(5.25)(3)(0.5) = 7.875$	−0.25	2.625	−1.9688	20.672
II	Rectangular plate	$(3)(1.5)(0.75) = 3.375$	0.75	1.5	2.5313	5.0625
III	–(Half cylinder)	$-\frac{\pi}{2}(0.95)^2(0.75) = -1.063$	1.097	1.5	−1.1664	−1.595
IV	Half cylinder	$\frac{\pi}{2}(1.5)^2(0.5) = 1.767$	−0.25	5.887	−0.4418	10.403
V	–(Cylinder)	$-\pi(0.95)^2(0.5) = -1.418$	−0.25	5.25	0.3544	−7.443
	Σ	10.536			−0.6910	27.10

Now $$\bar{Z}\Sigma V = \bar{z}V$$

$$\bar{Z}(10.536 \text{ in}^3) = 27.10 \text{ in}^4$$

$\bar{Z} = 2.57$ in. ◀

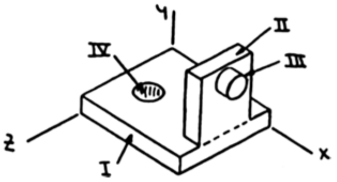


PROBLEM 5.104

For the machine element shown, locate the *y* coordinate of the center of gravity.

SOLUTION

First assume that the machine element is homogeneous so that its center of gravity will coincide with the centroid of the corresponding volume.

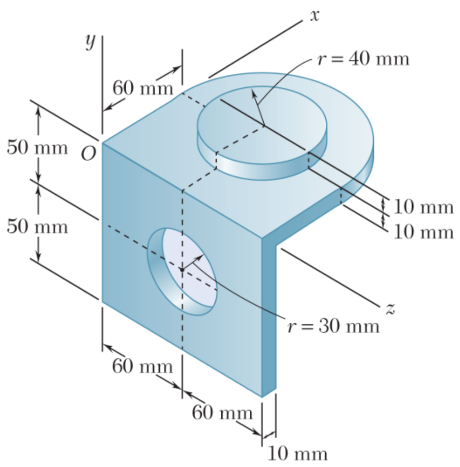


	V, mm^3	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}V$, mm^4	$\bar{y}V$, mm^4
I	$(120)(100)(10) = 120\times10^3$	5	-50	0.60×10^6	-6.00×10^6
II	$(120)(50)(10) = 60\times10^3$	35	-5	2.10×10^6	-0.30×10^6
III	$\frac{\pi}{2}(60)^2(10) = 56.549\times10^3$	85.5	-5	4.8349×10^6	-0.28274×10^6
IV	$\pi(40)^2(10) = 50.266\times10^3$	60	5	3.0160×10^6	0.25133×10^6
V	$-\pi\left(30\right)^2\left(10\right) = -28.274\times10^3$	5	-50	-0.141370×10^6	1.41370×10^6
Σ	258.54×10^3			10.4095×10^6	-4.9177×10^6

We have

$$\bar{Y}\Sigma V = \Sigma\bar{y}V$$

$$\bar{Y}(258.54\times10^3\ \text{mm}^3) = -4.9177\times10^6\ \text{mm}^4 \qquad \text{or} \quad \bar{Y} = -19.02\ \text{mm} \blacktriangleleft$$

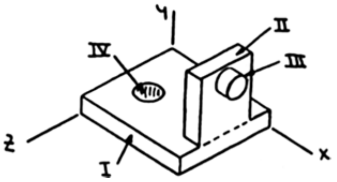


PROBLEM 5.105

For the machine element shown, locate the *x* coordinate of the center of gravity.

SOLUTION

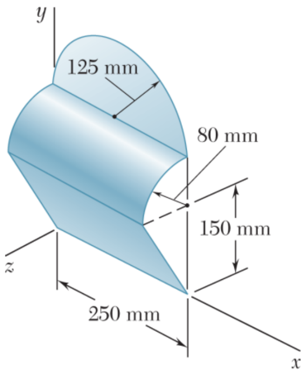
First assume that the machine element is homogeneous so that its center of gravity will coincide with the centroid of the corresponding volume.



	V, mm^3	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}V$, mm^4	$\bar{y}V$, mm^4
I	$(120)(100)(10) = 120\times10^3$	5	-50	0.60×10^6	-6.00×10^6
II	$(120)(50)(10) = 60\times10^3$	35	-5	2.10×10^6	-0.30×10^6
III	$\frac{\pi}{2}(60)^2(10) = 56.549\times10^3$	85.5	-5	4.8349×10^6	-0.28274×10^6
IV	$\pi(40)^2(10) = 50.266\times10^3$	60	5	3.0160×10^6	0.25133×10^6
V	$-\pi(30)^2(10) = -28.274\times10^3$	5	-50	-0.141370×10^6	1.41370×10^6
Σ	258.54×10^3			10.4095×10^6	-4.9177×10^6

We have $\bar{X}\Sigma V = \Sigma\bar{x}V$

$\bar{Y}(258.54\times10^3 \text{ mm}^3) = 10.4095\times10^6 \text{ mm}^4$ or $\bar{X} = 40.3$ mm ◀



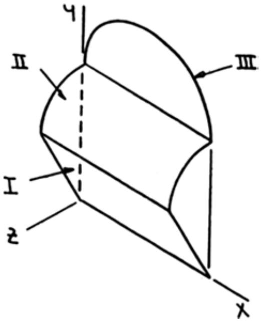
PROBLEM 5.106

Locate the center of gravity of the sheet-metal form shown.

SOLUTION

First assume that the sheet metal is homogeneous so that the center of gravity of the form will coincide with the centroid of the corresponding area. Now note that symmetry implies

$$\overline{X} = 125.0 \text{ mm} \blacktriangleleft$$

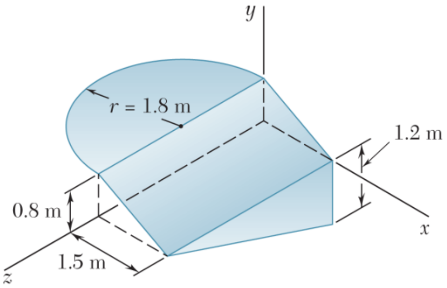


$$\overline{y}_{\text{II}} = 150 + \frac{2\times 80}{\pi}$$
$$= 200.93 \text{ mm}$$
$$\overline{z}_{\text{II}} = \frac{2\times 80}{\pi}$$
$$= 50.930 \text{ mm}$$
$$\overline{y}_{\text{III}} = 230 + \frac{4\times 125}{3\pi}$$
$$= 283.05 \text{ mm}$$

	A, mm^2	$\overline{y}$, mm	$\overline{z}$, mm	$\overline{y}A$, mm^3	$\overline{z}A$, mm^3
I	$(250)(170) = 42500$	75	40	3187500	1700000
II	$\frac{\pi}{2}(80)(250) = 31416$	200.93	50930	6312400	1600000
III	$\frac{\pi}{2}(125)^2 = 24544$	283.05	0	6947200	0
Σ	98460			16447100	3300000

We have $\overline{Y}\Sigma A = \Sigma\overline{y}A$: $\overline{Y}(98460 \text{ mm}^2) = 16447100 \text{ mm}^3$ or $\overline{Y} = 167.0$ mm ◀

$\overline{Z}\Sigma A = \Sigma\overline{z}A$: $\overline{Z}(98460 \text{ mm}^2) = 3.300\times 10^6 \text{ mm}^3$ or $\overline{Z} = 33.5$ mm ◀

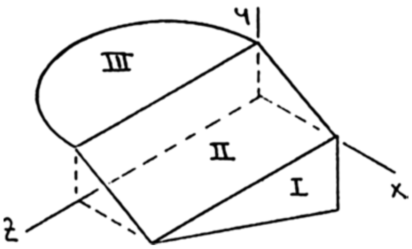


PROBLEM 5.107

Locate the center of gravity of the sheet-metal form shown.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the form will coincide with the centroid of the corresponding area.



$$\bar{y}_{\text{I}} = -\frac{1}{3}(1.2) = -0.4 \text{ m}$$

$$\bar{z}_{\text{I}} = \frac{1}{3}(3.6) = 1.2 \text{ m}$$

$$\bar{x}_{\text{III}} = -\frac{4(1.8)}{3\pi} = -\frac{2.4}{\pi} \text{ m}$$

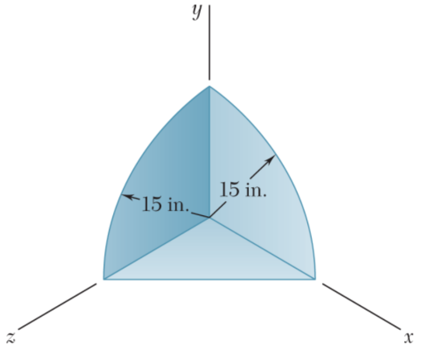
	A, m^2	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}A$, m^3	$\bar{y}A$, m^3	$\bar{z}A$, m^3
I	$\frac{1}{2}(3.6)(1.2) = 2.16$	1.5	−0.4	1.2	3.24	−0.864	2.592
II	$(3.6)(1.7) = 6.12$	0.75	0.4	1.8	4.59	2.448	11.016
III	$\frac{\pi}{2}(1.8)^2 = 5.0894$	$-\frac{2.4}{\pi}$	0.8	1.8	−3.888	4.0715	9.1609
Σ	13.3694				3.942	5.6555	22.769

We have

$\bar{X}\Sigma V = \Sigma \bar{x} V$: $\quad \bar{X}(13.3694 \text{ m}^2) = 3.942 \text{ m}^3$ $\quad$ or $\bar{X} = 0.295$ m ◀

$\bar{Y}\Sigma V = \Sigma \bar{y} V$: $\quad \bar{Y}(13.3694 \text{ m}^2) = 5.6555 \text{ m}^3$ $\quad$ or $\bar{Y} = 0.423$ m ◀

$\bar{Z}\Sigma V = \Sigma \bar{z} V$: $\quad \bar{Z}(13.3694 \text{ m}^2) = 22.769 \text{ m}^3$ $\quad$ or $\bar{Z} = 1.703$ m ◀



PROBLEM 5.108

A corner reflector for tracking by radar has two sides in the shape of a quarter circle with a radius of 15 in. and one side in the shape of a triangle. Locate the center of gravity of the reflector, knowing that it is made of sheet metal of uniform thickness.

SOLUTION

By symmetry, $\bar{X} = \bar{Z}$

For I and II (Quarter-circle), $\bar{x} = \bar{y} = \dfrac{4r}{3\pi} = \dfrac{4(15)}{3\pi} = 6.3662 \text{ in.}$

$$A = \frac{\pi}{4} r^2 = \frac{\pi}{4}(15)^2 = 1.76715 \text{ in}^2$$

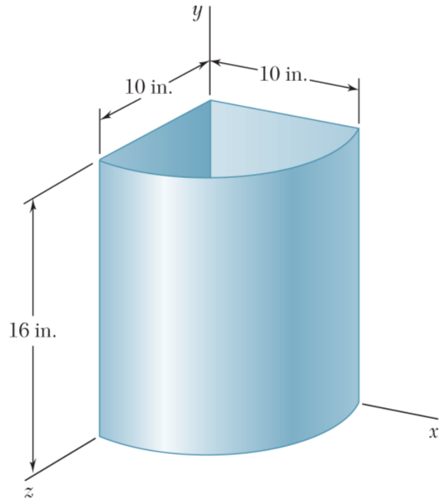
	A, in^2	$\bar{x}$, in.	$\bar{x}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
I	176.715	6.3662	6.3662	1125.0	1125.0
II	176.715	0	6.3662	0	1125.0
III	$\frac{1}{2}(15)^2 = 112.50$	5.0	0	562.50	0
Σ	465.93			1687.50	2250.0

$\bar{X}\Sigma A = \Sigma \bar{x} A$: $\bar{X}(465.93 \text{ in}^2) = 1687.50 \text{ in}^3$

$\bar{X} = \bar{Z} = 3.62 \text{ in.}$ ◀

$\bar{Y}\Sigma A = \Sigma \bar{y} A$: $\bar{Y}(465.93 \text{ in}^2) = 2250.0 \text{ in}^3$

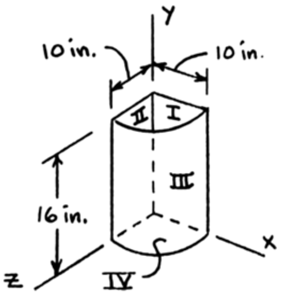
$\bar{Y} = 4.83 \text{ in.}$ ◀



PROBLEM 5.109

A wastebasket, designed to fit in the corner of a room, is 16 in. high and has a base in the shape of a quarter circle of radius 10 in. Locate the center of gravity of the wastebasket, knowing that it is made of sheet metal of uniform thickness.

SOLUTION



By symmetry, $\bar{X} = \bar{Z}$

For III (Cylindrical surface), $\bar{x} = \dfrac{2r}{\pi} = \dfrac{2(10)}{\pi} = 6.3662 \text{ in.}$

$$A = \frac{\pi}{2} rh = \frac{\pi}{2}(10)(16) = 251.33 \text{ in}^2$$

For IV (Quarter-circle bottom), $\bar{x} = \dfrac{4r}{3\pi} = \dfrac{4(10)}{3\pi} = 4.2441 \text{ in.}$

$$A = \frac{\pi}{4} r^2 = \frac{\pi}{4}(10)^2 = 78.540 \text{ in}^2$$

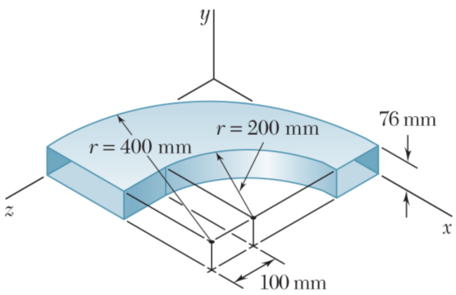
	A, in^2	$\bar{x}$, in.	$\bar{x}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
I	$(10)(16) = 160$	5	8	800	1280
II	$(10)(16) = 160$	0	8	0	1280
III	251.33	6.3662	8	1600.0	2010.6
IV	78.540	4.2441	0	333.33	0
Σ	649.87			2733.3	4570.6

$\bar{X}\Sigma A = \Sigma \bar{x}A$: $\bar{X}(649.87 \text{ in}^2) = 2733.3 \text{ in}^3$

$\bar{X} = 4.2059 \text{ in.}$ $\bar{X} = \bar{Z} = 4.21 \text{ in.}$ ◀

$\bar{Y}\Sigma A = \Sigma \bar{y}A$: $\bar{Y}(649.87 \text{ in}^2) = 4570.6 \text{ in}^3$

$\bar{Y} = 7.0331 \text{ in.}$ $\bar{Y} = 7.03 \text{ in.}$ ◀



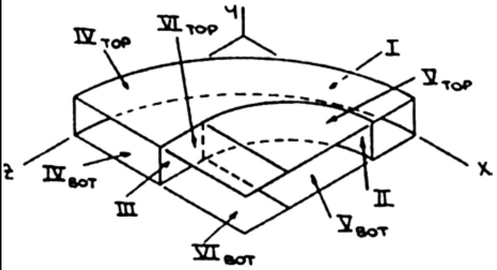
PROBLEM 5.110

An elbow for the duct of a ventilating system is made of sheet metal of uniform thickness. Locate the center of gravity of the elbow.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the duct coincides with the centroid of the corresponding area. Also, note that the shape of the duct implies

$$\overline{Y} = 38.0 \text{ mm} \blacktriangleleft$$



Note that

$$\overline{x}_{\text{I}} = \overline{z}_{\text{I}} = 400 - \frac{2}{\pi}(400) = 145.352 \text{ mm}$$

$$\overline{x}_{\text{II}} = 400 - \frac{2}{\pi}(200) = 272.68 \text{ mm}$$

$$\overline{z}_{\text{II}} = 300 - \frac{2}{\pi}(200) = 172.676 \text{ mm}$$

$$\overline{x}_{\text{IV}} = \overline{z}_{\text{IV}} = 400 - \frac{4}{3\pi}(400) = 230.23 \text{ mm}$$

$$\overline{x}_{\text{V}} = 400 - \frac{4}{3\pi}(200) = 315.12 \text{ mm}$$

$$\overline{z}_{\text{V}} = 300 - \frac{4}{3\pi}(200) = 215.12 \text{ mm}$$

Also note that the corresponding top and bottom areas will contribute equally when determining $\overline{x}$ and $\overline{z}$.

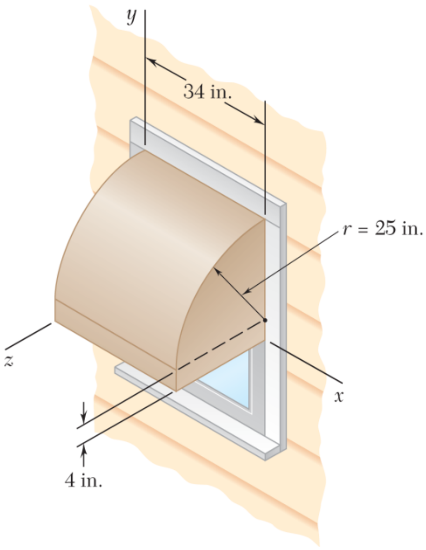
Thus,

	A, mm^2	$\overline{x}$, mm	$\overline{z}$, mm	$\overline{x}A$, mm^3	$\overline{z}A$, mm^3
I	$\frac{\pi}{2}(400)(76) = 47{,}752$	145.352	145.352	6,940,850	6,940,850
II	$\frac{\pi}{2}(200)(76) = 23{,}876$	272.68	172.676	6,510,510	4,122,810
III	$100(76) = 7600$	200	350	1,520,000	2,660,000
IV	$2\left(\frac{\pi}{4}\right)(400)^2 = 251{,}327$	230.23	230.23	57,863,020	57,863,020
V	$-2\left(\frac{\pi}{4}\right)(200)^2 = -62{,}832$	315.12	215.12	–19,799,620	–13,516,420
VI	$-2(100)(200) = -40{,}000$	300	350	–12,000,000	–14,000,000
Σ	227,723			41,034,760	44,070,260

PROBLEM 5.110 (Continued)

We have $\bar{X}\Sigma A = \Sigma\bar{x}A$: $\bar{X}(227,723\text{ mm}^2) = 41,034,760\text{ mm}^3$ or $\bar{X} = 180.2\text{ mm}$ ◀

$\bar{Z}\Sigma A = \Sigma\bar{z}A$: $\bar{Z}(227,723\text{ mm}^2) = 44,070,260\text{ mm}^3$ or $\bar{Z} = 193.5\text{ mm}$ ◀

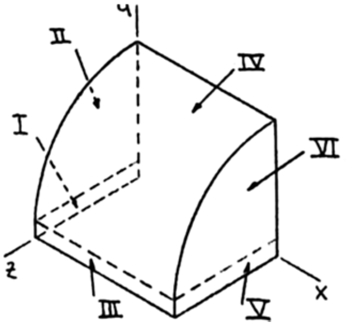


PROBLEM 5.111

A window awning is fabricated from sheet metal of uniform thickness. Locate the center of gravity of the awning.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the awning coincides with the centroid of the corresponding area.



$$\bar{y}_{\text{II}} = \bar{y}_{\text{VI}} = 4 + \frac{(4)(25)}{3\pi} = 14.6103 \text{ in.}$$

$$\bar{z}_{\text{II}} = \bar{z}_{\text{VI}} = \frac{(4)(25)}{3\pi} = \frac{100}{3\pi} \text{ in.}$$

$$\bar{y}_{\text{IV}} = 4 + \frac{(2)(25)}{\pi} = 19.9155 \text{ in.}$$

$$\bar{z}_{\text{IV}} = \frac{(2)(25)}{\pi} = \frac{50}{\pi} \text{ in.}$$

$$A_{\text{II}} = A_{\text{VI}} = \frac{\pi}{4}(25)^2 = 490.87 \text{ in}^2$$

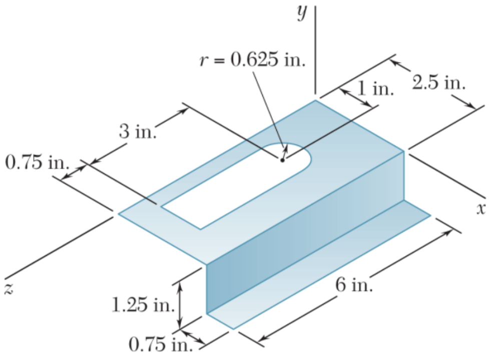
$$A_{\text{IV}} = \frac{\pi}{2}(25)(34) = 1335.18 \text{ in}^2$$

	A, in^2	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{y}A$, in^3	$\bar{z}A$, in^3
I	$(4)(25) = 100$	2	12.5	200	1250
II	490.87	14.6103	$\frac{100}{3\pi}$	7171.8	5208.3
III	$(4)(34) = 136$	2	25	272	3400
IV	1335.18	19.9155	$\frac{50}{\pi}$	26,591	21,250
V	$(4)(25) = 100$	2	12.5	200	1250
VI	490.87	14.6103	$\frac{100}{3\pi}$	7171.8	5208.3
Σ	2652.9			41,607	37,567

Now, symmetry implies $\bar{X} = 17.00\text{ in.}$ ◀

and $\bar{Y}\Sigma A = \Sigma\bar{y}A$: $\bar{Y}(2652.9\text{ in}^2) = 41{,}607\text{ in}^3$ or $\bar{Y} = 15.68\text{ in.}$ ◀

$\bar{Z}\Sigma A = \Sigma\bar{z}A$: $\bar{Z}(2652.9\text{ in}^2) = 37{,}567\text{ in}^3$ or $\bar{Z} = 14.16\text{ in.}$ ◀

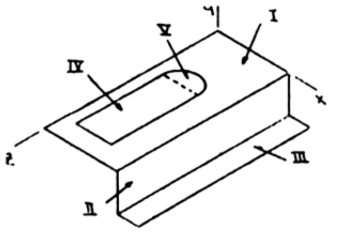


PROBLEM 5.112

A mounting bracket for electronic components is formed from sheet metal of uniform thickness. Locate the center of gravity of the bracket.

SOLUTION

First, assume that the sheet metal is homogeneous so that the center of gravity of the bracket coincides with the centroid of the corresponding area. Then (see diagram)



$$\bar{z}_V = 2.25 - \frac{4(0.625)}{3\pi} = 1.98474 \text{ in.}$$

$$A_V = -\frac{\pi}{2}(0.625)^2 = -0.61359 \text{ in}^2$$

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3	$\bar{z}A$, in^3
I	$(2.5)(6) = 15$	1.25	0	3	18.75	0	45
II	$(1.25)(6) = 7.5$	2.5	–0.625	3	18.75	–4.6875	22.5
III	$(0.75)(6) = 4.5$	2.875	–1.25	3	12.9375	–5.625	13.5
IV	$-\left(\frac{5}{4}\right)(3) = -3.75$	1.0	0	3.75	3.75	0	–14.0625
V	-0.61359	1.0	0	1.9847	0.61359	0	–1.21782
Σ	22.6364				46.0739	10.3125	65.7197

We have $\bar{X}\Sigma A = \Sigma \bar{x}A$

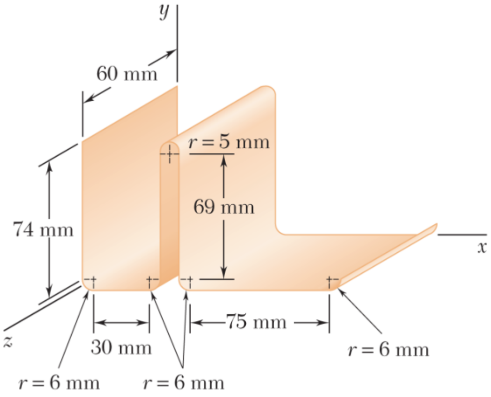
$\bar{X}(22.6364 \text{ in}^2) = 46.0739 \text{ in}^3$ or $\bar{X} = 2.04$ in. ◀

$\bar{Y}\Sigma A = \Sigma \bar{y}A$

$\bar{Y}(22.6364 \text{ in}^2) = -10.3125 \text{ in}^3$ or $\bar{Y} = -0.456$ in. ◀

$\bar{Z}\Sigma A = \Sigma \bar{z}A$

$\bar{Z}(22.6364 \text{ in}^2) = 65.7197 \text{ in}^3$ or $\bar{Z} = 2.90$ in. ◀



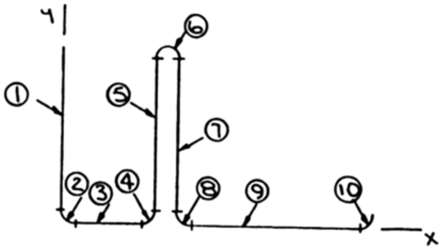
PROBLEM 5.113

A thin sheet of plastic of uniform thickness is bent to form a desk organizer. Locate the center of gravity of the organizer.

SOLUTION

First assume that the plastic is homogeneous so that the center of gravity of the organizer will coincide with the centroid of the corresponding area. Now note that symmetry implies

$$\bar{Z} = 30.0 \text{ mm} \blacktriangleleft$$



$$\bar{x}_2 = 6 - \frac{2 \times 6}{\pi} = 2.1803 \text{ mm}$$

$$\bar{x}_4 = 36 + \frac{2 \times 6}{\pi} = 39.820 \text{ mm}$$

$$\bar{x}_8 = 58 - \frac{2 \times 6}{\pi} = 54.180 \text{ mm}$$

$$\bar{x}_{10} = 133 + \frac{2 \times 6}{\pi} = 136.820 \text{ mm}$$

$$\bar{y}_2 = \bar{y}_4 = \bar{y}_8 = \bar{y}_{10} = 6 - \frac{2 \times 6}{\pi} = 2.1803 \text{ mm}$$

$$\bar{y}_6 = 75 + \frac{2 \times 5}{\pi} = 78.183 \text{ mm}$$

$$A_2 = A_4 = A_8 = A_{10} = \frac{\pi}{2} \times 6 \times 60 = 565.49 \text{ mm}^2$$

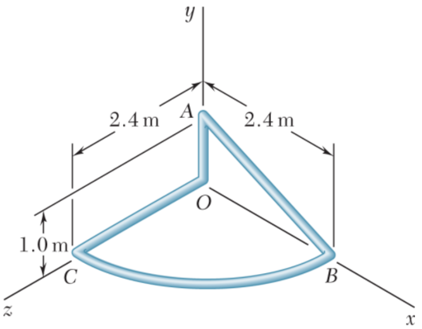
$$A_6 = \pi \times 5 \times 60 = 942.48 \text{ mm}^2$$

PROBLEM 5.113 (Continued)

	A, mm^2	$\overline{x}$, mm	$\overline{y}$, mm	$\overline{x}A$, mm^3	$\overline{y}A$, mm^3
1	$(74)(60) = 4440$	0	43	0	190,920
2	565.49	2.1803	2.1803	1233	1233
3	$(30)(60) = 1800$	21	0	37,800	0
4	565.49	39.820	2.1803	22,518	1233
5	$(69)(60) = 4140$	42	40.5	173,880	167,670
6	942.48	47	78.183	44,297	73,686
7	$(69)(60) = 4140$	52	40.5	215,280	167,670
8	565.49	54.180	2.1803	30,638	1233
9	$(75)(60) = 4500$	95.5	0	429,750	0
10	565.49	136.820	2.1803	77,370	1233
Σ	22,224.44			1,032,766	604,878

We have $\overline{X}\Sigma A = \Sigma\overline{x}A$: $\overline{X}(22,224.44\text{ mm}^2) = 1,032,766\text{ mm}^3$ or $\overline{X} = 46.5\text{ mm}$ ◀

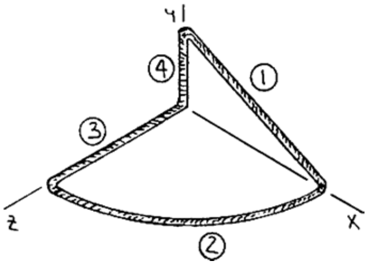
$\overline{Y}\Sigma A = \Sigma\overline{y}A$: $\overline{Y}(22,224.44\text{ mm}^2) = 604,878\text{ mm}^3$ or $\overline{Y} = 27.2\text{ mm}$ ◀



PROBLEM 5.114

A thin steel wire of uniform cross section is bent into the shape shown. Locate its center of gravity.

SOLUTION



First assume that the wire is homogeneous so that its center of gravity will coincide with the centroid of the corresponding line.

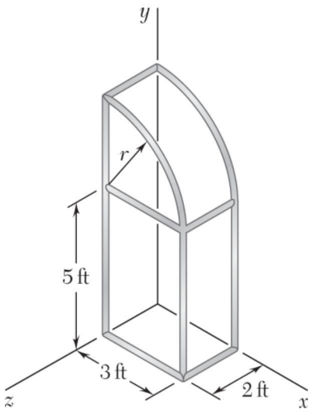
$$\bar{x}_2 = \bar{z}_2 = \frac{2 \times 2.4}{\pi} = \frac{4.8}{\pi} \text{ m}$$

	L, m	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}L$, m^2	$\bar{y}L$, m^2	$\bar{z}L$, m^2
1	2.6	1.2	0.5	0	3.12	1.3	0
2	$\frac{\pi}{2} \times 2.4 = 1.2\pi$	$\frac{4.8}{\pi}$	0	$\frac{4.8}{\pi}$	5.76	0	5.76
3	2.4	0	0	1.2	0	0	2.88
4	1.0	0	0.5	0	0	0.5	0
Σ	9.7699				8.88	1.8	8.64

We have $\bar{X}\Sigma L = \Sigma \bar{x} L$: $\bar{X}(9.7699 \text{ m}) = 8.88 \text{ m}^2$ or $\bar{X} = 0.909$ m ◀

$\bar{Y}\Sigma L = \Sigma \bar{y} L$: $\bar{Y}(9.7699 \text{ m}) = 1.8 \text{ m}^2$ or $\bar{Y} = 0.1842$ m ◀

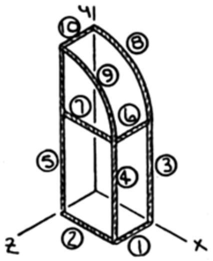
$\bar{Z}\Sigma L = \Sigma \bar{z} L$: $\bar{Z}(9.7699 \text{ m}) = 8.64 \text{ m}^2$ or $\bar{Z} = 0.884$ m ◀



PROBLEM 5.115

The frame of a greenhouse is constructed from uniform aluminum channels. Locate the center of gravity of the portion of the frame shown.

SOLUTION



First assume that the channels are homogeneous so that the center of gravity of the frame will coincide with the centroid of the corresponding line.

$$\bar{x}_8 = \bar{x}_9 = \frac{2\times 3}{\pi} = \frac{6}{\pi}\text{ ft}$$

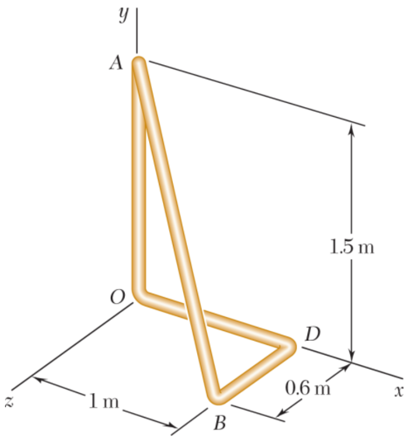
$$\bar{y}_8 = \bar{y}_9 = 5 + \frac{2\times 3}{\pi} = 6.9099\text{ ft}$$

	L, ft	$\bar{x}$, ft	$\bar{y}$, ft	$\bar{z}$, ft	$\bar{x}L$, ft^2	$\bar{y}L$, ft^2	$\bar{z}L$, ft^2
1	2	3	0	1	6	0	2
2	3	1.5	0	2	4.5	0	6
3	5	3	2.5	0	15	12.5	0
4	5	3	2.5	2	15	12.5	10
5	8	0	4	2	0	32	16
6	2	3	5	1	6	10	2
7	3	1.5	5	2	4.5	15	6
8	$\frac{\pi}{2}\times 3 = 4.7124$	$\frac{6}{\pi}$	6.9099	0	9	32.562	0
9	$\frac{\pi}{2}\times 3 = 4.7124$	$\frac{6}{\pi}$	6.9099	2	9	32.562	9.4248
10	2	0	8	1	0	16	2
Σ	39.4248				69	163.124	53.4248

We have $\bar{X}\Sigma L = \Sigma\bar{x}L$: $\bar{X}(39.4248\text{ ft}) = 69\text{ ft}^2$ or $\bar{X} = 1.750\text{ ft}$ ◀

$\bar{Y}\Sigma L = \Sigma\bar{y}L$: $\bar{Y}(39.4248\text{ ft}) = 163.124\text{ ft}^2$ or $\bar{Y} = 4.14\text{ ft}$ ◀

$\bar{Z}\Sigma L = \Sigma\bar{z}L$: $\bar{Z}(39.4248\text{ ft}) = 53.4248\text{ ft}^2$ or $\bar{Z} = 1.355\text{ ft}$ ◀

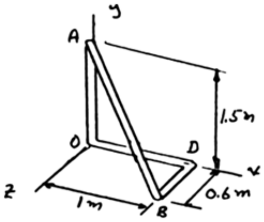


PROBLEM 5.116

Locate the center of gravity of the figure shown, knowing that it is made of thin brass rods of uniform diameter.

SOLUTION

Uniform rod:



$$AB^2 = (1\text{ m})^2 + (0.6\text{ m})^2 + (1.5\text{ m})^2$$

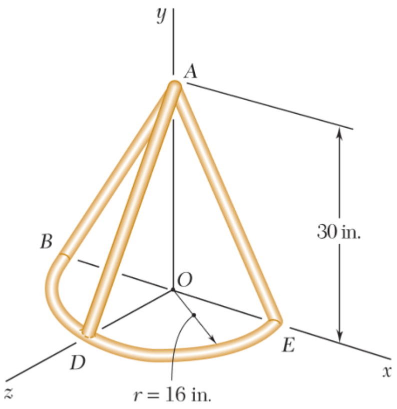
$$AB = 1.9\text{ m}$$

	L, m	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$\bar{x}L$, m^2	$\bar{y}L$, m^2	ΣL, m
AB	1.9	0.5	0.75	0.3	0.95	1.425	0.57
BD	0.6	1.0	0	0.3	0.60	0	0.18
DO	1.0	0.5	0	0	0.50	0	0
OA	1.5	0	0.75	0	0	1.125	0
Σ	5.0				2.05	2.550	0.75

$\bar{X}\Sigma L = \Sigma\bar{x}L$: $\bar{X}(5.0\text{ m}) = 2.05\text{ m}^2$ $\qquad \bar{X} = 0.410\text{ m}$ ◀

$\bar{Y}\Sigma L = \Sigma\bar{y}L$: $\bar{Y}(5.0\text{ m}) = 2.55\text{ m}^2$ $\qquad \bar{Y} = 0.510\text{ m}$ ◀

$\bar{Z}\Sigma L = \Sigma\bar{z}L$: $\bar{Z}(5.0\text{ m}) = 0.75\text{ m}^2$ $\qquad \bar{Z} = 0.1500\text{ m}$ ◀

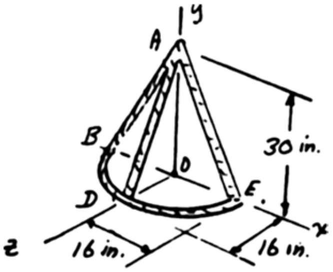


PROBLEM 5.117

Locate the center of gravity of the figure shown, knowing that it is made of thin brass rods of uniform diameter.

SOLUTION

By symmetry, $\bar{X} = 0$ ◀



	L, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{y}L$, in^2	$\bar{z}L$, in^2
AB	$\sqrt{30^2+16^2}=34$	15	0	510	0
AD	$\sqrt{30^2+16^2}=34$	15	8	510	272
AE	$\sqrt{30^2+16^2}=34$	15	0	510	0
BDE	$\pi(16)=50.265$	0	$\frac{2(16)}{\pi}=10.186$	0	512
Σ	152.265			1530	784

$$\bar{Y}\Sigma L = \Sigma\bar{y}L: \quad \bar{Y}(152.265 \text{ in.}) = 1530 \text{ in}^2$$

$$\bar{Y} = 10.048 \text{ in.}$$

$\bar{Y} = 10.05$ in. ◀

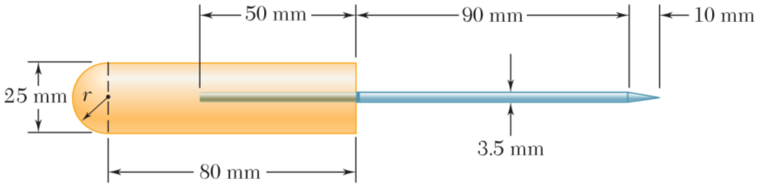
$$\bar{Z}\Sigma L = \Sigma\bar{z}L: \quad \bar{Z}(152.265 \text{ in.}) = 784 \text{ in}^2$$

$$\bar{Z} = 5.149 \text{ in.}$$

$\bar{Z} = 5.15$ in. ◀

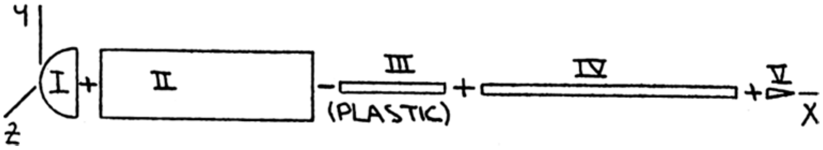
PROBLEM 5.118

A scratch awl has a plastic handle and a steel blade and shank. Knowing that the density of plastic is 1030 kg/m^3 and of steel is 7860 kg/m^3, locate the center of gravity of the awl.



SOLUTION

First, note that symmetry implies $\bar{Y} = \bar{Z} = 0$ ◀



$$\bar{x}_{\text{I}} = \frac{5}{8}(12.5\text{ mm}) = 7.8125\text{ mm}$$

$$W_{\text{I}} = (1030\text{ kg/m}^3)\left(\frac{2\pi}{3}\right)(0.0125\text{ m})^3$$

$$= 4.2133 \times 10^{-3}\text{ kg}$$

$$\bar{x}_{\text{II}} = 52.5\text{ mm}$$

$$W_{\text{II}} = (1030\text{ kg/m}^3)\left(\frac{\pi}{4}\right)(0.025\text{ m})^2(0.08\text{ m})$$

$$= 40.448 \times 10^{-3}\text{ kg}$$

$$\bar{x}_{\text{III}} = 92.5\text{ mm} - 25\text{ mm} = 67.5\text{ mm}$$

$$W_{\text{III}} = -(1030\text{ kg/m}^3)\left(\frac{\pi}{4}\right)(0.0035\text{ m})^2(0.05\text{ m})$$

$$= -0.49549 \times 10^{-3}\text{ kg}$$

$$\bar{x}_{\text{IV}} = 182.5\text{ mm} - 70\text{ mm} = 112.5\text{ mm}$$

$$W_{\text{IV}} = (7860\text{ kg/m}^3)\left(\frac{\pi}{4}\right)(0.0035\text{ m})^2(0.14\text{ m})^2 = 10.5871 \times 10^{-3}\text{ kg}$$

$$\bar{x}_{\text{V}} = 182.5\text{ mm} + \frac{1}{4}(10\text{ mm}) = 185\text{ mm}$$

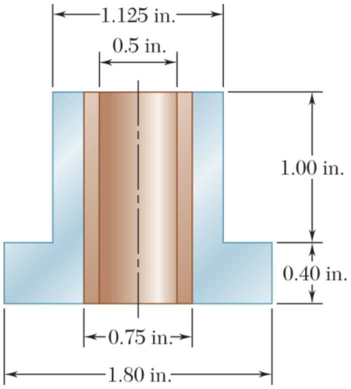
$$W_{\text{V}} = (7860\text{ kg/m}^3)\left(\frac{\pi}{3}\right)(0.00175\text{ m})^2(0.01\text{ m}) = 0.25207 \times 10^{-3}\text{ kg}$$

	W, kg	$\bar{x}$, mm	$\bar{x}W$, kg·mm
I	4.123×10^{-3}	7.8125	32.916×10^{-3}
II	40.948×10^{-3}	52.5	2123.5×10^{-3}
III	-0.49549×10^{-3}	67.5	-33.447×10^{-3}
IV	10.5871×10^{-3}	112.5	1191.05×10^{-3}
V	0.25207×10^{-3}	185	46.633×10^{-3}
Σ	55.005×10^{-3}		3360.7×10^{-3}

We have $\bar{X}\Sigma W = \Sigma\bar{x}W: \quad \bar{X}(55.005\times10^{-3}\text{ kg}) = 3360.7\times10^{-3}\text{ kg}\cdot\text{mm}$

or $\bar{X} = 61.1\text{ mm}$ ◀

(from the end of the handle)

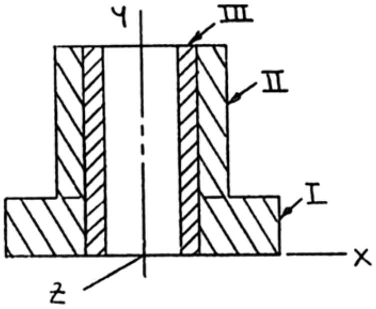


PROBLEM 5.119

A bronze bushing is mounted inside a steel sleeve. Knowing that the specific weight of bronze is 0.318 lb/in^3 and of steel is 0.284 lb/in^3, determine the location of the center of gravity of the assembly.

SOLUTION

First, note that symmetry implies $\overline{X} = \overline{Z} = 0$ ◀



Now $W = (\rho g)V$

$$\bar{y}_{\text{I}} = 0.20 \text{ in.} \quad W_{\text{I}} = (0.284 \text{ lb/in}^3)\left\{\left(\frac{\pi}{4}\right)[(1.8^2 - 0.75^2) \text{ in}^2](0.4 \text{ in.})\right\} = 0.23889 \text{ lb}$$

$$\bar{y}_{\text{II}} = 0.90 \text{ in.} \quad W_{\text{II}} = (0.284 \text{ lb/in}^3)\left\{\left(\frac{\pi}{4}\right)[(1.125^2 - 0.75^2) \text{ in}^2](1 \text{ in.})\right\} = 0.156834 \text{ lb}$$

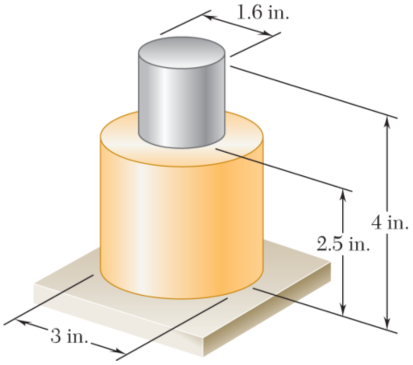
$$\bar{y}_{\text{III}} = 0.70 \text{ in.} \quad W_{\text{III}} = (0.318 \text{ lb/in}^3)\left\{\left(\frac{\pi}{4}\right)[(0.75^2 - 0.5^2) \text{ in}^2](1.4 \text{ in.})\right\} = 0.109269 \text{ lb}$$

We have $\overline{Y}\Sigma W = \Sigma \bar{y} W$

$$\overline{Y} = \frac{(0.20 \text{ in.})(0.23889 \text{ lb}) + (0.90 \text{ in.})(0.156834 \text{ lb}) + (0.70 \text{ in.})(0.109269 \text{ lb})}{0.23889 \text{ lb} + 0.156834 \text{ lb} + 0.109269 \text{ lb}}$$

or $\overline{Y} = 0.526 \text{ in.}$ ◀

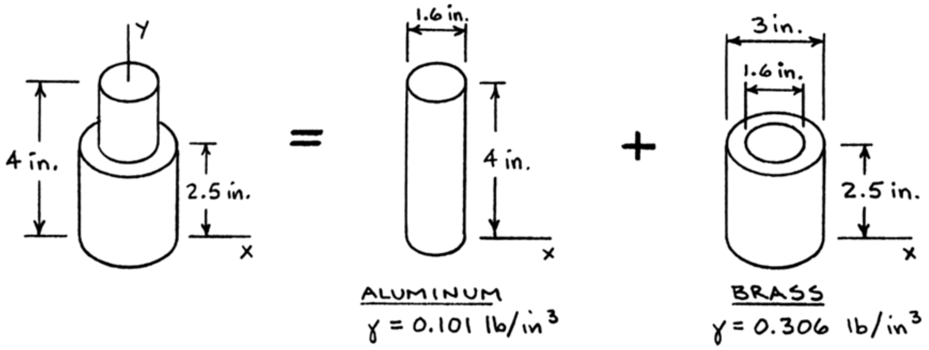
(above base)


1.6 in.
4 in.
2.5 in.
3 in.

PROBLEM 5.120

A brass collar, of length 2.5 in., is mounted on an aluminum rod of length 4 in. Locate the center of gravity of the composite body. (Specific weights: brass = 0.306 lb/in^3, aluminum = 0.101 lb/in^3)

SOLUTION


Y
4 in.
2.5 in.
x
=
1.6 in.
4 in.
x
ALUMINUM
γ = 0.101 lb/in³
+
3 in.
1.6 in.
2.5 in.
x
BRASS
γ = 0.306 lb/in³

Aluminum rod:

$$W = \gamma V$$
$$= (0.101\ \text{lb/in}^3)\left[\frac{\pi}{4}(1.6\ \text{in.})^2(4\ \text{in.})\right]$$
$$= 0.81229\ \text{lb}$$

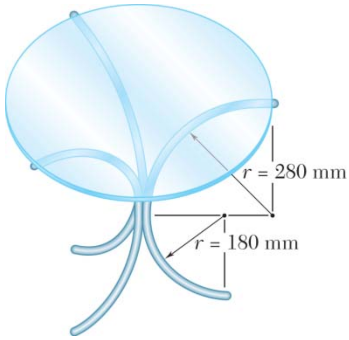
Brass collar:

$$W = \gamma V$$
$$= (0.306\ \text{lb/in.}^3)\frac{\pi}{4}[(3\ \text{in.})^2 - (1.6\ \text{in.})^2](2.5\ \text{in.})$$
$$= 3.8693\ \text{lb}$$

Component	W(lb)	$\bar{y}$ (in.)	$\bar{y}W$ (lb · in.)
Rod	0.81229	2	1.62458
Collar	3.8693	1.25	4.8366
Σ	4.6816		6.4612

$$\bar{Y}\Sigma W = \Sigma \bar{y} W: \quad \bar{Y}(4.6816\ \text{lb}) = 6.4612\ \text{lb} \cdot \text{in.}$$

$$\bar{Y} = 1.38013\ \text{in.} \qquad \bar{Y} = 1.380\ \text{in.} \blacktriangleleft$$



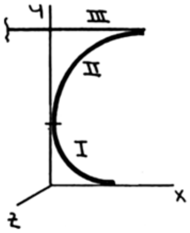
PROBLEM 5.121

The three legs of a small glass-topped table are equally spaced and are made of steel tubing, which has an outside diameter of 24 mm and a cross-sectional area of 150 mm^2. The diameter and the thickness of the table top are 600 mm and 10 mm, respectively. Knowing that the density of steel is $7860\ \text{kg/m}^3$ and of glass is $2190\ \text{kg/m}^3$, locate the center of gravity of the table.

SOLUTION

First note that symmetry implies $\bar{X} = \bar{Z} = 0$ ◀

Also, to account for the three legs, the masses of components I and II will each bex multiplied by three



$$\bar{y}_{\text{I}} = 12 + 180 - \frac{2\times 180}{\pi} = 77.408\ \text{mm} \qquad m_{\text{I}} = \rho_{ST}V_{\text{I}} = 7860\ \text{kg/m}^3 \times (150\times 10^{-6}\,\text{m}^2)\times \frac{\pi}{2}(0.180\ \text{m}) = 0.33335\ \text{kg}$$

$$\bar{y}_{\text{II}} = 12 + 180 + \frac{2\times 280}{\pi} = 370.25\ \text{mm} \qquad m_{\text{II}} = \rho_{ST}V_{\text{II}} = 7860\ \text{kg/m}^3 \times (150\times 10^{-6}\ \text{m}^2)\times \frac{\pi}{2}(0.280\ \text{m}) = 0.51855\ \text{kg}$$

$$\bar{y}_{\text{III}} = 24 + 180 + 280 + 5 = 489\ \text{mm} \qquad m_{\text{III}} = \rho_{GL}V_{\text{III}} = 2190\ \text{kg/m}^3 \times \frac{\pi}{4}(0.6\ \text{m})^2 \times (0.010\ \text{m}) = 6.1921\ \text{kg}$$

	m, kg	$\bar{y}$, mm	$\bar{y}m$, kg · mm
I	3(0.33335)	77.408	77.412
II	3(0.51855)	370.25	515.98
III	6.1921	489	3027.9
Σ	8.7478		3681.3

We have $\quad Y\Sigma m = \Sigma \bar{y}m: \quad \bar{Y}(8.7478\ \text{kg}) = 3681.3\ \text{kg}\cdot\text{mm}$

or $\quad \bar{Y} = 420.8\ \text{mm}$

The center of gravity is 421 mm ◀

(above the floor)

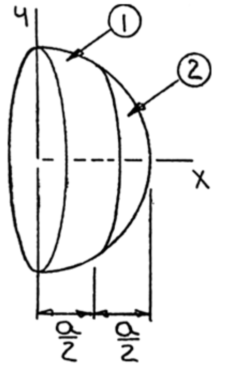
PROBLEM 5.122

Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Figure 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

A hemisphere

SOLUTION

Choose as the element of volume a disk of radius r and thickness dx. Then



$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

The equation of the generating curve is $x^2 + y^2 = a^2$ so that $r^2 = a^2 - x^2$ and then

$$dV = \pi(a^2 - x^2)dx$$

Component 1:

$$V_1 = \int_0^{a/2} \pi(a^2 - x^2)dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_0^{a/2}$$

$$= \frac{11}{24}\pi a^3$$

and

$$\int_1 \bar{x}_{EL} dV = \int_0^{a/2} x\left[\pi(a^2 - x^2)dx\right]$$

$$= \pi\left[a^2 \frac{x^2}{2} - \frac{x^4}{4}\right]_0^{a/2}$$

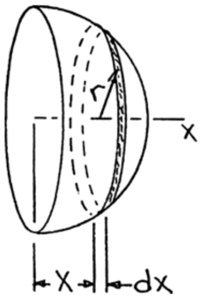
$$= \frac{7}{64}\pi a^4$$

Now

$$\bar{x}_1 V_1 = \int_1 \bar{x}_{EL} dV: \quad \bar{x}_1\left(\frac{11}{24}\pi a^3\right) = \frac{7}{64}\pi a^4$$

or $\bar{x}_1 = \dfrac{21}{88}a$ ◀

Component 2:



$$V_2 = \int_{a/2}^{a} \pi(a^2 - x^2)dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{a/2}^{a}$$

$$= \pi\left\{\left[a^2(a) - \frac{a^3}{3}\right] - \left[a^2\left(\frac{a}{2}\right) - \frac{\left(\frac{a}{2}\right)^3}{3}\right]\right\}$$

$$= \frac{5}{24}\pi a^3$$

and
$$\int_2 \overline{x}_{EL}\,dV = \int_{a/2}^{a} x\left[\pi(a^2 - x^2)dx\right] = \pi\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{a/2}^{a}$$

$$= \pi\left\{\left[a^2\frac{(a)^2}{2} - \frac{(a)^4}{4}\right] - \left[a^2\frac{\left(\frac{a}{2}\right)^2}{2} - \frac{\left(\frac{a}{2}\right)^4}{4}\right]\right\}$$

$$= \frac{9}{64}\pi a^4$$

Now
$$\overline{x}_2 V_2 = \int_2 \overline{x}_{EL}\,dV: \quad \overline{x}_2\left(\frac{5}{24}\pi a^3\right) = \frac{9}{64}\pi a^4$$

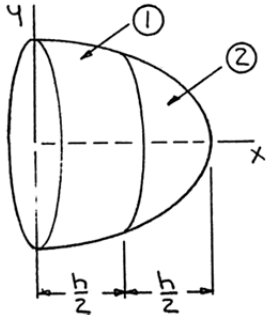
or $\overline{x}_2 = \frac{27}{40}a$ ◀

PROBLEM 5.123

Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Figure 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

A semiellipsoid of revolution

SOLUTION



Choose as the element of volume a disk of radius *r* and thickness *dx*. Then

$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

The equation of the generating curve is $\dfrac{x^2}{h^2} + \dfrac{y^2}{a^2} = 1$ so that

$$r^2 = \frac{a^2}{h^2}(h^2 - x^2)$$

and then
$$dV = \pi \frac{a^2}{h^2}(h^2 - x^2)dx$$

Component 1:
$$V_1 = \int_0^{h/2} \pi \frac{a^2}{h^2}(h^2 - x^2)dx = \pi \frac{a^2}{h^2}\left[h^2 x - \frac{x^3}{3}\right]_0^{h/2}$$

$$= \frac{11}{24}\pi a^2 h$$

and
$$\int_1 \bar{x}_{EL} dV = \int_0^{h/2} x\left[\pi \frac{a^2}{h^2}(h^2 - x^2)dx\right]$$

$$= \pi \frac{a^2}{h^2}\left[h^2 \frac{x^2}{2} - \frac{x^4}{4}\right]_0^{h/2}$$

$$= \frac{7}{64}\pi a^2 h^2$$

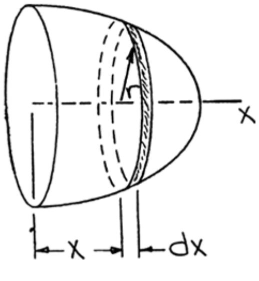
Now
$$\bar{x}_1 V_1 = \int_1 \bar{x}_{EL} dV: \quad \bar{x}_1\left(\frac{11}{24}\pi a^2 h\right) = \frac{7}{64}\pi a^2 h^2$$

or $\bar{x}_1 = \dfrac{21}{88}h$ ◀

Component 2:
$$V_2 = \int_{h/2}^{h} \pi \frac{a^2}{h^2}(h^2 - x^2)dx = \pi \frac{a^2}{h^2}\left[h^2 x - \frac{x^3}{3}\right]_{h/2}^{h}$$

$$= \pi \frac{a^2}{h^2}\left\{\left[h^2(h) - \frac{(h)^3}{3}\right] - \left[h^2\left(\frac{h}{2}\right) - \frac{\left(\frac{h}{2}\right)^3}{3}\right]\right\}$$

$$= \frac{5}{24}\pi a^2 h$$



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PROBLEM 5.123 (Continued)

and
$$\int_2 \overline{x}_{EL}\,dV = \int_{h/2}^{h} x\left[\pi\frac{a^2}{h^2}(h^2 - x^2)dx\right]$$

$$= \pi\frac{a^2}{h^2}\left[h^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{h/2}^{h}$$

$$= \pi\frac{a^2}{h^2}\left\{\left[h^2\frac{(h)^2}{2} - \frac{(h)^4}{4}\right] - \left[h^2\frac{\left(\frac{h}{2}\right)^2}{2} - \frac{\left(\frac{h}{2}\right)^4}{4}\right]\right\}$$

$$= \frac{9}{64}\pi a^2 h^2$$

Now
$$\overline{x}_2 V_2 = \int_2 \overline{x}_{EL}\,dV: \quad \overline{x}_2\left(\frac{5}{24}\pi a^2 h\right) = \frac{9}{64}\pi a^2 h^2$$

or $\overline{x}_2 = \frac{27}{40}h$ ◀

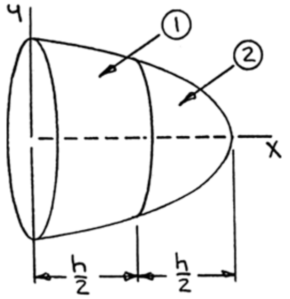
PROBLEM 5.124

Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Figure 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

A paraboloid of revolution

SOLUTION

Choose as the element of volume a disk of radius *r* and thickness *dx*. Then



$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

The equation of the generating curve is $x = h - \frac{h}{a^2}y^2$ so that $r^2 = \frac{a^2}{h}(h-x)$.

and then $$dV = \pi\frac{a^2}{h}(h-x)dx$$

Component 1: $$V_1 = \int_0^{h/2} \pi\frac{a^2}{h}(h-x)dx$$

$$= \pi\frac{a^2}{h}\left[hx - \frac{x^2}{2}\right]_0^{h/2}$$

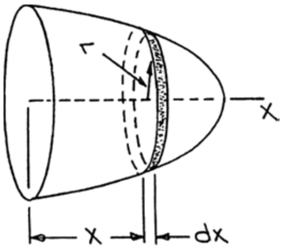
$$= \frac{3}{8}\pi a^2 h$$

and $$\int_1 \bar{x}_{EL} dV = \int_0^{h/2} x\left[\pi\frac{a^2}{h}(h-x)dx\right]$$

$$= \pi\frac{a^2}{h}\left[h\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{h/2} = \frac{1}{12}\pi a^2 h^2$$

Now $$\bar{x}_1 V_1 = \int_1 \bar{x}_{EL} dV: \quad \bar{x}_1\left(\frac{3}{8}\pi a^2 h\right) = \frac{1}{12}\pi a^2 h^2$$ or $\bar{x}_1 = \frac{2}{9}h$ ◀

Component 2:



$$V_2 = \int_{h/2}^{h} \pi\frac{a^2}{h}(h-x)dx = \pi\frac{a^2}{h}\left[hx - \frac{x^2}{2}\right]_{h/2}^{h}$$

$$= \pi\frac{a^2}{h}\left\{\left[h(h) - \frac{(h)^2}{2}\right] - \left[h\left(\frac{h}{2}\right) - \frac{\left(\frac{h}{2}\right)^2}{2}\right]\right\}$$

$$= \frac{1}{8}\pi a^2 h$$

PROBLEM 5.124 (Continued)

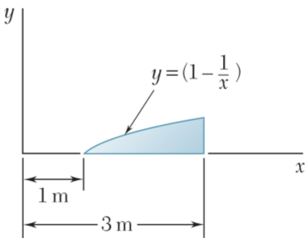
and
$$\int_2 \overline{x}_{EL}\,dV = \int_{h/2}^{h} x\left[\pi\frac{a^2}{h}(h-x)dx\right] = \pi\frac{a^2}{h}\left[h\frac{x^2}{2}-\frac{x^3}{3}\right]_{h/2}^{h}$$

$$= \pi\frac{a^2}{h}\left\{\left[h\frac{(h)^2}{2}-\frac{(h)^3}{3}\right]-\left[h\frac{\left(\frac{h}{2}\right)^2}{2}-\frac{\left(\frac{h}{2}\right)^3}{3}\right]\right\}$$

$$= \frac{1}{12}\pi a^2h^2$$

Now
$$\overline{x}_2V_2 = \int_2 \overline{x}_{EL}\,dV: \quad \overline{x}_2\left(\frac{1}{8}\pi a^2h\right) = \frac{1}{12}\pi a^2h^2$$

or $\overline{x}_2 = \frac{2}{3}h$ ◀



PROBLEM 5.125

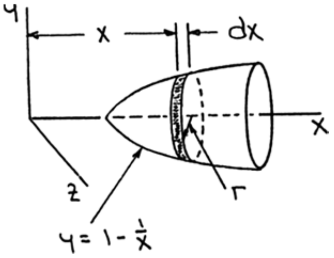
Locate the centroid of the volume obtained by rotating the shaded area about the *x*-axis.

SOLUTION

First, note that symmetry implies $\bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius *r* and thickness *dx*.



Then $dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$

Now $r = 1 - \dfrac{1}{x}$ so that

$$dV = \pi\left(1 - \frac{1}{x}\right)^2 dx$$

$$= \pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx$$

Then

$$V = \int_1^3 \pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx = \pi\left[x - 2\ln x - \frac{1}{x}\right]_1^3$$

$$= \pi\left[\left(3 - 2\ln 3 - \frac{1}{3}\right) - \left(1 - 2\ln 1 - \frac{1}{1}\right)\right]$$

$$= (0.46944\pi)\ \text{m}^3$$

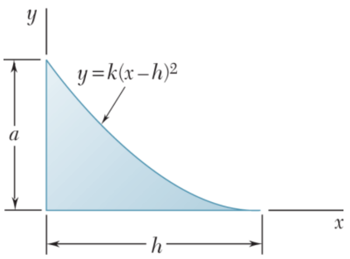
and

$$\int \bar{x}_{EL}\,dV = \int_1^3 x\left[\pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx\right] = \pi\left[\frac{x^2}{2} - 2x + \ln x\right]_1^3$$

$$= \pi\left\{\left[\frac{3^2}{2} - 2(3) + \ln 3\right] - \left[\frac{1^3}{2} - 2(1) + \ln 1\right]\right\}$$

$$= (1.09861\pi)\ \text{m}$$

Now $\bar{x}V = \int \bar{x}_{EL}\,dV: \quad \bar{x}(0.46944\pi\ \text{m}^3) = 1.09861\pi\ \text{m}^4$ or $\bar{x} = 2.34\ \text{m}$ ◀



PROBLEM 5.126

Locate the centroid of the volume obtained by rotating the shaded area about the *x*-axis.

SOLUTION

First note that symmetry implies $\bar{y} = 0$ ◀

and $\bar{z} = 0$ ◀

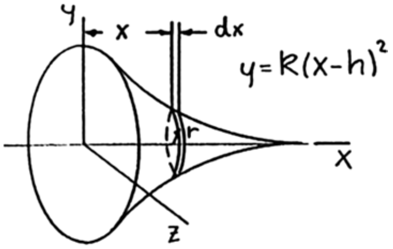
We have $$y = k(X - h)^2$$

At $x = 0$, $y = a$, $$a = k(-h)^2$$

or $$k = \frac{a}{h^2}$$

Choose as the element of volume a disk of radius *r* and thickness *dx*. Then

$$dV = \pi r^2 dx, \quad \bar{X}_{EL} = x$$



Now $$r = \frac{a}{h^2}(x-h)^2$$

so that $$dV = \pi \frac{a^2}{h^4}(x-h)^4 dx$$

Then $$V = \int_0^h \pi \frac{a^2}{h^4}(x-h)^4 dx = \frac{\pi}{5}\frac{a^2}{h^4}[(x-h)^5]_0^h$$

$$= \frac{1}{5}\pi a^2 h$$

and $$\int \bar{x}_{EL} dV = \int_0^h x\left[\pi \frac{a^2}{h^4}(x-h)^4 dx\right]$$

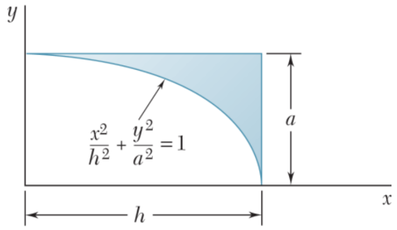
$$= \pi \frac{a^2}{h^4}\int_0^h (x^5 - 4hx^4 + 6h^2x^3 - 4h^3x^2 + h^4x)dx$$

$$= \pi \frac{a^2}{h^4}\left[\frac{1}{6}x^6 - \frac{4}{5}hx^5 + \frac{3}{2}h^2x^4 - \frac{4}{3}h^3x^3 + \frac{1}{2}h^4x^2\right]_0^h$$

$$= \frac{1}{30}\pi a^2 h^2$$

Now $$\bar{x}V = \int \bar{x}_{EL} dV: \quad \bar{x}\left(\frac{\pi}{5}a^2 h\right) = \frac{\pi}{30}a^2h^2$$

or $\bar{x} = \frac{1}{6}h$ ◀



PROBLEM 5.127

Locate the centroid of the volume obtained by rotating the shaded area about the line $x = h$.

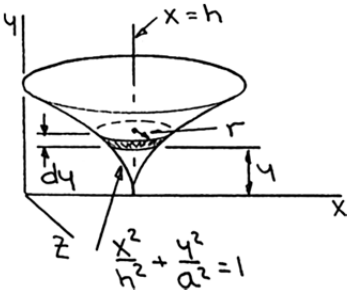
SOLUTION

First, note that symmetry implies $\bar{x} = h$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dy, \quad \bar{y}_{EL} = y$$



Now $x^2 = \frac{h^2}{a^2}(a^2 - y^2)$ so that $r = h - \frac{h}{a}\sqrt{a^2 - y^2}$.

Then
$$dV = \pi \frac{h^2}{a^2}\left(a - \sqrt{a^2 - y^2}\right)^2 dy$$

and
$$V = \int_0^a \pi \frac{h^2}{a^2}\left(a - \sqrt{a^2 - y^2}\right)^2 dy$$

Let
$$y = a \sin\theta \Rightarrow dy = a\cos\theta\, d\theta$$

Then
$$\begin{aligned}
V &= \pi \frac{h^2}{a^2}\int_0^{\pi/2}\left(a - \sqrt{a^2 - a^2\sin^2\theta}\right)^2 a\cos\theta\, d\theta \\
&= \pi \frac{h^2}{a^2}\int_0^{\pi/2}\left[a^2 - 2a(a\cos\theta) + (a^2 - a^2\sin^2\theta)\right] a\cos\theta\, d\theta \\
&= \pi a h^2 \int_0^{\pi/2}(2\cos\theta - 2\cos^2\theta - \sin^2\theta\cos\theta)\, d\theta \\
&= \pi a h^2 \left[2\sin\theta - 2\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} \\
&= \pi a h^2 \left[2 - 2\left(\frac{\frac{\pi}{2}}{2}\right) - \frac{1}{3}\right] \\
&= 0.095870\pi a h^2
\end{aligned}$$

and

$$\int \bar{y}_{EL}\,dV = \int_0^a y\left[\pi\frac{h^2}{a^2}\left(a-\sqrt{a^2-y^2}\right)^2 dy\right]$$

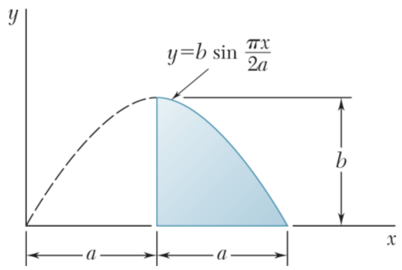
$$= \pi\frac{h^2}{a^2}\int_0^a\left(2a^2y - 2ay\sqrt{a^2-y^2} - y^3\right)dy$$

$$= \pi\frac{h^2}{a^2}\left[a^2y^2 + \frac{2}{3}a(a^2-y^2)^{3/2} - \frac{1}{4}y^4\right]_0^a$$

$$= \pi\frac{h^2}{a^2}\left\{\left[a^2(a)^2 - \frac{1}{4}a^4\right] - \left[\frac{2}{3}a(a^2)^{3/2}\right]\right\}$$

$$= \frac{1}{12}\pi a^2h^2$$

Now $\bar{y}V = \int \bar{y}_{EL}\,dV$: $\bar{y}(0.095870\pi ah^2) = \frac{1}{12}\pi a^2h^2$ or $\bar{y} = 0.869a$ ◀



PROBLEM 5.128*

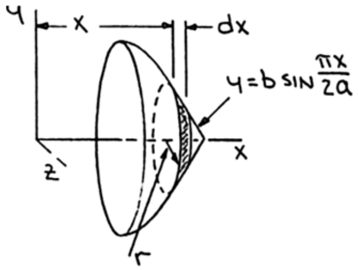
Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the *x*-axis.

SOLUTION

First, note that symmetry implies $\bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius *r* and thickness *dx*.



Then $dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$

Now $r = b\sin\dfrac{\pi x}{2a}$

so that $dV = \pi b^2 \sin^2\dfrac{\pi x}{2a}dx$

Then
$$V = \int_a^{2a} \pi b^2 \sin^2\frac{\pi x}{2a}dx$$
$$= \pi b^2\left[\frac{x}{2} - \frac{\sin\frac{\pi x}{2a}}{2\frac{\pi}{a}}\right]_a^{2a}$$
$$= \pi b^2\left[\left(\tfrac{2a}{2}\right) - \left(\tfrac{a}{2}\right)\right]$$
$$= \frac{1}{2}\pi ab^2$$

and
$$\int \bar{x}_{EL}dV = \int_a^{2a} x\left(\pi b^2 \sin^2\frac{\pi x}{2a}dx\right)$$

Use integration by parts with

$$u = x \qquad dV = \sin^2\frac{\pi x}{2a}$$
$$du = dx \qquad V = \frac{x}{2} - \frac{\sin\frac{\pi x}{a}}{\frac{2\pi}{a}}$$

PROBLEM 5.128* (Continued)

Then
$$\int \overline{x}_{EL}\,dV = \pi b^2\left\{\left[x\left(\frac{x}{2}-\frac{\sin\frac{\pi x}{a}}{\frac{2\pi}{a}}\right)\right]_a^{2a} - \int_a^{2a}\left(\frac{x}{2}-\frac{\sin\frac{\pi x}{a}}{\frac{2\pi}{a}}\right)dx\right\}$$

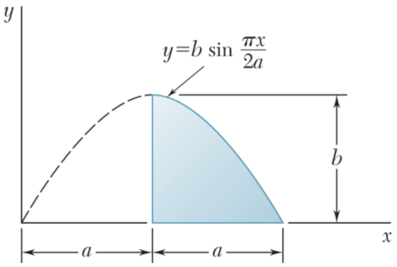
$$= \pi b^2\left\{\left[2a\left(\frac{2a}{2}\right)-a\left(\frac{a}{2}\right)\right]-\left[\frac{1}{4}x^2+\frac{a^2}{2\pi^2}\cos\frac{\pi x}{a}\right]_a^{2a}\right\}$$

$$= \pi b^2\left\{\left(\frac{3}{2}a^2\right)-\left[\frac{1}{4}(2a)^2+\frac{a^2}{2\pi^2}-\frac{1}{4}(a)^2+\frac{a^2}{2\pi^2}\right]\right\}$$

$$= \pi a^2 b^2\left(\frac{3}{4}-\frac{1}{\pi^2}\right)$$

$$= 0.64868\pi a^2 b^2$$

Now
$$\overline{x}V = \int \overline{x}_{EL}\,dV: \quad \overline{x}\left(\frac{1}{2}\pi a b^2\right) = 0.64868\pi a^2 b^2 \qquad \text{or} \quad \overline{x} = 1.297a \blacktriangleleft$$



PROBLEM 5.129*

Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the *y*-axis. (*Hint:* Use a thin cylindrical shell of radius *r* and thickness *dr* as the element of volume.)

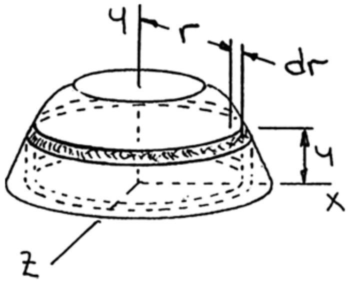
SOLUTION

First note that symmetry implies $\bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a cylindrical shell of radius *r* and thickness *dr*.

Then $dV = (2\pi r)(y)(dr), \quad \bar{y}_{EL} = \frac{1}{2}y$



Now $y = b\sin\frac{\pi r}{2a}$

so that $dV = 2\pi br\sin\frac{\pi r}{2a}dr$

Then $V = \int_a^{2a} 2\pi br\sin\frac{\pi r}{2a}dr$

Use integration by parts with

$$u = rd \qquad dv = \sin\frac{\pi r}{2a}dr$$

$$du = dr \qquad v = -\frac{2a}{\pi}\cos\frac{\pi r}{2a}$$

Then

$$V = 2\pi b\left\{\left[(r)\left(-\frac{2a}{\pi}\cos\frac{\pi r}{2a}\right)\right]_a^{2a} - \int_a^{2a}\left(\frac{2a}{\pi}\cos\frac{\pi r}{2a}\right)dr\right\}$$

$$= 2\pi b\left\{-\frac{2a}{\pi}\left[(2a)(-1)\right] + \left[\frac{4a^2}{\pi^2}\sin\frac{\pi r}{2a}\right]_a^{2a}\right\}$$

$$V = 2\pi b\left(\frac{4a^2}{\pi} - \frac{4a^2}{\pi^2}\right)$$

$$= 8a^2b\left(1 - \frac{1}{\pi}\right)$$

$$= 5.4535a^2b$$

PROBLEM 5.129* (Continued)

Also
$$\int \bar{y}_{EL}\,dV = \int_a^{2a}\left(\tfrac{1}{2}b\sin\frac{\pi r}{2a}\right)\left(2\pi br\sin\frac{\pi r}{2a}dr\right)$$
$$= \pi b^2\int_a^{2a} r\sin^2\frac{\pi r}{2a}dr$$

Use integration by parts with

$$u = r \qquad dv = \sin^2\frac{\pi r}{2a}dr$$
$$du = dr \qquad v = \frac{r}{2} - \frac{\sin\frac{\pi r}{a}}{\frac{2\pi}{a}}$$

Then
$$\int \bar{y}_{EL}\,dV = \pi b^2\left\{\left[(r)\left(\frac{r}{2}-\frac{\sin\frac{\pi r}{a}}{\frac{2\pi}{a}}\right)\right]_a^{2a} - \int_a^{2a}\left(\frac{r}{2}-\frac{\sin\frac{\pi r}{a}}{\frac{2\pi}{a}}\right)dr\right\}$$
$$= \pi b^2\left\{\left[(2a)\left(\frac{2a}{2}\right)-(a)\left(\frac{a}{2}\right)\right]-\left[\frac{r^2}{4}+\frac{a^2}{2\pi^2}\cos\frac{\pi r}{a}\right]_a^{2a}\right\}$$
$$= \pi b^2\left\{\frac{3}{2}a^2 - \left[\frac{(2a)^2}{4}+\frac{a^2}{2\pi^2}-\frac{(a)^2}{4}+\frac{a^2}{2\pi^2}\right]\right\}$$
$$= \pi a^2b^2\left(\frac{3}{4}-\frac{1}{\pi^2}\right)$$
$$= 2.0379a^2b^2$$

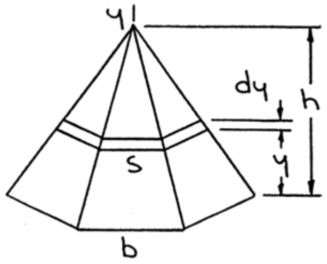
Now $\bar{y}V = \int \bar{y}_{EL}\,dV\colon \quad \bar{y}(5.4535a^2b) = 2.0379a^2b^2$ or $\bar{y} = 0.374b$ ◀

PROBLEM 5.130*

Show that for a regular pyramid of height h and n sides $(n = 3, 4, \ldots)$ the centroid of the volume of the pyramid is located at a distance $h/4$ above the base.

SOLUTION

Choose as the element of a horizontal slice of thickness dy. For any number N of sides, the area of the base of the pyramid is given by



$$A_{\text{base}} = kb^2$$

where $k = k(N);$ see note below. Using similar triangles, we have

$$\frac{s}{b} = \frac{h-y}{h}$$

or $$s = \frac{b}{h}(h-y)$$

Then $$dV = A_{\text{slice}}dy = ks^2dy = k\frac{b^2}{h^2}(h-y)^2\,dy$$

and $$V = \int_0^h k\frac{b^2}{h^2}(h-y)^2\,dy = k\frac{b^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_0^h$$

$$= \frac{1}{3}kb^2h$$

Also, $$\overline{y}_{EL} = y$$

so that $$\int \overline{y}_{EL}dV = \int_0^h y\left[k\frac{b^2}{h^2}(h-y)^2\,dy\right] = k\frac{b^2}{h^2}\int_0^h (h^2y - 2hy^2 + y^3)\,dy$$

$$= k\frac{b^2}{h^2}\left[\frac{1}{2}h^2y^2 - \frac{2}{3}hy^3 + \frac{1}{4}y^4\right]_0^h = \frac{1}{12}kb^2h^2$$

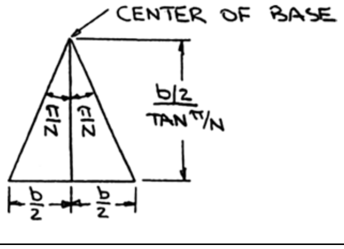
Now $$\overline{y}V = \int \overline{y}_{EL}dV: \quad \overline{y}\left(\frac{1}{3}kb^2h\right) = \frac{1}{12}kb^2h^2$$

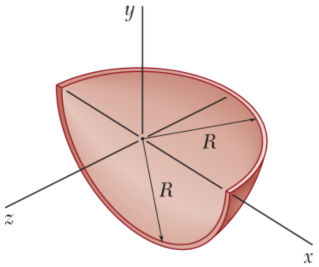
or $y = \dfrac{1}{4}h$ Q.E.D. ◀

Note: $$A_{\text{base}} = N\left(\frac{1}{2}\times b\times\frac{\frac{b}{2}}{\tan\frac{\pi}{N}}\right)$$

$$= \frac{N}{4\tan\frac{\pi}{N}}b^2$$

$$= k(N)b^2$$





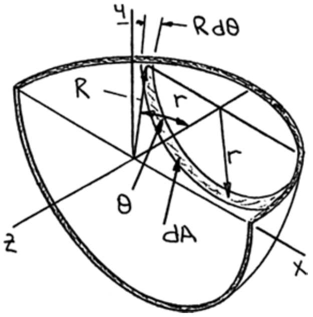
PROBLEM 5.131

Determine by direct integration the location of the centroid of one-half of a thin, uniform hemispherical shell of radius *R*.

SOLUTION

First note that symmetry implies $\bar{x} = 0$ ◀

The element of area *dA* of the shell shown is obtained by cutting the shell with two planes parallel to the *xy* plane. Now



$$dA = (\pi r)(R\,d\theta)$$

$$\bar{y}_{EL} = -\frac{2r}{\pi}$$

where $r = R\sin\theta$

so that $dA = \pi R^2 \sin\theta\, d\theta$

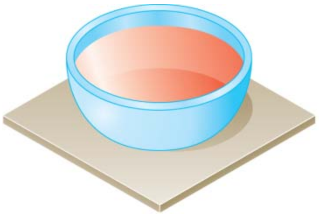
$$\bar{y}_{EL} = -\frac{2R}{\pi}\sin\theta$$

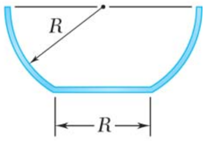
Then
$$A = \int_0^{\pi/2} \pi R^2 \sin\theta\, d\theta = \pi R^2[-\cos\theta]_0^{\pi/2}$$
$$= \pi R^2$$

and
$$\int \bar{y}_{EL}\, dA = \int_0^{\pi/2}\left(-\frac{2R}{\pi}\sin\theta\right)(\pi R^2 \sin\theta\, d\theta)$$
$$= -2R^3\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$
$$= -\frac{\pi}{2}R^3$$

Now $\bar{y}A = \int \bar{y}_{EL}\, dA$: $\bar{y}(\pi R^2) = -\frac{\pi}{2}R^3$ or $\bar{y} = -\frac{1}{2}R$ ◀

Symmetry implies $\bar{z} = \bar{y}$ $\bar{z} = -\frac{1}{2}R$ ◀





PROBLEM 5.132

The sides and the base of a punch bowl are of uniform thickness t. If $t << R$ and $R = 250$ mm, determine the location of the center of gravity of (*a*) the bowl, (*b*) the punch.

SOLUTION

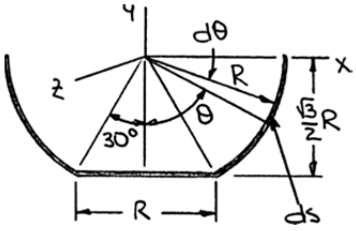
(*a*) Bowl:

First note that symmetry implies $\bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

for the coordinate axes shown below. Now assume that the bowl may be treated as a shell; the center of gravity of the bowl will coincide with the centroid of the shell. For the walls of the bowl, an element of area is obtained by rotating the arc ds about the y-axis. Then

$$dA_{\text{wall}} = (2\pi R\sin\theta)(R\,d\theta)$$



and $$(\bar{y}_{EL})_{\text{wall}} = -R\cos\theta$$

Then
$$A_{\text{wall}} = \int_{\pi/6}^{\pi/2} 2\pi R^2 \sin\theta\,d\theta$$
$$= 2\pi R^2[-\cos\theta]_{\pi/6}^{\pi/2}$$
$$= \pi\sqrt{3}R^2$$

and
$$\bar{y}_{\text{wall}}A_{\text{wall}} = \int(\bar{y}_{EL})_{\text{wall}}\,dA$$
$$= \int_{\pi/6}^{\pi/2}(-R\cos\theta)(2\pi R^2\sin\theta d\theta)$$
$$= \pi R^3[\cos^2\theta]_{\pi/6}^{\pi/2}$$
$$= -\frac{3}{4}\pi R^3$$

By observation,
$$A_{\text{base}} = \frac{\pi}{4}R^2, \quad \bar{y}_{\text{base}} = -\frac{\sqrt{3}}{2}R$$

Now
$$\bar{y}\Sigma A = \Sigma\bar{y}A$$

or
$$\bar{y}\left(\pi\sqrt{3}R^2 + \frac{\pi}{4}R^2\right) = -\frac{3}{4}\pi R^3 + \frac{\pi}{4}R^2\left(-\frac{\sqrt{3}}{2}R\right)$$

or $\bar{y} = -0.48763R \quad R = 250$ mm $\qquad \bar{y} = -121.9$ mm ◀

PROBLEM 5.132 (Continued)

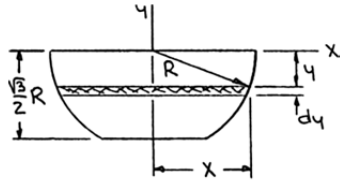
(*b*) Punch:

First note that symmetry implies $\overline{x} = 0$ ◀

$\overline{z} = 0$ ◀

and that because the punch is homogeneous, its center of gravity will coincide with the centroid of the corresponding volume. Choose as the element of volume a disk of radius *x* and thickness *dy*. Then

$$dV = \pi x^2 dy, \quad \overline{y}_{EL} = y$$



Now $x^2 + y^2 = R^2$

so that $dV = \pi(R^2 - y^2)dy$

Then

$$V = \int_{-\sqrt{3}/2R}^{0} \pi(R^2 - y^2)\,dy$$

$$= \pi\left[R^2 y - \frac{1}{3}y^3\right]_{-\sqrt{3}/2R}^{0}$$

$$= -\pi\left[R^2\left(-\frac{\sqrt{3}}{2}R\right) - \frac{1}{3}\left(-\frac{\sqrt{3}}{2}R\right)^3\right] = \frac{3}{8}\pi\sqrt{3}R^3$$

and

$$\int \overline{y}_{EL}\,dV = \int_{-\sqrt{3}/2R}^{0} (y)\left[\pi\left(R^2 - y^2\right)dy\right]$$

$$= \pi\left[\frac{1}{2}R^2 y^2 - \frac{1}{4}y^4\right]_{-\sqrt{3}/2R}^{0}$$

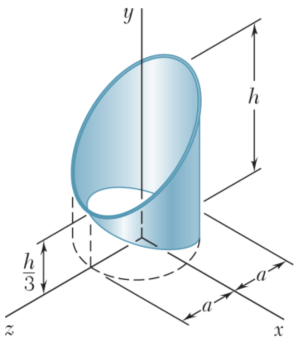
$$= -\pi\left[\frac{1}{2}R^2\left(-\frac{\sqrt{3}}{2}R\right)^2 - \frac{1}{4}\left(-\frac{\sqrt{3}}{2}R\right)^4\right] = -\frac{15}{64}\pi R^4$$

Now

$$\overline{y}V = \int \overline{y}_{EL}\,dV: \quad \overline{y}\left(\frac{3}{8}\pi\sqrt{3}R^3\right) = -\frac{15}{64}\pi R^4$$

or

$$\overline{y} = -\frac{5}{8\sqrt{3}}R \quad R = 250\text{ mm} \quad \overline{y} = -90.2\text{ mm}$$ ◀



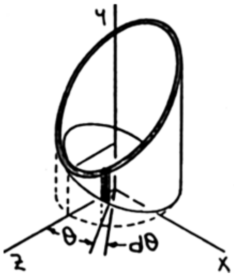
PROBLEM 5.133

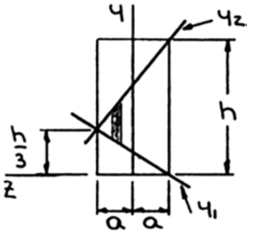
Locate the centroid of the section shown, which was cut from a thin circular pipe by two oblique planes.

SOLUTION

First note that symmetry implies $\overline{x} = 0$ ◀

Assume that the pipe has a uniform wall thickness t and choose as the element of volume a vertical strip of width $a\,d\theta$ and height $(y_2 - y_1)$. Then





$$dV = (y_2 - y_1)ta\,d\theta, \quad \overline{y}_{EL} = \frac{1}{2}(y_1 + \overline{y}_2)\,\overline{z}_{EL} = z$$

Now
$$y_1 = \frac{\frac{h}{3}}{2a}z + \frac{h}{6} \qquad y_2 = -\frac{\frac{2h}{3}}{2a}z + \frac{2}{3}h$$
$$= \frac{h}{6a}(z + a) \qquad = \frac{h}{3a}(-z + 2a)$$

and
$$z = a\cos\theta$$

Then
$$(y_2 - y_1) = \frac{h}{3a}(-a\cos\theta + 2a) - \frac{h}{6a}(a\cos\theta + a)$$
$$= \frac{h}{2}(1 - \cos\theta)$$

and
$$(y_1 + y_2) = \frac{h}{6a}(a\cos\theta + a) + \frac{h}{3a}(-a\cos\theta + 2a)$$
$$= \frac{h}{6}(5 - \cos\theta)$$
$$dV = \frac{aht}{2}(1 - \cos\theta)d\theta \quad \overline{y}_{EL} = \frac{h}{12}(5 - \cos\theta), \quad \overline{z}_{EL} = a\cos\theta$$

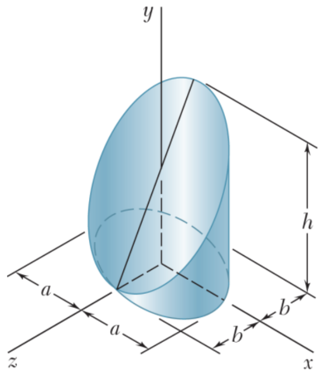
Then
$$V = 2\int_0^{\pi} \frac{aht}{2}(1-\cos\theta)d\theta = aht[\theta - \sin\theta]_0^{\pi}$$
$$= \pi aht$$

and
$$\int \overline{y}_{EL}dV = 2\int_0^{\pi} \frac{h}{12}(5-\cos\theta)\left[\frac{aht}{2}(1-\cos\theta)d\theta\right]$$
$$= \frac{ah^2t}{12}\int_0^{\pi}(5 - 6\cos\theta + \cos^2\theta)d\theta$$
$$= \frac{ah^2t}{12}\left[5\theta - 6\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$
$$= \frac{11}{24}\pi ah^2t$$

$$\int \overline{z}_{EL}dV = 2\int_0^{\pi} a\cos\theta\left[\frac{aht}{2}(1-\cos\theta)d\theta\right]$$
$$= a^2ht\left[\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}$$
$$= -\frac{1}{2}\pi a^2ht$$

Now
$$\overline{y}V = \int \overline{y}_{EL}dV: \quad \overline{y}(\pi aht) = \frac{11}{24}\pi ah^2t$$
or $\overline{y} = \frac{11}{24}h$ ◀

and
$$\overline{z}V = \int \overline{z}_{EL}dV: \quad \overline{z}(\pi aht) = -\frac{1}{2}\pi a^2ht$$
or $\overline{z} = -\frac{1}{2}a$ ◀

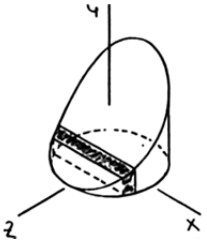


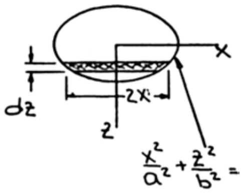
PROBLEM 5.134*

Locate the centroid of the section shown, which was cut from an elliptical cylinder by an oblique plane.

SOLUTION

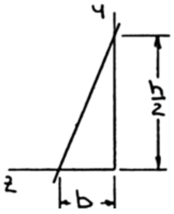
First note that symmetry implies $x = 0$ ◀





Choose as the element of volume a vertical slice of width *zx*, thickness *dz*, and height *y*. Then

$$dV = 2xy\,dz, \quad \overline{y}_{EL} = \frac{1}{24}, \quad \overline{z}_{EL} = z$$



Now

$$x = \frac{a}{b}\sqrt{b^2 - z^2}$$

and

$$y = -\frac{h/2}{b}z + \frac{h}{2} = \frac{h}{2b}(b - z)$$

Then

$$V = \int_{-b}^{b}\left(2\frac{a}{b}\sqrt{b^2 - z^2}\right)\left[\frac{h}{2b}(b - z)\right]dz$$

Let

$$z = b\sin\theta \quad dz = b\cos\theta\, d\theta$$

Then

$$V = \frac{ah}{b^2}\int_{\pi/2}^{\pi/2}(b\cos\theta)[b(1 - \sin\theta)]b\cos\theta\, d\theta$$

$$= abh\int_{-\pi/2}^{\pi/2}(\cos^2\theta - \sin\theta\cos^2\theta)\,d\theta$$

$$= abh\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} + \frac{1}{3}\cos^3\theta\right]_{-\pi/2}^{\pi/2}$$

$$V = \frac{1}{2}\pi abh$$

PROBLEM 5.134* (Continued)

and
$$\int \bar{y}_{EL}\,dV = \int_{-b}^{b}\left[\frac{1}{2}\times\frac{h}{2b}(b-z)\right]\left\{\left(2\frac{a}{b}\sqrt{b^2-z^2}\right)\left[\frac{h}{2b}(b-z)\right]dz\right\}$$
$$=\frac{1}{4}\frac{ah^2}{b^3}\int_{-b}^{b}(b-z)^2\sqrt{b^2-z^2}\,dz$$

Let
$$z=b\sin\theta\quad dz=b\cos\theta\,d\theta$$

Then
$$\int \bar{y}_{EL}\,dV=\frac{1}{4}\frac{ah^2}{b^3}\int_{-\pi/2}^{\pi/2}[b(1-\sin\theta)]^2(b\cos\theta)\times(b\cos\theta\,d\theta)$$
$$=\frac{1}{4}abh^2\int_{-\pi/2}^{\pi/2}(\cos^2\theta-2\sin\theta\cos^2\theta+\sin^2\theta\cos^2\theta)\,d\theta$$

Now
$$\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)\quad \cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$$

so that
$$\sin^2\theta\cos^2\theta=\frac{1}{4}(1-\cos^2 2\theta)$$

Then
$$\int \bar{y}_{EL}\,dV=\frac{1}{4}abh^2\int_{-\pi/2}^{\pi/2}\left[\cos^2\theta-2\sin\theta\cos^2\theta+\frac{1}{4}(1-\cos^2 2\theta)\right]d\theta$$
$$=\frac{1}{4}abh^2\left[\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)+\frac{1}{3}\cos^3\theta+\frac{1}{4}\theta-\frac{1}{4}\left(\frac{\theta}{2}+\frac{\sin 4\theta}{8}\right)\right]_{-\pi/2}^{\pi/2}$$
$$=\frac{5}{32}\pi abh^2$$

Also,
$$\int \bar{z}_{EL}\,dV=\int_{-b}^{b}z\left\{2\frac{a}{b}\sqrt{a^2-z^2}\left[\frac{h}{2b}(b-z)\right]dz\right\}$$
$$=\frac{ah}{b^2}\int_{-b}^{b}z(b-z)\sqrt{b^2-z^2}\,dz$$

Let
$$z=b\sin\theta\quad dz=b\cos\theta\,d\theta$$

Then
$$\int \bar{z}_{EL}\,dV=\frac{ah}{b^2}\int_{-\pi/2}^{\pi/2}(b\sin\theta)[b(1-\sin\theta)](b\cos\theta)\times(b\cos\theta\,d\theta)$$
$$=ab^2h\int_{-\pi/2}^{\pi/2}(\sin\theta\cos^2\theta-\sin^2\theta\cos^2\theta)d\theta$$

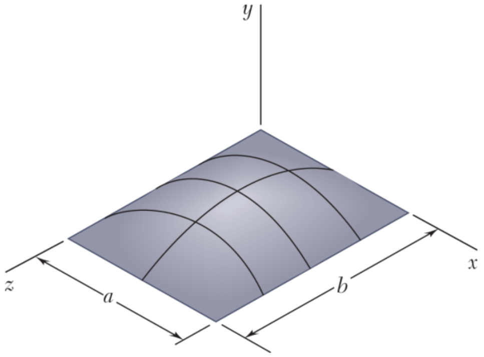
Using
$$\sin^2\theta\cos^2\theta=\frac{1}{4}(1-\cos^2 2\theta)\ \text{ from above,}$$

$$\int z_{EL}\,dV=ab^2h\int_{-\pi/2}^{\pi/2}\left[\sin\theta\cos^2\theta-\frac{1}{4}(1-\cos^2 2\theta)\right]d\theta$$
$$=ab^2h\left[-\frac{1}{3}\cos^3\theta-\frac{1}{4}\theta+\frac{1}{4}\left(\frac{\theta}{2}+\frac{\sin 4\theta}{8}\right)\right]_{-\pi/2}^{\pi/2}=-\frac{1}{8}\pi ab^2h$$

PROBLEM 5.134* (Continued)

Now $\bar{y}V = \int \bar{y}_{EL}\,dV: \quad \bar{y}\left(\frac{1}{2}\pi\,abh\right) = \frac{5}{32}\pi\,abh^2$ or $\bar{y} = \frac{5}{16}h$ ◀

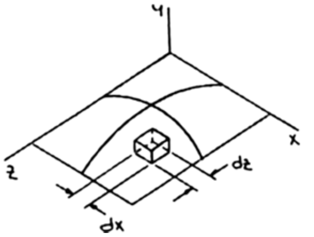
and $\bar{z}\,V = \int \bar{z}_{EL}\,dV: \quad \bar{z}\left(\frac{1}{2}\pi\,abh\right) = -\frac{1}{8}\pi\,ab^2h$ or $\bar{z} = -\frac{1}{4}b$ ◀



PROBLEM 5.135

Determine by direct integration the location of the centroid of the volume between the *xz* plane and the portion shown of the surface $y = 16h(ax - x^2)(bz - z^2)/a^2b^2$.

SOLUTION



First note that symmetry implies $\bar{x} = \frac{a}{2}$ ◀

$\bar{z} = \frac{b}{2}$ ◀

Choose as the element of volume a filament of base $dx \times dz$ and height y. Then

$$dV = y\,dx\,dz, \quad \bar{y}_{EL} = \frac{1}{2}y$$

or $$dV = \frac{16h}{a^2b^2}(ax - x^2)(bz - z^2)dx\,dz$$

Then $$V = \int_0^b \int_0^a \frac{16h}{a^2b^2}(ax - x^2)(bz - z^2)dx\,dz$$

$$V = \frac{16h}{a^2b^2}\int_0^b (bz - z^2)\left[\frac{a}{z}x^2 - \frac{1}{3}x^3\right]_0^a dz$$

$$= \frac{16h}{a^2b^2}\left[\frac{a}{2}(a^2) - \frac{1}{3}(a)^3\right]\left[\frac{b}{2}z^2 - \frac{1}{3}z^3\right]_0^b$$

$$= \frac{8ah}{3b^2}\left[\frac{b}{2}(b)^2 - \frac{1}{3}(b)^3\right]$$

$$= \frac{4}{9}abh$$

and $$\int \bar{y}_{EL}\,dV = \int_0^b \int_0^a \frac{1}{2}\left[\frac{16h}{a^2b^2}(ax - x^2)(bz - z^2)\right]\left[\frac{16h}{a^2b^2}(ax - x^2)(bz - z^2)dx\,dz\right]$$

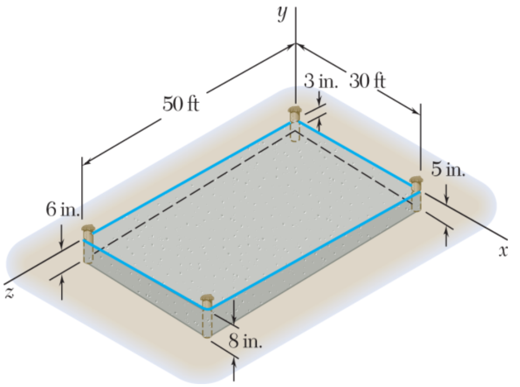
$$= \frac{128h^2}{a^4b^4}\int_0^b \int_0^a (a^2x^2 - 2ax^3 + x^4)(b^2z^2 - 2bz^3 + z^4)dx\,dz$$

$$= \frac{128h^2}{a^2b^4}\int_0^b (b^2z^2 - 2bz^3 + z^4)\left[\frac{a^2}{3}x^3 - \frac{a}{2}x^4 + \frac{1}{5}x^5\right]_0^a dz$$

$$= \frac{128h^2}{a^4b^4}\left[\frac{a^2}{3}(a)^3 - \frac{a}{2}(a)^4 + \frac{1}{5}(a)^5\right]\left[\frac{b^2}{3}z^3 - \frac{b}{z}z^4 + \frac{1}{5}z^5\right]_0^b$$

$$= \frac{64ah^2}{15b^4}\left[\frac{b^3}{3}(b)^3 - \frac{b}{2}(b)^4 + \frac{1}{5}(b)^5\right] = \frac{32}{225}abh^2$$

Now $$\bar{y}V = \int \bar{y}_{EL}\,dV: \quad \bar{y}\left(\frac{4}{9}abh\right) = \frac{32}{225}abh^2$$ or $\bar{y} = \dfrac{8}{25}h$ ◀

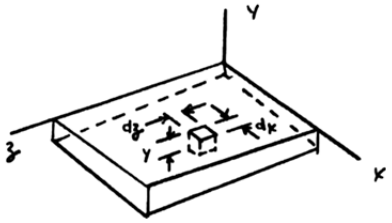


PROBLEM 5.136

After grading a lot, a builder places four stakes to designate the corners of the slab for a house. To provide a firm, level base for the slab, the builder places a minimum of 3 in. of gravel beneath the slab. Determine the volume of gravel needed and the x coordinate of the centroid of the volume of the gravel. (*Hint:* The bottom surface of the gravel is an oblique plane, which can be represented by the equation $y = a + bx + cz$.)

SOLUTION

The centroid can be found by integration. The equation for the bottom of the gravel is $y = a + bx + cz$, where the constants a, b, and c can be determined as follows:



For $x = 0$ and $z = 0$, $y = -3$ in., and therefore,

$$-\frac{3}{12}\text{ ft} = a, \quad \text{or} \quad a = -\frac{1}{4}\text{ ft}$$

For $x = 30$ ft and $z = 0$, $y = -5$ in., and therefore,

$$-\frac{5}{12}\text{ ft} = -\frac{1}{4}\text{ ft} + b(30\text{ ft}), \quad \text{or} \quad b = -\frac{1}{180}$$

For $x = 0$ and $z = 50$ ft, $y = -6$ in., and therefore,

$$-\frac{6}{12}\text{ ft} = -\frac{1}{4}\text{ ft} + c(50\text{ ft}), \quad \text{or} \quad c = -\frac{1}{200}$$

Therefore,

$$y = -\frac{1}{4}\text{ ft} - \frac{1}{180}x - \frac{1}{200}z$$

Now

$$\overline{x} = \frac{\int x_{EL}\,dV}{V}$$

A volume element can be chosen as

$$dV = |y|\,dx\,dz$$

or

$$dV = \frac{1}{4}\left(1 + \frac{1}{45}x + \frac{1}{50}z\right)dx\,dz$$

and

$$\overline{x}_{EL} = x$$

PROBLEM 5.136 (Continued)

Then
$$\int x_{EL}\,dV = \int_0^{50}\int_0^{30}\frac{x}{4}\left(1+\frac{1}{45}x+\frac{1}{50}z\right)dx\,dz$$

$$=\frac{1}{4}\int_0^{50}\left[\frac{x^2}{2}+\frac{1}{135}x^3+\frac{z}{100}x^2\right]_0^{30}dz$$

$$=\frac{1}{4}\int_0^{50}(650+9z)\,dz$$

$$=\frac{1}{4}\left[650\,z+\frac{9}{2}z^2\right]_0^{50}$$

$$=10937.5\text{ ft}^4$$

The volume is
$$V\int dV = \int_0^{50}\int_0^{30}\frac{1}{4}\left(1+\frac{1}{45}x+\frac{1}{50}z\right)dx\,dz$$

$$=\frac{1}{4}\int_0^{50}\left[x+\frac{1}{90}x^2+\frac{z}{50}x\right]_0^{30}dz$$

$$=\frac{1}{4}\int_0^{50}\left(40+\frac{3}{5}z\right)dz$$

$$=\frac{1}{4}\left[40z+\frac{3}{10}z^2\right]_0^{50}$$

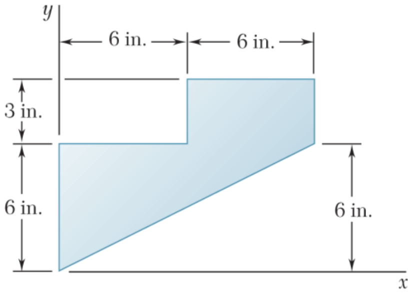
$$=687.50\text{ ft}^3$$

Then
$$\bar{x}=\frac{\int \bar{x}_{EL}\,dV}{V}=\frac{10937.5\,\text{ft}^4}{687.5\text{ ft}^3}=15.9091\text{ ft}$$

Therefore,

$V = 688\text{ ft}^3$ ◀

$\bar{x} = 15.91\text{ ft}$ ◀

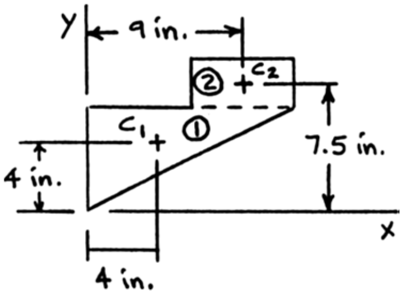


y
6 in.
6 in.
3 in.
6 in.
6 in.
x

PROBLEM 5.137

Locate the centroid of the plane area shown.

SOLUTION



y
9 in.
②
C_2
C_1
①
7.5 in.
4 in.
x
4 in.

	A, in^2	$\bar{x}$, in	$\bar{y}$, in	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$\frac{1}{2}(12)(6) = 36$	4	4	144	144
2	$(6)(3) = 18$	9	7.5	162	135
Σ	54			306	279

Then

$$\bar{X}A = \Sigma\bar{x}A$$

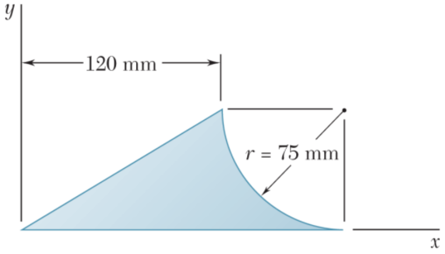
$$\bar{X}(54) = 306$$

$\bar{X} = 5.67$ in. ◀

$$\bar{Y}A = \Sigma\bar{y}A$$

$$\bar{Y}(54) = 279$$

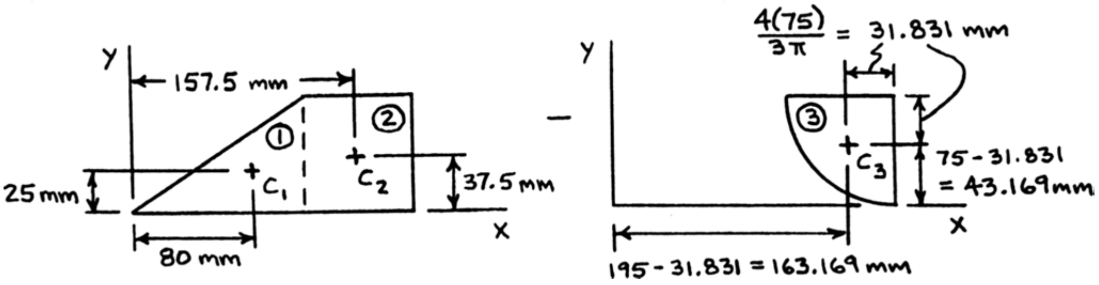
$\bar{Y} = 5.17$ in. ◀


y
120 mm
r = 75 mm
x

PROBLEM 5.138

Locate the centroid of the plane area shown.

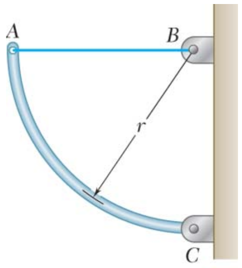
SOLUTION


y
157.5 mm
①
②
C_1
C_2
25 mm
37.5 mm
80 mm
x
−
y
$\frac{4(75)}{3\pi} = 31.831$ mm
③
C_3
75 − 31.831
= 43.169 mm
x
195 − 31.831 = 163.169 mm

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{1}{2}(120)(75) = 4500$	80	25	360×10^3	112.5×10^3
2	$(75)(75) = 5625$	157.5	37.5	885.94×10^3	210.94×10^3
3	$-\frac{\pi}{4}(75)^2 = -4417.9$	163.169	43.169	-720.86×10^3	-190.716×10^3
Σ	5707.1			525.08×10^3	132.724×10^3

Then $\bar{X}A = \Sigma \bar{x}A \qquad \bar{X}(5707.1) = 525.08 \times 10^3 \qquad \bar{X} = 92.0$ mm ◀

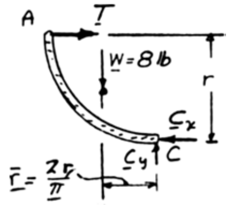
$\bar{Y}A = \Sigma \bar{y}A \qquad \bar{Y}(5707.1) = 132.724 \times 10^3 \qquad \bar{Y} = 23.3$ mm ◀



PROBLEM 5.139

A uniform circular rod of weight 8 lb and radius 10 in. is attached to a pin at *C* and to the cable *AB*. Determine (*a*) the tension in the cable, (*b*) the reaction at *C*.

SOLUTION



For quarter circle, $\bar{r} = \dfrac{2r}{\pi}$

(*a*) $\overset{+}{\circlearrowleft}\Sigma M_C = 0\text{:}\quad W\left(\dfrac{2r}{\pi}\right) - Tr = 0$

$$T = W\left(\frac{2}{\pi}\right) = (8\text{ lb})\left(\frac{2}{\pi}\right)$$

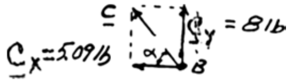
$T = 5.09$ lb ◀

(*b*) $\xrightarrow{+}\Sigma F_x = 0\text{:}\quad T - C_x = 0 \quad 5.09\text{ lb} - C_x = 0$

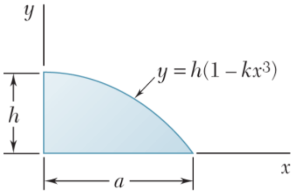
$\mathbf{C}_x = 5.09$ lb ⟵

$+\!\uparrow\Sigma F_y = 0\text{:}\quad C_y - W = 0 \qquad C_y - 8\text{ lb} = 0$

$\mathbf{C}_y = 8$ lb ↑



$\mathbf{C} = 9.48$ lb ∠ 57.5° ◀



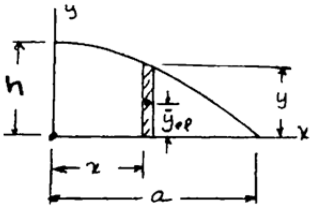
PROBLEM 5.140

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *h*.

SOLUTION

$$y = h(1 - kx^3)$$

For $x = a, \ y = 0.$



$$0 = h(1 - k\,a^3)$$

$$\therefore \quad k = \frac{1}{a^3}$$

$$y = h\left(1 - \frac{x^3}{a^3}\right)$$

$$\overline{x}_{EL} = x, \quad \overline{y}_{EL} = \frac{1}{2}y \quad dA = ydx$$

$$A = \int dA = \int_0^a ydx = \int_0^a h\left(1 - \frac{x^3}{a^3}\right)dx = h\left[x - \frac{x^4}{4a^3}\right]_0^a = \frac{3}{4}ah$$

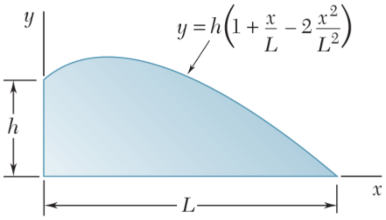
$$\int \overline{x}_{EL}dA = \int_0^a xydx = \int_0^a h\left(x - \frac{x^4}{a^3}\right)dx = h\left[\frac{x^2}{2} - \frac{x^5}{5a^3}\right]_0^a = \frac{3}{10}a^2h$$

$$\int \overline{y}_{EL}dA = \int_0^a \left(\frac{1}{2}y\right)ydx = \frac{1}{2}\int_0^a h^2\left(1 - \frac{x^3}{a^3}\right)dx = \frac{h^2}{2}\int_0^a \left(1 - \frac{2x^3}{a^3} + \frac{x^6}{a^6}\right)dx$$

$$= \frac{h^2}{2}\left[x - \frac{x^4}{2a^3} + \frac{x^7}{7a^6}\right]_0^a = \frac{9}{28}ah^2$$

$$\overline{x}A = \int \overline{x}_{EL}dA: \quad \overline{x}\left(\frac{3}{4}ah\right) = \frac{3}{10}a^2h \qquad \overline{x} = \frac{2}{5}a \ ◀$$

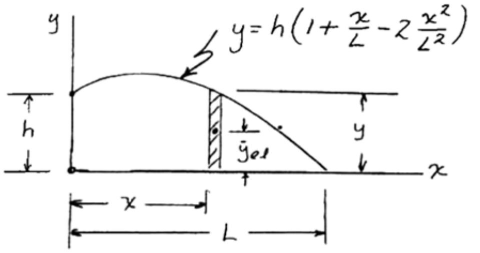
$$\overline{y}A = \int \overline{y}_{EL}dA: \quad \overline{y}\left(\frac{3}{4}ah\right) = \frac{9}{28}ah^2 \qquad \overline{y} = \frac{3}{7}h \ ◀$$



PROBLEM 5.141

Determine by direct integration the centroid of the area shown.

SOLUTION



$$\bar{x}_{EL} = x \quad \bar{y}_{EL} = \frac{1}{2}y \quad dA = y\,dx$$

$$A = \int dA = \int_0^L h\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)dx = h\left[x + \frac{x^2}{2L} - \frac{2}{3}\frac{x^3}{L^2}\right]_0^L = \frac{5}{6}hL$$

$$\int x_{EL}\,dA = \int_0^L xh\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)dx = h\int_0^L \left(x + \frac{x^2}{L} - 2\frac{x^3}{L^2}\right)dx$$

$$= h\left[\frac{x^2}{2} + \frac{1}{3}\frac{x^3}{L} - \frac{2}{4}\frac{x^4}{L^2}\right]_0^L = \frac{1}{3}hL^2$$

$$\bar{x}A = \int x_{EL}\,dA: \quad \bar{x}\left(\frac{5}{6}hL\right) = \frac{1}{3}hL^2 \qquad \bar{x} = \frac{2}{5}L \blacktriangleleft$$

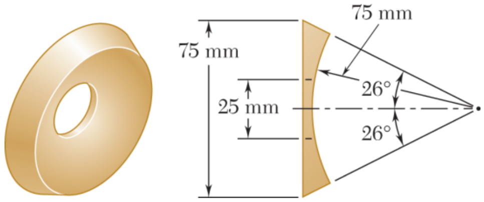
$$A = \frac{5}{6}hL \quad \bar{y}_{EL} = \frac{1}{2}y \quad y = h\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)$$

$$\int \bar{y}_{EL}\,dA = \frac{1}{2}\int y^2 dx = \frac{h^2}{2}\int_0^L \left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)^2 dx$$

$$= \frac{h^2}{2}\int_0^L \left(1 + \frac{x^2}{L^2} + 4\frac{x^4}{L^4} + 2\frac{x}{L} - 4\frac{x^2}{L^2} - 4\frac{x^3}{L^3}\right)dx$$

$$= \frac{h^2}{2}\left[x + \frac{x^3}{3L^2} + \frac{4x^5}{5L^4} + \frac{x^2}{L} - \frac{4x^3}{3L^2} - \frac{x^4}{L^3}\right]_0^L = \frac{4}{10}h^2L$$

$$\bar{y}A = \int y_{EL}\,dA: \quad \bar{y}\left(\frac{5}{6}hL\right) = \frac{4}{10}h^2L \qquad \bar{y} = \frac{12}{25}h \blacktriangleleft$$

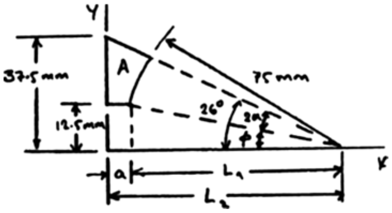


PROBLEM 5.142

The escutcheon (a decorative plate placed on a pipe where the pipe exits from a wall) shown is cast from brass. Knowing that the density of brass is 8470 kg/m^3, determine the mass of the escutcheon.

SOLUTION

The mass of the escutcheon is given by $m = (\text{density})V$, where V is the volume. V can be generated by rotating the area A about the x-axis.



From the figure:

$$L_1 = \sqrt{75^2 - 12.5^2} = 73.9510 \text{ m}$$

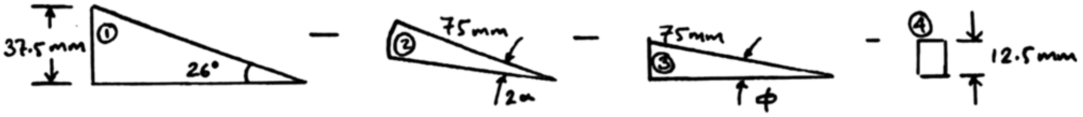
$$L_2 = \frac{37.5}{\tan 26°} = 76.8864 \text{ mm}$$

$$a = L_2 - L_1 = 2.9324 \text{ mm}$$

$$\phi = \sin^{-1}\frac{12.5}{75} = 9.5941°$$

$$\alpha = \frac{26° - 9.5941°}{2} = 8.2030° = 0.143168 \text{ rad}$$

Area A can be obtained by combining the following four areas:



Applying the second theorem of Pappus-Guldinus and using Figure 5.8a, we have

$$V = 2\pi\bar{y}A = 2\pi\Sigma\bar{y}A$$

Seg.	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$\frac{1}{2}(76.886)(37.5) = 1441.61$	$\frac{1}{3}(37.5) = 12.5$	18,020.1
2	$-\alpha(75)^2 = -805.32$	$\frac{2(75)\sin\alpha}{3\alpha}\sin(\alpha+\phi) = 15.2303$	−12,265.3
3	$-\frac{1}{2}(73.951)(12.5) = -462.19$	$\frac{1}{3}(12.5) = 4.1667$	−1925.81
4	$-(2.9354)(12.5) = -36.693$	$\frac{1}{2}(12.5) = 6.25$	−229.33
Σ			3599.7

PROBLEM 5.142 (Continued)

Then $V = 2\pi\Sigma\bar{y}A$

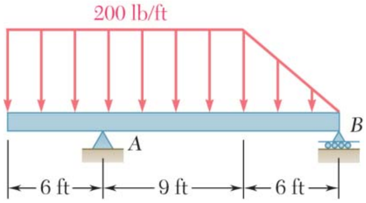
$= 2\pi(3599.7 \text{ mm}^3)$

$= 22,618 \text{ mm}^3$

$m = (\text{density})V$

$= (8470 \text{ kg/m}^3)(22.618 \times 10^{-6} \text{ m}^3)$

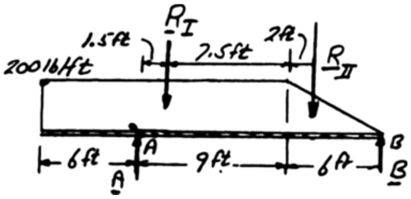
$= 0.191574 \text{ kg}$ or $m = 0.1916 \text{ kg}$ ◀


200 lb/ft
A
B
6 ft
9 ft
6 ft

PROBLEM 5.143

Determine the reactions at the beam supports for the given loading.

SOLUTION


R_I
1.5ft
2ft
7.5ft
R_{II}
200lb/ft
6ft
A
9ft
6ft
B
A
B

$$R_{\text{I}} = (200 \text{ lb/ft})(15 \text{ ft})$$

$$R_{\text{I}} = 3000 \text{ lb}$$

$$R_{\text{II}} = \frac{1}{2}(200 \text{ lb/ft})(6 \text{ ft})$$

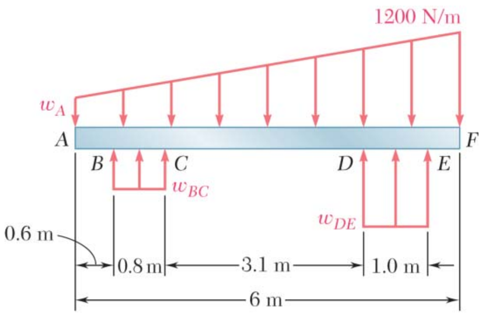
$$R_{\text{II}} = 600 \text{ lb}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad -(3000 \text{ lb})(1.5 \text{ ft}) - (600 \text{ lb})(9 \text{ ft} + 2 \text{ ft}) + B(15 \text{ ft}) = 0$$

$B = 740$ lb **B** $= 740 \text{ lb}\uparrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad A + 740 \text{ lb} - 3000 \text{ lb} - 600 \text{ lb} = 0$$

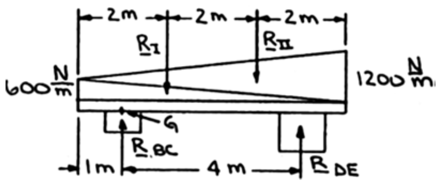
$A = 2860$ lb **A** $= 2860 \text{ lb}\uparrow$ ◀


1200 N/m
w_A
A
F
B
C
D
E
w_{BC}
w_{DE}
0.6 m
0.8 m
3.1 m
1.0 m
6 m

PROBLEM 5.144

A beam is subjected to a linearly distributed downward load and rests on two wide supports *BC* and *DE*, which exert uniformly distributed upward loads as shown. Determine the values of w_{BC} and w_{DE} corresponding to equilibrium when $w_A = 600$ N/m.

SOLUTION


2m
2m
2m
R_I
R_{II}
600 N/m
1200 N/m
G
R_{BC}
1m
4 m
R_{DE}

We have $$R_{\text{I}} = \frac{1}{2}(6\text{ m})(600\text{ N/m}) = 1800\text{ N}$$

$$R_{\text{II}} = \frac{1}{2}(6\text{ m})(1200\text{ N/m}) = 3600\text{ N}$$

$$R_{BC} = (0.8\text{ m})(w_{BC}\text{ N/m}) = (0.8w_{BC})\text{ N}$$

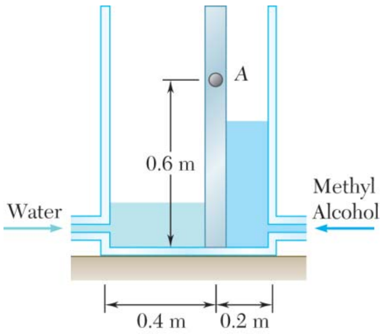
$$R_{DE} = (1.0\text{ m})(w_{DE}\text{ N/m}) = (w_{DE})\text{ N}$$

Then $$+\circlearrowleft \Sigma M_G = 0: \quad -(1\text{ m})(1800\text{ N}) - (3\text{ m})(3600\text{ N}) + (4\text{ m})(w_{DE}\text{ N}) = 0$$

or $$w_{DE} = 3150\text{ N/m} \blacktriangleleft$$

and $$+\uparrow \Sigma F_y = 0: \quad (0.8w_{BC})\text{ N} - 1800\text{ N} - 3600\text{ N} + 3150\text{ N} = 0$$

or $$w_{BC} = 2812.5\text{ N/m} \qquad w_{BC} = 2810\text{ N/m} \blacktriangleleft$$



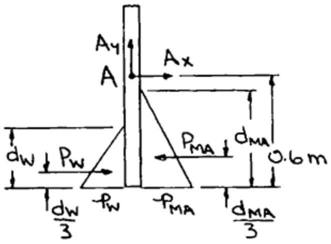
PROBLEM 5.145

A tank is divided into two sections by a 1 × 1-m square gate that is hinged at *A*. A couple of magnitude 490 N · m is required for the gate to rotate. If one side of the tank is filled with water at the rate of 0.1 $\mathrm{m^3}$/min and the other side is filled simultaneously with methyl alcohol (density ρ_{ma} = 789 kg/$\mathrm{m^3}$) at the rate of 0.2 $\mathrm{m^3}$/min, determine at what time and in which direction the gate will rotate.

SOLUTION

Consider the free-body diagram of the gate.

First note $V = A_{\text{base}}d$ and $V = rt$.



Then
$$d_W = \frac{0.1\ \mathrm{m^3/min} \times t(\mathrm{min})}{(0.4\ \mathrm{m})(1\ \mathrm{m})} = 0.25t(\mathrm{m})$$

$$d_{MA} = \frac{0.2\ \mathrm{m^3/min} \times t(\mathrm{min})}{(0.2\ \mathrm{m})(1\ \mathrm{m})} = t(\mathrm{m})$$

Now
$$P = \frac{1}{2}Ap = \frac{1}{2}A(\rho g h) \text{ so that}$$

$$P_W = \frac{1}{2}[(0.25t)\ \mathrm{m} \times (1\ \mathrm{m})][(10^3\ \mathrm{kg/m^3})(9.81\ \mathrm{m/s^2})(0.25t)\ \mathrm{m}]$$
$$= 306.56t^2\ \mathrm{N}$$

$$P_{MA} = \frac{1}{2}[(t)\ \mathrm{m} \times (1\ \mathrm{m})][(789\ \mathrm{kg/m^3})(9.81\ \mathrm{m/s^2})(t)\ \mathrm{m}]$$
$$= 3870t^2\ \mathrm{N}$$

Now assume that the gate will rotate clockwise and when $d_{MA} \le 0.6$ m. When rotation of the gate is impending, we require

$$\Sigma M_A: \quad M_R = \left(0.6\ \mathrm{m} - \frac{1}{3}d_{MA}\right)P_{MA} - \left(0.6\ \mathrm{m} - \frac{1}{3}d_W\right)P_W$$

Substituting

$$490\ \mathrm{N \cdot m} = \left(0.6 - \frac{1}{3}t\right)\mathrm{m} \times (3870t^2)\ \mathrm{N} - \left(0.6 - \frac{1}{3} \times 0.25t\right)\mathrm{m} \times (306.56t^2)\ \mathrm{N}$$

PROBLEM 5.145 (Continued)

Simplifying $$1264.45t^3 - 2138.1t^2 + 490 = 0$$

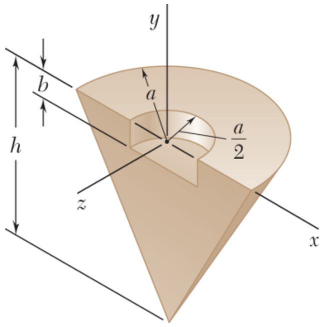
Solving (positive roots only)

$$t = 0.59451 \text{ min} \quad \text{and} \quad t = 1.52411 \text{ min}$$

Now check assumption using the smaller root. We have

$$d_{MA} = (t) \text{ m} = 0.59451 \text{ m} < 0.6 \text{ m} \qquad \therefore \quad t = 0.59451 \text{min} = 35.7 \text{ s} \blacktriangleleft$$

and the gate rotates clockwise. ◀

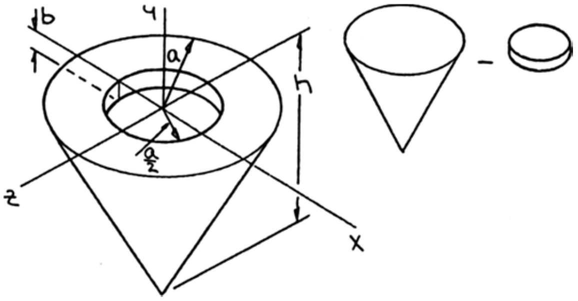


PROBLEM 5.146

Determine the *y* coordinate of the centroid of the body shown.

SOLUTION

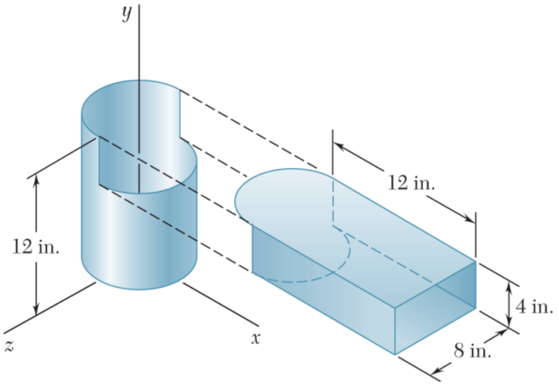
First note that the values of $\bar{Y}$ will be the same for the given body and the body shown below. Then



	V	$\bar{y}$	$\bar{y}V$
Cone	$\frac{1}{3}\pi a^2 h$	$-\frac{1}{4}h$	$-\frac{1}{12}\pi a^2 h^2$
Cylinder	$-\pi\left(\frac{a}{2}\right)^2 b = -\frac{1}{4}\pi a^2 b$	$-\frac{1}{2}b$	$\frac{1}{8}\pi a^2 b^2$
Σ	$\frac{\pi}{12}a^2(4h-3b)$		$-\frac{\pi}{24}a^2(2h^2-3b^2)$

We have $$\bar{Y}\Sigma V = \Sigma\bar{y}V$$

Then $$\bar{Y}\left[\frac{\pi}{12}a^2(4h-3b)\right] = -\frac{\pi}{24}a^2(2h^2-3b^2)$$ or $$\bar{Y} = -\frac{2h^2-3b^2}{2(4h-3b)}$$ ◀



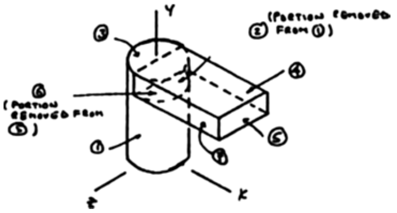
PROBLEM 5.147

An 8-in.-diameter cylindrical duct and a 4 × 8-in. rectangular duct are to be joined as indicated. Knowing that the ducts were fabricated from the same sheet metal, which is of uniform thickness, locate the center of gravity of the assembly.

SOLUTION

Assume that the body is homogeneous so that its center of gravity coincides with the centroid of the area.

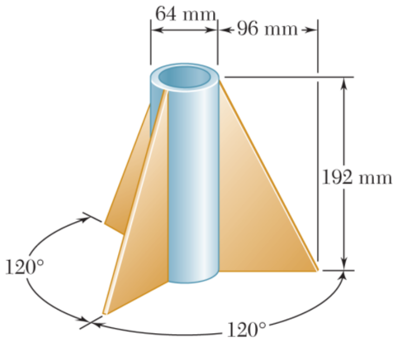
By symmetry, $\overline{Z} = 0$.



	A, in^2	$\overline{x}$, in.	$\overline{y}$, in.	$\overline{x}A$, in^3	$\overline{y}A$, in^3
1	$\pi(8)(12) = 96\pi$	0	6	0	576π
2	$-\frac{\pi}{2}(8)(4) = -16\pi$	$\frac{2(4)}{\pi} = \frac{8}{\pi}$	10	-128	-160π
3	$\frac{\pi}{2}(4)^2 = 8\pi$	$-\frac{4(4)}{3\pi} = -\frac{16}{3\pi}$	12	-42.667	96π
4	$(8)(12) = 96$	6	12	576	1152
5	$(8)(12) = 96$	6	8	576	768
6	$-\frac{\pi}{2}(4)^2 = -8\pi$	$\frac{4(4)}{3\pi} = \frac{16}{3\pi}$	8	-42.667	-64π
7	$(4)(12) = 48$	6	10	288	480
8	$(4)(12) = 48$	6	10	288	480
Σ	539.33			1514.6	4287.4

Then $$\overline{X} = \frac{\Sigma \overline{x}A}{\Sigma A} = \frac{1514.67}{539.33}\text{ in.}$$ or $\overline{X} = 2.81$ in. ◀

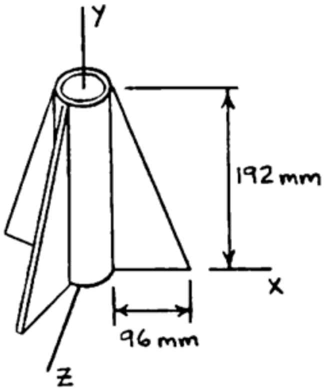
$$\overline{Y} = \frac{\Sigma \overline{y}A}{\Sigma A} = \frac{4287.4}{539.33}\text{ in.}$$ or $\overline{Y} = 7.95$ in. ◀



PROBLEM 5.148

Three brass plates are brazed to a steel pipe to form the flagpole base shown. Knowing that the pipe has a wall thickness of 8 mm and that each plate is 6 mm thick, determine the location of the center of gravity of the base. (Densities: brass = 8470 kg/m^3; steel = 7860 kg/m^3.)

SOLUTION



Since brass plates are equally spaced, we note that the center of gravity lies on the *y*-axis.

Thus, $\bar{X} = \bar{Z} = 0$ ◀

Steel pipe:

$$V = \frac{\pi}{4}[(0.064\text{ m})^2 - (0.048\text{ m})^2](0.192\text{ m})$$

$$= 270.22 \times 10^{-6}\text{ m}^3$$

$$m = \rho V = (7860\text{ kg/m}^3)(270.22 \times 10^{-6}\text{ m}^3)$$

$$= 2.1239\text{ kg}$$

Each brass plate:

$$V = \frac{1}{2}(0.096\text{ m})(0.192\text{ m})(0.006\text{ m}) = 55.296 \times 10^{-6}\text{ m}^3$$

$$m = \rho V = (8470\text{ kg/m}^3)(55.296 \times 10^{-6}\text{ m}^3) = 0.46836\text{ kg}$$

Flagpole base:

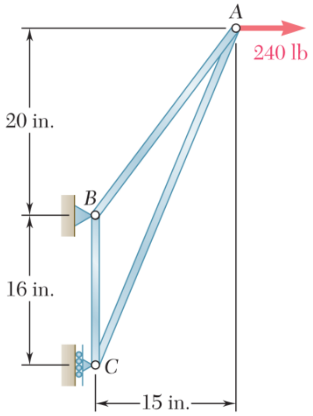
$$\Sigma m = 2.1239\text{ kg} + 3(0.46836\text{ kg}) = 3.5290\text{ kg}$$

$$\Sigma \bar{y} m = (0.096\text{ m})(2.1239\text{ kg}) + 3[(0.064\text{ m})(0.46836\text{ kg})] = 0.29382\text{ kg} \cdot \text{m}$$

$$\bar{Y}\Sigma m = \Sigma \bar{y} m: \qquad \bar{Y}(3.5290\text{ kg}) = 0.29382\text{ kg} \cdot \text{m}$$

$$\bar{Y} = 0.083259\text{ m} \qquad \bar{Y} = 83.3\text{ mm above the base} ◀$$

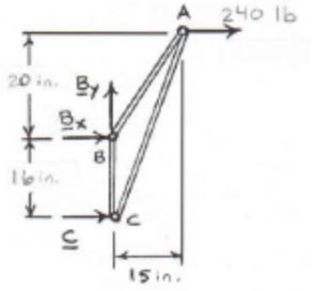
CHAPTER 6


A
240 lb
20 in.
B
16 in.
C
15 in.

PROBLEM 6.1

Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION


A
240 lb
20 in.
By
Bx
B
16 in.
C
C
15 in.

Free body: Entire truss:

$$+\uparrow \Sigma F_y = 0: \quad B_y = 0 \qquad \mathbf{B}_y = 0$$

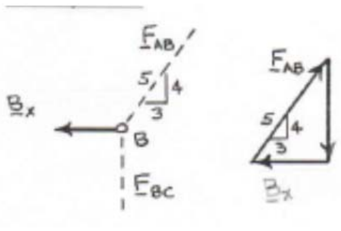
$$+\circlearrowleft \Sigma M_C = 0: \quad -B_x(16 \text{ in.}) - (240 \text{ lb})(36 \text{ in.}) = 0$$

$$B_x = -540 \text{ kN} \quad \mathbf{B}_x = 540 \text{ lb} \longleftarrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad C - 540 \text{ lb} + 240 \text{ lb} = 0$$

$$C = 300 \text{ lb} \qquad \mathbf{C} = 300 \text{ lb} \longrightarrow$$

Free body: Joint *B*:


F_AB
5
4
3
Bx
B
F_BC
F_AB
5
4
3
Bx

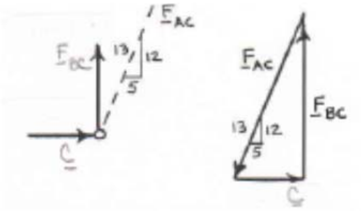
$$\frac{F_{AB}}{5} = \frac{F_{BC}}{4} = \frac{540 \text{ lb}}{3}$$

$$F_{AB} = 900 \text{ lb} \quad T \blacktriangleleft$$

$$F_{BC} = 720 \text{ lb} \quad T \blacktriangleleft$$

PROBLEM 6.1 (Continued)

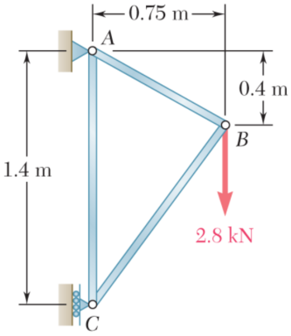
Free body: Joint *C*:


F_AC
F_BC
13
12
5
C
F_AC
F_BC
13
12
5
C

$$\frac{F_{AC}}{13} = \frac{F_{BC}}{12} = \frac{300 \text{ lb}}{5}$$

$F_{AC} = 780 \text{ lb} \quad C$ ◀

$F_{BC} = 720 \text{ lb}$ (checks)

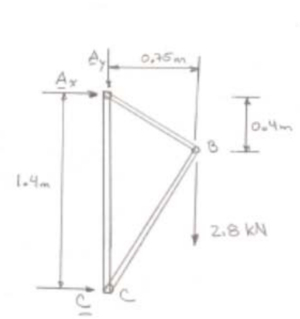

0.75 m
A
0.4 m
B
1.4 m
2.8 kN
C

PROBLEM 6.2

Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Entire truss:


A_y
A_x
0.75m
0.4m
B
1.4m
2.8 kN
C
C

$$+\uparrow \Sigma F_y = 0: \quad -A_y - 2.8 \text{ kN} = 0 \qquad A_y = -2.8 \text{ kN}$$

$$\mathbf{A}_y = 2.8 \text{ kN} \uparrow$$

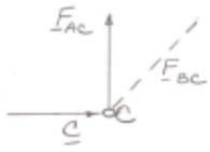
$$+\circlearrowleft \Sigma M_A = 0: \quad C(1.4 \text{ m}) - (2.8 \text{ kN})(0.75 \text{ m}) = 0$$

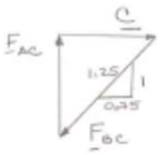
$$C = 1.500 \text{ kN} \quad \mathbf{C} = 1.500 \text{ kN} \longrightarrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + 1.500 \text{ kN} = 0$$

$$A_x = -1.500 \text{ kN} \quad \mathbf{A}_x = 1.500 \text{ kN} \longleftarrow$$

Free body: Joint *C*:




F_{AC}
C
1.25
1
0.75
F_{BC}

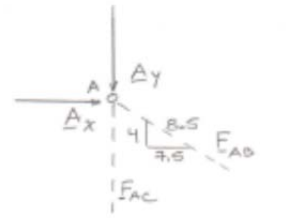
$$\frac{F_{BC}}{1.25} = \frac{F_{AC}}{1} = \frac{1.500 \text{ kN}}{0.75}$$

$F_{AC} = 2.00$ kN *T* ◀

$F_{BC} = 2.50$ kN *T* ◀

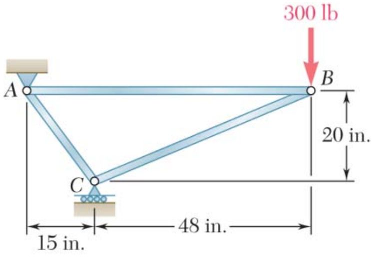
PROBLEM 6.2 (Continued)

Free body: Joint *A*:


A
Ay
Ax
4
8.5
7.5
FAB
FAC

$$\xrightarrow{+} \Sigma F_x = 0: \quad \left(\frac{7.5}{8.5}\right) F_{AB} - 1.500 \text{ kN} = 0$$

$$F_{AB} = 1.700 \text{ kN} \quad T$$ ◀



300 lb
A
B
C
20 in.
15 in.
48 in.

PROBLEM 6.3

Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

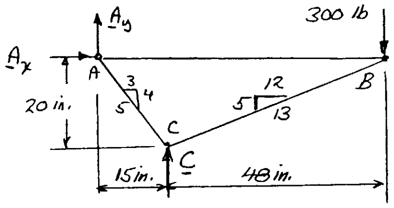
SOLUTION

Reactions:

$$\Sigma M_A = 0: \quad \mathbf{C} = 1260 \text{ lb} \uparrow$$

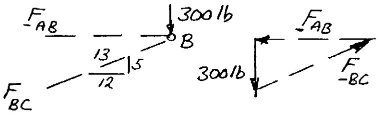
$$\Sigma F_x = 0: \quad \mathbf{A}_x = 0$$

$$\Sigma F_y = 0: \quad \mathbf{A}_y = 960 \text{ lb} \downarrow$$



A_y
300 lb
A_x
A
3
4
5
12
5
13
B
C
20 in.
15 in.
C
48 in.

Joint *B*:



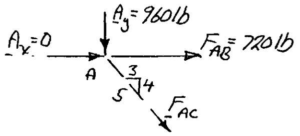
F_AB
300 lb
B
13
5
12
F_BC
300 lb
F_AB
F_BC

$$\frac{F_{AB}}{12} = \frac{F_{BC}}{13} = \frac{300 \text{ lb}}{5}$$

$F_{AB} = 720 \text{ lb} \quad T$ ◀

$F_{BC} = 780 \text{ lb} \quad C$ ◀

Joint *A*:

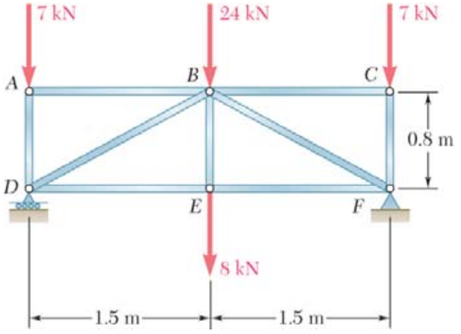


A_y = 960 lb
A_x = 0
F_AB = 720 lb
A
3
4
5
F_AC

$$+\uparrow \Sigma F_y = 0: \quad -960 \text{ lb} - \frac{4}{5} F_{AC} = 0$$

$F_{AC} = 1200 \text{ lb} \quad F_{AC} = 1200 \text{ lb} \quad C$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad 720 \text{ lb} - (1200 \text{ lb})\frac{3}{5} = 0 \quad \text{(checks)}$$

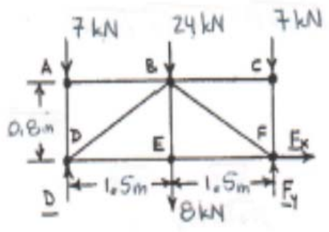

7 kN
24 kN
7 kN
A
B
C
0.8 m
D
E
F
8 kN
1.5 m
1.5 m

PROBLEM 6.4

Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION

Reactions:


7 kN
24 kN
7 kN
A
B
C
0.8 m
D
E
F
E_x
1.5 m
1.5 m
8 kN
F_y
D

$$+\circlearrowleft \Sigma M_D = 0: \quad F_y(3\text{ m}) - (24\text{ kN} + 8\text{ kN})(1.5\text{ m}) - (7\text{ kN})(3\text{ m}) = 0$$

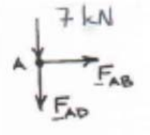
$$\mathbf{F}_y = 23.0\text{ kN}\uparrow$$

$$\Sigma F_x = 0: \quad \mathbf{F}_x = 0$$

$$+\uparrow\Sigma F_y = 0: \quad D - (7 + 24 + 8 + 7) + 23 = 0$$

$$\mathbf{D} = 23.0\text{ kN}\uparrow$$

Joint *A*:


7 kN
A
F_{AB}
F_{AD}

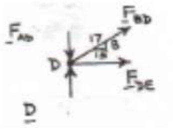
$$\Sigma F_x = 0: \quad F_{AB} = 0 \qquad F_{AB} = 0 \blacktriangleleft$$

$$+\uparrow\Sigma F_y = 0: \quad -7 - F_{AD} = 0$$

$$F_{AD} = -7\text{ kN} \qquad F_{AD} = 7.00\text{ kN} \quad C \blacktriangleleft$$

Joint *D*:

$$+\uparrow\Sigma F_y = 0: \quad -7 + 23.0 + \frac{8}{17}F_{BD} = 0$$


F_{AD}
F_{BD}
17
8
15
D
F_{DE}
D

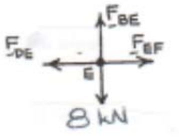
$$F_{BD} = -34.0\text{ kN} \qquad F_{BD} = 34.0\text{ kN} \quad C \blacktriangleleft$$

PROBLEM 6.4 (Continued)

$$\overset{+}{\rightarrow} \Sigma F_x = 0: \quad \frac{15}{17}(-34.0) + F_{DE} = 0$$

$F_{DE} = +30.0 \text{ kN}$ $\qquad F_{DE} = 30.0 \text{ kN} \quad T$ ◀

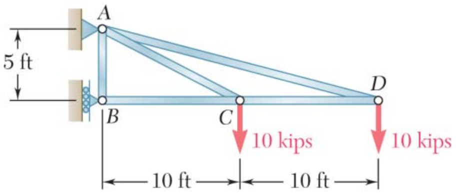
Joint *E*:



$$+\!\uparrow \Sigma F_y = 0: \quad F_{BE} - 8 = 0$$

$F_{BE} = +8.00 \text{ kN}$ $\qquad F_{BE} = 8.00 \text{ kN} \quad T$ ◀

Truss and loading symmetrical about ℄.

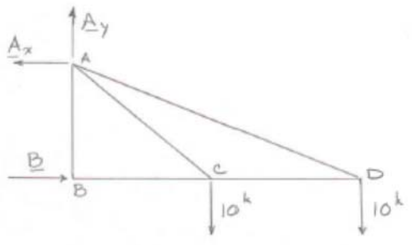

A
5 ft
D
B
C
10 kips
10 kips
10 ft
10 ft

PROBLEM 6.5

Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION

Reactions:


A_y
A_x
A
B
B
C
D
10^k
10^k

$$+\circlearrowleft \Sigma M_B = 0: \quad A_x(5 \text{ ft}) - (10 \text{ kips})(10 \text{ ft}) - (10 \text{ kips})(20 \text{ ft}) = 0$$

$$\mathbf{A}_x = 60.0 \text{ kips} \longleftarrow$$

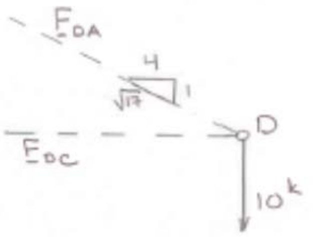
$$\Sigma F_x = 0: \quad A_x - B = 0$$

$$\mathbf{B} = 60 \text{ kips} \longrightarrow$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 10 \text{ kips} - 10 \text{ kips} = 0$$

$$\mathbf{A}_y = 20.0 \text{ kips} \uparrow$$

Joint *D*:


F_DA
4
√17
1
F_DC
D
10^k

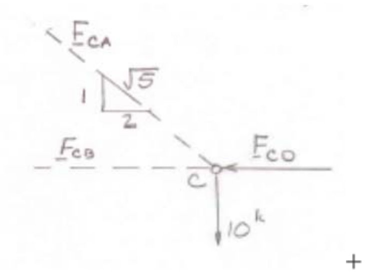
$$\frac{10 \text{ kips}}{1} = \frac{F_{DC}}{4} = \frac{F_{DA}}{\sqrt{17}}$$

$$F_{DA} = 41.2 \text{ kips} \quad T \blacktriangleleft$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$F_{DC} = 40.0 \text{ kips} \quad C \blacktriangleleft$$

PROBLEM 6.5 (Continued)



F_{CA}
$\sqrt{5}$
1
2
F_{CB}
F_{CO}
C
10^k

Joint *C*:

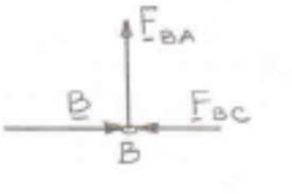
$$+\uparrow \Sigma F_y = 0: \quad \frac{1}{\sqrt{5}} F_{CA} - 10 \text{ kips} = 0$$

$F_{CA} = 22.4 \text{ kips}$ $\qquad F_{CA} = 22.4 \text{ kips} \quad T$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{2}{\sqrt{5}}(22.4 \text{ kips}) - 40 \text{ kips} - F_{CB} = 0$$

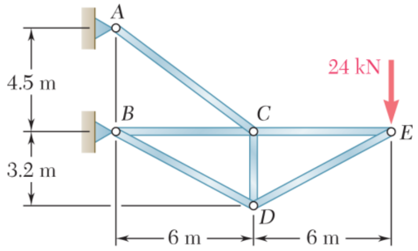
$F_{CB} = -60.0 \text{ kips}$ $\qquad F_{CB} = 60.0 \text{ kips} \quad C$ ◀

Joint *B*:



$$+\uparrow \Sigma F_y = 0: \quad F_{BA} = 0$$

$F_{BA} = 0$ ◀



PROBLEM 6.6

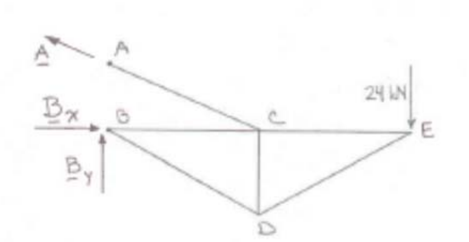
Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION

Reflections:

Reactions:

$$DE = \sqrt{(6\text{ m})^2 + (3.2\text{ m})^2} = 6.80\text{ m}$$



$$+\circlearrowleft \Sigma M_B = 0\!:\quad \left(\frac{4}{5}\right)F_{AC}(4.5\text{ m}) - (24\text{ kN})(12\text{ m}) = 0$$

$$\mathbf{F}_{AC} = 80.0\text{ kN}\quad T \blacktriangleleft$$

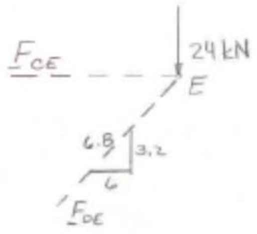
$$\xrightarrow{+} \Sigma F_x = 0\!:\quad B_x - \left(\frac{4}{5}\right)(80\text{ kN}) = 0$$

$$\mathbf{B}_x = 64\text{ kN} \longrightarrow$$

$$+\uparrow \Sigma F_y = 0\!:\quad B_y + \left(\frac{3}{5}\right)(80\text{ kN}) - 24\text{ kN} = 0$$

$$\mathbf{B}_y = 24\text{ kN} \uparrow$$

Joint *E*:



$$\frac{F_{CE}}{6} = \frac{F_{DE}}{6.8} = \frac{24\text{ kN}}{3.2}$$

$$F_{CE} = 45.0\text{ kN}\quad T \blacktriangleleft$$

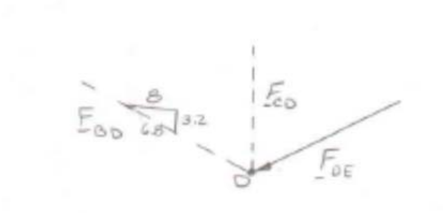
$$F_{DE} = 51.0\text{ kN}\quad C \blacktriangleleft$$

PROBLEM 6.6 (Continued)

Joint *D*:

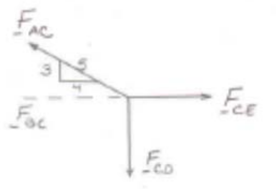
$$\overset{+}{\longleftarrow}\Sigma F_x = 0:\quad \left(\frac{6}{6.8}\right)F_{BD} - \left(\frac{6}{6.8}\right)(51.0\text{ kN}) = 0$$

$F_{BD} = 51.0\text{ kN}\quad C$ ◀



$F_{CD} = 48.0\text{ kN}\quad T$ ◀

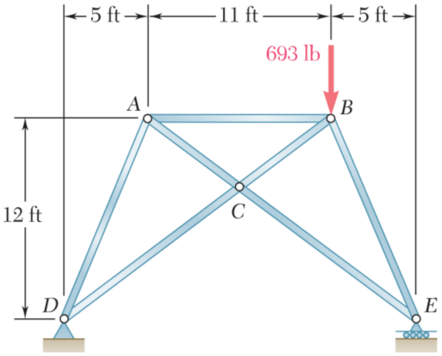
Joint *C*:



$$\overset{+}{\longrightarrow}\Sigma F_x = 0:\quad -F_{BC} - \left(\frac{4}{5}\right)(80\text{ kN}) + (45.0\text{ kN}) = 0$$

$F_{BC} = 19.00\text{ kN}\quad C$ ◀

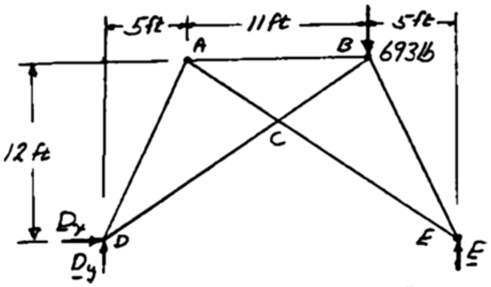
$$+\!\uparrow \Sigma F_y = 0:\quad \left(\frac{3}{5}\right)(80\text{ kN}) - 48.0\text{ kN} = 0\ \text{(checks)}$$


5 ft
11 ft
5 ft
693 lb
A
B
C
12 ft
D
E

PROBLEM 6.7

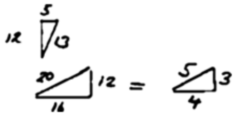
Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION


5ft
11ft
5ft
A
B
693lb
12 ft
C
D_x
D
D_y
E
E

$$AD = \sqrt{5^2 + 12^2} = 13 \text{ ft}$$

$$BCD = \sqrt{12^2 + 16^2} = 20 \text{ ft}$$


5
12
13
20
12
16
=
5
3
4

Reactions: $\Sigma F_x = 0: \quad D_x = 0$

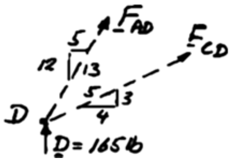
$+\circlearrowleft \Sigma M_E = 0: \quad D_y(21 \text{ ft}) - (693 \text{ lb})(5 \text{ ft}) = 0$ $\qquad \mathbf{D}_y = 165 \text{ lb} \uparrow$

$+\uparrow \Sigma F_y = 0: \quad 165 \text{ lb} - 693 \text{ lb} + E = 0$ $\qquad \mathbf{E} = 528 \text{ lb} \uparrow$

Joint *D*:

$$\overset{+}{\longrightarrow} \Sigma F_x = 0: \quad \frac{5}{13}F_{AD} + \frac{4}{5}F_{DC} = 0 \qquad (1)$$

$$+\uparrow \Sigma F_y = 0: \quad \frac{12}{13}F_{AD} + \frac{3}{5}F_{DC} + 165 \text{ lb} = 0 \qquad (2)$$


F_{AD}
F_{CD}
5
12
13
5
3
4
D
D = 165lb

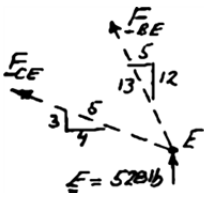
Solving Eqs. (1) and (2) simultaneously,

$F_{AD} = -260 \text{ lb}$ $\qquad F_{AD} = 260 \text{ lb} \quad C$ ◀

$F_{DC} = +125 \text{ lb}$ $\qquad F_{DC} = 125.0 \text{ lb} \quad T$ ◀

PROBLEM 6.7 (Continued)

Joint *E*:


F_{BE}
5
13
12
F_{CE}
3
5
4
E
E = 528 lb

$$\xleftarrow{+}\Sigma F_x = 0:\quad \frac{5}{13}F_{BE} + \frac{4}{5}F_{CE} = 0 \tag{3}$$

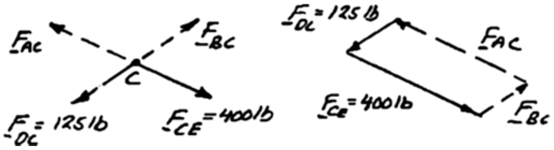
$$+\uparrow\Sigma F_y = 0:\quad \frac{12}{13}F_{BE} + \frac{3}{5}F_{CE} + 528\text{ lb} = 0 \tag{4}$$

Solving Eqs. (3) and (4) simultaneously,

$F_{BE} = -832$ lb $\qquad F_{BE} = 832$ lb C ◀

$F_{CE} = +400$ lb $\qquad F_{CE} = 400$ lb T ◀

Joint *C*:

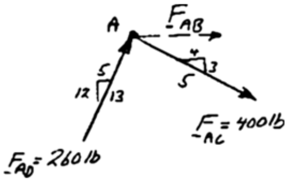

F_{AC}
F_{BC}
C
F_{DC} = 125 lb
F_{CE} = 400 lb
F_{DC} = 125 lb
F_{AC}
F_{CE} = 400 lb
F_{BC}

Force polygon is a parallelogram (see Fig. 6.11, p. 290).

$F_{AC} = 400$ lb T ◀

$F_{BC} = 125.0$ lb T ◀

Joint *A*:

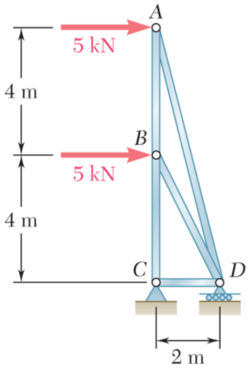

A
F_{AB}
5
12
13
4
3
5
F_{AC} = 400 lb
F_{AD} = 260 lb

$$\xrightarrow{+}\Sigma F_x = 0:\quad \frac{5}{13}(260\text{ lb}) + \frac{4}{5}(400\text{ lb}) + F_{AB} = 0$$

$F_{AB} = -420$ lb $\qquad F_{AB} = 420$ lb C ◀

$$+\uparrow\Sigma F_y = 0:\quad \frac{12}{13}(260\text{ lb}) - \frac{3}{5}(400\text{ lb}) = 0$$

$0 = 0$ (Checks)


A
5 kN
4 m
B
5 kN
4 m
C
D
2 m

PROBLEM 6.8

Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Entire truss:

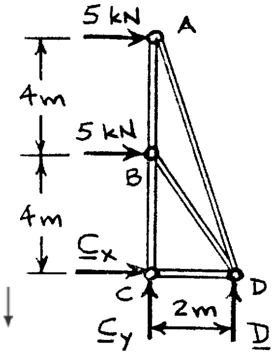
$$\overset{+}{\rightarrow}\Sigma F_x = 0: \quad C_x + 2(5\text{ kN}) = 0$$

$$C_x = -10\text{ kN} \qquad \mathbf{C}_x = 10\text{ kN} \longleftarrow$$

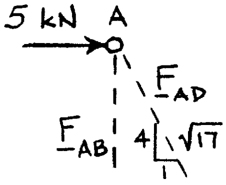
$$+\circlearrowleft \Sigma M_C = 0: \quad D(2\text{ m}) - (5\text{ kN})(8\text{ m}) - (5\text{ kN})(4\text{ m}) = 0$$

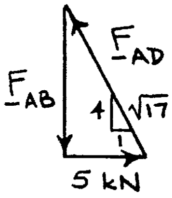
$$D = +30\text{ kN} \qquad \mathbf{D} = 30\text{ kN}\uparrow$$

$$+\uparrow\Sigma F_y = 0: \quad C_y + 30\text{ kN} = 0 \quad C_y = -30\text{ kN} \quad \mathbf{C}_y = 30\text{ kN}\downarrow$$


5 kN
A
4m
5 kN
B
4m
C_x
C
C_y
2m
D
D

Free body: Joint A:


5 kN
A
F_{AD}
F_{AB}
4
$\sqrt{17}$
1

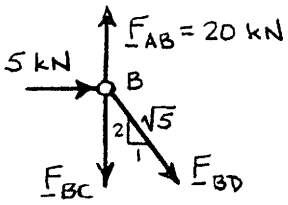

F_{AD}
F_{AB}
4
$\sqrt{17}$
1
5 kN

$$\frac{F_{AB}}{4} = \frac{F_{AD}}{\sqrt{17}} = \frac{5\text{ kN}}{1}$$

$F_{AB} = 20.0$ kN T ◀

$F_{AD} = 20.6$ kN C ◀

Free body: Joint B:


F_{AB} = 20 kN
5 kN
B
2
$\sqrt{5}$
1
F_{BC}
F_{BD}

$$\overset{+}{\rightarrow}\Sigma F_x = 0: \quad 5\text{ kN} + \frac{1}{\sqrt{5}}F_{BD} = 0$$

$$F_{BD} = -5\sqrt{5}\text{ kN}$$

$F_{BD} = 11.18$ kN C ◀

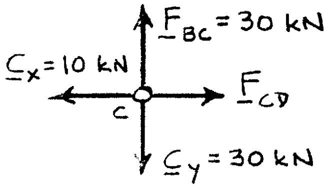
$$+\uparrow\Sigma F_y = 0: \quad 20\text{ kN} - F_{BC} - \frac{2}{\sqrt{5}}(-5\sqrt{5}\text{ kN}) = 0$$

$$F_{BC} = +30\text{ kN}$$

$F_{BC} = 30.0$ kN T ◀

PROBLEM 6.8 (Continued)

Free body: Joint *C*:

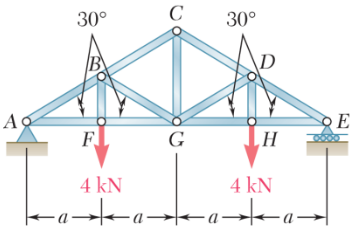


$F_{BC} = 30$ kN
$C_x = 10$ kN
F_{CD}
C
$C_y = 30$ kN

$\xrightarrow{+} \Sigma F_x = 0: \quad F_{CD} - 10 \text{ kN} = 0$

$F_{CD} = +10 \text{ kN} \qquad F_{CD} = 10.00 \text{ kN} \quad T$ ◀

$+\uparrow \Sigma F_y = 0: \quad 30 \text{ kN} - 30 \text{ kN} = 0 \quad \text{(checks)}$


30°
C
30°
B
D
A
E
F
G
H
4 kN
4 kN
a
a
a
a

PROBLEM 6.9

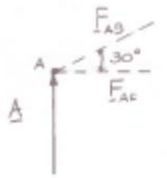
Determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION

Reactions for Entire truss:

By inspection: $\mathbf{A} = 4 \text{ kN} \uparrow \quad \mathbf{E} = 4 \text{ kN} \uparrow$

Free body: Joint *A*:


F_{AB}
A
30°
F_{AF}
A

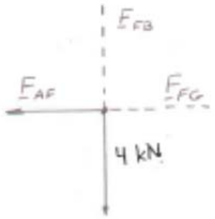
$$F_{AB} = \frac{4 \text{ kN}}{\sin 30^\circ}$$

$$F_{AF} = (4 \text{ kN})(\cot 30^\circ)$$

$F_{AB} = 8.00$ kN C ◀

$F_{AF} = 6.93$ kN T ◀

Free body: Joint *F*:


F_{FB}
F_{AF}
F_{FG}
4 kN

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{FG} = F_{AF}$$

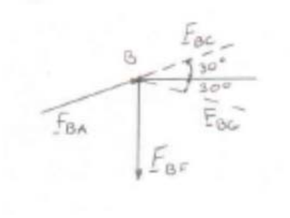
$F_{FG} = 6.93$ kN T ◀

$$+\uparrow \Sigma F_y = 0: \quad F_{FB} = 4.00 \text{ kN} \quad T$$

$F_{FB} = 4.00$ kN T ◀

PROBLEM 6.9 (Continued)

Free body: Joint *B*:



$$\xrightarrow{+} \Sigma F_x = 0: \quad 8\cos 30^\circ + F_{BC}\cos 30^\circ + F_{BG}\cos 30^\circ = 0 \quad (1)$$

$$+\!\uparrow \Sigma F_y = 0: \quad 8\sin 30^\circ - 4 + F_{BC}\sin 30^\circ - F_{BG}\sin 30^\circ = 0 \quad (2)$$

Solving (1) and (2) simultaneously yields:

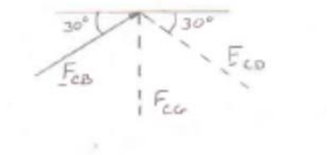
$F_{BC} = 4.00 \text{ kN} \quad C$ ◀

□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□ $F_{BG} = 4.00 \text{ kN} \quad C$ ◀

Free body: Joint C:

By symmetry, $F_{CD} = F_{CB} = 4.00 \text{ kN} \quad C$ ◀

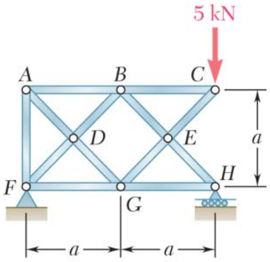


$$+\!\uparrow \Sigma F_y = 0: \quad 4\sin 30^\circ - 4\sin 30^\circ - F_{CG} = 0$$

$F_{CG} = 4.00 \text{ kN} \quad T$ ◀

By symmetry about the truss centerline, we note:

$F_{GD} = 4.00 \text{ kN} \quad C,\ F_{GH} = 6.93 \text{ kN} \quad T,\ F_{DH} = 4.00 \text{ kN} \quad T,\ F_{DE} = 8.00 \text{ kN} \ C,\ F_{EH} = 6.93 \text{ kN} \ T$ ◀

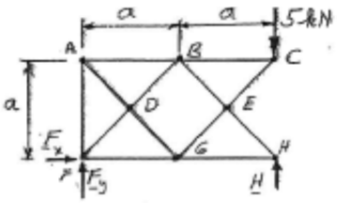

5 kN
A
B
C
D
E
a
F
H
G
a
a

PROBLEM 6.10

Determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Entire truss:

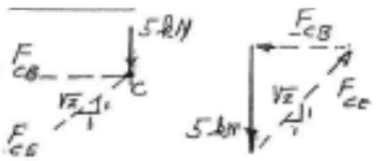

a
a
5 kN
A
B
C
a
D
E
F_x
G
H
F
F_y
H

$$\xrightarrow{+} \Sigma F_x = 0: \quad \mathbf{F}_x = 0$$

$$+\circlearrowleft \Sigma M_F = 0: \quad \mathbf{H} = 5 \text{ kN} \uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad \mathbf{F}_y = 0$$

Free body: Joint *C*:


5 kN
F_{CB}
C
$\sqrt{2}$
1
1
F_{CE}
F_{CB}
$\sqrt{2}$
F_{CE}
5 kN
1
1

$$F_{CB} = 5.00 \text{ kN}, \quad F_{CE} = 5\sqrt{2} = 7.07 \text{ kN}$$

$F_{CB} = 5.00$ kN T ◀

$F_{CE} = 7.07$ kN C ◀

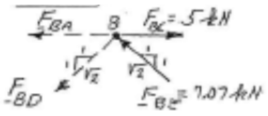
By symmetry about the horizontal centerline: $F_{HG} = 5.00$ kN T, $F_{HE} = 7.07$ kN C ◀

Joint E: Members lie in two intersecting straight lines, thus we have:

$F_{EG} = F_{EC} = 7.07$ kN C; $F_{EB} = F_{EH} = 7.07$ kN C ◀

PROBLEM 6.10 (Continued)

Free body: Joint *B*:


F_BA
B
F_BC = 5 kN
1
√2
√2
1
F_BD
F_BE = 7.07 kN

$$+\uparrow \Sigma F_y = 0: \quad \frac{1}{\sqrt{2}}(7.07) - \frac{1}{\sqrt{2}} F_{BD} = 0$$

$F_{BD} = 7.07 \text{ kN} \quad T$ ◀

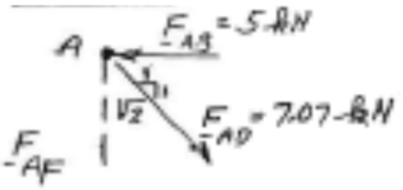
$$+\uparrow \Sigma F_x = 0: \quad 5 - F_{BA} - \frac{1}{\sqrt{2}}(7.07) - \frac{1}{\sqrt{2}}(7.07) = 0$$

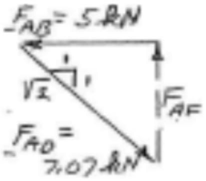
$F_{BA} = 5.00 \text{ kN} \quad C$ ◀

By symmetry about the horizontal centerline: $F_{DG} = 7.07 \text{ kN} \quad T, \; F_{GF} = 5.00 \text{ kN} \quad C$ ◀

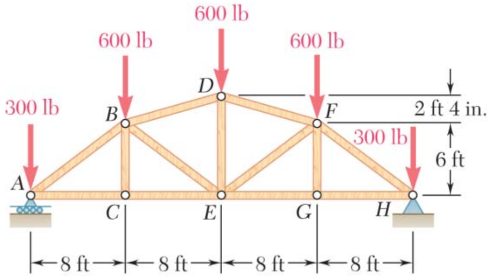
Joint D: $F_{DF} = F_{DB} = F_{DA} = F_{DG} = 7.07 \text{ kN} \quad T$ ◀

Free body: Joint *A*:


A
F_AB = 5 kN
1
1
√2
F_AD = 7.07 kN
F_AF


F_AB = 5 kN
1
√2
1
F_AF
F_AD = 7.07 kN

$F_{AF} = 5.00 \text{ kN} \quad C$ ◀



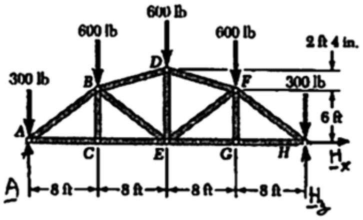
600 lb
600 lb
600 lb
300 lb
D
B
F
2 ft 4 in.
300 lb
6 ft
A
C
E
G
H
8 ft
8 ft
8 ft
8 ft

PROBLEM 6.11

Determine the force in each member of the Gambrel roof truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss:



600 lb
600 lb
600 lb
300 lb
2 ft 4 in.
B
D
F
300 lb
6 ft
A
C
E
G
H
H_x
A
8 ft
8 ft
8 ft
8 ft
H_y

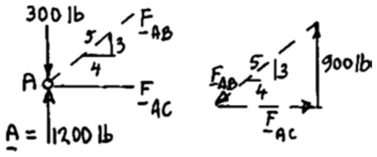
$$\Sigma F_x = 0: \quad \mathbf{H}_x = 0$$

Because of the symmetry of the truss and loading,

$$\mathbf{A} = \mathbf{H}_y = \frac{1}{2}\text{total load}$$

$$\mathbf{A} = \mathbf{H}_y = 1200 \text{ lb}\uparrow$$

Free body: Joint *A*:



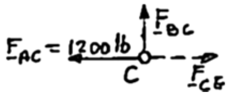
300 lb
F_{AB}
5
3
4
A
F_{AC}
A = 1200 lb
F_{AB}
5
3
4
900 lb
F_{AC}

$$\frac{F_{AB}}{5} = \frac{F_{AC}}{4} = \frac{900 \text{ lb}}{3}$$

$\mathbf{F}_{AB} = 1500$ lb *C* ◀

$\mathbf{F}_{AC} = 1200$ lb *T* ◀

Free body: Joint *C*:



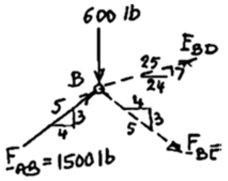
F_{BC}
F_{AC} = 1200 lb
C
F_{CE}

BC is a zero-force member.

$$\mathbf{F}_{BC} = 0$$

$F_{CE} = 1200$ lb *T* ◀

Free body: Joint *B*:



600 lb
F_{BD}
25
7
24
B
5
3
4
4
3
5
F_{BE}
F_{AB} = 1500 lb

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{24}{25}F_{BD} + \frac{4}{5}F_{BE} + \frac{4}{5}(1500 \text{ lb}) = 0$$

or $\qquad 24F_{BD} + 20F_{BE} = -30{,}000 \text{ lb} \qquad (1)$

$$+\!\uparrow \Sigma F_y = 0: \quad \frac{7}{25}F_{BD} - \frac{3}{5}F_{BE} + \frac{3}{5}(1500) - 600 = 0$$

or $\qquad 7F_{BD} - 15F_{BE} = -7{,}500 \text{ lb} \qquad (2)$

PROBLEM 6.11 (Continued)

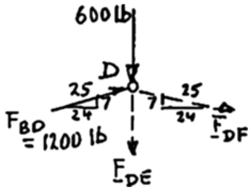
Multiply Eq. (1) by 3, Eq. (2) by 4, and add:

$$100F_{BD} = -120,000 \text{ lb} \qquad F_{BD} = 1200 \text{ lb} \quad C \blacktriangleleft$$

Multiply Eq. (1) by 7, Eq. (2) by –24, and add:

$$500F_{BE} = -30,000 \text{ lb} \qquad F_{BE} = 60.0 \text{ lb} \quad C \blacktriangleleft$$

Free body: Joint *D*:



$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{24}{25}(1200 \text{ lb}) + \frac{24}{25}F_{DF} = 0$$

$$F_{DF} = -1200 \text{ lb} \qquad F_{DF} = 1200 \text{ lb} \quad C \blacktriangleleft$$

$$+\big\uparrow \Sigma F_y = 0: \quad \frac{7}{25}(1200 \text{ lb}) - \frac{7}{25}(-1200 \text{ lb}) - 600 \text{ lb} - F_{DE} = 0$$

$$F_{DE} = 72.0 \text{ lb} \qquad F_{DE} = 72.0 \text{ lb} \quad T \blacktriangleleft$$

Because of the symmetry of the truss and loading, we deduce that

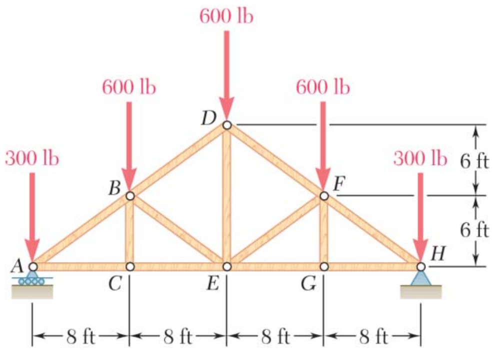
$$F_{EF} = F_{BE} \qquad F_{EF} = 60.0 \text{ lb} \quad C \blacktriangleleft$$

$$F_{EG} = F_{CE} \qquad F_{EG} = 1200 \text{ lb} \quad T \blacktriangleleft$$

$$F_{FG} = F_{BC} \qquad F_{FG} = 0 \blacktriangleleft$$

$$F_{FH} = F_{AB} \qquad F_{FH} = 1500 \text{ lb} \quad C \blacktriangleleft$$

$$F_{GH} = F_{AC} \qquad F_{GH} = 1200 \text{ lb} \quad T \blacktriangleleft$$

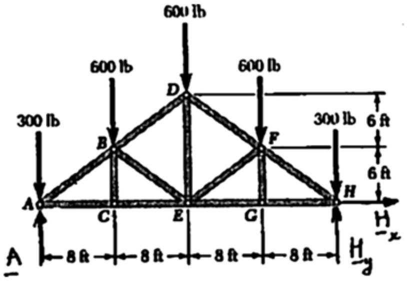

600 lb
600 lb
600 lb
300 lb
300 lb
D
B
F
A
C
E
G
H
6 ft
6 ft
8 ft
8 ft
8 ft
8 ft

PROBLEM 6.12

Determine the force in each member of the Howe roof truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss:


600 lb
600 lb
600 lb
300 lb
300 lb
6 ft
6 ft
D
B
F
A
C
E
G
H
H_x
H_y
A
8 ft
8 ft
8 ft
8 ft

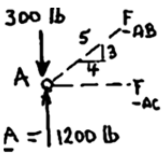
$$\Sigma F_x = 0: \quad \mathbf{H}_x = 0$$

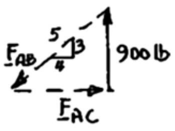
Because of the symmetry of the truss and loading,

$$A = H_y = \frac{1}{2}\text{total load}$$

$$\mathbf{A} = \mathbf{H}_y = 1200\text{ lb}\uparrow$$

Free body: Joint *A*:


300 lb
F_{AB}
5
3
4
A
F_{AC}
A = 1200 lb


F_{AB}
5
3
4
900 lb
F_{AC}

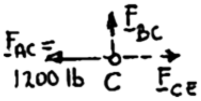
$$\frac{F_{AB}}{5} = \frac{F_{AC}}{4} = \frac{900\text{ lb}}{3}$$

$\mathbf{F}_{AB} = 1500\text{ lb}$ *C* ◀

$\mathbf{F}_{AC} = 1200\text{ lb}$ *T* ◀

Free body: Joint *C*:

BC is a zero-force member.

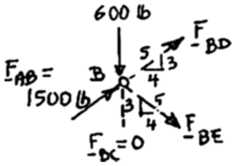

F_{BC}
F_{AC} =
1200 lb
C
F_{CE}

$\mathbf{F}_{BC} = 0$

$\mathbf{F}_{CE} = 1200\text{ lb}$ *T* ◀

PROBLEM 6.12 (Continued)

Free body: Joint *B*:



$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{4}{5}F_{BD} + \frac{4}{5}F_{BC} + \frac{4}{5}(1500 \text{ lb}) = 0$$

or $\quad F_{BD} + F_{BE} = -1500 \text{ lb} \quad (1)$

$$+\uparrow \Sigma F_y = 0: \quad \frac{3}{5}F_{BD} - \frac{3}{5}F_{BE} + \frac{3}{5}(1500 \text{ lb}) - 600 \text{ lb} = 0$$

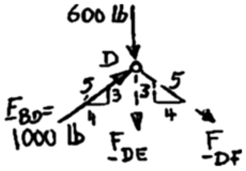
or $\quad F_{BD} - F_{BE} = -500 \text{ lb} \quad (2)$

Add Eqs. (1) and (2): $\quad 2F_{BD} = -2000 \text{ lb} \qquad F_{BD} = 1000 \text{ lb} \quad C$ ◀

Subtract Eq. (2) from Eq. (1): $\quad 2F_{BE} = -1000 \text{ lb} \qquad F_{BE} = 500 \text{ lb} \quad C$ ◀

Free Body: Joint *D*:

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{4}{5}(1000 \text{ lb}) + \frac{4}{5}F_{DF} = 0$$



$F_{DF} = -1000 \text{ lb} \qquad F_{DF} = 1000 \text{ lb} \quad C$ ◀

$$+\uparrow \Sigma F_y = 0: \quad \frac{3}{5}(1000 \text{ lb}) - \frac{3}{5}(-1000 \text{ lb}) - 600 \text{ lb} - F_{DE} = 0$$

$F_{DE} = +600 \text{ lb} \qquad F_{DE} = 600 \text{ lb} \quad T$ ◀

Because of the symmetry of the truss and loading, we deduce that

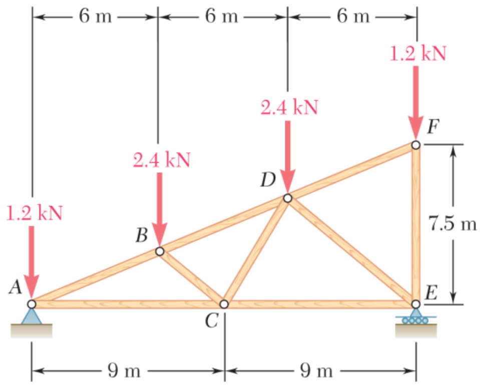
$F_{EF} = F_{BE} \qquad F_{EF} = 500 \text{ lb} \quad C$ ◀

$F_{EG} = F_{CE} \qquad F_{EG} = 1200 \text{ lb} \quad T$ ◀

$F_{FG} = F_{BC} \qquad F_{FG} = 0$ ◀

$F_{FH} = F_{AB} \qquad F_{FH} = 1500 \text{ lb} \quad C$ ◀

$F_{GH} = F_{AC} \qquad F_{GH} = 1200 \text{ lb} \quad T$ ◀


6 m
6 m
6 m
1.2 kN
2.4 kN
F
2.4 kN
D
1.2 kN
B
7.5 m
A
C
E
9 m
9 m

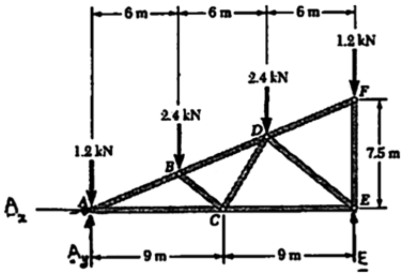
PROBLEM 6.13

Using the method of joints, determine the force in each member of the roof truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss:

$$\Sigma F_x = 0: \quad \mathbf{A}_x = 0$$


6 m
6 m
6 m
1.2 kN
2.4 kN
F
2.4 kN
D
1.2 kN
B
7.5 m
A_x
A
C
E
A_y
9 m
9 m
E

From symmetry of loading:

$$\mathbf{A}_y = \mathbf{E} = \frac{1}{2} \quad \text{total load}$$

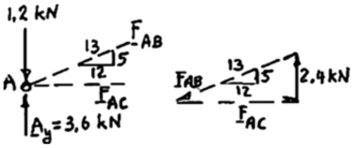
$$\mathbf{A}_y = \mathbf{E} = 3.6\text{ kN}\uparrow$$

We note that *DF* is a zero-force member and that *EF* is aligned with the load. Thus,

$$F_{DF} = 0 \blacktriangleleft$$

$$F_{EF} = 1.2\text{ kN} \quad C \blacktriangleleft$$

Free body: Joint *A*:

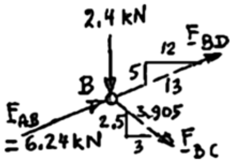

1,2 kN
F_AB
13
5
12
A
F_AC
A_y = 3,6 kN
F_AB
13
5
12
2,4 kN
F_AC

$$\frac{F_{AB}}{13} = \frac{F_{AC}}{12} = \frac{2.4\text{ kN}}{5}$$

$$F_{AB} = 6.24\text{ kN} \quad C \blacktriangleleft$$

$$F_{AC} = 2.76\text{ kN} \quad T \blacktriangleleft$$

Free body: Joint *B*:


2,4 kN
12
F_BD
5
13
B
F_AB = 6.24 kN
2.5
3.905
3
F_BC

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{3}{3.905}F_{BC} + \frac{12}{13}F_{BD} + \frac{12}{13}(6.24\text{ kN}) = 0 \qquad (1)$$

$$+\uparrow \Sigma F_y = 0: \quad -\frac{2.5}{3.905}F_{BC} + \frac{5}{13}F_{BD} + \frac{5}{13}(6.24\text{ kN}) - 2.4\text{ kN} = 0 \qquad (2)$$

PROBLEM 6.13 (Continued)

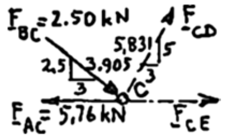
Multiply Eq. (1) by 2.5, Eq. (2) by 3, and add:

$$\frac{45}{13}F_{BD} + \frac{45}{13}(6.24\text{ kN}) - 7.2\text{ kN} = 0, \quad F_{BD} = -4.16\text{ kN}, \quad F_{BD} = 4.16\text{ kN} \quad C \blacktriangleleft$$

Multiply Eq. (1) by 5, Eq. (2) by –12, and add:

$$\frac{45}{3.905}F_{BC} + 28.8\text{ kN} = 0, \quad F_{BC} = -2.50\text{ kN}, \quad F_{BC} = 2.50\text{ kN} \quad C \blacktriangleleft$$

Free body: Joint *C*:



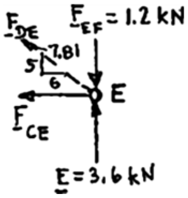
$$+\uparrow \Sigma F_y = 0: \quad \frac{5}{5.831}F_{CD} - \frac{2.5}{3.905}(2.50\text{ kN}) = 0$$

$$F_{CD} = 1.867\text{ kN} \quad T \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{CE} - 5.76\text{ kN} + \frac{3}{3.905}(2.50\text{ kN}) + \frac{3}{5.831}(1.867\text{ kN}) = 0$$

$$F_{CE} = 2.88\text{ kN} \quad T$$

Free body: Joint *E*:

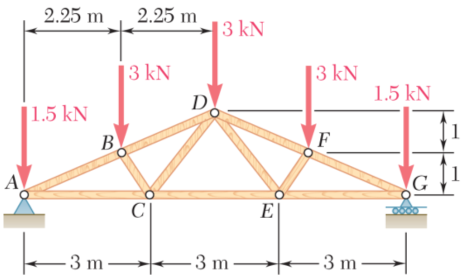


$$+\uparrow \Sigma F_y = 0: \quad \frac{5}{7.81}F_{DE} + 3.6\text{ kN} - 1.2\text{ kN} = 0$$

$$F_{DE} = -3.75\text{ kN} \quad F_{DE} = 3.75\text{ kN} \quad C \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -F_{CE} - \frac{6}{7.81}(-3.75\text{ kN}) = 0$$

$$F_{CE} = +2.88\text{ kN} \quad F_{CE} = 2.88\text{ kN} \quad T \qquad \text{(Checks)}$$

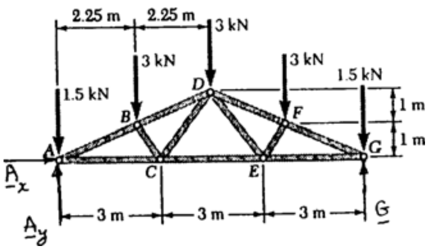

2.25 m
2.25 m
3 kN
3 kN
3 kN
1.5 kN
1.5 kN
D
B
F
A
G
C
E
1 m
1 m
3 m
3 m
3 m

PROBLEM 6.14

Determine the force in each member of the Fink roof truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss:


2.25 m
2.25 m
3 kN
3 kN
3 kN
1.5 kN
1.5 kN
D
B
F
A
G
C
E
1 m
1 m
A_x
A_y
3 m
3 m
3 m
G

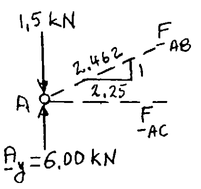
$$\Sigma F_x = 0: \quad \mathbf{A}_x = 0$$

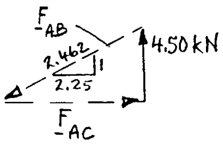
Because of the symmetry of the truss and loading,

$$\mathbf{A}_y = \mathbf{G} = \frac{1}{2} \text{ total load}$$

$$\mathbf{A}_y = \mathbf{G} = 6.00 \text{ kN} \uparrow$$

Free body: Joint A:


1,5 kN
F_{AB}
2,462
1
2,25
A
F_{AC}
A_y = 6,00 kN

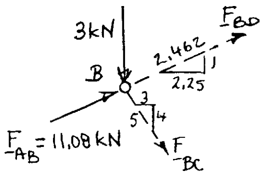

F_{AB}
2,462
1
2,25
4,50 kN
F_{AC}

$$\frac{F_{AB}}{2.462} = \frac{F_{AC}}{2.25} = \frac{4.50 \text{ kN}}{1}$$

$$F_{AB} = 11.08 \text{ kN} \quad C \blacktriangleleft$$

$$F_{AC} = 10.125 \text{ kN} \qquad F_{AC} = 10.13 \text{ kN} \quad T \blacktriangleleft$$

Free body: Joint B:


3 kN
F_{BD}
2,462
1
2,25
B
3
5
4
F_{AB} = 11,08 kN
F_{BC}

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{3}{5} F_{BC} + \frac{2.25}{2.462} F_{BD} + \frac{2.25}{2.462}(11.08 \text{ kN}) = 0 \tag{1}$$

$$+\!\uparrow \Sigma F_y = 0: \quad -\frac{4}{5} F_{BC} + \frac{F_{BD}}{2.462} + \frac{11.08 \text{ kN}}{2.462} - 3 \text{ kN} = 0 \tag{2}$$

Multiply Eq. (2) by –2.25 and add to Eq. (1):

$$\frac{12}{5} F_{BC} + 6.75 \text{ kN} = 0 \qquad F_{BC} = -2.8125 \qquad F_{BC} = 2.81 \text{ kN} \quad C \blacktriangleleft$$

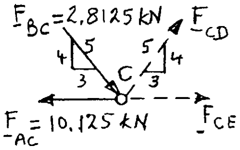
Multiply Eq. (1) by 4, Eq. (2) by 3, and add:

$$\frac{12}{2.462} F_{BD} + \frac{12}{2.462}(11.08 \text{ kN}) - 9 \text{ kN} = 0$$

$$F_{BD} = -9.2335 \text{ kN} \qquad F_{BD} = 9.23 \text{ kN} \quad C \blacktriangleleft$$

PROBLEM 6.14 (Continued)

Free body: Joint *C*:



$$+\uparrow \Sigma F_y = 0\colon\quad \frac{4}{5}F_{CD} - \frac{4}{5}(2.8125\text{ kN}) = 0$$

$$F_{CD} = 2.8125\text{ kN},\quad F_{CD} = 2.81\text{ kN}\quad T \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0\colon\quad F_{CE} - 10.125\text{ kN} + \frac{3}{5}(2.8125\text{ kN}) + \frac{3}{5}(2.8125\text{ kN}) = 0$$

$$F_{CE} = +6.7500\text{ kN}\quad F_{CE} = 6.75\text{ kN}\quad T \blacktriangleleft$$

Because of the symmetry of the truss and loading, we deduce that

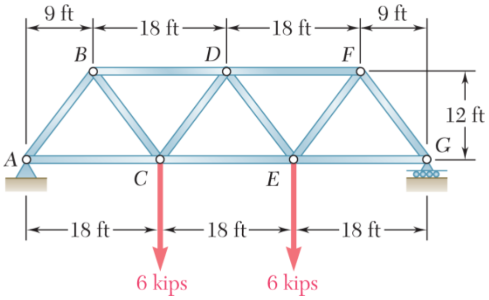
$$F_{DE} = F_{CD}\qquad F_{CD} = 2.81\text{ kN}\quad T \blacktriangleleft$$

$$F_{DF} = F_{BD}\qquad F_{DF} = 9.23\text{ kN}\quad C \blacktriangleleft$$

$$F_{EF} = F_{BC}\qquad F_{EF} = 2.81\text{ kN}\quad C \blacktriangleleft$$

$$F_{EG} = F_{AC}\qquad F_{EG} = 10.13\text{ kN}\quad T \blacktriangleleft$$

$$F_{FG} = F_{AB}\qquad F_{FG} = 11.08\text{ kN}\quad C \blacktriangleleft$$



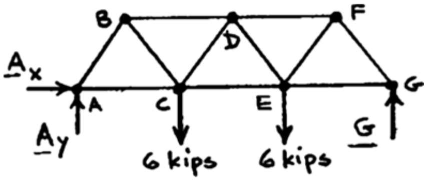
9 ft
18 ft
18 ft
9 ft
B
D
F
12 ft
A
G
C
E
18 ft
18 ft
18 ft
6 kips
6 kips

PROBLEM 6.15

Determine the force in each member of the Warren bridge truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss: $\Sigma F_x = 0: \quad A_x = 0$



B
D
F
A_x
A
C
E
G
A_y
6 kips
6 kips
G

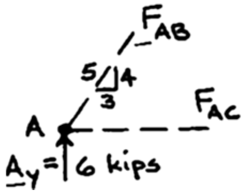
Due to symmetry of truss and loading,

$$A_y = G = \frac{1}{2} \text{ total load} = 6 \text{ kips}\uparrow$$

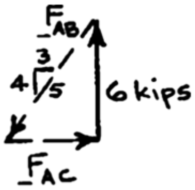
Free body: Joint *A*: $\frac{F_{AB}}{5} = \frac{F_{AC}}{3} = \frac{6 \text{ kips}}{4}$

$F_{AB} = 7.50$ kips *C* ◀

$F_{AC} = 4.50$ kips *T* ◀



F_{AB}
5
4
3
A
F_{AC}
A_y = 6 kips

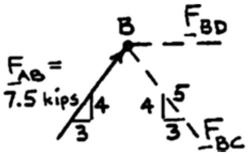


F_{AB}
3
4
5
6 kips
F_{AC}

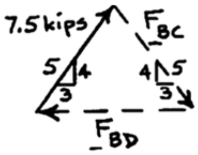
Free body: Joint *B*: $\frac{F_{BC}}{5} = \frac{F_{BD}}{6} = \frac{7.5 \text{ kips}}{5}$

$F_{BC} = 7.50$ kips *T* ◀

$F_{BD} = 9.00$ kips *C* ◀

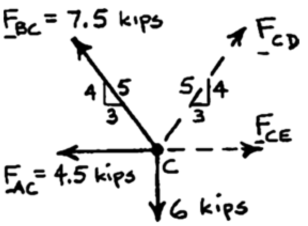


B
F_{BD}
F_{AB} =
7.5 kips
4
3
4
5
3
F_{BC}



7.5 kips
F_{BC}
5
4
3
4
5
3
F_{BD}

Free body: Joint *C*:



F_{BC} = 7.5 kips
F_{CD}
4
5
3
5
4
3
F_{CE}
F_{AC} = 4.5 kips
C
6 kips

$$+\uparrow \Sigma F_y = 0: \quad \frac{4}{5}(7.5) + \frac{4}{5}F_{CD} - 6 = 0$$

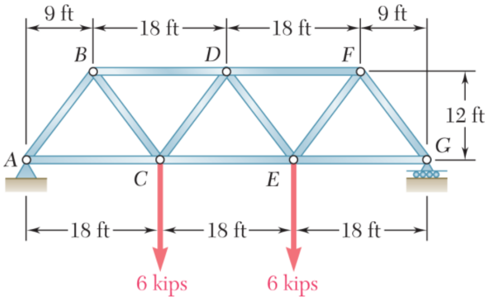
$F_{CD} = 0$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{CE} - 4.5 - \frac{3}{5}(7.5) = 0$$

$+\uparrow F_{CE} = +9$ kips

$F_{CE} = 9.00$ kips *T* ◀

Truss and loading is symmetrical about ℄.


9 ft
18 ft
18 ft
9 ft
B
D
F
12 ft
A
G
C
E
18 ft
18 ft
18 ft
6 kips
6 kips

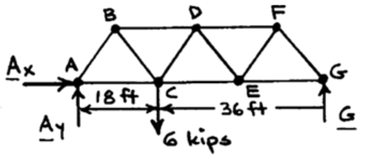
PROBLEM 6.16

Solve Problem 6.15 assuming that the load applied at *E* has been removed.

PROBLEM 6.15 Determine the force in each member of the Warren bridge truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss:

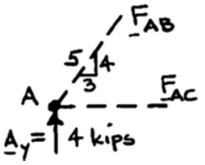

B
D
F
A
G
A_x
A_y
18 ft
C
E
36 ft
6 kips
G

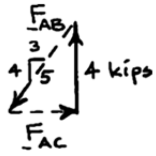
$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\circlearrowleft \Sigma M_G = 0: \quad 6(36) - A_y(54) = 0 \qquad \mathbf{A}_y = 4 \text{ kips}\uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad 4 - 6 + G = 0 \qquad \mathbf{G} = 2 \text{ kips}\uparrow$$

Free body: Joint *A*:


F_{AB}
5
4
3
A
F_{AC}
A_y = 4 kips

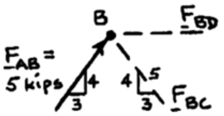

F_{AB}
3
4
5
4 kips
F_{AC}

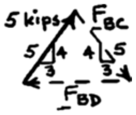
$$\frac{F_{AB}}{5} = \frac{F_{AC}}{3} = \frac{4 \text{ kips}}{4}$$

$F_{AB} = 5.00$ kips C ◀

$F_{AC} = 3.00$ kips T ◀

Free body Joint *B*:


B
F_{BD}
F_{AB} =
5 kips
4
3
4
5
3
F_{BC}

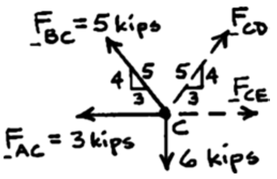

5 kips
F_{BC}
5
4
3
4
5
3
F_{BD}

$$\frac{F_{BC}}{5} = \frac{F_{BD}}{6} = \frac{5 \text{ kips}}{5}$$

$F_{BC} = 5.00$ kips T ◀

$F_{BD} = 6.00$ kips C ◀

Free body Joint *C*:


F_{BC} = 5 kips
F_{CD}
4
5
3
5
4
3
F_{CE}
F_{AC} = 3 kips
C
6 kips

$$+\uparrow \Sigma M_y = 0: \quad \frac{4}{5}(5) + \frac{4}{5}F_{CD} - 6 = 0$$

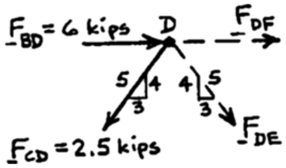
$F_{CD} = 2.50$ kips T ◀

$$+\uparrow \Sigma F_x = 0: \quad F_{CE} + \frac{3}{5}(2.5) - \frac{3}{5}(5) - 3 = 0$$

$F_{CE} = 4.50$ kips T ◀

PROBLEM 6.16 (Continued)

Free body: Joint *D*:


F_{BD} = 6 kips
D
F_{DF}
5
4
3
4
5
3
F_{DE}
F_{CD} = 2.5 kips

$$+\uparrow \Sigma F_y = 0: \quad -\frac{4}{5}(2.5) - \frac{4}{5}F_{DE} = 0$$

$$F_{DE} = -2.5 \text{ kips} \qquad F_{DE} = 2.50 \text{ kips} \quad C \ ◀$$

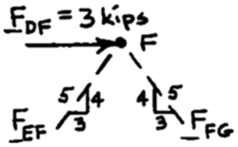
$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{DF} + 6 - \frac{3}{5}(2.5) - \frac{3}{5}(2.5) = 0$$

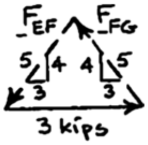
$$F_{DF} = -3 \text{ kips} \qquad F_{DF} = 3.00 \text{ kips} \quad C \ ◀$$

Free body: Joint *F*:

$$\frac{F_{EF}}{5} = \frac{F_{FG}}{5} = \frac{3 \text{ kips}}{6} \qquad F_{EF} = 2.50 \text{ kips} \quad T \ ◀$$

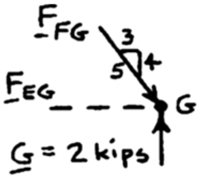
$$F_{FG} = 2.50 \text{ kips} \quad C \ ◀$$

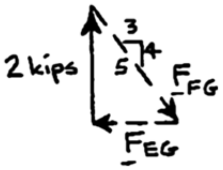

F_{DF} = 3 kips
F
5
4
3
4
5
3
F_{EF}
F_{FG}


F_{EF}
F_{FG}
5
4
4
5
3
3
3 kips

Free body: Joint *G*:

$$\frac{F_{EG}}{3} = \frac{2 \text{ kips}}{4} \qquad F_{EG} = 1.500 \text{ kips} \quad T \ ◀$$

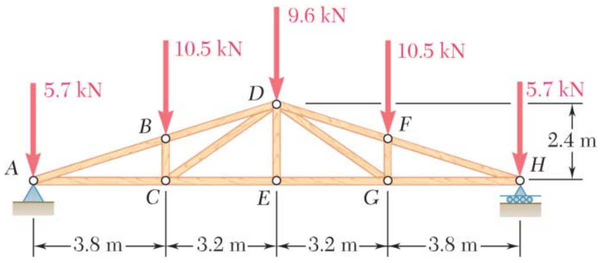

F_{FG}
3
4
5
F_{EG}
G
G = 2 kips


2 kips
3
4
5
F_{FG}
F_{EG}

Also,

$$\frac{F_{FG}}{5} = \frac{2 \text{ kips}}{4}$$

$$F_{FG} = 2.50 \text{ kips} \quad C \quad \text{(Checks)}$$



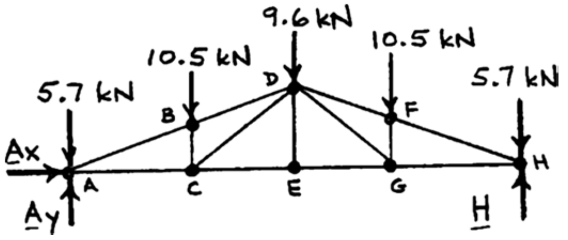
9.6 kN
10.5 kN
10.5 kN
5.7 kN
5.7 kN
D
B
F
A
H
2.4 m
C
E
G
3.8 m
3.2 m
3.2 m
3.8 m

PROBLEM 6.17

Determine the force in each member of the Pratt roof truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss:



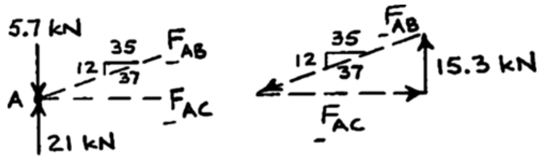
9.6 kN
10.5 kN
10.5 kN
5.7 kN
5.7 kN
D
B
F
Ax
A
C
E
G
H
Ay
H

$$\Sigma F_x = 0: \quad A_x = 0$$

Due to symmetry of truss and load,

$$A_y = H = \frac{1}{2} \text{ total load} = 21 \text{ kN}\uparrow$$

Free body: Joint *A*:



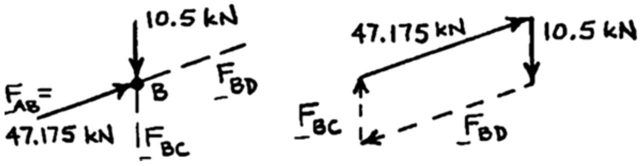
5.7 kN
35
12
37
F_AB
A
F_AC
21 kN
F_AB
35
12
37
15.3 kN
F_AC

$$\frac{F_{AB}}{37} = \frac{F_{AC}}{35} = \frac{15.3 \text{ kN}}{12}$$

$F_{AB} = 47.175 \text{ kN} \quad F_{AC} = 44.625 \text{ kN}$ $\qquad F_{AB} = 47.2 \text{ kN} \quad C$ ◀

$F_{AC} = 44.6 \text{ kN} \quad T$ ◀

Free body: Joint *B*:



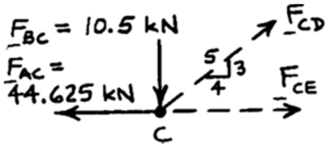
10.5 kN
F_BD
F_AB =
47.175 kN
B
F_BC
47.175 kN
10.5 kN
F_BC
F_BD

From force polygon: $F_{BD} = 47.175 \text{ kN}, \quad F_{BC} = 10.5 \text{ kN}$ $\qquad F_{BC} = 10.50 \text{ kN} \quad C$ ◀

$F_{BD} = 47.2 \text{ kN} \quad C$ ◀

PROBLEM 6.17 (Continued)

Free body: Joint *C*:


F_BC = 10.5 kN
F_AC = 44.625 kN
F_CD
5
3
4
F_CE
C

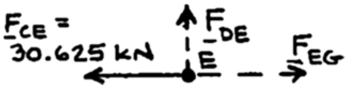
$+\uparrow \Sigma F_y = 0\text{:}\quad \frac{3}{5}F_{CD} - 10.5 = 0$ $\qquad F_{CD} = 17.50\text{ kN}\quad T$ ◀

$\xrightarrow{+} \Sigma F_x = 0\text{:}\quad F_{CE} + \frac{4}{5}(17.50) - 44.625 = 0$

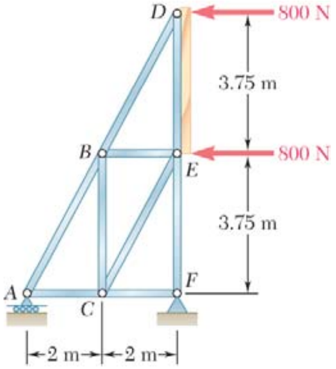
$F_{CE} = 30.625\text{ kN}$ $\qquad F_{CE} = 30.6\text{ kN}\quad T$ ◀

Free body: Joint *E*:

DE is a zero-force member. $\qquad F_{DE} = 0$ ◀


F_CE = 30.625 kN
F_DE
E
F_EG

Truss and loading symmetrical about ℄.

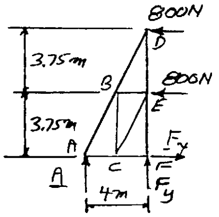

D
800 N
3.75 m
B
800 N
E
3.75 m
A
C
F
2 m
2 m

PROBLEM 6.18

The truss shown is one of several supporting an advertising panel. Determine the force in each member of the truss for a wind load equivalent to the two forces shown. State whether each member is in tension or compression.

SOLUTION

Free body: Entire truss:


800N
D
3.75m
B
800N
E
3.75m
A
F_x
C
F
A
F_y
4m

$$+\circlearrowleft \Sigma M_F = 0: \quad (800\text{ N})(7.5\text{ m}) + (800\text{ N})(3.75\text{ m}) - A(2\text{ m}) = 0$$

$$A = +2250\text{ N} \qquad \mathbf{A} = 2250\text{ N}\uparrow$$

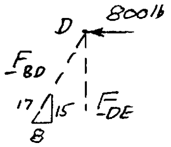
$$+\uparrow \Sigma F_y = 0: \quad 2250\text{ N} + F_y = 0$$

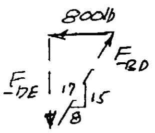
$$F_y = -2250\text{ N} \quad \mathbf{F}_y = 2250\text{ N}\downarrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -800\text{ N} - 800\text{ N} + F_x = 0$$

$$F_x = +1600\text{ N} \quad \mathbf{F}_x = 1600\text{ N}\longrightarrow$$

Joint D:


D
800lb
F_BD
17
15
8
F_DE

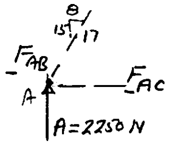

800lb
F_BD
F_DE
17
15
8

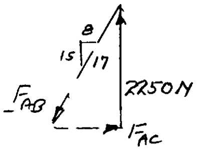
$$\frac{800\text{ N}}{8} = \frac{F_{DE}}{15} = \frac{F_{BD}}{17}$$

$F_{BD} = 1700\text{ N} \quad C$ ◀

$F_{DE} = 1500\text{ N} \quad T$ ◀

Joint A:


8
15
17
F_AB
A
F_AC
A=2250 N

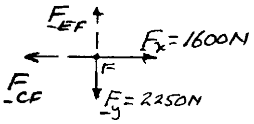

8
15
17
2250N
F_AB
F_AC

$$\frac{2250\text{ N}}{15} = \frac{F_{AB}}{17} = \frac{F_{AC}}{8}$$

$F_{AB} = 2250\text{ N} \quad C$ ◀

$F_{AC} = 1200\text{ N} \quad T$ ◀

Joint F:


F_EF
F_x = 1600N
F
F_CF
F_y = 2250N

$$\xrightarrow{+} \Sigma F_x = 0: \quad 1600\text{ N} - F_{CF} = 0$$

$$F_{CF} = +1600\text{ N}$$

$F_{CF} = 1600\text{ N} \quad T$ ◀

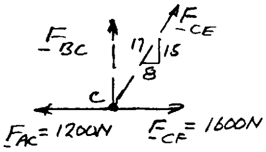
$$+\uparrow \Sigma F_y = 0: \quad F_{EF} - 2250\text{ N} = 0$$

$$F_{EF} = +2250\text{ N}$$

$F_{EF} = 2250\text{ N} \quad T$ ◀

PROBLEM 6.18 (Continued)

Joint *C*:


F_{BC}
17
15
8
C
F_{CE}
$F_{AC} = 1200N$
$F_{CF} = 1600N$

$$\xrightarrow{+} \Sigma F_x = 0\text{:}\quad \frac{8}{17}F_{CE} - 1200\text{ N} + 1600\text{ N} = 0$$

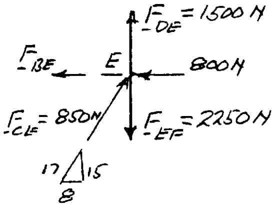
$$F_{CE} = -850\text{ N} \qquad F_{CE} = 850\text{ N} \quad C \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0\text{:}\quad F_{BC} + \frac{15}{17}F_{CE} = 0$$

$$F_{BC} = -\frac{15}{17}F_{CE} = -\frac{15}{17}(-850\text{ N})$$

$$F_{BC} = +750\text{ N} \qquad F_{BC} = 750\text{ N} \quad T \blacktriangleleft$$

Joint *E*:

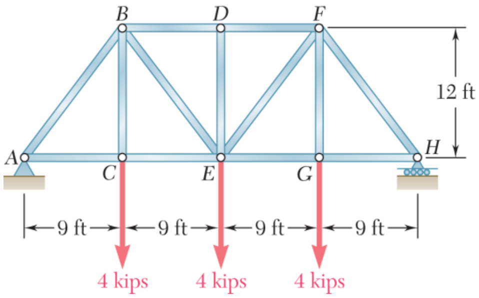

$F_{DE} = 1500 N$
F_{BE}
E
800N
$F_{CE} = 850N$
$F_{EF} = 2250N$
17
15
8

$$\xrightarrow{+} \Sigma F_x = 0\text{:}\quad -F_{BE} - 800\text{ N} + \frac{8}{17}(850\text{ N}) = 0$$

$$F_{BE} = -400\text{ N} \qquad F_{BE} = 400\text{ N} \quad C \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0\text{:}\quad 1500\text{ N} - 2250\text{ N} + \frac{15}{17}(850\text{ N}) = 0$$

$$0 = 0 \quad \text{(checks)}$$

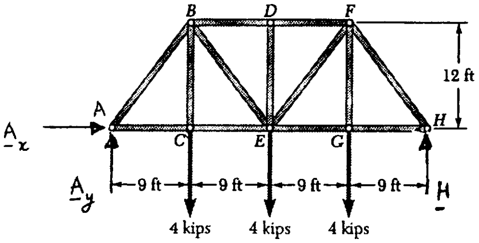

B
D
F
12 ft
A
H
C
E
G
9 ft
9 ft
9 ft
9 ft
4 kips
4 kips
4 kips

PROBLEM 6.19

Determine the force in each member of the Pratt bridge truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss:


B
D
F
12 ft
A
H
A_x
C
E
G
A_y
9 ft
9 ft
9 ft
9 ft
H
4 kips
4 kips
4 kips

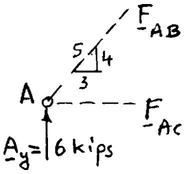
$$\overset{+}{\rightarrow} \Sigma F_z = 0: \quad \mathbf{A}_x = 0$$

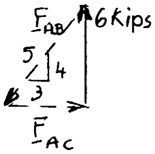
$$+\circlearrowleft \Sigma M_A = 0: \quad H(36\text{ ft}) - (4\text{ kips})(9\text{ ft}) - (4\text{ kips})(18\text{ ft}) - (4\text{ kips})(27\text{ ft}) = 0$$

$$\mathbf{H} = 6\text{ kips} \uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + 6\text{ kips} - 12\text{ kips} = 0 \qquad \mathbf{A}_y = 6\text{ kips} \uparrow$$

Free body: Joint *A*:


F_{AB}
5
4
3
A
F_{AC}
A_y = 6 kips

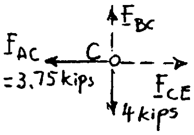

F_{AB}
6 kips
5
4
3
F_{AC}

$$\frac{F_{AB}}{5} = \frac{F_{AC}}{3} = \frac{6\text{ kips}}{4}$$

$F_{AB} = 7.50\text{ kips}$ *C* ◀

$F_{AC} = 4.50\text{ kips}$ *T* ◀

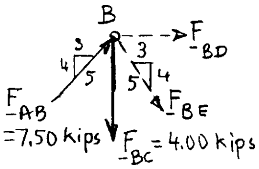
Free body: Joint *C*:


F_{BC}
F_{AC} = 3.75 kips
C
F_{CE}
4 kips

$\Sigma F_x = 0$: $F_{CE} = 4.50\text{ kips}$ *T* ◀

$\Sigma F_y = 0$: $F_{BC} = 4.00\text{ kips}$ *T* ◀

Free body: Joint *B*:


B
F_{BD}
3
4
5
3
5
4
F_{AB} = 7.50 kips
F_{BE}
F_{BC} = 4.00 kips

$$+\uparrow \Sigma F_y = 0: \quad -\frac{4}{5}F_{BE} + \frac{4}{5}(7.50\text{ kips}) - 4.00\text{ kips} = 0$$

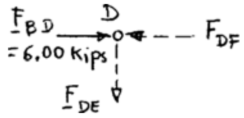
$F_{BE} = 2.50\text{ kips}$ *T* ◀

$$\overset{+}{\rightarrow} \Sigma F_x = 0: \quad \frac{8}{5}(7.50\text{ kips}) + \frac{3}{5}(2.50\text{ kips}) + F_{BD} = 0$$

$F_{BD} = -6.00\text{ kips}$ $F_{BD} = 6.00\text{ kips}$ *C* ◀

PROBLEM 6.19 (Continued)

Free body: Joint D:



We note that DE is a zero-force member: $F_{DE} = 0$ ◀

Also, $F_{DF} = 6.00$ kips C ◀

From symmetry:

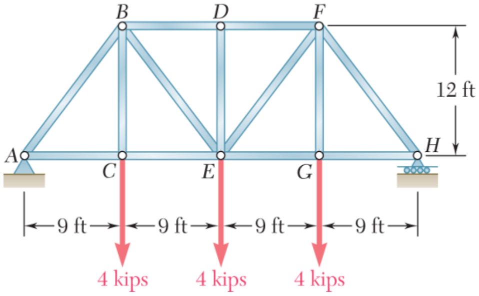
$F_{FE} = F_{BE}$ $F_{EF} = 2.50$ kips T ◀

$F_{EG} = F_{CE}$ $F_{EG} = 4.50$ kips T ◀

$F_{FG} = F_{BC}$ $F_{FG} = 4.00$ kips T ◀

$F_{FH} = F_{AB}$ $F_{FH} = 7.50$ kips C ◀

$F_{GH} = F_{AC}$ $F_{GH} = 4.50$ kips T ◀


B
D
F
12 ft
A
H
C
E
G
9 ft
9 ft
9 ft
9 ft
4 kips
4 kips
4 kips

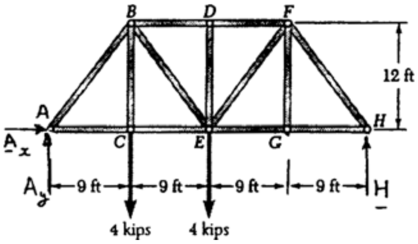
PROBLEM 6.20

Solve Problem 6.19 assuming that the load applied at *G* has been removed.

PROBLEM 6.19 Determine the force in each member of the Pratt bridge truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss:


B
D
F
12 ft
A
H
A_x
C
E
G
A_y
9 ft
9 ft
9 ft
9 ft
H
4 kips
4 kips

$$\Sigma F_x = 0: \quad \mathbf{A}_x = 0$$

$$+\circlearrowleft \Sigma \mathbf{M}_A = 0: \quad H(36\text{ ft}) - (4\text{ kips})(9\text{ ft}) - (4\text{ kips})(18\text{ ft}) = 0$$

$$\mathbf{H} = 3.00\text{ kips}\uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad \mathbf{A}_y + 5.00\text{ kips}\uparrow$$

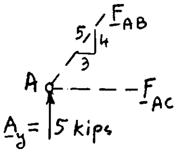
We note that *DE* and *FG* are zero-force members.

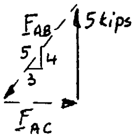
Therefore, $F_{DE} = 0, \quad F_{FG} = 0$ ◀

Also, $F_{BD} = F_{DF}$ (1)

and $F_{EG} = F_{GH}$ (2)

Free body: Joint *A*:


F_{AB}
5
4
3
A
F_{AC}
A_y = 5 kips

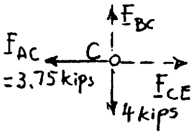

F_{AB}
5 kips
5
4
3
F_{AC}

$$\frac{F_{AB}}{5} = \frac{F_{AC}}{3} = \frac{5\text{ kips}}{4}$$

$F_{AB} = 6.25\text{ kips}$ *C* ◀

$F_{AC} = 3.75\text{ kips}$ *T* ◀

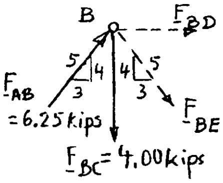
Free body: Joint *C*:


F_{BC}
F_{AC}
= 3.75 kips
C
F_{CE}
4 kips

$\Sigma F_x = 0: \quad F_{CE} = 3.75\text{ kips}$ *T* ◀

$\Sigma F_y = 0: \quad F_{BC} = 4.00\text{ kips}$ *T* ◀

Free body: Joint *B*:


B
F_{BD}
5
4
3
4
5
3
F_{AB}
= 6.25 kips
F_{BE}
F_{BC} = 4.00 kips

$$+\uparrow \Sigma F_x = 0: \quad \frac{4}{5}(6.25\text{ kips}) - 4.00\text{ kips} - \frac{4}{5}F_{BE} = 0$$

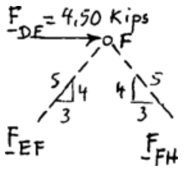
$F_{BE} = 1.250\text{ kips}$ *T* ◀

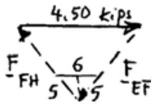
$$\overset{+}{\longrightarrow} \Sigma F_x = 0: \quad F_{BD} + \frac{3}{5}(6.25\text{ kips}) + \frac{3}{5}(1.250\text{ kips}) = 0$$

$F_{BD} = -4.50\text{ kips}$ $\quad F_{BD} = 4.50\text{ kips}$ *C* ◀

PROBLEM 6.20 (Continued)

Free body: Joint *F*:


F_DF = 4.50 kips
F
5
4
3
4
5
3
F_EF
F_FH


4.50 kips
F_FH
6
F_EF
5
5

We recall that $F_{FG} = 0$, and from Eq. (1) that

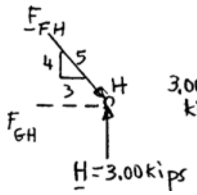
$F_{DF} = F_{BD}$ $\qquad$ $F_{DF} = 4.50 \text{ kips} \quad C \blacktriangleleft$

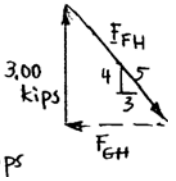
$$\frac{F_{EF}}{5} = \frac{F_{FH}}{5} = \frac{4.50 \text{ kips}}{6}$$

$F_{EF} = 3.75 \text{ kips} \quad T \blacktriangleleft$

$F_{FH} = 3.75 \text{ kips} \quad C \blacktriangleleft$

Free body: Joint *H*:


F_FH
4
5
3
H
F_GH
H = 3.00 kips


F_FH
3.00 kips
4
5
3
F_GH

$$\frac{F_{GH}}{3} = \frac{3.00 \text{ kips}}{4}$$

$F_{GH} = 2.25 \text{ kips} \quad T \blacktriangleleft$

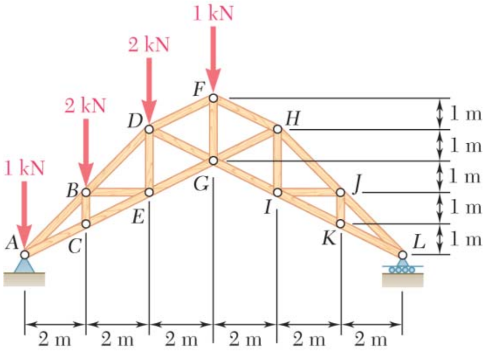
Also,

$$\frac{F_{FH}}{5} = \frac{3.00 \text{ kips}}{4}$$

$F_{FH} = 3.75 \text{ kips} \quad C \quad \text{(checks)}$

From Eq. (2):

$F_{EG} = F_{GH}$ $\qquad$ $F_{EG} = 2.25 \text{ kips} \quad T \blacktriangleleft$

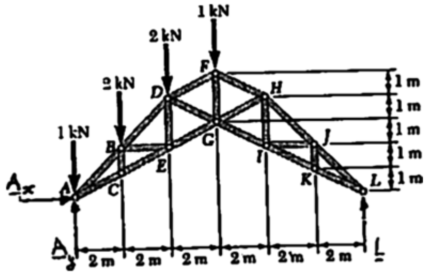

1 kN
2 kN
2 kN
1 kN
F
D
H
B
G
J
E
I
A
C
K
L
1 m
1 m
1 m
1 m
1 m
2 m
2 m
2 m
2 m
2 m
2 m

PROBLEM 6.21

Determine the force in each of the members located to the left of *FG* for the scissors roof truss shown. State whether each member is in tension or compression.

SOLUTION

Free Body: Truss:


1 kN
2 kN
2 kN
1 kN
F
D
H
B
G
J
E
I
A
C
K
L
A_x
A_y
L
1 m
1 m
1 m
1 m
1 m
2 m
2 m
2 m
2 m
2 m
2 m

$$\Sigma F_x = 0: \quad \mathbf{A}_x = 0$$

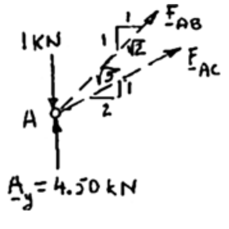
$$+\circlearrowleft \Sigma M_L = 0: \quad (1\text{ kN})(12\text{ m}) + (2\text{ kN})(10\text{ m}) + (2\text{ kN})(8\text{ m}) + (1\text{ kN})(6\text{ m}) - A_y(12\text{ m}) = 0$$

$$\mathbf{A}_y = 4.50\text{ kN}\uparrow$$

We note that *BC* is a zero-force member: $F_{BC} = 0$ ◀

Also, $F_{CE} = F_{AC}$ (1)

Free body: Joint *A*:


1 KN
1
$\sqrt{2}$
F_{AB}
1
F_{AC}
$\sqrt{5}$
1
2
A
$A_y = 4.50$ kN

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{1}{\sqrt{2}}F_{AB} + \frac{2}{\sqrt{5}}F_{AC} = 0 \tag{2}$$

$$+\!\uparrow \Sigma F_y = 0: \quad \frac{1}{\sqrt{2}}F_{AB} + \frac{1}{\sqrt{5}}F_{AC} + 3.50\text{ kN} = 0 \tag{3}$$

Multiply Eq. (3) by –2 and add Eq. (2):

$$-\frac{1}{\sqrt{2}}F_{AB} - 7\text{ kN} = 0$$

$F_{AB} = 9.90\text{ kN}$ *C* ◀

PROBLEM 6.21 (Continued)

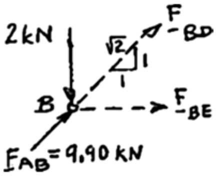
Subtract Eq. (3) from Eq. (2):

$$\frac{1}{\sqrt{5}}F_{AC} - 3.50\text{ kN} = 0 \quad F_{AC} = 7.826\text{ kN} \qquad F_{AC} = 7.83\text{ kN} \quad T$$ ◀

From Eq. (1):

$$F_{CE} = F_{AC} = 7.826\text{ kN} \qquad F_{CE} = 7.83\text{ kN} \quad T$$ ◀

Free body: Joint *B*:



2kN
F_{BD}
$\sqrt{2}$
1
1
B
F_{BE}
F_{AB}=9.90 kN

$$+\uparrow \Sigma F_y = 0: \quad \frac{1}{\sqrt{2}}F_{BD} + \frac{1}{\sqrt{2}}(9.90\text{ kN}) - 2\text{ kN} = 0$$

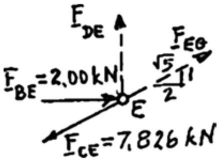
$$F_{BD} = -7.071\text{ kN} \qquad F_{BD} = 7.07\text{ kN} \quad C$$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{BE} + \frac{1}{\sqrt{2}}(9.90 - 7.071)\text{ kN} = 0$$

$$F_{BE} = -2.000\text{ kN} \qquad F_{BE} = 2.00\text{ kN} \quad C$$ ◀

Free body: Joint *E*:

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{2}{\sqrt{5}}(F_{EG} - 7.826\text{ kN}) + 2.00\text{ kN} = 0$$



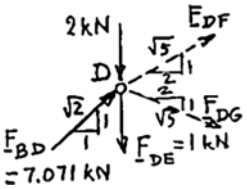
F_{DE}
F_{EG}
$\sqrt{5}$
1
2
F_{BE}=2.00kN
E
F_{CE}=7.826 kN

$$F_{EG} = 5.590\text{ kN} \quad F_{EG} = 5.59\text{ kN} \quad T$$ ◀

$$+\uparrow \Sigma F_y = 0: \quad F_{DE} - \frac{1}{\sqrt{5}}(7.826 - 5.590)\text{ kN} = 0$$

$$F_{DE} = 1.000\text{ kN} \qquad F_{DE} = 1.000\text{ kN} \quad T$$ ◀

Free body: Joint *D*:



2kN
F_{DF}
$\sqrt{5}$
1
2
D
$\sqrt{2}$
1
1
$\sqrt{5}$
1
2
F_{DG}
F_{BD}
=7.071 kN
F_{DE}=1 kN

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{2}{\sqrt{5}}(F_{DF} + F_{DG}) + \frac{1}{\sqrt{2}}(7.071\text{ kN})$$

or

$$F_{DF} + F_{DG} = -5.590\text{ kN} \tag{4}$$

$$+\uparrow \Sigma F_y = 0: \quad \frac{1}{\sqrt{5}}(F_{DF} - F_{DG}) + \frac{1}{\sqrt{2}}(7.071\text{ kN}) = 2\text{ kN} - 1\text{ kN} = 0$$

or

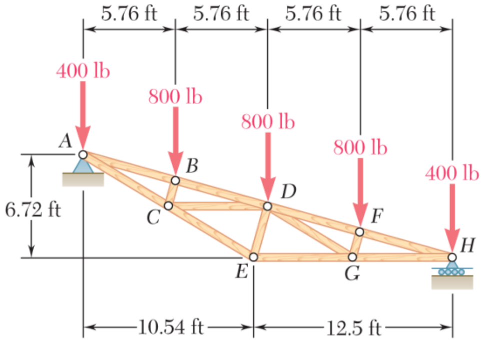
$$F_{DE} - F_{DG} = -4.472 \tag{5}$$

Add Eqs. (4) and (5):

$$2F_{DF} = -10.062\text{ kN} \qquad F_{DF} = 5.03\text{ kN} \quad C$$ ◀

Subtract Eq. (5) from Eq. (4):

$$2F_{DG} = -1.1180\text{ kN} \qquad F_{DG} = 0.559\text{ kN} \quad C$$ ◀

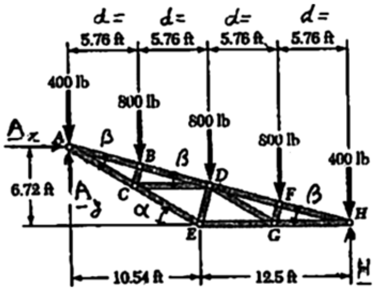

5.76 ft
5.76 ft
5.76 ft
5.76 ft
400 lb
800 lb
800 lb
800 lb
400 lb
A
B
C
D
E
F
G
H
6.72 ft
10.54 ft
12.5 ft

PROBLEM 6.22

Determine the force in member *DE* and in each of the members located to the left of *DE* for the inverted Howe roof truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss:


d = 5.76 ft
d = 5.76 ft
d = 5.76 ft
d = 5.76 ft
400 lb
800 lb
800 lb
800 lb
400 lb
A_x
A
A_y
β
B
β
C
α
D
E
F
β
G
H
6.72 ft
10.54 ft
12.5 ft
H

$$\Sigma F_x = 0: \quad \mathbf{A}_x = 0$$

$$+\circlearrowleft \Sigma M_H = 0: \quad (400\text{ lb})(4d) + (800\text{ lb})(3d) + (800\text{ lb})(2d) + (800\text{ lb})d - A_y(4d) = 0$$

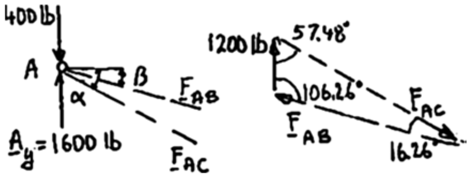
$$A_y = 1600\text{ lb}\uparrow$$

Angles:

$$\tan\alpha = \frac{6.72}{10.54} \qquad \alpha = 32.52^\circ$$

$$\tan\beta = \frac{6.72}{23.04} \qquad \beta = 16.26^\circ$$

Free body: Joint *A*:


400 lb
A
β
F_AB
α
A_y = 1600 lb
F_AC
1200 lb
57.48°
106.26°
F_AC
F_AB
16.26°

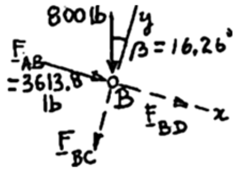
$$\frac{F_{AB}}{\sin 57.48^\circ} = \frac{F_{AC}}{\sin 106.26^\circ} = \frac{1200\text{ lb}}{\sin 16.26^\circ}$$

$$F_{AB} = 3613.8\text{ lb} \quad C$$

$$F_{AC} = 4114.3\text{ lb} \quad T \qquad F_{AB} = 3610\text{ lb} \quad C, \quad F_{AC} = 4110\text{ lb} \quad T \blacktriangleleft$$

PROBLEM 6.22 (Continued)

Free body: Joint *B*:


800 lb
y
β = 16.26°
F_AB = 3613.8 lb
B
F_BD
x
F_BC

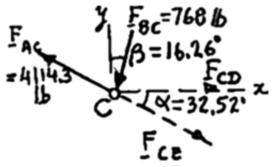
$$+\nearrow \Sigma F_y = 0: \quad -F_{BC} - (800 \text{ lb})\cos 16.26° = 0$$

$$F_{BC} = -768.0 \text{ lb} \qquad F_{BC} = 768 \text{ lb} \quad C$$ ◀

$$\searrow^{+} \Sigma F_x = 0: \quad F_{BD} + 3613.8 \text{ lb} + (800 \text{ lb})\sin 16.26° = 0$$

$$F_{BD} = -3837.8 \text{ lb} \qquad F_{BD} = 3840 \text{ lb} \quad C$$ ◀

Free body: Joint *C*:


y
F_BC = 768 lb
β = 16.26°
F_AC = 4114.3 lb
F_CD
x
α = 32.52°
C
F_CE

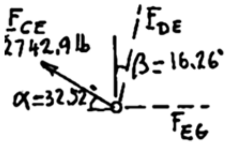
$$+\uparrow \Sigma F_y = 0: \quad -F_{CE}\sin 32.52° + (4114.3 \text{ lb})\sin 32.52° - (768 \text{ lb})\cos 16.26° = 0$$

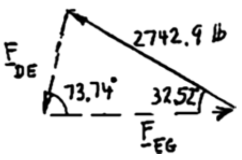
$$F_{CE} = 2742.9 \text{ lb} \qquad F_{CE} = 2740 \text{ lb} \quad T$$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{CD} - (4114.3 \text{ lb})\cos 32.52° + (2742.9 \text{ lb})\cos 32.52° - (768 \text{ lb})\sin 16.26° = 0$$

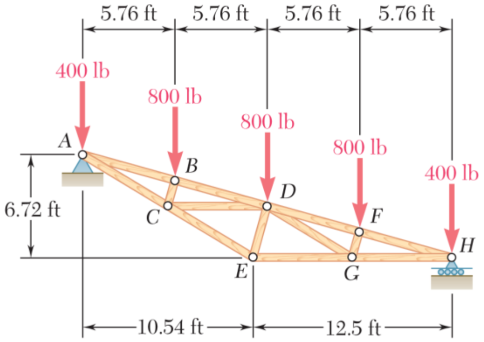
$$F_{CD} = 1371.4 \text{ lb} \qquad F_{CD} = 1371 \text{ lb} \quad T$$ ◀

Free body: Joint *E*:


F_CE
2742.9 lb
F_DE
β = 16.26°
α = 32.52°
F_EG


2742.9 lb
F_DE
73.74°
32.52°
F_EG

$$\frac{F_{DE}}{\sin 32.52°} = \frac{2742.9 \text{ lb}}{\sin 73.74°} \qquad F_{DE} = 1536 \text{ lb} \quad C$$ ◀

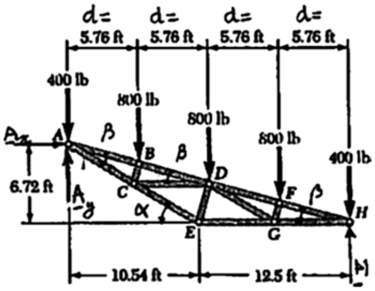

5.76 ft
5.76 ft
5.76 ft
5.76 ft
400 lb
800 lb
800 lb
800 lb
400 lb
A
B
C
D
E
F
G
H
6.72 ft
10.54 ft
12.5 ft

PROBLEM 6.23

Determine the force in each of the members located to the right of *DE* for the inverted Howe roof truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss


d= 5.76 ft
d= 5.76 ft
d= 5.76 ft
d= 5.76 ft
400 lb
800 lb
800 lb
800 lb
400 lb
A_x
A
A_y
B
C
D
E
F
G
H
β
β
β
α
6.72 ft
10.54 ft
12.5 ft
H

$$+\circlearrowleft \Sigma M_A = 0: \quad H(4d) - (800\text{ lb})d - (800\text{ lb})(2d) - (800\text{ lb})(3d) - (400\text{ lb})(4d) = 0$$

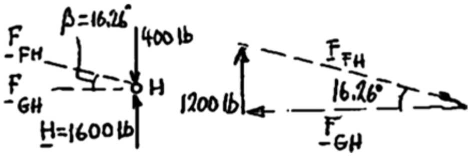
$$\mathbf{H} = 1600\text{ lb}\uparrow$$

Angles:

$$\tan\alpha = \frac{6.72}{10.54} \qquad \alpha = 32.52°$$

$$\tan\beta = \frac{6.72}{23.04} \qquad \beta = 16.26°$$

Free body: Joint *H*:


β=16.26°
400 lb
F_{FH}
F_{GH}
H
H=1600 lb
1200 lb
F_{FH}
16.26°
F_{GH}

$$F_{GH} = (1200\text{ lb})\cot 16.26°$$

$$F_{GH} = 4114.3\text{ lb} \quad T$$

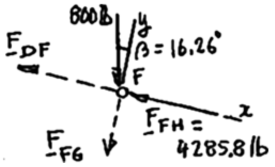
$$F_{GH} = 4110\text{ lb} \quad T \blacktriangleleft$$

$$F_{FH} = \frac{1200\text{ lb}}{\sin 16.26°} = 4285.8\text{ lb}$$

$$F_{FH} = 4290\text{ lb} \quad C \blacktriangleleft$$

PROBLEM 6.23 (Continued)

Free body Joint *F*:


800 lb
y
β = 16.26°
F
F_DF
F_FH = 4285.8 lb
x
F_FG

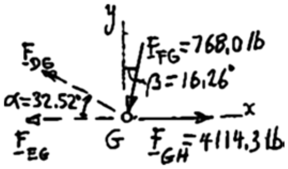
$$+\nearrow \Sigma F_y = 0: \quad -F_{FG} - (800\text{ lb})\cos 16.26^\circ = 0$$

$$F_{FG} = -768.0\text{ lb} \qquad F_{FG} = 768\text{ lb} \quad C \blacktriangleleft$$

$$\overset{+}{\searrow} \Sigma F_x = 0: \quad -F_{DF} - 4285.8\text{ lb} + (800\text{ lb})\sin 16.26^\circ = 0$$

$$F_{DF} = -4061.8\text{ lb} \qquad F_{DF} = 4060\text{ lb} \quad C \blacktriangleleft$$

Free body: Joint *G*:


y
F_FG = 768.0 lb
β = 16.26°
F_DG
α = 32.52°
x
F_EG
G
F_GH = 4114.3 lb

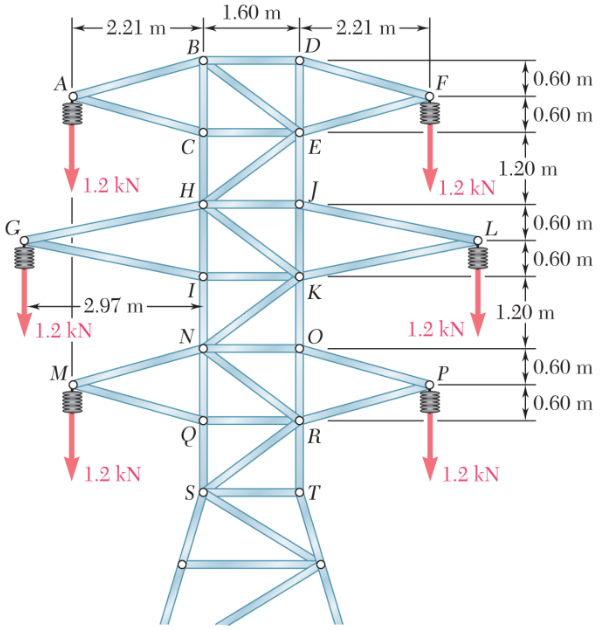
$$+\uparrow \Sigma F_y = 0: \quad F_{DG}\sin 32.52^\circ - (768.0\text{ lb})\cos 16.26^\circ = 0$$

$$F_{DG} = +1371.4\text{ lb}$$

$$F_{DG} = 1371\text{ lb} \quad T \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -F_{EG} + 4114.3\text{ lb} - (768.0\text{ lb})\sin 16.26^\circ - (1371.4\text{ lb})\cos 32.52^\circ = 0$$

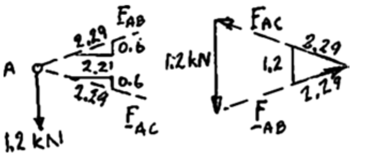
$$F_{EG} = +2742.9\text{ lb} \qquad F_{EG} = 2740\text{ lb} \quad T \blacktriangleleft$$


2.21 m
1.60 m
2.21 m
A
B
D
F
C
E
H
J
G
L
I
K
N
O
M
P
Q
R
S
T
0.60 m
0.60 m
1.20 m
0.60 m
0.60 m
1.20 m
0.60 m
0.60 m
2.97 m
1.2 kN
1.2 kN
1.2 kN
1.2 kN
1.2 kN
1.2 kN

PROBLEM 6.24

The portion of truss shown represents the upper part of a power transmission line tower. For the given loading, determine the force in each of the members located above *HJ*. State whether each member is in tension or compression.

SOLUTION

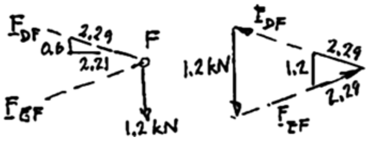

A
F_{AB}
2.29
0.6
2.21
0.6
2.29
F_{AC}
1.2 kN
F_{AC}
1.2 kN
2.29
1.2
2.29
F_{AB}

Free body: Joint *A*:

$$\frac{F_{AB}}{2.29} = \frac{F_{AC}}{2.29} = \frac{1.2\text{ kN}}{1.2}$$

$F_{AB} = 2.29\text{ kN}\quad T$ ◀

$F_{AC} = 2.29\text{ kN}\quad C$ ◀

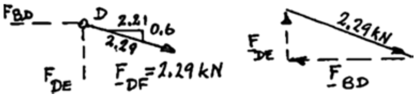

F_{DF}
2.29
0.6
2.21
F
F_{EF}
1.2 kN
F_{DF}
1.2 kN
2.29
1.2
2.29
F_{EF}

Free body: Joint *F*:

$$\frac{F_{DF}}{2.29} = \frac{F_{EF}}{2.29} = \frac{1.2\text{ kN}}{2.1}$$

$F_{DF} = 2.29\text{ kN}\quad T$ ◀

$F_{EF} = 2.29\text{ kN}\quad C$ ◀

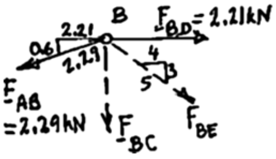

F_{BD}
D
2.21
0.6
2.29
F_{DE}
F_{DF} = 2.29 kN
2.29 kN
F_{DE}
F_{BD}

Free body: Joint *D*:

$$\frac{F_{BD}}{2.21} = \frac{F_{DE}}{0.6} = \frac{2.29\text{ kN}}{2.29}$$

$F_{BD} = 2.21\text{ kN}\quad T$ ◀

$F_{DE} = 0.600\text{ kN}\quad C$ ◀


B
F_{BD} = 2.21 kN
2.21
0.6
2.29
4
3
5
F_{AB} = 2.29 kN
F_{BC}
F_{BE}

Free body: Joint *B*:

$$\xrightarrow{+}\Sigma F_x = 0:\quad \frac{4}{5}F_{BE} + 2.21\text{ kN} - \frac{2.21}{2.29}(2.29\text{ kN}) = 0$$

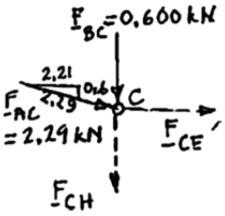
$F_{BE} = 0$ ◀

$$+\uparrow\Sigma F_y = 0:\quad -F_{BC} - \frac{3}{5}(0) - \frac{0.6}{2.29}(2.29\text{ kN}) = 0$$

$F_{BC} = -0.600\text{ kN}\qquad F_{BC} = 0.600\text{ kN}\quad C$ ◀

PROBLEM 6.24 (Continued)

Free body: Joint *C*:



$F_{BC} = 0.600$ kN
2.21
0.6
2.29
C
F_{AC}
$= 2.29$ kN
F_{CE}
F_{CH}

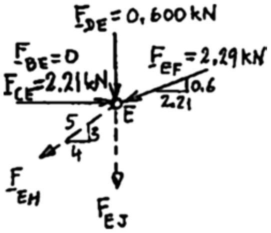
$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{CE} + \frac{2.21}{2.29}(2.29 \text{ kN}) = 0$$

$$F_{CE} = -2.21 \text{ kN} \qquad F_{CE} = 2.21 \text{ kN} \quad C \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -F_{CH} - 0.600 \text{ kN} - \frac{0.6}{2.29}(2.29 \text{ kN}) = 0$$

$$F_{CH} = -1.200 \text{ kN} \quad F_{CH} = 1.200 \text{ kN} \quad C \blacktriangleleft$$

Free body: Joint *E*:



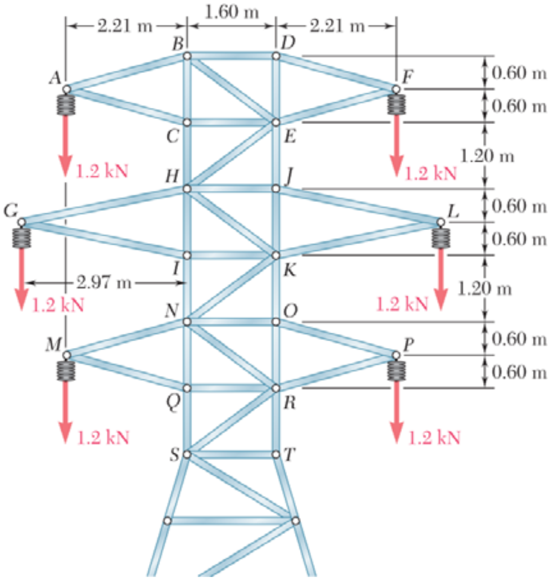
$F_{DE} = 0.600$ kN
$F_{BE} = 0$
$F_{EF} = 2.29$ kN
$F_{CE} = 2.21$ kN
0.6
2.21
E
5
3
4
F_{EH}
F_{EJ}

$$\xrightarrow{+} \Sigma F_x = 0: \quad 2.21 \text{ kN} - \frac{2.21}{2.29}(2.29 \text{ kN}) - \frac{4}{5}F_{EH} = 0$$

$$F_{EH} = 0 \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -F_{EJ} - 0.600 \text{ kN} - \frac{0.6}{2.29}(2.29 \text{ kN}) - 0 = 0$$

$$F_{EJ} = -1.200 \text{ kN} \quad F_{EJ} = 1.200 \text{ kN} \quad C \blacktriangleleft$$

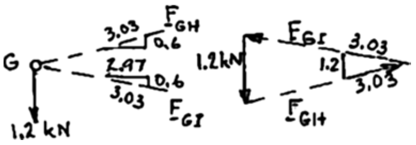

2.21 m
1.60 m
2.21 m
A
B
D
F
C
E
H
J
G
L
I
K
N
O
M
P
Q
R
S
T
0.60 m
0.60 m
1.20 m
0.60 m
0.60 m
1.20 m
0.60 m
0.60 m
2.97 m
1.2 kN
1.2 kN
1.2 kN
1.2 kN
1.2 kN
1.2 kN

PROBLEM 6.25

For the tower and loading of Problem 6.24 and knowing that $F_{CH} = F_{EJ} = 1.2$ kN *C* and $F_{EH} = 0$, determine the force in member *HJ* and in each of the members located between *HJ* and *NO*. State whether each member is in tension or compression.

PROBLEM 6.24 The portion of truss shown represents the upper part of a power transmission line tower. For the given loading, determine the force in each of the members located above *HJ*. State whether each member is in tension or compression.

SOLUTION

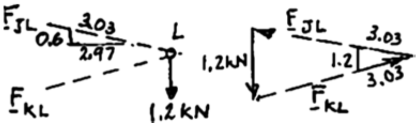

G
3.03
F_GH
0.6
2.97
0.6
3.03
F_GI
1.2 kN
1.2 kN
F_GI
3.03
1.2
3.03
F_GH

Free body: Joint *G*:

$$\frac{F_{GH}}{3.03} = \frac{F_{GI}}{3.03} = \frac{1.2\text{ kN}}{1.2}$$

$F_{GH} = 3.03\text{ kN}$ *T* ◀

$F_{GI} = 3.03\text{ kN}$ *C* ◀

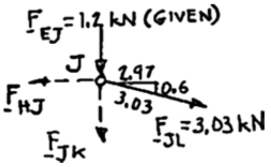

F_JL
0.6
3.03
2.97
L
F_KL
1.2 kN
1.2 kN
F_JL
3.03
1.2
3.03
F_KL

Free body: Joint *L*:

$$\frac{F_{JL}}{3.03} = \frac{F_{KL}}{3.03} = \frac{1.2\text{ kN}}{1.2}$$

$F_{JL} = 3.03\text{ kN}$ *T* ◀

$F_{KL} = 3.03\text{ kN}$ *C* ◀


F_EJ = 1.2 kN (GIVEN)
J
2.97
0.6
3.03
F_HJ
F_JL = 3.03 kN
F_JK

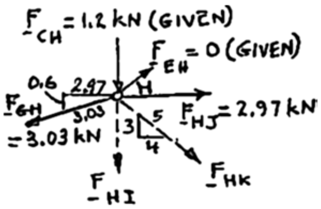
Free body: Joint *J*:

$$\xrightarrow{+} \Sigma F_x = 0: \quad -F_{HJ} + \frac{2.97}{3.03}(3.03\text{ kN}) = 0$$

$F_{HJ} = 2.97\text{ kN}$ *T* ◀

$$+\uparrow F_y = 0: \quad -F_{JK} - 1.2\text{ kN} - \frac{0.6}{3.03}(3.03\text{ kN}) = 0$$

$F_{JK} = -1.800\text{ kN}$ $\quad F_{JK} = 1.800\text{ kN}$ *C* ◀


F_CH = 1.2 kN (GIVEN)
F_EH = 0 (GIVEN)
0.6
2.97
H
F_GH = 3.03 kN
3.03
3
5
4
F_HJ = 2.97 kN
F_HI
F_HK

Free body: Joint *H*:

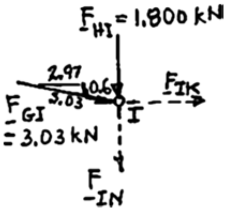
$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{4}{5}F_{HK} + 2.97\text{ kN} - \frac{2.97}{3.03}(3.03\text{ kN}) = 0$$

$F_{HK} = 0$ ◀

$$+\uparrow \Sigma F_y = 0: \quad -F_{HI} - 1.2\text{ kN} - \frac{0.6}{3.03}(3.03)\text{ kN} - \frac{3}{5}(0) = 0$$

$F_{HI} = -1.800\text{ kN}$ $\quad F_{HI} = 1.800\text{ kN}$ *C* ◀

PROBLEM 6.25 (Continued)



$F_{HI} = 1.800$ kN
2.97
0.6
3.03
$F_{GI} = 3.03$ kN
I
F_{IK}
F_{IN}

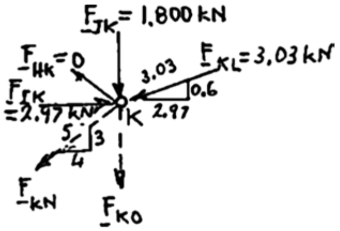
Free body: Joint *I*:

$$\xrightarrow{+}\Sigma F_x = 0:\quad F_{IK} + \frac{2.97}{3.03}(3.03\text{ kN}) = 0$$

$$F_{IK} = -2.97\text{ kN} \qquad F_{IK} = 2.97\text{ kN} \quad C$$ ◀

$$+\uparrow\Sigma F_y = 0:\quad -F_{IN} - 1.800\text{ kN} - \frac{0.6}{3.03}(3.03\text{ kN}) = 0$$

$$F_{IN} = -2.40\text{ kN} \qquad F_{IN} = 2.40\text{ kN} \quad C$$ ◀



$F_{JK} = 1.800$ kN
$F_{HK} = 0$
3.03
$F_{KL} = 3.03$ kN
0.6
2.97
$F_{IK} = 2.97$ kN
K
5
3
4
F_{KN}
F_{KO}

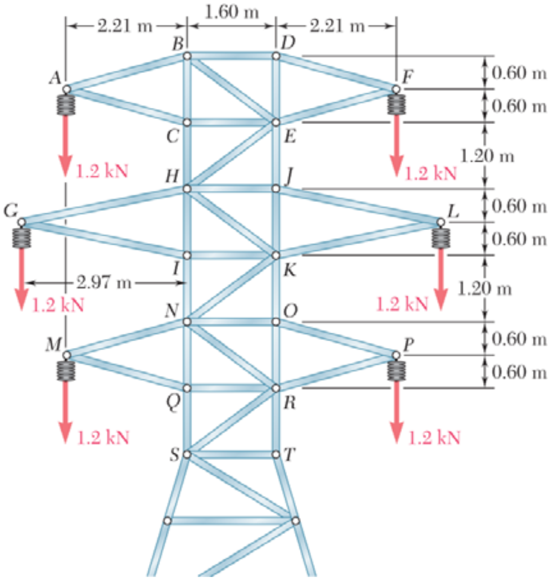
Free body: Joint *K*:

$$\xrightarrow{+}\Sigma F_x = 0:\quad -\frac{4}{5}F_{KN} + 2.97\text{ kN} - \frac{2.97}{3.03}(3.03\text{ kN}) = 0$$

$$F_{KN} = 0$$ ◀

$$+\uparrow\Sigma F_y = 0:\quad -F_{KO} - \frac{0.6}{3.03}(3.03\text{ kN}) - 1.800\text{ kN} - \frac{3}{5}(0) = 0$$

$$F_{KO} = -2.40\text{ kN} \qquad F_{KO} = 2.40\text{ kN} \quad C$$ ◀

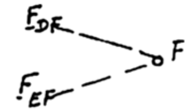

2.21 m
1.60 m
2.21 m
A
B
D
F
C
E
H
J
G
L
I
K
N
O
M
P
Q
R
S
T
0.60 m
0.60 m
1.20 m
0.60 m
0.60 m
1.20 m
0.60 m
0.60 m
2.97 m
1.2 kN
1.2 kN
1.2 kN
1.2 kN
1.2 kN
1.2 kN

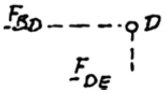
PROBLEM 6.26

Solve Problem 6.24 assuming that the cables hanging from the right side of the tower have fallen to the ground.

PROBLEM 6.24 The portion of truss shown represents the upper part of a power transmission line tower. For the given loading, determine the force in each of the members located above *HJ*. State whether each member is in tension or compression.

SOLUTION





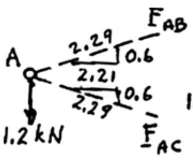
Zero-Force Members:

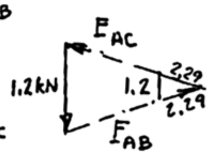
Considering joint *F*, we note that *DF* and *EF* are zero-force members:

$$F_{DF} = F_{EF} = 0 \blacktriangleleft$$

Considering next joint *D*, we note that *BD* and *DE* are zero-force members:

$$F_{BD} = F_{DE} = 0 \blacktriangleleft$$

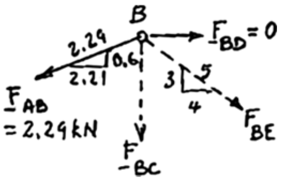

A
F_{AB}
2.29
0.6
2.21
0.6
2.29
1.2 kN
F_{AC}


F_{AC}
1.2 kN
1.2
2.29
2.29
F_{AB}

Free body: Joint *A*:

$$\frac{F_{AB}}{2.29} = \frac{F_{AC}}{2.29} = \frac{1.2\text{ kN}}{1.2} \qquad F_{AB} = 2.29\text{ kN} \quad T \blacktriangleleft$$

$$F_{AC} = 2.29\text{ kN} \quad C \blacktriangleleft$$


B
$F_{BD} = 0$
2.29
0.6
2.21
3
5
4
F_{AB}
= 2.29 kN
F_{BC}
F_{BE}

Free body: Joint *B*:

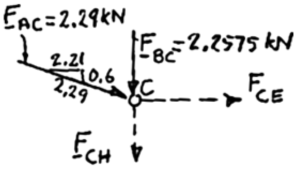
$$\xrightarrow{+} \Sigma F_x = 0\colon \quad \frac{4}{5}F_{BE} - \frac{2.21}{2.29}(2.29\text{ kN}) = 0$$

$$F_{BE} = 2.7625\text{ kN} \qquad F_{BE} = 2.76\text{ kN} \quad T \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0\colon \quad -F_{BC} - \frac{0.6}{2.29}(2.29\text{ kN}) - \frac{3}{5}(2.7625\text{ kN}) = 0$$

$$F_{BC} = -2.2575\text{ kN} \qquad F_{BC} = 2.26\text{ kN} \quad C \blacktriangleleft$$

PROBLEM 6.26 (Continued)



$F_{AC} = 2.29$ kN
$F_{BC} = 2.2575$ kN
2.21
0.6
2.29
C
F_{CE}
F_{CH}

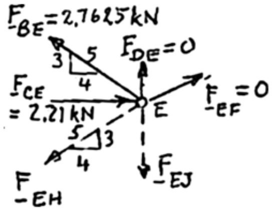
Free body: Joint *C*:

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{CE} + \frac{2.21}{2.29}(2.29\text{ kN}) = 0$$

$$F_{CE} = 2.21\text{ kN} \quad C \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -F_{CH} - 2.2575\text{ kN} - \frac{0.6}{2.29}(2.29\text{ kN}) = 0$$

$$F_{CH} = -2.8575\text{ kN} \qquad F_{CH} = 2.86\text{ kN} \quad C \blacktriangleleft$$



$F_{BE} = 2.7625$ kN
3
5
4
$F_{DE} = 0$
F_{CE}
$= 2.21$ kN
E
$F_{EF} = 0$
5
3
4
F_{EH}
F_{EJ}

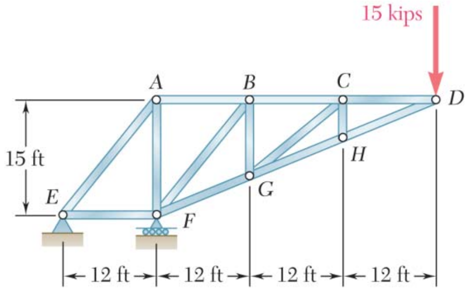
Free body: Joint *E*:

$$\xrightarrow{+} \Sigma F_x = 0: \quad -\frac{4}{5}F_{EH} - \frac{4}{5}(2.7625\text{ kN}) + 2.21\text{ kN} = 0$$

$$F_{EH} = 0 \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -F_{EJ} + \frac{3}{5}(2.7625\text{ kN}) - \frac{3}{5}(0) = 0$$

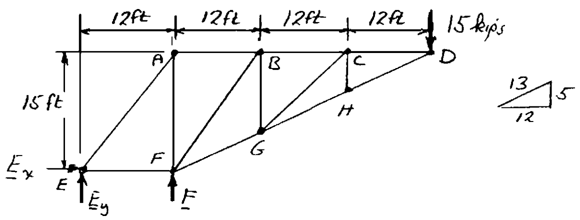
$$F_{EJ} = +1.6575\text{ kN} \qquad F_{EJ} = 1.658\text{ kN} \quad T \blacktriangleleft$$


15 kips
A
B
C
D
H
G
E
F
15 ft
12 ft
12 ft
12 ft
12 ft

PROBLEM 6.27

Determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION


12ft
12ft
12ft
12ft
15 kips
A
B
C
D
15 ft
H
G
F
E_x
E
E_y
F
13
5
12

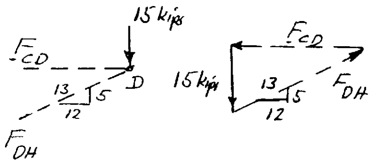
Reactions:

$\Sigma F_x = 0$: $\mathbf{E}_x = 0$

$\Sigma M_F = 0$: $\mathbf{E}_y = 45 \text{ kips}\downarrow$

$\Sigma F_y = 0$: $\mathbf{F} = 60 \text{ kips}\uparrow$

Joint *D*:

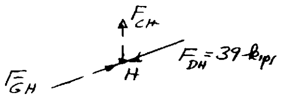

15 kips
F_{CD}
13
5
12
D
F_{DH}
15 kips
F_{CD}
13
5
12
F_{DH}

$$\frac{F_{CD}}{12} = \frac{F_{DH}}{13} = \frac{15 \text{ kips}}{5}$$

$F_{CD} = 36.0 \text{ kips}$ *T* ◀

$F_{DH} = 39.0 \text{ kips}$ *C* ◀

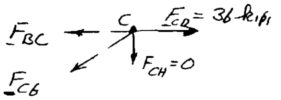
Joint *H*:


F_{CH}
$F_{DH} = 39$ kips
H
F_{GH}

↖ $\Sigma F = 0$: $F_{CH} = 0$ ◀

↗ $\Sigma F = 0$: $F_{GH} = 39.0 \text{ kips}$ *C* ◀

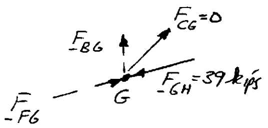
Joint *C*:


F_{BC}
C
$F_{CD} = 36$ kips
$F_{CH} = 0$
F_{CG}

↑ $\Sigma F = 0$: $F_{CG} = 0$ ◀

→ $\Sigma F = 0$: $F_{BC} = 36.0 \text{ kips}$ *T* ◀

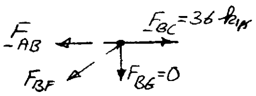
Joint *G*:


$F_{CG} = 0$
F_{BG}
$F_{GH} = 39$ kips
G
F_{FG}

↖ $\Sigma F = 0$: $F_{BG} = 0$ ◀

↗ $\Sigma F = 0$: $F_{FG} = 39.0 \text{ kips}$ *C* ◀

Joint *B*:

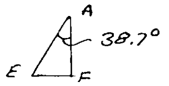

$F_{BC} = 36$ kips
F_{AB}
F_{BF}
$F_{BG} = 0$

↑ $\Sigma F = 0$: $F_{BF} = 0$ ◀

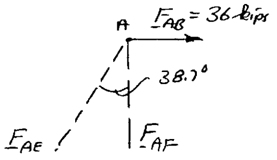
→ $\Sigma F = 0$: $F_{AB} = 36.0 \text{ kips}$ *T* ◀

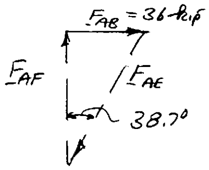
PROBLEM 6.27 (Continued)

Joint *A*:



$$AE = \sqrt{12^2 + 15^2} = 19.21 \text{ ft}$$

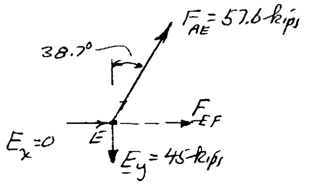

A
F_AB = 36 kips
38.7°
F_AE
F_AF


F_AB = 36 kips
F_AF
F_AE
38.7°

$$\tan 38.7° = \frac{36 \text{ kips}}{F_{AF}} \qquad F_{AF} = 45.0 \text{ kips} \quad C \blacktriangleleft$$

$$\sin 38.7° = \frac{36 \text{ kips}}{F_{AE}} \qquad F_{AE} = 57.6 \text{ kips} \quad T \blacktriangleleft$$

Joint *E*:

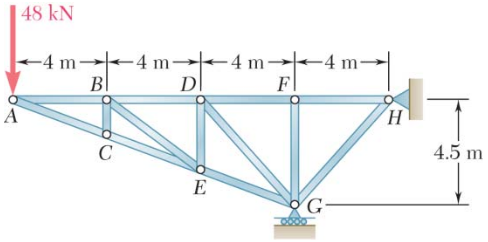

F_AE = 57.6 kips
38.7°
F_EF
E_x = 0
E
E_y = 45 kips

$$\xrightarrow{+} \Sigma F_x = 0: \quad +(57.6 \text{ kips})\sin 38.7° + F_{EF} = 0$$

$$F_{EF} = -36.0 \text{ kips} \qquad F_{EF} = 36.0 \text{ kips} \quad C \blacktriangleleft$$

$$\uparrow \Sigma F_y = 0: \quad (57.6 \text{ kips})\cos 38.7° - 45 \text{ kips} = 0$$

(Checks)

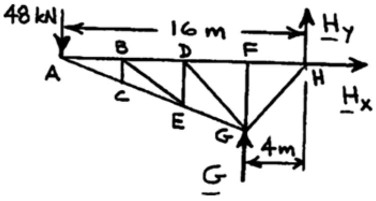

48 kN
4 m
4 m
4 m
4 m
B
D
F
A
H
C
E
G
4.5 m

PROBLEM 6.28

Determine the force in each member of the truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss


48 kN
16 m
H_y
B
D
F
A
H
H_x
C
E
G
4m
G

$$\Sigma F_x = 0: \quad \mathbf{H}_x = 0$$

$$+\circlearrowleft \Sigma M_H = 0: \quad 48(16) - G(4) = 0 \quad \mathbf{G} = 192 \text{ kN}\uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad 192 - 48 + H_y = 0$$

$$H_y = -144 \text{ kN} \quad \mathbf{H}_y = 144 \text{ kN}\downarrow$$

Zero-Force Members:

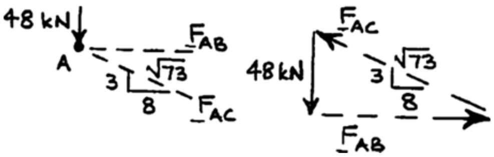
Examining successively joints C, B, E, D, and F, we note that the following are zero-force members: BC, BE, DE, DG, FG

Thus, $F_{BC} = F_{BE} = F_{DE} = F_{DG} = F_{FG} = 0$

Also note: $F_{AB} = F_{BD} = F_{DF} = F_{FH}$ (1)

$F_{AC} = F_{CE} = F_{EG}$ (2)

Free body: Joint A: $\dfrac{F_{AB}}{8} = \dfrac{F_{AC}}{\sqrt{73}} = \dfrac{48 \text{ kN}}{3}$


48 kN
F_{AB}
A
3
$\sqrt{73}$
8
F_{AC}
48 kN
F_{AC}
3
$\sqrt{73}$
8
F_{AB}

$F_{AB} = 128$ kN $F_{AB} = 128.0$ kN T ◀

$F_{AC} = 136.704$ kN $F_{AC} = 136.7$ kN C ◀

From Eq. (1): $F_{BD} = F_{DF} = F_{FH} = 128.0$ kN T ◀

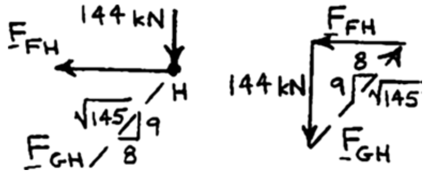
From Eq. (2): $F_{CE} = F_{EG} = 136.7$ kN C ◀

PROBLEM 6.28 (Continued)

Free body: Joint *H*

$$\frac{F_{GH}}{\sqrt{145}} = \frac{144 \text{ kN}}{9}$$

$F_{GH} = 192.7 \text{ kN} \quad C$ ◀

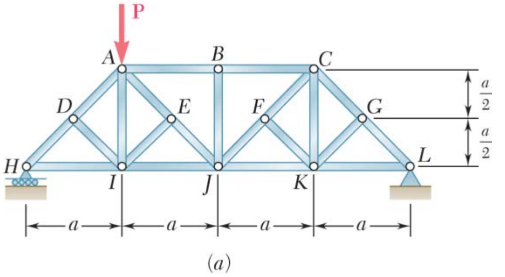


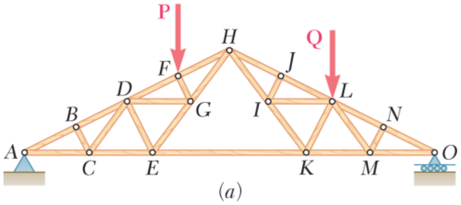
$\underline{F}_{FH}$
144 kN
H
√145
9
$\underline{F}_{GH}$
8
144 kN
$\underline{F}_{FH}$
8
9
√145
$\underline{F}_{GH}$

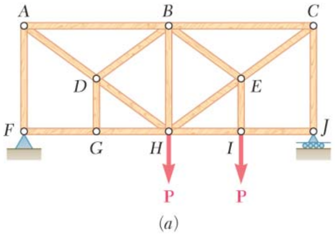
Also

$$\frac{F_{FH}}{8} = \frac{144 \text{ kN}}{9}$$

$F_{FH} = 128.0 \text{ kN} \quad T$ (Checks)


P
A
B
C
D
E
F
G
H
I
J
K
L
a/2
a/2
a
a
a
a
(a)

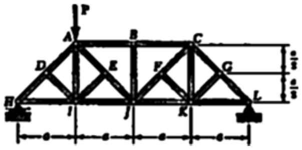

P
H
Q
J
F
D
L
B
G
I
N
A
C
E
K
M
O
(a)


A
B
C
D
E
F
G
H
I
J
P
P
(a)

PROBLEM 6.29

Determine whether the trusses of Problems 6.31*a*, 6.32*a*, and 6.33*a* are simple trusses.

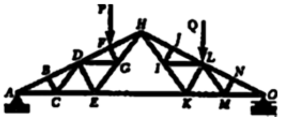
SOLUTION



Truss of Problem 6.31*a*:

Starting with triangle *HDI* and adding two members at a time, we obtain successively joints *A*, *E*, *J*, and *B*, but cannot go further. Thus, this truss

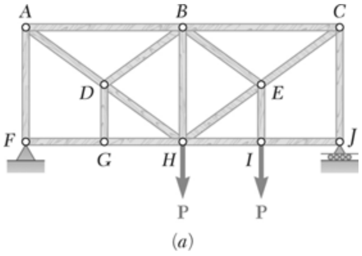
is not a simple truss. ◀



Truss of Problem 6.32*a*:

Starting with triangle *ABC* and adding two members at a time, we obtain joints *D*, *E*, *G*, *F*, and *H*, but cannot go further. Thus, this truss

is not a simple truss. ◀


A
B
C
D
E
F
G
H
I
J
P
P
(a)

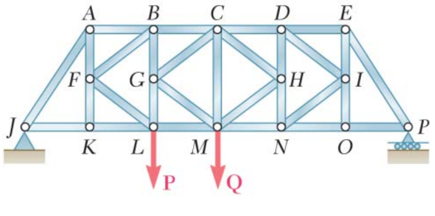
Truss of Problem 6.33*a*:

Starting with triangle *ABD* and adding two members at a time, we obtain successively joints *H*, *G*, *F*, *E*, *I*, *C*, and *J*, thus completing the truss.

Therefore, this is a simple truss. ◀

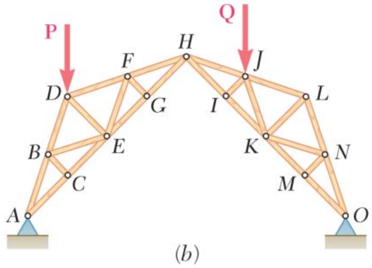
PROBLEM 6.30

Determine whether the trusses of Problems 6.31*b*, 6.32*b*, and 6.33*b* are simple trusses.



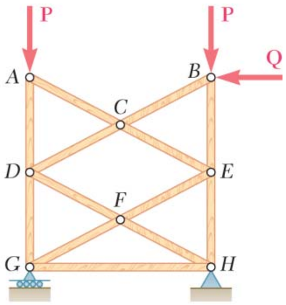
A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P
P
Q

(*b*)



P
Q
H
F
J
D
G
I
L
B
E
K
N
C
M
A
O

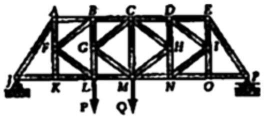
(*b*)



P
P
Q
A
B
C
D
E
F
G
H

(*b*)

SOLUTION

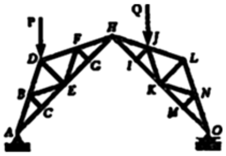


A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P
P
Q

Truss of Problem 6.31*b*:

Starting with triangle *CGM* and adding two members at a time, we obtain successively joints *B*, *L*, *F*, *A*, *K*, *J*, then *H*, *D*, *N*, *I*, *E*, *O*, and *P*, thus completing the truss.

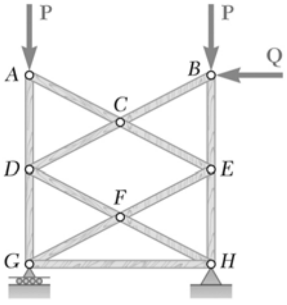
Therefore, this truss is a simple truss. ◀



P
Q
H
F
J
D
G
I
L
B
E
K
N
C
M
A
O

Truss of Problem 6.32*b*:

Starting with triangle *ABC* and adding two members at a time, we obtain successively joints *E*, *D*, *F*, *G*, and *H*, but cannot go further. Thus, this truss is not a simple truss. ◀



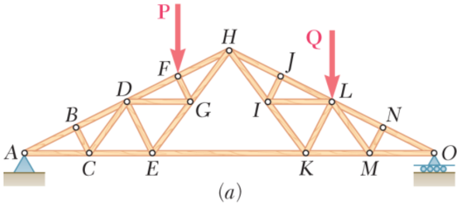
P
P
Q
A
B
C
D
E
F
G
H

(*b*)

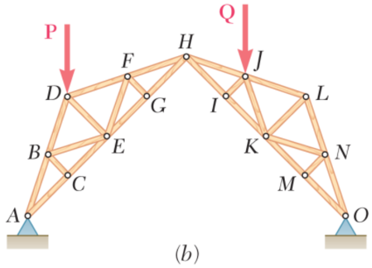
Truss of Problem 6.33*b*:

Starting with triangle *GFH* and adding two members at a time, we obtain successively joints *D*, *E*, *C*, *A*, and *B*, thus completing the truss.

Therefore, this is a simple truss. ◀


P
H
Q
F
J
D
G
I
L
B
N
A
O
C
E
K
M

(a)


Q
P
H
F
J
D
G
I
L
B
E
K
N
C
M
A
O

(b)

PROBLEM 6.31

For the given loading, determine the zero-force members in each of the two trusses shown.

SOLUTION

Truss (*a*):

FB: Joint *B*: $F_{BC} = 0$

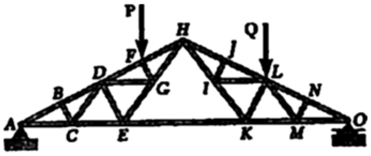
FB: Joint *C*: $F_{CD} = 0$

FB: Joint *J*: $F_{IJ} = 0$

FB: Joint *I*: $F_{IL} = 0$

FB: Joint *N*: $F_{MN} = 0$

FB: Joint *M*: $F_{LM} = 0$


P
H
Q
F
J
D
G
I
L
B
N
A
O
C
E
K
M

(*a*)

The zero-force members, therefore, are BC, CD, IJ, IL, LM, MN ◀

Truss (*b*):

FB: Joint *C*: $F_{BC} = 0$

FB: Joint *B*: $F_{BE} = 0$

FB: Joint *G*: $F_{FG} = 0$

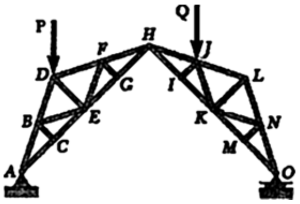
FB: Joint *F*: $F_{EF} = 0$

FB: Joint *E*: $F_{DE} = 0$

FB: Joint *I*: $F_{IJ} = 0$

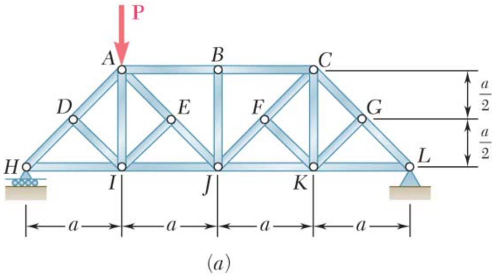
FB: Joint *M*: $F_{MN} = 0$

FB: Joint *N*: $F_{KN} = 0$

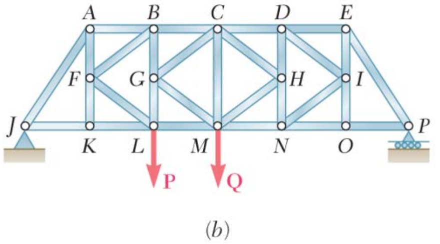

Q
P
H
F
J
D
G
I
L
B
E
K
N
C
M
A
O

(*b*)

The zero-force members, therefore, are $BC, BE, DE, EF, FG, IJ, KN, MN$ ◀


P
A
B
C
D
E
F
G
H
I
J
K
L
a/2
a/2
a
a
a
a

(a)


A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P
P
Q

(b)

PROBLEM 6.32

For the given loading, determine the zero-force members in each of the two trusses shown.

SOLUTION

Truss (*a*):

FB: Joint *B*: $F_{BJ} = 0$

FB: Joint *D*: $F_{DI} = 0$

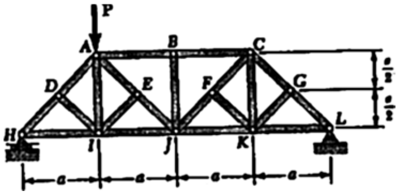
FB: Joint *E*: $F_{EI} = 0$

FB: Joint *I*: $F_{AI} = 0$

FB: Joint *F*: $F_{FK} = 0$

FB: Joint *G*: $F_{GK} = 0$

FB: Joint *K*: $F_{CK} = 0$


P
A
B
C
D
E
F
G
H
I
J
K
L
a
a
a
a

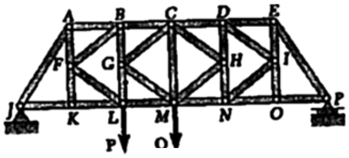
(*a*)

The zero-force members, therefore, are $AI, BJ, CK, DI, EI, FK, GK$ ◀

Truss (*b*):

FB: Joint *K*: $F_{FK} = 0$

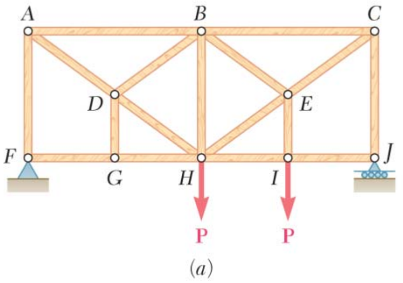
FB: Joint *O*: $F_{IO} = 0$


A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P
P
Q

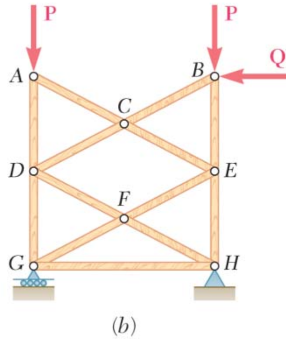
(*b*)

The zero-force members, therefore, are *FK* and *IO* ◀

All other members are either in tension or compression.


A
B
C
D
E
F
G
H
I
J
P
P

(a)


P
P
Q
A
B
C
D
E
F
G
H

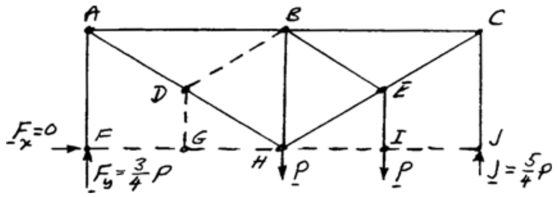
(b)

PROBLEM 6.33

For the given loading, determine the zero-force members in each of the two trusses shown.

SOLUTION

Truss (*a*):


A
B
C
D
E
$F_x = 0$
F
G
H
I
J
$F_y = \frac{3}{4}P$
P
P
$J = \frac{5}{4}P$

Note: Reation at F is vertical $(F_x = 0)$.

Joint G: $\downarrow \Sigma F = 0,$ $F_{DG} = 0$ ◀

Joint D: $\nearrow \Sigma F = 0,$ $F_{DB} = 0$ ◀

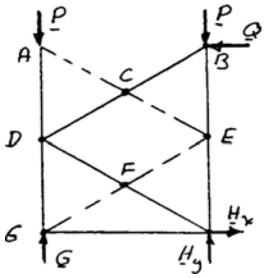
Joint F: $\longrightarrow \Sigma F = 0,$ $F_{FG} = 0$ ◀

Joint G: $\longrightarrow \Sigma F = 0,$ $F_{GH} = 0$ ◀

Joint J: $\longrightarrow \Sigma F = 0,$ $F_{IJ} = 0$ ◀

Joint I: $\longrightarrow \Sigma F = 0,$ $F_{HI} = 0$ ◀

Truss (*b*):


P
P
Q
A
B
C
D
E
F
G
H_x
G
H_y

Joint A: $\longrightarrow \Sigma F = 0,$ $F_{AC} = 0$ ◀

Joint C: $\searrow \Sigma F = 0,$ $F_{CE} = 0$ ◀

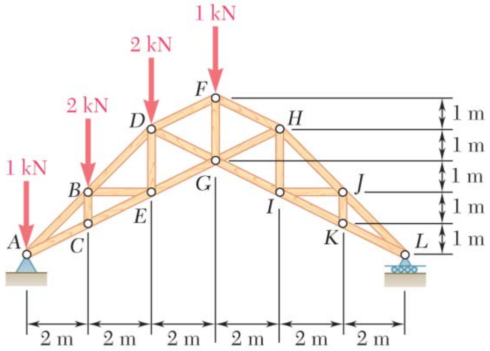
Joint E: $\longrightarrow \Sigma F = 0,$ $F_{EF} = 0$ ◀

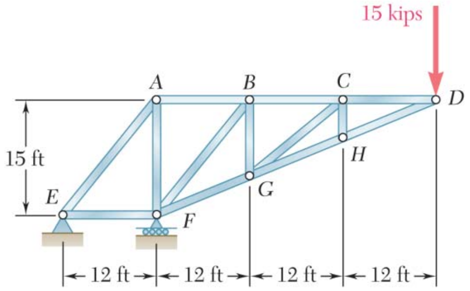
Joint F: $\swarrow \Sigma F = 0,$ $F_{FG} = 0$ ◀

Joint G: $\longrightarrow \Sigma F = 0,$ $F_{GH} = 0$ ◀

PROBLEM 6.34

Determine the zero-force members in the truss of (*a*) Problem 6.21, (*b*) Problem 6.27.


1 kN
2 kN
2 kN
1 kN
F
D
H
B
G
J
A
C
E
I
K
L
1 m
1 m
1 m
1 m
1 m
2 m
2 m
2 m
2 m
2 m
2 m


15 kips
A
B
C
D
H
G
E
F
15 ft
12 ft
12 ft
12 ft
12 ft

SOLUTION

(*a*) Truss of Problem 6.21:

$$FB\text{: Joint } C\text{: } F_{BC} = 0$$

$$FB\text{: Joint } J\text{: } F_{IJ} = 0$$

$$FB\text{: Joint } K\text{: } F_{JK} = 0$$

$$FB\text{: Joint } I\text{: } F_{HI} = 0$$

The zero-force members, therefore, are BC, HI, IJ, JK ◀

(*b*) Truss of Problem 6.27:

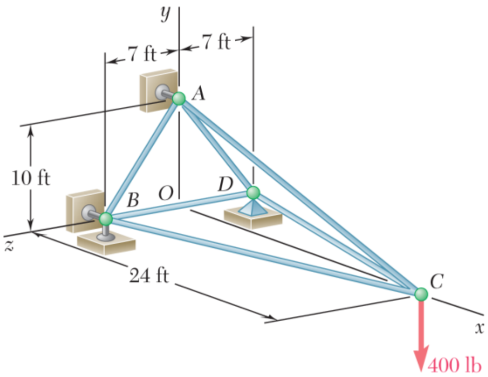
$$FB\text{: Joint } B\text{: } F_{BF} = 0$$

$$FB\text{: Joint } B\text{: } F_{BG} = 0$$

$$FB\text{: Joint } C\text{: } F_{CG} = 0$$

$$FB\text{: Joint } C\text{: } F_{CH} = 0$$

The zero-force members, therefore, are BF, BG, CG, CH ◀


y
7 ft
7 ft
A
10 ft
B
O
D
z
24 ft
C
x
400 lb

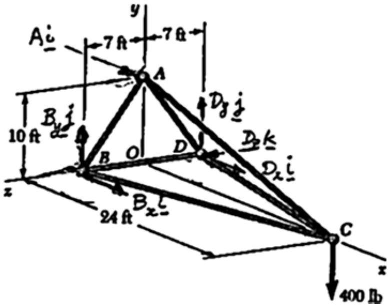
PROBLEM 6.35*

The truss shown consists of six members and is supported by a short link at *A*, two short links at *B*, and a ball and socket at *D*. Determine the force in each of the members for the given loading.

SOLUTION

Free body: Truss:

From symmetry:


$A\mathbf{i}$
y
7 ft
7 ft
A
$B_y\mathbf{j}$
$D_y\mathbf{j}$
10 ft
B
O
D
$D_z\mathbf{k}$
$D_x\mathbf{i}$
z
$B_x\mathbf{i}$
24 ft
C
x
400 lb

$$D_x = B_x \quad \text{and} \quad D_y = B_y$$

$$\Sigma M_z = 0: \quad -A(10\text{ ft}) - (400\text{ lb})(24\text{ ft}) = 0$$

$$A = -960\text{ lb}$$

$$\Sigma F_x = 0: \quad B_x + D_x + A = 0$$

$$2B_x - 960\text{ lb} = 0, \quad B_x = 480\text{ lb}$$

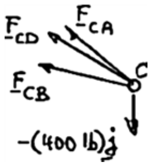
$$\Sigma F_y = 0: \quad B_y + D_y - 400\text{ lb} = 0$$

$$2B_y = 400\text{ lb}$$

$$B_y = +200\text{ lb}$$

Thus, $\mathbf{B} = (480\text{ lb})\mathbf{i} + (200\text{ lb})\mathbf{j}$ $\triangleleft$

Free body: *C*:


F_{CD}
F_{CA}
C
F_{CB}
$-(400\text{ lb})\mathbf{j}$

$$F_{CA} = F_{AC}\frac{\overrightarrow{CA}}{CA} = \frac{F_{AC}}{26}(-24\mathbf{i} + 10\mathbf{j})$$

$$F_{CB} = F_{BC}\frac{\overrightarrow{CB}}{CB} = \frac{F_{BC}}{25}(-24\mathbf{i} + 7\mathbf{k})$$

$$F_{CD} = F_{CD}\frac{\overrightarrow{CD}}{CD} = \frac{F_{CD}}{25}(-24\mathbf{i} - 7\mathbf{k})$$

$$\Sigma\mathbf{F} = 0: \quad \mathbf{F}_{CA} + \mathbf{F}_{CB} + \mathbf{F}_{CD} - (400\text{ lb})\mathbf{j} = 0$$

Substituting for $\mathbf{F}_{CA}, \mathbf{F}_{CB}, \mathbf{F}_{CD}$, and equating to zero the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

i: $$-\frac{24}{26}F_{AC} - \frac{24}{25}(F_{BC} + F_{CD}) = 0 \qquad (1)$$

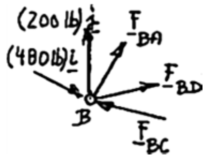
j: $$\frac{10}{26}F_{AC} - 400 \text{ lb} = 0 \qquad F_{AC} = 1040 \text{ lb} \quad T \blacktriangleleft$$

k: $$\frac{7}{25}(F_{BC} - F_{CD}) = 0 \quad F_{CD} = F_{BC}$$

Substitute for F_{AC} and F_{CD} in Eq. (1):

$$-\frac{24}{26}(10.40 \text{ lb}) - \frac{24}{25}(2F_{BC}) = 0 \quad F_{BC} = -500 \text{ lb} \qquad F_{BC} = F_{CD} = 500 \text{ lb} \quad C \blacktriangleleft$$

Free body: *B*:



$$F_{BC} = (500 \text{ lb})\frac{\overline{CB}}{CB} = -(480 \text{ lb})\mathbf{i} + (140 \text{ lb})\mathbf{k}$$

$$F_{BA} = F_{AB}\frac{\overrightarrow{BA}}{BA} = \frac{F_{AB}}{12.21}(10\mathbf{j} - 7\mathbf{k})$$

$$F_{BD} = -F_{BD}\mathbf{k}$$

$$\Sigma\mathbf{F} = 0: \quad \mathbf{F}_{BA} + \mathbf{F}_{BD} + \mathbf{F}_{BC} + (480 \text{ lb})\mathbf{i} + (200 \text{ lb})\mathbf{j} = 0$$

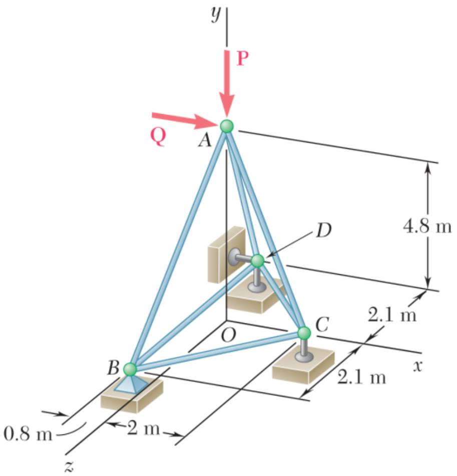
Substituting for $\mathbf{F}_{BA}, \mathbf{F}_{BD}, \mathbf{F}_{BC}$ and equating to zero the coefficients of $\mathbf{j}$ and $\mathbf{k}$:

j: $$\frac{10}{12.21}F_{AB} + 200 \text{ lb} = 0 \quad F_{AB} = -244.2 \text{ lb} \qquad F_{AB} = 244 \text{ lb} \quad C \blacktriangleleft$$

k: $$-\frac{7}{12.21}F_{AB} - F_{BD} + 140 \text{ lb} = 0$$

$$F_{BD} = -\frac{7}{12.21}(-244.2 \text{ lb}) + 140 \text{ lb} = +280 \text{ lb} \quad F_{BD} = 280 \text{ lb} \quad T \blacktriangleleft$$

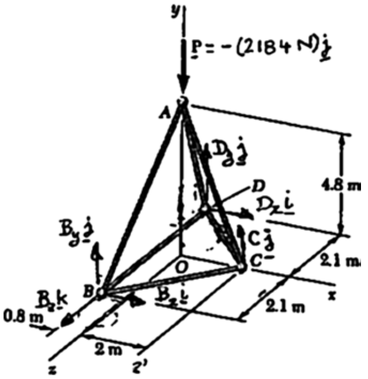
From symmetry: $$F_{AD} = F_{AB} \qquad F_{AD} = 244 \text{ lb} \quad C \blacktriangleleft$$


y
P
Q
A
D
4.8 m
2.1 m
O
C
B
2.1 m
x
0.8 m
2 m
z

PROBLEM 6.36*

The truss shown consists of six members and is supported by a ball and socket at *B*, a short link at *C*, and two short links at *D*. Determine the force in each of the members for $\mathbf{P} = (-2184\ \text{N})\mathbf{j}$ and $\mathbf{Q} = 0$.

SOLUTION


y
P = -(2184 N)j
A
D_y j
D
D_x i
4.8 m
B_y j
C j
C
O
B
2.1 m
B_x i
B_z k
0.8 m
2 m
2.1 m
x
z
z'

Free body: Truss:

From symmetry:

$$D_x = B_x \text{ and } D_y = B_y$$

$$\Sigma F_x = 0: \quad 2B_x = 0$$

$$B_x = D_x = 0$$

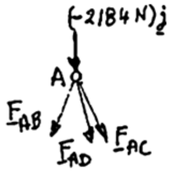
$$\Sigma F_z = 0: \quad B_z = 0$$

$$\Sigma M_{cz} = 0: \quad -2B_y(2.8\ \text{m}) + (2184\ \text{N})(2\ \text{m}) = 0$$

$$B_y = 780\ \text{N}$$

Thus, $\mathbf{B} = (780\ \text{N})\mathbf{j}$ ◁

Free body: *A*:


(-2184 N)j
A
F_AB
F_AD
F_AC

$$F_{AB} = F_{AB}\frac{\overrightarrow{AB}}{AB} = \frac{F_{AB}}{5.30}(-0.8\mathbf{i} - 4.8\mathbf{j} + 2.1\mathbf{k})$$

$$F_{AC} = F_{AC}\frac{\overrightarrow{AC}}{AC} = \frac{F_{AC}}{5.20}(2\mathbf{i} - 4.8\mathbf{j})$$

$$F_{AD} = F_{AD}\frac{\overrightarrow{AD}}{AD} = \frac{F_{AD}}{5.30}(+0.8\mathbf{i} - 4.8\mathbf{j} - 2.1\mathbf{k})$$

$$\Sigma\mathbf{F} = 0: \quad \mathbf{F}_{AB} + \mathbf{F}_{AC} + \mathbf{F}_{AD} - (2184\ \text{N})\mathbf{j} = 0$$

PROBLEM 6.36* (Continued)

Substituting for $\mathbf{F}_{AB}, \mathbf{F}_{AC}, \mathbf{F}_{AD}$, and equating to zero the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\mathbf{i}: \quad -\frac{0.8}{5.30}(F_{AB}+F_{AD})+\frac{2}{5.20}F_{AC}=0 \qquad (1)$$

$$\mathbf{j}: \quad -\frac{4.8}{5.30}(F_{AB}+F_{AD})-\frac{4.8}{5.20}F_{AC}-2184\text{ N}=0 \qquad (2)$$

$$\mathbf{k}: \quad \frac{2.1}{5.30}(F_{AB}-F_{AD})=0 \qquad F_{AD}=F_{AB}$$

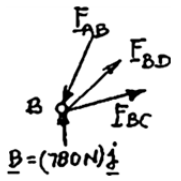
Multiply Eq. (1) by –6 and add Eq. (2):

$$-\left(\frac{16.8}{5.20}\right)F_{AC}-2184\text{ N}=0, \quad F_{AC}=-676\text{ N} \qquad F_{AC}=676\text{ N} \quad C \blacktriangleleft$$

Substitute for F_{AC} and F_{AD} in Eq. (1):

$$-\left(\frac{0.8}{5.30}\right)2F_{AB}+\left(\frac{2}{5.20}\right)(-676\text{ N})=0, \quad F_{AB}=-861.25\text{ N} \quad F_{AB}=F_{AD}=861\text{ N} \quad C \blacktriangleleft$$

Free body: *B*:



$$\mathbf{F}_{AB}=(861.25\text{ N})\frac{\overrightarrow{AB}}{AB}=-(130\text{ N})\mathbf{i}-(780\text{ N})\mathbf{j}+(341.25\text{ N})\mathbf{k}$$

$$\mathbf{F}_{BC}=F_{BC}\left(\frac{2.8\mathbf{i}-2.1\mathbf{k}}{3.5}\right)=F_{BC}(0.8\mathbf{i}-0.6\mathbf{k})$$

$$\mathbf{F}_{BD}=-F_{BD}\mathbf{k}$$

$$\Sigma\mathbf{F}=0: \quad \mathbf{F}_{AB}+\mathbf{F}_{BC}+\mathbf{F}_{BD}+(780\text{ N})\mathbf{j}=0$$

Substituting for $\mathbf{F}_{AB}, \mathbf{F}_{BC}, \mathbf{F}_{BD}$ and equating to zero the coefficients of $\mathbf{i}$ and $\mathbf{k}$,

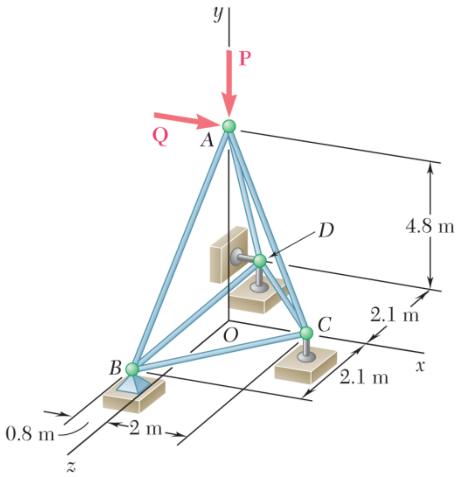
$$\mathbf{i}: \quad -130\text{ N}+0.8F_{BC}=0 \quad F_{BC}=+162.5\text{ N} \qquad F_{BC}=162.5\text{ N} \quad T \blacktriangleleft$$

$$\mathbf{k}: \quad 341.25\text{ N}-0.6F_{BC}-F_{BD}=0$$

$$F_{BD}=341.25-0.6(162.5)=+243.75\text{ N} \quad F_{BD}=244\text{ N} \quad T \blacktriangleleft$$

From symmetry:

$$F_{CD}=F_{BC} \qquad F_{CD}=162.5\text{ N} \quad T \blacktriangleleft$$

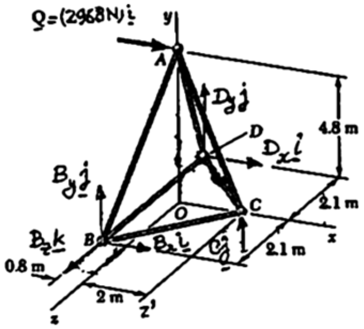

y
P
Q
A
D
4.8 m
2.1 m
C
O
B
2.1 m
x
0.8 m
2 m
z

PROBLEM 6.37*

The truss shown consists of six members and is supported by a ball and socket at *B*, a short link at *C*, and two short links at *D*. Determine the force in each of the members for $\mathbf{P} = 0$ and $\mathbf{Q} = (2968\text{ N})\mathbf{i}$.

SOLUTION

Free body: Truss:


Q=(2968 N)i
y
A
D_y j
D
D_x i
4.8 m
B_y j
O
C
2.1 m
B_z k
B
B_x i
C_y j
2.1 m
x
0.8 m
2 m
z
z'

From symmetry:

$$D_x = B_x \text{ and } D_y = B_y$$

$$\Sigma F_x = 0\text{:}\quad 2B_x + 2968\text{ N} = 0$$

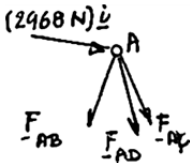
$$B_x = D_x = -1484\text{ N}$$

$$\Sigma M_{cz'} = 0\text{:}\quad -2B_y(2.8\text{ m}) - (2968\text{ N})(4.8\text{ m}) = 0$$

$$B_y = -2544\text{ N}$$

Thus, $\mathbf{B} = -(1484\text{ N})\mathbf{i} - (2544\text{ N})\mathbf{j}$ ◁

Free body: *A*:


(2968 N)i
A
F_AB
F_AD
F_AC

$$F_{AB} = F_{AB}\frac{\overrightarrow{AB}}{AB}$$

$$= \frac{F_{AB}}{5.30}(-0.8\mathbf{i} - 4.8\mathbf{j} + 2.1\mathbf{k})$$

$$F_{AC} = F_{AC}\frac{\overrightarrow{AC}}{AC} = \frac{F_{AC}}{5.20}(2\mathbf{i} - 4.8\mathbf{j})$$

$$F_{AD} = F_{AD}\frac{\overrightarrow{AD}}{AD}$$

$$= \frac{F_{AD}}{5.30}(-0.8\mathbf{i} - 4.8\mathbf{j} - 2.1\mathbf{k})$$

$$\Sigma\mathbf{F} = 0\text{:}\quad \mathbf{F}_{AB} + \mathbf{F}_{AC} + \mathbf{F}_{AD} + (2968\text{ N})\mathbf{i} = 0$$

PROBLEM 6.37* (Continued)

Substituting for $\mathbf{F}_{AB}, \mathbf{F}_{AC}, \mathbf{F}_{AD},$ and equating to zero the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$,

$$\mathbf{i}: \quad -\frac{0.8}{5.30}(F_{AB}+F_{AD})+\frac{2}{5.20}F_{AC}+2968\text{ N}=0 \qquad (1)$$

$$\mathbf{j}: \quad -\frac{4.8}{5.30}(F_{AB}+F_{AD})-\frac{4.8}{5.20}F_{AC}=0 \qquad (2)$$

$$\mathbf{k}: \quad \frac{2.1}{5.30}(F_{AB}-F_{AD})=0 \qquad F_{AD}=F_{AB}$$

Multiply Eq. (1) by –6 and add Eq. (2):

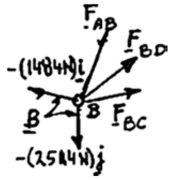
$$-\left(\frac{16.8}{5.20}\right)F_{AC}-6(2968\text{ N})=0, \quad F_{AC}=-5512\text{ N} \qquad F_{AC}=5510\text{ N} \quad C \blacktriangleleft$$

Substitute for F_{AC} and F_{AD} in Eq. (2):

$$-\left(\frac{4.8}{5.30}\right)2F_{AB}-\left(\frac{4.8}{5.20}\right)(-5512\text{ N})=0, \quad F_{AB}=+2809\text{ N}$$

$$F_{AB}=F_{AD}=2810\text{ N} \quad T \blacktriangleleft$$

Free body: *B*:



$$\mathbf{F}_{AB}=(2809\text{ N})\frac{\overrightarrow{BA}}{BA}=(424\text{ N})\mathbf{i}+(2544\text{ N})\mathbf{j}-(1113\text{ N})\mathbf{k}$$

$$\mathbf{F}_{BC}=F_{BC}\left(\frac{2.8\mathbf{i}-2.1\mathbf{k}}{3.5}\right)=F_{BC}(0.8\mathbf{i}-0.6\mathbf{k})$$

$$\mathbf{F}_{BD}=-F_{BD}\mathbf{k}$$

$$\Sigma\mathbf{F}=0: \quad \mathbf{F}_{AB}+\mathbf{F}_{BC}+\mathbf{F}_{BD}-(1484\text{ N})\mathbf{i}-(2544\text{ N})\mathbf{j}=0$$

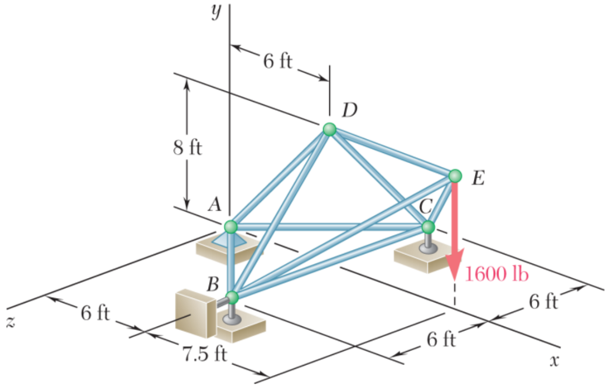
Substituting for $\mathbf{F}_{AB}, \mathbf{F}_{BC}, \mathbf{F}_{BD}$ and equating to zero the coefficients of $\mathbf{i}$ and $\mathbf{k}$,

$$\mathbf{i}: \quad +24\text{ N}+0.8F_{BC}-1484\text{ N}=0, \quad F_{BC}=+1325\text{ N} \qquad F_{BC}=1325\text{ N} \quad T \blacktriangleleft$$

$$\mathbf{k}: \quad -1113\text{ N}-0.6F_{BC}-F_{BD}=0$$

$$F_{BD}=-1113\text{ N}-0.6(1325\text{ N})=-1908\text{ N}, \quad F_{BD}=1908\text{ N} \quad C \blacktriangleleft$$

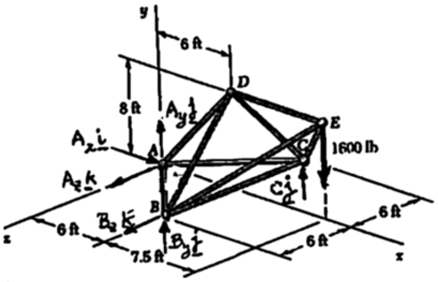
From symmetry: $\qquad F_{CD}=F_{BC} \qquad F_{CD}=1325\text{ N} \quad T \blacktriangleleft$


y
6 ft
D
8 ft
E
A
C
1600 lb
B
6 ft
z
6 ft
7.5 ft
6 ft
x

PROBLEM 6.38*

The truss shown consists of nine members and is supported by a ball and socket at *A*, two short links at *B*, and a short link at *C*. Determine the force in each of the members for the given loading.

SOLUTION


y
6 ft
D
8 ft
A_y j
E
A_x i
1600 lb
A
A_z k
C j
B
z
6 ft
B_z k
B_y j
7.5 ft
6 ft
6 ft
x

Free body: Truss:

From symmetry:

$$A_z = B_z = 0$$

$$\Sigma F_x = 0: \quad A_x = 0$$

$$\Sigma M_{BC} = 0: \quad A_y(6\text{ ft}) + (1600\text{ lb})(7.5\text{ ft}) = 0$$

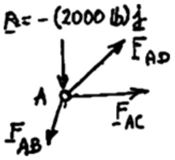
$$A_y = -2000\text{ lb} \qquad \mathbf{A} = -(2000\text{ lb})\mathbf{j} \ \triangleleft$$

From symmetry: $B_y = C$

$$\Sigma F_y = 0: \quad 2B_y - 2000\text{ lb} - 1600\text{ lb} = 0$$

$$B_y = 1800\text{ lb} \qquad \mathbf{B} = (1800\text{ lb})\mathbf{j} \ \triangleleft$$

Free body: *A*:


A = -(2000 lb)j
F_AD
A
F_AC
F_AB

$$\Sigma \mathbf{F} = 0: \quad \mathbf{F}_{AB} + \mathbf{F}_{AC} + \mathbf{F}_{AD} - (2000\text{ lb})\mathbf{j} = 0$$

$$F_{AB}\frac{\mathbf{i}+\mathbf{k}}{\sqrt{2}} + F_{AC}\frac{\mathbf{i}-\mathbf{k}}{\sqrt{2}} + F_{AD}(0.6\mathbf{i} + 0.8\mathbf{j}) - (2000\text{ lb})\mathbf{j} = 0$$

Factoring **i**, **j**, **k** and equating their coefficient to zero,

$$\frac{1}{\sqrt{2}}F_{AB} + \frac{1}{\sqrt{2}}F_{AC} + 0.6F_{AD} = 0 \qquad (1)$$

$$0.8F_{AD} - 2000\text{ lb} = 0 \qquad F_{AD} = 2500\text{ lb} \quad T \ \blacktriangleleft$$

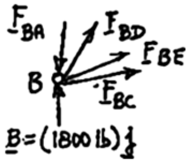
$$\frac{1}{\sqrt{2}}F_{AB} - \frac{1}{\sqrt{2}}F_{AC} = 0 \qquad F_{AC} = F_{AB}$$

PROBLEM 6.38* (Continued)

Substitute for F_{AD} and F_{AC} into Eq. (1):

$$\frac{2}{\sqrt{2}}F_{AB} + 0.6(2500\text{ lb}) = 0, \quad \mathbf{F}_{AB} = -1060.7\text{ lb}, \qquad F_{AB} = F_{AC} = 1061\text{ lb} \quad C \blacktriangleleft$$

Free body: *B*:



$$\mathbf{F}_{BA} = F_{AB}\frac{\overrightarrow{BA}}{BA} = +(1060.7\text{ lb})\frac{\mathbf{i}+\mathbf{k}}{\sqrt{2}} = (750\text{ lb})(\mathbf{i}+\mathbf{k})$$

$$\mathbf{F}_{BC} = -F_{BC}\mathbf{k}$$

$$\mathbf{F}_{BD} = F_{BD}(0.8\mathbf{j} - 0.6\mathbf{k})$$

$$\mathbf{F}_{BE} = F_{BE}\frac{\overrightarrow{BE}}{BE} = \frac{F_{BE}}{12.5}(7.5\mathbf{i} + 8\mathbf{j} - 6\mathbf{k})$$

$$\Sigma\mathbf{F} = 0: \quad \mathbf{F}_{BA} + \mathbf{F}_{BC} + \mathbf{F}_{BD} + \mathbf{F}_{BE} + (1800\text{ lb})\mathbf{j} = 0$$

Substituting for $\mathbf{F}_{BA}, \mathbf{F}_{BC}, \mathbf{F}_{BD}$, and $\mathbf{F}_{BE}$ and equating to zero the coefficients of **i**, **j**, **k**,

i: $750\text{ lb} + \left(\frac{7.5}{12.5}\right)F_{BE} = 0, \quad F_{BE} = -1250\text{ lb}$ $\qquad F_{BE} = 1250\text{ lb} \quad C \blacktriangleleft$

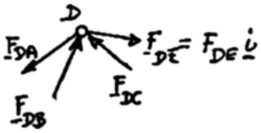
j: $0.8\,F_{BD} + \left(\frac{8}{12.5}\right)(-1250\text{ lb}) + 1800\text{ lb} = 0$ $\qquad F_{BD} = 1250\text{ lb} \quad C \blacktriangleleft$

k: $750\text{ lb} - F_{BC} - 0.6(-1250\text{ lb}) - \frac{6}{12.5}(-1250\text{ lb}) = 0$

$$F_{BC} = 2100\text{ lb} \quad T \blacktriangleleft$$

From symmetry: $\qquad F_{BD} = F_{CD} = 1250\text{ lb} \quad C \blacktriangleleft$

Free body: *D*:

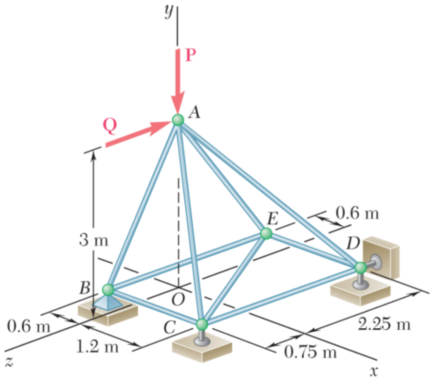


$$\Sigma\mathbf{F} = 0: \ \mathbf{F}_{DA} + \mathbf{F}_{DB} + \mathbf{F}_{DC} + F_{DE}\mathbf{i} = 0$$

We now substitute for $\mathbf{F}_{DA}, \mathbf{F}_{DB}, \mathbf{F}_{DC}$ and equate to zero the coefficient of **i**. Only $\mathbf{F}_{DA}$ contains **i** and its coefficient is

$$-0.6\,F_{AD} = -0.6(2500\text{ lb}) = -1500\text{ lb}$$

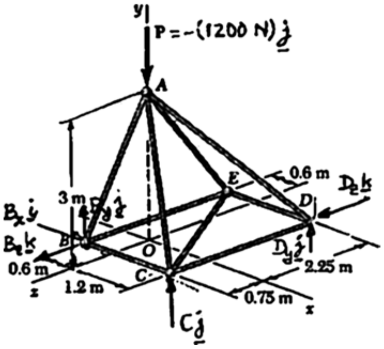
i: $-1500\text{ lb} + F_{DE} = 0$ $\qquad F_{DE} = 1500\text{ lb} \quad T \blacktriangleleft$



PROBLEM 6.39*

The truss shown consists of nine members and is supported by a ball and socket at *B*, a short link at *C*, and two short links at *D*. (*a*) Check that this truss is a simple truss, that it is completely constrained, and that the reactions at its supports are statically determinate. (*b*) Determine the force in each member for $\mathbf{P} = (-1200\text{ N})\mathbf{j}$ and $\mathbf{Q} = 0$.

SOLUTION



Free body: Truss:

$$\Sigma\mathbf{M}_B = 0:\quad 1.8\mathbf{i}\times C\mathbf{j} + (1.8\mathbf{i} - 3\mathbf{k})\times(D_y\mathbf{j} + D\mathbf{k})$$

$$+(0.6\mathbf{i} - 0.75\mathbf{k})\times(-1200\mathbf{j}) = 0$$

$$-1.8\,C\mathbf{k} + 1.8\,D_y\mathbf{k} - 1.8\,D_z\mathbf{j}$$

$$+3D_y\mathbf{i} - 720\mathbf{k} - 900\mathbf{i} = 0$$

Equate to zero the coefficients of **i**, **j**, **k**:

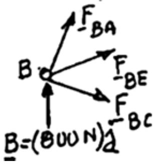
$$\mathbf{i}:\quad 3D_y - 900 = 0,\quad D_y = 300\text{ N}$$

$$\mathbf{j}:\quad D_z = 0, \qquad \mathbf{D} = (300\text{ N})\mathbf{j}\ \triangleleft$$

$$\mathbf{k}:\quad 1.8C + 1.8(300) - 720 = 0 \qquad \mathbf{C} = (100\text{ N})\mathbf{j}\ \triangleleft$$

$$\Sigma\mathbf{F} = 0:\quad \mathbf{B} + 300\mathbf{j} + 100\mathbf{j} - 1200\mathbf{j} = 0 \qquad \mathbf{B} = (800\text{ N})\mathbf{j}\ \triangleleft$$

Free body: *B*:



$$\Sigma\mathbf{F} = 0:\quad \mathbf{F}_{BA} + \mathbf{F}_{BC} + \mathbf{F}_{BE} + (800\text{ N})\mathbf{j} = 0,\ \text{with}$$

$$\mathbf{F}_{BA} = \mathbf{F}_{AB}\frac{\overrightarrow{BA}}{BA} = \frac{F_{AB}}{3.15}(0.6\mathbf{i} + 3\mathbf{j} - 0.75\mathbf{k})$$

$$\mathbf{F}_{BC} = F_{BC}\mathbf{i} \qquad \mathbf{F}_{BE} = -F_{BE}\mathbf{k}$$

Substitute and equate to zero the coefficient of **j**, **i**, **k**:

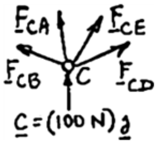
$$\mathbf{j}:\quad \left(\frac{3}{3.315}\right)F_{AB} + 800\text{ N} = 0,\quad F_{AB} = -840\text{ N}, \qquad F_{AB} = 840\text{ N}\quad C\ \blacktriangleleft$$

$$\mathbf{i}:\quad \left(\frac{0.6}{3.15}\right)(-840\text{ N}) + F_{BC} = 0 \qquad F_{BC} = 160.0\text{ N}\quad T\ \blacktriangleleft$$

$$\mathbf{k}:\quad \left(-\frac{0.75}{3.15}\right)(-840\text{ N}) - F_{BE} = 0 \qquad F_{BE} = 200\text{ N}\quad T\ \blacktriangleleft$$

PROBLEM 6.39* (Continued)

Free body: *C*:



$$\Sigma\mathbf{F} = 0: \quad \mathbf{F}_{CA} + \mathbf{F}_{CB} + \mathbf{F}_{CD} + \mathbf{F}_{CE} + (100\text{ N})\mathbf{j} = 0, \text{ with}$$

$$\mathbf{F}_{CA} = F_{AC}\frac{\overrightarrow{CA}}{CA} = \frac{F_{AC}}{3.317}(-1.2\mathbf{i} + 3\mathbf{j} - 0.75\mathbf{k})$$

$$F_{CB} = -(160\,\text{N})\,\mathbf{i}$$

$$\mathbf{F}_{CD} = -F_{CD}\mathbf{k} \quad \mathbf{F}_{CE} = \mathbf{F}_{CE}\frac{\overrightarrow{CE}}{CE} = \frac{F_{CE}}{3,499}(-1.8\mathbf{i} - 3\mathbf{k})$$

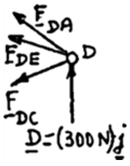
Substitute and equate to zero the coefficient of $\mathbf{j}, \mathbf{i}, \mathbf{k}$:

$\mathbf{j}$: $$\left(\frac{3}{3.317}\right)F_{AC} + 100\text{ N} = 0, \quad F_{AC} = -110.57\text{ N} \qquad F_{AC} = 110.6\text{ N} \quad C \blacktriangleleft$$

$\mathbf{i}$: $$-\frac{1.2}{3.317}(-110.57) - 160 - \frac{1.8}{3.499}F_{CE} = 0, \quad F_{CE} = -233.3 \qquad F_{CE} = 233\text{ N} \quad C \blacktriangleleft$$

$\mathbf{k}$: $$-\frac{0.75}{3.317}(-110.57) - F_{CD} - \frac{3}{3.499}(-233.3) = 0 \qquad F_{CD} = 225\text{ N} \quad T \blacktriangleleft$$

Free body: *D*:



$$\Sigma\mathbf{F} = 0: \quad \mathbf{F}_{DA} + \mathbf{F}_{DC} + \mathbf{F}_{DE} + (300\text{ N})\mathbf{j} = 0, \text{ with}$$

$$\mathbf{F}_{DA} = F_{AD}\frac{\overrightarrow{DA}}{DA} = \frac{F_{AD}}{3.937}(-1.2\mathbf{i} + 3\mathbf{j} + 2.25\mathbf{k})$$

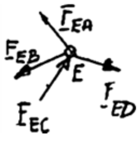
$$\mathbf{F}_{DC} = F_{CD}\mathbf{k} = (225\text{ N})\mathbf{k} \qquad F_{DE} = -F_{DE}\mathbf{i}$$

Substitute and equate to zero the coefficient of $\mathbf{j}, \mathbf{i}, \mathbf{k}$:

$\mathbf{j}$: $$\left(\frac{3}{3.937}\right)F_{AD} + 300\text{ N} = 0, \qquad F_{AD} = -393.7\text{ N}, \qquad F_{AD} = 394\text{ N} \quad C \blacktriangleleft$$

$\mathbf{i}$: $$\left(-\frac{1.2}{3.937}\right)(-393.7\text{ N}) - F_{DE} = 0 \qquad F_{DE} = 120.0\text{ N} \quad T \blacktriangleleft$$

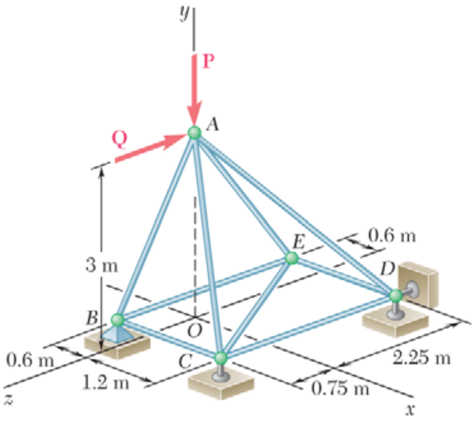
$\mathbf{k}$: $$\left(\frac{2.25}{3.937}\right)(-393.7\text{ N}) + 225\text{ N} = 0 \quad \text{(Checks)}$$



Free body: *E*:

Member *AE* is the only member at *E* which does not lie in the *xz* plane.
Therefore, it is a zero-force member.

$$F_{AE} = 0 \blacktriangleleft$$

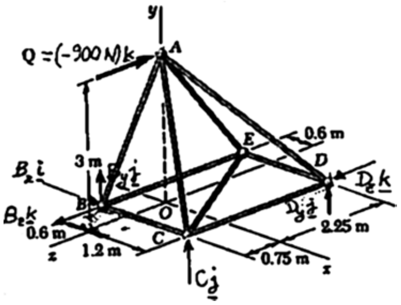

y
P
Q
A
3 m
E
0.6 m
D
B
O
0.6 m
C
2.25 m
1.2 m
0.75 m
z
x

PROBLEM 6.40*

Solve Problem 6.39 for **P** = 0 and **Q** = (−900 N)**k**.

PROBLEM 6.39* The truss shown consists of nine members and is supported by a ball and socket at *B*, a short link at *C*, and two short links at *D*. (*a*) Check that this truss is a simple truss, that it is completely constrained, and that the reactions at its supports are statically determinate. (*b*) Determine the force in each member for **P** = (−1200 N)**j** and **Q** = 0.

SOLUTION


y
Q =(-900 N)k
A
0.6 m
E
D
3 m
$B_x i$
$B_y j$
$B_z k$
$D_z k$
$D_y j$
O
2.25 m
0.6 m
1.2 m
C
0.75 m
z
x
$C j$

Free body: Truss:

$$\Sigma \mathbf{M}_B = 0: \quad 1.8\mathbf{i} \times C\mathbf{j} + (1.8\mathbf{i} - 3\mathbf{k}) \times (D_y\mathbf{j} + D_z\mathbf{k})$$

$$+ (0.6\mathbf{i} + 3\mathbf{j} - 0.75\mathbf{k}) \times (-900\text{N})\mathbf{k} = 0$$

$$1.8C\mathbf{k} + 1.8D_y\mathbf{k} - 1.8D_z\mathbf{j}$$

$$+ 3D_y\mathbf{i} + 540\mathbf{j} - 2700\mathbf{i} = 0$$

Equate to zero the coefficient of **i**, **j**, **k**:

$$3D_y - 2700 = 0 \qquad D_y = 900\text{ N}$$

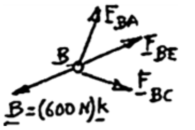
$$-1.8D_z + 540 = 0 \qquad D_z = 300\text{ N}$$

$$1.8C + 1.8D_y = 0 \qquad C = -D_y = -900\text{ N}$$

Thus, $\mathbf{C} = -(900\text{ N})\mathbf{j} \quad \mathbf{D} = (900\text{ N})\mathbf{j} + (300\text{ N})\mathbf{k}$ ◁

$\Sigma \mathbf{F} = 0: \quad \mathbf{B} - 900\mathbf{j} + 900\mathbf{j} + 300\mathbf{k} - 900\mathbf{k} = 0$ $\qquad \mathbf{B} = (600\text{ N})\mathbf{k}$ ◁

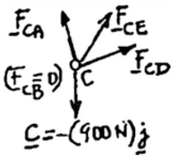
Free body: *B*:


F_{BA}
B
F_{BE}
F_{BC}
B=(600 N)k

Since **B** is aligned with member *BE*,

$F_{AB} = F_{BC} = 0,$ $\qquad F_{BE} = 600\text{ N}\quad T$ ◀

Free body: *C*:


F_{CA}
F_{CE}
F_{CD}
(F_{CB}=0)
C
C=-(900 N)j

$\Sigma \mathbf{F} = 0: \ \mathbf{F}_{CA} + \mathbf{F}_{CD} + \mathbf{F}_{CE} - (900\text{ N})\mathbf{j} = 0,$ with

PROBLEM 6.40* (Continued)

$$\mathbf{F}_{CA} = F_{AC}\frac{\overrightarrow{CA}}{CA} = \frac{F_{AC}}{3.317}(-1.2\mathbf{i} + 3\mathbf{j} - 0.75\mathbf{k})$$

$$\mathbf{F}_{CD} = -F_{CD}\mathbf{k} \quad \mathbf{F}_{CE} = F_{CE}\frac{\overrightarrow{CE}}{CE} = \frac{F_{CE}}{3.499}(-1.8\mathbf{i} - 3\mathbf{k})$$

Substitute and equate to zero the coefficient of **j**, **i**, **k**:

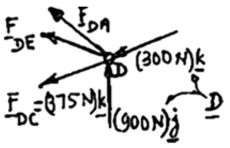
j: $\left(\frac{3}{3.317}\right)F_{AC} - 900\text{ N} = 0, \qquad F_{AC} = 995.1\text{ N} \qquad F_{AC} = 995\text{ N} \quad T$ ◀

i: $-\frac{1.2}{3.317}(995.1) - \frac{1.8}{3.499}F_{CE} = 0, \quad F_{CE} = -699.8\text{ N} \qquad F_{CE} = 700\text{ N} \quad C$ ◀

k: $-\frac{0.75}{3.317}(995.1) - F_{CD} - \frac{3}{3.499}(-699.8) = 0 \qquad F_{CD} = 375\text{ N} \quad T$ ◀

Free body: *D*:

$$\Sigma\mathbf{F} = 0: \quad \mathbf{F}_{DA} + \mathbf{F}_{DE} + (375\text{ N})\mathbf{k} + (900\text{ N})\mathbf{j} + (300\text{ N})\mathbf{k} = 0$$



with $\mathbf{F}_{DA} = \mathbf{F}_{AD}\frac{\overrightarrow{DA}}{DA} = \frac{F_{AD}}{3.937}(-1.2\mathbf{i} + 3\mathbf{j} + 2.25\mathbf{k})$

and $F_{DE} = -F_{DE}\,\mathbf{i}$

Substitute and equate to zero the coefficient **j**, **i**, **k**:

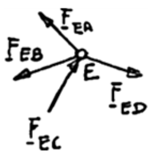
j: $\left(\frac{3}{3.937}\right)F_{AD} + 900\text{ N} = 0, \quad F_{AD} = -1181.1\text{ N} \qquad F_{AD} = 1181\text{ N} \quad C$ ◀

i: $-\left(\frac{1.2}{3.937}\right)(-1181.1\text{ N}) - F_{DE} = 0 \qquad F_{DE} = 360\text{ N} \quad T$ ◀

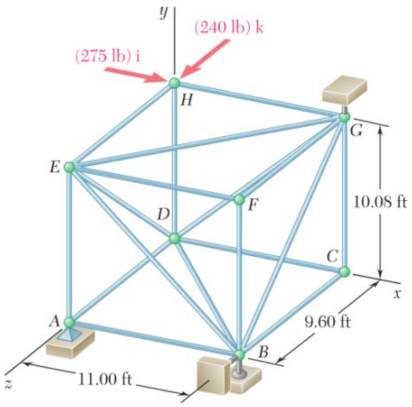
k: $\left(\frac{2.25}{3.937}\right)(-1181.1\text{ N} + 375\text{ N} + 300\text{ N} = 0)$ (Checks)

Free body: *E*:

Member *AE* is the only member at *E* which does not lie in the *xz* plane. Therefore, it is a zero-force member.



$F_{AE} = 0$ ◀



PROBLEM 6.41*

The truss shown consists of 18 members and is supported by a ball and socket at *A*, two short links at *B*, and one short link at *G*. (*a*) Check that this truss is a simple truss, that it is completely constrained, and that the reactions at its supports are statically determinate. (*b*) For the given loading, determine the force in each of the six members joined at *E*.

SOLUTION

(*a*) Check simple truss.

(1) Start with tetrahedron *BEFG*.

(2) Add members *BD*, *ED*, *GD* joining at *D*.

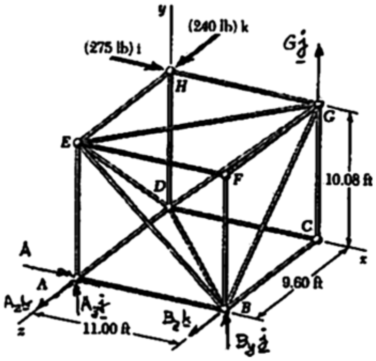
(3) Add members *BA*, *DA*, *EA* joining at *A*.

(4) Add members *DH*, *EH*, *GH* joining at *H*.

(5) Add members *BC*, *DC*, *GC* joining at *C*.

Truss has been completed: It is a simple truss.

Free body: Truss:



Check constraints and reactions.

Six unknown reactions—ok; however, supports at *A* and *B* constrain truss to rotate about *AB* and support at *G* prevents such a rotation. Thus,

Truss is completely constrained and reactions are statically determinate.

Determination of reactions:

$$\Sigma \mathbf{M}_A = 0: \quad 11\mathbf{i} \times (B_y\mathbf{j} + B_z\mathbf{k}) + (11\mathbf{i} - 9.6\mathbf{k}) \times G\mathbf{j} + (10.08\mathbf{j} - 9.6\mathbf{k}) \times (275\mathbf{i} + 240\mathbf{k}) = 0$$

$$11B_y\mathbf{k} - 11B_y\mathbf{j} + 11G\mathbf{k} + 9.6G\mathbf{i} - (10.08)(275)\mathbf{k} + (10.08)(240)\mathbf{i} - (9.6)(275)\mathbf{j} = 0$$

Equate to zero the coefficient of **i**, **j**, **k**:

$\mathbf{i}: \quad 9.6G + (10.08)(240) = 0 \quad G = -252 \text{ lb}$ $\qquad \mathbf{G} = (-252 \text{ lb})\mathbf{j}$ ◁

$\mathbf{j}: \quad -11B_z - (9.6)(275) = 0 \quad B_z = -240 \text{ lb}$

$\mathbf{k}: \quad 11B_y + 11(-252) - (10.08)(275) = 0, \quad B_y = 504 \text{ lb}$ $\qquad \mathbf{B} = (504 \text{ lb})\mathbf{j} - (240 \text{ lb})\mathbf{k}$ ◁

PROBLEM 6.41* (Continued)

$$\Sigma \mathbf{F} = 0: \quad \mathbf{A} + (504 \text{ lb})\mathbf{j} - (240 \text{ lb})\mathbf{k} - (252 \text{ lb})\mathbf{j}$$
$$+ (275 \text{ lb})\mathbf{i} + (240 \text{ lb})\mathbf{k} = 0$$

$\mathbf{A} = -(275 \text{ lb})\mathbf{i} - (252 \text{ lb})\mathbf{j}$ ◁

Zero-force members.

The determination of these members will facilitate our solution.

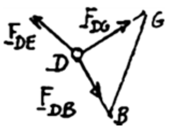
FB: C: Writing $\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma F_z = 0$ yields $F_{BC} = F_{CD} = F_{CG} = 0$ ◁

FB: F: Writing $\Sigma F_x = 0, \quad \Sigma F_y = 0, \quad \Sigma F_z = 0$ yields $F_{BF} = F_{EF} = F_{FG} = 0$ ◁

FB: A: Since $A_z = 0$, writing $\Sigma F_z = 0$ yields $F_{AD} = 0$ ◁

FB: H: Writing $\Sigma F_y = 0$ yields $F_{DH} = 0$ ◁

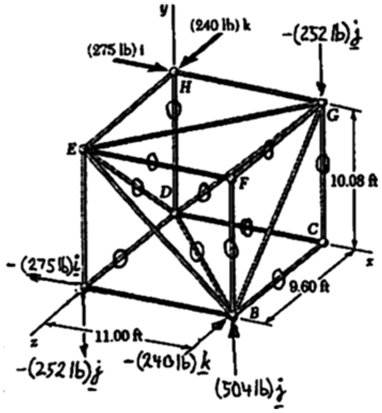
FB: D: Since $F_{AD} = F_{CD} = F_{DH} = 0$, we need consider only members DB, DE, and DG.



Since F_{DE} is the only force not contained in plane BDG, it must be zero. Simple reasonings show that the other two forces are also zero.

$F_{BD} = F_{DE} = F_{DG} = 0$ ◁

The results obtained for the reactions at the supports and for the zero-force members are shown on the figure below. Zero-force members are indicated by a zero ("0").



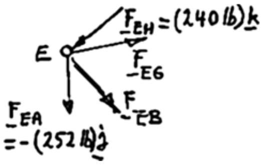
(*b*) Force in each of the members joined at *E*.

We already found that $F_{DE} = F_{EF} = 0$ ◀

Free body: *A*: $\Sigma F_y = 0$ yields $F_{AE} = 252 \text{ lb} \quad T$ ◀

Free body: *H*: $\Sigma F_z = 0$ yields $F_{EH} = 240 \text{ lb} \quad C$ ◀

PROBLEM 6.41* (Continued)


E
$\underline{F}_{EH} = (240\text{ lb})\underline{k}$
$\underline{F}_{EG}$
$\underline{F}_{EB}$
$\underline{F}_{EA}$
$= -(252\text{ lb})\underline{j}$

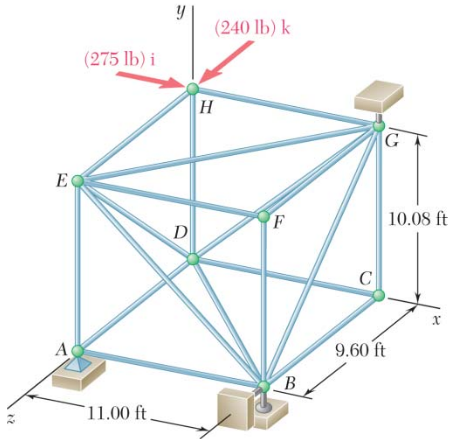
Free body: *E*: $\Sigma\mathbf{F} = 0: \quad \mathbf{F}_{EB} + \mathbf{F}_{EG} + (240\text{ lb})\mathbf{k} - (252\text{ lb})\mathbf{j} = 0$

$$\frac{F_{BE}}{14.92}(11\mathbf{i} - 10.08\mathbf{j}) + \frac{F_{EG}}{14.6}(11\mathbf{i} - 9.6\mathbf{k}) + 240\mathbf{k} - 252\mathbf{j} = 0$$

Equate to zero the coefficient of **y** and **k**:

$$\mathbf{j}: \quad -\left(\frac{10.08}{14.92}\right)F_{BE} - 252 = 0 \qquad F_{BE} = 373\text{ lb} \quad C \blacktriangleleft$$

$$\mathbf{k}: \quad -\left(\frac{9.6}{14.6}\right)F_{EG} + 240 = 0 \qquad F_{EG} = 365\text{ lb} \quad T \blacktriangleleft$$



PROBLEM 6.42*

The truss shown consists of 18 members and is supported by a ball and socket at *A*, two short links at *B*, and one short link at *G*. (*a*) Check that this truss is a simple truss, that it is completely constrained, and that the reactions at its supports are statically determinate. (*b*) For the given loading, determine the force in each of the six members joined at *G*.

SOLUTION

See solution to Problem 6.41 for part (a) and for reactions and zero-force members.

(*b*) Force in each of the members joined at *G*.

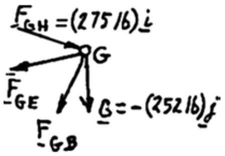
We already know that

$$F_{CG} = F_{DG} = F_{FG} = 0 \blacktriangleleft$$

Free body: *H*: $\Sigma F_x = 0$ yields $F_{GH} = 275 \text{ lb} \quad C \blacktriangleleft$

Free body: *G*: $\Sigma \mathbf{F} = 0{:}\quad \mathbf{F}_{GB} + \mathbf{F}_{GE} + (275 \text{ lb})\mathbf{i} - (252 \text{ lb})\mathbf{j} = 0$

$$\frac{F_{BG}}{13.92}(-10.08\mathbf{j} + 9.6\mathbf{k}) + \frac{F_{EG}}{14.6}(-11\mathbf{i} + 9.6\mathbf{k}) + 275\mathbf{i} - 252\mathbf{j} = 0$$

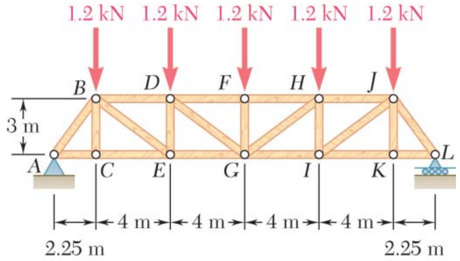


Equate to zero the coefficient of **i**, **j**, **k**:

$$\mathbf{i}{:}\quad -\left(\frac{11}{14.6}\right)F_{EG} + 275 = 0 \qquad F_{EG} = 365 \text{ lb} \quad T \blacktriangleleft$$

$$\mathbf{j}{:}\quad -\left(\frac{10.08}{13.92}\right)F_{BG} - 252 = 0 \qquad F_{BG} = 348 \text{ lb} \quad C \blacktriangleleft$$

$$\mathbf{k}{:}\quad \left(\frac{9.6}{13.92}\right)(-348) + \left(\frac{9.6}{14.6}\right)(365) = 0 \qquad \text{(Checks)}$$


1.2 kN
1.2 kN
1.2 kN
1.2 kN
1.2 kN
B
D
F
H
J
3 m
A
C
E
G
I
K
L
4 m
4 m
4 m
4 m
2.25 m
2.25 m

PROBLEM 6.43

A Mansard roof truss is loaded as shown. Determine the force in members *DF*, *DG*, and *EG*.

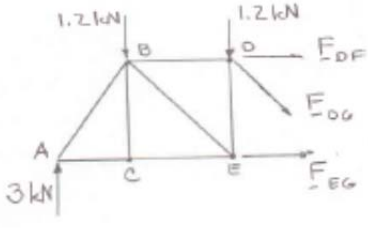
SOLUTION

Reactions:

Because of the symmetry of the truss and loadings, $A_x = 0,\ A_y = L = \frac{1}{2}(1.2\text{ kN})(5) = 3\text{ kN}\ \uparrow$

We pass a section through *DF, DG,* and *EG* and use the free body shown:

Member *DF*:


1.2 kN
1.2 kN
B
D
F_DF
F_DG
A
C
E
F_EG
3 kN

$$+\circlearrowleft \Sigma M_G = 0:\quad (1.2\text{ kN})(8\text{ m}) + (1.2\text{ kN})(4\text{ m}) - (3\text{ kN})(10.25\text{ m}) - F_{DF}(3\text{ m}) = 0$$

$F_{DF} = -5.45\text{ kN}$ $\qquad F_{DF} = 5.45\text{ kN}\quad C$ ◀

Member *DG*:

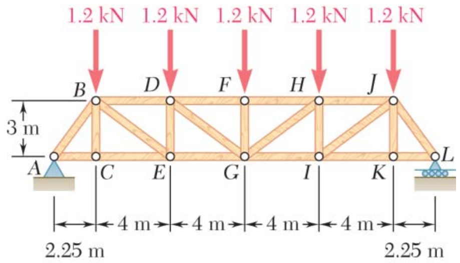
$$+\uparrow \Sigma F_y = 0:\quad 3\text{ kN} - 1.2\text{ kN} - 1.2\text{ kN} - \left(\frac{3}{5}\right)F_{DG} = 0$$

$F_{DG} = 1.000\text{ kN}$ $\qquad F_{DG} = 1.000\text{ kN}\quad T$ ◀

Member *EG*:

$$+\circlearrowleft \Sigma M_D = 0:\quad (1.2\text{ kN})(4\text{ m}) + F_{EG}(3\text{ m}) - (3\text{ kN})(6.25\text{ m}) = 0$$

$F_{EG} = 4.65\text{ kN}\quad T$ ◀


1.2 kN
1.2 kN
1.2 kN
1.2 kN
1.2 kN
B
D
F
H
J
3 m
A
C
E
G
I
K
L
4 m
4 m
4 m
4 m
2.25 m
2.25 m

PROBLEM 6.44

A Mansard roof truss is loaded as shown. Determine the force in members *GI*, *HI*, and *HJ*.

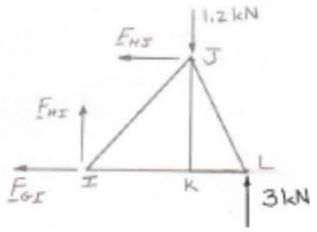
SOLUTION

Reactions:

Because of the symmetry of the truss and loadings, $A_x = 0,\ A_y = L = \frac{1}{2}(1.2\text{ kN})(5) = 3\text{ kN}\ \uparrow$

We pass a section through *GI, HI,* and *HJ* and use the free body shown:

Member *DF*:


1.2 kN
F_HJ
J
F_HI
I
K
L
F_GI
3 kN

$$+\circlearrowleft \Sigma M_H = 0{:}\quad (3\text{ kN})(6.25\text{ m}) - (1.2\text{ kN})(4\text{ m}) - F_{GI}(3\text{ m}) = 0$$

$$F_{GI} = 4.65\text{ kN} \qquad\qquad F_{GI} = 4.65\text{ kN}\ \ T\ ◀$$

Member *HI*:

$$+\uparrow \Sigma F_y = 0{:}\quad 3\text{ kN} - 1.2\text{ kN} + F_{HI} = 0$$

$$F_{HI} = -1.800\text{ kN} \qquad\qquad F_{HI} = 1.800\text{ kN}\ \ C\ ◀$$

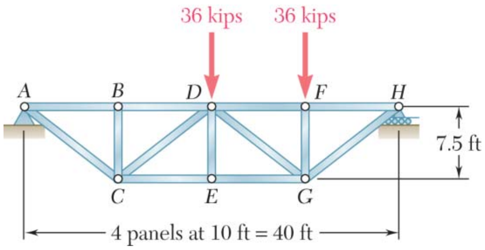
Member *HJ*:

$$+\circlearrowleft \Sigma M_I = 0{:}\quad (3\text{ kN})(6.25\text{ m}) + F_{HJ}(3\text{ m}) - (1.2\text{ kN})(4\text{ m}) = 0$$

$$F_{HJ} = -4.65\text{ kN}$$

$$F_{HJ} = 4.65\text{ kN}\ \ C\ ◀$$

Check $\Sigma \mathbf{F}_x = 0{:}\quad 4.65 - 4.65 = 0$ o.k.


36 kips
36 kips
A
B
D
F
H
7.5 ft
C
E
G
4 panels at 10 ft = 40 ft

PROBLEM 6.45

Determine the force in members *BD* and *CD* of the truss shown.

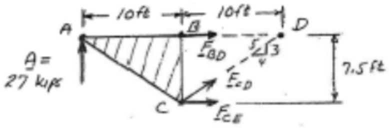
SOLUTION

Reactions from Free body of entire truss:

$$\mathbf{A} = \mathbf{A}_y = 27 \text{ kips}\uparrow \triangleleft$$

$$\mathbf{H} = 45 \text{ kips}\uparrow \triangleleft$$

We pass a section through members *BD*, *CD*, and *CE* and use the free body shown.

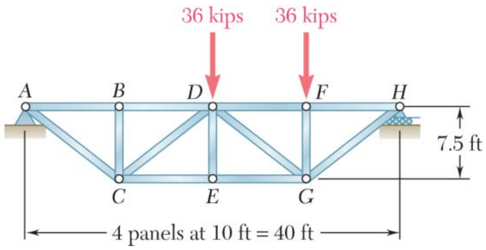

10 ft
10 ft
A
B
D
A =
27 kips
F_BD
5
3
4
F_CD
7.5 ft
C
F_CE

$$+\circlearrowleft \Sigma M_C = 0: \quad -F_{BD}(7.5 \text{ ft}) - (27 \text{ kips})(10 \text{ ft}) = 0$$

$$F_{BD} = -36.0 \text{ kips} \qquad F_{BD} = 36.0 \text{ kips} \quad C \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad 27 \text{ kips} + \left(\frac{3}{5}\right) F_{CD} = 0$$

$$F_{CD} = -45.0 \text{ kips} \qquad F_{CD} = 45.0 \text{ kips} \quad C \blacktriangleleft$$


36 kips
36 kips
A
B
D
F
H
7.5 ft
C
E
G
4 panels at 10 ft = 40 ft

PROBLEM 6.46

Determine the force in members *DF* and *DG* of the truss shown.

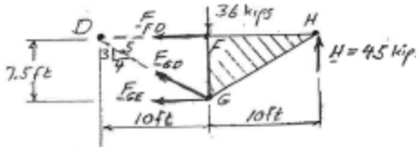
SOLUTION

Reactions from Free body of entire truss:

$$\mathbf{A} = \mathbf{A}_y = 27 \text{ kips}\uparrow \; \triangleleft$$

$$\mathbf{H} = 45 \text{ kips}\uparrow \; \triangleleft$$

We pass a section through members *DF*, *DG*, and *EG* and use the free body shown.

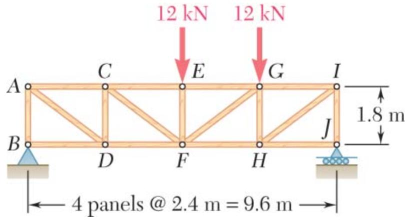

D
F
F_{FD}
36 kips
H
7.5 ft
3
5
4
F_{GD}
F_{GE}
G
H = 45 kip
10 ft
10 ft

$$+\circlearrowleft \Sigma M_G = 0: \quad F_{FD}(7.5 \text{ ft}) + (45 \text{ kips})(10 \text{ ft}) = 0$$

$$F_{FD} = -60.0 \text{ kips} \qquad F_{FD} = 60.0 \text{ kips} \quad C \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad 45 \text{ kips} + \left(\frac{3}{5}\right)F_{GD} - 36 \text{ kips} = 0$$

$$F_{GD} = -15.00 \text{ kips} \qquad F_{GD} = 15.00 \text{ kips} \quad C \blacktriangleleft$$

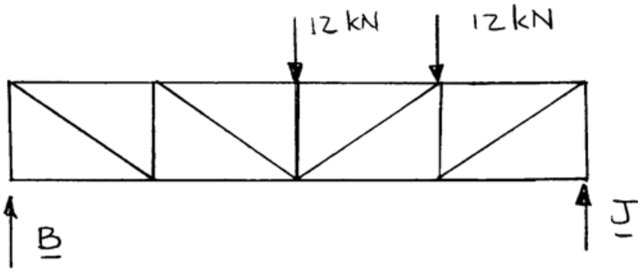

12 kN
12 kN
A
C
E
G
I
B
D
F
H
J
1.8 m
4 panels @ 2.4 m = 9.6 m

PROBLEM 6.47

Determine the force in members *CD* and *DF* of the truss shown.

SOLUTION

Reactions:


12 kN
12 kN
B
J

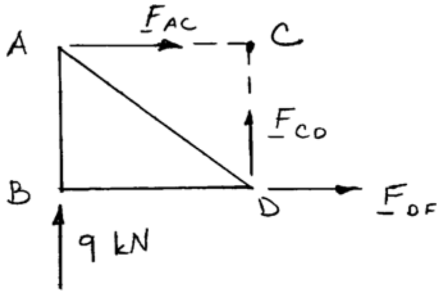
$$+\circlearrowleft \Sigma M_J = 0: \quad (12\text{ kN})(4.8\text{ m}) + (12\text{ kN})(2.4\text{ m}) - B(9.6\text{ m}) = 0$$

$$B = 9.00\text{ kN}$$

$$+\uparrow \Sigma F_y = 0: \quad 9.00\text{ kN} - 12.00\text{ kN} - 12.00\text{ kN} + J = 0$$

$$J = 15.00\text{ kN}$$

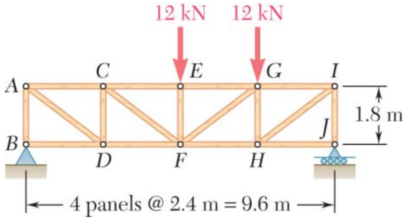
Member *CD*:


A
F_AC
C
F_CD
B
D
F_DF
9 kN

$+\uparrow \Sigma F_y = 0: \quad 9.00\text{ kN} + F_{CD} = 0 \qquad F_{CD} = 9.00\text{ kN} \quad C$ ◀

Member *DF*:

$+\circlearrowleft \Sigma M_C = 0: \quad F_{DF}(1.8\text{ m}) - (9.00\text{ kN})(2.4\text{ m}) = 0 \qquad F_{DF} = 12.00\text{ kN} \quad T$ ◀

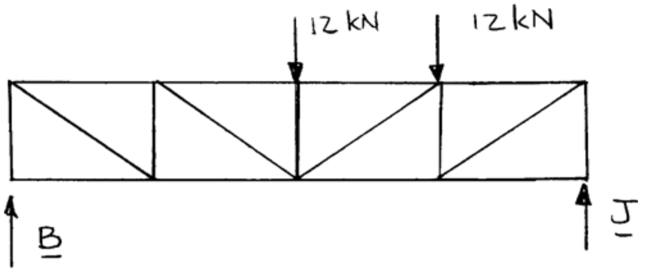

12 kN
12 kN
A
B
C
D
E
F
G
H
I
J
1.8 m
4 panels @ 2.4 m = 9.6 m

PROBLEM 6.48

Determine the force in members *FG* and *FH* of the truss shown.

SOLUTION

Reactions:


12 kN
12 kN
B
J

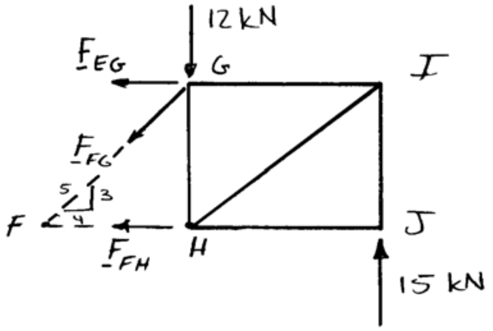
$$+\circlearrowleft \Sigma M_J = 0: \quad (12 \text{ kN})(4.8 \text{ m}) + (12 \text{ kN})(2.4 \text{ m}) - B(9.6 \text{ m}) = 0$$

$$B = 9.00 \text{ kN}$$

$$+\uparrow \Sigma F_y = 0: \quad 9.00 \text{ kN} - 12.00 \text{ kN} - 12.00 \text{ kN} + J = 0$$

$$J = 15.00 \text{ kN}$$

Member *FG*:


12 kN
F_EG
G
I
F_FG
5
3
4
F
F_FH
H
J
15 kN

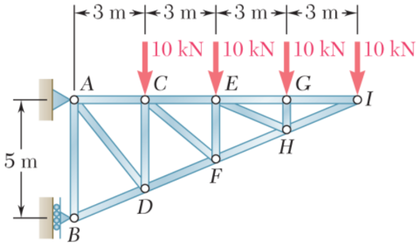
$$+\uparrow \Sigma F_y = 0: \quad -\frac{3}{5}F_{FG} - 12.00 \text{ kN} + 15.00 \text{ kN} = 0$$

$$F_{FG} = 5.00 \text{ kN} \quad T \blacktriangleleft$$

Member *FH*:

$$+\circlearrowright \Sigma M_G = 0: \quad (15.00 \text{ kN})(2.4 \text{ m}) - F_{FH}(1.8 \text{ m}) = 0$$

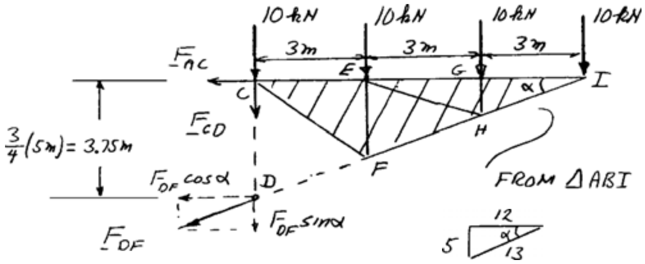
$$F_{FH} = 20.0 \text{ kN} \quad T \blacktriangleleft$$


3 m
3 m
3 m
3 m
10 kN
10 kN
10 kN
10 kN
A
C
E
G
I
5 m
H
F
D
B

PROBLEM 6.49

Determine the force in members *CD* and *DF* of the truss shown.

SOLUTION


10 kN
10 kN
10 kN
10 kN
3m
3m
3m
F_AC
C
E
G
I
α
F_CD
H
3/4(5m) = 3.75m
F_DF cos α
D
F
FROM Δ ABI
F_DF sin α
F_DF
12
5
13
α

$$\tan\alpha = \frac{5}{12} \qquad \alpha = 22.62°$$

$$\sin\alpha = \frac{5}{13} \quad \cos\alpha = \frac{12}{13}$$

Member *CD*:

$$+\circlearrowleft \Sigma M_I = 0: \quad F_{CD}(9\text{ m}) + (10\text{ kN})(9\text{ m}) + (10\text{ kN})(6\text{ m}) + (10\text{ kN})(3\text{ m}) = 0$$

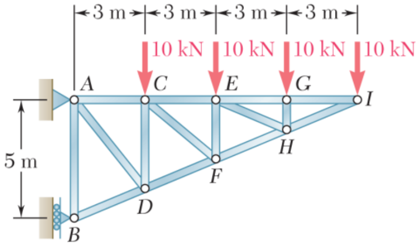
$$F_{CD} = -20\text{ kN} \qquad\qquad F_{CD} = 20.0\text{ kN} \quad C \blacktriangleleft$$

Member *DF*:

$$+\circlearrowright \Sigma M_C = 0: \quad (F_{DF}\cos\alpha)(3.75\text{ m}) + (10\text{ kN})(3\text{ m}) + (10\text{ kN})(6\text{ m}) + (10\text{ kN})(9\text{ m}) = 0$$

$$F_{DF}\cos\alpha = -48\text{ kN}$$

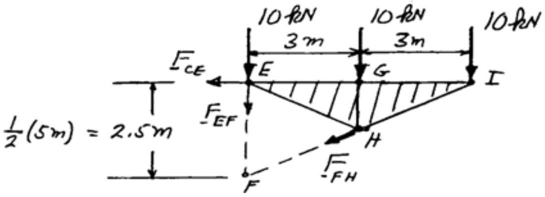
$$F_{DF}\left(\frac{12}{13}\right) = -48\text{ kN} \quad F_{DF} = -52.0\text{ kN} \qquad\qquad F_{DF} = 52.0\text{ kN} \quad C \blacktriangleleft$$


3 m
3 m
3 m
3 m
10 kN
10 kN
10 kN
10 kN
A
C
E
G
I
H
F
D
B
5 m

PROBLEM 6.50

Determine the force in members *CE* and *EF* of the truss shown.

SOLUTION


10 kN
10 kN
10 kN
3 m
3 m
F_{CE}
E
G
I
F_{EF}
H
F_{FH}
F
$\frac{1}{2}(5\text{ m}) = 2.5\text{ m}$

Member *CE*:

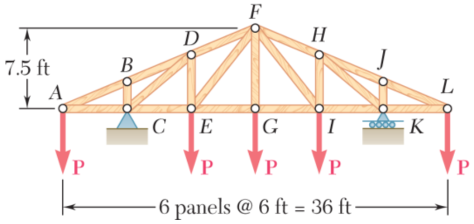
$$+\circlearrowleft \Sigma M_F = 0: \quad F_{CE}(2.5\text{ m}) - (10\text{ kN})(3\text{ m}) - (10\text{ kN})(6\text{ m}) = 0$$

$$F_{CE} = +36\text{ kN} \qquad\qquad F_{CE} = 36.0\text{ kN} \quad T \blacktriangleleft$$

Member *EF*:

$$+\circlearrowleft \Sigma M_I = 0: \quad F_{EF}(6\text{ m}) + (10\text{ kN})(6\text{ m}) + (10\text{ kN})(3\text{ m}) = 0$$

$$F_{EF} = -15\text{ kN} \qquad\qquad F_{EF} = 15.00\text{ kN} \quad C \blacktriangleleft$$


F
D
H
B
J
7.5 ft
A
L
C
E
G
I
K
P
P
P
P
P
6 panels @ 6 ft = 36 ft

PROBLEM 6.51

Determine the force in members *DE* and *DF* of the truss shown when $P = 20$ kips.

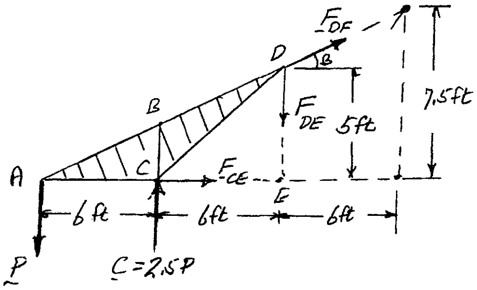
SOLUTION

Reactions:

$$\mathbf{C} = \mathbf{K} = 2.5P \uparrow$$

$$\tan\beta = \frac{7.5}{18}$$

$$\beta = 22.62^\circ$$


F_DF
β
D
B
F_DE
5ft
7.5ft
A
C
F_CE
E
6 ft
6ft
6ft
P
C=2.5P

Member *DE*:

$$+\circlearrowleft \Sigma M_A = 0: \quad (2.5P)(6\text{ ft}) - F_{DE}(12\text{ ft}) = 0$$

$$F_{DE} = +1.25P$$

For $P = 20$ kips, $F_{DE} = +1.25(20) = +25$ kips $\qquad F_{DE} = 25.0$ kips T ◀

Member *DF*:

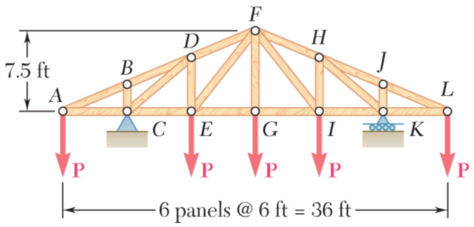
$$+\circlearrowleft \Sigma M_E = 0: \quad P(12\text{ ft}) - (2.5P)(6\text{ ft}) - F_{DF}\cos\beta(5\text{ ft}) = 0$$

$$12P - 15P - F_{DF}\cos 22.62^\circ(5\text{ ft}) = 0$$

$$F_{DF} = -0.65P$$

For $P = 20$ kips, $F_{DF} = -0.65(20) = -13$ kips

$F_{DF} = 13.00$ kips C ◀


F
D
H
B
J
7.5 ft
A
L
C
E
G
I
K
P
P
P
P
P
6 panels @ 6 ft = 36 ft

PROBLEM 6.52

Determine the force in members *EG* and *EF* of the truss shown when $P = 20$ kips.

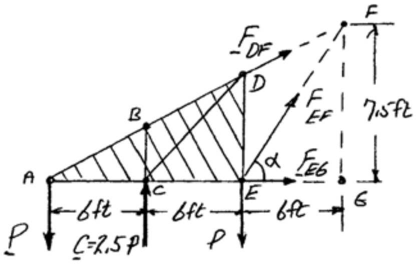
SOLUTION

Reactions:

$$\mathbf{C} = \mathbf{K} = 2.5P$$

$$\tan\alpha = \frac{7.5}{6}$$

$$\alpha = 51.34°$$


F_{DF}
F
D
B
F_{EF}
7.5ft
α
F_{EG}
A
C
E
G
6ft
6ft
6ft
P
C=2.5P
P

Member *EG*:

$$+\circlearrowleft \Sigma M_F = 0\colon \quad P(18\text{ ft}) - 2.5P(12\text{ ft}) - P(6\text{ ft}) + F_{EG}(7.5\text{ ft}) = 0$$

$$F_{EG} = +0.8P;$$

For $P = 20$ kips, $\quad F_{EG} = 0.8(20) = +16$ kips $\quad F_{EG} = 16.00$ kips T ◀

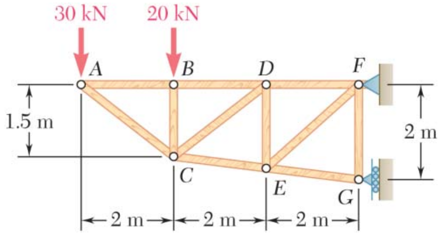
Member *EF*:

$$+\circlearrowleft \Sigma M_A = 0\colon \quad 2.5P(6\text{ ft}) - P(12\text{ ft}) + F_{EF}\sin 51.34°(12\text{ ft}) = 0$$

$$F_{EF} = -0.320P;$$

For $P = 20$ kips, $\quad F_{EF} = -0.320(20) = -6.4$ kips

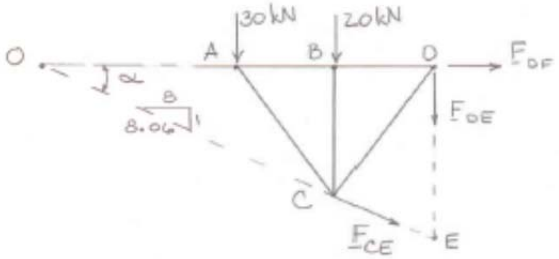
$F_{EF} = 6.40$ kips C ◀


30 kN
20 kN
A
B
D
F
1.5 m
2 m
C
E
G
2 m
2 m
2 m

PROBLEM 6.53

Determine the force in members *DF* and *DE* of the truss shown.

SOLUTION


30kN
20kN
O
A
B
D
F_{DF}
α
8
8.06
1
F_{DE}
C
F_{CE}
E

$$\alpha = \tan^{-1}\left(\frac{1}{8}\right) = 7.1^\circ$$

Member *CE*:

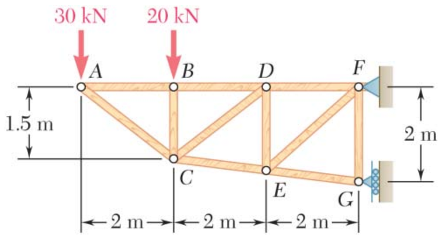
$$+\circlearrowleft \Sigma M_F = 0: \quad -F_{DF}(1.75 \text{ m}) + (30 \text{ kN})(4 \text{ m}) + (20 \text{ kN})(2 \text{ m}) = 0$$

$$F_{DF} = +91.4 \text{ kN} \qquad F_{DF} = 91.4 \text{ kN} \quad T \blacktriangleleft$$

Member *EF*:

$$+\circlearrowleft \Sigma M_O = 0: \quad -F_{DE}(14 \text{ m}) - (30 \text{ kN})(10 \text{ m}) - 20 \text{ kN})(12 \text{ m}) = 0$$

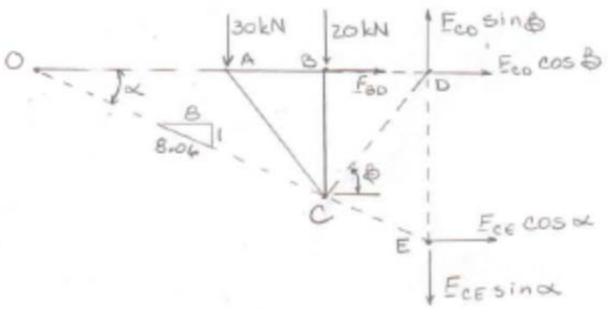
$$F_{DE} = -38.6 \text{ kN} \qquad F_{DE} = 38.6 \text{ kN} \quad C \blacktriangleleft$$


30 kN
20 kN
A
B
D
F
1.5 m
2 m
C
E
G
2 m
2 m
2 m

PROBLEM 6.54

Determine the force in members *CD* and *CE* of the truss shown.

SOLUTION


30kN
20kN
$F_{CD}\sin\beta$
O
A
B
F_{BD}
D
$F_{CD}\cos\beta$
α
8
1
8.06
β
C
E
$F_{CE}\cos\alpha$
$F_{CE}\sin\alpha$

$$\alpha = \tan^{-1}\left(\frac{1}{8}\right) = 7.1^\circ$$

$$\beta = \tan^{-1}\left(\frac{1.5}{2}\right) = 36.9^\circ$$

Member *CE*:

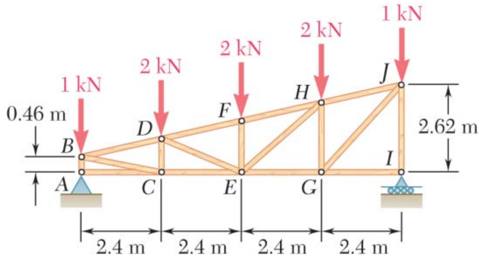
$$+\circlearrowleft \Sigma M_O = 0: \quad F_{CD}\sin\left(36.9^\circ\right)(14 \text{ m}) - (30 \text{ kN})(10 \text{ m}) - (20 \text{ kN})(12 \text{ m}) = 0$$

$F_{CD} = +64.2 \text{ kN}$ $\qquad$ $F_{CD} = 64.2 \text{ kN}$ T ◀

Member *EF*:

$$+\circlearrowleft \Sigma M_O = 0: \quad F_{CE}\cos\left(7.1^\circ\right)(1.75 \text{ m}) + (30 \text{ kN})(4 \text{ m}) + (20 \text{ kN})(2 \text{ m}) = 0$$

$F_{CE} = -92.1 \text{ kN}$ $\qquad$ $F_{CE} = 92.1 \text{ kN}$ C ◀


1 kN
2 kN
2 kN
2 kN
1 kN
0.46 m
B
D
F
H
J
2.62 m
A
C
E
G
I
2.4 m
2.4 m
2.4 m
2.4 m

PROBLEM 6.55

A monosloped roof truss is loaded as shown. Determine the force in members *CE*, *DE*, and *DF*.

SOLUTION

Reactions at supports: Because of the symmetry of the loading,

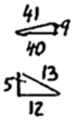
$$A_x = 0$$

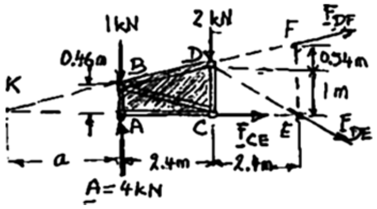
$$A_y = I = \frac{1}{2}\text{(Total load)} = \frac{1}{2}(8\text{ kN}) \qquad \mathbf{A} = \mathbf{I} = 4\text{ kN}\uparrow \triangleleft$$

We pass a section through members *CD*, *DE*, and *DF*, and use the free body shown.
(We moved F_{DE} to *E* and F_{DF} to *F*)

$$\text{Slope} \quad BJ = \frac{2.16\text{ m}}{9.6\text{ m}} = \frac{9}{40}$$

$$\text{Slope} \quad DE = \frac{-1\text{ m}}{2.4\text{ m}} = \frac{-5}{12}$$


41
9
40
5
13
12


1kN
2kN
F
F_DF
0.46m
B
D
0.54m
K
1m
A
C
F_CE
E
F_DE
a
2.4m
2.4m
A=4kN

$$a = \frac{0.46\text{ m}}{\text{Slope } BJ} = \frac{0.46\text{ m}}{\frac{9}{40}} = 2.0444\text{ m}$$

$$+\circlearrowleft \Sigma M_D = 0: \quad F_{CE}(1\text{ m}) + (1\text{ kN})(2.4\text{ m}) - (4\text{ kN})(2.4\text{ m}) = 0$$

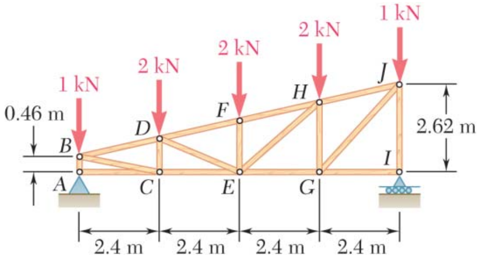
$$F_{CE} = +7.20\text{ kN} \qquad F_{CE} = 7.20\text{ kN} \quad T \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_K = 0: \quad (4\text{ kN})(2.0444\text{ m}) - (1\text{ kN})(2.0444\text{ m})$$
$$-(2\text{ kN})(4.4444\text{ m}) - \left(\frac{5}{13}F_{DE}\right)(6.8444\text{ m}) = 0$$

$$F_{DE} = -1.047\text{ kN} \qquad F_{DE} = 1.047\text{ kN} \quad C \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_E = 0: \quad (1\text{ kN})(4.8\text{ m}) + (2\text{kN})(2.4\text{ m}) - (4\text{ kN})(4.8\text{ m})$$
$$-\left(\frac{40}{41}F_{DF}\right)(1.54\text{ m}) = 0$$

$$F_{DF} = -6.39\text{ kN} \qquad F_{DF} = 6.39\text{ kN} \quad C \blacktriangleleft$$


1 kN
2 kN
2 kN
2 kN
1 kN
0.46 m
B
D
F
H
J
2.62 m
A
C
E
G
I
2.4 m
2.4 m
2.4 m
2.4 m

PROBLEM 6.56

A monosloped roof truss is loaded as shown. Determine the force in members *EG*, *GH*, and *HJ*.

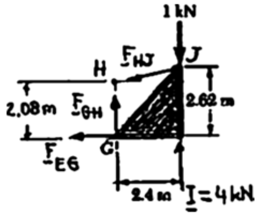
SOLUTION

Reactions at supports: Because of the symmetry of the loading,

$$A_x = 0$$

$$A_y = I = \frac{1}{2}\,(\text{Total load}) = \frac{1}{2}(8\text{ kN}) \qquad \mathbf{A} = \mathbf{I} = 4\text{ kN}\uparrow \triangleleft$$

We pass a section through members *EG*, *GH*, and *HJ*, and use the free body shown.


1 kN
H
F_{HJ}
J
2.08 m
F_{GH}
2.62 m
F_{EG}
G
2.4 m
I = 4 kN

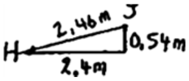
$$+\circlearrowleft \Sigma M_H = 0\colon \quad (4\text{ kN})(2.4\text{ m}) - (1\text{ kN})(2.4\text{ m}) - F_{EG}(2.08\text{ m}) = 0$$

$$F_{EG} = +3.4615\text{ kN} \qquad F_{EG} = 3.46\text{ kN} \quad T \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_J = 0\colon \quad -F_{GH}(2.4\text{ m}) - F_{EG}(2.62\text{ m}) = 0$$

$$F_{GH} = -\frac{2.62}{2.4}(3.4615\text{ kN})$$

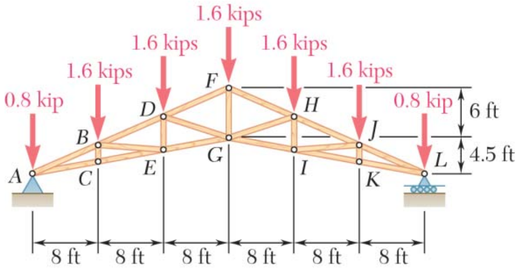
$$F_{GH} = -3.7788\text{ kN} \qquad F_{GH} = 3.78\text{ kN} \quad C \blacktriangleleft$$


2.46 m
J
0.54 m
H
2.4 m

$$\xrightarrow{+} \Sigma F_x = 0\colon \quad -F_{EG} - \frac{2.4}{2.46}F_{HJ} = 0$$

$$F_{HJ} = -\frac{2.46}{2.4}F_{EG} = -\frac{2.46}{2.4}(3.4615\text{ kN})$$

$$F_{HJ} = -3.548\text{ kN} \qquad F_{HJ} = 3.55\text{ kN} \quad C \blacktriangleleft$$


1.6 kips
1.6 kips
1.6 kips
1.6 kips
1.6 kips
0.8 kip
0.8 kip
A
B
C
D
E
F
G
H
I
J
K
L
6 ft
4.5 ft
8 ft
8 ft
8 ft
8 ft
8 ft
8 ft

PROBLEM 6.57

A Howe scissors roof truss is loaded as shown. Determine the force in members *DF*, *DG*, and *EG*.

SOLUTION

Reactions at supports.

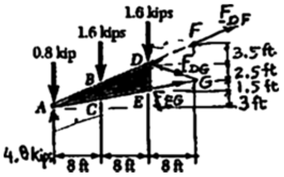
Because of symmetry of loading,

$$A_x = 0, \quad A_y = L = \frac{1}{2}(\text{total load}) = \frac{1}{2}(9.60\text{ kips}) = 4.80\text{ kips}$$

$$\mathbf{A} = \mathbf{L} = 4.80\text{ kips}\uparrow \blacktriangleleft$$

We pass a section through members *DF*, *DG*, and *EG*, and use the free body shown.

We slide $\mathbf{F}_{DF}$ to apply it at *F*:


1.6 kips
1.6 kips
0.8 kip
F_{DF}
F
D
B
A
C
E
F_{DG}
G
F_{EG}
3.5 ft
2.5 ft
1.5 ft
3 ft
4.8 kips
8 ft
8 ft
8 ft

$$+\circlearrowleft \Sigma M_G = 0: \quad (0.8\text{ kip})(24\text{ ft}) + (1.6\text{ kips})(16\text{ ft}) + (1.6\text{ kips})(8\text{ ft})$$

$$-(4.8\text{ kips})(24\text{ ft}) - \frac{8F_{DF}}{\sqrt{8^2+3.5^2}}(6\text{ ft}) = 0$$

$$F_{DF} = -10.48\text{ kips}, \quad F_{DF} = 10.48\text{ kips} \quad C \blacktriangleleft$$

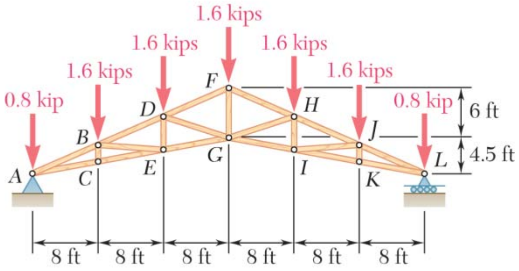
$$+\circlearrowleft \Sigma M_A = 0: \quad -(1.6\text{ kips})(8\text{ ft}) - (1.6\text{ kips})(16\text{ ft})$$

$$-\frac{2.5F_{DG}}{\sqrt{8^2+2.5^2}}(16\text{ ft}) - \frac{8F_{DG}}{\sqrt{8^2+2.5^2}}(7\text{ ft}) = 0$$

$$F_{DG} = -3.35\text{ kips}, \quad F_{DG} = 3.35\text{ kips} \quad C \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_D = 0: \quad (0.8\text{ kips})(16\text{ ft}) + (1.6\text{ kips})(8\text{ ft}) - (4.8\text{ kips})(16\text{ ft}) - \frac{8F_{EG}}{\sqrt{8^2+1.5^2}}(4\text{ ft}) = 0$$

$$F_{EG} = +13.02\text{ kips}, \quad F_{EG} = 13.02\text{ kips} \quad T \blacktriangleleft$$


0.8 kip
1.6 kips
1.6 kips
1.6 kips
1.6 kips
1.6 kips
0.8 kip
A
B
C
D
E
F
G
H
I
J
K
L
6 ft
4.5 ft
8 ft
8 ft
8 ft
8 ft
8 ft
8 ft

PROBLEM 6.58

A Howe scissors roof truss is loaded as shown. Determine the force in members *GI*, *HI*, and *HJ*.

SOLUTION

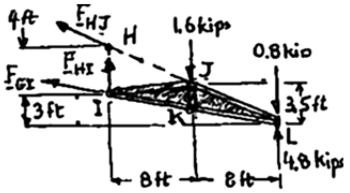
Reactions at supports. Because of symmetry of loading,

$$A_x = 0, \qquad A_y = L = \frac{1}{2}(\text{Total load})$$

$$= \frac{1}{2}(9.60 \text{ kips})$$

$$= 4.80 \text{ kips} \qquad \mathbf{A} = \mathbf{L} = 4.80 \text{ kips} \uparrow ◀$$

We pass a section through members *GI*, *HI*, and *HJ*, and use the free body shown.


4 ft
F_{HJ}
H
1.6 kips
0.8 kip
F_{HI}
F_{GI}
J
3 ft
I
K
3.5 ft
L
4.8 kips
8 ft
8 ft

$$+\circlearrowleft \Sigma M_H = 0: \quad -\frac{16F_{GI}}{\sqrt{16^2+3^2}}(4 \text{ ft}) + (4.8 \text{ kips})(16 \text{ ft}) - (0.8 \text{ kip})(16 \text{ ft}) - (1.6 \text{ kips})(8 \text{ ft}) = 0$$

$$F_{GI} = +13.02 \text{ kips} \qquad F_{GI} = 13.02 \text{ kips} \quad T ◀$$

$$+\circlearrowleft \Sigma M_L = 0: \quad (1.6 \text{ kips})(8 \text{ ft}) - F_{HI}(16 \text{ ft}) = 0$$

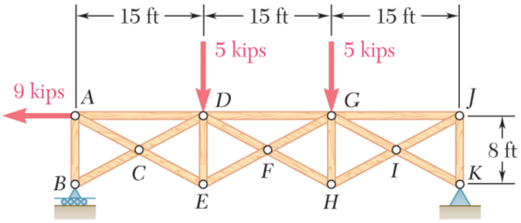
$$F_{HI} = +0.800 \text{ kips}$$

$$F_{HI} = 0.800 \text{ kips} \quad T ◀$$

We slide $\mathbf{F}_{HG}$ to apply it at *H*.

$$+\circlearrowleft \Sigma M_I = 0: \quad \frac{8F_{HJ}}{\sqrt{8^2+3.5^2}}(4 \text{ ft}) + (4.8 \text{ kips})(16 \text{ ft}) - (1.6 \text{ kips})(8 \text{ ft}) - (0.8 \text{ kip})(16 \text{ ft}) = 0$$

$$F_{HJ} = -13.97 \text{ kips} \qquad F_{HJ} = 13.97 \text{ kips} \quad C ◀$$

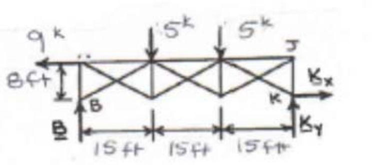

15 ft
15 ft
15 ft
5 kips
5 kips
9 kips
A
D
G
J
8 ft
B
C
E
F
H
I
K

PROBLEM 6.59

Determine the force in members *AD*, *CD*, and *CE* of the truss shown.

SOLUTION

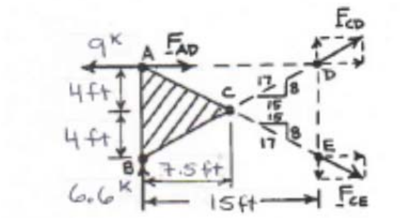
Reactions:


9k
5k
5k
J
8ft
B
K
Kx
B
Ky
15ft
15ft
15ft

$$+\circlearrowleft \Sigma M_k = 0: \quad 9 \text{ kips}(8 \text{ ft}) - B(45 \text{ ft}) + 5 \text{ kips}(30 \text{ ft}) + 5 \text{ kips}(15 \text{ ft}) = 0 \quad \mathbf{B} = 6.6 \text{ kips} \uparrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -9 \text{ kips} + K_x = 0 \qquad \mathbf{K}_x = 9 \text{ kips} \longrightarrow$$

$$+\uparrow \Sigma F_y = 0: \quad 6.6 \text{ kips} - 5 \text{ kips} - 5 \text{ kips} + K_y = 0 \quad \mathbf{K}_y = 3.4 \text{ kips} \uparrow$$


F_CD
9K
A
F_AD
D
4ft
C
17
8
15
15
4ft
17
8
E
B
7.5ft
6.6K
15ft
F_CE

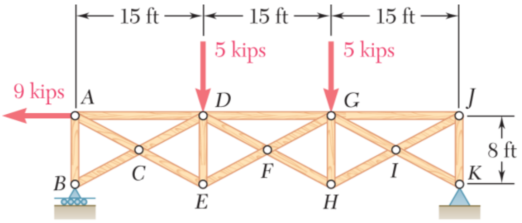
$$+\circlearrowleft \Sigma M_C = 0: \quad 9 \text{ kips}(4 \text{ ft}) - 6.6 \text{ kips}(7.5 \text{ ft}) - F_{AD}(4 \text{ ft}) = 0$$

$$F_{AD} = -3.375 \text{ kips} \qquad F_{AD} = 3.38 \text{ kN} \quad C \ ◀$$

$$+\circlearrowleft \Sigma M_A = 0: \quad \left(\frac{8}{17} F_{CD}\right)(15 \text{ ft}) = 0 \qquad F_{CD} = 0 \ ◀$$

$$+\circlearrowleft \Sigma M_D = 0: \quad \left(\frac{15}{17} F_{CE}\right)(8 \text{ ft}) - 6.6 \text{ kips}(15 \text{ ft}) = 0$$

$$F_{CE} = +14.025 \text{ kips} \qquad F_{CE} = 14.03 \text{ kips} \quad T \ ◀$$

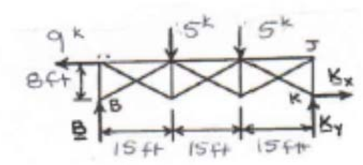

15 ft
15 ft
15 ft
5 kips
5 kips
9 kips
A
D
G
J
8 ft
B
C
E
F
H
I
K

PROBLEM 6.60

Determine the force in members *DG*, *FG*, and *FH* of the truss shown.

SOLUTION

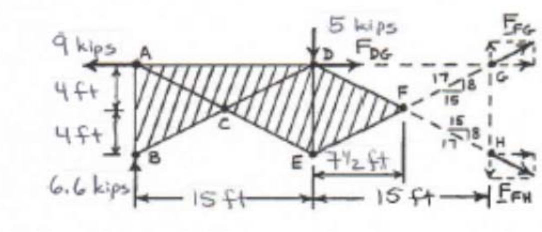
Reactions:


9k
5k
5k
J
8ft
Kx
B
K
B
Ky
15ft
15ft
15ft

$$+\circlearrowleft \Sigma M_k = 0: \quad 9 \text{ kips}(8 \text{ ft}) - B(45 \text{ ft}) + 5 \text{ kips}(30 \text{ ft}) + 5 \text{ kips}(15 \text{ ft}) = 0 \qquad \mathbf{B} = 6.6 \text{ kips} \uparrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -9 \text{ kips} + K_x = 0 \qquad \mathbf{K}_x = 9 \text{ kips} \longrightarrow$$

$$+\uparrow \Sigma F_y = 0: \quad 6.6 \text{ kips} - 5 \text{ kips} - 5 \text{ kips} + K_y = 0 \qquad \mathbf{K}_y = 3.4 \text{ kips} \uparrow$$


9 kips
A
5 kips
D
FDG
FFG
G
4 ft
4 ft
17
8
15
F
C
15
8
17
H
B
E
7½ ft
6.6 kips
15 ft
15 ft
FFH

$$+\circlearrowleft \Sigma M_F = 0: \quad 9 \text{ kips}(4 \text{ ft}) - 6.6 \text{ kips}(22.5 \text{ ft}) + 5 \text{ kips}(7.5 \text{ ft}) - F_{DG}(4 \text{ ft}) = 0$$

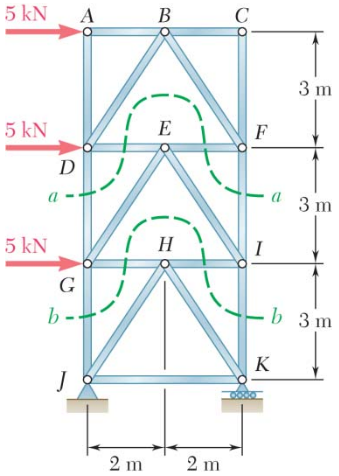
$$F_{DG} = -18.75 \text{ kips} \qquad F_{DG} = 18.75 \text{ kips} \quad C \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_D = 0: \quad -6.6 \text{ kips}(15 \text{ ft}) + \left(\frac{8}{17} F_{FG}\right)(15 \text{ ft}) = 0$$

$$F_{FG} = +14.025 \text{ kips} \qquad F_{FG} = 14.03 \text{ kips} \quad T \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_G = 0: \quad 5 \text{ kips}(15 \text{ ft}) - 6.6 \text{ kips}(30 \text{ ft}) + \left(\frac{15}{17} F_{FH}\right)(8 \text{ ft}) = 0$$

$$F_{FH} = +17.425 \text{ kips} \qquad F_{FH} = 17.43 \text{ kips} \quad T \blacktriangleleft$$

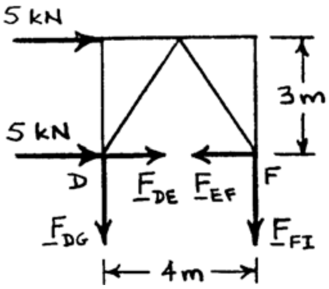


5 kN
A
B
C
3 m
5 kN
E
F
D
a
a
3 m
5 kN
H
I
G
b
b
3 m
K
J
2 m
2 m

PROBLEM 6.61

Determine the force in members *DG* and *FI* of the truss shown. (*Hint:* Use section *aa*.)

SOLUTION



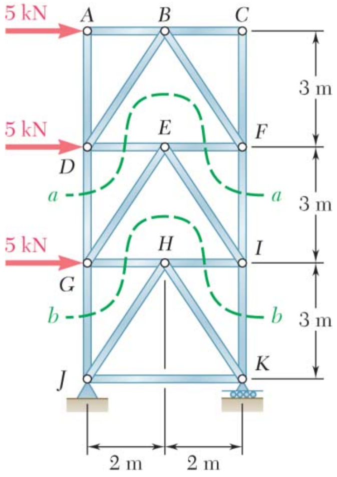
5 kN
3m
5 kN
D
F_{DE}
F_{EF}
F
F_{DG}
F_{FI}
4m

$$+\circlearrowleft \Sigma M_F = 0: \quad F_{DG}(4\text{ m}) - (5\text{ kN})(3\text{ m}) = 0$$

$$F_{DG} = +3.75\text{ kN} \qquad F_{DG} = 3.75\text{ kN} \quad T \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -3.75\text{ kN} - F_{FI} = 0$$

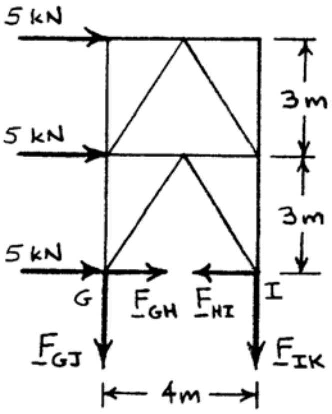
$$F_{FI} = -3.75\text{ kN} \qquad F_{FI} = 3.75\text{ kN} \quad C \blacktriangleleft$$


5 kN
A
B
C
3 m
5 kN
E
F
D
a
a
3 m
5 kN
H
I
G
b
b
3 m
K
J
2 m
2 m

PROBLEM 6.62

Determine the force in members *GJ* and *IK* of the truss shown. (*Hint:* Use section *bb*.)

SOLUTION

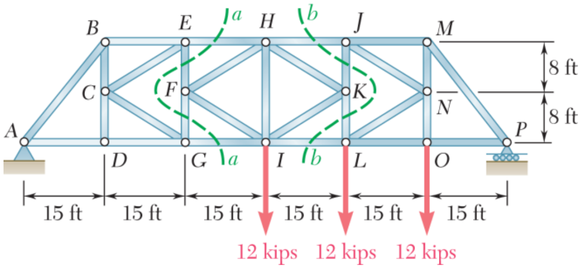

5 kN
3 m
5 kN
3 m
5 kN
G
F_{GH}
F_{HI}
I
F_{GJ}
F_{IK}
4 m

$+\circlearrowleft \Sigma M_I = 0$: $F_{GJ}(4\text{ m}) - (5\text{ kN})(6\text{ m}) - (5\text{ kN})(3\text{ m}) = 0$

$F_{GJ} = +11.25\text{ kN}$ $F_{GJ} = 11.25\text{ kN}$ T ◀

$+\uparrow \Sigma F_y = 0$: $-11.25\text{ kN} - F_{IK} = 0$

$F_{IK} = -11.25\text{ kN}$ $F_{IK} = 11.25\text{ kN}$ C ◀

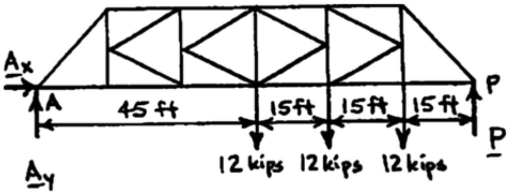

B
E
a
H
b
J
M
8 ft
8 ft
C
F
K
N
A
P
D
G
a
I
b
L
O
15 ft
15 ft
15 ft
15 ft
15 ft
15 ft
12 kips
12 kips
12 kips

PROBLEM 6.63

Determine the force in members *EH* and *GI* of the truss shown. (*Hint:* Use section *aa*.)

SOLUTION

Reactions:

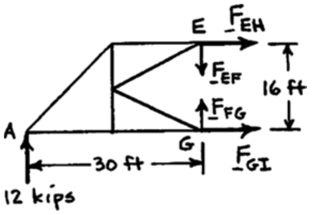

Ax
A
45 ft
15ft
15ft
15 ft
P
P
Ay
12 kips
12 kips
12 kips

$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\circlearrowleft \Sigma M_P = 0: \quad 12 \text{ kips}(45 \text{ ft}) + 12 \text{ kips}(30 \text{ ft}) + 12 \text{ kips}(15 \text{ ft}) - A_y(90 \text{ ft}) = 0$$

$$\mathbf{A}_y = 12 \text{ kips} \uparrow$$

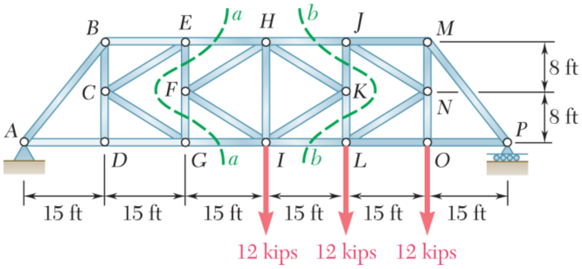
$$+\uparrow \Sigma F_y = 0: \quad 12 \text{ kips} - 12 \text{ kips} - 12 \text{ kips} - 12 \text{ kips} + P = 0 \qquad \mathbf{P} = 24 \text{ kips} \uparrow$$


E
F_EH
F_EF
16 ft
F_FG
A
G
F_GI
30 ft
12 kips

$$+\circlearrowleft \Sigma M_G = 0: \quad -(12 \text{ kips})(30 \text{ ft}) - F_{EH}(16 \text{ ft}) = 0$$

$$F_{EH} = -22.5 \text{ kips} \qquad F_{EH} = 22.5 \text{ kips} \quad C$$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{GI} - 22.5 \text{ kips} = 0 \qquad F_{GI} = 22.5 \text{ kips} \quad T$$ ◀

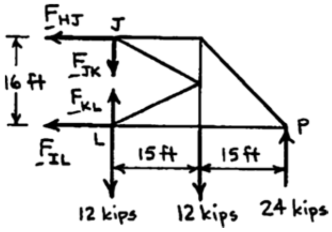

B
E
a
H
b
J
M
C
F
K
N
A
P
D
G
a
I
b
L
O
8 ft
8 ft
15 ft
15 ft
15 ft
15 ft
15 ft
15 ft
12 kips
12 kips
12 kips

PROBLEM 6.64

Determine the force in members *HJ* and *IL* of the truss shown. (*Hint:* Use section *bb*.)

SOLUTION

See the solution to Problem 6.63 for free body diagram and analysis to determine the reactions at supports *A* and *P*.


F_{HJ}
J
16 ft
F_{JK}
F_{KL}
F_{IL}
L
P
15 ft
15 ft
12 kips
12 kips
24 kips

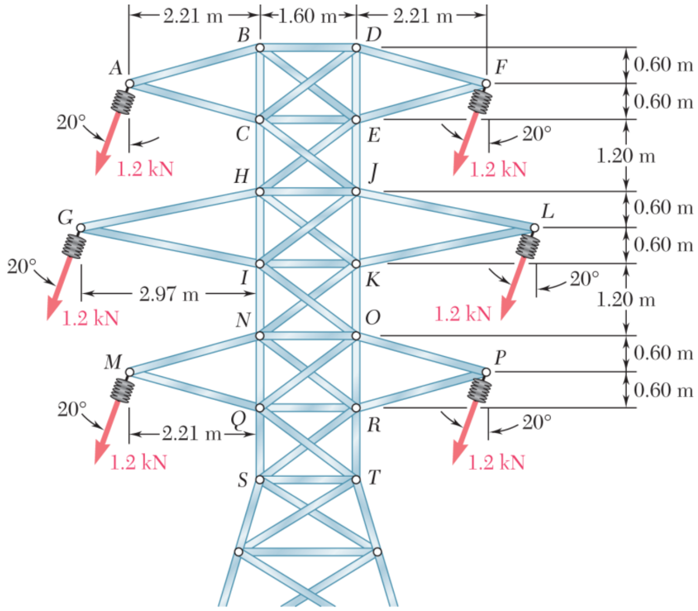
$$\mathbf{A}_x = 0; \quad \mathbf{A}_y = 12.00 \text{ kips} \uparrow; \quad \mathbf{P} = 24.0 \text{ kips} \uparrow$$

$$+\circlearrowleft \Sigma M_L = 0: \quad F_{HJ}(16 \text{ ft}) - (12 \text{ kips})(15 \text{ ft}) + (24 \text{ kips})(30 \text{ ft}) = 0$$

$$F_{HJ} = -33.75 \text{ kips} \qquad F_{HJ} = 33.8 \text{ kips} \quad C \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad 33.75 \text{ kips} - F_{IL} = 0$$

$$F_{IL} = +33.75 \text{ kips} \qquad F_{IL} = 33.8 \text{ kips} \quad T \blacktriangleleft$$


2.21 m
1.60 m
2.21 m
A
B
D
F
C
E
H
J
G
L
I
K
N
O
M
P
Q
R
S
T
0.60 m
0.60 m
1.20 m
0.60 m
0.60 m
1.20 m
0.60 m
0.60 m
20°
1.2 kN
2.97 m
2.21 m

PROBLEM 6.65

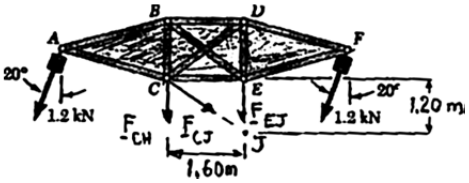
The diagonal members in the center panels of the power transmission line tower shown are very slender and can act only in tension; such members are known as *counters*. For the given loading, determine (*a*) which of the two counters listed below is acting, (*b*) the force in that counter.

Counters *CJ* and *HE*

SOLUTION

Free body: Portion *ABDFEC* of tower.

We assume that counter *CJ* is acting and show the forces exerted by that counter and by members *CH* and *EJ*.

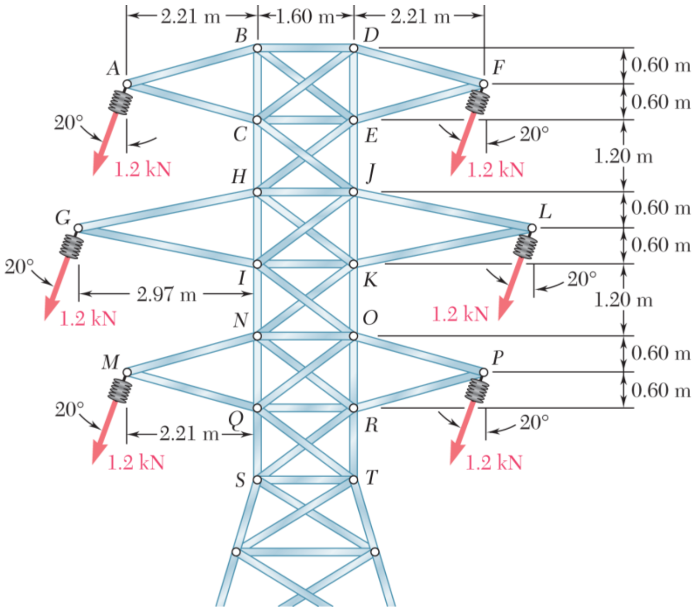

A
B
D
F
20°
C
E
1.2 kN
F_{CH}
F_{CJ}
F_{EJ}
J
20°
1.2 kN
1.20 m
1.60m

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{4}{5}F_{CJ} - 2(1.2\text{ kN})\sin 20° = 0 \quad F_{CJ} = +1.026\text{ kN}$$

Since *CJ* is found to be in tension, our assumption was correct. Thus, the answers are

(*a*) *CJ* ◀

(*b*) 1.026 kN *T* ◀


2.21 m
1.60 m
2.21 m
A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P
Q
R
S
T
0.60 m
0.60 m
1.20 m
0.60 m
0.60 m
1.20 m
0.60 m
0.60 m
20°
1.2 kN
2.97 m
2.21 m

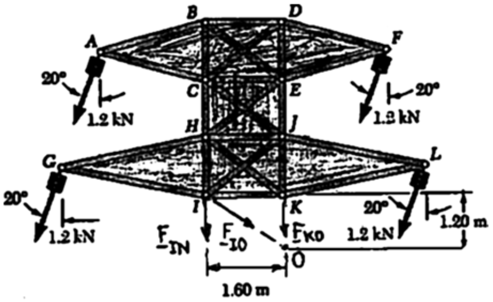
PROBLEM 6.66

The diagonal members in the center panels of the power transmission line tower shown are very slender and can act only in tension; such members are known as *counters*. For the given loading, determine (*a*) which of the two counters listed below is acting, (*b*) the force in that counter.

Counters *IO* and *KN*

SOLUTION

Free body: Portion of tower shown.


A
B
C
D
E
F
G
H
I
J
K
L
O
20°
1.2 kN
F_{IN}
F_{IO}
F_{KO}
1.20 m
1.60 m

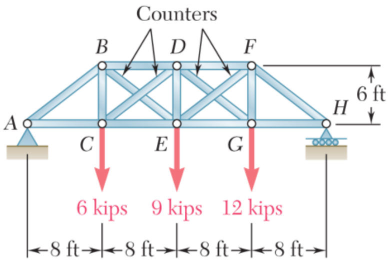
We assume that counter *IO* is acting and show the forces exerted by that counter and by members *IN* and *KO*.

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{4}{5}F_{IO} - 4(1.2 \text{ kN})\sin 20° = 0 \quad F_{IO} = +2.05 \text{ kN}$$

Since *IO* is found to be in tension, our assumption was correct. Thus, the answers are

(*a*) *IO* ◀

(*b*) 2.05 kN *T* ◀


Counters
B
D
F
6 ft
H
A
C
E
G
6 kips
9 kips
12 kips
8 ft
8 ft
8 ft
8 ft

PROBLEM 6.67

The diagonal members in the center panels of the truss shown are very slender and can act only in tension; such members are known as *counters*. Determine the force in member *DE* and in the counters that are acting under the given loading.

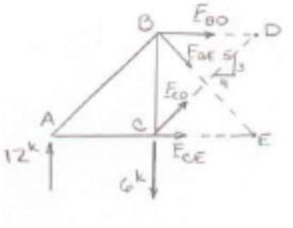
SOLUTION

Reactions from free body of entire truss:

$$+\circlearrowleft \Sigma M_H = 0: \mathbf{A} = 12 \text{ kips} \uparrow$$

$$+\uparrow \Sigma F_y = 0: \mathbf{H} = 15 \text{ kips} \uparrow$$

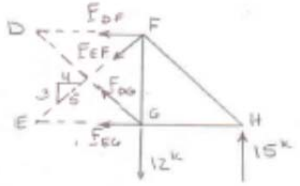
Portion to the left of vertical cut through panel *BCDE*:


B
F_{BD}
D
F_{BE}
5
3
4
F_{CD}
A
C
12^k
F_{CE}
E
6^k

Required vertical component of bar forces F_{BE} and F_{CD} is downward. We must have $F_{BE} =$ tension and $F_{CD} = 0$.

$$+\uparrow \Sigma F_y = 0: \quad 12 \text{ kips} - 6 \text{ kips} - \left(\frac{3}{5}\right) F_{BE} = 0$$

$F_{BE} = 10.00 \text{ kips} \quad T$ ◀


D
F_{DF}
F
F_{EF}
4
3
5
F_{DG}
E
G
H
F_{EG}
12^k
15^k

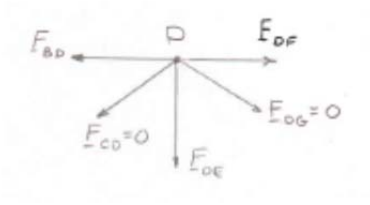
Required vertical component of bar forces F_{EF} and F_{DG} is downward. We must have $F_{EF} =$ tension and $F_{DG} = 0$.

PROBLEM 6.67 (Continued)

$$+\uparrow \Sigma F_y = 0: \quad -12 \text{ kips} + 15 \text{ kips} - \left(\frac{3}{5}\right) F_{EF} = 0$$

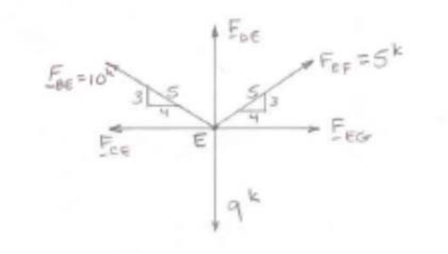
$F_{EF} = 5.00 \text{ kips}$ $\quad F_{EF} = 5.00 \text{ kips} \quad T$ ◀

Member *DE*: Free body of joint *D* $\quad +\uparrow \Sigma F_y = 0: \quad F_{DE} = 0$



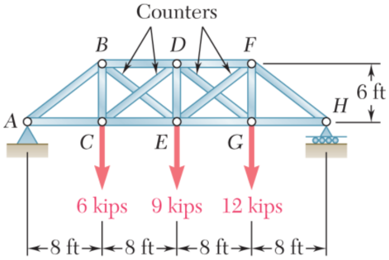
F_BD
D
F_DF
F_CD=0
F_DE
F_DG=0

Joint *E*: (check) $\quad +\uparrow \Sigma F_y = 0: \quad \left(\frac{3}{5}\right)(10 \text{ kips}) + \left(\frac{3}{5}\right)(5 \text{ kips}) - 9 \text{ kips} + F_{DE} = 0$



F_DE
F_BE=10^k
3
5
4
F_EF=5^k
5
3
4
F_CE
E
F_EG
9^k

$F_{DE} = 0$ (checks)


Counters
B
D
F
6 ft
H
A
C
E
G
6 kips
9 kips
12 kips
8 ft
8 ft
8 ft
8 ft

PROBLEM 6.68

Solve Prob. 6.67 assuming that the 9-kip load has been removed.

Problem 6.67 The diagonal members in the center panels of the truss shown are very slender and can act only in tension; such members are known as *counters*. Determine the force in member *DE* and in the counters that are acting under the given loading.

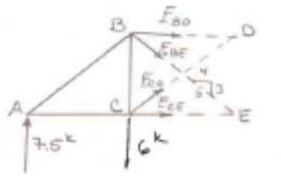
SOLUTION

Reactions from free body of entire truss:

$$+\circlearrowleft \Sigma M_H = 0: \mathbf{A} = 7.5 \text{ kips} \uparrow$$

$$+\uparrow \Sigma F_y = 0: \mathbf{H} = 10.5 \text{ kips} \uparrow$$

Portion to the left of vertical cut through panel *BCDE*:

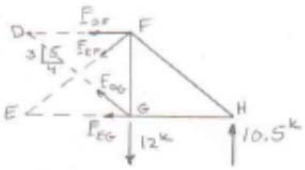

B
F_{BD}
D
F_{BE}
F_{CD}
4
5
3
A
C
F_{CE}
E
7.5^k
6^k

Required vertical component of bar forces F_{BE} and F_{CD} is downward. We must have $F_{BE} =$ tension and $F_{CD} = 0$.

$$+\uparrow \Sigma F_y = 0: \quad 7.5 \text{ kips} - 6 \text{ kips} - \left(\frac{3}{5}\right) F_{BE} = 0$$

$F_{BE} = 2.50$ kips T ◀

Portion to the right of vertical cut through panel *DFEG*:


F_{DF}
D
F
3
5
4
F_{EF}
F_{DG}
E
G
H
F_{EG}
12^k
10.5^k

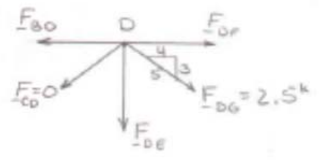
Required vertical component of bar forces F_{EF} and F_{DG} is downward. We must have $F_{DG} =$ tension and $F_{EF} = 0$.

PROBLEM 6.68 (Continued)

$$+\uparrow \Sigma F_y = 0: \quad -12 \text{ kips} + 10.5 \text{ kips} + \left(\frac{3}{5}\right) F_{DG} = 0$$

$F_{DG} = 2.5$ kips $\qquad F_{DG} = 2.50$ kips T ◀

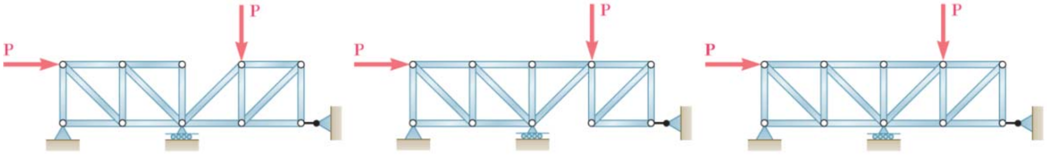
Member *DE*: Free body of joint *D* $\qquad +\downarrow \Sigma F_y = 0: \quad F_{DE} + \frac{3}{5}(2.5 \text{ kips}) = 0$


F_BD
D
F_DF
4
3
5
F_CD=0
F_DG=2.5^k
F_DE

$F_{DE} = -1.500$ kips $\qquad F_{DE} = 1.500$ kips C ◀

PROBLEM 6.69

Classify each of the structures shown as completely, partially, or improperly constrained; if completely constrained, further classify as determinate or indeterminate. (All members can act both in tension and in compression.)



(a) (b) (c)

SOLUTION

Structure (a)

Number of members:

$$m = 16$$

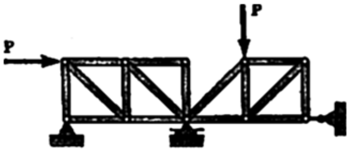
Number of joints:

$$n = 10$$

Reaction components:

$$r = 4$$

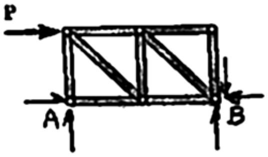
$$m + r = 20, \quad 2n = 20$$



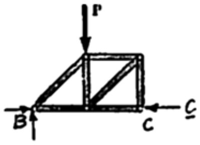
(a)

Thus, $m + r = 2n$ ◁

To determine whether the structure is actually completely constrained and determinate, we must try to find the reactions at the supports. We divide the structure into two simple trusses and draw the free-body diagram of each truss.



This is a properly supported simple truss – O.K.

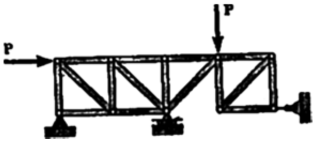


This is an improperly supported simple truss. (Reaction at *C* passes through *B*. Thus, Eq. $\Sigma M_B = 0$ cannot be satisfied.)

Structure is improperly constrained. ◀

Structure (b)

$$m = 16$$
$$n = 10$$
$$r = 4$$
$$m + r = 20, \quad 2n = 20$$

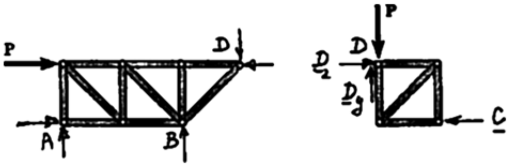


(b)

Thus, $m + r = 2n$ ◁

PROBLEM 6.69 (Continued)

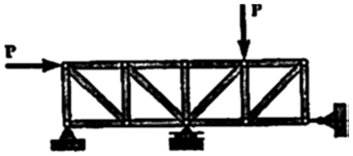
We must again try to find the reactions at the supports dividing the structure as shown.



Both portions are simply supported simple trusses.

Structure is completely constrained and determinate. ◀

Structure (c)



(c)

$$m = 17$$
$$n = 10$$
$$r = 4$$
$$m + r = 21, \quad 2n = 20$$

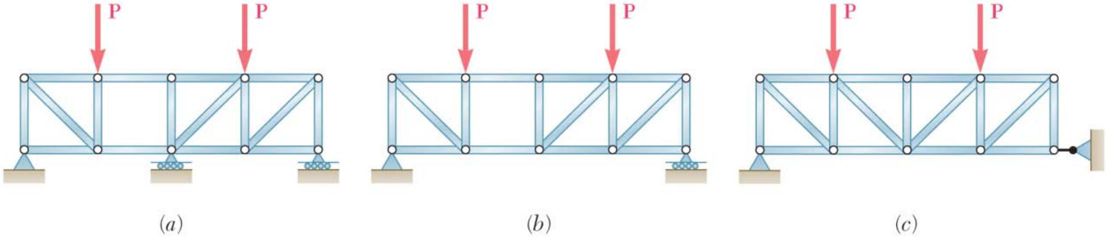
Thus, $m + r > 2n$ ◁

This is a simple truss with an extra support which causes reactions (and forces in members) to be indeterminate.

Structure is completely constrained and indeterminate. ◀

PROBLEM 6.70

Classify each of the structures shown as completely, partially, or improperly constrained; if completely constrained, further classify as determinate or indeterminate. (All members can act both in tension and in compression.)



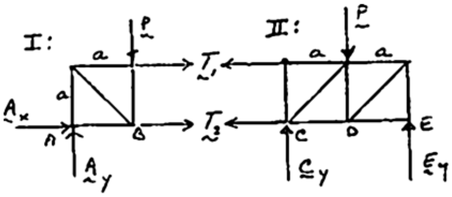
SOLUTION

Structure (a):

Nonsimple truss with $r = 4,\ \ m = 16,\ \ n = 10,$

so $m + r = 20 = 2n,$ but we must examine further.

FBD Sections:



FBD I: $\Sigma M_A = 0 \quad \Rightarrow \quad T_1$

II: $\Sigma F_x = 0 \quad \Rightarrow \quad T_2$

I: $\Sigma F_x = 0 \quad \Rightarrow \quad A_x$

I: $\Sigma F_y = 0 \quad \Rightarrow \quad A_y$

II: $\Sigma M_E = 0 \quad \Rightarrow \quad C_y$

II: $\Sigma F_y = 0 \quad \Rightarrow \quad E_y$

Since each section is a simple truss with reactions determined,

structure is completely constrained and determinate. ◀

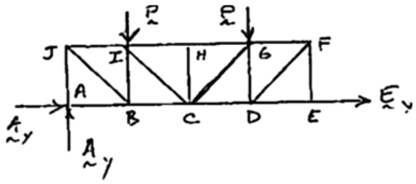
Structure (b):

Nonsimple truss with $r = 3,\ \ m = 16,\ \ n = 10,$

so $m + r = 19 < 2n = 20$ Structure is partially constrained. ◀

PROBLEM 6.70 (Continued)

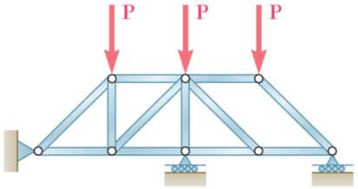
Structure (c):



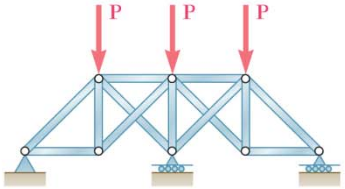
Simple truss with $r = 3$, $m = 17$, $n = 10$, $m + r = 20 = 2n$, but the horizontal reaction forces A_x and E_x are collinear and no equilibrium equation will resolve them, so the structure is improperly constrained and indeterminate. ◀

PROBLEM 6.71

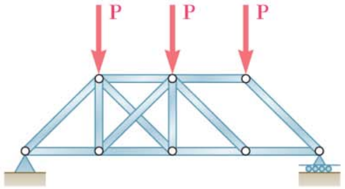
Classify each of the structures shown as completely, partially, or improperly constrained; if completely constrained, further classify as determinate or indeterminate. (All members can act both in tension and in compression.)



(a)



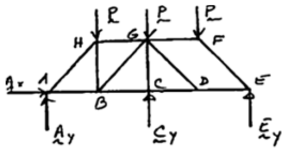
(b)

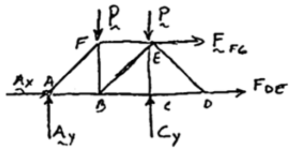


(c)

SOLUTION

Structure (a):


P
P
P
H
G
F
A_x
A
B
C
D
E
A_y
C_y
E_y


P
P
F
E
F_{FG}
A_x
A
F_{DE}
B
C
D
A_y
C_y

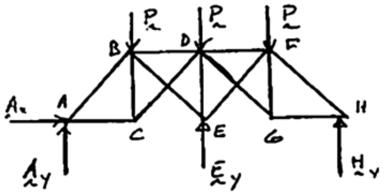
Nonsimple truss with $r = 4,\ m = 12,\ n = 8$ so $r + m = 16 = 2n.$
Check for determinacy:

One can solve joint *F* for forces in *EF*, *FG* and then solve joint *E* for $\mathbf{E}_y$ and force in *DE*.

This leaves a simple truss *ABCDGH* with

$r = 3, m = 9, n = 6$ so $r + m = 12 = 2n$ Structure is completely constrained and determinate. ◀

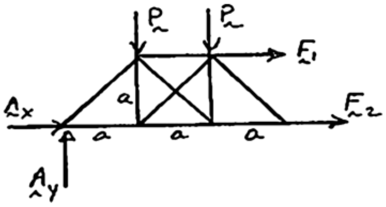
Structure (b):


P
P
P
B
D
F
A_x
A
H
C
E
G
A_y
E_y
H_y

Simple truss (start with *ABC* and add joints alphabetically to complete truss) with $r = 4,\ m = 13,\ n = 8$

so $r + m = 17 > 2n = 16$ Constrained but indeterminate ◀

Structure (c):



Nonsimple truss with $r = 3$, $m = 13$, $n = 8$ so $r + m = 16 = 2n$. To further examine, follow procedure in part (a) above to get truss at left.

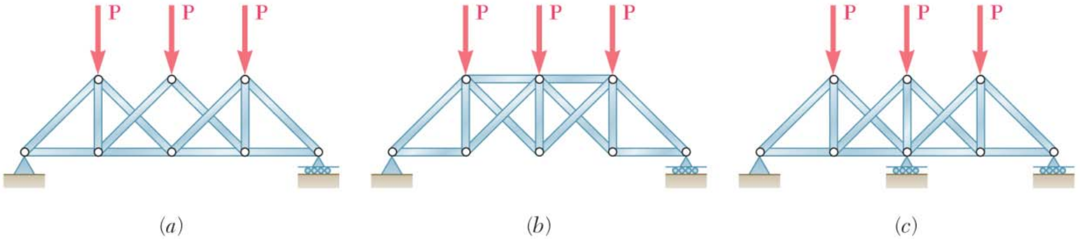
Since $\mathbf{F}_1 \neq 0$ (from solution of joint F),

$\Sigma M_A = aF_1 \circlearrowright \neq 0$ and there is no equilibrium.

Structure is improperly constrained. ◀

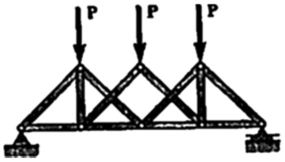
PROBLEM 6.72

Classify each of the structures shown as completely, partially, or improperly constrained; if completely constrained, further classify as determinate or indeterminate. (All members can act both in tension and in compression.)



SOLUTION

Structure (a)



Number of members:

$$m = 12$$

Number of joints:

$$n = 8$$

Reaction components:

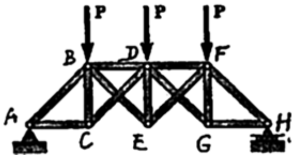
$$r = 3$$

$$m + r = 15, \quad 2n = 16$$

Thus, $m + r < 2n$ ◁

Structure is partially constrained. ◀

Structure (b)



$$m = 13, \; n = 8$$

$$r = 3$$

$$m + r = 16, \quad 2n = 16$$

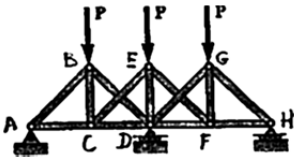
Thus, $m + r = 2n$ ◁

To verify that the structure is actually completely constrained and determinate, we observe that it is a simple truss (follow lettering to check this) and that it is simply supported by a pin-and-bracket and a roller. Thus,

structure is completely constrained and determinate. ◀

PROBLEM 6.72 (Continued)

Structure (c)



$$m = 13, \quad n = 8$$
$$r = 4$$
$$m + r = 17, \quad 2n = 16$$

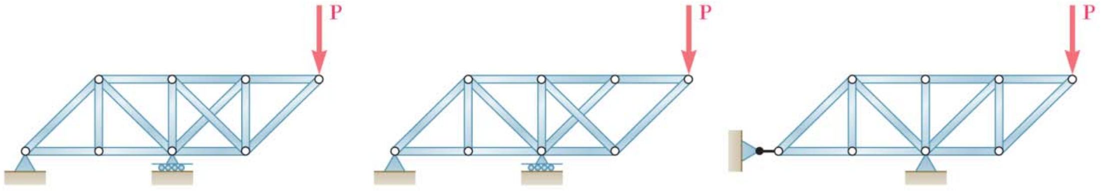
Thus, $m + r > 2n$ ◁

Structure is completely constrained and indeterminate. ◀

This result can be verified by observing that the structure is a simple truss (follow lettering to check this), therefore it is rigid, and that its supports involve four unknowns.

PROBLEM 6.73

Classify each of the structures shown as completely, partially, or improperly constrained; if completely constrained, further classify as determinate or indeterminate. (All members can act both in tension and in compression.)



(*a*) (*b*) (*c*)

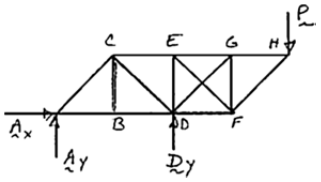
SOLUTION

Structure (a): Rigid truss with $r = 3, \quad m = 14, \quad n = 8,$

so $r + m = 17 > 2n = 16$

so completely constrained but indeterminate ◀

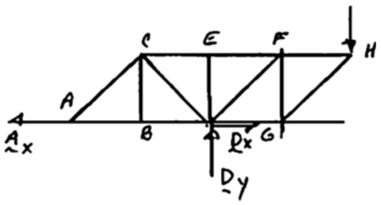
Structure (b): Simple truss (start with *ABC* and add joints alphabetically), with



$r = 3, m = 13, n = 8,$ so $r + m = 16 = 2n$

so completely constrained and determinate ◀

Structure (c):

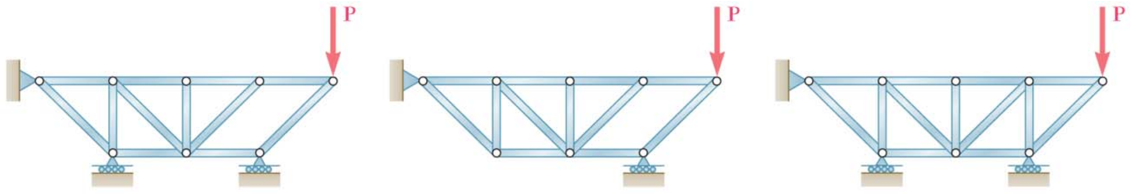


Simple truss with $r = 3, \quad m = 13, \quad n = 8,$ so $r + m = 16 = 2n,$ but horizontal reactions $(A_x$ and $D_x)$ are collinear, so cannot be resolved by any equilibrium equation.

Structure is improperly constrained. ◀

PROBLEM 6.74

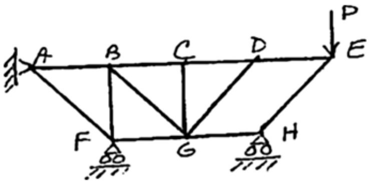
Classify each of the structures shown as completely, partially, or improperly constrained; if completely constrained, further classify as determinate or indeterminate. (All members can act both in tension and in compression.)



(*a*) (*b*) (*c*)

SOLUTION

Structure (a):

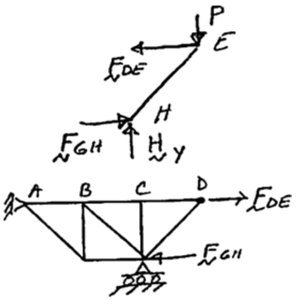


No. of members $m = 12$

No. of joints $n = 8$ $m + r = 16 = 2n$

No. of reaction components $r = 4$ unknows = equations

FBD of EH:

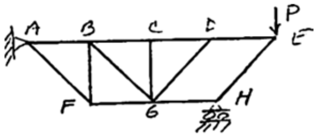


$$\Sigma M_H = 0 \longrightarrow F_{DE};\ \Sigma F_x = 0 \longrightarrow F_{GH};\ \Sigma F_y = 0 \longrightarrow H_y$$

Then *ABCDGF* is a simple truss and all forces can be determined.

This example is completely constrained and determinate. ◀

Structure (b):



No. of members $m = 12$

No. of joints $n = 8$ $m + r = 15 < 2n = 16$

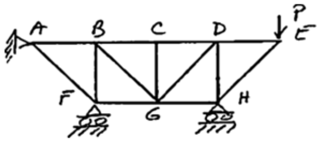
No. of reaction components $r = 3$ unknows < equations

partially constrained ◀

Note: Quadrilateral *DEHG* can collapse with joint *D* moving downward; in (a), the roller at *F* prevents this action.

PROBLEM 6.74 (Continued)

Structure (c):

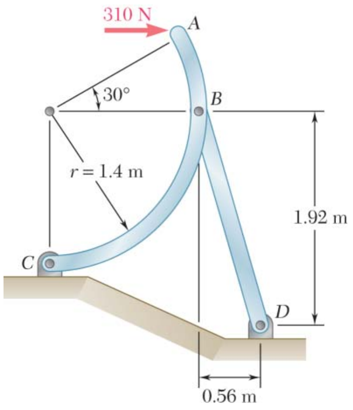


No. of members $m = 13$

No. of joints $n = 8$ $m + r = 17 > 2n = 16$

No. of reaction components $r = 4$ unknows > equations

completely constrained but indeterminate ◀



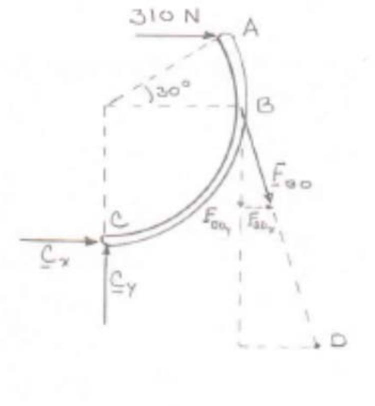
PROBLEM 6.75

Determine the force in member *BD* and the components of the reaction at *C*.

SOLUTION

We note that *BD* is a two-force member. The force it exerts on *ABC*, therefore, is directed along line *BD*.

Free body: *ABC*:



We have $BD = \sqrt{(1.92^2 + 0.56^2)} = 2\text{ m}$

$$+\circlearrowright \Sigma M_C = 0: \qquad (310\text{ N})(2.1\text{ m}) + \left(\frac{0.56}{2}F_{BD}\right)(1.4\text{ m}) + \left(\frac{1.92}{2}F_{BD}\right)(1.4\text{ m}) = 0$$

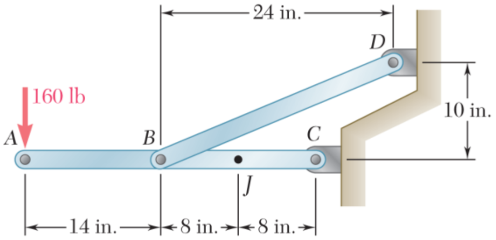
$$F_{BD} = -375\text{ N} \qquad F_{BD} = 375\text{ N}\quad C \;◀$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x + 310\text{ N} + \frac{0.56}{2}(-375\text{ N}) = 0$$

$$C_x = -205\text{ N} \qquad \mathbf{C}_x = 205\text{ N} \longleftarrow \;◀$$

$$+\!\uparrow \Sigma F_y = 0: \quad C_y - \frac{1.92}{2}(-375\text{ N}) = 0$$

$$C_y = -360\text{ N} \qquad \mathbf{C}_y = 360\text{ N}\downarrow \;◀$$


24 in.
D
160 lb
10 in.
A
B
C
J
14 in.
8 in.
8 in.

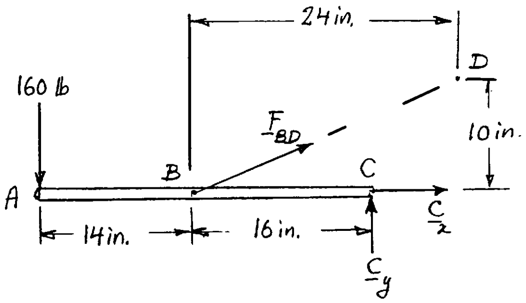
PROBLEM 6.76

Determine the force in member *BD* and the components of the reaction at *C*.

SOLUTION

We note that *BD* is a two-force member. The force it exerts on *ABC*, therefore, is directed along line *BD*.

Free body: *ABC*:


24 in.
D
160 lb
F_BD
10 in.
B
C
A
14 in.
16 in.
C_x
C_y

$$BD = \sqrt{(24)^2 + (10)^2} = 26 \text{ in.}$$

$$+\circlearrowleft \Sigma M_C = 0: \quad (160 \text{ lb})(30 \text{ in.}) - \left(\frac{10}{26} F_{BD}\right)(16 \text{ in.}) = 0$$

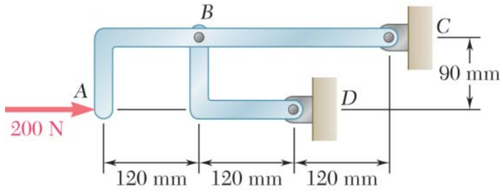
$$F_{BD} = +780 \text{ lb} \qquad F_{BD} = 780 \text{ lb} \quad T \blacktriangleleft$$

$$\xrightarrow{+} \Sigma M_x = 0: \quad C_x + \frac{24}{26}(780 \text{ lb}) = 0$$

$$C_x = -720 \text{ lb} \qquad \mathbf{C}_x = 720 \text{ lb} \longleftarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad C_y - 160 \text{ lb} + \frac{10}{26}(780 \text{ lb}) = 0$$

$$C_y = -140.0 \text{ lb} \qquad \mathbf{C}_y = 140.0 \text{ lb} \downarrow \blacktriangleleft$$

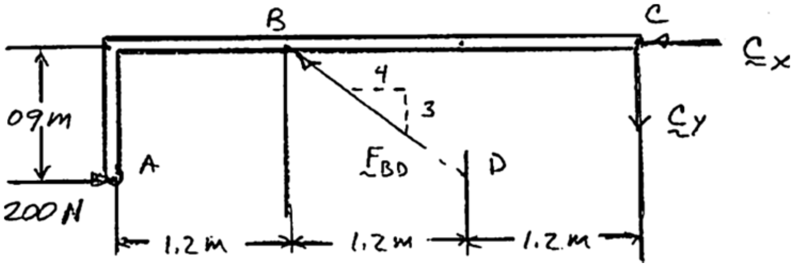

B
C
90 mm
A
D
200 N
120 mm
120 mm
120 mm

PROBLEM 6.77

For the frame and loading shown, determine the force acting on member *ABC* (*a*) at *B*, (*b*) at *C*.

SOLUTION

FBD ABC:


B
C
C_x
4
3
09m
C_y
A
F_{BD}
D
200 N
1.2 m
1.2 m
1.2 m

Note: *BD* is two-force member

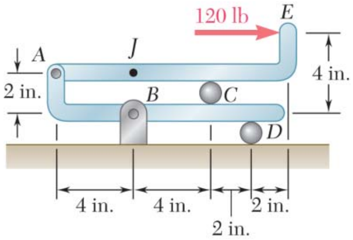
(*a*) $$\circlearrowleft \Sigma M_C = 0: \quad (0.09\text{ m})(200\text{ N}) - (2.4\text{ m})\left(\frac{3}{5}F_{BD}\right) = 0$$

$\mathbf{F}_{BD} = 125.0\text{ N} \nwarrow 36.9°$ ◀

(*b*) $$\xrightarrow{} \Sigma F_x = 0: \quad 200\text{ N} - \frac{4}{5}(125\text{ N}) - C_x = 0 \qquad \mathbf{C}_x = 100\text{ N} \longleftarrow$$

$$\uparrow \Sigma F_y = 0: \quad \frac{3}{5}F_{BD} - C_y = 0 \qquad \mathbf{C}_y = \frac{3}{5}(125\text{ N}) = 75\text{ N} \downarrow$$

$\mathbf{C} = 125.0\text{ N} \swarrow 36.9°$ ◀



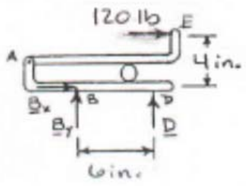
120 lb
E
A
J
2 in.
B
C
D
4 in.
4 in.
4 in.
2 in.
2 in.

PROBLEM 6.78

Determine the components of all forces acting on member *ABCD* of the assembly shown.

SOLUTION

Free body: Entire assembly:



120 lb
E
A
4 in.
B_x
B
D
B_y
D
6 in.

$$+\circlearrowleft \Sigma M_B = 0: \quad D(6\text{ in.}) - (120\text{ lb})(4\text{ in.}) = 0$$

$$\mathbf{D} = 80.0\text{ lb}\uparrow \blacktriangleleft$$

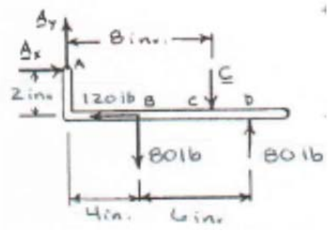
$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x + 120\text{ lb} = 0$$

$$\mathbf{B}_x = 120.0\text{ lb} \longleftarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad B_y + 80\text{ lb} = 0$$

$$\mathbf{B}_y = 80.0\text{ lb}\downarrow \blacktriangleleft$$

Free body: Member *ABCD*:



A_y
A_x
8 in.
A
C
2 in.
120 lb
B
C
D
80 lb
80 lb
4 in.
6 in.

$$+\circlearrowleft \Sigma M_A = 0: \quad (80\text{ lb})(10\text{ in.}) - C(8\text{ in.}) - (120\text{ lb})(2\text{ in.}) - (80\text{ lb})(4\text{ in.}) = 0$$

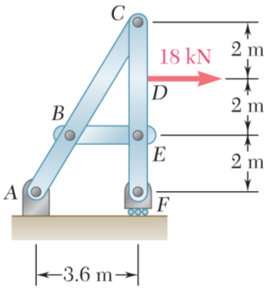
$$\mathbf{C} = 30.0\text{ lb}\downarrow \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - 120\text{ lb} = 0$$

$$\mathbf{A}_x = 120.0\text{ lb} \longrightarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 80\text{ lb} - 30\text{ lb} + 80\text{ lb} = 0$$

$$\mathbf{A}_y = 30.0\text{ lb}\uparrow \blacktriangleleft$$

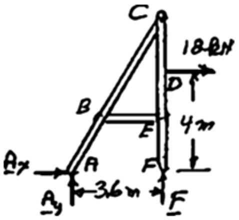


PROBLEM 6.79

For the frame and loading shown, determine the components of all forces acting on member *ABC*.

SOLUTION

Free body: Entire frame:



$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + 18\text{ kN} = 0$$

$$A_x = -18\text{ kN} \qquad \mathbf{A}_x = 18.00\text{ kN} \longleftarrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_E = 0: \quad -(18\text{ kN})(4\text{ m}) - A_y(3.6\text{ m}) = 0$$

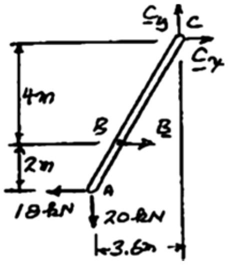
$$A_y = -20\text{ kN} \qquad \mathbf{A}_y = 20.0\text{ kN} \downarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -20\text{ kN} + F = 0$$

$$F = +20\text{ kN} \qquad \mathbf{F} = 20\text{ kN} \uparrow$$

Free body: Member *ABC*

Note: BE is a two-force member, thus **B** is directed along line *BE*.



$$+\circlearrowleft \Sigma M_C = 0: \quad B(4\text{ m}) - (18\text{ kN})(6\text{ m}) + (20\text{ kN})(3.6\text{ m}) = 0$$

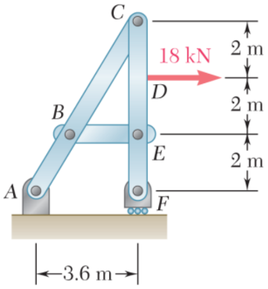
$$B = 9\text{ kN} \qquad \mathbf{B} = 9.00\text{ kN} \longrightarrow \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x - 18\text{ kN} + 9\text{ kN} = 0$$

$$C_x = 9\text{ kN} \qquad \mathbf{C}_x = 9.00\text{ kN} \longrightarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad C_y - 20\text{ kN} = 0$$

$$C_y = 20\text{ kN} \qquad \mathbf{C}_y = 20.0\text{ kN} \uparrow \blacktriangleleft$$



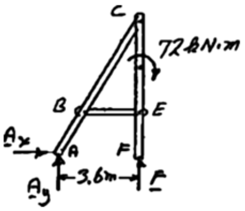
PROBLEM 6.80

Solve Problem 6.79 assuming that the 18-kN load is replaced by a clockwise couple of magnitude 72 kN · m applied to member *CDEF* at Point *D*.

PROBLEM 6.79 For the frame and loading shown, determine the components of all forces acting on member *ABC*.

SOLUTION

Free body: Entire frame



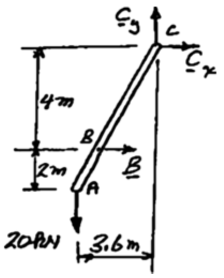
$\xrightarrow{+} \Sigma F_x = 0: \quad A_x = 0$

$+\circlearrowright \Sigma M_F = 0: \quad -72 \text{ kN}\cdot\text{m} - A_y(3.6 \text{ m}) = 0$

$A_y = -20 \text{ kN} \quad \mathbf{A}_y = 20 \text{ kN}\downarrow \qquad \mathbf{A} = 20.0 \text{ kN}\downarrow$ ◀

Free body: Member *ABC*

Note: BE is a two-force member. Thus **B** is directed along line *BE*.



$+\circlearrowright \Sigma M_C = 0: \quad B(4 \text{ m}) + (20 \text{ kN})(3.6 \text{ m}) = 0$

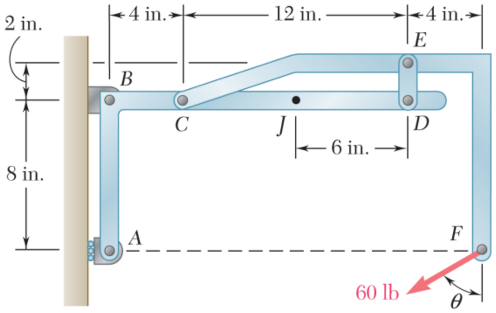
$B = -18 \text{ kN} \qquad \mathbf{B} = 18.00 \text{ kN} \longleftarrow$ ◀

$\xrightarrow{+} F_x = 0: \quad -18 \text{ kN} + C_x = 0$

$C_x = 18 \text{ kN} \qquad \mathbf{C}_x = 18.00 \text{ kN} \longrightarrow$ ◀

$+\uparrow \Sigma F_y = 0: \quad C_y - 20 \text{ kN} = 0$

$C_y = 20 \text{ kN} \qquad \mathbf{C}_y = 20.0 \text{ kN}\uparrow$ ◀

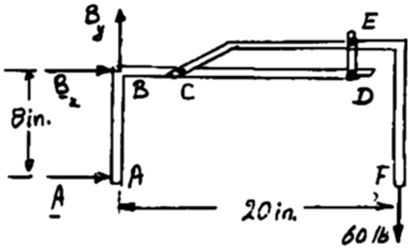

2 in.
4 in.
12 in.
4 in.
E
B
C
J
D
6 in.
8 in.
A
F
60 lb
θ

PROBLEM 6.81

Determine the components of all forces acting on member *ABCD* when $\theta = 0$.

SOLUTION

Free body: Entire assembly


B_y
E
B_x
B
C
D
8in.
A
A
F
20 in.
60 lb

$$+\circlearrowleft \Sigma M_B = 0: \quad A(8 \text{ in.}) - (60 \text{ lb})(20 \text{ in.}) = 0$$

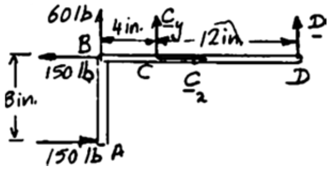
$A = 150 \text{ lb}$ $\qquad$ $\mathbf{A} = 150.0 \text{ lb} \longrightarrow$ ◀

$\xrightarrow{+} \Sigma F_x = 0: \quad B_x + 150.0 \text{ lb} = 0 \quad B_x = -150 \text{ lb}$ $\qquad$ $\mathbf{B}_x = 150.0 \text{ lb} \longleftarrow$ ◀

$+\uparrow \Sigma F_y = 0: \quad B_y - 60.0 \text{ lb} = 0 \quad B_y = +60.0 \text{ lb}$ $\qquad$ $\mathbf{B}_y = 60.0 \text{ lb} \uparrow$ ◀

Free body: Member *ABCD*

We note that **D** is directed along *DE*, since *DE* is a two-force member.

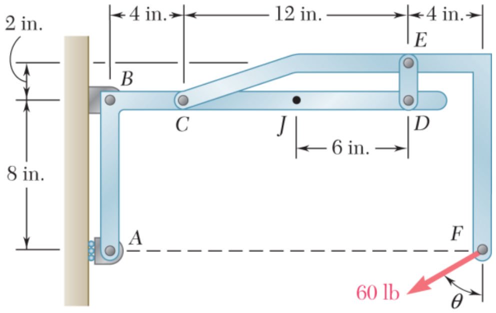

60 lb
4 in.
C_y
12 in.
D
B
150 lb
C
C_x
D
8 in.
150 lb
A

$$+\circlearrowleft \Sigma M_C = 0: \quad D(12) - (60 \text{ lb})(4) + (150 \text{ lb})(8) = 0$$

$D = -80 \text{ lb} \quad \mathbf{D} = 80.0 \text{ lb} \downarrow$ ◀

$\xrightarrow{+} \Sigma F_x = 0: \quad C_x + 150.0 - 150.0 = 0 \quad C_x = 0$

$+\uparrow \Sigma F_y = 0: \quad C_y + 60.0 - 80.0 = 0 \quad C_y = +20.0 \text{ lb} \quad \mathbf{C} = 20.0 \text{ lb} \uparrow$ ◀

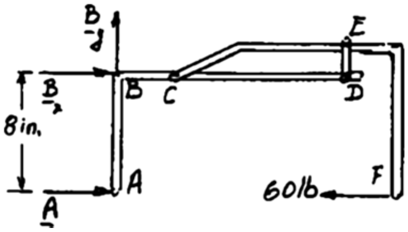

2 in.
4 in.
12 in.
4 in.
E
B
C
J
D
6 in.
8 in.
A
F
60 lb
θ

PROBLEM 6.82

Determine the components of all forces acting on member *ABCD* when $\theta = 90°$.

SOLUTION

Free body: Entire assembly


B_y
E
B_x
B
C
D
8 in.
A
A
60lb
F

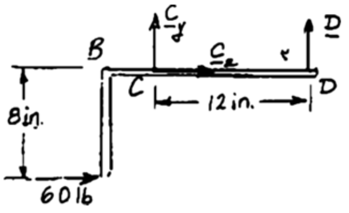
$$+\circlearrowleft \Sigma M_B = 0: \quad A(8\text{ in.}) - (60\text{ lb})(8\text{ in.}) = 0$$

$$A = +60.0\text{ lb} \qquad \mathbf{A} = 60.0\text{ lb} \longrightarrow$$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x + 60\text{ lb} - 60\text{ lb} = 0 \quad B_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad B_y = 0 \qquad \mathbf{B} = 0$$ ◀

Free body: Member *ABCD* We note that **D** is directed along *DE*, since *DE* is a two-force member.


C_y
D
B
C_x
C
12 in.
D
8 in.
60 lb

$$+\circlearrowleft \Sigma M_C = 0: \quad D(12\text{ in.}) + (60\text{ lb})(8\text{ in.}) = 0$$

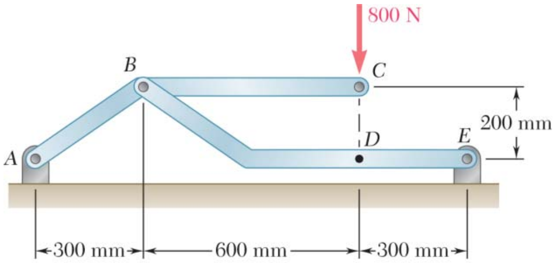
$$D = -40.0\text{ lb} \qquad \mathbf{D} = 40.0\text{ lb}\downarrow$$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x + 60\text{ lb} = 0$$

$$C_x = -60\text{ lb} \qquad \mathbf{C}_x = 60.0\text{ lb} \longleftarrow$$ ◀

$$+\uparrow \Sigma F_y = 0: \quad C_y - 40\text{ lb} = 0$$

$$C_y = +40\text{ lb} \qquad \mathbf{C}_y = 40.0\text{ lb}\uparrow$$ ◀

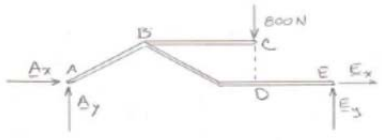


PROBLEM 6.83

Determine the components of the reactions at *A* and *E*, (*a*) if the 800-N load is applied as shown, (*b*) if the 800-N load is moved along its line of action and is applied at point *D*.

SOLUTION

Free body: Entire assembly. The analysis is valid for either (*a*) or (*b*), since position of the 800-N force on its line of action is immaterial.



$$+\circlearrowleft \Sigma M_A = 0: \quad E_y(1200) - (800\text{ N})(900) = 0$$

$$\mathbf{E}_y = 600\text{ N} \uparrow \blacktriangleleft$$

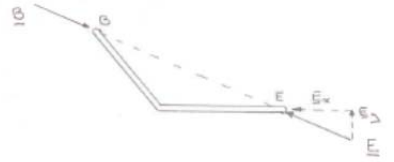
$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + E_x = 0 \quad (1)$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + 600 - 800 = 0 \qquad \mathbf{A}_y = 200\text{ N} \uparrow \blacktriangleleft$$

(*a*) Free body: Member *BE*:

We note that **E** is directed along *EB*, since *BE* is a two-force member.

$$\frac{E_x}{900} = \frac{E_y}{200} \quad \text{or} \quad E_x = \left(\frac{900}{200}\right)600\text{ N}$$



$$\mathbf{E}_x = 2700\text{ N} \leftarrow \blacktriangleleft$$

From equation (1): $A_x + 2700\text{ N} = 0$

$$A_x = 2700\text{ N} \qquad \mathbf{A}_x = 2700\text{ N} \rightarrow \blacktriangleleft$$

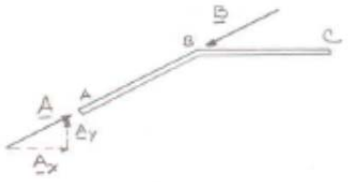
PROBLEM 6.83 (Continued)

(*b*) Free body: Member *ABC*:

We note that **A** is directed along *AB*, since *ABC* is a two-force member.

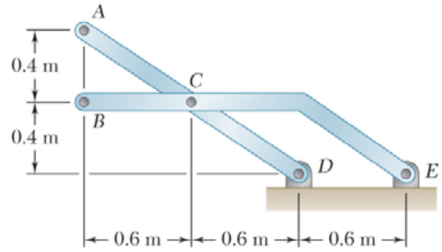
$$\frac{A_x}{300} = \frac{A_y}{200} \quad \text{or} \quad A_x = \left(\frac{300}{200}\right) 200 \text{ N}$$

$\mathbf{A}_x = 300 \text{ N} \longrightarrow$ ◀



From equation (1): $300 \text{ N} + E_x = 0$

$E_x = 300 \text{ N}$ $\mathbf{E}_x = 300 \text{ N} \longleftarrow$ ◀

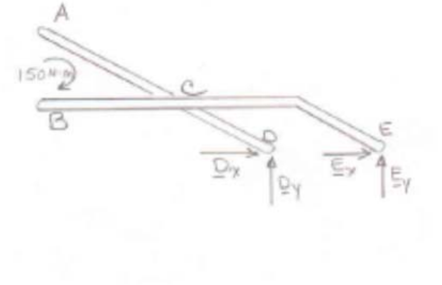


PROBLEM 6.84

Determine the components of the reactions at *D* and *E* if the frame is loaded by a clockwise couple of magnitude 150 N·m applied (*a*) at *A*, (*b*) at *B*.

SOLUTION

Free body: Entire assembly. The analysis is valid for either (*a*) or (*b*), since position of the couple is immaterial.



$$+\circlearrowleft \Sigma M_D = 0: \quad E_y(0.6\text{ m}) - (150\text{ N}\cdot\text{m}) = 0$$

$\mathbf{E}_y = 250\text{ N} \uparrow$ ◀

$$\overset{+}{\rightarrow} \Sigma F_x = 0: \quad D_x + E_x = 0 \quad (1)$$

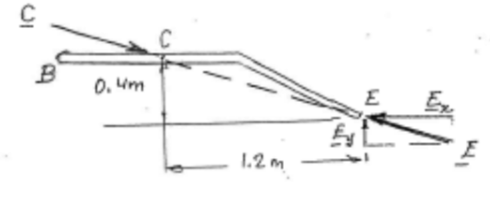
$$+\uparrow \Sigma F_y = 0: \quad D_y + 250\text{ N} = 0$$

$\mathbf{D}_y = 250\text{ N} \downarrow$ ◀

(*a*) Free body: Member *BCE*:

We note that **E** is directed along *EC*, since *BCE* is a two-force member.

$$\frac{E_x}{1.2} = \frac{E_y}{0.40} \quad \text{or} \quad E_x = \left(\frac{1.2}{0.40}\right) 250\text{ N}$$



$\mathbf{E}_x = 750\text{ N} \leftarrow$ ◀

From equation (1): $D_x - 750\text{ N} = 0$

$D_x = 750\text{ N}$ $\qquad$ $\mathbf{D}_x = 750\text{ N} \rightarrow$ ◀

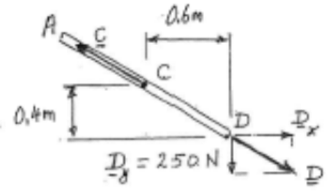
PROBLEM 6.84 (Continued)

(*b*) Free body: Member *ACD*:

We note that **D** is directed along *DC*, since *ACD* is a two-force member.

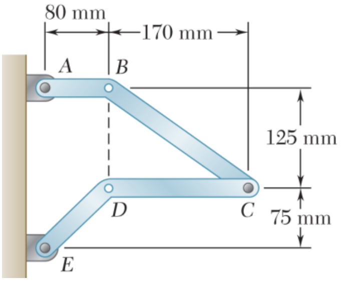
$$\frac{D_x}{0.6} = \frac{D_y}{0.4} \quad \text{or} \quad \mathrm{D}_x = \left(\frac{0.6}{0.4}\right) 250\ \mathrm{N}$$

$$\mathbf{D}_x = 375\ \mathrm{N} \rightarrow \blacktriangleleft$$



From equation (1): $375\ \mathrm{N} + E_x = 0$

$$E_x = -375\ \mathrm{N} \qquad \mathbf{E}_x = 375\ \mathrm{N} \leftarrow \blacktriangleleft$$



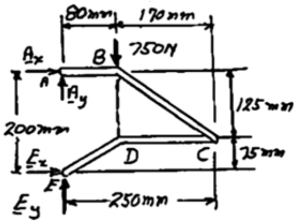
PROBLEM 6.85

Determine the components of the reactions at *A* and *E* if a 750-N force directed vertically downward is applied (*a*) at *B*, (*b*) at *D*.

SOLUTION

Free body: Entire frame:

The following analysis is valid for both parts (*a*) and (*b*) since position of load on its line of action is immaterial.



$$+\circlearrowleft \Sigma M_E = 0: \quad -(750\text{ N})(80\text{ mm}) - A_x(200\text{ mm}) = 0$$

$$A_x = -300\text{ N} \quad \mathbf{A}_x = 300\text{ N} \longleftarrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad E_x - 300\text{ N} = 0 \quad E_x = 300\text{ N} \quad \mathbf{E}_x = 300\text{ N} \longrightarrow$$

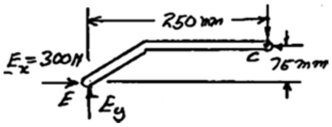
$$+\uparrow \Sigma F_y = 0: \quad A_y + E_y - 750\text{ N} = 0 \qquad (1)$$

(*a*) Load applied at *B*.

Free body: Member *CE*:

CE is a two-force member. Thus, the reaction at *E* must be directed along *CE*.

$$\frac{E_y}{300\text{ N}} = \frac{75\text{ mm}}{250\text{ mm}} \qquad E_y = 90\text{ N}\uparrow$$



From Eq. (1): $A_y + 90\text{ N} - 750\text{ N} = 0 \qquad A_y = 660\text{ N}\uparrow$

Thus, reactions are

$$\mathbf{A}_x = 300\text{ N} \longleftarrow, \ \mathbf{A}_y = 660\text{ N}\uparrow \blacktriangleleft$$

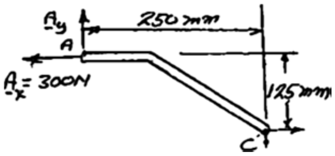
$$\mathbf{E}_x = 300\text{ N} \longrightarrow, \ \mathbf{E}_y = 90.0\text{ N}\uparrow \blacktriangleleft$$

(*b*) Load applied at *D*.

Free body: Member *AC*:

AC is a two-force member. Thus, the reaction at *A* must be directed along *AC*.

$$\frac{A_y}{300\text{ N}} = \frac{125\text{ mm}}{250\text{ mm}} \qquad A_y = 150\text{ N}\uparrow$$



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PROBLEM 6.85 (Continued)

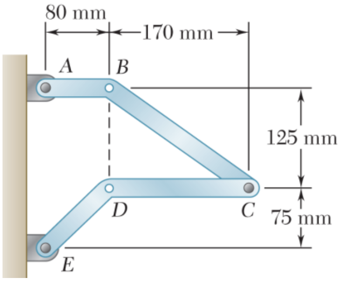
From Eq. (1): $A_y + E_y - 750\text{ N} = 0$

$$150\text{ N} + E_y - 750\text{ N} = 0$$

$$E_y = 600\text{ N} \quad \mathbf{E}_y = 600\text{ N}\uparrow$$

Thus, reactions are $\mathbf{A}_x = 300\text{ N} \longleftarrow, \quad \mathbf{A}_y = 150.0\text{ N}\uparrow$ ◀

$\mathbf{E}_x = 300\text{ N} \longrightarrow, \quad \mathbf{E}_y = 600\text{ N}\uparrow$ ◀



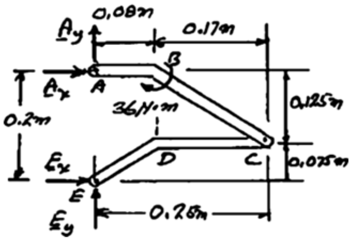
PROBLEM 6.86

Determine the components of the reactions at *A* and *E* if the frame is loaded by a clockwise couple of magnitude 36 N · m applied (*a*) at *B*, (*b*) at *D*.

SOLUTION

Free body: Entire frame:

The following analysis is valid for both parts (*a*) and (*b*) since the point of application of the couple is immaterial.



$$+\circlearrowleft \Sigma M_E = 0: \quad -36\ \text{N}\cdot\text{m} - A_x(0.2\ \text{m}) = 0$$

$$A_x = -180\ \text{N} \qquad \mathbf{A}_x = 180\ \text{N} \longleftarrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -180\ \text{N} + E_x = 0$$

$$E_x = 180\ \text{N} \qquad \mathbf{E}_x = 180\ \text{N} \longrightarrow$$

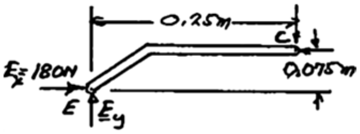
$$+\uparrow \Sigma F_y = 0: \quad A_y + E_y = 0 \tag{1}$$

(*a*) Couple applied at *B*.

Free body: Member *CE*:

AC is a two-force member. Thus, the reaction at *E* must be directed along *EC*.

$$\frac{E_y}{180\ \text{N}} = \frac{0.075\ \text{m}}{0.25\ \text{m}} \qquad \mathbf{E}_y = 54\ \text{N}\uparrow$$



From Eq. (1): $A_y + 54\ \text{N} = 0$

$$A_y = -54\ \text{N} \qquad \mathbf{A}_y = 54.0\ \text{N}\downarrow$$

Thus, reactions are

$$\mathbf{A}_x = 180.0\ \text{N} \longleftarrow, \quad \mathbf{A}_y = 54.0\ \text{N}\downarrow \blacktriangleleft$$

$$\mathbf{E}_x = 180.0\ \text{N} \longrightarrow, \quad \mathbf{E}_y = 54.0\ \text{N}\uparrow \blacktriangleleft$$

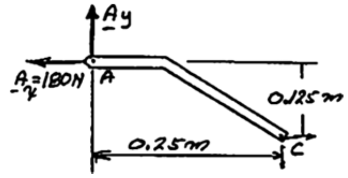
(*b*) Couple applied at *D*.

Free body: Member *AC*:

AC is a two-force member. Thus, the reaction at *A* must be directed along *EC*.

PROBLEM 6.86 (Continued)

$$\frac{A_y}{180\text{ N}} = \frac{0.125\text{ m}}{0.25\text{ m}} \quad A_y = 90\text{ N}\uparrow$$



From Eq. (1):

$$A_y + E_y = 0$$

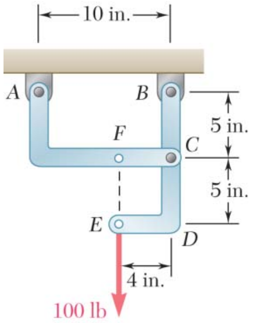
$$90\text{ N} + E_y = 0$$

$$E_y = -90\text{ N} \quad \mathbf{E}_y = 90\text{ N}\downarrow$$

Thus, reactions are

$$\mathbf{A}_x = 180.0\text{ N}\longleftarrow, \quad \mathbf{A}_y = 90.0\text{ N}\uparrow \blacktriangleleft$$

$$\mathbf{E}_x = -180.0\text{ N}\longrightarrow, \quad \mathbf{E}_y = 90.0\text{ N}\downarrow \blacktriangleleft$$

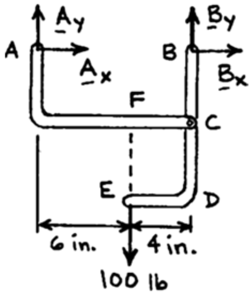


PROBLEM 6.87

Determine the components of the reactions at *A* and *B*, (*a*) if the 500-N load is applied as shown, (*b*) if the 500-N load is moved along its line of action and is applied at Point *F*.

SOLUTION

Free body: Entire frame:



Analysis is valid for either parts (*a*) or (*b*), since position of 100-lb load on its line of action is immatERIAL.

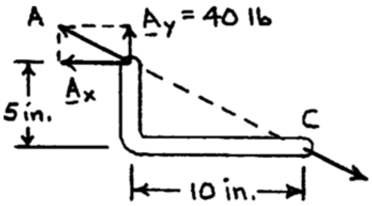
$$+\circlearrowleft \Sigma M_A = 0: \quad B_y(10) - (100\text{ lb})(6) = 0 \qquad B_y = +60\text{ lb}$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + 60 - 100 = 0 \qquad A_y = +40\text{ lb}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + B_x = 0 \tag{1}$$

(*a*) Load applied at *E*.

Free body: Member *AC*:



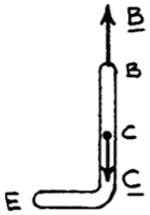
Since *AC* is a two-force member, the reaction at *A* must be directed along *CA*. We have

$$\frac{A_x}{10\text{ in.}} = \frac{40\text{ lb}}{5\text{ in.}} \qquad \mathbf{A}_x = 80.0\text{ lb} \longleftarrow, \quad \mathbf{A}_y = 40.0\text{ lb}\uparrow \blacktriangleleft$$

From Eq. (1): $-80 + B_x = 0 \quad B_x = +80\text{ lb}$

Thus, $\mathbf{B}_x = 80.0\text{ lb} \longrightarrow, \quad \mathbf{B}_y = 60.0\text{ lb}\uparrow \blacktriangleleft$

PROBLEM 6.87 (Continued)



(*b*) Load applied at *F*.

Free body: Member *BCD*:

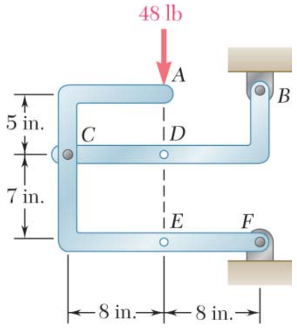
Since *BCD* is a two-force member (with forces applied at *B* and *C* only), the reaction at *B* must be directed along *CB*. We have, therefore,

$$B_x = 0$$

The reaction at *B* is $\mathbf{B}_x = 0$ $\mathbf{B}_y = 60.0\text{ lb}\uparrow$ ◀

From Eq. (1): $A_x + 0 = 0 \quad A_x = 0$

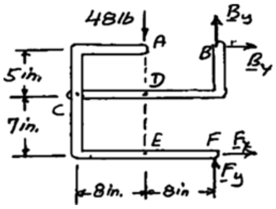
The reaction at *A* is $\mathbf{A}_x = 0$ $\mathbf{A}_y = 40.0\text{ lb}\uparrow$ ◀



PROBLEM 6.88

The 48-lb load can be moved along the line of action shown and applied at *A*, *D*, or *E*. Determine the components of the reactions at *B* and *F* if the 48-lb load is applied (*a*) at *A*, (*b*) at *D*, (*c*) at *E*.

SOLUTION



Free body: Entire frame

The following analysis is valid for (*a*), (*b*) and (*c*) since the position of the load along its line of action is immaterial.

$$+\circlearrowleft \Sigma M_F = 0: \quad (48\,\text{lb})(8\text{ in.}) - B_x(12\text{ in.}) = 0$$

$$B_x = 32\text{ lb} \quad \mathbf{B}_x = 32\text{ lb} \longrightarrow$$

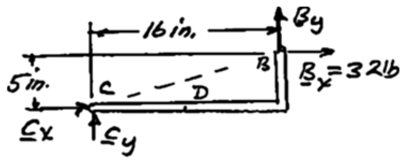
$$\xrightarrow{+} \Sigma F_x = 0: \quad 32\text{ lb} + F_x = 0$$

$$F_x = -32\text{ lb} \quad \mathbf{F}_x = 32\text{ lb} \longleftarrow$$

$$+\uparrow \Sigma F_y = 0: \quad B_y + F_y - 48\text{ lb} = 0 \qquad (1)$$

(*a*) Load applied at *A*.

Free body: Member *CDB*



CDB is a two-force member. Thus, the reaction at *B* must be directed along *BC*.

$$\frac{B_y}{32\text{ lb}} = \frac{5\text{ in.}}{16\text{ in.}} \qquad \mathbf{B}_y = 10\text{ lb}\uparrow$$

From Eq. (1): $10\text{ lb} + F_y - 48\text{ lb} = 0$

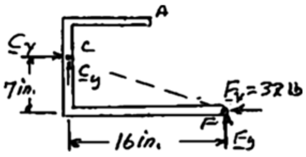
$$F_y = 38\text{ lb} \quad \mathbf{F}_y = 38\text{ lb}\uparrow$$

Thus reactions are:

$\mathbf{B}_x = 32.0\text{ lb} \longrightarrow, \ \mathbf{B}_y = 10.00\text{ lb}\uparrow$ ◀

$\mathbf{F}_x = 32.0\text{ lb} \longleftarrow, \ \mathbf{F}_y = 38.0\text{ lb}\uparrow$ ◀

PROBLEM 6.88 (Continued)



(*b*) Load applied at *D*.

Free body: Member *ACF*.

ACF is a two-force member. Thus, the reaction at *F* must be directed along *CF*.

$$\frac{F_y}{32 \text{ lb}} = \frac{7 \text{ in.}}{16 \text{ in.}} \qquad \mathbf{F}_y = 14 \text{ lb}\uparrow$$

From Eq. (1): $B_y + 14 \text{ lb} - 48 \text{ lb} = 0$

$$B_y = 34 \text{ lb} \quad B_y = 34 \text{ lb}\uparrow$$

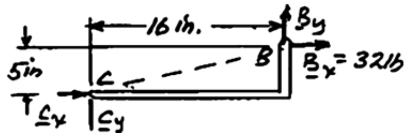
Thus, reactions are:

$\mathbf{B}_x = 32.0 \text{ lb} \leftarrow,\ \mathbf{B}_y = 34.0 \text{ lb}\uparrow$ ◀

$\mathbf{F}_x = 32.0 \text{ lb} \rightarrow,\ \mathbf{F}_y = 14.00 \text{ lb}\uparrow$ ◀

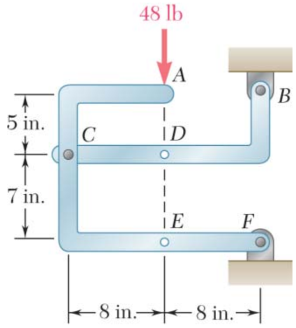
(*c*) Load applied at *E*.

Free body: Member *CDB*



This is the same free body as in Part (*a*).

Reactions are same as (*a*) ◀



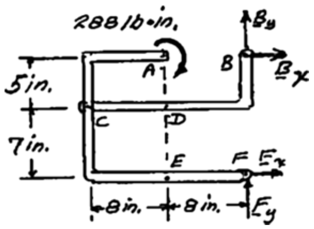
PROBLEM 6.89

The 48-lb load is removed and a 288-lb · in. clockwise couple is applied successively at *A*, *D*, and *E*. Determine the components of the reactions at *B* and *F* if the couple is applied (*a*) at *A*, (*b*) at *D*, (*c*) at *E*.

SOLUTION

Free body: Entire frame

The following analysis is valid for (*a*), (*b*), and (*c*), since the point of application of the couple is immaterial.



$$+\circlearrowleft \Sigma M_F = 0: \quad -288 \text{ lb}\cdot\text{in.} - B_x(12 \text{ in.}) = 0$$

$$B_x = -24 \text{ lb} \quad \mathbf{B}_x = 24 \text{ lb} \longleftarrow$$

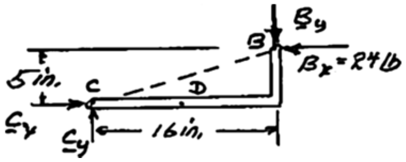
$$\xrightarrow{+} \Sigma F_x = 0: \quad -24 \text{ lb} + F_x = 0$$

$$F_x = 24 \text{ lb} \quad \mathbf{F}_x = 24 \text{ lb} \longrightarrow$$

$$+\!\uparrow \Sigma F_y = 0: \quad B_y + F_y = 0 \qquad (1)$$

(*a*) Couple applied at *A*.

Free body: Member *CDB*



CDB is a two-force member. Thus, reaction at *B* must be directed along *BC*.

$$\frac{B_y}{24 \text{ lb}} = \frac{5 \text{ in.}}{16 \text{ in.}} \qquad \mathbf{B}_y = 7.5 \text{ lb}\downarrow$$

From Eq. (1): $-7.5 \text{ lb} + F_y = 0$

$$F_y = 7.5 \text{ lb} \quad \mathbf{F}_y = 7.5 \text{ lb}\uparrow$$

Thus, reactions are:

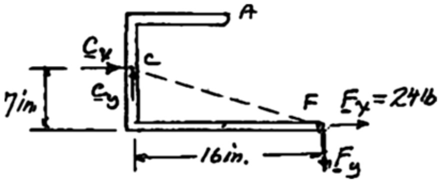
$\mathbf{B}_x = 24.0 \text{ lb} \longleftarrow, \ \mathbf{B}_y = 7.50 \text{ lb}\downarrow$ ◀

$\mathbf{F}_x = 24.0 \text{ lb} \longrightarrow, \mathbf{F}_y = 7.50 \text{ lb}\uparrow$ ◀

PROBLEM 6.89 (Continued)

(*b*) Couple applied at *D*.

Free body: Member *ACF*.



ACF is a two-force member. Thus, the reaction at *F* must be directed along *CF*.

$$\frac{F_y}{24 \text{ lb}} = \frac{7 \text{ in.}}{16 \text{ in.}} \qquad \mathbf{F}_y = 10.5 \text{ lb}\downarrow$$

From Eq. (1): $B_y - 10.5$ lb:

$$B_y = +10.5 \text{ lb} \quad \mathbf{B}_y = 10.5 \text{ lb}\uparrow$$

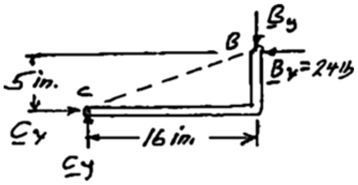
Thus, reactions are:

$\mathbf{B}_x = 24.0 \text{ lb} \longleftarrow,\ \mathbf{B}_y = 10.50 \text{ lb}\uparrow$ ◀

$\mathbf{F}_x = 24.0 \text{ lb} \longrightarrow, \mathbf{F}_y = 10.50 \text{ lb}\downarrow$ ◀

(*c*) Couple applied at *E*.

Free body: Member *CDB*

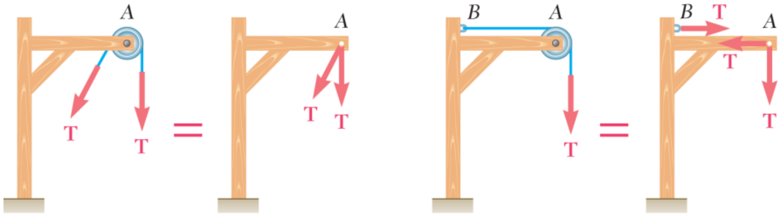


This is the same free body as in Part (*a*).

Reactions are same as in (*a*) ◀

PROBLEM 6.90

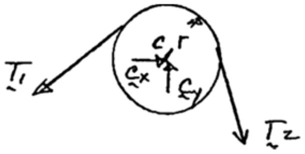
(*a*) Show that when a frame supports a pulley at *A*, an equivalent loading of the frame and of each of its component parts can be obtained by removing the pulley and applying at *A* two forces equal and parallel to the forces that the cable exerted on the pulley. (*b*) Show that if one end of the cable is attached to the frame at a Point *B*, a force of magnitude equal to the tension in the cable should also be applied at *B*.



(*a*) (*b*)

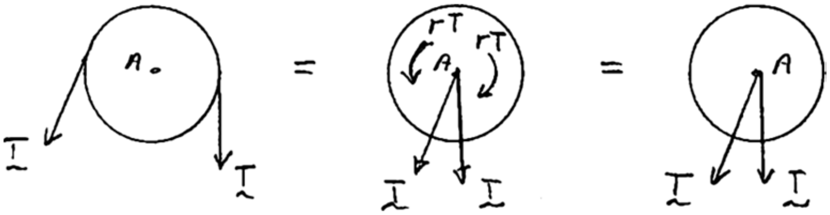
SOLUTION

First note that, when a cable or cord passes over a *frictionless, motionless* pulley, the tension is unchanged.

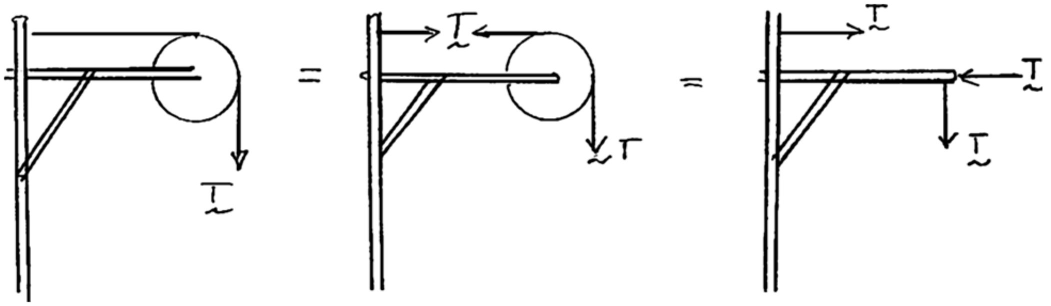


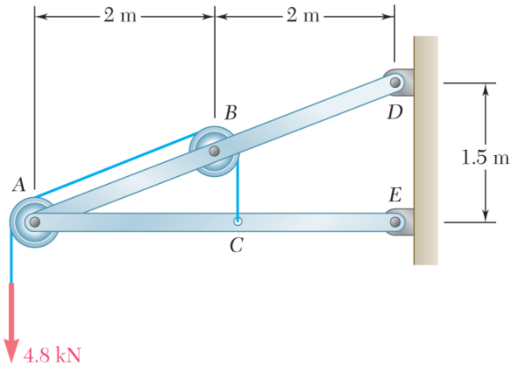
$$\circlearrowleft + \Sigma M_C = 0: \quad rT_1 - rT_2 = 0 \quad T_1 = T_2$$

(*a*) Replace each force with an equivalent force-couple.



(*b*) Cut the cable and replace the forces on pulley with equivalent pair of forces at *A* as above.



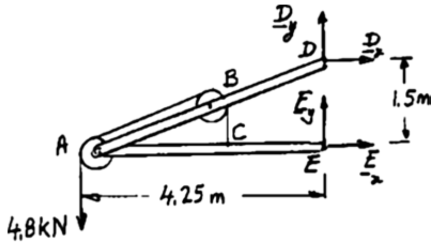

2 m
2 m
B
D
1.5 m
A
E
C
4.8 kN

PROBLEM 6.91

Knowing that each pulley has a radius of 250 mm, determine the components of the reactions at *D* and *E*.

SOLUTION

Free body: Entire assembly:


D_y
D
D_x
B
1.5m
E_y
C
A
E
E_x
4.25 m
4.8kN

$+\circlearrowleft \Sigma M_E = 0: \quad (4.8\text{ kN})(4.25\text{ m}) - D_x(1.5\text{ m}) = 0$

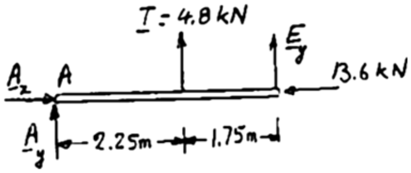
$D_x = +13.60\text{ kN}$ $\qquad \mathbf{D}_x = 13.60\text{ kN} \longrightarrow$ ◀

$\xrightarrow{+} \Sigma F_x = 0: \quad E_x + 13.60\text{ kN} = 0$

$E_x = -13.60\text{ kN}$ $\qquad \mathbf{E}_x = 13.60\text{ kN} \longleftarrow$ ◀

$+\uparrow \Sigma F_y = 0: \quad D_y + E_y - 4.8\text{ kN} = 0 \qquad (1)$

Free body: Member *ACE*:

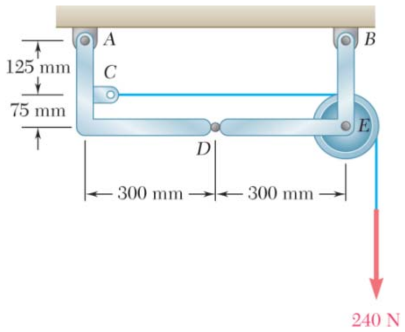

T = 4.8 kN
E_y
A_x
A
13.6 kN
A_y
2.25m
1.75m

$+\circlearrowleft \Sigma M_A = 0: \quad (4.8\text{ kN})(2.25\text{ m}) + E_y(4\text{ m}) = 0$

$E_y = -2.70\text{ kN}$ $\qquad \mathbf{E}_y = 2.70\text{ kN}\downarrow$ ◀

From Eq. (1): $\quad D_y - 2.70\text{ kN} - 4.80\text{ kN} = 0$

$D_y = +7.50\text{ kN}$ $\qquad \mathbf{D}_y = 7.50\text{ kN}\uparrow$ ◀

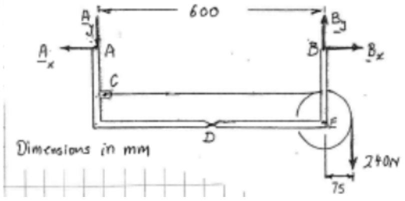

A
B
C
125 mm
75 mm
E
D
300 mm
300 mm
240 N

PROBLEM 6.92

Knowing that the pulley has a radius of 75 mm, determine the components of the reactions at *A* and *B*.

SOLUTION

Free body: Entire frame and pulley:


600
A_y
A_x
A
B
B_y
B_x
C
D
E
Dimensions in mm
240N
75

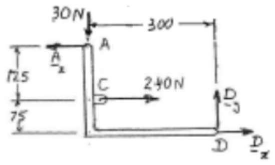
$$+\circlearrowleft \Sigma M_B = 0: \quad -(240\text{ N})(75\text{ mm}) + A_y(600\text{ mm}) = 0$$

$$A_y = -30.0\text{ N} \quad \mathbf{A}_y = 30.0\text{ N} \uparrow$$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x - A_x = 0 \quad B_x = A_x \quad (1)$$

$$+\uparrow \Sigma F_y = 0: \quad B_y - 30\text{ N} - 240\text{ N} = 0 \qquad \mathbf{B}_y = 270\text{ N} \uparrow$$ ◀

Free body: Member *ACD*:


30N
300
A
A_x
125
75
C
240N
D_y
D
D_x

$$\Sigma M_D = 0: \quad A_x(200) + (30\text{ N})(300) - (240\text{ N})(75) = 0$$

$$A_x = 45.0\text{ N}$$

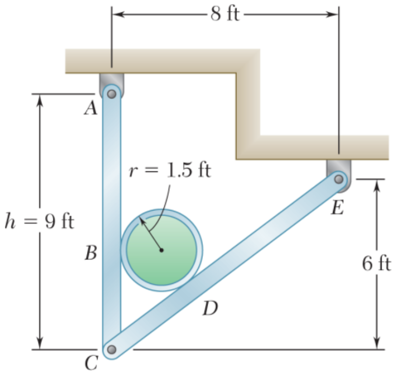
$$\mathbf{A}_x = 45.0\text{ N} \longleftarrow$$ ◀

$$\text{From Eq. (1):} \quad A_X = B_x = 45.0\text{ N} \qquad \mathbf{B}_x = 45.0\text{ N} \longleftarrow$$ ◀

Thus, reactions are

$$\mathbf{A}_x = 45.0\text{ N} \longleftarrow, \quad \mathbf{A}_y = 30.0\text{ N} \downarrow$$ ◀

$$\mathbf{B}_x = 45.0\text{ N} \longrightarrow, \quad \mathbf{B}_y = 270\text{ N} \uparrow$$ ◀



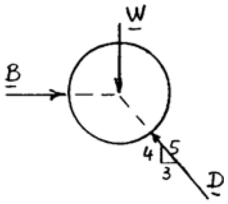
PROBLEM 6.93

A 3-ft-diameter pipe is supported every 16 ft by a small frame like that shown. Knowing that the combined weight of the pipe and its contents is 500 lb/ft and assuming frictionless surfaces, determine the components (*a*) of the reaction at *E*, (*b*) of the force exerted at *C* on member *CDE*.

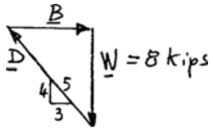
SOLUTION

Free body: 16-ft length of pipe:

$$W = (500\text{ lb/ft})(16\text{ ft}) = 8\text{ kips}$$



Force Triangle



$$\frac{B}{3} = \frac{D}{5} = \frac{8\text{ kips}}{4}$$

$$B = 6\text{ kips} \qquad D = 10\text{ kips}$$

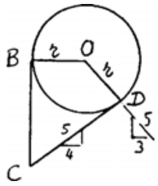
Determination of *CB* = *CD*.

We note that horizontal projection of $\overrightarrow{\mathbf{BO}} + \overrightarrow{\mathbf{OD}}$ = horizontal projection of $\overrightarrow{\mathbf{CD}}$

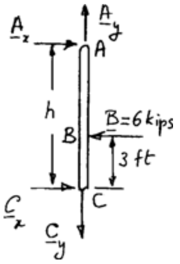
$$r + \frac{3}{5}r = \frac{4}{5}(CD)$$

Thus,

$$CB = CD = \frac{8}{4}r = 2(1.5\text{ ft}) = 3\text{ ft}$$



Free body: Member *ABC*:



$$+\circlearrowleft \Sigma M_A = 0: \quad C_x h - (6\text{ kips})(h-3)$$

$$C_x = \frac{h-3}{h}(6\text{ kips}) \qquad (1)$$

For $h = 9$ ft,

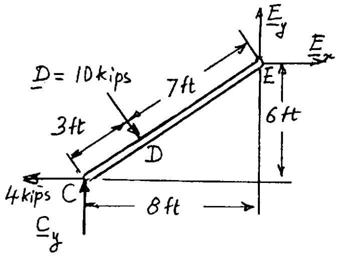
$$C_x = \frac{9-3}{9}(6\text{ kips}) = 4\text{ kips}$$

PROBLEM 6.93 (Continued)

Free body: Member *CDE*:

From above, we have

$$\mathbf{C}_x = 4.00 \text{ kips} \longleftarrow \blacktriangleleft$$



$$+\circlearrowleft \Sigma M_E = 0: \quad (10 \text{ kips})(7 \text{ ft}) - (4 \text{ kips})(6 \text{ ft}) - C_y(8 \text{ ft}) = 0$$

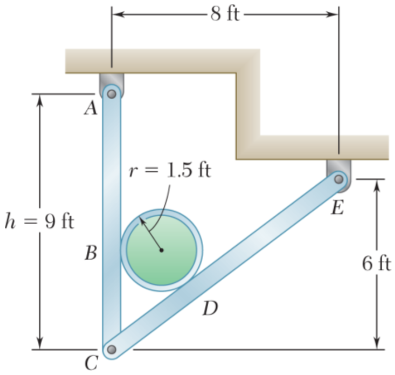
$$C_y = +5.75 \text{ kips}, \qquad \mathbf{C}_y = 5.75 \text{ kips} \uparrow \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -4 \text{ kips} + \frac{3}{5}(10 \text{ kips}) + E_x = 0$$

$$E_x = -2 \text{ kips}, \qquad \mathbf{E}_x = 2.00 \text{ kips} \longleftarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad 5.75 \text{ kips} - \frac{4}{5}(10 \text{ kips}) + E_y = 0,$$

$$\mathbf{E}_y = 2.25 \text{ kips} \uparrow \blacktriangleleft$$



PROBLEM 6.94

Solve Problem 6.93 for a frame where $h = 6$ ft.

PROBLEM 6.93 A 3-ft-diameter pipe is supported every 16 ft by a small frame like that shown. Knowing that the combined weight of the pipe and its contents is 500 lb/ft and assuming frictionless surfaces, determine the components (*a*) of the reaction at *E*, (*b*) of the force exerted at *C* on member *CDE*.

SOLUTION

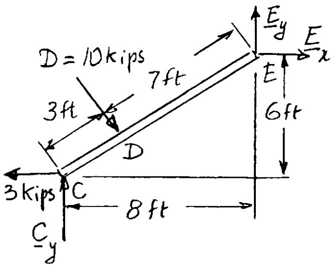
See solution of Problem 6.91 for derivation of Eq. (1).

For $h = 6$ ft, $\quad C_x = \dfrac{h-3}{h}(6 \text{ kips}) = \dfrac{6-3}{6} = 3 \text{ kips}$

Free body: Member *CDE*:

From above, we have

$$\mathbf{C}_x = 3.00 \text{ kips} \longleftarrow \blacktriangleleft$$



$$+\circlearrowleft \Sigma M_E = 0: \quad (10 \text{ kips})(7 \text{ ft}) - (3 \text{ kips})(6 \text{ ft}) - C_y(8 \text{ ft}) = 0$$

$$C_y = +6.50 \text{ kips}, \qquad \mathbf{C}_y = 6.50 \text{ kips} \uparrow \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -3 \text{ kips} + \frac{3}{5}(10 \text{ kips}) + E_x = 0$$

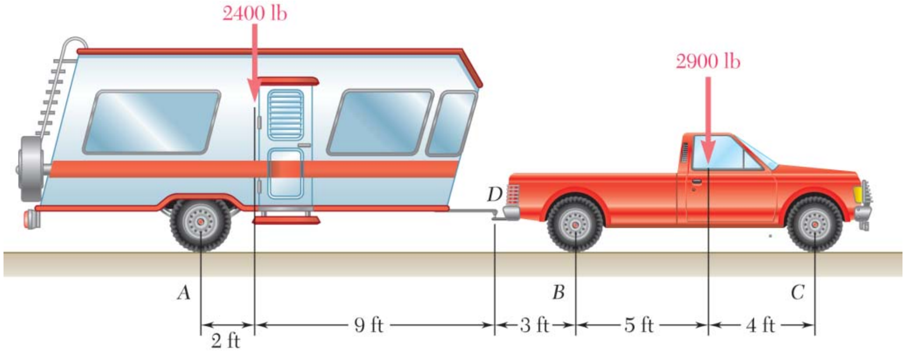
$$E_x = -3.00 \text{ kips}, \qquad \mathbf{E}_x = 3.00 \text{ kips} \longleftarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad 6.5 \text{ kips} - \frac{4}{5}(10 \text{ kips}) + E_y = 0$$

$$E_y = 1.500 \text{ kips} \qquad \mathbf{E}_y = 1.500 \text{ kips} \uparrow \blacktriangleleft$$

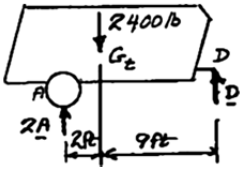
PROBLEM 6.95

A trailer weighing 2400 lb is attached to a 2900-lb pickup truck by a ball-and-socket truck hitch at *D*. Determine (*a*) the reactions at each of the six wheels when the truck and trailer are at rest, (*b*) the additional load on each of the truck wheels due to the trailer.



SOLUTION

(*a*) Free body: Trailer:



(We shall denote by *A*, *B*, *C* the reaction at one wheel.)

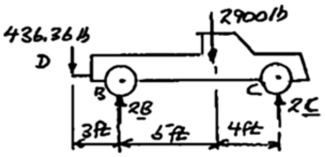
$$+\circlearrowleft \Sigma M_A = 0: \quad -(2400 \text{ lb})(2 \text{ ft}) + D(11 \text{ ft}) = 0$$

$$D = 436.36 \text{ lb}$$

$$+\uparrow \Sigma F_y = 0: \quad 2A - 2400 \text{ lb} + 436.36 \text{ lb} = 0$$

$A = 981.82 \text{ lb}$ $\quad\quad$ $\mathbf{A} = 982 \text{ lb}\uparrow$ ◀

Free body: Truck.



$$+\circlearrowleft \Sigma M_B = 0: \quad (436.36 \text{ lb})(3 \text{ ft}) - (2900 \text{ lb})(5 \text{ ft}) + 2C(9 \text{ ft}) = 0$$

$C = 732.83 \text{ lb}$ $\quad\quad$ $\mathbf{C} = 733 \text{ lb}\uparrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad 2B - 436.36 \text{ lb} - 2900 \text{ lb} + 2(732.83 \text{ lb}) = 0$$

$B = 935.35 \text{ lb}$ $\quad\quad$ $\mathbf{B} = 935 \text{ lb}\uparrow$ ◀

(*b*) Additional load on truck wheels.

Use free body diagram of truck without 2900 lb.

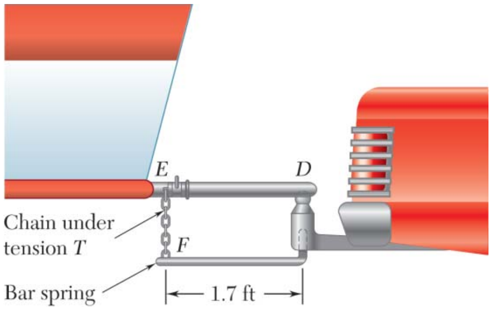
$$+\circlearrowleft \Sigma M_B = 0: \quad (436.36 \text{ lb})(3 \text{ ft}) + 2C(9 \text{ ft}) = 0$$

$C = -72.73 \text{ lb}$ $\quad\quad$ $\Delta C = -72.7 \text{ lb}$ ◀

$$+\uparrow \Sigma F_y = 0: \quad 2B - 436.36 \text{ lb} - 2(72.73 \text{ lb}) = 0$$

$B = 290.9 \text{ lb}$ $\quad\quad$ $\Delta B = +291 \text{ lb}$ ◀

PROBLEM 6.96



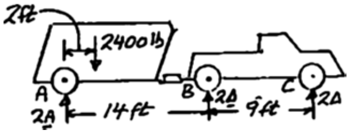
In order to obtain a better weight distribution over the four wheels of the pickup truck of Problem 6.95, a compensating hitch of the type shown is used to attach the trailer to the truck. The hitch consists of two bar springs (only one is shown in the figure) that fit into bearings inside a support rigidly attached to the truck. The springs are also connected by chains to the trailer frame, and specially designed hooks make it possible to place both chains in tension. (*a*) Determine the tension *T* required in each of the two chains if the additional load due to the trailer is to be evenly distributed over the four wheels of the truck. (*b*) What are the resulting reactions at each of the six wheels of the trailer-truck combination?

PROBLEM 6.95 A trailer weighing 2400 lb is attached to a 2900-lb pickup truck by a ball-and-socket truck hitch at *D*. Determine (*a*) the reactions at each of the six wheels when the truck and trailer are at rest, (*b*) the additional load on each of the truck wheels due to the trailer.

SOLUTION

(*a*) We small first find the additional reaction Δ at each wheel due to the trailer.

Free body diagram: (Same Δ at each truck wheel)



$$+\circlearrowleft \Sigma M_A = 0: \quad -(2400 \text{ lb})(2 \text{ ft}) + 2\Delta(14 \text{ ft}) + 2\Delta(23 \text{ ft}) = 0$$

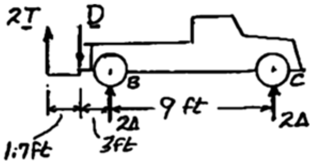
$$\Delta = 64.86 \text{ lb}$$

$$+\uparrow \Sigma F_y = 0: \quad 2A - 2400 \text{ lb} + 4(64.86 \text{ lb}) = 0;$$

$$A = 1070 \text{ lb}; \qquad \mathbf{A} = 1070 \text{ lb} \uparrow$$

Free body: Truck:

(Trailer loading only)



$$+\circlearrowleft \Sigma M_D = 0: \quad 2\Delta(12 \text{ ft}) + 2\Delta(3 \text{ ft}) - 2T(1.7 \text{ ft}) = 0$$

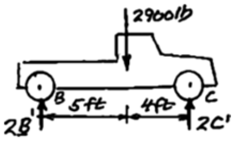
$$T = 8.824\Delta$$

$$= 8.824(64.86 \text{ lb})$$

$$T = 572.3 \text{ lb} \qquad T = 572 \text{ lb} \blacktriangleleft$$

Free body: Truck:

(Truck weight only)



$$+\circlearrowleft \Sigma M_B = 0: \quad -(2900 \text{ lb})(5 \text{ ft}) + 2C'(9 \text{ ft}) = 0$$

$$C' = 805.6 \text{ lb} \qquad \mathbf{C}' = 805.6 \text{ lb} \uparrow$$

PROBLEM 6.96 (Continued)

$$+\uparrow \Sigma F_y = 0: \quad 2B' - 2900\text{ lb} + 2(805.6\text{ lb}) = 0$$

$B' = 644.4\text{ lb}$ $\qquad$ $\mathbf{B}' = 644.4\text{ lb}\uparrow$

Actual reactions:

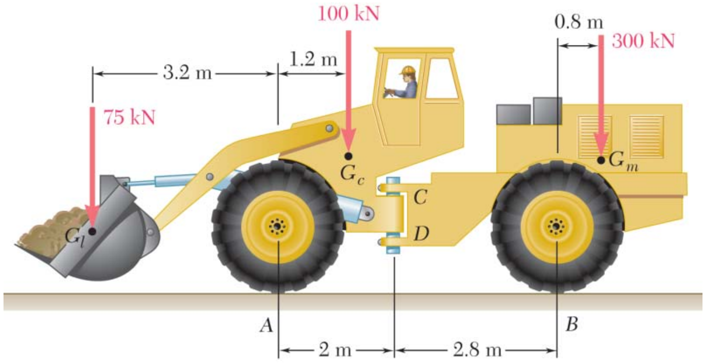
$B = B' + \Delta = 644.4\text{ lb} + 64.86 = 709.2\text{ lb}$ $\qquad$ $\mathbf{B} = 709\text{ lb}\uparrow$ ◀

$C = C' + \Delta = 805.6\text{ lb} + 64.86 = 870.46\text{ lb}$ $\qquad$ $\mathbf{C} = 870\text{ lb}\uparrow$ ◀

From part *a*: $\qquad$ $\mathbf{A} = 1070\text{ lb}\uparrow$ ◀

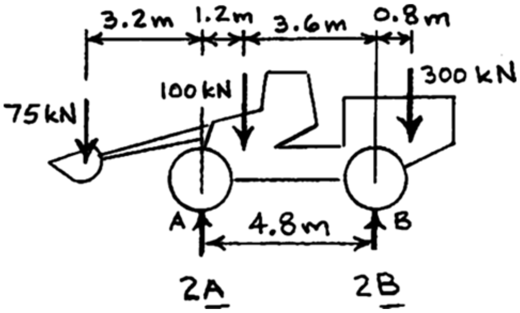
PROBLEM 6.97

The cab and motor units of the front-end loader shown are connected by a vertical pin located 2 m behind the cab wheels. The distance from *C* to *D* is 1 m. The center of gravity of the 300-kN motor unit is located at G_m, while the centers of gravity of the 100-kN cab and 75-kN load are located, respectively, at G_c and G_l. Knowing that the machine is at rest with its brakes released, determine (*a*) the reactions at each of the four wheels, (*b*) the forces exerted on the motor unit at *C* and *D*.


100 kN
0.8 m
300 kN
1.2 m
3.2 m
75 kN
G_c
G_m
G_l
C
D
A
B
2 m
2.8 m

SOLUTION

(*a*) Free body: Entire machine:


3.2m
1.2m
3.6m
0.8m
100 kN
300 kN
75 kN
A
B
4.8m
2A
2B

A = Reaction at each front wheel

B = Reaction at each rear wheel

$$+\circlearrowleft \Sigma M_A = 0: \quad 75(3.2 \text{ m}) - 100(1.2 \text{ m}) + 2B(4.8 \text{ m}) - 300(5.6 \text{ m}) = 0$$

$2B = 325 \text{ kN}$ **B** $= 162.5 \text{ kN}\uparrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad 2A + 325 - 75 - 100 - 300 = 0$$

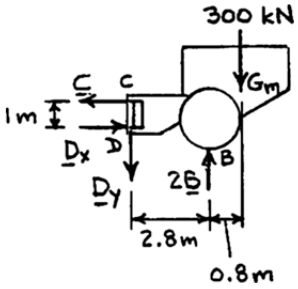
$2A = 150 \text{ kN}$ **A** $= 75.0 \text{ kN}\uparrow$ ◀

PROBLEM 6.97 (Continued)

(*b*) Free body: Motor unit:

$$+\circlearrowleft \Sigma M_D = 0: \quad C(1\text{ m}) + 2B(2.8\text{ m}) - 300(3.6\text{ m}) = 0$$

$$C = 1080 - 5.6B \qquad (1)$$



Recalling $\quad B = 162.5\text{ kN}, \quad C = 1080 - 5.6(162.5) = 170\text{ kN}$

$$\mathbf{C} = 170.0\text{ kN} \longleftarrow \blacktriangleleft$$

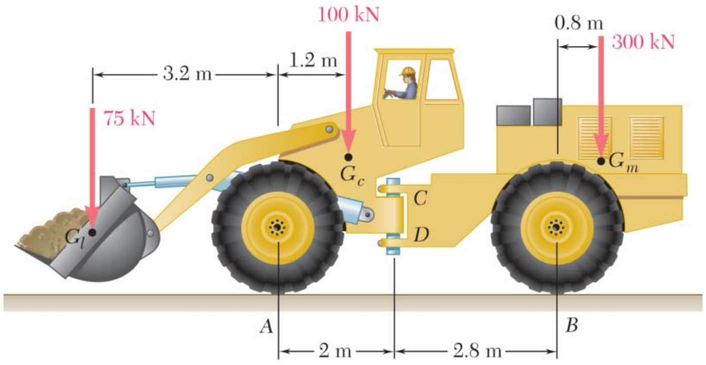
$$\xrightarrow{+} \Sigma F_x = 0: \quad D_x - 170 = 0 \qquad \mathbf{D}_x = 170.0\text{ kN} \longrightarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad 2(162.5) - D_y - 300 = 0 \qquad \mathbf{D}_y = 25.0\text{ kN}\downarrow \blacktriangleleft$$

PROBLEM 6.98

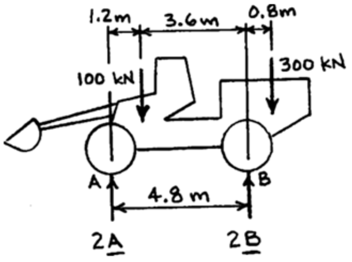
Solve Problem 6.97 assuming that the 75-kN load has been removed.

PROBLEM 6.97 The cab and motor units of the front-end loader shown are connected by a vertical pin located 2 m behind the cab wheels. The distance from *C* to *D* is 1 m. The center of gravity of the 300-kN motor unit is located at G_m, while the centers of gravity of the 100-kN cab and 75-kN load are located, respectively, at G_c and G_l. Knowing that the machine is at rest with its brakes released, determine (*a*) the reactions at each of the four wheels, (*b*) the forces exerted on the motor unit at *C* and *D*.



SOLUTION

(*a*) Free body: Entire machine:



A = Reaction at each front wheel

B = Reaction at each rear wheel

$$+\circlearrowleft \Sigma M_A = 0: \quad 2B(4.8 \text{ m}) - 100(1.2 \text{ m}) - 300(5.6 \text{ m}) = 0$$

$$2B = 375 \text{ kN} \qquad \mathbf{B} = 187.5 \text{ kN}\uparrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad 2A + 375 - 100 - 300 = 0$$

$$2A = 25 \text{ kN} \qquad \mathbf{A} = 12.50 \text{ kN}\uparrow \blacktriangleleft$$

(*b*) Free body: Motor unit:

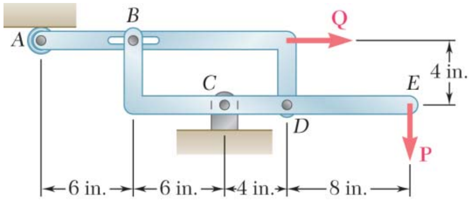
See solution of Problem 6.97 for free body diagram and derivation of Eq. (1). With $B = 187.5$ kN, we have

$$C = 1080 - 5.6(187.5) = 30 \text{ kN}$$

$$\mathbf{C} = 30.0 \text{ kN} \longleftarrow \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad D_x - 30 = 0 \qquad \mathbf{D}_x = 30.0 \text{ kN} \longrightarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad 2(187.5) - D_y - 300 = 0 \qquad \mathbf{D}_y = 75.0 \text{ kN}\downarrow \blacktriangleleft$$

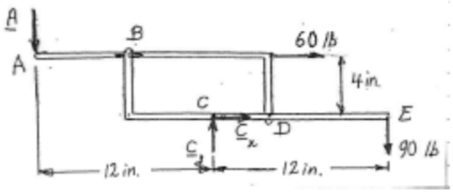

A
B
Q
C
E
4 in.
D
P
6 in.
6 in.
4 in.
8 in.

PROBLEM 6.99

Knowing that $P = 90$ lb and $Q = 60$ lb, determine the components of all forces acting on member *BCDE* of the assembly shown.

SOLUTION

Free body: Entire assembly:


A
A
B
60 lb
4 in
C
C_x
D
E
C_y
90 lb
12 in.
12 in.

$$+\circlearrowleft \Sigma M_C = 0: \quad A(12 \text{ in.}) - (60 \text{ lb})(4 \text{ in.}) - (90 \text{ lb})(12 \text{ in.}) = 0$$

$$A = +110.0 \text{ lb}$$

$$\mathbf{A} = 110.0 \text{ lb} \downarrow$$

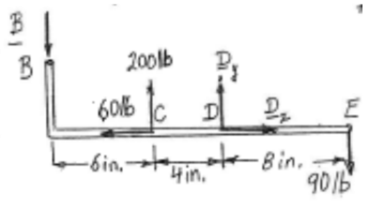
$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x + 60 \text{ lb} = 0$$

$$C_x = -60 \text{ lb} \qquad \mathbf{C}_x = 60.0 \text{ lb} \longleftarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad C_y - 110 \text{ lb} - 90 \text{ lb} = 0$$

$$C_y = 200 \text{ lb} \qquad \mathbf{C}_y = 200 \text{ lb} \uparrow \blacktriangleleft$$

Free body: Member *BCDE*:


B
B
200 lb
D_y
60 lb
C
D
D_x
E
6 in.
4 in.
8 in.
90 lb

$$+\circlearrowleft \Sigma M_D = 0: \quad B(10 \text{ in.}) - (200 \text{ lb})(4 \text{ in.}) - (90 \text{ lb})(8 \text{ in.}) = 0$$

$$B = +152.0 \text{ lb} \qquad \mathbf{B} = 152.0 \text{ lb} \downarrow \blacktriangleleft$$

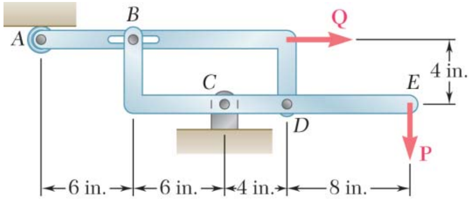
PROBLEM 6.99 (Continued)

$$\xrightarrow{+} \Sigma F_x = 0: \quad D_x - 60\text{ lb} = 0$$

$$D_x = 60.0\text{ lb} \qquad \mathbf{D} = 60.0\text{ lb} \longrightarrow \blacktriangleleft$$

$$+\!\uparrow \Sigma F_y = 0: \quad D_y + 200\text{ lb} - 152\text{ lb} - 90\text{ lb} = 0$$

$$D_y = +42.0\text{ lb} \qquad \mathbf{D}_y = 42.0\text{ lb} \uparrow \blacktriangleleft$$

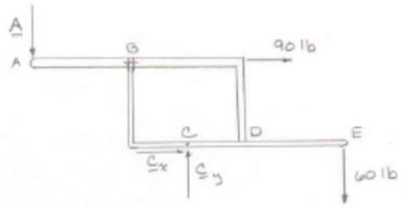

A
B
Q
C
E
4 in.
D
P
6 in.
6 in.
4 in.
8 in.

PROBLEM 6.100

Knowing that $P = 60$ lb and $Q = 90$ lb, determine the components of all forces acting on member *BCDE* of the assembly shown.

SOLUTION

Free body: Entire assembly:


A
A
B
90 lb
C
D
E
C_x
C_y
60 lb

$$+\circlearrowleft \Sigma M_C = 0: \quad A(12 \text{ in.}) - (60 \text{ lb})(12 \text{ in.}) - (90 \text{ lb})(4 \text{ in.}) = 0$$

$$A = +90.0 \text{ lb}$$

$$\mathbf{A} = 90.0 \text{ lb} \downarrow$$

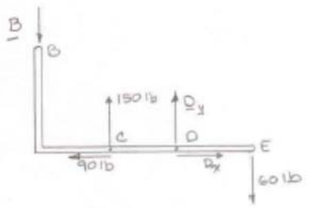
$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x + 90 \text{ lb} = 0$$

$$C_x = -90 \text{ lb} \qquad \mathbf{C}_x = 90.0 \text{ lb} \longleftarrow$$ ◀

$$+\uparrow \Sigma F_y = 0: \quad C_y - 60 \text{ lb} - 90 \text{ lb} = 0$$

$$C_y = 150 \text{ lb} \qquad \mathbf{C}_y = 150.0 \text{ lb} \uparrow$$ ◀

Free body: Member *BCDE*:


B
B
150 lb
D_y
C
D
E
90 lb
D_x
60 lb

PROBLEM 6.100 (Continued)

$$+\circlearrowleft \Sigma M_D = 0: \quad B(10\text{ in.}) - (150\text{ lb})(4\text{ in.}) - (60\text{ lb})(8\text{ in.}) = 0$$

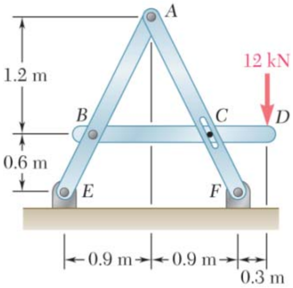
$$B = +108.0\text{ lb} \quad \mathbf{B} = 108.0\text{ lb} \downarrow \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad D_x - 90\text{ lb} = 0$$

$$D_x = 90.0\text{ lb} \qquad \mathbf{D} = 90.0\text{ lb} \longrightarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad D_y + 150\text{ lb} - 108\text{ lb} - 60\text{ lb} = 0$$

$$D_y = +18.00\text{ lb} \qquad \mathbf{D}_y = 18.00\text{ lb} \uparrow \blacktriangleleft$$

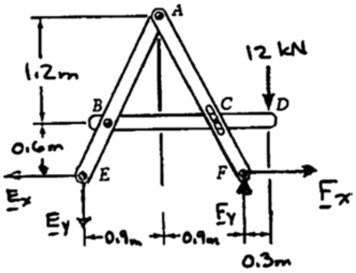

A
1.2 m
12 kN
B
C
D
0.6 m
E
F
0.9 m
0.9 m
0.3 m

PROBLEM 6.101

For the frame and loading shown, determine the components of all forces acting on member *ABE*.

SOLUTION

FBD Frame:


A
1.2m
12 kN
B
C
D
0.6m
E
F
Fx
Ex
Ey
Fy
0.9m
0.9m
0.3m

$$\circlearrowleft + \Sigma M_E = 0\colon \quad (1.8\text{ m})F_y - (2.1\text{ m})(12\text{ kN}) = 0$$

$$\mathbf{F}_y = 14.00\text{ kN}\uparrow$$

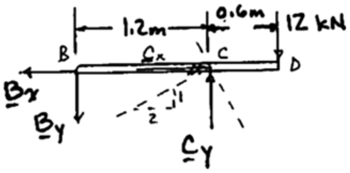
$$+\!\uparrow \Sigma F_y = 0\colon \quad -E_y + 14.00\text{ kN} - 12\text{ kN} = 0$$

$$E_y = 2\text{ kN}\downarrow$$

$$\mathbf{E}_y = 2.00\text{ kN}\downarrow$$ ◀

FBD member *BCD*:

$$\circlearrowleft + \Sigma M_B = 0\colon \quad (1.2\text{ m})C_y - (12\text{ kN})(1.8\text{ m}) = 0 \quad C_y = 18.00\text{ kN}\uparrow$$


1.2m
0.6m
12 kN
B
Cx
C
D
Bx
By
2
1
Cy

But **C** is $\perp$ *ACF*, so $C_x = 2C_y$; $C_x = 36.0\text{ kN}\longrightarrow$

$$\overset{+}{\longrightarrow} \Sigma F_x = 0\colon \quad -B_x + C_x = 0 \quad B_x = C_x = 36.0\text{ kN}$$

$$B_x = 36.0\text{ kN}\longleftarrow \text{ on } BCD$$

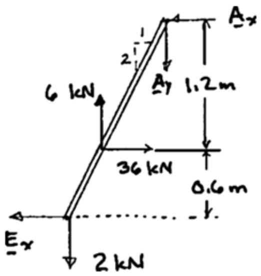
$$+\!\uparrow \Sigma F_y = 0\colon \quad -B_y + 18.00\text{ kN} - 12\text{ kN} = 0 \quad B_y = 6.00\text{ kN}\downarrow \text{ on } BCD$$

On *ABE*:

$$\mathbf{B}_x = 36.0\text{ kN}\longrightarrow$$ ◀

$$\mathbf{B}_y = 6.00\text{ kN}\uparrow$$ ◀

FBD member *ABE*:

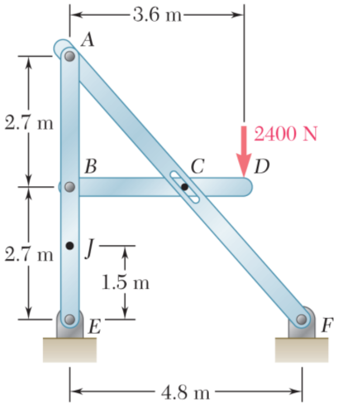

Ax
1
2
Ay
1.2m
6 kN
36 kN
0.6 m
Ex
2 kN

$$\circlearrowleft + \Sigma M_A = 0\colon \quad (1.2\text{ m})(36.0\text{ kN}) - (0.6\text{ m})(6.00\text{ kN}) + (0.9\text{ m})(2.00\text{ kN}) - (1.8\text{ m})(E_x) = 0$$

$$\mathbf{E}_x = 23.0\text{ kN}\longleftarrow$$ ◀

$$\overset{+}{\longrightarrow} \Sigma F_x = 0\colon \quad -23.0\text{ kN} + 36.0\text{ kN} - A_x = 0 \qquad \mathbf{A}_x = 13.00\text{ kN}\longleftarrow$$ ◀

$$+\!\uparrow \Sigma F_y = 0\colon \quad -2.00\text{ kN} + 6.00\text{ kN} - A_y = 0 \qquad \mathbf{A}_y = 4.00\text{ kN}\downarrow$$ ◀

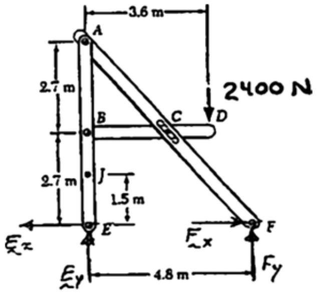

3.6 m
A
2.7 m
2400 N
B
C
D
2.7 m
J
1.5 m
E
F
4.8 m

PROBLEM 6.102

For the frame and loading shown, determine the components of all forces acting on member *ABE*.

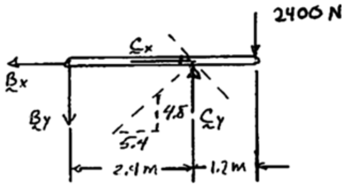
SOLUTION

FBD Frame:


3.6 m
A
2.7 m
2400 N
B
C
D
2.7 m
J
1.5 m
E_x
E
F_x
F
E_y
4.8 m
F_y

$$\circlearrowleft + \Sigma M_F = 0: \quad (1.2 \text{ m})(2400 \text{ N}) - (4.8 \text{ m})E_y = 0 \qquad \mathbf{E}_y = 600 \text{ N}\uparrow \blacktriangleleft$$

FBD member *BC*:


2400 N
C_x
B_x
B_y
4.8
C_y
5.4
2.4 m
1.2 m

$$C_y = \frac{4.8}{5.4}C_x = \frac{8}{9}C_x$$

$$\circlearrowleft + \Sigma M_C = 0: \quad (2.4 \text{ m})B_y - (1.2 \text{ m})(2400 \text{ N}) = 0 \qquad B_y = 1200 \text{ N}\downarrow$$

On *ABE*: $\mathbf{B}_y = 1200 \text{ N}\uparrow \blacktriangleleft$

$$+\uparrow \Sigma F_y = 0: \quad -1200 \text{ N} + C_y - 2400 \text{ N} = 0 \qquad C_y = 3600 \text{ N}\uparrow$$

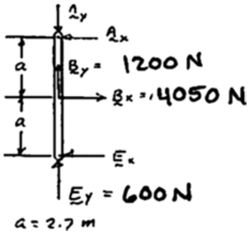
so $$C_x = \frac{9}{8}C_y \qquad C_x = 4050 \text{ N}\longrightarrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -B_x + C_x = 0 \qquad B_x = 4050 \text{ N}\longleftarrow \text{ on } BC$$

On *ABE*: $\mathbf{B}_x = 4050 \text{ N}\longrightarrow \blacktriangleleft$

PROBLEM 6.102 (Continued)

FBD member *AB0E*:

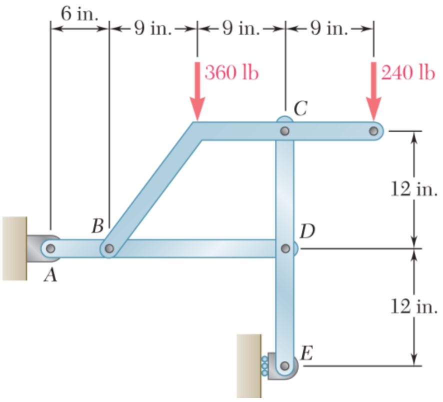


$$\circlearrowright\!+\Sigma M_A = 0: \quad a(4050\text{ N}) - 2aE_x = 0$$

$$E_x = 2025\text{ N} \qquad \mathbf{E}_x = 2025\text{ N} \longleftarrow \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -A_x + (4050 - 2025)\text{ N} = 0 \qquad \mathbf{A}_x = 2025\text{ N} \longleftarrow \blacktriangleleft$$

$$+\!\uparrow \Sigma F_y = 0: \quad 600\text{ N} + 1200\text{ N} - A_y = 0 \qquad \mathbf{A}_y = 1800\text{ N}\downarrow \blacktriangleleft$$

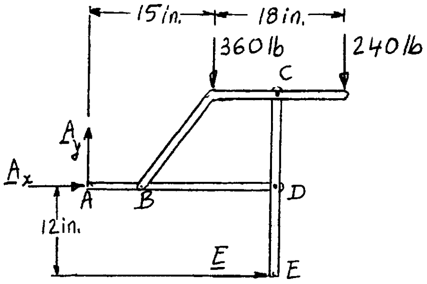

6 in.
9 in.
9 in.
9 in.
360 lb
240 lb
C
12 in.
B
D
A
12 in.
E

PROBLEM 6.103

For the frame and loading shown, determine the components of all forces acting on member *ABD*.

SOLUTION

Free body: Entire frame:


15 in.
18 in.
360 lb
240 lb
C
A_y
A_x
A
B
D
12 in.
E
E

$+\circlearrowleft \Sigma M_A = 0\!: \quad E(12\text{ in.}) - (360\text{ lb})(15\text{ in.}) - (240\text{ lb})(33\text{ in.}) = 0$

$E = +1110\text{ lb}$ $\qquad$ $\mathbf{E} = +1110\text{ lb} \longrightarrow$

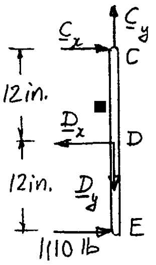
$\overset{+}{\longrightarrow} \Sigma F_x = 0\!: \quad A_x + 1110\text{ lb} = 0$

$A_x = -1110\text{ lb}$ $\qquad$ $\mathbf{A}_x = 1110\text{ lb} \longleftarrow$ ◀

$+\uparrow \Sigma F_y = 0\!: \quad A_y - 360\text{ lb} - 240\text{ lb} = 0$

$A_y = +600\text{ lb}$ $\qquad$ $\mathbf{A}_y = 600\text{ lb}\uparrow$ ◀

Free body: Member *CDE*:


C_x
C_y
C
12 in.
D_x
D
12 in.
D_y
E
1110 lb

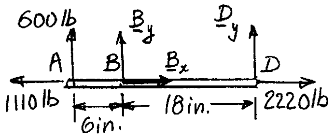
$+\circlearrowleft \Sigma M_C = 0\!: \quad (1110\text{ lb})(24\text{ in.}) - D_x(12\text{ in.}) = 0$

$D_x = +2220\text{ lb}$

PROBLEM 6.103 (Continued)

Free body: Member *ABD*:

From above: $\mathbf{D}_x = 2220 \text{ lb} \longrightarrow$ ◀



$$+\circlearrowleft \Sigma M_B = 0\text{:} \quad D_y(18 \text{ in.}) - (600 \text{ lb})(6 \text{ in.}) = 0$$

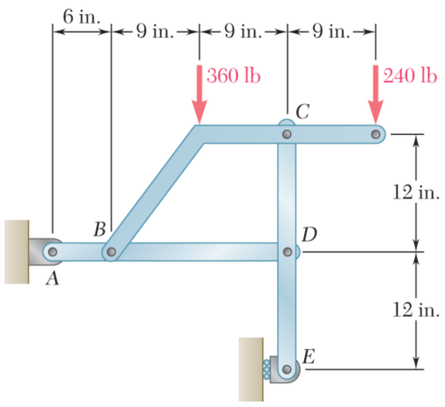
$$D_y = +200 \text{ lb} \quad \mathbf{D}_y = 200 \text{ lb} \uparrow$$ ◀

$$\xrightarrow{+} \Sigma F_x = 0\text{:} \quad B_x + 2220 \text{ lb} - 1110 \text{ lb} = 0$$

$$B_x = -1110 \text{ lb} \quad \mathbf{B}_x = 1110 \text{ lb} \longleftarrow$$ ◀

$$+\uparrow \Sigma F_y = 0\text{:} \quad B_y + 200 \text{ lb} + 600 \text{ lb} = 0$$

$$B_y = -800 \text{ lb} \qquad \mathbf{B}_y = 800 \text{ lb} \downarrow$$ ◀


6 in.
9 in.
9 in.
9 in.
360 lb
240 lb
C
12 in.
B
D
A
12 in.
E

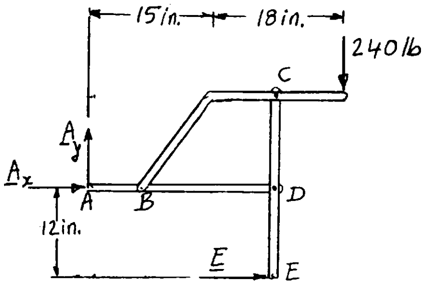
PROBLEM 6.104

Solve Problem 6.103 assuming that the 360-lb load has been removed.

PROBLEM 6.103 For the frame and loading shown, determine the components of all forces acting on member *ABD*.

SOLUTION

Free body diagram of entire frame.


15 in.
18 in.
240 lb
C
A_y
A_x
A
B
D
12 in.
E
E

$$+\circlearrowleft \Sigma M_A = 0: \quad E(12\text{ in.}) - (240\text{ lb})(33\text{ in.}) = 0$$

$E = +660$ lb $\qquad \mathbf{E} = 660$ lb →

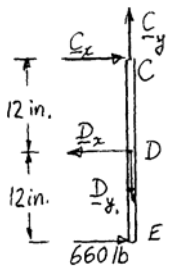
$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + 660\text{ lb} = 0$$

$A_x = -660$ lb $\qquad \mathbf{A}_x = 660$ lb ← ◀

$$+\uparrow \Sigma F_y = 0: \quad A_y - 240\text{ lb} = 0$$

$A_y = +240$ lb $\qquad \mathbf{A}_y = 240$ lb ↑ ◀

Free body: Member *CDE*:


C_x
C_y
C
12 in.
D_x
D
12 in.
D_y
E
660 lb

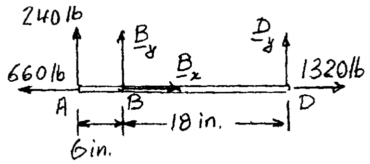
$$+\circlearrowleft \Sigma M_C = 0: \quad (660\text{ lb})(24\text{ in.}) - D_x(12\text{ in.}) = 0$$

$D_x = +1320$ lb

PROBLEM 6.104 (Continued)

Free body: Member *ABD*:

From above: $\mathbf{D}_x = 1320 \text{ lb} \longrightarrow$ ◀



$$+\circlearrowleft \Sigma M_B = 0: \quad D_y(18 \text{ in.}) - (240 \text{ lb})(6 \text{ in.}) = 0$$

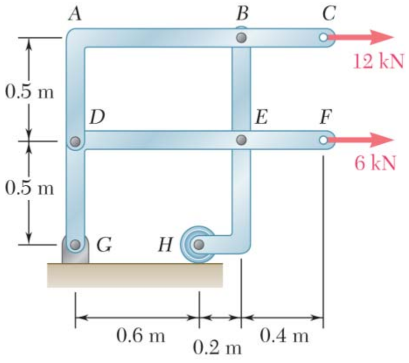
$D_y = +80 \text{ lb}$ $\quad \mathbf{D}_y = 80.0 \text{ lb} \uparrow$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x + 1320 \text{ lb} - 660 \text{ lb} = 0$$

$B_x = -660 \text{ lb}$ $\quad \mathbf{B}_x = 660 \text{ lb} \longleftarrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad B_y + 80 \text{ lb} + 240 \text{ lb} = 0$$

$B_y = -320 \text{ lb}$ $\quad \mathbf{B}_y = 320 \text{ lb} \downarrow$ ◀

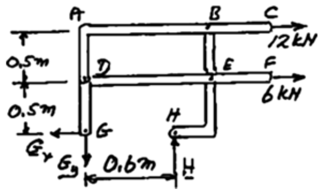

A
B
C
12 kN
0.5 m
D
E
F
6 kN
0.5 m
G
H
0.6 m
0.2 m
0.4 m

PROBLEM 6.105

For the frame and loading shown, determine the components of the forces acting on member *DABC* at *B* and *D*.

SOLUTION

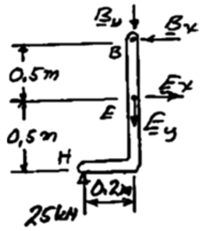
Free body: Entire frame:


A
B
C
12 kN
0.5 m
D
E
F
6 kN
0.5 m
H
G_x
G
G_y
0.6 m
H

$$+\circlearrowleft \Sigma M_G = 0: \quad H(0.6\text{ m}) - (12\text{ kN})(1\text{ m}) - (6\text{ kN})(0.5\text{ m}) = 0$$

$$H = 25\text{ kN} \quad \mathbf{H} = 25\text{ kN}\uparrow$$

Free body: Member *BEH*:

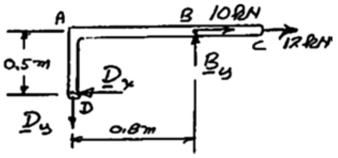

B_y
B_x
B
0.5 m
E_x
E
E_y
0.5 m
H
0.2 m
25 kN

$$+\circlearrowleft \Sigma M_F = 0: \quad B_x(0.5\text{ m}) - (25\text{ kN})(0.2\text{ m}) = 0$$

$$B_x = +10\text{ kN}$$

Free body: Member *DABC*:

From above: $\mathbf{B}_x = 10.00\text{ kN} \longrightarrow$ ◀


A
B
10 kN
C
12 kN
0.5 m
D_x
B_y
D
D_y
0.8 m

$$+\circlearrowleft \Sigma M_D = 0: \quad -B_y(0.8\text{ m}) + (10\text{ kN} + 12\text{ kN})(0.5\text{ m}) = 0$$

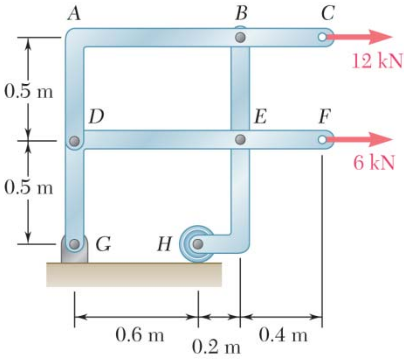
$$B_y = +13.75\text{ kN} \qquad \mathbf{B}_y = 13.75\text{ kN}\uparrow$$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad -D_x + 10\text{ kN} + 12\text{ kN} = 0$$

$$D_x = +22\text{ kN} \qquad \mathbf{D}_x = 22.0\text{ kN} \longleftarrow$$ ◀

$$+\uparrow \Sigma F_y = 0: \quad -D_y + 13.75\text{ kN} = 0$$

$$D_y = +13.75\text{ kN} \qquad \mathbf{D}_y = 13.75\text{ kN}\downarrow$$ ◀


A
B
C
12 kN
0.5 m
D
E
F
6 kN
0.5 m
G
H
0.6 m
0.2 m
0.4 m

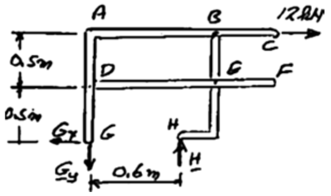
PROBLEM 6.106

Solve Problem 6.105 assuming that the 6-kN load has been removed.

PROBLEM 6.105 For the frame and loading shown, determine the components of the forces acting on member *DABC* at *B* and *D*.

SOLUTION

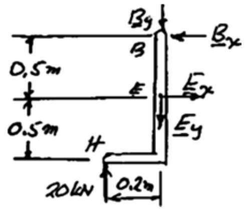
Free body: Entire frame


A
B
12 kN
C
0.5m
D
E
F
0.5m
Gx
G
H
Gy
0.6m
H

$$+\circlearrowleft \Sigma M_G = 0: \quad H(0.6\text{ m}) - (12\text{ kN})(1\text{ m}) = 0$$

$$H = 20\text{ kN} \qquad \mathbf{H} = 20\text{ kN}\uparrow$$

Free body: Member *BEH*

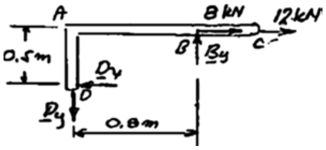

By
Bx
B
0.5m
E
Ex
Ey
0.5m
H
20kN
0.2m

$$+\circlearrowleft \Sigma M_E = 0: \quad B_x(0.5\text{ m}) - (20\text{ kN})(0.2\text{ m}) = 0$$

$$B_x = +8\text{ kN}$$

Free body: Member *DABC*

From above: $\mathbf{B}_x = -8.00\text{ kN} \longrightarrow$ ◀


A
8 kN
12kN
0.5m
B
By
C
Dx
D
Dy
0.8m

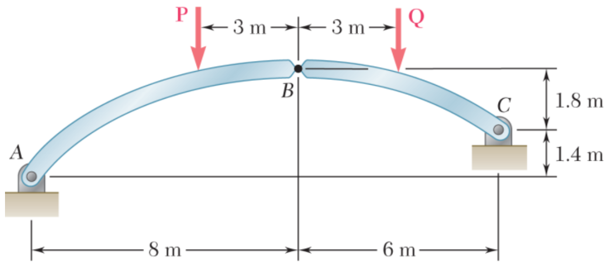
$$+\circlearrowleft \Sigma M_D = 0: \quad -B_y(0.8\text{ m}) + (8\text{ kN} + 12\text{ kN})(0.5\text{ m}) = 0$$

$$B_y = +12.5\text{ kN} \qquad \mathbf{B}_y = 12.50\text{ kN}\uparrow \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -D_x + 8\text{ kN} + 12\text{ kN} = 0$$

$$D_x = +20\text{ kN} \qquad \mathbf{D}_x = 20.0\text{ kN} \longleftarrow \blacktriangleleft$$

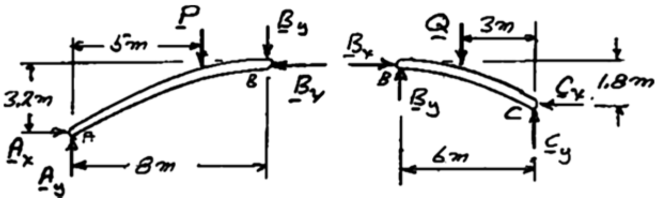
$$+\uparrow \Sigma F_y = 0: \quad -D_y + 12.5\text{ kN} = 0; \quad D_y = +12.5\text{ kN} \qquad \mathbf{D}_y = 12.50\text{ kN}\downarrow \blacktriangleleft$$



PROBLEM 6.107

The axis of the three-hinge arch *ABC* is a parabola with the vertex at *B*. Knowing that *P* = 112 kN and *Q* = 140 kN, determine (*a*) the components of the reaction at *A*, (*b*) the components of the force exerted at *B* on segment *AB*.

SOLUTION



Free body: Segment *AB*:

$$+\circlearrowleft \Sigma M_A = 0: \quad B_x(3.2\text{ m}) - B_y(8\text{ m}) - P(5\text{ m}) = 0 \tag{1}$$

0.75 (Eq. 1): $$B_x(2.4\text{ m}) - B_y(6\text{ m}) - P(3.75\text{ m}) = 0 \tag{2}$$

Free body: Segment *BC*:

$$+\circlearrowleft \Sigma M_C = 0: \quad B_x(1.8\text{ m}) + B_y(6\text{ m}) - Q(3\text{ m}) = 0 \tag{3}$$

Add Eqs. (2) and (3): $$4.2B_x - 3.75P - 3Q = 0$$

$$B_x = (3.75P + 3Q)/4.2 \tag{4}$$

From Eq. (1): $$(3.75P + 3Q)\frac{3.2}{4.2} - 8B_y - 5P = 0$$

$$B_y = (-9P + 9.6Q)/33.6 \tag{5}$$

given that $P = 112$ kN and $Q = 140$ kN.

(*a*) Reaction at *A*.

Considering again *AB* as a free body,

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - B_x = 0; \quad A_x = B_x = 200\text{ kN} \qquad \mathbf{A}_x = 200\text{ kN} \longrightarrow ◀$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - P - B_y = 0$$

$$A_y - 112\text{ kN} - 10\text{ kN} = 0$$

$$A_y = +122\text{ kN} \qquad \mathbf{A}_y = 122.0\text{ kN} \uparrow ◀$$

PROBLEM 6.107 (Continued)

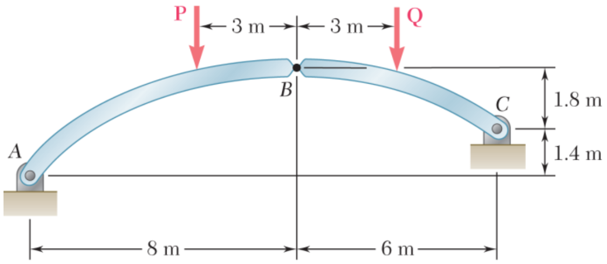
(*b*) Force exerted at *B* on *AB*.

From Eq. (4): $B_x = (3.75 \times 112 + 3 \times 140)/4.2 = 200 \text{ kN}$

$$\mathbf{B}_x = 200 \text{ kN} \longleftarrow \blacktriangleleft$$

From Eq. (5): $B_y = (-9 \times 112 + 9.6 \times 140)/33.6 = +10 \text{ kN}$

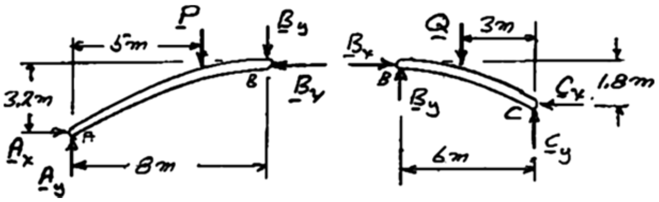
$$\mathbf{B}_y = 10.00 \text{ kN} \downarrow \blacktriangleleft$$



PROBLEM 6.108

The axis of the three-hinge arch *ABC* is a parabola with the vertex at *B*. Knowing that $P = 140$ kN and $Q = 112$ kN, determine (*a*) the components of the reaction at *A*, (*b*) the components of the force exerted at *B* on segment *AB*.

SOLUTION



Free body: Segment *AB*:

$$+\circlearrowleft \Sigma M_A = 0: \quad B_x(3.2\text{ m}) - B_y(8\text{ m}) - P(5\text{ m}) = 0 \qquad (1)$$

0.75 (Eq. 1):

$$B_x(2.4\text{ m}) - B_y(6\text{ m}) - P(3.75\text{ m}) = 0 \qquad (2)$$

Free body: Segment *BC*:

$$+\circlearrowleft \Sigma M_C = 0: \quad B_x(1.8\text{ m}) + B_y(6\text{ m}) - Q(3\text{ m}) = 0 \qquad (3)$$

Add Eqs. (2) and (3):

$$4.2B_x - 3.75P - 3Q = 0$$

$$B_x = (3.75P + 3Q)/4.2 \qquad (4)$$

From Eq. (1):

$$(3.75P + 3Q)\frac{3.2}{4.2} - 8B_y - 5P = 0$$

$$B_y = (-9P + 9.6Q)/33.6 \qquad (5)$$

given that $P = 140$ kN and $Q = 112$ kN.

(*a*) Reaction at *A*.

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - B_x = 0; \quad A_x = B_x = 205\text{ kN} \qquad \mathbf{A}_x = 205\text{ kN} \longrightarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - P - B_y = 0$$

$$A_y - 140\text{ kN} - (-5.5\text{ kN}) = 0$$

$$A_y = 134.5\text{ kN} \qquad \mathbf{A}_y = 134.5\text{ kN}\uparrow \blacktriangleleft$$

PROBLEM 6.108 (Continued)

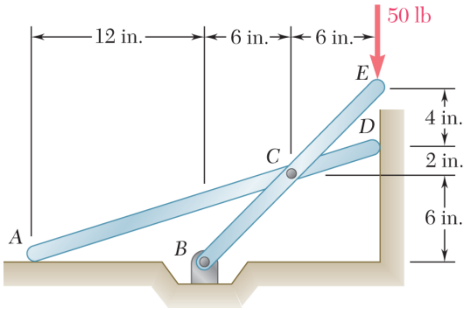
(*b*) Force exerted at *B* on *AB*.

From Eq. (4): $B_x = (3.75 \times 140 + 3 \times 112)/4.2 = 205 \text{ kN}$

$$\mathbf{B}_x = 205 \text{ kN} \longleftarrow \blacktriangleleft$$

From Eq. (5): $B_y = (-9 \times 140 + 9.6 \times 112)/33.6 = -5.5 \text{ kN}$

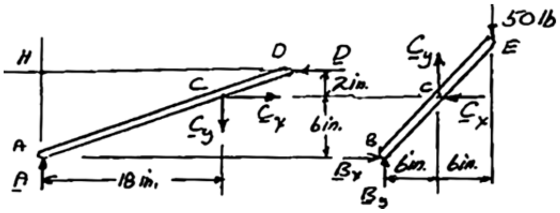
$$\mathbf{B}_y = 5.50 \text{ kN} \uparrow \blacktriangleleft$$


12 in.
6 in.
6 in.
50 lb
E
D
C
A
B
4 in.
2 in.
6 in.

PROBLEM 6.109

Neglecting the effect of friction at the horizontal and vertical surfaces, determine the forces exerted at *B* and *C* on member *BCE*.

SOLUTION


H
D
D
2 in.
C
C_y
C_x
6 in.
A
A
18 in.
B
B_x
B_y
6 in.
6 in.
C_y
C
C_x
50 lb
E

Free body of Member *ACD*

$$+\circlearrowleft \Sigma M_H = 0: \quad C_x(2\text{ in.}) - C_y(18\text{ in.}) = 0 \quad C_x = 9C_y \qquad (1)$$

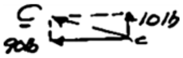
Free body of Member *BCE*

$$+\circlearrowleft \Sigma M_B = 0: \quad C_x(6\text{ in.}) + C_y(6\text{ in.}) - (50\text{ lb})(12\text{ in.}) = 0$$

Substitute from (1):

$$9C_y(6) + C_y(6) - 600 = 0$$

$$C_y = +10\text{ lb}; \quad C_x = 9C_y = 9(10) = +90\text{ lb}$$

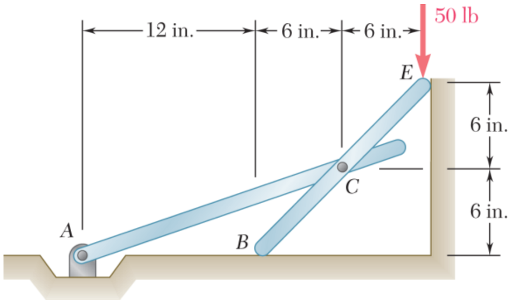

C
90 lb
10 lb
C

C = 90.6 lb ⦣ 6.34° ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x - 90\text{ lb} = 0 \qquad B_x = 90\text{ lb}$$

$$+\uparrow \Sigma F_y = 0: \quad B_y + 10\text{ lb} - 50\text{ lb} = 0 \qquad B_y = 40\text{ lb}$$

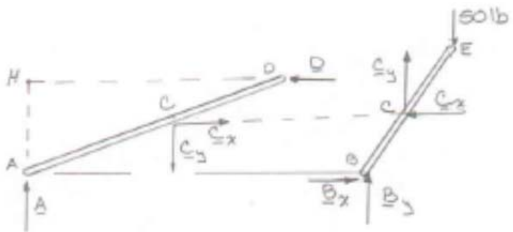
B = 98.5 lb ∠ 24.0° ◀


12 in.
6 in.
6 in.
50 lb
E
6 in.
6 in.
C
A
B

PROBLEM 6.110

Neglecting the effect of friction at the horizontal and vertical surfaces, determine the forces exerted at *B* and *C* on member *BCE*.

SOLUTION


50 lb
E
H
D
D
C
C_y
C_x
C
C_y
C_x
A
A
B
B_x
B_y

Free body of Member *ACD*

We note that *AC* is a two-force member.

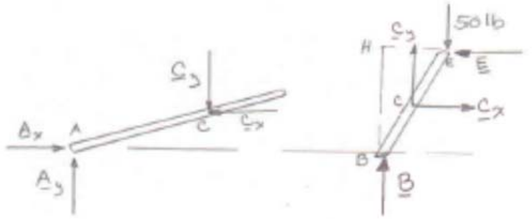
$$\frac{C_x}{18} = \frac{C_y}{6} \quad \text{or} \quad C_x = 3C_y \quad (1)$$

On free body of Member *BCE*

$$+\circlearrowleft \Sigma M_H = 0: \quad C_x(6\text{ in.}) + C_y(6\text{ in.}) - (50.0\text{ lb})(12\text{ in.}) = 0$$

Substitute from (1):

$$3C_y(6) + C_y(6) - 600 = 0$$

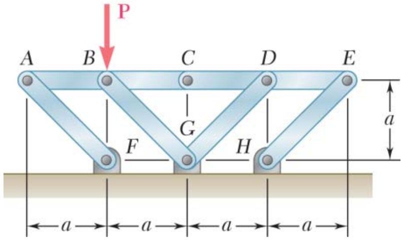

C_y
A_x
A
C
C_x
A_y
C_y
50 lb
H
E
C
C_x
B
B

$$C_y = +25.0\text{ lb}; \quad C_x = 3C_y = 3(25.0) = +75.0\text{ lb}$$

C = 79.1 lb ⦪ 18.43° ◀

$$+\uparrow \Sigma F_y = 0: \quad B + 25.0\text{ lb} - 50.0\text{ lb} = 0 \qquad B = 25.0\text{ lb}$$

B = 25.0 lb ↑ ◀

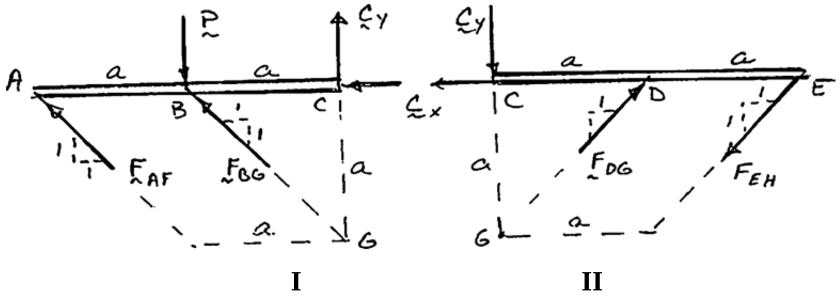


PROBLEM 6.111

Members *ABC* and *CDE* are pin-connected at *C* and supported by four links. For the loading shown, determine the force in each link.

SOLUTION

Member FBDs:



FBD I: $\circlearrowleft \Sigma M_B = 0: \quad aC_y - a\frac{1}{\sqrt{2}}F_{AF} = 0 \quad F_{AF} = \sqrt{2}C_y$

FBD II: $\circlearrowleft M_D = 0: \quad aC_y - a\frac{1}{\sqrt{2}}F_{EH} = 0 \quad F_{EH} = \sqrt{2}C_y$

FBDs combined: $\circlearrowleft \Sigma M_G = 0: \quad aP - a\frac{1}{\sqrt{2}}F_{AF} - a\frac{1}{\sqrt{2}}F_{EH} = 0 \quad P = \frac{1}{\sqrt{2}}\sqrt{2}C_y + \frac{1}{\sqrt{2}}\sqrt{2}C_y$

$$C_y = \frac{P}{2} \qquad \text{so } F_{AF} = \frac{\sqrt{2}}{2}P \quad C \blacktriangleleft$$

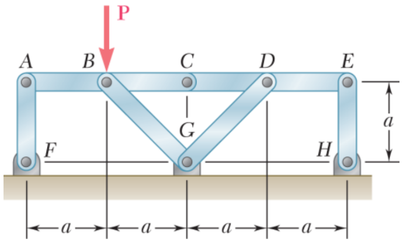
$$F_{EH} = \frac{\sqrt{2}}{2}P \quad T \blacktriangleleft$$

FBD I: $\uparrow \Sigma F_y = 0: \quad \frac{1}{\sqrt{2}}F_{AF} + \frac{1}{\sqrt{2}}F_{BG} - P + C_y = 0 \quad \frac{P}{2} + \frac{1}{\sqrt{2}}F_{BG} - P + \frac{P}{2} = 0$

$$F_{BG} = 0 \blacktriangleleft$$

FBD II: $\uparrow \Sigma F_y = 0: \quad -C_y + \frac{1}{\sqrt{2}}F_{DG} - \frac{1}{\sqrt{2}}F_{EH} = 0 \quad -\frac{P}{2} + \frac{1}{\sqrt{2}}F_{DG} - \frac{P}{2} = 0$

$$F_{DG} = \sqrt{2}P \quad C \blacktriangleleft$$

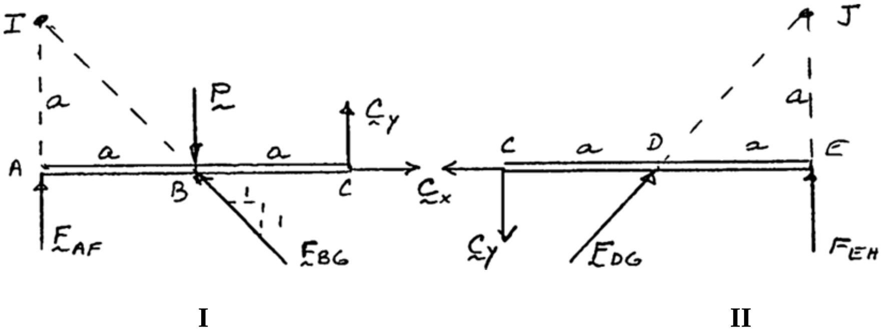


PROBLEM 6.112

Members *ABC* and *CDE* are pin-connected at *C* and supported by four links. For the loading shown, determine the force in each link.

SOLUTION

Member FBDs:



FBD I: $\circlearrowleft \Sigma M_I = 0{:}\quad 2aC_y + aC_x - aP = 0 \quad 2C_y + C_x = P$

FBD II: $\circlearrowleft \Sigma M_J = 0{:}\quad 2aC_y - aC_x = 0 \quad 2C_y - C_x = 0$

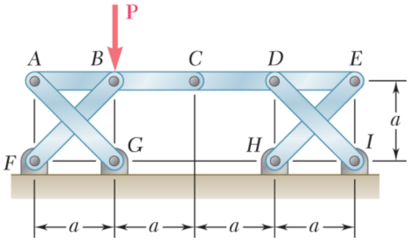
Solving, $C_x = \dfrac{P}{2};\quad C_y = \dfrac{P}{4}$ as shown.

FBD I: $\xrightarrow{} \Sigma F_x = 0{:}\quad -\dfrac{1}{\sqrt{2}}F_{BG} + C_x = 0 \quad F_{BG} = C_x\sqrt{2}$ $\qquad F_{BG} = \dfrac{P}{\sqrt{2}}\ C$ ◀

$\uparrow \Sigma F_y = 0{:}\quad F_{AF} - P + \dfrac{1}{\sqrt{2}}\left(\dfrac{\sqrt{2}}{2}P\right) + \dfrac{P}{4} = 0$ $\qquad F_{AF} = \dfrac{P}{4}\ C$ ◀

FBD II: $\xrightarrow{} \Sigma F_x = 0{:}\quad -C_x + \dfrac{1}{\sqrt{2}}F_{DG} = 0 \quad F_{DG} = C_x\sqrt{2}$ $\qquad F_{DG} = \dfrac{P}{\sqrt{2}}\ C$ ◀

$\uparrow \Sigma F_y = 0{:}\quad -C_y + \dfrac{1}{\sqrt{2}}\left(\dfrac{\sqrt{2}}{2}P\right) + F_{EH} = 0 \quad F_{EH} = \dfrac{P}{4} - \dfrac{P}{2} = -\dfrac{P}{4}$ $\qquad F_{EH} = \dfrac{P}{4}\ T$ ◀

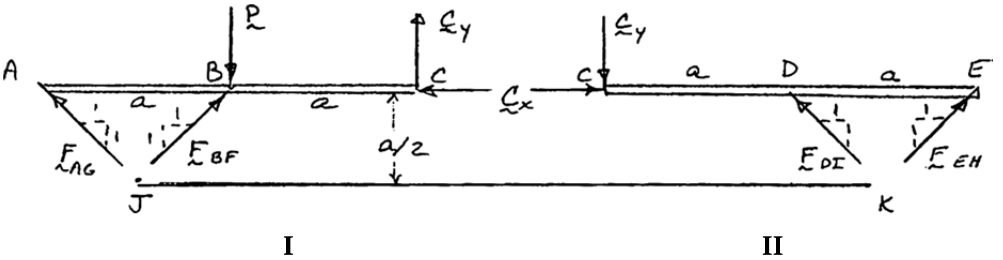


PROBLEM 6.113

Members *ABC* and *CDE* are pin-connected at *C* and supported by four links. For the loading shown, determine the force in each link.

SOLUTION

Member FBDs:



From FBD I: $\circlearrowleft \Sigma M_J = 0\text{:}\quad \frac{a}{2}C_x + \frac{3a}{2}C_y - \frac{a}{2}P = 0 \quad C_x + 3C_y = P$

FBD II: $\circlearrowleft \Sigma M_K = 0\text{:}\quad \frac{a}{2}C_x - \frac{3a}{2}C_y = 0 \quad C_x - 3C_y = 0$

Solving, $C_x = \frac{P}{2};\quad C_y = \frac{P}{6}$ as drawn.

FBD I: $\circlearrowleft \Sigma M_B = 0\text{:}\quad aC_y - a\frac{1}{\sqrt{2}}F_{AG} = 0 \quad F_{AG} = \sqrt{2}C_y = \frac{\sqrt{2}}{6}P$

$$F_{AG} = \frac{\sqrt{2}}{6}P \quad C \blacktriangleleft$$

$\xrightarrow{} \Sigma F_x = 0\text{:}\quad -\frac{1}{\sqrt{2}}F_{AG} + \frac{1}{\sqrt{2}}F_{BF} - C_x = 0 \quad F_{BF} = F_{AG} + C_x\sqrt{2} = \frac{\sqrt{2}}{6}P + \frac{\sqrt{2}}{2}P$

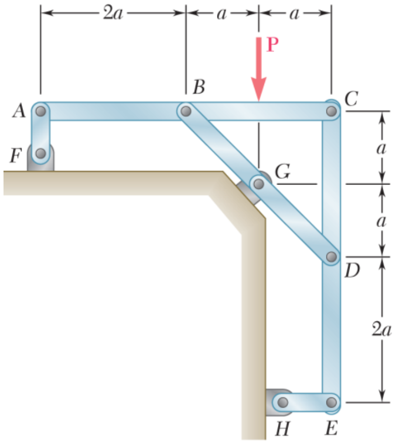
$$F_{BF} = \frac{2\sqrt{2}}{3}P \quad C \blacktriangleleft$$

FBD II: $\circlearrowleft \Sigma M_D = 0\text{:}\quad a\frac{1}{\sqrt{2}}F_{EH} + aC_y = 0 \quad F_{EH} = -\sqrt{2}C_y = -\frac{\sqrt{2}}{6}P$

$$F_{EH} = \frac{\sqrt{2}}{6}P \quad T \blacktriangleleft$$

$\xrightarrow{} \Sigma F_x = 0\text{:}\quad C_x - \frac{1}{\sqrt{2}}F_{DI} + \frac{1}{\sqrt{2}}F_{EH} = 0 \quad F_{DI} = F_{EH} + C_x\sqrt{2} = -\frac{\sqrt{2}}{6}P + \frac{\sqrt{2}}{2}P$

$$F_{DI} = \frac{\sqrt{2}}{3}P \quad C \blacktriangleleft$$

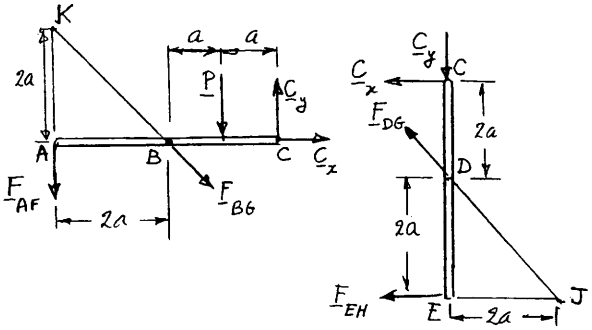

2a
a
a
P
A
B
C
F
G
a
a
D
2a
H
E

PROBLEM 6.114

Members *ABC* and *CDE* are pin-connected at *C* and supported by the four links *AF*, *BG*, *DG*, and *EH*. For the loading shown, determine the force in each link.

SOLUTION

We consider members *ABC* and *CDE*:


K
2a
a
a
P
C_y
A
B
C
C_x
F_{AF}
2a
F_{BG}
C_y
C
C_x
F_{DG}
2a
D
2a
F_{EH}
E
2a
J

Free body: *CDE*:

$$+\circlearrowleft \Sigma M_J = 0: \quad C_x(4a) + C_y(2a) = 0 \qquad C_y = -2C_x \qquad (1)$$

Free body: *ABC*:

$$+\circlearrowleft \Sigma M_K = 0: \quad C_x(2a) + C_y(4a) - P(3a) = 0$$

Substituting for C_y from Eq. (1): $C_x(2a) - 2C_x(4a) - P(3a) = 0$

$$C_x = -\frac{1}{2}P \qquad C_y = -2\left(-\frac{1}{2}P\right) = +P$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{1}{\sqrt{2}}F_{BG} + C_x = 0$$

$$F_{BG} = -\sqrt{2}C_x = -\sqrt{2}\left(-\frac{1}{2}P\right) = +\frac{P}{\sqrt{2}}, \qquad F_{BG} = \frac{P}{\sqrt{2}} \quad T \blacktriangleleft$$

PROBLEM 6.114 (Continued)

$$+\uparrow \Sigma F_y = 0: \quad -F_{AF} - \frac{1}{\sqrt{2}} F_{BG} - P + C_y = 0$$

$$F_{AF} = -\frac{1}{\sqrt{2}}\frac{P}{\sqrt{2}} - P + P = -\frac{P}{2} \qquad F_{AF} = \frac{P}{2} \quad C \blacktriangleleft$$

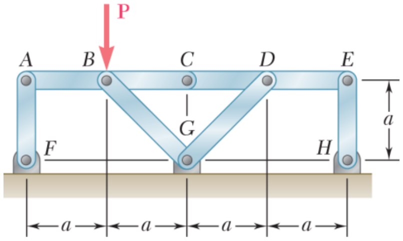
Free body: *CDE*:

$$+\uparrow \Sigma F_y = 0: \quad \frac{1}{\sqrt{2}} F_{DG} - C_y = 0$$

$$F_{DG} = \sqrt{2} C_y = +\sqrt{2} P \qquad F_{DG} = \sqrt{2} P \quad T \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -F_{EH} - C_x - \frac{1}{\sqrt{2}} F_{DG} = 0$$

$$F_{EH} = -\left(-\frac{1}{2}P\right) - \frac{1}{\sqrt{2}}\sqrt{2}P = -\frac{P}{2} \qquad F_{EH} = \frac{P}{2} \quad C \blacktriangleleft$$



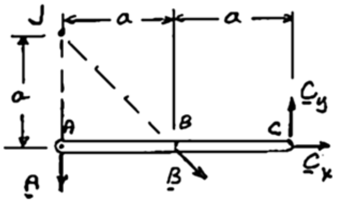
PROBLEM 6.115

Solve Problem 6.112 assuming that the force **P** is replaced by a clockwise couple of moment $\mathbf{M}_0$ applied to member *CDE* at *D*.

PROBLEM 6.112 Members *ABC* and *CDE* are pin-connected at *C* and supported by four links. For the loading shown, determine the force in each link.

SOLUTION

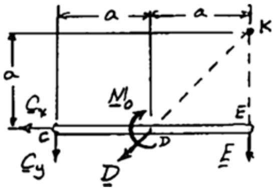
Free body: Member *ABC*:



$$+\circlearrowleft \Sigma M_J = 0: \quad C_y(2a) + C_x(a) = 0$$

$$C_x = -2C_y$$

Free body: Member *CDE*:



$$+\circlearrowleft \Sigma M_K = 0: \quad C_y(2a) - C_x(a) - M_0 = 0$$

$$C_y(2a) - (-2C_y)(a) - M_0 = 0 \qquad C_y = \frac{M_0}{4a}$$ v

$$C_x = -2C_y: \qquad C_x = -\frac{M_0}{2a}$$ v

$$\overset{+}{\longleftarrow} \Sigma F_x = 0: \quad \frac{D}{\sqrt{2}} + C_x = 0; \quad \frac{D}{\sqrt{2}} - \frac{M_0}{2a} = 0$$

$$D = \frac{M_0}{\sqrt{2}a} \qquad F_{DG} = \frac{M_0}{\sqrt{2}a} \quad T$$ ◀

$$+\circlearrowright \Sigma M_D = 0: \quad E(a) - C_y(a) + M_0 = 0$$

$$E(a) - \left(\frac{M_0}{4a}\right)(a) + M_0 = 0$$

$$E = -\frac{3}{4}\frac{M_0}{a} \qquad F_{EH} = \frac{3}{4}\frac{M_0}{a} \quad C$$ ◀

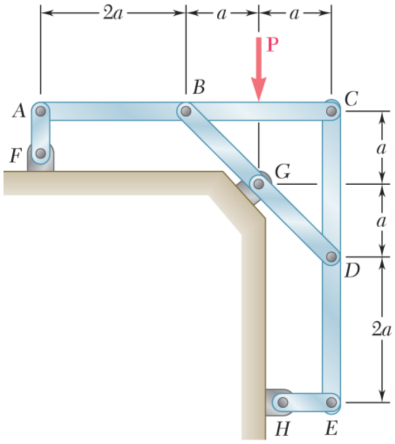
Return to free body of *ABC*:

$$\overset{+}{\longrightarrow} \Sigma F_x = 0: \quad \frac{B}{\sqrt{2}} + C_x = 0; \quad \frac{B}{\sqrt{2}} - \frac{M_0}{2a} = 0$$

$$B = \frac{M_0}{\sqrt{2}a} \qquad F_{BG} = \frac{M_0}{\sqrt{2}a} \quad T$$ ◀

$$+\circlearrowleft \Sigma M_B = 0: \quad A(a) + C_y(a); \quad A(a) + \frac{M_0}{4a}(a) = 0$$

$$A = -\frac{M_0}{4a} \qquad F_{AF} = \frac{M_0}{4a} \quad C$$ ◀



PROBLEM 6.116

Solve Problem 6.114 assuming that the force **P** is replaced by a clockwise couple of moment $\mathbf{M}_0$ applied at the same point.

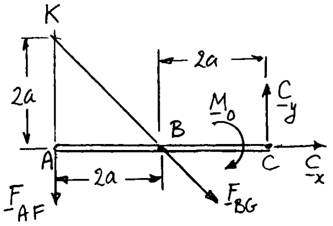
PROBLEM 6.114 Members *ABC* and *CDE* are pin-connected at *C* and supported by the four links *AF*, *BG*, *DG*, and *EH*. For the loading shown, determine the force in each link.

SOLUTION

Free body: *CDE*: (same as for Problem 6.114)

$$+\circlearrowleft \Sigma M_J = 0: \quad C_x(4a) + C_y(2a) = 0 \qquad C_y = -2C_x \qquad (1)$$

Free body: *ABC*:



$$+\circlearrowleft \Sigma M_K = 0: \quad C_x(2a) + C_y(4a) - M_0 = 0$$

Substituting for C_y from Eq. (1): $C_x(2a) - 2C_x(4a) - M_0 = 0$

$$C_x = -\frac{M_0}{6a}$$

$$C_y = -2\left(-\frac{M_0}{6a}\right) = +\frac{M_0}{3a}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{1}{\sqrt{2}} F_{BG} + C_x = 0$$

$$F_{BG} = -\sqrt{2}C_x = +\frac{\sqrt{2}M_0}{6a} \qquad F_{BG} = \frac{\sqrt{2}M_0}{6a} \quad T \ ◀$$

$$+\uparrow \Sigma F_y = 0: \quad -F_{AF} - \frac{1}{\sqrt{2}} F_{BG} + C_y = 0$$

$$F_{AF} = -\frac{1}{\sqrt{2}} \frac{\sqrt{2}M_0}{6a} + \frac{M_0}{3a} = +\frac{M_0}{6a} \qquad F_{AF} = \frac{M_0}{6a} \quad T \ ◀$$

PROBLEM 6.116 (Continued)

Free body: *CDE*: (Use F.B. diagram of Problem 6.114.)

$$+\uparrow \Sigma F_y = 0: \quad \frac{1}{\sqrt{2}} F_{DG} - C_y = 0$$

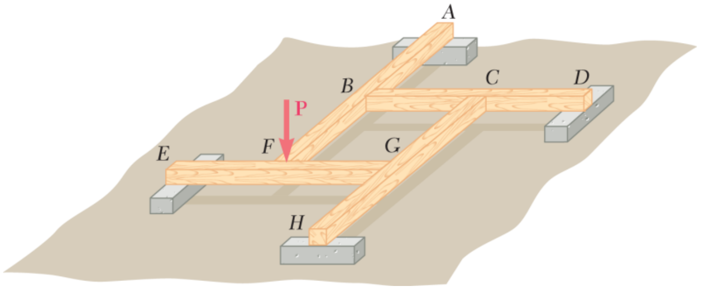
$$F_{DG} = \sqrt{2} C_y = +\frac{\sqrt{2} M_0}{3a} \qquad F_{DG} = \frac{\sqrt{2} M_0}{3a} \quad T \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -F_{EH} - C_x - \frac{1}{\sqrt{2}} F_{DG} = 0$$

$$F_{EH} = -\left(-\frac{M_0}{6a}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{2} M_0}{3a}\right) = -\frac{M_0}{6a}, \quad F_{EH} = \frac{M_0}{6a} \quad C \blacktriangleleft$$

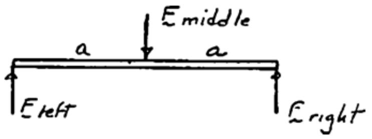
PROBLEM 6.117

Four beams, each of length $2a$, are nailed together at their midpoints to form the support system shown. Assuming that only vertical forces are exerted at the connections, determine the vertical reactions at A, D, E, and H.



SOLUTION

Note that, if we assume P is applied to EG, each individual member FBD looks like



so $$2F_{\text{left}} = 2F_{\text{right}} = F_{\text{middle}}$$

Labeling each interaction force with the letter corresponding to the joint of its application, we see that

$$B = 2A = 2F$$

$$C = 2B = 2D$$

$$G = 2C = 2H$$

$$P + F = 2G(= 4C = 8B = 16F) = 2E$$

From $P + F = 16F$, $$F = \frac{P}{15}$$ so $\mathbf{A} = \dfrac{P}{15}\uparrow$ ◀

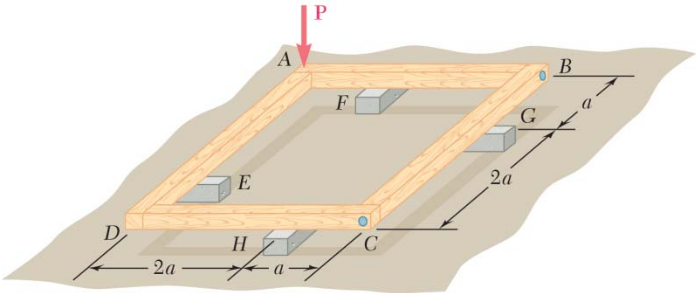
$\mathbf{D} = \dfrac{2P}{15}\uparrow$ ◀

$\mathbf{H} = \dfrac{4P}{15}\uparrow$ ◀

$\mathbf{E} = \dfrac{8P}{15}\uparrow$ ◀

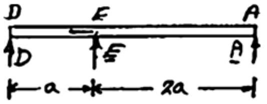
PROBLEM 6.118

Four beams, each of length 3*a*, are held together by single nails at *A*, *B*, *C*, and *D*. Each beam is attached to a support located at a distance *a* from an end of the beam as shown. Assuming that only vertical forces are exerted at the connections, determine the vertical reactions at *E*, *F*, *G*, and *H*.



SOLUTION

We shall draw the free body of each member. Force **P** will be applied to member *AFB*. Starting with member *AED*, we shall express all forces in terms of reaction *E*.



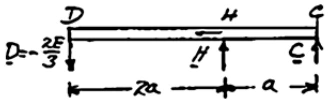
Member *AED*:

$$+\circlearrowleft \Sigma M_D = 0: \quad A(3a) + E(a) = 0$$

$$A = -\frac{E}{3}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad -D(3a) - E(2a) = 0$$

$$D = -\frac{2E}{3}$$



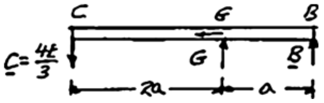
Member *DHC*:

$$+\circlearrowleft \Sigma M_C = 0: \quad \left(-\frac{2E}{3}\right)(3a) - H(a) = 0$$

$$H = -2E \qquad (1)$$

$$+\circlearrowleft \Sigma M_H = 0: \quad \left(-\frac{2E}{3}\right)(2a) + C(a) = 0$$

$$C = +\frac{4E}{3}$$



Member *CGB*:

$$+\circlearrowleft \Sigma M_B = 0: \quad +\left(\frac{4E}{3}\right)(3a) - G(a) = 0$$

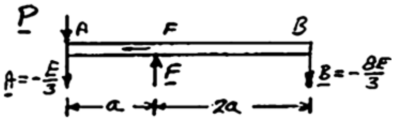
$$G = +4E \qquad (2)$$

$$+\circlearrowleft \Sigma M_G = 0: \quad +\left(\frac{4E}{3}\right)(2a) + B(a) = 0$$

$$B = -\frac{8E}{3}$$

PROBLEM 6.118 (Continued)

Member *AFB*:



$$+\uparrow \Sigma F_y = 0: \quad F - A - B - P = 0$$

$$F - \left(-\frac{E}{3}\right) - \left(-\frac{8E}{3}\right) - P = 0$$

$$F = P - 3E \tag{3}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad F(a) - B(3a) = 0$$

$$(P - 3E)(a) - \left(-\frac{8E}{3}\right)(3a) = 0$$

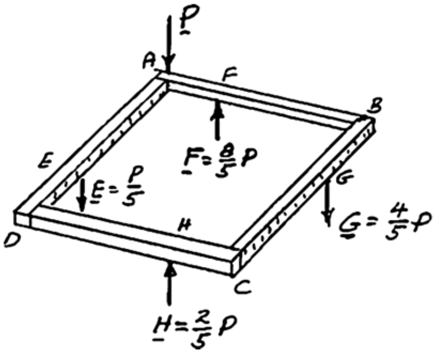
$$P - 3E + 8E = 0; \quad E = -\frac{P}{5} \qquad \mathbf{E} = \frac{P}{5}\downarrow \blacktriangleleft$$

Substitute $E = -\frac{P}{5}$ into Eqs. (1), (2), and (3).

$$H = -2E = -2\left(-\frac{P}{5}\right) \qquad H = +\frac{2P}{5} \qquad \mathbf{H} = \frac{2P}{5}\uparrow \blacktriangleleft$$

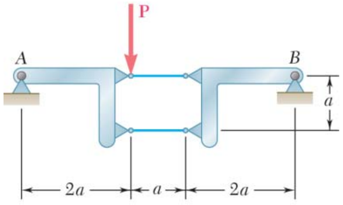
$$G = +4E = 4\left(-\frac{P}{5}\right) \qquad G = -\frac{4P}{5} \qquad \mathbf{G} = \frac{4P}{5}\downarrow \blacktriangleleft$$

$$F = P - 3E = P - 3\left(-\frac{P}{5}\right) \qquad F = +\frac{8P}{5} \qquad \mathbf{F} = \frac{8P}{5}\uparrow \blacktriangleleft$$

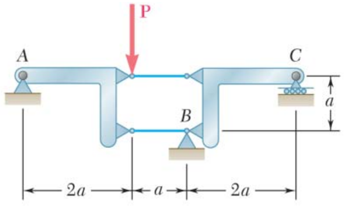


PROBLEM 6.119

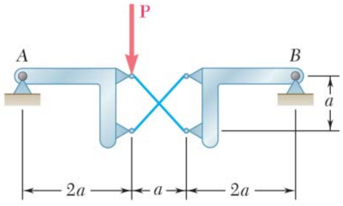
Each of the frames shown consists of two L-shaped members connected by two rigid links. For each frame, determine the reactions at the supports and indicate whether the frame is rigid.



(a)


P
A
C
B
a
2a
a
2a

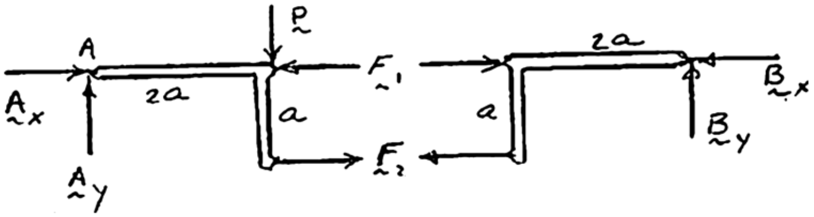
(b)



(c)

SOLUTION

(*a*) **Member FBDs:**


P
A
A_x
A_y
2a
a
F_1
F_2
2a
a
B
B_x
B_y

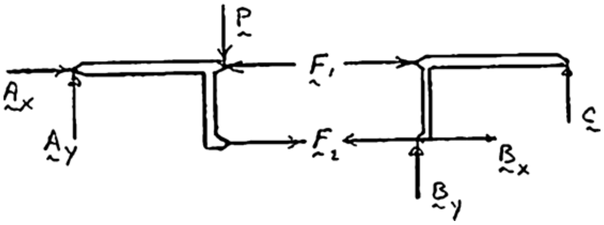
I **II**

FBD II: $\uparrow \Sigma F_y = 0: \quad B_y = 0 \qquad \circlearrowleft \Sigma M_B = 0: \quad aF_2 = 0 \qquad F_2 = 0$

FBD I: $\circlearrowleft \Sigma M_A = 0: aF_2 - 2aP = 0, \text{ but } F_2 = 0$

so $P = 0$ not rigid for $P \neq 0$ ◀

(*b*) **Member FBDs:**


P
A_x
A_y
F_1
F_2
C
B_x
B_y

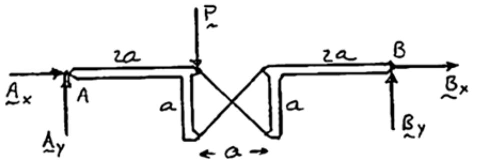
Note: Seven unknowns $(A_x, A_y, B_x, B_y, F_1, F_2, C)$, but only six independent equations

System is statically indeterminate. ◀

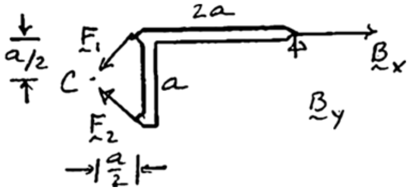
System is, however, rigid. ◀

PROBLEM 6.119 (Continued)

(*c*) **FBD whole:** **FBD right:**



I



II

FBD I: $\circlearrowleft \Sigma M_A = 0{:}\quad 5aB_y - 2aP = 0$ $\qquad \mathbf{B}_y = \frac{2}{5}P \uparrow$

$\uparrow \Sigma F_y = 0{:}\quad A_y - P + \frac{2}{5}P = 0$ $\qquad \mathbf{A}_y = \frac{3}{5}P \uparrow$

FBD II: $\circlearrowright \Sigma M_C = 0{:}\quad \frac{a}{2}B_x - \frac{5a}{2}B_y = 0 \quad B_x = 5B_y$ $\qquad \mathbf{B}_x = 2P \longrightarrow$

FBD I: $\longrightarrow \Sigma F_x = 0{:}\quad A_x + B_x = 0 \quad A_x = -B_x$ $\qquad \mathbf{A}_x = 2P \longleftarrow$

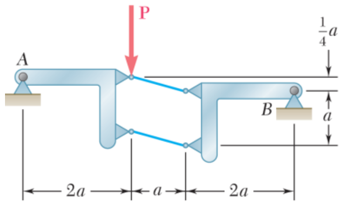
$\mathbf{A} = 2.09P \searrow 16.70°$ ◀

$\mathbf{B} = 2.04P \nearrow 11.31°$ ◀

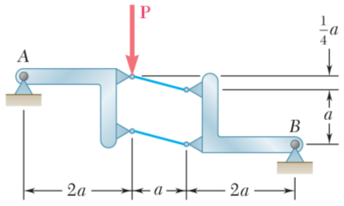
System is rigid. ◀

PROBLEM 6.120

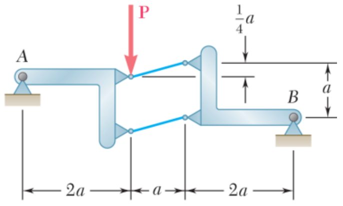
Each of the frames shown consists of two L-shaped members connected by two rigid links. For each frame, determine the reactions at the supports and indicate whether the frame is rigid.


P
A
B
$\frac{1}{4}a$
a
2a
a
2a

(*a*)


P
A
B
$\frac{1}{4}a$
a
2a
a
2a

(*b*)

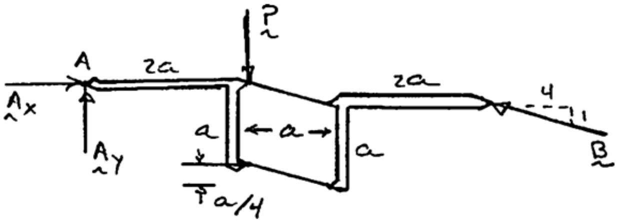

P
A
B
$\frac{1}{4}a$
a
2a
a
2a

(*c*)

SOLUTION

Note: In all three cases, the right member has only three forces acting, two of which are parallel. Thus, the third force, at *B*, must be parallel to the link forces.

(*a*) **FBD whole:**


P
A
2a
2a
4
1
A_x
a
a
a
A_y
a/4
B

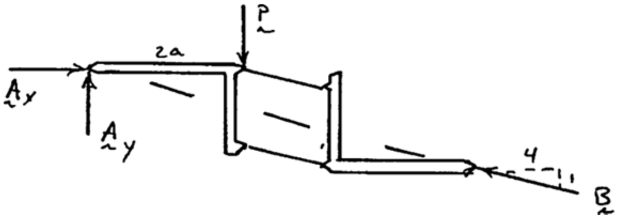
$\circlearrowleft \Sigma M_A = 0: \quad -2aP - \frac{a}{4}\frac{4}{\sqrt{17}}B + 5a\frac{1}{\sqrt{17}}B = 0 \quad B = 2.06P$ $\qquad$ $\mathbf{B} = 2.06P \searrow 14.04°$ ◀

$\xrightarrow{} \Sigma F_x = 0: \quad A_x - \frac{4}{\sqrt{17}}B = 0 \qquad \mathbf{A}_x = 2P \leftarrow$

$\uparrow \Sigma F_y = 0: \quad A_y - P + \frac{1}{\sqrt{17}}B = 0 \quad \mathbf{A}_y = \frac{P}{2}\uparrow$ $\qquad$ $\mathbf{A} = 2.06P \measuredangle 14.04°$ ◀

rigid ◀

(*b*) **FBD whole:**

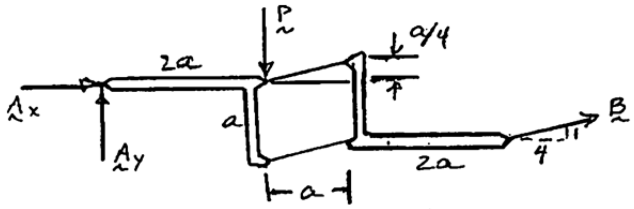

P
2a
A_x
A_y
4
1
B

Since **B** passes through A, $\circlearrowleft \Sigma M_A = 2aP = 0$ only if $P = 0$.

no equilibrium if $P \neq 0$ $\qquad$ not rigid ◀

PROBLEM 6.120 (Continued)

(*c*) **FBD whole:**



$\circlearrowleft \Sigma M_A = 0: \quad 5a\frac{1}{\sqrt{17}}B + \frac{3a}{4}\frac{4}{\sqrt{17}}B - 2aP = 0 \quad B = \frac{\sqrt{17}}{4}P$ $\qquad \mathbf{B} = 1.031P \nearrow 14.04°$ ◀

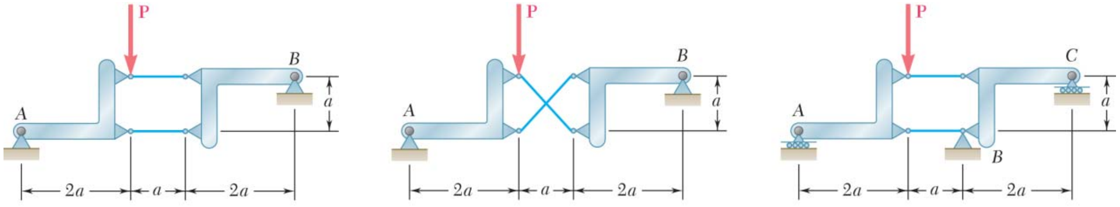
$\xrightarrow{} \Sigma F_x = 0: \quad A_x + \frac{4}{\sqrt{17}}B = 0 \quad A_x = -P$

$\uparrow \Sigma F_y = 0: \quad A_y - P + \frac{1}{\sqrt{17}}B = 0 \quad A_y = P - \frac{P}{4} = \frac{3P}{4}$ $\qquad \mathbf{A} = 1.250P \nwarrow 36.9°$ ◀

System is rigid. ◀

PROBLEM 6.121

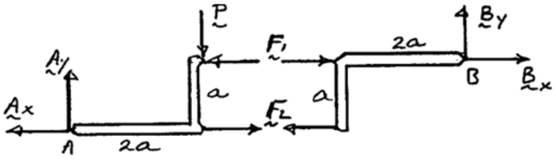
Each of the frames shown consists of two L-shaped members connected by two rigid links. For each frame, determine the reactions at the supports and indicate whether the frame is rigid.



(*a*) (*b*) (*c*)

SOLUTION

(*a*) **Member FBDs:**



I II

FBD I: $\circlearrowleft \Sigma M_A = 0\text{:}\quad aF_1 - 2aP = 0 \quad F_1 = 2P;\quad \uparrow \Sigma F_y = 0\text{:}\quad A_y - P = 0 \quad \mathbf{A}_y = P\uparrow$

FBD II: $\circlearrowleft \Sigma M_B = 0\text{:}\quad -aF_2 = 0 \quad F_2 = 0$

$\longrightarrow \Sigma F_x = 0\text{:}\quad B_x + F_1 = 0,\quad B_x = -F_1 = -2P \qquad \mathbf{B}_x = 2P \longrightarrow$

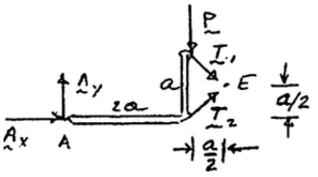
$\uparrow \Sigma F_y = 0\text{:}\quad B_y = 0 \qquad$ so $\mathbf{B} = 2P \longrightarrow$ ◀

FBD I: $\longrightarrow \Sigma F_x = 0\text{:}\quad -A_x - F_1 + F_2 = 0 \quad A_x = F_2 - F_1 = 0 - 2P \qquad \mathbf{A}_x = 2P \longrightarrow$

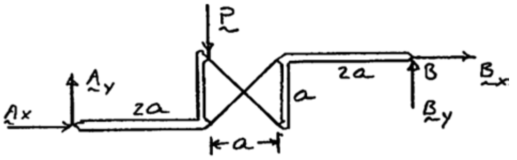
so $\mathbf{A} = 2.24P \measuredangle 26.6°$ ◀

Frame is rigid. ◀

(*b*) **FBD left:** **FBD whole:**



I



II

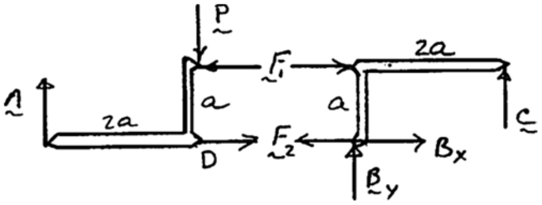
FBD I: $\circlearrowleft \Sigma M_E = 0\text{:}\quad \frac{a}{2}P + \frac{a}{2}A_x - \frac{5a}{2}A_y = 0 \quad A_x - 5A_y = -P$

FBD II: $\circlearrowleft \Sigma M_B = 0\text{:}\quad 3aP + aA_x - 5aA_y = 0 \quad A_x - 5A_y = -3P$

This is impossible unless $P = 0$. not rigid ◀

PROBLEM 6.121 (Continued)

(*c*) **Member FBDs:**



I **II**

FBD I: $\Sigma F_y = 0: \quad A - P = 0$ $\qquad \mathbf{A} = P \uparrow$ ◀

$\circlearrowleft \Sigma M_D = 0: \quad aF_1 - 2aA = 0 \quad F_1 = 2P$

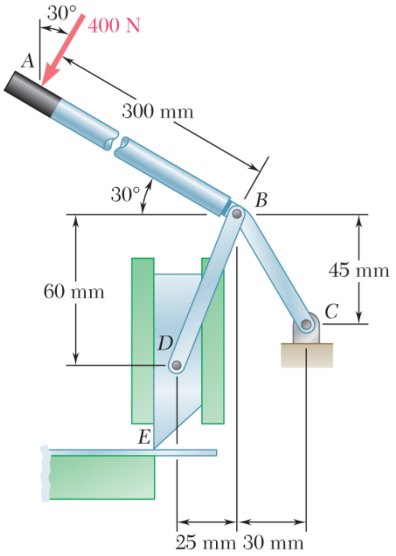
$\xrightarrow{} \Sigma F_x = 0: \quad F_2 - F_1 = 0 \quad F_2 = 2P$

FBD II: $\circlearrowleft \Sigma M_B = 0: \quad 2aC - aF_1 = 0 \quad C = \frac{F_1}{2} = P$ $\qquad \mathbf{C} = P \uparrow$ ◀

$\xrightarrow{} \Sigma F_x = 0: \quad F_1 - F_2 + B_x = 0 \quad B_x = P - P = 0$

$\uparrow \Sigma F_x = 0: \quad B_y + C = 0 \quad B_y = -C = -P$ $\qquad \mathbf{B} = P$ ◀

Frame is rigid. ◀

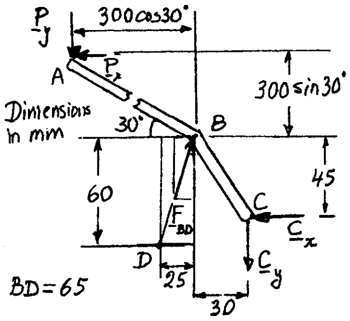

30°
400 N
A
300 mm
30°
B
45 mm
60 mm
C
D
E
25 mm
30 mm

PROBLEM 6.122

The shear shown is used to cut and trim electronic circuit board laminates. For the position shown, determine (*a*) the vertical component of the force exerted on the shearing blade at *D*, (*b*) the reaction at *C*.

SOLUTION

We note that *BD* is a two-force member.


P_y
300cos30°
A
P_x
300sin30°
Dimensions in mm
30°
B
45
60
F_{BD}
C
C_x
D
C_y
BD=65
25
30

Free body: Member *ABC*: We have the components:

$$\mathbf{P}_x = (400\text{ N})\sin 30° = 200\text{ N} \longleftarrow$$

$$\mathbf{P}_y = (400\text{ N})\cos 30° = 346.41\text{ N}\big\downarrow$$

$$(\mathbf{F}_{BD})_x = \frac{25}{65}F_{BD} \longrightarrow$$

$$(\mathbf{F}_{BD})_y = \frac{60}{65}F_{BD}\big\uparrow$$

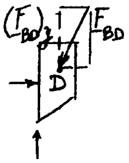
$$+\circlearrowleft \Sigma M_C = 0\text{:}\quad (F_{BD})_x(45) + (F_{BD})_y(30) - P_x(45 + 300\sin 30°) - P_y(30 + 300\cos 30°) = 0$$

$$\left(\frac{25}{65}F_{BD}\right)(45) + \left(\frac{60}{65}F_{BD}\right)(30) = (200)(195) + (346.41)(289.81)$$

$$45F_{BD} = 139.39\times 10^3$$

$$F_{BD} = 3097.6\text{ N}$$

(*a*) Vertical component of force exerted on shearing blade at *D*.



$$(F_{BD})_y = \frac{60}{65}F_{BD} = \frac{60}{65}(3097.6\text{ N}) = 2859.3\text{ N}$$

$$(\mathbf{F}_{BD})_y = 2860\text{ N}\big\downarrow \blacktriangleleft$$

PROBLEM 6.122 (Continued)

(*b*) Returning to *FB* diagram of member *ABC*,

$$\xrightarrow{+} \Sigma F_x = 0: \quad (F_{BD})_x - P_x - C_x = 0$$

$$C_x = (F_{BD})_x - P_x = \frac{25}{65} F_{BD} - P_x$$

$$= \frac{25}{65}(3097.6) - 200$$

$$C_x = +991.39 \qquad \mathbf{C}_x = 991.39 \text{ N} \longleftarrow$$

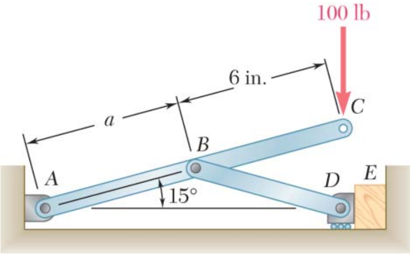
$$+\uparrow \Sigma F_y = 0: \quad (F_{BD})_y - P_y - C_y = 0$$

$$C_y = (F_{BD})_y - P_y = \frac{60}{65} F_{BD} - P_y = \frac{60}{65}(3097.6) - 346.41$$

$$C_y = +2512.9 \text{ N} \qquad \mathbf{C}_y = 2512.9 \downarrow$$

$$C = 2295 \text{ N} \qquad \mathbf{C} = 2700 \text{ N} \measuredangle 68.5° \blacktriangleleft$$

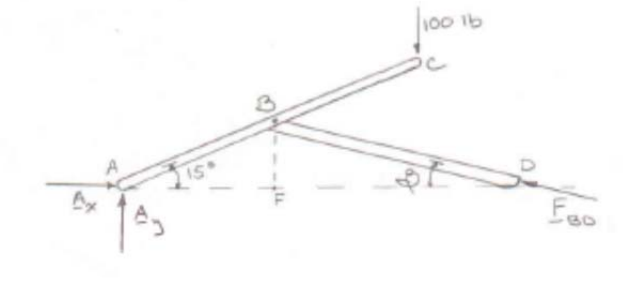
PROBLEM 6.123



A 100-lb force directed vertically downward is applied to the toggle vise at *C*. Knowing that link *BD* is 6 in. long and that *a* = 4 in., determine the horizontal force exerted on block *E*.

SOLUTION

Free body: Entire Frame



We have

$$BF = (4\text{ in.})\sin 15^\circ = 1.03528\text{ in.}$$

$$\sin\beta = \frac{1.03528\text{ in.}}{6\text{ in.}} \qquad \beta = 9.9359^\circ$$

$$AD = AG + FD = 4\cos 15^\circ + 6\cos 9.9359^\circ = 9.7737\text{ in.}$$

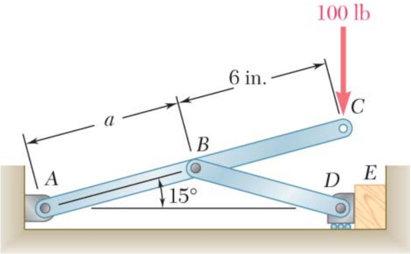
$$+\circlearrowleft \Sigma M_A = 0: \quad (100\text{ lb})(10\text{ in.})\cos(15^\circ) - (F_{BD})\sin\beta(9.7737\text{ in.}) = 0$$

$$F_{BD} = 572.77\text{ lb}$$

Horizontal force exerted on block.

$$(F_{BD})_x = F_{BD}\cos\beta = (572.77\text{ lb})\cos(9.9359^\circ)$$

$$(\mathbf{F}_{BD})_x = 564\text{ lb} \longrightarrow \blacktriangleleft$$

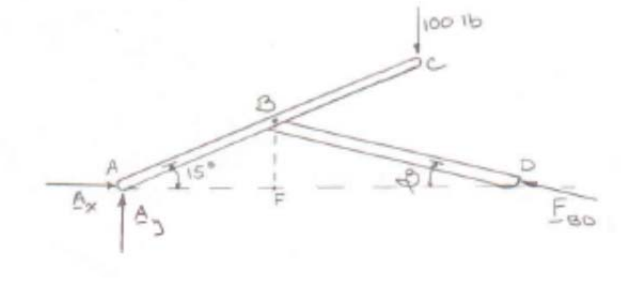


PROBLEM 6.124

A 100-lb force directed vertically downward is applied to the toggle vise at *C*. Knowing that link *BD* is 6 in. long and that $a = 8$ in., determine the horizontal force exerted on block *E*.

SOLUTION

Free body: Entire Frame



$$BF = (8 \text{ in.})\sin 15^\circ = 2.0706 \text{ in.}$$

We have

$$\sin\beta = \frac{2.0706 \text{ in.}}{6 \text{ in.}} \qquad \beta = 20.188^\circ$$

$$AF = (8 \text{ in.})\cos 15^\circ = 7.7274 \text{ in.}$$

$$FD = (6 \text{ in.})\cos 20.188^\circ = 5.6314 \text{ in.}$$

$$AD = AF + FD = 13.359 \text{ in.}$$

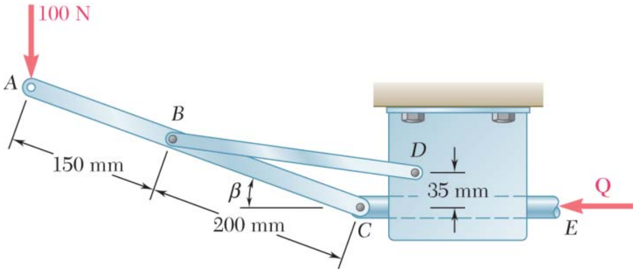
$$+\circlearrowleft \Sigma M_A = 0: \quad (100 \text{ lb})(14 \text{ in.})\cos(15^\circ) - (F_{BD})\sin\beta(13.359 \text{ in.}) = 0$$

$$F_{BD} = 293.33 \text{ lb}$$

Horizontal force exerted on block.

$$(F_{BD})_x = F_{BD}\cos\beta = (293.33 \text{ lb})\cos(20.188^\circ)$$

$$(\mathbf{F}_{BD})_x = 275 \text{ lb} \longrightarrow \blacktriangleleft$$


100 N
A
B
D
150 mm
β
35 mm
Q
200 mm
C
E

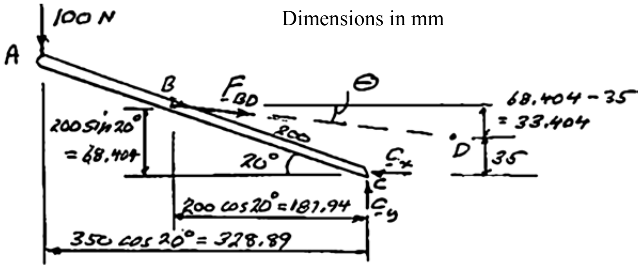
PROBLEM 6.125

The control rod *CE* passes through a horizontal hole in the body of the toggle system shown. Knowing that link *BD* is 250 mm long, determine the force **Q** required to hold the system in equilibrium when $\beta = 20°$.

SOLUTION

We note that *BD* is a two-force member.

Free body: Member *ABC*:


100 N
Dimensions in mm
A
B
F_{BD}
θ
68.404 − 35
= 33.404
200 sin 20°
= 68.404
200
20°
C_x
C
D
35
C_y
200 cos 20° = 187.94
350 cos 20° = 328.89

Since $BD = 250$, $\quad \theta = \sin^{-1}\dfrac{33.404}{250}; \quad \theta = 7.679°$

$$+\circlearrowright \Sigma M_C = 0: \quad (F_{BD}\sin\theta)187.94 - (F_{BD}\cos\theta)68.404 + (100\text{ N})328.89 = 0$$

$$F_{BD}[187.94\sin 7.679° - 68.404\cos 7.679°] = 32{,}889$$

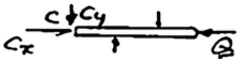
$$F_{BD} = 770.6\text{ N}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad (770.6\text{ N})\cos 7.679° = C_x = 0$$

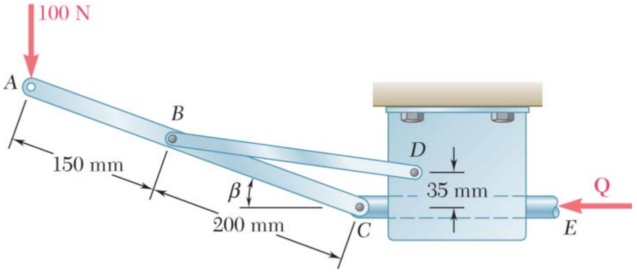
$$C_x = +763.7\text{ N}$$

Member *CQ*:

$$\Sigma F_x = 0: \quad Q = C_x = 763.7\text{ N}$$



$\mathbf{Q} = 764\text{ N} \leftarrow$ ◀


100 N
A
B
150 mm
β
200 mm
C
D
35 mm
Q
E

PROBLEM 6.126

Solve Problem 6.125 when (*a*) $\beta = 0$, (*b*) $\beta = 6°$.

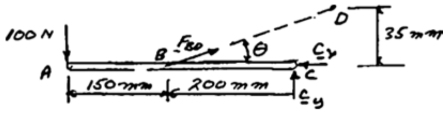
PROBLEM 6.125 The control rod *CE* passes through a horizontal hole in the body of the toggle system shown. Knowing that link *BD* is 250 mm long, determine the force **Q** required to hold the system in equilibrium when $\beta = 20°$.

SOLUTION

We note that *BD* is a two-force member.

(*a*) $\underline{\beta = 0}$:

Free body: Member *ABC*


100 N
A
B
F_{BD}
θ
D
C_x
C
C_y
35 mm
150 mm
200 mm

Since $\quad BD = 250 \text{ mm}, \quad \sin\theta = \dfrac{35 \text{ mm}}{250 \text{ mm}}; \quad \theta = 8.048°$

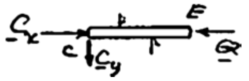
$$+\circlearrowleft \Sigma M_C = 0: \quad (100 \text{ N})(350 \text{ mm}) - F_{BD}\sin\theta(200 \text{ mm}) = 0$$

$$F_{BD} = 1250 \text{ N}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{BD}\cos\theta - C_x = 0$$

$$(1250 \text{ N})(\cos 8.048°) - C_x = 0 \qquad \mathbf{C}_x = 1237.7 \text{ N}$$

Member *CE*:



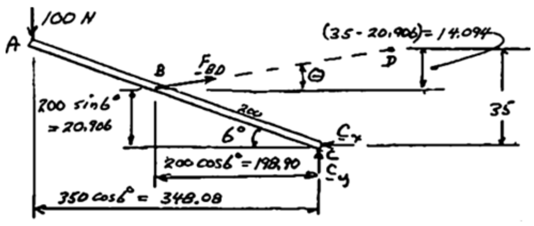
$$\xrightarrow{+} \Sigma F_x = 0: \quad (1237.7 \text{ N}) - Q = 0$$

$$Q = 1237.7 \text{ N}$$

$$\mathbf{Q} = 1238 \text{ N} \longleftarrow \blacktriangleleft$$

(*b*) $\underline{\beta = 6°}$

Free body: Member *ABC*


100 N
A
B
F_{BD}
θ
D
(35 - 20.906) = 14.094
200 sin 6°
= 20.906
200
6°
C_x
C
C_y
200 cos 6° = 198.90
350 cos 6° = 348.08
35

Dimensions in mm

Since $\quad BD = 250 \text{ mm}, \quad \theta = \sin^{-1}\dfrac{14.094 \text{ mm}}{250 \text{ mm}}$

$$\theta = 3.232°$$

PROBLEM 6.126 (Continued)

$$+\circlearrowright \Sigma M_C = 0: \quad (F_{BD}\sin\theta)198.90 + (F_{BD}\cos\theta)20.906 - (100\text{ N})348.08 = 0$$

$$F_{BD}[198.90\sin 3.232^\circ + 20.906\cos 3.232^\circ] = 34808$$

$$F_{BD} = 1084.8\text{ N}$$

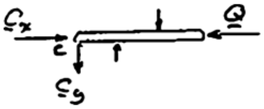
$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{BD}\cos\theta - C_x = 0$$

$$(1084.8\text{ N})\cos 3.232^\circ - C_x = 0$$

$$C_x = +1083.1\text{ N}$$

Member *DE*:

$$\Sigma F_x = 0: \quad Q = C_x$$



$$Q = 1083.1\text{ N}$$

$\mathbf{Q} = 1083\text{ N} \longleftarrow$ ◀

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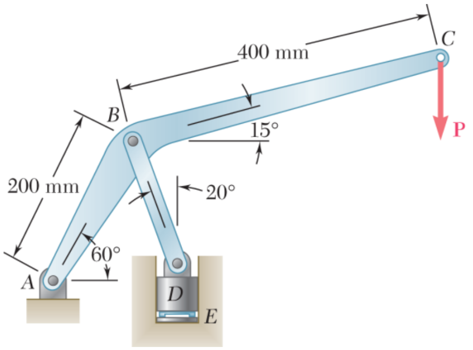
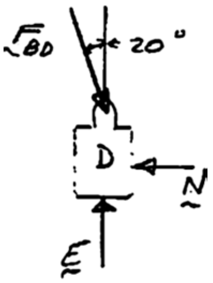


PROBLEM 6.127

The press shown is used to emboss a small seal at *E*. Knowing that $P = 250$ N, determine (*a*) the vertical component of the force exerted on the seal, (*b*) the reaction at *A*.

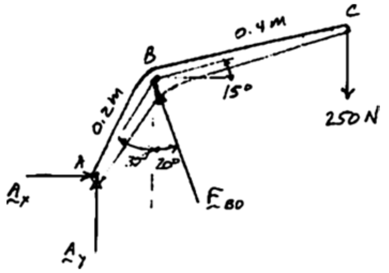
SOLUTION

FBD Stamp *D*:



$$+\!\uparrow \Sigma F_y = 0: \quad E - F_{BD}\cos 20° = 0, \quad E = F_{BD}\cos 20°$$

FBD *ABC*:



$$\circlearrowleft \Sigma M_A = 0: \quad (0.2\text{ m})(\sin 30°)(F_{BD}\cos 20°) + (0.2\text{ m})(\cos 30°)(F_{BD}\sin 20°)$$

$$-[(0.2\text{ m})\sin 30° + (0.4\text{ m})\cos 15°](250\text{ N}) = 0$$

$$F_{BD} = 793.64\text{ N} \quad C$$

and, from above,

$$E = (793.64\text{ N})\cos 20°$$

(*a*) $\mathbf{E} = 746\text{ N}\downarrow$ ◀

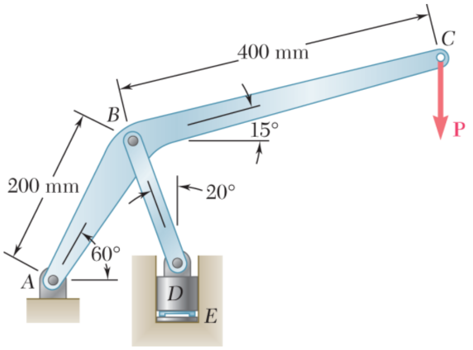
$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - (793.64\text{ N})\sin 20° = 0$$

$$\mathbf{A}_x = 271.44\text{ N} \longrightarrow$$

$$+\!\uparrow \Sigma F_y = 0: \quad A_y + (793.64\text{ N})\cos 20° - 250\text{ N} = 0$$

$$\mathbf{A}_y = 495.78\text{ N}\downarrow$$

(*b*) $\mathbf{A} = 565\text{ N} \searrow 61.3°$ ◀



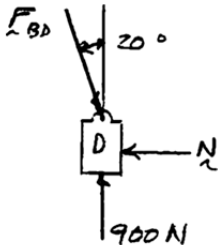
PROBLEM 6.128

The press shown is used to emboss a small seal at *E*. Knowing that the vertical component of the force exerted on the seal must be 900 N, determine (*a*) the required vertical force **P**, (*b*) the corresponding reaction at *A*.

SOLUTION

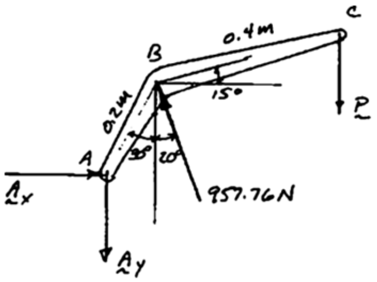
FBD Stamp *D*:

$$\uparrow \Sigma F_y = 0: \quad 900\text{ N} - F_{BD}\cos 20° = 0, \quad F_{BD} = 957.76\text{ N} \quad C$$



(*a*)

FBD *ABC*:



$$\circlearrowleft \Sigma M_A = 0: \quad [(0.2\text{ m})(\sin 30°)](957.76\text{ N})\cos 20° + [(0.2\text{ m})(\cos 30°)](957.76\text{ N})\sin 20°$$

$$-[(0.2\text{ m})\sin 30° + (0.4\text{ m})\cos 15°]P = 0$$

$$P = 301.70\text{ N}, \qquad \mathbf{P} = 302\text{ N}\downarrow ◀$$

(*b*)

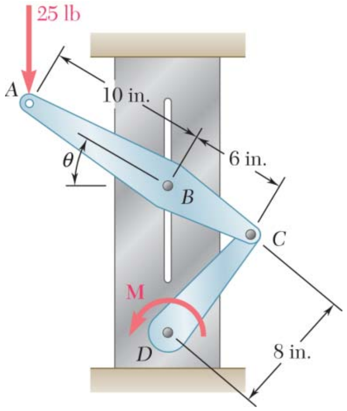
$$\rightarrow \Sigma F_x = 0: \quad A_x - (957.76\text{ N})\sin 20° = 0$$

$$\mathbf{A}_x = 327.57\text{ N} \rightarrow$$

$$\uparrow \Sigma F_y = 0: \quad -A_y + (957.76\text{ N})\cos 20° - 301.70\text{ N} = 0$$

$$\mathbf{A}_y = 598.30\text{ N}\downarrow$$

$$\mathbf{A} = 682\text{ N} ⦪ 61.3° ◀$$

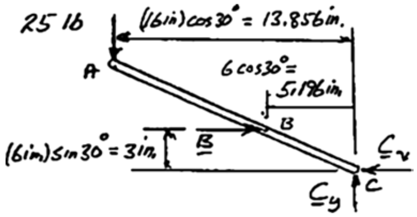

25 lb
A
10 in.
6 in.
θ
B
C
M
D
8 in.

PROBLEM 6.129

The pin at *B* is attached to member *ABC* and can slide freely along the slot cut in the fixed plate. Neglecting the effect of friction, determine the couple **M** required to hold the system in equilibrium when $\theta = 30°$.

SOLUTION

Free body: Member *ABC*:


25 lb
(16 in) cos 30° = 13.856 in.
A
6 cos 30° =
5.196 in
B
(6 in) sin 30° = 3 in.
B
C_x
C
C_y

$$\overset{\curvearrowleft}{+}\Sigma M_C = 0: \quad (25\text{ lb})(13.856\text{ in.}) - B(3\text{ in.}) = 0$$

$$B = +115.47\text{ lb}$$

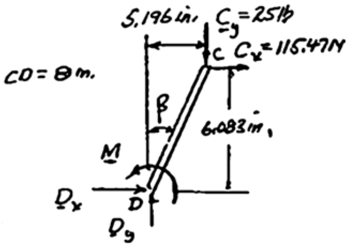
$$+\!\uparrow \Sigma F_y = 0: \quad -25\text{ lb} + C_y = 0$$

$$C_y = +25\text{ lb}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad 115.47\text{ lb} - C_x = 0$$

$$C_x = +115.47\text{ lb}$$

Free body: Member *CD*:


5.196 in.
C_y = 25 lb
C
C_x = 115.47N
CD = 8 in.
β
6.083 in.
M
D_x
D
D_y

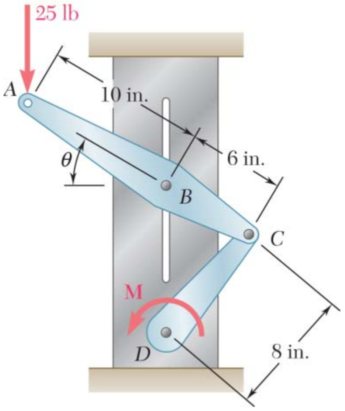
$$\beta = \sin^{-1}\frac{5.196}{8}; \quad \beta = 40.505°$$

$$CD\cos\beta = (8\text{ in.})\cos 40.505° = 6.083\text{ in.}$$

$$\overset{\curvearrowleft}{+}\Sigma M_D = 0: \quad M - (25\text{ lb})(5.196\text{ in.}) - (115.47\text{ lb})(6.083\text{ in.}) = 0$$

$$M = +832.3\text{ lb}\cdot\text{in.}$$

$$\mathbf{M} = 832\text{ lb}\cdot\text{in.} \curvearrowleft \blacktriangleleft$$

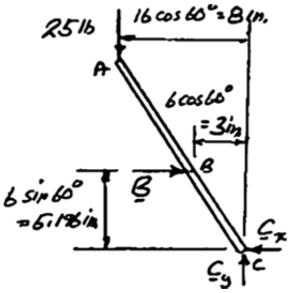

25 lb
A
10 in.
6 in.
θ
B
C
M
D
8 in.

PROBLEM 6.130

The pin at B is attached to member ABC and can slide freely along the slot cut in the fixed plate. Neglecting the effect of friction, determine the couple **M** required to hold the system in equilibrium when $\theta = 60°$.

SOLUTION

Free body: Member *ABC*:


25 lb
16 cos 60° = 8 in.
A
6 cos 60°
= 3 in.
B
B
6 sin 60° = 5.196 in.
C_x
C_y
C

$$+\circlearrowleft \Sigma M_C = 0: \quad (25\text{ lb})(8\text{ in.}) - B(5.196\text{ in.}) = 0$$

$$B = +38.49\text{ lb}$$

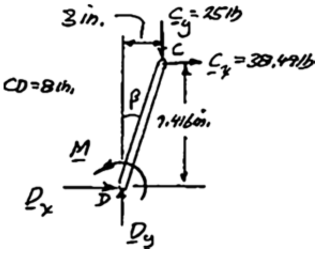
$$\xrightarrow{+} \Sigma F_x = 0: \quad 38.49\text{ lb} - C_x = 0$$

$$C_x = +38.49\text{ lb}$$

$$+\uparrow \Sigma F_y = 0: \quad -25\text{ lb} + C_y = 0$$

$$C_y = +25\text{ lb}$$

Free body: Member *CD*:

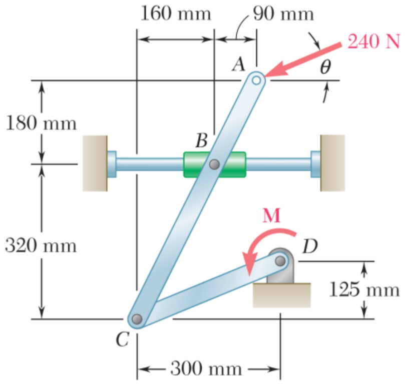

3 in.
C_y = 25 lb
C
C_x = 38.49 lb
CD = 8 in.
β
7.416 in.
M
D_x
D
D_y

$$\beta = \sin^{-1}\frac{3}{8}; \quad \beta = 22.024°$$

$$CD\cos\beta = (8\text{ in.})\cos 22.024° = 7.416\text{ in.}$$

$$+\circlearrowleft \Sigma M_D = 0: \quad M - (25\text{ lb})(3\text{ in.}) - (38.49\text{ lb})(7.416\text{ in.}) = 0$$

$$M = +360.4\text{ lb}\cdot\text{in.} \qquad \mathbf{M} = 360\text{ lb}\cdot\text{in.}\circlearrowleft \blacktriangleleft$$

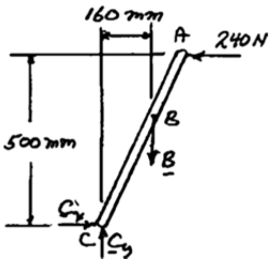

160 mm
90 mm
240 N
A
θ
180 mm
B
M
D
320 mm
125 mm
C
300 mm

PROBLEM 6.131

Arm *ABC* is connected by pins to a collar at *B* and to crank *CD* at *C*. Neglecting the effect of friction, determine the couple **M** required to hold the system in equilibrium when $\theta = 0$.

SOLUTION

Free body: Member *ABC*


160 mm
A
240 N
B
500 mm
B
C_x
C
C_y

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x - 240\text{ N} = 0$$

$$C_x = +240\text{ N}$$

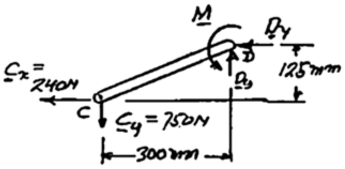
$$+\circlearrowleft \Sigma M_C = 0: \quad (240\text{ N})(500\text{ mm}) - B(160\text{ mm}) = 0$$

$$B = +750\text{ N}$$

$$+\uparrow \Sigma F_y = 0: \quad C_y - 750\text{ N} = 0$$

$$C_y = +750\text{ N}$$

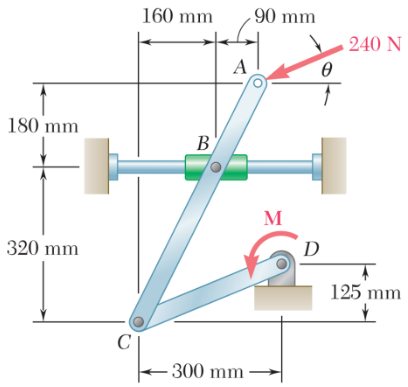
Free body: Member *CD*


M
D_x
D
$C_x = 240N$
D_y
125 mm
C
$C_y = 750N$
300 mm

$$+\circlearrowleft \Sigma M_D = 0: \quad M + (750\text{ N})(300\text{ mm}) - (240\text{ N})(125\text{ mm}) = 0$$

$$M = -195 \times 10^3\text{ N}\cdot\text{mm}$$

$$\mathbf{M} = 195.0\text{ kN}\cdot\text{m} \circlearrowright$$ ◀

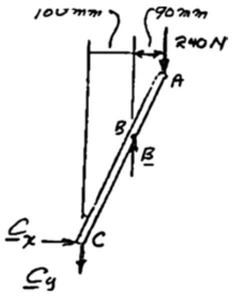

160 mm
90 mm
240 N
A
θ
180 mm
B
M
320 mm
D
125 mm
C
300 mm

PROBLEM 6.132

Arm *ABC* is connected by pins to a collar at *B* and to crank *CD* at *C*. Neglecting the effect of friction, determine the couple **M** required to hold the system in equilibrium when $\theta = 90°$.

SOLUTION

Free body: Member *ABC*

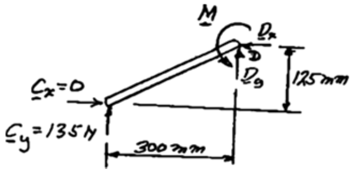

160mm
90mm
240N
A
B
B
Cx
C
Cy

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \quad C_y(160\text{ mm}) - (240\text{ N})(90\text{ mm}) = 0$$

$$C_y = +135\text{ N}$$

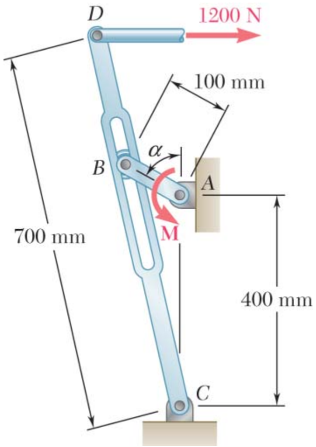
Free body: Member *CD*


M
Dx
D
Dy
125mm
Cx=0
Cy=135N
300mm

$$+\circlearrowleft \Sigma M_D = 0: \quad M - (135\text{ N})(300\text{ mm}) = 0$$

$$M = +40.5 \times 10^3\text{ N} \cdot \text{mm}$$

$$\mathbf{M} = 40.5\text{ kN} \cdot \text{m} \circlearrowleft \blacktriangleleft$$

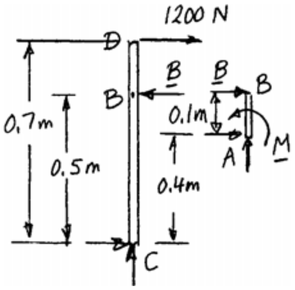

D
1200 N
100 mm
α
B
A
M
700 mm
400 mm
C

PROBLEM 6.133

The Whitworth mechanism shown is used to produce a quick-return motion of Point *D*. The block at *B* is pinned to the crank *AB* and is free to slide in a slot cut in member *CD*. Determine the couple **M** that must be applied to the crank *AB* to hold the mechanism in equilibrium when (*a*) $\alpha = 0$, (*b*) $\alpha = 30°$.

SOLUTION

(*a*) Free body: Member *CD*:


1200 N
D
B
B
B
B
0.7m
0.5m
0.1m
A
A
M
0.4m
C

$$+\circlearrowleft \Sigma M_C = 0: \quad B(0.5\text{ m}) - (1200\text{ N})(0.7\text{ m}) = 0$$

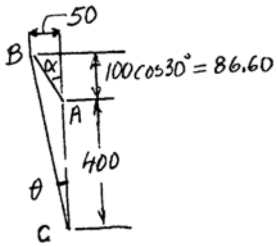
$$B = 1680\text{ N}$$

Free body: Crank *AB*:

$$+\circlearrowleft \Sigma M_A = 0: \quad M - (1680\text{ N})(0.1\text{ m}) = 0$$

$$M = 168.0\text{ N}\cdot\text{m} \qquad \mathbf{M} = 168.0\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

(*b*) Geometry:

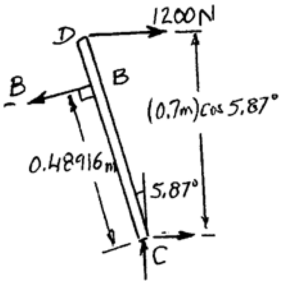

50
B
α
100cos30° = 86.60
A
400
θ
C

$$AB = 100\text{ mm}, \quad \alpha = 30°$$

$$\tan\theta = \frac{50}{486.6} \qquad \theta = 5.87°$$

$$BC = \sqrt{(50)^2 + (486.6)^2}$$
$$= 489.16\text{ mm}$$

Free body: Member *CD*:

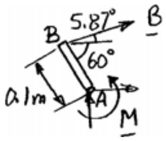

D
1200N
B
B
(0.7m)cos 5.87°
0.48916m
5.87°
C

$$+\circlearrowleft \Sigma M_C = 0: \quad B(0.48916) - (1200\text{ N})(0.7)\cos 5.87° = 0$$

$$B = 1708.2\text{ N}$$

PROBLEM 6.133 (Continued)

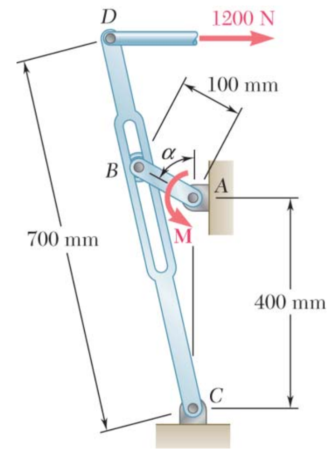
Free body: Crank *AB*:



$$+\circlearrowleft \Sigma M_A = 0: \quad M - (B \sin 65.87°)(0.1\text{ m}) = 0$$

$$M = (1708.2\text{ N})(0.1\text{ m}) \sin 65.87°$$

$$M = 155.90\text{ N}\cdot\text{m} \qquad \mathbf{M} = 155.9\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$


D
1200 N
100 mm
α
B
A
M
700 mm
400 mm
C

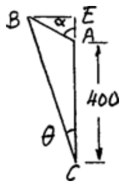
PROBLEM 6.134

Solve Problem 6.133 when (*a*) $\alpha = 60°$, (*b*) $\alpha = 90°$.

PROBLEM 6.133 The Whitworth mechanism shown is used to produce a quick-return motion of Point *D*. The block at *B* is pinned to the crank *AB* and is free to slide in a slot cut in member *CD*. Determine the couple **M** that must be applied to the crank *AB* to hold the mechanism in equilibrium when (*a*) $\alpha = 0$, (*b*) $\alpha = 30°$.

SOLUTION

(*a*) Geometry:



$$AB = 100 \text{ mm}, \quad \alpha = 60°$$

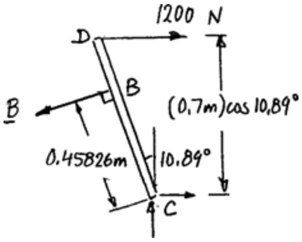
$$BE = 100\cos 60° = 86.60$$

$$CE = 400 + 100\sin 60° = 450$$

$$\tan\theta = \frac{86.60}{450} \qquad \theta = 10.89°$$

$$BC = \sqrt{(86.60)^2 + (450)^2} = 458.26 \text{ mm}$$

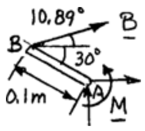
Free body: Member *CD*:


1200 N
D
B
B
(0.7m)cos 10.89°
0.45826m
10.89°
C

$$+\circlearrowleft \Sigma M_C = 0: \quad B(0.45826 \text{ m}) - (1200 \text{ N})(0.7 \text{ m})\cos 10.89° = 0$$

$$B = 1800.0 \text{ N}$$

Free body: Crank *AB*:


10.89°
B
B
30°
0.1m
A
M

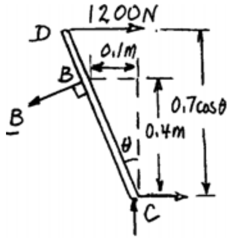
$$+\circlearrowleft \Sigma M_A = 0: \quad M - (B\sin 40.89°)(0.1 \text{ m}) = 0$$

$$M = (1800.0 \text{ N})(0.1 \text{ m})\sin 40.89°$$

$$M = 117.83 \text{ N}\cdot\text{m} \qquad \mathbf{M} = 117.8 \text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

PROBLEM 6.134 (Continued)

(*b*) Free body: Member *CD*:



$$\tan\theta = \frac{0.1\text{ m}}{0.4\text{ m}}$$

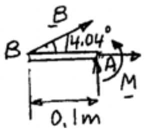
$$\theta = 14.04°$$

$$BC = \sqrt{(0.1)^2 + (0.4)^2} = 0.41231\text{ m}$$

$$+\circlearrowleft \Sigma M_C = 0: \quad B(0.41231) - (1200\text{ N})(0.7\cos 14.04°) = 0$$

$$B = 1976.4\text{ N}$$

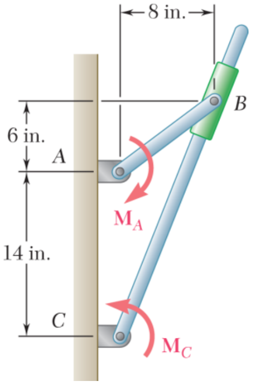
Free body: Crank *AB*:



$$+\circlearrowleft \Sigma M_A = 0: \quad M - (B\sin 14.04°)(0.1\text{ m}) = 0$$

$$M = (1976.4\text{ N})(0.1\text{ m})\sin 14.04°$$

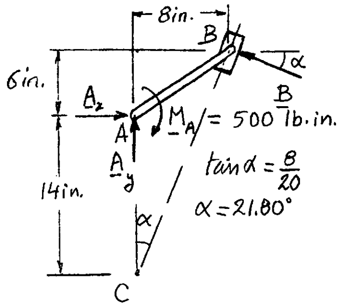
$$M = 47.948\text{ N}\cdot\text{m} \qquad \mathbf{M} = 47.9\text{ N}\cdot\text{m} \circlearrowleft ◀$$

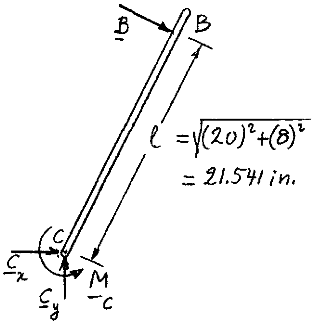

8 in.
6 in.
A
B
$\mathbf{M}_A$
14 in.
C
$\mathbf{M}_C$

PROBLEM 6.135

Two rods are connected by a frictionless collar *B*. Knowing that the magnitude of the couple $\mathbf{M}_A$ is 500 lb · in., determine (*a*) the couple $\mathbf{M}_C$ required for equilibrium, (*b*) the corresponding components of the reaction at *C*.

SOLUTION


8in.
B
6in.
A_x
A
A_y
$M_A = 500$ lb·in.
B
α
$\tan\alpha = \frac{8}{20}$
$\alpha = 21.80°$
14in.
α
C


B
B
$\ell = \sqrt{(20)^2 + (8)^2}$
= 21.541 in.
C
C_x
C_y
M_C

(*a*) Free body: Rod *AB* & collar:

$$+\circlearrowleft \Sigma M_A = 0: \quad (B\cos\alpha)(6\text{ in.}) + (B\sin\alpha)(8\text{ in.}) - M_A = 0$$

$$B = (6\cos 21.8° + 8\sin 21.8°) - 500 = 0$$

$$B = 58.535\text{ lb}$$

Free body: Rod *BC*:

$$+\Sigma M_C = 0: \quad M_C - B\ell = 0$$

$$M_C = B\ell = (58.535\text{ lb})(21.541\text{ in.}) = 1260.9\text{ lb}\cdot\text{in.}$$

$$\mathbf{M}_C = 1261\text{ lb}\cdot\text{in.} \circlearrowright$$ ◀

(*b*)

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x + B\cos\alpha = 0$$

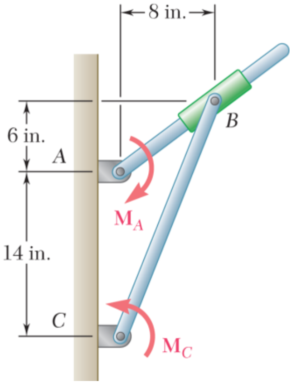
$$C_x = -B\cos\alpha = -(58.535\text{ lb})\cos 21.8° = -54.3\text{ lb}$$

$$\mathbf{C}_x = 54.3\text{ lb} \longleftarrow$$ ◀

$$+\uparrow \Sigma F_y = 0: \quad C_y - B\sin\alpha = 0$$

$$C_y = B\sin\alpha = (58.535\text{ lb})\sin 21.8° = +21.7\text{ lb}$$

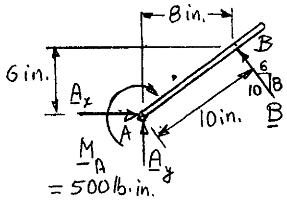
$$\mathbf{C}_y = 21.7\text{ lb}\uparrow$$ ◀

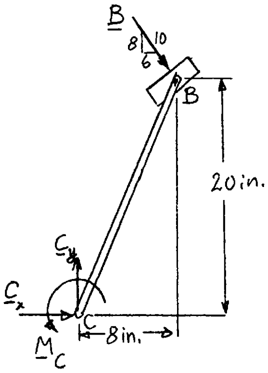


PROBLEM 6.136

Two rods are connected by a frictionless collar B. Knowing that the magnitude of the couple $\mathbf{M}_A$ is 500 lb · in., determine (a) the couple $\mathbf{M}_C$ required for equilibrium, (b) the corresponding components of the reaction at C.

SOLUTION





(a) Free body: Rod AB:

$$+\circlearrowright \Sigma M_A = 0: \quad B(10 \text{ in.}) - 500 \text{ lb} \cdot \text{in.} = 0$$

$$B = 50.0$$

Free body: Rod BC & collar:

$$+\circlearrowright \Sigma M_C = 0: \quad M_C - (0.6B)(20 \text{ in.}) - (0.8B)(8 \text{ in.}) = 0$$

$$M_C - (30 \text{ lb})(20 \text{ in.}) - (40 \text{ lb})(8 \text{ in.}) = 0$$

$$M_C = 920 \text{ lb} \cdot \text{in.} \qquad \mathbf{M}_C = 920 \text{ lb} \cdot \text{in.} \circlearrowright \blacktriangleleft$$

(b)

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x + 0.6B = 0$$

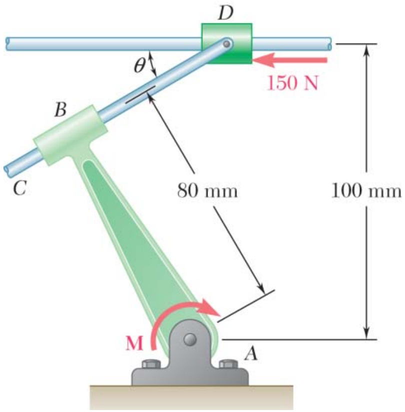
$$C_x = -0.6B = -0.6(50.0 \text{ lb}) = -30.0 \text{ lb}$$

$$\mathbf{C}_x = 30.0 \text{ lb} \longleftarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad C_y - 0.8B = 0$$

$$C_y = 0.8B = 0.8(50.0 \text{ lb}) = +40.0 \text{ lb}$$

$$\mathbf{C}_y = 40.0 \text{ lb} \uparrow \blacktriangleleft$$

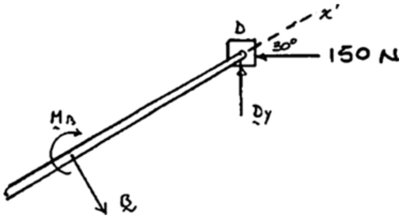

D
θ
150 N
B
C
80 mm
100 mm
M
A

PROBLEM 6.137

Rod *CD* is attached to the collar *D* and passes through a collar welded to end *B* of lever *AB*. Neglecting the effect of friction, determine the couple **M** required to hold the system in equilibrium when $\theta = 30°$.

SOLUTION

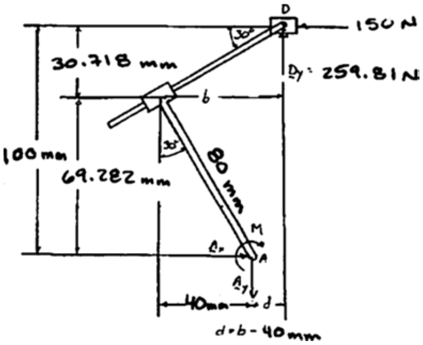
FBD *DC*:


D
x'
30°
150 N
M_B
D_y
B

$$\nearrow \Sigma F_{x'} = 0: \quad D_y \sin 30° - (150\text{ N})\cos 30° = 0$$

$$D_y = (150\text{ N})\text{ ctn }30° = 259.81\text{ N}$$

FBD machine:


D
150 N
30°
30.718 mm
D_y = 259.81 N
b
100 mm
30°
69.282 mm
80 mm
M
C_x
A
A_y
40 mm
d
d = b - 40 mm

$$\circlearrowleft \Sigma M_A = 0: \quad (0.100\text{ m})(150\text{ N}) + d(259.81\text{ N}) - M = 0$$

$$d = b - 0.040\text{ m}$$

$$b = \frac{0.030718\text{ m}}{\tan 30}$$

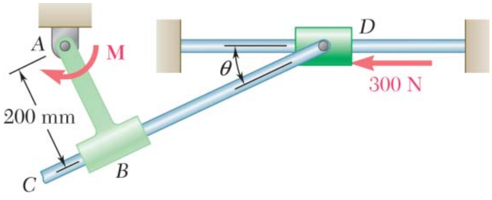
so

$$b = 0.053210\text{ m}$$

$$d = 0.0132100\text{ m}$$

$$M = 18.4321\text{ N}\cdot\text{m}$$

$$\mathbf{M} = 18.43\text{ N}\cdot\text{m} \circlearrowright \blacktriangleleft$$


A
M
D
θ
300 N
200 mm
C
B

PROBLEM 6.138

Rod *CD* is attached to the collar *D* and passes through a collar welded to end *B* of lever *AB*. Neglecting the effect of friction, determine the couple **M** required to hold the system in equilibrium when $\theta = 30°$.

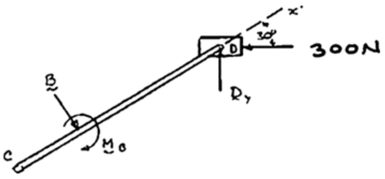
SOLUTION

Note: $\mathbf{B} \perp CD$

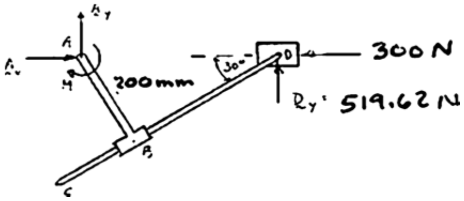
FBD *DC*:

$$\nearrow \Sigma F_{x'} = 0: \quad D_y \sin 30° - (300\text{ N})\cos 30° = 0$$

$$D_y = \frac{300\text{ N}}{\tan 30°} = 519.62\text{ N}$$

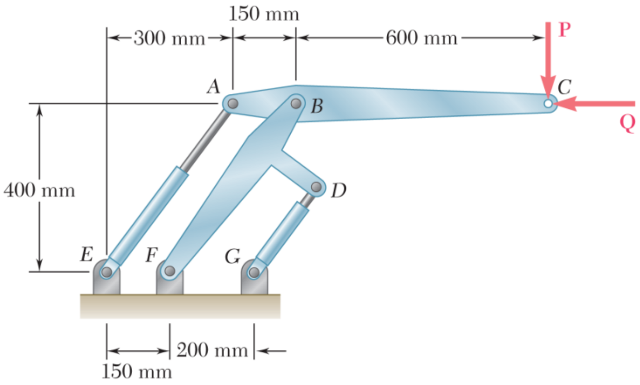

x'
30°
D
300N
D_y
B
M_B
C

FBD machine:


A_y
A
A_x
M
200mm
30°
D
300 N
D_y: 519.62 N
B
C

$$\circlearrowleft \Sigma M_A = 0: \quad \frac{0.200\text{ m}}{\sin 30°} 519.62\text{ N} - M = 0$$

$$M = 207.85\text{ N}\cdot\text{m} \qquad \mathbf{M} = 208\text{ N}\cdot\text{m} \circlearrowright \blacktriangleleft$$

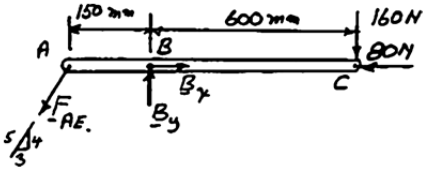

150 mm
300 mm
600 mm
P
A
B
C
Q
400 mm
D
E
F
G
200 mm
150 mm

PROBLEM 6.139

Two hydraulic cylinders control the position of the robotic arm *ABC*. Knowing that in the position shown the cylinders are parallel, determine the force exerted by each cylinder when $P = 160$ N and $Q = 80$ N.

SOLUTION

Free body: Member *ABC*:


150 mm
600 mm
160 N
80 N
A
B
C
B_x
B_y
F_{AE}
5
4
3

$$+\circlearrowleft \Sigma M_B = 0: \quad \frac{4}{5}F_{AE}(150 \text{ mm}) - (160 \text{ N})(600 \text{ mm}) = 0$$

$$F_{AE} = +800 \text{ N} \qquad F_{AE} = 800 \text{ N} \quad T \blacktriangleleft$$

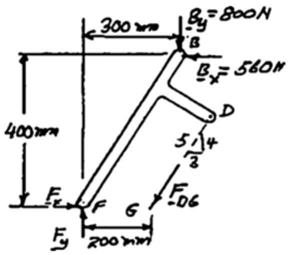
$$\xrightarrow{+} \Sigma F_x = 0: \quad -\frac{3}{5}(800 \text{ N}) + B_x - 80 \text{ N} = 0$$

$$B_x = +560 \text{ N}$$

$$+\uparrow \Sigma F_y = 0: \quad -\frac{4}{5}(800 \text{ N}) + B_y - 160 \text{ N} = 0$$

$$B_y = +800 \text{ N}$$

Free body: Member *BDF*:

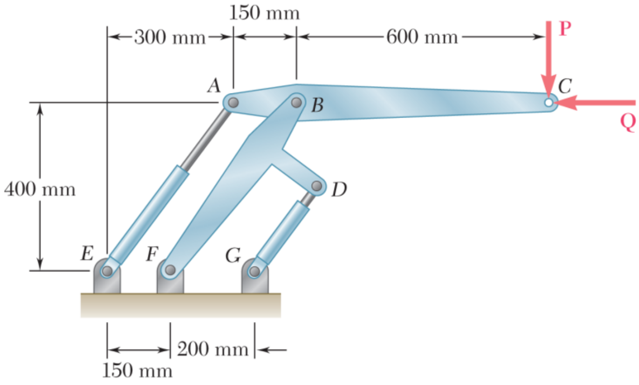

300 mm
$B_y = 800$ N
B
$B_x = 560$ N
D
400 mm
5
4
3
F_{DG}
F_x
F
G
F_y
200 mm

$$+\circlearrowleft \Sigma M_F = 0: \quad (560 \text{ N})(400 \text{ mm}) - (800 \text{ N})(300 \text{ mm}) - \frac{4}{5}F_{DG}(200 \text{ mm}) = 0$$

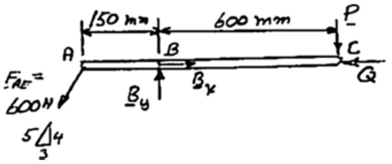
$$F_{DG} = -100 \text{ N} \qquad F_{DG} = 100.0 \text{ N} \quad C \blacktriangleleft$$

PROBLEM 6.140

Two hydraulic cylinders control the position of the robotic arm *ABC*. In the position shown, the cylinders are parallel and both are in tension. Knowing the $F_{AE} = 600 \text{ N}$ and $F_{DG} = 50 \text{ N}$, determine the forces **P** and **Q** applied at *C* to arm *ABC*.


150 mm
300 mm
600 mm
P
A
B
C
Q
400 mm
D
E
F
G
200 mm
150 mm

SOLUTION


150 mm
600 mm
P
A
B
C
Q
F_{AE} = 600N
B_y
B_x
5
4
3

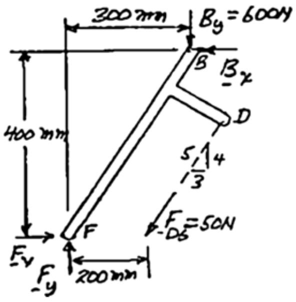
Free body: Member *ABC*:

$$+\circlearrowleft \Sigma M_B = 0\colon \quad \frac{4}{5}(600 \text{ N})(150 \text{ mm}) - P(600 \text{ mm}) = 0$$

$$P = +120 \text{ N} \qquad \mathbf{P} = 120.0 \text{ N} \downarrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_C = 0\colon \quad \frac{4}{5}(600)(750 \text{ mm}) - B_y(600 \text{ mm}) = 0$$

$$B_y = +600 \text{ N}$$


300 mm
B_y = 600N
B
B_x
D
400 mm
5
4
3
F_{DG} = 50N
F
F_x
F_y
200 mm

Free body: Member *BDF*:

$$+\circlearrowleft \Sigma M_F = 0\colon \quad B_x(400 \text{ mm}) - (600 \text{ N})(300 \text{ mm})$$

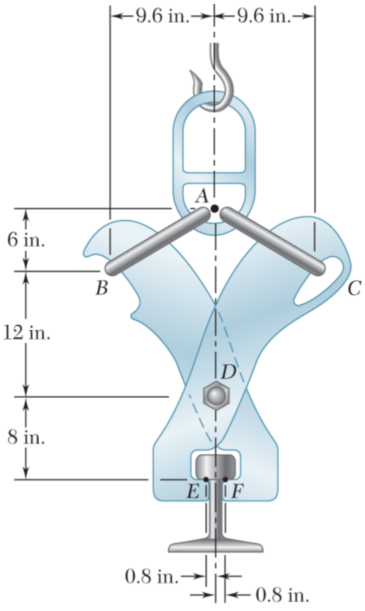
$$-\frac{4}{5}(50 \text{ N})(200 \text{ mm}) = 0$$

$$B_x = +470 \text{ N}$$

Return to free body: Member *ABC*:

$$\xrightarrow{+} \Sigma F_x = 0\colon \quad -\frac{3}{5}(600 \text{ N}) + 470 \text{ N} - Q = 0$$

$$Q = +110 \text{ N} \qquad \mathbf{Q} = 110.0 \text{ N} \longleftarrow \blacktriangleleft$$


9.6 in.
9.6 in.
A
6 in.
B
C
12 in.
D
8 in.
E
F
0.8 in.
0.8 in.

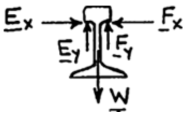
PROBLEM 6.141

A 39-ft length of railroad rail of weight 44 lb/ft is lifted by the tongs shown. Determine the forces exerted at *D* and *F* on tong *BDF*.

SOLUTION

$$W = (39\text{ ft})(44\text{ lb/ft}) = 1716\text{ lb}$$

Free body: Rail:



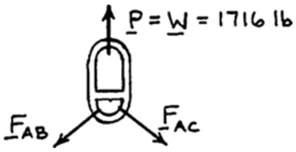
By symmetry, $E_y = F_y = \frac{1}{2}W = 858\text{ lb}$

By symmetry, $(F_{AB})_y = (F_{AC})_y = \frac{1}{2}W = 858\text{ lb}$

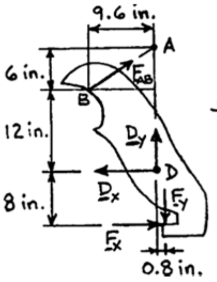
Since *AB* is a two-force member,

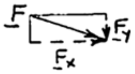
$$\frac{(F_{AB})_x}{9.6} = \frac{(F_{AB})_y}{6} \qquad (F_{AB})_x = \frac{9.6}{6}(858) = 1372.8\text{ lb}$$

Free body: Upper link:


$P = W = 1716$ lb
F_{AB}
F_{AC}

Free Body: Tong *BDF*:


9.6 in.
A
6 in.
F_{AB}
B
12 in.
D_y
D
D_x
8 in.
F_y
F_x
0.8 in.



$+\circlearrowleft \Sigma M_D = 0$: (Attach F_{AB} at *A*.)

$$F_x(8) - (F_{AB})_x(18) - F_y(0.8) = 0$$

$$F_x(8) - (1372.8\text{ lb})(18) - (858\text{ lb})(0.8) = 0$$

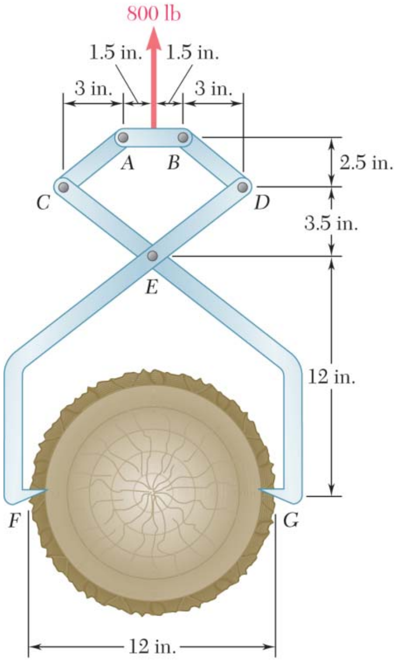
$F_x = +3174.6\text{ lb}$ **F** = 3290 lb ↘ 15.12° ◀

$\xrightarrow{+} \Sigma F_x = 0$: $-D_x + (F_{AB})_x + F_x = 0$

$$D_x = (F_{AB})_x + F_x = 1372.8 + 3174.6 = 4547.4\text{ lb}$$

$+\uparrow \Sigma F_y = 0$: $D_y + (F_{AB})_y - F_y = 0$

$D_y = 0$ **D** = 4550 lb ← ◀

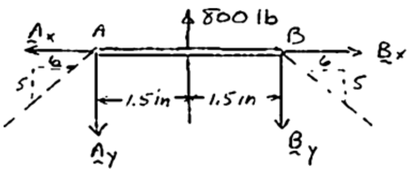

800 lb
1.5 in.
1.5 in.
3 in.
3 in.
A
B
C
D
2.5 in.
3.5 in.
E
12 in.
F
G
12 in.

PROBLEM 6.142

A log weighing 800 lb is lifted by a pair of tongs as shown. Determine the forces exerted at E and F on tong DEF.

SOLUTION

FBD *AB*:


A_x
A
800 lb
B
B_x
6
5
6
5
1.5 in
1.5 in
A_y
B_y

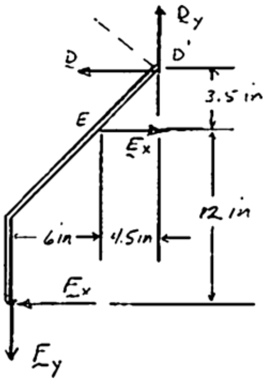
By symmetry: $A_y = B_y = 400\text{ lb}$

and
$$A_x = B_x$$
$$= \frac{6}{5}(400\text{ lb})$$
$$= 480\text{ lb}$$

Note: $\mathbf{D} = -\mathbf{B}$

so
$$D_x = 480\text{ lb}$$
$$D_y = 400\text{ lb}$$

FBD *DEF*:


D_y
D_x
D'
3.5 in
E
E_x
12 in
6 in
4.5 in
F_x
F_y

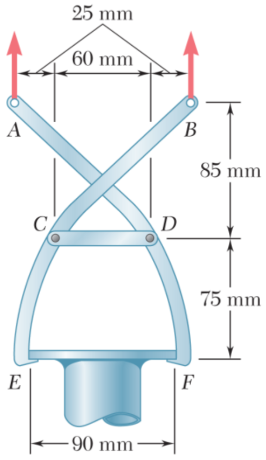
$$\circlearrowleft \Sigma M_F = (10.5\text{ in.})(400\text{ lb}) + (15.5\text{ in.})(480\text{ lb}) - (12\text{ in.})E_x = 0$$

$$E_x = 970\text{ lb} \qquad \mathbf{E} = 970\text{ lb} \longrightarrow \blacktriangleleft$$

$$\xrightarrow{} \Sigma F_x = 0{:} \quad -480\text{ lb} + 970\text{ lb} - F_x = 0$$
$$F_x = 490\text{ lb}$$

$$\uparrow \Sigma F_y = 0{:} \quad 400\text{ lb} - F_y = 0$$
$$F_y = 400\text{ lb}$$

$$\mathbf{F} = 633\text{ lb} \nearrow 39.2^\circ \blacktriangleleft$$

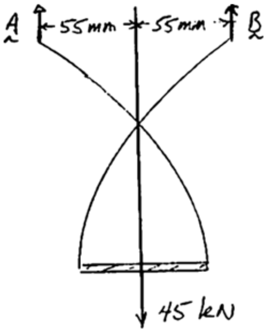

25 mm
60 mm
A
B
85 mm
C
D
75 mm
E
F
90 mm

PROBLEM 6.143

The tongs shown are used to apply a total upward force of 45 kN on a pipe cap. Determine the forces exerted at *D* and *F* on tong *ADF*.

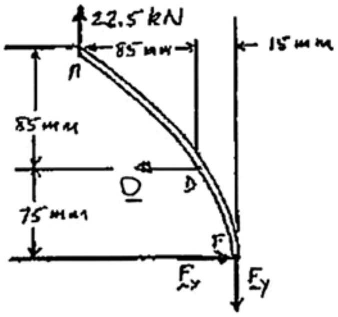
SOLUTION

FBD whole:


A
55mm
55mm
B
45 kN

By symmetry, $\mathbf{A} = \mathbf{B} = 22.5\text{ kN}\uparrow$

FBD *ADF*:


22.5 kN
85 mm
15 mm
A
85 mm
75 mm
D
D
F
F_x
F_y

$$\circlearrowleft \Sigma M_F = 0: \quad (75\text{ mm})D - (100\text{ mm})(22.5\text{ kN}) = 0$$

$$\mathbf{D} = 30.0\text{ kN} \longleftarrow \blacktriangleleft$$

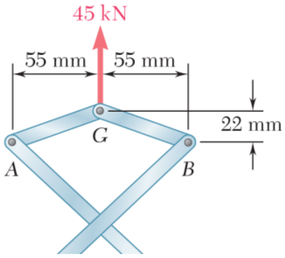
$$\longrightarrow \Sigma F_x = 0: \quad F_x - D = 0$$

$$F_x = D = 30\text{ kN}$$

$$\uparrow \Sigma F_y = 0: \quad 22.5\text{ kN} - F_y = 0$$

$$F_y = 22.5\text{ kN}$$

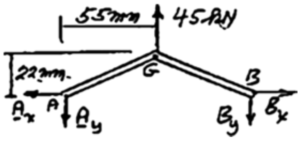
so $$\mathbf{F} = 37.5\text{ kN} \searrow 36.9^\circ \blacktriangleleft$$


45 kN
55 mm
55 mm
22 mm
G
A
B

PROBLEM 6.144

If the toggle shown is added to the tongs of Problem 6.141 and a single vertical force is applied at *G*, determine the forces exerted at *D* and *F* on tong *ADF*.

SOLUTION


55mm
45kN
22mm
G
A
B
A_x
A_y
B_y
B_x

Free body: Toggle:

By symmetry,

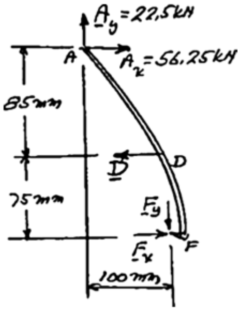
$$A_y = \frac{1}{2}(45\text{ kN}) = 22.5\text{ kN}$$

AG is a two-force member.

$$\frac{22.5\text{ kN}}{22\text{ mm}} = \frac{A_x}{55\text{ mm}}$$

$$A_x = 56.25\text{ kN}$$

Free body: Tong *ADF*:


A_y=22.5kN
A
A_x=56.25kN
85mm
D
D
75mm
F_y
F_x
F
100mm

$$+\uparrow \Sigma F_y = 0: \quad 22.5\text{ kN} - F_y = 0$$

$$F_y = +22.5\text{ kN}$$

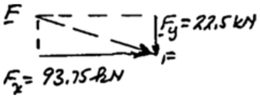
$$+\circlearrowleft \Sigma M_F = 0: \quad D(75\text{ mm}) - (22.5\text{ kN})(100\text{ mm}) - (56.25\text{ kN})(160\text{ mm}) = 0$$

$$D = +150\text{ kN}$$

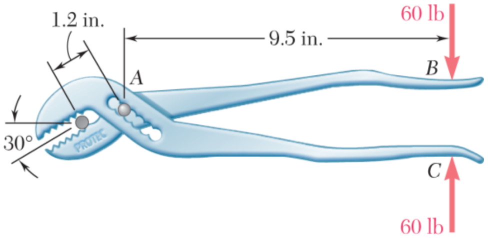
D = 150.0 kN ← ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad 56.25\text{ kN} - 150\text{ kN} + F_x = 0$$

$$F_x = 93.75\text{ kN}$$


F
F_y=22.5kN
F
F_x= 93.75kN

F = 96.4 kN ↖ 13.50° ◀


1.2 in.
9.5 in.
60 lb
A
B
30°
PROTEC
C
60 lb

PROBLEM 6.145

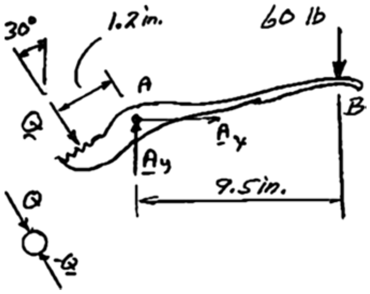
The pliers shown are used to grip a 0.3-in.-diameter rod. Knowing that two 60-lb forces are applied to the handles, determine (*a*) the magnitude of the forces exerted on the rod, (*b*) the force exerted by the pin at *A* on portion *AB* of the pliers.

SOLUTION

Free body: Portion *AB*:

(*a*) $+\circlearrowleft \Sigma M_A = 0: \quad Q(1.2 \text{ in.}) - (60 \text{ lb})(9.5 \text{ in.}) = 0$

$$Q = 475 \text{ lb} \blacktriangleleft$$


30°
1.2 in.
60 lb
A
B
Q
A_x
A_y
9.5 in.
Q
-Q

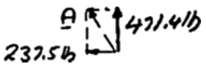
(*b*) $\xrightarrow{+} \Sigma F_x = 0: \quad Q(\sin 30°) + A_x = 0$

$$(475 \text{ lb})(\sin 30°) + A_x = 0$$

$$A_x = -237.5 \text{ lb} \qquad \mathbf{A}_x = 237.5 \text{ lb} \longleftarrow$$

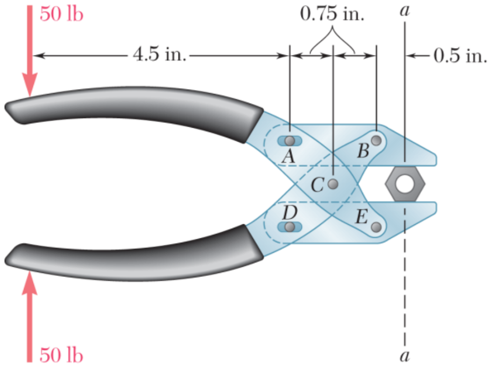
$+\uparrow \Sigma F_y = 0: \quad -Q(\cos 30°) + A_y - 60 \text{ lb} = 0$

$$-(475 \text{ lb})(\cos 30°) + A_y - 60 \text{ lb} = 0$$


A
471.4 lb
237.5 lb

$$A_y = +471.4 \text{ lb} \qquad \mathbf{A}_y = 471.4 \text{ lb} \uparrow$$

$$\mathbf{A} = 528 \text{ lb} \measuredangle 63.3° \blacktriangleleft$$

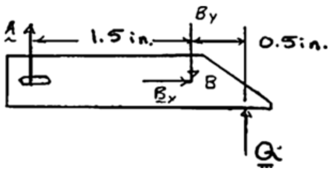

50 lb
0.75 in.
a
4.5 in.
0.5 in.
A
B
C
D
E
50 lb
a

PROBLEM 6.146

Determine the magnitude of the gripping forces exerted along line *aa* on the nut when two 50-lb forces are applied to the handles as shown. Assume that pins *A* and *D* slide freely in slots cut in the jaws.

SOLUTION

FBD jaw *AB*:


A
1.5 in.
B_y
0.5 in.
B_x
B
Q

$$\xrightarrow{} \Sigma F_x = 0: \quad B_x = 0$$

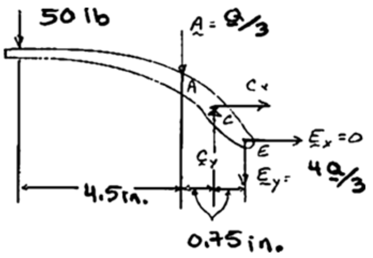
$$\circlearrowleft \Sigma M_B = 0: \quad (0.5 \text{ in.})Q - (1.5 \text{ in.})A = 0$$

$$A = \frac{Q}{3}$$

$$\uparrow \Sigma F_y = 0: \quad A + Q - B_y = 0$$

$$B_y = A + Q = \frac{4Q}{3}$$

FBD handle *ACE*:


50 lb
$A = Q/3$
A
C_x
C
C_y
E
$E_x = 0$
$E_y = 4Q/3$
4.5 in.
0.75 in.

By symmetry and FBD jaw *DE*,

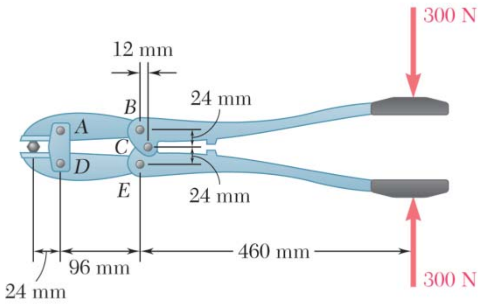
$$D = A = \frac{Q}{3}$$

$$E_x = B_x = 0$$

$$E_y = B_y = \frac{4Q}{3}$$

$$\circlearrowleft \Sigma M_C = 0: \quad (5.25 \text{ in.})(50 \text{ lb}) + (0.75 \text{ in.})\frac{Q}{3} - (0.75 \text{ in.})\frac{4Q}{3} = 0$$

$$Q = 350 \text{ lb} \blacktriangleleft$$

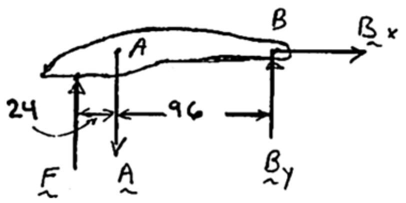

300 N
12 mm
B
24 mm
A
C
D
E
24 mm
96 mm
460 mm
24 mm
300 N

PROBLEM 6.147

In using the bolt cutter shown, a worker applies two 300-N forces to the handles. Determine the magnitude of the forces exerted by the cutter on the bolt.

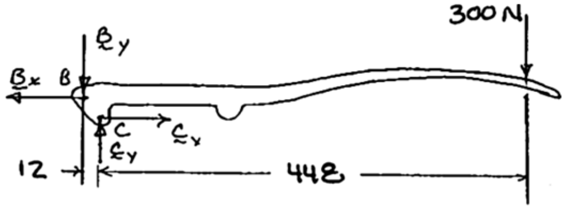
SOLUTION

FBD cutter *AB*:


B
$\underset{\sim}{B}_x$
A
24
96
$\underset{\sim}{F}$
$\underset{\sim}{A}$
$\underset{\sim}{B}_y$

I

FBD handle *BC*:


$\underset{\sim}{B}_y$
300 N
$\underset{\sim}{B}_x$
B
C
$\underset{\sim}{C}_x$
$\underset{\sim}{C}_y$
12
448

II

Dimensions in mm

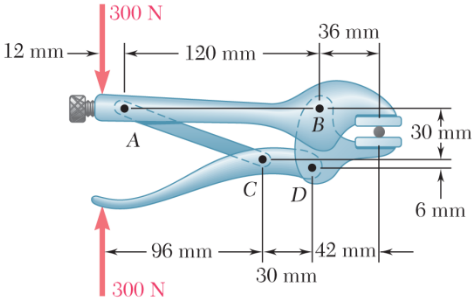
FBD I: $\xrightarrow{} \Sigma F_x = 0\!: \quad B_x = 0$

FBD II: $\circlearrowleft \Sigma M_C = 0\!: \quad (12 \text{ mm})B_y - (448 \text{ mm})300 \text{ N} = 0$

$$B_y = 11{,}200.0 \text{ N}$$

Then FBD I: $\circlearrowleft \Sigma M_A = 0\!: \quad (96 \text{ mm})B_y - (24 \text{ mm})F = 0 \qquad F = 4B_y$

$$F = 44{,}800 \text{ N} = 44.8 \text{ kN} \blacktriangleleft$$


300 N
12 mm
120 mm
36 mm
B
A
30 mm
C
D
6 mm
96 mm
42 mm
30 mm
300 N

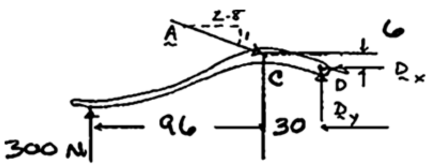
PROBLEM 6.148

Determine the magnitude of the gripping forces produced when two 300-N forces are applied as shown.

SOLUTION

We note that *AC* is a two-force member.

FBD handle *CD*:

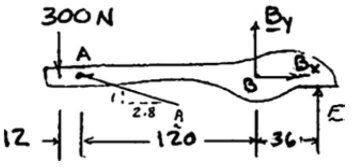

A
2.8
1
6
C
D
D_x
D_y
300 N
96
30

Dimensions in mm

$$\circlearrowleft \Sigma M_D = 0: \quad -(126\text{ mm})(300\text{ N}) - (6\text{ mm})\frac{2.8}{\sqrt{8.84}}A + (30\text{ mm})\left(\frac{1}{\sqrt{8.84}}A\right) = 0$$

$$A = 2863.6\sqrt{8.84}\text{ N}$$

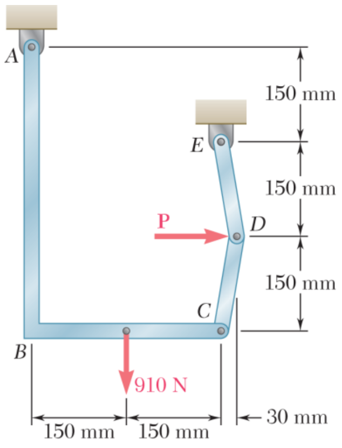
FBD handle *AB*:


300 N
A
B_y
B_x
B
2.8
1
A
E
12
120
36

Dimensions in mm

$$\circlearrowleft \Sigma M_B = 0: \quad (132\text{ mm})(300\text{ N}) - (120\text{ mm})\frac{1}{\sqrt{8.84}}(2863.6\sqrt{8.84}\text{ N}) + (36\text{ mm})F = 0$$

$$F = 8.45\text{ kN} \blacktriangleleft$$


A
E
150 mm
150 mm
P
D
150 mm
C
B
910 N
150 mm
150 mm
30 mm

PROBLEM 6.149

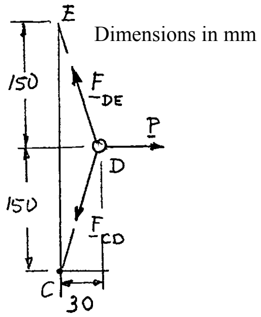
Determine the force **P** that must be applied to the toggle *CDE* to maintain bracket *ABC* in the position shown.

SOLUTION

We note that *CD* and *DE* are two-force members.

Free body: Joint *D*:

$$\frac{(F_{CD})_x}{30} = \frac{(F_{CD})_y}{150} \qquad (F_{CD})_y = 5(F_{CD})_x$$


E
150
F_DE
P
D
150
F_CD
C
30

Dimensions in mm

Similarly, $$(F_{DE})_y = 5(F_{DE})_x$$

$$+\uparrow \Sigma F_y = 0: \quad (F_{DE})_y = (F_{CD})_y$$

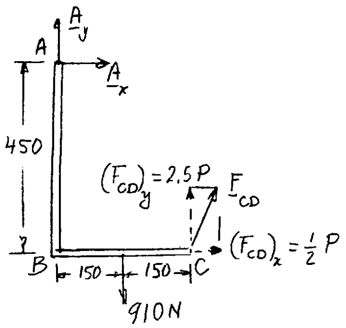
It follows that $$(F_{DE})_x = (F_{CD})_x$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad P - (F_{DE})_x - (F_{CD})_x = 0$$

$$(F_{DE})_x = (F_{CD})_x = \frac{1}{2}P$$

Also, $$(F_{DE})_y = (F_{CD})_y = 5\left(\frac{1}{2}P\right) = 2.5P$$

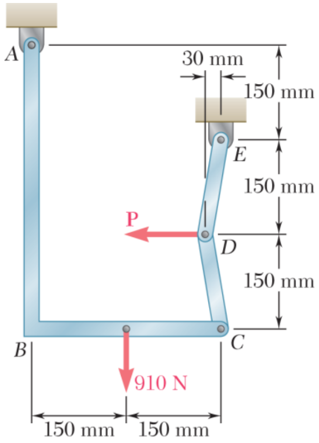
Free body: Member *ABC*:


A_y
A
A_x
450
$(F_{CD})_y = 2.5P$
F_CD
$(F_{CD})_x = \frac{1}{2}P$
B
150
150
C
910N

$$+\circlearrowleft \Sigma M_A = 0: \quad (2.5P)(300) + \left(\frac{1}{2}P\right)(450) - (910\text{ N})(150) = 0$$

$$(750 + 225)P = (910\text{ N})(150)$$

$$P = 140.0\text{ N} \blacktriangleleft$$


A
30 mm
150 mm
E
150 mm
P
D
150 mm
B
C
910 N
150 mm
150 mm

PROBLEM 6.150

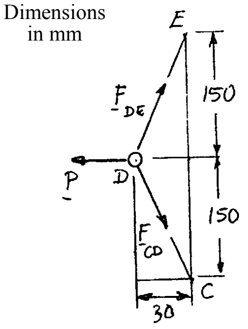
Determine the force **P** that must be applied to the toggle *CDE* to maintain bracket *ABC* in the position shown.

SOLUTION

We note that *CD* and *DE* are two-force members.

Free body: Joint *D*:

$$\frac{(F_{CD})_x}{30} = \frac{(F_{CD})_y}{150} \qquad (F_{CD})_y = 5(F_{CD})_x$$


Dimensions in mm
E
F_{DE}
150
P
D
150
F_{CD}
C
30

Similarly, $(F_{DE})_y = 5(F_{DE})_x$

$$+\uparrow \Sigma F_y = 0: \quad (F_{DE})_y = (F_{CD})_y$$

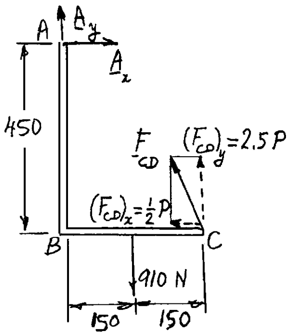
It follows that $(F_{DE})_x = (F_{CD})_x$

$$\xrightarrow{+} \Sigma F_x = 0: \quad (F_{DE})_x + (F_{CD})_x - P = 0$$

$$(F_{DE})_x = (F_{CD})_x = \frac{1}{2}P$$

Also, $(F_{DE})_y = (F_{CD})_y = 5\left(\frac{1}{2}P\right) = 2.5P$

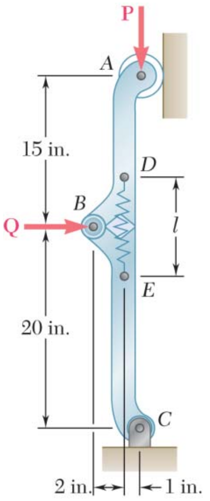
Free body: Member *ABC*:


A
A_y
A_x
450
F_{CD}
$(F_{CD})_y = 2.5P$
$(F_{CD})_x = \frac{1}{2}P$
B
C
910 N
150
150

$$+\circlearrowleft \Sigma M_A = 0: \quad (2.5P)(300) - \left(\frac{1}{2}P\right)(450) - (910 \text{ N})(150) = 0$$

$$(750 - 225)P = (910 \text{ N})(150)$$

$P = 260$ N ◀

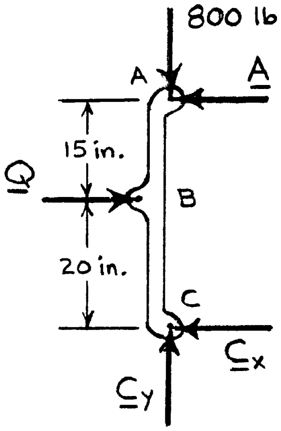


PROBLEM 6.151

Since the brace shown must remain in position even when the magnitude of **P** is very small, a single safety spring is attached at *D* and *E*. The spring *DE* has a constant of 50 lb/in. and an unstretched length of 7 in. Knowing that *l* = 10 in. and that the magnitude of **P** is 800 lb, determine the force **Q** required to release the brace.

SOLUTION

Free body:



$$+\circlearrowright \Sigma M_A = 0: \quad Q(15 \text{ in.}) - C_x(35 \text{ in.}) = 0$$

$$Q = \frac{7}{3}C_x \quad (1)$$

$$+\uparrow \Sigma F_y = 0: \quad C_y - 800 \text{ lb} = 0 \quad \text{or} \quad C_y = 800 \text{ lb}$$

Spring force.

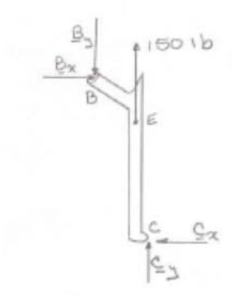
Unstretched length: $\ell_0 = 7$ in.

Stretched length: $\ell = 10$ in.

$k = 50$ lb/in.

$$F = kx = k(\ell - \ell_0)$$
$$= 50(10-7) = 150 \text{ lb}$$

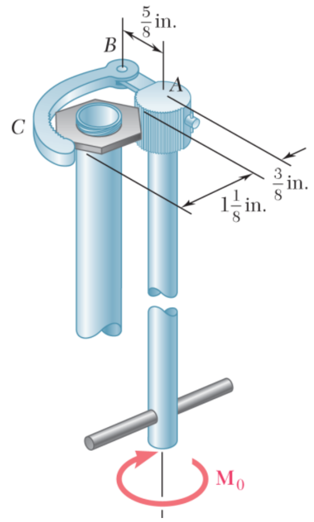
Free body: Portion *BC*



$$+\circlearrowright \Sigma M_B = 0: \quad (150 \text{ lb})(2 \text{ in.}) + (800 \text{ lb})(3 \text{ in.}) - C_x(20 \text{ in.}) = 0$$

$$C_x = 135.0 \text{ lb}$$

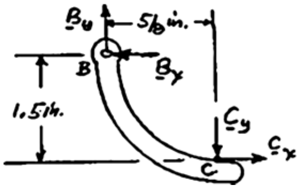
Substituting into equation (1) gives $\mathbf{Q} = 315$ lb ◀



PROBLEM 6.152

The specialized plumbing wrench shown is used in confined areas (e.g., under a basin or sink). It consists essentially of a jaw *BC* pinned at *B* to a long rod. Knowing that the forces exerted on the nut are equivalent to a clockwise (when viewed from above) couple of magnitude 135 lb · in., determine (*a*) the magnitude of the force exerted by pin *B* on jaw *BC*, (*b*) the couple $\mathbf{M}_0$ that is applied to the wrench.

SOLUTION



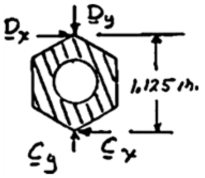
Free body: Jaw *BC*:

This is a two-force member.

$$\frac{C_y}{1.5\text{ in.}} = \frac{C_x}{\frac{5}{8}\text{ in.}} \qquad C_y = 2.4C_x$$

$$\Sigma F_x = 0: \quad B_x = C_x \qquad (1)$$

$$\Sigma F_y = 0: \quad B_y = C_y = 2.4\,C_x \qquad (2)$$



Free body: Nut: $\Sigma F_x = 0: \quad C_x = D_x$

$$\circlearrowright \Sigma M = 135\text{ lb}\cdot\text{in.}$$

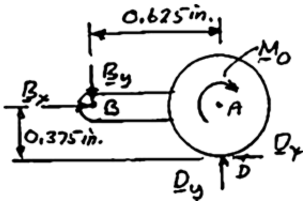
$$C_x(1.125\text{ in.}) = 135\text{ lb}\cdot\text{in.}$$

$$C_x = 120\text{ lb}$$

(*a*) Eq. (1): $B_x = C_x = 120\text{ lb}$

Eq. (2): $B_y = C_y = 2.4(120\text{ lb}) = 288\text{ lb}$

$$B = \left(B_x^2 + B_y^2\right)^{1/2} = (120^2 + 288^2)^{1/2} \qquad B = 312\text{ lb} \blacktriangleleft$$

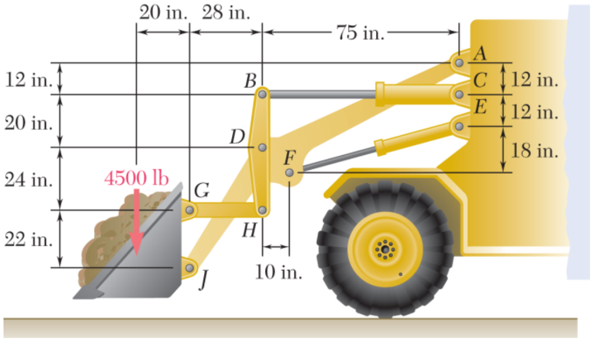


(*b*) Free body: Rod:

$$+\circlearrowright \Sigma M_D = 0: \quad -M_0 + B_y(0.625\text{ in.}) - B_x(0.375\text{ in.}) = 0$$

$$-M_0 + (288)(0.625) - (120)(0.375) = 0$$

$$\mathbf{M}_0 = 135.0\text{ lb}\cdot\text{in.} \circlearrowright \blacktriangleleft$$

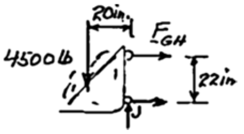

20 in.
28 in.
75 in.
12 in.
20 in.
24 in.
22 in.
4500 lb
A
B
C
D
E
F
G
H
J
12 in.
12 in.
18 in.
10 in.

PROBLEM 6.153

The motion of the bucket of the front-end loader shown is controlled by two arms and a linkage that are pin-connected at *D*. The arms are located symmetrically with respect to the central, vertical, and longitudinal plane of the loader; one arm *AFJ* and its control cylinder *EF* are shown. The single linkage *GHDB* and its control cylinder *BC* are located in the plane of symmetry. For the position and loading shown, determine the force exerted (*a*) by cylinder *BC*, (*b*) by cylinder *EF*.

SOLUTION

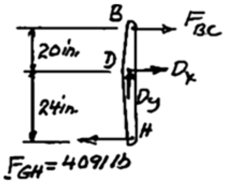
Free body: Bucket


20in.
4500lb
F_{GH}
22in.
J

$$+\circlearrowleft \Sigma M_J = 0: \quad (4500\text{ lb})(20\text{ in.}) - F_{GH}(22\text{ in.}) = 0$$

$$F_{GH} = 4091\text{ lb}$$

Free body: Arm *BDH*


B
F_{BC}
20in.
D
D_x
24in.
D_y
H
F_{GH} = 4091 lb

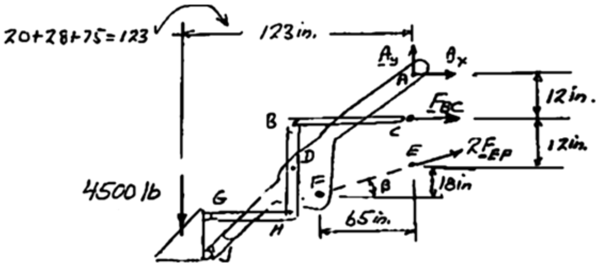
$$+\circlearrowleft \Sigma M_D = 0: \quad -(4091\text{ lb})(24\text{ in.}) - F_{BC}(20\text{ in.}) = 0$$

$$F_{BC} = -4909\text{ lb}$$

$F_{BC} = 4.91$ kips C ◀

Free body: Entire mechanism

(Two arms and cylinders *AFJE*)


20+28+75=123
123in.
A_y
A_x
A
12in.
B
F_{BC}
C
12in.
E
$2F_{EF}$
D
F
β
18in
4500 lb
G
H
J
65in.

Note: Two arms thus $2F_{EF}$

$$\tan\beta = \frac{18\text{ in.}}{65\text{ in.}}$$

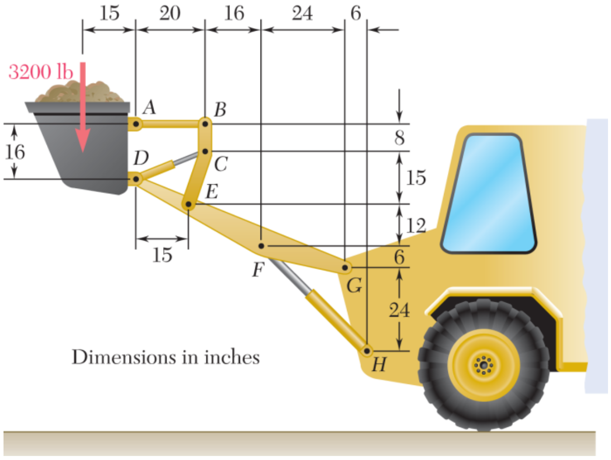
$$\beta = 15.48°$$

$$+\circlearrowleft \Sigma M_A = 0: \quad (4500\text{ lb})(123\text{ in.}) + F_{BC}(12\text{ in.}) + 2F_{EF}\cos\beta(24\text{ in.}) = 0$$

$$(4500\text{ lb})(123\text{ in.}) - (4909\text{ lb})(12\text{ in.}) + 2F_{EF}\cos 15.48°(24\text{ in.}) = 0$$

$F_{EF} = -10.690$ lb

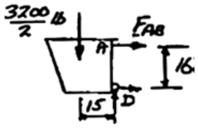
$F_{EF} = 10.69$ kips C ◀


15
20
16
24
6
3200 lb
A
B
16
D
C
E
8
15
12
6
24
15
F
G
H
Dimensions in inches

PROBLEM 6.154

The bucket of the front-end loader shown carries a 3200-lb load. The motion of the bucket is controlled by two identical mechanisms, only one of which is shown. Knowing that the mechanism shown supports one-half of the 3200-lb load, determine the force exerted (*a*) by cylinder *CD*, (*b*) by cylinder *FH*.

SOLUTION

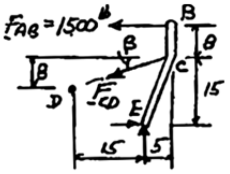

3200/2 lb
F_AB
A
16
15
D

Free body: Bucket: (one mechanism)

$$+\circlearrowleft \Sigma M_D = 0: \quad (1600\text{ lb})(15\text{ in.}) - F_{AB}(16\text{ in.}) = 0$$

$$F_{AB} = 1500\text{ lb}$$

Note: There are two identical support mechanisms.


F_AB = 1500 lb
B
β
8
C
β
D
F_CD
E
15
15
5

Free body: One arm *BCE*:

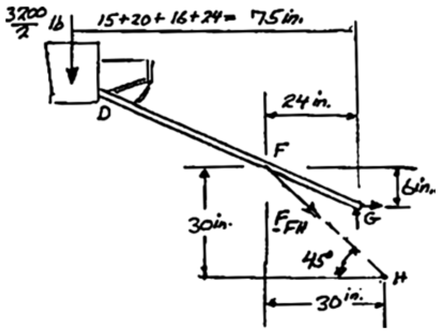
$$\tan\beta = \frac{8}{20}$$

$$\beta = 21.8°$$

$$+\circlearrowleft \Sigma M_E = 0: \quad (1500\text{ lb})(23\text{ in.}) + F_{CD}\cos 21.8°(15\text{ in.}) - F_{CD}\sin 21.8°(5\text{ in.}) = 0$$

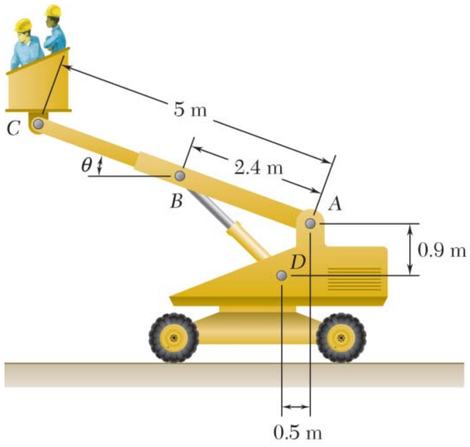
$$F_{CD} = -2858\text{ lb} \qquad F_{CD} = 2.86\text{ kips} \quad C$$ ◀

Free body: Arm *DFG*:


3200/2 lb
15+20+16+24= 75 in.
24 in.
D
F
6 in.
30 in.
F_FH
G
45°
H
30 in.

$$+\circlearrowleft \Sigma M_G = 0: \quad (1600\text{ lb})(75\text{ in.}) + F_{FH}\sin 45°(24\text{ in.}) - F_{FH}\cos 45°(6\text{ in.}) = 0$$

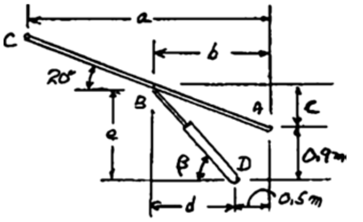
$$F_{FH} = -9.428\text{ kips} \qquad F_{FH} = 9.43\text{ kips} \quad C$$ ◀



PROBLEM 6.155

The telescoping arm *ABC* is used to provide an elevated platform for construction workers. The workers and the platform together have a mass of 200 kg and have a combined center of gravity located directly above *C*. For the position when $\theta = 20°$, determine (*a*) the force exerted at *B* by the single hydraulic cylinder *BD*, (*b*) the force exerted on the supporting carriage at *A*.

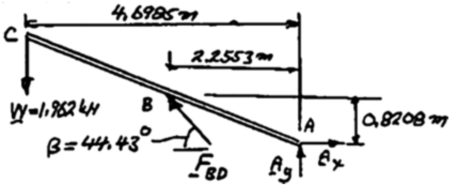
SOLUTION



Geometry:

$$a = (5\text{ m})\cos 20° = 4.6985\text{ m}$$
$$b = (2.4\text{ m})\cos 20° = 2.2553\text{ m}$$
$$c = (2.4\text{ m})\sin 20° = 0.8208\text{ m}$$
$$d = b - 0.5 = 1.7553\text{ m}$$
$$e = c + 0.9 = 1.7208\text{ m}$$

$$\tan\beta = \frac{e}{d} = \frac{1.7208}{1.7553}; \quad \beta = 44.43°$$



Free body: Arm *ABC*:

We note that *BD* is a two-force member.

$$W = (200\text{ kg})(9.81\text{ m/s}^2) = 1.962\text{ kN}$$

(*a*) $+\circlearrowleft \Sigma M_A = 0\colon \quad (1.962\text{ kN})(4.6985\text{ m}) - F_{BD}\sin 44.43°(2.2553\text{ m}) + F_{BD}\cos 44.43(0.8208\text{ m}) = 0$

$$9.2185 - F_{BD}(0.9927) = 0\colon \quad F_{BD} = 9.2867\text{ kN}$$

(*b*) $\mathbf{F}_{BD} = 9.29\text{ kN} \searrow 44.4°$ ◀

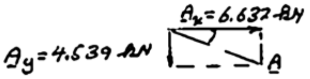
$$\xrightarrow{+} \Sigma F_x = 0\colon \quad A_x - F_{BD}\cos\beta = 0$$

$$A_x = (9.2867\text{ kN})\cos 44.43° = 6.632\text{ kN}$$

$\mathbf{A}_x = 6.632\text{ kN} \longrightarrow$

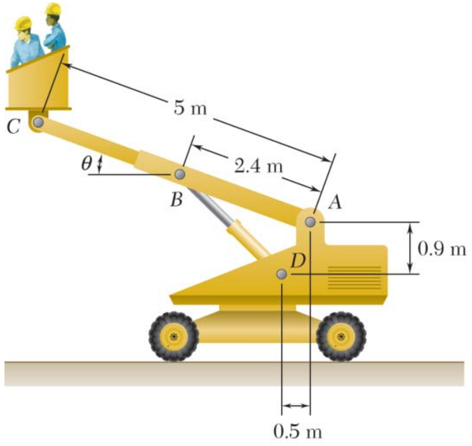
$$+\uparrow \Sigma F_y = 0\colon \quad A_y - 1.962\text{ kN} + F_{BD}\sin\beta = 0$$

$$A_y = 1.962\text{ kN} - (9.2867\text{ kN})\sin 44.43° = -4.539\text{ kN}$$



$\mathbf{A}_y = 4.539\text{ kN}\downarrow$

$\mathbf{A} = 8.04\text{ kN} \searrow 34.4°$ ◀



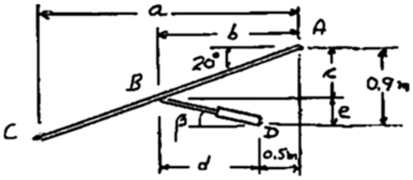
PROBLEM 6.156

The telescoping arm *ABC* of Prob. 6.155 can be lowered until end *C* is close to the ground, so that workers can easily board the platform. For the position when $\theta = -20°$, determine (*a*) the force exerted at *B* by the single hydraulic cylinder *BD*, (*b*) the force exerted on the supporting carriage at *A*.

SOLUTION

Geometry:

$$a = (5\text{ m})\cos 20° = 4.6985\text{ m}$$
$$b = (2.4\text{ m})\cos 20° = 2.2552\text{ m}$$
$$c = (2.4\text{ m})\sin 20° = 0.8208\text{ m}$$
$$d = b - 0.5 = 1.7553\text{ m}$$
$$e = 0.9 - c = 0.0792\text{ m}$$

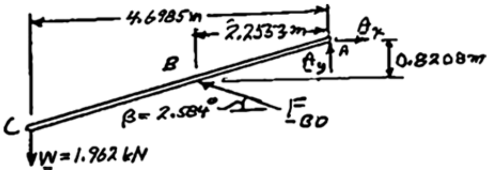


$$\tan\beta = \frac{e}{d} = \frac{0.0792}{1.7552}; \quad \beta = 2.584°$$

Free body: Arm *ABC*:

We note that *BD* is a two-force member.

$$W = (200\text{ kg})(9.81\text{ m/s}^2)$$
$$W = 1962\text{ N} = 1.962\text{ kN}$$



(*a*) $\quad +\circlearrowleft \Sigma M_A = 0: \quad (1.962\text{ kN})(4.6985\text{ m}) - F_{BD}\sin 2.584°(2.2553\text{ m}) - F_{BD}\cos 2.584°(0.8208\text{ m}) = 0$

$$9.2185 - F_{BD}(0.9216) = 0 \quad F_{BD} = 10.003\text{ kN}$$

(*b*) $\quad \mathbf{F}_{BD} = 10.00\text{ kN} \searrow 2.58°$ ◀

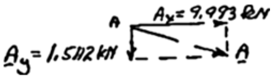
$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - F_{BD}\cos\beta = 0$$

$$A_x = (10.003\text{ kN})\cos 2.583° = 9.993\text{ kN} \qquad \mathbf{A}_x = 9.993\text{ kN} \longrightarrow$$

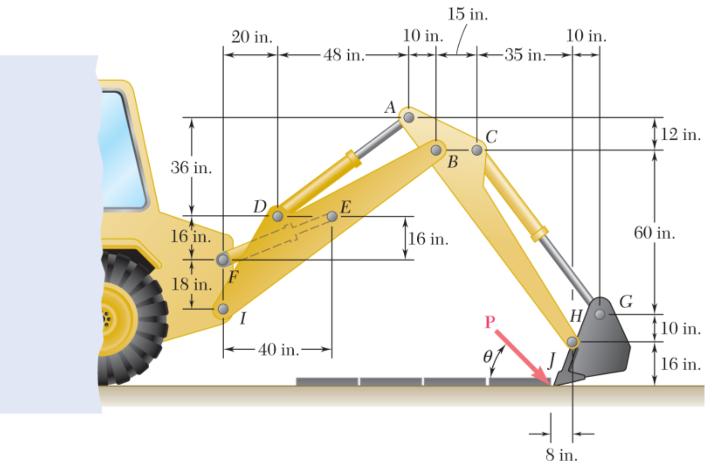
$$+\uparrow \Sigma F_y = 0: \quad A_y - 1.962\text{ kN} + F_{BD}\sin\beta = 0$$

$$A_y = 1.962\text{ kN} - (10.003\text{ kN})\sin 2.583° = -1.5112\text{ kN}$$

$$\mathbf{A}_y = 1.5112\text{ kN}\downarrow$$



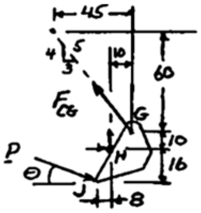
$\mathbf{A} = 10.11\text{ kN} \searrow 8.60°$ ◀


15 in.
20 in.
10 in.
48 in.
35 in.
10 in.
A
C
B
12 in.
36 in.
D
E
16 in.
16 in.
60 in.
18 in.
F
I
G
H
P
10 in.
40 in.
θ
J
16 in.
8 in.

PROBLEM 6.157

The motion of the backhoe bucket shown is controlled by the hydraulic cylinders *AD*, *CG*, and *EF*. As a result of an attempt to dislodge a portion of a slab, a 2-kip force **P** is exerted on the bucket teeth at *J*. Knowing that $\theta = 45°$, determine the force exerted by each cylinder.

SOLUTION


45
4
5
3
10
60
F_{CG}
G
P
10
H
16
θ
J
8

Free body: Bucket:

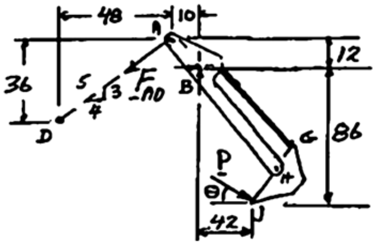
$$+\circlearrowleft \Sigma M_H = 0: \quad \text{(Dimensions in inches)}$$

$$\frac{4}{5}F_{CG}(10) + \frac{3}{5}F_{CG}(10) + P\cos\theta(16) + P\sin\theta(8) = 0$$

$$F_{CG} = -\frac{P}{14}(16\cos\theta + 8\sin\theta) \qquad (1)$$

Free body: Arm *ABH* and bucket:

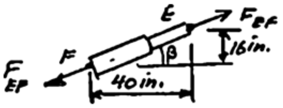
(Dimensions in inches)


48
10
A
12
36
5
3
4
F_{AD}
B
D
G
86
P
θ
H
42
J

$$+\circlearrowleft \Sigma M_B = 0: \quad \frac{4}{5}F_{AD}(12) + \frac{3}{5}F_{AD}(10) + P\cos\theta(86) - P\sin\theta(42) = 0$$

$$F_{AD} = -\frac{P}{15.6}(86\cos\theta - 42\sin\theta) \qquad (2)$$

Free body: Bucket and arms *IEB* + *ABH*:

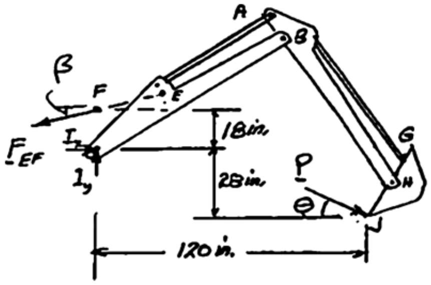

E
F_{EF}
F
β
16 in.
F_{EF}
40 in.

Geometry of cylinder *EF*:

$$\tan\beta = \frac{16 \text{ in.}}{40 \text{ in.}}$$

$$\beta = 21.801°$$

PROBLEM 6.157 (Continued)



$$+\circlearrowleft \Sigma M_I = 0: \quad F_{EF}\cos\beta(18\text{ in.}) + P\cos\theta(28\text{ in.}) - P\sin\theta(120\text{ in.}) = 0$$

$$F_{EF} = \frac{P(120\sin\theta - 28\cos\theta)}{\cos 21.8°(18)}$$

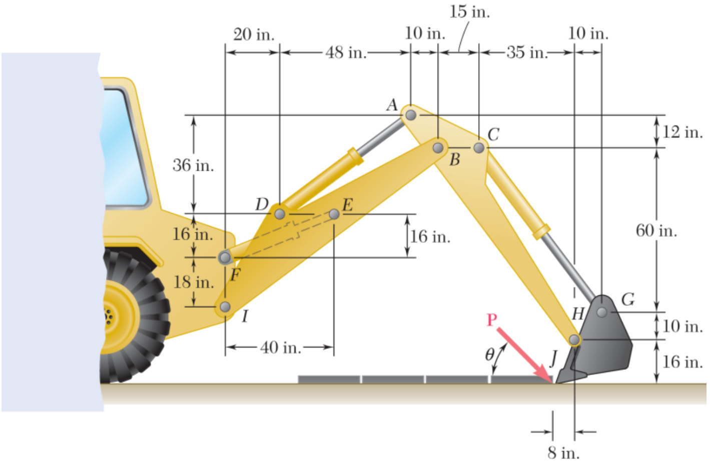
$$= \frac{P}{16.7126}(120\sin\theta - 28\cos\theta) \qquad (3)$$

For $P = 2$ kips, $\theta = 45°$

From Eq. (1): $F_{CG} = -\frac{2}{14}(16\cos 45° + 8\sin 45°) = -2.42\text{ kips}$ $\qquad F_{CG} = 2.42\text{ kips}\ C$ ◀

From Eq. (2): $F_{AD} = -\frac{2}{15.6}(86\cos 45° - 42\sin 45°) = -3.99\text{ kips}$ $\qquad F_{AD} = 3.99\text{ kips}\ C$ ◀

From Eq. (3): $F_{EF} = \frac{2}{16.7126}(120\sin 45° - 28\cos 45°) = +7.79\text{ kips}$ $\qquad F_{EF} = 7.79\text{ kips}\ T$ ◀

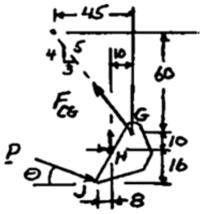

15 in.
20 in.
48 in.
10 in.
35 in.
10 in.
A
C
B
12 in.
36 in.
D
E
16 in.
16 in.
F
18 in.
I
60 in.
G
H
P
10 in.
θ
J
16 in.
40 in.
8 in.

PROBLEM 6.158

Solve Problem 6.157 assuming that the 2-kip force **P** acts horizontally to the right ($\theta = 0$).

PROBLEM 6.157 The motion of the backhoe bucket shown is controlled by the hydraulic cylinders *AD*, *CG*, and *EF*. As a result of an attempt to dislodge a portion of a slab, a 2-kip force **P** is exerted on the bucket teeth at *J*. Knowing that $\theta = 45°$, determine the force exerted by each cylinder.

SOLUTION


45
4
5
3
10
60
F_{CG}
G
P
H
10
16
θ
J
8

Free body: Bucket:

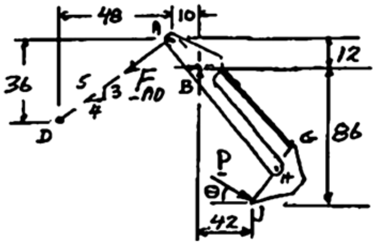
$+\circlearrowleft \Sigma M_H = 0$: (Dimensions in inches)

$$\frac{4}{5}F_{CG}(10) + \frac{3}{5}F_{CG}(10) + P\cos\theta(16) + P\sin\theta(8) = 0$$

$$F_{CG} = -\frac{P}{14}(16\cos\theta + 8\sin\theta) \qquad (1)$$

Free body: Arm *ABH* and bucket:

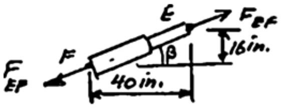
(Dimensions in inches)


48
10
A
12
36
5
3
4
F_{AD}
B
D
G
86
P
θ
H
42
J

$+\circlearrowleft \Sigma M_B = 0$: $\frac{4}{5}F_{AD}(12) + \frac{3}{5}F_{AD}(10) + P\cos\theta(86) - P\sin\theta(42) = 0$

$$F_{AD} = -\frac{P}{15.6}(86\cos\theta - 42\sin\theta) \qquad (2)$$

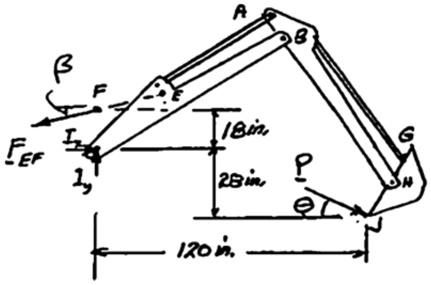
Free body: Bucket and arms *IEB* + *ABH*:


E
F_{EF}
F
F_{EF}
β
16 in.
40 in.

Geometry of cylinder *EF*: $\tan\beta = \frac{16 \text{ in.}}{40 \text{ in.}}$

$$\beta = 21.801°$$

PROBLEM 6.158 (Continued)



$$+\circlearrowleft \Sigma M_I = 0: \quad F_{EF}\cos\beta(18 \text{ in.}) + P\cos\theta(28 \text{ in.}) - P\sin\theta(120 \text{ in.}) = 0$$

$$F_{EF} = \frac{P(120\sin\theta - 28\cos\theta)}{\cos 21.8°(18)}$$

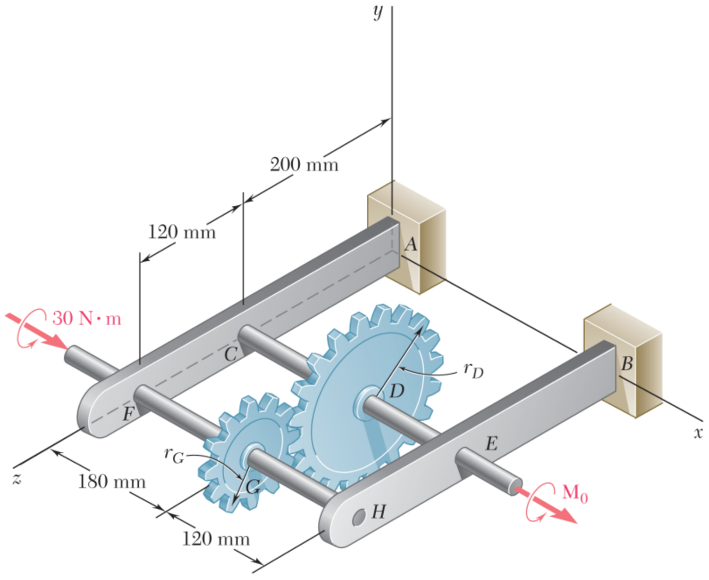
$$= \frac{P}{16.7126}(120\sin\theta - 28\cos\theta) \qquad (3)$$

For $P = 2$ kips, $\theta = 0$

From Eq. (1): $F_{CG} = -\frac{2}{14}(16\cos 0 + 8\sin 0) = -2.29$ kips $\qquad F_{CG} = 2.29$ kips C ◀

From Eq. (2): $F_{AD} = -\frac{2}{15.6}(86\cos 0 - 42\sin 0) = -11.03$ kips $\qquad F_{AD} = 11.03$ kips C ◀

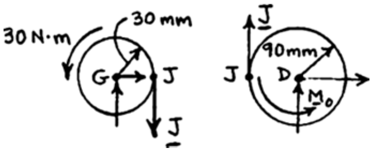
From Eq. (3): $F_{EF} = \frac{2}{16.7126}(120\sin 0 - 28\cos 0) = -3.35$ kips $\qquad F_{EF} = 3.35$ kips C ◀


y
200 mm
120 mm
A
30 N · m
C
r_D
D
B
x
F
r_G
G
E
H
M_0
z
180 mm
120 mm

PROBLEM 6.159

The gears D and G are rigidly attached to shafts that are held by frictionless bearings. If $r_D = 90$ mm and $r_G = 30$ mm, determine (*a*) the couple $\mathbf{M}_0$ that must be applied for equilibrium, (*b*) the reactions at A and B.

SOLUTION


30 N·m
30 mm
G
J
J
J
90mm
J
D
M_0

(*a*) Projections on *yz* plane.

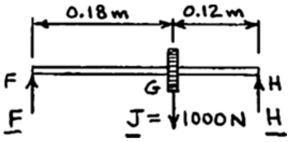
Free body: Gear *G*:

$$+\circlearrowleft \Sigma M_G = 0: \quad 30\ \text{N}\cdot\text{m} - J(0.03\ \text{m}) = 0; \quad J = 1000\ \text{N}$$

Free body: Gear *D*:

$$+\circlearrowleft \Sigma M_D = 0: \quad M_0 - (1000\ \text{N})(0.09\ \text{m}) = 0$$

$$M_0 = 90\ \text{N}\cdot\text{m} \qquad \mathbf{M}_0 = (90.0\ \text{N}\cdot\text{m})\mathbf{i} \ \blacktriangleleft$$


0.18 m
0.12 m
F
G
H
F
J = 1000 N
H

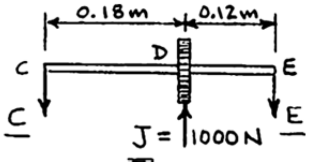
(*b*) Gear *G* and axle *FH*:

$$+\circlearrowleft \Sigma M_F = 0: \quad H(0.3\ \text{m}) - (1000\ \text{N})(0.18\ \text{m}) = 0$$

$$H = 600\ \text{N}$$

$$+\uparrow \Sigma F_y = 0: \quad F + 600 - 1000 = 0$$

$$F = 400\ \text{N}$$


0.18 m
0.12 m
C
D
E
C
J = 1000 N
E

Gear *D* and axle *CE*:

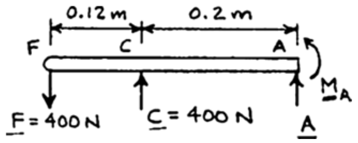
$$+\circlearrowleft \Sigma M_C = 0: \quad (1000\ \text{N})(0.18\ \text{m}) - E(0.3\ \text{m}) = 0$$

$$E = 600\ \text{N}$$

$$+\uparrow \Sigma F_y = 0: \quad 1000 - C - 600 = 0$$

$$C = 400\ \text{N}$$

PROBLEM 6.159 (Continued)

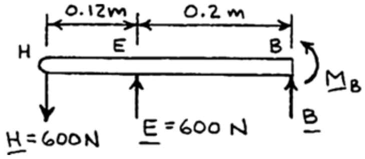


Free body: Bracket *AE*:

$$+\uparrow \Sigma F_y = 0: \quad A - 400 + 400 = 0 \qquad \mathbf{A} = 0 \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_A = 0: \quad M_A + (400\text{ N})(0.32\text{ m}) - (400\text{ N})(0.2\text{ m}) = 0$$

$$M_A = -48\text{ N}\cdot\text{m} \qquad \mathbf{M}_A = -(48.0\text{ N}\cdot\text{m})\mathbf{i} \blacktriangleleft$$

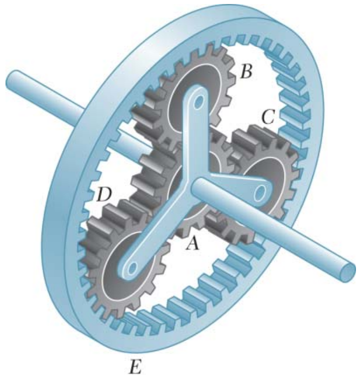


Free body: Bracket *BH*:

$$+\uparrow \Sigma F_y = 0: \quad B - 600 + 600 = 0 \qquad \mathbf{B} = 0 \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_B = 0: \quad M_B + (600\text{ N})(0.32\text{ m}) - (600\text{ N})(0.2\text{ m}) = 0$$

$$M_B = -72\text{ N}\cdot\text{m} \qquad \mathbf{M}_B = -(72.0\text{ N}\cdot\text{m})\mathbf{i} \blacktriangleleft$$

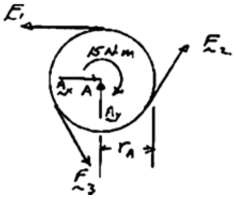


PROBLEM 6.160

In the planetary gear system shown, the radius of the central gear A is a = 18 mm, the radius of each planetary gear is b, and the radius of the outer gear E is $(a + 2b)$. A clockwise couple of magnitude $M_A = 10\ \text{N}\cdot\text{m}$ is applied to the central gear A and a counterclockwise couple of magnitude $M_S = 50\ \text{N}\cdot\text{m}$ is applied to the spider BCD. If the system is to be in equilibrium, determine (a) the required radius b of the planetary gears, (b) the magnitude M_E of the couple that must be applied to the outer gear E.

SOLUTION

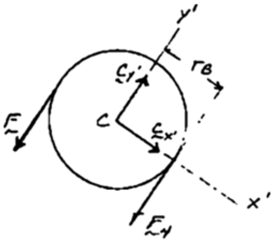
FBD Central Gear:



By symmetry, $F_1 = F_2 = F_3 = F$

$$\circlearrowleft \Sigma M_A = 0{:}\quad 3(r_A F) - 10\ \text{N}\cdot\text{m} = 0, \qquad F = \frac{10}{3r_A}\text{N}\cdot\text{m}$$

FBD Gear *C*:



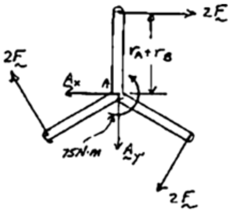
$$\circlearrowleft \Sigma M_C = 0{:}\quad r_B(F - F_4) = 0, \quad F_4 = F$$

$$\searrow \Sigma F_{x'} = 0{:}\quad C_{x'} = 0$$

$$\nearrow \Sigma F_{y'} = 0{:}\quad C_{y'} - 2F = 0, \qquad C_{y'} = 2F$$

Gears B and D are analogous, each having a central force of $2F$.

FBD Spider:



$$\circlearrowleft \Sigma M_A = 0{:}\quad 50\ \text{N}\cdot\text{m} - 3(r_A + r_B)2F = 0$$

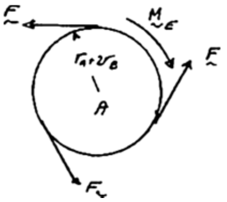
$$50\ \text{N}\cdot\text{m} - 3(r_A + r_B)\frac{20}{r_A}\text{N}\cdot\text{m} = 0$$

$$\frac{r_A + r_B}{r_A} = 2.5 = 1 + \frac{r_B}{r_A}, \qquad r_B = 1.5r_A$$

Since $r_A = 18$ mm,

(a) $r_B = 27.0$ mm ◀

FBD Outer Gear:



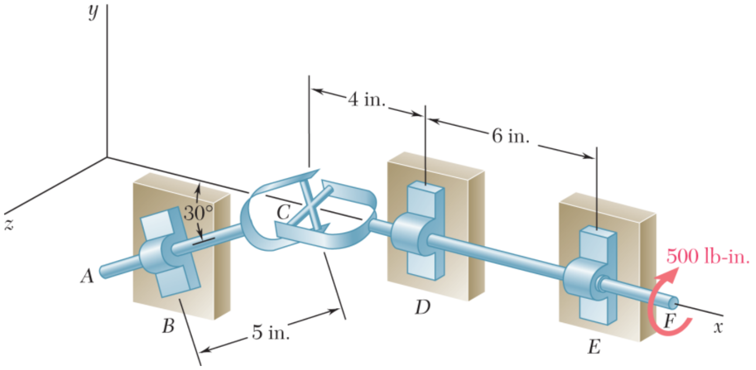
$$\circlearrowleft \Sigma M_A = 0{:}\quad 3(r_A + 2r_B)F - M_E = 0$$

$$3(18\ \text{mm} + 54\ \text{mm})\frac{10\ \text{N}\cdot\text{m}}{54\ \text{mm}} - M_E = 0$$

(b) $\mathbf{M}_E = 40.0\ \text{N}\cdot\text{m} \circlearrowright$ ◀

PROBLEM 6.161*

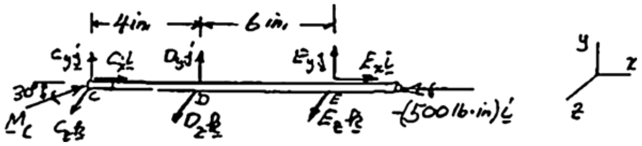
Two shafts *AC* and *CF*, which lie in the vertical *xy* plane, are connected by a universal joint at *C*. The bearings at *B* and *D* do not exert any axial force. A couple of magnitude 500 lb · in. (clockwise when viewed from the positive *x*-axis) is applied to shaft *CF* at *F*. At a time when the arm of the crosspiece attached to shaft *CF* is horizontal, determine (*a*) the magnitude of the couple that must be applied to shaft *AC* at *A* to maintain equilibrium, (*b*) the reactions at *B*, *D*, and *E*. (*Hint:* The sum of the couples exerted on the crosspiece must be zero.)


y
4 in.
6 in.
z
30°
C
A
B
5 in.
D
500 lb-in.
F
x
E

SOLUTION

We recall from Figure 4.10 that a universal joint exerts on members it connects a force of unknown direction and a couple about an axis perpendicular to the crosspiece.

Free body: Shaft *DF*:

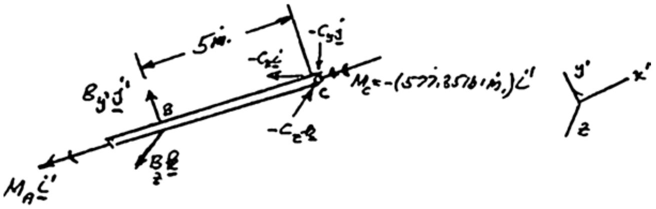

4 in
6 in.
$C_y\mathbf{j}$
$C_x\mathbf{i}$
$D_y\mathbf{j}$
$E_y\mathbf{j}$
$E_x\mathbf{i}$
30°
C
$\mathbf{M}_C$
$C_z\mathbf{k}$
D
$D_z\mathbf{k}$
E
$E_z\mathbf{k}$
$-(500\text{ lb}\cdot\text{in})\mathbf{i}$
y
z
z

$$\Sigma M_x = 0: \quad M_C \cos 30° - 500 \text{ lb} \cdot \text{in.} = 0$$

$$M_C = 577.35 \text{ lb} \cdot \text{in.}$$

Free body: Shaft *BC*:

We use here x', y', z with x' along BC.


5 in.
$-C_y\mathbf{j}$
$-C_x\mathbf{i}$
$B_{y'}\mathbf{j}'$
B
C
$\mathbf{M}_C = -(577.35\text{ lb}\cdot\text{in.})\mathbf{i}'$
$-C_z\mathbf{k}$
$B_z\mathbf{k}$
$M_A\mathbf{i}'$
y'
x'
z

$$\Sigma M_C = 0: \quad -M_R i' - (577.35 \text{ lb} \cdot \text{in.})i' + (-5 \text{ in.})i' \times (B_y j' + B_z k) = 0$$

PROBLEM 6.161* (Continued)

Equate coefficients of unit vectors to zero:

i: $M_A - 577.35\text{ lb}\cdot\text{in.} = 0$ $\quad M_A = 577.35\text{ lb}\cdot\text{in.}$

j: $B_z = 0$ $\quad M_A = 577\text{ lb}\cdot\text{in.}$ ◀

k: $B_y = 0$ $\quad B = 0$ $\quad \mathbf{B} = 0$

$$\left.\begin{array}{l} B_z = 0 \\ B_y = 0 \end{array}\right\} \rightarrow B = 0$$

$\Sigma\mathbf{F} = 0{:}\quad B + C = 0,\quad$ since $B = 0$, $\quad \mathbf{C} = 0$

Return to free body of shaft *DF*.

$\Sigma\mathbf{M}_D = 0$ $\qquad$ (Note that $C = 0$ and $M_C = 577.35\text{ lb}\cdot\text{in.}$)

$$(577.35\text{ lb}\cdot\text{in.})(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) - (500\text{ lb}\cdot\text{in.})\mathbf{i}$$

$$+(6\text{ in.})\mathbf{i}\times(E_x\mathbf{i} + E_y\mathbf{j} + E_z\mathbf{k}) = 0$$

$$(500\text{ lb}\cdot\text{in.})\mathbf{i} + (288.68\text{ lb}\cdot\text{in.})\mathbf{j} - (500\text{ lb}\cdot\text{in.})\mathbf{i}$$

$$+(6\text{ in.})E_y\mathbf{k} - (6\text{ in.})E_z\mathbf{j} = 0$$

Equate coefficients of unit vectors to zero:

j: $288.68\text{ lb}\cdot\text{in.} - (6\text{ in.})E_z = 0 \qquad E_z = 48.1\text{ lb}$

k: $E_y = 0$

$\Sigma\mathbf{F} = 0{:}\quad \mathbf{C} + \mathbf{D} + \mathbf{E} = 0$

$0 + D_y\mathbf{j} + D_z\mathbf{k} + E_x\mathbf{i} + (48.1\text{ lb})\mathbf{k} = 0$

i: $E_x = 0$

j: $D_y = 0$

k: $D_z + 48.1\text{ lb} = 0 \quad D_z = -48.1\text{ lb}$

Reactions are:

$\mathbf{B} = 0$ ◀

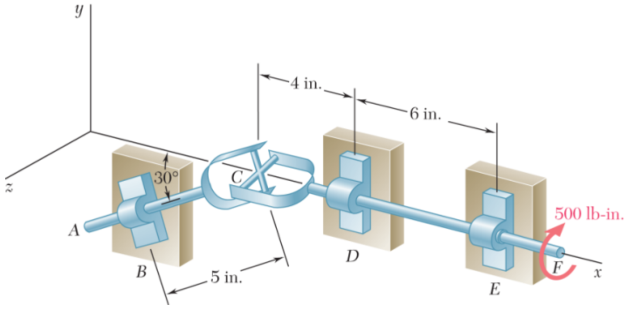
$\mathbf{D} = -(48.1\text{ lb})\mathbf{k}$ ◀

$\mathbf{E} = (48.1\text{ lb})\mathbf{k}$ ◀

PROBLEM 6.162*

Solve Problem 6.161 assuming that the arm of the crosspiece attached to shaft *CF* is vertical.

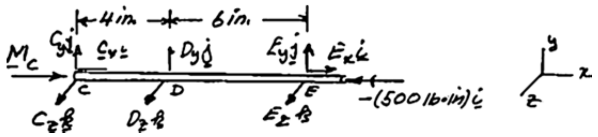
PROBLEM 6.161 Two shafts *AC* and *CF*, which lie in the vertical *xy* plane, are connected by a universal joint at *C*. The bearings at *B* and *D* do not exert any axial force. A couple of magnitude 500 lb · in. (clockwise when viewed from the positive *x*-axis) is applied to shaft *CF* at *F*. At a time when the arm of the crosspiece attached to shaft *CF* is horizontal, determine (*a*) the magnitude of the couple that must be applied to shaft *AC* at *A* to maintain equilibrium, (*b*) the reactions at *B*, *D*, and *E*. (*Hint:* The sum of the couples exerted on the crosspiece must be zero.)



y
z
4 in.
6 in.
30°
C
A
B
5 in.
D
E
500 lb-in.
F
x

SOLUTION

Free body: Shaft *DF*.

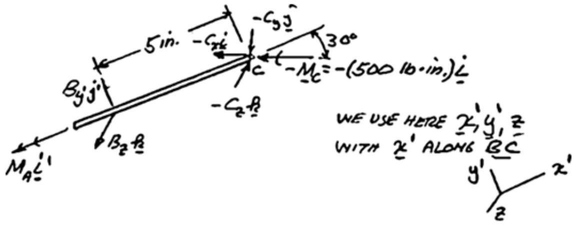


4 in.
6 in.
M_C
$C_y\mathbf{j}$
$C_y\mathbf{i}$
$D_y\mathbf{j}$
$E_y\mathbf{j}$
$E_x\mathbf{i}$
C
D
E
$-(500\text{ lb}\cdot\text{in})\mathbf{i}$
$C_z\mathbf{k}$
$D_z\mathbf{k}$
$E_z\mathbf{k}$
y
x
z

$$\Sigma M_x = 0: \quad M_C - 500\text{ lb}\cdot\text{in.} = 0$$

$$M_C = 500\text{ lb}\cdot\text{in.}$$

Free body: Shaft *BC*:



5 in.
$-C_y\mathbf{j}$
$-C_x\mathbf{i}$
30°
C
$-M_C = -(500\text{ lb}\cdot\text{in.})\mathbf{i}$
$B_{y'}\mathbf{j}'$
$-C_z\mathbf{k}$
$B_z\mathbf{k}$
$M_A\mathbf{i}'$
WE USE HERE x', y', z
WITH x' ALONG BC
y'
x'
z

We resolve $-(520\text{ lb}\cdot\text{in.})\mathbf{i}$ into components along x' and y' axes:

$$-\mathbf{M}_C = -(500\text{ lb}\cdot\text{in.})(\cos 30°\mathbf{i}' + \sin 30°\mathbf{j}')$$

$$\Sigma\mathbf{M}_C = 0: \quad M_A\mathbf{i}' - (500\text{ lb}\cdot\text{in.})(\cos 30°\mathbf{i}' + \sin 30°\mathbf{j}') + (5\text{ in.})\mathbf{i}' \times (B_{y'}\mathbf{j}' + B_z\mathbf{k}) = 0$$

$$M_A\mathbf{i}' - (433\text{ lb}\cdot\text{in.})\mathbf{i}' - (250\text{ lb}\cdot\text{in.})\mathbf{j} + (5\text{ in.})B_{y'}\mathbf{k} - (5\text{ in.})B_z\mathbf{j}' = 0$$

PROBLEM 6.162* (Continued)

Equate to zero coefficients of unit vectors:

$\mathbf{i}'$: $M_A - 433 \text{ lb}\cdot\text{in.} = 0$ $\qquad M_A = 433 \text{ lb}\cdot\text{in.}$ ◀

$\mathbf{j}'$: $-250 \text{ lb}\cdot\text{in.} - (5 \text{ in.})B_z = 0$ $\qquad B_z = -50 \text{ lb}$

$\mathbf{k}$: $\mathbf{B}_{y'} = 0$

Reactions at *B*. $\qquad \mathbf{B} = -(50 \text{ lb})\mathbf{k}$

$$\Sigma\mathbf{F} = 0: \quad \mathbf{B} - \mathbf{C} = 0$$

$$-(50 \text{ lb})\mathbf{k} - C = 0 \qquad \mathbf{C} = -(50 \text{ lb})\mathbf{k}$$

Return to free body of shaft *DF*.

$$\Sigma\mathbf{M}_D = 0: \quad (6 \text{ in.})\mathbf{i} \times (E_x\mathbf{i} + E_y\mathbf{j} + E_z\mathbf{k}) - (4 \text{ in.})\mathbf{i} \times (-50 \text{ lb})\mathbf{k}$$
$$- (500 \text{ lb}\cdot\text{in.})\mathbf{i} + (500 \text{ lb}\cdot\text{in.})\mathbf{i} = 0$$
$$(6 \text{ in.})E_y\mathbf{k} - (6 \text{ in.})E_z\mathbf{j} - (200 \text{ lb}\cdot\text{in.})\mathbf{j} = 0$$

$\mathbf{k}$: $E_y = 0$

$\mathbf{j}$: $-(6 \text{ in.})E_z - 200 \text{ lb}\cdot\text{in.} = 0$ $\qquad E_z = -33.3 \text{ lb}$

$$\Sigma F = 0: \quad \mathbf{C} + \mathbf{D} + \mathbf{E} = 0$$

$$-(50 \text{ lb})\mathbf{k} + D_y\mathbf{j} + D_z\mathbf{k} + E_x\mathbf{i} - (33.3 \text{ lb})\mathbf{k} = 0$$

$\mathbf{i}$: $E_x = 0$

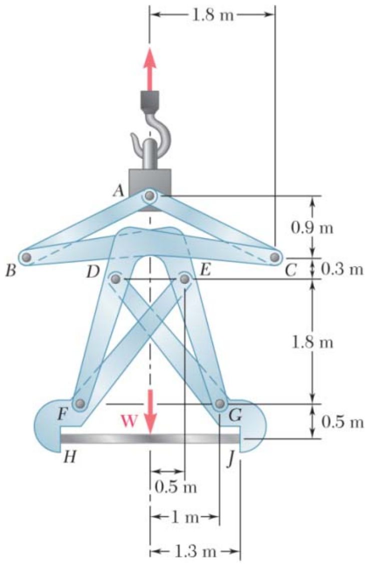
$\mathbf{k}$: $-50 \text{ lb} - 33.3 \text{ lb} + D_z = 0$

$$D_z = 83.3 \text{ lb}$$

Reactions are $\qquad \mathbf{B} = -(50 \text{ lb})\mathbf{k}$ ◀

$\mathbf{D} = (83.3 \text{ lb})\mathbf{k}$ ◀

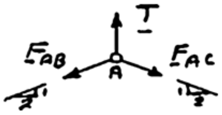
$\mathbf{E} = -(33.3 \text{ lb})\mathbf{k}$ ◀


1.8 m
A
B
D
E
C
0.9 m
0.3 m
1.8 m
F
W
G
0.5 m
H
J
0.5 m
1 m
1.3 m

PROBLEM 6.163*

The large mechanical tongs shown are used to grab and lift a thick 7500-kg steel slab *HJ*. Knowing that slipping does not occur between the tong grips and the slab at *H* and *J*, determine the components of all forces acting on member *EFH*. (*Hint:* Consider the symmetry of the tongs to establish relationships between the components of the force acting at *E* on *EFH* and the components of the force acting at *D* on *DGJ*.)

SOLUTION


T
F_AB
A
F_AC
2
1
1
2

Free body: Pin *A*:

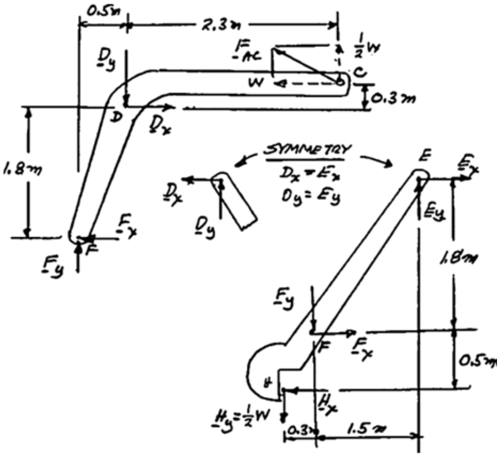
$$T = W = mg = (7500\text{ kg})(9.81\text{ m/s}^2) = 73.575\text{ kN}$$

$$\Sigma F_x = 0: \quad (F_{AB})_x = (F_{AC})_x$$

$$\Sigma F_y = 0: \quad (F_{AB})_y = (F_{AC})_y = \frac{1}{2}W$$

Also,

$$(F_{AC})_x = 2(F_{AC})_y = W$$


0.5m
2.3m
D_y
F_AC
1/2 W
W
C
0.3m
D
D_x
1.8m
F_x
F
F_y
SYMMETRY
D_x = E_x
D_y = E_y
D_x
D_y
E
E_x
E_y
1.8m
F_y
F
F_x
0.5m
H
H_x
H_y = 1/2 W
0.3m
1.5m

Free body: Member *CDF*:

$$\overset{+}{\circlearrowleft}\Sigma M_D = 0: \quad W(0.3) + \frac{1}{2}W(2.3) - F_x(1.8) - F_y(0.5\text{ m}) = 0$$

or

$$1.8F_x + 0.5F_y = 1.45W \qquad (1)$$

PROBLEM 6.163* (Continued)

$$\xrightarrow{+}\Sigma F_x = 0:\quad D_x - F_x - W = 0$$

or $$E_x - F_x = W \qquad (2)$$

$$+\!\uparrow\Sigma F_y = 0:\quad F_y - D_y + \frac{1}{2}W = 0$$

or $$E_y - F_y = \frac{1}{2}W \qquad (3)$$

Free body: Member *EFH*:

$$+\circlearrowright\Sigma M_E = 0:\quad F_x(1.8) + F_y(1.5) - H_x(2.3) + \frac{1}{2}W(1.8\text{ m}) = 0$$

or $$1.8F_x + 1.5F_y = 2.3H_x - 0.9W \qquad (4)$$

$$\xrightarrow{+}\Sigma F_x = 0:\quad E_x + F_x - H_x = 0$$

or $$E_x + F_x = H_x \qquad (5)$$

Subtract Eq. (2) from Eq. (5): $$2F_x = H_x - W \qquad (6)$$

Subtract Eq. (4) from 3 × (1): $$3.6F_x = 5.25W - 2.3H_x \qquad (7)$$

Add Eq. (7) to 2.3 × Eq. (6): $$8.2F_x = 2.95W$$

$$F_x = 0.35976W \qquad (8)$$

Substitute from Eq. (8) into Eq. (1):

$$(1.8)(0.35976W) + 0.5F_y = 1.45W$$

$$0.5F_y = 1.45W - 0.64756W = 0.80244W$$

$$F_y = 1.6049W \qquad (9)$$

Substitute from Eq. (8) into Eq. (2): $E_x - 0.35976W = W;\quad E_x = 1.35976W$

Substitute from Eq. (9) into Eq. (3): $E_y - 1.6049W = \frac{1}{2}W \quad E_y = 2.1049W$

From Eq. (5): $H_x = E_x + F_x = 1.35976W + 0.35976W = 1.71952W$

Recall that $H_y = \frac{1}{2}W$

PROBLEM 6.163* (Continued)

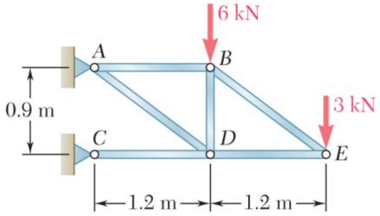
Since all expressions obtained are positive, all forces are directed as shown on the free-body diagrams.

Substitute $W = 73.575 \text{ kN}$

$\mathbf{E}_x = 100.0 \text{ kN} \longrightarrow \quad \mathbf{E}_y = 154.9 \text{ kN} \uparrow$ ◀

$\mathbf{F}_x = 26.5 \text{ kN} \longrightarrow \quad \mathbf{F}_y = 118.1 \text{ kN} \downarrow$ ◀

$\mathbf{H}_x = 126.5 \text{ kN} \longleftarrow \quad \mathbf{H}_y = 36.8 \text{ kN} \downarrow$ ◀


6 kN
A
B
3 kN
0.9 m
C
D
E
1.2 m
1.2 m

PROBLEM 6.164

Using the method of joints, determine the force in each member of the truss shown. State whether each member is in tension or compression.

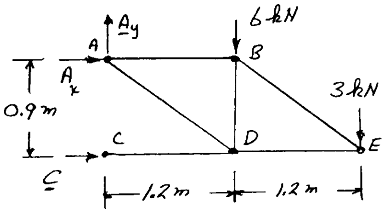
SOLUTION

Reactions:

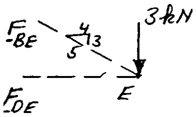
$$\Sigma M_C = 0: \quad \mathbf{A}_x = 16\text{ kN} \longleftarrow$$

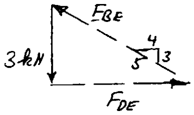
$$\Sigma F_y = 0: \quad \mathbf{A}_y = 9\text{ kN}\uparrow$$

$$\Sigma F_x = 0: \quad \mathbf{C} = 16\text{ kN} \longrightarrow$$


A_y
6 kN
A
B
A_x
3 kN
0.9 m
C
D
E
C
1.2 m
1.2 m

Joint E:


F_{BE}
4
3
5
3 kN
F_{DE}
E

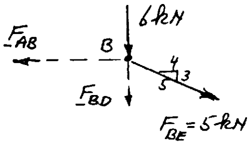

F_{BE}
4
3
5
3 kN
F_{DE}

$$\frac{F_{BE}}{5} = \frac{F_{DE}}{4} = \frac{3\text{ kN}}{3}$$

$F_{BE} = 5.00\text{ kN} \quad T$ ◀

$F_{DE} = 4.00\text{ kN} \quad C$ ◀

Joint B:


6 kN
F_{AB}
B
4
3
5
F_{BD}
$F_{BE} = 5$ kN

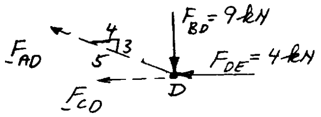
$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{4}{5}(5\text{ kN}) - F_{AB} = 0$$

$F_{AB} = +4\text{ kN} \qquad F_{AB} = 4.00\text{ kN} \quad T$ ◀

$$+\uparrow \Sigma F_y = 0: \quad -6\text{ kN} - \frac{3}{5}(5\text{ kN}) - F_{BD} = 0$$

$F_{BD} = -9\text{ kN} \qquad F_{BD} = 9.00\text{ kN} \quad C$ ◀

Joint D:

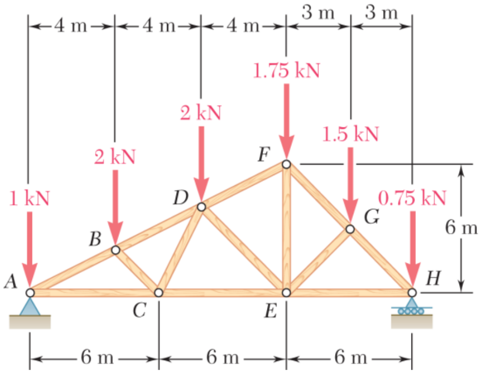

$F_{BD} = 9$ kN
F_{AD}
4
3
5
$F_{DE} = 4$ kN
F_{CD}
D

$$+\uparrow \Sigma F_y = 0: \quad -9\text{ kN} + \frac{3}{5}F_{AD} = 0$$

$F_{AD} = +15\text{ kN} \quad F_{AD} = 15.00\text{ kN} \quad T$ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad -4\text{ kN} - \frac{4}{5}(15\text{ kN}) - F_{CD} = 0$$

$F_{CD} = -16\text{ kN} \quad F_{CD} = 16.00\text{ kN} \quad C$ ◀

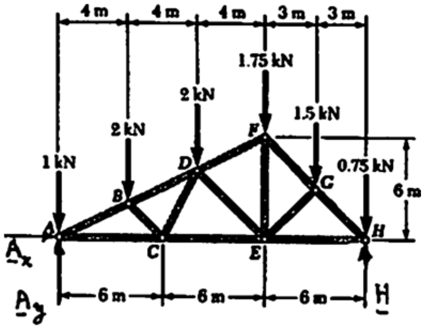

4 m
4 m
4 m
3 m
3 m
1.75 kN
2 kN
1.5 kN
2 kN
F
1 kN
D
0.75 kN
G
6 m
B
A
H
C
E
6 m
6 m
6 m

PROBLEM 6.165

Using the method of joints, determine the force in each member of the double-pitch roof truss shown. State whether each member is in tension or compression.

SOLUTION

Free body: Truss:


4 m
4 m
4 m
3 m
3 m
1.75 kN
2 kN
1.5 kN
2 kN
F
1 kN
D
0.75 kN
G
6 m
B
A
H
$\underline{A}_x$
C
E
$\underline{A}_y$
6 m
6 m
6 m
$\underline{H}$

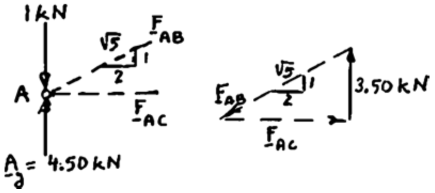
$$+\circlearrowleft \Sigma M_A = 0:\quad H(18\text{ m}) - (2\text{ kN})(4\text{ m}) - (2\text{ kN})(8\text{ m}) - (1.75\text{ kN})(12\text{ m})$$
$$-(1.5\text{ kN})(15\text{ m}) - (0.75\text{ kN})(18\text{ m}) = 0$$

$\mathbf{H} = 4.50\text{ kN}\uparrow$

$$\Sigma F_x = 0:\quad A_x = 0$$
$$\Sigma F_y = 0:\quad A_y + H - 9 = 0$$
$$A_y = 9 - 4.50$$

$\mathbf{A}_y = 4.50\text{ kN}\uparrow$

Free body: Joint *A*:


1 kN
F_{AB}
$\sqrt{5}$
1
2
A
F_{AC}
$A_y = 4.50$ kN
F_{AB}
$\sqrt{5}$
1
2
3.50 kN
F_{AC}

$$\frac{F_{AB}}{\sqrt{5}} = \frac{F_{AC}}{2} = \frac{3.50\text{ kN}}{1}$$

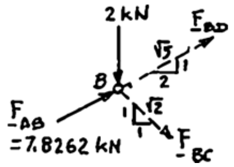
$$F_{AB} = 7.8262\text{ kN}\quad C$$

$F_{AB} = 7.83\text{ kN}\quad C$ ◀

$F_{AC} = 7.00\text{ kN}\quad T$ ◀

PROBLEM 6.165 (Continued)

Free body: Joint *B*:


2 kN
F_{BD}
$\sqrt{5}$
1
2
B
$\sqrt{2}$
1
1
F_{AB} = 7.8262 kN
F_{BC}

$$\xrightarrow{+}\Sigma F_x = 0:\quad \frac{2}{\sqrt{5}}F_{BD} + \frac{2}{\sqrt{5}}(7.8262\text{ kN}) + \frac{1}{\sqrt{2}}F_{BC} = 0$$

or $$F_{BD} + 0.79057F_{BC} = -7.8262\text{ kN} \qquad (1)$$

$$+\!\uparrow\Sigma F_y = 0:\quad \frac{1}{\sqrt{5}}F_{BD} + \frac{1}{\sqrt{5}}(7.8262\text{ kN}) - \frac{1}{\sqrt{2}}F_{BC} - 2\text{ kN} = 0$$

or $$F_{BD} - 1.58114F_{BC} = -3.3541 \qquad (2)$$

Multiply Eq. (1) by 2 and add Eq. (2):

$$3F_{BD} = -19.0065$$

$$F_{BD} = -6.3355\text{ kN}$$

$F_{BD} = 6.34\text{ kN}$ *C* ◀

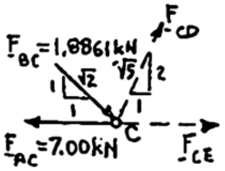
Subtract Eq. (2) from Eq. (1):

$$2.37111F_{BC} = -4.4721$$

$$F_{BC} = -1.8861\text{ kN}$$

$F_{BC} = 1.886\text{ kN}$ *C* ◀

Free body: Joint *C*:


F_{CD}
F_{BC}=1.8861kN
1
$\sqrt{2}$
1
$\sqrt{5}$
2
1
C
F_{AC}=7.00kN
F_{CE}

$$+\!\uparrow\Sigma F_y = 0:\quad \frac{2}{\sqrt{5}}F_{CD} - \frac{1}{\sqrt{2}}(1.8861\text{ kN}) = 0$$

$$F_{CD} = +1.4911\text{ kN}$$

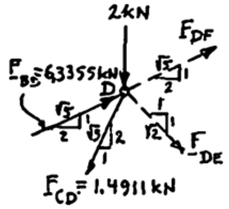
$F_{CD} = 1.491\text{ kN}$ *T* ◀

$$\xrightarrow{+}\Sigma F_x = 0:\quad F_{CE} - 7.00\text{ kN} + \frac{1}{\sqrt{2}}(1.8861\text{ kN}) + \frac{1}{\sqrt{5}}(1.4911\text{ kN}) = 0$$

$$F_{CE} = +5.000\text{ kN}$$

$F_{CE} = 5.00\text{ kN}$ *T* ◀

Free body: Joint *D*:


2kN
F_{DF}
$\sqrt{5}$
1
2
F_{BD}=6.3355kN
D
$\sqrt{5}$
1
2
1
$\sqrt{5}$
2
1
$\sqrt{2}$
1
F_{DE}
F_{CD}= 1.4911 kN

$$\xrightarrow{+}\Sigma F_x = 0:\quad \frac{2}{\sqrt{5}}F_{DF} + \frac{1}{\sqrt{2}}F_{DE} + \frac{2}{\sqrt{5}}(6.3355\text{ kN}) - \frac{1}{\sqrt{5}}(1.4911\text{ kN}) = 0$$

or $$F_{DF} + 0.79057F_{DE} = -5.5900\text{ kN} \qquad (1)$$

$$+\!\uparrow\Sigma F_y = 0:\quad \frac{1}{\sqrt{5}}F_{DF} - \frac{1}{\sqrt{2}}F_{DE} + \frac{1}{\sqrt{5}}(6.3355\text{ kN}) - \frac{2}{\sqrt{5}}(1.4911\text{ kN}) - 2\text{ kN} = 0$$

or $$F_{DF} - 0.79057F_{DE} = -1.1188\text{ kN} \qquad (2)$$

Add Eqs. (1) and (2): $$2F_{DF} = -6.7088\text{ kN}$$

$$F_{DF} = -3.3544\text{ kN}$$

$F_{DF} = 3.35\text{ kN}$ *C* ◀

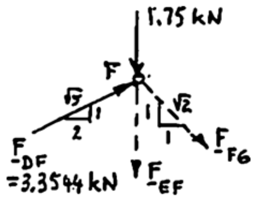
Subtract Eq. (2) from Eq. (1): $$1.58114F_{DE} = -4.4712\text{ kN}$$

$$F_{DE} = -2.8278\text{ kN}$$

$F_{DE} = 2.83\text{ kN}$ *C* ◀

PROBLEM 6.165 (Continued)

Free body: Joint *F*:


1.75 kN
F
√5
1
2
1
√2
1
F_FG
F_DF
=3.3544 kN
F_EF

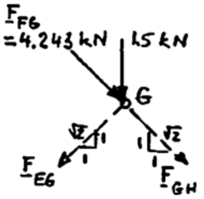
$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{1}{\sqrt{2}} F_{FG} + \frac{2}{\sqrt{5}}(3.3544 \text{ kN}) = 0$$

$$F_{FG} = -4.243 \text{ kN} \qquad F_{FG} = 4.24 \text{ kN} \quad C \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad -F_{EF} - 1.75 \text{ kN} + \frac{1}{\sqrt{5}}(3.3544 \text{ kN}) - \frac{1}{\sqrt{2}}(-4.243 \text{ kN}) = 0$$

$$F_{EF} = 2.750 \text{ kN} \qquad F_{EF} = 2.75 \text{ kN} \quad T \blacktriangleleft$$

Free body: Joint *G*:


F_FG
=4.243 kN
1.5 kN
G
√2
1
1
1
√2
1
F_EG
F_GH

$$\xrightarrow{+} \Sigma F_x = 0: \quad \frac{1}{\sqrt{2}} F_{GH} - \frac{1}{\sqrt{2}} F_{EG} + \frac{1}{\sqrt{2}}(4.243 \text{ kN}) = 0$$

or

$$F_{GH} - F_{EG} = -4.243 \text{ kN} \qquad (1)$$

$$+\uparrow \Sigma F_y = 0: \quad -\frac{1}{\sqrt{2}} F_{GH} - \frac{1}{\sqrt{2}} F_{EG} - \frac{1}{\sqrt{2}}(4.243 \text{ kN}) - 1.5 \text{ kN} = 0$$

or

$$F_{GH} + F_{EG} = -6.364 \text{ kN} \qquad (2)$$

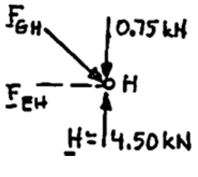
Add Eqs. (1) and (2): $2F_{GH} = -10.607$

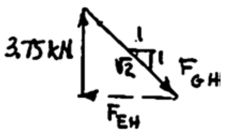
$$F_{GH} = -5.303 \qquad F_{GH} = 5.30 \text{ kN} \quad C \blacktriangleleft$$

Subtract Eq. (1) from Eq. (2): $2F_{EG} = -2.121 \text{ kN}$

$$F_{EG} = -1.0605 \text{ kN} \qquad F_{EG} = 1.061 \text{ kN} \quad C \blacktriangleleft$$

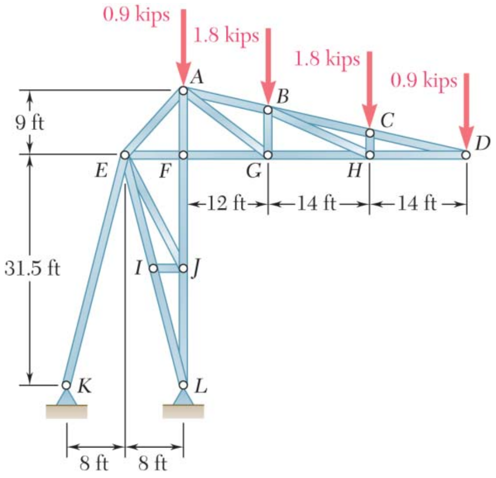
Free body: Joint *H*:


F_GH
0.75 kN
H
F_EH
H = 4.50 kN


3.75 kN
1
√2
1
F_GH
F_EH

$$\frac{F_{EH}}{1} = \frac{3.75 \text{ kN}}{1} \qquad F_{EH} = 3.75 \text{ kN} \quad T \blacktriangleleft$$

We can also write $\frac{F_{GH}}{\sqrt{2}} = \frac{3.75 \text{ kN}}{1}$ $\qquad F_{GH} = 5.30 \text{ kN} \quad C$ (Checks)

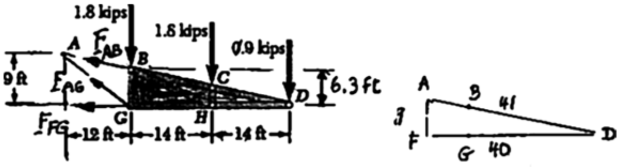

0.9 kips
1.8 kips
1.8 kips
0.9 kips
A
B
C
D
9 ft
E
F
G
H
12 ft
14 ft
14 ft
31.5 ft
I
J
K
L
8 ft
8 ft

PROBLEM 6.166

A stadium roof truss is loaded as shown. Determine the force in members *AB*, *AG*, and *FG*.

SOLUTION

We pass a section through members *AB*, *AG*, and *FG*, and use the free body shown.


1.8 kips
1.8 kips
0.9 kips
A
F_{AB}
B
C
D
9 ft
F_{AG}
6.3 ft
F_{FG}
G
H
12 ft
14 ft
14 ft
A
B
41
3
F
G
40
D

$$+\circlearrowleft \Sigma M_G = 0: \quad \left(\frac{40}{41}F_{AB}\right)(6.3\text{ ft}) - (1.8\text{ kips})(14\text{ ft})$$

$$-(0.9\text{ kips})(28\text{ ft}) = 0$$

$$F_{AB} = +8.20\text{ kips} \qquad F_{AB} = 8.20\text{ kips} \quad T \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_D = 0: \quad -\left(\frac{3}{5}F_{AG}\right)(28\text{ ft}) + (1.8\text{ kips})(28\text{ ft})$$

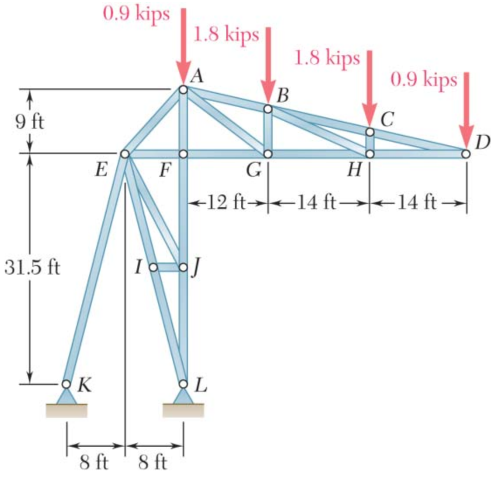
$$+(1.8\text{ kips})(14\text{ ft}) = 0$$

$$F_{AG} = +4.50\text{ kips} \qquad F_{AG} = 4.50\text{ kips} \quad T \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_A = 0: \quad -F_{FG}(9\text{ ft}) - (1.8\text{ kips})(12\text{ ft}) - (1.8\text{ kips})(26\text{ ft})$$

$$-(0.9\text{ kips})(40\text{ ft}) = 0$$

$$F_{FG} = -11.60\text{ kips} \qquad F_{FG} = 11.60\text{ kips} \quad C \blacktriangleleft$$

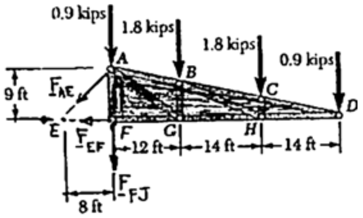

0.9 kips
1.8 kips
1.8 kips
0.9 kips
A
B
C
D
9 ft
E
F
G
H
12 ft
14 ft
14 ft
31.5 ft
I
J
K
L
8 ft
8 ft

PROBLEM 6.167

A stadium roof truss is loaded as shown. Determine the force in members *AE*, *EF*, and *FJ*.

SOLUTION

We pass a section through members *AE*, *EF*, and *FJ*, and use the free body shown.


0.9 kips
1.8 kips
1.8 kips
0.9 kips
A
B
C
D
9 ft
F_{AE}
E
F_{EF}
F
G
H
12 ft
14 ft
14 ft
F_{FJ}
8 ft

$$+\circlearrowleft \Sigma M_F = 0: \quad \left(\frac{8}{\sqrt{8^2+9^2}}F_{AE}\right)(9\text{ ft}) - (1.8\text{ kips})(12\text{ ft}) - (1.8\text{ kips})(26\text{ ft}) - (0.9\text{ kips})(40\text{ ft}) = 0$$

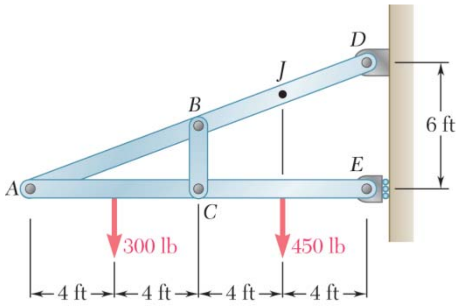
$$F_{AE} = +17.46\text{ kips} \qquad F_{AE} = 17.46\text{ kips} \quad T \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_A = 0: \quad -F_{EF}(9\text{ ft}) - (1.8\text{ kips})(12\text{ ft}) - (1.8\text{ kips})(26\text{ ft}) - (0.9\text{ kips})(40\text{ ft}) = 0$$

$$F_{EF} = -11.60\text{ kips} \qquad F_{EF} = 11.60\text{ kips} \quad C \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_E = 0: \quad -F_{FJ}(8\text{ ft}) - (0.9\text{ kips})(8\text{ ft}) - (1.8\text{ kips})(20\text{ ft}) - (1.8\text{ kips})(34\text{ ft}) - (0.9\text{ kips})(48\text{ ft}) = 0$$

$$F_{FJ} = -18.45\text{ kips} \qquad F_{FJ} = 18.45\text{ kips} \quad C \blacktriangleleft$$

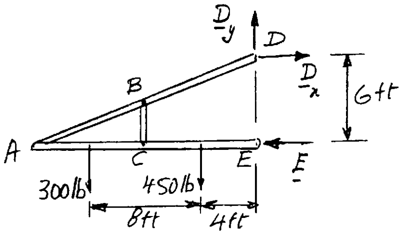

D
J
B
6 ft
E
A
C
300 lb
450 lb
4 ft
4 ft
4 ft
4 ft

PROBLEM 6.168

Determine the components of all forces acting on member *ABD* of the frame shown.

SOLUTION

Free body: Entire frame:


D_y
D
D_x
6ft
B
A
C
E
E
300lb
450lb
8ft
4ft

$$+\circlearrowright \Sigma M_D = 0: \quad (300 \text{ lb}) - (12 \text{ ft}) + (450 \text{ lb})(4 \text{ ft}) - E(6 \text{ ft}) = 0$$

$\mathbf{E} = +900$ lb $\mathbf{E} = 900$ lb ← ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad D_x - 900 \text{ lb} = 0$$

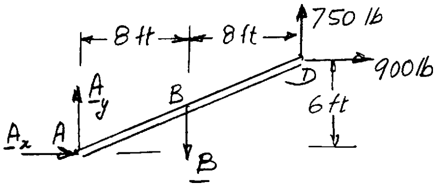
$\mathbf{D}_x = 900$ lb → ◀

$$+\uparrow \Sigma F_y = 0: \quad D_y - 300 \text{ lb} - 450 \text{ lb} = 0$$

$\mathbf{D}_y = 750$ lb ↑ ◀

Free body: Member *ABD*:

We note that *BC* is a two-force member and that **B** is directed along *BC*.


8 ft
8 ft
750 lb
900lb
A_y
B
D
6 ft
A_x
A
B

$$+\circlearrowright \Sigma M_A = 0: \quad (750 \text{ lb})(16 \text{ ft}) - (900 \text{ lb})(6 \text{ ft}) - B(8 \text{ ft}) = 0$$

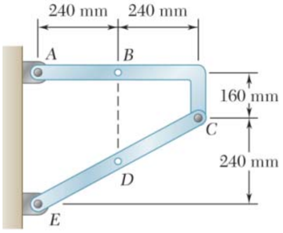
$B = +825$ lb $\mathbf{B} = 825$ lb ↓ ◀

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + 900 \text{ lb} = 0$$

$A_x = -900$ lb $\mathbf{A}_x = 900$ lb ← ◀

$$+\uparrow \Sigma F_y = 0: \quad A_y + 750 \text{ lb} - 825 \text{ lb} = 0$$

$A_y = +75$ lb $\mathbf{A}_y = 75.0$ lb ↑ ◀



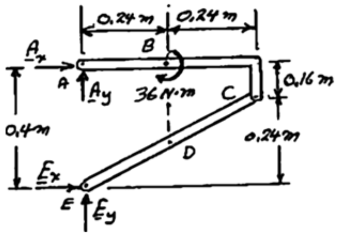
PROBLEM 6.169

Determine the components of the reactions at *A* and *E* if the frame is loaded by a clockwise couple of magnitude 36 N · m applied (*a*) at *B*, (*b*) at *D*.

SOLUTION

Free body: Entire frame:

The following analysis is valid for both parts (*a*) and (*b*) since the point of application of the couple is immaterial.



$$+\circlearrowright \Sigma M_E = 0: \quad -36\ \text{N}\cdot\text{m} - A_x(0.4\ \text{m}) = 0$$

$$A_x = -90\ \text{N} \qquad \mathbf{A}_x = 90.0\ \text{N} \longleftarrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -90 + E_x = 0$$

$$E_x = 90\ \text{N} \qquad \mathbf{E}_x = 90.0\ \text{N} \longrightarrow$$

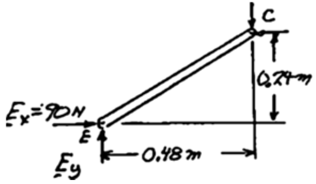
$$+\uparrow \Sigma F_y = 0: \quad A_y + E_y = 0 \qquad (1)$$

(*a*) Couple applied at *B*.

Free body: Member *CE*:

AC is a two-force member. Thus, the reaction at *E* must be directed along *EC*.

$$\frac{E_y}{90\ \text{N}} = \frac{0.24\ \text{m}}{0.48\ \text{m}}; \quad \mathbf{E}_y = 45.0\ \text{N}\uparrow$$



From Eq. (1): $A_y + 45\ \text{N} = 0$

$$A_y = -45\ \text{N} \qquad \mathbf{A}_y = 45.0\ \text{N}\downarrow$$

Thus, reactions are

$\mathbf{A}_x = 90.0\ \text{N} \longleftarrow, \quad \mathbf{A}_y = 45.0\ \text{N}\downarrow$ ◀

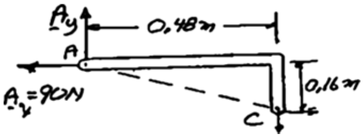
$\mathbf{E}_x = 90.0\ \text{N} \longrightarrow, \quad \mathbf{E}_y = 45.0\ \text{N}\uparrow$ ◀

(*b*) Couple applied at *D*.

Free body: Member *AC*:

AC is a two-force member. Thus, the reaction at *A* must be directed along *AC*.

$$\frac{A_y}{90\ \text{N}} = \frac{0.16\ \text{m}}{0.48\ \text{m}}; \quad \mathbf{A}_y = 30\ \text{N}\uparrow$$



PROBLEM 6.169 (Continued)

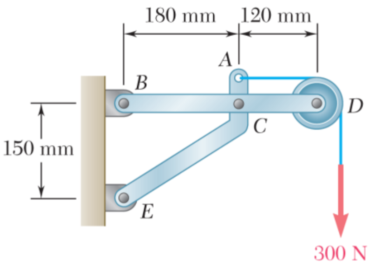
From Eq. (1): $$A_y + E_y = 0$$

$$30\text{ N} + E_y = 0$$

$$E_y = -30\text{ N} \quad \mathbf{E}_y = 30\text{ N}\downarrow$$

Thus, reactions are $\mathbf{A}_x = 90.0\text{ N} \leftarrow, \quad \mathbf{A}_y = 30.0\text{ N}\uparrow$ ◀

$\mathbf{E}_x = -90.0\text{ N} \rightarrow, \quad \mathbf{E}_y = 30.0\text{ N}\downarrow$ ◀

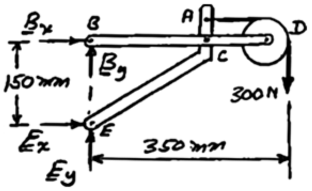

180 mm
120 mm
A
B
D
C
150 mm
E
300 N

PROBLEM 6.170

Knowing that the pulley has a radius of 50 mm, determine the components of the reactions at *B* and *E*.

SOLUTION

Free body: Entire assembly:


B_x
B
A
D
B_y
C
150 mm
300 N
E_x
E
350 mm
E_y

$+\circlearrowleft \Sigma M_E = 0: \quad -(300\text{ N})(350\text{ mm}) - B_x(150\text{ mm}) = 0$

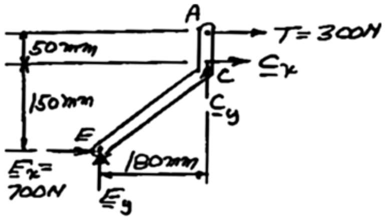
$B_x = -700\text{ N} \quad \mathbf{B}_x = 700\text{ N} \longleftarrow$

$\xrightarrow{+} \Sigma F_x = 0: \quad -700N + E_x = 0$

$E_x = 700\text{ N} \qquad \mathbf{E}_x = 700\text{ N} \longrightarrow$

$+\uparrow \Sigma F_y = 0: \quad B_y + E_y - 300\text{ N} = 0 \qquad (1)$

Free body: Member *ACE*:


A
T= 300N
50 mm
C
C_x
150 mm
C_y
E
E_x=
700N
180 mm
E_y

$+\circlearrowleft \Sigma M_C = 0: \quad (700\text{ N})(150\text{ mm}) - (300\text{ N})(50\text{ mm}) - E_y(180\text{ mm}) = 0$

$E_y = 500\text{ N} \qquad \mathbf{E}_y = 500\text{ N}\uparrow$

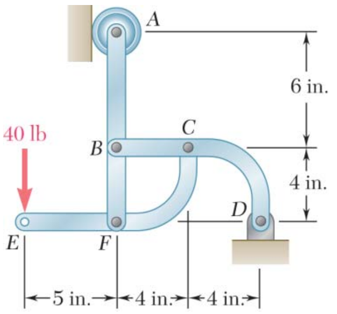
From Eq. (1): $B_y + 500\text{ N} - 300\text{ N} = 0$

$B_y = -200\text{ N} \qquad \mathbf{B}_y = 200\text{ N}\downarrow$

Thus, reactions are

$\mathbf{B}_x = 700\text{ N} \longleftarrow, \quad \mathbf{B}_y = 200\text{ N}\downarrow$ ◀

$\mathbf{E}_x = 700\text{ N} \longrightarrow, \quad \mathbf{E}_y = 500\text{ N}\uparrow$ ◀

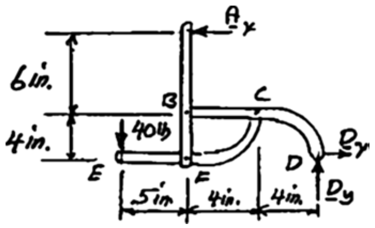

A
6 in.
40 lb
C
B
4 in.
D
E
F
5 in.
4 in.
4 in.

PROBLEM 6.171

For the frame and loading shown, determine the components of the forces acting on member *CFE* at *C* and *F*.

SOLUTION

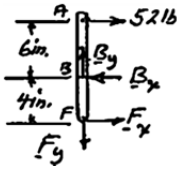
Free body: Entire frame:


A_x
6 in.
B
C
4 in.
40 lb
D_x
E
F
D
5 in
4 in.
4 in
D_y

$$+\circlearrowleft \Sigma M_D = 0: \quad (40\text{ lb})(13\text{ in.}) + A_x(10\text{ in.}) = 0$$

$A_x = -52\text{ lb},$ $\mathbf{A}_x = 52\text{ lb} \longrightarrow$

Free body: Member *ABF*:

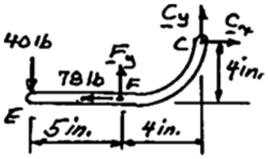

A
52 lb
6 in.
B_y
B
B_x
4 in.
F
F_x
F_y

$$+\circlearrowleft \Sigma M_B = 0: \quad -(52\text{ lb})(6\text{ in.}) + F_x(4\text{ in.}) = 0$$

$F_x = +78\text{ lb}$

Free body: Member *CFE*:

From above: $\mathbf{F}_x = 78.0\text{ lb} \longleftarrow$ ◀


C_y
C_x
40 lb
C
F_y
4 in.
78 lb
F
E
5 in.
4 in.

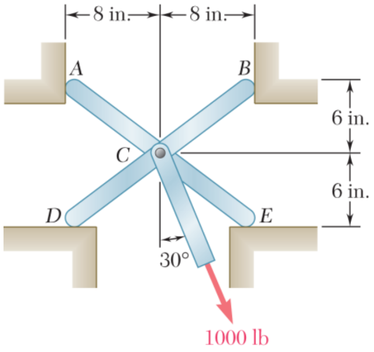
$$+\circlearrowleft \Sigma M_C = 0: \quad (40\text{ lb})(9\text{ in.}) - (78\text{ lb})(4\text{ in.}) - F_y(4\text{ in.}) = 0$$

$F_y = +12\text{ lb}$ $\mathbf{F}_y = 12.00\text{ lb} \uparrow$ ◀

$$\overset{+}{\longrightarrow} \Sigma F_x = 0: \quad C_x - 78\text{ lb} = 0$$

$C_x = +78\text{ lb}$ $\mathbf{C}_x = 78.0\text{ lb} \longrightarrow$ ◀

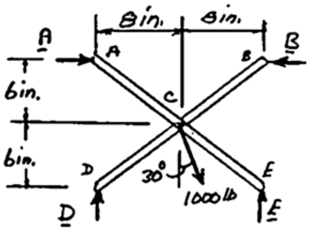
$+\uparrow \Sigma F_y = 0: \quad -40\text{ lb} + 12\text{ lb} + C_y = 0; \quad C_y = +28\text{ lb}$ $\mathbf{C}_y = 28.0\text{ lb} \uparrow$ ◀


8 in.
8 in.
A
B
6 in.
C
6 in.
D
E
30°
1000 lb

PROBLEM 6.172

For the frame and loading shown, determine the reactions at A, B, D, and E. Assume that the surface at each support is frictionless.

SOLUTION


8in.
8in.
A
A
B
B
6in.
C
6in.
D
30°
1000lb
E
D
E

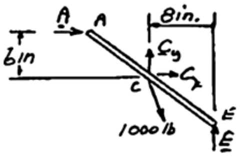
Free body: Entire frame:

$$\xrightarrow{+} \Sigma F_x = 0: \quad A - B + (1000\text{ lb})\sin 30° = 0$$

$$A - B + 500 = 0 \tag{1}$$

$$+\uparrow \Sigma F_y = 0: \quad D + E - (1000\text{ lb})\cos 30° = 0$$

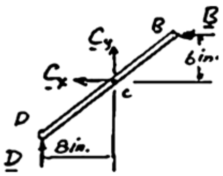
$$D + E - 866.03 = 0 \tag{2}$$


A
A
8in.
6in
C_y
C_x
C
1000lb
E
E

Free body: Member ACE:

$$+\circlearrowleft \Sigma M_C = 0: \quad -A(6\text{ in.}) + E(8\text{ in.}) = 0$$

$$E = \frac{3}{4}A \tag{3}$$


C_y
B
B
6in.
C_x
C
D
D
8in.

Free body: Member BCD:

$$+\circlearrowleft \Sigma M_C = 0: \quad -D(8\text{ in.}) + B(6\text{ in.}) = 0$$

$$D = \frac{3}{4}B \tag{4}$$

Substitute E and D from Eqs. (3) and (4) into Eq. (2):

$$-\frac{3}{4}A + \frac{3}{4}B - 866.06 = 0$$

$$A + B - 1154.71 = 0 \tag{5}$$

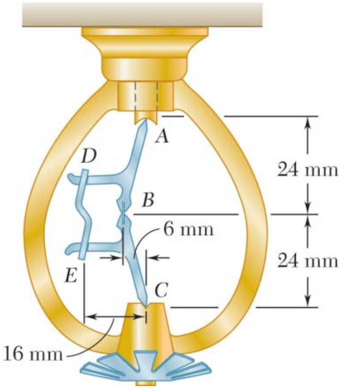
From Eq. (1): $A - B + 500 = 0$ (6)

Eqs. (5) + (6): $2A - 654.71 = 0$ $\quad A = 327.4\text{ lb}$ $\quad \mathbf{A} = 327\text{ lb} \longrightarrow$ ◀

Eqs. (5) − (6): $2B - 1654.71 = 0$ $\quad B = 827.4\text{ lb}$ $\quad \mathbf{B} = 827\text{ lb} \longleftarrow$ ◀

From Eq. (4): $D = \frac{3}{4}(827.4)$ $\quad D = 620.5\text{ lb}$ $\quad \mathbf{D} = 621\text{ lb}\uparrow$ ◀

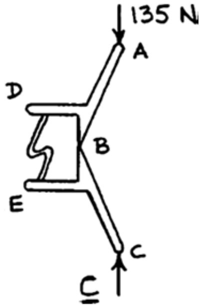
From Eq. (3): $E = \frac{3}{4}(327.4)$ $\quad E = 245.5\text{ lb}$ $\quad \mathbf{E} = 246\text{ lb}\uparrow$ ◀


A
D
B
6 mm
24 mm
24 mm
E
C
16 mm

PROBLEM 6.173

Water pressure in the supply system exerts a downward force of 135 N on the vertical plug at *A*. Determine the tension in the fusible link *DE* and the force exerted on member *BCE* at *B*.

SOLUTION

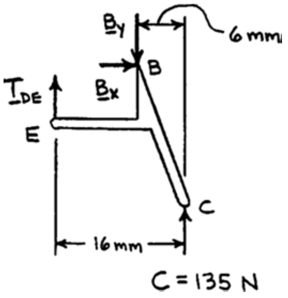

135 N
A
D
B
E
C
C

Free body: Entire linkage:

$$+\Sigma F_y = 0: \quad C - 135 = 0$$

$$C = +135 \text{ N}$$

Free body: Member *BCE*:


By
6 mm
Bx
B
TDE
E
C
16 mm
C = 135 N

$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (135 \text{ N})(6 \text{ mm}) - T_{DE}(10 \text{ mm}) = 0$$

$$T_{DE} = 81.0 \text{ N}$$ ◀

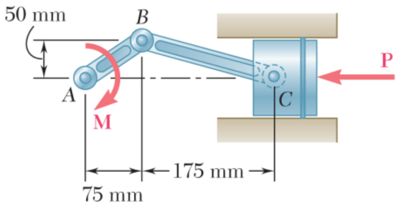
$$+\uparrow \Sigma F_y = 0: \quad 135 + 81 - B_y = 0$$

$$B_y = +216 \text{ N}$$

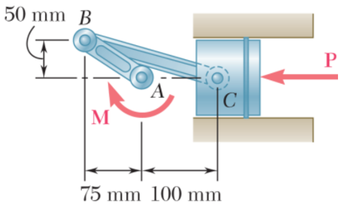
$$\mathbf{B} = 216 \text{ N} \downarrow$$ ◀

PROBLEM 6.174

A couple **M** of magnitude 1.5 kN · m is applied to the crank of the engine system shown. For each of the two positions shown, determine the force **P** required to hold the system in equilibrium.


50 mm
B
A
M
P
C
175 mm
75 mm

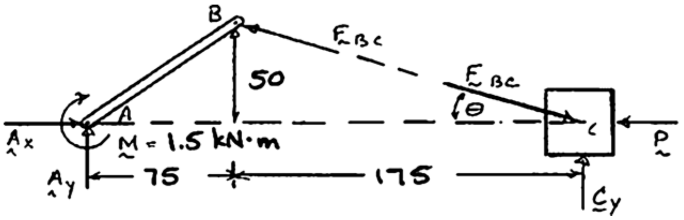
(*a*)


50 mm
B
M
A
C
P
75 mm
100 mm

(*b*)

SOLUTION

(*a*) **FBDs:**


B
F_BC
50
F_BC
θ
A
Ax
M = 1.5 kN·m
Ay
75
175
C
P
Cy

Dimensions in mm

Note: $\tan\theta = \dfrac{50 \text{ mm}}{175 \text{ mm}}$

$$= \frac{2}{7}$$

FBD whole: $\circlearrowright \Sigma M_A = 0\text{:} \quad (0.250 \text{ m})C_y - 1.5 \text{ kN}\cdot\text{m} = 0 \quad C_y = 6.00 \text{ kN}$

FBD piston: $\uparrow \Sigma F_y = 0\text{:} \quad C_y - F_{BC}\sin\theta = 0 \quad F_{BC} = \dfrac{C_y}{\sin\theta} = \dfrac{6.00 \text{ kN}}{\sin\theta}$

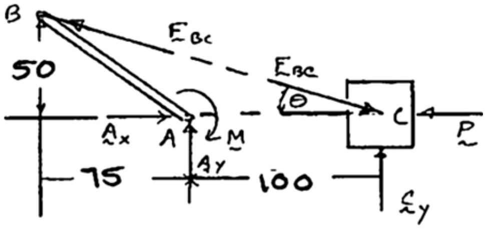
$$\xrightarrow{} \Sigma F_x = 0\text{:} \quad F_{BC}\cos\theta - P = 0$$

$$P = F_{BC}\cos\theta = \frac{6.00 \text{ kN}}{\tan\theta} = 7 \text{ kips}$$

P = 21.0 kN ⟵ ◀

PROBLEM 6.174 (Continued)

(*b*) **FBDs:**



Dimensions in mm

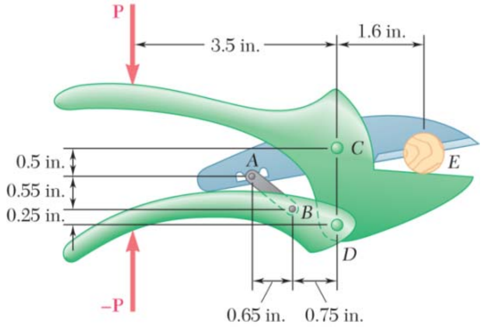
Note: $\tan\theta = \frac{2}{7}$ as above

FBD whole: $\circlearrowleft \Sigma M_A = 0{:}\quad (0.100\text{ m})C_y - 1.5\text{ kN}\cdot\text{m} = 0 \quad C_y = 15\text{ kN}$

$$\Sigma F_y = 0{:}\quad C_y - F_{BC}\sin\theta = 0 \quad F_{BC} = \frac{C_y}{\sin\theta}$$

$$\xrightarrow{}\Sigma F_x = 0{:}\quad F_{BC}\cos\theta - P = 0$$

$$P = F_{BC}\cos\theta = \frac{C_y}{\tan\theta} = \frac{15\text{ kN}}{2/7} \quad \mathbf{P} = 52.5\text{ kN} \longleftarrow \blacktriangleleft$$

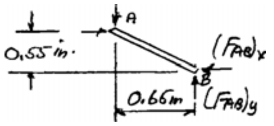

P
3.5 in.
1.6 in.
0.5 in.
0.55 in.
0.25 in.
A
B
C
D
E
–P
0.65 in.
0.75 in.

PROBLEM 6.175

The compound-lever pruning shears shown can be adjusted by placing pin A at various ratchet positions on blade ACE. Knowing that 300-lb vertical forces are required to complete the pruning of a small branch, determine the magnitude P of the forces that must be applied to the handles when the shears are adjusted as shown.

SOLUTION

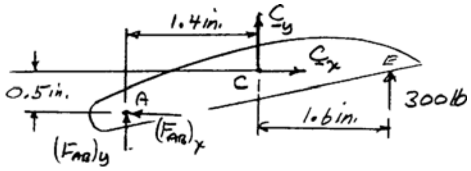
We note that AB is a two-force member.


A
0.55 in.
$(F_{AB})_x$
B
0.65 in
$(F_{AB})_y$

$$\frac{(F_{AB})_x}{0.65\text{ in.}} = \frac{(F_{AB})_y}{0.55\text{ in.}}$$

$$(F_{AB})_y = \frac{11}{13}(F_{AB})_x \qquad (1)$$

Free body: Blade *ACE*:


1.4 in.
C_y
C_x
E
0.5 in.
C
A
300 lb
1.6 in.
$(F_{AB})_x$
$(F_{AB})_y$

$$+\circlearrowleft \Sigma M_C = 0\text{:}\quad (300\text{ lb})(1.6\text{ in.}) - (F_{AB})_x(0.5\text{ in.}) - (F_{AB})_y(1.4\text{ in.}) = 0$$

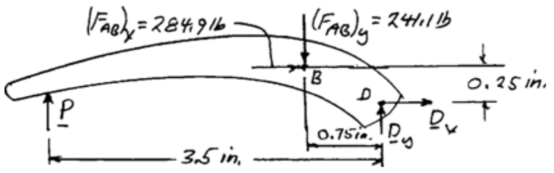
Use Eq. (1):

$$(F_{AB})_x(0.5\text{ in.}) + \frac{11}{13}(F_{AB})_x(1.4\text{ in.}) = 480\text{ lb}\cdot\text{in.}$$

$$1.6846(F_{AB})_x = 480 \qquad (F_{AB})_x = 284.9\text{ lb}$$

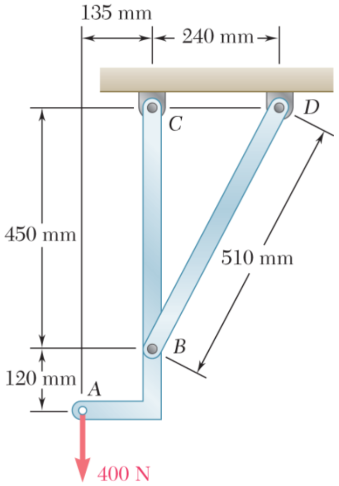
$$(F_{AB})_y = \frac{11}{13}(284.9\text{ lb}) \qquad (F_{AB})_y = 241.1\text{ lb}$$

Free body: Lower handle:


$(F_{AB})_x = 284.9$ lb
$(F_{AB})_y = 241.1$ lb
B
D
0.25 in.
P
0.75 in.
D_y
D_x
3.5 in.

$$+\circlearrowleft \Sigma M_D = 0\text{:}\quad (241.1\text{ lb})(0.75\text{ in.}) - (284.9\text{ lb})(0.25\text{ in.}) - P(3.5\text{ in.}) = 0$$

$$P = 31.3\text{ lb} \blacktriangleleft$$


135 mm
240 mm
D
C
450 mm
510 mm
B
120 mm
A
400 N

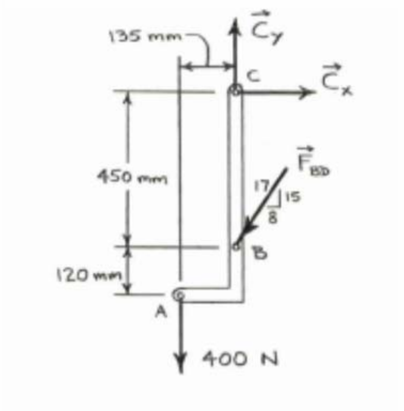
PROBLEM 6.F1

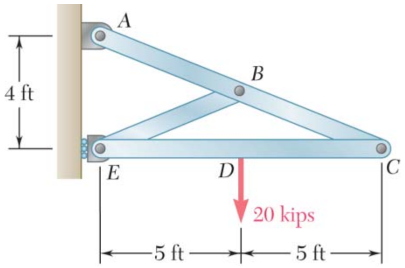
For the frame and loading shown, draw the free-body diagram(s) needed to determine the force in member *BD* and the components of the reaction at *C*.

SOLUTION

We note that *BD* is a two-force member.

Free body: *ABC*


135 mm
$\vec{C}_y$
C
$\vec{C}_x$
$\vec{F}_{BD}$
17
15
8
450 mm
B
120 mm
A
400 N


A
B
4 ft
E
D
C
20 kips
5 ft
5 ft

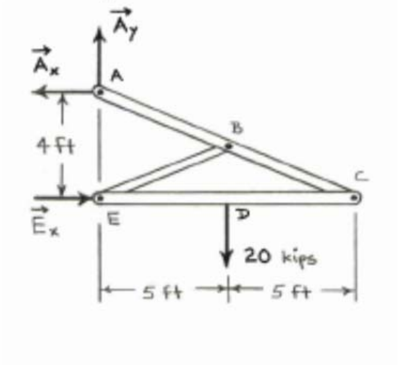
PROBLEM 6.F2

For the frame and loading shown, draw the free-body diagram(s) needed to determine the components of all forces acting on member *ABC*.

SOLUTION

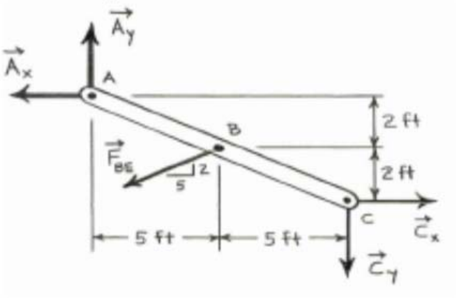
We note that *BE* is a two-force member.

Free body: Frame

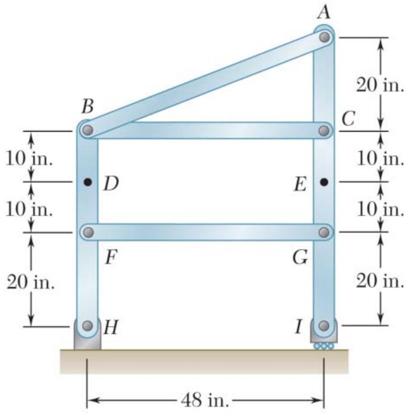

$\vec{A}_y$
$\vec{A}_x$
A
B
4 ft
C
$\vec{E}_x$
E
D
20 kips
5 ft
5 ft

Note: Sum forces in y about A to determine $\mathbf{A}_y$.
Sum moments about E to determine $\mathbf{A}_x$.

Free body: *ABC*


$\vec{A}_y$
$\vec{A}_x$
A
B
2 ft
2 ft
$\vec{F}_{BE}$
2
5
C
$\vec{C}_x$
$\vec{C}_y$
5 ft
5 ft

Note: Sum moments about C to determine $\mathbf{F}_{BE}$.
Sum forces in x to determine $\mathbf{C}_x$.
Sum forces in y to determine $\mathbf{C}_y$.


A
B
C
D
E
F
G
H
I
20 in.
10 in.
10 in.
10 in.
10 in.
20 in.
20 in.
48 in.

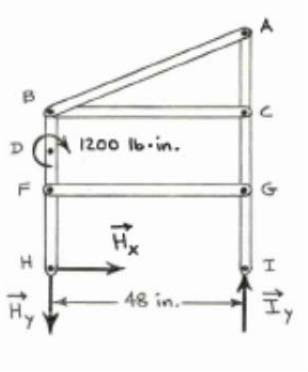
PROBLEM 6.F3

Draw the free-body diagram(s) needed to determine all the forces exerted on member *AI* if the frame is loaded by a clockwise couple of magnitude 1200 lb·in. applied at point *D*.

SOLUTION

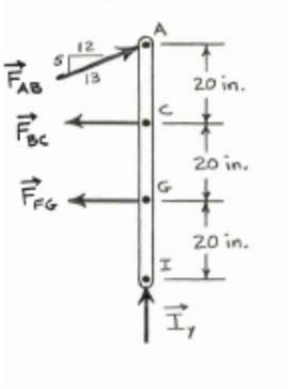
We note that *AB*, *BC*, and *FG* are two-force members.

Free body: Frame


A
B
C
D
1200 lb·in.
F
G
H
I
$\vec{H}_x$
$\vec{H}_y$
48 in.
$\vec{I}_y$

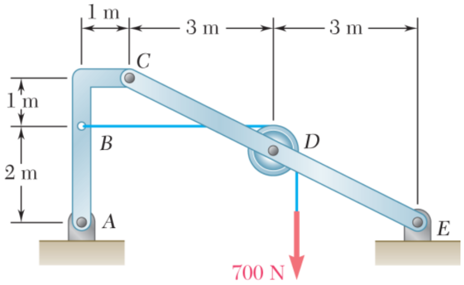
Note: Sum moments about *H* to determine $\mathbf{I}_y$.

Free body: *AI*


A
12
5
13
$\vec{F}_{AB}$
20 in.
C
$\vec{F}_{BC}$
20 in.
G
$\vec{F}_{FG}$
20 in.
I
$\vec{I}_y$

PROBLEM 6.F3 (Continued)

Note: Sum forces in y to determine $\mathbf{F}_{AB}$
Sum moments about G to determine $\mathbf{F}_{BC}$
Sum forces in x to determine $\mathbf{F}_{FG}$

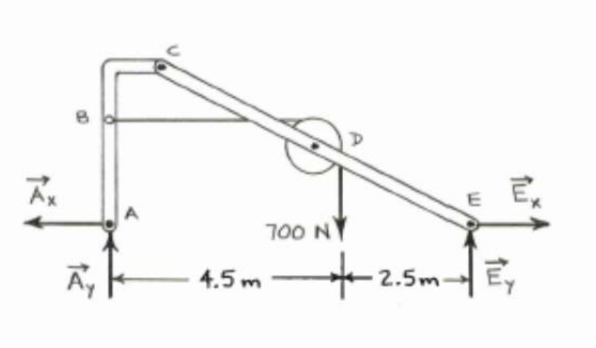

1 m
3 m
3 m
C
1 m
B
D
2 m
A
E
700 N

PROBLEM 6.F4

Knowing that the pulley has a radius of 0.5 m, draw the free-body diagram(s) needed to determine the components of the reactions at *A* and *E*.

SOLUTION

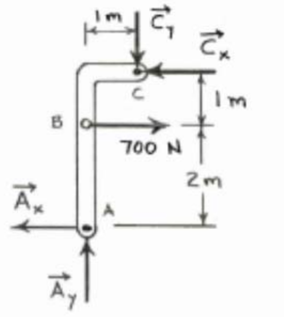
Free body: Frame


C
B
D
$\vec{A}_x$
A
E
$\vec{E}_x$
700 N
$\vec{A}_y$
4.5 m
2.5 m
$\vec{E}_y$

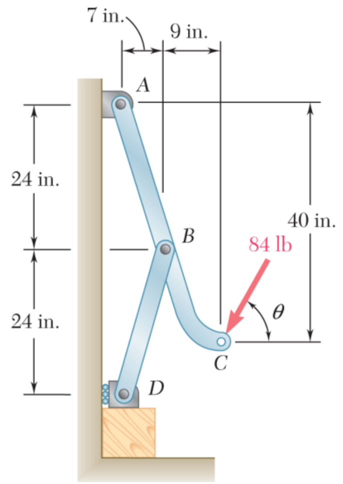
Note: Sum moments about *E* to determine $\mathbf{A}_y$.
Sum forces in *y* to determine $\mathbf{E}_y$.

Sum forces in *x* to determine $\mathbf{A}_x$ and $\mathbf{E}_x$.

Free body: *ABC*


1 m
$\vec{C}_y$
$\vec{C}_x$
C
1 m
B
700 N
2 m
$\vec{A}_x$
A
$\vec{A}_y$

Note: Sum moments about *C* to determine $\mathbf{A}_x$.
Use previously obtained relation between $\mathbf{A}_x$ and $\mathbf{E}_x$ to determine $\mathbf{E}_x$.


7 in.
9 in.
A
24 in.
B
40 in.
84 lb
24 in.
θ
C
D

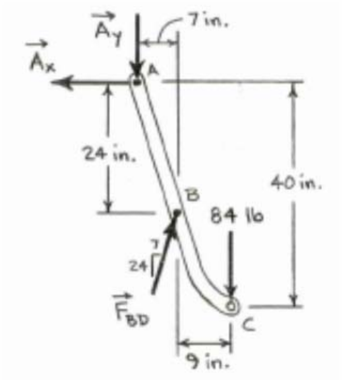
PROBLEM 6.F5

An 84-lb force is applied to the toggle vise at *C*. Knowing that $\theta = 90°$, draw the free-body diagram(s) needed to determine the vertical force exerted on the block at *D*.

SOLUTION

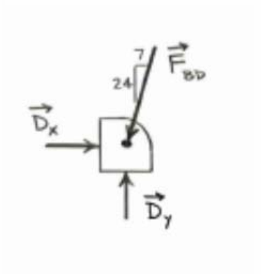
We note that *BD* is a two-force member.

Free body: *ABC*

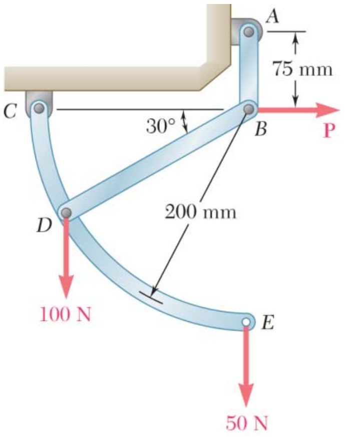

$\vec{A}_y$
7 in.
$\vec{A}_x$
A
24 in.
40 in.
B
84 lb
7
24
$\vec{F}_{BD}$
C
9 in.

Note: Sum moments about *A* to determine $\mathbf{F}_{BD}$.

Free body: *D*


7
24
$\vec{F}_{BD}$
$\vec{D}_x$
$\vec{D}_y$

Note: Sum forces in *y* to determine $\mathbf{D}_y$.


A
75 mm
C
30°
B
P
200 mm
D
100 N
E
50 N

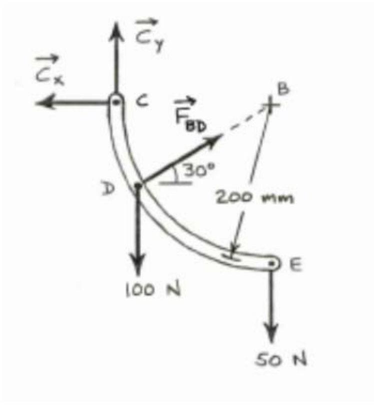
PROBLEM 6.F6

For the system and loading shown, draw the free-body diagram(s) needed to determine the force **P** required for equilibrium.

SOLUTION

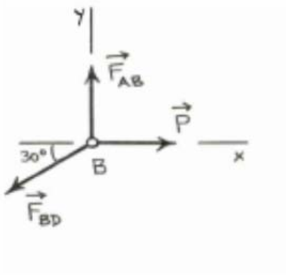
We note that *AB* and *BD* are two-force members.

Free body: *CDE*

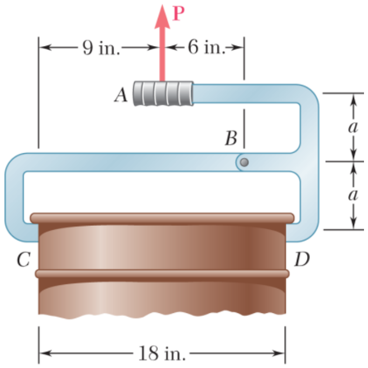

$\vec{C}_y$
$\vec{C}_x$
C
$\vec{F}_{BD}$
B
30°
D
200 mm
E
100 N
50 N

Note: Sum moments about *C* to determine $\mathbf{F}_{BD}$.

Free body: Pin *B*


y
$\vec{F}_{AB}$
$\vec{P}$
x
30°
B
$\vec{F}_{BD}$

Note: Sum forces in *x* to determine **P**.


P
9 in.
6 in.
A
B
a
a
C
D
18 in.

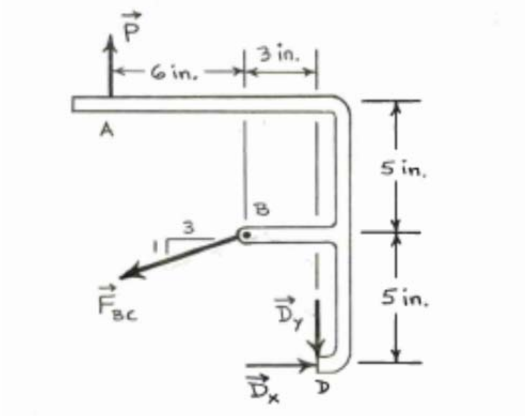
PROBLEM 6.F7

A small barrel weighing 60 lb is lifted by a pair of tongs as shown. Knowing that a = 5 in., draw the free-body diagram(s) needed to determine the forces exerted at B and D on tong ABD.

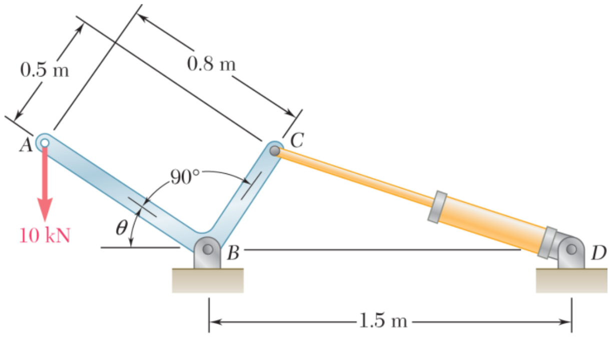
SOLUTION

We note that BC is a two-force member.

Free body: *ABD*


$\vec{P}$
6 in.
3 in.
A
5 in.
B
3
1
$\vec{F}_{BC}$
$\vec{D}_y$
5 in.
$\vec{D}_x$
D

Note: Sum moments about D to determine $\mathbf{F}_{BC}$.
Sum forces in x to determine $\mathbf{D}_x$.
Sum forces in y to determine $\mathbf{D}_y$.


0.5 m
0.8 m
A
C
90°
θ
10 kN
B
D
1.5 m

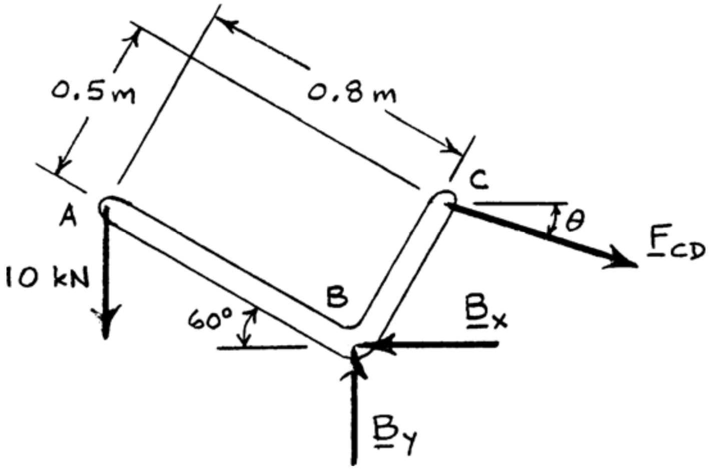
PROBLEM 6.F8

The position of member *ABC* is controlled by the hydraulic cylinder *CD*. Knowing that $\theta = 30°$, draw the free-body diagram(s) needed to determine the force exerted by the hydraulic cylinder on pin *C*, and the reaction at *B*.

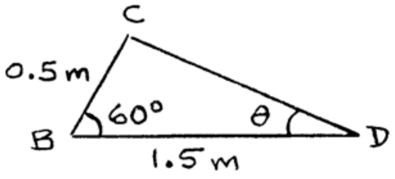
SOLUTION

We note that *CD* is a two-force member.

Free body: *ABC*


0.5 m
0.8 m
A
C
θ
F_{CD}
10 kN
B
60°
B_x
B_y

Note: To find θ, consider geometry of triangle *BCD*:


C
0.5 m
60°
θ
B
D
1.5 m

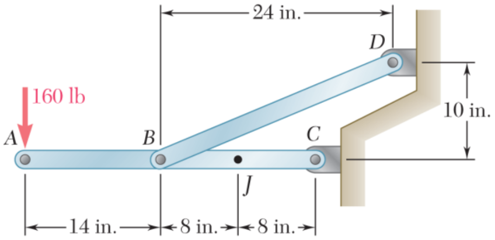
Law of cosines: $(CD)^2 = (0.5)^2 + (1.5)^2 - 2(0.5)(1.5)\cos 60°$

$$CD = 1.32288 \text{ m}$$

Law of sines: $\dfrac{\sin\theta}{0.5 \text{ m}} = \dfrac{\sin 60°}{1.32288 \text{ m}}$

$$\theta = 19.1066°$$

CHAPTER 7



PROBLEM 7.1

Determine the internal forces (axial force, shearing force, and bending moment) at Point *J* of the structure indicated.

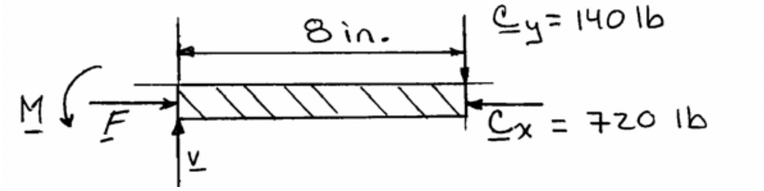
Frame and loading of Problem 6.76.

SOLUTION

From Problem 6.76: $\mathbf{C}_x = 720 \text{ lb} \longleftarrow$

$\mathbf{C}_y = 140 \text{ lb}\downarrow$

FBD of *JC*:



$$\xrightarrow{+} \Sigma F_x = 0: \quad F - 720 \text{ lb} = 0$$

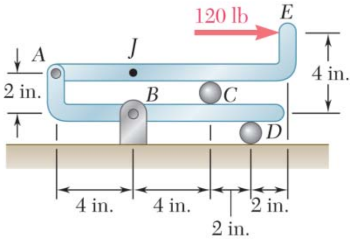
$$F = +720 \text{ lb} \qquad \mathbf{F} = 720 \text{ lb} \longrightarrow \blacktriangleleft$$

$$+\!\uparrow \Sigma F_y = 0: \quad V - 140 \text{ lb} = 0$$

$$V = 140 \text{ lb} \qquad \mathbf{V} = 140.0 \text{ lb}\uparrow \blacktriangleleft$$

$$+\circlearrowright \Sigma M_J = 0: \quad M - (140 \text{ lb})(8 \text{ in.}) = 0$$

$$M = +1120 \text{ lb}\cdot\text{in.} \qquad \mathbf{M} = 1120 \text{ lb}\cdot\text{in.} \circlearrowright \blacktriangleleft$$


120 lb
E
A
J
2 in.
B
C
D
4 in.
4 in.
4 in.
2 in.
2 in.

PROBLEM 7.2

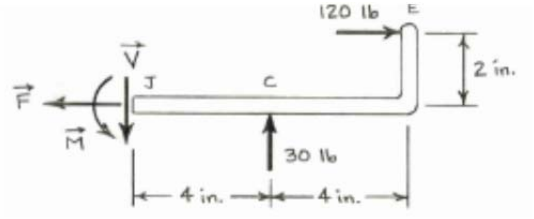
Determine the internal forces (axial force, shearing force, and bending moment) at Point *J* of the structure indicated.

Frame and loading of Problem 6.78.

SOLUTION

From Problem 6.78: $\mathbf{C} = 30 \text{ lb}\uparrow$

FBD of *AE*:


120 lb
E
2 in.
$\vec{V}$
J
C
$\vec{F}$
$\vec{M}$
30 lb
4 in.
4 in.

$$\xrightarrow{+} \Sigma F_x = 0: \quad -F + 120 \text{ lb} = 0$$

$$F = +120 \text{ lb} \qquad \mathbf{F} = 120.0 \text{ lb} \longleftarrow \blacktriangleleft$$

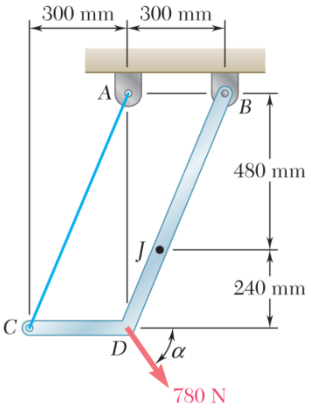
$$+\uparrow \Sigma F_y = 0: \quad -V + 30 \text{ lb} = 0$$

$$V = 30 \text{ lb} \qquad \mathbf{V} = 30.0 \text{ lb}\downarrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_J = 0: \quad M + (30 \text{ lb})(4 \text{ in.}) - (120 \text{ lb})(2 \text{ in.}) = 0$$

$$M = +120.0 \text{ lb}\cdot\text{in.}$$

$$\mathbf{M} = 120.0 \text{ lb}\cdot\text{in.} \circlearrowleft \blacktriangleleft$$

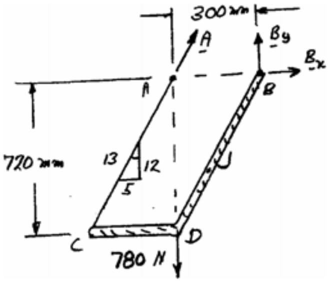

300 mm
300 mm
A
B
480 mm
J
240 mm
C
D
α
780 N

PROBLEM 7.3

Determine the internal forces at Point J when $\alpha = 90°$.

SOLUTION

Reactions $(\alpha = 90°)$


300 mm
A
B_y
B_x
A
B
720 mm
13
12
5
J
C
D
780 N

$$\Sigma M_A = 0: \quad \mathbf{B}_y = 0$$

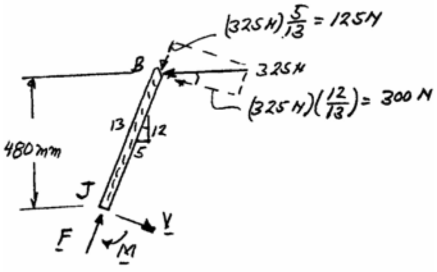
$$+\uparrow \Sigma F_y = 0: \quad A\left(\frac{12}{13}\right) - 780 \text{ N} = 0$$

$$A = 845 \text{ N} \qquad \mathbf{A} = 845 \text{ N} \nearrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad (845 \text{ N})\frac{5}{13} + B_x = 0$$

$$B_x = -325 \text{ N} \qquad \mathbf{B}_x = 325 \text{ N} \longleftarrow$$

FBD BJ:

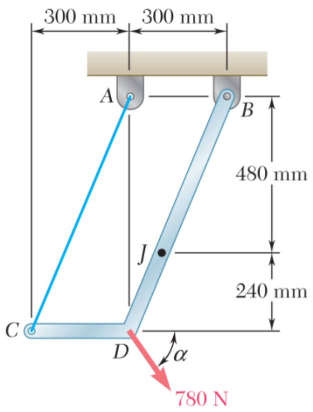

$(325 N)\frac{5}{13} = 125 N$
B
325 N
$(325 N)(\frac{12}{13}) = 300 N$
480 mm
13
12
5
J
F
V
M

$$+\nearrow \Sigma F = 0: \quad 125 \text{ N} - F = 0 \qquad \mathbf{F} = 125.0 \text{ N} \measuredangle 67.4° \blacktriangleleft$$

$$+\searrow \Sigma F = 0: \quad V - 300 \text{ N} = 0 \qquad \mathbf{V} = 300 \text{ N} \nwarrow 22.6° \blacktriangleleft$$

$$+\circlearrowright \Sigma M = 0: \quad (325 \text{ N})(0.480 \text{ m}) - M = 0$$

$$M = +156 \text{ N}\cdot\text{m} \qquad \mathbf{M} = 156.0 \text{ N}\cdot\text{m} \circlearrowright \blacktriangleleft$$

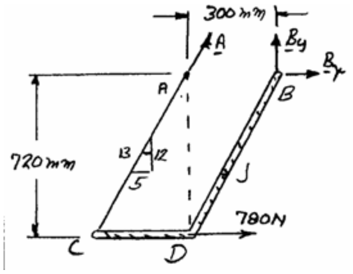

300 mm
300 mm
A
B
480 mm
J
240 mm
C
D
α
780 N

PROBLEM 7.4

Determine the internal forces at Point J when $\alpha = 0$.

SOLUTION

Reactions $(\alpha = 0)$


300 mm
A
A
By
Bx
B
720 mm
13
12
5
J
780N
C
D

$$+\circlearrowleft \Sigma M_A = 0: \quad (780\text{ N})(0.720\text{ m}) + B_y(0.3\text{ m}) = 0$$

$$B_y = -1872\text{ N} \qquad \mathbf{B}_y = 1872\text{ N}\downarrow$$

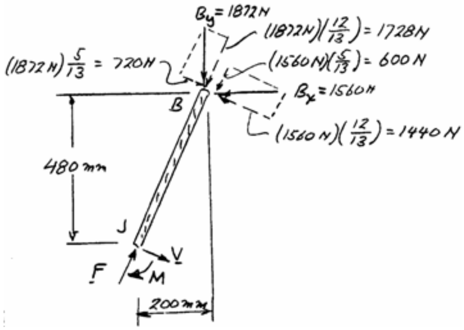
$$+\uparrow \Sigma F_y = 0: \quad A\left(\frac{12}{13}\right) - 1872\text{ N} = 0$$

$$\mathbf{A} = 2028\text{ N} \nearrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad (2028\text{ N})\left(\frac{5}{13}\right) + 780\text{ N} + B_x = 0$$

$$B_x = -1560\text{ N} \qquad \mathbf{B}_x = 1560\text{ N} \longleftarrow$$

FBD BJ:


By = 1872N
(1872N)(12/13) = 1728N
(1872N)5/13 = 720N
(1560N)(5/13) = 600N
Bx = 1560N
B
(1560N)(12/13) = 1440N
480mm
J
V
F
M
200mm

$$\swarrow \Sigma F = 0: \quad 1728\text{ N} + 600\text{ N} - F = 0$$

$$F = +2328\text{ N} \qquad \mathbf{F} = 2330\text{ N} \measuredangle 67.4° \blacktriangleleft$$

$$\overset{+}{\searrow} \Sigma F = 0: \quad 720\text{ N} - 1440\text{ N} + V = 0$$

$$V = +720\text{ N} \qquad \mathbf{V} = 720\text{ N} \searrow 22.6° \blacktriangleleft$$

$$BJ = \sqrt{480^2 + 200^2} = 520\text{ mm}$$

$$+\circlearrowleft \Sigma M_J = 0: \quad (1440\text{ N})(0.520\text{ m}) - (720\text{ N})(0.520\text{ m}) - M = 0$$

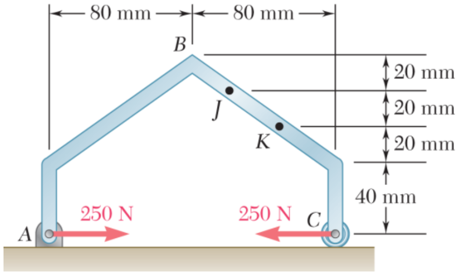
$$M = +374.4\text{ N}\cdot\text{m} \qquad \mathbf{M} = 374\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

Alternate

Computation of M using $\mathbf{B}_x + \mathbf{B}_y$:

$$+\circlearrowleft \Sigma M_J = 0: \quad (1560\text{ N})(0.48\text{ m}) - (1872\text{ N})(0.2\text{ m}) - M = 0$$

$$M = +374.4\text{ N}\cdot\text{m} \qquad \mathbf{M} = 374\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$


80 mm
80 mm
B
J
K
20 mm
20 mm
20 mm
40 mm
250 N
250 N
A
C

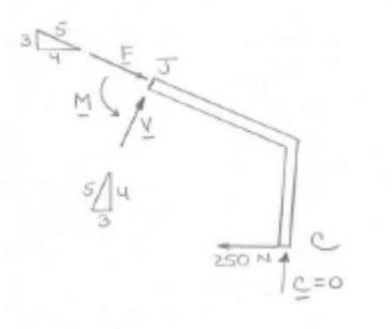
PROBLEM 7.5

For the frame and loading shown, determine the internal forces at the point indicated:

Point *J*.

SOLUTION

From free body diagram, all reactions = 0


3
5
4
F
J
M
V
5
4
3
250 N
C
C=0

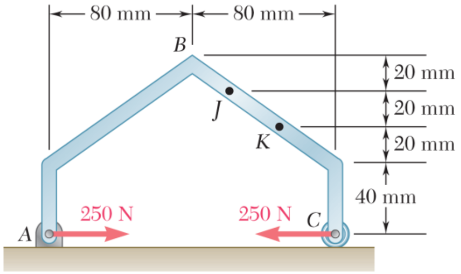
$$\Sigma F = 0 \text{(Force Triangle)} \quad \frac{V}{3} = \frac{F}{4} = \frac{250}{5}$$

$V = +150 \text{ N}$ $\qquad$ $\mathbf{V} = 150.0 \text{ N} \measuredangle 53.1°$ ◀

$F = +200 \text{ N}$ $\qquad$ $\mathbf{F} = 200 \text{ N} \searrow 36.9°$ ◀

$+\circlearrowleft \Sigma M_J = 0: \quad M - (250 \text{ N})(0.080 \text{ m}) = 0$

$M = +20 \text{ N}\cdot\text{m}$ $\qquad$ $\mathbf{M} = 20.0 \text{ N}\cdot\text{m} \circlearrowleft$ ◀



PROBLEM 7.6

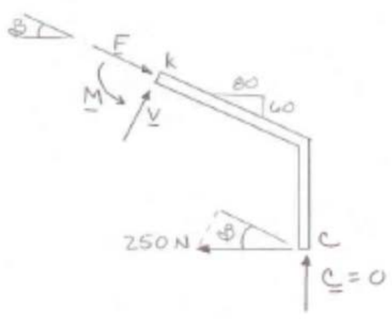
For the frame and loading shown, determine the internal forces at the point indicated:

Point *K*.

SOLUTION

From free body diagram, all reaction = 0

Free body *KC*:



$$\tan\beta = \frac{60}{80} = \frac{3}{4}$$

$$\sin\beta = \frac{3}{5}; \qquad \cos\beta = \frac{4}{5}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad F - (250\text{ N})\cos\beta = 250\text{ N}\left(\frac{4}{5}\right)$$

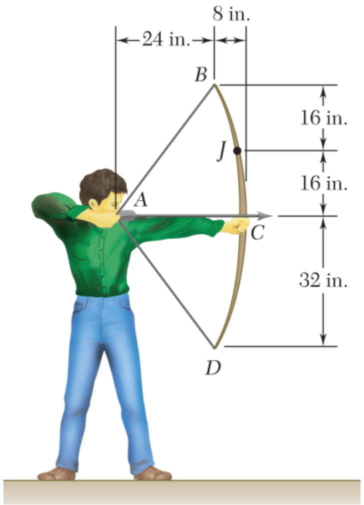
$$F = +200\text{ N} \qquad \mathbf{F} = 200\text{ N} \searrow 36.9° \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad V - (250\text{ N})\sin\beta = 250\text{ N}\left(\frac{3}{5}\right)$$

$$V = +150\text{ N} \qquad \mathbf{V} = 150.0\text{ N} \nearrow 53.1° \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_K = 0: \quad M - (250\text{ N})(0.060\text{ m}) = 0$$

$$M = +15.00\text{ N}\cdot\text{m} \qquad \mathbf{M} = 15.00\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

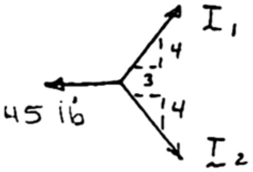

8 in.
24 in.
B
16 in.
J
16 in.
A
C
32 in.
D

PROBLEM 7.7

An archer aiming at a target is pulling with a 45-lb force on the bowstring. Assuming that the shape of the bow can be approximated by a parabola, determine the internal forces at Point *J*.

SOLUTION

FBD Point A:

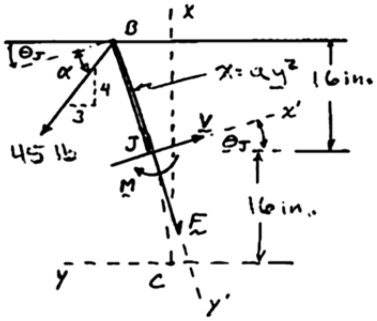

$\underset{\sim}{T}_1$
4
3
4
45 lb
$\underset{\sim}{T}_2$

By symmetry $T_1 = T_2$

$$\xrightarrow{\quad}\Sigma F_x = 0:\quad 2\left(\frac{3}{5}T_1\right) - 45\text{ lb} = 0 \qquad T_1 = T_2 = 37.5\text{ lb}$$

Curve *CJB* is parabolic: $x = ay^2$

FBD BJ:


x
B
θJ
α
4
3
45 lb
x = ay²
16 in.
V
x′
θJ
J
M
E
16 in.
y
C
y′

At B: $x = 8\text{ in.}$ $\qquad y = 32\text{ in.}$

$$a = \frac{8\text{ in.}}{(32\text{ in.})^2} = \frac{1}{128\text{ in.}}$$

$$x = \frac{y^2}{128}$$

$$\text{Slope of parabola} = \tan\theta = \frac{dx}{dy} = \frac{2y}{128} = \frac{y}{64}$$

At *J*: $$\theta_J = \tan^{-1}\left[\frac{16}{64}\right] = 14.036°$$

So $$\alpha = \tan^{-1}\frac{4}{3} - 14.036° = 39.094°$$

$$\nearrow \Sigma F_{x'} = 0:\quad V - (37.5\text{ lb})\cos(39.094°) = 0$$

$\mathbf{V} = 29.1\text{ lb} \measuredangle 14.04°$ ◀

PROBLEM 7.7 (Continued)

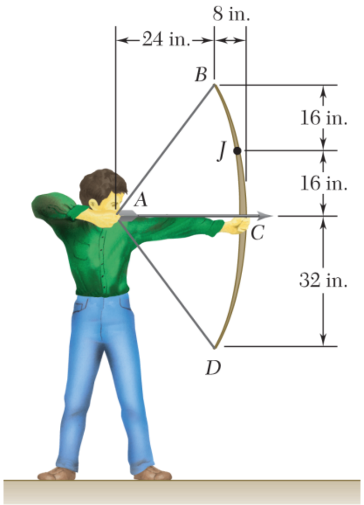
$$\searrow \Sigma F_{y'} = 0: \quad F + (37.5 \text{ lb}) \sin(39.094°) = 0$$

$$F = -23.647$$

$\mathbf{F} = 23.6 \text{ lb} \measuredangle 76.0°$ ◀

$$\circlearrowleft \Sigma M_J = 0: \quad M + (16 \text{ in.})\left[\frac{3}{5}(37.5 \text{ lb})\right] + [(8-2) \text{ in.}]\left[\frac{4}{5}(37.5 \text{ lb})\right] = 0$$

$\mathbf{M} = 540 \text{ lb} \cdot \text{in.} \circlearrowright$ ◀



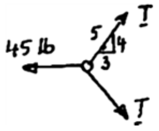
PROBLEM 7.8

For the bow of Problem 7.7, determine the magnitude and location of the maximum (*a*) axial force, (*b*) shearing force, (*c*) bending moment.

PROBLEM 7.7 An archer aiming at a target is pulling with a 45-lb force on the bowstring. Assuming that the shape of the bow can be approximated by a parabola, determine the internal forces at Point *J*.

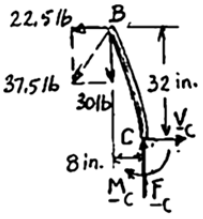
SOLUTION

Free body: Point *A*



$\xrightarrow{+} \Sigma F_x = 0: \quad 2\left(\frac{3}{5}T\right) - 45 \text{ lb} = 0$ $\qquad T = 37.5 \text{ lb}$ ◁

Free body: Portion of bow *BC*



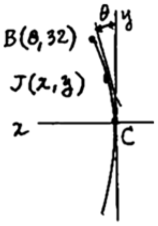
$+\uparrow \Sigma F_y = 0: \quad F_C - 30 \text{ lb} = 0$ $\qquad \mathbf{F}_C = 30 \text{ lb} \uparrow$ ◁

$\xrightarrow{+} \Sigma F_x = 0: \quad V_C - 22.5 \text{ lb} = 0$ $\qquad \mathbf{V}_C = 22.5 \text{ lb} \longrightarrow$ ◁

$\uparrow \Sigma M_C = 0: \quad (22.5 \text{ lb})(32 \text{ in.}) + (30 \text{ lb})(8 \text{ in.}) - M_C = 0$

$\mathbf{M}_C = 960 \text{ lb} \cdot \text{in.}$ ↻ ◁

Equation of parabola



$$x = ky^2$$

At *B*: $\qquad 8 = k(32)^2 \quad k = \frac{1}{128}$

Therefore, equation is $\qquad x = \frac{y^2}{128} \qquad (1)$

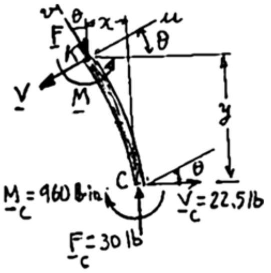
The slope at *J* is obtained by differentiating (1):

$$d_x = \frac{2y\,dy}{128}, \quad \tan\theta = \frac{dx}{dy} = \frac{y}{64} \qquad (2)$$

PROBLEM 7.8 (Continued)

(*a*) Maximum axial force

$$+\nwarrow \Sigma F_V = 0: \quad -F + (30\text{ lb})\cos\theta - (22.5\text{ lb})\sin\theta = 0$$



Free body: Portion bow *CK*

$$F = 30\cos\theta - 22.5\sin\theta$$

F is largest at $C(\theta = 0)$

$F_m = 30.0$ lb at *C* ◀

(*b*) Maximum shearing force

$$+\nearrow \Sigma F_V = 0: \quad -V + (30\text{ lb})\sin\theta + (22.5\text{ lb})\cos\theta = 0$$

$$V = 30\sin\theta + 22.5\cos\theta$$

V is largest at *B* (and *D*)

Where $\theta = \theta_{\max} = \tan^{-1}\left(\frac{1}{2}\right) = 26.56°$

$$V_m = 30\sin 26.56° + 22.5\cos 26.56°$$

$V_m = 33.5$ lb at *B* and *D* ◀

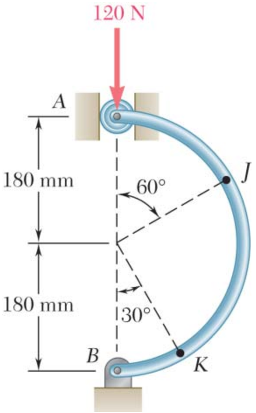
(*c*) Maximum bending moment

$$+\circlearrowleft \Sigma M_K = 0: \quad M - 960\text{ lb}\cdot\text{in.} + (30\text{ lb})x + (22.5\text{ lb})y = 0$$

$$M = 960 - 30x - 22.5y$$

M is largest at *C*, where $x = y = 0$.

$M_m = 960$ lb·in. at *C* ◀

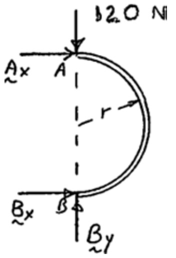

120 N
A
180 mm
60°
J
180 mm
30°
B
K

PROBLEM 7.9

A semicircular rod is loaded as shown. Determine the internal forces at Point *J*.

SOLUTION

FBD Rod:


120 N
Ax
A
r
Bx
B
By

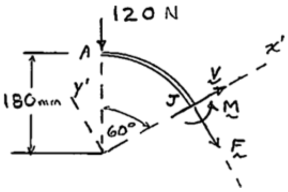
↺ $\Sigma M_B = 0: \quad A_x(2r) = 0$

$\mathbf{A}_x = 0$

↗ $\Sigma F_{x'} = 0: \quad V - (120 \text{ N})\cos 60° = 0$

$\mathbf{V} = 60.0 \text{ N}$ ↗ ◀

FBD AJ:


120 N
A
180mm
y'
60°
J
V
M
F
x'

↘ $\Sigma F_{y'} = 0: \quad F + (120 \text{ N})\sin 60° = 0$

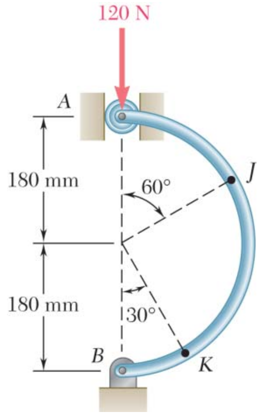
$F = -103.923 \text{ N}$

$\mathbf{F} = 103.9 \text{ N}$ ↖ ◀

↺ $\Sigma M_J = 0: \quad M - [(0.180 \text{ m})\sin 60°](120 \text{ N}) = 0$

$M = 18.7061$

$\mathbf{M} = 18.71$ ↻ ◀

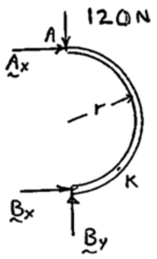

120 N
A
180 mm
60°
J
180 mm
30°
B
K

PROBLEM 7.10

A semicircular rod is loaded as shown. Determine the internal forces at Point *K*.

SOLUTION

FBD Rod:


120N
A
Ax
r
K
Bx
By

$$\uparrow \Sigma F_y = 0: \quad B_y - 120\text{ N} = 0 \quad \mathbf{B}_y = 120\text{ N}\uparrow$$

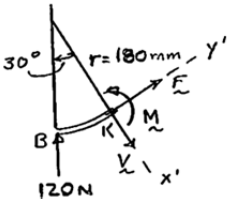
$$\circlearrowleft \Sigma M_A = 0: \quad 2rB_x = 0 \quad \mathbf{B}_x = 0$$

$$\searrow \Sigma F_{x'} = 0: \quad V - (120\text{ N})\cos 30° = 0$$

$$V = 103.923\text{ N}$$

$$\mathbf{V} = 103.9\text{ N} \searrow \blacktriangleleft$$

FBD BK:


30°
r = 180mm
y'
F
M
K
B
V
x'
120N

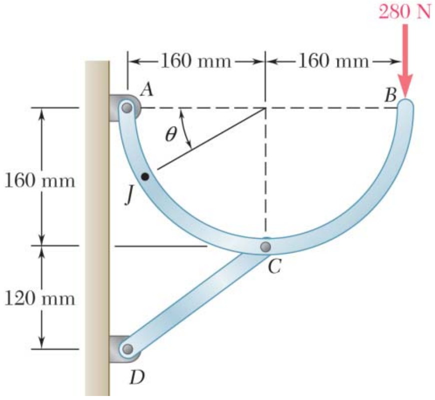
$$\nearrow \Sigma F_{y'} = 0: \quad F + (120\text{ N})\sin 30° = 0$$

$$F = -60\text{ N}$$

$$\mathbf{F} = 60.0\text{ N} \swarrow \blacktriangleleft$$

$$\circlearrowleft \Sigma M_K = 0: \quad M - [(0.180\text{ m})\sin 30°](120\text{ N}) = 0$$

$$\mathbf{M} = 10.80\text{ N}\cdot\text{m} \circlearrowright \blacktriangleleft$$

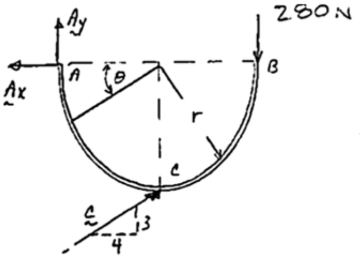


PROBLEM 7.11

A semicircular rod is loaded as shown. Determine the internal forces at Point *J* knowing that $\theta = 30°$.

SOLUTION

FBD AB:



$$\circlearrowleft \Sigma M_A = 0: \quad r\left(\frac{4}{5}C\right) + r\left(\frac{3}{5}C\right) - 2r(280\text{ N}) = 0$$

$$\mathbf{C} = 400\text{ N} \nearrow$$

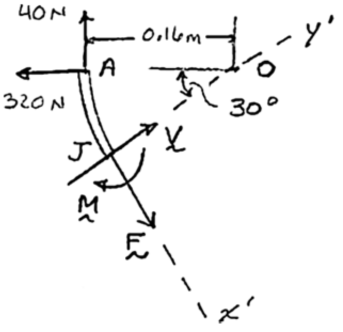
$$\longrightarrow \Sigma F_x = 0: \quad -A_x + \frac{4}{5}(400\text{ N}) = 0$$

$$\mathbf{A}_x = 320\text{ N} \longleftarrow$$

$$\uparrow \Sigma F_y = 0: \quad A_y + \frac{3}{5}(400\text{ N}) - 280\text{ N} = 0$$

$$\mathbf{A}_y = 40.0\text{ N} \uparrow$$

FBD AJ:



$$\searrow \Sigma F_{x'} = 0: \quad F - (320\text{ N})\sin 30° - (40.0\text{ N})\cos 30° = 0$$

$$F = 194.641\text{ N}$$

$$\mathbf{F} = 194.6\text{ N} \measuredangle 60.0° \blacktriangleleft$$

$$\nearrow \Sigma F_{y'} = 0: \quad V - (320\text{ N})\cos 30° + (40\text{ N})\sin 30° = 0$$

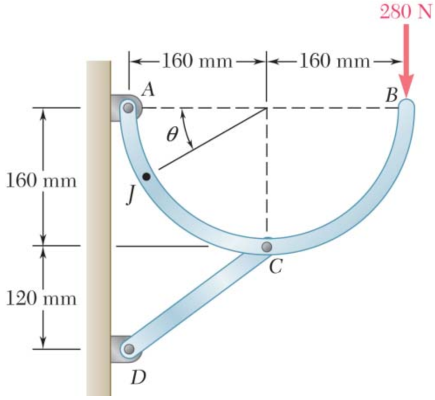
$$V = 257.13\text{ N}$$

$$\mathbf{V} = 257\text{ N} \measuredangle 30.0° \blacktriangleleft$$

$$\circlearrowleft \Sigma M_0 = 0: \quad (0.160\text{ m})(194.641\text{ N}) - (0.160\text{ m})(40.0\text{ N}) - M = 0$$

$$M = 24.743$$

$$\mathbf{M} = 24.7\text{ N}\cdot\text{m} \circlearrowright \blacktriangleleft$$

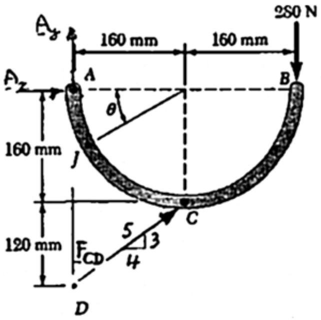

280 N
160 mm
160 mm
A
B
θ
J
C
160 mm
120 mm
D

PROBLEM 7.12

A semicircular rod is loaded as shown. Determine the magnitude and location of the maximum bending moment in the rod.

SOLUTION

Free body: Rod *ACB*


A_y
A_x
160 mm
160 mm
280 N
A
B
θ
160 mm
J
120 mm
F_CD
5
3
4
C
D

$$+\circlearrowleft \Sigma M_A = 0: \quad \left(\frac{4}{5}F_{CD}\right)(0.16\text{ m}) + \left(\frac{3}{5}F_{CD}\right)(0.16\text{ m})$$

$$-(280\text{ N})(0.32\text{ m}) = 0$$

$$\mathbf{F}_{CD} = 400\text{ N} \nearrow \quad \triangleleft$$

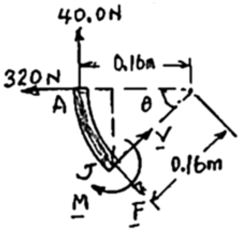
$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + \frac{4}{5}(400\text{ N}) = 0$$

$$A_x = -320\text{ N} \qquad \mathbf{A}_x = 320\text{ N} \longleftarrow \quad \triangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + \frac{3}{5}(400\text{ N}) - 280\text{ N} = 0$$

$$A_y = +40.0\text{ N} \qquad \mathbf{A}_y = 40.0\text{ N}\uparrow \quad \triangleleft$$

Free body: *AJ* (For $\theta < 90°$)


40.0N
320N
0.16m
A
θ
V
0.16m
J
M
F

$$+\circlearrowleft$$

$$\Sigma M_J = 0: \quad (320\text{ N})(0.16\text{ m})\sin\theta - (40.0\text{ N})(0.16\text{ m})(1-\cos\theta) - M = 0$$

$$M = 51.2\sin\theta + 6.4\cos\theta - 6.4 \qquad (1)$$

For maximum value between *A* and *C*:

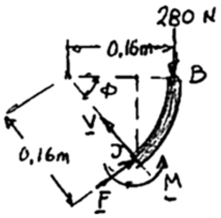
$$\frac{dM}{d\theta} = 0: \quad 51.2\cos\theta - 6.4\sin\theta = 0$$

$$\tan\theta = \frac{51.2}{6.4} = 8 \qquad \theta = 82.87°$$

Carrying into (1):

$$M = 51.2\sin 82.87° + 6.4\cos 82.87° - 6.4 = +45.20\text{ N}\cdot\text{m}$$

PROBLEM 7.12 (Continued)



Free body: *BJ* (For $\theta > 90°$)

$$+\circlearrowleft \Sigma M_J = 0: \quad M - (280\text{ N})(0.16\text{ m})(1 - \cos\phi) = 0$$

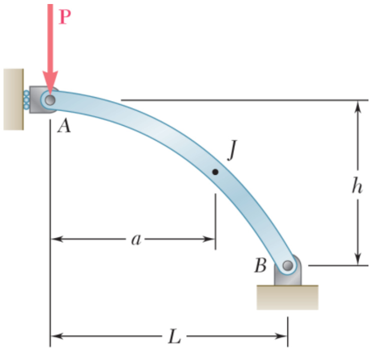
$$M = (44.8\text{ N}\cdot\text{m})(1 - \cos\phi)$$

Largest value occurs for $\phi = 90°$, that is, at *C*, and is

$$M_C = 44.8\text{ N}\cdot\text{m} \triangleleft$$

We conclude that

$$M_{\max} = 45.2\text{ N}\cdot\text{m} \quad \text{for} \quad \theta = 82.9° \blacktriangleleft$$

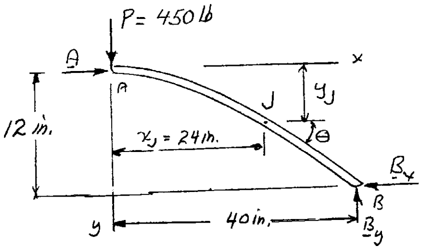


PROBLEM 7.13

The axis of the curved member *AB* is a parabola with vertex at *A*. If a verticalload **P** of magnitude 450 lb is applied at *A*, determine the internal forces at *J* when *h*= 12 in., *L*= 40 in., and *a*= 24 in.

SOLUTION

Free body *AB*



$$\uparrow \Sigma F_y = 0: \quad -450 \text{ lb} + B_y = 0$$

$$\mathbf{B}_y = 450 \text{ lb} \uparrow$$

$$+\circlearrowleft \Sigma M_A = 0: \quad B_x(12 \text{ in.}) - (450 \text{ lb})(40 \text{ in.}) = 0$$

$$\mathbf{B}_x = 1500 \text{ lb} \longleftarrow$$

$$\Sigma F_x = 0: \quad \mathbf{A} = 1500 \text{ lb} \longrightarrow$$

Parabola: $y = kx^2$

At *B*: $12 \text{ in.} = k(40 \text{ in.})^2 \qquad k = 0.0075$

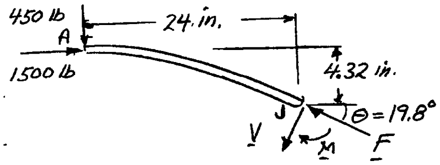
Equation of parabola: $y = 0.0075x^2$

$$\text{slope} = \frac{dy}{dx} = 0.015x$$

At *J*: $x_J = 24 \text{ in.} \qquad y_J = 0.0075(24)^2 = 4.32 \text{ in.}$

$$\text{slope} = 0.015(24) = 0.36, \quad \tan\theta = 0.36, \quad \theta = 19.8°$$

Free body *AJ*



$$+\circlearrowleft \Sigma M_J = 0: \quad (450 \text{ lb})(24 \text{ in.}) - (1500 \text{ lb})(4.32 \text{ in.}) - M = 0$$

$$M = 4320 \text{ lb} \cdot \text{in.} \quad \mathbf{M} = 4320 \text{ lb} \cdot \text{in.} \circlearrowright \blacktriangleleft$$

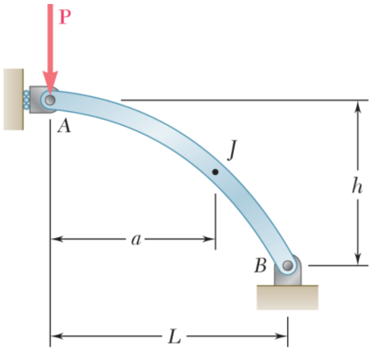
PROBLEM 7.13 (Continued)

$$+\nwarrow \Sigma F = 0: \quad F - (450\text{ lb})\sin 19.8^\circ - (1500\text{ lb})\cos 19.8^\circ = 0$$

$$F = +1563.8\text{ lb} \qquad \mathbf{F} = 1564\text{ lb} \nwarrow 19.8^\circ \blacktriangleleft$$

$$+\nearrow \Sigma F = 0: \quad -V - (450\text{ lb})\cos 19.8^\circ + (1500\text{ lb})\sin 19.8^\circ = 0$$

$$V = +84.71\text{ lb} \qquad \mathbf{V} = 84.7\text{ lb} \swarrow 70.2^\circ \blacktriangleleft$$



PROBLEM 7.14

Knowing that the axis of the curved member *AB* is a parabola with vertex at *A*, determine the magnitude and location of the maximum bending moment.

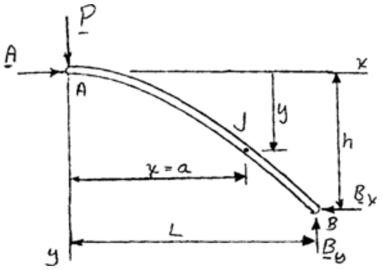
SOLUTION

Parabola $y = kx^2$

At *B*: $h = kL^2$

$k = h/L^2$

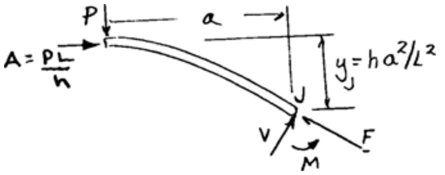
Equation of parabola $y = hx^2/L^2$



$$+\circlearrowleft \Sigma M_B = 0: \quad P(L) - A(h) = 0$$

$\mathbf{A} = PL/h \longrightarrow$

Free body *AJ* At *J*: $x_J = a \qquad y_J = ha^2/L^2$



$$+\circlearrowleft \Sigma M_J = 0: \quad P_a - (PL/h)(ha^2/L^2) + M = 0$$

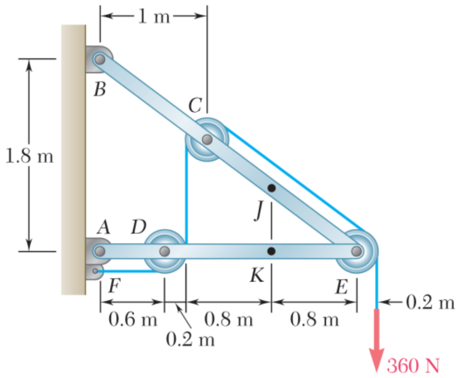
$$M = P\left(\frac{a^2}{L} - a\right)$$

For maximum: $\dfrac{dM}{da} = P\left(\dfrac{2a}{L} - 1\right) = 0$

$M_{\max}$ occurs at: $a = \dfrac{1}{2}L$ ◀

$$M_{\max} = P\left[\frac{(L/2)^2}{L} - \frac{L}{2}\right] = -\frac{PL}{4}$$

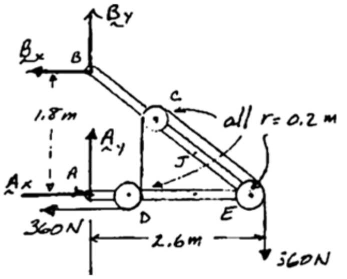
$|M|_{\max} = \dfrac{1}{4}PL$ ◀


1 m
B
C
1.8 m
J
A
D
K
F
E
0.6 m
0.2 m
0.8 m
0.8 m
0.2 m
360 N

PROBLEM 7.15

Knowing that the radius of each pulley is 200 mm and neglecting friction, determine the internal forces at Point *J* of the frame shown.

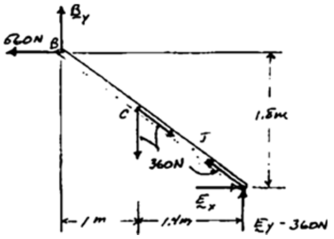
SOLUTION


B_y
B_x
B
1.8m
C
all r= 0.2 m
A_y
A_x
A
J
360N
D
E
2.6m
360N

FBD Frame with pulley and cord:

$$\circlearrowleft \Sigma M_A = 0\text{:}\quad (1.8\text{ m})B_x - (2.6\text{ m})(360\text{ N}) - (0.2\text{ m})(360\text{ N}) = 0$$

$$\mathbf{B}_x = 560\text{ N} \longleftarrow$$

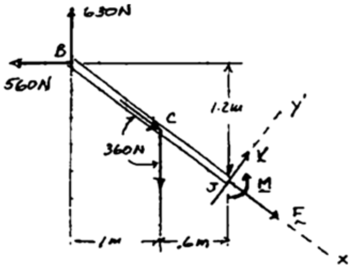

B_y
560N
B
C
1.8m
J
360N
E_x
1m
1.4m
E_y - 360N

FBD BE:

Note: Cord forces have been moved to pulley hub as per Problem 6.91.

$$\circlearrowleft \Sigma M_E = 0\text{:}\quad (1.4\text{ m})(360\text{ N}) + (1.8\text{ m})(560\text{ N}) - (2.4\text{ m})B_y = 0$$

$$\mathbf{B}_y = 630\text{ N}\uparrow$$


630N
B
560N
1.2m
y'
C
360N
V
M
J
F
1m
.6m
x

FBD BJ:

$$\searrow \Sigma F_{x'} = 0\text{:}\quad F + 360\text{ N} - \frac{3}{5}(630\text{ N} - 360\text{ N}) - \frac{4}{5}(560\text{ N}) = 0$$

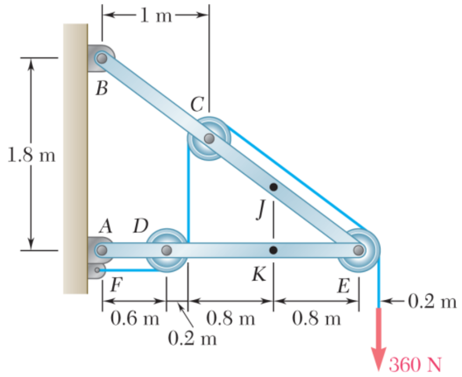
$$\mathbf{F} = 250\text{ N} \nwarrow 36.9^\circ \blacktriangleleft$$

$$\nearrow \Sigma F_{y'} = 0\text{:}\quad V + \frac{4}{5}(630\text{ N} - 360\text{ N}) - \frac{3}{5}(560\text{ N}) = 0$$

$$\mathbf{V} = 120.0\text{ N} \nearrow 53.1^\circ \blacktriangleleft$$

$$\circlearrowleft \Sigma M_J = 0\text{:}\quad M + (0.6\text{ m})(360\text{ N}) + (1.2\text{ m})(560\text{ N}) - (1.6\text{ m})(630\text{ N}) = 0$$

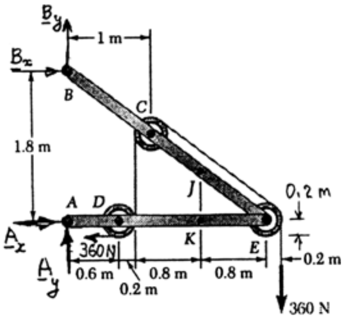
$$\mathbf{M} = 120.0\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$


1 m
B
C
1.8 m
J
A
D
K
F
E
0.6 m
0.2 m
0.8 m
0.8 m
0.2 m
360 N

PROBLEM 7.16

Knowing that the radius of each pulley is 200 mm and neglecting friction, determine the internal forces at Point *K* of the frame shown.

SOLUTION


B_y
B_x
1 m
B
C
1.8 m
J
A
D
0.2 m
A_x
360 N
K
E
0.2 m
A_y
0.6 m
0.8 m
0.8 m
0.2 m
360 N

Free body: frame and pulleys

$$+\circlearrowleft \Sigma M_A = 0: \quad -B_x(1.8 \text{ m}) - (360 \text{ N})(0.2 \text{ m}) - (360 \text{ N})(2.6 \text{ m}) = 0$$

$$B_x = -560 \text{ N} \qquad \mathbf{B}_x = 560 \text{ N} \longleftarrow \lhd$$

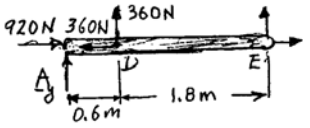
$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x - 560 \text{ N} - 360 \text{ N} = 0$$

$$A_x = +920 \text{ N} \quad \mathbf{A}_x = +920 \text{ N} \longrightarrow \lhd$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + B_y - 360 \text{ N} = 0$$

$$A_y + B_y = 360 \text{ N} \qquad (1)$$

Free body: member *AE*


920 N
360 N
360 N
D
E
A_y
0.6 m
1.8 m

We recall that the forces applied to a pulley may be applied directly to the axle of the pulley.

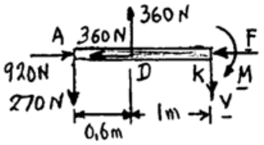
$$+\circlearrowleft \Sigma M_E = 0: \quad -A_y(2.4 \text{ m}) - (360 \text{ N})(1.8 \text{ m}) = 0$$

$$A_y = -270 \text{ N} \qquad \mathbf{A}_y = 270 \text{ N} \downarrow \lhd$$

From (1): $B_y = 360 \text{ N} + 270 \text{ N}$

$$B_y = 630 \text{ N} \qquad \mathbf{B}_y = 630 \text{ N} \uparrow \lhd$$

PROBLEM 7.16 (Continued)



Free body: *AK*

$\xrightarrow{+} \Sigma F_x = 0$: $920\text{ N} - 360\text{ N} - F = 0$

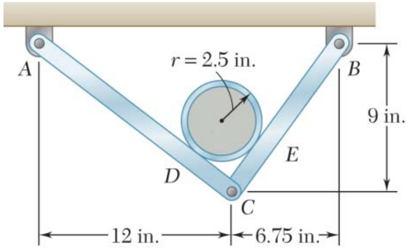
$F = +560\text{ N}$ $\quad\mathbf{F} = 560\text{ N} \leftarrow$ ◀

$+\uparrow \Sigma F_y = 0$: $360\text{ N} - 270\text{ N} - V = 0$

$V = +90.0\text{ N}$ $\quad\mathbf{V} = 90.0\text{ N}\downarrow$ ◀

$+\circlearrowleft \Sigma M_K = 0$: $(270\text{ N})(1.6\text{ m}) - (360\text{ N})(1\text{ m}) - M = 0$

$M = +72.0\text{ N}\cdot\text{m}$ $\quad\mathbf{M} = 72.0\text{ N}\cdot\text{m} \circlearrowright$ ◀

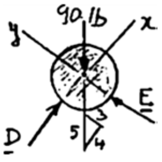


PROBLEM 7.17

A 5-in.-diameter pipe is supported every 9 ft by a small frame consisting of two members as shown. Knowing that the combined weight of the pipe and its contents is 10 lb/ft and neglecting the effect of friction, determine the magnitude and location of the maximum bending moment in member AC.

SOLUTION

Free body: 10-ft section of pipe



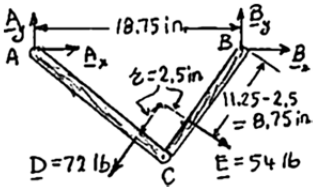
$$+\nearrow \Sigma F_x = 0: \quad D - \frac{4}{5}(90\text{ lb}) = 0 \qquad \mathbf{D} = 72\text{ lb} \nearrow \triangleleft$$

$$\nwarrow + \Sigma F_y = 0: \quad E - \frac{3}{5}(90\text{ lb}) = 0 \qquad \mathbf{E} = 54\text{ lb} \nwarrow \triangleleft$$

Free body: Frame

$$+\circlearrowleft \Sigma M_B = 0: \quad -A_y(18.75\text{ in.}) + (72\text{ lb})(2.5\text{ in.}) + (54\text{ lb})(8.75\text{ in.}) = 0$$

$$A_y = +34.8\text{ lb} \qquad \mathbf{A}_y = 34.8\text{ lb}\uparrow \triangleleft$$



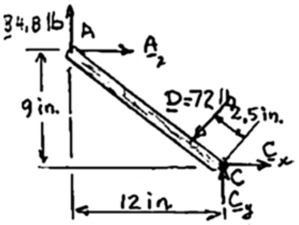
$$+\uparrow \Sigma F_y = 0: \quad B_y + 34.8\text{ lb} - \frac{4}{5}(72\text{ lb}) - \frac{3}{5}(54\text{ lb}) = 0$$

$$B_y = +55.2\text{ lb} \qquad \mathbf{B}_y = 55.2\text{ lb}\uparrow \triangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + B_x - \frac{3}{5}(72\text{ lb}) + \frac{4}{5}(54\text{ lb}) = 0$$

$$A_x + B_x = 0 \qquad (1)$$

Free body: Member AC



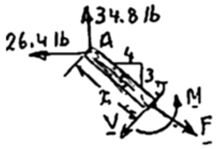
$$+\circlearrowleft \Sigma M_C = 0: \quad (72\text{ lb})(2.5\text{ in.}) - (34.8\text{ lb})(12\text{ in.}) - A_x(9\text{ in.}) = 0$$

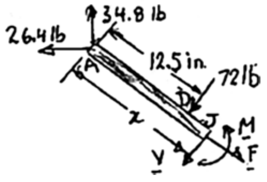
$$A_x = -26.4\text{ lb} \qquad \mathbf{A}_x = 26.4\text{ lb} \longleftarrow \triangleleft$$

From (1): $B_x = -A_x = +26.4\text{ lb}$

$$\mathbf{B}_x = 26.4\text{ lb} \longrightarrow \triangleleft$$

PROBLEM 7.17 (Continued)





Free body: Portion *AJ*

For $x \le 12.5$ in. $(AJ \le AD)$:

$$+\circlearrowleft \Sigma M_J = 0: \quad (26.4\text{ lb})\frac{3}{5}x - (34.8\text{ lb})\frac{4}{5}x + M = 0$$

$$M = 12x$$

$$M_{\max} = 150\text{ lb}\cdot\text{in. for } x = 12.5\text{ in.}$$

$$M_{\max} = 150.0\text{ lb}\cdot\text{in. at } D \quad \triangleleft$$

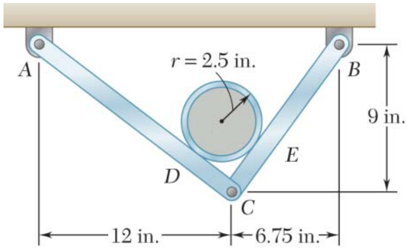
For $x > 12.5$ in. $(AJ > AD)$:

$$+\circlearrowleft \Sigma M_J = 0: \quad (26.4\text{ lb})\frac{3}{5}x - (34.8\text{ lb})\frac{4}{5}x + (72\text{ lb})(x - 12.5) + M = 0$$

$$M = 900 - 60x$$

$$M_{\max} = 150\text{ lb}\cdot\text{in. for } x = 12.5\text{ in.}$$

Thus: $M_{\max} = 150.0\text{ lb}\cdot\text{in. at } D$ ◀



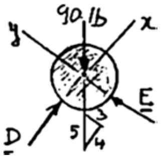
PROBLEM 7.18

For the frame of Problem 7.17, determine the magnitude and location of the maximum bending moment in member *BC*.

PROBLEM 7.17 A 5-in.-diameter pipe is supported every 9 ft by a small frame consisting of two members as shown. Knowing that the combined weight of the pipe and its contents is 10 lb/ft and neglecting the effect of friction, determine the magnitude and location of the maximum bending moment in member *AC*.

SOLUTION

Free body: 10-ft section of pipe



$$+\nearrow \Sigma F_x = 0: \quad D - \frac{4}{5}(90 \text{ lb}) = 0 \qquad \mathbf{D} = 72 \text{ lb} \nearrow \quad \triangleleft$$

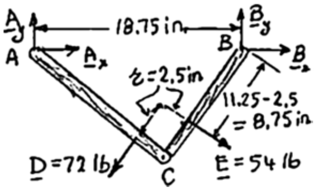
$$\nwarrow + \Sigma F_y = 0: \quad E - \frac{3}{5}(90 \text{ lb}) = 0 \qquad \mathbf{E} = 54 \text{ lb} \nwarrow \quad \triangleleft$$

Free body: Frame

$$+\circlearrowleft \Sigma M_B = 0: \quad -A_y(18.75 \text{ in.}) + (72 \text{ lb})(2.5 \text{ in.})$$

$$+(54 \text{ lb})(8.75 \text{ in.}) = 0$$

$$A_y = +34.8 \text{ lb} \qquad \mathbf{A}_y = 34.8 \text{ lb} \uparrow \quad \triangleleft$$



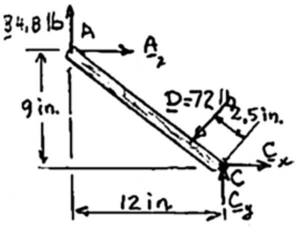
$$+\uparrow \Sigma F_y = 0: \quad B_y + 34.8 \text{ lb} - \frac{4}{5}(72 \text{ lb}) - \frac{3}{5}(54 \text{ lb}) = 0$$

$$B_y = +55.2 \text{ lb} \qquad \mathbf{B}_y = 55.2 \text{ lb} \uparrow \quad \triangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad A_x + B_x - \frac{3}{5}(72 \text{ lb}) + \frac{4}{5}(54 \text{ lb}) = 0$$

$$A_x + B_x = 0 \qquad (1)$$

Free body: Member *AC*



$$+\circlearrowleft \Sigma M_C = 0: \quad (72 \text{ lb})(2.5 \text{ in.}) - (34.8 \text{ lb})(12 \text{ in.})$$

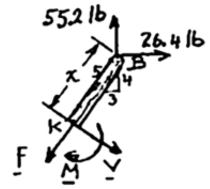
$$-A_x(9 \text{ in.}) = 0$$

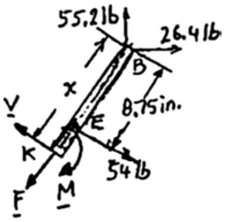
$$A_x = -26.4 \text{ lb} \qquad \mathbf{A}_x = 26.4 \text{ lb} \longleftarrow \quad \triangleleft$$

From (1): $B_x = -A_x = +26.4 \text{ lb}$

$$\mathbf{B}_x = 26.4 \text{ lb} \longrightarrow \quad \triangleleft$$

PROBLEM 7.18 (Continued)


55.2 lb
26.4 lb
x
5
4
3
B
K
F
M
V


55.2 lb
26.4 lb
x
B
8.75 in.
V
E
K
54 lb
F
M

Free body: Portion *BK*

For $x \le 8.75$ in. $(BK \le BE)$:

$$+\circlearrowleft \Sigma M_K = 0: \quad (55.2 \text{ lb})\frac{3}{5}x - (26.4 \text{ lb})\frac{4}{5}x - M = 0$$

$$M = 12x$$

$$M_{\max} = 105.0 \text{ lb} \cdot \text{in.} \quad \text{for} \quad x = 8.75 \text{ in.}$$

$$M_{\max} = 105.0 \text{ lb} \cdot \text{in. at } E \triangleleft$$

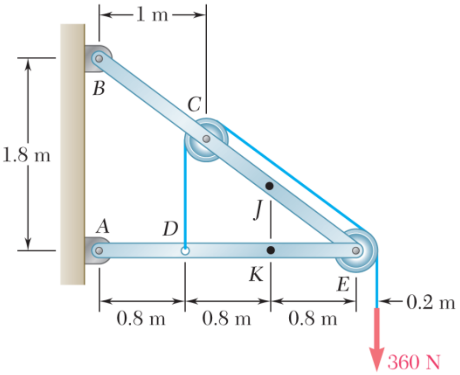
For $x > 8.75$ in. $(BK > BE)$:

$$+\circlearrowleft \Sigma M_K = 0: \quad (55.2 \text{ lb})\frac{3}{5}x - (26.4 \text{ lb})\frac{4}{5}x - (54 \text{ lb})(x - 8.75 \text{ in.}) - M = 0$$

$$M = 472.5 - 42x$$

$$M_{\max} = 105.0 \text{ lb} \cdot \text{in.} \quad \text{for} \quad x = 8.75 \text{ in.}$$

Thus $M_{\max} = 105.0 \text{ lb} \cdot \text{in. at } E$ ◀

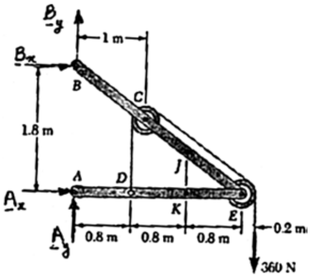

1 m
B
C
1.8 m
A
D
J
K
E
0.8 m
0.8 m
0.8 m
0.2 m
360 N

PROBLEM 7.19

Knowing that the radius of each pulley is 200 mm and neglecting friction, determine the internal forces at Point J of the frame shown.

SOLUTION

Free body: Frame and pulleys


B_y
1 m
B_x
B
C
1.8 m
A
D
J
A_x
K
E
0.2 m
A_y
0.8 m
0.8 m
0.8 m
360 N

$+\circlearrowleft \Sigma M_A = 0\text{:}\quad -B_x(1.8\text{ m}) - (360\text{ N})(2.6\text{ m}) = 0$

$B_x = -520\text{ N}$ $\qquad \mathbf{B}_x = 520\text{ N} \longleftarrow \triangleleft$

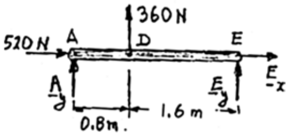
$\xrightarrow{+} \Sigma F_x = 0\text{:}\quad A_x - 520\text{ N} = 0$

$A_x = +520\text{ N}$ $\qquad \mathbf{A}_x = 520\text{ N} \longrightarrow \triangleleft$

$+\uparrow \Sigma F_y = 0\text{:}\quad A_y + B_y - 360\text{ N} = 0$

$A_y + B_y = 360\text{ N} \qquad (1)$

Free body: Member *AE*


360 N
520 N
A
D
E
A_y
E_y
E_x
0.8 m
1.6 m

$+\circlearrowleft \Sigma M_E = 0\text{:}\quad -A_y(2.4\text{ m}) - (360\text{ N})(1.6\text{ m}) = 0$

$A_y = -240\text{ N}$ $\qquad \mathbf{A}_y = 240\text{ N}\downarrow \triangleleft$

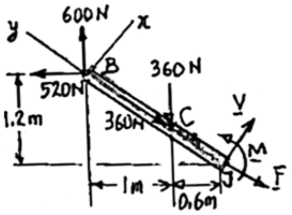
From (1):

$B_y = 360\text{ N} + 240\text{ N}$

$B_y = +600\text{ N}$ $\qquad \mathbf{B}_y = 600\text{ N}\uparrow \triangleleft$

Free body: *BJ*

We recall that the forces applied to a pulley may be applied directly to its axle.


y
600 N
x
B
360 N
520 N
1.2 m
360 N
C
V
M
F
J
1 m
0.6 m

$\overset{+}{\nwarrow} \Sigma F_y = 0\text{:}\quad \frac{3}{5}(600\text{ N}) + \frac{4}{5}(520\text{ N})$

$-360\text{ N} - \frac{3}{5}(360\text{ N}) - F = 0$

$F = +200\text{ N}$ $\qquad \mathbf{F} = 200\text{ N} \nwarrow 36.9° \blacktriangleleft$

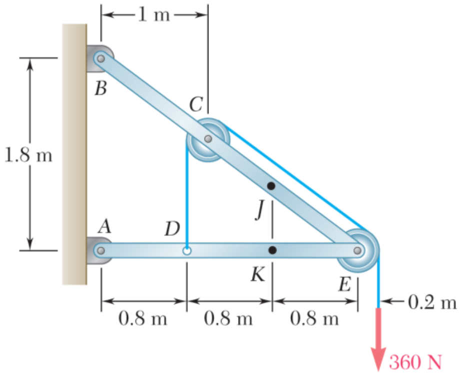
PROBLEM 7.19 (Continued)

$$+\nearrow \Sigma F_x = 0: \quad \frac{4}{5}(600\text{ N}) - \frac{3}{5}(520\text{ N}) - \frac{4}{5}(360\text{ N}) + V = 0$$

$$V = +120.0\text{ N} \qquad \mathbf{V} = 120.0\text{ N} \measuredangle 53.1° \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_J = 0: \quad (520\text{ N})(1.2\text{ m}) - (600\text{ N})(1.6\text{ m}) + (360\text{ N})(0.6\text{ m}) + M = 0$$

$$M = +120.0\text{ N}\cdot\text{m} \qquad \mathbf{M} = 120.0\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

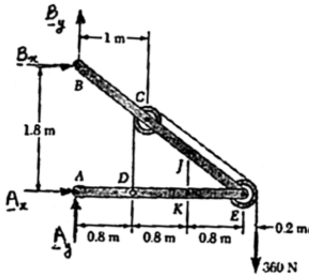

1 m
B
C
1.8 m
J
A
D
K
E
0.8 m
0.8 m
0.8 m
0.2 m
360 N

PROBLEM 7.20

Knowing that the radius of each pulley is 200 mm and neglecting friction, determine the internal forces at Point *K* of the frame shown.

SOLUTION

Free body: Frame and pulleys


B_y
B_x
1 m
B
C
1.8 m
A
D
J
A_x
K
E
A_y
0.8 m
0.8 m
0.8 m
0.2 m
360 N

$+\circlearrowleft \Sigma M_A = 0\text{:}\quad -B_x(1.8\text{ m}) - (360\text{ N})(2.6\text{ m}) = 0$

$B_x = -520\text{ N}$ $\qquad \mathbf{B}_x = 520\text{ N} \longleftarrow$ ◁

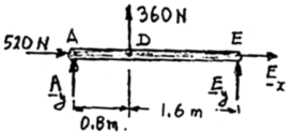
$\xrightarrow{+} \Sigma F_x = 0\text{:}\quad A_x - 520\text{ N} = 0$

$A_x = +520\text{ N}$ $\qquad \mathbf{A}_x = 520\text{ N} \longrightarrow$ ◁

$+\!\uparrow \Sigma F_y = 0\text{:}\quad A_y + B_y - 360\text{ N} = 0$

$$A_y + B_y = 360\text{ N} \qquad (1)$$

Free body: Member *AE*


360 N
520 N
A
D
E
A_y
E_y
E_x
0.8 m.
1.6 m

$+\circlearrowleft \Sigma M_E = 0\text{:}\quad -A_y(2.4\text{ m}) - (360\text{ N})(1.6\text{ m}) = 0$

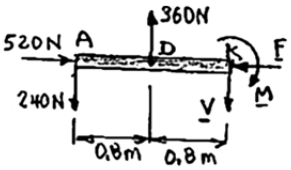
$A_y = -240\text{ N}$ $\qquad \mathbf{A}_y = 240\text{ N}\downarrow$ ◁

From (1):

$B_y = 360\text{ N} + 240\text{ N}$

$B_y = +600\text{ N}$ $\qquad \mathbf{B}_y = 600\text{ N}\uparrow$ ◁

Free body: *AK*


360N
520N
A
D
K
F
240N
V
M
0.8m
0.8 m

$\xrightarrow{+} \Sigma F_x = 0\text{:}\quad 520\text{ N} - F = 0$

$F = +520\text{ N}$ $\qquad \mathbf{F} = 520\text{ N} \longleftarrow$ ◀

$+\!\uparrow \Sigma F_y = 0\text{:}\quad 360\text{ N} - 240\text{ N} - V = 0$

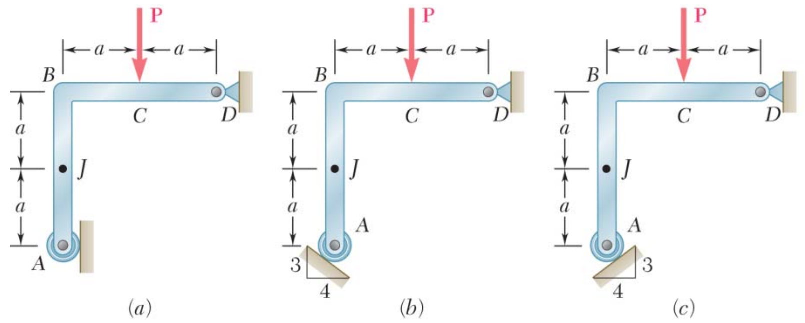
$V = +120.0\text{ N}$ $\qquad \mathbf{V} = 120.0\text{ N}\downarrow$ ◀

$+\circlearrowleft \Sigma M_K = 0\text{:}\quad (240\text{ N})(1.6\text{ m}) - (360\text{ N})(0.8\text{ m}) - M = 0$

$M = +96.0\text{ N}\cdot\text{m}$ $\qquad \mathbf{M} = 96.0\text{ N}\cdot\text{m}\circlearrowright$ ◀

PROBLEM 7.21

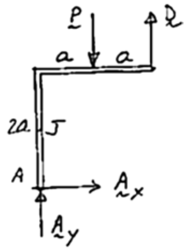
A force **P** is applied to a bent rod that is supported by a roller and a pin and bracket. For each of the three cases shown, determine the internal forces at Point *J*.



(*a*) (*b*) (*c*)

SOLUTION

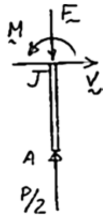
(*a*) **FBD Rod:**



$$\rightarrow \Sigma F_x = 0: \quad A_x = 0$$

$$\circlearrowleft \Sigma M_D = 0: \quad aP - 2aA_y = 0 \qquad A_y = \frac{P}{2}$$

FBD AJ:

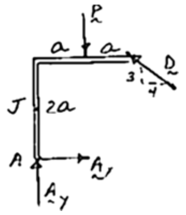


$$\rightarrow \Sigma F_x = 0: \quad \mathbf{V} = 0$$ ◀

$$\uparrow \Sigma F_y = 0: \quad \frac{P}{2} - F = 0 \qquad \mathbf{F} = \frac{P}{2}\downarrow$$ ◀

$$\circlearrowleft \Sigma M_J = 0: \quad \mathbf{M} = 0$$ ◀

(*b*) **FBD Rod:**



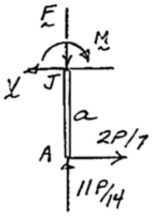
$$\circlearrowleft \Sigma M_A = 0: \quad 2a\left(\frac{4}{5}D\right) + 2a\left(\frac{3}{5}D\right) - aP = 0 \qquad D = \frac{5P}{14}$$

$$\rightarrow \Sigma F_x = 0: \quad A_x - \frac{4}{5}\frac{5}{14}P = 0 \qquad A_x = \frac{2P}{7}$$

$$\uparrow \Sigma F_y = 0: \quad A_y - P + \frac{3}{5}\frac{5}{14}P = 0 \qquad A_y = \frac{11P}{14}$$

PROBLEM 7.21 (Continued)

FBD AJ:

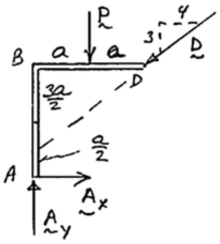


$$\xrightarrow{} \Sigma F_x = 0: \quad \frac{2}{7}P - V = 0 \qquad \mathbf{V} = \frac{2P}{7} \longleftarrow ◀$$

$$\uparrow \Sigma F_y = 0: \quad \frac{11P}{14} - F = 0 \qquad \mathbf{F} = \frac{11P}{14} \downarrow ◀$$

$$\circlearrowleft \Sigma M_J = 0: \quad a\frac{2P}{7} - M = 0 \qquad \mathbf{M} = \frac{2}{7}aP \circlearrowright ◀$$

(*c*) **FBD Rod:**

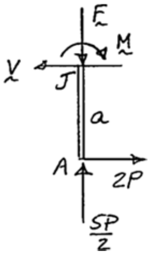


$$\circlearrowleft \Sigma M_A = 0: \quad \frac{a}{2}\left(\frac{4D}{5}\right) - aP = 0 \qquad D = \frac{5P}{2}$$

$$\xrightarrow{} \Sigma F_x = 0: \quad A_x - \frac{4}{5}\frac{5P}{2} = 0 \qquad A_x = 2P$$

$$\uparrow \Sigma F_y = 0: \quad A_y - P - \frac{3}{5}\frac{5P}{2} = 0 \qquad A_y = \frac{5P}{2}$$

FBD AJ:



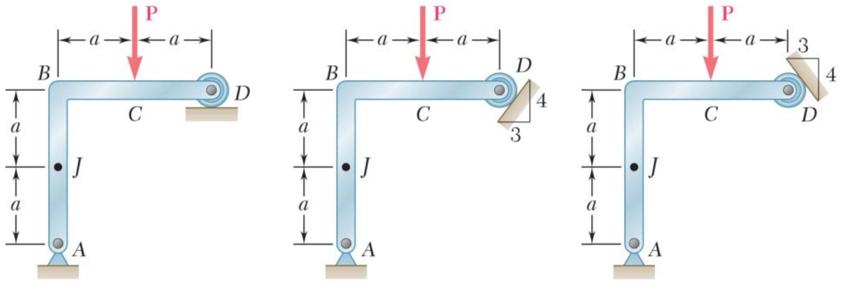
$$\xrightarrow{} \Sigma F_x = 0: \quad 2P - V = 0 \qquad \mathbf{V} = 2P \longleftarrow ◀$$

$$\uparrow \Sigma F_y = 0: \quad \frac{5P}{2} - F = 0 \qquad \mathbf{F} = \frac{5P}{2} \downarrow ◀$$

$$\circlearrowleft \Sigma M_J = 0: \quad a(2P) - M = 0 \qquad \mathbf{M} = 2aP \circlearrowright ◀$$

PROBLEM 7.22

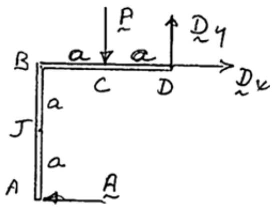
A force **P** is applied to a bent rod that is supported by a roller and a pin and bracket. For each of the three cases shown, determine the internal forces at Point *J*.


P
a
a
B
C
D
J
A
a
a
4
3

(*a*) (*b*) (*c*)

SOLUTION

(*a*) **FBD Rod:**


P
Dy
B
a
a
C
D
Dx
a
J
a
A
A

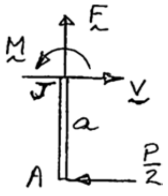
$$\circlearrowleft \Sigma M_D = 0: \quad aP - 2aA = 0$$

$$\mathbf{A} = \frac{P}{2} \longleftarrow$$

$$\longrightarrow \Sigma F_x = 0: \quad V - \frac{P}{2} = 0$$

$$\mathbf{V} = \frac{P}{2} \longrightarrow$$ ◀

FBD AJ:



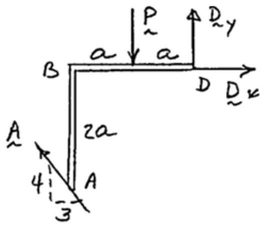
$$\uparrow \Sigma F_y = 0:$$

$$\mathbf{F} = 0$$ ◀

$$\circlearrowleft \Sigma M_J = 0: \quad M - a\frac{P}{2} = 0$$

$$\mathbf{M} = \frac{aP}{2} \circlearrowleft$$ ◀

(*b*) **FBD Rod:**

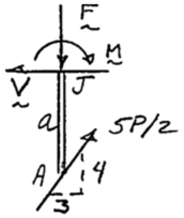

P
Dy
B
a
a
D
Dx
A
2a
4
3
A

$$\circlearrowleft \Sigma M_D = 0: \quad aP - \frac{a}{2}\left(\frac{4}{5}A\right) = 0$$

$$\mathbf{A} = \frac{5P}{2} \nearrow$$

PROBLEM 7.22 (Continued)

FBD AJ:

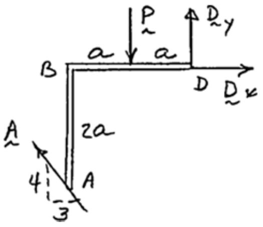


$$\xrightarrow{\quad}\Sigma F_x = 0:\quad \frac{3}{5}\frac{5P}{2} - V = 0 \qquad \mathbf{V} = \frac{3P}{2} \longleftarrow$$ ◀

$$\uparrow \Sigma F_y = 0:\quad \frac{4}{5}\frac{5P}{2} - F = 0 \qquad \mathbf{F} = 2P \downarrow$$ ◀

$$\mathbf{M} = \frac{3}{2}aP \circlearrowright$$ ◀

(*c*) **FBD Rod:**



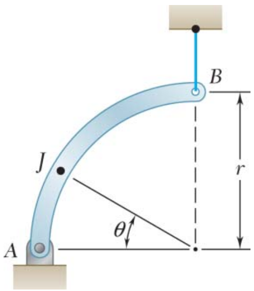
$$\circlearrowleft \Sigma M_D = 0:\quad aP - 2a\left(\frac{3}{5}A\right) - 2a\left(\frac{4}{5}A\right) = 0$$

$$A = \frac{5P}{14}$$

$$\xrightarrow{\quad}\Sigma F_x = 0:\quad V - \left(\frac{3}{5}\frac{5P}{14}\right) = 0 \qquad \mathbf{V} = \frac{3P}{14} \longrightarrow$$ ◀

$$\uparrow \Sigma F_y = 0:\quad \frac{4}{5}\frac{5P}{14} - F = 0 \qquad \mathbf{F} = \frac{2P}{7} \downarrow$$ ◀

$$\circlearrowleft \Sigma M_J = 0:\quad M - a\left(\frac{3}{5}\frac{5P}{14}\right) = 0 \qquad \mathbf{M} = \frac{3}{14}aP \circlearrowleft$$ ◀

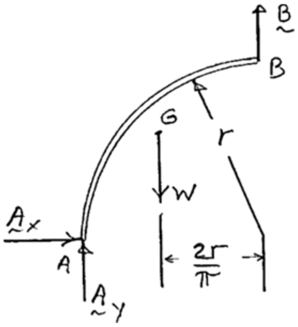


PROBLEM 7.23

A quarter-circular rod of weight W and uniform cross section is supported as shown. Determine the bending moment at Point J when $\theta = 30°$.

SOLUTION

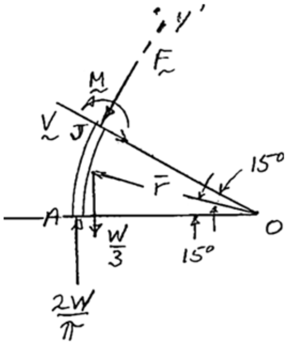
FBD Rod:



$$\xrightarrow{} \Sigma F_x = 0: \quad \mathbf{A}_x = 0$$

$$\circlearrowleft \Sigma M_B = 0: \quad \frac{2r}{\pi}W - rA_y = 0 \qquad \mathbf{A}_y = \frac{2W}{\pi}\uparrow$$

FBD AJ:



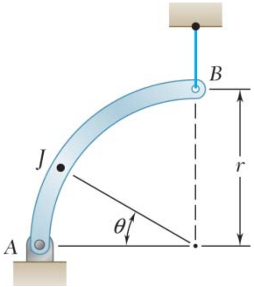
$$\alpha = 15°, \text{ weight of segment} = W\frac{30°}{90°} = \frac{W}{3}$$

$$\bar{r} = \frac{r}{\alpha}\sin\alpha = \frac{r}{\frac{\pi}{12}}\sin 15° = 0.9886r$$

$$\nearrow \Sigma F_{y'} = 0: \quad \frac{2W}{\pi}\cos 30° - \frac{W}{3}\cos 30° - F = 0$$

$$\mathbf{F} = \frac{W\sqrt{3}}{2}\left(\frac{2}{\pi} - \frac{1}{3}\right) \swarrow$$

$$\circlearrowleft \Sigma M_0 = M + r\left(F - \frac{2W}{\pi}\right) + \bar{r}\cos 15°\frac{W}{3} = 0 \qquad \mathbf{M} = 0.0557Wr \circlearrowright \blacktriangleleft$$



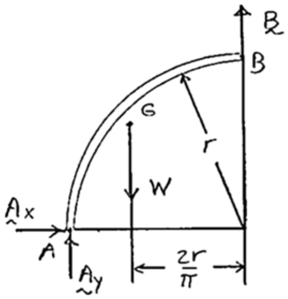
PROBLEM 7.24

For the rod of Problem 7.23, determine the magnitude and location of the maximum bending moment.

PROBLEM 7.23 A quarter-circular rod of weight W and uniform cross section is supported as shown. Determine the bending moment at Point J when $\theta = 30°$.

SOLUTION

FBD Rod:



$$\xrightarrow{} \Sigma F_x = 0: \quad A_x = 0$$

$$\circlearrowleft \Sigma M_B = 0: \quad \frac{2r}{\pi}W - rA_y = 0 \qquad A_y = \frac{2W}{\pi}$$

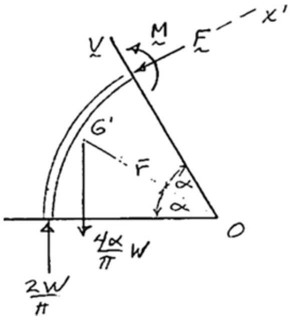
$$\alpha = \frac{\theta}{2}, \qquad \bar{r} = \frac{r}{\alpha}\sin\alpha$$

$$\text{Weight of segment } = W\frac{2\alpha}{\frac{\pi}{2}} = \frac{4\alpha}{\pi}W$$

$$\nearrow \Sigma F_{x'} = 0: \quad -F - \frac{4\alpha}{\pi}W\cos 2\alpha + \frac{2W}{\pi}\cos 2\alpha = 0$$

$$F = \frac{2W}{\pi}(1-2\alpha)\cos 2\alpha = \frac{2W}{\pi}(1-\theta)\cos\theta$$

FBD AJ: $\circlearrowleft \Sigma M_0 = 0: \quad M + \left(F - \frac{2W}{\pi}\right)r + (\bar{r}\cos\alpha)\frac{4\alpha}{\pi}W = 0$



$$M = \frac{2W}{\pi}(1+\theta\cos\theta - \cos\theta)r - \frac{4\alpha W}{\pi}\frac{r}{\alpha}\sin\alpha\cos\alpha$$

But, $\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha = \frac{1}{2}\sin\theta$

so $M = \frac{2r}{\pi}W(1 - \cos\theta + \theta\cos\theta - \sin\theta)$

$$\frac{dM}{d\theta} = \frac{2rW}{\pi}(\sin\theta - \theta\sin\theta + \cos\theta - \cos\theta) = 0$$

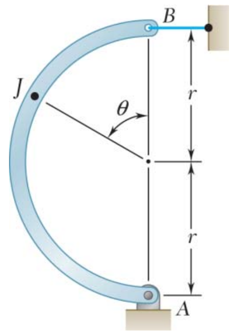
for $(1-\theta)\sin\theta = 0$

$$\frac{dM}{d\theta} = 0 \quad \text{for} \quad \theta = 0, 1, n\pi\,(n = 1, 2, \cdots)$$

Only 0 and 1 in valid range

At $\theta = 0 \quad M = 0,$ at $\theta = 1 \text{ rad}$

at $\theta = 57.3°$ $\qquad M = M_{\max} = 0.1009Wr$ ◀

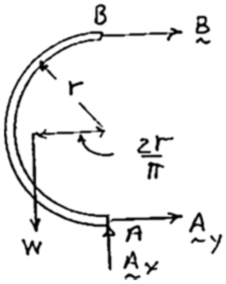


PROBLEM 7.25

A semicircular rod of weight W and uniform cross section is supported as shown. Determine the bending moment at Point J when $\theta = 60°$.

SOLUTION

FBD Rod:



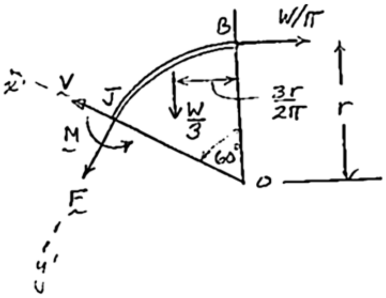
$$\circlearrowleft \Sigma M_A = 0: \quad \frac{2r}{\pi}W - 2rB = 0$$

$$\mathbf{B} = \frac{W}{\pi} \longrightarrow$$

$$\swarrow \Sigma F_{y'} = 0: \quad F + \frac{W}{3}\sin 60° - \frac{W}{\pi}\cos 60° = 0$$

$$F = -0.12952W$$

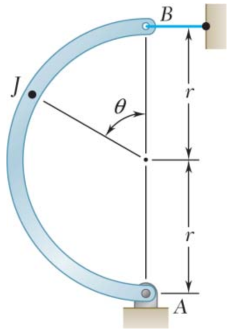
FBD BJ:



$$\circlearrowleft \Sigma M_0 = 0: \quad r\left(F - \frac{W}{\pi}\right) + \frac{3r}{2\pi}\left(\frac{W}{3}\right) + M = 0$$

$$M = Wr\left(0.12952 + \frac{1}{\pi} - \frac{1}{2\pi}\right) = 0.28868Wr$$

On BJ $\quad \mathbf{M}_J = 0.289Wr \circlearrowright$ ◀

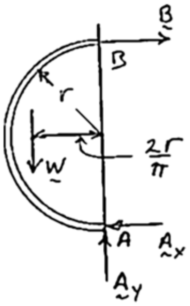


PROBLEM 7.26

A semicircular rod of weight W and uniform cross section is supported as shown. Determine the bending moment at Point J when $\theta = 150°$.

SOLUTION

FBD Rod:

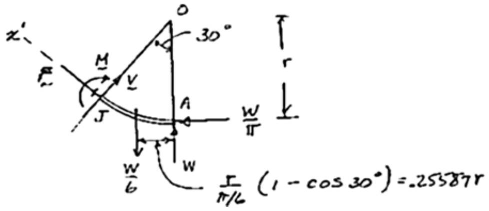

B
B
r
$\frac{2r}{\pi}$
W
A
A_x
A_y

$$\uparrow \Sigma F_y = 0: \quad A_y - W = 0 \qquad \mathbf{A}_y = W \uparrow$$

$$\Sigma M_B = 0: \quad \frac{2r}{\pi} W - 2rA_x = 0$$

$$\mathbf{A}_x = \frac{W}{\pi} \longleftarrow$$

FBD AJ:


x'
O
30°
M
V
F
J
A
$\frac{W}{\pi}$
r
$\frac{W}{6}$
W
$\frac{r}{\pi/6}(1 - \cos 30°) = .25587r$

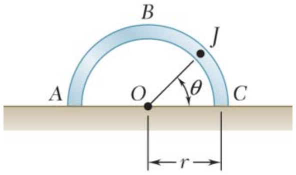
$$\nwarrow \Sigma F_{x'} = 0: \quad \frac{W}{\pi}\cos 30° + \frac{5W}{6}\sin 30° - F = 0 \quad F = 0.69233W \searrow$$

$$\circlearrowleft \Sigma M_0 = 0: \quad 0.25587r\left(\frac{W}{6}\right) + r\left(F - \frac{W}{\pi}\right) - M = 0$$

$$M = Wr\left[\frac{0.25587}{6} + 0.69233 - \frac{1}{\pi}\right]$$

$$M = Wr(0.4166)$$

On AJ $\qquad \mathbf{M} = 0.417Wr \circlearrowright$ ◀



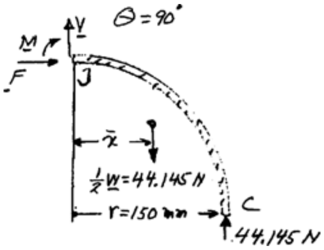
PROBLEM 7.27

A half section of pipe rests on a frictionless horizontal surface as shown. If the half section of pipe has a mass of 9 kg and a diameter of 300 mm, determine the bending moment at Point *J* when $\theta = 90°$.

SOLUTION

For half section $m = 9 \text{ kg}$

$$W = mg = (9)(9.81) = 88.29 \text{ N}$$



Portion *JC*:

$$\text{Weight} = \frac{1}{2}W = 44.145 \text{ N}$$

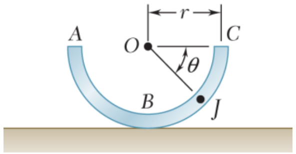
From Fig. 5.8B:

$$x = \frac{2r}{\pi} = \frac{2(150)}{\pi}$$

$$x = 95.49 \text{ mm}$$

$$\circlearrowleft + \Sigma M_J = 0: \quad (44.145 \text{ N})(0.15 \text{ m}) - (44.145 \text{ N})(0.09549 \text{ m}) - M = 0$$

$$M = +2.406 \text{ N}\cdot\text{m} \qquad \mathbf{M} = 2.41 \text{ N}\cdot\text{m} \circlearrowright \blacktriangleleft$$



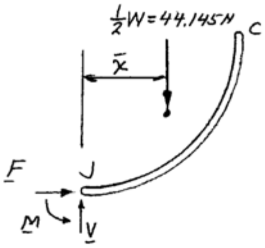
PROBLEM 7.28

A half section of pipe rests on a frictionless horizontal surface as shown. If the half section of pipe has a mass of 9 kg and a diameter of 300 mm, determine the bending moment at point J when $\theta = 90°$.

SOLUTION

For half section $m = 9 \text{ kg}$

$$W = mg = (9 \text{ kg})(9.81 \text{ m/s}^2) = 88.29 \text{ N}$$



Free body JC

Weight of portion JC, $= \dfrac{1}{2}W = 44.145 \text{ N}$

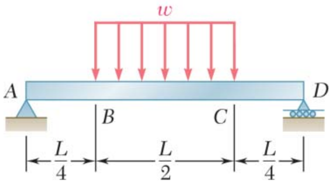
$$r = 150 \text{ mm}$$

From Fig. 5.8B:

$$x = \frac{2r}{\pi} = \frac{2(150)}{\pi} = 95.49 \text{ mm}$$

$$+\circlearrowleft \Sigma M_J = 0: \quad M - (44.145 \text{ N})(0.09549 \text{ m}) = 0$$

$$M = +4.2154 \text{ N}\cdot\text{m} \qquad \mathbf{M} = 4.22 \text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$


w
A
D
B
C
$\frac{L}{4}$
$\frac{L}{2}$
$\frac{L}{4}$

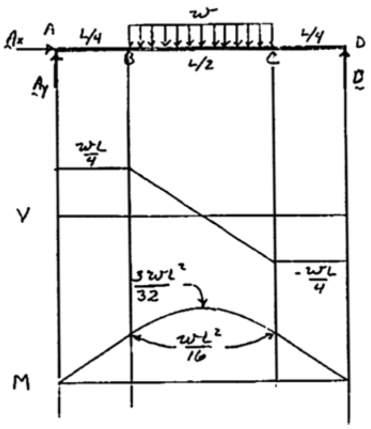
PROBLEM 7.29

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

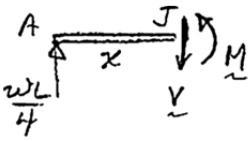
SOLUTION

FBD beam:

(*a*) By symmetry: $A_y = D = \frac{1}{2}(w)\frac{L}{2}$ $\mathbf{A}_y = \mathbf{D} = \frac{wL}{4}\uparrow$


A_x
A
L/4
w
L/4
D
B
L/2
C
A_y
D
$\frac{wL}{4}$
V
$\frac{3wL^2}{32}$
$-\frac{wL}{4}$
$\frac{wL^2}{16}$
M

Along AB:


A
J
x
M
$\frac{wL}{4}$
V

$$\uparrow \Sigma F_y = 0: \quad \frac{wL}{4} - V = 0 \qquad V = \frac{wL}{4}$$

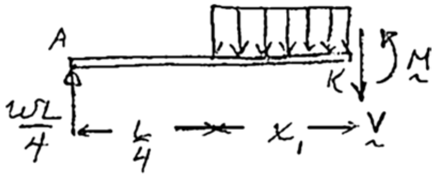
$$\circlearrowleft \Sigma M_J = 0: \quad M - x\frac{wL}{4} = 0$$

$$M = \frac{wL}{4}x \text{ (straight)}$$

Along BC:

$$\uparrow \Sigma F_y = 0: \quad \frac{wL}{4} - wx_1 - V = 0$$

$$V = \frac{wL}{4} - wx_1$$


A
M
K
$\frac{wL}{4}$
$\frac{L}{4}$
x_1
V

Straight with $V = 0$ at $x_1 = \frac{L}{4}$

$$\circlearrowleft \Sigma M_k = 0: \quad M + \frac{x_1}{2}wx_1 - \left(\frac{L}{4} + x_1\right)\frac{wL}{4} = 0$$

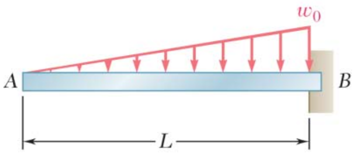
$$M = \frac{w}{2}\left(\frac{L^2}{8} + \frac{L}{2}x_1 - x_1^2\right)$$

Parabola with $M = \frac{3}{32}wL^2$ at $x_1 = \frac{L}{4}$

Section *CD* by symmetry

(*b*) From diagrams: $|V|_{\max} = \frac{wL}{4}$ on *AB* and *CD* ◀

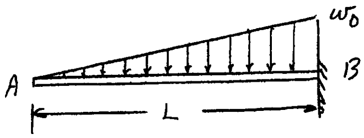
$|M|_{\max} = \frac{3wL^2}{32}$ at center ◀



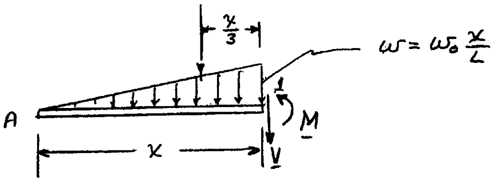
PROBLEM 7.30

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

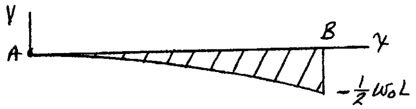


$$\frac{1}{2}wx = \frac{1}{2}\left(w_0\frac{x}{L}\right)x = \frac{1}{2}w_0\frac{x^2}{L} \qquad \text{By similar } \Delta\text{'s}$$

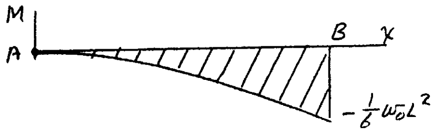


$+\!\uparrow \Sigma F_y = 0\text{:} \quad -\frac{1}{2}w_0\frac{x^2}{L} - V = 0 \qquad V = -\frac{1}{2}w_0\frac{x^2}{L}$

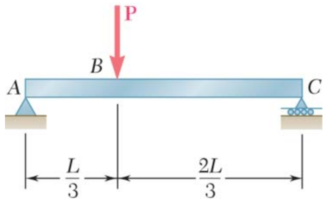
$+\circlearrowleft \Sigma F_y = 0\text{:} \quad \left(\frac{1}{2}w_0\frac{x^2}{L}\right)\frac{x}{3} + M = 0 \qquad M = -\frac{1}{6}w_0\frac{x^3}{L}$



$$|V|_{\max} = \frac{1}{2}w_0L \blacktriangleleft$$



$$|M|_{\max} = \frac{1}{6}w_0L^2 \blacktriangleleft$$


P
B
A
C
$\frac{L}{3}$
$\frac{2L}{3}$

PROBLEM 7.31

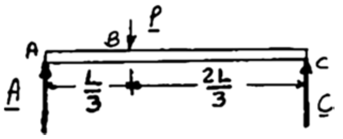
For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

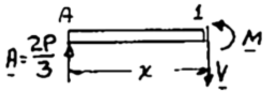
(*a*) Reactions:

$$\mathbf{A} = \frac{2P}{3}\uparrow$$

$$\mathbf{C} = \frac{P}{3}\uparrow$$


P
B
A
C
A
$\frac{L}{3}$
$\frac{2L}{3}$
C

From *A* to *B*:


A
1
M
A = $\frac{2P}{3}$
x
V

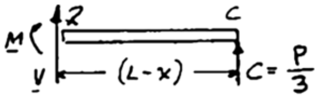
$$+\uparrow \Sigma F_y = 0: \quad V = +\frac{2P}{3}$$

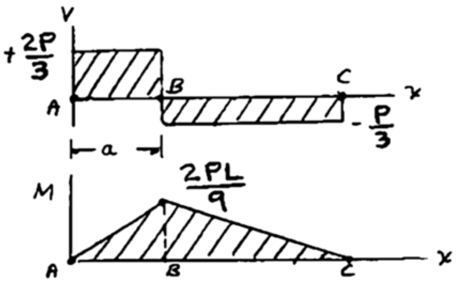
$$+\circlearrowleft \Sigma M_1 = 0: \quad M = +\frac{2P}{3}x$$

From *B* to *C*:

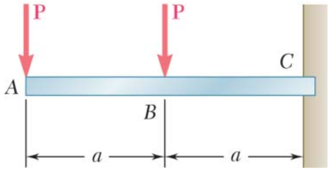
$$+\uparrow \Sigma F_y = 0: \quad V = -\frac{P}{3}$$

$$+\circlearrowleft \Sigma M_2 = 0: \quad M = +\frac{P}{3}(L-x)$$


M
2
C
V
(L-x)
C = $\frac{P}{3}$


V
+$\frac{2P}{3}$
A
B
C
x
-$\frac{P}{3}$
a
M
$\frac{2PL}{9}$
A
B
C
x

(*b*) $$|V|_{\max} = \frac{2P}{3}; \quad |M|_{\max} = \frac{2PL}{9}$$ ◀

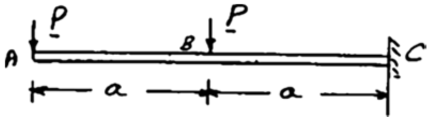


PROBLEM 7.32

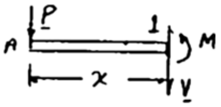
For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

(*a*)



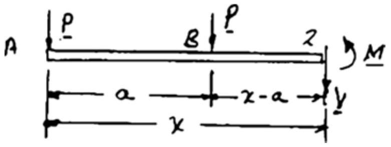
From *A* to *B*:



$+\uparrow \Sigma F_y = 0$: $V = -P$

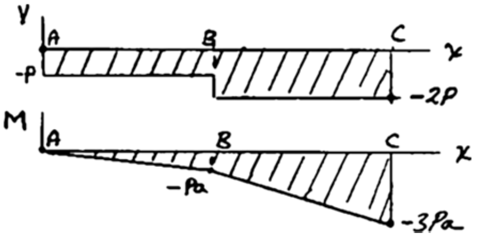
$+\circlearrowleft \Sigma M_1 = 0$: $M = -Px$

From *B* to *C*:

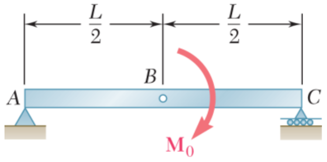

A
P
B
P
2
M
a
x-a
V
x

$+\uparrow \Sigma F_y = 0$: $-P - P - V = 0$ $V = -2P$

$+\circlearrowleft \Sigma M_2 = 0$: $Px + P(x-a) + M = 0$ $M = -2Px + Pa$



(*b*) $|V|_{\max} = 2P$; $|M|_{\max} = 3Pa$ ◀


$\frac{L}{2}$
$\frac{L}{2}$
B
A
C
$\mathbf{M}_0$

PROBLEM 7.33

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

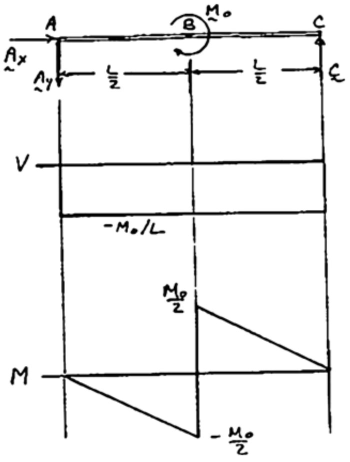
(*a*) **FBD Beam:**

$$\circlearrowleft \Sigma M_C = 0: \quad LA_y - M_0 = 0$$

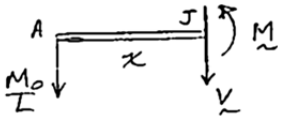
$$\mathbf{A}_y = \frac{M_0}{L} \downarrow$$

$$\uparrow \Sigma F_y = 0: \quad -A_y + C = 0$$

$$\mathbf{C} = \frac{M_0}{L} \uparrow$$


A
B
M_0
C
A_x
A_y
$\frac{L}{2}$
$\frac{L}{2}$
C
V
$-M_0/L$
$\frac{M_0}{2}$
M
$-\frac{M_0}{2}$

Along AB:


A
J
M
$\frac{M_0}{L}$
x
V

$$\uparrow \Sigma F_y = 0: \quad -\frac{M_0}{L} - V = 0$$

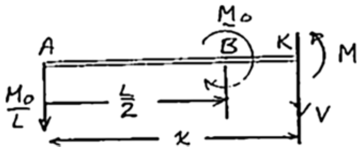
$$V = -\frac{M_0}{L}$$

$$\circlearrowleft \Sigma M_J = 0: \quad x\frac{M_0}{L} + M = 0$$

$$M = -\frac{M_0}{L}x$$

Straight with $M = -\dfrac{M_0}{2}$ at B

Along BC:


M_0
A
B
K
M
$\frac{M_0}{L}$
$\frac{L}{2}$
V
x

$$\uparrow \Sigma F_y = 0: \quad -\frac{M_0}{L} - V = 0 \qquad V = -\frac{M_0}{L}$$

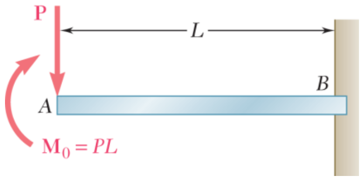
$$\circlearrowleft \Sigma M_K = 0: \quad M + x\frac{M_0}{L} - M_0 = 0 \qquad M = M_0\left(1 - \frac{x}{L}\right)$$

Straight with $M = \dfrac{M_0}{2}$ at B $\qquad M = 0$ at C

(*b*) From diagrams:

$$|V|_{\max} = M_0/L \blacktriangleleft$$

$$|M|_{\max} = \frac{M_0}{2} \text{ at } B \blacktriangleleft$$

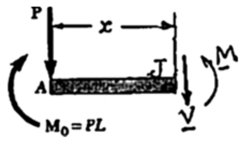


PROBLEM 7.34

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

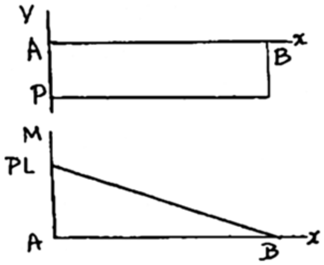
Free body: Portion *AJ*



$+\uparrow \Sigma F_y = 0: \quad -P - V = 0 \qquad V = -P \triangleleft$

$+\circlearrowleft \Sigma M_J = 0: \quad M + P_x - PL = 0 \qquad M = P(L - x) \triangleleft$

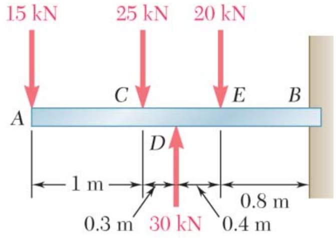
(*a*) The *V* and *M* diagrams are obtained by plotting the functions *V* and *M*.



(*b*)

$|V|_{\max} = P \blacktriangleleft$

$|M|_{\max} = PL \blacktriangleleft$

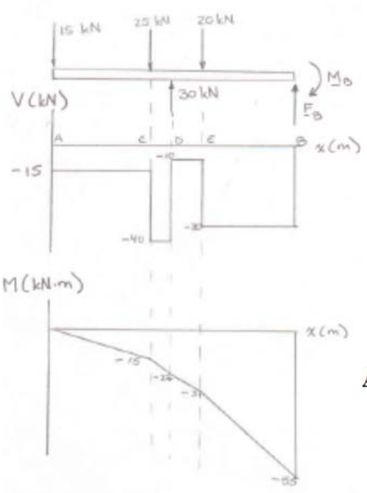

15 kN
25 kN
20 kN
C
E
B
A
D
1 m
0.3 m
30 kN
0.4 m
0.8 m

PROBLEM 7.35

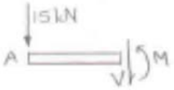
For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

(*a*)



Along AC:

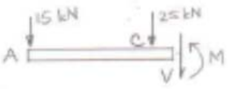


$$+\uparrow \Sigma F_y = 0: \quad -15\text{ kN} - V = 0 \qquad V = -15\text{ kN}$$

$$+\circlearrowleft \Sigma M_J = 0: \quad M + x(15\text{ kN}) = 0 \qquad M = (15\text{ kN})x$$

$$M = -15\text{ kN}\cdot\text{m at } C$$

Along CD:


15 kN
25 kN
A
C
V
M

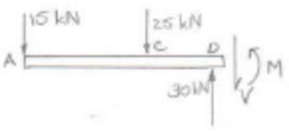
$$+\uparrow \Sigma F_y = 0: \quad -15\text{ kN} - 25\text{ kN} - V = 0 \qquad V = -40\text{ kN}$$

$$+\circlearrowleft \Sigma M_K = 0: \quad M + (x - 1\text{ m})(25\text{ kN}) + x(15\text{ kN}) = 0$$

$$M = +25\text{ kN}\cdot\text{m} - (40\text{ kN})x$$

$$M = -27\text{ kN}\cdot\text{m at } D \ (x = 1.3\text{ m})$$

Along DE:


15 kN
25 kN
A
C
D
30 kN
M
V

$$+\uparrow \Sigma F_y = 0: \quad -15\text{ kN} - 25\text{ kN} + 30\text{ kN} - V = 0$$

$$V = -10\text{ kN}$$

PROBLEM 7.35 (Continued)

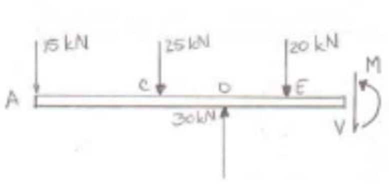
$$\circlearrowleft \Sigma M_L = 0: \qquad M - x_1(30\ \text{kN}) + (0.3\ \text{m} + x_1)(25\ \text{kN})$$

$$+ (1.3\ \text{m} + x_1)(15\ \text{kN}) = 0$$

$$M = -27\ \text{kN}\cdot\text{m} - (10\ \text{kN})x_1$$

$$M = -31.0\ \text{kN}\cdot\text{m at } E \quad (x_1 = 0.4\ \text{m})$$

Along EB:



$$\uparrow \Sigma F_y = 0: -15\ \text{kN} - 25\ \text{kN} + 30\ \text{kN} - 20\ \text{kN} - V = 0 \qquad V = -30\ \text{kN}$$

$$\circlearrowleft \Sigma M_N = 0: \qquad M + (20\ \text{kN})x_2 + (0.7\ \text{m} + x_2)(25\ \text{kN})$$

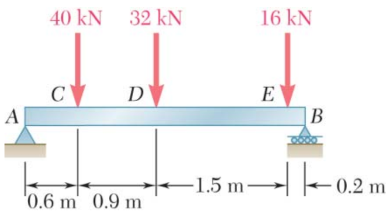
$$+ (1.7\ \text{m} + x_2)(15\ \text{kN}) - (0.4\ \text{m} + x_2)(30\ \text{kN}) = 0$$

$$M = -31\ \text{kN}\cdot\text{m} - (30\ \text{kN})x_2$$

$$M = -55\ \text{kN}\cdot\text{m at } B \quad (x_2 = 0.8\ \text{m})$$

From diagrams: $|V|_{\max} = 40.0\ \text{kN on } CD$ ◀

$|M|_{\max} = 55.0\ \text{kN}\cdot\text{m at } B$ ◀

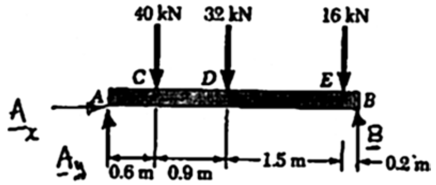

40 kN
32 kN
16 kN
C
D
E
A
B
0.6 m
0.9 m
1.5 m
0.2 m

PROBLEM 7.36

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

Free body: Entire beam


40 kN
32 kN
16 kN
C
D
E
A
B
A_x
A_y
B
0.6 m
0.9 m
1.5 m
0.2 m

$$+\circlearrowleft \Sigma M_A = 0: \quad B(3.2\text{ m}) - (40\text{ kN})(0.6\text{ m}) - (32\text{ kN})(1.5\text{ m}) - (16\text{ kN})(3\text{ m}) = 0$$

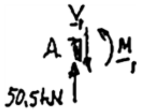
$B = +37.5\text{ kN}$ $\qquad$ $\mathbf{B} = 37.5\text{ kN}\uparrow$ ◁

$\Sigma F_x = 0: \quad A_x = 0$

$$+\uparrow \Sigma F_y = 0: \quad A_y + 37.5\text{ kN} - 40\text{ kN} - 32\text{ kN} - 16\text{ kN} = 0$$

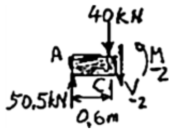
$A_y = +50.5\text{ kN}$ $\qquad$ $\mathbf{A} = 50.5\text{ kN}\uparrow$ ◁

(*a*) Shear and bending moment


V_1
A
M_1
50.5kN

Just to the right of *A*: $\qquad$ $V_1 = 50.5\text{ kN}$ $\qquad$ $M_1 = 0$ ◁

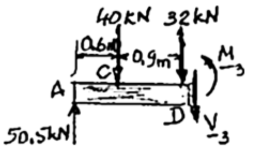
Just to the right of *C*:


40kN
A
M_2
C
V_2
50.5kN
0.6m

$+\uparrow \Sigma F_y = 0: \quad 50.5\text{ kN} - 40\text{ kN} - V_2 = 0$ $\qquad$ $V_2 = +10.5\text{ kN}$ ◁

$+\circlearrowleft \Sigma M_2 = 0: \quad M_2 - (50.5\text{ kN})(0.6\text{ m}) = 0$ $\qquad$ $M_2 = +30.3\text{ kN}\cdot\text{m}$ ◁

Just to the right of *D*:

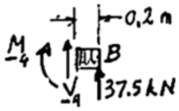

40kN
32kN
0.6m
0.9m
M_3
C
A
D
50.5kN
V_3

$+\uparrow \Sigma F_y = 0: \quad 50.5 - 40 - 32 - V_3 = 0$ $\qquad$ $V_3 = -21.5\text{ kN}$ ◁

$+\circlearrowleft \Sigma M_3 = 0: \quad M_3 - (50.5)(1.5) + (40)(0.9) = 0$ $\qquad$ $M_3 = +39.8\text{ kN}\cdot\text{m}$ ◁

PROBLEM 7.36 (Continued)

Just to the right of *E*:

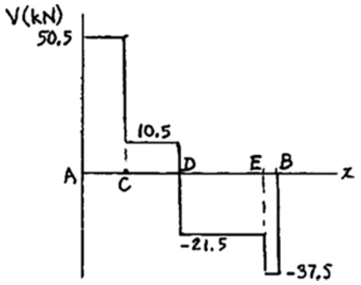

0.2 m
M_4
B
V_4
37.5 kN

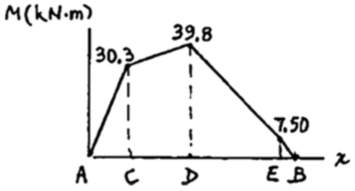
$+\uparrow \Sigma F_y = 0: \quad V_4 + 37.5 = 0$ $\qquad V_4 = -37.5 \text{ kN}$ ◁

$+\circlearrowleft \Sigma M_4 = 0: \quad -M_4 + (37.5)(0.2) = 0$ $\qquad M_4 = +7.50 \text{ kN}\cdot\text{m}$ ◁

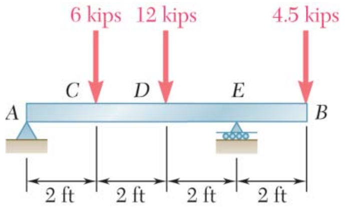
At *B*: $\qquad V_B = M_B = 0$ ◁

(*b*) $\qquad |V|_{\max} = 50.5 \text{ kN}$ ◀


V(kN)
50.5
10.5
A
C
D
E
B
x
-21.5
-37.5


M(kN·m)
30.3
39.8
7.50
x
A
C
D
E
B

$|M|_{\max} = 39.8 \text{ kN}\cdot\text{m}$ ◀

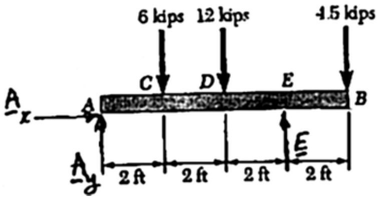

6 kips
12 kips
4.5 kips
C
D
E
A
B
2 ft
2 ft
2 ft
2 ft

PROBLEM 7.37

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

Free body: Entire beam


6 kips
12 kips
4.5 kips
C
D
E
A_x
A
B
E
A_y
2 ft
2 ft
2 ft
2 ft

$+\circlearrowleft \Sigma M_A = 0$: $E(6 \text{ ft}) - (6 \text{ kips})(2 \text{ ft}) - (12 \text{ kips})(4 \text{ ft}) - (4.5 \text{ kips})(8 \text{ ft}) = 0$

$E = +16 \text{ kips}$ $\quad\quad$ $\mathbf{E} = 16 \text{ kips} \uparrow \triangleleft$

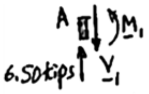
$\xrightarrow{+} \Sigma F_x = 0$: $A_x = 0$

$+\uparrow \Sigma F_y = 0$: $A_y + 16 \text{ kips} - 6 \text{ kips} - 12 \text{ kips} - 4.5 \text{ kips} = 0$

$A_y = +6.50 \text{ kips}$ $\quad\quad$ $\mathbf{A} = 6.50 \text{ kips} \uparrow \triangleleft$

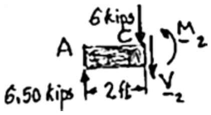
(*a*) Shear and bending moment

Just to the right of *A*:


A
M_1
6.50 kips
V_1

$V_1 = +6.50 \text{ kips} \quad M_1 = 0$ $\quad\quad \triangleleft$

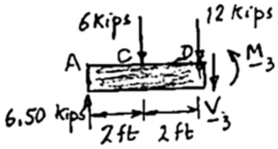
Just to the right of *C*:


6 kips
A
C
M_2
6.50 kips
2 ft
V_2

$+\uparrow \Sigma F_y = 0$: $6.50 \text{ kips} - 6 \text{ kips} - V_2 = 0$ $\quad\quad$ $V_2 = +0.50 \text{ kips} \triangleleft$

$+\circlearrowleft \Sigma M_2 = 0$: $M_2 - (6.50 \text{ kips})(2 \text{ ft}) = 0$ $\quad\quad$ $M_2 = +13 \text{ kip} \cdot \text{ft} \triangleleft$

Just to the right of *D*:

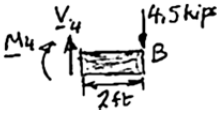

6 Kips
12 Kips
A
C
D
M_3
6.50 kips
2 ft
2 ft
V_3

$+\uparrow \Sigma F_y = 0$: $6.50 - 6 - 12 - V_3 = 0$ $\quad\quad$ $V_3 = +11.5 \text{ kips} \triangleleft$

$+\circlearrowleft \Sigma M_3 = 0$: $M_3 - (6.50)(4) - (6)(2) = 0$ $\quad\quad$ $M_3 = +14 \text{ kip} \cdot \text{ft} \triangleleft$

PROBLEM 7.37 (Continued)

Just to the right of *E*:

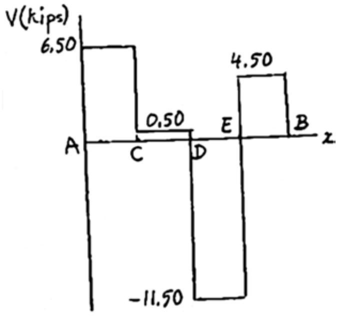


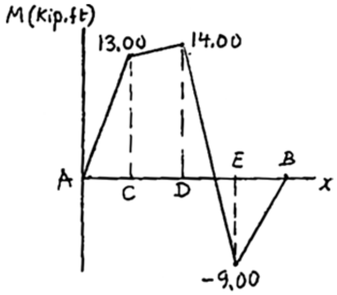
$+\uparrow \Sigma F_y = 0: \quad V_4 - 4.5 = 0$ $\qquad V_4 = +4.5 \text{ kips}$ ◁

$+\circlearrowright \Sigma M_4 = 0: \quad -M_4 - (4.5)2 = 0$ $\qquad M_4 = -9 \text{ kip} \cdot \text{ft}$ ◁

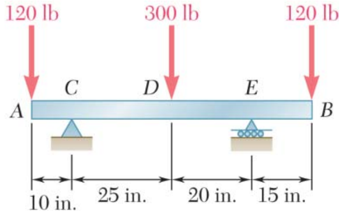
At *B*: $\qquad V_B = M_B = 0$ ◁

(*b*) $\qquad |V|_{\max} = 11.50 \text{ kips}$ ◀





$|M|_{\max} = 14.00 \text{ kip} \cdot \text{ft}$ ◀

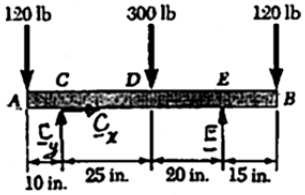

120 lb
300 lb
120 lb
C
D
E
A
B
10 in.
25 in.
20 in.
15 in.

PROBLEM 7.38

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

Free body: Entire beam


120 lb
300 lb
120 lb
C
D
E
A
B
C_x
C_y
E
10 in.
25 in.
20 in.
15 in.

$$+\circlearrowleft \Sigma M_C = 0: \quad (120\text{ lb})(10\text{ in.}) - (300\text{ lb})(25\text{ in.}) + E(45\text{ in.}) - (120\text{ lb})(60\text{ in.}) = 0$$

$$E = +300\text{ lb} \qquad \mathbf{E} = 300\text{ lb}\uparrow ◁$$

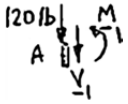
$$\Sigma F_x = 0: \quad C_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad C_y + 300\text{ lb} - 120\text{ lb} - 300\text{ lb} - 120\text{ lb} = 0$$

$$C_y = +240\text{ lb} \qquad \mathbf{C} = 240\text{ lb}\uparrow ◁$$

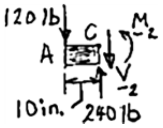
(*a*) Shear and bending moment

Just to the right of *A*:


120 lb
A
M
1
V
1

$$+\uparrow \Sigma F_y = 0: \quad -120\text{ lb} - V_1 = 0 \qquad V_1 = -120\text{ lb}, \quad M_1 = 0 ◁$$

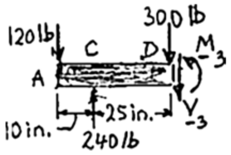
Just to the right of *C*:


120 lb
C
A
M
2
V
2
10 in.
240 lb

$$+\uparrow \Sigma F_y = 0: \quad 240\text{ lb} - 120\text{ lb} - V_2 = 0 \qquad V_2 = +120\text{ lb} ◁$$

$$+\circlearrowleft \Sigma M_C = 0: \quad M_2 + (120\text{ lb})(10\text{ in.}) = 0 \qquad M_2 = -1200\text{ lb}\cdot\text{in.} ◁$$

Just to the right of *D*:

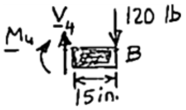

300 lb
120 lb
C
D
A
M
3
V
3
10 in.
25 in.
240 lb

$$+\uparrow \Sigma F_y = 0: \quad 240 - 120 - 300 - V_3 = 0 \qquad V_3 = -180\text{ lb} ◁$$

$$+\circlearrowleft \Sigma M_3 = 0: \quad M_3 + (120)(35) - (240)(25) = 0, \qquad M_3 = +1800\text{ lb}\cdot\text{in.} ◁$$

PROBLEM 7.38 (Continued)

Just to the right of *E*:

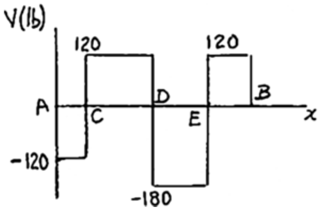


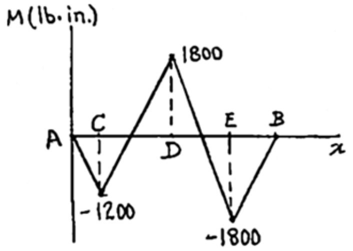
$+\Sigma F_y = 0$: $V_4 - 120\text{ lb} = 0$ $\quad V_4 = +120\text{ lb}$ ◁

$+\circlearrowleft \Sigma M_4 = 0$: $-M_4 - (120\text{ lb})(15\text{ in.}) = 0$ $\quad M_4 = -1800\text{ lb}\cdot\text{in.}$ ◁

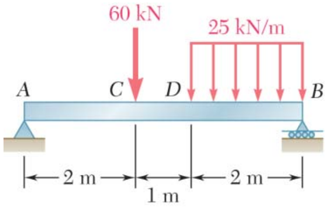
At *B*: $V_B = M_B = 0$ ◁

(*b*) $|V|_{\max} = 180.0\text{ lb}$ ◀





$|M|_{\max} = 1800\text{ lb}\cdot\text{in.}$ ◀

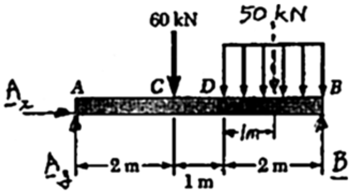

60 kN
25 kN/m
A
C
D
B
2 m
1 m
2 m

PROBLEM 7.39

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

Free body: Entire beam


60 kN
50 kN
A_x
A
C
D
B
1m
A_y
2 m
1 m
2 m
B

$+\circlearrowleft \Sigma M_A = 0$: $B(5\text{ m}) - (60\text{ kN})(2\text{ m}) - (50\text{ kN})(4\text{ m}) = 0$

$B = +64.0\text{ kN}$ **B** = 64.0 kN↑ ◁

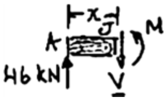
$\Sigma F_x = 0$: $A_x = 0$

$+\uparrow \Sigma F_y = 0$: $A_y + 64.0\text{ kN} - 6.0\text{ kN} - 50\text{ kN} = 0$

$A_y = +46.0\text{ kN}$ **A** = 46.0 kN↑ ◁

(*a*) Shear and bending-moment diagrams.

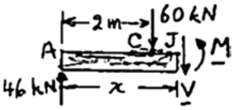
From *A* to *C*:



$+\uparrow \Sigma F_y = 0$: $46 - V = 0$ $V = +46\text{ kN}$ ◁

$+\circlearrowleft \Sigma M_y = 0$: $M - 46x = 0$ $M = (46x)\text{kN}\cdot\text{m}$ ◁

From *C* to *D*:


2 m
60 kN
C
J
M
A
46 kN
x
V

$+\uparrow \Sigma F_y = 0$: $46 - 60 - V = 0$ $V = -14\text{ kN}$ ◁

$+\circlearrowleft \Sigma M_j = 0$: $M - 46x + 60(x-2) = 0$

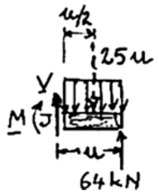
$M = (120 - 14x)\text{kN}\cdot\text{m}$

For $x = 2\text{ m}$: $M_C = +92.0\text{ kN}\cdot\text{m}$ ◁

For $x = 3\text{ m}$: $M_D = +78.0\text{ kN}\cdot\text{m}$ ◁

PROBLEM 7.39 (Continued)

From *D* to *B*:



$$+\uparrow \Sigma F_y = 0: \quad V + 64 - 25\mu = 0$$

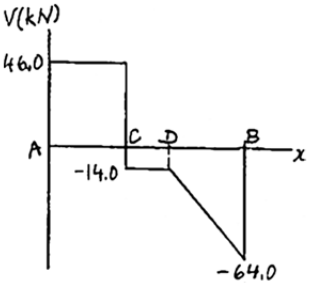
$$V = (25\mu - 64)\text{kN}$$

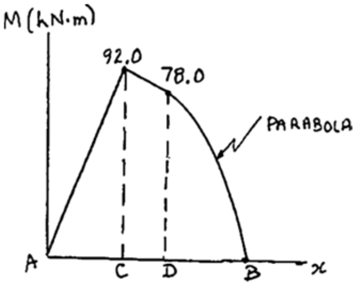
$$+\circlearrowleft \Sigma M_j = 0: \quad 64\mu - (25\mu)\left(\frac{\mu}{2}\right) - M = 0$$

$$M = (64\mu - 12.5\mu^2)\text{kN}\cdot\text{m}$$

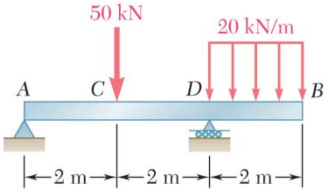
For $\mu = 0$: $V_B = -64$ kN $\qquad M_B = 0$ ◁

(*b*) $|V|_{\max} = 64.0$ kN ◀





$|M|_{\max} = 92.0$ kN · m ◀

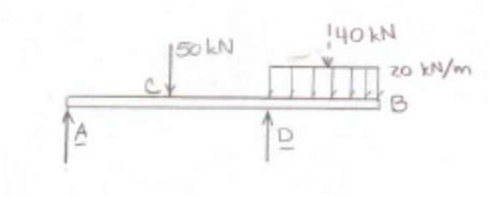


PROBLEM 7.40

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

Free body: Entire beam

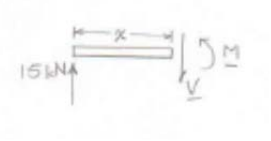


$+\circlearrowleft \Sigma M_A = 0: \quad -(50 \text{ kN})(2 \text{ m}) + D(4 \text{ m}) - (40 \text{ kN})(5 \text{ m}) = 0$ $\qquad \mathbf{D} = 75 \text{ kN} \uparrow$

$+\uparrow \Sigma F_y = 0: \quad 75 - 50 - 40 + A = 0$

$A = -15 \text{ kN}$ $\qquad \mathbf{A} = 15 \text{ kN} \uparrow$

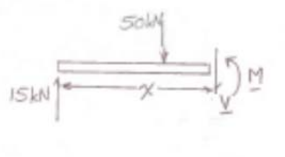
From *A* to *C*:



$+\uparrow \Sigma F_y = 0: \quad 15 - V = 0 \quad V = 15 \text{ kN}$

$+\circlearrowleft \Sigma M_1 = 0: \quad M - (15)(x) = 0$ $\qquad M = (15x) \text{ kN} \cdot \text{m}$

From *C* to *D*:



$+\uparrow \Sigma F_y = 0: \quad 15 - 50 - V = 0 \quad V = -35 \text{ kN}$

$+\circlearrowleft \Sigma M_2 = 0: \quad M - (15)(x) + 50(x - 2) = 0$

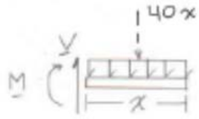
$M = (100 - 35x) \text{kN} \cdot \text{m}$

$M_C = +30 \text{ kN} \cdot \text{m at } x = 2 \text{ m}$

$M_D = -40 \text{ kN} \cdot \text{m at } x = 4 \text{ m}$

PROBLEM 7.40 (Continued)

From *D* to *B*:



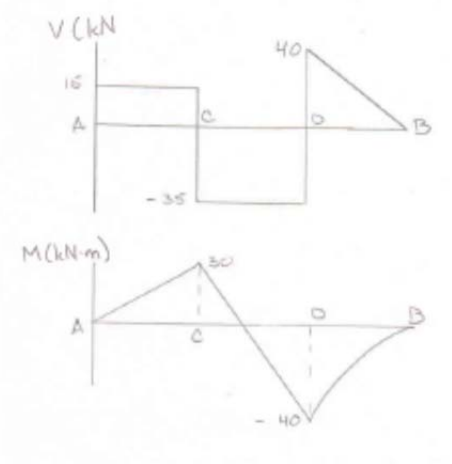
$$+\uparrow \Sigma F_y = 0: \quad V - 20x = 0 \quad V = +20x \text{ kN}$$

$$+\circlearrowleft \Sigma M_3 = 0: \quad -M - (20x)\left(\frac{x}{2}\right) = 0 \qquad M = -10x^2 \text{ kN}\cdot\text{m}$$

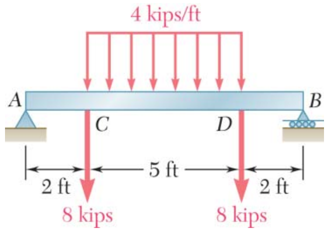
$V_D = 40$ kN, $M_D = -40$ kN · m at $x = 2$ m

$V_B = M_B = 0$ at $x = 0$

(*b*) $|V|_{\max} = 40.0$ kN ◀



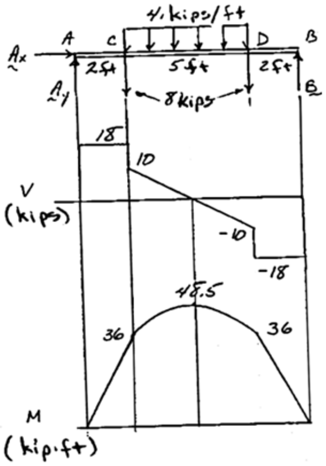
$|M|_{\max} = 40.0$ kN · m ◀


4 kips/ft
A
B
C
D
2 ft
5 ft
2 ft
8 kips
8 kips

PROBLEM 7.41

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

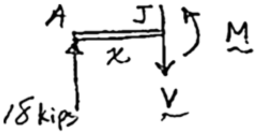
SOLUTION


4 kips/ft
A
C
D
B
A_x
A_y
2 ft
5 ft
2 ft
B
8 kips
18
10
V
(kips)
-10
-18
48.5
36
36
M
(kip·ft)

(*a*) By symmetry:

$$A_y = B = 8\text{ kips} + \frac{1}{2}(4\text{ kips})(5\text{ ft}) \qquad \mathbf{A}_y = \mathbf{B} = 18\text{ kips}\uparrow$$

Along AC:

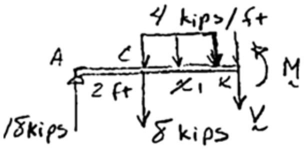

A
J
M
x
18 kips
V

$$\uparrow\Sigma F_y = 0\text{:}\quad 18\text{ kips} - V = 0 \quad V = 18\text{ kips}$$

$$\circlearrowleft \Sigma M_J = 0\text{:}\quad M - x(18\text{ kips}) \quad M = (18\text{ kips})x$$

$$M = 36\text{ kip}\cdot\text{ft at } C(x = 2\text{ ft})$$

Along CD:


4 kips/ft
A
C
K
M
2 ft
x_1
18kips
8 kips
V

$$\uparrow\Sigma F_y = 0\text{:}\quad 18\text{ kips} - 8\text{ kips} - (4\text{ kips/ft})x_1 - V = 0$$

$$V = 10\text{ kips} - (4\text{ kips/ft})x_1$$

$$V = 0\text{ at } x_1 = 2.5\text{ ft (at center)}$$

$$\circlearrowleft \Sigma M_K = 0\text{:}\quad M + \frac{x_1}{2}(4\text{ kips/ft})x_1 + (8\text{ kips})x_1 - (2\text{ ft} + x_1)(18\text{ kips}) = 0$$

$$M = 36\text{ kip}\cdot\text{ft} + (10\text{ kips/ft})x_1 - (2\text{ kips/ft})x_1^2$$

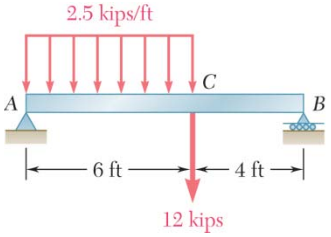
$$M = 48.5\text{ kip}\cdot\text{ft at } x_1 = 2.5\text{ ft}$$

Complete diagram by symmetry

(*b*) From diagrams:

$$|V|_{\max} = 18.00\text{ kips} \blacktriangleleft$$

$$|M|_{\max} = 48.5\text{ kip}\cdot\text{ft} \blacktriangleleft$$

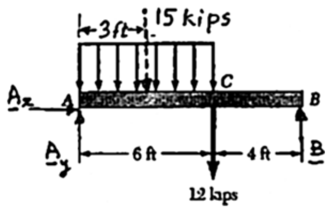

2.5 kips/ft
C
A
B
6 ft
4 ft
12 kips

PROBLEM 7.42

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

Free body: Entire beam


3ft
15 kips
C
A_x
A
B
A_y
6 ft
4 ft
B
12 kips

$$+\circlearrowleft \Sigma M_A = 0: \quad B(10\text{ ft}) - (15\text{ kips})(3\text{ ft}) - (12\text{ kips})(6\text{ ft}) = 0$$

$$B = +11.70\text{ kips} \qquad \mathbf{B} = 11.70\text{ kips}\uparrow \triangleleft$$

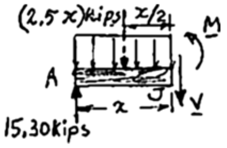
$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 15 - 12 + 11.70 = 0$$

$$A_y = +15.30\text{ kips} \qquad \mathbf{A} = 15.30\text{ kips}\uparrow \triangleleft$$

(*a*) Shear and bending-moment diagrams

From *A* to *C*:


(2.5 x)kips
x/2
M
A
x
J
V
15.30kips

$$+\uparrow \Sigma F_y = 0: \quad 15.30 - 2.5x - V = 0$$

$$V = (15.30 - 2.5x)\text{ kips}$$

$$+\circlearrowleft \Sigma M_J = 0: \quad M + (2.5x)\left(\frac{x}{2}\right) - 15.30x = 0$$

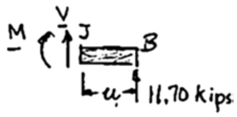
$$M = 15.30x - 1.25x^2$$

For $x = 0$: $\quad V_A = +15.30\text{ kips} \qquad M_A = 0 \triangleleft$

For $x = 6\text{ ft}$: $\quad V_C = +0.300\text{ kip} \qquad M_C = +46.8\text{ kip}\cdot\text{ft} \triangleleft$

PROBLEM 7.42 (Continued)

From *C* to *B*:



$+\uparrow \Sigma F_y = 0: \quad V + 11.70 = 0$ $\qquad V = -11.70 \text{ kips}$ ◁

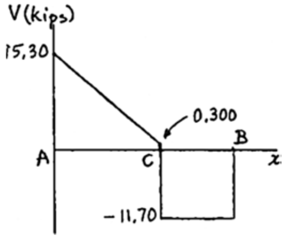
$+\circlearrowleft \Sigma M_J = 0: \quad 11.70\mu - M = 0$

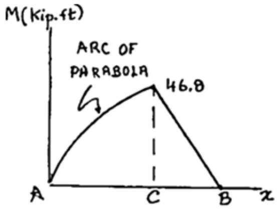
$$M = (11.70\mu) \text{ kip} \cdot \text{ft}$$

For $\mu = 4$ ft: $\qquad M_C = +46.8 \text{ kip} \cdot \text{ft}$ ◁

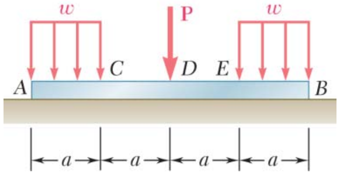
For $\mu = 0$: $\qquad M_B = 0$ ◁

(*b*) $\quad |V|_{\max} = 15.30 \text{ kips}$ ◀





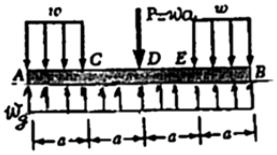
$|M|_{\max} = 46.8 \text{ kip} \cdot \text{ft}$ ◀


w
P
w
A
C
D
E
B
a
a
a
a

PROBLEM 7.43

Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed and knowing that $P = wa$, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION


w
P=wa
w
A
C
D
E
B
w_g
a
a
a
a

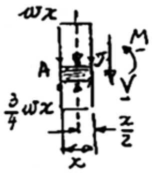
Free body: Entire beam

$$+\uparrow \Sigma F_y = 0: \quad w_g(4a) - 2wa - wa = 0$$

$$w_g = \frac{3}{4}w \triangleleft$$

(*a*) Shear and bending-moment diagrams

From *A* to *C*:


wx
M
A
J
V
$\frac{3}{4}wx$
$\frac{x}{2}$
x

$$+\uparrow \Sigma F_y = 0: \quad \frac{3}{4}wx - wx - V = 0$$

$$V = -\frac{1}{4}wx$$

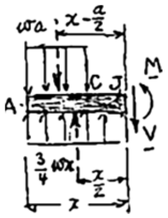
$$+\circlearrowleft \Sigma M_J = 0: \quad M + (wx)\frac{x}{2} - \left(\frac{3}{4}wx\right)\frac{x}{2} = 0$$

$$M = -\frac{1}{8}wx^2$$

For $x = 0$: $\qquad V_A = M_A = 0 \triangleleft$

For $x = a$: $\qquad V_C = -\frac{1}{4}wa \qquad M_C = -\frac{1}{8}wa^2 \triangleleft$

From *C* to *D*:


wa
$x-\frac{a}{2}$
M
A
C
J
V
$\frac{3}{4}wx$
$\frac{x}{2}$
x

$$+\uparrow \Sigma F_y = 0: \quad \frac{3}{4}wx - wa - V = 0$$

$$V = \left(\frac{3}{4}x - a\right)w$$

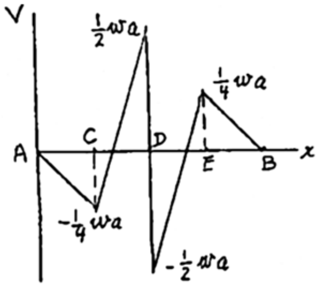
$$+\circlearrowleft \Sigma M_J = 0: \quad M + wa\left(x - \frac{a}{2}\right) - \frac{3}{4}wx\left(\frac{x}{2}\right) = 0$$

$$M = \frac{3}{8}wx^2 - wa\left(x - \frac{a}{2}\right) \qquad (1)$$

PROBLEM 7.43 (Continued)

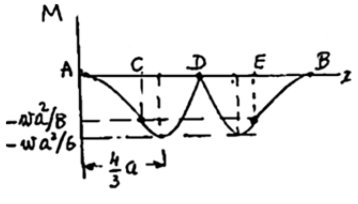
For $x = a$: $V_C = -\frac{1}{4}wa$ $M_C = -\frac{1}{8}wa^2$ ◁

For $x = 2a$: $V_D = +\frac{1}{2}wa$ $M_D = 0$ ◁


V
$\frac{1}{2}wa$
$\frac{1}{4}wa$
A
C
D
E
B
x
$-\frac{1}{4}wa$
$-\frac{1}{2}wa$

Because of the symmetry of the loading, we can deduce the values of V and M for the right-hand half of the beam from the values obtained for its left-hand half.

(*b*) $|V|_{\max} = \frac{1}{2}wa$ ◀


M
A
C
D
E
B
x
$-wa^2/8$
$-wa^3/6$
$\frac{4}{3}a$

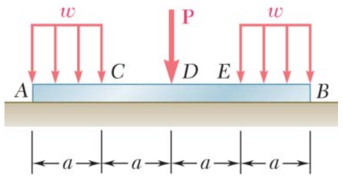
To find $|M|_{\max}$, we differentiate Eq. (1) and set $\frac{dM}{dx} = 0$:

$$\frac{dM}{dx} = \frac{3}{4}wx - wa = 0, \quad x = \frac{4}{3}a$$

$$M = \frac{3}{8}w\left(\frac{4}{3}a\right)^2 - wa^2\left(\frac{4}{3} - \frac{1}{2}\right) = -\frac{wa^2}{6}$$

$|M|_{\max} = \frac{1}{6}wa^2$ ◀

Bending-moment diagram consists of four distinct arcs of parabola.

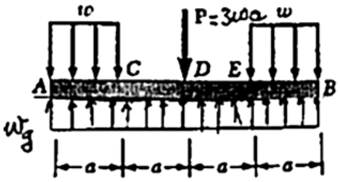

w
P
w
A
C
D
E
B
a
a
a
a

PROBLEM 7.44

Solve Problem 7.43 knowing that $P=3wa$.

PROBLEM 7.43 Assuming the upward reaction of the ground on beam AB to be uniformly distributed and knowing that $P=wa$, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION


w
P=3wa
w
A
C
D
E
B
w_g
a
a
a
a

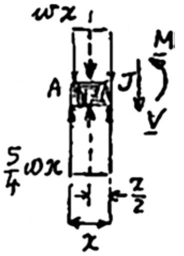
Free body: Entire beam

$$+\!\uparrow\Sigma F_y=0:\quad w_g(4a)-2wa-3wa=0$$

$$w_g=\frac{5}{4}w \triangleleft$$

(*a*) Shear and bending-moment diagrams

From *A* to *C*:


wx
M
A
J
V
$\frac{5}{4}wx$
$\frac{x}{2}$
x

$$+\!\uparrow\Sigma F_y=0:\quad \frac{5}{4}wx-wx-V=0$$

$$V=+\frac{1}{4}wx$$

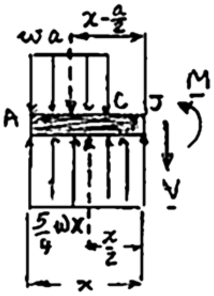
$$+\circlearrowleft\Sigma M_J=0:\quad M+(wx)\frac{x}{2}-\left(\frac{5}{4}wx\right)\frac{x}{2}=0$$

$$M=+\frac{1}{8}wx^2$$

For $x=0$: $V_A=M_A=0 \triangleleft$

For $x=a$: $V_C=+\frac{1}{4}wa$ $M_C=+\frac{1}{8}wa^2 \triangleleft$

From *C* to *D*:


wa
$x-\frac{a}{2}$
M
A
C
J
V
$\frac{5}{4}wx$
$\frac{x}{2}$
x

$$+\!\uparrow\Sigma F_y=0:\quad \frac{5}{4}wx-wa-V=0$$

$$V=\left(\frac{5}{4}x-a\right)w$$

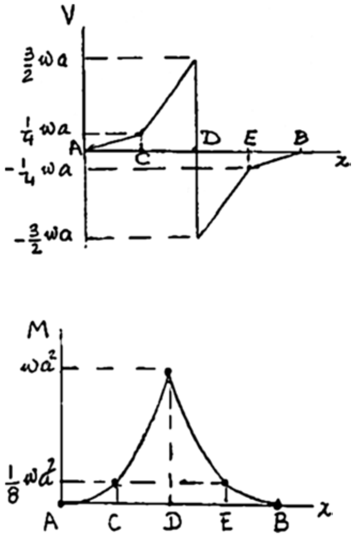
$$+\circlearrowleft\Sigma M_J=0:\quad M+wa\left(x-\frac{a}{2}\right)-\frac{5}{4}wx\left(\frac{x}{2}\right)=0$$

$$M=\frac{5}{8}wx^2-wa\left(x-\frac{a}{2}\right) \qquad (1)$$

PROBLEM 7.44 (Continued)

For $x = a$: $$V_C = +\frac{1}{4}wa,\ M_C = +\frac{1}{8}wa^2 \triangleleft$$

For $x = 2a$: $$V_D = +\frac{3}{2}wa,\ M_D = +wa^2 \triangleleft$$


V
$\frac{3}{2}wa$
$\frac{1}{4}wa$
A
C
D
E
B
x
$-\frac{1}{4}wa$
$-\frac{3}{2}wa$
M
wa^2
$\frac{1}{8}wa^2$
A
C
D
E
B
x

(*b*)

Because of the symmetry of the loading, we can deduce the values of *V* and *M* for the right-hand half of the beam from the values obtained for its left-hand half.

$$|V|_{\max} = \frac{3}{2}wa \blacktriangleleft$$

To find $|M|_{\max}$, we differentiate Eq. (1) and set $\frac{dM}{dx} = 0$:

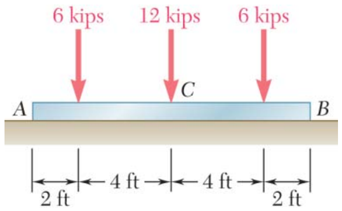
$$\frac{dM}{dx} = \frac{5}{4}wx - wa = 0$$

$$x = \frac{4}{5}a < a \quad \text{(outside portion } CD\text{)}$$

The maximum value of $|M|$ occurs at *D*:

$$|M|_{\max} = wa^2 \blacktriangleleft$$

Bending-moment diagram consists of four distinct arcs of parabola.



PROBLEM 7.45

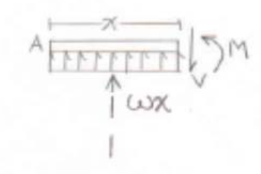
Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

(*a*) For equilibrium, $+\uparrow \Sigma F_y = 0: \quad -6-12-6+w(12\text{ ft})=0$

$$w = 2.0\text{ kips/ft}$$

Along AD:



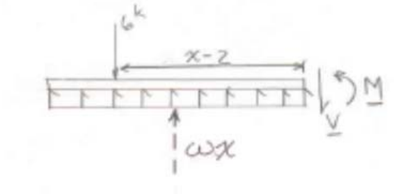
$$+\uparrow \Sigma F_y = 0: \quad (2.0\text{ kips/ft})x - V = 0$$

$$V = 2x$$

$$+\circlearrowleft \Sigma M_1 = 0: \quad M - (2x)\frac{x}{2} = 0$$

$$M = x^2$$

Along DC:



$+\uparrow \Sigma F_y = 0: \quad 2x - 6 - V = 0$

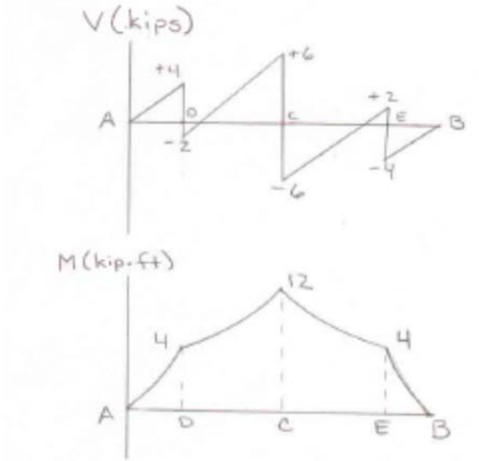
$V = 2x - 6$

$$+\circlearrowleft \Sigma M_2 = 0: \quad M + 6(x-2) - (2x)\frac{x}{2} = 0$$

$$M = x^2 - 6x + 12$$

PROBLEM 7.45 (Continued)

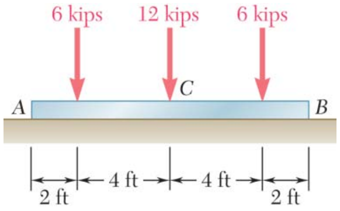
Complete diagrams using symmetry.



(*b*)

$|V|_{\max} = 6.00 \text{ kips}$ ◀

$|M|_{\max} = 12.00 \text{ kip} \cdot \text{ft}$ ◀



PROBLEM 7.46

Solve Prob. 7.45 assuming that the 12-kip load has been removed.

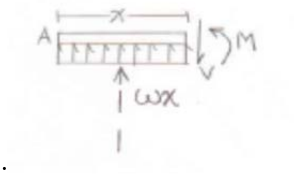
PROBLEM 7.45 Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

For equilibrium,

$$+\uparrow \Sigma F_y = 0: \quad 12w_g - 6 - 6 = 0 \qquad\qquad w_g = 1 \text{ kip/ft} \ \triangleleft$$

(*a*) Shear and bending moment

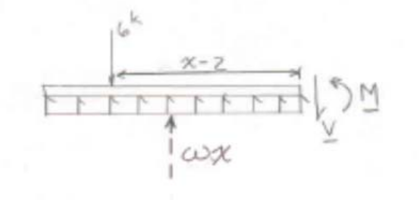


From *A* to *D*:

$$+\uparrow \Sigma F_y = 0: \quad x - V = 0 \quad V = x$$

$$+\circlearrowleft \Sigma M_1 = 0: \quad M - (x)\frac{x}{2} = 0, \quad M = \frac{x^2}{2}$$

From *D* to *C*: $+\uparrow \Sigma F_y = 0: \quad x - 6 - V = 0$

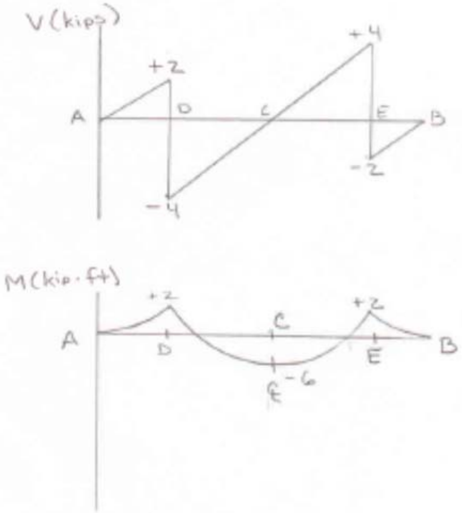


$$V = x - 6$$

$$+\circlearrowleft \Sigma M_2 = 0: \quad M + 6(x-2) - (x)\frac{x}{2} = 0$$

$$M = 12 - 6x + \frac{x^2}{2}$$

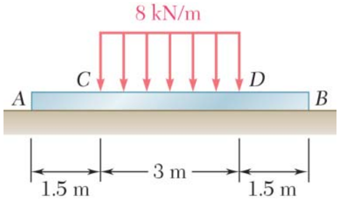
PROBLEM 7.46 (Continued)


V (kips)
+2
+4
A
D
C
E
B
-2
-4
M (kip·ft)
+2
+2
A
C
D
E
B
-6
℄

(*b*)

$|V|_{\max} = 4.00 \text{ kips}$ ◀

$|M|_{\max} = 6.00 \text{ kip} \cdot \text{ft}$ ◀



PROBLEM 7.47

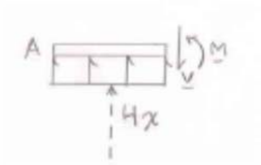
Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed, (*a*) draw the shear and bending-moment diagrams,(*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

For equilibrium,

$$+\uparrow \Sigma F_y = 0: \quad w_g(6 \text{ m}) - (8 \text{ kN/m})(3 \text{ m}) = 0$$

$$w_g = 4 \text{ kN/m} \triangleleft$$



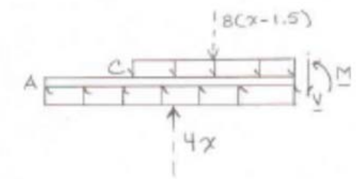
(*a*) Shear and bending-moment diagrams from *A* to *C*:

$$+\uparrow \Sigma F_y = 0: \quad 4x - V = 0 \qquad V = (4x) \text{ kN}$$

$$+\circlearrowleft \Sigma M_J = 0: \quad M - (4x)\frac{x}{2} \qquad M = (2x^2) \text{ kN}\cdot\text{m}$$

For $x = 0$: $V_A = M_A = 0 \triangleleft$

For $x = 1.5$ m: $V_C = 6$ kN, $M_C = 4.5 \text{ kN}\cdot\text{m} \triangleleft$



From *C* to *D*:

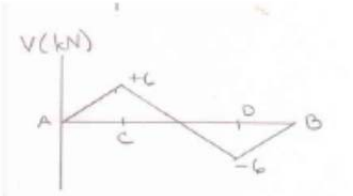
$$+\uparrow \Sigma F_y = 0: \quad 4x - 8(x - 1.5) - V = 0$$

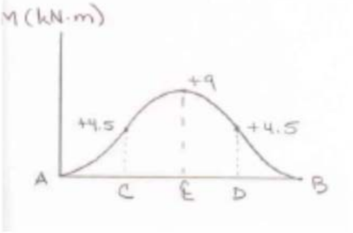
$$V = (12 - 4x) \text{ kN}$$

$$+\circlearrowleft \Sigma M_J = 0: \quad M + 8(x-1.5)\frac{x-1.5}{2} - (4x)\frac{x}{2} = 0$$

$$M = [2x^2 - 4(x - 1.5)^2]\text{kN}\cdot\text{m}$$

$$M = -9 + 12x - 2x^2$$





For $x = 3$ m: $V_{℄} = 0$, $M = 9 \text{ kN}\cdot\text{m} \triangleleft$

For $x = 4.5$ m: $V_D = -6$ kN, $M_D = 4.5 \text{ kN}\cdot\text{m} \triangleleft$

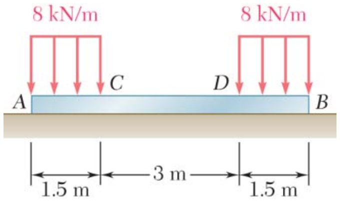
At *B*: $V_B = M_B = 0 \triangleleft$

(*b*) $|V|_{\max} = 6.00 \text{ kN}$ ◀

$|M|_{\max} = 9.00 \text{ kN}\cdot\text{m}$ ◀

Bending-moment diagram consists or three distinct arcs of parabola, all located above the *x* axis.

Thus: $M \geq 0$ everywhere ◀



PROBLEM 7.48

Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

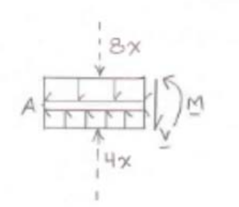
SOLUTION

For equilibrium,

$$+\uparrow \Sigma F_y = 0: \quad w_g(6\text{ m}) - (8\text{ kN/m})(3\text{ m}) = 0 \qquad w_g = 4\text{ kN/m} \triangleleft$$

(*a*) Shear and bending-moment diagrams.

From *A* to *C*:



$$\Sigma F_y = 0\uparrow: \quad -8x + 4x - V = 0$$

$$V = (-4x)\text{ kN}$$

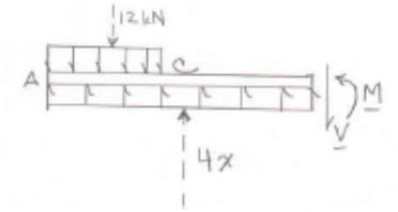
$$+\circlearrowleft \Sigma M_1 = 0: \quad M + (8x)\frac{x}{2} - (4x)\frac{x}{2} = 0$$

$$M = (-2x^2)\text{ kN}\cdot\text{m}$$

For $x = 0$: $\qquad V_A = M_A = 0 \triangleleft$

For $x = 1.5\text{ m}$: $\qquad V_C = -6\text{ kN} \quad M_C = -4.5\text{ kN}\cdot\text{m} \triangleleft$

From *A* to *D*:



$$+\uparrow \Sigma F_y = 0: \quad 4x - 12 - V = 0, \quad V = (4x - 12)\text{ kN}$$

$$+\circlearrowleft \Sigma M_2 = 0: \quad M + 12(x - 0.75) - (4x)\frac{x}{2} = 0$$

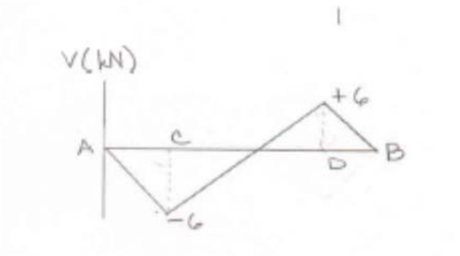
$$M = 2x^2 - 12x + 9$$

PROBLEM 7.48 (Continued)

For $x = 3$ m: $V = 0$, $M = -9.00\ \text{kN}\cdot\text{m} \triangleleft$

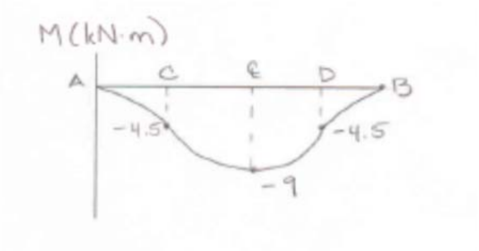
For $x = 4.5$ m: $V_D = +6$ kN, $M_D = -4.50\ \text{kN}\cdot\text{m} \triangleleft$

At *B*: $V_B = M_B = 0 \triangleleft$



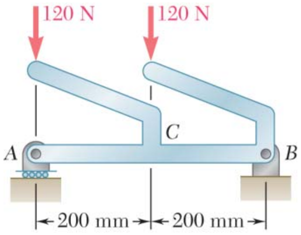
(*b*) $|V|_{\max} = 6.00$ kN ◀

Bending-moment diagram consists of three distinct arcs of parabola.



$|M|_{\max} = 9.00\ \text{kN}\cdot\text{m}$ ◀

Since entire diagram is below the *x* axis: $M \leq 0$ everywhere ◀

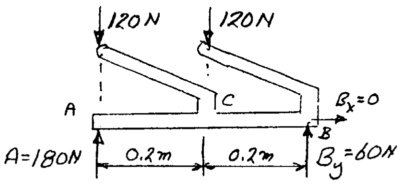

120 N
120 N
C
A
B
200 mm
200 mm

PROBLEM 7.49

Draw the shear and bending-moment diagrams for the beam *AB*, and determine the maximum absolute values of the shear and bending moment.

SOLUTION

Reactions:


120 N
120 N
A
C
$B_x=0$
B
A=180N
0.2m
0.2m
$B_y=60N$

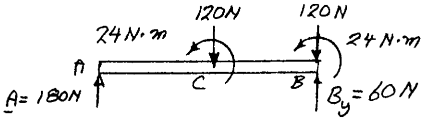
$$+\circlearrowleft \Sigma M_A = 0: \quad B_y(0.4) - (120)(0.2) = 0$$

$$\mathbf{B}_y = 60\text{ N}\uparrow$$

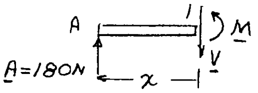
$$\Sigma F_x = 0: \qquad \mathbf{B}_x = 0$$

$$\Sigma F_y = 0: \qquad \mathbf{A} = 180\text{ N}\uparrow$$

Equivalent loading on straight part of beam *AB*.


120N
120 N
24 N·m
24 N·m
A
C
B
A= 180N
$B_y=60N$

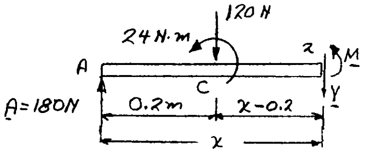
From *A* to *C*:


A
1
M
V
A=180N
x

$$+\uparrow \Sigma F_y = 0: \quad V = +180\text{ N}$$

$$+\circlearrowleft \Sigma M_1 = 0: \quad M = +180x$$

From *C* to *B*:

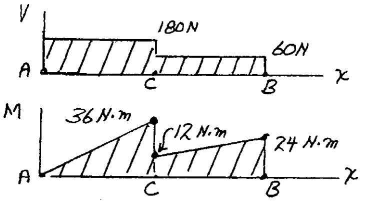

120 N
24 N·m
x
M
A
C
V
A=180N
0.2m
x-0.2
x

$$+\uparrow \Sigma F_y = 0: \quad 180 - 120 - V = 0$$

$$V = 60\text{ N}$$

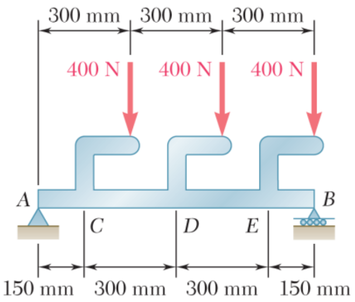
$$+\circlearrowleft \Sigma M_x = 0: \quad -(180\text{ N})(x) + 24\text{ N}\cdot\text{m} + (120\text{ N})(x-0.2) + M = 0$$

$$M = +60x$$



$$|V|_{\max} = 180.0\text{ N} \blacktriangleleft$$

$$|M|_{\max} = 36.0\text{ N}\cdot\text{m} \blacktriangleleft$$

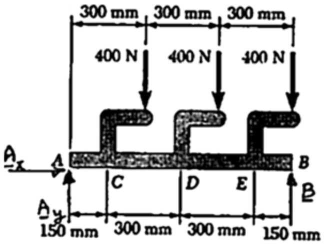

300 mm
300 mm
300 mm
400 N
400 N
400 N
A
B
C
D
E
150 mm
300 mm
300 mm
150 mm

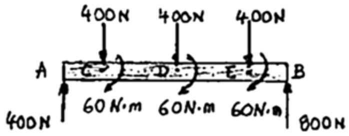
PROBLEM 7.50

Draw the shear and bending-moment diagrams for the beam *AB*, and determine the maximum absolute values of the shear and bending moment.

SOLUTION

Free body: Entire beam


300 mm
300 mm
300 mm
400 N
400 N
400 N
A_x
A
B
C
D
E
A_y
B
150 mm
300 mm
300 mm
150 mm


400N
400N
400N
A
C
D
E
B
400N
60 N·m
60N·m
60N·m
800N

$$+\circlearrowleft \Sigma M_A = 0: \quad B(0.9\text{ m}) - (400\text{ N})(0.3\text{ m}) - (400\text{ N})(0.6\text{ m}) - (400\text{ N})(0.9\text{ m}) = 0$$

$$B = +800\text{ N} \qquad \mathbf{B} = 800\text{ N}\uparrow \triangleleft$$

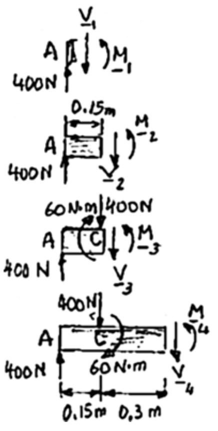
$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + 800\text{ N} - 3(400\text{ N}) = 0$$

$$A_y = +400\text{ N} \qquad \mathbf{A} = 400\text{ N}\uparrow \triangleleft$$

We replace the loads by equivalent force-couple systems at *C*, *D*, and *E*.

We consider successively the following *F*-*B* diagrams.


V_1
A
M_1
400N
0.15m
A
M_2
400N
V_2
60N·m
400N
A
C
M_3
400 N
V_3
400N
A
C
M_4
400N
60N·m
V_4
0.15m
0.3 m

$V_1 = +400$ N

$M_1 = 0$

$V_2 = +400$ N

$M_2 = +60$ N · m

$V_3 = 0$

$M_3 = +120$ N · m

$V_4 = 0$

$M_4 = +120$ N · m

$V_5 = -400$ N

$M_5 = +180$ N · m

$V_6 = -400$ N

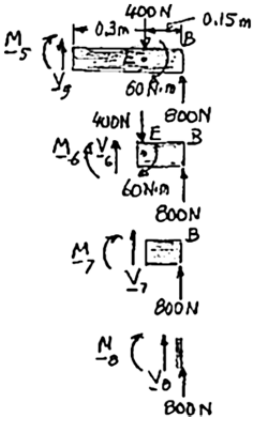
$M_6 = +60$ N · m

$V_7 = -800$ N

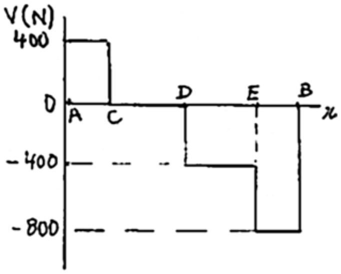
$M_7 = +120$ N · m

$V_8 = -800$ N

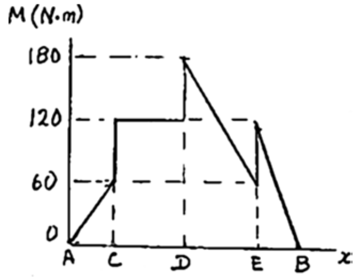
$M_8 = 0$


400 N
0.15m
0.3m
B
M_5
V_5
60 N·m
400N
800N
E
B
M_6
V_6
60N·m
800N
B
M_7
V_7
800N
M_8
V_8
800 N

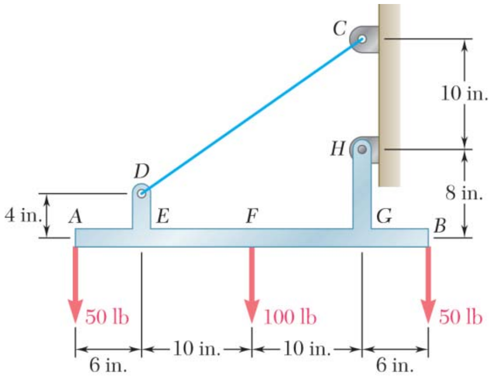
PROBLEM 7.50 (Continued)



(*b*) $|V|_{\max} = 800 \text{ N}$ ◀



$|M|_{\max} = 180.0 \text{ N} \cdot \text{m}$ ◀

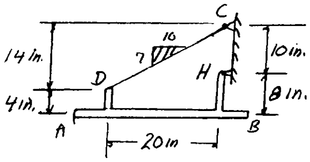

C
10 in.
H
8 in.
D
4 in.
A
E
F
G
B
50 lb
100 lb
50 lb
6 in.
10 in.
10 in.
6 in.

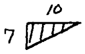
PROBLEM 7.51

Draw the shear and bending-moment diagrams for the beam *AB*, and determine the maximum absolute values of the shear and bending moment.

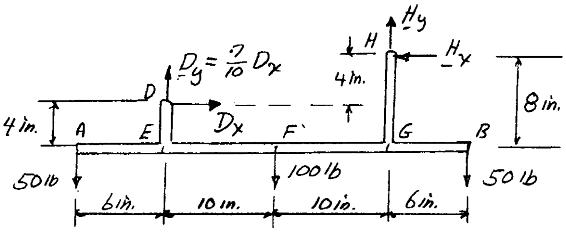
SOLUTION

Slope of cable *CD* is


C
14 in.
10
7
10 in.
D
H
4 in.
8 in.
A
B
20 in



$$\therefore \frac{D_y}{7} = \frac{D_x}{10}$$


H_y
H
H_x
$D_y = \frac{7}{10} D_x$
4 in.
8 in.
D
D_x
4 in.
A
E
F
G
B
50 lb
100 lb
50 lb
6 in.
10 in.
10 in.
6 in.

$$+\circlearrowright$$

$$\Sigma M_H = 0\!: \quad (50\text{ lb})(26\text{ in.}) + D_x(4\text{ in.}) - \left(\frac{7}{10} D_x\right)(20\text{ in.}) + (100\text{ lb})(10\text{ in.}) - (50\text{ lb})(6\text{ in.}) = 0$$

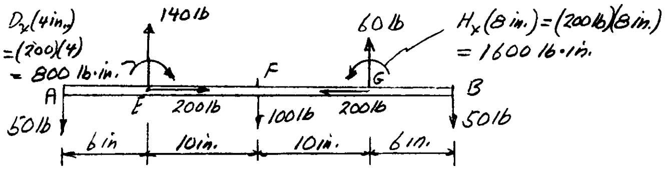
$$2000 + (4 - 14)D_x = 0 \qquad \mathbf{D}_x = 200\text{ lb} \longrightarrow$$

$$D_y = \frac{7}{10} D_x = \frac{7}{10}(200) \qquad \mathbf{D}_y = 140\text{ lb} \uparrow$$

$$\overset{+}{\longrightarrow} \Sigma F_x = 0\!: \quad 200\text{ lb} - H_x = 0 \qquad H_x = 200\text{ lb} \longleftarrow$$

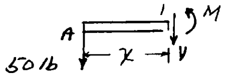
$$+\!\uparrow \Sigma F_y = 0\!: \quad 140 - 50 - 100 - 50 + H_y = 0 \qquad H_y = 60\text{ lb} \uparrow$$

Equivalent loading on straight part of beam *AB*.


$D_x(4\text{ in.})$
$=(200)(4)$
$= 800$ lb·in.
140 lb
60 lb
$H_x(8\text{ in.}) = (200\text{ lb})(8\text{ in.})$
$= 1600$ lb·in.
A
E
F
G
B
200 lb
100 lb
200 lb
50 lb
50 lb
6 in
10 in.
10 in.
6 in.

PROBLEM 7.51 (Continued)

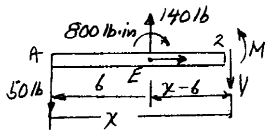
From *A* to *E*:



$$\Sigma F_y = 0: \quad V = -50 \text{ lb}$$

$$\Sigma M_1 = 0: \quad M = -50x$$

From *E* to *F*:


800 lb·in
140 lb
2
M
A
50 lb
6
E
x−6
V
x

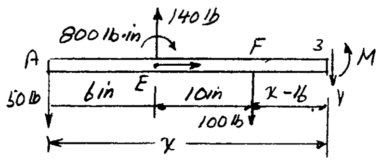
$$+\uparrow \Sigma F_y = 0: \quad -50 + 140 - V = 0$$

$$V = +90 \text{ lb}$$

$$+\circlearrowleft \Sigma M_2 = 0: \quad (50 \text{ lb})x - (140 \text{ lb})(x - 6) - 800 \text{ lb} \cdot \text{in.} + M = 0$$

$$M = -40 + 90x$$

From *F* to *G*:


140 lb
800 lb·in
F
3
M
A
50 lb
6 in
E
10 in
x−16
V
100 lb
x

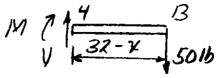
$$+\uparrow \Sigma F_y = 0: \quad -50 + 140 - 100 - V = 0$$

$$V = -10 \text{ lb}$$

$$+\circlearrowleft \Sigma M_3 = 0: \quad 50x - 140(x - 6) + 100(x - 16) - 800 + M = 0$$

$$M = 1560 - 10x$$

From *G* to *B*:

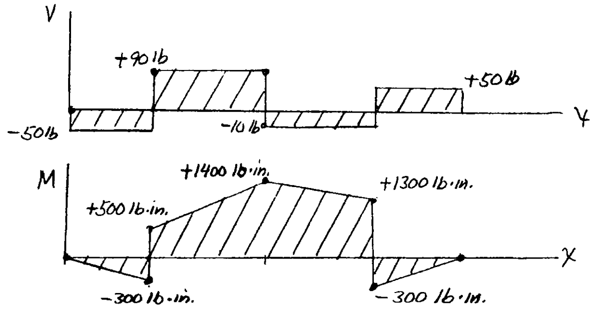

M
4
B
V
32 − x
50 lb

$$+\uparrow \Sigma F_y = 0: \quad V - 50 = 0$$

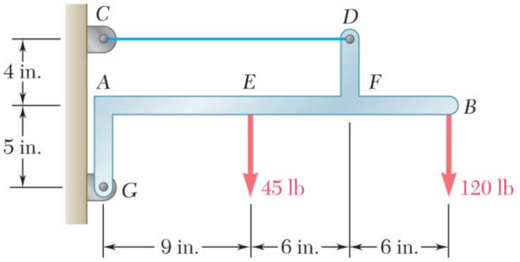
$$V = 50 \text{ lb}$$

$$+\circlearrowleft \Sigma M_4 = 0: \quad -M - (50)(32 - x) = 0$$

$$M = -1600 + 50x$$



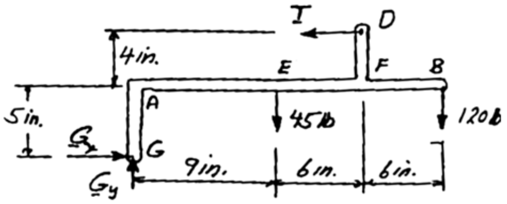
$$|V|_{\max} = 90.0 \text{ lb}; \quad |M|_{\max} = 1400 \text{ lb} \cdot \text{in.} \blacktriangleleft$$


C
D
4 in.
A
E
F
B
5 in.
G
45 lb
120 lb
9 in.
6 in.
6 in.

PROBLEM 7.52

Draw the shear and bending-moment diagrams for the beam *AB*, and determine the maximum absolute values of the shear and bending moment.

SOLUTION

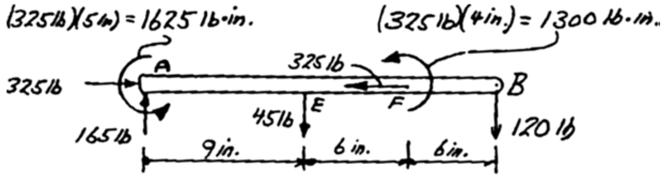

T
D
4in.
E
F
B
A
5in.
45lb
120lb
Gx
G
9in.
6in.
6in.
Gy

$$+\circlearrowleft \Sigma M_G = 0: \quad T(9\text{ in.}) - (45\text{ lb})(9\text{ in.}) - (120\text{ lb})(21\text{ in.}) = 0 \qquad T = 325\text{ lb}$$

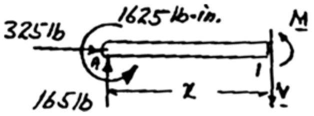
$$\xrightarrow{+} \Sigma F_x = 0: \quad -325\text{ lb} + G_x = 0 \qquad \mathbf{G}_x = 325\text{ lb} \longrightarrow$$

$$+\uparrow \Sigma F_y = 0: \quad G_y - 45\text{ lb} - 120\text{ lb} = 0 \qquad \mathbf{G}_y = 165\text{ lb}\uparrow$$

Equivalent loading on straight part of beam *AB*


(325lb)(5in) = 1625 lb·in.
(325lb)(4in.) = 1300 lb·in.
A
325lb
325lb
B
E
F
45lb
120lb
165lb
9in.
6 in.
6in.

From *A* to *E*:

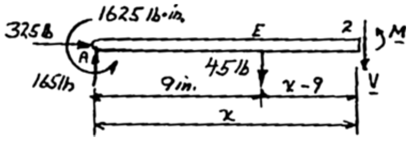

1625 lb·in.
M
325lb
A
1
165lb
x
V

$$\Sigma F_y = 0: \quad V = +165\text{ lb}$$

$$+\circlearrowleft \Sigma M_1 = 0: \quad +1625\text{ lb}\cdot\text{in.} - (165\text{ lb})x + M = 0$$

$$M = -1625 + 165x$$

From *E* to *F*:


1625 lb·in.
325lb
E
2
M
A
45lb
165lb
9in.
x-9
V
x

$$+\uparrow \Sigma F_y = 0: \quad 165 - 45 - V = 0$$

$$V = +120\text{ lb}$$

$$+\circlearrowleft \Sigma M_2 = 0: \quad +1625\text{ lb}\cdot\text{in.} - (165\text{ lb})x + (45\text{ lb})(x-9) + M = 0$$

$$M = -1220 + 120x$$

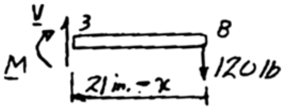
PROBLEM 7.52 (Continued)

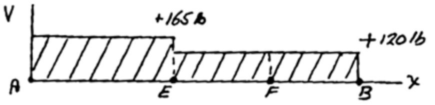
From *F* to *B*:

$$\Sigma F_y = 0 \quad V = +120\text{ lb}$$

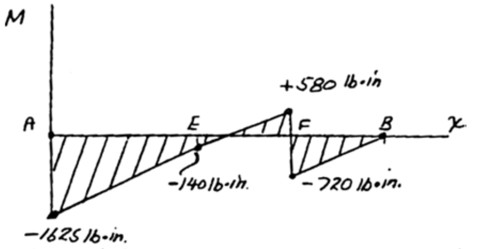
$$+\circlearrowleft \Sigma M_3 = 0: \quad -(120)(21-x) - M = 0$$

$$M = -2520 + 120x$$

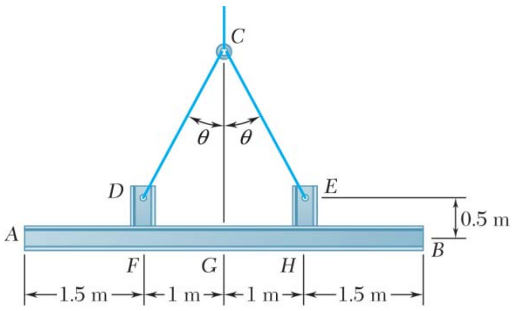




$$|V|_{\max} = 165.0\text{ lb} \blacktriangleleft$$



$$|M|_{\max} = 1625\text{ lb}\cdot\text{in.} \blacktriangleleft$$

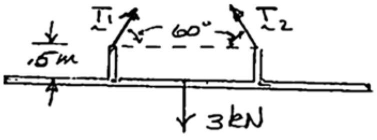


PROBLEM 7.53

Two small channel sections *DF* and *EH* have been welded to the uniform beam *AB* of weight *W*= 3 kN to form the rigid structural member shown. This member is being lifted by two cables attached at *D* and *E*. Knowing that $\theta = 30°$ and neglecting the weight of the channel sections, (*a*) draw the shear and bending-moment diagrams for beam *AB*, (*b*) determine the maximum absolute values of the shear and bending moment in the beam.

SOLUTION

FBD Beam + channels:



(*a*) By symmetry: $T_1 = T_2 = T$

$$+\uparrow \Sigma F_y = 0: \quad 2T \sin 60° - 3 \text{ kN} = 0$$

$$T = \frac{3}{\sqrt{3}} \text{ kN}$$

$$T_{1x} = \frac{3}{2\sqrt{3}}$$

$$T_{1y} = \frac{3}{2} \text{ kN}$$

FBD Beam:

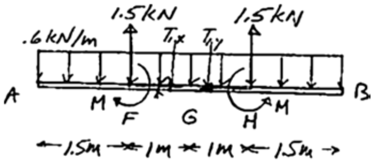
$$M = (0.5 \text{ m})\frac{3}{2\sqrt{3}} \text{ kN}$$

$$= 0.433 \text{ kN} \cdot \text{m}$$

With cable force replaced by equivalent force-couple system at *F* and *G*

Shear Diagram: *V* is piecewise linear

$$\left(\frac{dV}{dx} = -0.6 \text{ kN/m}\right) \text{with } 1.5 \text{ kN}$$



discontinuities at *F* and *H*.

$$V_{F^-} = -(0.6 \text{ kN/m})(1.5 \text{ m}) = 0.9 \text{ kN}$$

V increases by 1.5 kN to $+0.6$ kN at F^+

$$V_G = 0.6 \text{ kN} - (0.6 \text{ kN/m})(1 \text{ m}) = 0$$

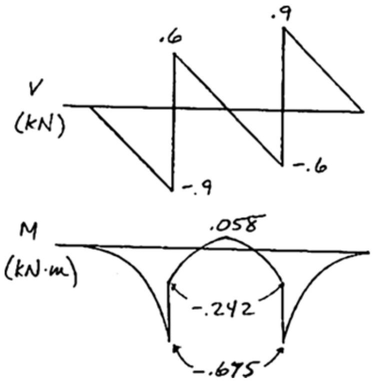
Finish by invoking symmetry

PROBLEM 7.53 (Continued)

Moment diagram: M is piecewise parabolic

$$\left(\frac{dM}{dx}\text{ decreasing with } V\right)$$

with discontinuities of .433 kN at F and H.



$$M_{F^-} = -\frac{1}{2}(0.9\text{ kN})(1.5\text{ m})$$
$$= -0.675\text{ kN}\cdot\text{m}$$

M increases by 0.433 kN m to –0.242 kN · m at F^+

$$M_G = -0.242\text{ kN}\cdot\text{m} + \frac{1}{2}(0.6\text{ kN})(1\text{ m})$$
$$= 0.058\text{ kN}\cdot\text{m}$$

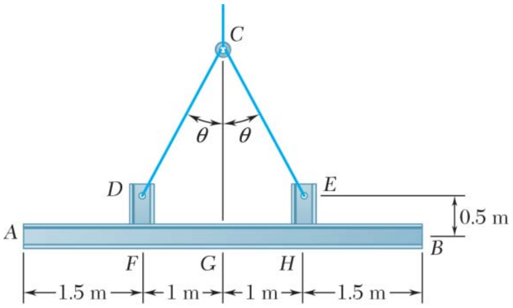
Finish by invoking symmetry

(*b*) $|V|_{\max} = 900\text{ N}$ ◀

at F^- and G^+

$|M|_{\max} = 675\text{ N}\cdot\text{m}$ ◀

at F and G

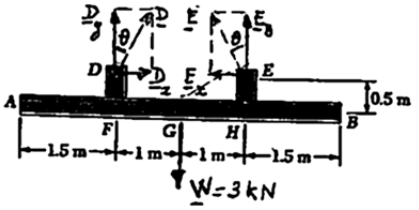


PROBLEM 7.54

Solve Problem 7.53 when $\theta = 60°$.

PROBLEM 7.53 Two small channel sections *DF* and *EH* have been welded to the uniform beam *AB* of weight $W = 3$ kN to form the rigid structural member shown. This member is being lifted by two cables attached at *D* and *E*. Knowing that $\theta = 30°$ and neglecting the weight of the channel sections, (*a*) draw the shear and bending-moment diagrams for beam *AB*, (*b*) determine the maximum absolute values of the shear and bending moment in the beam.

SOLUTION



Free body: Beam and channels

From symmetry:

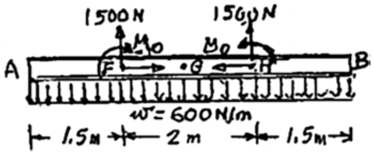
$$E_y = D_y$$

Thus: $$E_x = D_x = D_y \tan\theta \qquad (1)$$

$$+\!\uparrow \Sigma F_y = 0{:}\quad D_y + E_y - 3\text{ kN} = 0 \qquad \mathbf{D}_y = \mathbf{E}_y = 1.5\text{ kN}\uparrow \triangleleft$$

From (1): $$\mathbf{D}_x = (1.5\text{ kN})\tan\theta \longrightarrow \quad \mathbf{E} = (1.5\text{ kN})\tan\theta \longleftarrow \triangleleft$$

We replace the forces at *D* and *E* by equivalent force-couple systems at *F* and *H*, where



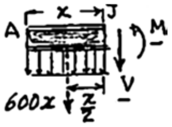
$$M_0 = (1.5\text{ kN}\tan\theta)(0.5\text{ m}) = (750\text{ N}\cdot\text{m})\tan\theta \qquad (2)$$

We note that the weight of the beam per unit length is

$$w = \frac{W}{L} = \frac{3\text{ kN}}{5\text{ m}} = 0.6\text{ kN/m} = 600\text{ N/m}$$

(*a*) Shear and bending moment diagrams

From *A* to *F*:



$$+\!\uparrow \Sigma F_y = 0{:}\quad -V - 600x = 0 \quad V = (-600x)\text{ N}$$

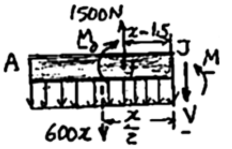
$$+\circlearrowleft \Sigma M_J = 0{:}\quad M + (600x)\frac{x}{2} = 0, \quad M = (-300x^2)\text{ N}\cdot\text{m}$$

For $x = 0$: $$V_A = M_A = 0 \triangleleft$$

For $x = 1.5$ m: $$V_F = -900\text{ N}, \quad M_F = -675\text{ N}\cdot\text{m} \triangleleft$$

PROBLEM 7.54 (Continued)

From *F* to *H*:



$$+\uparrow \Sigma F_y = 0: \quad 1500 - 600x - V = 0$$

$$V = (1500 - 600x)\ \text{N}$$

$$+\circlearrowleft \Sigma M_J = 0: \quad M + (600x)\frac{x}{2} - 1500(x - 1.5) - M_0 = 0$$

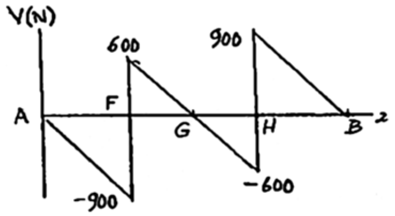
$$M = M_0 - 300x^2 + 1500(x - 1.5)\ \text{N}\cdot\text{m}$$

For $x = 1.5$ m: $V_F = +600$ N, $M_F = (M_0 - 675)$ N·m ◁

For $x = 2.5$ m: $V_G = 0$, $M_G = (M_0 - 375)$ N·m ◁

From *G* to *B*, The *V* and *M* diagrams will be obtained by symmetry,

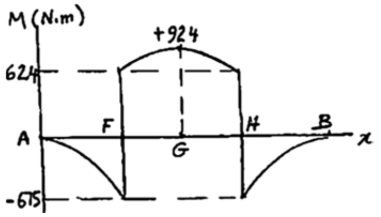
(*b*) $|V|_{\max} = 900$ N ◀



Making $\theta = 60°$ in Eq. (2): $M_0 = 750 \tan 60° = 1299$ N·m

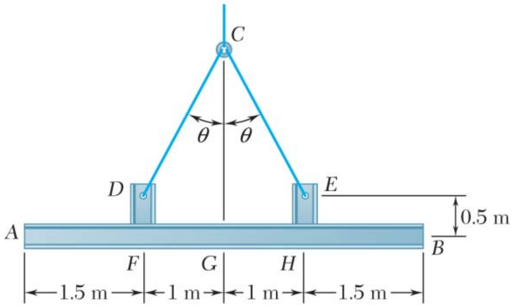
Thus, just to the right of *F*: $M = 1299 - 675 = 624$ N·m ◁

and $M_G = 1299 - 375 = 924$ N·m ◁



(*b*) $|V|_{\max} = 900$ N ◀

$|M|_{\max} = 924$ N·m ◀



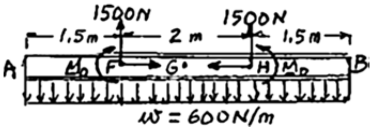
PROBLEM 7.55

For the structural member of Problem 7.53, determine (*a*) the angle θ for which the maximum absolute value of the bending moment in beam *AB* is as small as possible, (*b*) the corresponding value of $|M|_{\max}$. (*Hint:* Draw the bending-moment diagram and then equate the absolute values of the largest positive and negative bending moments obtained.)

PROBLEM 7.53 Two small channel sections *DF* and *EH* have been welded to the uniform beam *AB* of weight $W = 3$ kN to form the rigid structural member shown. This member is being lifted by two cables attached at *D* and *E*. Knowing that $\theta = 30°$ and neglecting the weight of the channel sections, (*a*) draw the shear and bending-moment diagrams for beam *AB*, (*b*) determine the maximum absolute values of the shear and bending moment in the beam.

SOLUTION

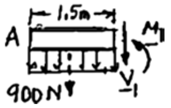
See solution of Problem 7.50 for reduction of loading or beam *AB* to the following:



where $$M_0 = (750\ \text{N}\cdot\text{m})\tan\theta \triangleleft$$

[Equation (2)]

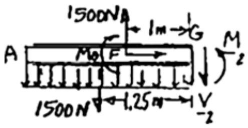
The largest negative bending moment occurs Just to the left of *F*:



$$+\circlearrowleft \Sigma M_1 = 0: \quad M_1 + (900\ \text{N})\left(\frac{1.5\ \text{m}}{2}\right) = 0 \qquad M_1 = -675\ \text{N}\cdot\text{m} \triangleleft$$

The largest positive bending moment occurs

At *G*:



$$+\circlearrowleft \Sigma M_2 = 0: \quad M_2 - M_0 + (1500\ \text{N})(1.25\ \text{m} - 1\ \text{m}) = 0$$

$$M_2 = M_0 - 375\ \text{N}\cdot\text{m} \triangleleft$$

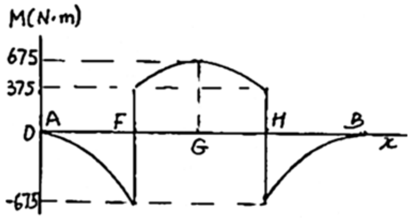
Equating M_2 and $-M_1$:

$$M_0 - 375 = +675$$

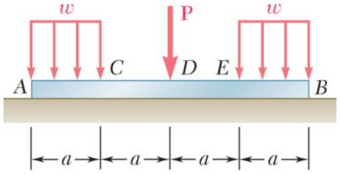
$$M_0 = 1050\ \text{N}\cdot\text{m}$$

PROBLEM 7.55 (Continued)

(*a*) From Equation (2): $\tan\theta = \dfrac{1050}{750} = 1.400$ $\theta = 54.5°$ ◀



(*b*) $|M|_{\max} = 675\ \text{N}\cdot\text{m}$ ◀

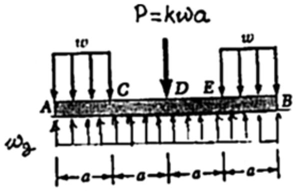


PROBLEM 7.56

For the beam of Problem 7.43, determine (*a*) the ratio *k*=*P*/*wa* for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding value of $|M|_{\max}$. (See hint for Problem 7.55.)

PROBLEM 7.43 Assuming the upward reaction of the ground on beam *AB* to be uniformly distributed and knowing that *P* =*wa*, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION



Free body: Entire beam

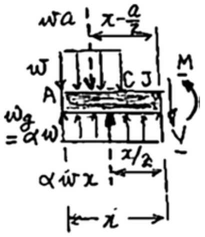
$$+\uparrow \Sigma F_y = 0: \quad w_g(4a) - 2wa - kwa = 0$$

$$w_g = \frac{w}{4}(2+k)$$

Setting $$\frac{w_g}{w} = \alpha \quad (1)$$

We have $$k = 4\alpha - 2 \quad (2)$$

Minimum value of B.M. For *M* to have negative values, we must have $w_g < w$. We verify that *M* will then be negative and keep decreasing in the portion *AC* of the beam. Therefore, $M_{\min}$ will occur between *C* and *D*.



From *C* to *D*:

$$+\circlearrowleft \Sigma M_J = 0: \quad M + wa\left(x - \frac{a}{2}\right) - \alpha wx\left(\frac{x}{2}\right) = 0$$

$$M = \frac{1}{2}w(\alpha x^2 - 2ax + a^2) \quad (3)$$

We differentiate and set $\frac{dM}{dx} = 0$: $$\alpha x - a = 0 \qquad x_{\min} = \frac{a}{\alpha} \quad (4)$$

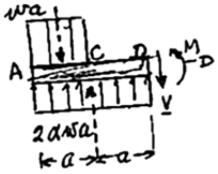
Substituting in (3):

$$M_{\min} = \frac{1}{2}wa^2\left(\frac{1}{\alpha} - \frac{2}{\alpha} + 1\right)$$

$$M_{\min} = -wa^2\frac{1-\alpha}{2\alpha} \quad (5)$$

PROBLEM 7.56 (Continued)

Maximum value of bending moment occurs at D



$$\circlearrowleft + \Sigma M_D = 0: \quad M_D + wa\left(\frac{3a}{2}\right) - (2\alpha wa)a = 0$$

$$M_{\max} = M_D = wa^2\left(2\alpha - \frac{3}{2}\right) \tag{6}$$

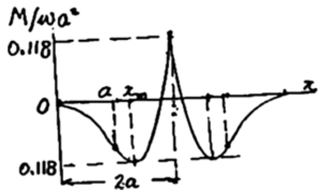
Equating $-M_{\min}$ and $M_{\max}$:

$$wa^2\frac{1-\alpha}{2\alpha} = wa^2\left(2\alpha - \frac{3}{2}\right)$$

$$4\alpha^2 - 2\alpha - 1 = 0$$

$$\alpha = \frac{2+\sqrt{20}}{8} \qquad \alpha = \frac{1+\sqrt{5}}{4} = 0.809$$

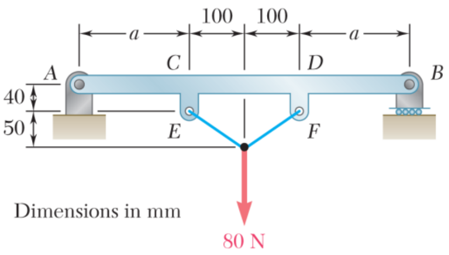
(a) Substitute in (2): $k = 4(0.809) - 2$ $k = 1.236$ ◀



(b) Substitute for α in (5):

$$|M|_{\max} = -M_{\min} = -wa^2\frac{1-0.809}{2(0.809)} \qquad |M|_{\max} = 0.1180wa^2$$ ◀

Substitute for α in (4): $x_{\min} = \frac{a}{0.809}1.236a$ ◁


100
100
a
a
A
C
D
B
40
50
E
F
Dimensions in mm
80 N

PROBLEM 7.57

Determine (*a*) the distance *a* for which the maximum absolute value of the bending moment in beam *AB* is as small as possible, (*b*) the corresponding value of $|M|_{\max}$. (See hint for Prob. 7.55.)

SOLUTION

Tension in cords supporting load:

$$\uparrow \Sigma F_y = 0: \quad 2T_y - 80\text{ N} = 0; \qquad \frac{T_x}{T_y} = \frac{100\text{ mm}}{50\text{ mm}}$$

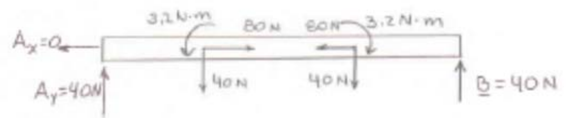
$$T_y = 40\text{ N}$$

$$T_x = 80\text{ N}$$

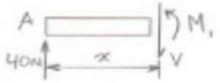
Translate cord tensions to straight part of beam *AB*at *C* and *D*:

$$M_C = M_D = (80\text{ N})(0.04\text{ m}) = 3.2\text{ N}\cdot\text{m}$$

FBD of beam *AB*:


$A_x=0$
$A_y=40N$
3.2 N·m
80 N
80 N
3.2 N·m
40 N
40 N
B = 40 N

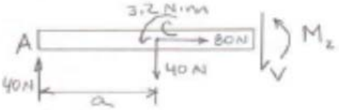
From *A* to *C*:



$$\uparrow \Sigma F_y = 0: \quad 40 - V = 0 \qquad V = 40\text{ N}$$

$$+\circlearrowleft \Sigma M = 0: \quad M_1 - (40\text{ N})x = 0 \qquad M_1 = 40x$$

From *C* to *D*:


3.2 N·m
C
A
80 N
40 N
40N
a
M_2
V

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PROBLEM 7.57 (Continued)

$\uparrow \Sigma F_y = 0$: $\quad V = 0$

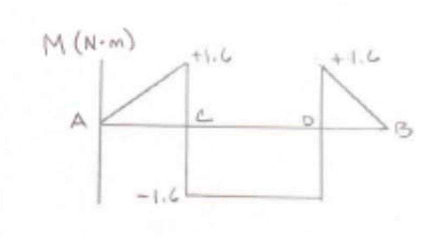
$+\circlearrowleft \Sigma M = 0$: $\quad 3.2\ \text{N}\cdot\text{m} + M_2 - (40\ \text{N})a = 0 \qquad M_2 = 40a - 3.2$

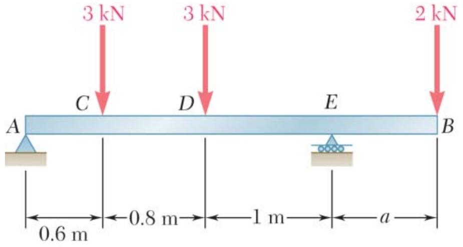
Equating the absolute values of M_1 and M_2 :

$$|40a| = |40a - 3.2|$$
$$|40a| = |-(-40a + 3.2)|$$
$$40a = -40a + 3.2$$
$$80a = 3.2$$
$$a = 0.04\ \text{m}$$

(*a*) $a = 40.0\ \text{mm}$ ◀

(*b*) Substituting: $\quad |M_{\text{max}}| = M_2 = 40\text{N}(.04\ \text{m}) - 3.2\ \text{N}\cdot\text{m} \qquad |M|_{\text{max}} = 1.600\ \text{N}\cdot\text{m}$ ◀



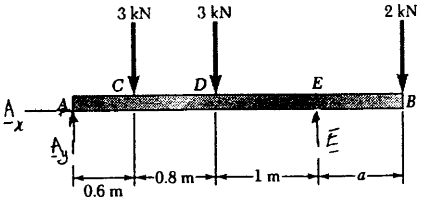


PROBLEM 7.58

For the beam and loading shown, determine (*a*) the distance *a* for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding value of $|M|_{\max}$. (See hint for Problem 7.55.)

SOLUTION

Free body: Entire beam

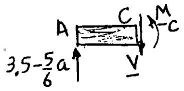


$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\circlearrowleft \Sigma M_E = 0: \quad -A_y(2.4) + (3)(1.8) + 3(1) - (2)a = 0$$

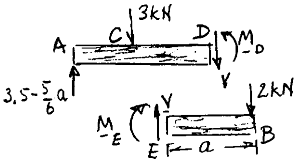
$$A_y = 3.5\text{ kN} - \frac{5}{6}a \qquad \mathbf{A} = 3.5\text{ kN} - \frac{5}{6}a \uparrow \triangleleft$$

Free body: *AC*



$$+\circlearrowleft \Sigma M_C = 0: \quad M_C - \left(3.5 - \frac{5}{6}a\right)(0.6\text{ m}) = 0, \qquad M_C = +2.1 - \frac{a}{2} \triangleleft$$

Free body: *AD*



$$+\circlearrowleft \Sigma M_D = 0: \quad M_D - \left(3.5 - \frac{5}{6}a\right)(1.4\text{ m}) + (3\text{ kN})(0.8\text{ m}) = 0$$

$$M_D = +2.5 - \frac{7}{6}a \triangleleft$$

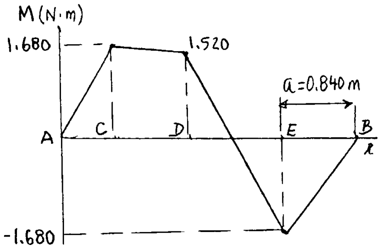
Free body: *EB*

$$+\circlearrowleft \Sigma M_E = 0: \quad -M_E - (2\text{ kN})a = 0 \qquad M_E = -2a \triangleleft$$

We shall assume that $M_C > M_D$ and, thus, that $M_{\max} = M_C$.

We set $M_{\max} = |M_{\min}|$ or $M_C = |M_E| = 2.1 - \frac{a}{2} = 2a$ $\qquad a = 0.840\text{ m} \triangleleft$

$$|M|_{\max} = M_C = |M_E| = 2a = 2(0.840) \qquad |M|_{\max} = 1.680\text{ N}\cdot\text{m} \triangleleft$$



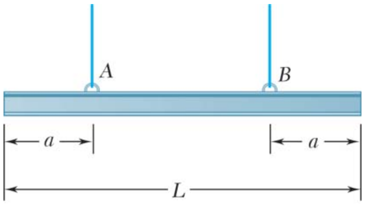
We must check our assumption.

$$M_D = 2.5 - \frac{7}{6}(0.840) = 1.520\text{ N}\cdot\text{m}$$

Thus, $M_C > M_D$, O.K.

The answers are (*a*) $a = 0.840\text{ m} \blacktriangleleft$

(*b*) $|M|_{\max} = 1.680\text{ N}\cdot\text{m} \blacktriangleleft$

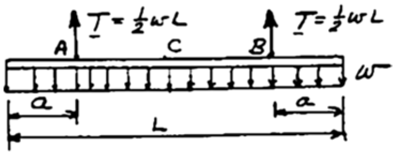


PROBLEM 7.59

A uniform beam is to be picked up by crane cables attached at *A* and *B*. Determine the distance *a* from the ends of the beam to the points where the cables should be attached if the maximum absolute value of the bending moment in the beam is to be as small as possible. (*Hint:* Draw the bending-moment diagram in terms of *a*, *L*, and the weight *w* per unit length, and then equate the absolute values of the largest positive and negative bending moments obtained.)

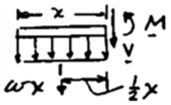
SOLUTION

w= weight per unit length


T = ½wL
T = ½wL
A
C
B
w
a
a
L

To the left of *A*:

$$+\circlearrowleft \Sigma M_1 = 0: \quad M + wx\left(\frac{x}{2}\right) = 0$$

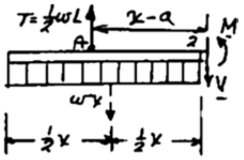


$$M = -\frac{1}{2}wx^2$$

$$M_A = -\frac{1}{2}wa^2$$

Between *A* and *B*:

$$+\circlearrowleft \Sigma M_2 = 0: \quad M - \left(\frac{1}{2}wL\right)(x-a) + (wx)\left(\frac{1}{2}x\right) = 0$$

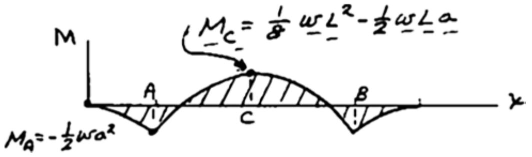

T=½wL
x-a
2
M
A
V
wx
½x
½x

$$M = -\frac{1}{2}wx^2 + \frac{1}{2}wLx - \frac{1}{2}wLa$$

At center *C*:

$$x = \frac{L}{2}$$

$$M_C = -\frac{1}{2}w\left(\frac{L}{2}\right)^2 + \frac{1}{2}wL\left(\frac{L}{2}\right) - \frac{1}{2}wLa$$


M
$M_C = \frac{1}{8}wL^2 - \frac{1}{2}wLa$
A
C
B
x
$M_A = -\frac{1}{2}wa^2$

PROBLEM 7.59 (Continued)

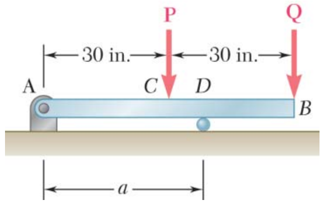
We set $|M_A| = |M_C|$: $\left|-\frac{1}{2}wa^2\right| = \left|\frac{1}{8}wL^2 - \frac{1}{2}wLa\right| + \frac{1}{2}wa^2 = \frac{1}{8}wL^2 - \frac{1}{2}wLa$

$$a^2 + La - 0.25L^2 = 0$$

$$a = \frac{1}{2}(L \pm \sqrt{L^2 + L^2}) = \frac{1}{2}(\sqrt{2} - 1)L$$

$$M_{\max} = \frac{1}{2}w(0.207L)^2 = 0.0214wL^2$$

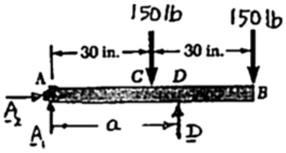
$a = 0.207L$ ◀


P
Q
30 in.
30 in.
A
C
D
B
a

PROBLEM 7.60

Knowing that $P = Q = 150$ lb, determine (*a*) the distance *a* for which the maximum absolute value of the bending moment in beam *AB* is as small as possible, (*b*) the corresponding value of $|M|_{\max}$. (See hint for Problem 7.55.)

SOLUTION

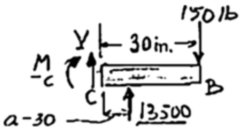

150 lb
150 lb
30 in.
30 in.
A
C
D
B
A_x
A_y
a
D

Free body: Entire beam

$$+\circlearrowleft \Sigma M_A = 0: \quad Da - (150)(30) - (150)(60) = 0$$

$$D = \frac{13{,}500}{a} \qquad ◁$$

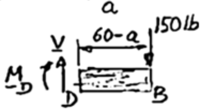
Free body: *CB*


150 lb
V
30 in.
M_C
C
B
a−30
$\frac{13{,}500}{a}$

$$+\circlearrowleft \Sigma M_C = 0: \quad -M_C - (150)(30) + \frac{13{,}500}{a}(a - 30) = 0$$

$$M_C = 9000\left(1 - \frac{45}{a}\right) \qquad ◁$$

Free body: *DB*


V
60−a
150 lb
M_D
D
B

$$+\circlearrowleft \Sigma M_D = 0: \quad -M_D - (150)(60 - a) = 0$$

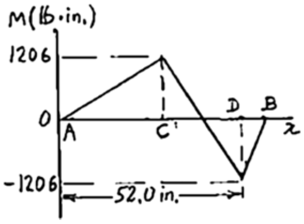
$$M_D = -150(60 - a) \qquad ◁$$

(*a*) We set

$$M_{\max} = |M_{\min}| \quad \text{or} \quad M_C = -M_D: \quad 9000\left(1 - \frac{45}{a}\right) = 150(60 - a)$$

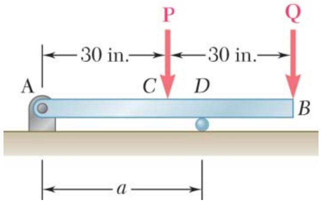
$$60 - \frac{2700}{a} = 60 - a$$

$$a^2 = 2700 \quad a = 51.96 \text{ in.} \qquad a = 52.0 \text{ in.} ◀$$



(*b*) $|M|_{\max} = -M_D = 150(60 - 51.96)$

$$|M|_{\max} = 1206 \text{ lb} \cdot \text{in.} ◀$$

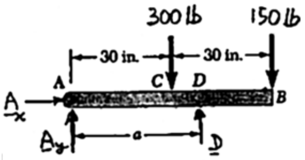

P
Q
30 in.
30 in.
A
C
D
B
a

PROBLEM 7.61

Solve Problem 7.60 assuming that $P = 300$ lb and $Q = 150$lb.

PROBLEM 7.60 Knowing that $P = Q = 150$ lb, determine (*a*) the distance *a* for which the maximum absolute value of the bending moment in beam *AB* is as small as possible, (*b*) the corresponding value of $|M|_{\max}$. (See hint for Problem 7.55.)

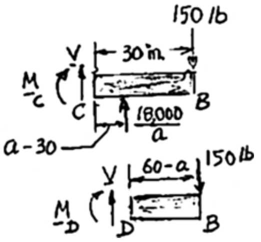
SOLUTION


300 lb
150 lb
30 in.
30 in.
A
C
D
B
A_x
A_y
a
D

Free body: Entire beam

$$+\circlearrowleft \Sigma M_A = 0: \quad Da - (300)(30) - (150)(60) = 0$$

$$D = \frac{18{,}000}{a}$$ ◁


150 lb
V
30 in.
M_C
C
B
18,000/a
a-30
V
60-a
150 lb
M_D
D
B

Free body: *CB*

$$+\circlearrowleft \Sigma M_C = 0: \quad -M_C - (150)(30) + \frac{18{,}000}{a}(a-30) = 0$$

$$M_C = 13{,}500\left(1 - \frac{40}{a}\right)$$ ◁

Free body: *DB*

$$+\circlearrowleft \Sigma M_D = 0: \quad -M_D - (150)(60-a) = 0$$

$$M_D = -150(60-a)$$ ◁

(*a*) We set

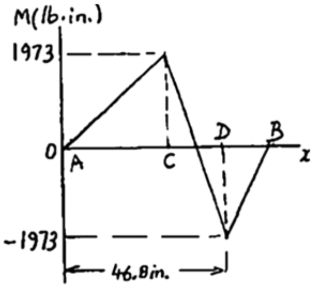
$$M_{\max} = |M_{\min}| \quad \text{or} \quad M_C = -M_D: \quad 13{,}500\left(1 - \frac{40}{a}\right) = 150(60-a)$$

$$90 - \frac{3600}{a} = 60 - a$$

$$a^2 + 30a - 3600 = 0$$

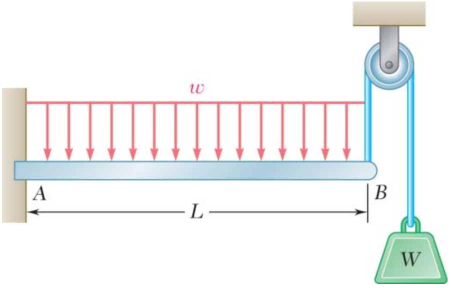
$$a = \frac{-30 + \sqrt{15.300}}{2} = 46.847$$

$$a = 46.8 \text{ in.}$$ ◀



(*b*) $$|M|_{\max} = -M_D = 150(60 - 46.847)$$

$$|M|_{\max} = 1973 \text{ lb} \cdot \text{in.}$$ ◀

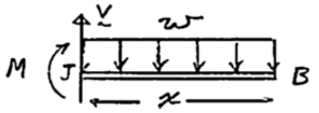


PROBLEM 7.62*

In order to reduce the bending moment in the cantilever beam *AB*, a cable and counterweight are permanently attached at end *B*. Determine the magnitude of the counterweight for which the maximum absolute value of the bending moment in the beam is as small as possible and the corresponding value of $|M|_{\max}$. Consider (*a*) the case when the distributed load is permanently applied to the beam, (*b*) the more general case when the distributed load may either be applied or removed.

SOLUTION

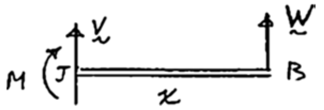
M due to distributed load:



$$\circlearrowleft \Sigma M_J = 0: \quad -M - \frac{x}{2}wx = 0$$

$$M = -\frac{1}{2}wx^2$$

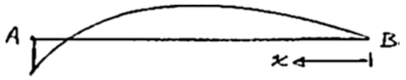
M due to counter weight:



$$\circlearrowleft \Sigma M_J = 0: \quad -M + xw = 0$$

$$M = wx$$

(*a*) **Both applied:**



$$M = W_x - \frac{w}{2}x^2$$

$$\frac{dM}{dx} = W - wx = 0 \text{ at } x = \frac{W}{w}$$

And here $M = \frac{W^2}{2w} > 0$ so $M_{\max}$; $M_{\min}$ must be at $x = L$

So $M_{\min} = WL - \frac{1}{2}wL^2$. For minimum $|M|_{\max}$ set $M_{\max} = -M_{\min}$,

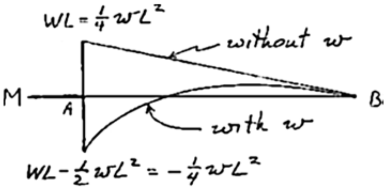
so $$\frac{W^2}{2w} = -WL + \frac{1}{2}wL^2 \quad \text{or} \quad W^2 + 2wLW - w^2L^2 = 0$$

$$W = -wL \pm \sqrt{2w^2L^2} \text{ (need+)} \qquad W = (\sqrt{2} - 1)wL = 0.414\, wL \blacktriangleleft$$

PROBLEM 7.62* (Continued)

(*b*) **w may be removed**

$$M_{\max} = \frac{W^2}{2w} = \frac{(\sqrt{2}-1)^2}{2} wL^2 \qquad M_{\max} = 0.0858\, wL^2 \blacktriangleleft$$


$WL = \frac{1}{4}wL^2$
without w
M
A
B
with w
$WL - \frac{1}{2}wL^2 = -\frac{1}{4}wL^2$

Without w, $M = Wx$

$$M_{\max} = WL \text{ at } A$$

With w (see Part *a*) $M = Wx - \frac{w}{2}x^2$

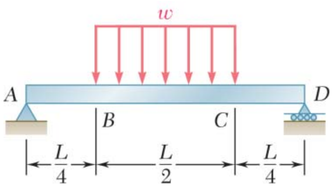
$$M_{\max} = \frac{W^2}{2w} \text{ at } x = \frac{W}{w}$$

$$M_{\min} = WL - \frac{1}{2}wL^2 \text{ at } x = L$$

For minimum $M_{\max}$, set $M_{\max}(\text{no } w) = -M_{\min}(\text{with } w)$

$$WL = -WL + \frac{1}{2}wL^2 \rightarrow W = \frac{1}{4}wL \rightarrow \qquad M_{\max} = \frac{1}{4}wL^2 \blacktriangleleft$$

With $W = \frac{1}{4}wL \blacktriangleleft$

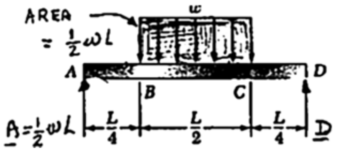

w
A
D
B
C
$\frac{L}{4}$
$\frac{L}{2}$
$\frac{L}{4}$

PROBLEM 7.63

Using the method of Section 7.6, solve Problem 7.29.

PROBLEM 7.29 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

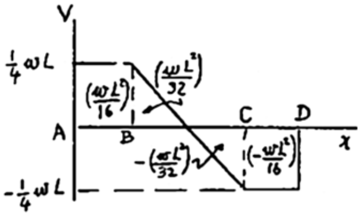

AREA
$= \frac{1}{2}wL$
w
A
D
B
C
$A = \frac{1}{2}wL$
$\frac{L}{4}$
$\frac{L}{2}$
$\frac{L}{4}$
D

Reactions at *A* and *D*

Because of the symmetry of the supports and loading.

$$A = D = \frac{1}{2}\left(w\frac{L}{2}\right) = \frac{1}{4}wL$$

$$\mathbf{A} = \mathbf{D} = \frac{1}{4}wL\uparrow \triangleleft$$

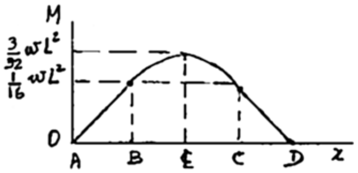

V
$\frac{1}{4}wL$
$\left(\frac{wL^2}{32}\right)$
$\left(\frac{wL^2}{16}\right)$
A
B
C
D
x
$-\left(\frac{wL^2}{32}\right)$
$\left(-\frac{wL^2}{16}\right)$
$-\frac{1}{4}wL$

Shear diagram

At *A*: $V_A = +\frac{1}{4}wL$

From *B* to *C*:

Oblique straight line $|V|_{\max} = \frac{1}{4}wL$ ◀

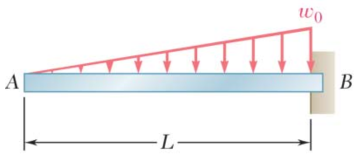

M
$\frac{3}{32}wL^2$
$\frac{1}{16}wL^2$
0
A
B
℄
C
D
x

Bending-moment diagram

At *A*: $M_A = 0$

From *B* to *C*: *ARC* of parabola $|M|_{\max} = \frac{3}{32}wL^2$ ◀

Since *V* has no discontinuity at *B*nor*C*, the slope of the parabola at these points is the same as the slope of the adjoining straight line segment.

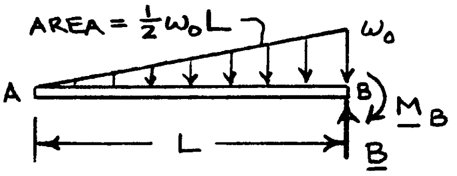


PROBLEM 7.64

Using the method of Section 7.6, solve Problem 7.30.

PROBLEM 7.30 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

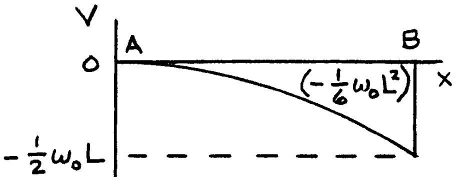
SOLUTION



Free body: Entire beam

$$+\uparrow \Sigma F_y = 0: \quad B - \frac{1}{2}(w_0)(L) = 0$$

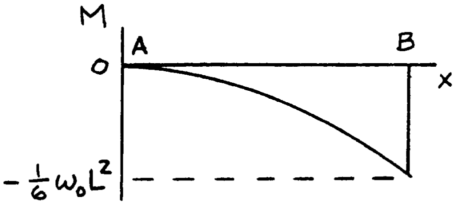
$$\mathbf{B} = \frac{1}{2}w_0L \uparrow$$



Shear diagram

$$+\circlearrowleft \Sigma M_B = 0: \quad \frac{1}{2}(w_0)(L)\left(\frac{L}{3}\right) - M_B = 0$$

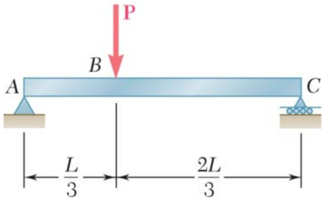
$$\mathbf{M}_B = \frac{1}{6}w_0L^2 \circlearrowright$$



Moment diagram

(*b*) $|V|_{\max} = w_0L/2$ ◀

$|M|_{\max} = w_0L^2/6$ ◀

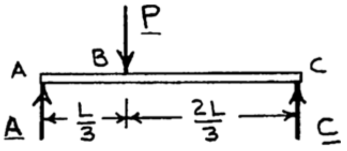

P
B
A
C
$\frac{L}{3}$
$\frac{2L}{3}$

PROBLEM 7.65

Using the method of Section 7.6, solve Problem 7.31.

PROBLEM 7.31 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION


P
A
B
C
A
$\frac{L}{3}$
$\frac{2L}{3}$
C

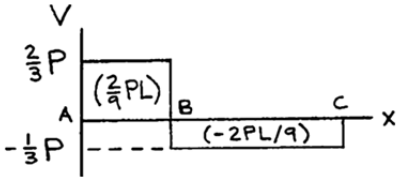
Free body: Entire beam

$$+\circlearrowleft \Sigma M_C = 0: \quad P\left(\frac{2L}{3}\right) - A(L) = 0$$

$$\mathbf{A} = \frac{2}{3}P \uparrow$$

$$+\uparrow \Sigma F_Y = 0: \quad \frac{2}{3}P - P + C = 0$$

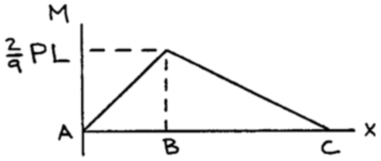
$$\mathbf{C} = \frac{1}{3}P \uparrow$$


V
$\frac{2}{3}P$
$(\frac{2}{9}PL)$
A
B
C
x
$(-2PL/9)$
$-\frac{1}{3}P$

Shear diagram

At *A*: $V_A = \frac{2}{3}P$

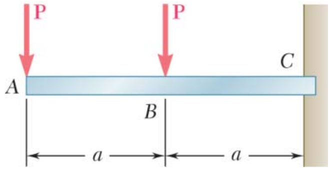
$$|V|_{\max} = \frac{2}{3}P \blacktriangleleft$$


M
$\frac{2}{9}PL$
A
B
C
x

Bending-moment diagram

At *A*: $M_A = 0$

$$|M|_{\max} = \frac{2}{9}PL \blacktriangleleft$$

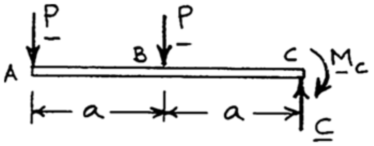


PROBLEM 7.66

Using the method of Section 7.6, solve Problem 7.32.

PROBLEM 7.32 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION



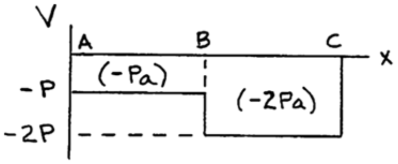
Free body: Entire beam

$$+\uparrow \Sigma F_y = 0: \quad C - P - P = 0$$

$$\mathbf{C} = 2P \uparrow$$

$$+\circlearrowleft \Sigma M_C = 0: \quad P(2a) + P(a) - M_C = 0$$

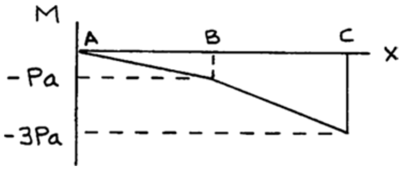
$$\mathbf{M}_C = 3Pa \circlearrowright$$



Shear diagram

At *A*: $V_A = -P$

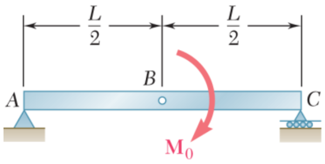
$$|V|_{\max} = 2P \blacktriangleleft$$



Bending-moment diagram

At *A*: $M_A = 0$

$$|M|_{\max} = 3Pa \blacktriangleleft$$

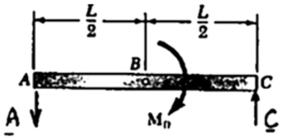


PROBLEM 7.67

Using the method of Section 7.6, solve Problem 7.33.

PROBLEM 7.33 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

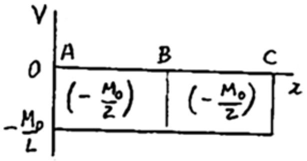

L/2
L/2
B
A
C
A
M_0
C

Free body: Entire beam

$$\Sigma F_y = 0: \quad A = C$$

$$+\circlearrowleft \Sigma M_C = 0: \quad Al - M_0 = 0$$

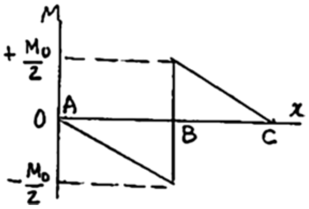
$$A = C = \frac{M_0}{L}$$


V
0
A
B
C
$(-\frac{M_0}{2})$
$(-\frac{M_0}{2})$
x
$-\frac{M_0}{L}$

Shear diagram

At *A*: $V_A = -\frac{M_0}{L}$

$$|V|_{\max} = \frac{M_0}{L} \blacktriangleleft$$

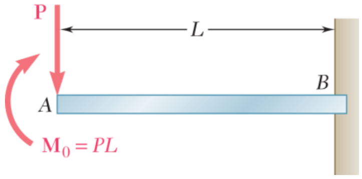

M
$+\frac{M_0}{2}$
0
A
x
B
C
$-\frac{M_0}{2}$

Bending-moment diagram

At *A*: $M_A = 0$

At *B*, *M* increases by M_0 on account of applied couple.

$$|M|_{\max} = M_0/2 \blacktriangleleft$$

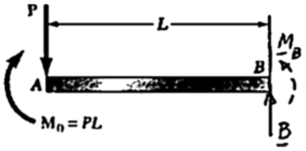


PROBLEM 7.68

Using the method of Section 7.6, solve Problem 7.34.

PROBLEM 7.34 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION



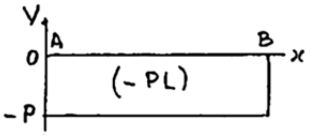
Free body: Entire beam

$$+\uparrow \Sigma F_y = 0: \quad B - P = 0$$

$$\mathbf{B} = P \uparrow$$

$$+\circlearrowleft \Sigma M_B = 0: \quad M_B - M_0 + PL = 0$$

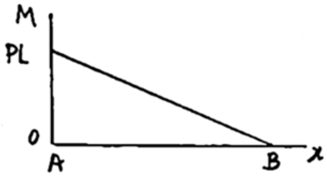
$$\mathbf{M}_B = 0$$



Shear diagram

At A: $V_A = -P$

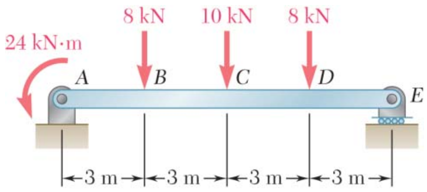
$$|V|_{\max} = P \blacktriangleleft$$



Bending-moment diagram

At A: $M_A = M_0 = PL$

$$|M|_{\max} = PL \blacktriangleleft$$


24 kN·m
8 kN
10 kN
8 kN
A
B
C
D
E
3 m
3 m
3 m
3 m

PROBLEM 7.69

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

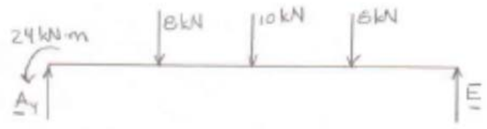
Reactions: $\Sigma F_x = 0 \quad A_x = 0$

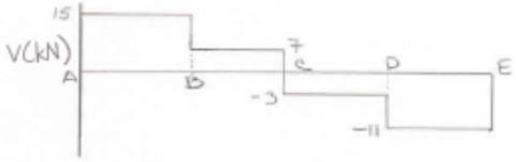
$$\overset{\curvearrowleft}{+}\Sigma M_A = 0\text{:} \quad 24\text{ kN}\cdot\text{m} - (8\text{ kN})(3\text{ m}) - (10\text{ kN})(6\text{ m}) - (8\text{ kN})(9\text{ m}) + E(12\text{ m}) = 0$$

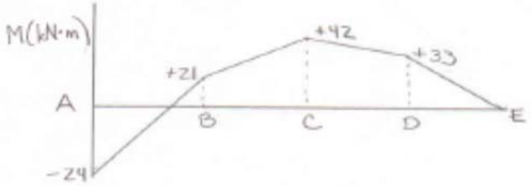
$$E = 11\text{ kN}\uparrow$$

$$\Sigma F_y = 0$$

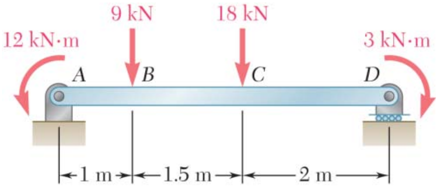
$$A = 15\text{ kN}\uparrow$$


24 kN·m
8kN
10kN
8kN
A_y
E





(*b*) $|V|_{\max} = 15.00\text{ kN};$ $|M|_{\max} = 42.0\text{ kN}\cdot\text{m}$ ◀


12 kN·m
9 kN
18 kN
3 kN·m
A
B
C
D
1 m
1.5 m
2 m

PROBLEM 7.70

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

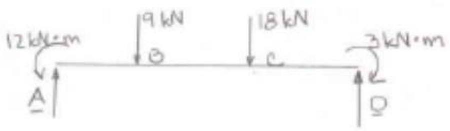
Reactions $\xrightarrow{+} \Sigma F_x = 0 \quad A_x = 0$

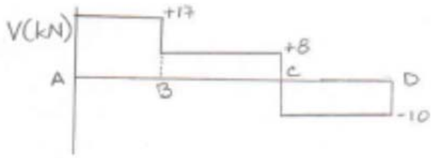
$$+\circlearrowleft \Sigma M_D = 0: \quad +12\text{ kN}\cdot\text{m} + (9\text{ kN})(3.5\text{ m}) + (18\text{ kN})(2\text{ m}) - 3\text{ kN}\cdot\text{m} - A_y(4.5\text{ m}) = 0$$

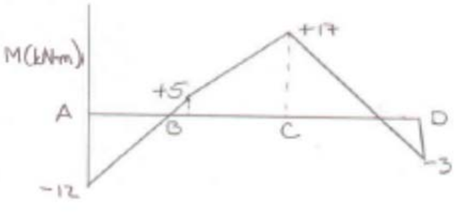
$$A_y = 17.00\text{ kN}\uparrow$$

$$+\uparrow \Sigma F_y = 0 \quad 17\text{ kN} - 9\text{ kN} - 18\text{ kN} + D = 0$$

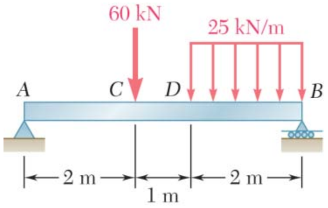
$$D = 10.00\text{kN}\uparrow$$


12 kN·m
9 kN
18 kN
3 kN·m
A
B
C
D





(*b*) $|V|_{\max} = 17.00\text{ kN};$ $|M|_{\max} = 17.00\text{ kN}\cdot\text{m}$ ◀

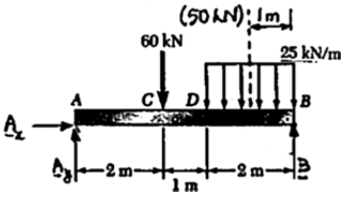


PROBLEM 7.71

Using the method of Section 7.6, solve Problem 7.39.

PROBLEM 7.39 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION



Free body: Beam

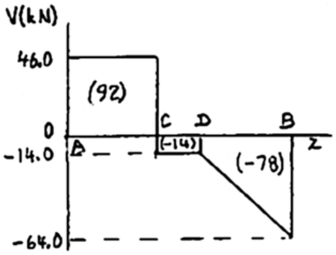
$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (60\text{ kN})(3\text{ m}) + (50\text{ kN})(1\text{ m}) - A_y(5\text{ m}) = 0$$

$$A_y = +46.0\text{ kN}\uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad B + 46.0\text{ kN} - 60\text{ kN} - 50\text{ kN} = 0$$

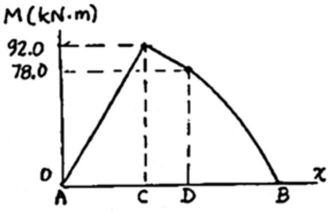
$$B = +64.0\text{ kN}\uparrow$$



Shear diagram

At *A*: $V_A = A_y = +46.0\text{ kN}$

$$|V|_{\max} = 64.0\text{ kN} \blacktriangleleft$$

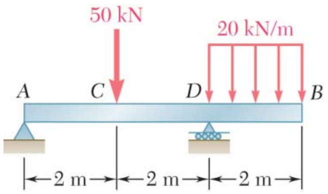


Bending-moment diagram

At *A*: $M_A = 0$

$$|M|_{\max} = 92.0\text{ kN}\cdot\text{m} \blacktriangleleft$$

Parabola from *D* to *B*. Its slope at *D* is same as that of straight-line segment *CD* since *V* has no discontinuity at *D*.



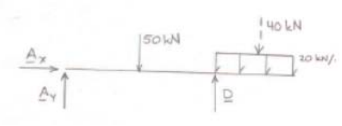
PROBLEM 7.72

Using the method of Section 7.6, solve Problem 7.40.

PROBLEM 7.40 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

Free body: Beam



$$\Sigma F_x = 0: \quad A_x = 0$$

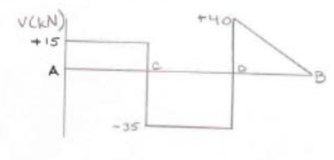
$$+\circlearrowleft \Sigma M_D = 0: \quad (50\text{ kN})(2\text{ m}) - (40\text{ kN})(1\text{ m}) - A_y(4\text{ m}) = 0$$

$$A_y = +15.00\text{ kN}\uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad D + 15\text{ kN} - 40\text{ kN} - 50\text{ kN} = 0$$

$$D = +75.0\text{ kN}\uparrow$$

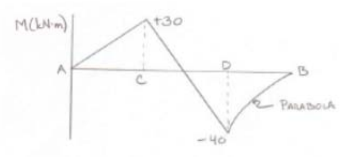
Shear diagram



At *A*: $V_A = A_y = +15.00\text{ kN}$

$$|V|_{\max} = 40.0\text{ kN} ◀$$

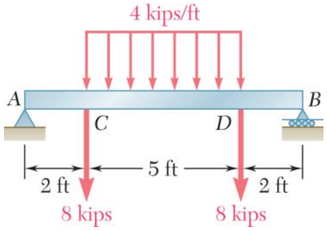
Bending-moment diagram



At *A*: $M_A = 0$

$$|M|_{\max} = 40.0\text{ kN}\cdot\text{m} ◀$$

Parabola from *D* to *B*.

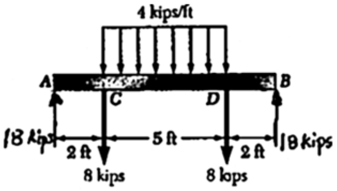

4 kips/ft
A
B
C
D
2 ft
5 ft
2 ft
8 kips
8 kips

PROBLEM 7.73

Using the method of Section 7.6, solve Problem 7.41.

PROBLEM 7.41 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

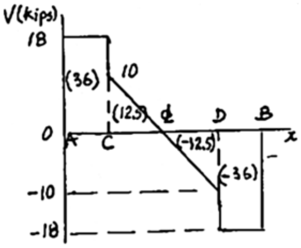

4 kips/ft
A
B
C
D
18 kips
2 ft
5 ft
2 ft
18 kips
8 kips
8 kips

Reactions at supports.

Because of the symmetry:

$$A = B = \frac{1}{2}(8 + 8 + 4 \times 5)\,\text{kips}$$

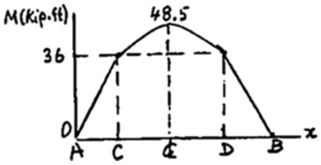
$$\mathbf{A} = \mathbf{B} = 18\ \text{kips}\uparrow \triangleleft$$



Shear diagram

At A: $V_A = +18$ kips

$$|V|_{\max} = 18.00\ \text{kips} \blacktriangleleft$$

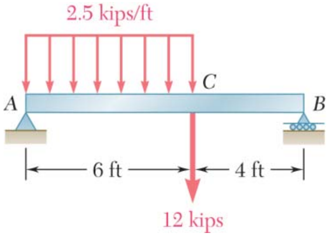


Bending-moment diagram

At A: $M_A = 0$

$$|M|_{\max} = 48.5\ \text{kip}\cdot\text{ft} \blacktriangleleft$$

Discontinuities in slope at C and D, due to the discontinuities of V.

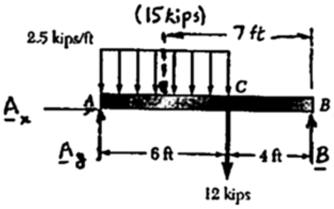

2.5 kips/ft
C
A
B
6 ft
4 ft
12 kips

PROBLEM 7.74

Using the method of Section 7.6, solve Problem 7.42.

PROBLEM 7.42 For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION


(15 kips)
2.5 kips/ft
7 ft
C
A_x
A
B
A_y
6 ft
4 ft
B
12 kips

Free body: Beam

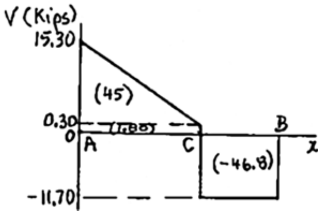
$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (12 \text{ kips})(4 \text{ ft}) + (15 \text{ kips})(7 \text{ ft}) - A_y(10 \text{ ft}) = 0$$

$$A_y = +15.3 \text{ kips} \triangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad B + 15.3 - 15 - 12 = 0$$

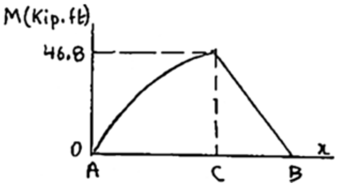
$$B = +11.7 \text{ kips} \triangleleft$$



Shear diagram

At A: $V_A = A_y = 15.3 \text{ kips}$

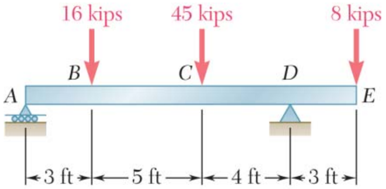
$$|V|_{\max} = 15.30 \text{ kips} \blacktriangleleft$$



Bending-moment diagram

At A: $M_A = 0$

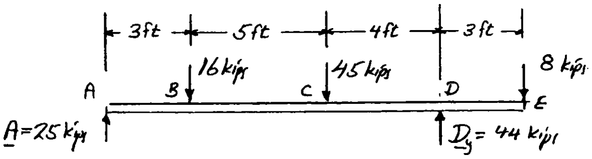
$$|M|_{\max} = 46.8 \text{ kip} \cdot \text{ft} \blacktriangleleft$$


16 kips
45 kips
8 kips
A
B
C
D
E
3 ft
5 ft
4 ft
3 ft

PROBLEM 7.75

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION


3ft
5ft
4ft
3ft
16kips
45kips
8 kips
A
B
C
D
E
A=25kips
Dy= 44 kips

Reactions

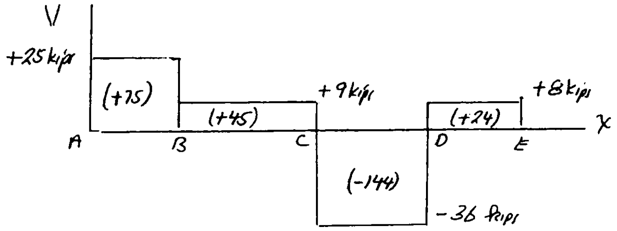
$$+\circlearrowleft \Sigma M_D = 0: \quad (16)(9) + (45)(4) - (8)(3) - A(12) = 0$$

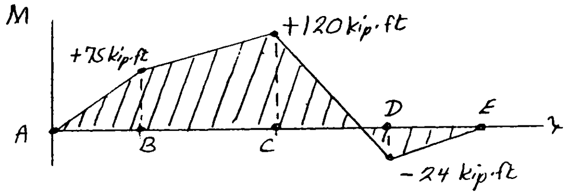
$$A = +25 \text{ kips} \qquad \mathbf{A} = 25 \text{ kips} \uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad 25 - 16 - 45 - 8 + D_y = 0$$

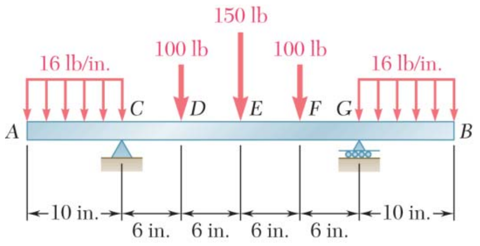
$$D_y = +44, \qquad D_y = 44 \text{ kips} \uparrow$$

$$\Sigma F_x = 0: \quad D_x = 0$$





(*b*) $|V|_{\max} = 36.0$ kips; $|M|_{\max} = 120.0$ kip · ft ◀


150 lb
100 lb
100 lb
16 lb/in.
16 lb/in.
A
C
D
E
F
G
B
10 in.
6 in.
6 in.
6 in.
6 in.
10 in.

PROBLEM 7.76

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

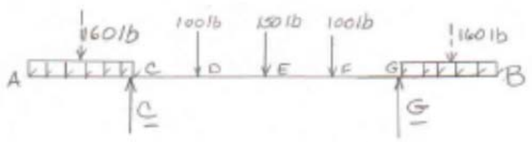
SOLUTION

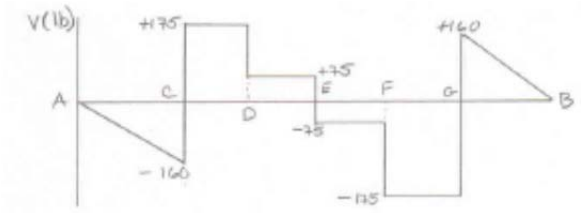
Reactions

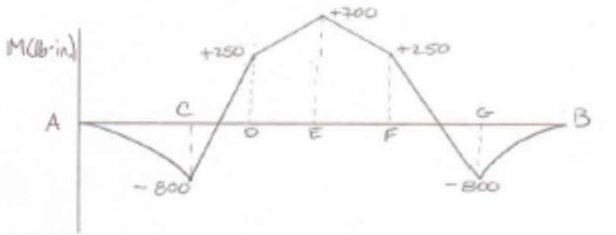
From symmetry, $C = G$

$$+\uparrow \Sigma F_y = 0: \quad -2(16\text{ lb/in.})(10\text{ in.}) - 100\text{ lb} - 150\text{ lb} - 100\text{ lb} + 2C = 0$$

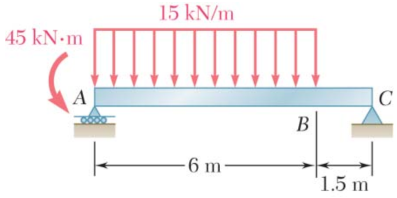
$C = G = +335$ lb $\quad \mathbf{C} = \mathbf{G} = 335\text{ lb}\uparrow$


160 lb
100 lb
150 lb
100 lb
160 lb
A
C
D
E
F
G
B
C
G





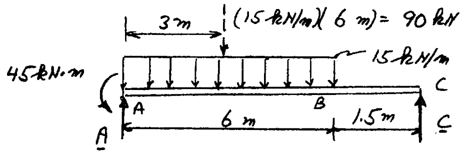
(*b*) $\quad |V|_{\max} = 175.0$ lbs; $\quad |M|_{\max} = 800\text{ lb}\cdot\text{in.}$ ◀


15 kN/m
45 kN·m
A
C
B
6 m
1.5 m

PROBLEM 7.77

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the magnitude and location of the maximum absolute value of the bending moment.

SOLUTION


3 m
(15 kN/m)(6 m) = 90 kN
15 kN/m
45 kN·m
A
B
C
6 m
1.5 m
A
C

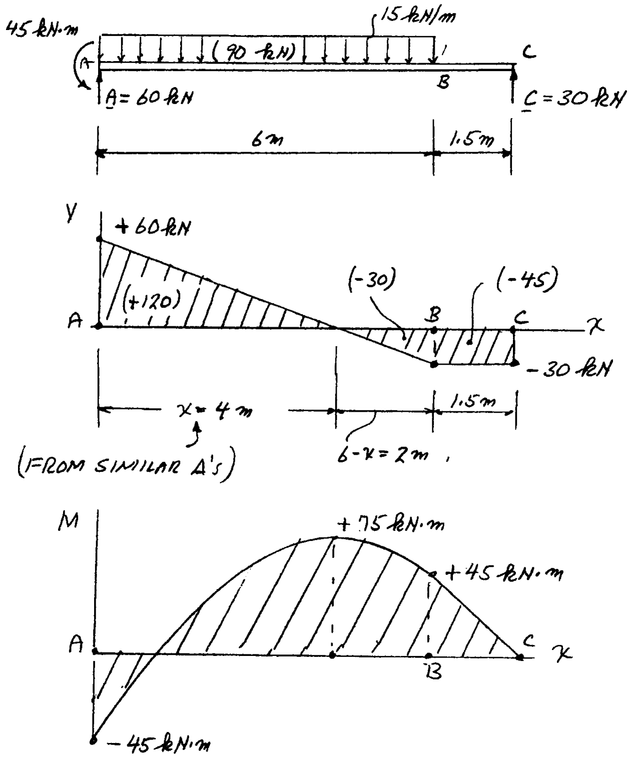
Reactions

$$+\circlearrowleft \Sigma M_A = 0: \quad +45 \text{ kN}\cdot\text{m} - (90 \text{ kN})(3 \text{ m}) + C(7.5 \text{ m}) = 0$$

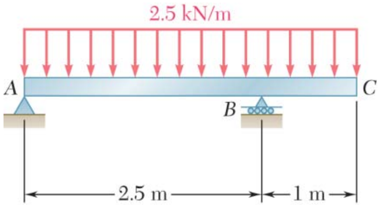
$C = +30$ kN **C** = 30 kN↑

$$+\uparrow \Sigma F_y = 0: \quad A - 90 \text{ kN} + 30 \text{ kN} = 0$$

A = 60 kN↑


45 kN·m
15 kN/m
(90 kN)
C
B
A = 60 kN
C = 30 kN
6 m
1.5 m
V
+60 kN
(+120)
(-30)
(-45)
A
B
C
x
-30 kN
x = 4 m
1.5 m
(FROM SIMILAR Δ's)
6-x = 2 m
M
+75 kN·m
+45 kN·m
A
C
x
B
-45 kN·m

(*b*) 75.0 kN·m, 4.00 m from *A* ◀

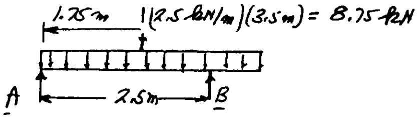


2.5 kN/m
A
B
C
2.5 m
1 m

PROBLEM 7.78

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the magnitude and location of the maximum absolute value of the bending moment.

SOLUTION



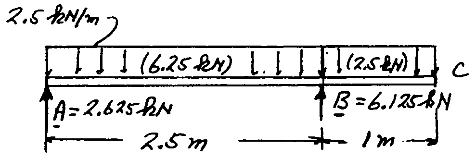
1.75 m
(2.5 kN/m)(3.5 m) = 8.75 kN
A
2.5 m
B

$+\circlearrowleft \Sigma M_A = 0$: $(8.75)(1.75) - B(2.5) = 0$

$\mathbf{B} = 6.125\text{ kN}\uparrow$

$\Sigma F_y = 0$:

$\mathbf{A} = 2.625\text{ kN}\uparrow$

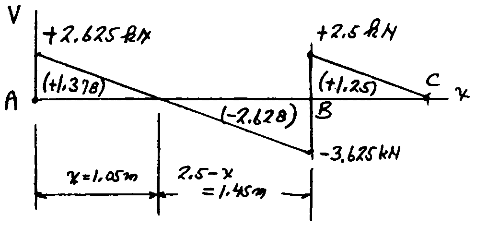


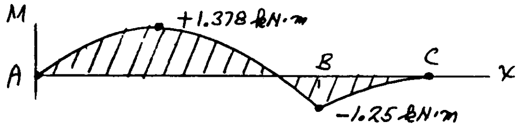
2.5 kN/m
(6.25 kN)
(2.5 kN)
C
A = 2.625 kN
B = 6.125 kN
2.5 m
1 m

Similar Δ's

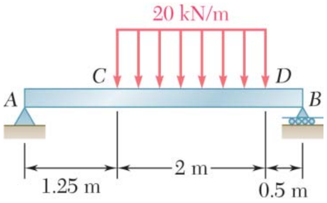
$$\underbrace{\frac{x}{2.625}=\frac{2.5-x}{3.625}}_{\text{Add numerators and denominators}}=\frac{2.5}{6.25}$$

$x = 1.05\text{ m}$





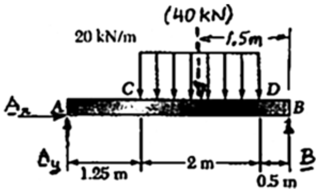
(*b*) 1.378 kN · m, 1.050 m from *A* ◀



PROBLEM 7.79

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the magnitude and location of the maximum bending moment.

SOLUTION

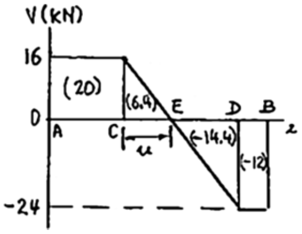


Free body: Beam

$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (40\text{ kN})(1.5\text{ m}) - A_y(3.75\text{ m}) = 0$$

$$A_y = +16.00\text{ kN} \triangleleft$$



Shear diagram

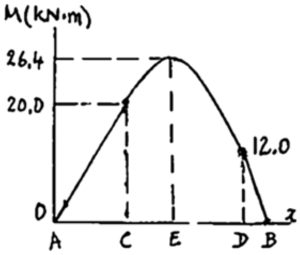
$$V_A = V_C = A_y = +16.00\text{ kN}$$

To determine Point *E* where $V = 0$, we write

$$V_E - V_C = -wu$$

$$0 - 16\text{ kN} = -(20\text{ kN/m})u \qquad u = 0.800\text{ m} \triangleleft$$

We next compute all areas



Bending-moment diagram

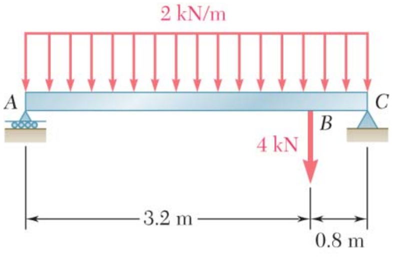
At *A*: $M_A = 0$

Largest value occurs at *E* with

$$AE = 1.25 + 0.8 = 2.05\text{ m} \qquad |M|_{\max} = 26.4\text{ kN}\cdot\text{m} \blacktriangleleft$$

$$2.05\text{ m from } A \blacktriangleleft$$

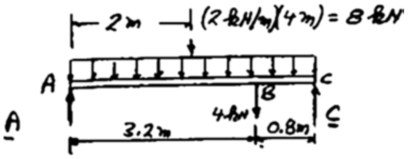
From *A* to *C* and *D* to *B*: Straight line segments. From *C* to *D*: Parabola


2 kN/m
A
C
B
4 kN
3.2 m
0.8 m

PROBLEM 7.80

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the magnitude and location of the maximum bending moment.

SOLUTION

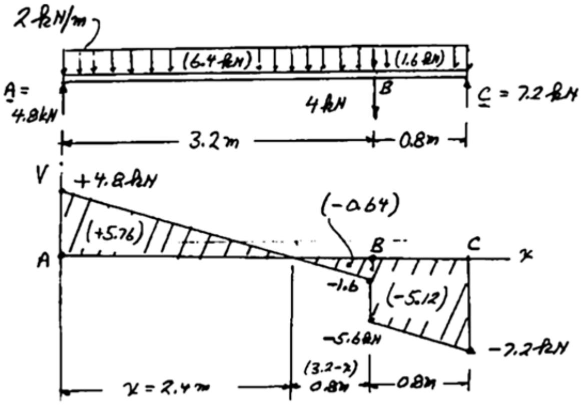

2 m
(2 kN/m)(4 m) = 8 kN
A
C
B
4 kN
3.2 m
0.8 m
A
C

$$+\circlearrowleft\ \Sigma M_A = 0: \quad (8)(2) + (4)(3.2) - 4C = 0$$

$$\mathbf{C} = 7.2\text{ kN}\uparrow$$

$$\Sigma F_y = 0: \quad \mathbf{A} = 4.8\text{ kN}\uparrow$$

(*a*) Shear diagram

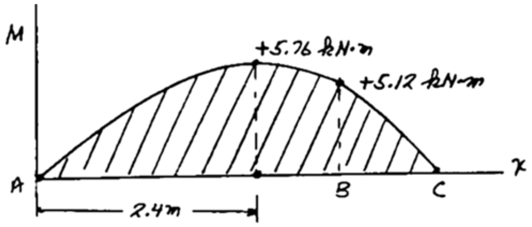

2 kN/m
(6.4 kN)
(1.6 kN)
A = 4.8 kN
4 kN
B
C = 7.2 kN
3.2 m
0.8 m
V
+4.8 kN
(+5.76)
(−0.64)
B
C
x
A
−1.6
(−5.12)
−5.6 kN
−7.2 kN
x = 2.4 m
(3.2−x) 0.8 m
0.8 m

Similar Triangles:
$$\frac{x}{4.8} = \frac{3.2 - x}{1.6} = \frac{3.2}{6.4}; \quad x = 2.4\text{ m}$$

$\uparrow$

Add num. & den.

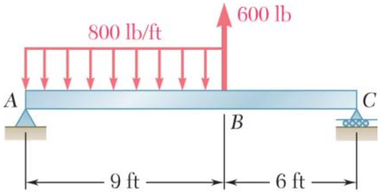
Bending-moment diagram


M
+5.76 kN·m
+5.12 kN·m
x
A
B
C
2.4 m

(*b*)

$$|M|_{\max} = 5.76\text{ kN}\cdot\text{m} ◀$$

2.40from A ◀



PROBLEM 7.81

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the magnitude and location of the maximum absolute value of the bending moment.

SOLUTION

Reactions

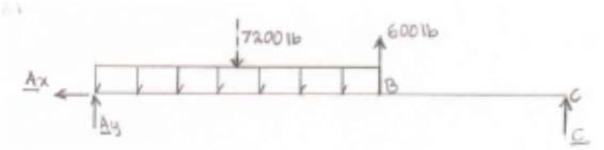
$$\Sigma F_x = 0: \quad A_x = 0$$

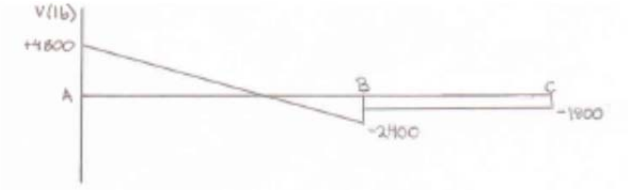
$$+\circlearrowleft \Sigma M_A = 0: \quad -(800\text{ lb/ft})(9\text{ ft})(4.5\text{ ft}) + (600\text{ lb})(9\text{ ft}) + C(15\text{ ft}) = 0$$

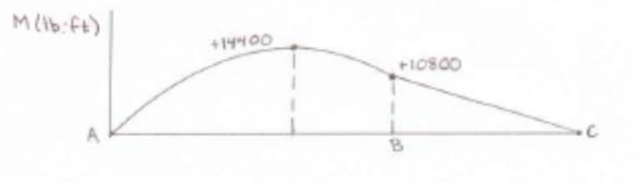
$$C = +1800\text{ lb} \quad \mathbf{C} = 1.800\text{ kips}\uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad -(800\text{ lb/ft})(9\text{ ft}) + 600\text{ lb} + 1800\text{ lb} + A = 0$$

$$A = +4800\text{ lb} \qquad \mathbf{A} = 4.80\text{ kips}\uparrow$$



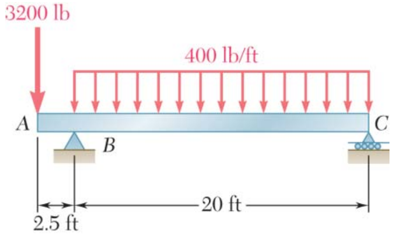




Maximum moment occurs at zero shear,

$$4800\text{ lb} = (800\text{ lb/ft})x \qquad x = 6.00\text{ ft}$$

(*b*) $|M|_{\max} = 14.40\text{ kip}\cdot\text{ft, }6.00\text{ ft from }A$ ◀



PROBLEM 7.82

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the magnitude and location of the maximum absolute value of the bending moment.

SOLUTION

Reactions

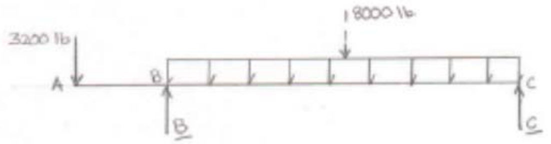
$$\Sigma F_x = 0: \quad B_x = 0$$

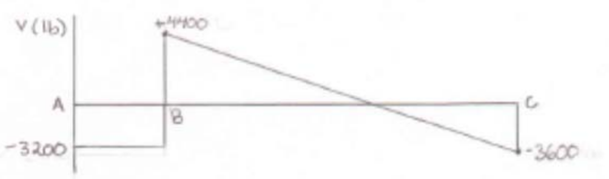
$$+\circlearrowleft \Sigma M_C = 0: \quad (400 \text{ lb/ft})(20 \text{ ft})(10 \text{ ft}) + (3200 \text{ lb})(22.5 \text{ ft}) - B(20 \text{ ft}) = 0$$

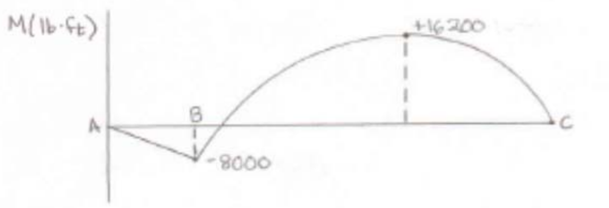
$$B = +7600 \text{ lb} \quad \mathbf{B} = 7.60 \text{ kips} \uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad -(400 \text{ lb/ft})(20 \text{ ft}) - 3200 \text{ lb} + 7600 \text{ lb} + C = 0$$

$$C = +3600 \text{ lb} \qquad \mathbf{C} = 3.60 \text{ kips} \uparrow$$



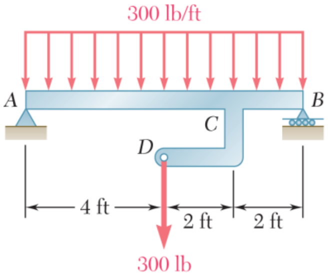




Maximum moment occurs at zero shear,

$$x = \frac{4400 \text{ lb}}{400 \text{ lb/ft}} + 2.5 \text{ ft} \qquad x = 13.50 \text{ ft}$$

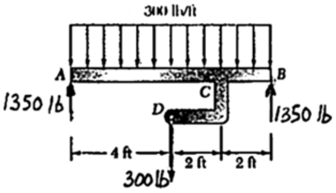
(*b*) $|M|_{\max} = 16.20 \text{ kip} \cdot \text{ft}$, 13.50 ft from A ◀


300 lb/ft
A
B
C
D
4 ft
2 ft
2 ft
300 lb

PROBLEM 7.83

(*a*) Draw the shear and bending-moment diagrams for beam *AB*, (*b*) determine the magnitude and location of the maximum absolute value of the bending moment.

SOLUTION

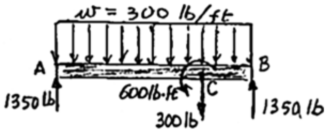

300 lb/ft
A
B
C
D
1350 lb
1350 lb
4 ft
2 ft
2 ft
300 lb

Reactions at supports

Because of symmetry of load

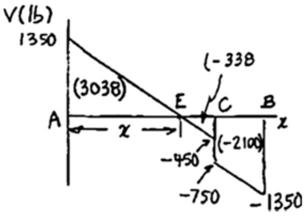
$$A = B = \frac{1}{2}(300 \times 8 + 300)$$

$$\mathbf{A} = \mathbf{B} = 1350 \text{ lb}\uparrow \triangleleft$$


w = 300 lb/ft
A
B
1350 lb
600 lb·ft
C
300 lb
1350 lb

Load diagram for *AB*

The 300-lb force at *D* is replaced by an equivalent force-couple system at *C*.



Shear diagram

At *A*: $V_A = A = 1350 \text{ lb}$

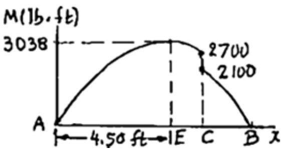
To determine Point *E* where $V = 0$:

$$V_E - V_A = -wx$$
$$0 - 1350 \text{ lb} = -(300 \text{ lb/ft})x$$

$$x = 4.50 \text{ ft} \triangleleft$$

We compute all areas

Bending-moment diagram

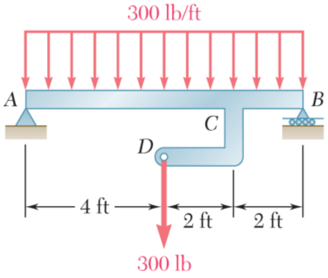


At *A*: $M_A = 0$

Note $600 - \text{lb}\cdot\text{ft}$ drop at *C* due to couple

$$|M|_{\max} = 3040 \text{ lb}\cdot\text{ft} \blacktriangleleft$$

4.50 ft from *A* ◀

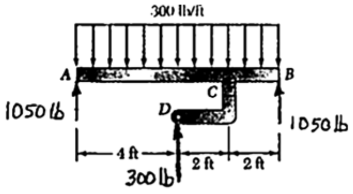

300 lb/ft
A
B
C
D
4 ft
2 ft
2 ft
300 lb

PROBLEM 7.84

Solve Problem 7.83 assuming that the 300-lb force applied at *D* is directed upward.

PROBLEM 7.83 (*a*) Draw the shear and bending-moment diagrams for beam *AB*, (*b*) determine the magnitude and location of the maximum absolute value of the bending moment.

SOLUTION

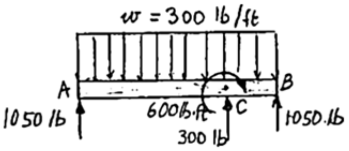

300 lb/ft
A
B
C
D
1050 lb
1050 lb
4 ft
2 ft
2 ft
300 lb

Reactions at supports

Because of symmetry of load:

$$A = B = \frac{1}{2}(300 \times 8 - 300)$$

$$\mathbf{A} = \mathbf{B} = 1050 \text{ lb} \uparrow \triangleleft$$


w = 300 lb/ft
A
B
C
1050 lb
600 lb·ft
300 lb
1050 lb

Load diagram

The 300-lb force at *D* is replaced by an equivalent force-couple system at *C*.

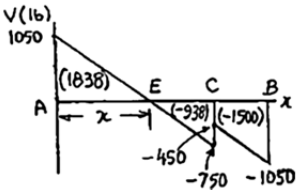
Shear diagram

At *A*: $V_A = A = 1050 \text{ lb}$

To determine Point *E* where $V = 0$:

$$V_E - V_A = -wx$$
$$0 - 1050 \text{ lb} = -(300 \text{ lb/ft})x$$

$$x = 3.50 \text{ ft} \triangleleft$$

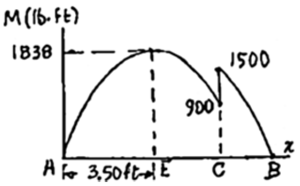


We compute all areas

Bending-moment diagram

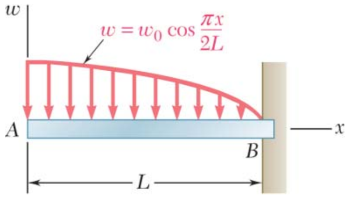
At *A*: $M_A = 0$

Note $600 - \text{lb} \cdot \text{ft}$ increase at *C* due to couple



$$|M|_{\max} = 1838 \text{ lb} \cdot \text{ft} \blacktriangleleft$$

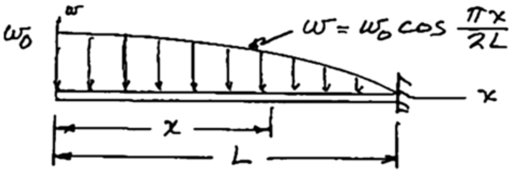
3.50 ft from *A* ◀



PROBLEM 7.85

For the beam and loading shown, (*a*) write the equations of the shear and bending-moment curves, (*b*) determine the magnitude and location of the maximum bending moment.

SOLUTION



$$\frac{dv}{dx} = -w = w_0 \cos \frac{\pi}{2}\frac{x}{L}$$

$$V = -\int w dx = -w_0 \left(\frac{2L}{\pi}\right) \sin \frac{\pi x}{2L} + C_1 \qquad (1)$$

$$\frac{dM}{dx} = V = -w_0 \left(\frac{2L}{\pi}\right) \sin \frac{\pi x}{2L} + C_1$$

$$M = \int V dx = +w_0 \left(\frac{2L}{\pi}\right)^2 \cos \frac{\pi x}{2L} + C_1 x + C_2 \qquad (2)$$

Boundary conditions

At $x = 0$: $V = C_1 = 0 \quad C_1 = 0$

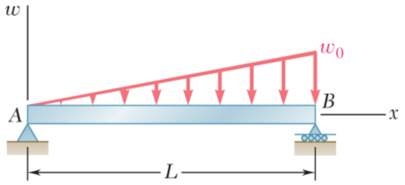
At $x = 0$: $M = +w_0 \left(\frac{2L}{\pi}\right)^2 \cos(0) + C_2 = 0$

$$C_2 = -w_0 \left(\frac{2L}{\pi}\right)^2$$

Eq. (1) $V = -w_0 \left(\frac{2L}{\pi}\right) \sin \frac{\pi x}{2L}$ ◀

$$M = w_0 \left(\frac{2L}{\pi}\right)^2 \left(-1 + \cos \frac{\pi x}{2L}\right) \text{ ◀}$$

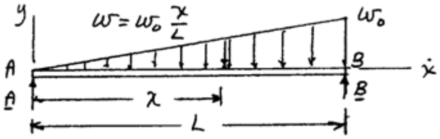
$M_{\max}$ at $x = L$: $|M_{\max}| = w_0 \left(\frac{2L}{\pi}\right)^2 |-1 + 0| = \frac{4}{\pi^2} w_0 L^2$ ◀



PROBLEM 7.86

For the beam and loading shown, (*a*) write the equations of the shear and bending-moment curves, (*b*) determine the magnitude and location of the maximum bending moment.

Solution



Eq. (7.1): $$\frac{dV}{dx} = -w = -w_0 \frac{x}{L}$$

$$V = \int -w_0 \frac{x}{L} dx = -\frac{1}{2} w_0 \frac{x^2}{L} + C_1 \quad (1)$$

Eq. (7.3): $$\frac{dM}{dx} = V = -\frac{1}{2} w_0 \frac{x^2}{L} + C_1$$

$$M = \int \left(-\frac{1}{2} w_0 \frac{x^2}{L} + C_1 \right) dx = -\frac{1}{6} w_0 = \frac{x^3}{L} + C_1 x + C_2 \qquad (2)$$

(*a*) Boundary conditions

At $x = 0$: $M = 0 = 0 + 0 + C_2 \quad C_2 = 0$

$x = L$: $M = 0 = -\frac{1}{6} w_0 L^2 + C_1 L \quad C_1 = \frac{1}{6} w_0 L$

Substituting C_1 and C_2 into (1) and (2):

$$V = -\frac{1}{2} w_0 \frac{x^2}{L} + \frac{1}{6} w_0 L \qquad V = \frac{1}{2} w_0 L \left(\frac{1}{3} - \frac{x^2}{L^2} \right) \blacktriangleleft$$

$$M = -\frac{1}{6} w_0 \frac{x^3}{L} + \frac{1}{6} w_0 L x \qquad M = \frac{1}{6} w_0 L^2 \left(\frac{x}{L} - \frac{x^3}{L^3} \right) \blacktriangleleft$$

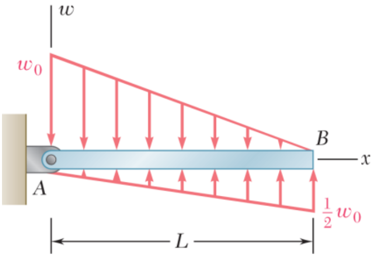
(*b*) Max moment occurs when $V = 0$:

$$1 - 3\frac{x^2}{L^2} = 0 \qquad \frac{x}{L} = \frac{1}{\sqrt{3}} \qquad x = 0.577L \blacktriangleleft$$

PROBLEM 7.86 (Continued)

$$M_{\max} = \frac{1}{6} w_0 L^2 \left[\frac{1}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}} \right)^3 \right] \qquad M_{\max} = 0.0642 w_0 L^2 \blacktriangleleft$$

Note: At $x = 0$, $A = V_A = \frac{1}{2} w_0 L \left(\frac{1}{3} \right)$ $\qquad$ $\mathbf{A} = \frac{1}{6} w_0 L \uparrow$



PROBLEM 7.87

For the beam and loading shown, (*a*) write the equations of the shear and bending-moment curves, (*b*) determine the magnitude and location of the maximum bending moment.

SOLUTION

(*a*) We check that beam is in equilibrium $-\frac{1}{2}(w_0L)\left(\frac{L}{3}\right)+\frac{1}{2}\left(\frac{1}{2}w_0L\right)\left(\frac{2}{3}L\right)=0$ (ok)

Load: $w(x)=w_0\left(1-\frac{x}{L}\right)-\frac{w_0}{2}\left(\frac{x}{L}\right)=w_0\left(1-\frac{3x}{2L}\right)$

Shear: We use Eq. (7.1)

$$\frac{dV}{dx}=-w=-w_0\left(1-\frac{3x}{2L}\right)$$

$$V=-\int w_0(1-\frac{3x}{2L})dx=-w_0\left(x-\frac{3x^2}{4L}\right)+C_1$$

Boundary Condition: at $x=L$, $V=0$ therefore $0=-w_0\left(L-\frac{3}{4}L\right)+C_1$ $\quad C_1=\frac{1}{4}w_0L$

Bending moment: Using Eq. (7.3)

$$\frac{dM}{dx}=V \quad M=\int Vdx=\int\left(-w_0x+\frac{3}{4}w_0\frac{x^2}{L}+\frac{1}{4}w_0L\right)dx$$

$$M(x)=-\frac{1}{2}w_0x^2+\frac{1}{4}w_0\frac{x^3}{L}+\frac{1}{4}w_0Lx+C_2$$

Boundary Condition: at x=0, $M=0=C_2$ therefore $C_2=0$

$$V(x)=\frac{1}{4}w_0L\left[3\left(\frac{x}{L}\right)^2-4\left(\frac{x}{L}\right)+1\right] \blacktriangleleft$$

$$M(x)=\frac{1}{4}w_0L^2\left[\left(\frac{x}{L}\right)^3-2\left(\frac{x}{L}\right)^2+\left(\frac{x}{L}\right)\right] \blacktriangleleft$$

PROBLEM 7.87 (Continued)

(*b*) Maximum bending moment

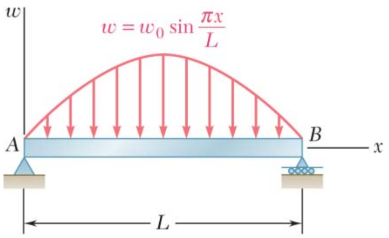
$$\frac{dM}{dx} = V = 0$$

$$0 = \left[3\left(\frac{x}{L}\right)^2 - 4\left(\frac{x}{L}\right) + 1\right]$$

$$0 = \left(3\frac{x}{L} - 1\right)\left(\frac{x}{L} - 1\right); \qquad x = \frac{L}{3} \text{ and } x = L$$

$$M_{\max} = \frac{1}{4} w_0 L^2 \left(\frac{1}{27} - \frac{2}{9} + \frac{1}{3}\right)$$

$$M_{\max} = \frac{1}{27} w_0 L^2, \quad \text{at } x = \frac{L}{3} \blacktriangleleft$$



PROBLEM 7.88

For the beam and loading shown, (*a*) write the equations of the shear and bending-moment curves, (*b*) determine the magnitude and location of the maximum bending moment.

SOLUTION

Eq. (7.1): $$\frac{dV}{dx} = -w = -w_0 \sin \frac{\pi x}{L}$$

$$V = \int -w_0 \frac{\sin \pi x}{L} dx = w_0 \frac{L}{\pi} \cos \frac{\pi x}{L} + C_1 \qquad (1)$$

Eq. (7.3): $$\frac{dM}{dx} = V = w_0 \frac{L}{\pi} \cos \frac{\pi x}{L} + C_1$$

$$M = \int \left(w_0 \frac{L}{\pi} \cos \frac{\pi x}{L} + C_1 \right) dx = w_0 \frac{L^2}{\pi^2} \sin \frac{\pi x}{L} + C_1 x + C_2 \qquad (2)$$

(*a*) Boundary conditions

At $x = 0$: $M = 0$ therefore, $C_2 = 0$

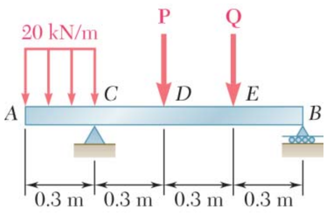
$x = L$: $M = 0$ therefore, $C_1 = 0$

$$V(x) = w_0 \frac{L}{\pi} \cos\left(\frac{\pi x}{L}\right) \blacktriangleleft$$

$$M = w_0 \frac{L^2}{\pi^2} \sin\left(\frac{\pi x}{L}\right) \blacktriangleleft$$

(*b*) Max moment occurs when $V = 0$ at $x = \frac{L}{2}$

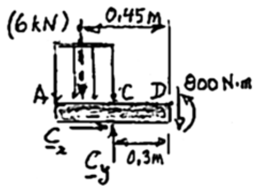
$$M_{\max} = w_0 \frac{L^2}{\pi^2} \blacktriangleleft$$


20 kN/m
P
Q
A
C
D
E
B
0.3 m
0.3 m
0.3 m
0.3 m

PROBLEM 7.89

The beam *AB* is subjected to the uniformly distributed load shown and to two unknown forces **P** and **Q**. Knowing that it has been experimentally determined that the bending moment is $+800\ \text{N}\cdot\text{m}$ at *D* and $+1300\ \text{N}\cdot\text{m}$ at *E*, (*a*) determine **P** and **Q**, (*b*) draw the shear and bending-moment diagrams for the beam.

SOLUTION

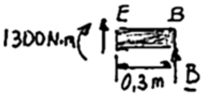

(6 kN)
0.45 m
A
C
D
800 N·m
C_x
C_y
0.3 m

(*a*) Free body: Portion *AD*

$$\Sigma F_x = 0: \quad C_x = 0$$

$$+\circlearrowleft \Sigma M_D = 0: \quad -C_y(0.3\ \text{m}) + 0.800\ \text{kN}\cdot\text{m} + (6\ \text{kN})(0.45\ \text{m}) = 0$$

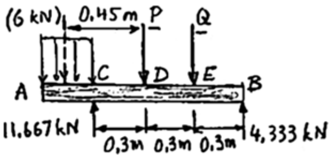
$$C_y = +11.667\ \text{kN} \qquad \mathbf{C} = 11.667\ \text{kN} \uparrow ◁$$


1300 N·m
E
B
0.3 m
B

Free body: Portion *EB*

$$+\circlearrowleft \Sigma M_E = 0: \quad B(0.3\ \text{m}) - 1.300\ \text{kN}\cdot\text{m} = 0$$

$$\mathbf{B} = 4.333\ \text{kN} \uparrow ◁$$


(6 kN)
0.45 m
P
Q
A
C
D
E
B
11.667 kN
0.3 m
0.3 m
0.3 m
4.333 kN

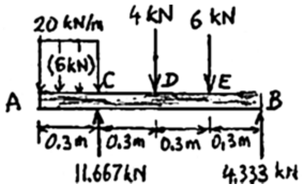
Free body: Entire beam

$$+\circlearrowleft \Sigma M_D = 0: \quad (6\ \text{kN})(0.45\ \text{m}) - (11.667\ \text{kN})(0.3\ \text{m})$$
$$-Q(0.3\ \text{m}) + (4.333\ \text{kN})(0.6\ \text{m}) = 0$$

$$\mathbf{Q} = 6.00\ \text{kN} \downarrow ◀$$

$$+\uparrow \Sigma M_y = 0: \quad 11.667\ \text{kN} + 4.333\ \text{kN}$$
$$-6\ \text{kN} - P - 6\ \text{kN} = 0$$

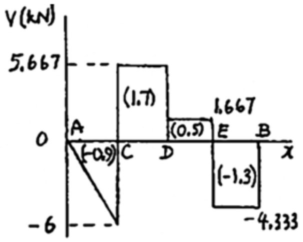
$$\mathbf{P} = 4.00\ \text{kN} \downarrow ◀$$


20 kN/m
4 kN
6 kN
(6 kN)
A
C
D
E
B
0.3 m
0.3 m
0.3 m
0.3 m
11.667 kN
4.333 kN

Load diagram

(*b*) Shear diagram

At *A*: $V_A = 0$



$$|V|_{\max} = 6\ \text{kN} ◁$$

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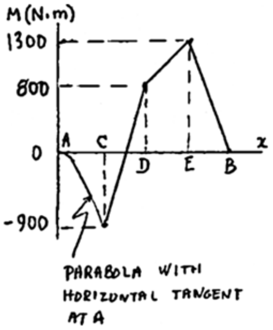
PROBLEM 7.89 (Continued)



Bending-moment diagram

At *A*: $M_A = 0$

$$|M|_{\max} = 1300 \text{ N}\cdot\text{m} \triangleleft$$

We check that

$$M_D = +800 \text{ N}\cdot\text{m} \quad \text{and} \quad M_E = +1300 \text{ N}\cdot\text{m}$$

As given:

At *C*: $M_C = -900 \text{ N}\cdot\text{m}$ ◀

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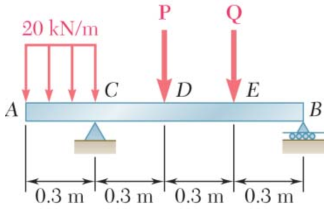
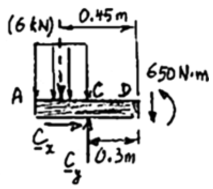

20 kN/m
P
Q
A
C
D
E
B
0.3 m
0.3 m
0.3 m
0.3 m

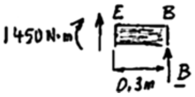
PROBLEM 7.90

Solve Problem 7.89 assuming that the bending moment was found to be $+650\ \text{N}\cdot\text{m}$ at D and $+1450\ \text{N}\cdot\text{m}$ at E.

PROBLEM 7.89 The beam *AB* is subjected to the uniformly distributed load shown and to two unknown forces **P** and **Q**. Knowing that it has been experimentally determined that the bending moment is $+800\ \text{N}\cdot\text{m}$ at D and $+1300\ \text{N}\cdot\text{m}$ at E, (*a*) determine **P** and **Q**, (*b*) draw the shear and bending-moment diagrams for the beam.

SOLUTION


(6 kN)
0.45 m
650 N·m
A
C
D
C_x
C_y
0.3 m


1450 N·m
E
B
0.3 m
B

(*a*) Free body: Portion *AD*

$$\Sigma F_x = 0: \quad C_x = 0$$

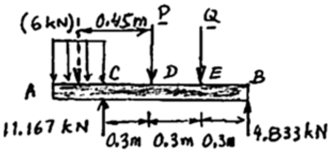
$$+\circlearrowleft \Sigma M_D = 0: \quad -C(0.3\ \text{m}) + 0.650\ \text{kN}\cdot\text{m} + (6\ \text{kN})(0.45\ \text{m}) = 0$$

$$C_y = +11.167\ \text{kN} \qquad \mathbf{C} = 11.167\ \text{kN} \uparrow ◁$$

Free body: Portion *EB*

$$+\circlearrowleft \Sigma M_E = 0: \quad B(0.3\ \text{m}) - 1.450\ \text{kN}\cdot\text{m} = 0$$

$$\mathbf{B} = 4.833\ \text{kN} \uparrow ◁$$


(6 kN)
0.45 m
P
Q
A
C
D
E
B
11.167 kN
0.3 m
0.3 m
0.3 m
4.833 kN

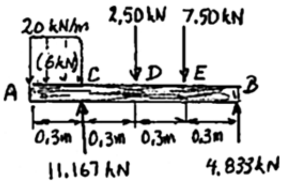
Free body: Entire beam

$$+\circlearrowleft \Sigma M_D = 0: \quad (6\ \text{kN})(0.45\ \text{m}) - (11.167\ \text{kN})(0.3\ \text{m})$$
$$-Q(0.3\ \text{m}) + (4.833\ \text{kN})(0.6\ \text{m}) = 0$$

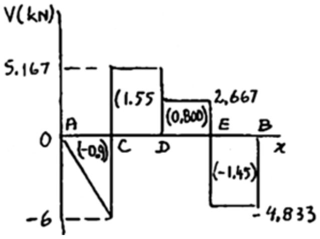
$$\mathbf{Q} = 7.50\ \text{kN} \downarrow ◀$$

$$+\uparrow \Sigma M_y = 0: \quad 11.167\ \text{kN} + 4.833\ \text{kN}$$
$$-6\ \text{kN} - P - 7.50\ \text{kN} = 0$$

$$\mathbf{P} = 2.50\ \text{kN} \downarrow ◀$$


20 kN/m
2.50 kN
7.50 kN
(6 kN)
A
C
D
E
B
0.3 m
0.3 m
0.3 m
0.3 m
11.167 kN
4.833 kN

Load diagram

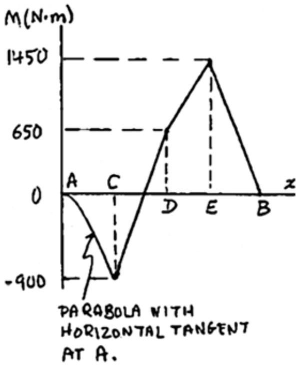


(*b*) Shear diagram

At *A*: $\quad V_A = 0$

$$|V|_{\max} = 6\ \text{kN} ◁$$

PROBLEM 7.90 (Continued)



M(N·m)
1450
650
0
-900
A
C
D
E
B
x
PARABOLA WITH HORIZONTAL TANGENT AT A.

Bending-moment diagram

At A: $M_A = 0$

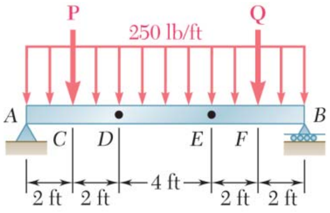
$$|M|_{\max} = 1450\ \text{N}\cdot\text{m} \ \triangleleft$$

We check that

$$M_D = +650\ \text{N}\cdot\text{m} \quad \text{and} \quad M_E = +1450\ \text{N}\cdot\text{m}$$

As given:

At C: $M_C = -900\ \text{N}\cdot\text{m}$ ◀

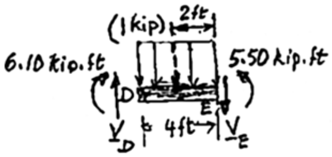

P
Q
250 lb/ft
A
B
C
D
E
F
2 ft
2 ft
4 ft
2 ft
2 ft

PROBLEM 7.91*

The beam *AB* is subjected to the uniformly distributed load shown and to two unknown forces **P** and **Q**. Knowing that it has been experimentally determined that the bending moment is $+6.10\ \text{kip}\cdot\text{ft}$ at *D* and $+5.50\ \text{kip}\cdot\text{ft}$ at *E*, (*a*) determine **P** and **Q**, (*b*) draw the shear and bending-moment diagrams for the beam.

SOLUTION

(*a*) Free body: Portion *DE*


(1 kip)
2ft
6.10 kip.ft
5.50 kip.ft
D
E
V_D
4ft
V_E

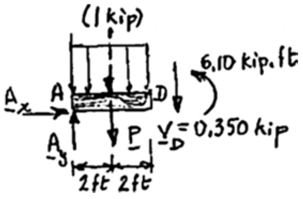
$$+\circlearrowleft \Sigma M_E = 0: \quad 5.50\ \text{kip}\cdot\text{ft} - 6.10\ \text{kip}\cdot\text{ft} + (1\ \text{kip})(2\ \text{ft}) - V_D(4\ \text{ft}) = 0$$

$$V_D = +0.350\ \text{kip}$$

$$+\uparrow \Sigma F_y = 0: \quad 0.350\ \text{kip} - 1\ \text{kip} - V_E = 0$$

$$V_E = -0.650\ \text{kip}$$

Free body: Portion *AD*


(1 kip)
6.10 kip.ft
A_x
A
D
A_y
P
V_D = 0.350 kip
2ft 2ft

$$+\circlearrowleft \Sigma M_A = 0: \quad 6.10\ \text{kip}\cdot\text{ft} - P(2\ \text{ft}) - (1\ \text{kip})(2\ \text{ft}) - (0.350\ \text{kip})(4\ \text{ft}) = 0$$

$\mathbf{P} = 1.350\ \text{kips}\downarrow$ ◀

$$\Sigma F_x = 0: \quad A_x = 0$$

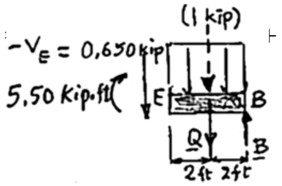
$$+\uparrow \Sigma F_y = 0: \quad A_y - 1\ \text{kip} - 1.350\ \text{kip} - 0.350\ \text{kip} = 0$$

$$A_y = +2.70\ \text{kips}$$

$\mathbf{A} = 2.70\ \text{kips}\uparrow$ ◀

Free body: Portion *EB*

$+\circlearrowleft \Sigma M_B$


−V_E = 0.650 kip
(1 kip)
5.50 kip.ft
E
B
Q
B
2ft 2ft

$$+ (1\ \text{kip})(2\ \text{ft}) + Q(2\ \text{ft}) - 5.50\ \text{kip}\cdot\text{ft} = 0$$

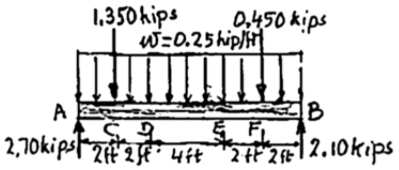
$\mathbf{Q} = 0.450\ \text{kip}\downarrow$ ◀

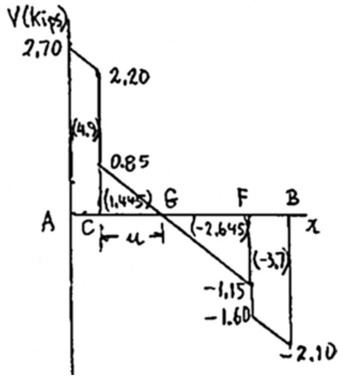
$$+\uparrow \Sigma F_y = 0: \quad B - 0.450 - 1 - 0.650 = 0$$

$\mathbf{B} = 2.10\ \text{kips}\uparrow$ ◀

PROBLEM 7.91* (Continued)

(*b*) Load diagram


1.350 kips
0.450 kips
w = 0.25 kip/ft
A
B
C
D
E
F
2.70 kips
2.10 kips
2 ft
2 ft
4 ft
2 ft
2 ft



Shear diagram

At *A*: $V_A = A = +2.70 \text{ kips}$

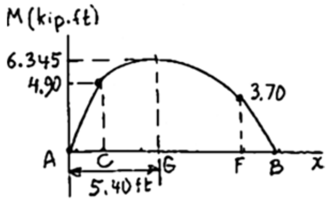
To determine Point *G* where $V = 0$, we write

$$V_G - V_C = -w\mu$$
$$0 - 0.85 \text{ kips} = -(0.25 \text{ kip/ft})\mu$$

$\mu = 3.40 \text{ ft}$ ◀

We next compute all areas

$|V|_{\max} = 2.70 \text{ kips at } A$ ◀



Bending-moment diagram

At *A*: $M_A = 0$

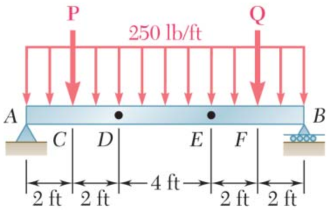
Largest value occurs at *G* with

$$AG = 2 + 3.40 = 5.40 \text{ ft}$$

$|M|_{\max} = 6.345 \text{ kip} \cdot \text{ft}$ ◀

5.40 ft from *A* ◀

Bending-moment diagram consists of 3 distinct arcs of parabolas.


P
Q
250 lb/ft
A
B
C
D
E
F
2 ft
2 ft
4 ft
2 ft
2 ft

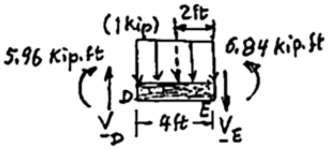
PROBLEM 7.92*

Solve Problem 7.91 assuming that the bending moment was found to be +5.96 kip·ft at *D* and +6.84 kip·ft at *E*.

PROBLEM 7.91* The beam *AB* is subjected to the uniformly distributed load shown and to two unknown forces **P** and **Q**. Knowing that it has been experimentally determined that the bending moment is +6.10 kip·ft at *D* and +5.50 kip·ft at *E*, (*a*) determine **P**and **Q**, (*b*) draw the shear and bending-moment diagrams for the beam.

SOLUTION

(*a*) Free body: Portion *DE*


(1 kip)
2ft
5.96 kip·ft
6.84 kip·ft
D
E
V_D
4ft
V_E

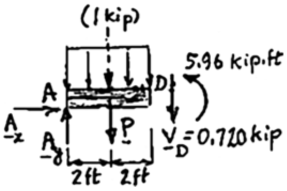
$$+\circlearrowleft \Sigma M_E = 0: \quad 6.84 \text{ kip}\cdot\text{ft} - 5.96 \text{ kip}\cdot\text{ft} + (1 \text{ kip})(2 \text{ ft}) - V_D(4 \text{ ft}) = 0$$

$$V_D = +0.720 \text{ kip}$$

$$+\uparrow \Sigma F_y = 0: \quad 0.720 \text{ kip} - 1 \text{kip} - V_E = 0$$

$$V_E = -0.280 \text{ kip}$$

Free body: Portion *AD*


(1 kip)
5.96 kip·ft
A
D
A_x
A_y
P
V_D = 0.720 kip
2ft
2ft

$$+\circlearrowleft \Sigma M_A = 0: \quad 5.96 \text{ kip}\cdot\text{ft} - P(2 \text{ ft}) - (1 \text{ kip})(2 \text{ ft}) - (0.720 \text{ kip})(4 \text{ ft}) = 0$$

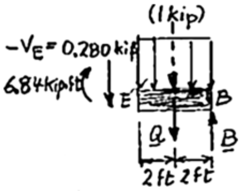
P = 0.540 kip ↓ ◀

$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - 1 \text{ kip} - 0.540 \text{ kip} - 0.720 \text{ kip} = 0$$

$A_y = +2.26$ kips **A** = 2.26 kips ↑ ◁

Free body: Portion *EB*


(1 kip)
-V_E = 0.280 kip
6.84 kip·ft
E
B
Q
B
2 ft
2 ft

$$+\circlearrowleft \Sigma M_B = 0: \quad (0.280 \text{ kip})(4 \text{ ft}) + (1 \text{ kip})(2 \text{ ft}) + Q(2 \text{ ft}) - 6.84 \text{ kip}\cdot\text{ft} = 0$$

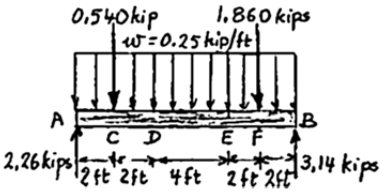
Q = 1.860 kips ↓ ◀

$$+\uparrow \Sigma F_y = 0: \quad B - 1.860 - 1 - 0.280 = 0$$

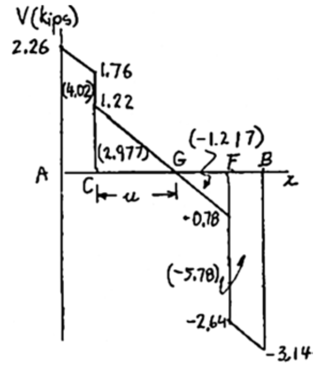
B = 3.14 kips ↑ ◀

PROBLEM 7.92* (Continued)

(*b*) Load diagram


0.540 kip
1.860 kips
w = 0.25 kip/ft
A
B
C
D
E
F
2.26 kips
3.14 kips
2 ft
2 ft
4 ft
2 ft
2 ft

Shear diagram


V(kips)
2.26
1.76
(4.02)
1.22
(2.977)
G
(−1.217)
F
B
A
x
C
u
−0.78
(−5.78)
−2.64
−3.14

At A: $$V_A = A = +2.26 \text{ kips}$$

To determine Point G where $V = 0$, we write

$$V_G - V_C = -w\mu$$

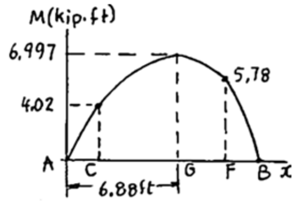
$$0 - (1.22 \text{ kips}) = -(0.25 \text{ kip/ft})\mu$$

$\mu = 4.88$ ft ◁

We next compute all areas

$|V|_{\max} = 3.14$ kips at B ◀

Bending-moment diagram


M(kip·ft)
6.997
5.78
4.02
A
C
G
F
B
x
6.88 ft

At A: $M_A = 0$

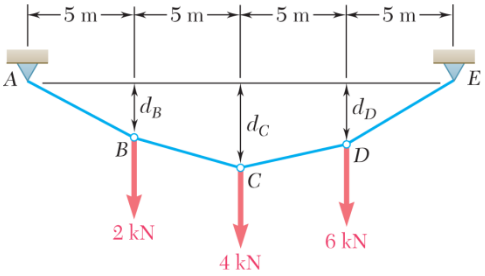
Largest value occurs at G with

$$AG = 2 + 4.88 = 6.88 \text{ ft}$$

$|M|_{\max} = 6.997$ kip·ft ◀

6.88 ft from A ◀

Bending-moment diagram consists of 3 distinct arcs of parabolas.

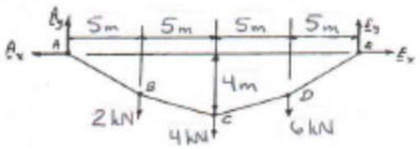

5 m
5 m
5 m
5 m
A
E
d_B
d_C
d_D
B
C
D
2 kN
4 kN
6 kN

PROBLEM 7.93

Three loads are suspended as shown from the cable *ABCDE*. Knowing that $d_C = 4$ m, determine (*a*) the components of the reaction at *E*, (*b*) the maximum tension in the cable.

SOLUTION

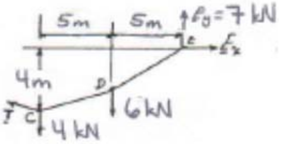
(*a*)


A_y
5m
5m
5m
5m
E_y
A_x
A
E
E_x
B
4m
D
2 kN
4 kN
C
6 kN

$$\overset{\curvearrowleft}{+}\Sigma M_A = 0: \quad E_y(20\text{ m}) - (2\text{ kN})(5\text{ m}) - (4\text{ kN})(10\text{ m}) - (6\text{ kN})(15\text{ m}) = 0$$

$E_y = +7$ kN $\qquad$ $\mathbf{E}_y = 7.00\text{ kN}\uparrow$ ◀

Portion *CDE*:


5m
5m
$E_y = 7$ kN
E
E_x
4m
D
T
C
6kN
4 kN

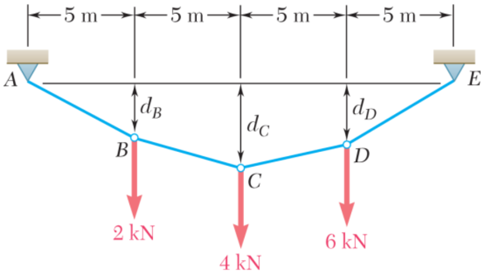
$$\overset{\curvearrowleft}{+}\Sigma M_C = 0: \quad (7\text{ kN})(10\text{ m}) - (6\text{ kN})(5\text{ m}) - E_x(4\text{ m}) = 0$$

$E_x = 10$ kN $\qquad$ $\mathbf{E}_x = 10.00\text{ kN}\longrightarrow$ ◀

(*b*) Maximum tension occurs in *DE*:

$$T_m = \sqrt{E_x^2 + E_y^2} = \sqrt{10^2 + 7^2}$$

$T_m = 12.21$ kN ◀

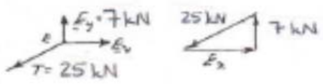

5 m
5 m
5 m
5 m
A
E
d_B
d_C
d_D
B
C
D
2 kN
4 kN
6 kN

PROBLEM 7.94

Knowing that the maximum tension in cable *ABCDE* is 25 kN, determine the distance d_C.

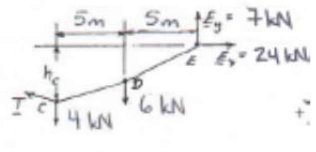
SOLUTION

Maximum *T* of 25 kN occurs in *DE*. See solution of Problem 7.93 for the determination of $\mathbf{E}_y = 7.00\text{ kN}\uparrow$


$E_y = 7$ kN
E
E_x
T = 25 kN
25 kN
7 kN
E_x

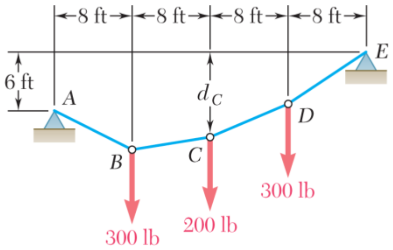
From force triangle $\qquad E_x^2 + 7^2 = 25^2$

Portion *CDE*: $\qquad E_x = 24\text{ kN}$


5m
5m
$E_y = 7$ kN
E
$E_x = 24$ kN
h_C
D
T
C
4 kN
6 kN

$$+\circlearrowleft \Sigma M_C = 0: \quad (6\text{ kN})(5\text{ m}) - (24\text{ kN})h_C - (7\text{ kN})(10\text{ m}) = 0$$

$$h_C = 1.667\text{ m} \blacktriangleleft$$

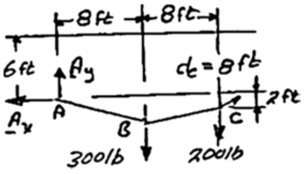

8 ft
8 ft
8 ft
8 ft
6 ft
A
B
C
d_C
D
E
300 lb
200 lb
300 lb

PROBLEM 7.95

If $d_C = 8$ ft, determine (*a*) the reaction at *A*, (*b*) the reaction at *E*.

SOLUTION

Free body: Portion *ABC*

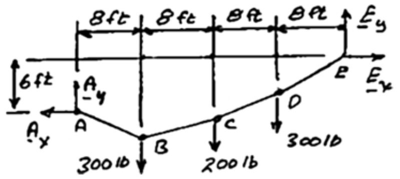

8 ft
8 ft
6 ft
A_y
$d_C = 8$ ft
2 ft
A_x
A
B
C
300 lb
200 lb

$$+\circlearrowleft \Sigma M_C = 0$$

$$2A_x - 16A_y + 300(8) = 0$$

$$A_x = 8A_y - 1200 \qquad (1)$$

Free body: Entire cable


8 ft
8 ft
8 ft
8 ft
E_y
6 ft
A_y
E
E_x
D
A_x
A
C
300 lb
B
300 lb
200 lb

$$+\circlearrowright \Sigma M_E = 0: \quad +6A_x + 32A_y - (300\text{ lb} + 200\text{ lb} + 300\text{ lb})16\text{ ft} = 0$$

$$3A_x + 16A_y - 6400 = 0$$

Substitute from Eq. (1):

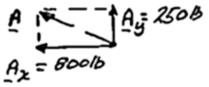
$$3(8A_y - 1200) + 16A_y - 6400 = 0 \qquad \mathbf{A}_y = 250\text{ lb}\uparrow$$

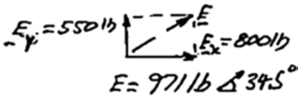
Eq. (1) $\quad A_x = 8(250) - 1200 \qquad \mathbf{A}_x = 800\text{ lb}$

←

$$\xrightarrow{+} \Sigma F_x = 0: \quad -A_x + E_x = 0 \quad -800\text{ lb} + E_x = 0 \qquad \mathbf{E}_x = 800\text{ lb}\rightarrow$$

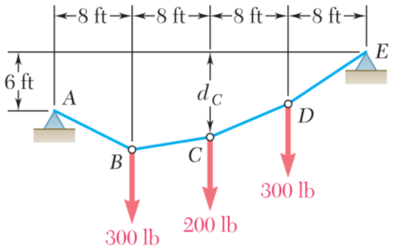
$$+\uparrow \Sigma F_y = 0: \quad 250 + E_y - 300 - 200 - 300 = 0 \qquad \mathbf{E}_y = 550\text{ lb}\uparrow$$


A
$A_y = 250$ lb
$A_x = 800$ lb


$E_y = 550$ lb
E
$E_x = 800$ lb
E = 971 lb ∠ 34.5°

(*a*) $\quad \mathbf{A} = 838\text{ lb} \measuredangle 17.35°$ ◀

(*b*) $\quad \mathbf{E} = 971\text{ lb} \measuredangle 34.5°$ ◀

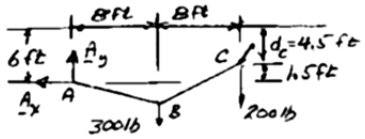

8 ft
8 ft
8 ft
8 ft
6 ft
A
d_C
E
D
B
C
300 lb
300 lb
200 lb

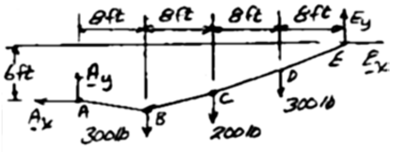
PROBLEM 7.96

If $d_C = 4.5$ ft, determine (*a*) the reaction at *A*, (*b*) the reaction at *E*.

SOLUTION

Free body: Portion *ABC*


8ft
8ft
6ft
A_y
$d_c = 4.5$ ft
1.5ft
C
A_x
A
B
300lb
200lb


8ft
8ft
8ft
8ft
E_y
6ft
A_y
E
E_x
D
C
300lb
A_x
A
B
300lb
200lb

$$+\circlearrowleft \Sigma M_C = 0: \quad -1.5A_x - 16A_y + 300 \times 8 = 0$$

$$A_x = \frac{(2400 - 16A_y)}{1.5} \qquad (1)$$

Free body: Entire cable $\quad +\circlearrowleft \Sigma M_E = 0: \quad +6A_x + 32A_y - (300\text{ lb} + 200\text{ lb} + 300\text{ lb})16\text{ ft} = 0$

$$3A_x + 16A_y - 6400 = 0$$

Substitute from Eq. (1): $\quad \dfrac{3(2400 - 16A_y)}{1.5} + 16A_y - 6400 = 0$

$$A_y = -100\text{ lb}$$

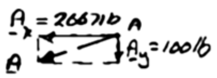
Thus A_y acts downward $\qquad \mathbf{A}_y = 100\text{ lb}\downarrow$

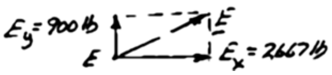
Eq. (1) $\quad A_x = \dfrac{(2400 - 16(-100))}{1.5} = 2667\text{ lb} \qquad \mathbf{A}_x = 2667\text{ lb}\leftarrow$

$\xrightarrow{+} \Sigma F_x = 0: \quad -A_x + E_x = 0 \quad -2667 + E_x = 0 \qquad \mathbf{E}_x = 2667\text{ lb}\rightarrow$

$+\uparrow \Sigma F_y = 0: \quad A_y + E_y - 300 - 200 - 300 = 0$

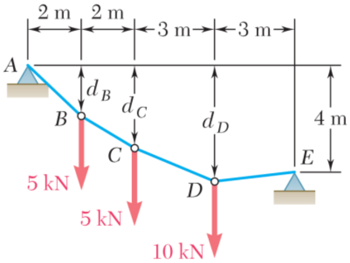
$-100\text{ lb} + E_y - 800\text{ lb} = 0 \qquad \mathbf{E}_y = 900\text{ lb}\uparrow$


$A_x = 2667$lb
A
A
$A_y = 100$lb


$E_y = 900$lb
E
E
$E_x = 2667$lb

(*a*) $\qquad \mathbf{A} = 2670\text{ lb} \measuredangle 2.10°$ ◀

(*b*) $\qquad \mathbf{E} = 2810\text{ lb} \measuredangle 18.65°$ ◀

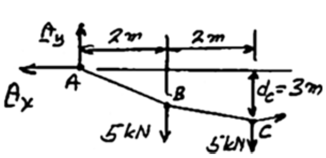

2 m
2 m
3 m
3 m
A
d_B
d_C
d_D
4 m
B
C
E
D
5 kN
5 kN
10 kN

PROBLEM 7.97

Knowing that $d_C = 3$ m, determine (*a*) the distances d_B and d_D (*b*) the reaction at *E*.

SOLUTION

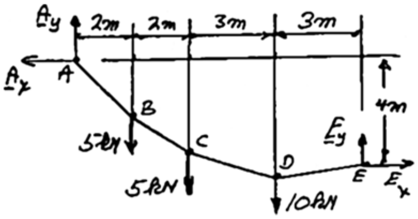
Free body: Portion *ABC*


A_y
2m
2m
A_x
A
B
$d_C=3m$
5kN
5kN
C

$$+\circlearrowleft \Sigma M_C = 0: \quad 3A_x - 4A_y + (5 \text{ kN})(2 \text{ m}) = 0$$

$$A_x = \frac{4}{3}A_y - \frac{10}{3} \qquad (1)$$

Free body: Entire cable


A_y
2m
2m
3m
3m
A_x
A
B
4m
5kN
C
E_y
D
E
E_x
5kN
10kN

$$+\circlearrowleft \Sigma M_E = 0: \quad 4A_x - 10A_y + (5 \text{ kN})(8 \text{ m}) + (5 \text{ kN})(6 \text{ m}) + (10 \text{ kN})(3 \text{ m}) = 0$$

$$4A_x - 10A_y + 100 = 0$$

Substitute from Eq. (1):

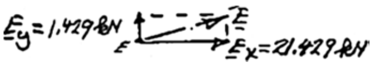
$$4\left(\frac{4}{3}A_y - \frac{10}{3}\right) - 10A_y + 100 = 0$$

$$A_y = +18.571 \text{ kN} \qquad \mathbf{A}_y = 18.571 \text{ kN} \uparrow$$

Eq. (1) $$A_x = \frac{4}{3}(18.511) - \frac{10}{3} = +21.429 \text{ kN} \qquad \mathbf{A}_x = 21.429 \text{ kN} \leftarrow$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad -A_x + E_x = 0 \quad -21.429 + E_x = 0 \qquad \mathbf{E}_x = 21.429 \text{ kN} \rightarrow$$

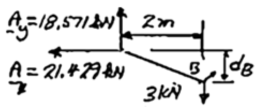
$$+\uparrow \Sigma F_y = 0: \quad 18.571 \text{ kN} + E_y + 5 \text{ kN} + 5 \text{ kN} + 10 \text{ kN} = 0 \qquad \mathbf{E}_y = 1.429 \text{ kN} \uparrow$$


$E_y=1.429kN$
E
E
$E_x=21.429kN$

(*b*) $\mathbf{E} = 21.5$ kN ∠3.81° ◀

PROBLEM 7.97 (Continued)

(*a*) Portion *AB*



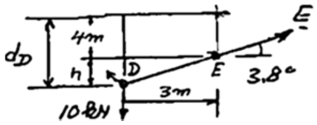
$A_y = 18.571$ kN
2m
$A = 21.429$ kN
B
d_B
3kN

$$+\circlearrowleft \Sigma M_B = 0: \quad (18.571 \text{ kN})(2 \text{ m}) - (21.429 \text{ kN})d_B = 0$$

$d_B = 1.733$ m ◀

Portion *DE*

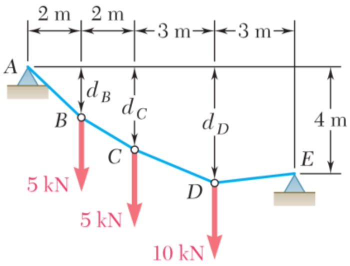
Geometry



d_D
4m
h
D
E
3m
10kN
3.8°
E

$$h = (3 \text{ m}) \tan 3.8°$$
$$= 0.199 \text{ m}$$
$$d_D = 4 \text{ m} + 0.199 \text{ m}$$

$d_D = 4.20$ m ◀

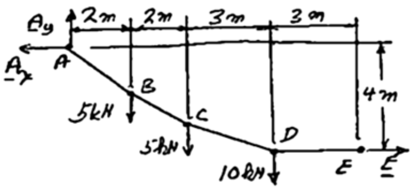

2 m
2 m
3 m
3 m
A
d_B
d_C
d_D
4 m
B
C
E
D
5 kN
5 kN
10 kN

PROBLEM 7.98

Determine (*a*) distance d_C for which portion *DE* of the cable is horizontal, (*b*) the corresponding reactions at *A* and *E*.

SOLUTION

Free body: Entire cable

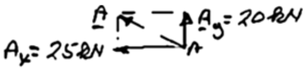

A_y
2m
2m
3m
3m
A_x
A
B
4m
5kN
C
D
5kN
10kN
E
E

(*b*) $+\uparrow \Sigma F_y = 0\text{:}\quad A_y - 5\text{ kN} - 5\text{ kN} - 10\text{ kN} = 0$ $\qquad \mathbf{A}_y = 20\text{ kN}\uparrow$

$+\circlearrowleft \Sigma M_A = 0\text{:}\quad E(4\text{ m}) - (5\text{ kN})(2\text{ m}) - (5\text{ kN})(4\text{ m}) - (10\text{ kN})(7\text{ m}) = 0$

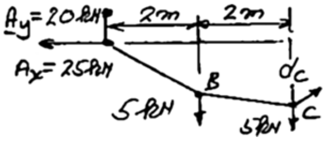
$E = +25\text{ kN}$ $\qquad \mathbf{E} = 25.0\text{ kN} \longrightarrow$ ◀

$\Sigma F_x = 0\text{:}\quad -A_x + 25\text{ kN} = 0$ $\qquad \mathbf{A}_x = 25\text{ kN} \longleftarrow$


A
$A_y = 20\text{kN}$
$A_x = 25\text{kN}$
A

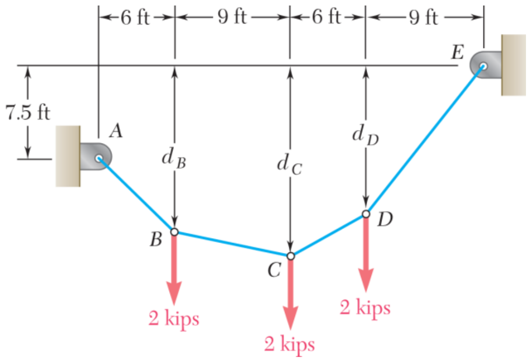
$\mathbf{A} = 32.0\text{ kN} \measuredangle 38.7°$ ◀

(*a*) Free body: Portion *ABC*


$A_y = 20\text{kN}$
2m
2m
$A_x = 25\text{kN}$
d_C
B
5kN
5kN
C

$+\circlearrowleft \Sigma M_C = 0\text{:}\quad (25\text{ kN})d_C - (20\text{ kN})(4\text{ m}) + (5\text{ kN})(2\text{ m}) = 0$

$25d_C - 70 = 0$ $\qquad d_C = 2.80\text{ m}$ ◀

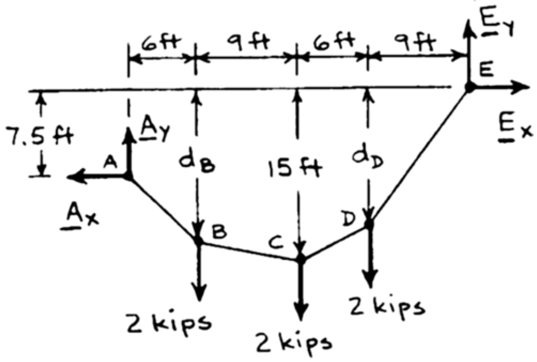

6 ft
9 ft
6 ft
9 ft
E
7.5 ft
A
d_B
d_C
d_D
D
B
C
2 kips
2 kips
2 kips

PROBLEM 7.99

If d_C = 15 ft, determine (*a*) the distances d_B and d_D, (*b*) the maximum tension in the cable.

SOLUTION

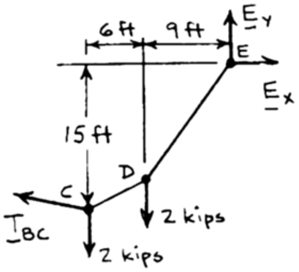
Free body: Entire cable


6 ft
9 ft
6 ft
9 ft
E_y
E
E_x
7.5 ft
A_y
A
A_x
d_B
15 ft
d_D
B
C
D
2 kips
2 kips
2 kips

$$+\circlearrowleft \Sigma M_A = 0: \quad E_y(30\text{ ft}) - E_x(7.5\text{ ft}) - (2\text{ kips})(6\text{ ft}) - (2\text{ kips})(15\text{ ft}) - (2\text{ kips})(21\text{ ft}) = 0$$

$$7.5E_x - 30E_y + 84 = 0 \qquad (1)$$

Free body: Portion *CDE*


6 ft
9 ft
E_y
E
E_x
15 ft
D
C
T_{BC}
2 kips
2 kips

$$+\circlearrowleft \Sigma M_C = 0: \quad E_y(15\text{ ft}) - E_x(15\text{ ft}) - (2\text{ kips})(6\text{ ft}) = 0$$

$$15E_x - 15E_y + 12 = 0 \qquad (2)$$

$$\text{Eq. (1)} \times \frac{1}{2}: \quad 3.75E_x - 15E_y + 42 = 0 \qquad (3)$$

$$(2) - (3): \quad 11.25E_x - 30 = 0$$

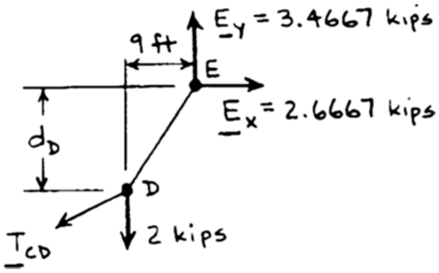
$$E_x = 2.6667\text{ kips}$$

Eq. (1): $7.5(2.6667) - 30E_y + 84 = 0 \quad E_y = 3.4667\text{ kips}$

$$T_m = \sqrt{E_x^2 + E_y^2} = \sqrt{(2.6667)^2 + (3.4667)^2} \qquad T_m = 4.37\text{ kips} \blacktriangleleft$$

PROBLEM 7.99 (Continued)

Free body: Portion *DE*


E_y = 3.4667 kips
9 ft
E
E_x = 2.6667 kips
d_D
D
T_{CD}
2 kips

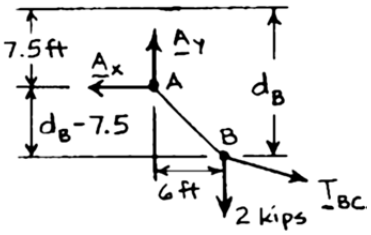
$$+\circlearrowleft\Sigma M_D = 0: \quad (3.4667 \text{ kips})(9 \text{ ft}) - (2.6667 \text{ kips})d_D = 0 \qquad d_D = 11.70 \text{ ft} \blacktriangleleft$$

Return to free body of entire cable (with $E_x = 2.6667$ kips, $E_y = 3.4667$ kips)

$$+\uparrow \Sigma F_y = 0: \quad A_y - 3(2 \text{ kips}) + 3.4667 \text{ kips} = 0 \qquad A_y = 2.5333 \text{ kips}$$

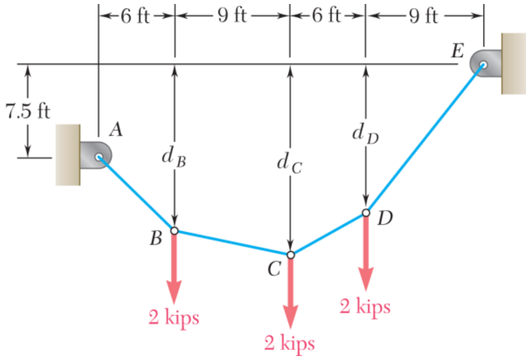
$$\xrightarrow{+} \Sigma F_x = 0: \quad 2.6667 \text{ kips} - A_x = 0 \qquad A_x = 2.6667 \text{ kips}$$

Free body: Portion *AB*


7.5 ft
A_x
A_y
A
d_B
d_B-7.5
B
6 ft
2 kips
T_{BC}

$$+\circlearrowleft\Sigma M_B = 0: \quad A_x(d_B - 7.5) - A_y(6) = 0$$

$$(2.6667)(d_B - 7.5) - (2.5333)(6) = 0 \qquad d_B = 13.20 \text{ ft} \blacktriangleleft$$



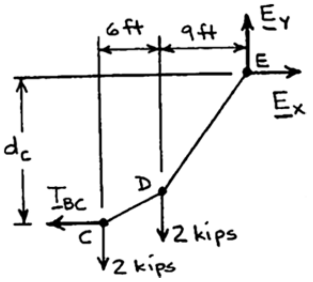
6 ft
9 ft
6 ft
9 ft
E
7.5 ft
A
d_B
d_C
d_D
D
B
C
2 kips
2 kips
2 kips

PROBLEM 7.100

Determine (*a*) the distance d_C for which portion *BC* of the cable is horizontal, (*b*) the corresponding components of the reaction at *E*.

SOLUTION

Free body: Portion *CDE*



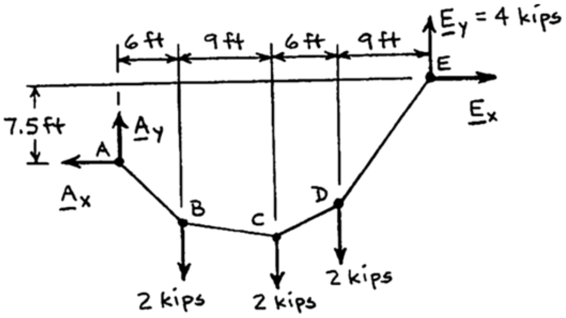
6 ft
9 ft
E_y
E
E_x
d_c
T_{BC}
D
C
2 kips
2 kips

$$+\uparrow \Sigma F_y = 0: \quad E_y - 2(2 \text{ kips}) = 0 \quad E_y = 4 \text{ kips}$$

$$+\circlearrowleft \Sigma M_C = 0: \quad (4 \text{ kips})(15 \text{ ft}) - E_x d_C - (2 \text{ kips})(6 \text{ ft}) = 0$$

$$E_x d_C = 48 \text{ kip} \cdot \text{ft} \qquad (1)$$

Free body: Entire cable



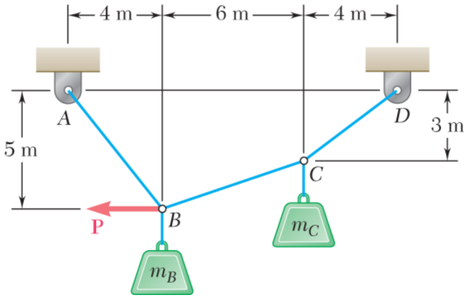
$E_y = 4$ kips
6 ft
9 ft
6 ft
9 ft
E
E_x
7.5 ft
A_y
A
A_x
B
C
D
2 kips
2 kips
2 kips

$$+\circlearrowleft \Sigma M_A = 0: \quad (4 \text{ kips})(30 \text{ ft}) - E_x(7.5 \text{ ft}) - (2 \text{ kips})(6 \text{ ft}) - (2 \text{ kips})(15 \text{ ft}) - (2 \text{ kips})(21 \text{ ft}) = 0$$

$$E_x = 4.8 \text{ kips}$$

From Eq. (1): $$d_C = \frac{48}{E_x} = \frac{48}{4.8}$$ $d_C = 10.00$ ft ◀

Components of reaction at *E*: $\mathbf{E}_x = 4.80$ kips $\longrightarrow$; $\mathbf{E}_y = 4.00$ kips $\uparrow$ ◀

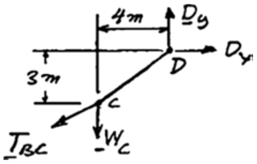

4 m
6 m
4 m
A
D
5 m
3 m
C
B
P
m_C
m_B

PROBLEM 7.101

Knowing that $m_B = 70$ kg and $m_C = 25$ kg, determine the magnitude of the force **P** required to maintain equilibrium.

SOLUTION

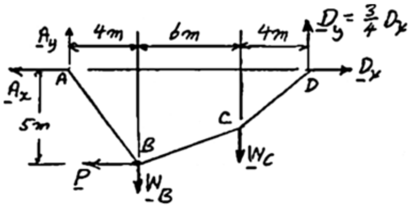
Free body: Portion *CD*


4m
D_y
D_x
3m
D
C
T_{BC}
W_C

$$+\circlearrowleft \Sigma M_C = 0: \quad D_y(4\text{ m}) - D_x(3\text{ m}) = 0$$

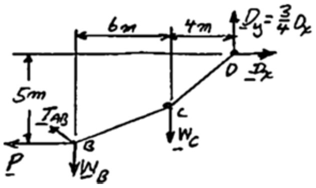
$$D_y = \frac{3}{4}D_x$$

Free body: Entire cable


A_y
4m
6m
4m
$D_y = \frac{3}{4}D_x$
D_x
A_x
A
D
5m
C
B
P
W_B
W_C

$$+\circlearrowleft \Sigma M_A = 0: \quad \frac{3}{4}D_x(14\text{ m}) - W_B(4\text{ m}) - W_C(10\text{ m}) - P(5\text{ m}) = 0 \qquad (1)$$

Free body: Portion *BCD*


6m
4m
$D_y = \frac{3}{4}D_x$
D
D_x
5m
T_{AB}
C
W_C
B
P
W_B

$$+\circlearrowleft \Sigma M_B = 0: \quad \frac{3}{4}D_x(10\text{ m}) - D_x(5\text{ m}) - W_C(6\text{ m}) = 0$$

$$D_x = 2.4W_C \qquad (2)$$

For $\quad m_B = 70\text{ kg} \quad m_C = 25\text{ kg}$

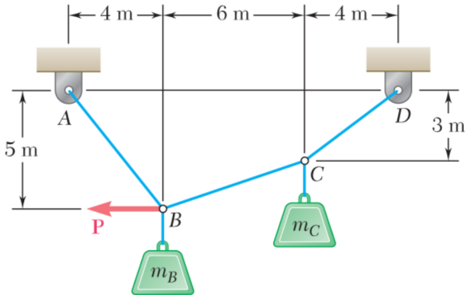
$g = 9.81\text{ m/s}^2$: $\quad W_B = 70g \quad W_C = 25g$

Eq. (2): $\quad D_x = 2.4W_C = 2.4(25g) = 60g$

Eq. (1): $\quad \frac{3}{4}60g(14) - 70g(4) - 25g(10) - 5P = 0$

$$100g - 5P = 0: \quad P = 20g$$

$$P = 20(9.81) = 196.2\text{ N} \qquad P = 196.2\text{ N} ◀$$

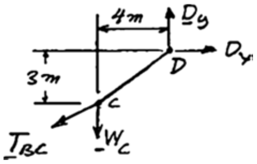

4 m
6 m
4 m
A
D
3 m
5 m
C
B
P
m_C
m_B

PROBLEM 7.102

Knowing that m_B= 18 kg and m_C= 10 kg, determine the magnitude of the force **P** required to maintain equilibrium.

SOLUTION

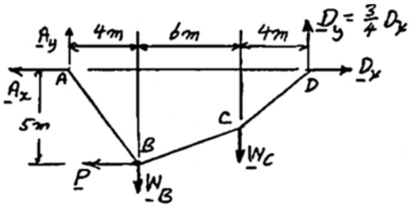
Free body: Portion *CD*


4m
D_y
D_x
D
3m
C
T_{BC}
W_C

$$+\circlearrowleft \Sigma M_C = 0: \quad D_y(4\text{ m}) - D_x(3\text{ m}) = 0$$

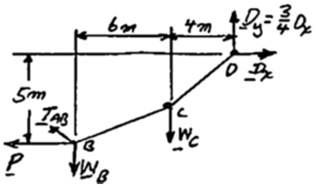
$$D_y = \frac{3}{4}D_x$$

Free body: Entire cable


A_y
4m
6m
4m
$D_y = \frac{3}{4}D_x$
D_x
A_x
A
D
C
5m
B
P
W_C
W_B

$$+\circlearrowleft \Sigma M_A = 0: \quad \frac{3}{4}D_x(14\text{ m}) - W_B(4\text{ m}) - W_C(10\text{ m}) - P(5\text{ m}) = 0 \tag{1}$$

Free body: Portion *BCD*


6m
4m
$D_y = \frac{3}{4}D_x$
D
D_x
5m
T_{AB}
C
B
P
W_C
W_B

$$+\circlearrowleft \Sigma M_B = 0: \quad \frac{3}{4}D_x(10\text{ m}) - D_x(5\text{ m}) - W_C(6\text{ m}) = 0$$

$$D_x = 2.4W_C \tag{2}$$

For $\quad m_B = 18\text{ kg} \quad m_C = 10\text{ kg}$

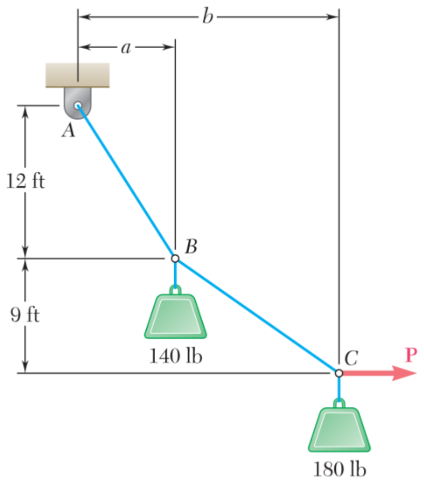
$g = 9.81\text{ m/s}^2$: $\quad W_B = 18g \quad W_C = 10g$

Eq. (2): $\quad D_x = 2.4W_C = 2.4(10g) = 24g$

Eq. (1): $\quad \frac{3}{4}24g(14) - (18g)(4) - (10g)(10) - 5P = 0$

$80g - 5P$: $\quad P = 16g$

$P = 16(9.81) = 156.96\text{ N}$ $\qquad P = 157.0\text{ N}$ ◀

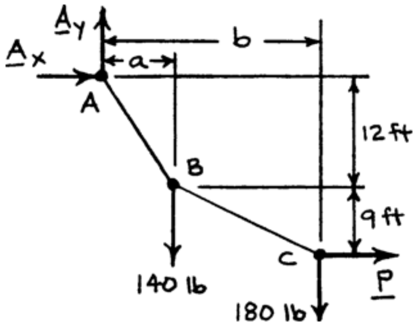

b
a
A
12 ft
B
9 ft
140 lb
C
P
180 lb

PROBLEM 7.103

Cable *ABC* supports two loads as shown. Knowing that $b = 21$ ft, determine (*a*) the required magnitude of the horizontal force **P**, (*b*) the corresponding distance *a*.

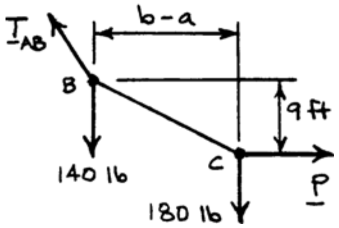
SOLUTION

Free body: *ABC*


Ay
Ax
b
a
A
12 ft
B
9 ft
140 lb
C
P
180 lb

$$+\circlearrowleft \Sigma M_A = 0: \quad P(12\text{ ft}) - (140\text{ lb})a - (180\text{ lb})b = 0 \qquad (1)$$

Free body: *BC*


TAB
b − a
B
9 ft
140 lb
C
P
180 lb

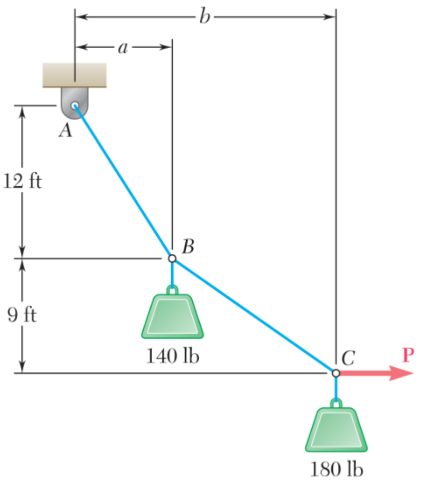
$$+\circlearrowleft \Sigma M_B = 0: \quad P(9\text{ ft}) - (180\text{ lb})(b-a) = 0 \qquad (2)$$

Data $\quad b = 21$ ft $\quad$ Eq. (1): $\quad 21P - 140a - 180(21) = 0 \qquad P = \frac{20}{3}a + 180 \qquad (3)$

Eq. (2): $\quad 9P - 180(21-a) = 0 \qquad P = -20a + 420 \qquad (4)$

Equate (3) and (4) through P: $\quad \frac{20}{3}a + 180 = -20a + 420 \qquad$ (*b*) $\quad a = 9.00$ ft ◀

Eq. (3): $\quad P = \frac{20}{3}(9.00) + 180 \qquad$ (*a*) $\quad P = 240$ lb ◀

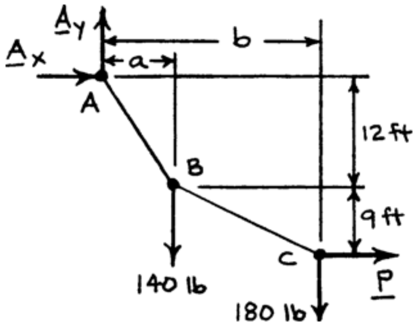

b
a
A
12 ft
B
9 ft
140 lb
C
P
180 lb

PROBLEM 7.104

Cable *ABC* supports two loads as shown. Determine the distances *a* and *b* when a horizontal force **P** of magnitude 200 lb is applied at *A*.

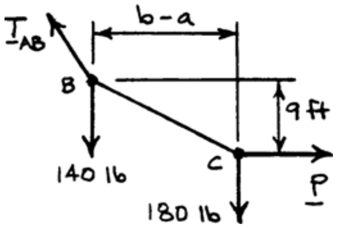
SOLUTION

Free body: *ABC*


A_y
A_x
b
a
A
12 ft
B
9 ft
140 lb
C
P
180 lb

$$+\circlearrowleft \Sigma M_A = 0: \quad P(12 \text{ ft}) - (140 \text{ lb})a - (180 \text{ lb})b = 0 \qquad (1)$$

Free body: *BC*


T_{AB}
b − a
B
9 ft
140 lb
C
P
180 lb

$$+\circlearrowleft \Sigma M_B = 0: \quad P(9 \text{ ft}) - (180 \text{ lb})(b-a) = 0 \qquad (2)$$

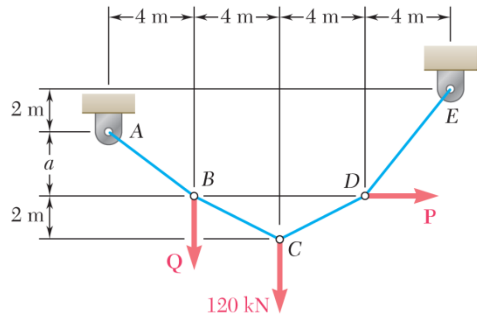
Data $P = 200 \text{ lb}$

Eq. (1): $200(21) - 140a - 180b = 0$ (3)

Eq. (2): $200(9) + 180a - 180b = 0$ (4)

(3) − (4): $2400 - 320a = 0$ $a = 7.50 \text{ ft}$ ◀

Eq. (2): $(200 \text{ lb})(9 \text{ ft}) - (180 \text{ lb})(b - 7.50 \text{ ft}) = 0$ $b = 17.50 \text{ ft}$ ◀

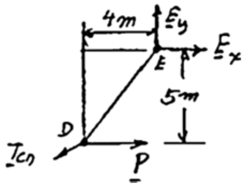

4 m
4 m
4 m
4 m
2 m
a
2 m
A
B
C
D
E
P
Q
120 kN

PROBLEM 7.105

If a= 3 m, determine the magnitudes of **P** and **Q** required to maintain the cable in the shape shown.

SOLUTION

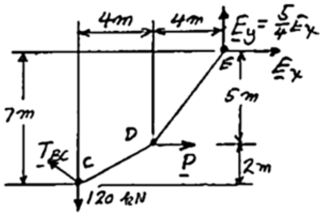
Free body: Portion DE


4m
E_y
E_x
E
5m
D
T_CD
P

$$+\circlearrowleft \Sigma M_D = 0: \quad E_y(4\text{ m}) - E_x(5\text{ m}) = 0$$

$$E_y = \frac{5}{4}E_x$$

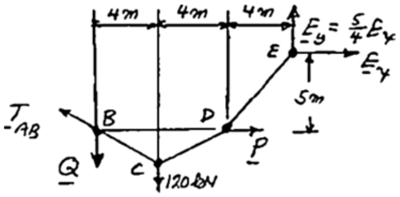
Free body: Portion CDE


4m
4m
E_y = 5/4 E_x
E
E_x
5m
7m
D
T_BC
C
P
2m
120 kN

$$+\circlearrowleft \Sigma M_C = 0: \quad \frac{5}{4}E_x(8\text{ m}) - E_x(7\text{ m}) - P(2\text{ m}) = 0$$

$$E_x = \frac{2}{3}P \qquad (1)$$

Free body: Portion $BCDE$


4m
4m
4m
E_y = 5/4 E_x
E
E_x
5m
T_AB
B
D
P
Q
C
120 kN

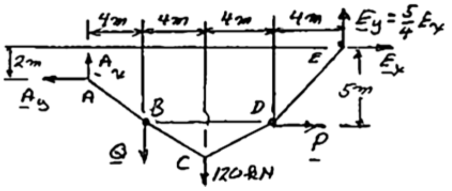
$$+\circlearrowleft \Sigma M_B = 0: \quad \frac{5}{4}E_x(12\text{ m}) - E_x(5\text{ m}) - (120\text{ kN})(4\text{ m}) = 0$$

$$10E_x - 480 = 0; \quad E_x = 48\text{ kN}$$

Eq. (1):

$$48\text{ kN} = \frac{2}{3}P \qquad P = 72.0\text{ kN} \blacktriangleleft$$

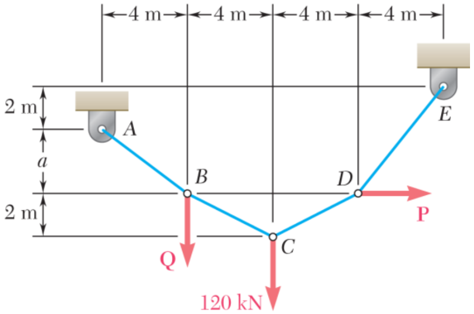
Free body: Entire cable


4m
4m
4m
4m
$E_y = \frac{5}{4}E_x$
2m
A_x
E_x
E
A_y
A
5m
B
D
P
Q
C
120 kN

$$+\circlearrowleft \Sigma M_A = 0: \quad \frac{5}{4}E_x(16\text{ m}) - E_x(2\text{ m}) + P(3\text{ m}) - Q(4\text{ m}) - (120\text{ kN})(8\text{ m}) = 0$$

$$(48\text{ kN})(20\text{ m} - 2\text{ m}) + (72\text{ kN})(3\text{ m}) - Q(4\text{ m}) - 960\text{ kN}\cdot\text{m} = 0$$

$$4Q = 120 \qquad Q = 30.0\text{ kN} \blacktriangleleft$$


4 m
4 m
4 m
4 m
2 m
a
2 m
A
B
C
D
E
P
Q
120 kN

PROBLEM 7.106

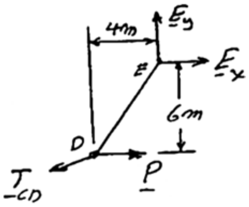
If $a = 4$ m, determine the magnitudes of **P** and **Q** required to maintain the cable in the shape shown.

SOLUTION

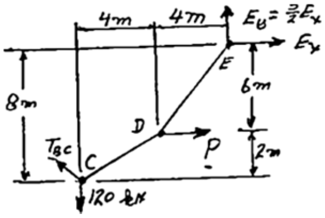
Free body: Portion *DE*

$$+\circlearrowleft \Sigma M_D = 0: \quad E_y(4\text{ m}) - E_x(6\text{ m}) = 0$$

$$E_y = \frac{3}{2}E_x$$


4m
E_y
E_x
E
6m
D
T_CD
P

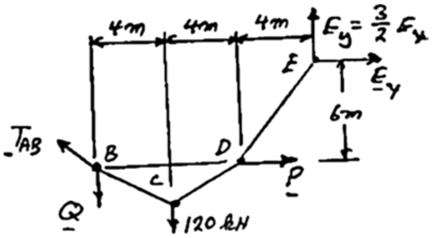
Free body: Portion *CDE*


4m
4m
E_y = 3/2 E_x
E_x
E
6m
8m
D
T_BC
C
P
2m
120 kN

$$+\circlearrowleft \Sigma M_C = 0: \quad \frac{3}{2}E_x(8\text{ m}) - E_x(8\text{ m}) - P(2\text{ m}) = 0$$

$$E_x = \frac{1}{2}P \qquad (1)$$

Free body: Portion *BCDE*


4m
4m
4m
E_y = 3/2 E_x
E
E_x
6m
T_AB
B
D
C
P
Q
120 kN

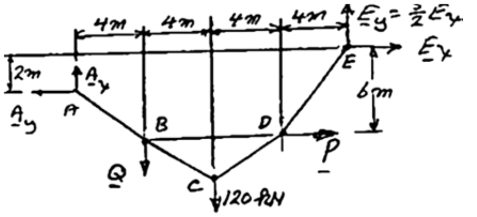
$$+\circlearrowleft \Sigma M_B = 0: \quad \frac{3}{2}E_x(12\text{ m}) - E_x(6\text{ m}) + (120\text{ kN})(4\text{ m}) = 0$$

$$12E_x = 480 \qquad E_x = 40\text{ kN}$$

Eq (1): $E_x = \frac{1}{2}P; \qquad 40\text{ kN} = \frac{1}{2}P$ $\qquad P = 80.0\text{ kN}$ ◀

PROBLEM 7.106 (Continued)

Free body: Entire cable


4m
4m
4m
4m
$E_y = \frac{3}{2}E_x$
E_x
E
2m
A_y
A_y
A
B
D
6m
P
Q
C
120 kN

$$+\circlearrowleft \Sigma M_A = 0: \quad \frac{3}{2}E_x(16\text{ m}) - E_x(2\text{ m}) + P(4\text{ m}) - Q(4\text{ m}) - (120\text{ kN})(8\text{ m}) = 0$$

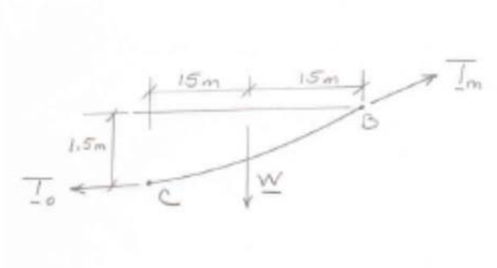
$$(40\text{ kN})(24\text{ m} - 2\text{ m}) + (80\text{ kN})(4\text{ m}) - Q(4\text{ m}) - 960\text{ kN}\cdot\text{m} = 0$$

$$4Q = 240 \qquad Q = 60.0\text{ kN} \blacktriangleleft$$

PROBLEM 7.107

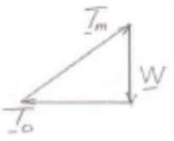
An electric wire having a mass per unit length of 0.6 kg/m is strung between two insulators at the same elevation that are 60 m apart. Knowing that the sag of the cable is 1.5 m, determine (*a*) the maximum tension in the wire, (*b*) the length of the wire.

SOLUTION



$$w = (0.6\text{ kg/m})(9.81\text{ m/s}^2)$$
$$= 5.886\text{ N/m}$$
$$W = (5.886\text{ N/m})(30\text{ m})$$
$$W = 176.580\text{ N}$$

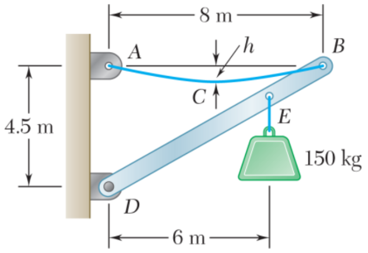
(*a*) $\circlearrowleft + \Sigma M_B = 0\!: \quad -T_0(1.5\text{ m}) - (176.580\text{ N})(15\text{ m}) = 0$



$$T_0 = 1765.80\text{ N}$$
$$T_m^2 = (176.580\text{ N})^2 + (1765.80\text{ N})^2 \qquad T_m = 1775\text{ N} ◀$$

(*b*)
$$s_B = x_B\left[1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 + \cdots\right]$$
$$= 30\text{ m}\left[1 + \frac{2}{3}\left(\frac{1.5\text{ m}}{30\text{ m}}\right)^2 + \cdots\right]$$
$$= 30.05\text{ m}$$

Length $= 2s_B = 2(30.05\text{ m})$ Length = 60.1 m ◀

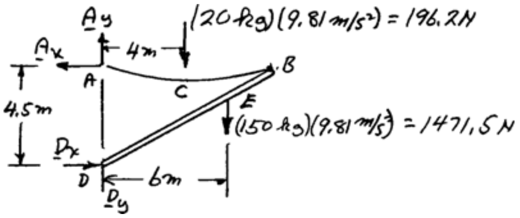

8 m
A
h
B
C
E
4.5 m
150 kg
D
6 m

PROBLEM 7.108

The total mass of cable *ACB* is 20 kg. Assuming that the mass of the cable is distributed uniformly along the horizontal, determine (*a*) the sag *h*, (*b*) the slope of the cable at *A*.

SOLUTION

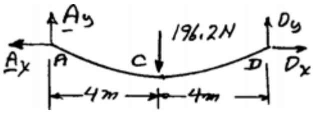
Free body: Entire frame


Ay
(20 kg)(9.81 m/s²) = 196.2N
Ax
4m
A
B
C
E
4.5m
(150 kg)(9.81 m/s²) = 1471.5N
Dx
D
6m
Dy

$$+\circlearrowleft \Sigma M_D = 0: \quad A_x(4.5\text{ m}) - (196.2\text{ N})(4\text{ m}) - (1471.5\text{ N})(6\text{ m}) = 0$$

$$A_x = 2136.4\text{ N}$$

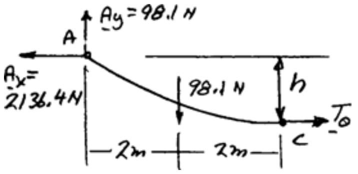
Free body: Entire cable


Ay
196.2N
Dy
Ax
A
C
D
Dx
4m
4m

$$+\circlearrowleft \Sigma M_D = 0: \quad A_y(8\text{ m}) - (196.2\text{ N})(4\text{ m}) = 0$$

$$A_y = 98.1\text{ N}$$

(*a*) Free body: Portion *AC*


Ay = 98.1 N
A
Ax = 2136.4N
98.1 N
h
T0
C
2m
2m

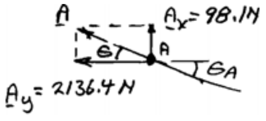
$$\Sigma F_x = 0: \quad T_0 = A_x = 2136.4\text{ N}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad T_0 h - (98.1\text{ N})(2\text{ m}) = 0$$

$$(2136.4\text{ N})h - 196.2\text{ N}\cdot\text{m} = 0$$

$$h = 0.09183\text{ m} \qquad h = 91.8\text{ mm} \blacktriangleleft$$

(*b*)


A
Ax = 98.1N
θA
A
θA
Ay = 2136.4N

$$\tan\theta_A = \frac{A_x}{A_y} = \frac{98.1\text{ N}}{2136.4\text{ N}}$$

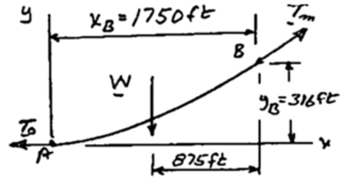
$$\tan\theta_A = 0.045918$$

$$\theta_A = 2.629^\circ \qquad \theta_A = 2.63^\circ \blacktriangleleft$$

PROBLEM 7.109

The center span of the George Washington Bridge, as originally constructed, consisted of a uniform roadway suspended from four cables. The uniform load supported by each cable was $w = 9.75$ kips/ft along the horizontal. Knowing that the span L is 3500 ft and that the sag h is 316 ft, determine for the original configuration (*a*) the maximum tension in each cable, (*b*) the length of each cable.

SOLUTION



$$W = wx_B = (9.75 \text{ kips/ft})(1750 \text{ ft})$$

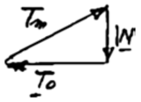
$$W = 17{,}063 \text{ kips}$$

$$+\circlearrowleft \Sigma M_B = 0: \quad T_0(316 \text{ ft}) - (17063 \text{ kips})(875 \text{ ft}) = c$$

$$T_0 = 47{,}247 \text{ kips}$$

(*a*)

$$T_m = \sqrt{T_0^2 + W^2}$$



$$= \sqrt{(47{,}247 \text{ kips})^2 + (17{,}063 \text{ kips})^2}$$

$$T_m = 50{,}200 \text{ kips} \blacktriangleleft$$

(*b*)

$$s_B = x_B\left[1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 - \frac{2}{5}\left(\frac{y_B}{x_B}\right)^4 + \cdots\right]$$

$$\frac{y_B}{x_B} = \frac{316 \text{ ft}}{1750 \text{ ft}} = 0.18057$$

$$= (1750 \text{ ft})\left[1 + \frac{2}{3}(0.18057)^2 - \frac{2}{5}(0.18057)^4 + \cdots\right]$$

$s_B = 1787.3$ ft; Length $= 2s_B = 3579.6$ ft $\qquad$ Length $= 3580$ ft ◀

PROBLEM 7.110

The center span of the Verrazano-Narrows Bridge consists of two uniform roadways suspended from four cables. The design of the bridge allows for the effect of extreme temperature changes that cause the sag of the center span to vary from $h_w = 386$ ft in winter to $h_s = 394$ ft in summer. Knowing that the span is $L = 4260$ ft, determine the change in length of the cables due to extreme temperature changes.

SOLUTION

Eq. 7.10:

$$s_B = x_B\left[1+\frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 - \frac{2}{5}\left(\frac{y_B}{x_B}\right)^4 + \cdots\right]$$

Winter:

$$y_B = h = 386 \text{ ft}, \quad x_B = \frac{1}{2}L = 2130 \text{ ft}$$

$$s_B = (2130)\left[1+\frac{2}{3}\left(\frac{386}{2130}\right)^2 - \frac{2}{5}\left(\frac{386}{2130}\right)^4 + \cdots\right] = 2175.715 \text{ ft}$$

Summer:

$$y_B = h = 394 \text{ ft}, \quad x_B = \frac{1}{2}L = 2130 \text{ ft}$$

$$s_B = (2130)\left[1+\frac{2}{3}\left(\frac{394}{2130}\right)^2 - \frac{2}{5}\left(\frac{394}{2130}\right)^4 + \cdots\right] = 2177.59 \text{ ft}$$

$$\Delta = 2(\Delta s_B) = 2(2177.59 \text{ ft} - 2175.715 \text{ ft}) = 2(1.875 \text{ ft})$$

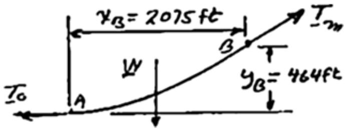
Change in length = 3.75 ft ◀

PROBLEM 7.111

Each cable of the Golden Gate Bridge supports a load w= 11.1 kips/ft along the horizontal. Knowing that the span L is 4150 ft and that the sag h is 464 ft, determine (*a*) the maximum tension in each cable, (*b*) the length of each cable.

SOLUTION

Eq. (7.8) Page 386:

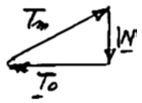


At B:

$$y_B = \frac{wx_B^2}{2T_0}$$

$$T_0 = \frac{wx_B^2}{2y_B} = \frac{(11.1\text{ kip/ft})(2075\text{ ft})^2}{2(464\text{ ft})}$$

(*a*)



$$T_0 = 51.500\text{ kips}$$

$$W = wx_B = (11.1\text{ kips/ft})(2075\text{ ft}) = 23.033\text{ kips}$$

$$T_m = \sqrt{T_0^2 + W^2} = \sqrt{(51.500\text{ kips})^2 + (23.033\text{ kips})^2}$$

$$T_m = 56,400\text{ kips} \blacktriangleleft$$

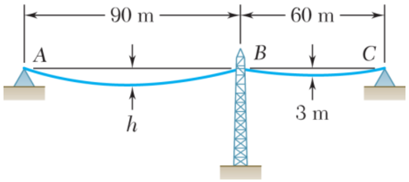
(*b*)

$$s_B = x_B\left[1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 - \frac{2}{5}\left(\frac{y_B}{y_B}\right)^4 + \cdots\right] \qquad \frac{y_B}{x_B} = \frac{464\text{ ft}}{2075\text{ ft}} = 0.22361$$

$$s_B = (2075\text{ ft})\left[1 + \frac{2}{3}(0.22361)^2 - \frac{2}{5}(0.22361)^4 + \cdots\right] = 2142.1\text{ ft}$$

Length $= 2s_B = 2(2142.1\text{ ft})$

Length $= 4284$ ft ◀

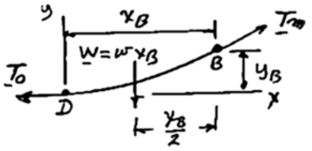


PROBLEM 7.112

Two cables of the same gauge are attached to a transmission tower at *B*. Since the tower is slender, the horizontal component of the resultant of the forces exerted by the cables at *B* is to be zero. Knowing that the mass per unit length of the cables is 0.4 kg/m, determine (*a*) the required sag *h*, (*b*) the maximum tension in each cable.

SOLUTION

$$W = wx_B$$



$$+\circlearrowleft \Sigma M_B = 0: \quad T_0 \; y_B - (wx_B)\frac{y_B}{2} = 0$$

(*a*) $$\text{Horiz. comp.} = T_0 = \frac{wx_B^2}{2y_B}$$

Cable *AB*: $x_B = 45 \text{ m}$

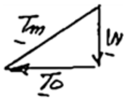
$$T_0 = \frac{w(45 \text{ m})^2}{2h}$$

Cable *BC*: $x_B = 30 \text{ m}, \quad y_B = 3 \text{ m}$

$$T_0 = \frac{w(30 \text{ m})^2}{2(3 \text{ m})}$$

Equate $T_0 = T_0$ $$\frac{w(45 \text{ m})^2}{2h} = \frac{w(30 \text{ m})^2}{2(3 \text{ m})}$$ $h = 6.75 \text{ m}$ ◀

(*b*) $$T_m^2 = T_0^2 + W^2$$



Cable *AB*: $w = (0.4 \text{ kg/m})(9.81 \text{ m/s}) = 3.924 \text{ N/m}$

$$x_B = 45 \text{ m}, \quad y_B = h = 6.75 \text{ m}$$

$$T_0 = \frac{wx_B^2}{2y_B} = \frac{(3.924 \text{ N/m})(45 \text{ m})^2}{2(6.75 \text{ m})} = 588.6 \text{ N}$$

$$W = wx_B = (3.924 \text{ N/m})(45 \text{ m}) = 176.58 \text{ N}$$

$$T_m^2 = (588.6 \text{ N})^2 + (176.58 \text{ N})^2$$

For *AB*: $T_m = 615 \text{ N}$ ◀

PROBLEM 7.112 (Continued)

Cable *BC*: $x_B = 30 \text{ m}, \quad y_B = 3 \text{ m}$

$$T_0 = \frac{wx_B^2}{2y_B} = \frac{(3.924 \text{ N/m})(30 \text{ m})^2}{2(3 \text{ m})} = 588.6 \text{ N} \quad \text{(Checks)}$$

$$W = wx_B = (3.924 \text{ N/m})(30 \text{ m}) = 117.72 \text{ N}$$

$$T_m^2 = (588.6 \text{ N})^2 + (117.72 \text{ N})^2$$

For *BC*: $T_m = 600 \text{ N}$ ◀

PROBLEM 7.113

A 76-m length of wire having a mass per unit length of 2.2 kg/m is used to span a horizontal distance of 75 m. Determine (a) the approximate sag of the wire, (b) the maximum tension in the wire. [Hint: Use only the first two terms of Eq. (7.10).]

SOLUTION

First two terms of Eq. 7.10

(*a*)

$$s_B = \frac{1}{2}(76\text{ m}) = 38\text{ m},$$

$$x_B = \frac{1}{2}(75\text{ m}) = 37.5\text{ m}$$

$$y_B = h$$

$$s_B = x_B\left[1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2\right]$$

$$38\text{ m} = 37.5\text{ m}\left[1 - \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2\right]$$

$$\left(\frac{y_B}{x_B}\right)^2 = 0.02$$

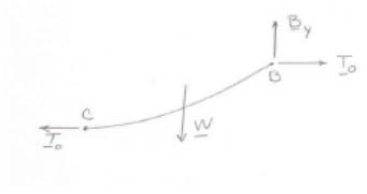
$$\frac{y_B}{x_B} = 0.141421$$

$$\frac{h}{37.5\text{ m}} = 0.141421$$

$$h = 5.3033\text{ m}$$

$h = 5.30\text{ m}$ ◀

(*b*) Free body: Portion *CB*



$$w = (2.2\text{ kg/m})(9.81\text{m}) = 21.582\text{ N/m}$$

$$W = s_B\ w = (38\text{ m})(21.582\text{ N/m})$$

$$W = 820.12\text{ N}$$

$$y_B = \frac{w{x_B}^2}{2y_B}: \quad T_0 = \frac{(2.2\text{ kg/m})(9.81\text{ m/s}^2)(37.5\text{ m})^2}{2(5.3033\text{ m})}$$

$$T_0 = 2861.6\text{ N}$$

$$B_x = T_0 = 2861.6\text{ N}$$

PROBLEM 7.113 (Continued)

$+\uparrow \Sigma F_y = 0: \quad B_y - 820.12 \text{ N} = 0 \quad B_y = 820.12 \text{ N}$

$T_m = \sqrt{B_x^2 + B_y^2} = \sqrt{(2861.6 \text{ N})^2 + (820.12 \text{ N})^2} \qquad T_m = 2980 \text{ N}$ ◀

PROBLEM 7.114

A cable of length $L + \Delta$ is suspended between two points that are at the same elevation and a distance L apart. (*a*) Assuming that Δ is small compared to L and that the cable is parabolic, determine the approximate sag in terms of L and Δ. (*b*) If $L = 100$ ft and $\Delta = 4$ ft, determine the approximate sag. [*Hint:* Use only the first two terms of Eq. (7.10).

SOLUTION

Eq. 7.10 (First two terms)

(*a*)

$$s_B = x_B\left[1+\frac{2}{3}\left(\frac{y_B}{x_B}\right)^2\right]$$

$$x_B = L/2$$

$$s_B = \frac{1}{2}(L+\Delta)$$

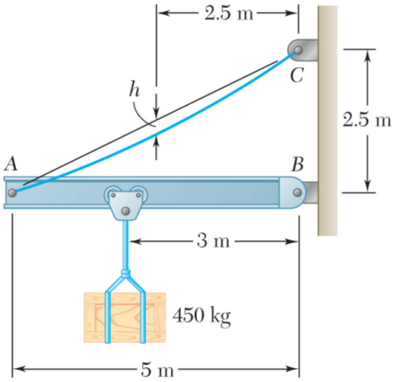
$$y_B = h$$

$$\frac{1}{2}(L+\Delta) = \frac{L}{2}\left[1+\frac{2}{3}\left(\frac{h}{\frac{L}{2}}\right)^2\right]$$

$$\frac{\Delta}{2} = \frac{4}{3}\frac{h^2}{L}; \quad h^2 = \frac{3}{8}L\Delta; \qquad h = \sqrt{\frac{3}{8}L\Delta} \blacktriangleleft$$

(*b*)

$$L = 100 \text{ ft}, \quad h = 4 \text{ ft}. \qquad h = \sqrt{\frac{3}{8}(100)(4)}; \qquad h = 12.25 \text{ ft} \blacktriangleleft$$


2.5 m
h
C
2.5 m
A
B
3 m
450 kg
5 m

PROBLEM 7.115

The total mass of cable *AC* is 25 kg. Assuming that the mass of the cable is distributed uniformly along the horizontal, determine the sag *h* and the slope of the cable at *A* and *C*.

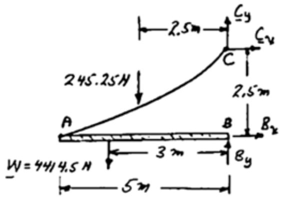
SOLUTION

Cable:

$$m = 25 \text{ kg}$$

$$W = 25\,(9.81)$$

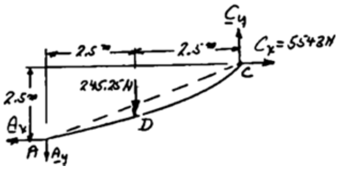
$$= 245.25 \text{ N}$$


2.5m
Cy
Cx
C
245.25N
2.5m
A
B
Bx
3m
By
W = 4414.5 N
5m

Block:

$$m = 450 \text{ kg}$$

$$W = 4414.5 \text{ N}$$


Cy
2.5m
2.5m
Cx = 5543N
C
2.5m
245.25N
D
Ax
A
Ay

$$+\circlearrowleft \Sigma M_B = 0\!: \quad (245.25)(2.5) + (4414.5)(3) - C_x(2.5) = 0$$

$$C_x = 5543 \text{ N}$$

$$\Sigma F_x = 0\!: \quad A_x = C_x = 5543 \text{ N}$$

$$+\circlearrowleft \Sigma M_A = 0\!: \quad C_y(5) - (5543)(2.5) - (245.25)(2.5) = 0$$

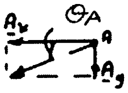
$$\mathbf{C}_y = 2894 \text{ N} \uparrow$$

$$+\uparrow \Sigma F_y = 0\!: \quad C_y - A_y - 245.25 \text{ N} = 0$$

$$2894\; N - A_y - 245.25 \text{ N} = 0 \qquad \mathbf{A}_y = 2649 \text{ N} \downarrow$$

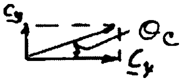
Point *A*:

$$\tan\theta_A = \frac{A_y}{A_x} = \frac{2649}{5543} = 0.4779; \qquad \theta_A = 25.5° ◀$$



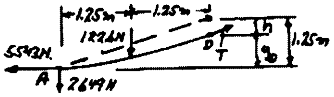
Point *C*:

$$\tan\theta_C = \frac{C_y}{C_x} = \frac{2894}{5543} = 0.5221; \qquad \theta_A = 27.6° ◀$$



Free body: Half cable

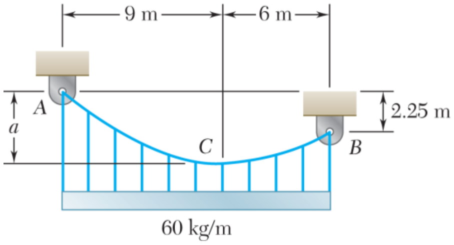
$$W = (12.5 \text{ kg})\, g = 122.6$$


1.25m
1.25m
122.6N
5543N
D
T
h
yD
1.25m
A
2649N

$$+\circlearrowleft \Sigma M_0 = 0\!: \quad (122.6 \text{ N})(1.25 \text{ M}) + (2649 \text{ N})(2.5 \text{ m}) - (5543 \text{ N})\, y_d = 0$$

$$y_d = 1.2224 \text{ m}; \text{ sag} = h = 1.25 \text{ m} - 1.2224 \text{ m}$$

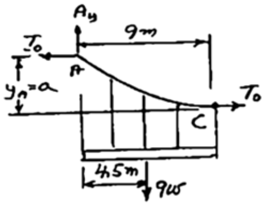
$$h = 0.0276 \text{ m} = 27.6 \text{ mm} ◀$$


9 m
6 m
A
a
C
2.25 m
B
60 kg/m

PROBLEM 7.116

Cable *ACB* supports a load uniformly distributed along the horizontal as shown. The lowest Point *C* is located 9 m to the right of *A*. Determine (*a*) the vertical distance *a*, (*b*) the length of the cable, (*c*) the components of the reaction at *A*.

SOLUTION

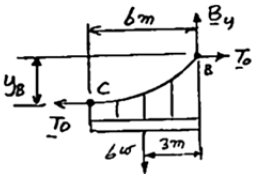

A_y
9m
T_o
A
y_A=a
T_o
C
4.5m
9w

Free body: Portion *AC*

$$+\uparrow \Sigma F_y = 0: \quad A_y - 9w = 0$$

$$\mathbf{A}_y = 9w \uparrow$$

$$+\circlearrowleft \Sigma M_A = 0: \quad T_0 a - (9w)(4.5 \text{ m}) = 0 \qquad (1)$$

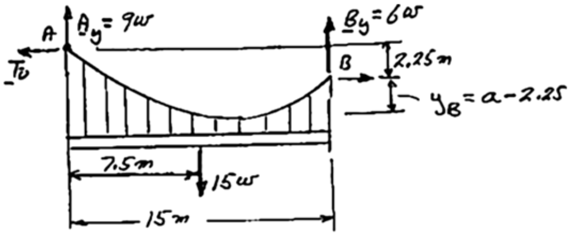

6m
B_y
T_o
y_B
C
B
T_o
6w
3m

Free body: Portion *CB*

$$+\uparrow \Sigma F_y = 0: \quad B_y - 6w = 0$$

$$\mathbf{B}_y = 6w \uparrow$$

Free body: Entire cable


A
A_y = 9w
T_o
B_y = 6w
B
2.25m
y_B = a - 2.25
7.5m
15w
15m

$$+\circlearrowleft \Sigma M_A = 0: \quad 15w(7.5 \text{ m}) - 6w(15 \text{ m}) - T_0(2.25 \text{ m}) = 0$$

(*a*) $T_0 = 10w$

Eq. (1): $T_0 a - (9w)(4.5 \text{ m}) = 0$

$10wa = (9w)(4.5) = 0$ $\qquad a = 4.05 \text{ m}$ ◀

(*b*) Length $=AC+CB$

Portion AC:

$$x_A = 9\text{ m},\quad y_A = a = 4.05\text{ m};\quad \frac{y_A}{x_A} = \frac{4.05}{9} = 0.45$$

$$s_{AC} = x_B\left[1+\frac{2}{3}\left(\frac{y_A}{x_A}\right)^2 - \frac{2}{5}\left(\frac{y_B}{x_A}\right)^4 + \cdots\right]$$

$$s_{AC} = 9\text{ m}\left(1+\frac{2}{3}0.45^2 - \frac{2}{5}0.45^4 + \cdots\right) = 10.067\text{ m}$$

Portion CB:

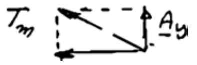
$$x_B = 6\text{ m},\quad y_B = 4.05 - 2.25 = 1.8\text{ m};\quad \frac{y_B}{x_B} = 0.3$$

$$s_{CB} = 6\text{ m}\left(1+\frac{2}{3}0.3^2 - \frac{2}{5}0.3^4 + \cdots\right) = 6.341\text{ m}$$

Total length $= 10.067\text{ m} + 6.341\text{ m}$

Total length $= 16.41\text{ m}$ ◀

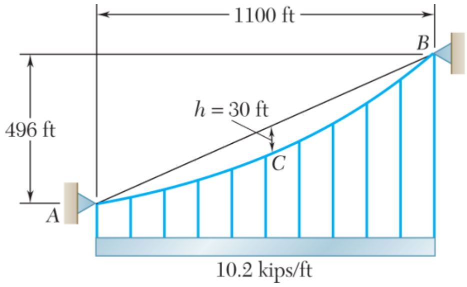
(*c*) Components of reaction at A.



$$A_y = 9w = 9(60\text{ kg/m})(9.81\text{ m/s}^2)$$
$$= 5297.4\text{ N}$$
$$A_x = T_0 = 10w = 10(60\text{ kg/m})(9.81\text{ m/s}^2)$$
$$= 5886\text{ N}$$

$\mathbf{A}_x = 5890\text{ N}$ ← ◀

$\mathbf{A}_y = 5300\text{ N}$ ↑ ◀

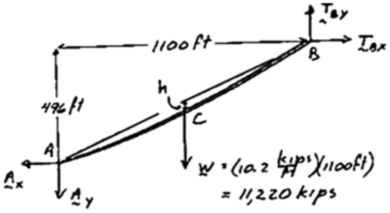

1100 ft
B
496 ft
$h = 30$ ft
C
A
10.2 kips/ft

PROBLEM 7.117

Each cable of the side spans of the Golden Gate Bridge supports a load *w*= 10.2 kips/ft along the horizontal. Knowing that for the side spans the maximum vertical distance *h* from each cable to the chord *AB* is 30 ft and occurs at midspan, determine (*a*) the maximum tension in each cable, (*b*) the slope at *B*.

SOLUTION

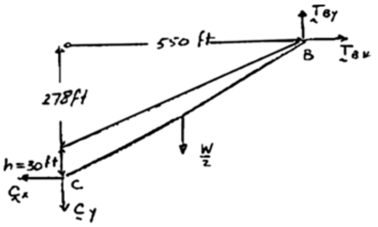
FBD AB:


T_{By}
1100 ft
T_{Bx}
B
h
496 ft
C
A
A_x
A_y
$W = (10.2 \frac{kips}{ft})(1100 ft)$
$= 11,220$ kips

$$\circlearrowleft \Sigma M_A = 0\text{:}\quad (1100\text{ ft})T_{By} - (496\text{ ft})T_{Bx} - (550\text{ ft})W = 0$$

$$11T_{By} - 4.96T_{Bx} = 5.5W \qquad (1)$$

FBD CB:


T_{By}
550 ft
B
T_{Bx}
278 ft
$h = 30$ ft
$\frac{W}{2}$
C_x
C
C_y

$$\circlearrowleft \Sigma M_C = 0\text{:}\quad (550\text{ ft})T_{By} - (278\text{ ft})T_{Bx} - (275\text{ ft})\frac{W}{2} = 0$$

$$11T_{By} - 5.56T_{Bx} = 2.75W \qquad (2)$$

Solving (1) and (2) $\quad T_{By} = 28,798$ kips

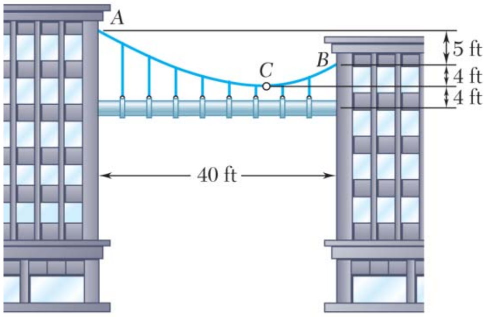
$$T_{Bx} = 51,425\text{ kips}$$

$$T_{\max} = T_B = \sqrt{T_{B_x}^2 + T_{B_y}^2} \qquad \tan\theta_B = \frac{T_{B_y}}{T_{B_x}}$$

So that

(*a*) $T_{\max} = 58,900$ kips ◀

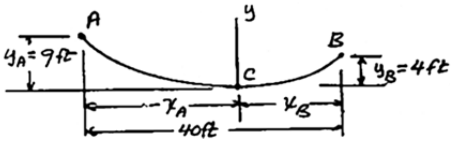
(*b*) $\theta_B = 29.2°$ ◀



PROBLEM 7.118

A steam pipe weighting 45 lb/ft that passes between two buildings 40 ft apart is supported by a system of cables as shown. Assuming that the weight of the cable system is equivalent to a uniformly distributed loading of 5 lb/ft, determine (*a*) the location of the lowest Point *C* of the cable, (*b*) the maximum tension in the cable.

SOLUTION



Note: $x_B - x_A = 40$ ft

or $x_A = x_B - 40$ ft

(*a*) Use Eq. 7.8

Point *A*: $y_A = \dfrac{wx_A^2}{2T_0}; \quad 9 = \dfrac{w(x_B - 40)^2}{2T_0}$ (1)

Point *B*: $y_B = \dfrac{wx_B^2}{2T_0}; \quad 4 = \dfrac{wx_B^2}{2T_0}$ (2)

Dividing (1) by (2): $\dfrac{9}{4} = \dfrac{(x_B - 40)^2}{x_B^2}; \quad x_B = 16$ ft

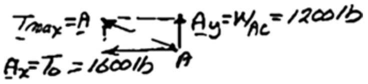
Point *C* is 16.00 ft to left of *B* ◀

(*b*) Maximum slope and thus $T_{\max}$ is at *A*

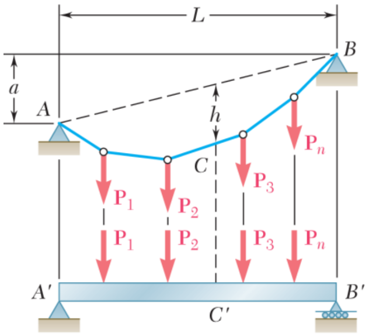
$$x_A = x_B - 40 = 16 - 40 = -24 \text{ ft}$$

$$y_A = \frac{wx_A^2}{2T_0}; \quad 9 \text{ ft} = \frac{(50 \text{ lb/ft})(-24 \text{ ft})^2}{2T_0}; \quad T_0 = 1600 \text{ lb}$$

$$W_{AC} = (50 \text{ lb/ft})(24 \text{ ft}) = 1200 \text{ lb}$$



$T_{\max} = 2000$ lb ◀



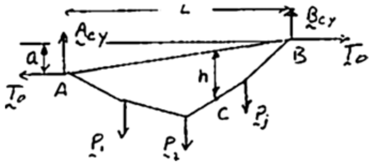
PROBLEM 7.119*

A cable *AB* of span *L* and a simple beam *A′B′* of the same span are subjected to identical vertical loadings as shown. Show that the magnitude of the bending moment at a point *C′* in the beam is equal to the product T_0h, where T_0 is the magnitude of the horizontal component of the tension force in the cable and *h* is the vertical distance between Point *C* and the chord joining the points of support *A* and *B*.

SOLUTION

$$\curvearrowleft \Sigma M_B = 0: \quad LA_{Cy} + aT_0 - \Sigma \overset{\curvearrowright}{M}_{B\text{ loads}} = 0 \qquad (1)$$

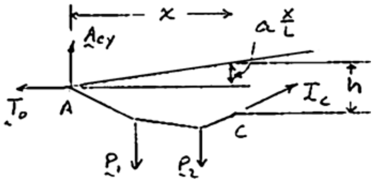
FBD Cable:



(Where $\Sigma M_{B\text{ loads}}$ includes all applied loads)

$$\curvearrowleft \Sigma M_C = 0: \quad xA_{Cy} - \left(h - a\frac{x}{L}\right)T_0 - \Sigma \overset{\curvearrowright}{M}_{C\text{ left}} = 0 \qquad (2)$$

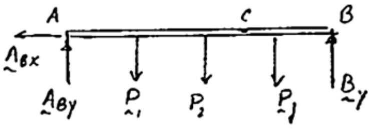
FBD AC:



(Where $\Sigma M_{C\text{ left}}$ includes all loads left of *C*)

$$\frac{x}{L}(1) - (2): \quad hT_0 - \frac{x}{L}\Sigma \overset{\curvearrowright}{M}_{B\text{ loads}} + \Sigma \overset{\curvearrowright}{M}_{C\text{ left}} = 0 \qquad (3)$$

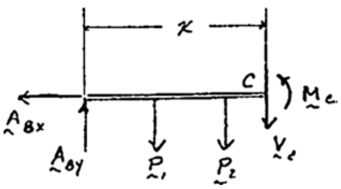
FBD Beam:



$$\curvearrowleft \Sigma M_B = 0: \quad LA_{By} - \Sigma \overset{\curvearrowright}{M}_{B\text{ loads}} = 0 \qquad (4)$$

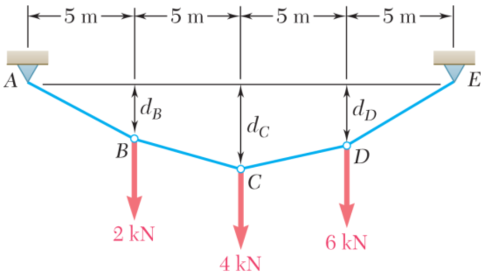
$$\curvearrowleft \Sigma M_C = 0: \quad xA_{By} - \Sigma \overset{\curvearrowright}{M}_{C\text{ left}} - M_C = 0 \qquad (5)$$

FBD AC:



$$\frac{x}{L}(4) - (5): \quad -\frac{x}{L}\Sigma \overset{\curvearrowright}{M}_{B\text{ loads}} + \Sigma \overset{\curvearrowright}{M}_{C\text{ left}} + M_C = 0 \qquad (6)$$

Comparing (3) and (6) $\qquad M_C = hT_0 \qquad$ Q.E.D.


5 m
5 m
5 m
5 m
A
E
d_B
d_C
d_D
B
C
D
2 kN
4 kN
6 kN

PROBLEM 7.120

Making use of the property established in Problem 7.119, solve the problem indicated by first solving the corresponding beam problem.

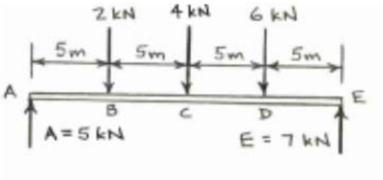
PROBLEM 7.94 Knowing that the maximum tension in cable *ABCDE* is 25 kN, determine the distance d_C.

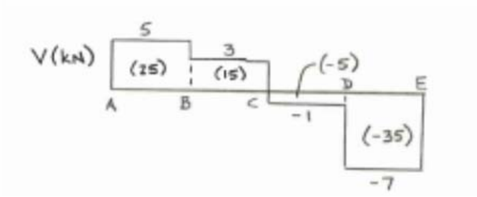
SOLUTION

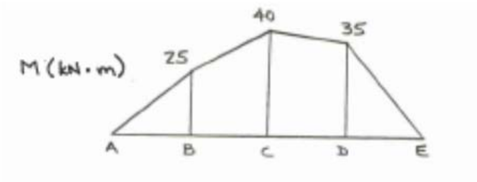
Free body: beam *AE*

$$+\circlearrowright \Sigma M_E = 0: \quad -A(20) + 2(15) + 4(10) + 6(5) = 0 \qquad \mathbf{A} = 5\text{ kN}\uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad 5 - 2 - 4 - 6 + E = 0 \qquad \mathbf{E} = 7\text{ kN}\uparrow$$


2 kN
4 kN
6 kN
5m
5m
5m
5m
A
B
C
D
E
A = 5 kN
E = 7 kN





At *E*: $\quad T_m^2 = T_0^2 + E^2 \qquad 25^2 = T_0^2 + 7^2 \qquad T_0 = 24\text{ kN}$

At *C*: $\quad M_C = T_0 h_C; \qquad 40\text{ kN}\cdot\text{m} = (24\text{ kN})d_C$

$$d_C = 1.667\text{ m}$$

1.667 m ◀

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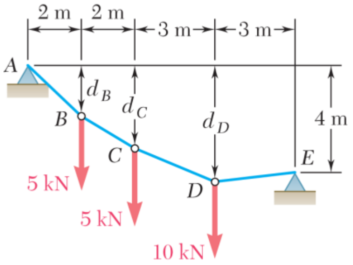
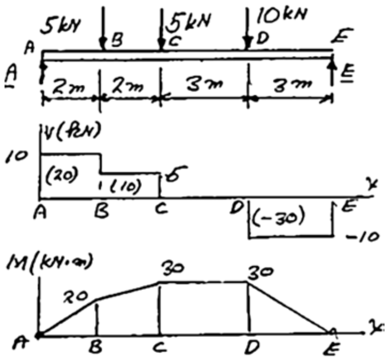

2 m
2 m
3 m
3 m
A
d_B
d_C
d_D
4 m
B
C
D
E
5 kN
5 kN
10 kN

PROBLEM 7.121

Making use of the property established in Problem 7.119, solve the problem indicated by first solving the corresponding beam problem.

PROBLEM 7.97 (*a*) Knowing that $d_C = 3$ m, determine the distances $d_B$and d_D.

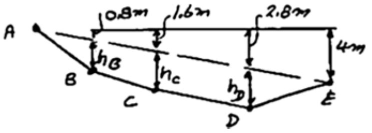
SOLUTION


5kN
B
5kN
C
10kN
D
A
E
A
E
2m
2m
3m
3m
V(kN)
10
(20)
(10)
5
x
A
B
C
D
(−30)
E
−10
M(kN·m)
20
30
30
A
B
C
D
E
x

$$+\circlearrowleft \Sigma M_B = 0: \quad A(10\text{ m}) - (5\text{ kN})(8\text{ m}) - (5\text{ kN})(6\text{ m}) - (10\text{ kN})(3\text{ m}) = 0$$

$$A = 10\text{ kN}$$

Geometry:


A
0.8m
1.6m
2.8m
4m
h_B
h_c
h_D
B
C
D
E

$$d_C = 1.6\text{ m} + h_C$$
$$3\text{ m} = 1.6\text{ m} + h_C$$
$$h_C = 1.4\text{ m}$$

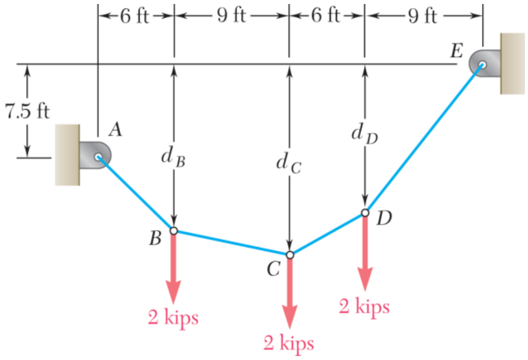
Since $M = T_0 h$, h is proportional to M, thus

$$\frac{h_B}{M_B} = \frac{h_C}{M_C} = \frac{h_D}{M_D}; \quad \frac{h_B}{20\text{ kN}\cdot\text{m}} = \frac{1.4\text{ m}}{30\text{ kN}\cdot\text{m}} = \frac{h_D}{30\text{ kN}\cdot\text{m}}$$

$$h_B = 1.4\left(\frac{20}{30}\right) = 0.9333\text{ m} \qquad \| \qquad h_D = 1.4\left(\frac{30}{30}\right) = 1.4\text{ m}$$

$$d_B = 0.8\text{ m} + 0.9333\text{ m} \qquad \| \qquad d_D = 2.8\text{ m} + 1.4\text{ m}$$

$$d_B = 1.733\text{ m} \blacktriangleleft \qquad d_D = 4.20\text{ m} \blacktriangleleft$$


6 ft
9 ft
6 ft
9 ft
E
7.5 ft
A
d_B
d_C
d_D
D
B
C
2 kips
2 kips
2 kips

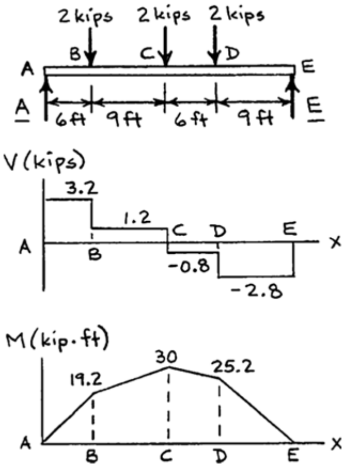
PROBLEM 7.122

Making use of the property established in Problem 7.119, solve the problem indicated by first solving the corresponding beam problem.

PROBLEM 7.99 (*a*) If $d_C = 15$ ft, determine the distances d_B and d_D.

SOLUTION

Free body: Beam *AE*


2 kips
2 kips
2 kips
A
B
C
D
E
A
E
6 ft
9 ft
6 ft
9 ft
V (kips)
3.2
1.2
-0.8
-2.8
A
B
C
D
E
x
M (kip·ft)
19.2
30
25.2
A
B
C
D
E
x

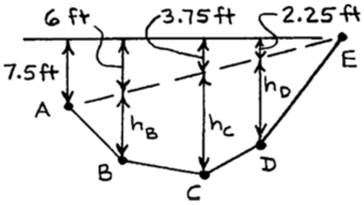
$$+\circlearrowleft \Sigma M_E = 0: \quad -A(30) + 2(24) + 2(15) + 2(9) = 0$$

$$\mathbf{A} = 3.2 \text{ kips} \uparrow$$

$$+\Sigma F_y = 0: \quad 3.2 - 3(2) + B = 0$$

$$\mathbf{B} = 2.8 \text{ kips} \uparrow$$

Geometry:


6 ft
3.75 ft
2.25 ft
7.5 ft
E
A
h_B
h_C
h_D
B
C
D

Given: $d_C = 15$ ft

Then, $h_C = d_C - 3.75 \text{ ft} = 11.25 \text{ ft}$

Since $M = T_0 h$, h is proportional to M. Thus,

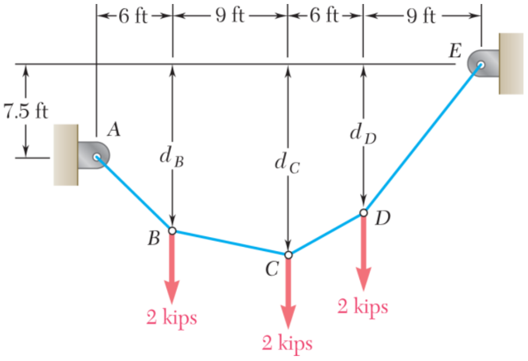
$$\frac{h_B}{M_B} = \frac{h_C}{M_C} = \frac{h_D}{M_D}$$

or, $$\frac{h_B}{19.2} = \frac{11.25 \text{ ft}}{30} = \frac{h_D}{25.2}$$

or, $h_B = 7.2$ ft $\quad h_D = 9.45$ ft

Then, $d_B = 6 + h_B = 6 + 7.2$ $\quad d_B = 13.20$ ft ◀

$d_D = 2.25 + h_D = 2.25 + 9.45$ $\quad d_D = 11.70$ ft ◀


6 ft
9 ft
6 ft
9 ft
7.5 ft
A
E
d_B
d_C
d_D
B
C
D
2 kips
2 kips
2 kips

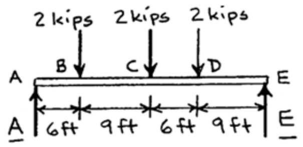
PROBLEM 7.123

Making use of the property established in Problem 7.119, solve the problem indicated by first solving the corresponding beam problem.

PROBLEM 7.100 (*a*) Determine the distance d_C for which portion *BC* of the cable is horizontal.

SOLUTION

Free body: Beam *AE*

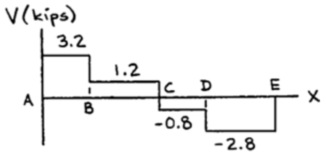

2 kips
2 kips
2 kips
A
B
C
D
E
6 ft
9 ft
6 ft
9 ft
A
E

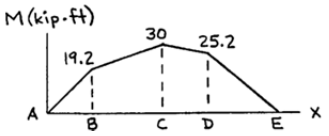
$$+\circlearrowleft \Sigma M_E = 0: \quad -A(30)+2(24)+2(15)+2(9)=0$$

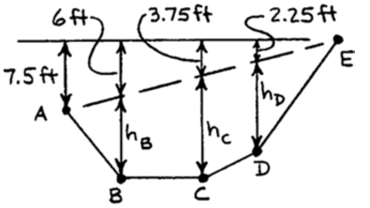
$$\mathbf{A} = 3.2 \text{ kips} \uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad 3.2 - 3(2) + B = 0$$

$$\mathbf{B} = 2.8 \text{ kips} \uparrow$$






6 ft
3.75 ft
2.25 ft
7.5 ft
A
E
h_B
h_C
h_D
B
C
D

Geometry:

Given: $d_C = d_B$

Then, $3.75 + h_C = 6 + h_B$

$$h_C = 2.25 + h_B \qquad (1)$$

Since $M=T_0h$, h is proportional to M. Thus,

$$\frac{h_B}{M_B} = \frac{h_C}{M_C} \quad \text{or,} \quad \frac{h_B}{19.2} = \frac{h_C}{30}$$

$$h_B = 0.64 h_C \qquad (2)$$

Substituting (2) into (1):

$$h_C = 2.25 + 0.64 h_C \qquad h_C = 6.25 \text{ ft}$$

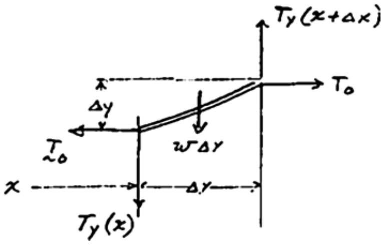
Then, $d_C = 3.75 + h_C = 3.75 + 6.25$ $\quad d_C = 10.00 \text{ ft}$ ◀

PROBLEM 7.124*

Show that the curve assumed by a cable that carries a distributed load $w(x)$ is defined by the differential equation $d^2y/dx^2 = w(x)/T_0$, where T_0 is the tension at the lowest point.

SOLUTION

FBD Elemental segment:



$$+\!\uparrow \Sigma F_y = 0: \quad T_y(x+\Delta x) - T_y(x) - w(x)\Delta x = 0$$

So $$\frac{T_y(x+\Delta x)}{T_0} - \frac{T_y(x)}{T_0} = \frac{w(x)}{T_0}\Delta x$$

But $$\frac{T_y}{T_0} = \frac{dy}{dx}$$

So $$\frac{\left.\frac{dy}{dx}\right|_{x+\Delta x} - \left.\frac{dy}{dx}\right|_{x}}{\Delta x} = \frac{w(x)}{T_0}$$

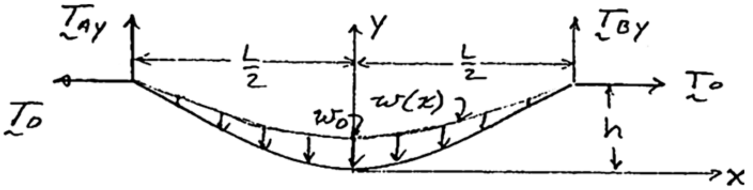
In $\lim\limits_{\Delta x \to 0}$: $$\frac{d^2 y}{dx^2} = \frac{w(x)}{T_0}$$ Q.E.D.

PROBLEM 7.125*

Using the property indicated in Problem 7.124, determine the curve assumed by a cable of span L and sag h carrying a distributed load $w=w_0 \cos(\pi x/L)$, where x is measured from mid-span. Also determine the maximum and minimum values of the tension in the cable.

PROBLEM 7.124 Show that the curve assumed by a cable that carries a distributed load $w(x)$ is defined by the differential equation $d^2y/dx^2=w(x)/T_0$, where $T_0$is the tension at the lowest point.

SOLUTION



$$w(x) = w_0 \cos\frac{\pi x}{L}$$

From Problem 7.124

$$\frac{d^2y}{dx^2} = \frac{w(x)}{T_0} = \frac{w_0}{T_0}\cos\frac{\pi x}{L}$$

So
$$\frac{dy}{dx} = \frac{W_0 L}{T_0 \pi}\sin\frac{\pi x}{L} \quad \left(\text{using } \left.\frac{dy}{dx}\right|_0 = 0\right)$$

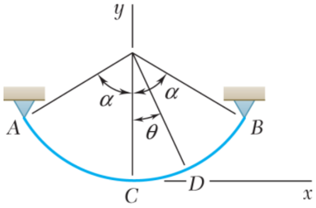
$$y = \frac{w_0 L^2}{T_0 \pi^2}\left(1-\cos\frac{\pi x}{L}\right) \quad [\text{using } y(0)=0] \blacktriangleleft$$

But
$$y\left(\frac{L}{2}\right) = h = \frac{w_0 L^2}{T_0 \pi^2}\left(1-\cos\frac{\pi}{2}\right) \quad \text{so} \quad T_0 = \frac{w_0 L^2}{\pi^2 h}$$

And
$$T_0 = T_{\min} \quad \text{so} \qquad T_{\min} = \frac{w_0 L^2}{\pi^2 h} \blacktriangleleft$$

$$T_{\max} = T_A = T_B: \quad \frac{T_{By}}{T_0} = \left.\frac{dy}{dx}\right|_{x=L/2} = \frac{w_0 L}{T_0 \pi}$$

$$T_{By} = \frac{w_0 L}{\pi} \qquad T_{\max} = \sqrt{T_{By}^2 + T_0^2} = \frac{w_0 L}{\pi}\sqrt{1+\left(\frac{L}{\pi h}\right)^2} \blacktriangleleft$$



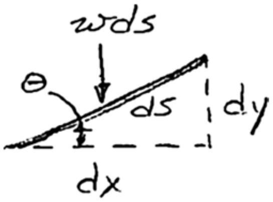
PROBLEM 7.126*

If the weight per unit length of the cable *AB* is $w_0/\cos^2\theta$, prove that the curve formed by the cable is a circular arc. (*Hint:* Use the property indicated in Problem 7.124.)

PROBLEM 7.124 Show that the curve assumed by a cable that carries a distributed load $w(x)$ is defined by the differential equation $d^2y/dx^2 = w(x)/T_0$, where T_0 is the tension at the lowest point.

SOLUTION

Elemental Segment:



Load on segment* $$w(x)dx = \frac{w_0}{\cos^2\theta}ds$$

But $$dx = \cos\theta ds, \quad \text{so} \quad w(x) = \frac{w_0}{\cos^3\theta}$$

From Problem 7.119 $$\frac{d^2y}{dx^2} = \frac{w(x)}{T_0} = \frac{w_0}{T_0\cos^3\theta}$$

In general $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta) = \sec^2\theta\frac{d\theta}{dx}$$

So $$\frac{d\theta}{dx} = \frac{w_0}{T_0\cos^3\theta\sec^2\theta} = \frac{w_0}{T_0\cos\theta}$$

or $$\frac{T_0}{w_0}\cos\theta d\theta = dx = rd\theta\cos\theta$$

Giving $r = \dfrac{T_0}{w_0} = \text{constant.}$ So curve is circular arc Q.E.D.

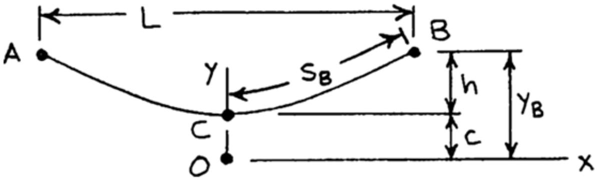
*For large sag, it is not appropriate to approximate *ds* by *dx*.

PROBLEM 7.127

A 25-ft chain with a weight of 30 lb is suspended between two points at the same elevation. Knowing that the sag is 10 ft, determine (*a*) the distance between the supports, (*b*) the maximum tension in the chain.

SOLUTION

$$w = (30\text{ lb})/(25\text{ ft}) = 1.2\text{ lb/ft}$$



Eq. 7.17:
$$y_B^2 - s_B^2 = c^2; \quad (10\text{ ft} + c)^2 - (12.5\text{ ft})^2 = c^2$$

$$100 + 20c + c^2 - 156.25 = c^2$$

$$c = 2.8125\text{ ft}$$

Eq. 7.18:
$$T_m = wy_B = (1.2\text{ lb/ft})(10\text{ ft} + 2.8125\text{ ft})$$

$$T_m = 15.375\text{ lb} \qquad T_m = 15.38\text{ lb} \blacktriangleleft$$

Eq. 7.15:
$$s_B = c\sinh\frac{x_B}{c}; \quad 12.5 = (2.8125\text{ ft})\sinh\frac{x_B}{c}$$

$$\sinh\frac{x_B}{c} = 4.444; \quad \frac{x_B}{c} = 2.1972$$

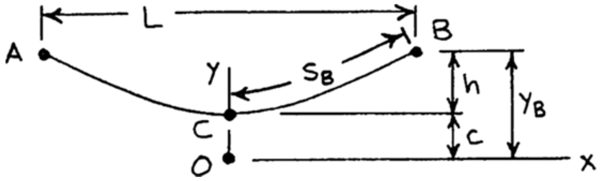
$$x_B = 2.1972(2.8125\text{ ft}) = 6.1796\text{ ft}; \quad L = 2x_B = 2(6.1796\text{ ft}) = 12.3592\text{ ft}$$

$$L = 12.36\text{ ft} \blacktriangleleft$$

PROBLEM 7.128

A 500-ft-long aerial tramway cable having a weight per unit length of 2.8 lb/ft is suspended between two points at the same elevation. Knowing that the sag is 125 ft, find (*a*) the horizontal distance between the supports, (*b*) the maximum tension in the cable.

SOLUTION



Given:

$$\text{Length} = 500 \text{ ft}$$
$$\text{Unit mass} = 2.8 \text{ lb/ft}$$
$$h = 125 \text{ ft}$$

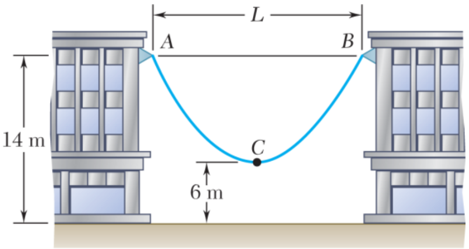
Then,

$$s_B = 250 \text{ ft}$$
$$y_B = h + c = 125 \text{ ft} + c$$
$$y_B^2 - s_B^2 = c^2; \quad (125 + c)^2 - (250)^2 = c^2$$
$$125^2 + 250c + c^2 - 250^2 = c^2$$
$$c = 187.50 \text{ ft}$$

$$s_B = c \sinh \frac{x_B}{c}; \quad 250 = 187.50 \sinh \frac{x_B}{187.50}$$
$$x_B = 206.06 \text{ ft}$$

$$\text{span} = L = 2x_B = 2(206.06 \text{ ft})$$ $L = 412$ ft ◀

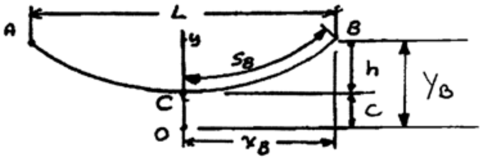
$$T_m = wy_B = (2.8 \text{ lb/ft})(125 \text{ ft} + 187.50 \text{ ft})$$ $T_m = 875$ lb ◀



PROBLEM 7.129

A 40-m cable is strung as shown between two buildings. The maximum tension is found to be 350 N, and the lowest point of the cable is observed to be 6 m above the ground. Determine (*a*) the horizontal distance between the buildings, (*b*) the total mass of the cable.

SOLUTION



$$s_B = 20 \text{ m}$$

$$T_m = 350 \text{ N}$$

$$h = 14 \text{ m} - 6 \text{ m} = 8 \text{ m}$$

$$y_B = h + c = 8 \text{ m} + c$$

Eq. 7.17:

$$y_B^2 - s_B^2 = c^2; \quad (8+c)^2 - 20^2 = c^2$$

$$64 + 16c + c^2 - 400 = c^2$$

$$c = 21.0 \text{ m}$$

Eq. 7.15:

$$s_B = c \sinh \frac{x_B}{c}; \quad 20 = (21.0)\sinh \frac{x_B}{21.0}$$

(*a*) $x_B = 17.7933 \text{ m}; \quad L = 2x_B$ $\quad L = 35.6 \text{ m}$ ◀

(*b*) Eq. 7.18:

$$T_m = wy_B; \quad 350 \text{ N} = w(8 + 21.0)$$

$$w = 12.0690 \text{ N/m}$$

$$W = 2s_B w = (40 \text{ m})(12.0690 \text{ N/m})$$

$$= 482.76 \text{ N}$$

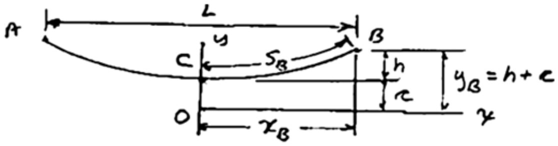
$$m = \frac{W}{g} = \frac{482.76 \text{ N}}{9.81 \text{ m/s}^2}$$

Total mass = 49.2 kg ◀

PROBLEM 7.130

A 50-m steel surveying tape has a mass of 1.6 kg. If the tape is stretched between two points at the same elevation and pulled until the tension at each end is 60 N, determine the horizontal distance between the ends of the tape. Neglect the elongation of the tape due to the tension.

SOLUTION



$$s_B = 25\text{ m}$$

$$w = \left(\frac{1.6\text{ kg}}{50\text{ m}}\right)\left(9.81\text{ m/s}^2\right) = 0.31392\text{ N/m} \quad T_m = 60\text{ N}$$

Eq. 7.18: $$T_m = wy_B; \quad 60\text{ N} = (0.31392\text{ N/m})y_B; \quad y_B = 191.131\text{ m}$$

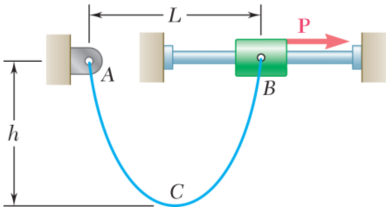
Eq. 7.17: $$y_B^2 - s_B^2 = c^2; \quad (191.131)^2 - (25)^2 = c^2; \quad c = 189.489\text{ m}$$

Eq. 7.15: $$s_B = c\sinh\frac{x_B}{c}; \quad 25 = 189.489\sinh\frac{x_B}{c}$$

$$\sinh\frac{x_B}{c} = 0.131933; \quad \frac{x_B}{c} = 0.131553\text{ m}$$

$$x_B = (0.131553)(189.489\text{ m}) = 24.928\text{ m}$$

$$L = 2x_B = 2(24.928\text{ ft})$$ $L = 49.86$ ft ◀

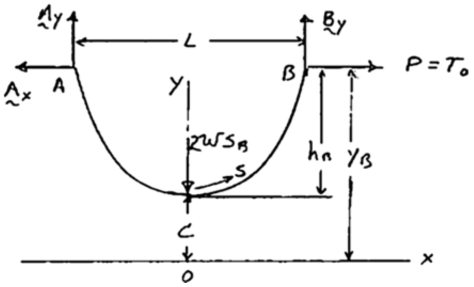


PROBLEM 7.131

A 20-m length of wire having a mass per unit length of 0.2 kg/m is attached to a fixed support at *A* and to a collar at *B*. Neglecting the effect of friction, determine (*a*) the force **P** for which *h*= 8 m, (*b*) the corresponding span *L*.

SOLUTION

FBD Cable:



$$s_T = 20\text{ m} \quad \left(\text{so } s_B = \frac{20\text{ m}}{2} = 10\text{ m}\right)$$

$$w = (0.2\text{ kg/m})(9.81\text{ m/s}^2)$$
$$= 1.96200\text{ N/m}$$

$$h_B = 8\text{ m}$$

$$y_B^2 = (c + h_B)^2 = c^2 + s_B^2$$

So $$c = \frac{s_B^2 - h_B^2}{2h_B}$$

$$c = \frac{(10\text{ m})^2 - (8\text{ m})^2}{2(8\text{ m})}$$
$$= 2.250\text{ m}$$

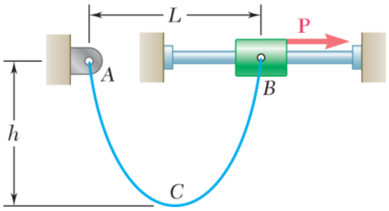
Now $$s_B = c\sinh\frac{x_B}{c} \rightarrow x_B = c\sinh^{-1}\frac{s_B}{c}$$

$$= (2.250\text{ m})\sinh^{-1}\left(\frac{10\text{ m}}{2.250\text{ m}}\right)$$

$$x_B = 4.9438\text{ m}$$

$P = T_0 = wc = (1.96200\text{ N/m})(2.250\text{ m})$ (*a*) $\mathbf{P} = 4.41\text{ N} \longrightarrow$ ◀

$L = 2x_B = 2(4.9438\text{ m})$ (*b*) $L = 9.89\text{ m}$ ◀

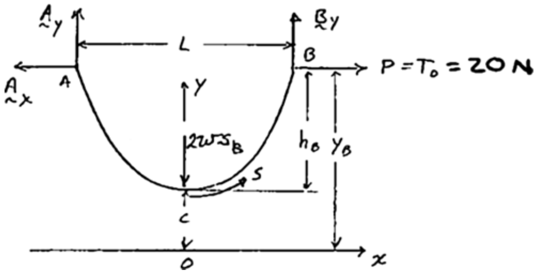


PROBLEM 7.132

A 20-m length of wire having a mass per unit length of 0.2 kg/m is attached to a fixed support at *A* and to a collar at *B*. Knowing that the magnitude of the horizontal force applied to the collar is $P = 20$ N, determine (*a*) the sag *h*, (*b*) the span *L*.

SOLUTION

FBD Cable:



$$s_T = 20 \text{ m}, \quad w = (0.2 \text{ kg/m})(9.81 \text{ m/s}^2) = 1.96200 \text{ N/m}$$

$$P = T_0 = wc \quad c = \frac{P}{w}$$

$$c = \frac{20 \text{ N}}{1.9620 \text{ N/m}} = 10.1937 \text{ m}$$

$$y_B^2 = (h_B + c)^2 = c^2 + s_B^2$$

$$h^2 + 2ch - s_B^2 = 0 \quad s_B = \frac{20 \text{ m}}{2} = 10 \text{ m}$$

$$h^2 + 2(10.1937 \text{ m})h - 100 \text{ m}^2 = 0$$

$$h = 4.0861 \text{ m}$$

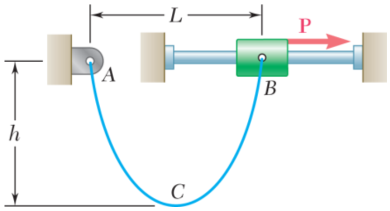
(*a*) $h = 4.09 \text{ m}$ ◀

$$s_B = c\sinh\frac{x_A}{c} \rightarrow x_B = c\sinh^{-1}\frac{s_B}{c} = (10.1937 \text{ m})\sinh^{-1}\left(\frac{10 \text{ m}}{10.1937 \text{ m}}\right)$$

$$= 8.8468 \text{ m}$$

$$L = 2x_B = 2(8.8468 \text{ m})$$

(*b*) $L = 17.69 \text{ m}$ ◀

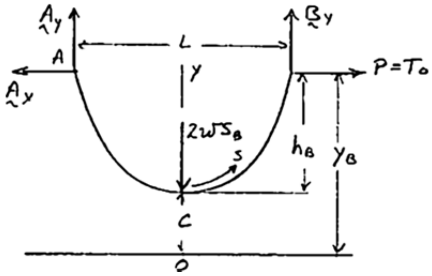


PROBLEM 7.133

A 20-m length of wire having a mass per unit length of 0.2 kg/m is attached to a fixed support at *A* and to a collar at *B*. Neglecting the effect of friction, determine (*a*) the sag *h* for which *L*= 15 m, (*b*) the corresponding force **P**.

SOLUTION

FBD Cable:



$$s_T = 20\text{ m} \rightarrow s_B = \frac{20\text{ m}}{2} = 10\text{ m}$$

$$w = (0.2\text{ kg/m})(9.81\text{ m/s}^2) = 1.96200\text{ N/m}$$

$$L = 15\text{ m}$$

$$s_B = c\sinh\frac{x_B}{c} = c\sinh\frac{\frac{L}{2}}{c}$$

$$10\text{ m} = c\sinh\frac{7.5\text{ m}}{c}$$

Solving numerically: $c = 5.5504\text{ m}$

$$y_B = c\cosh\left(\frac{x_B}{c}\right) = (5.5504)\cosh\left(\frac{7.5}{5.5504}\right)$$

$$y_B = 11.4371\text{ m}$$

$$h_B = y_B - c = 11.4371\text{ m} - 5.5504\text{ m}$$

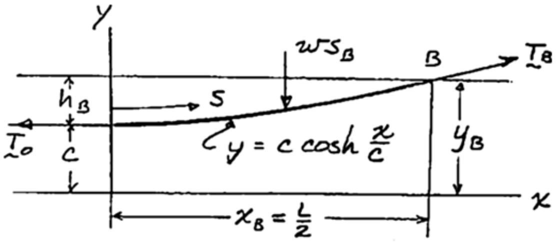
(*a*) $h_B = 5.89\text{ m}$ ◀

$P = wc = (1.96200\text{ N/m})(5.5504\text{ m})$ (*b*) $\mathbf{P} = 10.89\text{ N} \longrightarrow$ ◀

PROBLEM 7.134

Determine the sag of a 30-ft chain that is attached to two points at the same elevation that are 20 ft apart.

SOLUTION



$$s_B = \frac{30\text{ ft}}{2} = 15\text{ ft} \quad L = 20\text{ ft}$$

$$x_B = \frac{L}{2} = 10\text{ ft}$$

$$s_B = c\sinh\frac{x_B}{c}$$

$$15\text{ ft} = c\sinh\frac{10\text{ ft}}{c}$$

Solving numerically: $c = 6.1647\text{ ft}$

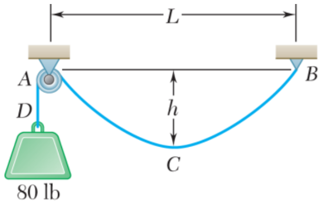
$$y_B = c\cosh\frac{x_B}{c}$$

$$= (6.1647\text{ ft})\cosh\frac{10\text{ ft}}{6.1647\text{ ft}}$$

$$= 16.2174\text{ ft}$$

$$h_B = y_B - c = 16.2174\text{ ft} - 6.1647\text{ ft}$$

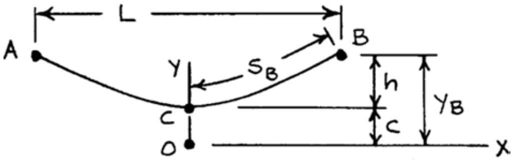
$h_B = 10.05$ ft ◀



PROBLEM 7.135

A counterweight D is attached to a cable that passes over a small pulley at A and is attached to a support at B. Knowing that $L = 45$ ft and $h = 15$ ft, determine (*a*) the length of the cable from A to B, (*b*) the weight per unit length of the cable. Neglect the weight of the cable from A to D.

SOLUTION



Given:

$$L = 45 \text{ ft}$$

$$h = 15 \text{ ft}$$

$$T_A = 80 \text{ lb}$$

$$x_B = 22.5 \text{ ft}$$

By symmetry: $T_B = T_A = T_m = 80 \text{ lb}$

We have $y_B = c\cosh\frac{x_B}{c} = c\cosh\frac{22.5}{c}$

and $y_B = h + c = 15 + c$

Then $c\cosh\frac{22.5}{c} = 15 + c$

or $\cosh\frac{22.5}{c} = \frac{15}{c} + 1$

Solve by trial for c: $c = 18.9525 \text{ ft}$

(*a*)

$$s_B = c\sinh\frac{x_B}{c}$$

$$= (18.9525 \text{ ft})\sinh\frac{22.5}{18.9525}$$

$$= 28.170 \text{ ft}$$

Length $= 2s_B = 2(28.170 \text{ ft}) = 56.3 \text{ ft}$ ◀

(*b*)

$$T_m = wy_B = w(h + c)$$

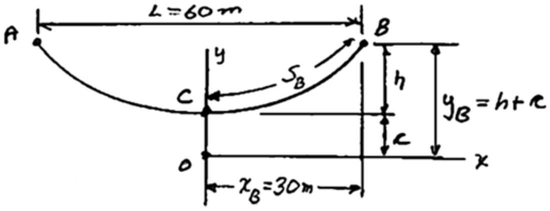
$$80 \text{ lb} = w(15 \text{ ft} + 18.9525 \text{ ft})$$

$w = 2.36 \text{ lb/ft}$ ◀

PROBLEM 7.136

A 90-m wire is suspended between two points at the same elevation that are 60 m apart. Knowing that the maximum tension is 300 N, determine (*a*) the sag of the wire, (*b*) the total mass of the wire.

SOLUTION



$$s_B = 45 \text{ m}$$

Eq. 7.17:
$$s_B = c \sinh \frac{x_B}{c}$$
$$45 = c \sinh \frac{30}{c}; \quad c = 18.494 \text{ m}$$

Eq. 7.16:
$$y_B = c \cosh \frac{x_B}{c}$$
$$y_B = (18.494) \cosh \frac{30}{18.494}$$
$$y_B = 48.652 \text{ m}$$
$$y_B = h + c$$
$$48.652 = h + 18.494$$
$$h = 30.158 \text{ m} \qquad\qquad h = 30.2 \text{ m} \blacktriangleleft$$

Eq. 7.18:
$$T_m = w y_B$$
$$300 \text{ N} = w(48.652 \text{ m})$$
$$w = 6.166 \text{ N/m}$$

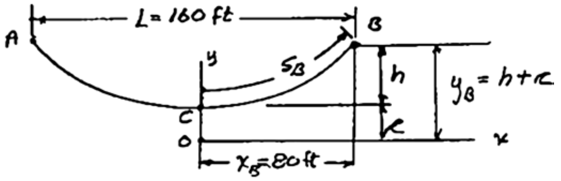
Total weight of cable
$$W = w(\text{Length})$$
$$= (6.166 \text{ N/m})(90 \text{ m})$$
$$= 554.96 \text{ N}$$

Total mass of cable
$$m = \frac{W}{g} = \frac{554.96 \text{ N}}{9.81 \text{ m/s}} = 56.57 \text{ kg} \qquad\qquad m = 56.6 \text{ kg} \blacktriangleleft$$

PROBLEM 7.137

A cable weighing 2 lb/ft is suspended between two points at the same elevation that are 160 ft apart. Determine the smallest allowable sag of the cable if the maximum tension is not to exceed 400 lb.

SOLUTION



Eq. 7.18: $T_m = wy_B; \quad 400 \text{ lb} = (2 \text{ lb/ft})y_B; \quad y_B = 200 \text{ ft}$

Eq. 7.16: $y_B = c\cosh\dfrac{x_B}{c}$

$$200 \text{ ft} = c\cosh\frac{80 \text{ ft}}{c}$$

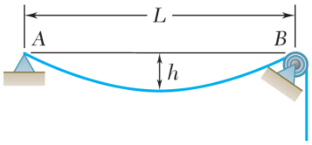
Solve for c: $c = 182.148$ ft and $c = 31.592$ ft

$$y_B = h + c; \quad h = y_B - c$$

For $c = 182.148$ ft; $h = 200 - 182.147 = 17.852$ ft ◁

For $c = 31.592$ ft; $h = 200 - 31.592 = 168.408$ ft ◁

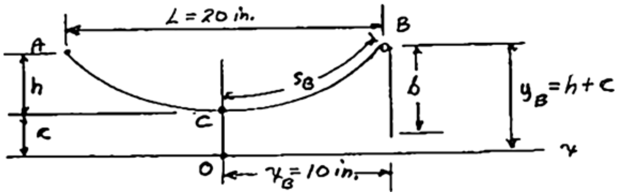
For $T_m \le 400$ lb: smallest $h = 17.85$ ft ◀



PROBLEM 7.138

A uniform cord 50 in. long passes over a pulley at *B* and is attached to a pin support at *A*. Knowing that *L*= 20 in. and neglecting the effect of friction, determine the smaller of the two values of *h* for which the cord is in equilibrium.

SOLUTION



Length of overhang: $b = 50\text{ in.} - 2s_B$

Weight of overhang equals max. tension

$$T_m = T_B = wb = w(50\text{ in.} - 2s_B)$$

Eq. 7.15: $s_B = c\sinh\dfrac{x_B}{c}$

Eq. 7.16: $y_B = c\cosh\dfrac{x_B}{c}$

Eq. 7.18: $T_m = wy_B$

$$w(50\text{ in.} - 2s_B) = wy_B$$

$$w\left(50\text{ in.} - 2c\sinh\frac{x_B}{c}\right) = wc\cosh\frac{x_B}{c}$$

$$x_B = 10: \quad 50 - 2c\sinh\frac{10}{c} = c\cosh\frac{10}{c}$$

Solve by trial and error: $c = 5.549\text{ in.}$ and $c = 27.742\text{ in.}$

For $c = 5.549$ in.

$$y_B = (5.549\text{ in.})\cosh\frac{10\text{ in.}}{5.549\text{ in.}} = 17.277\text{ in.}$$

$$y_B = h + c; \quad 17.277\text{ in.} = h + 5.549\text{ in.}$$

$$h = 11.728\text{ in.}$$

$h = 11.73$ in. ◀

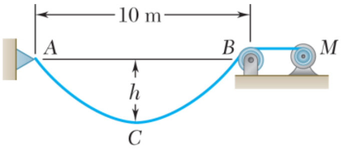
For $c = 27.742$ in.

$$y_B = (27.742\text{ in.})\cosh\frac{10\text{ in.}}{27.742\text{ in.}} = 29.564\text{ in.}$$

$$y_B = h + c; \quad 29.564\text{ in.} = h + 27.742\text{ in.}$$

$$h = 1.8219\text{ in.}$$

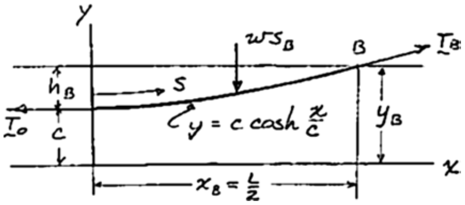
$h = 1.822$ in. ◀



PROBLEM 7.139

A motor *M* is used to slowly reel in the cable shown. Knowing that the mass per unit length of the cable is 0.4 kg/m, determine the maximum tension in the cable when *h*= 5 m.

SOLUTION


y
ws_B
B
T_B
h_B
s
T_0
c
$y = c \cosh \frac{x}{c}$
y_B
x
$x_B = \frac{L}{2}$

$$w = 0.4 \text{ kg/m} \quad L = 10 \text{ m} \quad h_B = 5 \text{ m}$$

$$y_B = c \cosh \frac{x_B}{c}$$

$$h_B + c = c \cosh \frac{L}{2c}$$

$$5 \text{ m} = c\left(\cosh \frac{5 \text{ m}}{c} - 1\right)$$

Solving numerically:

$$c = 3.0938 \text{ m}$$

$$y_B = h_B + c = 5 \text{ m} + 3.0938 \text{ m}$$

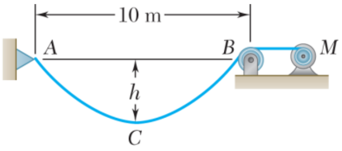
$$= 8.0938 \text{ m}$$

$$T_{\max} = T_B = w y_B$$

$$= (0.4 \text{ kg/m})(9.81 \text{ m/s}^2)(8.0938 \text{ m})$$

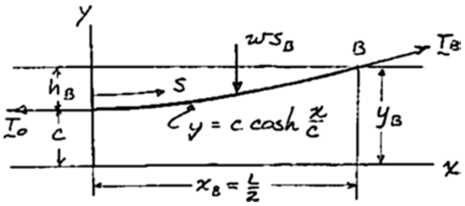
$$T_{\max} = 31.8 \text{ N} \blacktriangleleft$$

PROBLEM 7.140



A motor *M* is used to slowly reel in the cable shown. Knowing that the mass per unit length of the cable is 0.4 kg/m, determine the maximum tension in the cable when *h*= 3 m.

SOLUTION


y
ws_B
B
T_B
h_B
s
T_0
c
$y = c \cosh \frac{x}{c}$
y_B
x
$x_B = \frac{L}{2}$

$$w = 0.4 \text{ kg/m}, \quad L = 10 \text{ m}, \quad h_B = 3 \text{ m}$$

$$y_B = h_B + c = c\cosh\frac{x_B}{c} = c\cosh\frac{L}{2c}$$

$$3 \text{ m} = c\left(c\cosh\frac{5 \text{ m}}{c} - 1\right)$$

Solving numerically: $c = 4.5945 \text{ m}$

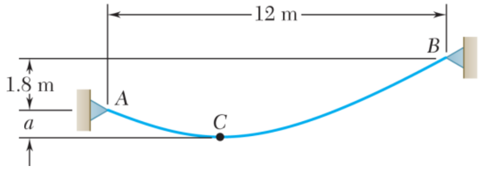
$$y_B = h_B + c = 3 \text{ m} + 4.5945 \text{ m}$$

$$= 7.5945 \text{ m}$$

$$T_{\max} = T_B = wy_B$$

$$= (0.4 \text{ kg/m})(9.81 \text{ m/s}^2)(7.5945 \text{ m})$$

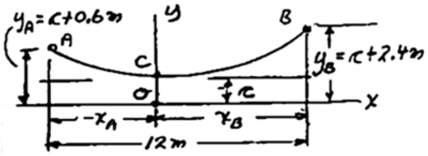
$T_{\max} = 29.8 \text{ N}$ ◀



PROBLEM 7.141

The cable *ACB* has a mass per unit length of 0.45 kg/m. Knowing that the lowest point of the cable is located at a distance *a*= 0.6 m below the support *A*, determine (*a*) the location of the lowest Point *C*, (*b*) the maximum tension in the cable.

SOLUTION



Note: $x_B - x_A = 12\text{ m}$

or, $-x_A = 12\text{ m} - x_B$

Point *A*: $$y_A = c\cosh\frac{-x_A}{c};\quad c + 0.6 = c\cosh\frac{12 - x_B}{c} \tag{1}$$

Point *B*: $$y_B = c\cosh\frac{x_B}{c};\quad c + 2.4 = c\cosh\frac{x_B}{c} \tag{2}$$

From (1): $$\frac{12}{c} - \frac{x_B}{c} = \cosh^{-1}\left(\frac{c+0.6}{c}\right) \tag{3}$$

From (2): $$\frac{x_B}{c} = \cosh^{-1}\left(\frac{c+2.4}{c}\right) \tag{4}$$

Add (3) + (4): $$\frac{12}{c} = \cosh^{-1}\left(\frac{c+0.6}{c}\right) + \cosh^{-1}\left(\frac{c+2.4}{c}\right)$$

Solve by trial and error: $c = 13.6214\text{ m}$

Eq. (2): $$13.6214 + 2.4 = 13.6214\cosh\frac{x_B}{c}$$

$$\cosh\frac{x_B}{c} = 1.1762;\quad \frac{x_B}{c} = 0.58523$$

$$x_B = 0.58523(13.6214\text{ m}) = 7.9717\text{ m}$$

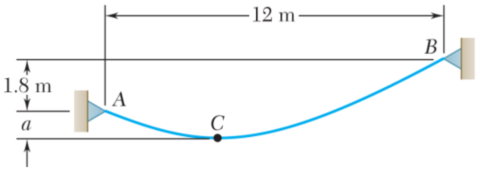
Point *C* is 7.97 m to left of *B* ◀

$$y_B = c + 2.4 = 13.6214 + 2.4 = 16.0214\text{ m}$$

Eq. 7.18: $$T_m = wy_B = (0.45\text{ kg/m})(9.81\text{ m/s}^2)(16.0214\text{ m})$$

$$T_m = 70.726\text{ N}$$

$T_m = 70.7$ N ◀

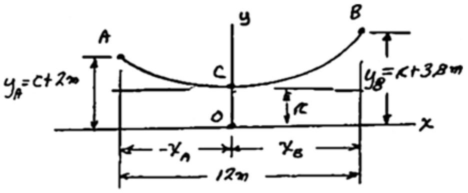


PROBLEM 7.142

The cable *ACB* has a mass per unit length of 0.45 kg/m. Knowing that the lowest point of the cable is located at a distance *a*= 2 m below the support *A*, determine (*a*) the location of the lowest Point *C*, (*b*) the maximum tension in the cable.

SOLUTION

Note:



$$x_B - x_A = 12 \text{ m}$$

or

$$-x_A = 12 \text{ m} - x_B$$

Point *A*: $$y_A = c\cosh\frac{-x_A}{c}; \quad c+2 = c\cosh\frac{12-x_B}{c} \qquad (1)$$

Point *B*: $$y_B = c\cosh\frac{x_B}{c}; \quad c+3.8 = c\cosh\frac{x_B}{c} \qquad (2)$$

From (1): $$\frac{12}{c} - \frac{x_B}{c} = \cosh^{-1}\left(\frac{c+2}{c}\right) \qquad (3)$$

From (2): $$\frac{x_B}{c} = \cosh^{-1}\left(\frac{c+3.8}{c}\right) \qquad (4)$$

Add (3) + (4): $$\frac{12}{c} = \cosh^{-1}\left(\frac{c+2}{c}\right) + \cosh^{-1}\left(\frac{c+3.8}{c}\right)$$

Solve by trial and error: $c = 6.8154 \text{ m}$

Eq. (2): $$6.8154 \text{ m} + 3.8 \text{ m} = (6.8154 \text{ m})\cosh\frac{x_B}{c}$$

$$\cosh\frac{x_B}{c} = 1.5576 \quad \frac{x_B}{c} = 1.0122$$

$$x_B = 1.0122(6.8154 \text{ m}) = 6.899 \text{ m}$$

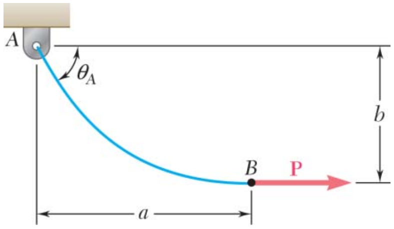
Point *C* is 6.90 m to left of *B* ◀

$$y_B = c + 3.8 = 6.8154 + 3.8 = 10.6154 \text{ m}$$

Eq. (7.18): $$T_m = wy_B = (0.45 \text{ kg/m})(9.81 \text{ m/s}^2)(10.6154 \text{ m})$$

$$T_m = 46.86 \text{ N}$$

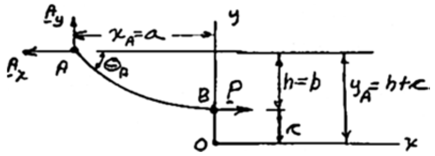
$T_m = 46.9$ N ◀



PROBLEM 7.143

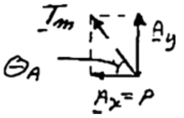
A uniform cable weighing 3 lb/ft is held in the position shown by a horizontal force **P** applied at *B*. Knowing that *P*=180lbandθ_A =60°, determine (*a*) the location of Point *B*, (*b*) the length of the cable.

SOLUTION



Eq. 7.18: $T_0 = P = cw$

$$c = \frac{P}{w} = \frac{180 \text{ lb}}{3 \text{ lb/ft}} \quad c = 60 \text{ ft}$$



At *A*:

$$T_m = \frac{P}{\cos 60°}$$

$$= \frac{cw}{0.5} = 2cw$$

(*a*) Eq. 7.18:

$$T_m = w(h+c)$$

$$2cw = w(h+c)$$

$$2c = h + c \quad h = b = c$$

$b = 60.0$ ft ◀

Eq. 7.16:

$$y_A = c\cosh\frac{x_A}{c}$$

$$h + c = c\cosh\frac{x_A}{c}$$

$$(60 \text{ ft} + 60 \text{ ft}) = (60 \text{ ft}) \quad \cosh\frac{x_A}{60}$$

$$\cosh\frac{x_A}{60 \text{ m}} = 2 \quad \frac{x_A}{60 \text{ m}} = 1.3170$$

$$x_A = 79.02 \text{ ft}$$

$a = 79.0$ ft ◀

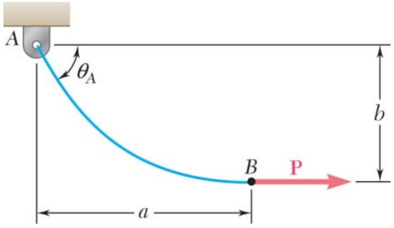
(*b*) Eq. 7.15:

$$s_A = c\sinh\frac{x_B}{c} = (60 \text{ ft})\sinh\frac{79.02 \text{ ft}}{60 \text{ ft}}$$

$$s_A = 103.92 \text{ ft}$$

$$\text{length} = s_A$$

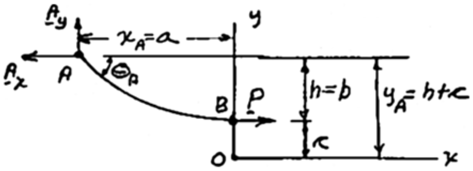
$s_A = 103.9$ ft ◀



PROBLEM 7.144

A uniform cable weighing 3 lb/ft is held in the position shown by a horizontal force **P** applied at B. Knowing that P =150lbandθ_A=60°, determine (*a*) the location of Point B, (*b*) the length of the cable.

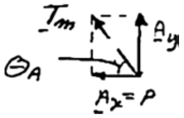
SOLUTION



Eq. 7.18:

$$T_0 = P = cw$$

$$c = \frac{P}{w} = \frac{150 \text{ lb}}{3 \text{ lb/ft}} = 50 \text{ ft}$$



At A:

$$T_m = \frac{P}{\cos 60°}$$

$$= \frac{cw}{0.5} = 2\,cw$$

(*a*) Eq. 7.18:

$$T_m = w(h+c)$$

$$2cw = w(h+c)$$

$$2c = h + c \quad h = c = b$$

$b = 50.0$ ft ◀

Eq. 7.16:

$$y_A = c\cosh\frac{x_A}{c}$$

$$h + c = c\cosh\frac{x_A}{c}$$

$$(50 \text{ ft} + 50 \text{ ft}) = (50 \text{ ft})\cosh\frac{x_A}{c}$$

$$\cosh\frac{x_A}{c} = 2 \quad \frac{x_A}{c} = 1.3170$$

$$x_A = 1.3170(50 \text{ ft}) = 65.85 \text{ ft}$$

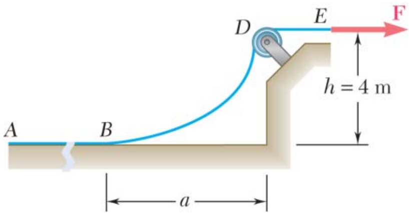
$a = 65.8$ ft ◀

(*b*) Eq. 7.15:

$$s_A = c\sinh\frac{x_A}{c} = (50 \text{ ft})\sinh\frac{65.85 \text{ ft}}{50 \text{ ft}}$$

$$s_A = 86.6 \text{ ft}$$

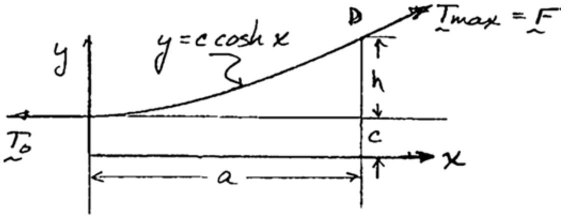
length $= s_A = 86.6$ ft ◀



PROBLEM 7.145

To the left of Point *B* the long cable *ABDE* rests on the rough horizontal surface shown. Knowing that the mass per unit length of the cable is 2 kg/m, determine the force **F** when *a*= 3.6 m.

SOLUTION



$$x_D = a = 3.6 \text{ m} \quad h = 4 \text{ m}$$

$$y_D = c\cosh\frac{x_D}{c}$$

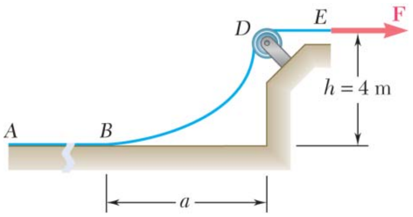
$$h + c = c\cosh\frac{a}{c}$$

$$4 \text{ m} = c\left(\cosh\frac{3.6 \text{ m}}{c} - 1\right)$$

Solving numerically: $c = 2.0712 \text{ m}$

Then $y_B = h + c = 4 \text{ m} + 2.0712 \text{ m} = 6.0712 \text{ m}$

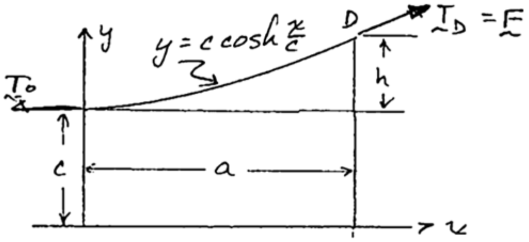
$$F = T_{\max} = wy_B = (2 \text{ kg/m})(9.81 \text{ m/s}^2)(6.0712 \text{ m}) \qquad \mathbf{F} = 119.1 \text{ N} \longrightarrow ◀$$



PROBLEM 7.146

To the left of Point *B* the long cable *ABDE* rests on the rough horizontal surface shown. Knowing that the mass per unit length of the cable is 2 kg/m, determine the force **F** when *a*= 6 m.

SOLUTION



$$x_D = a = 6\text{ m} \quad h = 4\text{ m}$$

$$y_D = c\cosh\frac{x_D}{c}$$

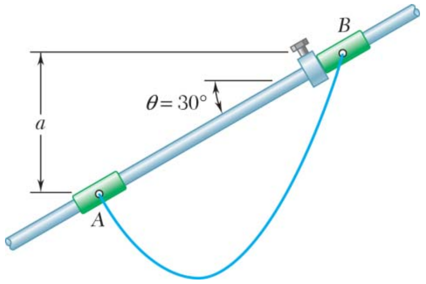
$$h + c = c\cosh\frac{a}{c}$$

$$4\text{ m} = c\left(\cosh\frac{6\text{ m}}{c} - 1\right)$$

Solving numerically: $c = 5.054\text{ m}$

$$y_B = h + c = 4\text{ m} + 5.054\text{ m} = 9.054\text{ m}$$

$$F = T_D = wy_D = (2\text{ kg/m})(9.81\text{ m/s}^2)(9.054\text{ m}) \qquad \mathbf{F} = 177.6\text{ N} \longrightarrow ◀$$

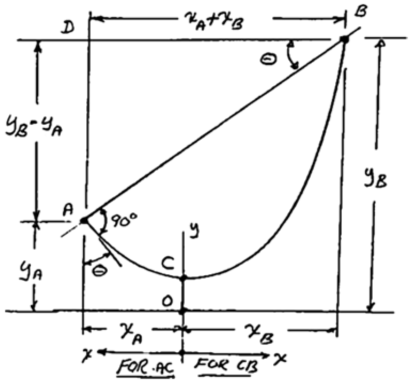


PROBLEM 7.147*

The 10-ft cable *AB* is attached to two collars as shown. The collar at *A* can slide freely along the rod; a stop attached to the rod prevents the collar at *B* from moving on the rod. Neglecting the effect of friction and the weight of the collars, determine the distance *a*.

SOLUTION

Collar at *A*: Since $\mu = 0,$ cable ' rod



Point *A*:

$$y = c\cosh\frac{x}{c}; \quad \frac{dy}{dx} = \sinh\frac{x}{c}$$

$$\tan\theta = \left|\frac{dy}{dx}\right|_A = \sinh\frac{x_A}{c}$$

$$\frac{x_A}{c} = \sinh(\tan(90° - \theta))$$

$$x_A = c\sinh(\tan(90° - \theta)) \qquad (1)$$

Length of cable= 10 ft

$$10 \text{ ft} = AC + CB$$

$$10 = c\sinh\frac{x_A}{c} + c\sinh\frac{x_B}{c}$$

$$\sinh\frac{x_B}{c} = \frac{10}{c} - \sinh\frac{x_A}{c}$$

$$x_B = c\sinh^{-1}\left[\frac{10}{c} - \sinh\frac{x_A}{c}\right] \qquad (2)$$

$$y_A = c\cosh\frac{x_A}{c} \quad y_B = c\cosh\frac{x_B}{c} \qquad (3)$$

In ΔABD:

$$\tan\theta = \frac{y_B - y_A}{x_B + x_A} \qquad (4)$$

PROBLEM 7.147* (Continued)

Method of solution:

For given value of θ, choose trial value of c and calculate:

From Eq. (1): x_A

Using value of x_A and c, calculate:

From Eq. (2): x_B

From Eq. (3): y_A and y_B

Substitute values obtained for x_A, x_B, y_A, y_B into Eq. (4) and calculate θ

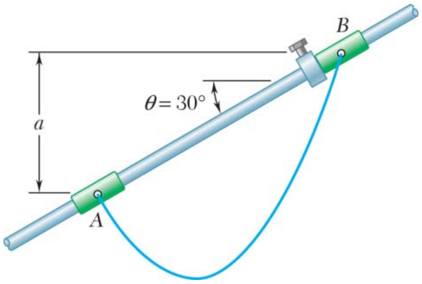
Choose new trial value of θ and repeat above procedure until calculated value of θ is equal to given value of θ.

For $\theta = 30°$

Result of trial and error procedure:

$$c = 1.803 \text{ ft}$$
$$x_A = 2.3745 \text{ ft}$$
$$x_B = 3.6937 \text{ ft}$$
$$y_A = 3.606 \text{ ft}$$
$$y_B = 7.109 \text{ ft}$$
$$\begin{aligned} a &= y_B - y_A \\ &= 7.109 \text{ ft} - 3.606 \text{ ft} \\ &= 3.503 \text{ ft} \end{aligned}$$

$a = 3.50$ ft ◀



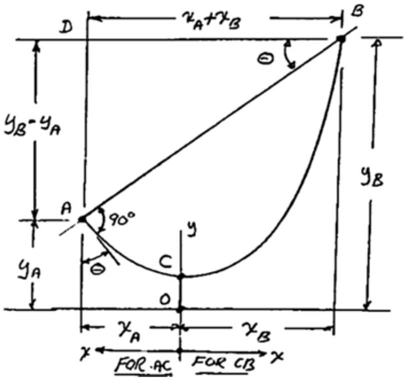
PROBLEM 7.148*

Solve Problem 7.147 assuming that the angle θ formed by the rod and the horizontal is 45°.

PROBLEM 7.147 The 10-ft cable *AB* is attached to two collars as shown. The collar at *A* can slide freely along the rod; a stop attached to the rod prevents the collar at *B* from moving on the rod. Neglecting the effect of friction and the weight of the collars, determine the distance *a*.

SOLUTION

Collar at *A*: Since $\mu = 0,$ cable ' rod



Point *A*:

$$y = c\cosh\frac{x}{c}; \quad \frac{dy}{dx} = \sinh\frac{x}{c}$$

$$\tan\theta = \left|\frac{dy}{dx}\right|_A = \sinh\frac{x_A}{c}$$

$$\frac{x_A}{c} = \sinh(\tan(90° - \theta))$$

$$x_A = c\sinh(\tan(90° - \theta)) \qquad (1)$$

Length of cable= 10 ft

$$10 \text{ ft} = AC + CB$$

$$10 = c\sinh\frac{x_A}{c} + c\sinh\frac{x_B}{c}$$

$$\sinh\frac{x_B}{c} = \frac{10}{c} - \sinh\frac{x_A}{c}$$

$$x_B = c\sinh^{-1}\left[\frac{10}{c} - \sinh\frac{x_A}{c}\right] \qquad (2)$$

$$y_A = c\cosh\frac{x_A}{c} \quad y_B = c\cosh\frac{x_B}{c} \qquad (3)$$

PROBLEM 7.148* (Continued)

In ΔABD: $$\tan\theta = \frac{y_B - y_A}{x_B + x_A} \tag{4}$$

Method of solution:

For given value of θ, choose trial value of c and calculate:

From Eq. (1): x_A

Using value of x_A and c, calculate:

From Eq. (2): x_B

From Eq. (3): y_A and y_B

Substitute values obtained for x_A, x_B, y_A, $y_B$into Eq. (4) and calculate θ

Choose new trial value of $\theta$and repeat above procedure until calculated value of θ is equal to given value of θ.

For $\theta = 45°$

Result of trial and error procedure:

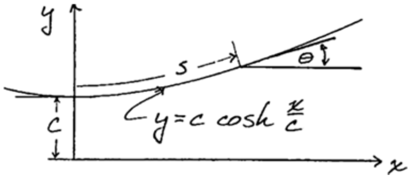
$$c = 1.8652 \text{ ft}$$
$$x_A = 1.644 \text{ ft}$$
$$x_B = 4.064 \text{ ft}$$
$$y_A = 2.638 \text{ ft}$$
$$y_B = 8.346 \text{ ft}$$
$$\begin{aligned} a &= y_B - y_A \\ &= 8.346 \text{ ft} - 2.638 \text{ ft} \\ &= 5.708 \text{ ft} \end{aligned}$$

$a = 5.71$ ft ◀

PROBLEM 7.149

Denoting by θ the angle formed by a uniform cable and the horizontal, show that at any point (*a*) $s = c\tan\theta$, (*b*) $y = c\sec\theta$.

SOLUTION



(*a*) $$\tan\theta = \frac{dy}{dx} = \sinh\frac{x}{c}$$

$$s = c\sinh\frac{x}{c} = c\tan\theta \qquad \text{Q.E.D.}$$

(*b*) Also $$y^2 = s^2 + c^2(\cosh^2 x = \sinh^2 x + 1)$$

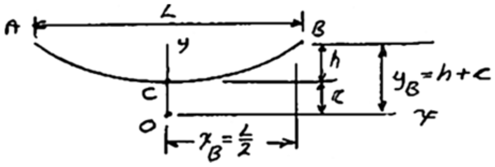
so $$y^2 = c^2(\tan^2\theta + 1)c^2\sec^2\theta$$

and $$y = c\sec\theta \qquad \text{Q.E.D.}$$

PROBLEM 7.150*

(*a*) Determine the maximum allowable horizontal span for a uniform cable of weight per unit length *w* if the tension in the cable is not to exceed a given value T_m. (*b*) Using the result of part *a*, determine the maximum span of a steel wire for which $w = 0.25$ lb/ft and $T_m = 8000$ lb.

SOLUTION



(*a*)

$$T_m = wy_B$$

$$= wc\cosh\frac{x_B}{c}$$

$$= wx_B\left(\frac{1}{\frac{x_B}{c}}\right)\cosh\frac{x_B}{c}$$

We shall find ratio $\left(\frac{x_B}{c}\right)$ for when T_m is minimum

$$\frac{dT_m}{d\left(\frac{x_B}{c}\right)} = wx_B\left[\frac{1}{\frac{x_B}{c}}\sinh\frac{x_B}{c} - \left(\frac{1}{\frac{x_B}{c}}\right)^2\cosh\frac{x_B}{c}\right] = 0$$

$$\frac{\sinh\frac{x_B}{c}}{\cosh\frac{x_B}{c}} = \frac{1}{\frac{x_B}{c}}$$

$$\tanh\frac{x_B}{c} = \frac{c}{x_B}$$

Solve by trial and error for: $\frac{x_B}{c} = 1.200$ (1)

$s_B = c\sinh\frac{x_B}{c} = c\sinh(1.200)$: $\frac{s_B}{c} = 1.509$

Eq. 7.17: $y_B^2 - s_B^2 = c^2$

$$y_B^2 = c^2\left[1 + \left(\frac{s_B}{c}\right)^2\right] = c^2(1 + 1.509^2)$$

$$y_B = 1.810c$$

PROBLEM 7.150* (Continued)

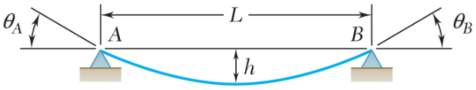
Eq. 7.18:
$$T_m = wy_B$$
$$= 1.810\,wc$$
$$c = \frac{T_m}{1.810\,w}$$

Eq. (1):
$$x_B = 1.200c = 1.200\frac{T_m}{1.810w} = 0.6630\frac{T_m}{w}$$

Span:
$$L = 2x_B = 2(0.6630)\frac{T_m}{w}$$
$$L = 1.326\frac{T_m}{w} \blacktriangleleft$$

(*b*) For $w = 0.25\,\text{lb/ft}$ and $T_m = 8000\text{ lb}$

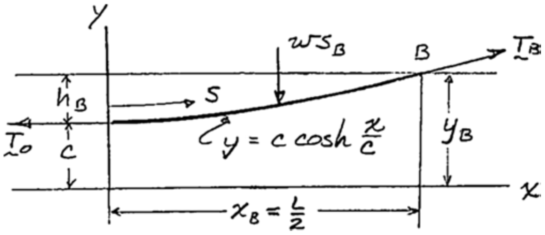
$$L = 1.326\frac{8000\text{ lb}}{0.25\,\text{lb/ft}}$$
$$= 42,432\,\text{ft}$$
$$L = 8.04\ \text{miles} \blacktriangleleft$$


θ_A
L
A
B
θ_B
h

PROBLEM 7.151*

A cable has a mass per unit length of 3 kg/m and is supported as shown. Knowing that the span L is 6 m, determine the *two* values of the sag h for which the maximum tension is 350 N.

SOLUTION


y
ws_B
B
T_B
h_B
s
T_0
c
$y = c\cosh\frac{x}{c}$
y_B
x
$x_B = \frac{L}{2}$

$$y_{\max} = c\cosh\frac{L}{2c} = h + c$$

$$w = (3\text{ kg/m})(9.81\text{ m/s}^2) = 29.43\text{ N/m}$$

$$T_{\max} = wy_{\max}$$

$$y_{\max} = \frac{T_{\max}}{w}$$

$$y_{\max} = \frac{350\text{ N}}{29.43\text{ N/m}} = 11.893\text{ m}$$

$$c\cosh\frac{3\text{ m}}{c} = 11.893\text{ m}$$

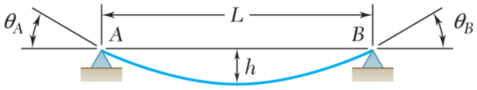
Solving numerically: $c_1 = 0.9241\text{ m}$

$$c_2 = 11.499\text{ m}$$

$$h = y_{\max} - c$$

$$h_1 = 11.893\text{ m} - 0.9241\text{ m} \qquad h_1 = 10.97\text{ m} \blacktriangleleft$$

$$h_2 = 11.893\text{ m} - 11.499\text{ m} \qquad h_2 = 0.394\text{ m} \blacktriangleleft$$

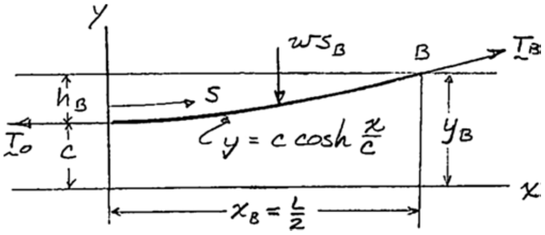


θ_A
L
A
B
θ_B
h

PROBLEM 7.152*

Determine the sag-to-span ratio for which the maximum tension in the cable is equal to the total weight of the entire cable *AB*.

SOLUTION



y
ws_B
B
T_B
h_B
s
T_0
c
$y = c\cosh\frac{x}{c}$
y_B
x
$x_B = \frac{L}{2}$

$$T_{\max} = wy_B = 2ws_B$$

$$y_B = 2s_B$$

$$c\cosh\frac{L}{2c} = 2c\sinh\frac{L}{2c}$$

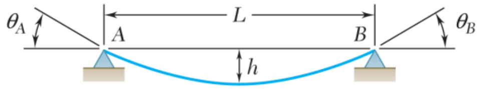
$$\tanh\frac{L}{2c} = \frac{1}{2}$$

$$\frac{L}{2c} = \tanh^{-1}\frac{1}{2} = 0.549306$$

$$\frac{h_B}{c} = \frac{y_B - c}{c} = \cosh\frac{L}{2c} - 1 = 0.154701$$

$$\frac{h_B}{L} = \frac{\frac{h_B}{c}}{2\left(\frac{L}{2c}\right)} = \frac{0.5(0.154701)}{0.549306} = 0.14081$$

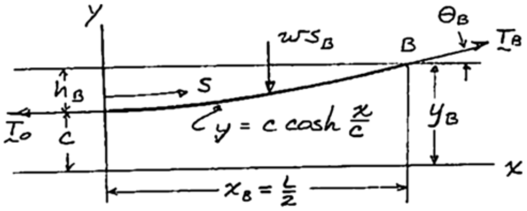
$$\frac{h_B}{L} = 0.1408 \blacktriangleleft$$



PROBLEM 7.153*

A cable of weight per unit length w is suspended between two points at the same elevation that are a distance L apart. Determine (a) the sag-to-span ratio for which the maximum tension is as small as possible, (b) the corresponding values of θ_B and T_m.

SOLUTION



(a)

$$T_{\max} = wy_B = wc\cosh\frac{L}{2c}$$

$$\frac{dT_{\max}}{dc} = w\left(\cosh\frac{L}{2c} - \frac{L}{2c}\sinh\frac{L}{2c}\right)$$

For $\quad \min T_{\max}, \quad \frac{dT_{\max}}{dc} = 0$

$$\tanh\frac{L}{2c} = \frac{2c}{L} \rightarrow \frac{L}{2c} = 1.1997$$

$$\frac{y_B}{c} = \cosh\frac{L}{2c} = 1.8102$$

$$\frac{h}{c} = \frac{y_B}{c} - 1 = 0.8102$$

$$\frac{h}{L} = \left[\frac{1}{2}\frac{h}{c}\left(\frac{2c}{L}\right)\right] = \frac{0.8102}{2(1.1997)} = 0.3375 \qquad \frac{h}{L} = 0.338 \blacktriangleleft$$

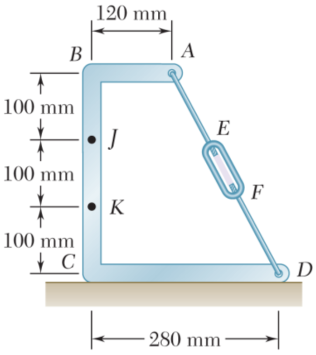
(b)

$$T_0 = wc \quad T_{\max} = wc\cosh\frac{L}{2c} \quad \frac{T_{\max}}{T_0} = \cosh\frac{L}{2c} = \frac{y_B}{c}$$

But $\quad T_0 = T_{\max}\cos\theta_B \quad \frac{T_{\max}}{T_0} = \sec\theta_B$

So $\quad \theta_B = \sec^{-1}\left(\frac{y_B}{c}\right) = \sec^{-1}(1.8102) = 56.46° \qquad \theta_B = 56.5° \blacktriangleleft$

$$T_{\max} = wy_B = w\frac{y_B}{c}\left(\frac{2c}{L}\right)\left(\frac{L}{2}\right) = w(1.8102)\frac{L}{2(1.1997)} \qquad T_{\max} = 0.755wL \blacktriangleleft$$

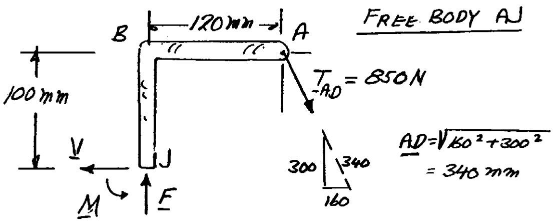

120 mm
B
A
100 mm
J
E
100 mm
K
F
100 mm
C
D
280 mm

PROBLEM 7.154

Knowing that the turnbuckle has been tightened until the tension in wire *AD* is 850 N, determine the internal forces at point indicated:

Point *J*.

SOLUTION


FREE BODY AJ
B
120 mm
A
100 mm
$T_{AD} = 850$ N
V
J
M
F
300
340
160
$AD = \sqrt{160^2 + 300^2}$
= 340 mm

$$\xrightarrow{+} \Sigma F_x = 0: \quad -V + \left(\frac{160}{340}\right)(850\text{ N}) = 0$$

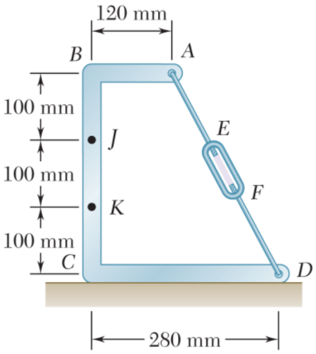
$$V = +400\text{ N} \qquad \mathbf{V} = 400\text{ N} \leftarrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad F - \left(\frac{300}{340}\right)(850\text{ N}) = 0$$

$$F = +750\text{ N} \qquad \mathbf{F} = 750\text{ N} \uparrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_J = 0: \quad M - \left(\frac{300}{340}\right)(850\text{ N})(120\text{ mm}) - \left(\frac{160}{340}\right)(850\text{ N})(100\text{ mm}) = 0$$

$$M = +130\text{ N}\cdot\text{m} \qquad \mathbf{M} = 130\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$



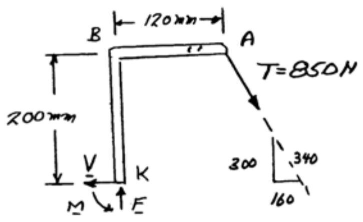
PROBLEM 7.155

Knowing that the turnbuckle has been tightened until the tension in wire *AD* is 850 N, determine the internal forces at point indicated:

Point *K*.

SOLUTION

Free body *AK*:



$$AD = \sqrt{160^2 + 300^2}$$
$$= 340 \text{ mm}$$

On portion *KBA*:

$$\xrightarrow{+} \Sigma F_x = 0: \quad -V + \left(\frac{160}{340}\right)(850 \text{ N}) = 0$$

$$V = +400 \text{ N} \qquad \mathbf{V} = 400 \text{ N} \longleftarrow \blacktriangleleft$$

$$+\!\uparrow \Sigma F_y = 0: \quad F - \left(\frac{300}{340}\right)(850 \text{ N}) = 0$$

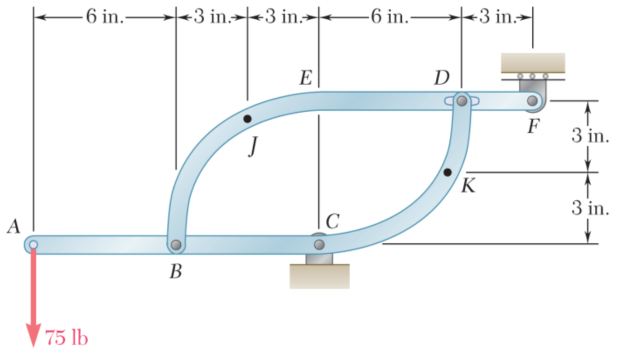
$$F = +750 \text{ N} \qquad \mathbf{F} = 750 \text{ N} \uparrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_J = 0: \quad M - \left(\frac{300}{340}\right)(850 \text{ N})(120 \text{ mm}) - \left(\frac{160}{340}\right)(850 \text{ N})(200 \text{ mm}) = 0$$

$$M = +170 \text{ N}\cdot\text{m} \qquad \mathbf{M} = 170.0 \text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

Internal forces acting on *KCD* are equal and opposite

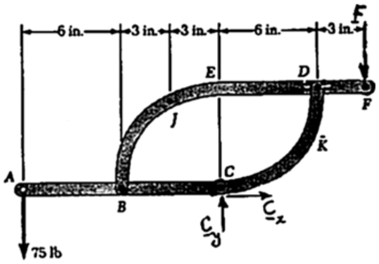
$$\mathbf{F} = 750 \text{ N} \downarrow, \quad \mathbf{V} = 400 \text{ N} \longrightarrow, \quad \mathbf{M} = 170.0 \text{ N}\cdot\text{m} \circlearrowright \blacktriangleleft$$


6 in.
3 in.
3 in.
6 in.
3 in.
E
D
F
J
K
3 in.
3 in.
A
C
B
75 lb

PROBLEM 7.156

Two members, each consisting of a straight and a quarter- circular portion of rod, are connected as shown and support a 75-lb load at *A*. Determine the internal forces at Point *J*.

SOLUTION


6 in.
3 in.
3 in.
6 in.
3 in.
F
E
D
F
J
K
A
C
B
C_y
C_x
75 lb

Free body: Entire frame

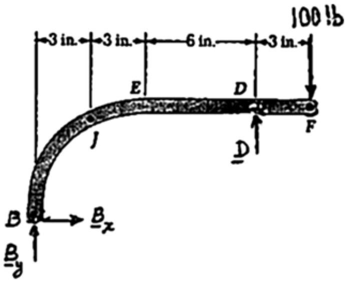
$$+\circlearrowleft \Sigma M_C = 0: \quad (75 \text{ lb})(12 \text{ in.}) - F(9 \text{ in.}) = 0$$

$$\mathbf{F} = 100 \text{ lb} \downarrow \triangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad C_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad C_y - 75 \text{ lb} - 100 \text{ lb} = 0$$

$$C_y = +175 \text{ lb} \qquad \mathbf{C} = 175 \text{ lb} \uparrow \triangleleft$$


100 lb
3 in.
3 in.
6 in.
3 in.
E
D
J
D
F
B
B_x
B_y

Free body: Member *BEDF*

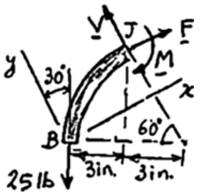
$$+\circlearrowleft \Sigma M_B = 0: \quad D(12 \text{ in.}) - (100 \text{ lb})(15 \text{ in.}) = 0$$

$$\mathbf{D} = 125 \text{ lb} \uparrow \triangleleft$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad B_y + 125 \text{ lb} - 100 \text{ lb} = 0$$

$$B_y = -25 \text{ lb} \qquad \mathbf{B} = 25 \text{ lb} \downarrow \triangleleft$$


y
30°
V
J
F
M
x
60°
B
25 lb
3in.
3in.

Free body: *BJ*

$$+\nearrow \Sigma F_x = 0: \quad F - (25 \text{ lb})\sin 30° = 0$$

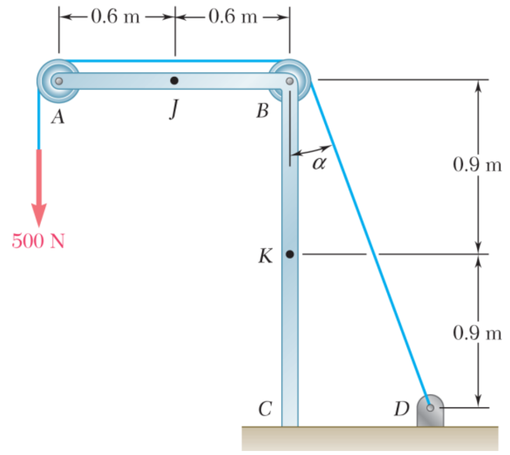
$$\mathbf{F} = 12.50 \text{ lb} \angle 30.0° \blacktriangleleft$$

$$+\nwarrow \Sigma F_y = 0: \quad V - (25 \text{ lb})\cos 30° = 0$$

$$\mathbf{V} = 21.7 \text{ lb} \searrow 60.0° \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_J = 0: \quad -M + (25 \text{ lb})(3 \text{ in.}) = 0$$

$$\mathbf{M} = 75.0 \text{ lb} \cdot \text{in.} \circlearrowright \blacktriangleleft$$


0.6 m
0.6 m
A
J
B
α
0.9 m
500 N
K
0.9 m
C
D

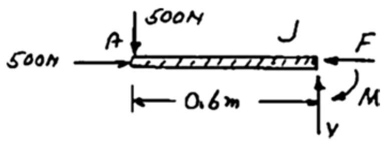
PROBLEM 7.157

Knowing that the radius of each pulley is 150 mm, that $\alpha = 20°$, and neglecting friction, determine the internal forces at (*a*) Point *J*, (*b*) Point *K*.

SOLUTION

Tension in cable = 500 N. Replace cable tension by forces at pins *A* and *B*. Radius does not enter computations: (cf. Problem 6.90)

(*a*) Free body: *AJ*


500N
A
J
F
500N
0.6m
M
V

$$\xrightarrow{+} \Sigma F_x = 0: \quad 500\ \text{N} - F = 0$$

$$F = 500\ \text{N} \qquad \mathbf{F} = 500\ \text{N} \longleftarrow ◀$$

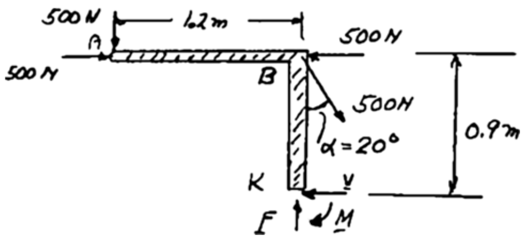
$$+\uparrow \Sigma F_y = 0: \quad V - 500\ \text{N} = 0$$

$$V = 500\ \text{N} \qquad \mathbf{V} = 500\ \text{N} \uparrow ◀$$

$$+\circlearrowleft \Sigma M_J = 0: \quad (500\ \text{N})(0.6\ \text{m}) = 0$$

$$M = 300\ \text{N} \cdot \text{m} \qquad \mathbf{M} = 300\ \text{N} \cdot \text{m} \circlearrowright ◀$$

(*b*) Free body: *ABK*


500 N
1.2 m
A
500 N
500 N
B
500N
α = 20°
0.9m
K
V
F
M

$$\xrightarrow{+} \Sigma F_x = 0: \quad 500\ \text{N} - 500\ \text{N} + (500\ \text{N})\sin 20° - V = 0$$

$$V = 171.01\ \text{N} \qquad \mathbf{V} = 171.0\ \text{N} \longleftarrow ◀$$

PROBLEM 7.157 (Continued)

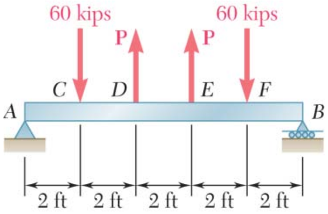
$$+\uparrow \Sigma F_y = 0: \quad -500\text{ N} - (500\text{ N})\cos 20^\circ + F = 0$$

$$F = 969.8\text{ N} \qquad \mathbf{F} = 970\text{ N}\uparrow$$ ◀

$$+\circlearrowleft \Sigma M_K = 0: \quad (500\text{ N})(1.2\text{ m}) - (500\text{ N})\sin 20^\circ(0.9\text{ m}) - M = 0$$

$$M = 446.1\text{ N}\cdot\text{m}$$

$$\mathbf{M} = 446\text{ N}\cdot\text{m}\circlearrowright$$ ◀

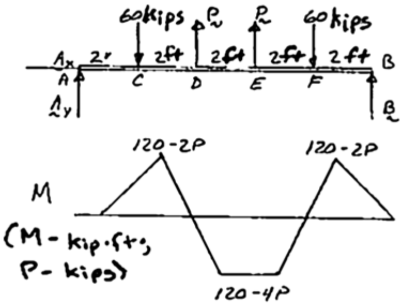

60 kips
60 kips
P
P
A
C
D
E
F
B
2 ft
2 ft
2 ft
2 ft
2 ft

PROBLEM 7.158

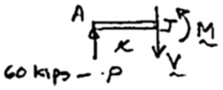
For the beam shown, determine (*a*) the magnitude *P* of the two upward forces for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding value of $|M|_{\max}$.

SOLUTION

By symmetry: $A_y = B = 60\text{ kips} - P$


60kips
P
P
60kips
A_x
2'
2ft
2ft
2ft
2ft
B
A
C
D
E
F
A_y
B
120-2P
120-2P
M
(M - kip·ft,
P - kips)
120-4P

Along AC:


A
J
M
x
V
60 kips - P

$$\circlearrowleft \Sigma M_J = 0{:}\quad M - x(60\text{ kips} - P) = 0$$

$$M = (60\text{ kips} - P)x$$

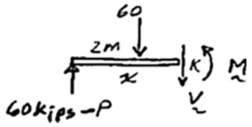
$$M = 120\text{ kips}\cdot\text{ft} - (2\text{ ft})P \quad \text{at} \quad x = 2\text{ ft}$$

Along CD:

$$\circlearrowleft \Sigma M_K = 0{:}\quad M + (x - 2\text{ ft})(60\text{ kips}) - x(60\text{ kips} - P) = 0$$

$$M = 120\text{ kip}\cdot\text{ft} - Px$$

$$M = 120\text{ kip}\cdot\text{ft} - (4\text{ ft})P \quad \text{at} \quad x = 4\text{ ft}$$

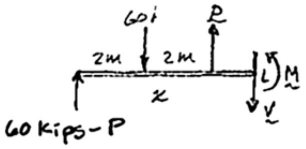

60
2m
K
M
x
V
60kips - P

Along DE:

$$\circlearrowleft \Sigma M_L = 0{:}\quad M - (x - 4\text{ ft})P + (x - 2\text{ ft})(60\text{ kips})$$

$$-x(60\text{ kips} - P) = 0$$

$$M = 120\text{ kip}\cdot\text{ft} - (4\text{ ft})P \quad \text{(const)}$$


60i
P
2m
2m
L
M
x
V
60 Kips - P

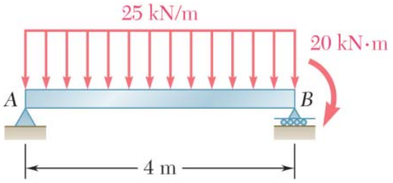
Complete diagram by symmetry

For minimum $|M|_{\max}$, set $M_{\max} = -M_{\min}$

$$120\text{ kip}\cdot\text{ft} - (2\text{ ft})P = -[120\text{ kip}\cdot\text{ft} - (4\text{ ft})P]$$

(*a*) $P = 40.0\text{ kips}$ ◀

$M_{\min} = 120\text{ kip}\cdot\text{ft} - (4\text{ ft})P$ (*b*) $|M|_{\max} = 40.0\text{ kip}\cdot\text{ft}$ ◀

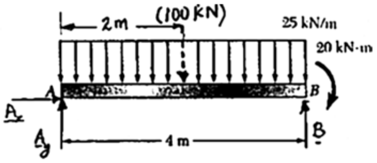

25 kN/m
20 kN·m
A
B
4 m

PROBLEM 7.159

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the magnitude and location of the maximum absolute value of the bending moment.

SOLUTION

Free body: Beam


2m
(100 kN)
25 kN/m
20 kN·m
A
B
A_x
A_y
4 m
B

$$+\circlearrowleft \Sigma M_A = 0: \quad B(4\text{ m}) - (100\text{ kN})(2\text{ m}) - 20\text{ kN}\cdot\text{m} = 0$$

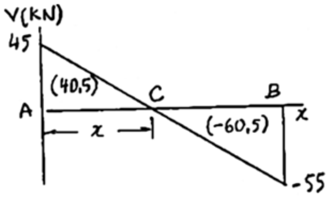
$B = +55$ kN ◁

$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0: \quad A_y + 55 - 100 = 0$$

$A_y = +45$ kN ◁

Shear diagram


V(kN)
45
(40.5)
C
B
A
x
x
(-60.5)
-55

At *A*: $V_A = A_y = +45\text{ kN}$

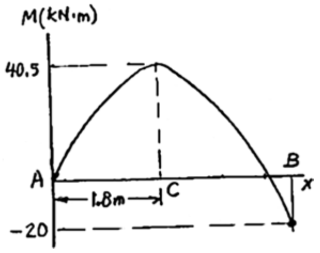
To determine Point *C* where $V = 0$:

$$V_C - V_A = -wx$$
$$0 - 45\text{ kN} = -(25\text{ kN}\cdot\text{m})x$$

$x = 1.8$ m ◁

We compute all areas bending-moment

Bending-moment diagram


M(kN·m)
40.5
A
B
x
1.8m
C
-20

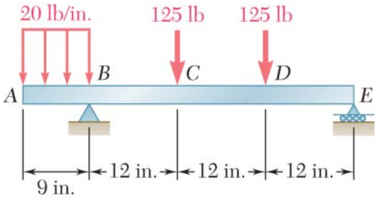
At *A*: $M_A = 0$

At *B*: $M_B = -20\text{ kN}\cdot\text{m}$

$|M|_{\max} = 40.5\text{ kN}\cdot\text{m}$ ◀

1.800 m from *A* ◀

Single arc of parabola


20 lb/in.
125 lb
125 lb
A
B
C
D
E
9 in.
12 in.
12 in.
12 in.

PROBLEM 7.160

For the beam and loading shown, (*a*) draw the shear and bending-moment diagrams, (*b*) determine the maximum absolute values of the shear and bending moment.

SOLUTION

Reactions

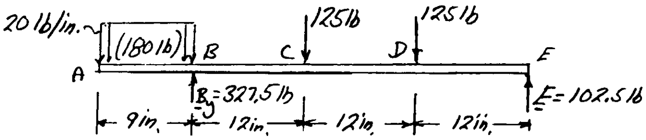
$$\Sigma F_x = 0: \quad B_x = 0$$

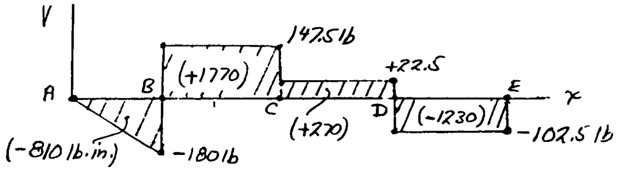
$$+\circlearrowleft \Sigma M_E = 0: \quad (20\text{ lb/in.})(9\text{ in.})(40.5\text{ in.}) + (125\text{ in.})(24\text{ in.}) + (125\text{ in.})(12\text{ in.}) - B(36\text{ in.}) = 0$$

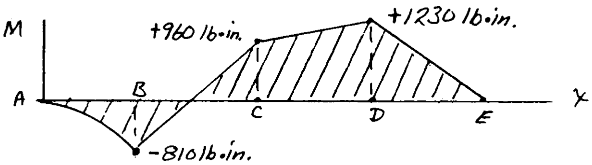
$$B_y = +327.5\text{ lb} \qquad \mathbf{B}_y = 327.5\text{ lb}\uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad -(20\text{ lb/in.})(9\text{ in.}) - 125\text{ lb} - 125\text{ lb} + 327.5\text{ lb} + E = 0$$

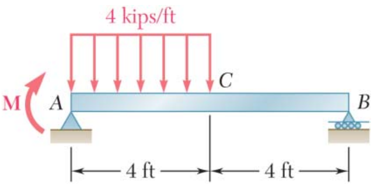
$$E = +102.5\text{ lb} \qquad \mathbf{E} = 102.5\text{ lb}\uparrow$$


20 lb/in.
(180 lb)
125 lb
125 lb
A
B
C
D
E
B_y = 327.5 lb
E = 102.5 lb
9 in.
12 in.
12 in.
12 in.





(*b*) $|V|_{\max} = 180.0\text{ lb}; \ |M|_{\max} = 1230\text{ lb}\cdot\text{in.}$ ◀

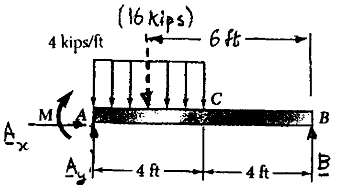

4 kips/ft
M
A
C
B
4 ft
4 ft

PROBLEM 7.161

For the beam shown, draw the shear and bending-moment diagrams, and determine the magnitude and location of the maximum absolute value of the bending moment, knowing that (*a*) $M = 0$, (*b*) $M = 24$ kip · ft.

SOLUTION

Free body: Beam


(16 kips)
6 ft
4 kips/ft
M
A
C
B
A_x
A_y
4 ft
4 ft
B

$$\Sigma F_x = 0: \quad A_x = 0$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (16 \text{ kips})(6 \text{ ft}) - A_y(8 \text{ ft}) - M = 0$$

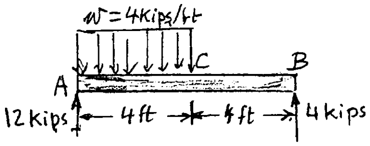
$$A_y = 12 \text{ kips} - \frac{1}{8}M \qquad (1) \ \triangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad B + 12 - \frac{1}{8}M - 16 = 0$$

$$B = 4 \text{ kips} + \frac{1}{8}M \qquad (2) \ \triangleleft$$

(*a*) $M = 0$:

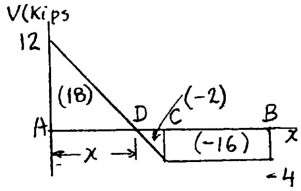
Load diagram


w = 4 kips/ft
C
B
A
12 kips
4 ft
4 ft
4 kips

Making M= 0 in. (1) and (2).

$$A_y = +12 \text{ kips}$$
$$B = +4 \text{ kips}$$

Shear diagram $\qquad V_A = A_y = +12$ kips



To determine Point D where $V = 0$:

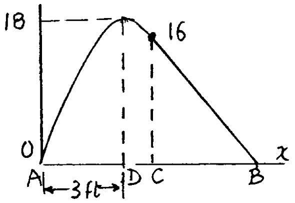
$$V_D - V_A = -wx$$
$$0 - 12 \text{ kips} = -(4 \text{ kips/ft})x$$

$x = 3$ ft ◁

We compute all areas

B. M. Diagram

At A: $\qquad M_A = 0$



$|M|_{\max} = 18.00$ kip·ft, ◀

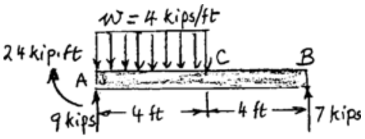
3.00 ft from A ◀

Parabola from A to C

PROBLEM 7.161 (Continued)

(*b*) $\underline{M = 24 \text{ kip} \cdot \text{ft}}$

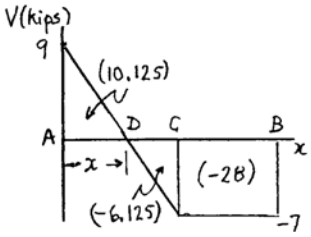
Load diagram


w = 4 kips/ft
24 kip·ft
A
C
B
9 kips
4 ft
4 ft
7 kips

Making $M = 24 \text{ kip} \cdot \text{ft}$ in (1) and (2)

$$A = 12 - \frac{1}{8}(24) = +9 \text{ kips}$$

$$B = 4 + \frac{1}{8}(24) = +7 \text{ kips}$$


V(kips)
9
(10.125)
A
D
C
B
x
x
(-28)
(-6.125)
-7

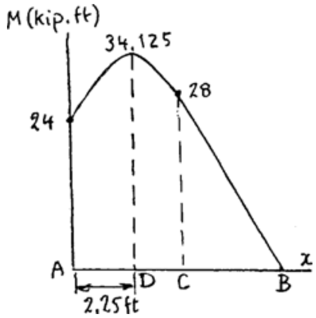
Shear diagram $V_A = A_y = +9 \text{ kips}$

To determine Point *D* where $V = 0$:

$$V_D = V_A = -wx$$

$$0 - 9 \text{ kips} = -(4 \text{ kips/ft})\, x$$

$x = 2.25 \text{ ft}$ ◁


M (kip. ft)
34.125
28
24
A
D
C
B
x
2.25 ft

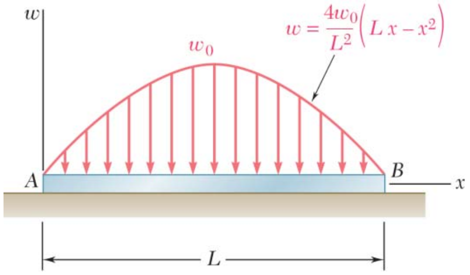
B. M. Diagram

At *A*: $M_A = +24 \text{ kip} \cdot \text{ft}$

$|M|_{\max} = 34.1 \text{ kip} \cdot \text{ft}$, ◀

2.25 ft from *A* ◀

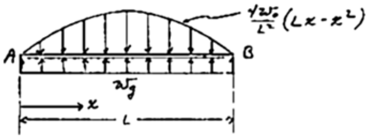
Parabola from *A* to *C*.



PROBLEM 7.162

The beam AB, which lies on the ground, supports the parabolic load shown. Assuming the upward reaction of the ground to be uniformly distributed, (*a*) write the equations of the shear and bending-moment curves, (*b*) determine the maximum bending moment.

SOLUTION



(*a*) $$\uparrow \Sigma F_y = 0: \quad w_g L - \int_0^L \frac{4w_0}{L^2}(Lx - x^2)dx = 0$$

$$w_g L = \frac{4w_0}{L^2}\left(\frac{1}{2}LL^2 - \frac{1}{3}L^3\right) = \frac{2}{3}w_0 L$$

$$w_g = \frac{2w_0}{3}$$

Define $$\xi = \frac{x}{L} \text{ so } d\xi = \frac{dx}{L} \longrightarrow \text{net load } w = 4w_0\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right] - \frac{2}{3}w_0$$

or $$w = 4w_0\left(-\frac{1}{6} + \xi - \xi^2\right)$$

$$V = V(0) - \int_0^{\xi} 4w_0 L\left(-\frac{1}{6} + \xi - \xi^2\right)d\xi$$

$$= 0 + 4w_0 L\left(\frac{1}{6}\xi + \frac{1}{2}\xi^2 - \frac{1}{3}\xi^3\right)$$

$$V = \frac{2}{3}w_0 L\,(\xi - 3\xi^2 + 2\xi^3) \blacktriangleleft$$

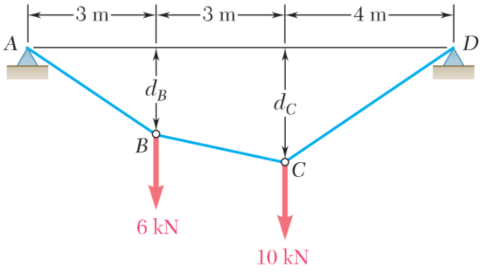
$$M = M_0 + \int_0^x V dx = 0 + \frac{2}{3}w_0 L^2 \int_0^{\xi}(\xi - 3\xi^2 + 2\xi^3)d\xi$$

$$= \frac{2}{3}w_0 L^2\left(\frac{1}{2}\xi^2 - \xi^3 + \frac{1}{2}\xi^4\right) = \frac{1}{3}w_0 L^2(\xi^2 - 2\xi^3 + \xi^4) \blacktriangleleft$$

(*b*) Max M occurs where $$V = 0 \longrightarrow 1 - 3\xi + 2\xi^2 = 0 \longrightarrow \xi = \frac{1}{2}$$

$$M\left(\xi = \frac{1}{2}\right) = \frac{1}{3}w_0 L^2\left(\frac{1}{4} - \frac{2}{8} + \frac{1}{16}\right) = \frac{w_0 L^2}{48}$$

$$M_{\max} = \frac{w_0 L^2}{48} \text{ at center of beam} \blacktriangleleft$$

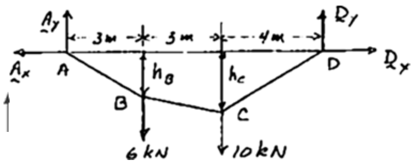

3 m
3 m
4 m
A
D
d_B
d_C
B
C
6 kN
10 kN

PROBLEM 7.163

Two loads are suspended as shown from the cable *ABCD*. Knowing that $d_B = 1.8$ m, determine (*a*) the distance d_C, (*b*) the components of the reaction at *D*, (*c*) the maximum tension in the cable.

SOLUTION

FBD Cable:


A_y
3m
3m
4m
D_y
A_x
A
h_B
h_C
D
D_x
B
C
6 kN
10 kN

$$\xrightarrow{} \Sigma F_x = 0: \quad -A_x + D_x = 0 \quad A_x = D_x$$

$$\circlearrowleft \Sigma M_A = 0: \quad (10\text{ m})D_y - (6\text{ m})(10\text{ kN}) - (3\text{ m})(6\text{ kN}) = 0$$

$$\mathbf{D}_y = 7.8\text{ kN}$$

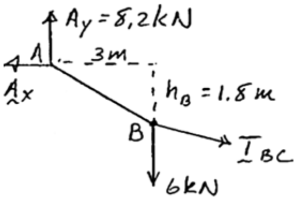
$$\uparrow \Sigma F_y = 0: \quad A_y - 6\text{ kN} - 10\text{ kN} + 7.8\text{ kN} = 0$$

$$\mathbf{A}_y = 8.2\text{ kN} \uparrow$$

FDB AB:

$$\circlearrowleft \Sigma M_B = 0: \quad (1.8\text{ m})A_x - (3\text{ m})(8.2\text{ kN}) = 0$$

$$\mathbf{A}_x = \frac{41}{3}\text{ kN} \longleftarrow$$

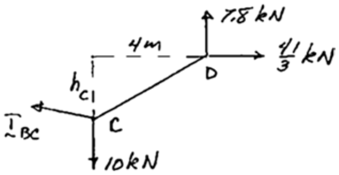

$A_y = 8.2$ kN
A
3m
A_x
$h_B = 1.8$ m
B
T_{BC}
6kN

From above $D_x = A_x = \dfrac{41}{3}$ kN

FBD CD:

$$\circlearrowleft \Sigma M_C = 0: \quad (4\text{ m})(7.8\text{ kN}) - d_C\left(\frac{41}{3}\text{ kN}\right) = 0$$

$$d_C = 2.283\text{ m}$$


7.8 kN
4m
$\frac{41}{3}$ kN
h_C
D
T_{BC}
C
10kN

(*a*) $d_C = 2.28$ m ◀

(*b*) $\mathbf{D}_x = 13.67$ kN ⟶ ◀

$\mathbf{D}_y = 7.80$ kN ↑ ◀

Since $A_x = B_x$ and $A_y > B_y$, max T is T_{AB}

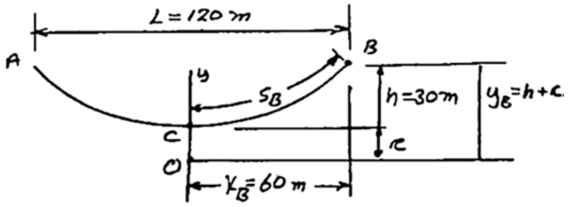
$$T_{AB} = \sqrt{A_x^2 + A_y^2} = \sqrt{\left(\frac{41}{3}\text{ kN}\right)^2 + (8.2\text{ kN})^2}$$

(*c*) $T_{\max} = 15.94$ kN ◀

PROBLEM 7.164

A wire having a mass per unit length of 0.65 kg/m is suspended from two supports at the same elevation that are 120 m apart. If the sag is 30 m, determine (*a*) the total length of the wire, (*b*) the maximum tension in the wire.

SOLUTION



Eq. 7.16:

$$y_B = c\cosh\frac{x_B}{c}$$

$$30m + c = c\cosh\frac{60}{c}$$

Solve by trial and error: $c = 64.459\text{ m}$

Eq. 7.15:

$$s_B = c\sin h\frac{x_B}{c}$$

$$s_B = (64.456\text{ m})\sinh\frac{60\text{ m}}{64.459\text{ m}}$$

$$s_B = 69.0478\text{ m}$$

Length $= 2s_B = 2(69.0478\text{ m}) = 138.0956\text{ m}$ $L = 138.1\text{ m}$ ◀

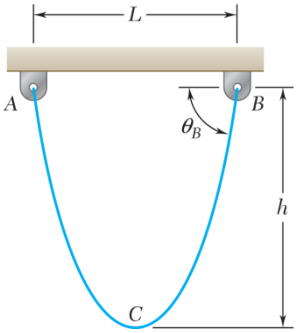
Eq. 7.18:

$$T_m = wy_B = w(h+c)$$

$$= (0.65\text{ kg/m})(9.81\text{ m/s}^2)(30\text{ m} + 64.459\text{ m})$$

$$T_m = 602.32\text{ N}$$

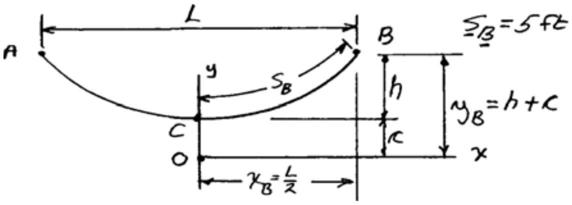
$T_m = 602\text{ N}$ ◀



PROBLEM 7.165

A 10-ft rope is attached to two supports A and B as shown. Determine (a) the span of the rope for which the span is equal to the sag, (b) the corresponding angle θ_B.

SOLUTION



Note: Since $L = h$,

$$x_B = \frac{L}{2} = \frac{h}{2}$$

Eq. 7.16:

$$y_B = \cosh\frac{x_B}{c}$$

$$h + c = c\cosh\frac{h/2}{c}$$

$$\frac{h}{c} + 1 = \cosh\left(\frac{1}{2}\frac{h}{c}\right)$$

Solve for h/c:

$$\frac{h}{c} = 4.933$$

Eq. 7.16:

$$y = c\cosh\frac{x}{c}$$

$$\frac{dy}{dx} = \sinh\frac{x}{c}$$

At B:

$$\tan\theta_B = \left.\frac{dy}{dx}\right|_B = \sinh\frac{x_B}{c}$$

Substitute

$$x_B = \frac{h}{2}: \quad \tan\theta_B = \sinh\left(\frac{1}{2}\frac{h}{c}\right) = \sinh\left(\frac{1}{2} \times 4.933\right)$$

$$\tan\theta_B = 5.848$$

$\theta_B = 80.3°$ ◀

PROBLEM 7.165 (Continued)

Eq. 7.17:

$$s_B = c\sinh\frac{x_B}{c} = c\sinh\left(\frac{1}{2}\frac{h}{c}\right)$$

$$5\text{ ft} = c\sinh\left(\frac{1}{2}\times 4.933\right)$$

$$5\text{ ft} = c(5.848)$$

$$c = 0.855$$

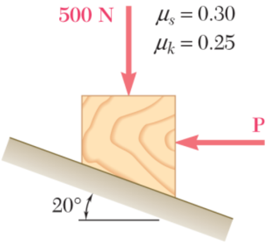
Recall that

$$\frac{h}{c} = 4.933$$

$$h = 4.933(0.855) = 4.218$$

$$h = 4.22\text{ ft} \blacktriangleleft$$

CHAPTER 8

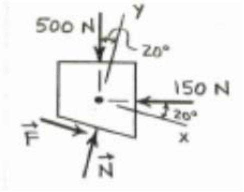


PROBLEM 8.1

Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when $P = 150$ N.

SOLUTION

Assume equilibrium:



$$\overset{+}{\searrow} \Sigma F_x = 0: \quad F + (500 \text{ N})\sin 20° - (150 \text{ N})\cos 20° = 0$$

$$F = -30.056 \text{ N} \quad \mathbf{F} = 30.056 \text{ N} \nwarrow$$

$$+\nearrow \Sigma F_y = 0: \quad N - (500 \text{ N})\cos 20° - (150 \text{ N})\sin 20° = 0$$

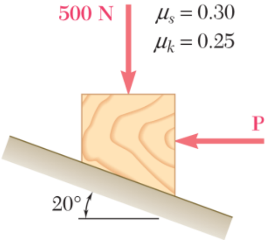
$$N = +521.15 \text{ N} \quad \mathbf{N} = 521.15 \text{ N} \nearrow$$

Maximum friction force:

$$\begin{aligned} F_m &= \mu_s N \\ &= 0.30(521.15 \text{ N}) \\ &= 156.345 \text{ N} \end{aligned}$$

Since $\mathbf{F}$ is $\nwarrow$ and $F < F_m$, block is in equilibrium ◀

$\mathbf{F} = 30.1 \text{ N} \nwarrow 20.0°$ ◀

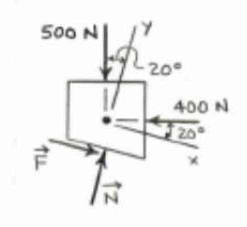


PROBLEM 8.2

Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when $P = 400$ N.

SOLUTION

Assume equilibrium:



$\overset{+}{\searrow}\ \Sigma F_x = 0\colon\quad F + (500\text{ N})\sin 20° - (400\text{ N})\cos 20° = 0$

$F = +204.87\text{ N} \qquad \mathbf{F} = 204.87\text{ N} \searrow$

$+\nearrow\ \Sigma F_y = 0\colon\quad N - (500\text{ N})\cos 20° - (400\text{ N})\sin 20° = 0$

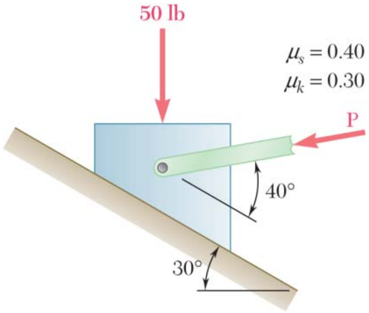
$N = +606.65\text{ N} \quad \mathbf{N} = 606.65\text{ N} \nearrow$

Maximum friction force:

$$\begin{aligned} F_m &= \mu_s N \\ &= 0.30(606.65\text{ N}) \\ &= 181.995\text{ N} \end{aligned}$$

Since $F > F_m$, block moves up ◀

Actual friction force: $F = F_k = \mu_k N = 0.25(606.65\text{ N})$ $\qquad \mathbf{F} = 151.7\text{ N} \nwarrow 20.0°$ ◀

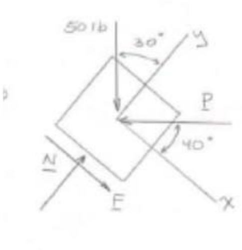


PROBLEM 8.3

Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when $P = 120$ lb.

SOLUTION

Assume equilibrium:



$$\overset{+}{\searrow} \Sigma F_x = 0: \quad F + (50 \text{ lb})\sin 30° - (120 \text{ lb})\cos 40° = 0$$

$$F = +66.925 \text{ lb}$$

$$+\nearrow \Sigma F_y = 0: \quad N - (50 \text{ lb})\cos 30° - (120 \text{ lb})\sin 40° = 0$$

$$N = +120.436 \text{ lb}$$

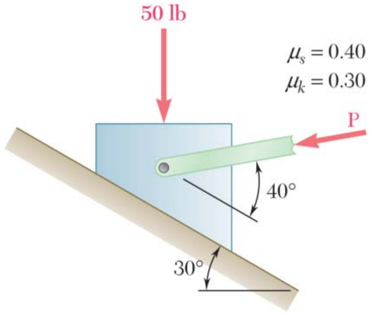
Maximum friction force:

$$\begin{aligned} F_m &= \mu_s N \\ &= 0.40(120.436 \text{ lb}) \\ &= 48.174 \text{ lb} \end{aligned}$$

We note that $F > F_m$. Thus, block moves up◀

Actual friction force:

$F = F_k = \mu_k N = 0.30(120.436 \text{ lb}) = 36.131$ lb, **F** = 36.1 lb ↖ 30.0° ◀

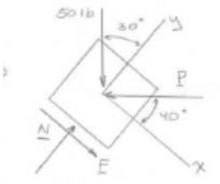


PROBLEM 8.4

Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when $P = 80$ lb.

SOLUTION

Assume equilibrium:



$$\overset{+}{\searrow}\Sigma F_x = 0: \quad F + (50\text{ lb})\sin 30° - (80\text{ lb})\cos 40° = 0$$

$$F = +36.284\text{ lb}$$

$$+\nearrow \Sigma F_y = 0: \quad N - (50\text{ lb})\cos 30° - (80\text{ lb})\sin 40° = 0$$

$$N = +94.724\text{ lb}$$

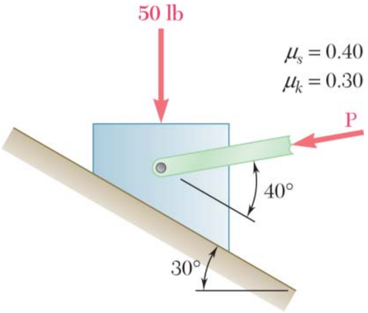
Maximum friction force:

$$\begin{aligned} F_m &= \mu_s N \\ &= 0.40(94.724\text{ lb}) \\ &= 37.890\text{ lb} \end{aligned}$$

We note that $F < F_m$. Thus, block is in equilibrium◀

Thus $\mathbf{F} = 36.3\text{ lb} \nwarrow 30.0°$ ◀

Note: $F_k = \mu_k N = 0.30(94.724\text{ lb}) = 28.417\text{ lb}$, $F > F_K$. If block is originally in motion, it will keep moving with $F_K = 28.4$ lb.


50 lb
$\mu_s = 0.40$
$\mu_k = 0.30$
P
40°
30°

PROBLEM 8.5

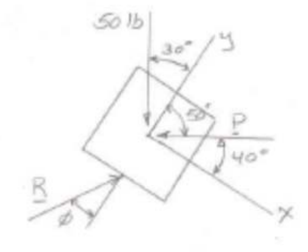
Determine the smallest value of P required to (*a*) start the block up the incline, (*b*) keep it moving up.

SOLUTION

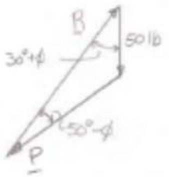
(*a*) To start block up the incline:

$$\mu_s = 0.40$$

$$\phi_s = \tan^{-1} 0.40 = 21.80°$$


50 lb
30°
y
50°
P
40°
R
ϕ
x

From force triangle:


B
50 lb
30°+ϕ
50°−ϕ
P

$$\frac{P}{\sin 51.80°} = \frac{50 \text{ lb}}{\sin 28.20°}$$

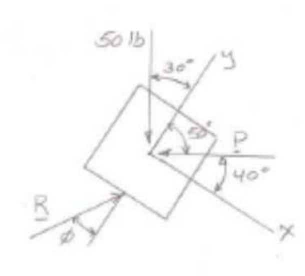
$P = 83.2$ lb ◀

(*b*) To keep block moving up:

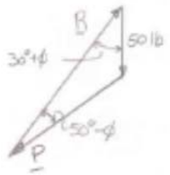
$$\mu_k = 0.30$$

$$\phi_k = \tan^{-1} 0.30 = 16.70°$$

PROBLEM 8.5 (Continued)

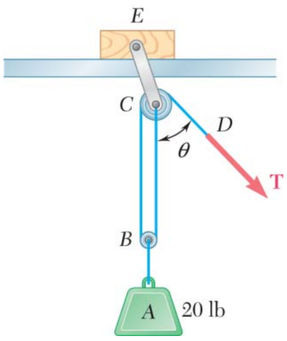

50 lb
30°
y
50°
P
40°
x
R
ϕ

From force triangle:


B
50 lb
30°+ϕ
50°−ϕ
P

$$\frac{P}{\sin 46.70°} = \frac{50 \text{ lb}}{\sin 33.30°}$$

$P = 66.3$ lb ◀



PROBLEM 8.6

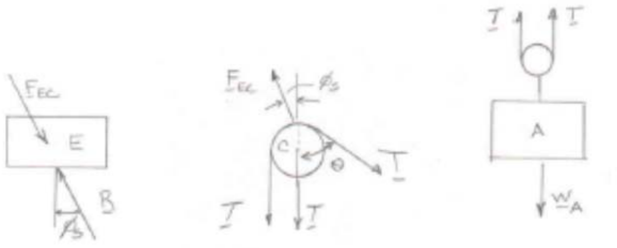
The 20-lb block A hangs from a cable as shown. Pulley C is connected by a short link to block E, which rests on a horizontal rail. Knowing that the coefficient of static friction between block E and the rail is μs = 0.35 and neglecting the weight of block E and the friction in the pulleys, determine the maximum allowable value of θ if the system is to remain in equilibrium.

SOLUTION

Free-body diagrams:

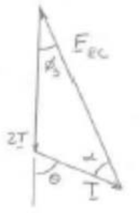
$\mu_s = 0.35$

$\phi_s = \tan^{-1} 0.35 = 19.29°$



Force triangle for pulley C:

Law of sines:



$$\frac{\sin\alpha}{2T} = \frac{\sin\phi_s}{T}$$

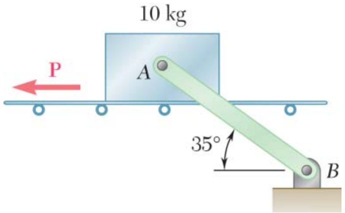
$$\sin\alpha = 2\sin\phi_s = 2\sin 19.29^\circ$$

$$\alpha = 41.35^\circ$$

$$\theta = \phi_s + \alpha = 19.29 + 41.35$$

$\theta = 60.6^\circ$ ◀

Note: Answer is independent of W_A.



PROBLEM 8.7

The 10-kg block is attached to link *AB* and rests on a moving belt. Knowing that $\mu_s = 0.30$ and $\mu_k = 0.25$ and neglecting the weight of the link, determine the magnitude of the horizontal force **P** that should be applied to the belt to maintain its motion (*a*) to the left as shown, (*b*) to the right.

SOLUTION

Link *AB* is a two-force member.

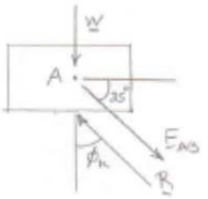
(a) Free-body diagram of block:

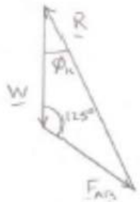
Since there is motion between block and belt, use

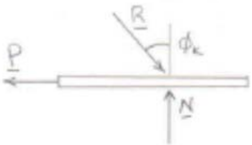
$$\mu_k = 0.25$$

$$\phi_k = \tan^{-1} 0.25 = 14.04°$$ Free-body diagram of belt:

$$W = 98.1 \text{ N}$$







Force triangle:

$$\frac{F_{AB}}{\sin 14.04°} = \frac{R}{\sin 125^\circ} = \frac{98.1 \text{ N}}{\sin 40.96}$$

$F_{AB} = 36.3 \text{ N } T$

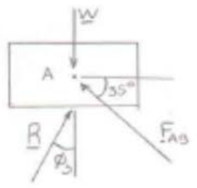
$$R = 122.586 \text{ N}$$

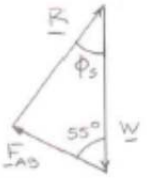
From free-body diagram of belt: $\Sigma F_x = 0: \quad (122.586 \text{ N})\sin 14.04^\circ - P = 0$

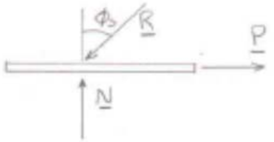
P = 29.7 N ⟵ ◀

PROBLEM 8.7 (Continued)

(*b*) Free-body diagram of block:Free-body diagram of belt:


W
A
35°
R
ϕ_s
F_{AB}


R
ϕ_s
55°
W
F_{AB}



Force triangle:

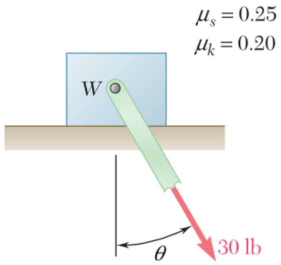
$$\frac{F_{AB}}{\sin 14.04°} = \frac{R}{\sin 55^\circ} = \frac{98.1 \text{ N}}{\sin 110.96}$$

$F_{AB} = 25.5 \text{ N } C$

$$R = 86.053 \text{ N}$$

Free-body diagram of belt: $\Sigma F_x = 0: \quad P - (86.053 \text{ N})\sin 14.04^\circ = 0$

$\mathbf{P} = 20.9 \text{ N} \longrightarrow$ ◀

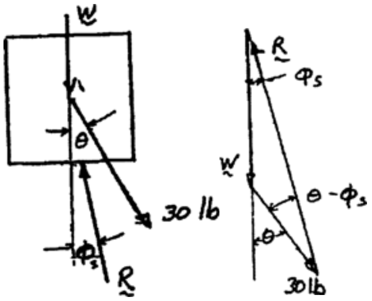

$\mu_s = 0.25$
$\mu_k = 0.20$
W
30 lb
θ

PROBLEM 8.8

Considering only values of θ less than 90°, determine the smallest value of θ required to start the block moving to the right when (*a*) $W = 75$ lb, (*b*) $W = 100$ lb.

SOLUTION

FBD block (Motion impending):


W
N
θ
30 lb
ϕ_s
R
R
ϕ_s
W
$\theta - \phi_s$
θ
30 lb

$$\phi_s = \tan^{-1}\mu_s = 14.036°$$

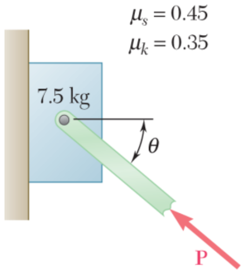
$$\frac{30 \text{ lb}}{\sin\phi_s} = \frac{W}{\sin(\theta - \phi_s)}$$

$$\sin(\theta - \phi_s) = \frac{W \sin 14.036°}{30 \text{ lb}}$$

or
$$\sin(\theta - 14.036°) = \frac{W}{123.695 \text{ lb}}$$

(*a*) $W = 75 \text{ lb}: \quad \theta = 14.036° + \sin^{-1}\frac{75 \text{ lb}}{123.695 \text{ lb}}$ $\qquad \theta = 51.4°$ ◀

(*b*) $W = 100 \text{ lb}: \quad \theta = 14.036° + \sin^{-1}\frac{100 \text{ lb}}{123.695 \text{ lb}}$ $\qquad \theta = 68.0°$ ◀


$\mu_s = 0.45$
$\mu_k = 0.35$
7.5 kg
θ
P

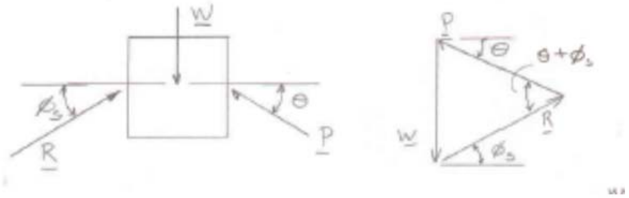
PROBLEM 8.9

Knowing that $\theta = 40°$, determine the smallest force **P** for which equilibrium of the 7.5-kg block is maintained.

SOLUTION

Free-body diagram of block and force triangle:

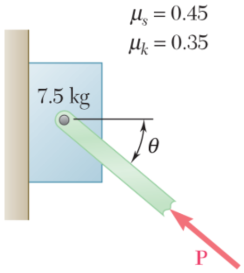
Since we seek smallest P downward motion impends. $W = (7.5 \text{ kg})(9.81 \text{ m/s}^2) = 73.575 \text{ N}$


W
ϕ_s
R
θ
P
P
θ
$\theta+\phi_s$
W
R
ϕ_s

$$\phi_s = \tan^{-1} \mu_s = \tan^{-1}(0.45) = 24.23°$$

$$\frac{P}{\sin 65.77^\circ} = \frac{73.575 \text{ N}}{\sin 64.23^\circ}$$

$P = 74.5$ N ◀


$\mu_s = 0.45$
$\mu_k = 0.35$
7.5 kg
θ
P

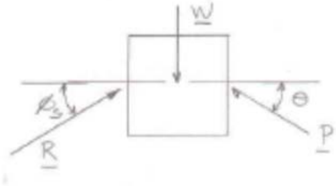
PROBLEM 8.10

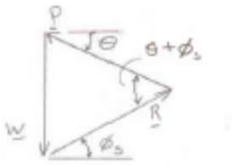
Knowing that P = 100 N, determine the range of values of θ for which equilibrium of the 7.5-kg block is maintained.

SOLUTION

Free-body diagram of block and force triangle:

For motion impending downward, $W = (7.5 \text{ kg})(9.81 \text{ m/s}^2) = 73.575 \text{ N}$


W
ϕ_s
R
θ
P


P
θ
$\theta+\phi_s$
W
R
ϕ_s

$$\phi_s = \tan^{-1}\mu_s = \tan^{-1}(0.45) = 24.23°$$

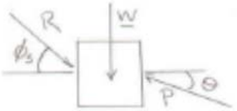
$$\frac{100 \text{ N}}{\sin 65.77^\circ} = \frac{73.575 \text{ N}}{\sin(\theta + \phi_s)}$$

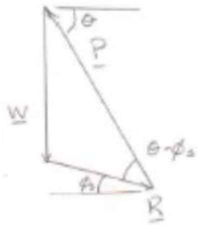
$$\sin(\theta + 24.23^\circ) = 0.67093$$

$$\theta + 24.23^\circ = 42.14^\circ$$

$$\theta = 17.91^\circ$$

For motion impending upward,


R
W
ϕ_s
P
θ


θ
P
W
$\theta-\phi_s$
ϕ_s
R

PROBLEM 8.10 (Continued)

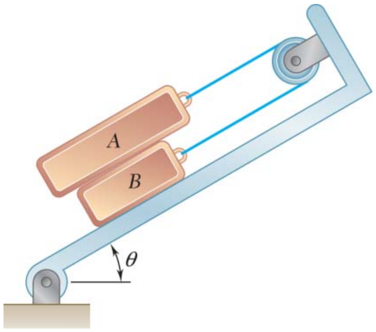
$$\frac{100\text{ N}}{\sin 114.23^\circ} = \frac{73.575\text{ N}}{\sin\left(\theta - \phi_s\right)}$$

$$\sin\left(\theta - 24.23^\circ\right) = 0.67093$$

$$\theta - 24.23^\circ = 42.14^\circ$$

$$\theta = 66.37^\circ$$

$$17.91^\circ \le \theta \le 66.4^\circ \blacktriangleleft$$



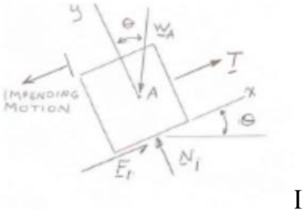
PROBLEM 8.11

The 50-lb block *A* and the 25-lb block *B* are supported by an incline that is held in the position shown. Knowing that the coefficient of static friction is 0.15 between the two blocks and zero between block *B* and the incline, determine the value of θ for which motion is impending.

SOLUTION

Since motion impends, $F = \mu_s N$ between *A*+*B*

Free body: Block *A*



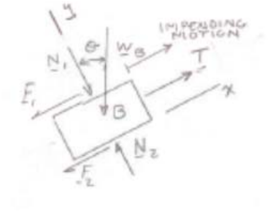
$$\text{Impending motion: } \Sigma F_y = 0: \quad N_1 - 50\cos\theta = 0$$

$N_1 = 50\cos\theta$

$$\Sigma F_x = 0: \quad T - 50\sin\theta + \mu_1 N_1 = 0$$

$$T = 50\sin\theta - \mu_1(50)\cos\theta \qquad (1)$$

Free body: Block *B*



$$\text{Impending motion: } \Sigma F_y = 0: \quad N_2 - N_1 - 25\cos\theta = 0$$

$$N_2 = 75\cos\theta$$

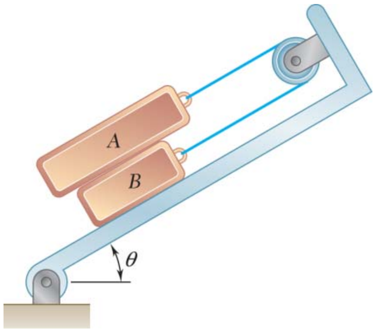
$$\Sigma F_x: \quad T - \mu_1 N_1 - \mu_2 N_2 - 25\sin\theta$$

$$T = \mu_1(50)\cos\theta + \mu_2(75)\cos\theta + 25\sin\theta \qquad (2)$$

Eq. (1)-Eq. (2): $$0 = 25\sin\theta - \mu_1(100)\cos\theta - \mu_2(75)\cos\theta = 0$$

Substituting in for $\mu_1 = 0.15,\ \mu_2 = 0,$ we have:

$$15\cos\theta = 25\sin\theta: \quad \tan\theta = \frac{15}{25}; \qquad \theta = 31.0° ◀$$



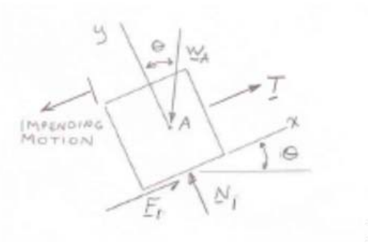
PROBLEM 8.12

The 50-lb block *A* and the 25-lb block *B* are supported by an incline that is held in the position shown. Knowing that the coefficient of static friction is 0.15 between all surfaces of contact, determine the value of θ for which motion is impending.

SOLUTION

Since motion impends, $F = \mu_s N$ between *A*+*B*

Free body: Block *A*



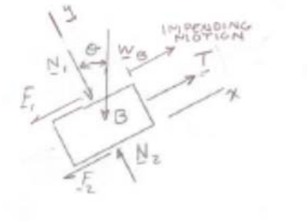
Impending motion: $\Sigma F_y = 0\colon \quad N_1 - 50\cos\theta = 0$

$N_1 = 50\cos\theta$

$$\Sigma F_x = 0\colon \quad T - 50\sin\theta + \mu_1 N_1 = 0$$

$$T = 50\sin\theta - \mu_1(50)\cos\theta \tag{1}$$

Free body: Block *B*



Impending motion: $\Sigma F_y = 0\colon \quad N_2 - N_1 - 25\cos\theta = 0$

$$N_2 = 75\cos\theta$$

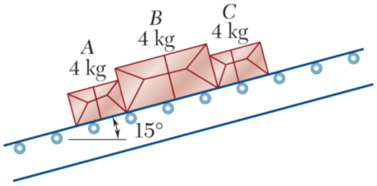
$$\Sigma F_x\colon \quad T - \mu_1 N_1 - \mu_2 N_2 - 25\sin\theta$$

$$T = \mu_1(50)\cos\theta + \mu_2(75)\cos\theta + 25\sin\theta \tag{2}$$

Eq. (1)-Eq. (2): $\quad 0 = 25\sin\theta - \mu_1(100)\cos\theta - \mu_2(75)\cos\theta = 0$

Substituting in for $\mu_1 = \mu_2 = 0.15$, we have:

$$175(0.15)\cos\theta = 25\sin\theta\colon \quad \tan\theta = \frac{26.25}{25}; \qquad \theta = 46.4° \blacktriangleleft$$

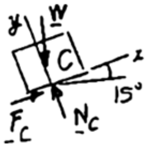


PROBLEM 8.13

Three 4-kg packages *A*, *B*, and *C* are placed on a conveyor belt that is at rest. Between the belt and both packages *A* and *C* the coefficients of friction are $\mu_s = 0.30$ and $\mu_k = 0.20$; between package *B* and the belt the coefficients are $\mu_s = 0.10$ and $\mu_k = 0.08$. The packages are placed on the belt so that they are in contact with each other and at rest. Determine which, if any, of the packages will move and the friction force acting on each package.

SOLUTION

Consider *C* by itself: Assume equilibrium



$$+\nwarrow \Sigma F_y = 0: \quad N_C - W\cos 15° = 0$$

$$N_C = W\cos 15° = 0.966W$$

$$+\nearrow \Sigma F_x = 0: \quad F_C - W\sin 15° = 0$$

$$F_C = W\sin 15° = 0.259W$$

But
$$\begin{aligned} F_m &= \mu_s N_C \\ &= 0.30(0.966W) \\ &= 0.290W \end{aligned}$$

Thus, $F_C < F_m$ Package *C* does not move ◀

$$\begin{aligned} F_C &= 0.259W \\ &= 0.259(4\text{ kg})(9.81\text{ m/s}^2) \\ &= 10.16\text{ N} \end{aligned}$$

$\mathbf{F}_C = 10.16\text{ N} \nearrow$ ◀

Consider *B* by itself: Assume equilibrium. We find,

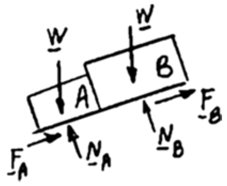
$$F_B = 0.259W$$
$$N_B = 0.966W$$

But
$$\begin{aligned} F_m &= \mu_s N_B \\ &= 0.10(0.966W) \\ &= 0.0966W \end{aligned}$$

Thus, $F_B > F_m$. Package *B* would move if alone ◀

PROBLEM 8.13 (Continued)

Consider *A* and *B* together: Assume equilibrium



$$F_A = F_B = 0.259W$$

$$N_A = N_B = 0.966W$$

$$F_A + F_B = 2(0.259W) = 0.518W$$

$$(F_A)_m + (F_B)_m = 0.3N_A + 0.1N_B = 0.386W$$

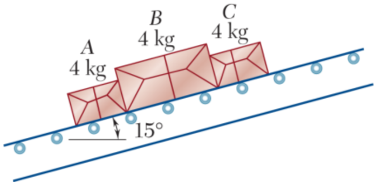
Thus, $F_A + F_B > (F_A)_m + (F_B)_m$ *A* and *B* move◀

$$F_A = \mu_k N_A = 0.2(0.966)(4)(9.81)$$

$\mathbf{F}_A = 7.58$ N ↗ ◀

$$F_B = \mu_k N_B = 0.08(0.966)(4)(9.81)$$

$\mathbf{F}_B = 3.03$ N ↗ ◀



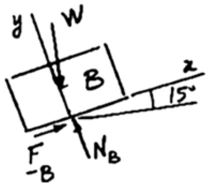
PROBLEM 8.14

Solve Problem 8.13 assuming that package *B* is placed to the right of both packages *A* and *C*.

PROBLEM 8.13 Three 4-kg packages*A*, *B*, and *C* are placed on a conveyor belt that is at rest. Between the belt and both packages *A* and *C* the coefficients of friction are $\mu_s = 0.30$ and $\mu_k = 0.20$; between package *B* and the belt the coefficients are $\mu_s = 0.10$ and $\mu_k = 0.08$. The packages are placed on the belt so that they are in contact with each other and at rest. Determine which, if any, of the packages will move and the friction force acting on each package.

SOLUTION

Consider package *B* by itself: Assume equilibrium



$$+\nwarrow \Sigma F_y = 0\colon \quad N_B - W\cos 15° = 0$$

$$N_B = W\cos 15° = 0.966W$$

$$+\nearrow \Sigma F_x = 0\colon \quad F_B - W\sin 15° = 0$$

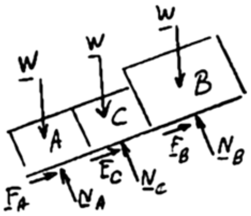
$$F_B = W\sin 15° = 0.259W$$

But
$$F_m = \mu_s N_B$$
$$= 0.10(0.966W)$$
$$= 0.0966W$$

Thus, F_B . F_m. Package *B* would move if alone.

Consider all packages together: Assume equilibrium. In a manner similar to above, we find

$$N_A = N_B = N_C = 0.966W$$
$$F_A = F_B = F_C = 0.259W$$
$$F_A + F_B + F_C = 3(0.259W)$$
$$= 0.777W$$



But
$$(F_A)_m = (F_C)_m = \mu_s N$$
$$= 0.30(0.966W)$$
$$= 0.290W$$

and
$$(F_B)_m = 0.10(0.966W)$$
$$= 0.0966W$$

PROBLEM 8.14 (Continued)

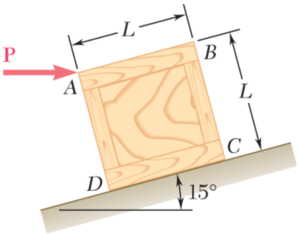
Thus, $$(F_A)_m + (F_C)_m + (F_B)_m = 2(0.290W) + 0.0966W$$
$$= 0.677W$$

and we note that $$F_A + F_B + F_C \;.\; (F_A)_m + (F_C)_m + (F_B)_m$$

All packages move◀

$$F_A = F_C = \mu_k N$$
$$= 0.20(0.966)(4\text{ kg})(9.81\text{ m/s}^2)$$
$$= 7.58\text{ N}$$
$$F_B = \mu_k N$$
$$= 0.08(0.966)(4\text{ kg})(9.81\text{ m/s}^2)$$
$$= 3.03\text{ N}$$

$\mathbf{F}_A = \mathbf{F}_C = 7.58\text{ N}$ ↗ ; $\mathbf{F}_B = 3.03\text{ N}$ ↗ ◀



PROBLEM 8.15

A uniform crate with a massof 30 kg must be moved up along the 15° incline without tipping. Knowing that force **P** is horizontal, determine (*a*) the largest allowable coefficient of static friction between the crate and the incline, (*b*) the corresponding magnitude of force **P**.

SOLUTION

(*a*) Free-body diagram

For tipping to be impending, reaction is at *C*,

Since the crate is a 3-force body. **C** must pass through *E* where **P** and **W** intersect.

Geometry analysis:

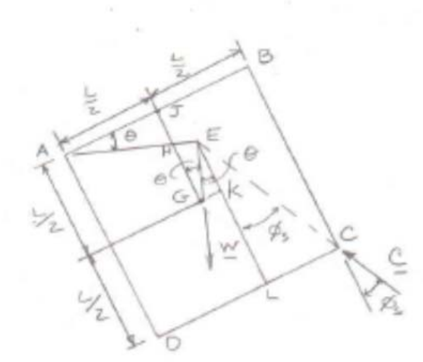
$$HJ = \frac{L}{2}\tan\theta \quad HG = \frac{L}{2} - HJ = \frac{L}{2}(1-\tan\theta) \quad GE = HG\cos\theta = \frac{L}{2}(1-\tan\theta)\cos\theta$$

$$EK = GE\cos\theta = \frac{L}{2}(1-\tan\theta)\cos^2\theta \quad GK = GE\sin\theta = \frac{L}{2}(1-\tan\theta)\cos\theta\sin\theta$$

$$EL = \frac{L}{2} + EK = \frac{L}{2}\left(1+\cos^2\theta - \sin\theta\cos\theta\right)$$

$$EL = \frac{L}{2}\left(1+\cos^2 15^\circ - \sin 15^\circ \cos 15^\circ\right)$$

$$EL = 0.84151L$$



$$LC = \frac{L}{2} - GK$$

$$LC = \frac{L}{2} - \frac{L}{2}(1-\tan\theta)\cos\theta\sin\theta$$

$$LC = \frac{L}{2} - \frac{L}{2}\left(\cos\theta\sin\theta - \frac{\sin\theta}{\cos\theta}\cos\theta\sin\theta\right)$$

$$LC = \frac{L}{2}\left(1-\cos\theta\sin\theta + \sin^2\theta\right)$$

$$LC = \frac{L}{2}\left(1-\cos 15^\circ \sin 15^\circ + \sin^2 15^\circ\right)$$

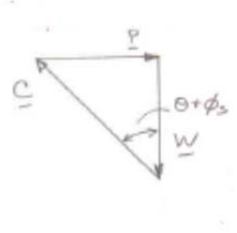
$$LC = 0.40849L$$

PROBLEM 8.15 (Continued)

$$\tan\phi_s = \frac{LC}{EL} = \frac{0.40849L}{0.84151L} = 0.48543, \quad \phi_s = 25.89^\circ$$

$\mu_s = \tan\phi_s$ $\mu_s = 0.485$ ◀

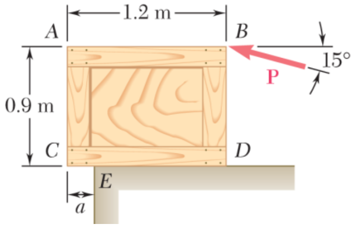
(b) Force Triangle



$$P = W\tan(\theta + \phi_s)$$

$$P = (30\text{ kg})(9.81\text{ m/s}^2)\tan(15^\circ + 25.89^\circ)$$

$P = 255\text{ N}$ ◀


1.2 m
A
B
15°
P
0.9 m
C
D
E
a

PROBLEM 8.16

A worker slowly moves a 50-kg crate to the left along a loading dock by applying a force **P** at corner *B* as shown. Knowing that the crate starts to tip about the edge *E* of the loading dock when *a*= 200 mm, determine (*a*) the coefficient of kinetic friction between the crate and the loading dock, (*b*) the corresponding magnitude *P* of the force.

SOLUTION

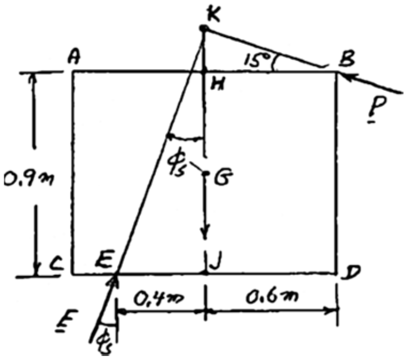
Free body: Crate Three-force body.

Reaction *E* must pass through *K* where **P** and **W** intersect.

Geometry:

(*a*)

$$HK = (0.6\text{ m})\tan 15° = 0.16077\text{ m}$$


K
A
15°
B
H
P
0.9m
ϕ_s
G
C
E
J
D
0.4m
0.6m
E
ϕ_s

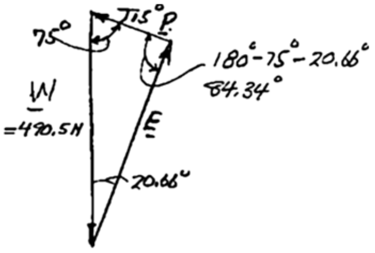
$$JK = 0.9\text{ m} + HK = 1.06077\text{ m}$$

$$\tan\phi_s = \frac{0.4\text{ m}}{1.06077\text{ m}} = 0.37708$$

$$\mu_s = \tan\phi_s = 0.377 \blacktriangleleft$$

$$\phi_s = 20.66°$$

Force triangle:


75°
15°
P
180°−75°−20.66°
84.34°
W
=490.5N
E
20.66°

(*b*)

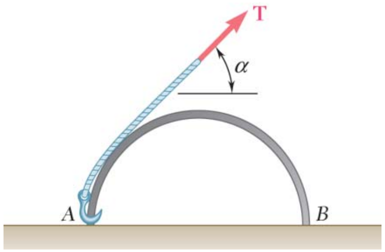
$$W = (50\text{ kg})(9.81\text{ m/s}) = 490.5\text{ N}$$

Law of sines:

$$\frac{P}{\sin 20.66°} = \frac{490.5\text{ N}}{\sin 84.34°}$$

$$P = 173.91\text{ N}$$

$$P = 173.9\text{ N} \blacktriangleleft$$



PROBLEM 8.17

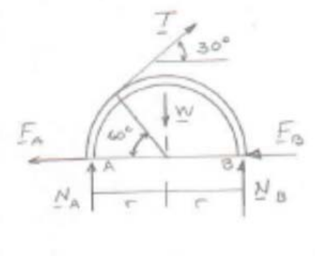
A half-section of pipe weighing 200 lb is pulled by a cable as shown. The coefficient of static friction between the pipe and the floor is 0.40. If $\alpha = 30°$, determine (*a*) the tension T required to move the pipe, (*b*) whether the pipe will slide or tip.

SOLUTION

(*a*)FBD pipe:*Note:* assume that pipe slides

$$\uparrow \Sigma F_y = 0; \quad N_A + N_B - 200 + T\sin 30^\circ = 0$$

$$N_A + N_B = 200 - 0.5T$$



$$F_A + F_B = \mu_s N_A + \mu_s N_B = \mu_s\left(N_A + N_B\right)$$

$$F_A + F_B = 0.4\left(200 - 0.5T\right) \qquad (1)$$

$$\longrightarrow \Sigma F_x = 0: \quad T\cos 30^\circ - F_A - F_B = 0$$

$$0.866T - \left(80 - 0.2T\right) = 0$$

$T = 75.0$ lb ◀

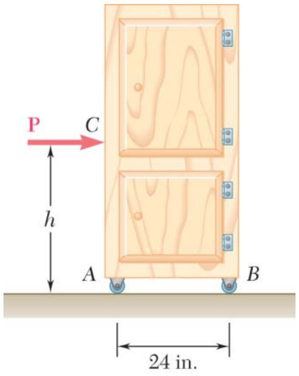
(*b*) Check that pipe does not tip about B: $\circlearrowleft$

$$\Sigma M_B = 0: \quad (200\text{ lb})r - N_A(2r) - \left(T\cos 30^\circ\right)\left(r\sin 60^\circ\right) - \left(T\sin 30^\circ\right)\left(r + r\cos 60^\circ\right) = 0$$

$$2N_A = 200\text{ lb} - T(\cos 30^\circ \sin 60^\circ + \sin 30^\circ + \sin 30^\circ \sin 30^\circ \cos 60^\circ)$$

$$N_A = 43.7\text{ lb}$$

Since $N_A > 0$, pipe does not tip. ◀

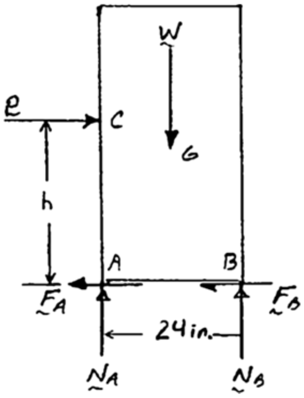


PROBLEM 8.18

A 120-lb cabinet is mounted on casters that can be locked to prevent their rotation. The coefficient of static friction between the floor and each caster is 0.30. Assuming that the casters at both *A* and *B* are locked, determine (*a*) the force **P** required to move the cabinet to the right, (*b*) the largest allowable value of *h* if the cabinet is not to tip over.

SOLUTION

FBD cabinet:



(*a*)

$$\uparrow \Sigma F_y = 0: \quad N_A + N_B - W = 0$$

$$N_A + N_B = W$$

Impending slip:

$$F_A = \mu_s N_A$$

$$F_B = \mu_s N_B$$

So

$$F_A + F_B = \mu_s W$$

$$\xrightarrow{} \Sigma F_x = 0: \quad P - F_A - F_B = 0$$

$$P = F_A + F_B = \mu_s W$$

$W = 120$ lb

$\mu_s = 0.3$

$$P = 0.3(120 \text{ lb}) = 141.26 \text{ N}$$

$\mathbf{P} = 36.0 \text{ lb} \longrightarrow$ ◀

(*b*) For tipping, $N_A = F_A = 0$

$$\circlearrowleft \Sigma M_B = 0: \quad hP - (12 \text{ in.})W = 0$$

$$h_{\max} = (12 \text{ in.})\frac{W}{P} = (12 \text{ in.})\frac{1}{\mu_s} = \frac{12 \text{ in.}}{0.3}$$

$h_{\max} = 40.0$ in. ◀

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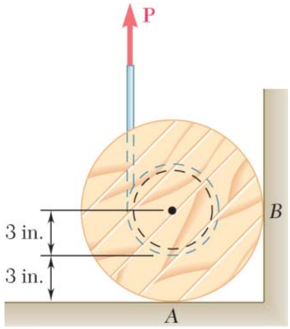
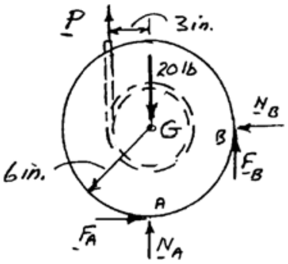


PROBLEM 8.19

Wire is being drawn at a constant rate from a spool by applying a vertical force **P** to the wire as shown. The spool and the wire wrapped on the spool have a combined weight of 20 lb. Knowing that the coefficients of friction at both *A* and *B* are $\mu_s = 0.40$ and $\mu_k = 0.30$, determine the required magnitude of the force **P**.

SOLUTION



Since spool is rotating

$$F_A = \mu_k N_A \qquad F_B = \mu_k N_B$$

$$+\circlearrowright \Sigma M_G = 0: \quad P(3 \text{ in.}) - F_A(6 \text{ in.}) - F_B(6 \text{ in.}) = 0$$

$$3P - 6\mu_k(N_A + N_B) = 0 \tag{1}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_A - N_B = 0$$

$$N_B = \mu_k N_A \tag{2}$$

$$+\uparrow \Sigma F_y = 0: \quad P + N_A + F_B - 20 \text{ lb} = 0$$

$$P + N_A + \mu_k N_B - 20 = 0$$

Substitute for N_B from (2):

$$P + N_A + \mu_k N_A - 20 = 0$$

$$N_A = \frac{20 - P}{1 + \mu_{k^2}} \tag{3}$$

Substitute from (2) into (1):

$$3P - 6\mu_k(N_A + \mu_k N_A) = 0$$

$$N_A = \frac{1}{2}\frac{P}{\mu_k(1 + \mu_k)} \tag{4}$$

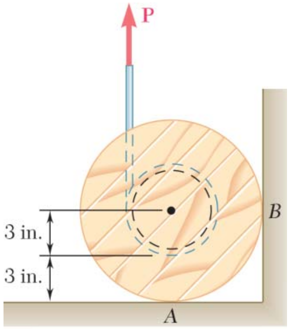
(3)=(4):

$$\frac{20 - P}{1 + \mu_k^2} = \frac{P}{2(\mu_k + \mu_k^2)}$$

Substitute $\mu_k = 0.30$:

$$\frac{20 - P}{1 + (0.3)^2} \; \frac{P}{2(0.3)(1.03)}$$

$20 - P = 1.3974P; \quad 2.3974P = 20;$ $\qquad P = 8.34 \text{ lb}$ ◀

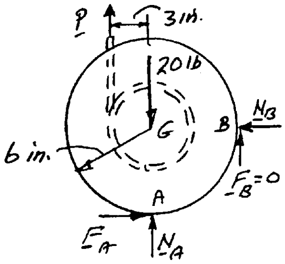


PROBLEM 8.20

Solve Problem 8.19 assuming that the coefficients of friction at *B* are zero.

PROBLEM 8.19 Wire is being drawn at a constant rate from a spool by applying a vertical force **P** to the wire as shown. The spool and the wire wrapped on the spool have a combined weight of 20 lb. Knowing that the coefficients of friction at both *A* and *B* are $\mu_s = 0.40$ and $\mu_k = 0.30$, determine the required magnitude of the force **P**.

SOLUTION



Since spool is rotating

$$F_A = \mu_k N_A$$

$$+\circlearrowright \Sigma M_G = 0: \quad P(3 \text{ in.}) - F_A(6 \text{ in.}) = 0$$

$$P = 2F_A = 2\mu_k N_A \qquad (1)$$

$$+\uparrow \Sigma F_y = 20: \quad P - 20 \text{ lb} + N_A = 0$$

$$N_A = 20 - P \qquad (2)$$

Substitute for N_A from (2) into (1) $\quad P = 2\mu_k(20 - P)$

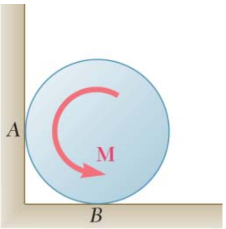
Substitute $\mu_k = 0.30$: $\quad P = 2(0.3)(20 - P)$

$$1.667P = 20 - P$$

$$2.667P = 20$$

$$P = 7.50 \text{ lb}$$

$P = 7.50$ lb ◀

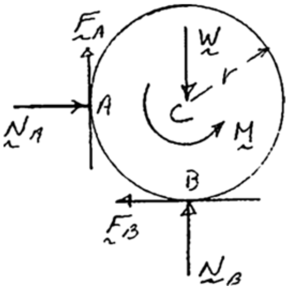


PROBLEM 8.21

The cylinder shown is of weight W and radius r. Express in terms W and r the magnitude of the largest couple **M** that can be applied to the cylinder if it is not to rotate, assuming the coefficient of static friction to be (*a*) zero at A and 0.30 at B, (*b*) 0.25 at A and 0.30 at B.

SOLUTION

FBD cylinder:



For maximum M, motion impends at both A and B

$$F_A = \mu_A N_A$$
$$F_B = \mu_B N_B$$

$$\xrightarrow{} \Sigma F_x = 0: \quad N_A - F_B = 0$$
$$N_A = F_B = \mu_B N_B$$

$$F_A = \mu_A N_A = \mu_A \mu_B N_B$$

$$\uparrow \Sigma F_y = 0: \quad N_B + F_A - W = 0$$
$$N_B(1 + \mu_A \mu_B) = W$$

or

$$N_B = \frac{1}{1 + \mu_A \mu_B} W$$

and

$$F_B = \mu_B N_B = \frac{\mu_B}{1 + \mu_A \mu_B} W$$

$$F_A = \mu_A \mu_B N_B = \frac{\mu_A \mu_B}{1 + \mu_A \mu_B} W$$

$$\circlearrowleft \Sigma M_C = 0: \quad M - r(F_A + F_B) = 0$$

$$M = Wr\mu_B \frac{1 + \mu_A}{1 + \mu_A \mu_B}$$

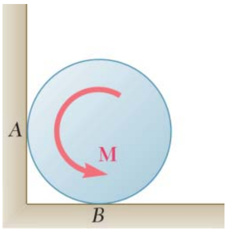
(*a*) For $\mu_A = 0$ and $\mu_B = 0.30$:

$M = 0.300Wr$ ◀

(*b*) For $\mu_A = 0.25$ and $\mu_B = 0.30$:

$M = 0.349Wr$ ◀

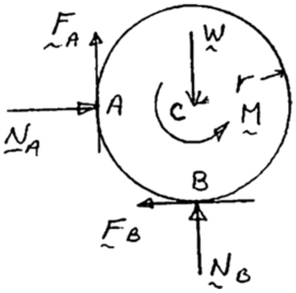
PROBLEM 8.22



The cylinder shown is of weight W and radius r, and the coefficient of static friction μ_s is the same at A and B. Determine the magnitude of the largest couple **M** that can be applied to the cylinder if it is not to rotate.

SOLUTION

FBD cylinder:



For maximum M, motion impends at both A and B

$$F_A = \mu_s N_A$$
$$F_B = \mu_s N_B$$

$$\rightarrow \Sigma F_x = 0: \quad N_A - F_B = 0$$
$$N_A = F_B = \mu_s N_B$$

$$F_A = \mu_s N_A = \mu_s^2 N_B$$

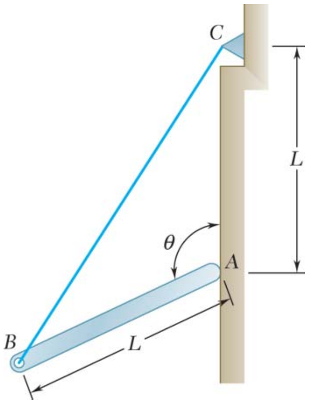
$$\uparrow \Sigma F_y = 0: \quad N_B + F_A - W = 0$$
$$N_B + \mu_s^2 N_B = W$$

or
$$N_B = \frac{W}{1+\mu_s^2}$$

and
$$F_B = \frac{\mu_s W}{1+\mu_s^2}$$
$$F_A = \frac{\mu_s^2 W}{1+\mu^2}$$

$$\circlearrowleft \Sigma M_C = 0: \quad M - r(F_A + F_B) = 0$$
$$M = r\left(\mu_s + \mu_s^2\right)\frac{W}{1+\mu_s^2}$$

$$M_{\max} = Wr\mu_s \frac{1+\mu_s}{1+\mu_s^2} \blacktriangleleft$$



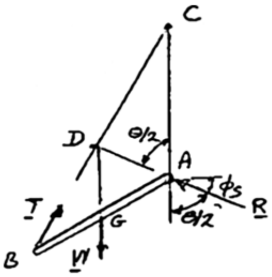
PROBLEM 8.23

End A of a slender, uniform rod of length L and weight W bears on a surface as shown, while end B is supported by a cord BC. Knowing that the coefficients of friction are $\mu_s = 0.40$ and $\mu_k = 0.30$, determine (*a*) the largest value of θ for which motion is impending, (*b*) the corresponding value of the tension in the cord.

SOLUTION

Free-body diagram

Three-force body. Line of action of **R** must pass through *D*, where **T** and **R** intersect.



Motion impends:

$$\tan\phi_s = 0.4$$

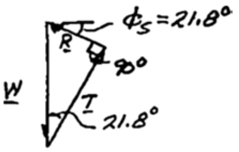
$$\phi_s = 21.80°$$

(*a*) Since *BG*=*GA*, it follows that *BD*=*DC* and *AD* bisects $\angle BAC$

$$\frac{\theta}{2} + \phi_s = 90°$$

$$\frac{\theta}{2} + 21.8° = 90°$$

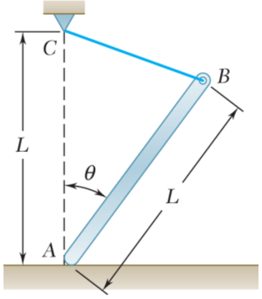
$\theta = 136.4°$ ◀



(*b*) Force triangle (right triangle):

$$T = W\cos 21.8°$$

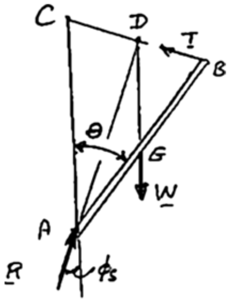
$T = 0.928W$ ◀



PROBLEM 8.24

End A of a slender, uniform rod of length L and weight W bears on a surface as shown, while end B is supported by a cord BC. Knowing that the coefficients of friction are $\mu_s = 0.40$ and $\mu_k = 0.30$, determine (*a*) the largest value of θ for which motion is impending, (*b*) the corresponding value of the tension in the cord.

SOLUTION



Free-body diagram

Rod AB is a three-force body. Thus, line of action of **R** must pass through D, where **W** and **T** intersect.

Since AG=GB, CD=DB and the median AD of the isosceles triangle ABC bisects the angle θ.

(*a*) Thus,

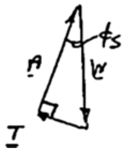
$$\phi_s = \frac{1}{2}\theta$$

Since motion impends,

$$\phi_s = \tan^{-1} 0.40 = 21.80^\circ$$
$$\theta = 2\phi_s = 2(21.8^\circ)$$

$\theta = 43.6^\circ$ ◀

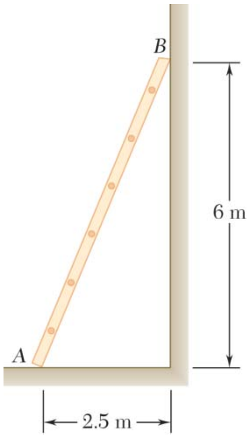
(*b*) Force triangle:



This is a right triangle.

$$T = W\sin\phi_s$$
$$= W\sin 21.8^\circ$$

$T = 0.371W$ ◀



PROBLEM 8.25

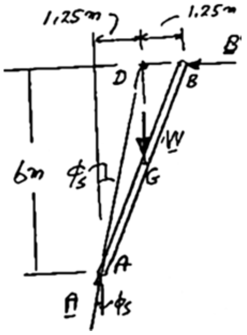
A 6.5-m ladder *AB* leans against a wall as shown. Assuming that the coefficient of static friction μ_s is zero at *B*, determine the smallest value of μ_s at *A* for which equilibrium is maintained.

SOLUTION

Free body: Ladder

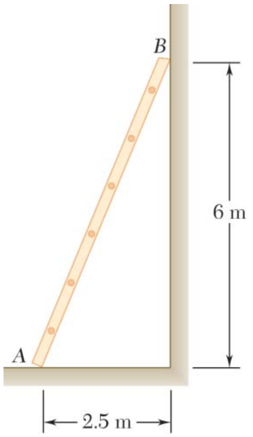
Three-force body.

Line of action of **A** must pass through *D*, where **W** and **B** intersect.



At *A*: $$\mu_s = \tan\phi_s = \frac{1.25\text{ m}}{6\text{ m}} = 0.2083$$

$\mu_s = 0.208$ ◀



PROBLEM 8.26

A 6.5-m ladder *AB* leans against a wall as shown. Assuming that the coefficient of static friction μ_s is the same at *A* and *B*, determine the smallest value of μ_s for which equilibrium is maintained.

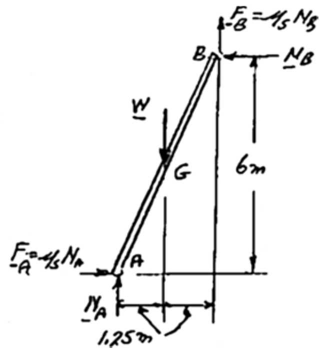
SOLUTION

Free body: Ladder

Motion impending:

$$F_A = \mu_s N_A$$

$$F_B = \mu_s N_B$$



$$+\circlearrowleft \Sigma M_A = 0: \quad W(1.25\text{ m}) - N_B(6\text{ m}) - \mu_s N_B(2.5\text{ m}) = 0$$

$$N_B = \frac{1.25W}{6+2.5\mu_s} \qquad (1)$$

$$+\uparrow \Sigma F_y = 0: \quad N_A + \mu_s N_B - W = 0$$

$$N_A = W - \mu_s N_B$$

$$N_A = W - \frac{1.25\mu_s W}{6+2.5\mu_s} \qquad (2)$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad \mu_s N_A - N_B = 0$$

Substitute for N_A and N_B from Eqs. (1) and (2):

$$\mu_s W - \frac{1.25\mu_s^2 W}{6+2.5\mu_s} = \frac{1.25W}{6+2.5\mu_s}$$

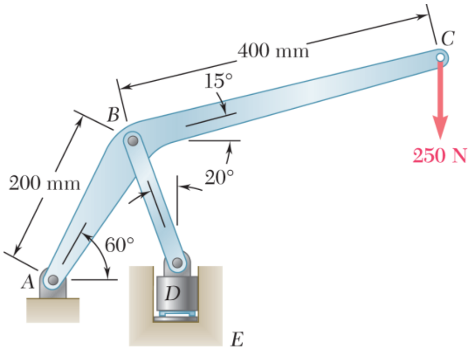
$$6\mu_s + 2.5\mu_s^2 - 1.25\mu_s^2 = 1.25$$

$$1.25\mu_s^2 + 6\mu_s - 1.25 = 0$$

$$\mu_s = 0.2$$

and $\mu_s = -5$ (Discard)

$\mu_s = 0.200$ ◀

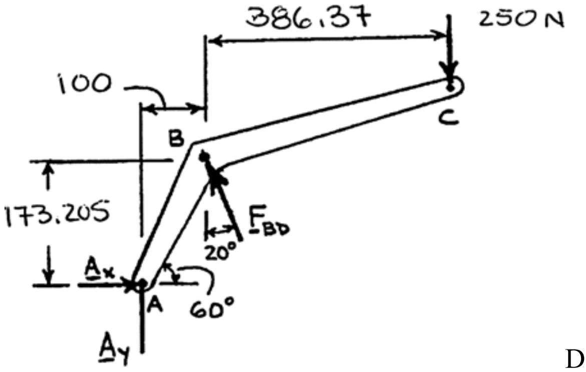

400 mm
C
15°
B
250 N
200 mm
20°
60°
A
D
E

PROBLEM 8.27

The press shown is used to emboss a small seal at *E*. Knowing that the coefficient of static friction between the vertical guide and the embossing die *D* is 0.30, determine the force exerted by the die on the seal.

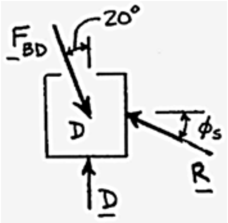
SOLUTION

Free body: Member *ABC*


386.37
250 N
100
C
B
173.205
F_BD
20°
A_x
A
60°
A_y

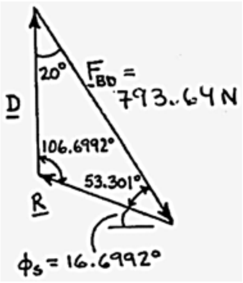
Dimensions in mm

$$+\circlearrowleft \Sigma M_A = 0: \quad F_{BD}\cos 20°(100) + F_{BD}\sin 20°(173.205)$$
$$-(250\text{ N})(100+386.37) = 0$$
$$F_{BD} = 793.64\text{ N}$$


F_BD
20°
D
φ_s
R
D

Free body: Die *D*

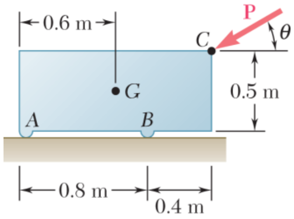
$$\phi_s = \tan^{-1}\mu_s$$
$$= \tan^{-1} 0.3$$
$$= 16.6992°$$


20°
F_BD = 793.64 N
D
106.6992°
53.301°
R
φ_s = 16.6992°

Force triangle:

$$\frac{D}{\sin 53.301°} = \frac{793.64\text{ N}}{\sin 106.6992°}$$
$$D = 664.35\text{ N}$$

On seal: **D** = 664 N ↓ ◀



PROBLEM 8.28

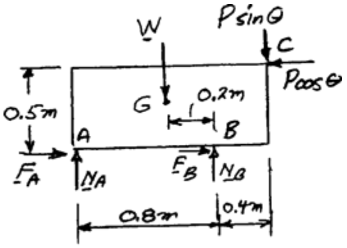
The machine base shown has a mass of 75 kg and is fitted with skids at *A* and *B*. The coefficient of static friction between the skids and the floor is 0.30. If a force **P** of magnitude 500 N is applied at corner *C*, determine the range of values of *θ* for which the base will not move.

SOLUTION

Free-body: Machine base

$$m = (75\text{ kg})(9.81\text{ m/s}^2) = 735.75\text{ N}$$

Assume sliding impends



$$F_A = \mu_s N_A \qquad F_B = \mu_s N_B$$

$$+\uparrow \Sigma F_y = 0: \quad N_A + N_B - W - P\sin\theta = 0$$

$$(N_A + N_B) = W + P\sin\theta \tag{1}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_A + F_B - P\cos\theta = 0$$

$$\mu_s(N_A + N_B) = P\cos\theta = 0 \tag{2}$$

$\dfrac{\text{Eq. (2)}}{\text{Eq. (1)}}$:

$$\mu_s = \frac{P\cos\theta}{W + P\sin\theta}$$

$$\mu_s W + \mu_s P\sin\theta = P\cos\theta$$

$$0.30(735.75\text{ N}) + 0.30(500\text{ N})\sin\theta = 500\cos\theta$$

$$500\cos\theta - 150\sin\theta = 220.73$$

Solve for θ: $\quad \theta = 48.28°$

Assume tipping about *B* impends: $\quad \therefore N_A = 0$

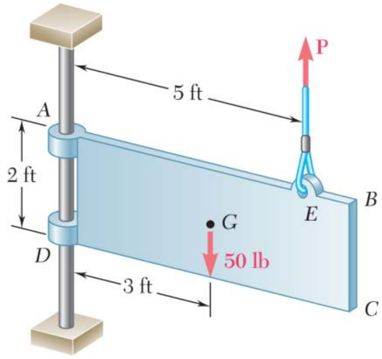
$$+\circlearrowright \Sigma M_B = 0: \quad P\sin\theta(0.4\text{ m}) - P\cos\theta(0.5\text{ m}) - W(0.2\text{ m}) = 0$$

$$500\sin\theta(0.4) - 500\cos\theta(0.5) - 735.75(0.2\text{ m}) = 0$$

$$200\sin\theta - 250\cos\theta = 147.15$$

Solve for θ: $\quad \theta = 78.70°$

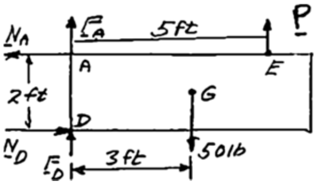
Range for no motion: $\quad 48.3° \le \theta \le 78.7°$ ◀



PROBLEM 8.29

The 50-lb plate *ABCD* is attached at *A* and *D* to collars that can slide on the vertical rod. Knowing that the coefficient of static friction is 0.40 between both collars and the rod, determine whether the plate is in equilibrium in the position shown when the magnitude of the vertical force applied at *E* is (*a*) $P = 0$, (*b*) $P = 20$ lb.

SOLUTION



(*a*) P= 0

$$+\circlearrowleft \Sigma M_D = 0: \quad N_A(2\text{ ft}) - (50\text{ lb})(3\text{ ft}) = 0$$

$$N_A = 75\text{ lb}$$

$$\Sigma F_x = 0: \quad N_D = N_A = 75\text{ lb}$$

$$+\uparrow \Sigma F_y = 0: \quad F_A + F_D - 50\text{ lb} = 0$$

$$F_A + F_D = 50\text{ lb}$$

But:

$$(F_A)_m = \mu_s N_A = 0.40(75\text{ lb}) = 30\text{ lb}$$
$$(F_D)_m = \mu_s N_D = 0.40(75\text{ lb}) = 30\text{ lb}$$

Thus:

$$(F_A)_m + (F_D)_m = 60\text{ lb}$$

and

$$(F_A)_m + (F_D)_m > F_A + F_D$$

Plate is in equilibrium ◀

(*b*) P= 20 lb

$$+\circlearrowleft \Sigma M_D = 0: \quad N_A(2\text{ ft}) - (50\text{ lb})(3\text{ ft}) + (20\text{ lb})(5\text{ ft}) = 0$$

$$N_A = 25\text{ lb}$$

$$\Sigma F_x = 0: \quad N_D = N_A = 25\text{ lb}$$

$$+\uparrow \Sigma F_y = 0: \quad F_A + F_D - 50\text{ lb} + 20\text{ lb} = 0$$

$$F_A + F_D = 30\text{ lb}$$

But:

$$(F_A)_m = \mu_s N_A = 0.4(25\text{ lb}) = 10\text{ lb}$$
$$(F_D)_m = \mu_s N_D = 0.4(25\text{ lb}) = 10\text{ lb}$$

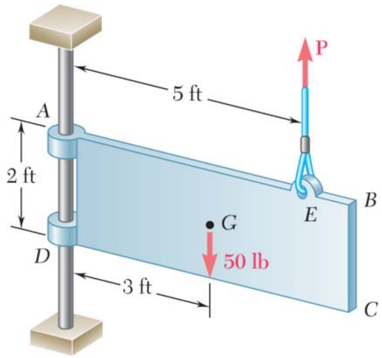
Thus:

$$(F_A)_m + (F_D)_m = 20\text{ lb}$$

and

$$F_A + F_D > (F_A)_m + (F_D)_m$$

Plate moves downward ◀

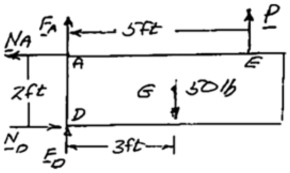


PROBLEM 8.30

In Problem 8.29, determine the range of values of the magnitude P of the vertical force applied at E for which the plate will move downward.

PROBLEM 8.29 The 50-lb plate *ABCD* is attached at *A* and *D* to collars that can slide on the vertical rod. Knowing that the coefficient of static friction is 0.40 between both collars and the rod, determine whether the plate is in equilibrium in the position shown when the magnitude of the vertical force applied at *E* is (*a*) $P = 0$, (*b*) $P = 20$ lb.

SOLUTION



We shall consider the following two cases:

(1) $0 < P < 30$ lb

$$+\circlearrowleft \Sigma M_D = 0: \quad N_A(2\text{ ft}) - (50\text{ lb})(3\text{ ft}) + P(5\text{ ft}) = 0$$

$$N_A = 75\text{ lb} - 2.5P$$

(*Note:* $N_A \geq 0$ and directed ⟵ for $P \leq 30$ lb as assumed here)

$$\Sigma F_x = 0: \quad N_A = N_D$$

$$+\uparrow \Sigma F_y = 0: \quad F_A + F_D + P - 50 = 0$$

$$F_A + F_D = 50 - P$$

But:

$$(F_A)_m = (F_0)_m = \mu_s N_A$$
$$= 0.40(75 - 2.5P)$$
$$= 30 - P$$

Plate moves $\downarrow$ if:

$$F_A + F_D > (F_A)_m + (F_D)_m$$

or

$$50 - P > (30 - P) + (30 - P)$$

$P > 10$ lb ◁

(2) $30\text{ lb} < P < 50$ lb

$$+\circlearrowleft \Sigma M_D = 0: \quad -N_A(2\text{ ft}) - (50\text{ lb})(3\text{ ft}) + P(5\text{ ft}) = 0$$

$$N_A = 2.5P - 75$$

(*Note:* $N_A >$ and directed ⟶ for $P > 30$ lb as assumed)

$$\Sigma F_x = 0: \quad N_A = N_D$$

$$+\uparrow \Sigma F_y = 0: \quad F_A + F_D + P - 50 = 0$$

$$F_A + F_D = 50 - P$$

PROBLEM 8.30 (Continued)

But:

$$\begin{aligned}(F_A)_m &= (F_D)_m = \mu_s N_A \\ &= 0.40(2.5P - 75) \\ &= P - 30 \text{ lb}\end{aligned}$$

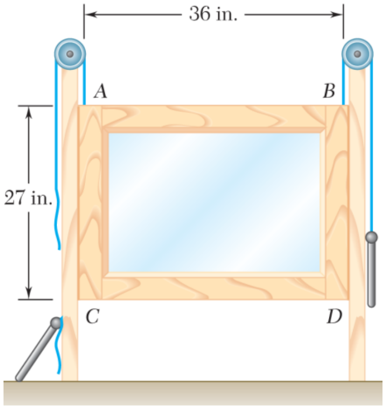
Plate moves ↓ if:

$$F_A + F_D > (F_A)_m + (F_D)_m$$

$$50 - P > (P - 30) + (P - 30) \qquad P < \frac{110}{3} = 36.7 \text{ lb} \ \triangleleft$$

Thus, plate moves downward for: $10.00 \text{ lb} < P < 36.7 \text{ lb}$ ◀

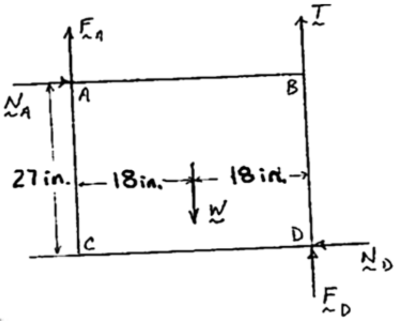
(*Note:* For $P > 50$ lb, plate is in equilibrium)



PROBLEM 8.31

A window sash weighing 10 lb is normally supported by two 5-lb sash weights. Knowing that the window remains open after one sash cord has broken, determine the smallest possible value of the coefficient of static friction. (Assume that the sash is slightly smaller than the frame and will bind only at Points *A* and *D*.)

SOLUTION



FBD window: $T = 5\text{ lb}$

$$\xrightarrow{\quad} \Sigma F_x = 0: \quad N_A - N_D = 0$$

$$N_A = N_D$$

Impending motion:

$$F_A = \mu_s N_A$$

$$F_D = \mu_s N_D$$

$$\circlearrowleft \Sigma M_D = 0: \quad (18\text{ in.})W - (27\text{ in.})N_A - (36\text{ in.})F_A = 0 \qquad W = 10\text{ lb}$$

$$W = \frac{3}{2}N_A + 2\mu_s N_A$$

$$N_A = \frac{2W}{3 + 4\mu_s}$$

$$\uparrow \Sigma F_y = 0: \quad F_A - W + T + F_D = 0$$

$$F_A + F_D = W - T = \frac{W}{2}$$

Now

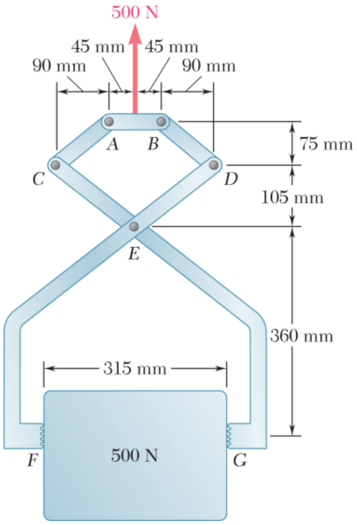
$$F_A + F_D = \mu_s(N_A + N_D)$$
$$= 2\mu_s N_A$$

Then

$$\frac{W}{2} = 2\mu_s \frac{2W}{3 + 4\mu_s}$$

or

$$\mu_s = 0.750 \blacktriangleleft$$

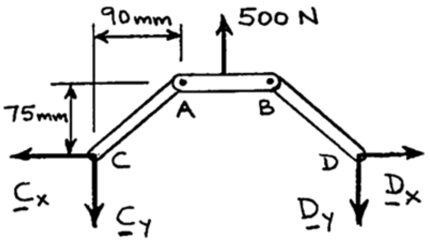

500 N
45 mm
45 mm
90 mm
90 mm
A
B
C
D
75 mm
105 mm
E
360 mm
315 mm
500 N
F
G

PROBLEM 8.32

A 500-N concrete block is to be lifted by the pair of tongs shown. Determine the smallest allowable value of the coefficient of static friction between the block and the tongs at *F* and *G*.

SOLUTION

Free body: Members *CA*, *AB*, *BD*


90mm
500 N
75mm
A
B
C
D
C_x
C_y
D_y
D_x

By symmetry: $C_y = D_y = \frac{1}{2}(500) = 250 \text{ N}$

Since *CA* is a two-force member,

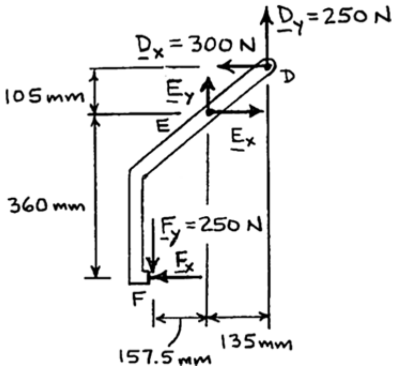
$$\frac{C_x}{90 \text{ mm}} = \frac{C_y}{75 \text{ mm}} = \frac{250 \text{ N}}{75 \text{ mm}}$$

$$C_x = 300 \text{ N}$$

$$\Sigma F_x = 0: \quad D_x = C_x$$

$$D_x = 300 \text{ N}$$

Free body: Tong *DEF*


$D_y = 250$ N
$D_x = 300$ N
D
105mm
E_y
E
E_x
360mm
$F_y = 250$ N
F_x
F
157.5mm
135mm

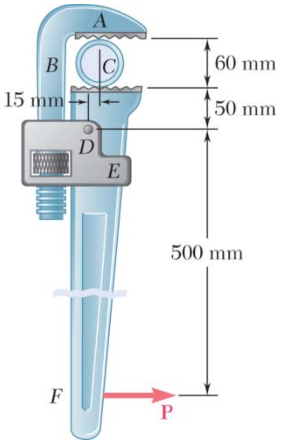
$$+\circlearrowleft \Sigma M_E = 0: \quad (300 \text{ N})(105 \text{ mm}) + (250 \text{ N})(135 \text{ mm})$$

$$+ (250 \text{ N})(157.5 \text{ mm}) - F_x(360 \text{ mm}) = 0$$

$$F_x = +290.625 \text{ N}$$

Minimum value of μ_s: $\mu_s = \frac{F_y}{F_x} = \frac{250 \text{ N}}{290.625 \text{ N}}$

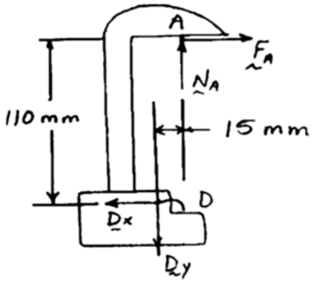
$\mu_s = 0.860$ ◀


A
B
C
15 mm
60 mm
50 mm
D
E
500 mm
F
P

PROBLEM 8.33

A pipe of diameter 60 mm is gripped by the stillson wrench shown. Portions *AB* and *DE* of the wrench are rigidly attached to each other, and portion *CF* is connected by a pin at *D*. If the wrench is to grip the pipe and be self-locking, determine the required minimum coefficients of friction at *A* and *C*.

SOLUTION


A
F_A
N_A
110 mm
15 mm
D
D_x
D_y

FBD *ABD*:

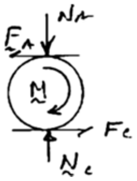
$$\circlearrowleft \Sigma M_D = 0: \quad (15 \text{ mm})N_A - (110 \text{ mm})F_A = 0$$

Impending motion: $F_A = \mu_A N_A$

Then $15 - 110\mu_A = 0$

or $\mu_A = 0.13636$ $\mu_A = 0.1364$ ◀

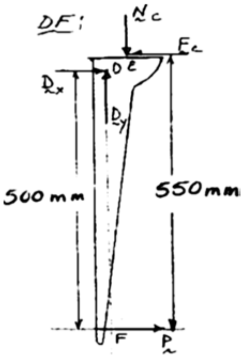
$$\xrightarrow{} \Sigma F_x = 0: \quad F_A - D_x = 0 \quad D_x = F_A$$



FBD pipe:

$$\uparrow \Sigma F_y = 0: \quad N_C - N_A = 0$$

$$N_C = N_A$$


DF:
N_C
F_C
D_x
D
C
D_y
500 mm
550 mm
F
P

FBD *DF*:

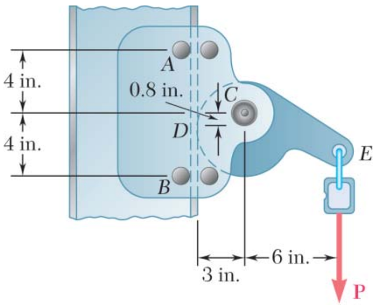
$$\circlearrowleft \Sigma M_F = 0: \quad (550 \text{ mm})F_C - (15 \text{ mm})N_C - (500 \text{ mm})D_x = 0$$

Impending motion: $F_C = \mu_C N_C$

Then $$550\mu_C - 15 = 500\frac{F_A}{N_C}$$

But $$N_C = N_A \quad \text{and} \quad \frac{F_A}{N_A} = \mu_A = 0.13636$$

So $550\mu_C = 15 + 500(0.13636)$ $\mu_C = 0.1512$ ◀

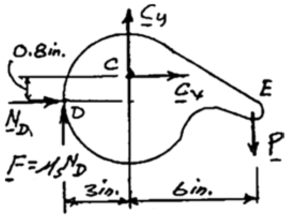


PROBLEM 8.34

A safety device used by workers climbing ladders fixed to high structures consists of a rail attached to the ladder and a sleeve that can slide on the flange of the rail. A chain connects the worker's belt to the end of an eccentric cam that can be rotated about an axle attached to the sleeve at *C*. Determine the smallest allowable common value of the coefficient of static friction between the flange of the rail, the pins at *A* and *B*, and the eccentric cam if the sleeve is not to slide down when the chain is pulled vertically downward.

SOLUTION

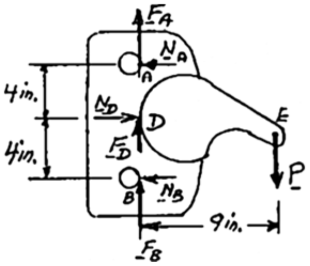
Free body: Cam



$$+\circlearrowleft \Sigma M_C = 0: \quad N_D(0.8 \text{ in.}) - \mu_s N_D(3 \text{in.}) - P(6 \text{ in.}) = 0$$

$$N_D = \frac{6P}{0.8 - 3\mu_s} \qquad (1)$$

Free body: Sleeve and cam



$$\xrightarrow{+} \Sigma F_x = 0: \quad N_D - N_A - N_B = 0$$

$$N_A + N_B = N_D \qquad (2)$$

$$+\!\uparrow \Sigma F_y = 0: \quad F_A + F_B + F_D - P = 0$$

or

$$\mu_s(N_A + N_B + N_D) = P \qquad (3)$$

Substitute from Eq. (2) into Eq. (3): $\mu_s(2N_D) = P \qquad N_D = \dfrac{P}{2\mu_s}$ (4)

Equate expressions for N_D from Eq. (1) and Eq. (4):

$$\frac{P}{2\mu_s} = \frac{6P}{0.8 - 3\mu_s}; \quad 0.8 - 3\mu_s = 12\mu_s$$

$$\mu_s = \frac{0.8}{15} \qquad \mu_s = 0.0533 \blacktriangleleft$$

(*Note:* To verify that contact at pins *A* and *B* takes places as assumed, we shall check that $N_A > 0$ and $N_B = 0$.)

PROBLEM 8.34 (Continued)

From Eq. (4): $$N_D = \frac{P}{2\mu_s} = \frac{P}{2(0.0533)} = 9.375P$$

From free body of cam and sleeve:

$$+\circlearrowleft \Sigma M_B = 0: \quad N_A(8 \text{ in.}) - N_D(4 \text{ in.}) - P(9 \text{ in.}) = 0$$

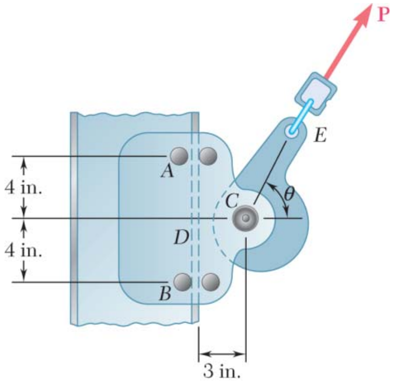
$$8N_A = (9.375P)(4) + 9P$$

$$N_A = 5.8125P > 0 \quad \text{OK}$$

From Eq. (2): $$N_A + N_B = N_D$$

$$5.8125P + N_B = 9.375P$$

$$N_B = 3.5625P > 0 \quad \text{OK}$$



PROBLEM 8.35

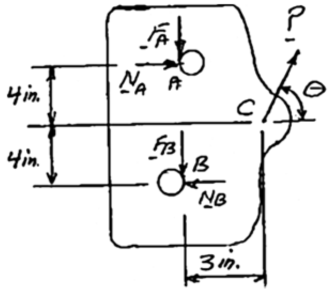
To be of practical use, the safety sleeve described in Problem 8.34 must be free to slide along the rail when pulled upward. Determine the largest allowable value of the coefficient of static friction between the flange of the rail and the pins at *A* and *B* if the sleeve is to be free to slide when pulled as shown in the figure, assuming (*a*) $\theta = 60°$, (*b*) $\theta = 50°$, (*c*) $\theta = 40°$.

SOLUTION

Note the cam is a two-force member.

Free body: Sleeve

We assume contact between rail and pins as shown.



$$+\circlearrowleft \Sigma M_C = 0: \quad F_A(3\text{ in.}) + F_B(3\text{ in.}) - N_A(4\text{ in.}) - N_B(4\text{ in.}) = 0$$

But $\quad F_A = \mu_s N_A$

$$F_B = \mu_s N_B$$

We find $\quad 3\mu_s(N_A + N_B) - 4(N_A + N_B) = 0$

$$\mu_s = \frac{4}{3} = 1.33333$$

We now verify that our assumption was correct.

$$\xrightarrow{+} \Sigma F_x = 0: \quad N_A - N_B + P\cos\theta = 0$$

$$N_B - N_A = P\cos\theta \qquad (1)$$

$$+\uparrow \Sigma F_y = 0: \quad -F_A - F_B + P\sin\theta = 0$$

$$\mu_s N_A + \mu_s N_B = P\sin\theta$$

$$N_A + N_B = \frac{P\sin\theta}{\mu_s} \qquad (2)$$

Add Eqs. (1) and (2): $\quad 2N_B = P\left(\cos\theta + \frac{\sin\theta}{\mu_s}\right) > 0 \quad$ OK

PROBLEM 8.35 (Continued)

Subtract Eq. (1) from Eq. (2):

$$2N_A = P\left(\frac{\sin\theta}{\mu_s} - \cos\theta\right)$$

$N_A > 0$ only if

$$\frac{\sin\theta}{\mu_s} - \cos\theta > 0$$

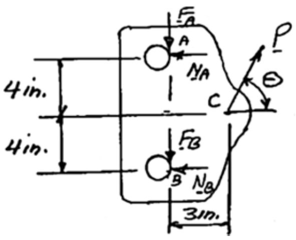
$$\tan\theta > \mu_s = 1.33333$$

$$\theta = 53.130°$$

(*a*) For case (*a*): Condition is satisfied, contact takes place as shown. Answer is correct.

$\mu_s = 1.333$ ◀

But for (*b*) and (*c*): $\theta < 53.130°$ and our assumption is wrong, N_A is directed to left.



$$\xrightarrow{+} \Sigma F_x = 0: \quad -N_A - N_B + P\cos\theta = 0$$

$$N_A + N_B = P\cos\theta \tag{3}$$

$$+\!\uparrow \Sigma F_y = 0: \quad -F_A - F_A + P\sin\theta = 0$$

$$\mu_s(N_A + N_B) = P\sin\theta \tag{4}$$

Divide Eq. (4) by Eq. (3):

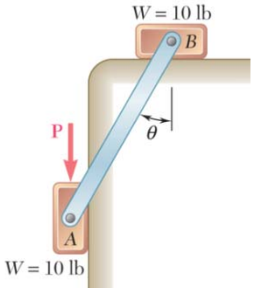
$$\mu_s = \tan\theta \tag{5}$$

(*b*) We make $\theta = 50°$ in Eq. (5):

$\mu_s = \tan 50°$ $\mu_s = 1.192$ ◀

(*c*) We make $\theta = 40°$ in Eq. (5):

$\mu_s = \tan 40°$ $\mu_s = 0.839$ ◀



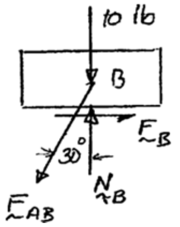
PROBLEM 8.36

Two 10-lb blocks A and B are connected by a slender rod of negligible weight. The coefficient of static friction is 0.30 between all surfaces of contact, and the rod forms an angle $\theta = 30°$. with the vertical. (a) Show that the system is in equilibrium when $P = 0$. (b) Determine the largest value of P for which equilibrium is maintained.

SOLUTION

FBD block$B$:

(a) Since $P = 2.69$ lb to initiate motion, equilibrium exists with $P = 0$ ◀



(b) For $P_{\max}$, motion impends at both surfaces:

Block B: $+\uparrow \Sigma F_y = 0$: $N_B - 10\,\text{lb} - F_{AB}\cos 30° = 0$

$$N_B = 10\text{ lb} + \frac{\sqrt{3}}{2}F_{AB} \quad (1)$$

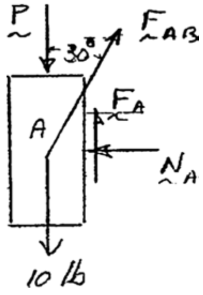
Impending motion: $F_B = \mu_s N_B = 0.3N_B$

$\xrightarrow{+} \Sigma F_x = 0$: $F_B - F_{AB}\sin 30° = 0$

$$F_{AB} = 2F_B = 0.6N_B \quad (2)$$

Solving Eqs. (1) and (2): $N_B = 10\text{ lb} + \dfrac{\sqrt{3}}{2}(0.6N_B) = 20.8166$ lb

FBD block$A$:



Then $F_{AB} = 0.6N_B = 12.4900$ lb

Block A: $\xrightarrow{+} \Sigma F_x = 0$: $F_{AB}\sin 30° - N_A = 0$

$$N_A = \frac{1}{2}F_{AB} = \frac{1}{2}(12.4900\text{ lb}) = 6.2450\text{ lb}$$

Impending motion: $F_A = \mu_s N_A = 0.3(6.2450\text{ lb}) = 1.8735$ lb

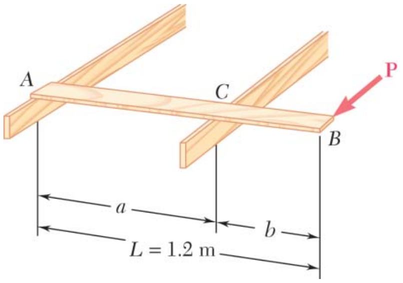
$+\uparrow \Sigma F_y = 0$: $F_A + F_{AB}\cos 30° - P - 10\text{ lb} = 0$

$$P = F_A + \frac{\sqrt{3}}{2}F_{AB} - 10\text{ lb}$$

$$= 1.8735\text{ lb} + \frac{\sqrt{3}}{2}(12.4900\text{ lb}) - 10\text{ lb}$$

$$= 2.69\text{ lb}$$

$P = 2.69$ lb ◀



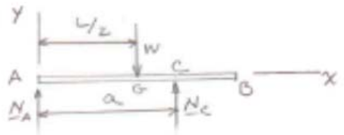
PROBLEM 8.37

A 1.2-m plank with a mass of 3 kg rests on two joists. Knowing that the coefficient of static friction between the plank and the joists is 0.30, determine the magnitude of the horizontal force required to move the plank when (*a*) $a = 750$ mm, (*b*) $a = 900$ mm.

SOLUTION

Free body member *AB*:

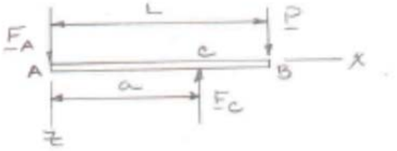
In the vertical plane,



$$+\circlearrowleft \Sigma M_A = 0: \quad N_C a - W\frac{L}{2} = 0 \quad \text{hence } N_C = W\frac{L}{2a} \qquad (1)$$

$$+\uparrow \Sigma F_y = 0: \quad N_A + N_C - W = 0 \quad \text{hence} \quad N_A = W\frac{2a - L}{2a} \qquad (2)$$

In the horizontal plane,



$$+\circlearrowleft \Sigma M_A = 0: \quad F_C a - PL = 0 \quad \text{hence} \quad \mathrm{F}_C = P\frac{L}{a} \qquad (3)$$

$$+\downarrow \Sigma F_Z = 0: \quad F_A + P - F_C = 0 \quad \text{hence} \quad F_A = P\frac{L - a}{a} \qquad (4)$$

(a) Substituting given data: $a = 0.75$ m, $L = 1.2$ m, $\mu_s = 0.30$, $W = (3\text{ kg})(9.81\text{ m/s}^2) = 29.430$ N

$$N_C = 0.8W, \quad N_A = 0.2W, \quad F_C = 1.6P, \quad F_A = 0.6P$$

To slip at *A*, $\quad F_A = \mu_s N_A \quad$ or $\quad 0.6P = 0.30(0.2W) \qquad P = 0.1W$

To slip at *C*, $\quad F_C = \mu_s N_C \quad$ or $\quad 1.6P = 0.30(0.8W) \qquad P = 0.15W$

The plank will slip at *A* $\qquad P = 2.94$ N ◀

PROBLEM 8.37 (Continued)

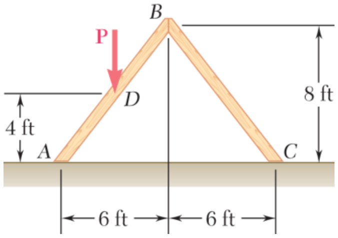
(b) Substituting given data: $a = 0.9\text{ m},\ L = 1.2\text{ m},\ \mu_s = 0.30,\ W = (3\text{ kg})(9.81\text{ m/s}^2) = 29.430\text{ N}$

$$N_C = \frac{2}{3}W,\ N_A = \frac{1}{3}W,\ F_C = \frac{4}{3}P,\ F_A = \frac{1}{3}P$$

To slip at A, $F_A = \mu_s N_A$ or $\frac{1}{3}P = 0.30\left(\frac{1}{3}W\right)$ $P = 0.3W$

To slip at C, $F_C = \mu_s N_C$ or $\frac{4}{3}P = 0.30\left(\frac{2}{3}W\right)$ $P = 0.15W$

The plank will slip at C for $P = 0.15(29.43\text{ N})$ $P = 4.41\text{ N}$ ◀

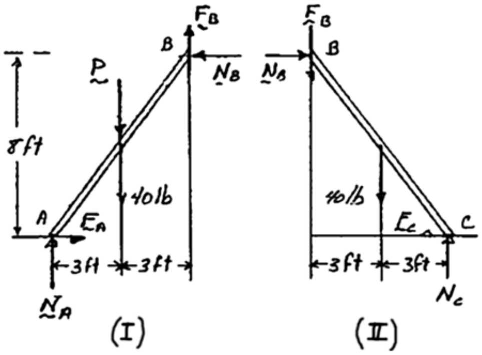


PROBLEM 8.38

Two identical uniform boards, each of weight 40 lb, are temporarily leaned against each other as shown. Knowing that the coefficient of static friction between all surfaces is 0.40, determine (*a*) the largest magnitude of the force **P** for which equilibrium will be maintained, (*b*) the surface at which motion will impend.

SOLUTION

Board FBDs:



Assume impending motion at C, so

$$F_C = \mu_s N_C = 0.4 N_C$$

FBD II: $\circlearrowleft \Sigma M_B = 0\colon \quad (6\text{ ft})N_C - (8\text{ ft})F_C - (3\text{ ft})(40\text{ lb}) = 0$

$$[6\text{ ft} - 0.4(8\text{ ft})]N_C = (3\text{ ft})(40\text{ lb})$$

or $$N_C = 42.857\text{ lb}$$

and $$F_C = 0.4N_C = 17.143\text{ lb}$$

$$\xrightarrow{} \Sigma F_x = 0\colon \quad N_B - F_C = 0$$

$$N_B = F_C = 17.143\text{ lb}$$

$$\uparrow \Sigma M_y = 0\colon \quad -F_B - 40\text{ lb} + N_C = 0$$

$$F_B = N_C - 40\text{ lb} = 2.857\text{ lb}$$

Check for motion at B: $\dfrac{F_B}{N_B} = \dfrac{2.857\text{ lb}}{17.143\text{ lb}} = 0.167 < \mu_s$, OK, no motion.

FBD I: $\circlearrowleft \Sigma M_A = 0\colon \quad (8\text{ ft})N_B + (6\text{ ft})F_B - (3\text{ ft})(P + 40\text{ lb}) = 0$

$$P = \frac{(8\text{ ft})(17.143\text{ lb}) + (6\text{ ft})(2.857\text{ lb})}{3\text{ ft}} - 40\text{ lb}$$

$$= 11.429\text{ lb}$$

PROBLEM 8.38 (Continued)

Check for slip at *A* (unlikely because of *P*):

$$\xrightarrow{} \Sigma F_x = 0: \quad F_A - N_B = 0 \quad \text{or} \quad F_A = N_B = 17.143 \text{ lb}$$

$$\uparrow \Sigma F_y = 0: \quad N_A - P - 40 \text{ lb} + F_B = 0$$

or

$$N_A = 11.429 \text{ lb} + 40 \text{ lb} - 2.857 \text{ lb}$$
$$= 48.572 \text{ lb}$$

Then

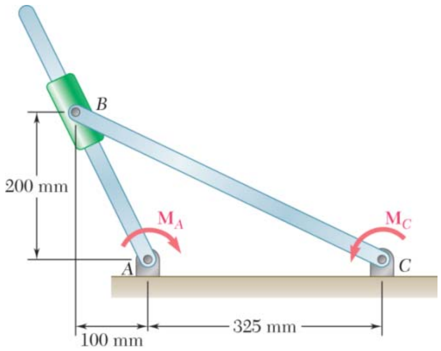
$$\frac{F_A}{N_A} = \frac{17.143 \text{ lb}}{48.572 \text{ lb}} = 0.353 < \mu_s$$

OK, no slip $\Rightarrow$ assumption is correct.

Therefore

(*a*) $P_{\max} = 11.43 \text{ lb}$ ◀

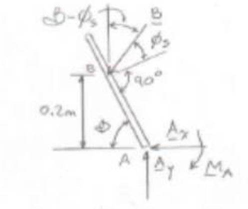
(*b*) Motion impends at *C* ◀



PROBLEM 8.39

Two rods are connected by a collar at *B*. A couple $\mathbf{M}_A$ with a magnitude of 15 N·m is applied to rod *AB*. Knowing that the coefficient of static friction between the collar and the rod is 0.30, determine the largest couple $\mathbf{M}_C$ for which equilibrium will be maintained.

SOLUTION



$$\tan\beta = \frac{0.2}{0.1} \qquad \beta = 63.43^\circ$$

$$\tan\theta = \frac{0.2}{0.425} \qquad \theta = 25.2^\circ$$

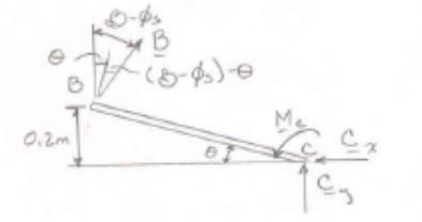
$$\tan\phi_s = \mu_s = 0.3 \qquad \phi_s = 16.7^\circ$$

For largest M_C, motion of *B* impends downward.

$$AB = \frac{0.2}{\sin 63.43^\circ} = 0.22362 \text{ m}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad B\cos\phi_s\left(AB\right) - M_A = 0$$

$$B\cos 16.7^\circ\left(0.22362 \text{ m}\right) - 15 \text{ N}\cdot\text{m} = 0 \qquad\qquad B = 70.032 \text{ N}$$

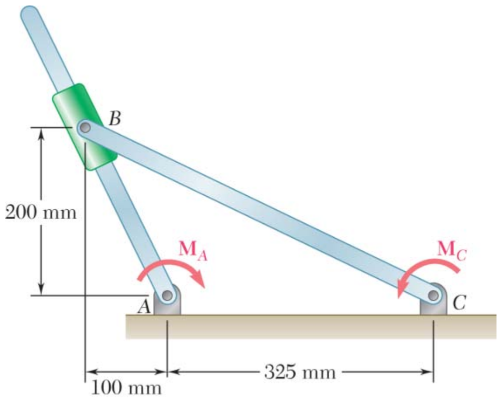


$$BC = \frac{0.2}{\sin 25.2^\circ} = 0.46973 \text{ m}$$

$$+\circlearrowleft \Sigma M_C = 0: \quad M_C - B\cos(\beta - \phi_s - \theta)\left(BC\right) = 0$$

$$M_C = (70.032 \text{ N})\cos\left(63.43^\circ - 16.7^\circ - 25.2^\circ\right)\left(0.46973 \text{ m}\right)$$

$$\mathbf{M}_C = 30.6 \text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$



PROBLEM 8.40

In Prob. 8.39, determine the smallest couple $\mathbf{M}_C$ for which equilibrium will be maintained.

PROBLEM 8.39

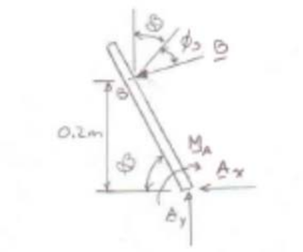
Two rods are connected by a collar at *B*. A couple $\mathbf{M}_A$ with a magnitude of 15 N·m is applied to rod *AB*. Knowing that the coefficient of static friction between the collar and the rod is 0.30, determine the largest couple $\mathbf{M}_C$ for which equilibrium will be maintained.

SOLUTION

$$\tan\beta = \frac{0.2}{0.1} \quad \beta = 63.43^\circ$$

$$\tan\theta = \frac{0.2}{0.425} \quad \theta = 25.2^\circ$$

$$\tan\phi_s = \mu_s = 0.3 \quad \phi_s = 16.7^\circ$$



For smallest M_C, motion of *B* impends upward.

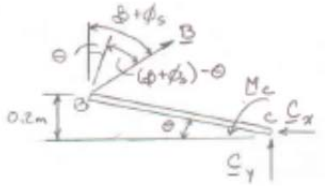
$$AB = \frac{0.2}{\sin 63.43^\circ} = 0.22362 \text{ m}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad B\cos\phi_s(AB) - M_A = 0$$

$$B\cos 16.7^\circ (0.22362 \text{ m}) - 15 \text{ N}\cdot\text{m} = 0 \qquad B = 70.032 \text{ N}$$

$$BC = \frac{0.2}{\sin 25.2^\circ} = 0.46973 \text{ m}$$

PROBLEM 8.40 (Continued)

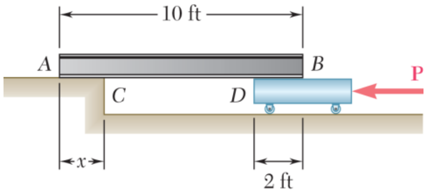


$$+\circlearrowleft \Sigma M_C = 0: \quad M_C - B\cos(\beta + \phi_s - \theta)\left(BC\right) = 0$$

$$M_C = (70.032\text{ N})\cos\left(63.43^\circ + 16.7^\circ - 25.2^\circ\right)\left(0.46973\text{ m}\right)$$

$$\mathbf{M}_C = 18.90\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

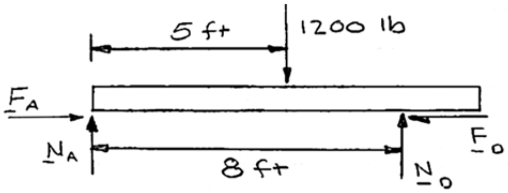
PROBLEM 8.41



A 10-ft beam, weighing 1200 lb, is to be moved to the left onto the platform. A horizontal force **P** is applied to the dolly, which is mounted on frictionless wheels. The coefficients of friction between all surfaces are $\mu_s = 0.30$ and $\mu_k = 0.25$, and initially $x = 2$ ft. Knowing that the top surface of the dolly is slightly higher than the platform, determine the force **P** required to start moving the beam. (*Hint:* The beam is supported at *A* and *D*.)

SOLUTION

FBD beam:



$$+\circlearrowleft \Sigma M_A = 0: \quad N_D(8\text{ ft}) - (1200\text{ lb})(5\text{ ft}) = 0$$

$$\mathbf{N}_D = 750\text{ lb}\uparrow$$

$$+\uparrow \Sigma F_y = 0: \quad N_A - 1200 + 750 = 0$$

$$\mathbf{N}_A = 450\text{ lb}\uparrow$$

$$(F_A)_m = \mu_s N_A = 0.3(450) = 135.0\text{ lb}$$

$$(F_D)_m = \mu_s N_D = 0.3(750) = 225\text{ lb}$$

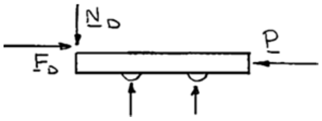
Since $(F_A)_m < (F_D)_m$, sliding first impends at *A* with

$$F_A = (F_A)_m = 135\text{ lb}$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_A - F_D = 0$$

$$F_D = F_A = 135.0\text{ lb}$$

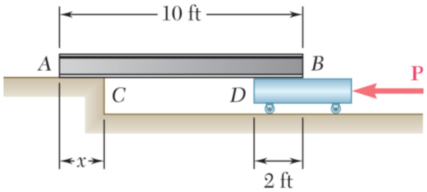
FBD dolly:



From *FBD* of dolly:

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_D - P = 0$$

$$P = F_D = 135.0\text{ lb} \qquad P = 135.0\text{ lb} ◀$$



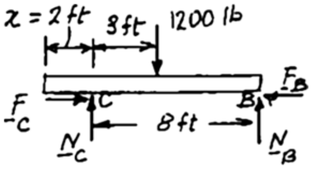
PROBLEM 8.42

(*a*) Show that the beam of Problem 8.41 *cannot* be moved if the top surface of the dolly is slightly *lower* than the platform. (*b*) Show that the beam *can* be moved if two 175-lb workers stand on the beam at *B* and determine how far to the left the beam can be moved.

PROBLEM 8.41 A 10-ft beam, weighing 1200 lb, is to be moved to the left onto the platform. A horizontal force **P** is applied to the dolly, which is mounted on frictionless wheels. The coefficients of friction between all surfaces are $\mu_s = 0.30$ and $\mu_k = 0.25$, and initially $x = 2$ ft. Knowing that the top surface of the dolly is slightly higher than the platform, determine the force **P** required to start moving the beam. (*Hint:* The beam is supported at *A* and *D*.)

SOLUTION

(*a*) Beam alone



$$+\circlearrowleft \Sigma M_C = 0: \quad N_B(8\text{ ft}) - (1200\text{ lb})(3\text{ ft}) = 0$$

$$\mathbf{N}_B = 450\text{ lb}\uparrow$$

$$+\uparrow F_y = 0: \quad N_C + 450 - 1200 = 0$$

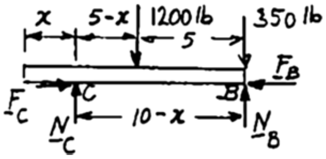
$$\mathbf{N}_C = 750\text{ lb}\uparrow$$

$$(F_C)_m = \mu_s N_C = 0.3(750) = 225\text{ lb}$$

$$(F_B)_m = \mu_s N_B = 0.3(450) = 135\text{ lb}$$

Since $(F_B)_m < (F_C)_m$, sliding first impends at *B*, and Beam cannot be moved ◀

(*b*) Beam with workers standing at *B*



$$+\circlearrowleft \Sigma M_C = 0: \quad N_B(10-x) - (1200)(5-x) - 350(10-x) = 0$$

$$N_B = \frac{9500 - 1550x}{10 - x}$$

$$+\circlearrowleft \Sigma M_B = 0: \quad (1200)(5) - N_C(10-x) = 0$$

$$N_C = \frac{6000}{10 - x}$$

PROBLEM 8.42 (Continued)

Check that beam starts moving for *x*= 2 ft:

For *x*= 2 ft: $N_B = \dfrac{9500 - 1550(2)}{10 - 2} = 800 \text{ lb}$

$$N_C = \frac{6000}{10 - 2} = 750 \text{ lb}$$

$$(F_C)_m = \mu_s N_C = 0.3(750) = 225 \text{ lb}$$

$$(F_B)_m = \mu_s N_B = 0.3(800) = 240 \text{ lb}$$

Since $(F_C)_m < (F_B)_m$, sliding first impends at *C*, Beam moves◀

How far does beam move?

Beam will stop moving when

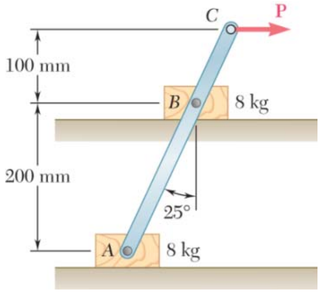
$$F_C = (F_B)_m$$

But $F_C = \mu_k N_C = 0.25 \dfrac{6000}{10 - x} = \dfrac{1500}{10 - x}$

and $(F_B)_m = \mu_s N_B = 0.30 \dfrac{9500 - 1550x}{10 - x} = \dfrac{2850 - 465x}{10 - x}$

Setting $F_C = (F_B)_m$: $1500 = 2850 - 465x$ $x = 2.90 \text{ ft}$ ◀

(*Note:* We have assumed that, once started, motion is continuous and uniform (no acceleration).)

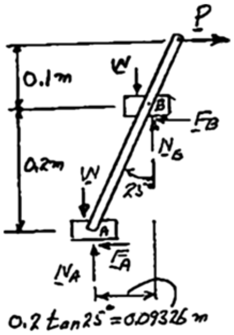


PROBLEM 8.43

Two 8-kg blocks *A* and *B* resting on shelves are connected by a rod of negligible mass. Knowing that the magnitude of a horizontal force **P** applied at *C* is slowly increased from zero, determine the value of *P* for which motion occurs, and what that motion is, when the coefficient of static friction between all surfaces is (*a*) $\mu_s = 0.40$, (*b*) $\mu_s = 0.50$.

SOLUTION

(*a*) $\underline{\mu_s = 0.40}$: Assume blocks slide to right.



$$W = mg = (8 \text{ kg})(9.81 \text{ m/s}^2) = 78.48 \text{ N}$$
$$F_A = \mu_s N_A$$
$$F_B = \mu_s N_B$$

$$+\uparrow \Sigma F_y = 0: \quad N_A + N_B - 2W = 0$$

$$N_A + N_B = 2W$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad P - F_A - F_B = 0$$

$$P = F_A + F_B = \mu_s (N_A + N_B) = \mu_s (2W) \qquad (1)$$
$$P = 0.40(2)(78.48 \text{ N}) = 62.78 \text{ N}$$

$$+\circlearrowright \Sigma M_B = 0: \quad P(0.1 \text{ m}) - (N_A - W)(0.09326 \text{ m}) + F_A(0.2 \text{ m}) = 0$$

$$(62.78)(0.1) - (N_A - 78.48)(0.09326) + (0.4)(N_B)(0.2) = 0$$

$$0.17326 N_A = 1.041$$

$$N_A = 6.01 \text{ N} > 0 \quad \text{OK}$$

System slides: $P = 62.8$ N ◀

(*b*) $\underline{\mu_s = 0.50}$: See part *a*.

Eq. (1): $P = 0.5(2)(78.48 \text{ N}) = 78.48 \text{ N}$

$$+\circlearrowright \Sigma M_B = 0: \quad P(0.1 \text{ m}) + (N_A - W)(0.09326 \text{ m}) + F_A(0.2 \text{ m}) = 0$$

$$(78.48)(0.1) + (N_A - 78.48)(0.09326) + (0.5)N_A(0.2) = 0$$

$$0.19326 N_A = -0.529$$

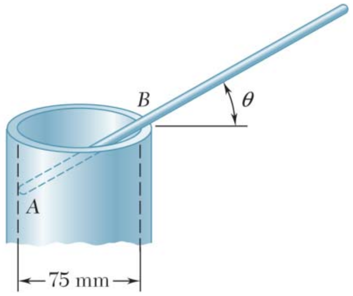
$$N_A = -2.73 \text{ N} < 0 \ \text{ uplift, rotation about } B$$

PROBLEM 8.43 (Continued)

For $N_A = 0$: $\circlearrowright \Sigma M_B = 0$: $P(0.1\text{ m}) - W(0.09326\text{ m}) = 0$

$$P = (78.48\text{ N})(0.09326\text{ m})/(0.1) = 73.19$$

System rotates about *B*: $P = 73.2\text{ N}$ ◀

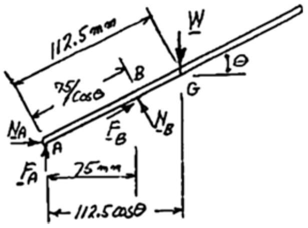


PROBLEM 8.44

A slender steel rod of length 225 mm is placed inside a pipe as shown. Knowing that the coefficient of static friction between the rod and the pipe is 0.20, determine the largest value of θ for which the rod will not fall into the pipe.

SOLUTION

Motion of rod impends down at *A* and to left at *B*.



$$F_A = \mu_s N_A \quad F_B = \mu_s N_B$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad N_A - N_B \sin\theta + F_B \cos\theta = 0$$

$$N_A - N_B \sin\theta + \mu_s N_B \cos\theta = 0$$

$$N_A = N_B(\sin\theta - \mu_s \cos\theta) \qquad (1)$$

$$+\!\uparrow \Sigma F_y = 0: \quad F_A + N_B \cos\theta + F_B \sin\theta - W = 0$$

$$\mu_s N_A + N_B \cos\theta + \mu_s N_B \sin\theta - W = 0 \qquad (2)$$

Substitute for N_A from Eq. (1) into Eq. (2):

$$\mu_s N_B(\sin\theta - \mu_s \cos\theta) + N_B \cos\theta + \mu_s N_B \sin\theta - W = 0$$

$$N_B = \frac{W}{(1-\mu_s^2)\cos\theta + 2\mu_s \sin\theta} = \frac{W}{(1-0.2^2)\cos\theta + 2(0.2)\sin\theta} \qquad (3)$$

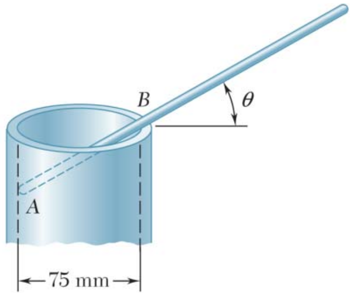
$$+\circlearrowleft \Sigma M_A = 0: \quad N_B\left(\frac{75}{\cos\theta}\right) - W(112.5\cos\theta) = 0$$

Substitute for N_B from Eq. (3), cancel *W*, and simplify to find

$$9.6\cos^3\theta + 4\sin\theta\cos^2\theta - 6.6667 = 0$$

$$\cos^3\theta(2.4 + \tan\theta) = 1.6667$$

Solve by trial & error: $\theta = 35.8° ◀$



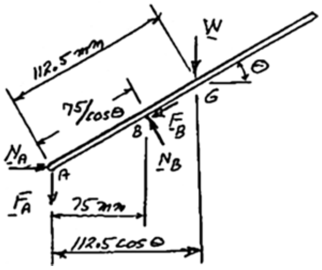
PROBLEM 8.45

In Problem 8.44, determine the smallest value of θ for which the rod will not fall out the pipe.

PROBLEM 8.44 A slender steel rod of length 225 mm is placed inside a pipe as shown. Knowing that the coefficient of static friction between the rod and the pipe is 0.20, determine the largest value of θ for which the rod will not fall into the pipe.

SOLUTION

Motion of rod impends up at A and right at B.



$$F_A = \mu_s N_A \qquad F_B = \mu_s N_B$$

$$\xrightarrow{+}\Sigma F_x = 0: \quad N_A - N_B \sin\theta - F_B \cos\theta = 0$$

$$N_A - N_B \sin\theta - \mu_s N_B \cos\theta = 0$$

$$N_A = N_B(\sin\theta + \mu_s \cos\theta) \tag{1}$$

$$+\!\uparrow \Sigma F_y = 0: \quad -F_A + N_B \cos\theta - F_B \sin\theta - W = 0$$

$$-\mu_s N_A + N_B \cos\theta - \mu_s N_B \sin\theta - W = 0 \tag{2}$$

Substitute for N_A from Eq. (1) into Eq. (2):

$$-\mu_s N_B(\sin\theta + \mu_s \cos\theta) + N_B \cos\theta - \mu_s N_B \sin\theta - W = 0$$

$$N_B = \frac{W}{(1-\mu_s^2)\cos\theta - 2\mu_s \sin\theta} = \frac{W}{(1-0.2^2)\cos\theta - 2(0.2)\sin\theta} \tag{3}$$

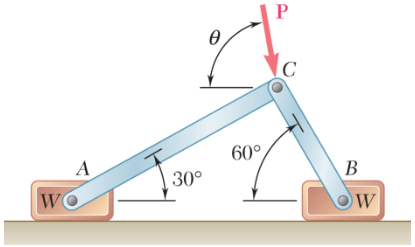
$$+\circlearrowleft \Sigma M_A = 0: \quad N_B\left(\frac{75}{\cos\theta}\right) - W(112.5\cos\theta) = 0$$

Substitute for N_B from Eq. (3), cancel W, and simplify to find

$$9.6\cos^3\theta - 4\sin\theta\cos^2\theta - 6.6667 = 0$$

$$\cos^3\theta(2.4 - \tan\theta) = 1.6667$$

Solve by trial + error: $\theta = 20.5°$ ◀

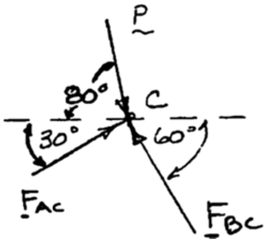


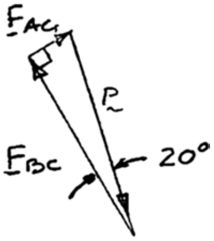
PROBLEM 8.46

Two slender rods of negligible weight are pin-connected at C and attached to blocks A and B, each of weight W. Knowing that $\theta = 80°$ and that the coefficient of static friction between the blocks and the horizontal surface is 0.30, determine the largest value of P for which equilibrium is maintained.

SOLUTION

FBD pin$C$:



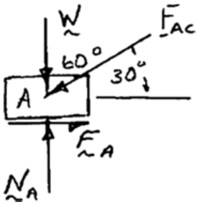


$$F_{AC} = P\sin 20° = 0.34202P$$

$$F_{BC} = P\cos 20° = 0.93969P$$

$\uparrow \Sigma F_y = 0$: $N_A - W - F_{AC}\sin 30° = 0$

or $N_A = W + 0.34202P\sin 30° = W + 0.171010P$



FBD block$A$: $\longrightarrow \Sigma F_x = 0$: $F_A - F_{AC}\cos 30° = 0$

or $F_A = 0.34202P\cos 30° = 0.29620P$

For impending motion at A: $F_A = \mu_s N_A$

Then $N_A = \dfrac{F_A}{\mu_s}$: $W + 0.171010P = \dfrac{0.29620}{0.3}P$

or $P = 1.22500W$

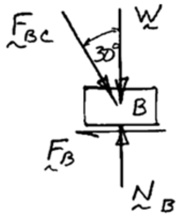
$\uparrow \Sigma F_y = 0$: $N_B - W - F_{BC}\cos 30° = 0$

$$N_B = W + 0.93969P\cos 30° = W + 0.81380P$$

$\longrightarrow \Sigma F_x = 0$: $F_{BC}\sin 30° - F_B = 0$

$$F_B = 0.93969P\sin 30° = 0.46985P$$

PROBLEM 8.46 (Continued)



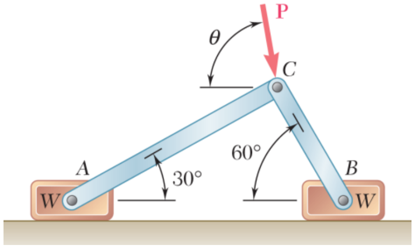
FBD block *B*:

For impending motion at *B*: $F_B = \mu_s N_B$

Then $N_B = \dfrac{F_B}{\mu_s}$: $W + 0.81380P = \dfrac{0.46985P}{0.3}$

or $P = 1.32914W$

Thus, maximum *P* for equilibrium $P_{\max} = 1.225W$ ◀



PROBLEM 8.47

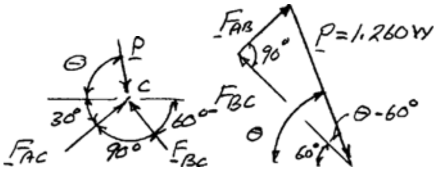
Two slender rods of negligible weight are pin-connected at *C* and attached to blocks *A* and *B*, each of weight *W*. Knowing that $P = 1.260W$ and that the coefficient of static friction between the blocks and the horizontal surface is 0.30, determine the range of values of θ, between 0 and 180°, for which equilibrium is maintained.

SOLUTION

AC and *BC* are two-force members

Free body: Joint *C*

Force triangle:



From Force triangle:

$$F_{AB} = P\sin(\theta - 60°) = 1.26W\sin(\theta - 60°) \qquad (1)$$

$$F_{BC} = P\cos(\theta - 60°) = 1.26W\cos(\theta - 60°) \qquad (2)$$

We shall, in turn, seek θ corresponding to impending motion of each block

For motion of *A* impending to left

from solution of Prob. 8.46: $F_{AC} = 0.419W$

EQ (1):

$$F_{AC} = 0.419W = 1.26W\sin(\theta - 60°)$$

$$\sin(\theta - 60°) = 0.33254$$

$$\theta - 60° = 19.423°$$

$$\theta = 79.42° \quad \triangleleft$$

For motion of *B* impending to right.

from solution of Prob. 8.46: $F_{BC} = 1.249W$

Eq. (2):

$$F_{BC} = 1.249W = 1.26W\cos(\theta - 60°)$$

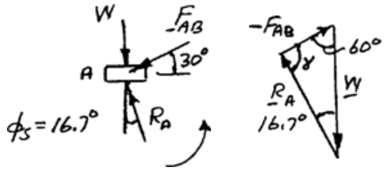
$$\cos(\theta - 60°) = 0.99127$$

$$\theta - 60° = \pm 7.58°$$

$$\theta - 60° = +7.58° \qquad \theta = 67.6° \triangleleft$$

$$\theta - 60° = -7.58° \qquad \theta = 52.4° \triangleleft$$

PROBLEM 8.47 (Continued)


W
F_AB
30°
A
φs = 16.7°
R_A
-F_AB
γ
60°
R_A
16.7°
W

For motion of *A* impending to right

$$\gamma = 180° - 60° - 16.7° = 103.3°$$

Law of sines:

$$\frac{-F_{AB}}{\sin 16.7°} = \frac{W}{\sin 103.3°}$$

$$F_{AB} = -0.29528W$$

Note: Direction of $+F_{AB}$ is kept same as in free body of Joint *C*.

Eq. (1):

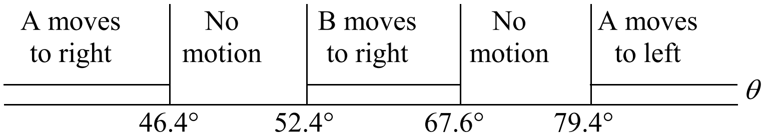
$$F_{AB} = -0.29528W = 1.26W\sin(\theta - 60°)$$

$$\sin(\theta - 60°) = -0.23435$$

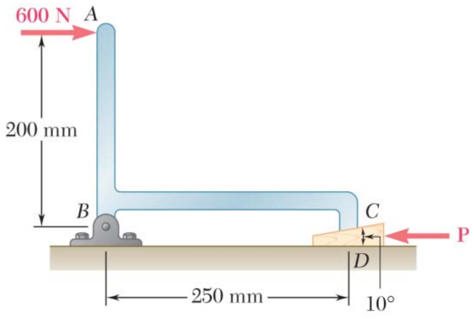
$$(\theta - 60°) = -13.553°$$

$\theta = 46.4°$ ◁

Summary:


A moves to right
No motion
B moves to right
No motion
A moves to left
θ
46.4°
52.4°
67.6°
79.4°

No motion for: $46.4° \le \theta \le 52.4°$ and $67.6° \le \theta \le 79.4°$ ◀


600 N
A
200 mm
B
C
P
D
250 mm
10°

PROBLEM 8.48

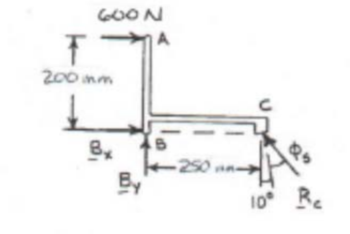
The machine part *ABC* is supported by a frictionless hinge at *B* and a 10° wedge at *C*. Knowing that the coefficient of static friction is 0.20 at both surfaces of the wedge, determine (*a*) the force **P** required to move the wedge to the left, (*b*) the components of the corresponding reaction at *B*.

SOLUTION

$$\mu_s = 0.20$$

$$\phi_s = \tan^{-1}\mu_s = \tan^{-1}0.20 = 11.3099^\circ$$

Free body: *ABC*

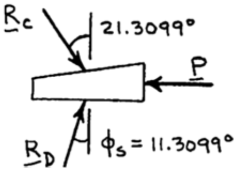

600 N
A
200 mm
C
B
B_x
B_y
250 mm
ϕ_s
10°
R_c

$$10^\circ + 11.31^\circ = 21.31^\circ$$

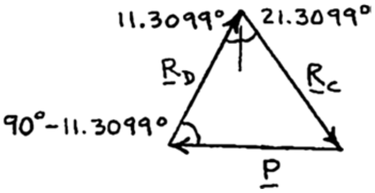
$$+\circlearrowleft \Sigma M_B = 0: \quad (R_C \cos 21.3099^\circ)(250 \text{ mm}) - (600 \text{ N})(200 \text{ mm}) = 0$$

$$R_C = 515.23 \text{ N}$$

Free body: Wedge


R_c
21.3099°
P
R_D
$\phi_s = 11.3099°$

Force triangle:


11.3099°
21.3099°
R_D
R_c
90°−11.3099°
P

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PROBLEM 8.48 (Continued)

Law of sines:

$$\frac{P}{\sin 32.6198^\circ} = \frac{515.23\text{ N}}{\sin 78.690^\circ}$$

(*a*) $\mathbf{P} = 283\text{ N} \longleftarrow$ ◀

(*b*) Returning to free body of *ABC*:

$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x + 600 - (515.23)\sin 21.3099^\circ = 0$$

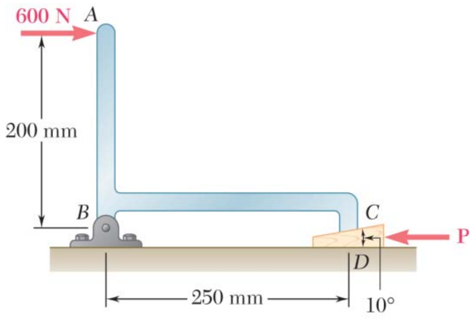
$$B_x = -412.76\text{ N}$$

$\mathbf{B}_x = 413\text{ N} \longleftarrow$ ◀

$$+\!\uparrow \Sigma F_y = 0: \quad B_y + (515.23)\cos 21.3099^\circ = 0$$

$$B_y = -480.0\text{ N}$$

$\mathbf{B}_y = 480\text{ N} \downarrow$ ◀


600 N
A
200 mm
B
C
D
P
250 mm
10°

PROBLEM 8.49

Solve Prob. 8.48 assuming that the wedge is moved to the right.

PROBLEM 8.48

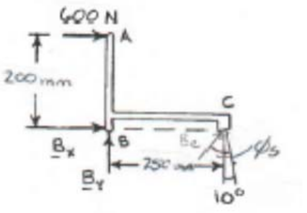
The machine part *ABC* is supported by a frictionless hinge at *B* and a 10° wedge at *C*. Knowing that the coefficient of static friction is 0.20 at both surfaces of the wedge, determine (*a*) the force **P** required to move the wedge to the left, (*b*) the components of the corresponding reaction at *B*.

SOLUTION

$$\mu_s = 0.20$$

$$\phi_s = \tan^{-1}\mu_s = \tan^{-1} 0.20 = 11.3099°$$

Free body: *ABC*

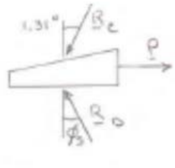

600 N
A
200 mm
C
B_x
B
R_C
ϕ_s
250 mm
B_y
10°

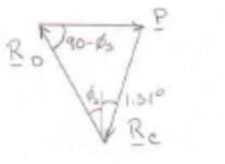
$$10° + 11.31° = 21.31°$$

$$+\circlearrowright \Sigma M_B = 0: \quad (R_C \cos 1.3099°)(250\text{ mm}) - (600\text{ N})(200\text{ mm}) = 0$$

$$R_C = 480.13\text{ N}$$

Free body: Wedge


1.31°
R_C
P
R_D
ϕ_s


P
R_D
$90-\phi_s$
ϕ_s
1.31°
R_C

Force triangle:

PROBLEM 8.49 (Continued)

Law of sines:

$$\frac{P}{\sin 12.6198^\circ} = \frac{480.13\text{ N}}{\sin 78.690^\circ}$$

(*a*) $\mathbf{P} = 107.0\text{ N} \leftarrow$ ◀

(*b*) Returning to free body of *ABC*:

$$\xrightarrow{+} \Sigma F_x = 0: \quad B_x + 600 + (480.13)\sin 1.3099^\circ = 0$$

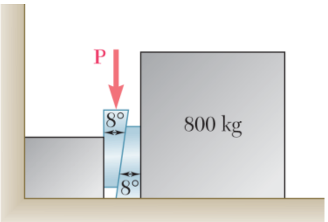
$$B_x = -610.98\text{ N}$$

$\mathbf{B}_x = 611\text{ N} \leftarrow$ ◀

$$+\uparrow \Sigma F_y = 0: \quad B_y + (480.13)\cos 1.3099^\circ = 0$$

$$B_y = -480.0\text{ N}$$

$\mathbf{B}_y = 480\text{ N} \downarrow$ ◀



PROBLEM 8.50

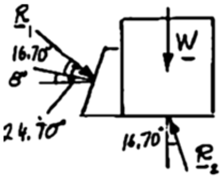
Two 8° wedges of negligible weight are used to move and position the 800-kg block. Knowing that the coefficient of static friction is 0.30 at all surfaces of contact, determine the smallest force **P** that should be applied as shown to one of the wedges.

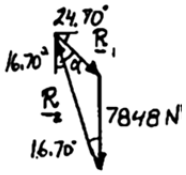
SOLUTION

$$\mu_s = 0.30 \quad \phi_s = \tan^{-1} 0.30 = 16.70°$$

Free body: 800-kg block and right-hand wedge

$$W = (800\text{ kg})(9.81\text{ m/s}^2) = 7848\text{ N}$$


R₁
16.70°
8°
24.70°
W
16.70°
R₂


24.70°
R₁
16.70°
α
R₂
7848 N
16.70°

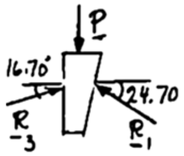
Force triangle:

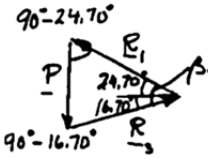
$$\alpha = 90° - 16.70° - 24.70°$$
$$= 48.60°$$

Law of sines:

$$\frac{R_1}{\sin 16.70°} = \frac{7848\text{ N}}{\sin 48.60°}$$
$$R_1 = 3006.5\text{ N}$$

Free body: Left-hand wedge


P
16.70°
24.70
R₃
R₁


90°-24.70°
R₁
P
24.70°
16.70°
β
90°-16.70°
R₃

Force triangle:

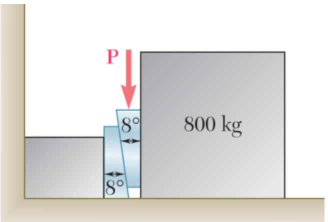
$$\beta = 16.70° + 24.70°$$
$$= 41.40°$$

Law of sines:

$$\frac{P}{\sin 41.40°} = \frac{R_1}{\sin(90° - 16.70°)}$$

$$\frac{P}{\sin 41.40°} = \frac{3006.5\text{ N}}{\cos 16.70°}$$

P = 2080 N ↓ ◀



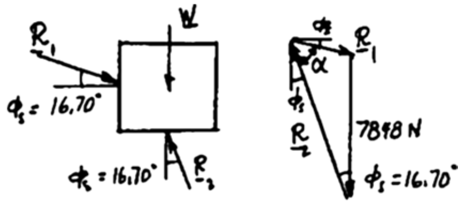
PROBLEM 8.51

Two 8° wedges of negligible weight are used to move and position the 800-kg block. Knowing that the coefficient of static friction is 0.30 at all surfaces of contact, determine the smallest force **P** that should be applied as shown to one of the wedges.

SOLUTION

$$\mu_s = 0.30 \quad \phi_s = \tan^{-1} 0.30 = 16.70°$$

Free body: 800-kg block


R1
W
φs = 16.70°
φs = 16.70°
R2
φs
R1
α
φs
R2
7848 N
φs = 16.70°

Force triangle:

$$W = (800 \text{ kg})(9.81 \text{ m/s}^2) = 7848 \text{ N}$$

$$\alpha = 90° - 2\phi_s = 90° - 2(16.70°)$$

$$= 56.60°$$

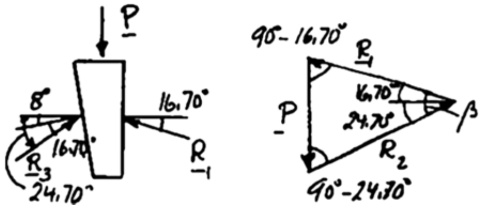
Law of sines:

$$\frac{R_1}{\sin 16.70°} = \frac{7848 \text{ N}}{\sin 56.60°}$$

$$R_1 = 2701 \text{ N}$$

Free body: Right-hand wedge

Force triangle:


P
8°
16.70°
R3
24.70°
16.70°
R1
90°- 16.70°
R1
P
16.70°
24.70°
β
R2
90°-24.70°

$$\beta = 16.70° + 24.70°$$

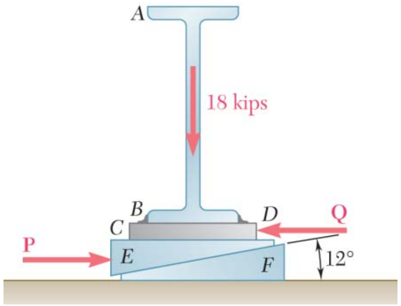
$$= 41.40°$$

Law of sines:

$$\frac{P}{\sin 41.40°} = \frac{R_1}{\sin(90° - 24.70°)}$$

$$\frac{P}{\sin 41.40°} = \frac{2701 \text{ N}}{\cos 24.70°}$$

P = 1966 N ↓ ◀

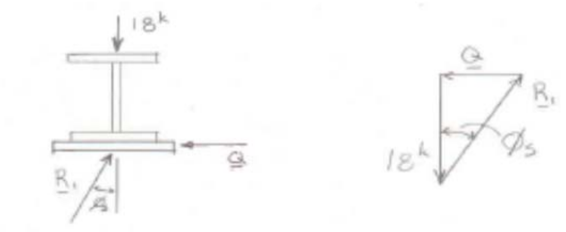


PROBLEM 8.52

The elevation of the end of the steel beam supported by a concrete floor is adjusted by means of the steel wedges *E* and *F*. The base plate *CD* has been welded to the lower flange of the beam, and the end reaction of the beam is known to be 18 kips. The coefficient of static friction is 0.30 between two steel surfaces and 0.60 between steel and concrete. If the horizontal motion of the beam is prevented by the force **Q**, determine (*a*) the force **P** required to raise the beam, (*b*) the corresponding force **Q**.

SOLUTION

$\mu_s = 0.30 \quad \phi_s = \tan^{-1} 0.30 = 16.70°$

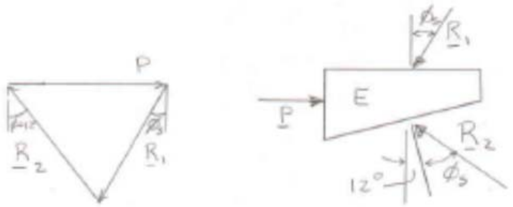


Free body: Beam and plate *CD*

$$R_1 = \frac{(18\text{ kips})}{\cos 16.7°}$$

$$R_1 = 18.7926\text{ kips}$$

Free body: Wedge *E*



(*a*)

$$\frac{P}{\sin 45.4°} = \frac{18.7926\text{ kips}}{\cos 28.7°}$$

P $= 15.26\text{ kips}$ → ◀

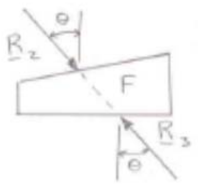
(*b*)

$$Q = (18\text{ kips})\tan 16.7°$$

Q $= 5.40\text{ kips}$ ← ◀

Free body: Wedge *F*(To check that it does not move.)

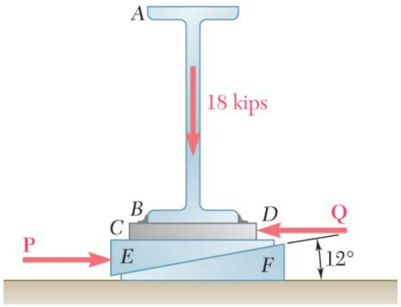
PROBLEM 8.52 (Continued)



Since wedge F is a two-force body, R_2 and R_3 are colinear

Thus $\theta = 28.7°$

But $\phi_{\text{concrete}} = \tan^{-1} 0.6 = 31.0° > \theta$ OK



PROBLEM 8.53

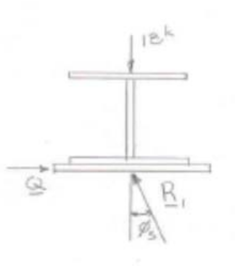
Solve Prob. 8.52 assuming that the end of the beam is to be lowered.

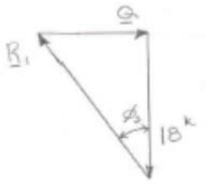
PROBLEM 8.52

The elevation of the end of the steel beam supported by a concrete floor is adjusted by means of the steel wedges *E* and *F*. The base plate *CD* has been welded to the lower flange of the beam, and the end reaction of the beam is known to be 18 kips. The coefficient of static friction is 0.30 between two steel surfaces and 0.60 between steel and concrete. If the horizontal motion of the beam is prevented by the force **Q**, determine (*a*) the force **P** required to raise the beam, (*b*) the corresponding force **Q**.

SOLUTION

Free body: Beam and plate *CD* $\mu_s = 0.30 \quad \phi_s = \tan^{-1} 0.30 = 16.70°$

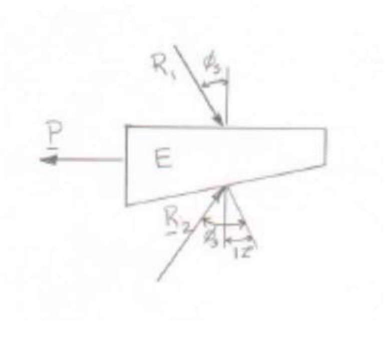




$$R_1 = \frac{(18 \text{ kips})}{\cos 16.7°}$$

$$R_1 = 18{,}792.6 \text{ lb}$$

Free body: Wedge *E*



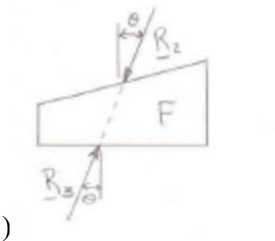
(*a*) $$\frac{P}{\sin 21.4°} = \frac{18.7926 \text{ kips}}{\sin 85.3°}$$ **P** = 6.88 kips ⟶ ◀

PROBLEM 8.53 (Continued)

(*b*)

$$Q = (18\text{ kips})\tan 16.7^\circ \qquad \mathbf{Q} = 5.40\text{ kips} \longleftarrow$$

Free body: Wedge *F*

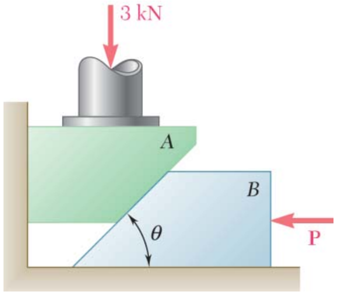


(To check that it does not move.)

Since wedge *F* is a two-force body, R_2 and R_3 are colinear

Thus $\theta = 4.7^\circ$

But $\phi_{\text{concrete}} = \tan^{-1} 0.6 = 31.0^\circ > \theta$ OK



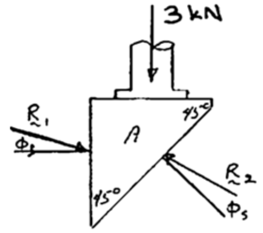
PROBLEM 8.54

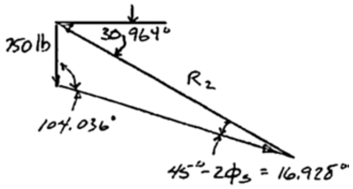
Block *A* supports a pipe column and rests as shown on wedge *B*. Knowing that the coefficient of static friction at all surfaces of contact is 0.25 and that $\theta = 45°$, determine the smallest force **P** required to raise block *A*.

SOLUTION

$$\phi_s = \tan^{-1}\mu_s = \tan^{-1}0.25 = 14.036°$$

FBD block ***A*****:**

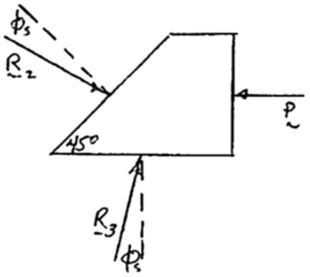

3 kN
R₁
φs
A
45°
45°
R₂
φs

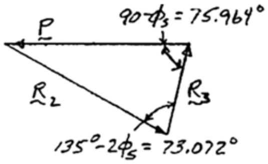

750 lb
30.964°
R₂
104.036°
45° − 2φs = 16.928°

$$\frac{R_2}{\sin 104.036°} = \frac{3\text{ kN}}{\sin 16.928°}$$

$$R_2 = 10.0000\text{ kN}$$

FBD wedge ***B*****:**

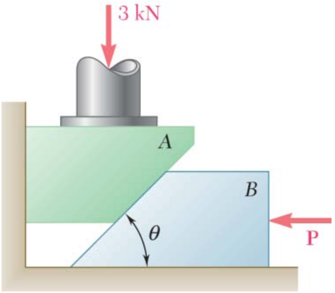

φs
R₂
45°
P
R₃
φs


P
90 − φs = 75.964°
R₂
R₃
135° − 2φs = 73.072°

$$\frac{P}{\sin 73.072°} = \frac{10.0000\text{ kN}}{\sin 75.964°}$$

$$P = 9.8611\text{ kN}$$

P = 9.86 kN ← ◀



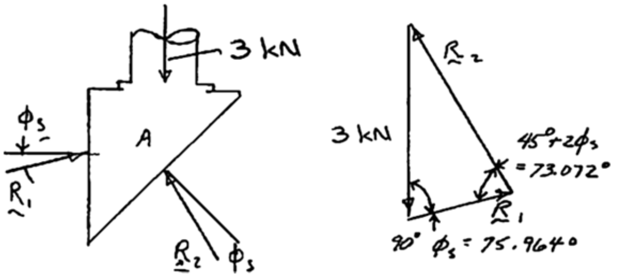
PROBLEM 8.55

Block *A* supports a pipe column and rests as shown on wedge *B*. Knowing that the coefficient of static friction at all surfaces of contact is 0.25 and that $\theta = 45°$, determine the smallest force **P** for which equilibrium is maintained.

SOLUTION

$$\phi_s = \tan^{-1}\mu_s = \tan^{-1} 0.25 = 14.036°$$

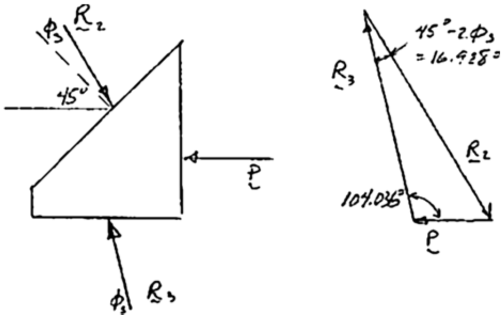
FBD block *A*:


3 kN
φs
A
R1
R2
φs
R2
3 kN
45°+2φs
=73.072°
R1
90° φs = 75.964°

$$\frac{R_2}{\sin(75.964°)} = \frac{3\text{ kN}}{\sin(73.072°)}$$

$$R_2 = 3.0420\text{ kN}$$

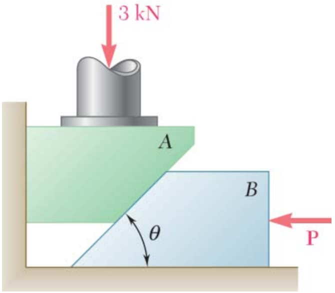
FBD wedge *B*:


φs
R2
45°
P
φs
R3
45°-2φs
=16.928°
R3
R2
104.036°
P

$$\frac{P}{\sin 16.928°} = \frac{3.0420\text{ kN}}{\sin 104.036°}$$

$$P = 0.91300\text{ kN}$$

P = 913 N ⟵◀

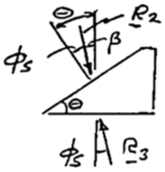


PROBLEM 8.56

Block *A* supports a pipe column and rests as shown on wedge *B*. The coefficient of static friction at all surfaces of contact is 0.25. If $\mathbf{P} = 0$, determine (*a*) the angle θ for which sliding is impending, (*b*) the corresponding force exerted on the block by the vertical wall.

SOLUTION

Free body: Wedge *B* $\qquad \phi_s = \tan^{-1} 0.25 = 14.04°$



(*a*) Since wedge is a two-force body, $\mathbf{R}_2$ and $\mathbf{R}_3$ must be equal and opposite.

Therefore, they form equal angles with vertical

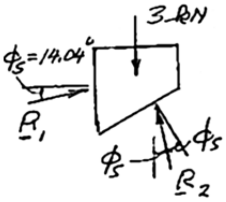
$$\beta = \phi_s$$

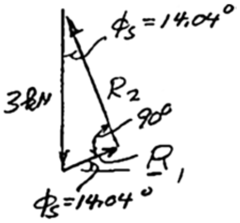
and

$$\theta - \phi_s = \phi_s$$
$$\theta = 2\phi_s = 2(14.04°)$$

$\theta = 28.1°$ ◀

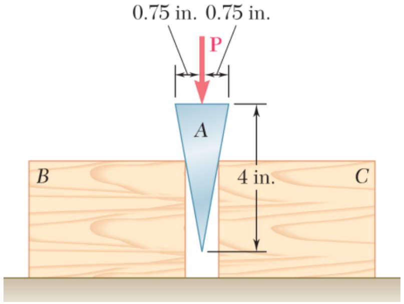
Free body: Block *A*





$$R_1 = (3 \text{ kN}) \sin 14.04° = 0.7278 \text{ kN}$$

(*b*) Force exerted by wall: $\qquad \mathbf{R}_1 = 728 \text{ N} \measuredangle\, 14.04°$ ◀


0.75 in. 0.75 in.
P
A
B
4 in.
C

PROBLEM 8.57

A wedge *A* of negligible weight is to be driven between two 100-lb blocks *B* and *C* resting on a horizontal surface. Knowing that the coefficient of static friction between all surfaces of contact is 0.35, determine the smallest force **P** required to start moving the wedge (*a*) if the blocks are equally free to move, (*b*) if block *C* is securely bolted to the horizontal surface.

SOLUTION

Wedge angle

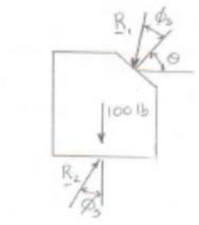
$$\theta = \tan^{-1}\frac{0.75\text{ in.}}{4\text{ in.}} = 10.62°$$

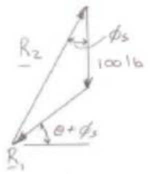
(*a*) Free body: Block *B*

$\phi_s = \tan^{-1}(0.35) = 19.29°$

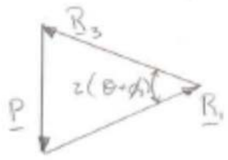
$$\frac{R_1}{\sin 19.29°} = \frac{100\text{ lb}}{\sin 40.8°}$$

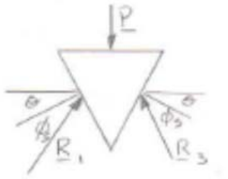
$R_1 = 50.557\text{ lb}$


R1
φs
θ
100 lb
R2
φs


R2
φs
100 lb
θ+φs
R1

Free body: Wedge   From symmetry $R_3 = R_1$



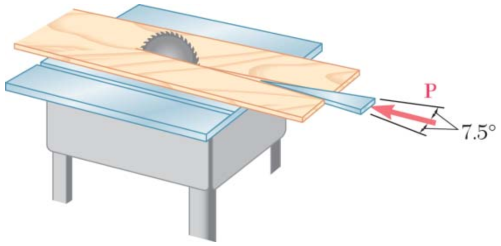

P
θ
φs
R1
θ
φs
R3

$$P = 2R_1 \sin(\theta + \phi_s)$$

$$P = 2(50.557\text{ lb})\sin(10.62° + 19.29°)$$

P = 50.4 lb ↓ ◀

(*b*) Free body diagrams remain unchanged; therefore, result is the same. **P** = 50.4 lb ↓ ◀



PROBLEM 8.58

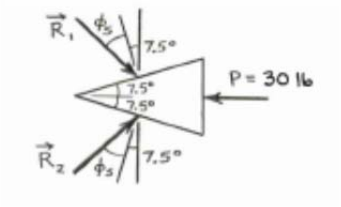
A 15° wedge is forced into a saw cut to prevent binding of the circular saw. The coefficient of static friction between the wedge and the wood is 0.25. Knowing that a horizontal force **P** with a magnitude of 30 lb was required to insert the wedge, determine the magnitude of the forces exerted on the board by the wedge after insertion.

SOLUTION

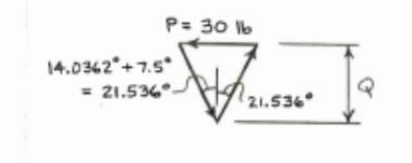
(*a*)Free body: Wedge being inserted,

$\mu_s = 0.25$

$\phi_s = \tan^{-1}(0.25) = 14.0362^\circ$


$\vec{R}_1$
ϕ_s
7.5°
7.5°
7.5°
P = 30 lb
$\vec{R}_2$
ϕ_s
7.5°

Force Triangle:


P = 30 lb
14.0362° + 7.5° = 21.536°
21.536°
Q

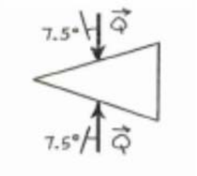
Q = component of $\mathbf{R}_1$ and $\mathbf{R}_2$perpendicular to **P**.

$$Q = \frac{1}{2} P \cot 21.536^\circ$$

$$Q = (15 \text{ lb}) \cot 21.536^\circ$$

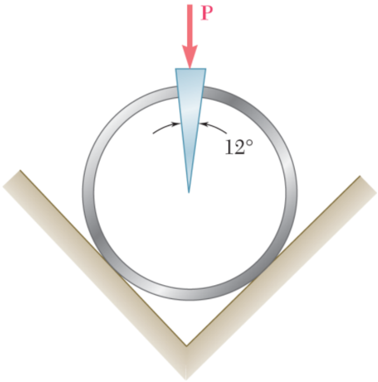
$$Q = 38.010 \text{ lb}$$

Free body: After P is removed only the components Q of $\mathbf{R}_1$ and $\mathbf{R}_2$ remain;


7.5°
$\vec{Q}$
7.5°
$\vec{Q}$

$Q = 38.0 \text{ lb}$ ◀

Note: since $7.5^\circ < 14.0362^\circ$, which is the angle of static friction, wedge remains in place



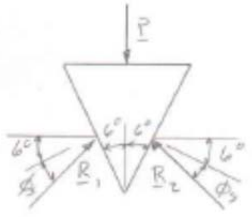
PROBLEM 8.59

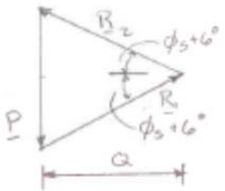
A 12° wedge is used to spread a split ring. The coefficient of static friction between the wedge and the ring is 0.30. Knowing that a force **P** with a magnitude of 120 N was required to insert the wedge, determine the magnitude of the forces exerted on the ring by the wedge after insertion.

SOLUTION

$$\phi_s = \tan^{-1}\mu_s = \tan^{-1} 0.30 = 16.70°$$

FBD wedge:

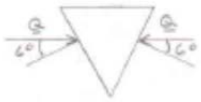




From force triangle: Q = horizontal component of **R** $\quad Q = \left(\frac{1}{2}\right)\frac{(120\text{ N})}{\tan 22.7^\circ} = 143.435\text{ N}$

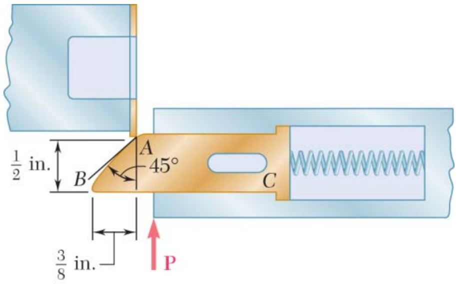
FBD after wedge has been inserted:

Wedge is now a two-force body with forces shown.



$Q = 143.4\text{ N}$ ◀

Note: since $6.0^\circ < 16.70^\circ$, which is the angle of static friction, wedge remains in place

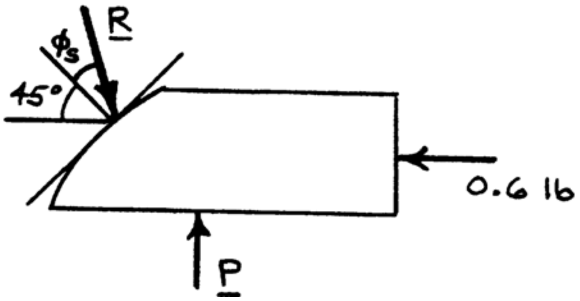

$\frac{1}{2}$ in.
B
A
45°
C
$\frac{3}{8}$ in.
P

PROBLEM 8.60

The spring of the door latch has a constant of 1.8 lb/in. and in the position shown exerts a 0.6-lb force on the bolt. The coefficient of static friction between the bolt and the strike plate is 0.40; all other surfaces are well lubricated and may be assumed frictionless. Determine the magnitude of the force **P** required to start closing the door.

SOLUTION

Free body: Bolt

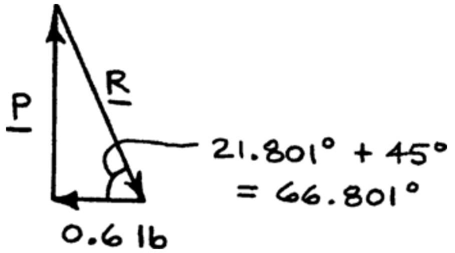

R
ϕ_s
45°
0.6 lb
P

$$\mu_s = 0.40$$

$$\phi_s = \tan^{-1} 0.40$$

$$= 21.801°$$

Force triangle:

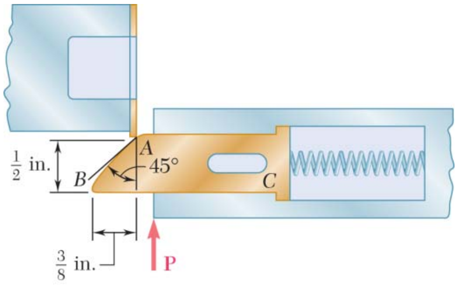

P
R
21.801° + 45°
= 66.801°
0.6 lb

From force triangle,

$$P = (0.6 \text{ lb}) \tan 66.801°$$

$$P = 1.39997 \text{ lb}$$

$P = 1.400$ lb ◀



PROBLEM 8.61

In Problem 8.60, determine the angle that the face of the bolt should form with the line *BC* if the force **P** required to close the door is to be the same for both the position shown and the position when *B* is almost at the strike plate.

PROBLEM 8.60 The spring of the door latch has a constant of 1.8 lb/in. and in the position shown exerts a 0.6-lb force on the bolt. The coefficient of static friction between the bolt and the strike plate is 0.40; all other surfaces are well lubricated and may be assumed frictionless. Determine the magnitude of the force **P** required to start closing the door.

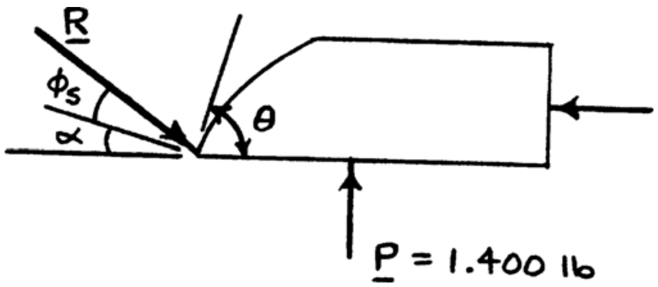
SOLUTION

For position shown in figure:

From Prob. 8.60: $P = 1.400 \text{ lb}$

For position when *B* reaches strike plate:

Free body: Bolt



$$\mu_s = 0.40$$

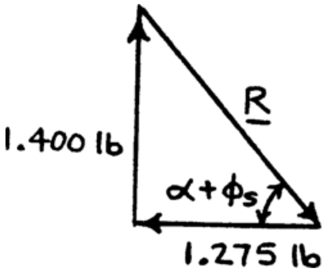
$$\phi_s = \tan^{-1} 0.40 = 21.801°$$

$$F = 0.6 \text{ lb} + kx$$

$$= 0.6 \text{ lb} + (1.8 \text{ lb/in.})\left(\frac{3}{8} \text{ in.}\right)$$

$$= 1.275 \text{ lb}$$

Force triangle:



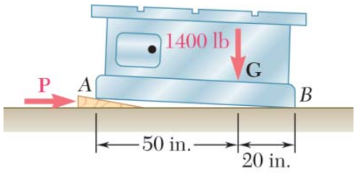
From force triangle,

$$\tan(\alpha - \phi_s) = \frac{1.400 \text{ lb}}{1.275 \text{ lb}} = 1.09804$$

$$\alpha + \phi_s = 47.675°$$

$$\alpha = 47.675° - 21.801° = 25.874°$$

$$\theta = 180° - 90° - \alpha = 90° - \alpha = 90° - 25.874° \qquad \theta = 64.1° ◀$$

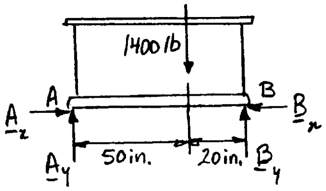

1400 lb
G
P
A
B
50 in.
20 in.

PROBLEM 8.62

A 5° wedge is to be forced under a 1400-lb machine base at *A*. Knowing that the coefficient of static friction at all surfaces is 0.20, (*a*) determine the force **P** required to move the wedge, (*b*) indicate whether the machine base will move.

SOLUTION

Free body: Machine base


1400 lb
A
B
A_x
B_x
50 in.
20 in.
A_y
B_y

$$+\circlearrowleft \Sigma M_B = 0: \quad (1400\text{ lb})(20\text{ in}) - A_y(70\text{ in}) = 0$$

$$A_y = 400\text{ lb}$$

$$\xrightarrow{+} \Sigma F_y = 0: \quad A_y + B_y - 1400\text{ lb} = 0$$

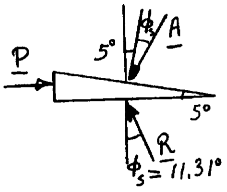
$$B_y = 1400 - 400 = 1000\text{ lb}$$

Free body: Wedge

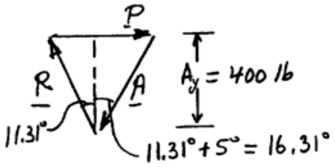
(Assume machine will not move)

$$\mu_s = 0.20, \quad \phi_s = \tan 0.20 = 11.31°$$

We know that $A_y = 400\text{ lb}$


P
5°
ϕ_s
A
5°
R
$\phi_s = 11.31°$

Force triangle:

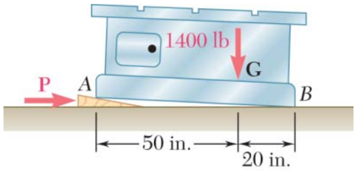

P
R
A
$A_y = 400$ lb
11.31°
11.31° + 5° = 16.31°

(*a*) $$P = (400\text{ lb})(\tan 11.31° + \tan 16.31°) = 197.0\text{ lb}$$ $\mathbf{P} = 197.0\text{ lb} \longrightarrow$ ◀

(*b*) Total maximum friction force at *A* and *B*:

$$F_m = \mu_c W = 0.20(1400\text{ lb}) = 280\text{ lb}$$

Since $P < F_m$: Machine base will not move ◀



PROBLEM 8.63

Solve Problem 8.62 assuming that the wedge is to be forced under the machine base at *B* instead of *A*.

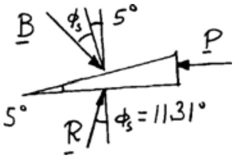
PROBLEM 8.62 A 5° wedge is to be forced under a 1400-lb machine base at *A*. Knowing that the coefficient of static friction at all surfaces is 0.20, (*a*) determine the force **P** required to move the wedge, (*b*) indicate whether the machine base will move.

SOLUTION

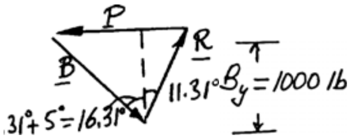
See solution to Prob. 8.62 for F.B.D. of machine base and determination of $B_y = 1000$ lb.

Free body: Wedge (Assume machine will not move)

$$\mu_s = 0.20 \qquad \phi_s = \tan^{-1} 0.20 = 11.31°$$



Force triangle:



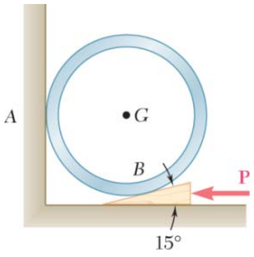
$$P = (1000 \text{ lb})(\tan 11.31° + \tan 16.31°) = 493 \text{ lb}$$

Total maximum friction force at *A* and *B*:

$$F_m = \mu_s W = 0.20(1400 \text{ lb}) = 280 \text{ lb} < 493 \text{ lb}$$

(*b*) Since $P > F_m$, machine base will move with wedge ◀

(*a*) We then have $P = F_m$, **P** = 280 lb ← ◀



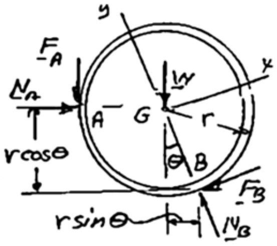
PROBLEM 8.64

A 15° wedge is forced under a 50-kg pipe as shown. The coefficient of static friction at all surfaces is 0.20. (*a*) Show that slipping will occur between the pipe and the vertical wall. (*b*) Determine the force **P** required to move the wedge.

SOLUTION

Free body: Pipe

$$+\circlearrowleft \Sigma M_B = 0: \quad Wr\sin\theta + F_A r(1+\sin\theta) - N_A r\cos\theta = 0$$



Assume slipping at A:

$$F_A = \mu_s N_A$$

$$N_A\cos\theta - \mu_s N_A(1+\sin\theta) = W\sin\theta$$

$$N_A = \frac{W\sin\theta}{\cos\theta - \mu_s(1+\sin\theta)}$$

$$N_A = \frac{W\sin 15^\circ}{\cos 15^\circ - (0.20)(1+\sin 15^\circ)}$$

$$= 0.36241W$$

$$+\nearrow \Sigma F_x = 0: \quad -F_B - W\sin\theta - F_A\sin\theta + N_A\cos\theta = 0$$

$$F_B = N_A\cos\theta - \mu_s N_A\sin\theta - W\sin\theta$$

$$F_B = 0.36241W\cos 15^\circ - 0.20(0.36241W)\sin 15^\circ - W\sin 15^\circ$$

$$F_B = 0.072482W$$

$$+\nwarrow \Sigma F_y = 0: \quad N_B - W\cos\theta - F_A\cos\theta - N_A\sin\theta = 0$$

$$N_B = N_A\sin\theta + \mu_s N_A\cos\theta + W\cos\theta$$

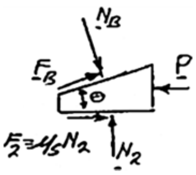
$$N_B = (0.36241W)\sin 15^\circ + 0.20(0.36241W)\cos 15^\circ + W\cos 15^\circ$$

$$N_B = 1.12974W$$

Maximum available: $F_B = \mu_s N_B = 0.22595W$

(*a*) We note that $F_B < F_{\max}$ No slip at B ◀

(*b*) Free body: Wedge $+\uparrow \Sigma F_y = 0: \quad N_2 - N_B\cos\theta + F_B\sin\theta = 0$



$$N_2 = N_B\cos\theta - F_B\sin\theta$$

$$N_2 = (1.12974W)\cos 15^\circ - (0.07248W)\sin 15^\circ$$

$$N_2 = 1.07249W$$

PROBLEM 8.64 (Continued)

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_B \cos\theta + N_B \sin\theta + \mu_s N_2 - P = 0$$

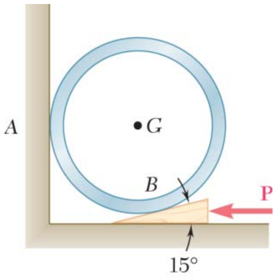
$$P = F_B \cos\theta + N_B \sin\theta + \mu_s N_2$$

$$P = (0.07248W)\cos 15° + (1.12974W)\sin 15° + 0.2(1.07249W)$$

$$P = 0.5769W$$

$$W = mg: \quad P = 0.5769(50 \text{ kg})(9.81 \text{ m/s}^2)$$

$\mathbf{P} = 283 \text{ N} \longleftarrow$ ◀

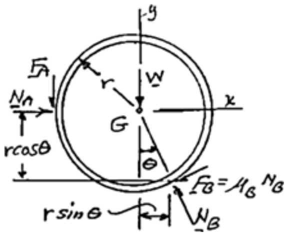


PROBLEM 8.65

A 15° wedge is forced under a 50-kg pipe as shown. Knowing that the coefficient of static friction at both surfaces of the wedge is 0.20, determine the largest coefficient of static friction between the pipe and the vertical wall for which slipping will occur at *A*.

SOLUTION

Free body: Pipe



$$+\circlearrowleft \Sigma M_A = 0: \quad N_B r\cos\theta - \mu_B N_B r - (\mu_B N_B \sin\theta)r - Wr = 0$$

$$N_B = \frac{W}{\cos\theta - \mu_B(1+\sin\theta)}$$

$$N_B = \frac{W}{\cos 15° - 0.2(1+\sin 15°)}$$

$$N_B = 1.4002W$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad N_A - N_B \sin\theta - \mu_B N_B \cos\theta = 0$$

$$N_A = N_B(\sin\theta + \mu_B\cos\theta)$$

$$= (1.4002W)(\sin 15° + 0.2\times\cos 15°)$$

$$N_A = 0.63293W$$

$$+\uparrow \Sigma F_y = 0: \quad -F_A - W + N_B\cos\theta - \mu_B N_B \sin\theta = 0$$

$$F_A = N_B(\cos\theta - \mu_B\sin\theta) - W$$

$$F_A = (1.4002W)(\cos 15° - 0.2\times\sin\theta) - W$$

$$F_A = 0.28001W$$

For slipping at *A*: $F_A = \mu_A N_A$

$$\mu_A = \frac{F_A}{N_A} = \frac{0.28001W}{0.63293W}$$

$\mu_A = 0.442$ ◀

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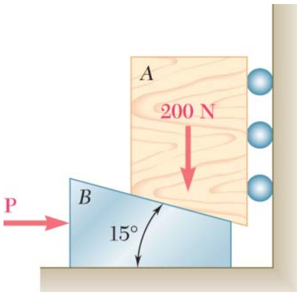


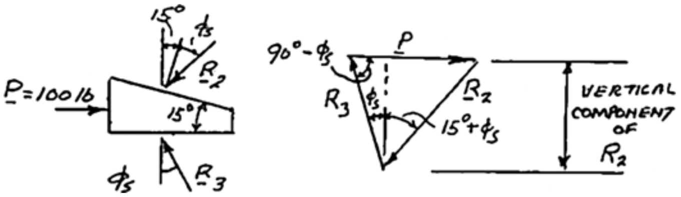
PROBLEM 8.66*

A 200-N block rests as shown on a wedge of negligible weight. The coefficient of static friction μ_s is the same at both surfaces of the wedge, and friction between the block and the vertical wall may be neglected. For P = 100 N, determine the value of μ_s for which motion is impending. (*Hint:* Solve the equation obtained by trial and error.)

SOLUTION

Free body: Wedge

Force triangle:

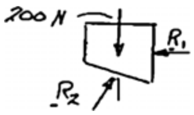

15°
φs
R2
P=100 lb
15°
φs
R3
90°-φs
P
R3
φs
R2
15°+φs
VERTICAL COMPONENT OF R2

Law of sines:

$$\frac{R_2}{\sin(90^\circ - \phi_s)} = \frac{P}{\sin(15^\circ + 2\phi_s)}$$

$$R_2 = P\frac{\sin(90^\circ - \phi_s)}{\sin(15^\circ + 2\phi_s)} \qquad (1)$$

Free body: Block



$$\Sigma F_y = 0$$

Vertical component of R_2 is 200 N

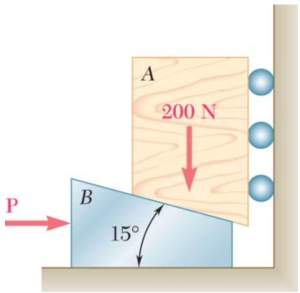
Return to force triangle of wedge. Note $P = 100$ N

$$100\,\text{N} = (200\,\text{N})\tan\phi + (200\,\text{N})\tan(15^\circ + \phi_s)$$

$$0.5 = \tan\phi + \tan(15^\circ + \phi_s)$$

Solve by trial and error $\qquad \phi_s = 6.292$

$$\mu_s = \tan\phi_s = \tan 6.292^\circ \qquad \mu_s = 0.1103 ◀$$



PROBLEM 8.67*

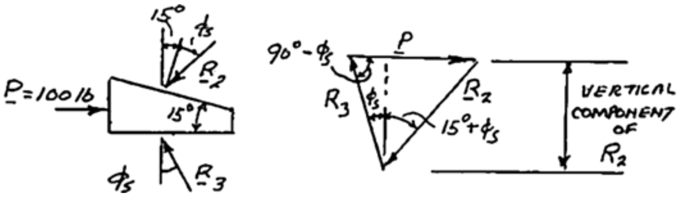
Solve Problem 8.66 assuming that the rollers are removed and that μ_s is the coefficient of friction at all surfaces of contact.

PROBLEM 8.66* A 200-N block rests as shown on a wedge of negligible weight. The coefficient of static friction μ_s is the same at both surfaces of the wedge, and friction between the block and the vertical wall may be neglected. For P= 100 N, determine the value of μ_s for which motion is impending. (*Hint:* Solve the equation obtained by trial and error.)

SOLUTION

Free body: Wedge

Force triangle:


15°
ϕs
R2
P=100 lb
15°
ϕs
R3
90°-ϕs
P
R3
ϕs
R2
15°+ϕs
VERTICAL COMPONENT OF R2

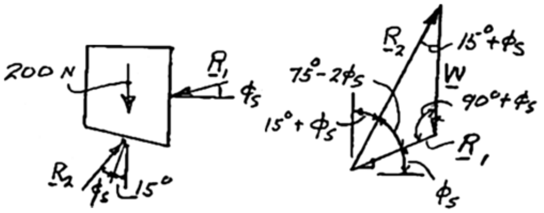
Law of sines:

$$\frac{R_2}{\sin(90°-\phi_s)} = \frac{P}{\sin(15°+2\phi_s)}$$

$$R_2 = P\frac{\sin(90°-\phi_s)}{\sin(15°+2\phi_s)} \qquad (1)$$

Free body: Block (Rollers removed)

Force triangle:


200 N
R1
ϕs
R2
ϕs
15°
R2
15°+ϕs
75°-2ϕs
W
90°+ϕs
15°+ϕs
R1
ϕs

Law of sines:

$$\frac{R_2}{\sin(90°+\phi_s)} = \frac{W}{\sin(75°-2\phi_s)}$$

$$R_2 = W\frac{\sin(90°+\phi_s)}{\sin(75°-2\theta_s)} \qquad (2)$$

PROBLEM 8.67* (Continued)

Equate R_2 from Eq. (1) and Eq. (2):

$$P\frac{\sin(90^\circ-\phi_s)}{\sin(15^\circ+2\phi_s)}=W\frac{\sin(90^\circ+\phi_s)}{\sin(75^\circ-2\phi_s)}$$

$$P=100\text{ lb}$$

$$W=200\text{ N}$$

$$0.5=\frac{\sin(90^\circ+\phi_s)\sin(15^\circ+2\phi_s)}{\sin(75^\circ-2\phi_s)\sin(90^\circ-\phi_s)}$$

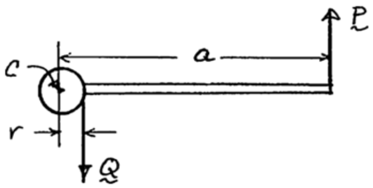
Solve by trial and error: $\phi_s=5.784^\circ$

$\mu_s=\tan\phi_s=\tan 5.784^\circ$ $\mu_s=0.1013$ ◀

PROBLEM 8.68

Derive the following formulas relating the load **W** and the force **P** exerted on the handle of the jack discussed in Section 8.6. (*a*) $P = (Wr/a)\tan(\theta + \phi_s)$, to raise the load; (*b*) $P = (Wr/a)\tan(\phi_s - \theta)$, to lower the load if the screw is self-locking; (*c*) $P = (Wr/a)\tan(\theta - \phi_s)$, to hold the load if the screw is not self-locking.

SOLUTION

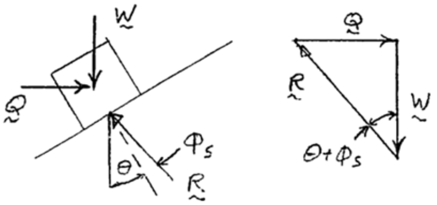


FBD jack handle:

See Section 8.6. $\circlearrowleft \Sigma M_C = 0: \quad aP - rQ = 0 \quad \text{or} \quad P = \dfrac{r}{a}Q$

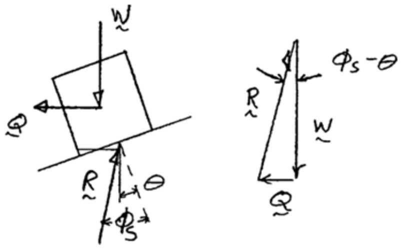
FBD block on incline:

(*a*) Raising load



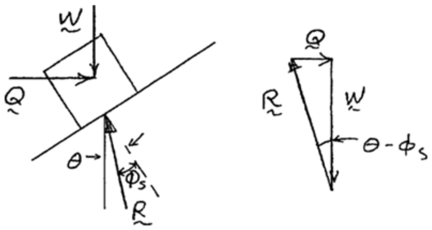
$Q = W\tan(\theta + \phi_s)$ $\qquad P = \dfrac{r}{a}W\tan(\theta + \phi_s)$ ◀

(*b*) Lowering load if screw is self-locking (i.e., if $\phi_s > \theta$)

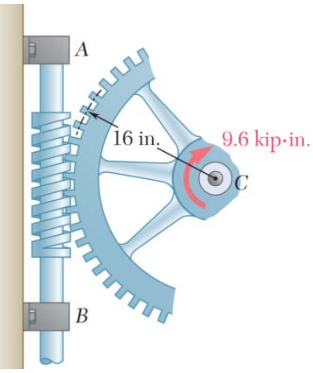


$Q = W\tan(\phi_s - \theta)$ $\qquad P = \dfrac{r}{a}W\tan(\phi_s - \theta)$ ◀

(*c*) Holding load is screw is not self-locking (i.e., if $\phi_s < \theta$)



$Q = W\tan(\theta - \phi_s)$ $\qquad P = \dfrac{r}{a}W\tan(\theta - \phi_s)$ ◀


A
16 in.
9.6 kip·in.
C
B

PROBLEM 8.69

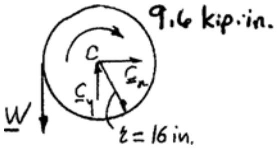
The square-threaded worm gear shown has a mean radius of 2 in. and a lead of 0.5 in. The large gear is subjected to a constant clockwise couple of 9.6 kip · in. Knowing that the coefficient of static friction between the two gears is 0.12, determine the couple that must be applied to shaft *AB* in order to rotate the large gear counterclockwise. Neglect friction in the bearings at *A*, *B*, and *C*.

SOLUTION

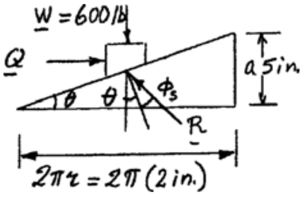
Free body: Large gear

$$+\circlearrowleft \Sigma M_C = 0: \quad W(16 \text{ in.}) - 9.6 \text{ kip} \cdot \text{in.} = 0$$

$$W = 0.6 \text{ kip} = 600 \text{ lb}$$


9.6 kip·in.
C
C_x
C_y
W
r = 16 in.

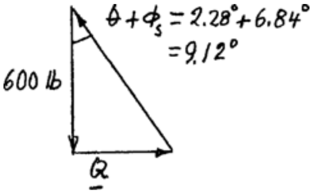
Block-and-Incline analysis of worn gear


W = 600 lb
Q
0.5 in.
θ
θ
ϕ_s
R
2πr = 2π(2 in.)

$$\tan\theta = \frac{0.5 \text{ in.}}{2\pi(2 \text{ in.})} = 0.039789$$

$$\theta = 2.28°$$

$$\mu_s = 0.12, \quad \phi_s = \tan^{-1} 0.12 = 6.84°$$


θ + ϕ_s = 2.28° + 6.84° = 9.12°
600 lb
Q

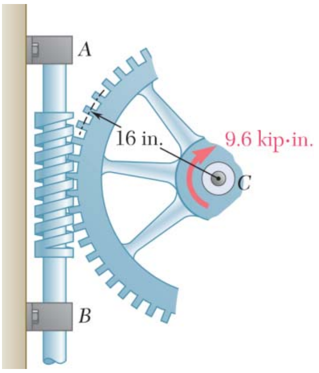
$$Q = (600 \text{ lb}) \tan 9.12°$$

$$= 96.32 \text{ lb}$$

$$\text{Torque} = Q_r = (96.32 \text{ lb})(2 \text{ in.})$$

$$= 192.6 \text{ lb} \cdot \text{in.}$$

$$\text{Torque} = 16.05 \text{ lb} \cdot \text{ft} \blacktriangleleft$$



PROBLEM 8.70

In Problem 8.69, determine the couple that must be applied to shaft *AB* in order to rotate the large gear clockwise.

PROBLEM 8.69 The square-threaded worm gear shown has a mean radius of 2 in. and a lead of 0.5 in. The large gear is subjected to a constant clockwise couple of 9.6 kip · in. Knowing that the coefficient of static friction between the two gears is 0.12, determine the couple that must be applied to shaft *AB* in order to rotate the large gear counterclockwise. Neglect friction in the bearings at *A*, *B*, and *C*.

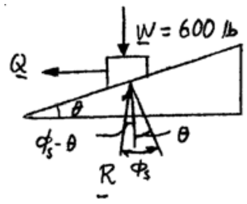
SOLUTION

Free body: Large gear

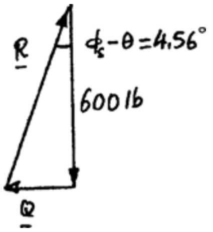
See solution to Prob. 8.69. We find $W = 600$ lb

Block-and-incline analysis of worm gear

From Prob. 8.69 we have $\theta = 2.28°$ and $\phi_s = 6.84°$. Since $\theta < \phi_s$, gear is self-looking and a torque must be applied to it to rotate large gear clockwise.



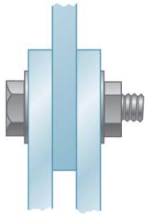
$$\phi_s - \theta = 6.84° - 2.28° = 4.56°$$



$$Q = (600 \text{ lb}) \tan 4.56°$$
$$= 47.85 \text{ lb}$$

$$\text{Torque} = Q_r = (47.85 \text{ lb})(2 \text{ in.})$$
$$= 95.70 \text{ lb} \cdot \text{in.}$$

$$\text{Torque} = 7.98 \text{ lb} \cdot \text{ft} \blacktriangleleft$$

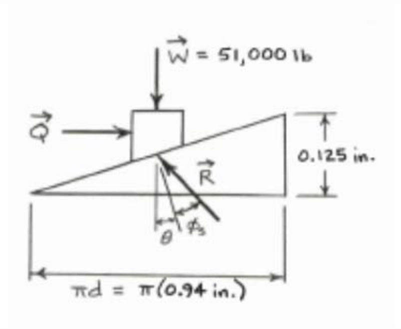


PROBLEM 8.71

High-strength bolts are used in the construction of many steel structures. For a 1-in.-nominal-diameter bolt, the required minimum bolt tension is 51 kips. Assuming the coefficient of friction to be 0.30, determine the required couple that should be applied to the bolt and nut. The mean diameter of the thread is 0.94 in., and the lead is 0.125 in. Neglect friction between the nut and washer, and assume the bolt to be square-threaded.

SOLUTION

Block-and-incline analysis of bolt and nut:



$\vec{W}$ = 51,000 lb
$\vec{Q}$
$\vec{R}$
0.125 in.
θ
ϕ_s
$\pi d = \pi(0.94 \text{ in.})$

$$\theta = \tan^{-1}\frac{0.125 \text{ in.}}{(0.94 \text{ in.})\pi}$$

$$= 2.4238°$$

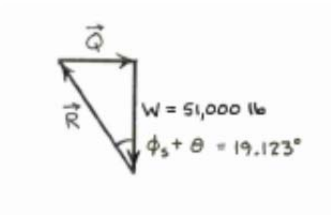
$$\phi_s = \tan^{-1}\mu_s = \tan^{-1} 0.30$$

$$\phi_s = 16.6992°$$

From force triangle:

$$Q = (51{,}000 \text{ lb}) \tan(19.123°)$$

$$Q = 17{,}683.3 \text{ lb}$$



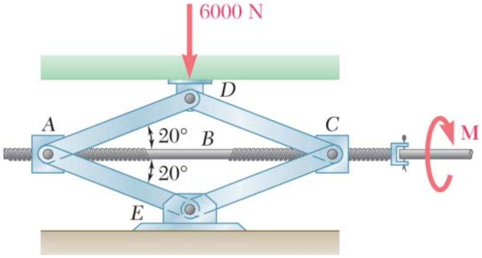
$\vec{Q}$
$\vec{R}$
W = 51,000 lb
$\phi_s + \theta = 19.123°$

$$\text{Torque} = Qr = Q\frac{d}{2}$$

$$= (17{,}683.3 \text{ lb})\frac{0.94 \text{ in.}}{2}$$

$$= 8311.1 \text{ lb} \cdot \text{in.}$$

$$\text{Torque} = 693 \text{ lb} \cdot \text{ft} \blacktriangleleft$$


6000 N
D
A
20°
B
C
M
20°
E

PROBLEM 8.72

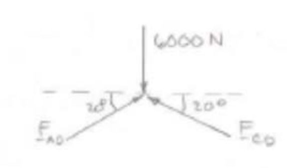
The position of the automobile jack shown is controlled by a screw *ABC* that is single-threaded at each end (right-handed thread at *A*, left-handed thread at *C*). Each thread has a pitch of 2.5 mm and a mean diameter of 9 mm. If the coefficient of static friction is 0.15, determine the magnitude of the couple **M** that must be applied to raise the automobile.

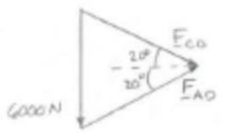
SOLUTION

All members of jack are two-force members; analyze joint equilibrium at *C* and *D*.

Joint *D*:

Symmetry: $F_{AD} = F_{CD}$


6000 N
20°
20°
F_{AD}
F_{CD}

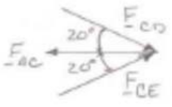

F_{CD}
20°
20°
F_{AD}
6000 N

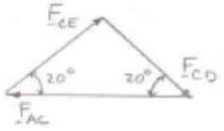
$$+\!\uparrow \Sigma F_y = 0: \quad 2F_{CD}\sin 20° - 6000 \text{ N} = 0$$

$$F_{CD} = 8771.4 \text{ N}$$

Joint *C*:

Symmetry: $F_{CE} = F_{CD}$


F_{CD}
20°
F_{AC}
20°
F_{CE}


F_{CE}
20°
20°
F_{CD}
F_{AC}

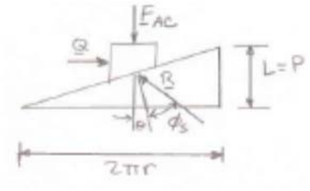
$$\xrightarrow{+} \Sigma F_x = 0: \quad 2F_{CD}\cos 25° - F_{AC} = 0$$

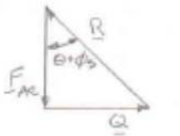
$$F_{AC} = 2(8771.4 \text{ N})\cos 20°$$

$$F_{AC} = 16{,}484.8 \text{ N}$$

PROBLEM 8.72 (Continued)

Block-and-incline analysis of *one* screw:





$$\tan\theta = \frac{2.5\text{ mm}}{\pi(9\text{ mm})}$$

$$\theta = 5.053°$$

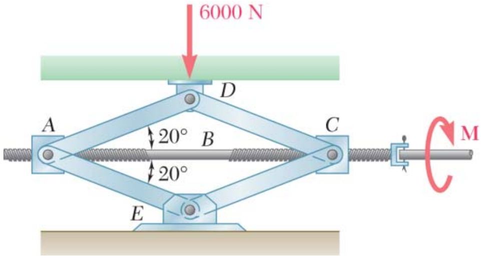
$$\phi_s = \tan^{-1} 0.15$$

$$= 8.531^\circ$$

$$Q = (16{,}484.8\text{ N})\tan 13.584^\circ$$

$$Q = 3983.3\text{ N}$$

But, we have *two* screws: $\text{Torque} = 2Qr = Qd = (3983.3\text{ N})(0.009\text{ m})$ $\text{Torque} = 35.8\text{ N}\cdot\text{m}$ ◀


6000 N
D
A
20°
B
C
M
20°
E

PROBLEM 8.73

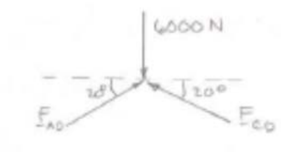
For the jack of Prob. 8.72, determine the magnitude of the couple **M** that must be applied to lower the automobile.

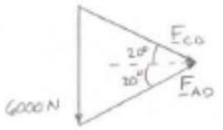
PROBLEM 8.72

The position of the automobile jack shown is controlled by a screw *ABC* that is single-threaded at each end (right-handed thread at *A*, left-handed thread at *C*). Each thread has a pitch of 2.5 mm and a mean diameter of 9 mm. If the coefficient of static friction is 0.15, determine the magnitude of the couple **M** that must be applied to raise the automobile.

SOLUTION All members of jack are two-force members; analyze joint equilibrium at *C* and *D*

Joint *D*:Symmetry: $F_{AD} = F_{CD}$


6000 N
20°
20°
F_{AD}
F_{CD}

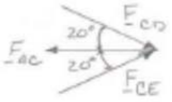

F_{CD}
20°
20°
6000 N
F_{AD}

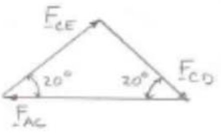
$$+\uparrow \Sigma F_y = 0: \quad 2F_{CD}\sin 20° - 6000\text{ N} = 0$$

$F_{CD} = 8771.4\text{ N}$

Joint *C*:

Symmetry: $F_{CE} = F_{CD} \quad \xrightarrow{+} \Sigma F_x = 0: \quad 2F_{CD}\cos 25° - F_{AC} = 0$


F_{AC}
20°
20°
F_{CD}
F_{CE}

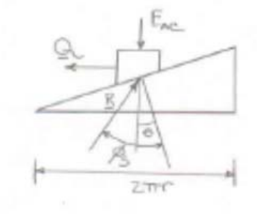

F_{CE}
20°
20°
F_{CD}
F_{AC}

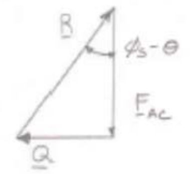
$$F_{AC} = 2(8771.4\text{ N})\cos 20°$$

$$F_{AC} = 16{,}484.8\text{ N}$$

PROBLEM 8.73 (Continued)

Block-and-incline analysis of *one* screw:


F_{AC}
Q
B
ϕ_s
θ
2πr


B
$\phi_s - \theta$
F_{AC}
Q

$$\tan\theta = \frac{2.5\text{ mm}}{\pi(9\text{ mm})}$$

$$\theta = 5.053°$$

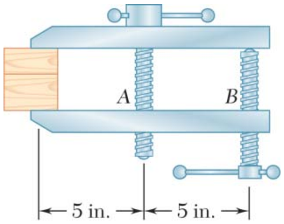
$$\phi_s = \tan^{-1} 0.15$$

$$= 8.531°$$

$$Q = (16{,}484.8\text{ N})\tan 3.478°$$

$$Q = 1001.87\text{ N}$$

But, we have *two* screws: $\text{Torque} = 2Qr = Qd = (1001.87\text{ N})(0.009\text{ m})$ $\text{Torque} = 9.02\text{ N}\cdot\text{m}$ ◀

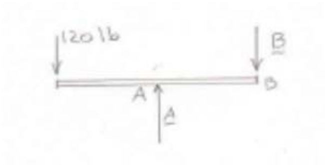


PROBLEM 8.74

The vise shown consists of two members connected by two double-threaded screws with a mean radius of 0.25 in. and pitch of 0.08 in. The lower member is threaded at *A* and *B* ($\mu_s = 0.35$), but the upper member is not threaded. It is desired to apply two equal and opposite forces of 120 lb on the blocks held between the jaws. (*a*) What screw should be adjusted first? (*b*) What is the maximum couple applied in tightening the second screw?

SOLUTION

Free body: Lower Jaw



$$+\circlearrowleft \Sigma M_B = 0: \quad (120 \text{ lb})(10 \text{ in.}) - A(5 \text{ in.}) = 0$$

$$A = 240 \text{ lb}$$

$$+\uparrow \Sigma F_y = 0: \quad A - B - 120 \text{ lb} = 0$$

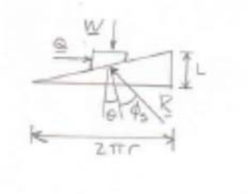
$$B = 120 \text{ lb}$$

Since $B < A$, Screw *A* should be adjusted first. ◀

Block-and-incline analysis of screw *B*:

For double-threaded screw: Lead = 2 × Pitch

$$L = 2(0.08 \text{ in.}) = 0.16 \text{ in.}$$

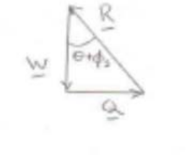


$$\tan\theta = \frac{L}{2\pi r} = \frac{0.16 \text{ in.}}{2\pi(0.25 \text{ in.})} = 0.101859 \quad \theta = 5.816^\circ$$

PROBLEM 8.74 (Continued)

$\phi_s = \tan^{-1}\mu_s = \tan^{-1}(0.35)$

$\phi_s = 19.29^\circ \ \theta + \phi_s = 25.11^\circ$



$Q = W\tan(25.11^\circ)$

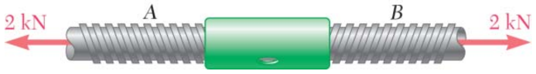
$Q = (120 \text{ lb})\tan 25.11^\circ$

$Q = 56.238 \text{ lb}$

$\text{Torque} = Qr = (56.238 \text{ lb})(0.25 \text{ in.})$ $\qquad \text{Torque} = 14.06 \text{ lb}\cdot\text{in.}$ ◀

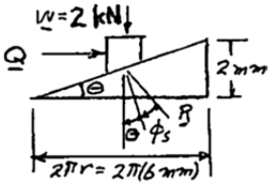
PROBLEM 8.75

The ends of two fixed rods *A* and *B* are each made in the form of a single-threaded screw of mean radius 6 mm and pitch 2 mm. Rod *A* has a right-handed thread and rod *B* has a left-handed thread. The coefficient of static friction between the rods and the threaded sleeve is 0.12. Determine the magnitude of the couple that must be applied to the sleeve in order to draw the rods closer together.



SOLUTION

To draw rods together:



Screw at *A*

$$\tan\theta = \frac{2\text{ mm}}{2\pi(6\text{ mm})}$$

$$\theta = 3.037°$$

$$\phi_s = \tan^{-1} 0.12$$

$$= 6.843°$$

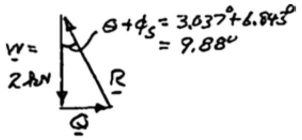
$$Q = (2\text{ kN})\tan 9.88°$$

$$= 348.3\text{ N}$$

$$\text{Torque at } A = Qr$$

$$= (348.3\text{ N})(0.006\text{ m})$$

$$= 2.0898\text{ N}\cdot\text{m}$$



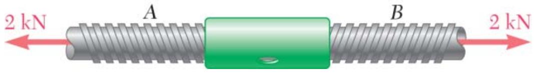
Same torque required at *B*

Total torque $= 4.18\text{ N}\cdot\text{m}$ ◀

PROBLEM 8.76

Assuming that in Problem 8.75 a right-handed thread is used on *both* rods *A* and *B*, determine the magnitude of the couple that must be applied to the sleeve in order to rotate it.

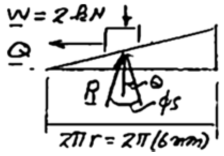
PROBLEM 8.75 The ends of two fixed rods *A* and *B* are each made in the form of a single-threaded screw of mean radius 6 mm and pitch 2 mm. Rod *A* has a right-handed thread and rod *B* has a left-handed thread. The coefficient of static friction between the rods and the threaded sleeve is 0.12. Determine the magnitude of the couple that must be applied to the sleeve in order to draw the rods closer together.



SOLUTION

From the solution to Problem 8.70,

$$\text{Torque at } A = 2.09 \text{ N}\cdot\text{m}$$



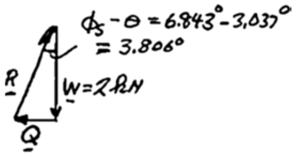
Screw at *B*: Loosening

$$\theta = 3.037°$$

$$\phi_s = 6.843°$$

$$Q = (2 \text{ kN}) \tan 3.806°$$

$$= 133.1 \text{ N}$$



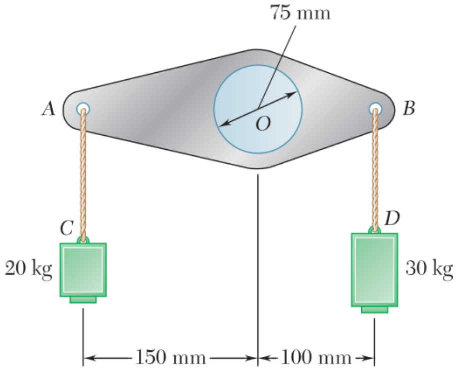
$$\text{Torque at } B = Qr$$

$$= (133.1 \text{ N})(0.006 \text{ m})$$

$$= 0.79860 \text{ N}\cdot\text{m}$$

$$\text{Total torque } = 2.0848 \text{ N}\cdot\text{m} + 0.79860 \text{ N}\cdot\text{m}$$

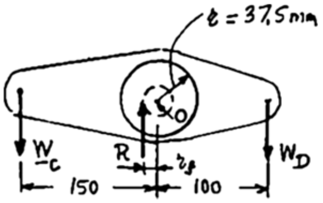
Total torque $= 2.89 \text{ N}\cdot\text{m}$ ◀


75 mm
A
B
O
C
D
20 kg
30 kg
150 mm
100 mm

PROBLEM 8.77

A lever of negligible weight is loosely fitted onto a 75-mm-diameter fixed shaft. It is observed that the lever will just start rotating if a 3-kg mass is added at C. Determine the coefficient of static friction between the shaft and the lever.

SOLUTION


r = 37.5 mm
O
W_C
R
r_f
W_D
150
100

$$+\circlearrowleft \Sigma M_O = 0: \quad W_C(150) - W_D(100) - Rr_f = 0$$

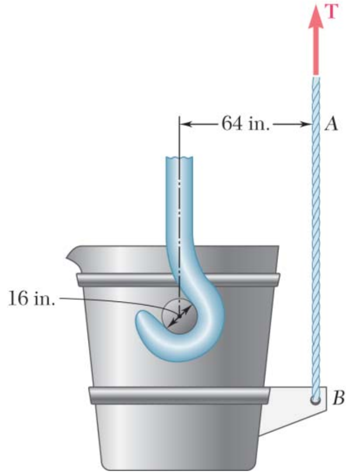
But
$$W_C = (23\text{ kg})(9.81\text{ m/s}^2)$$
$$W_D = (30\text{ kg})(9.81\text{ m/s}^2)$$
$$R = W_C + W_D = (53\text{ kg})(9.81)$$

Thus, after dividing by 9.81,

$$23(150) - 30(100) - 53\,r_f = 0$$
$$r_f = 8.49\text{ mm}$$

But
$$\mu_s \approx \frac{r_f}{r} = \frac{8.49\text{ mm}}{37.5\text{ mm}}$$

$\mu_s \approx 0.226$ ◀

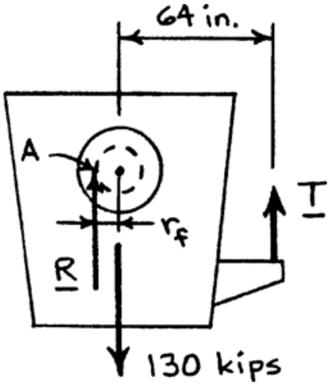


PROBLEM 8.78

A hot-metal ladle and its contents weigh 130 kips. Knowing that the coefficient of static friction between the hooks and the pinion is 0.30, determine the tension in cable *AB* required to start tipping the ladle.

SOLUTION

Free body: Ladle



$$\sin\phi_s \approx \tan\phi_s = \mu_s = 0.30$$

$$r_{\text{bearing}} = 8 \text{ in.}$$

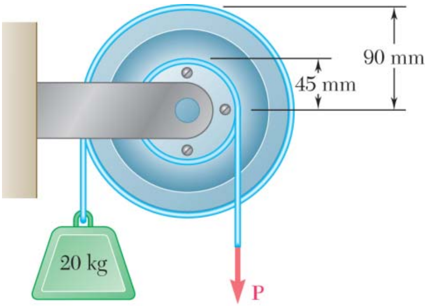
$$r_f = r_{\text{bearing}} \sin\phi_s = (8 \text{ in.})(0.30) = 2.4 \text{ in.}$$

R is tangent to friction circle at *A*.

$$+\circlearrowleft \Sigma M_A = 0: \quad T(64 \text{ in.} + r_f) - (130 \text{ kips})r_f = 0$$

$$T(64 + 2.4) - (130)(2.4) = 0$$

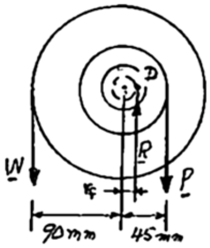
$$T = 4.70 \text{ kips} \blacktriangleleft$$



PROBLEM 8.79

The double pulley shown is attached to a 10-mm-radius shaft that fits loosely in a fixed bearing. Knowing that the coefficient of static friction between the shaft and the poorly lubricated bearing is 0.40, determine the magnitude of the force **P** required to start raising the load.

SOLUTION



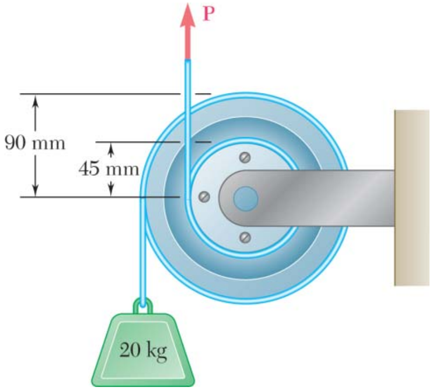
$$+\circlearrowright \Sigma M_D = 0: \quad P(45 - r_f) - W(90 + r_f) = 0$$

$$P = W\frac{90 + r_f}{45 - r_f}$$

$$= (196.2\text{ N})\frac{90\text{ mm} + 4\text{ mm}}{45\text{ mm} - 4\text{ mm}}$$

$$P = 449.82\text{ N}$$

$P = 450$ N ◀

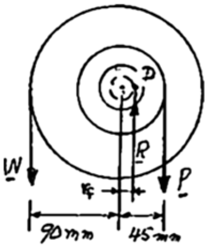


PROBLEM 8.80

The double pulley shown is attached to a 10-mm-radius shaft that fits loosely in a fixed bearing. Knowing that the coefficient of static friction between the shaft and the poorly lubricated bearing is 0.40, determine the magnitude of the force **P** required to start raising the load.

SOLUTION

Find **P** required to start raising load



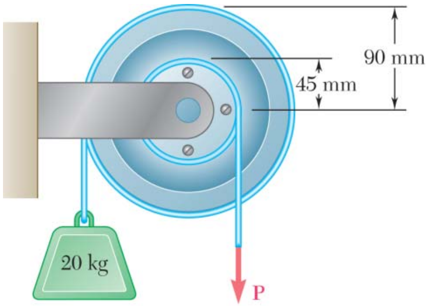
$$+\circlearrowright \Sigma M_D = 0: \quad P(45 - r_f) - W(90 - r_f) = 0$$

$$P = W\frac{90 - r_f}{45 - r_f}$$

$$= (196.2\text{ N})\frac{90\text{ mm} - 4\text{ mm}}{45\text{ mm} - 4\text{ mm}}$$

$$P = 411.54\text{ N}$$

$P = 412$ N ◀

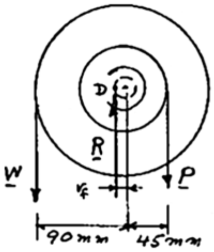


PROBLEM 8.81

The double pulley shown is attached to a 10-mm-radius shaft that fits loosely in a fixed bearing. Knowing that the coefficient of static friction between the shaft and the poorly lubricated bearing is 0.40, determine the magnitude of the smallest force **P** required to maintain equilibrium.

SOLUTION

Find smallest **P** to maintain equilibrium



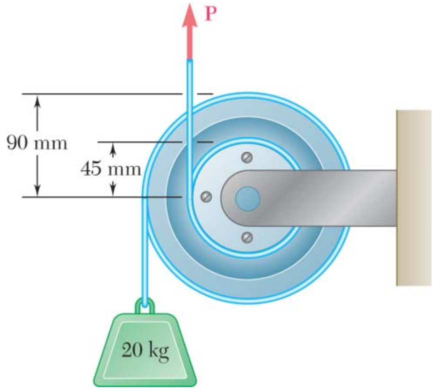
$$+\circlearrowright \Sigma M_D = 0: \quad P(45 + r_f) - W(90 - r_f) = 0$$

$$P = W\frac{90 - r_f}{45 + r_f}$$

$$= (196.2\text{ N})\frac{90\text{ mm} - 4\text{ mm}}{45\text{ mm} + 4\text{ mm}}$$

$$P = 344.35\text{ N}$$

$P = 344$ N ◀

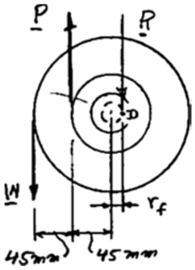


PROBLEM 8.82

The double pulley shown is attached to a 10-mm-radius shaft that fits loosely in a fixed bearing. Knowing that the coefficient of static friction between the shaft and the poorly lubricated bearing is 0.40, determine the magnitude of the smallest force **P** required to maintain equilibrium.

SOLUTION

Find smallest **P** to maintain equilibrium

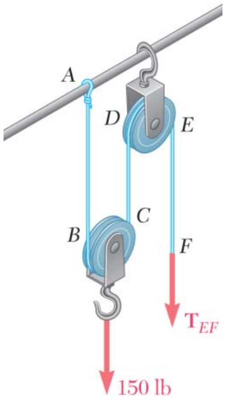


$$+\circlearrowleft \Sigma M_D = 0: \quad P(45 + r_f) - W(90 + r_f) = 0$$

$$P = W\frac{90 + r_f}{45 + r_f}$$

$$= (196.2 \text{ N})\frac{90 \text{ mm} + 4 \text{ mm}}{45 \text{ mm} + 4 \text{ mm}}$$

$$P = 376.38 \text{ N} \qquad\qquad P = 376 \text{ N} \blacktriangleleft$$



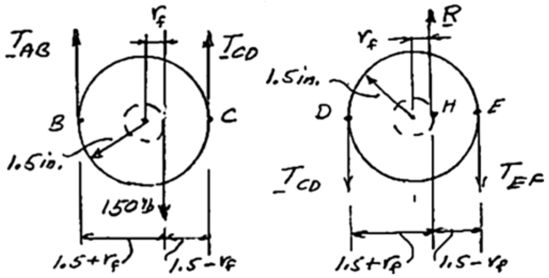
PROBLEM 8.83

The block and tackle shown are used to raise a 150-lb load. Each of the 3-in.-diameter pulleys rotates on a 0.5-in.-diameter axle. Knowing that the coefficient of static friction is 0.20, determine the tension in each portion of the rope as the load is slowly raised.

SOLUTION

For each pulley: Axle diameter $= 0.5$ in.

$$r_f = r \sin \phi_s \approx \mu_s r = 0.20\left(\frac{0.5\text{ in.}}{2}\right) = 0.05\text{ in.}$$



Pulley *BC*:

$$+\circlearrowleft \Sigma M_B = 0: \quad T_{CD}(3\text{ in.}) - (150\text{ lb})(1.5\text{ in.} + r_f) = 0$$

$$T_{CD} = \frac{1}{3}(150\text{ lb})(1.5\text{ in.} + 0.05\text{ in.}) \qquad T_{CD} = 77.5\text{ lb} ◀$$

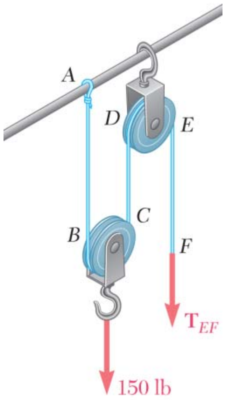
$$+\uparrow \Sigma F_y = 0: \quad T_{AB} + 77.5\text{ lb} - 150\text{ lb} = 0 \qquad T_{AB} = 72.5\text{ lb} ◀$$

Pulley *DE*:

$$+\circlearrowleft \Sigma M_B = 0: \quad T_{CD}(1.5 + r_f) - T_{EF}(1.5 - r_f) = 0$$

$$T_{EF} = T_{CD}\frac{1.5 + r_f}{1.5 - r_f}$$

$$= (77.5\text{ lb})\frac{1.5\text{ in.} + 0.05\text{ in.}}{1.5\text{ in.} - 0.05\text{ in.}} \qquad T_{EF} = 82.8\text{ lb} ◀$$


A
D
E
C
B
F
T_{EF}
150 lb

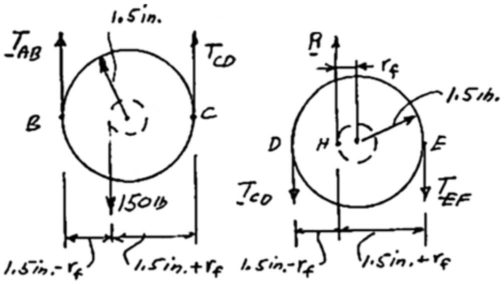
PROBLEM 8.84

The block and tackle shown are used to lower a 150-lb load. Each of the 3-in.-diameter pulleys rotates on a 0.5-in.-diameter axle. Knowing that the coefficient of static friction is 0.20, determine the tension in each portion of the rope as the load is slowly lowered.

SOLUTION

For each pulley:

$$r_f = r\mu_s = \left(\frac{0.5\text{ in.}}{2}\right)0.2 = 0.05\text{ in.}$$


T_{AB}
1.5 in.
T_{CD}
B
C
150 lb
1.5 in. − r_f
1.5 in. + r_f
R
r_f
1.5 in.
D
H
E
T_{CD}
T_{EF}
1.5 in. − r_f
1.5 in. + r_f

Pulley *BC*:

$$+\circlearrowleft \Sigma M_B = 0: \quad T_{CD}(3\text{ in.}) - (150\text{ lb})(1.5\text{ in.} - r_f) = 0$$

$$T_{CD} = \frac{(150\text{ lb})(1.5\text{ in.} - 0.05\text{ in.})}{3\text{ in.}}$$

$T_{CD} = 72.5\text{ lb}$ ◀

$$+\!\uparrow \Sigma F_y = 0: \quad T_{AB} + 72.5\text{ lb} - 150\text{ lb} = 0$$

$T_{AB} = 77.5\text{ lb}$ ◀

Pulley *DE*:

$$T_{CD}(1.5\text{ in.} - r_f) - T_{EF}(1.5\text{ in.} + r_f) = 0$$

$$T_{EF} = T_{CD}\frac{1.5\text{ in.} - r_f}{1.5\text{ in.} + r_f}$$

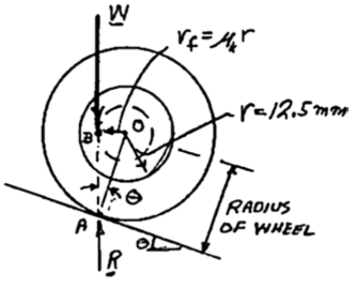
$$= (72.5\text{ lb})\frac{1.5\text{ in.} - 0.05\text{ in.}}{1.5\text{ in.} + 0.05\text{ in.}}$$

$T_{EF} = 67.8\text{ lb}$ ◀

PROBLEM 8.85

A scooter is to be designed to roll down a 2 percent slope at a constant speed. Assuming that the coefficient of kinetic friction between the 25-mm-diameter axles and the bearings is 0.10, determine the required diameter of the wheels. Neglect the rolling resistance between the wheels and the ground.

SOLUTION



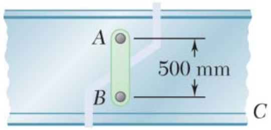
$$\tan\theta = \frac{2}{100} = 0.02$$

Since a scooter rolls at constant speed, each wheel is in equilibrium. Thus, **W** and **R** must have a common line of action tangent to the friction circle.

$$r_f = \mu_k r = (0.10)(12.5\text{ mm})$$
$$= 1.25\text{ mm}$$

$$OA = \frac{OB}{\tan\theta} = \frac{r_f}{\tan\theta} = \frac{1.25\text{ mm}}{0.02}$$
$$= 62.5\text{ mm}$$

$$\text{Diameter of wheel} = 2(OA) = 125.0\text{ mm} \blacktriangleleft$$

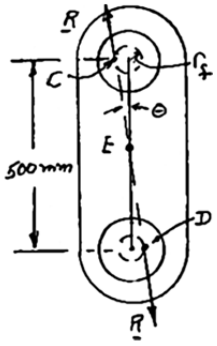


PROBLEM 8.86

The link arrangement shown is frequently used in highway bridge construction to allow for expansion due to changes in temperature. At each of the 60-mm-diameter pins *A* and *B* the coefficient of static friction is 0.20. Knowing that the vertical component of the force exerted by *BC* on the link is 200 kN, determine (*a*) the horizontal force that should be exerted on beam *BC* to just move the link, (*b*) the angle that the resulting force exerted by beam *BC* on the link will form with the vertical.

SOLUTION

Bearing:

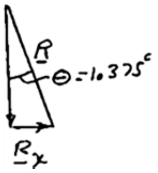


$$r = 30 \text{ mm}$$

$$r_f = \mu_s r$$
$$= 0.20(30 \text{ mm})$$
$$= 6 \text{ mm}$$

Resultant forces **R** must be tangent to friction circles at Points *C* and *D*.

(*a*)

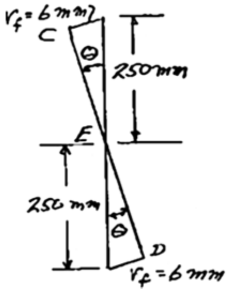


$$R_y = \text{Vertical component} = 200 \text{ kN}$$

$$R_x = R_y \tan\theta$$
$$= (200 \text{ kN}) \tan 1.375°$$
$$= 4.80 \text{ kN}$$

Horizontal force = 4.80 kN ◀

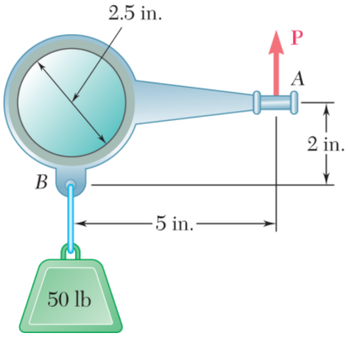
(*b*)



$$\sin\theta = \frac{6 \text{ mm}}{250 \text{ mm}}$$

$$\sin\theta = 0.024$$

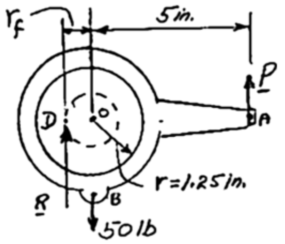
$\theta = 1.375°$ ◀



PROBLEM 8.87

A lever *AB* of negligible weight is loosely fitted onto a 2.5-in.-diameter fixed shaft. Knowing that the coefficient of static friction between the fixed shaft and the lever is 0.15, determine the force **P** required to start the lever rotating counterclockwise.

SOLUTION



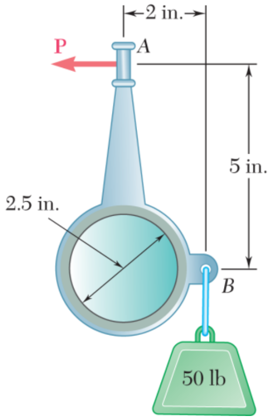
$$\begin{aligned} r_f &= \mu_s r \\ &= 0.15(1.25\text{ in.}) \\ &= 0.1875\text{ in.} \end{aligned}$$

$$+\circlearrowleft \Sigma M_D = 0: \quad P(5\text{ in.} + r_f) - (50\text{ lb})r_f = 0$$

$$P = \frac{50(0.1875)}{5.1875}$$

$$= 1.807\text{ lb}$$

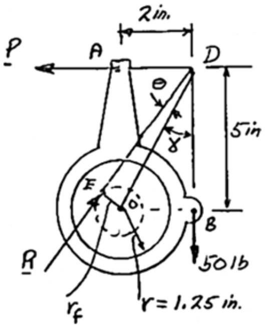
$\mathbf{P} = 1.807\text{ lb}\uparrow$ ◀


2 in.
P
A
5 in.
2.5 in.
B
50 lb

PROBLEM 8.88

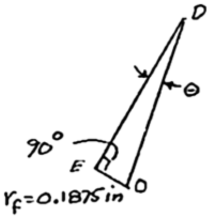
A lever *AB* of negligible weight is loosely fitted onto a 2.5-in.-diameter fixed shaft. Knowing that the coefficient of static friction between the fixed shaft and the lever is 0.15, determine the force **P** required to start the lever rotating counterclockwise.

SOLUTION


2in.
P
A
D
θ
γ
5in
E
O
B
R
50lb
r_f
r = 1.25 in.

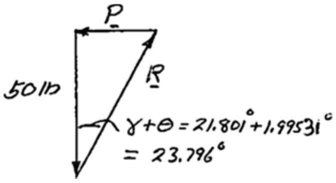
$$r_f = \mu_s r$$
$$= 0.15(1.25 \text{ in.})$$
$$r_f = 0.1875 \text{ in.}$$
$$\tan\gamma = \frac{2 \text{ in.}}{5 \text{ in.}}$$
$$\gamma = 21.801°$$

In ΔEOD:


D
θ
90°
E
r_f=0.1875 in
O

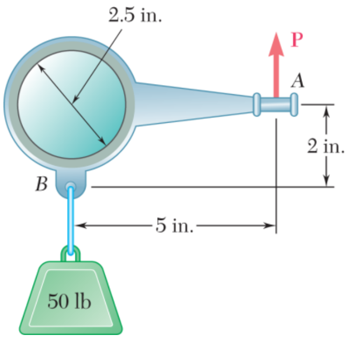
$$OD = \sqrt{(2 \text{ in.})^2 + (5 \text{ in.})^2}$$
$$= 5.3852 \text{ in.}$$
$$\sin\theta = \frac{OE}{OD} = \frac{r_f}{OD}$$
$$= \frac{0.1875 \text{ in.}}{5.3852 \text{ in.}}$$
$$\theta = 1.99531°$$

Force triangle:


P
50lb
R
γ+θ = 21.801°+1.99531°
= 23.796°

$$P = (50 \text{ lb})\tan(\gamma + \theta)$$
$$= (50 \text{ lb})\tan 23.796°$$
$$= 22.049 \text{ lb}$$

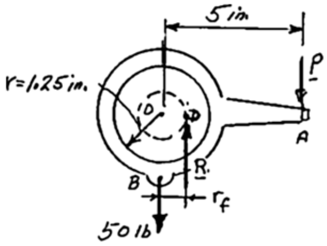
P = 22.0 lb ←◀



PROBLEM 8.89

A lever *AB* of negligible weight is loosely fitted onto a 2.5-in.-diameter fixed shaft. Knowing that the coefficient of static friction between the fixed shaft and the lever is 0.15, determine the force **P** required to start the lever rotating clockwise.

SOLUTION



$$r_f = \mu_s r$$

$$= 0.15(1.25\text{ in.})$$

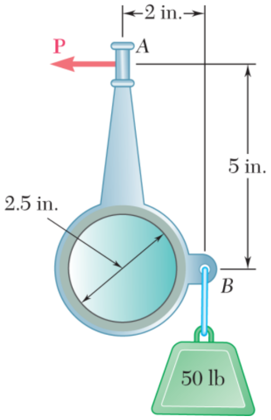
$$r_f = 0.1875\text{ in.}$$

$$+\circlearrowright \Sigma M_D = 0: \quad P(5\text{ in.} - r_f) - (50\text{ lb})r_f = 0$$

$$P = \frac{50(0.1875)}{5 - 0.1875}$$

$$= 1.948\text{ lb}$$

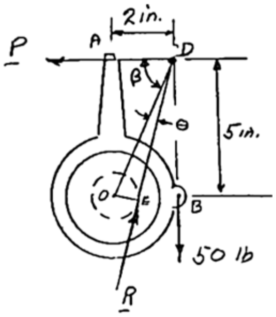
$\mathbf{P} = 1.948\text{ lb}\downarrow$ ◀


2 in.
P
A
5 in.
2.5 in.
B
50 lb

PROBLEM 8.90

A lever *AB* of negligible weight is loosely fitted onto a 2.5-in.-diameter fixed shaft. Knowing that the coefficient of static friction between the fixed shaft and the lever is 0.15, determine the force **P** required to start the lever rotating clockwise.

SOLUTION


2 in.
P
A
D
β
θ
5 in.
O
E
B
50 lb
R

$$r_f = \mu_s r$$

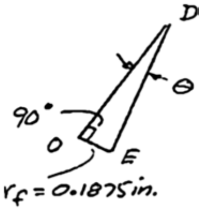
$$= 0.15(1.25\text{ in.})$$

$$= 0.1875\text{ in.}$$

$$\tan\beta = \frac{5\text{ in.}}{2\text{ in.}}$$

$$\beta = 68.198°$$

In ΔEOD:


D
θ
90°
O
E
$r_f = 0.1875$ in.

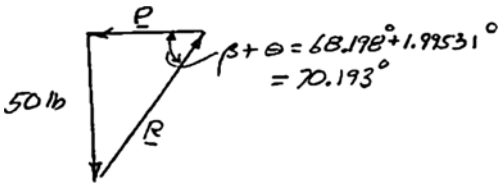
$$OD = \sqrt{(2\text{ in.})^2 + (5\text{ in.})^2}$$

$$OD = 5.3852\text{ in.}$$

$$\sin\theta = \frac{OE}{OD} = \frac{0.1875\text{ in.}}{5.3852\text{ in.}}$$

$$\theta = 1.99531°$$

Force triangle:


P
$\beta+\theta = 68.198° + 1.99531°$
$= 70.193°$
50 lb
R

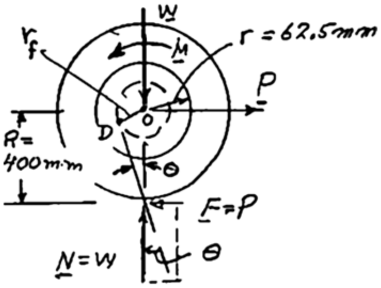
$$P = \frac{50}{\tan(\beta+\theta)} = \frac{50\text{ lb}}{\tan 70.193°}$$

P = 18.01 lb ◀

PROBLEM 8.91

A loaded railroad car has a mass of 30 Mg and is supported by eight 800-mm-diameter wheels with 125-mm-diameter axles. Knowing that the coefficients of friction are $\mu_s = 0.020$ and $\mu_k = 0.015$, determine the horizontal force required (*a*) to start the car moving, (*b*) to keep the car moving at a constant speed. Neglect rolling resistance between the wheels and the track.

SOLUTION



$$r_f = \mu r; \quad R = 400 \text{ mm}$$

$$\sin\theta = \tan\theta = \frac{r_f}{R} = \frac{\mu r}{R}$$

$$P = W\tan\theta = W\frac{\mu r}{R}$$

$$P = W\mu\frac{62.5 \text{ mm}}{400 \text{ mm}}$$

$$= 0.15625W\mu$$

For one wheel:

$$W = \frac{1}{8}(30 \text{ mg})(9.81 \text{ m/s}^2)$$

$$= \frac{1}{8}(294.3 \text{ kN})$$

For eight wheels of railroad car:

$$\Sigma P = 8(0.15625)\frac{1}{8}(294.3 \text{ kN})\mu$$

$$= (45.984\mu) \text{ kN}$$

(*a*) To start motion:

$$\mu_s = 0.020$$

$$\Sigma P = (45.984)(0.020)$$

$$= 0.9197 \text{ kN}$$

$\Sigma P = 920$ N ◀

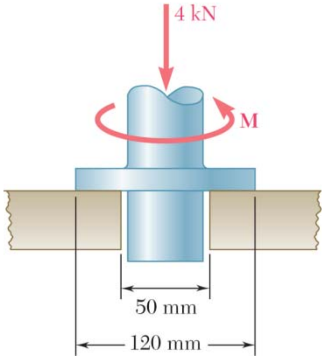
(*b*) To maintain motion:

$$\mu_k = 0.015$$

$$\Sigma P = (45.984)(0.015)$$

$$= 0.6897 \text{ kN}$$

$\Sigma P = 690$ N ◀



PROBLEM 8.92

Knowing that a couple of magnitude $30\ \text{N}\cdot\text{m}$ is required to start the vertical shaft rotating, determine the coefficient of static friction between the annular surfaces of contact.

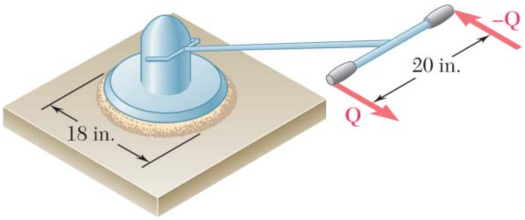
SOLUTION

For annular contact regions, use Equation 8.8 with impending slipping:

$$M = \frac{2}{3}\mu_s N \frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}$$

So,
$$30\ \text{N}\cdot\text{m} = \frac{2}{3}\mu_s (4000\ \text{N})\frac{(0.06\ \text{m})^3 - (0.025\ \text{m})^3}{(0.06\ \text{m})^2 - (0.025\ \text{m})^2}$$

$$\mu_s = 0.1670 \blacktriangleleft$$



PROBLEM 8.93

A 50-lb electric floor polisher is operated on a surface for which the coefficient of kinetic friction is 0.25. Assuming that the normal force per unit area between the disk and the floor is uniformly distributed, determine the magnitude Q of the horizontal forces required to prevent motion of the machine.

SOLUTION

See Figure 8.12 and Eq. (8.9).

Using:

$$R = 9\text{ in.}$$
$$P = 50\text{ lb}$$

and

$$\mu_k = 0.25$$
$$M = \frac{2}{3}\mu_k PR = \frac{2}{3}(0.25)(50\text{ lb})(9\text{ in.})$$
$$= 75\text{ lb}\cdot\text{in.}$$

$\Sigma M_y = 0$ yields:

$$M = Q(20\text{ in.})$$
$$75\text{ lb}\cdot\text{in.} = Q(20\text{ in.})$$

$Q = 3.75\text{ lb}$ ◀

PROBLEM 8.94*

The frictional resistance of a thrust bearing decreases as the shaft and bearing surfaces wear out. It is generally assumed that the wear is directly proportional to the distance traveled by any given point of the shaft and thus to the distance r from the point to the axis of the shaft. Assuming, then, that the normal force per unit area is inversely proportional to r, show that the magnitude M of the couple required to overcome the frictional resistance of a worn-out end bearing (with contact over the full circular area) is equal to 75 percent of the value given by Eq. (8.9) for a new bearing.

SOLUTION

Using Figure 8.12, we assume

$$\Delta N = \frac{k}{r}\Delta A: \quad \Delta A = r\Delta\theta\,\Delta r$$

$$\Delta N = \frac{k}{r} r\Delta\theta\Delta r = k\Delta\theta\Delta r$$

We write

$$P = \Sigma\Delta N \quad \text{or} \quad P = \int dN$$

$$P = \int_0^{2\pi}\int_0^R k\Delta\theta\Delta r = 2\pi Rk; \quad k = \frac{P}{2\pi R}$$

$$\Delta N = \frac{P\Delta\theta\Delta r}{2\pi R}$$

$$\Delta M = r\Delta F = r\mu_k \Delta N = r\mu_k \frac{P\Delta\theta\Delta r}{2\pi R}$$

$$M = \int_0^{2\pi}\int_0^R \frac{\mu_k P}{2\pi R} rdrd\theta = \frac{2\pi\mu_k P}{2\pi R}\cdot\frac{R^2}{2} = \frac{1}{2}\mu_k PR$$

From Eq. (8.9) for a new bearing,

$$M_{\text{new}} = \frac{2}{3}\mu_k PR$$

Thus,

$$\frac{M}{M_{\text{new}}} = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}$$

$M = 0.75M_{\text{new}}$ ◀

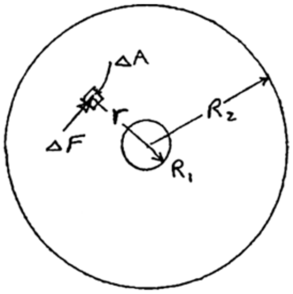
PROBLEM 8.95*

Assuming that bearings wear out as indicated in Problem 8.94, show that the magnitude M of the couple required to overcome the frictional resistance of a worn-out collar bearing is

$$M = \tfrac{1}{2}\mu_k P(R_1 + R_2)$$

where P= magnitude of the total axial force
R_1, R_2= inner and outer radii of collar

SOLUTION



Let normal force on ΔA be ΔN, and $\dfrac{\Delta N}{\Delta A} = \dfrac{k}{r}$.

As in the text, $\Delta F = \mu \Delta N \quad \Delta M = r\Delta F$

The total normal force P is

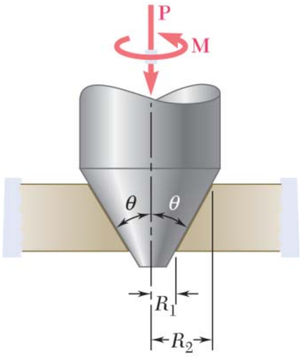
$$P = \lim_{\Delta A \to 0} \Sigma \Delta N = \int_0^{2\pi}\left(\int_{R_1}^{R_2} \frac{k}{r} r\,dr\right) d\theta$$

$$P = 2\pi \int_{R_1}^{R_2} k\,dr = 2\pi k(R_2 - R_1) \quad \text{or} \quad k = \frac{P}{2\pi(R_2 - R_1)}$$

Total couple: $$M_{\text{worn}} = \lim_{\Delta A \to 0} \Sigma \Delta M = \int_0^{2\pi}\left(\int_{R_1}^{R_2} r\mu \frac{k}{r} r\,dr\right) d\theta$$

$$M_{\text{worn}} = 2\pi\mu k \int_{R_1}^{R_2} (r\,dr) = \pi\mu k\left(R_2^2 - R_1^2\right) = \frac{\pi\mu P\left(R_2^2 - R_1^2\right)}{2\pi(R_2 - R_1)}$$

$$M_{\text{worn}} = \frac{1}{2}\mu P(R_2 + R_1) \blacktriangleleft$$



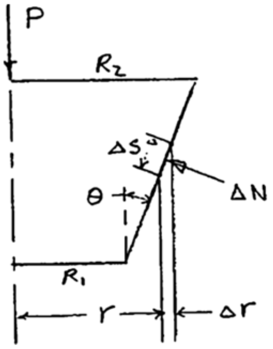
PROBLEM 8.96*

Assuming that the pressure between the surfaces of contact is uniform, show that the magnitude M of the couple required to overcome frictional resistance for the conical bearing shown is

$$M = \frac{2}{3}\frac{\mu_k P}{\sin\theta}\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}$$

SOLUTION

Let normal force on ΔA be ΔN and $\dfrac{\Delta N}{\Delta A} = k.$



So $\qquad \Delta N = k\Delta A \qquad \Delta A = r\Delta s\Delta\phi \qquad \Delta s = \dfrac{\Delta r}{\sin\theta}$

where ϕ is the azimuthal angle around the symmetry axis of rotation.

$$\Delta F_y = \Delta N\sin\theta = kr\Delta r\Delta\phi$$

Total vertical force: $\qquad P = \lim\limits_{\Delta A\to 0}\Sigma\Delta F_y$

$$P = \int_0^{2\pi}\left(\int_{R_1}^{R_2} krdr\right)d\phi = 2\pi k\int_{R_1}^{R_2} rdr$$

$$P = \pi k\left(R_2^2 - R_1^2\right) \quad \text{or} \quad k = \frac{P}{\pi\left(R_2^2 - R_1^2\right)}$$

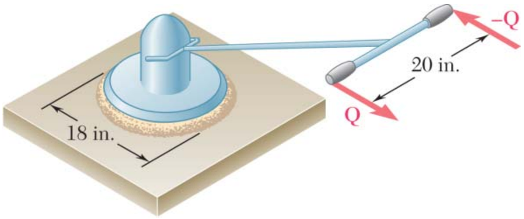
Friction force: $\qquad \Delta F = \mu\Delta N = \mu k\Delta A$

Moment: $\qquad \Delta M = r\Delta F = r\mu kr\dfrac{\Delta r}{\sin\theta}\Delta\phi$

Total couple: $\qquad M = \lim\limits_{\Delta A\to 0}\Sigma\Delta M = \displaystyle\int_0^{2\pi}\left(\int_{R_1}^{R_2}\frac{\mu k}{\sin\theta}r^2dr\right)d\phi$

$$M = 2\pi\frac{\mu k}{\sin\theta}\int_{R_1}^{R_2} r^2dr = \frac{2}{3}\frac{\pi\mu}{\sin\theta}\frac{P}{\pi\left(R_2^2 - R_3^2\right)}\left(R_2^3 - R_3^3\right)$$

$$M = \frac{2}{3}\frac{\mu P}{\sin\theta}\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2} \blacktriangleleft$$


–Q
20 in.
Q
18 in.

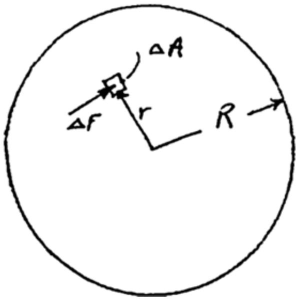
PROBLEM 8.97

Solve Problem 8.93 assuming that the normal force per unit area between the disk and the floor varies linearly from a maximum at the center to zero at the circumference of the disk.

PROBLEM 8.93 A 50-lb electric floor polisher is operated on a surface for which the coefficient of kinetic friction is 0.25. Assuming that the normal force per unit area between the disk and the floor is uniformly distributed, determine the magnitude Q of the horizontal forces required to prevent motion of the machine.

SOLUTION

Let normal force on ΔA be ΔN and $\dfrac{\Delta N}{\Delta A} = k\left(1-\dfrac{r}{R}\right)$.



$$\Delta F = \mu \Delta N = \mu k\left(1-\frac{r}{R}\right)\Delta A = \mu k\left(1-\frac{r}{R}\right) r\Delta r\Delta\theta$$

$$P = \lim_{\Delta A\to 0} \Sigma\Delta N = \int_0^{2\pi}\left[\int_0^R k\left(1-\frac{r}{R}\right) rdr\right]d\theta$$

$$P = 2\pi k\int_0^R \left(1-\frac{r}{R}\right) rdr = 2\pi k\left(\frac{R^2}{2}-\frac{R^3}{3R}\right)$$

$$P = \frac{1}{3}\pi kR^2 \quad \text{or} \quad k = \frac{3P}{\pi R^2}$$

$$M = \lim_{\Delta A\to 0} \Sigma r\Delta F = \int_0^{2\pi}\left[\int_0^R r\mu k\left(1-\frac{r}{R}\right) rdr\right]d\theta$$

$$= 2\pi\mu k\int_0^R \left(r^2-\frac{r^3}{R}\right)dr$$

$$= 2\pi\mu k\left(\frac{R^3}{3}-\frac{R^4}{4R}\right) = \frac{1}{6}\pi\mu kR^3$$

$$= \frac{\pi\mu}{6}\frac{3P}{\pi R^2}R^3 = \frac{1}{2}\mu PR$$

where $\mu = \mu_k = 0.25 \quad R = 9 \text{ in.}$

PROBLEM 8.97 (Continued)

$$P = W = 50 \text{ lb}$$

Then $$M = \frac{1}{2}(0.25)(50 \text{ lb})(9 \text{ in.})$$
$$= 56.250 \text{ lb}\cdot\text{in.}$$

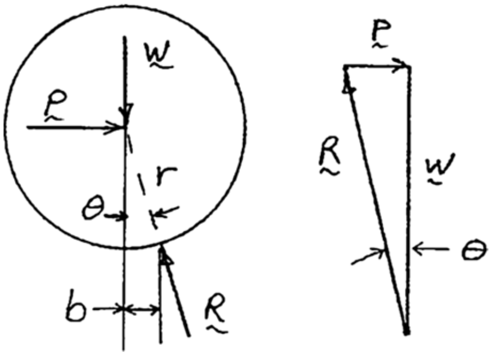
Finally $$Q = \frac{M}{d} = \frac{56.250 \text{ lb}\cdot\text{in.}}{20 \text{ in.}}$$ $Q = 2.81$ lb ◀

PROBLEM 8.98

Determine the horizontal force required to move a 2500-lb automobile with 23-in.-diameter tires along a horizontal road at a constant speed. Neglect all forms of friction except rolling resistance, and assume the coefficient of rolling resistance to be 0.05 in.

SOLUTION

FBD wheel:



$$r = 11.5 \text{ in.}$$

$$b = 0.05 \text{ in.}$$

$$\theta = \sin^{-1}\frac{b}{r}$$

$P = W\tan\theta = W\tan\left(\sin^{-1}\frac{b}{r}\right)$ for each wheel, so for total

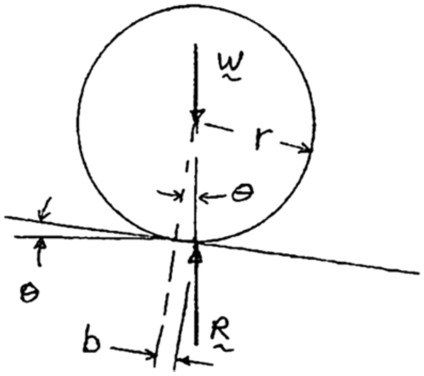
$$P = 2500 \text{ lb}\tan\left(\sin^{-1}\frac{0.05}{11.5}\right) \qquad P = 10.87 \text{ lb} ◀$$

PROBLEM 8.99

Knowing that a 6-in.-diameter disk rolls at a constant velocity down a 2 percent incline, determine the coefficient of rolling resistance between the disk and the incline.

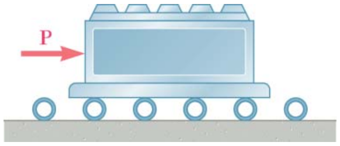
SOLUTION

FBD disk:



$$\tan\theta = \text{slope} = 0.02$$

$$b = r\tan\theta = (3\text{ in.})(0.02) \qquad b = 0.0600\text{ in.} \blacktriangleleft$$

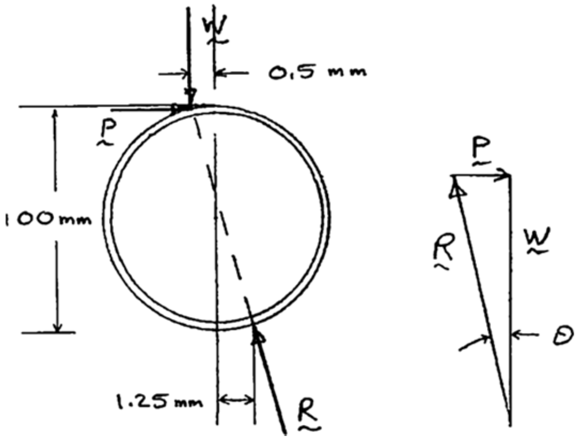


PROBLEM 8.100

A 900-kg machine base is rolled along a concrete floor using a series of steel pipes with outside diameters of 100 mm. Knowing that the coefficient of rolling resistance is 0.5 mm between the pipes and the base and 1.25 mm between the pipes and the concrete floor, determine the magnitude of the force **P** required to slowly move the base along the floor.

SOLUTION

FBD pipe:



$$W = mg$$
$$= (900\text{ kg})(9.81\text{ m/s}^2)$$
$$= 8829.0\text{ N}$$
$$\theta = \sin^{-1}\frac{.5\text{ mm} + 1.25\text{ mm}}{100\text{ mm}}$$
$$= 1.00273°$$

$P = W\tan\theta$ for each pipe, so also for total

$$P = (8829.0\text{ N})\tan(1.00273°)$$

$P = 154.4$ N ◀

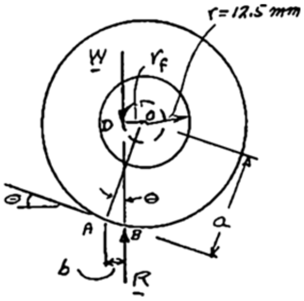
PROBLEM 8.101

Solve Problem 8.85 including the effect of a coefficient of rolling resistance of 1.75 mm.

PROBLEM 8.85 A scooter is to be designed to roll down a 2 percent slope at a constant speed. Assuming that the coefficient of kinetic friction between the 25-mm-diameter axles and the bearings is 0.10, determine the required diameter of the wheels. Neglect the rolling resistance between the wheels and the ground.

SOLUTION

Since the scooter rolls at a constant speed, each wheel is in equilibrium. Thus, **W** and **R** must have a common line of action tangent to the friction circle.



$$a = \text{Radius of wheel}$$

$$\tan\theta = \frac{2}{100} = 0.02$$

Since b and r_f are small compared to a,

$$\tan\theta \approx \frac{r_f + b}{a} = \frac{\mu_k r + b}{a} = 0.02$$

Data: $\mu_k = 0.10, \quad b = 1.75 \text{ mm}, \quad r = 12.5 \text{ mm}$

$$\frac{(0.10)(12.5 \text{ mm}) + 1.75 \text{ mm}}{a} = 0.02$$

$$a = 150 \text{ mm} \qquad \text{Diameter} = 2a = 300 \text{ mm} \blacktriangleleft$$

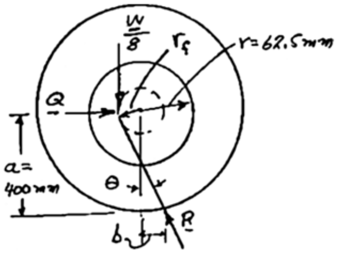
PROBLEM 8.102

Solve Problem 8.91 including the effect of a coefficient of rolling resistance of 0.5 mm.

PROBLEM 8.91 A loaded railroad car has a mass of 30 Mg and is supported by eight 800-mm-diameter wheels with 125-mm-diameter axles. Knowing that the coefficients of friction are $\mu_s = 0.020$ and $\mu_k = 0.015$, determine the horizontal force required (*a*) to start the car moving, (*b*) to keep the car moving at a constant speed. Neglect rolling resistance between the wheels and the track.

SOLUTION

For one wheel:



$$r_f = \mu r$$

$$\tan\theta \approx \sin\theta \approx \frac{r_f + b}{a}$$

$$\tan\theta = \frac{\mu r + b}{a}$$

$$Q = \frac{W}{8}\tan\theta = \frac{W}{8}\frac{\mu r + b}{a}$$

For eight wheels of car:

$$P = W\frac{\mu r + b}{a}$$

$$W = mg = (30 \text{ Mg})(9.81 \text{ m/s}^2) = 294.3 \text{ kN}$$

$$a = 400 \text{ mm}, \quad r = 62.5 \text{ mm}, \quad b = 0.5 \text{ mm}$$

(*a*) To start motion:

$$\mu = \mu_s = 0.02$$

$$P = (294.3 \text{ kN})\frac{(0.020)(62.5 \text{ mm}) + 0.5 \text{ mm}}{400 \text{ mm}}$$

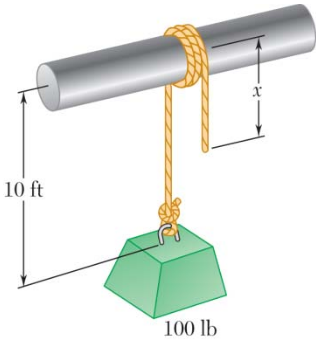
$P = 1.288$ kN ◀

(*b*) To maintain constant speed

$$\mu = \mu_k = 0.015$$

$$P = (294.3 \text{ kN})\frac{(0.015)(62.5 \text{ mm}) + 0.5 \text{ mm}}{400 \text{ mm}}$$

$P = 1.058$ kN ◀



PROBLEM 8.103

A rope having a weight per unit length of 0.4 lb/ft is wound $2\frac{1}{2}$ times around a horizontal rod. Knowing that the coefficient of static friction between the rope and the rod is 0.30, determine the minimum length *x* of rope that should be left hanging if a 100-lb load is to be supported.

SOLUTION

Use eq. (8.14) with

$$T_1 = (0.4 \text{ lb/ft})x$$

$$T_2 = 100 \text{ lb} + (0.4 \text{ lb/ft})(10 \text{ ft}) = 104 \text{ lb}$$

$$\mu_s = 0.30 \qquad \beta = 2.5(2\pi) = 5\pi$$

Then,

$$\frac{T_2}{T_1} = e^{\mu\beta}$$

$$\frac{104}{0.4x} = e^{0.30(5\pi)}$$

$x = 2.34$ ft ◀

PROBLEM 8.104

A hawser is wrapped two full turns around a bollard. By exerting an 80-lb force on the free end of the hawser, a dockworker can resist a force of 5000 lb on the other end of the hawser. Determine (*a*) the coefficient of static friction between the hawser and the bollard, (*b*) the number of times the hawser should be wrapped around the bollard if a 20,000-lb force is to be resisted by the same 80-lb force.

SOLUTION

(*a*)

$$\beta = 2 \text{ turns} = 2(2\pi) = 4\pi$$

$$T_1 = 80 \text{ lb}, \qquad T_2 = 5000 \text{ lb}$$

$$\ln\frac{T_2}{T_1} = \mu_s \beta$$

$$\mu_s = \frac{1}{\beta}\ln\frac{T_2}{T_1} = \frac{1}{4\pi}\ln\frac{5000 \text{ lb}}{80 \text{ lb}}$$

$$\mu_s = \frac{1}{4\pi}\ln 62.5 = \frac{4.1351}{4\pi}$$

$\mu_s = 0.329$ ◀

(*b*)

$$T_1 = 80 \text{ lb}, \quad T_2 = 20{,}000 \text{ lb}, \quad \mu_s = 0.329$$

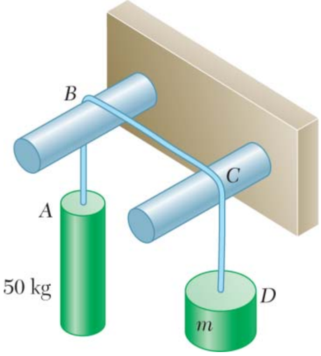
$$\ln\frac{T_2}{T_1} = \mu_s \beta$$

$$\beta = \frac{1}{\mu}\ln\frac{T_2}{T_1} = \frac{1}{0.329}\ln\frac{20{,}000 \text{ lb}}{80 \text{ lb}}$$

$$\beta = \frac{1}{0.329}\ln(250) = \frac{5.5215}{0.329} = 16.783$$

$$\text{Number of turns} = \frac{16.783}{2\pi}$$

Number of turns = 2.67 ◀

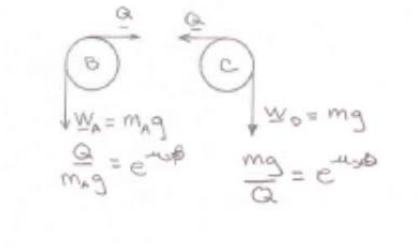


PROBLEM 8.105

Two cylinders are connected by a rope that passes over two fixed rods as shown. Knowing that the coefficient of static friction between the rope and the rods is 0.40, determine the range of the mass *m* of cylinder *D* for which equilibrium is maintained.

SOLUTION

For each rod, $\beta = \frac{\pi}{2}, \quad \mu_s = 0.4$


Q
Q
B
C
$W_A = m_A g$
$W_D = mg$
$\frac{Q}{m_A g} = e^{\mu_s \beta}$
$\frac{mg}{Q} = e^{\mu_s \beta}$

For impending motion of *A* downward,

$$\frac{Q}{m_A g} \cdot \frac{mg}{Q} = e^{\mu_s(\beta+\beta)}: \qquad \frac{m}{m_A} = e^{(0.4)(2)\left(\frac{\pi}{2}\right)} = 3.5136$$

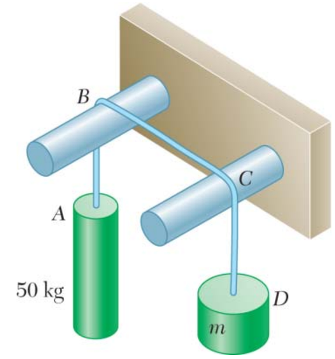
$$m = (50 \text{ kg})(3.5136) = 175.680 \text{ kg}$$

For impending motion of *A* upward,

$$\frac{m_A}{m} = e^{\mu_s(\beta+\beta)} = 3.1536$$

$$m = \frac{50 \text{ kg}}{5.5136} = 14.2304 \text{ kg}$$

Range for equilibrium, $14.23 \text{ kg} \le m \le 175.7 \text{ kg}$ ◀

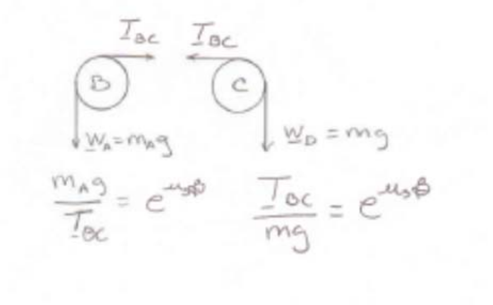


PROBLEM 8.106

Two cylinders are connected by a rope that passes over two fixed rods as shown. Knowing that for cylinder D upward motion impends when $m = 20$ kg, determine (*a*) the coefficient of static friction between the rope and the rods, (*b*) the corresponding tension in portion BC of the rope.

SOLUTION

For each rod, $\beta = \dfrac{\pi}{2}$


T_{BC}
T_{BC}
B
C
$W_A = m_A g$
$W_D = mg$
$\frac{m_A g}{T_{BC}} = e^{\mu_s \beta}$
$\frac{T_{BC}}{mg} = e^{\mu_s \beta}$

(*a*) $$\frac{m_A g}{T_{BC}} \cdot \frac{T_{BC}}{mg} = e^{\mu_s(\beta+\beta)}: \quad \frac{m_A}{m} = e^{\mu_s \pi}$$

$$\frac{50 \text{ kg}}{20 \text{ kg}} = e^{\mu_s \pi}$$
$$0.91629 = \mu_s \pi$$
$$\mu_s = 0.29166$$

$\mu_s = 0.292$ ◀

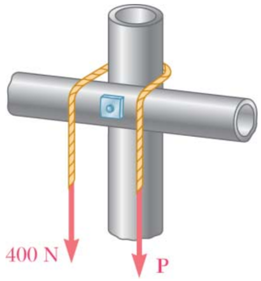
For impending motion of A upward,

$$\frac{m_A}{m} = e^{\mu_s(\beta+\beta)} = 3.1536$$

$$m = \frac{50 \text{ kg}}{5.5136} = 14.2304 \text{ kg}$$

(*b*) $$\frac{T_{BC}}{(20 \text{ kg})g} = e^{(0.29166)\frac{\pi}{2}}$$

$T_{BC} = 310$ N ◀



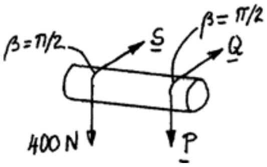
PROBLEM 8.107

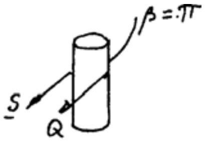
Knowing that the coefficient of static friction is 0.25 between the rope and the horizontal pipe and 0.20 between the rope and the vertical pipe, determine the range of values of *P* for which equilibrium is maintained.

SOLUTION

Horizontal pipe $\mu_s = \mu_h$

Vertical pipe $\mu_s = \mu_v$





For motion of *P* to be impending downward:

$$\frac{P}{Q} = e^{\mu_h(\pi/2)}, \quad \frac{Q}{S} = e^{\mu_v \pi}, \quad \frac{S}{400\text{ N}} = e^{\mu_h(\pi/2)}$$

Multiply equation member by member:

$$\frac{P}{Q}\frac{Q}{S}\frac{S}{400} = e^{(\mu_h + 2\mu_v + \mu_h)(\pi/2)}$$

or:

$$\frac{P}{400\text{ N}} = e^{(\mu_h + \mu_v)\pi} \qquad (1)$$

For motion of **P** to be impending upward, we find in a similar way

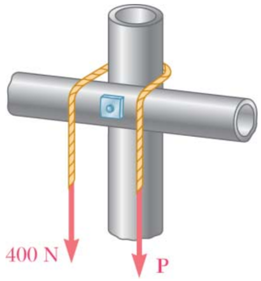
$$\frac{P}{400\text{ N}} = e^{-(\mu_h + \mu_v)\pi} \qquad (2)$$

Given data: $\mu_h = 0.25, \quad \mu_v = 0.20$

From (1): $P = (400\text{ N})e^{0.45\pi} = 1644\text{ N}$

From (2): $P = (400\text{ N})e^{-0.45\pi} = 97.3\text{ N}$

Range for equilibrium: $97.3\text{ N} \leq P \leq 1644\text{ N}$ ◀

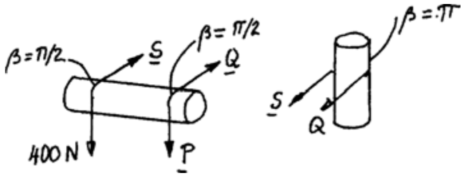


PROBLEM 8.108

Knowing that the coefficient of static friction is 0.30 between the rope and the horizontal pipe and that the smallest value of P for which equilibrium is maintained is 80 N, determine (*a*) the largest value of P for which equilibrium is maintained, (*b*) the coefficient of static friction between the rope and the vertical pipe.

SOLUTION

Horizontal pipe: $\mu_s = \mu_h$ Vertical pipe: $\mu_s = \mu_v$



For motion of P to be impending downward:

$$\frac{P}{Q} = e^{\mu_h(\pi/2)}, \quad \frac{Q}{S} = e^{\mu_v \pi}, \quad \frac{S}{400\text{ N}} = e^{\mu_h(\pi/2)}$$

Multiply equation member by member:

$$\frac{P}{Q}\frac{Q}{S}\frac{S}{400} = e^{(\mu_h + 2\mu_v + \mu_h)(\pi/2)}$$

or:

$$\frac{P}{400\text{ N}} = e^{(\mu_h + \mu_v)\pi} \qquad (1)$$

For motion of **P** to be impending upward, we find in a similar way

$$\frac{P}{400\text{ N}} = e^{-(\mu_h + \mu_v)\pi} \qquad (2)$$

Setting $P = P_{\max}$ in Eq. (1), $P = 80\text{ N}$ in Eq. (2) and $\mu_h = 0.30$ in both, we get

$$\frac{P_{\max}}{400\text{ N}} = e^{(0.30+\mu_v)\pi} \ (1'); \quad \frac{80\text{ N}}{400\text{ N}} = e^{-(0.30+\mu_v)\pi} \ (2')$$

(a) Multiplying $(1')$ by $(2')$:

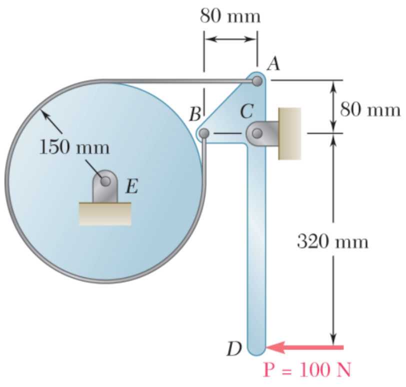
$$\frac{P_{\max}}{400} \cdot \frac{80}{400} = e^0 = 1$$

$P_{\max} = 2000\text{ N}$ ◀

(b) From $(2')$: $(0.30 + \mu_v)\pi = \ln\frac{400}{80} = \ln 5 = 1.60944$

$$0.30 + \mu_v = 0.512$$

$\mu_v = 0.212$ ◀

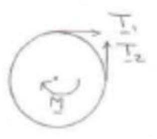


PROBLEM 8.109

A band belt is used to control the speed of a flywheel as shown. Determine the magnitude of the couple being applied to the flywheel, knowing that the coefficient of kinetic friction between the belt and the flywheel is 0.25 and that the flywheel is rotating clockwise at a constant speed. Show that the same result is obtained if the flywheel rotates counterclockwise.

SOLUTION

Free body: Flywheel

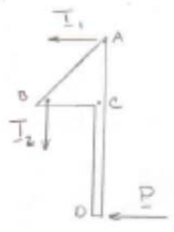


For clockwise rotation of flywheel, $\mathbf{T}_1$ and $\mathbf{T}_2$ are located as shown.

$$\beta = \left(\frac{3}{4}\right)(2\pi) = \frac{3}{2}\pi \text{ radians} \quad \frac{T_2}{T_1} = e^{\mu_k \beta} = e^{0.25\left(\frac{3\pi}{2}\right)} = 3.2482$$

$$T_2 = 3.2482T_1$$

Free body: Handle



$$+\circlearrowleft \Sigma M_C = 0: \quad (T_1 + T_2)(0.08 \text{ m}) - (100 \text{ N})(0.32 \text{ m}) = 0$$

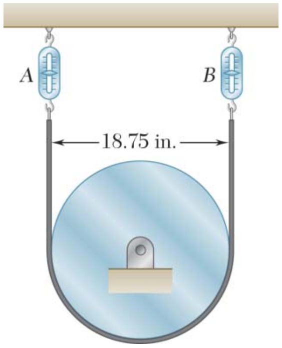
$$T_1 + 3.2482T_1 = 400 \text{ N}$$

$$T_1 = 94.157 \text{ N}, \qquad T_2 = 305.84 \text{ N}$$

From free body: Flywheel

$$+\circlearrowleft \Sigma M_E = 0: \quad M + (T_1 - T_2)(0.15 \text{ m}) = 0$$

$$M + (94.157 \text{ N} - 305.84 \text{ N})(0.15 \text{ m}) = 0 \quad \mathbf{M} = 31.8 \text{ N} \cdot \text{m} \circlearrowleft \blacktriangleleft$$



PROBLEM 8.110

The setup shown is used to measure the output of a small turbine. When the flywheel is at rest, the reading of each spring scale is 14 lb. If a 105-lb · in. couple must be applied to the flywheel to keep it rotating clockwise at a constant speed, determine (*a*) the reading of each scale at that time, (*b*) the coefficient of kinetic friction. Assume that the length of the belt does not change.

SOLUTION

(*a*) Since the length of the belt is constant, the spring in scale *B* will increase in length by $\delta$and the spring in scale *A* will decrease by the same amount. Thus, the sum of the readings in scales *A* and *B* remains constant:

$$T_A + T_B = 14\text{ lb} + 14\text{ lb} \qquad T_A + T_B = 28\text{ lb} \quad (1)$$

On the other hand, the sum of the moments of T_A and T_B about axle must be equal to moment of couple:

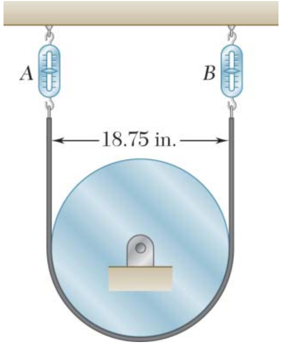
$$(T_B - T_A)(9.375\text{ in.}) = 105\text{ lb}\cdot\text{in.}, \qquad T_B - T_A = 11.2\text{ lb} \quad (2)$$

Solving (1) and (2) simultaneously

$$T_A = 8.40\text{ lb}; \qquad T_B = 19.60\text{ lb} \blacktriangleleft$$

(*b*) Apply Eq. (8.13) with $T_2 = T_B, \quad T_1 = T_A, \ \beta = 180° = \pi$ rad.

$$\ln\frac{T_2}{T_1} = \mu_s \beta: \qquad \mu_s \pi = \ln\frac{19.60}{8.40} = 0.84730 \qquad \mu_s = 0.270 \blacktriangleleft$$



PROBLEM 8.111

The setup shown is used to measure the output of a small turbine. The coefficient of kinetic friction is 0.20 and the reading of each spring scale is 16 lb when the flywheel is at rest. Determine (*a*) the reading of each scale when the flywheel is rotating clockwise at a constant speed, (*b*) the couple that must be applied to the flywheel. Assume that the length of the belt does not change.

SOLUTION

(*a*) Since the length of the belt is constant, the spring in scale *B* will increase in length by $\delta$and the spring in scale *A* will decrease by the same amount. Thus, the sum of the readings in scales *A* and *B* remains constant:

$$T_A + T_B = 16 \text{ lb} + 16 \text{ lb} \qquad T_A + T_B = 32 \text{ lb} \quad (1)$$

We now apply Eq. (8.14) with

$$T_1 = T_A, \quad T_2 = T_B, \quad \mu_k = 0.20, \quad \beta = 180° = \pi \text{ rad.}$$

$$\frac{T_2}{T_1} = e^{\mu_k B}: \quad \frac{T_B}{T_A} = e^{0.20\pi} = 1.87446$$

$$T_B = 1.87446 \; T_A \quad (2)$$

Substituting (2) into (1):

$$T_A + 1.87446 T_A = 32 \text{ lb}$$

$$T_A = 11.1325 \text{ lb} \qquad T_A = 11.13 \text{ lb} \blacktriangleleft$$

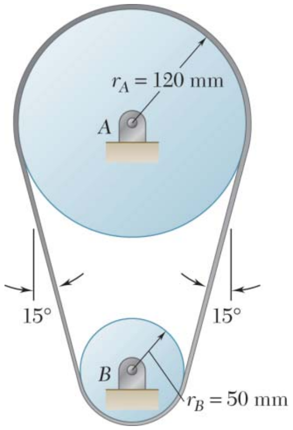
From (1): $T_B = 32 \text{ lb} - 11.1325 \text{ lb}$

$$T_B = 20.868 \text{ lb} \qquad T_B = 20.9 \text{ lb} \blacktriangleleft$$

(*b*) Couple applied to flywheel:

$$M = (T_B - T_A)r$$
$$= (20.868 \text{ lb} - 11.1325 \text{ lb})(9.375 \text{ in.})$$

$$M = 91.3 \text{ lb} \cdot \text{in.} \circlearrowright \blacktriangleleft$$

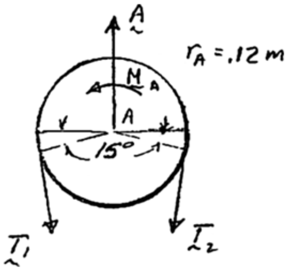


PROBLEM 8.112

A flat belt is used to transmit a couple from drum *B* to drum *A*. Knowing that the coefficient of static friction is 0.40 and that the allowable belt tension is 450 N, determine the largest couple that can be exerted on drum *A*.

SOLUTION

FBD's drums:



$$\beta_A = 180° + 30° = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

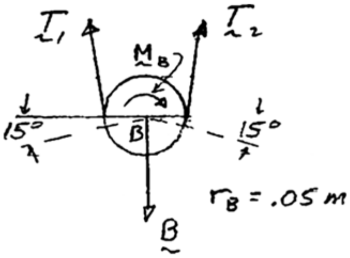
$$\beta_B = 180° - 30° = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Since $\beta_B < \beta_A$, slipping will impend first on *B* (friction coefficients being equal)

So

$$T_2 = T_{\max} = T_1 e^{\mu_s \beta_B}$$

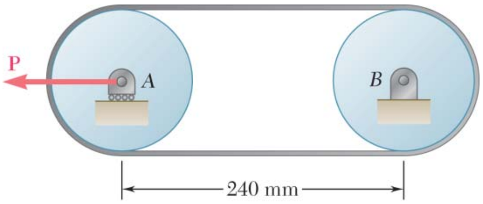
$$450 \text{ N} = T_1 e^{(0.4)5\pi/6} \quad \text{or} \quad T_1 = 157.914 \text{ N}$$



$$\circlearrowleft \Sigma M_A = 0: \quad M_A + (0.12 \text{ m})(T_1 - T_2) = 0$$

$$M_A = (0.12 \text{ m})(450 \text{ N} - 157.914 \text{ N}) = 35.05 \text{ N} \cdot \text{m}$$

$$M_A = 35.1 \text{ N} \cdot \text{m} \blacktriangleleft$$

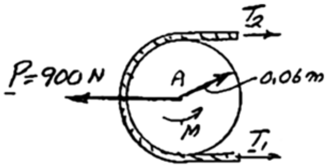


PROBLEM 8.113

A flat belt is used to transmit a couple from pulley *A* to pulley *B*. The radius of each pulley is 60 mm, and a force of magnitude *P*= 900 N is applied as shown to the axle of pulley *A*. Knowing that the coefficient of static friction is 0.35, determine (*a*) the largest couple that can be transmitted, (*b*) the corresponding maximum value of the tension in the belt.

SOLUTION

Drum *A*:



$$\frac{T_2}{T_1} = e^{\mu_s \pi} = e^{(0.35)\pi}$$

$$T_2 = 3.0028 T_1$$

$$\beta = 180° = \pi \text{ radians}$$

(*a*) Torque: $+\circlearrowleft \Sigma M_A = 0: \quad M - (675.15 \text{ N})(0.06 \text{ m}) + (224.84 \text{ N})(0.06 \text{ m})$

$$M = 27.0 \text{ N} \cdot \text{m} \blacktriangleleft$$

(*b*) $\xrightarrow{+} \Sigma F_x = 0: \quad T_1 + T_2 - 900 \text{ N} = 0$

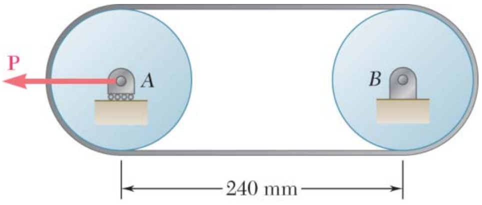
$$T_1 + 3.0028 T_1 - 900 \text{ N} = 0$$

$$4.00282 T_1 = 900$$

$$T_1 = 224.841 \text{ N}$$

$$T_2 = 3.0028(224.841 \text{ N}) = 675.15 \text{ N}$$

$$T_{\max} = 675 \text{ N} \blacktriangleleft$$



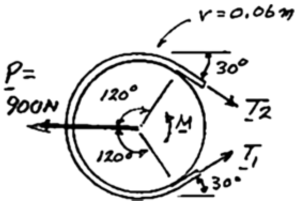
PROBLEM 8.114

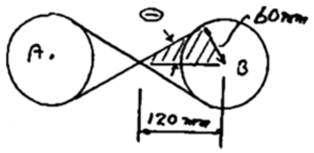
Solve Problem 8.113 assuming that the belt is looped around the pulleys in a figure eight.

PROBLEM 8.113 A flat belt is used to transmit a couple from pulley *A* to pulley *B*. The radius of each pulley is 60 mm, and a force of magnitude *P*= 900 N is applied as shown to the axle of pulley *A*. Knowing that the coefficient of static friction is 0.35, determine (*a*) the largest couple that can be transmitted, (*b*) the corresponding maximum value of the tension in the belt.

SOLUTION

Drum *A*:





$$\beta = 240° = 240°\frac{\pi}{180°} = \frac{4}{3}\pi$$

$$\sin\theta = \frac{60}{120} = \frac{1}{2}$$

$$\theta = 30°$$

$$\frac{T_2}{T_1} = e^{\mu_s \beta} = e^{0.35(4/3\pi)}$$

$$T_2 = 4.3322 T_1$$

(*a*) Torque: $+\circlearrowleft \Sigma M_B = 0: \quad M - (844.3\text{ N})(0.06\text{ m}) + (194.9\text{ N})(0.06\text{ m}) = 0$

$$M = 39.0\text{ N}\cdot\text{m} \blacktriangleleft$$

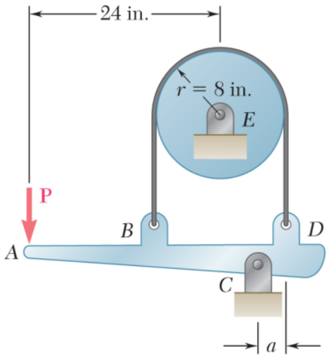
(*b*) $\xrightarrow{+} \Sigma F_x = 0: \quad (T_1 + T_2)\cos 30° - 900\text{ N}$

$$(T_1 + 4.3322T_1)\cos 30° = 900$$

$$T_1 = 194.90\text{ N}$$

$$T_2 = 4.3322(194.90\text{ N}) = 844.3\text{ N}$$

$$T_{\max} = 844\text{ N} \blacktriangleleft$$



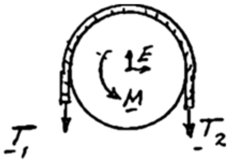
PROBLEM 8.115

The speed of the brake drum shown is controlled by a belt attached to the control bar *AD*. A force **P** of magnitude 25 lb is applied to the control bar at *A*. Determine the magnitude of the couple being applied to the drum, knowing that the coefficient of kinetic friction between the belt and the drum is 0.25, that *a*= 4 in., and that the drum is rotating at a constant speed (*a*) counterclockwise, (*b*) clockwise.

SOLUTION

(*a*) Counterclockwise rotation

Free body: Drum



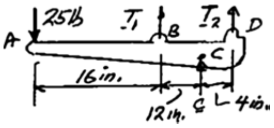
$$r = 8 \text{ in.} \quad \beta = 180° = \pi \text{ radians}$$

$$\frac{T_2}{T_1} = e^{\mu_k \beta} = e^{0.25\pi} = 2.1933$$

$$T_2 = 2.1933 T_1$$

Free body: Control bar

$$+\circlearrowleft \Sigma M_C = 0: \quad T_1(12 \text{ in.}) - T_2(4 \text{ in.}) - (25 \text{ lb})(28 \text{ in.}) = 0$$



$$T_1(12) - 2.1933 T_1(4) - 700 = 0$$

$$T_1 = 216.93 \text{ lb}$$

$$T_2 = 2.1933(216.93 \text{ lb}) = 475.80 \text{ lb}$$

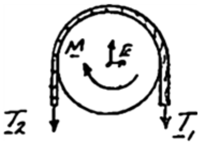
Return to free body of drum

$$+\circlearrowright \Sigma M_E = 0: \quad M + T_1(8 \text{ in.}) - T_2(8 \text{ in.}) = 0$$

$$M + (216.96 \text{ lb})(8 \text{ in.}) - (475.80 \text{ lb})(8 \text{ in.}) = 0$$

$$M = 2070.9 \text{ lb} \cdot \text{in.} \qquad M = 2070 \text{ lb} \cdot \text{in.} ◀$$

(*b*) Clockwise rotation



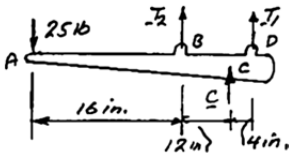
$$r = 8 \text{ in.} \quad \beta = \pi \text{ rad}$$

$$\frac{T_2}{T_1} = e^{\mu_k \beta} = e^{0.25\pi} = 2.1933$$

$$T_2 = 2.1933 T_1$$

PROBLEM 8.115 (Continued)

Free body: Control rod



$$+\circlearrowright \Sigma M_C = 0: \quad T_2(12\text{ in.}) - T_1(4\text{ in.}) - (25\text{ lb})(28\text{ in.}) = 0$$

$$2.1933T_1(12) - T_1(4) - 700 = 0$$

$$T_1 = 31.363\text{ lb}$$

$$T_2 = 2.1933(31.363\text{ lb})$$

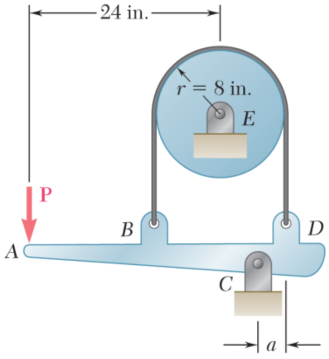
$$T_2 = 68.788\text{ lb}$$

Return to free body of drum

$$+\circlearrowright \Sigma M_E = 0: \quad M + T_1(8\text{ in.}) - T_2(8\text{ in.}) = 0$$

$$M + (31.363\text{ lb})(8\text{ in.}) - (68.788\text{ lb})(8\text{ in.}) = 0$$

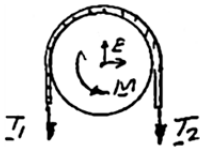
$$M = 299.4\text{ lb}\cdot\text{in.} \qquad M = 299\text{ lb}\cdot\text{in.} \blacktriangleleft$$



PROBLEM 8.116

The speed of the brake drum shown is controlled by a belt attached to the control bar *AD*. Knowing that *a*= 4 in., determine the maximum value of the coefficient of static friction for which the brake is not self-locking when the drum rotates counterclockwise.

SOLUTION

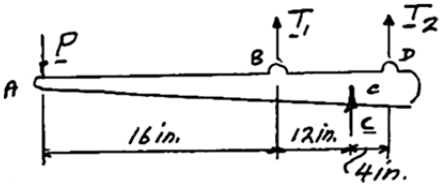


$$r = 8\text{ in.},\quad \beta = 180° = \pi\text{ radians}$$

$$\frac{T_2}{T_1} = e^{\mu_s\beta} = e^{\mu_s\pi}$$

$$T_2 = e^{\mu_s\pi}T_1$$

Free body: Control rod



$$+\circlearrowleft \Sigma M_C = 0:\quad P(28\text{ in.}) - T_1(12\text{ in.}) + T_2(4\text{ in.}) = 0$$

$$28P - 12T_1 + e^{\mu\pi}T_1(4) = 0$$

For self-locking brake: $P = 0$

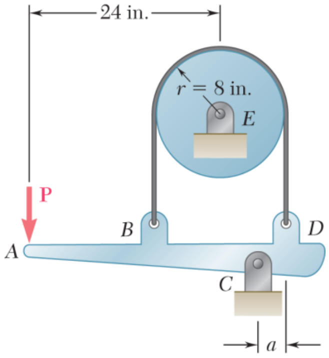
$$12T_1 = 4T_1e^{\mu_s\pi}$$

$$e^{\mu_s\pi} = 3$$

$$\mu_s\pi = \ln 3 = 1.0986$$

$$\mu_s = \frac{1.0986}{\pi} = 0.3497$$

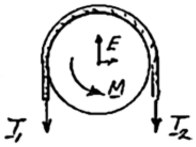
$\mu_s = 0.350$ ◀



PROBLEM 8.117

The speed of the brake drum shown is controlled by a belt attached to the control bar *AD*. Knowing that the coefficient of static friction is 0.30 and that the brake drum is rotating counterclockwise, determine the minimum value of *a* for which the brake is not self-locking.

SOLUTION

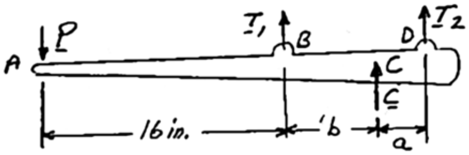


$$r = 8\text{ in.},\quad \beta = \pi \text{ radians}$$

$$\frac{T_2}{T_1} = e^{\mu_s \beta} = e^{0.30\pi} = 2.5663$$

$$T_2 = 2.5663 T_1$$

Free body: Control rod



$$b = 16\text{ in.} - a$$

$$+\circlearrowleft \Sigma M_C = 0:\quad P(16\text{ in.} + b) - T_1 b + T_2 a = 0$$

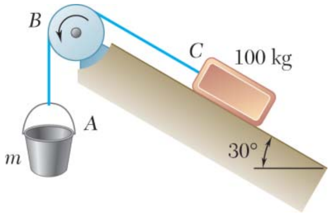
For brake to be self-locking, $P = 0$

$$T_2 a = T_1 b;\quad 2.5663 T_1 a = T_1(16 - a)$$

$$2.5663a = 16 - a$$

$$3.5663a = 16$$

$a = 4.49$ in. ◀

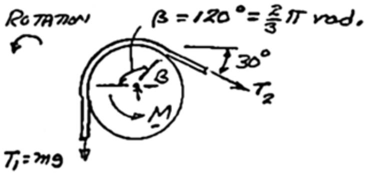


PROBLEM 8.118

Bucket *A* and block *C* are connected by a cable that passes over drum *B*. Knowing that drum *B* rotates slowly counterclockwise and that the coefficients of friction at all surfaces are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine the smallest combined mass *m* of the bucket and its contents for which block *C* will (*a*) remain at rest, (*b*) start moving up the incline, (*c*) continue moving up the incline at a constant speed.

SOLUTION

Free body: Drum



$$\frac{T_2}{mg} = e^{\mu \frac{2}{3}\pi}$$

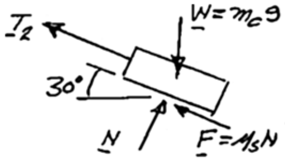
$$T_2 = mge^{2\mu\pi/3} \qquad (1)$$

(*a*) Smallest *m* for block *C* to remain at rest

Cable slips on drum.

Eq. (1) with $\mu_k = 0.25$; $T_2 = mge^{2(0.25)\pi/3} = 1.6881mg$

Block *C*: At rest, motion impending ↘



$$+\nearrow \Sigma F = 0: \quad N - m_C g\cos 30°$$

$$N = m_C g\cos 30°$$

$$F = \mu_s N = 0.35 m_C g\cos 30°$$

$$m_C = 100 \text{ kg}$$

$$+\nwarrow \Sigma F = 0: \quad T_2 + F - m_C g\sin 30° = C$$

$$1.6881mg + 0.35 m_C g\cos 30° - m_C g\sin 30° = 0$$

$$1.6881m = 0.19689m_C$$

$$m = 0.11663m_C = 0.11663(100 \text{ kg}); \qquad m = 11.66 \text{ kg} ◀$$

(*b*) Smallest *m* to start block moving up

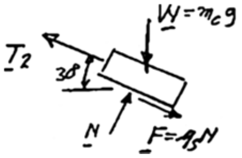
No slipping at both drum and block: $\mu_s = 0.35$

Eq. (1): $T_2 = mge^{2(0.35)\pi/3} = 2.0814mg$

PROBLEM 8.118 (Continued)

Block C:

Motion impending ↖



$$m_C = 100\,\text{kg}$$

$$+\nearrow \Sigma F = 0: \quad N - mg\cos 30°$$

$$N = m_C g \cos 30°$$
$$F = \mu_s N = 0.35 m_C g \cos 30°$$

$$+\nwarrow \Sigma F = 0: \quad T_2 - F - m_C g \sin 30° = 0$$

$$2.0814mg - 0.35 m_C g \cos 30° - m_C g \sin 30° = 0$$
$$2.0814m = 0.80311 m_C$$
$$m = 0.38585 m_C = 0.38585(100\,\text{kg})$$

$m = 38.6$ kg ◀

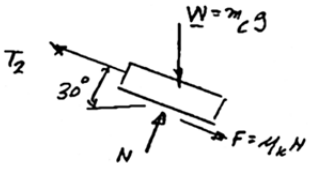
(*c*) Smallest *m* to keep block moving up drum: No slipping: $\mu_s = 0.35$

Eq. (1) with $\mu_s = 0.35$

$$T_2 = mg^{2\mu_s \pi/3} = mge^{2(0.35)\pi/3}$$
$$T_2 = 2.0814mg$$

Block C: Moving up plane, thus $\mu_k = 0.25$

Motion up ↖



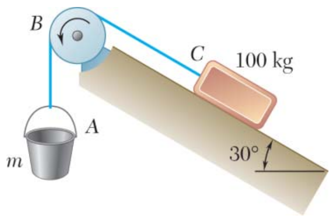
$$+\nearrow \Sigma F = 0: \quad N - m_C g \cos 30° = 0$$

$$N = m_C g \cos 30°$$
$$F = \mu_k N = 0.25\, m_C g \cos 30°$$

$$+\nwarrow \Sigma F = 0: \quad T_2 - F - m_C g \sin 30° = 0$$

$$2.0814mg - 0.25 m_C g \cos 30° - m_C g \sin 30° = 0$$
$$2.0814m = 0.71651 m_C$$
$$m = 0.34424 m_C = 0.34424(100\,\text{kg})$$

$m = 34.4$ kg ◀

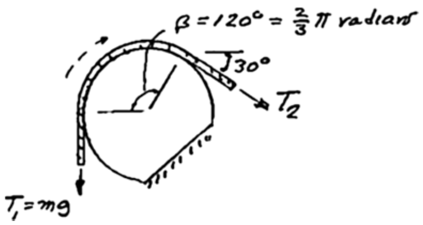

B
C
100 kg
A
m
30°

PROBLEM 8.119

Solve Problem 8.118 assuming that drum *B* is frozen and cannot rotate.

PROBLEM 8.118 Bucket *A* and block *C* are connected by a cable that passes over drum *B*. Knowing that drum *B* rotates slowly counterclockwise and that the coefficients of friction at all surfaces are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine the smallest combined mass *m* of the bucket and its contents for which block *C* will (*a*) remain at rest, (*b*) start moving up the incline, (*c*) continue moving up the incline at a constant speed.

SOLUTION


$\beta = 120° = \frac{2}{3}\pi$ radians
30°
T_2
$T_1 = mg$

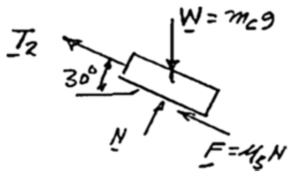
(*a*) Block *C* remains at rest: Motion impends ↘

Drum:

$$\frac{T_2}{mg} = e^{\mu_k \beta} = e^{0.35(2\pi/3)}$$

$$T_2 = 2.0814mg$$

Block *C*: Motion impends ↘


$W = m_C g$
T_2
30°
N
$F = \mu_s N$

$$\nearrow \Sigma F = 0: \quad N - m_C g \cos 30° = 0$$

$$N = m_C g \cos 30°$$

$$F = \mu_s N = 0.35 m_C g \cos 30°$$

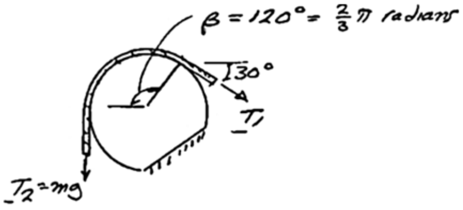
$$+\nwarrow \Sigma F = 0: \quad T_2 + F - m_C g \sin 30° = 0$$

$$2.0814mg + 0.35 m_C g \cos 30° - m_C g \sin 30° = 0$$

$$2.0814mg = 0.19689 m_C$$

$$m = 0.09459 m_C = 0.09459(100\text{ kg})$$

$m = 9.46$ kg ◀


$\beta = 120° = \frac{2}{3}\pi$ radians
30°
T_1
$T_2 = mg$

(*b*) Block *C*: Starts moving up $\mu_s = 0.35$

Drum: Impending motion of cable ↺

$$\frac{T_2}{T_1} = e^{\mu_s \beta}$$

$$\frac{mg}{T_1} = e^{0.35(2/3\pi)}$$

$$T_1 = \frac{mg}{2.0814}$$

$$= 0.48045mg$$

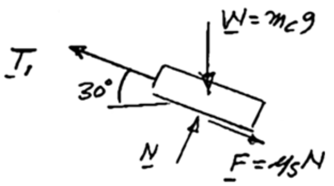
PROBLEM 8.119 (Continued)

Block C: Motion impends ↖

+↗ $\Sigma F = 0$: $N - m_C g \cos 30°$

$N = m_C g \cos 30°$

$F = \mu_s N = 0.35 m_C g \cos 30°$



+↖ $\Sigma F = 0$: $T_1 - F - m_C g \sin 30° = 0$

$0.48045mg - 0.35m_C g \cos 30° - 0.5m_C g = 0$

$0.48045m = 0.80311m_C$

$m = 1.67158m_C = 1.67158(100 \text{ kg})$ $\qquad m = 167.2 \text{ kg}$ ◀

(c) Smallest m to keep block moving ↖

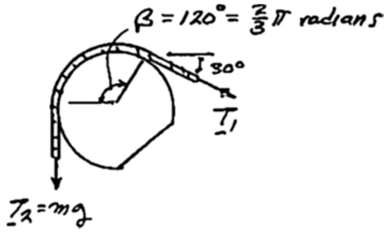
Drum: Motion of cable ↺

$\mu_k = 0.25$

$$\frac{T_2}{T_1} = e^{\mu_k \beta} = e^{0.25(2/3\pi)}$$

$$\frac{mg}{T_1} = 1.6881$$

$$T_1 = \frac{mg}{1.6881} = 0.59238mg$$

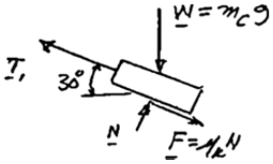


Block C: Block moves ↖

+↗ $\Sigma F = 0$: $N - m_C g \cos 30° = 0$

$N = m_C g \cos 30°$

$F = \mu_k N = 0.25 m_C g \cos 30°$

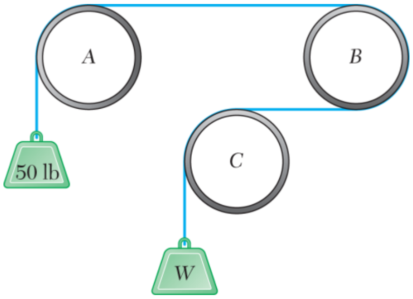


+↖ $\Sigma F = 0$: $T_1 - F - m_C g \sin 30° = 0$

$0.59238mg - 0.25m_C g \cos 30° - 0.5m_C g = 0$

$0.59238m = 0.71651m_C$

$m = 1.20954m_C = 1.20954(100 \text{ kg})$ $\qquad m = 121.0 \text{ kg}$ ◀

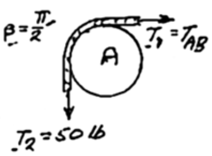


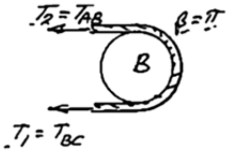
PROBLEM 8.120

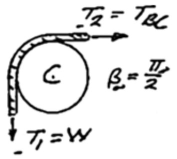
A cable is placed around three parallel pipes. Knowing that the coefficients of friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, determine (*a*) the smallest weight W for which equilibrium is maintained, (*b*) the largest weight W that can be raised if pipe B is slowly rotated counterclockwise while pipes A and C remain fixed.

SOLUTION

(*a*) $\mu = \mu_s = 0.25$ at all pipes.


$\beta = \frac{\pi}{2}$
$T_1 = T_{AB}$
A
$T_2 = 50$ lb


$T_2 = T_{AB}$
$\beta = \pi$
B
$T_1 = T_{BC}$


$T_2 = T_{BC}$
C
$\beta = \frac{\pi}{2}$
$T_1 = W$

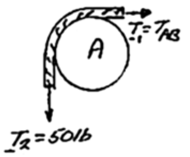
$$\frac{50\text{ lb}}{T_{AB}} = e^{0.25\pi/2} \qquad \frac{T_{AC}}{T_{BC}} = e^{0.25\pi} \qquad \frac{T_{BC}}{W} = e^{0.25\pi/2}$$

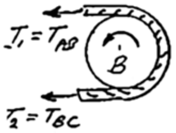
$$\frac{50\text{ lb}}{T_{AB}} \cdot \frac{T_{AB}}{T_{BC}} \cdot \frac{T_{BC}}{W} = e^{\pi/8} \cdot e^{\pi/4} \cdot e^{\pi/8} = e^{\pi/8+\pi/4+\pi/8} = e^{\pi/2} = 4.8105$$

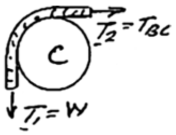
$$\frac{50\text{ lb}}{W} = 4.8015; \quad W = 10.394\text{ lb}$$

$W = 10.39$ lb ◀

(*b*) Pipe B rotated ↺ $\quad \beta = \frac{\pi}{2};\ \mu = \mu_k \qquad \beta = \pi;\ \mu = \mu_s \qquad \beta = \frac{\pi}{2};\ \mu = \mu_k$


$T_1 = T_{AB}$
A
$T_2 = 50$ lb


$T_1 = T_{AB}$
B
$T_2 = T_{BC}$



$$\frac{50\text{ lb}}{T_{AB}} = e^{0.2\pi/2} \qquad \frac{T_{BC}}{T_{AB}} = e^{0.25\pi} \qquad \frac{T_{BC}}{W} = e^{0.2\pi/2}$$

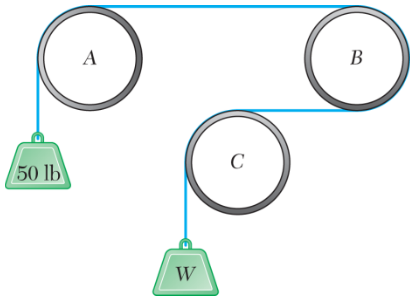
$$\frac{50\text{ lb}}{T_{AB}} \cdot \frac{T_{AB}}{T_{BC}} \cdot \frac{T_{BC}}{W} = e^{\pi/10} \cdot e^{-\pi/4} \cdot e^{\pi/10}$$

$$= e^{\pi/10-\pi/4+\pi/10} = e^{-\pi/20} = 0.85464$$

$$\frac{50\text{ lb}}{W} = 0.85464$$

$$W = \frac{50\text{ lb}}{0.85464} = 58.504\text{ lb}$$

$W = 58.5$ lb ◀

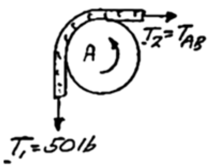


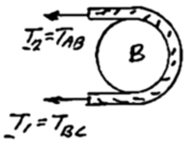
PROBLEM 8.121

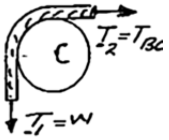
A cable is placed around three parallel pipes. Two of the pipes are fixed and do not rotate; the third pipe is slowly rotated. Knowing that the coefficients of friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, determine the largest weight W that can be raised (*a*) if only pipe *A* is rotated counterclockwise, (*b*) if only pipe *C* is rotated clockwise.

SOLUTION

(*a*) Pipe *A* rotates ↺ $\beta = \frac{\pi}{2};\ \mu = \mu_s$ $\beta = \pi,\ \mu = \mu_k$ $\beta = \frac{\pi}{2};\ \mu = \mu_k$







$$\frac{T_{AB}}{50\text{ lb}} = e^{0.25\pi/2} \qquad \frac{T_{AB}}{T_{BC}} = e^{0.2\pi} \qquad \frac{T_{BC}}{W} = e^{0.2\pi/2}$$

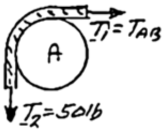
$$\frac{T_{AB}}{50\text{ lb}} \cdot \frac{T_{BC}}{T_{AB}} \cdot \frac{W}{T_{BC}} = e^{\pi/8} \cdot e^{-\pi/5} \cdot e^{-\pi/10}$$

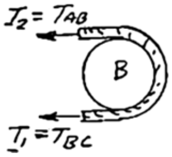
$$= e^{\pi(1/8-1/5-1/10)} = e^{-7\pi/40} = 0.57708$$

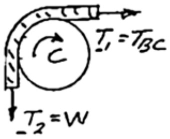
$$\frac{W}{50\text{ lb}} = 0.57708; \quad W = 28.854\text{ lb}$$

$W = 28.9$ lb ◀

(*b*) Pipe *C* rotates ↻ $\beta = \frac{\pi}{2};\ \mu = \mu_k$ $\beta = \pi;\ \mu = \mu_k$ $\beta = \frac{\pi}{2},\ \mu = \mu_s$







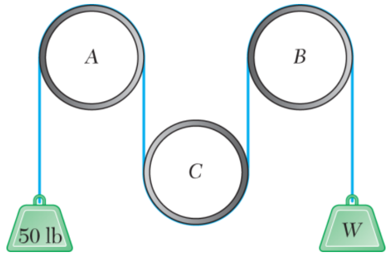
$$\frac{50\text{ lb}}{T_{AB}} = e^{0.2\pi/2} \qquad \frac{T_{AB}}{T_{BC}} = e^{0.2\pi} \qquad \frac{W}{T_{BC}} = e^{0.25\pi/2}$$

$$\frac{50\text{ lb}}{T_{AB}} \cdot \frac{T_{AB}}{T_{BC}} \cdot \frac{T_{BC}}{W} = e^{\pi/10} \cdot e^{\pi/5} \cdot e^{-\pi/8} = e^{7\pi/40} = 0.57708$$

$$\frac{50\text{ lb}}{W} = 0.57708$$

$$W = 28.854\text{ lb}$$

$W = 28.9$ lb ◀

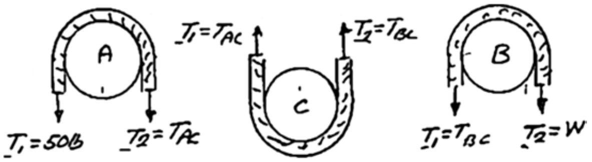


PROBLEM 8.122

A cable is placed around three parallel pipes. Knowing that the coefficients of friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, determine (*a*) the smallest weight W for which equilibrium is maintained, (*b*) the largest weight W that can be raised if pipe B is slowly rotated counterclockwise while pipes A and C remain fixed.

SOLUTION

(*a*) Smallest W for equilibrium $\qquad B = \pi, \quad \mu = \mu_s$



$$\frac{T_{AC}}{50\text{ lb}} = e^{0.25\pi} \qquad \frac{T_{BC}}{T_{AC}} = e^{0.25\pi} \qquad \frac{W}{T_{BC}} = e^{0.25\pi}$$

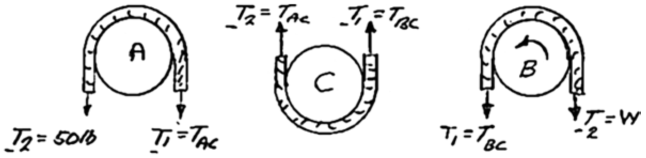
$$\frac{T_{AC}}{50\text{ lb}} \cdot \frac{T_{BC}}{T_{AC}} \cdot \frac{W}{T_{BC}} = e^{\pi/4} \cdot e^{\pi/4} \cdot e^{\pi/4} = e^{3\pi/4} = 10.551$$

$$\frac{W}{50\text{ lb}} = 10.551; \qquad W = 4.739\text{ lb}$$

$W = 4.74\text{ lb}$ ◀

(*b*) Largest W which can be raised by pipe B rotated ↺

$$\beta = \pi,\ \mu = \mu_k \qquad \beta = \pi,\ \mu = \mu_k \qquad \beta = \pi,\ \mu = \mu_s$$



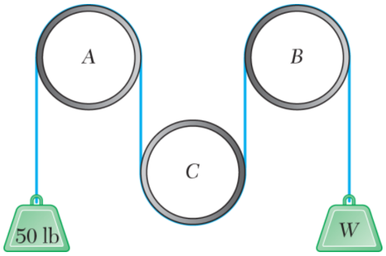
$$\frac{50\text{ lb}}{T_{AC}} = e^{0.2\pi} \qquad \frac{T_{AC}}{T_{BC}} = e^{0.2\pi} \qquad \frac{W}{T_{BC}} = e^{0.25\pi}$$

$$\frac{50\text{ lb}}{T_{AC}} \cdot \frac{T_{AC}}{T_{BC}} \cdot \frac{T_{BC}}{W} = e^{\pi/5} \cdot e^{\pi/5} \cdot e^{-\pi/4} = e^{\pi(1/5+1/5-1/4)}$$

$$= e^{3\pi/20} = 1.602$$

$$\frac{50\text{ lb}}{W} = 1.602; \qquad W = \frac{50\text{ lb}}{1.602} = 31.21\text{ lb}$$

$W = 31.2\text{ lb}$ ◀



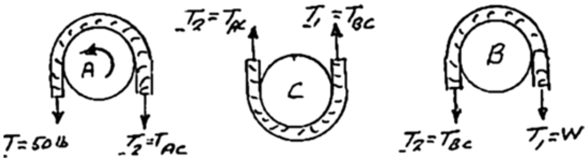
PROBLEM 8.123

A cable is placed around three parallel pipes. Two of the pipes are fixed and do not rotate; the third pipe is slowly rotated. Knowing that the coefficients of friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, determine the largest weight W that can be raised (*a*) if only pipe A is rotated counterclockwise, (*b*) if only pipe C is rotated clockwise.

SOLUTION

(*a*) Pipe *A* rotates $\circlearrowleft$

$\beta = \pi,\ \mu = \mu_s \qquad \beta = \pi,\ \mu = \mu_k \qquad \beta = \pi,\ \mu = \mu_k$



$$\frac{T_{AC}}{50\text{ lb}} = e^{0.25\pi} \qquad \frac{T_{AC}}{T_{BC}} = e^{0.2\pi} \qquad \frac{T_{BC}}{W} = e^{0.2\pi}$$

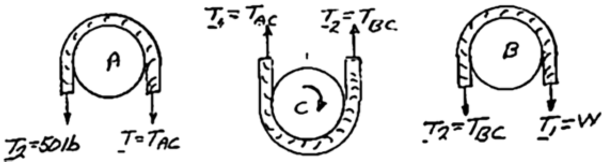
$$\frac{T_{AC}}{50\text{ lb}} \cdot \frac{T_{BC}}{T_{AC}} \cdot \frac{W}{T_{BC}} = e^{\pi/4} \cdot e^{-\pi/5} \cdot e^{-\pi/5}$$

$$= e^{\pi(1/4-1/5-1/5)} = e^{-3\pi/20} = 0.62423$$

$$\frac{W}{50\text{ lb}} = 0.62423; \quad W = 31.21\text{ lb} \qquad W = 31.2\text{ lb} \blacktriangleleft$$

(*b*) Pipe *C* rotates $\circlearrowright$

$\beta = \pi,\ \mu = \mu_k \qquad \beta = \pi,\ \mu = \mu_s \qquad \beta = \pi,\ \mu = \mu_k$

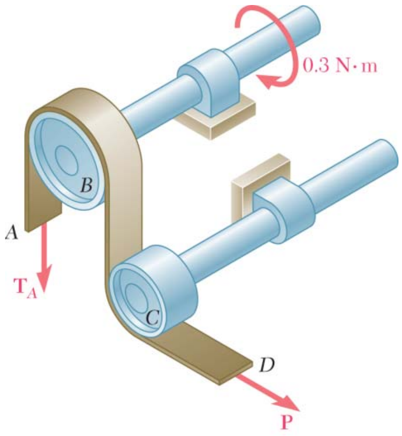


$$\frac{50\text{ lb}}{T_{AC}} = e^{0.2\pi} \qquad \frac{T_{BC}}{T_{AC}} = e^{0.25\pi} \qquad \frac{T_{BC}}{W} = e^{0.2\pi}$$

$$\frac{50\text{ lb}}{T_{AC}} \cdot \frac{T_{AC}}{T_{BC}} \cdot \frac{T_{BC}}{W} = e^{\pi/5} \cdot e^{-\pi/4} \cdot e^{\pi/5} = e^{\pi(1/5-1/4+1/5)} = e^{3\pi/20}$$

$$\frac{50\text{ lb}}{W} = e^{3\pi/20} = 1.602$$

$$W = \frac{50\text{ lb}}{1.602} = 31.21\text{ lb} \qquad W = 31.2\text{ lb} \blacktriangleleft$$

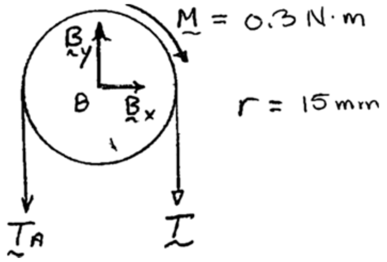


PROBLEM 8.124

A recording tape passes over the 20-mm-radius drive drum *B* and under the idler drum *C*. Knowing that the coefficients of friction between the tape and the drums are $\mu_s = 0.40$ and $\mu_k = 0.30$ and that drum *C* is free to rotate, determine the smallest allowable value of *P* if slipping of the tape on drum *B* is not to occur.

SOLUTION

FBD drive drum:



$$\circlearrowleft \Sigma M_B = 0: \quad r(T_A - T) - M = 0$$

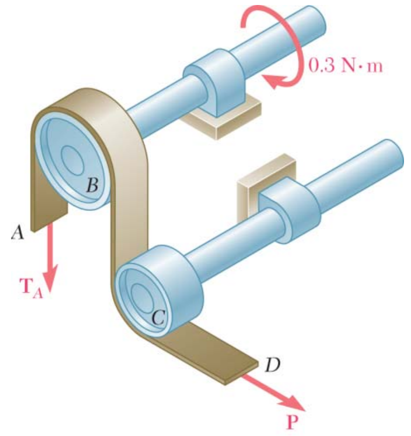
$$T_A - T = \frac{M}{r} = \frac{300\ \text{N}\cdot\text{mm}}{20\ \text{mm}} = 15.0000\ \text{N}$$

Impending slipping: $T_A = Te^{\mu_s \beta} = Te^{0.4\pi}$

So $T(e^{0.4\pi} - 1) = 15.0000\ \text{N}$

or $T = 5.9676\ \text{N}$

If *C* is free to rotate, $P = T$ $\qquad P = 5.97\ \text{N}$ ◀



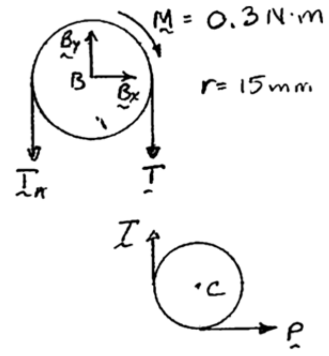
PROBLEM 8.125

Solve Problem 8.124 assuming that the idler drum *C* is frozen and cannot rotate.

PROBLEM 8.124 A recording tape passes over the 20-mm-radius drive drum *B* and under the idler drum *C*. Knowing that the coefficients of friction between the tape and the drums are $\mu_s = 0.40$ and $\mu_k = 0.30$ and that drum *C* is free to rotate, determine the smallest allowable value of *P* if slipping of the tape on drum *B* is not to occur.

SOLUTION

FBD drive drum:



$$\circlearrowleft \Sigma M_B = 0: \quad r(T_A - T) - M = 0$$

$$T_A - T = \frac{M}{r} = 300\ \text{N}\cdot\text{mm} = 15.0000\ \text{N}$$

Impending slipping: $T_A = Te^{\mu_s\beta} = Te^{0.4\pi}$

So $(e^{0.4\pi} - 1)T = 15.000\ \text{N}$

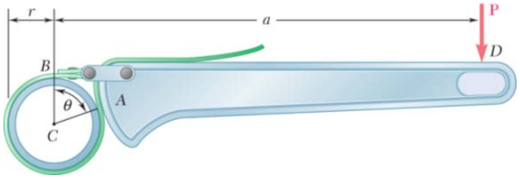
or $T = 5.9676\ \text{N}$

If *C* is fixed, the tape must slip $\circlearrowleft$

So $P = Te^{\mu_k\beta_C} = (5.9676\ \text{N})e^{0.3\pi/2} = 9.5600\ \text{N}$ $\quad P = 9.56\ \text{N}$ ◀

PROBLEM 8.126

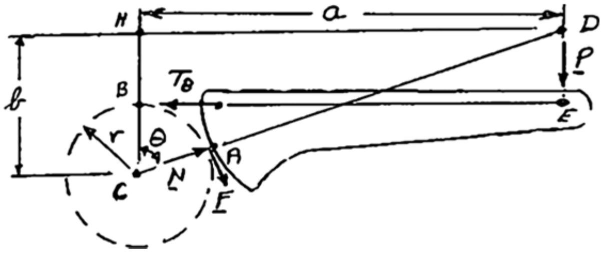
The strap wrench shown is used to grip the pipe firmly without marring the external surface of the pipe. Knowing that the coefficient of static friction is the same for all surfaces of contact, determine the smallest value of μ_s for which the wrench will be self-locking when $a = 200$ mm, $r = 30$ mm, and $\theta = 65°$.



SOLUTION

For wrench to be self-locking $(P = 0)$, the value of μ_s must prevent slipping of strap which is in contact with the pipe from Point *A* to Point *B*and must be large enough so that at Point *A* the strap tension can increase from zero to the minimum tension required to develop “belt friction” between strap and pipe.

Free body: Wrench handle



Geometry In ΔCDH:

$$CH = \frac{a}{\tan\theta}$$

$$CD = \frac{a}{\sin\theta}$$

$$DE = BH = CH - BC$$

$$DE = \frac{a}{\tan\theta} - r$$

$$AD = CD - CA = \frac{a}{\sin\theta} - r$$

On wrench handle

$$+\circlearrowleft \Sigma M_D = 0: \quad T_B(DE) - F(AD) = 0$$

$$\frac{T_B}{F} = \frac{AD}{DE} = \frac{\dfrac{a}{\sin\theta} - r}{\dfrac{a}{\tan\theta} - r} \qquad (1)$$

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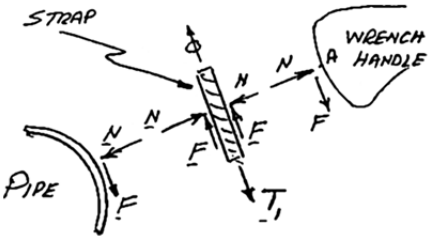
PROBLEM 8.126 (Continued)

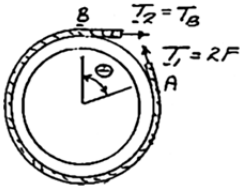
Free body: Strap at Point *A*



$$\overset{+}{\searrow} \Sigma F = 0: \quad T_1 - 2F = 0$$

$$T_1 = 2F \qquad (2)$$

Pipe and strap



$$\beta = (2\pi - \theta) \text{ radians}$$

Eq. (8.13):

$$\mu_s \beta = \ln \frac{T_2}{T_1}$$

$$\mu_s = \frac{1}{\beta} \ln \frac{T_B}{2F} \qquad (3)$$

Return to free body of wrench handle

$$\xrightarrow{+} \Sigma F_x = 0: \quad N \sin\theta + F \cos\theta - T_B = 0$$

$$\frac{N}{F} \sin\theta = \frac{T_B}{F} - \cos\theta$$

Since $F = \mu_s N$, we have

$$\frac{1}{\mu_s} \sin\theta = \frac{T_B}{F} - \cos\theta$$

or

$$\mu_s = \frac{\sin\theta}{\frac{T_B}{F} - \cos\theta} \qquad (4)$$

(*Note:* For a given set of data, we seek the larger of the values of μ_s from Eqs. (3) and (4).)

For $a = 200 \text{ mm}, \quad r = 30 \text{ mm}, \quad \theta = 65°$

Eq. (1):

$$\frac{T_B}{F} = \frac{\frac{200 \text{ mm}}{\sin 65°} - 30 \text{ mm}}{\frac{200 \text{ mm}}{\tan 65°} - 30 \text{ mm}}$$

$$= \frac{190.676 \text{ mm}}{63.262 \text{ mm}} = 3.0141$$

$$\beta = 2\pi - \theta = 2\pi - 65° \frac{\pi}{180°} = 5.1487 \text{ radians}$$

PROBLEM 8.126 (Continued)

Eq. (3):
$$\mu_s = \frac{1}{5.1487\text{ rad}} \ln \frac{3.0141}{2}$$
$$= \frac{0.41015}{5.1487}$$
$$= 0.0797 \quad \triangleleft$$

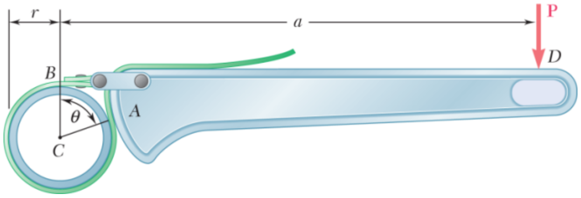
Eq. (4):
$$\mu_s = \frac{\sin 65^\circ}{3.0141 - \cos 65^\circ}$$
$$= \frac{0.90631}{2.1595}$$
$$= 0.3497 \quad \triangleleft$$

We choose the larger value: $\mu_s = 0.350$ ◀

PROBLEM 8.127

Solve Problem 8.126 assuming that $\theta = 75°$.

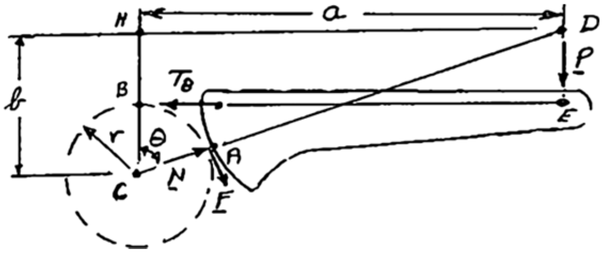
PROBLEM 8.126 The strap wrench shown is used to grip the pipe firmly without marring the external surface of the pipe. Knowing that the coefficient of static friction is the same for all surfaces of contact, determine the smallest value of μ_s for which the wrench will be self-locking when $a = 200$ mm, $r = 30$ mm, and $\theta = 65°$.



SOLUTION

For wrench to be self-locking $(P = 0)$, the value of μ_s must prevent slipping of strap which is in contact with the pipe from Point *A* to Point *B*and must be large enough so that at Point *A* the strap tension can increase from zero to the minimum tension required to develop "belt friction" between strap and pipe.

Free body: Wrench handle



Geometry In ΔCDH:

$$CH = \frac{a}{\tan\theta}$$

$$CD = \frac{a}{\sin\theta}$$

$$DE = BH = CH - BC$$

$$DE = \frac{a}{\tan\theta} - r$$

$$AD = CD - CA = \frac{a}{\sin\theta} - r$$

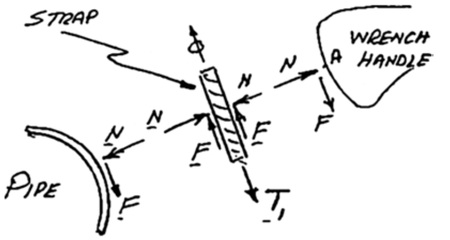
On wrench handle

$$+\circlearrowleft \Sigma M_D = 0: \quad T_B(DE) - F(AD) = 0$$

$$\frac{T_B}{F} = \frac{AD}{DE} = \frac{\dfrac{a}{\sin\theta} - r}{\dfrac{a}{\tan\theta} - r} \qquad (1)$$

PROBLEM 8.127 (Continued)

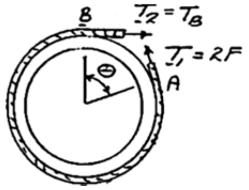
Free body: Strap at Point *A*



$$\overset{+}{\searrow}\ \Sigma F = 0: \quad T_1 - 2F = 0$$

$$T_1 = 2F \qquad (2)$$

Pipe and strap



$$\beta = (2\pi - \theta) \text{ radians}$$

Eq. (8.13):

$$\mu_s \beta = \ln \frac{T_2}{T_1}$$

$$\mu_s = \frac{1}{\beta} \ln \frac{T_B}{2F} \qquad (3)$$

Return to free body of wrench handle

$$\xrightarrow{+} \Sigma F_x = 0: \quad N \sin\theta + F\cos\theta - T_B = 0$$

$$\frac{N}{F}\sin\theta = \frac{T_B}{F} - \cos\theta$$

Since $F = \mu_s N$, we have

$$\frac{1}{\mu_s}\sin\theta = \frac{T_B}{F} - \cos\theta$$

or

$$\mu_s = \frac{\sin\theta}{\frac{T_B}{F} - \cos\theta} \qquad (4)$$

(*Note:* For a given set of data, we seek the larger of the values of μ_s from Eqs. (3) and (4).)

For $a = 200 \text{ mm}, \quad r = 30 \text{ mm}, \quad \theta = 75°$

Eq. (1):

$$\frac{T_B}{F} = \frac{\frac{200 \text{ mm}}{\sin 75°} - 30 \text{ mm}}{\frac{200 \text{ mm}}{\tan 75°} - 30 \text{ mm}}$$

$$= \frac{177.055 \text{ mm}}{23.590 \text{ mm}} = 7.5056$$

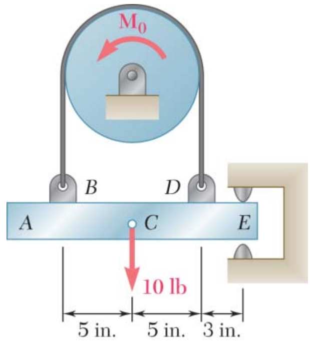
$$\beta = 2\pi - \theta = 2\pi - 75° \frac{\pi}{180°} = 4.9742$$

PROBLEM 8.127 (Continued)

Eq. (3):
$$\mu_s = \frac{1}{4.9742\text{ rad}}\ln\frac{7.5056}{2}$$
$$= \frac{1.3225}{4.9742}$$
$$= 0.2659 \quad \triangleleft$$

Eq. (4):
$$\mu_s = \frac{\sin 75^\circ}{7.5056 - \cos 75^\circ}$$
$$= \frac{0.96953}{7.2468}$$
$$= 0.1333 \quad \triangleleft$$

We choose the larger value: $\mu_s = 0.266$ ◀



PROBLEM 8.128

The 10-lb bar *AE* is suspended by a cable that passes over a 5-in.-radius drum. Vertical motion of end *E* of the bar is prevented by the two stops shown. Knowing that $\mu_s = 0.30$ between the cable and the drum, determine (*a*) the largest counterclockwise couple $\mathbf{M}_0$ that can be applied to the drum if slipping is not to occur, (*b*) the corresponding force exerted on end *E* of the bar.

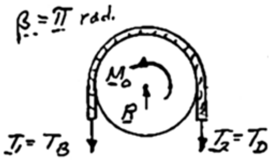
SOLUTION

Drum: Slipping impends

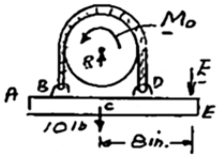
$$\mu_s = 0.30$$

$$\frac{T_2}{T_1} = e^{\mu\beta}: \quad \frac{T_D}{T_B} = e^{0.30\pi} = 2.5663$$

$$T_D = 2.5663T_B$$



(*a*) Free-body: Drum and bar

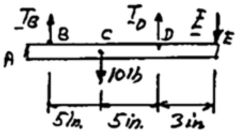


$$+\circlearrowleft \Sigma M_C = 0: \quad M_0 - E(8 \text{ in.}) = 0$$

$$M_0 = (3.78649 \text{ lb})(8 \text{ in.})$$

$$= 30.27 \text{ lb} \cdot \text{in.} \qquad \mathbf{M}_0 = 30.3 \text{ lb} \cdot \text{in.} \circlearrowleft \blacktriangleleft$$

(*b*) Bar *AE*:



$$+\uparrow \Sigma F_y = 0: \quad T_B + T_D - E - 10 \text{ lb} = 0$$

$$T_B + 2.5663T_B - E - 10 \text{ lb} = 0$$

$$3.5663T_B - E - 10 \text{ lb} = 0$$

$$E = 3.5663T_B - 10 \text{ lb} \qquad (1)$$

$$+\circlearrowright \Sigma M_D = 0: \quad E(3 \text{ in.}) - (10 \text{ lb})(5 \text{ in.}) + T_B(10 \text{ in.}) = 0$$

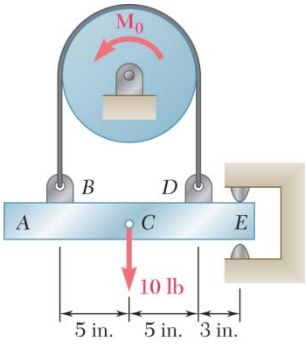
$$(3.5663T_B - 10 \text{ lb})(3 \text{ in.}) - 50 \text{ lb} \cdot \text{in.} + T_B(10 \text{ in.}) = 0$$

$$20.699T_B = 80 \quad T_B = 3.8649 \text{ lb}$$

Eq. (1):

$$E = 3.5663(3.8649 \text{ lb}) - 10 \text{ lb}$$

$$E = +3.78347 \text{ lb} \qquad \mathbf{E} = 3.78 \text{ lb} \downarrow \blacktriangleleft$$

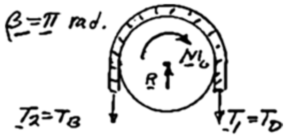


PROBLEM 8.129

Solve Problem 8.128 assuming that a clockwise couple $\mathbf{M}_0$ is applied to the drum.

PROBLEM 8.128 The 10-lb bar *AE* is suspended by a cable that passes over a 5-in.-radius drum. Vertical motion of end *E* of the bar is prevented by the two stops shown. Knowing that $\mu_s = 0.30$ between the cable and the drum, determine (*a*) the largest counterclockwise couple $\mathbf{M}_0$ that can be applied to the drum if slipping is not to occur, (*b*) the corresponding force exerted on end *E* of the bar.

SOLUTION



Drum: Slipping impends

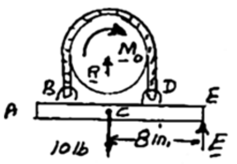
$$\mu_s = 0.30$$

$$\frac{T_2}{T_1} = e^{\mu\beta}$$

$$\frac{T_B}{T_D} = e^{0.30\pi} = 2.5663$$

$$T_B = 2.5663 T_D$$

(*a*) Free body: Drum and bar

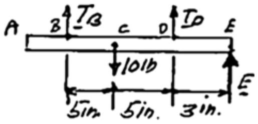


$$+\circlearrowright \Sigma M_C = 0: \quad M_0 - E(8 \text{ in.}) = 0$$

$$M_0 = (2.1538 \text{ lb})(8 \text{ in.})$$

$$\mathbf{M}_0 = 17.23 \text{ lb} \cdot \text{in.} \circlearrowright \blacktriangleleft$$

(*b*) Bar *AE*:



$$+\uparrow \Sigma F_y = 0: \quad T_B + T_D + E - 10 \text{ lb} = 0$$

$$= 2.5663 T_D + T_D + E - 10 \text{ lb}$$

$$E = -3.5663 T_D + 10 \text{ lb} \qquad (1)$$

$$+\circlearrowleft \Sigma M_B = 0: \quad T_D(10 \text{ in.}) - (10 \text{ lb})(5 \text{ in.}) + E(13 \text{ in.}) = 0$$

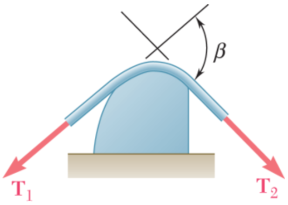
$$T_D(10 \text{ in.}) - 50 \text{ lb} \cdot \text{in.} + (-3.5663 T_D + 10 \text{ lb})(13 \text{ in.}) = 0$$

$$-36.362 T_D + 80 \text{ lb} \cdot \text{in.} = 0; \quad T_D = 2.200 \text{ lb}$$

Eq. (1):

$$E = -3.5633(2.200 \text{ lb}) + 10 \text{ lb}$$

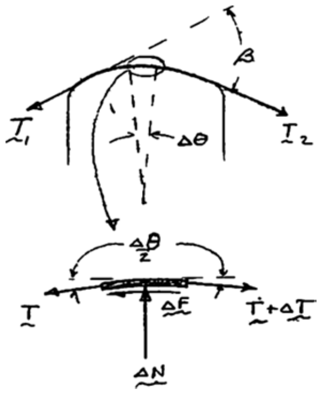
$$E = +2.1538 \text{ lb} \qquad \mathbf{E} = 2.15 \text{ lb} \uparrow \blacktriangleleft$$



PROBLEM 8.130

Prove that Eqs. (8.13) and (8.14) are valid for any shape of surface provided that the coefficient of friction is the same at all points of contact.

SOLUTION



$$\uparrow \Sigma F_n = 0: \quad \Delta N - [T + (T + \Delta T)]\sin\frac{\Delta\theta}{2} = 0$$

or
$$\Delta N = (2T + \Delta T)\sin\frac{\Delta\theta}{2}$$

$$\longrightarrow \Sigma F_t = 0: \quad [(T + \Delta T) - T]\cos\frac{\Delta\theta}{2} - \Delta F = 0$$

or
$$\Delta F = \Delta T \cos\frac{\Delta\theta}{2}$$

Impending slipping:
$$\Delta F = \mu_s \Delta N$$

So
$$\Delta T \cos\frac{\Delta\theta}{2} = \mu_s 2T \sin\frac{\Delta\theta}{2} + \mu_s \Delta T \frac{\sin\Delta\theta}{2}$$

In limit as
$$\Delta\theta \longrightarrow 0: \quad dT = \mu_s T d\theta \quad \text{or} \quad \frac{dT}{T} = \mu_s d\theta$$

So
$$\int_{T_1}^{T_2} \frac{dT}{T} = \int_0^{\beta} \mu_s d\theta$$

and
$$\ln\frac{T_2}{T_1} = \mu_s \beta \qquad \text{or} \quad T_2 = T_1 e^{\mu_s \beta} \blacktriangleleft$$

(*Note:* Nothing above depends on the shape of the surface, except it is assumed to be a smooth curve.)

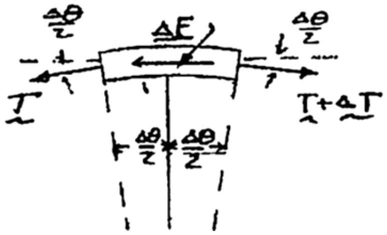
PROBLEM 8.131

Complete the derivation of Eq. (8.15), which relates the tension in both parts of a V belt.

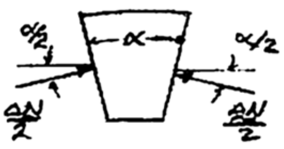
SOLUTION

Small belt section:

Side view:



End view:



$$\uparrow \Sigma F_y = 0: \quad 2\frac{\Delta N}{2}\sin\frac{\alpha}{2} - [T + (T + \Delta T)]\sin\frac{\Delta\theta}{2} = 0$$

$$\longrightarrow \Sigma F_x = 0: \quad [(T + \Delta T) - T]\cos\frac{\Delta\theta}{2} - \Delta F = 0$$

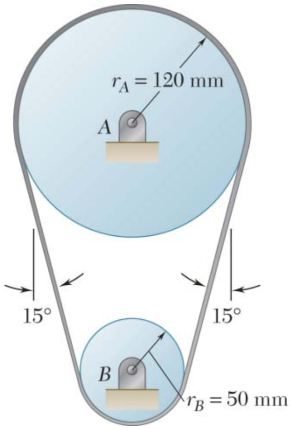
Impending slipping:
$$\Delta F = \mu_s \Delta N \Rightarrow \Delta T\cos\frac{\Delta\theta}{2} = \mu_s\frac{2T + \Delta T}{\sin\frac{\alpha}{2}}\sin\frac{\Delta\theta}{2}$$

In limit as
$$\Delta\theta \longrightarrow 0: \quad dT = \frac{\mu_s T d\theta}{\sin\frac{\alpha}{2}} \quad \text{or} \quad \frac{dT}{T} = \frac{\mu_s}{\sin\frac{\alpha}{2}}d\theta$$

so
$$\int_{T_1}^{T_2}\frac{dT}{T} = \frac{\mu_s}{\sin\frac{\alpha}{2}}\int_0^\beta d\theta$$

or
$$\ln\frac{T_2}{T_1} = \frac{\mu_s\beta}{\sin\frac{\alpha}{2}}$$

or
$$T_2 = T_1 e^{\mu_s\beta/\sin\frac{\alpha}{2}} \blacktriangleleft$$



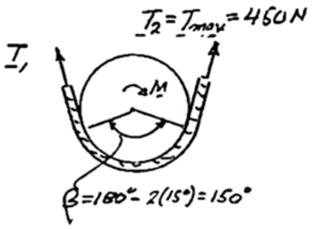
PROBLEM 8.132

Solve Problem 8.112 assuming that the flat belt and drums are replaced by a V belt and V pulleys with $\alpha = 36°$. (The angle α is as shown in Figure 8.15*a*.)

PROBLEM 8.112 A flat belt is used to transmit a couple from drum *B* to drum *A*. Knowing that the coefficient of static friction is 0.40 and that the allowable belt tension is 450 N, determine the largest couple that can be exerted on drum *A*.

SOLUTION

Since β is smaller for pulley *B*. The belt will slip first at *B*.



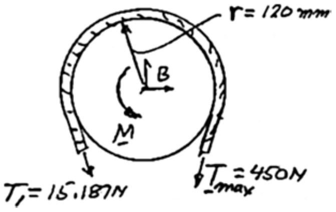
$$\beta = 150°\left(\frac{\pi \text{ rad}}{180°}\right) = \frac{5}{6}\pi \text{ rad}$$

$$\frac{T_2}{T_1} = e^{\mu_s \beta / \sin\frac{\alpha}{2}}$$

$$\frac{450 \text{ N}}{T_1} = e^{(0.4)\frac{5}{6}\pi/\sin 18°} = e^{3.389}$$

$$\frac{450 \text{ N}}{T_1} = 29.63; \quad T_1 = 15.187 \text{ N}$$

Torque on pulley *A*:

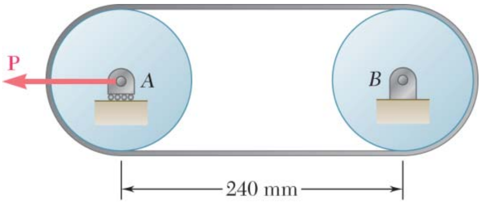


$$+\circlearrowleft \Sigma M_B = 0: \quad M - (T_{\text{max}} - T_1)(0.12 \text{ m}) = 0$$

$$M - (450 \text{ N} - 15.187 \text{ N})(0.12 \text{ m}) = 0$$

$$M = 52.18 \text{ N}\cdot\text{m}$$

$M = 52.2$ N · m ◀



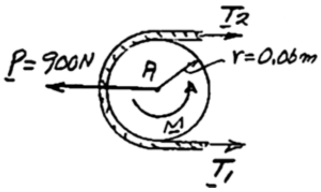
PROBLEM 8.133

Solve Problem 8.113 assuming that the flat belt and pulleys are replaced by a V belt and V pulleys with $\alpha = 36°$. (The angle α is as shown in Figure 8.15*a*.)

PROBLEM 8.113 A flat belt is used to transmit a couple from pulley *A* to pulley *B*. The radius of each pulley is 60 mm, and a force of magnitude $P = 900\text{ N}$ is applied as shown to the axle of pulley *A*. Knowing that the coefficient of static friction is 0.35, determine (*a*) the largest couple that can be transmitted, (*b*) the corresponding maximum value of the tension in the belt.

SOLUTION

Pulley *A*: $\beta = \pi\text{ rad}$



$$\frac{T_2}{T_1} = e^{\mu_s \beta / \sin\frac{\alpha}{2}}$$

$$\frac{T_2}{T_1} = e^{0.35\pi/\sin 18°}$$

$$\frac{T_2}{T_1} = e^{3.558} = 35.1$$

$$T_2 = 35.1T_1$$

$$\xrightarrow{+} \Sigma F_x = 0: \quad T_1 + T_2 - 900\text{ N} = 0$$

$$T_1 + 35.1T_1 - 900\text{ N} = 0$$

$$T_1 = 24.93\text{ N}$$

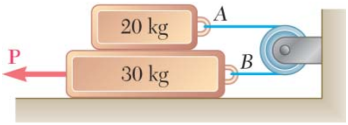
$$T_2 = 35.1(24.93\text{ N}) = 875.03\text{ N}$$

$$+\circlearrowleft \Sigma M_A = 0: \quad M - T_2(0.06\text{ m}) + T_1(0.06\text{ m}) = 0$$

$$M - (875.03\text{ N})(0.06\text{ m}) + (24.93\text{ N})(0.06\text{ m}) = 0$$

$$M = 51.0\text{ N}\cdot\text{m} \blacktriangleleft$$

$$T_{\max} = T_2 \qquad T_{\max} = 875\text{ N} \blacktriangleleft$$

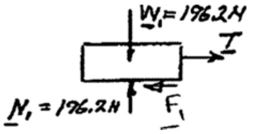

20 kg
A
P
30 kg
B

PROBLEM 8.134

The coefficients of friction are $\mu_s = 0.40$ and $\mu_k = 0.30$ between all surfaces of contact. Determine the smallest force **P** required to start the 30-kg block moving if cable *AB* (*a*) is attached as shown, (*b*) is removed.

SOLUTION

(*a*) Free body: 20-kg block

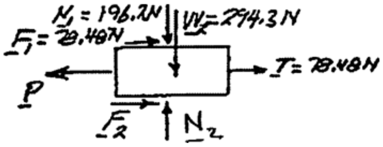

$W_1 = 196.2N$
T
$N_1 = 196.2N$
F_1

$$W_1 = (20\text{ kg})(9.81\text{ m/s}^2) = 196.2\text{ N}$$

$$F_1 = \mu_s N_1 = 0.4(196.2\text{ N}) = 78.48\text{ N}$$

$$\xrightarrow{+} \Sigma F = 0: \quad T - F_1 = 0 \quad T = F_1 = 78.48\text{ N}$$

Free body: 30-kg block


$N_1 = 196.2N$
$W_2 = 294.3N$
$F_1 = 78.48N$
$T = 78.48N$
P
F_2
N_2

$$W_2 = (30\text{ kg})(9.81\text{ m/s}^2) = 294.3\text{ N}$$

$$N_2 = 196.2\text{ N} + 294.3\text{ N} = 490.5\text{ N}$$

$$F_2 = \mu_s N_2 = 0.4(490.5\text{ N}) = 196.2\text{ N}$$

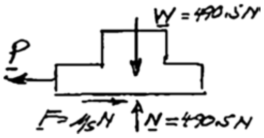
$$\xleftarrow{+} \Sigma F = 0: \quad P - F_1 - F_2 - T = 0$$

$$P = 78.48\text{ N} + 196.2\text{ N} + 78.48\text{ N} = 353.2\text{ N}$$

P = 353 N ← ◀

(*b*) Free body: Both blocks

Blocks move together

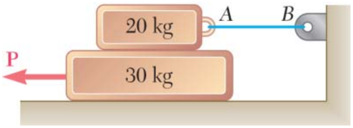

$W = 490.5N$
P
$F = \mu_s N$
$N = 490.5N$

$$W = (50\text{ kg})(9.81\text{ m/s}^2) = 490.5\text{ N}$$

$$\xleftarrow{+} \Sigma F = 0: \quad P - F = 0$$

$$P = \mu_s N = 0.4(490.5\text{ N}) = 196.2\text{ N}$$

P = 196.2 N ← ◀

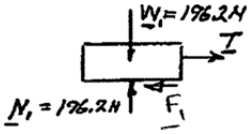

20 kg
30 kg
A
B
P

PROBLEM 8.135

The coefficients of friction are $\mu_s = 0.40$ and $\mu_k = 0.30$ between all surfaces of contact. Determine the smallest force **P** required to start the 30-kg block moving if cable AB (a) is attached as shown, (b) is removed.

SOLUTION

(a) Free body: 20-kg block

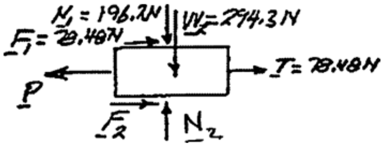

W₁=196.2N
T
N₁=196.2N
F₁

$$W_1 = (20\text{ kg})(9.81\text{ m/s}^2) = 196.2\text{ N}$$

$$F_1 = \mu_s N_1 = 0.4(196.2\text{ N}) = 78.48\text{ N}$$

$$\xrightarrow{+}\Sigma F = 0: \quad T - F_1 = 0 \quad T = F_1 = 78.48\text{ N}$$

Free body: 30-kg block


N₁=196.2N
W₂=294.3N
F₁=78.48N
P
T=78.48N
F₂
N₂

$$W_2 = (30\text{ kg})(9.81\text{ m/s}^2) = 294.3\text{ N}$$

$$N_2 = 196.2\text{ N} + 294.3\text{ N} = 490.5\text{ N}$$

$$F_2 = \mu_s N_2 = 0.4(490.5\text{ N}) = 196.2\text{ N}$$

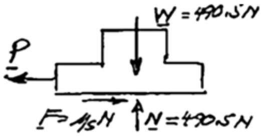
$$\xleftarrow{+}\Sigma F = 0: \quad P - F_1 - F_2 = 0$$

$$P = 78.48\text{ N} + 196.2\text{ N} = 274.7\text{ N}$$

$\mathbf{P} = 275\text{ N} \longleftarrow$ ◀

(b) Free body: Both blocks

Blocks move together


W=490.5N
P
F=μsN
N=490.5N

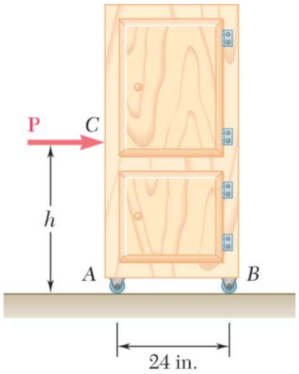
$$W = (50\text{ kg})(9.81\text{ m/s}^2)$$

$$= 490.5\text{ N}$$

$$\xleftarrow{+}\Sigma F = 0: \quad P - F = 0$$

$$P = \mu_s N = 0.4(490.5\text{ N}) = 196.2\text{ N}$$

$\mathbf{P} = 196.2\text{ N} \longleftarrow$ ◀

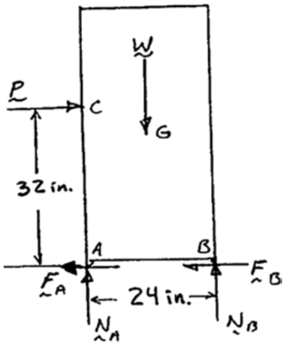


PROBLEM 8.136

A 120-lb cabinet is mounted on casters that can be locked to prevent their rotation. The coefficient of static friction between the floor and each caster is 0.30. If $h = 32$ in., determine the magnitude of the force **P** required to move the cabinet to the right (*a*) if all casters are locked, (*b*) if the casters at *B* are locked and the casters at *A* are free to rotate, (*c*) if the casters at *A* are locked and the casters at *B* are free to rotate.

SOLUTION

FBD cabinet: *Note:* for tipping,



$$N_A = F_A = 0$$

$$\circlearrowleft \Sigma M_B = 0: \quad (12 \text{ in.})W - (32 \text{ in.})P_{\text{tip}} = 0$$

$$P_{\text{tip}} = 2.66667$$

(*a*) All casters locked. Impending slip:

$$F_A = \mu_s N_A$$

$$F_B = \mu_s N_B$$

$$\uparrow \Sigma F_y = 0: \quad N_A + N_B - W = 0$$

$$N_A + N_B = W \qquad W = 120 \text{ lb}$$

So $$F_A + F_B = \mu_s W \qquad \mu_s = 0.3$$

$$\xrightarrow{} \Sigma F_x = 0: \quad P - F_A - F_B = 0$$

$$P = F_A + F_B = \mu_s W$$

$$P = 0.3(120 \text{ lb}) \qquad \text{or} \quad \mathbf{P} = 36.0 \text{ lb} \longrightarrow \blacktriangleleft$$

$$(P = 0.3W < P_{\text{tip}} \quad \text{OK})$$

(*b*) Casters at *A* free, so $F_A = 0$

Impending slip: $F_B = \mu_s N_B$

$$\xrightarrow{} \Sigma F_x = 0: \quad P - F_B = 0$$

$$P = F_B = \mu_s N_B \qquad N_B = \frac{P}{\mu_s}$$

PROBLEM 8.136 (Continued)

$$\circlearrowright \Sigma M_A = 0: \quad (32\text{ in.})P + (12\text{ in.})W - (24\text{ in.})N_B = 0$$

$$8P + 3W - 6\frac{P}{0.3} = 0 \quad P = 0.25W$$

$$(P = 0.25W < P_{\text{tip}} \quad \text{OK})$$

$$P = 0.25(120\text{ lb}) \qquad \text{or} \quad \mathbf{P} = 30.0\text{ lb} \longrightarrow$$ ◀

(*c*) Casters at *B* free, so $F_B = 0$

Impending slip: $F_A = \mu_s N_A$

$$\xrightarrow{} \Sigma F_x = 0: \quad P - F_A = 0 \quad P = F_A = \mu_s N_A$$

$$N_A = \frac{P}{\mu_s} = \frac{P}{0.3}$$

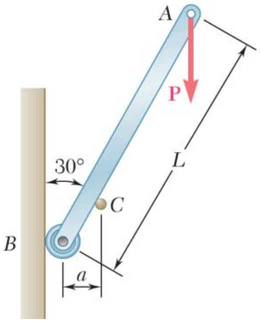
$$\circlearrowleft \Sigma M_B = 0: \quad (12\text{ in.})W - (32\text{ in.})P - (24\text{ in.})N_A = 0$$

$$3W - 8P - 6\frac{P}{0.3} = 0$$

$$P = 0.107143W = 12.8572$$

$$(P < P_{\text{tip}} \quad \text{OK})$$

$$\mathbf{P} = 12.86\text{ lb} \longrightarrow$$ ◀



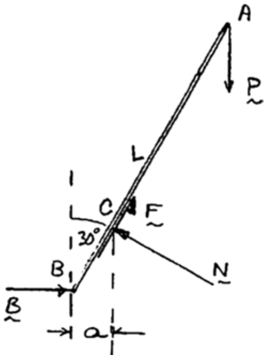
PROBLEM 8.137

A slender rod of length L is lodged between peg C and the vertical wall and supports a load **P** at end A. Knowing that the coefficient of static friction between the peg and the rod is 0.15 and neglecting friction at the roller, determine the range of values of the ratio L/a for which equilibrium is maintained.

SOLUTION

FBD rod:

Free-body diagram: For motion of B impending upward:



$$+\circlearrowleft \Sigma M_B = 0: \quad PL\sin\theta - N_C\left(\frac{a}{\sin\theta}\right) = 0$$

$$N_C = \frac{PL}{a}\sin^2\theta \qquad (1)$$

$$+\uparrow \Sigma F_y = 0: \quad N_C\sin\theta - \mu_s N_C\cos\theta - P = 0$$

$$N_C(\sin\theta - \mu\cos\theta) = P$$

Substitute for N_C from Eq. (1), and solve for a/L.

$$\frac{a}{L} = \sin^2\theta(\sin\theta - \mu_s\cos\theta) \qquad (2)$$

For $\theta = 30°$ and $\mu_s = 0.15$:

$$\frac{a}{L} = \sin^2 30°(\sin 30° - 0.15\cos 30°)$$

$$\frac{a}{L} = 0.092524 \qquad \frac{L}{a} = 10.808$$

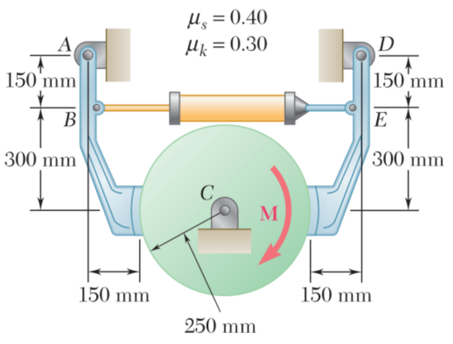
For motion of B impending downward, reverse sense of friction force F_C. To do this we make $\mu_s = -0.15$ in. Eq. (2).

Eq. (2):

$$\frac{a}{L} = \sin^2 30°(\sin 30° - (-0.15)\cos 30°)$$

$$\frac{a}{L} = 0.15748 \qquad \frac{L}{a} = 6.350$$

Range of values of L/a for equilibrium: $6.35 \le \frac{L}{a} \le 10.81$ ◀

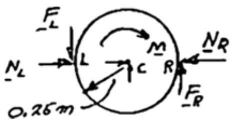

$\mu_s = 0.40$
$\mu_k = 0.30$
A
D
150 mm
150 mm
B
E
300 mm
300 mm
C
M
150 mm
150 mm
250 mm

PROBLEM 8.138

The hydraulic cylinder shown exerts a force of 3 kN directed to the right on Point *B* and to the left on Point *E*. Determine the magnitude of the couple **M** required to rotate the drum clockwise at a constant speed.

SOLUTION

Free body: Drum


F_L
N_R
M
N_L
L
C
R
F_R
0.25 m

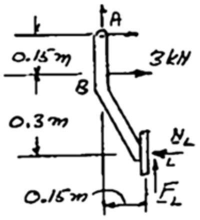
$$+\circlearrowright \Sigma M_C = 0: \quad M - (0.25 \text{ m})(F_L + F_R) = 0$$

$$M = (0.25 \text{ m})(F_L + F_R) \qquad (1)$$

Since drum is rotating

$$F_L = \mu_k N_L = 0.3 N_L$$
$$F_R = \mu_k N_R = 0.3 N_R$$

Free body: Left arm *ABL*

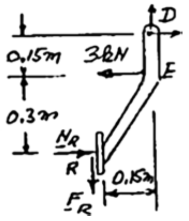

A
0.15 m
3 kN
B
0.3 m
N_L
L
0.15 m
F_L

$$+\circlearrowleft \Sigma M_A = 0: \quad (3 \text{ kN})(0.15 \text{ m}) + F_L(0.15 \text{ m}) - N_L(0.45 \text{ m}) = 0$$

$$0.45 \text{ kN}\cdot\text{m} + (0.3N_L)(0.15 \text{ m}) - N_L(0.45 \text{ m}) = 0$$
$$0.405 N_L = 0.45$$

$$N_L = 1.111 \text{ kN}$$
$$F_L = 0.3N_L = 0.3(1.111 \text{ kN})$$
$$= 0.3333 \text{ kN} \qquad (2)$$

Free body: Right arm *DER*


D
0.15 m
3 kN
E
0.3 m
N_R
R
0.15 m
F_R

$$+\circlearrowright \Sigma M_D = 0: \quad (3 \text{ kN})(0.15 \text{ m}) - F_R(0.15 \text{ m}) - N_R(0.45 \text{ m}) = 0$$

$$0.45 \text{ kN}\cdot\text{m} - (0.3N_R)(0.15 \text{ m}) - N_R(0.45 \text{ m}) = 0$$

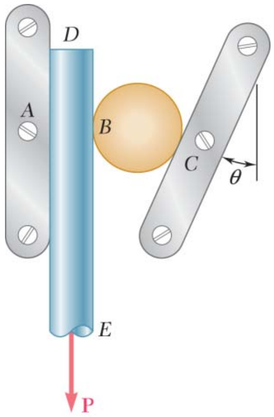
$$0.495 N_R = 0.45$$

$$N_R = 0.9091 \text{ kN}$$
$$F_R = \mu_k N_R = 0.3(0.9091 \text{ kN})$$
$$= 0.2727 \text{ kN} \qquad (3)$$

Substitute for F_L and F_R into (1): $M = (0.25 \text{ m})(0.333 \text{ kN} + 0.2727 \text{ kN})$

$$M = 0.1515 \text{ kN}\cdot\text{m}$$

$$M = 151.5 \text{ N}\cdot\text{m} \blacktriangleleft$$



PROBLEM 8.139

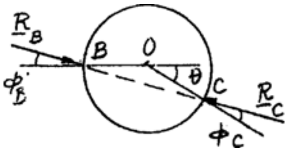
A rod *DE* and a small cylinder are placed between two guides as shown. The rod is not to slip downward, however large the force **P** may be; i.e., the arrangement is said to be self-locking. Neglecting the weight of the cylinder, determine the minimum allowable coefficients of static friction at *A*, *B*, and *C*.

SOLUTION

Free body: Cylinder

Since cylinder is a two-force body, $\mathbf{R}_B$ and $\mathbf{R}_C$ have the same line of action. Thus $\phi_B = \phi_C$:

From triangle *OBC*:



$$\phi_B + \phi_C = \theta$$

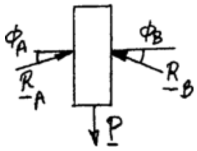
Thus: $$\phi_B = \phi_C = \frac{\theta}{2}$$

For no sliding, we must have $\tan\phi_B \le (\mu_B)_s$, $\tan\phi_C \le (\mu_C)_s$

Therefore: $$(\mu_B)_s \ge \tan\frac{\theta}{2}, \qquad (\mu_c)_s \ge \tan\frac{\theta}{2} \blacktriangleleft$$

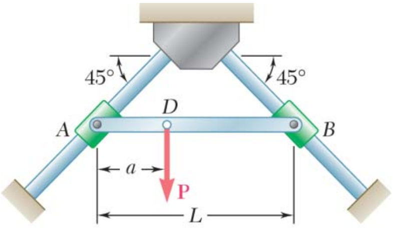
We also note that R_B and R_C are indeterminate.

Free body: Rod



Since R_B is indeterminate, it may be as large as necessary to satisfy equation $\Sigma F_y = 0$, no matter how large **P** is or how small ϕ_A is. Therefore

$(\mu_A)_s$ may have any value ◀

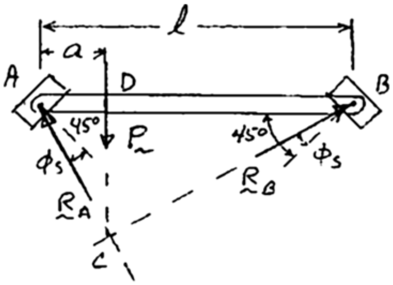


PROBLEM 8.140

Bar *AB* is attached to collars that can slide on the inclined rods shown. A force **P** is applied at Point *D* located at a distance *a* from end *A*. Knowing that the coefficient of static friction μ_s between each collar and the rod upon which it slides is 0.30 and neglecting the weights of the bar and of the collars, determine the smallest value of the ratio *a*/*L* for which equilibrium is maintained.

SOLUTION

FBD bar and collars:



Impending motion:

$$\phi_s = \tan^{-1}\mu_s = \tan^{-1} 0.3 = 16.6992°$$

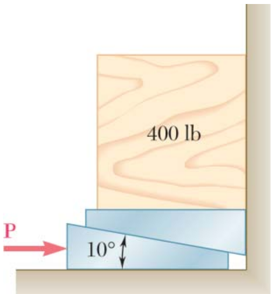
Neglect weights: 3-force *FBD* and $\measuredangle ACB = 90°$

so

$$AC = \frac{a}{\cos(45° + \phi_s)} = l\sin(45° - \phi_s)$$

$$\frac{a}{l} = \sin(45° - 16.6992°)\cos(45° + 16.6992°)$$

$$\frac{a}{l} = 0.225 \blacktriangleleft$$



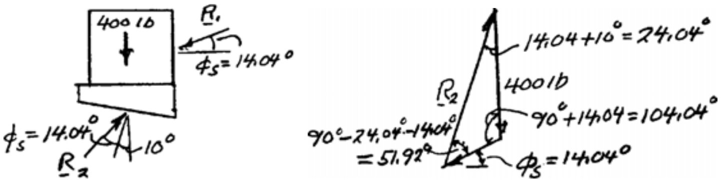
PROBLEM 8.141

Two 10° wedges of negligible weight are used to move and position the 400-lb block. Knowing that the coefficient of static friction is 0.25 at all surfaces of contact, determine the smallest force **P** that should be applied as shown to one of the wedges.

SOLUTION

Free body: Block and top wedge $\phi_s = \tan^{-1} 0.25 = 14.04°$

Force triangle

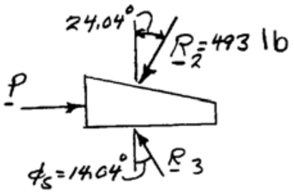

400 lb
R1
ϕs = 14.04°
ϕs = 14.04°
R2
10°
14.04 + 10° = 24.04°
400 lb
R2
90° + 14.04 = 104.04°
90° − 24.04° − 14.04°
= 51.92°
ϕs = 14.04°

Law of sines:

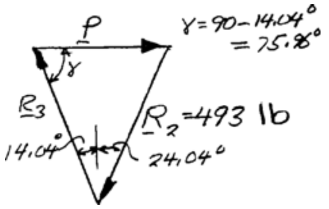
$$\frac{R_2}{\sin 104.04°} = \frac{400\text{ lb}}{\sin 51.92°}$$

$$R_2 = 493\text{ lb}$$

Free body: Lower wedge


24.04°
R2 = 493 lb
P
ϕs = 14.04°
R3

Force triangle

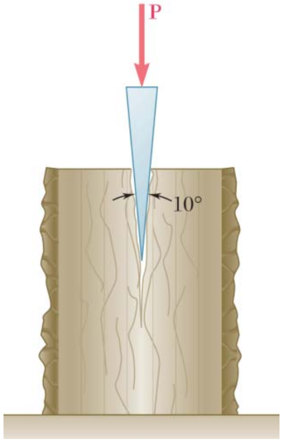

P
γ = 90 − 14.04°
= 75.96°
γ
R3
R2 = 493 lb
14.04°
24.04°

Law of sine:

$$\frac{P}{\sin(14.04° + 24.04°)} = \frac{493\text{ lb}}{\sin 75.96°}$$

$$P = 313.4\text{ N}$$

$$\mathbf{P} = 313\text{ lb} \longrightarrow$$ ◀

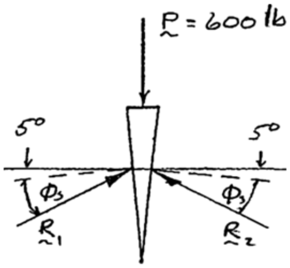


PROBLEM 8.142

A 10° wedge is used to split a section of a log. The coefficient of static friction between the wedge and the log is 0.35. Knowing that a force **P** of magnitude 600 lb was required to insert the wedge, determine the magnitude of the forces exerted on the wood by the wedge after insertion.

SOLUTION

FBD wedge (impending motion ↓):



$$\phi_s = \tan^{-1}\mu_s$$

$$= \tan^{-1} 0.35$$

$$= 19.29°$$

By symmetry: $R_1 = R_2$

$$\uparrow \Sigma F_y = 0: \quad 2R_1 \sin(5° + \phi_s) - 600 \text{ lb} = 0$$

or $$R_1 = R_2 = \frac{300 \text{ lb}}{\sin(5°+19.29°)} = 729.30 \text{ lb}$$

When P is removed, the vertical components of R_1 and R_2 vanish, leaving the horizontal components

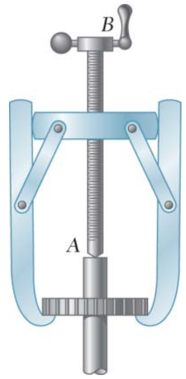
$$R_{1x} = R_{2x}$$

$$= R_1 \cos(5° + \phi_s)$$

$$= (729.30 \text{ lb})\cos(5° + 19.29°)$$

$$R_{1x} = R_{2x} = 665 \text{ lb} \blacktriangleleft$$

(Note that $\phi_s > 5°$, so wedge is self-locking.)

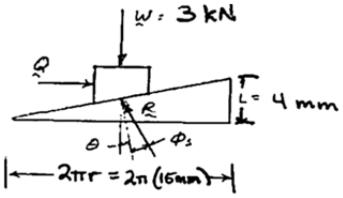


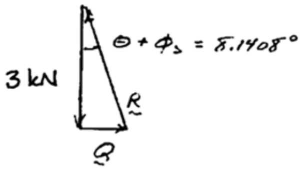
PROBLEM 8.143

In the gear-pulling assembly shown the square-threaded screw *AB* has a mean radius of 15 mm and a lead of 4 mm. Knowing that the coefficient of static friction is 0.10, determine the couple that must be applied to the screw in order to produce a force of 3 kN on the gear. Neglect friction at end *A* of the screw.

SOLUTION

Block/Incline:





$$\theta = \tan^{-1}\frac{4\text{ mm}}{30\pi\text{ mm}}$$

$$= 2.4302°$$

$$\phi_s = \tan^{-1}\mu_s$$

$$= \tan^{-1}(0.10)$$

$$= 5.7106°$$

$$Q = (3000\text{ N})\tan(8.1408°)$$

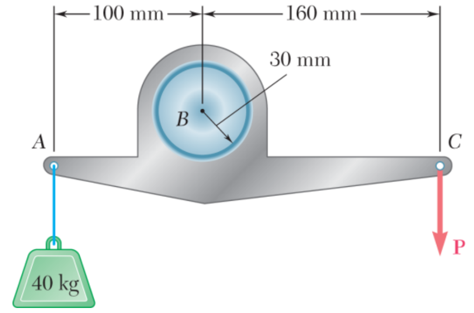
$$= 429.14\text{ N}$$

$$\text{Couple} = rQ$$

$$= (0.015\text{ m})(429.14\text{ N})$$

$$= 6.4371\text{ N}\cdot\text{m}$$

$M = 6.44\text{ N}\cdot\text{m}$ ◀

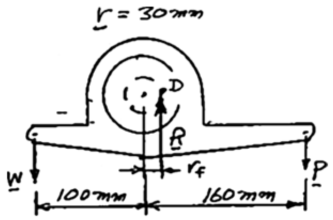


PROBLEM 8.144

A lever of negligible weight is loosely fitted onto a 30-mm-radius fixed shaft as shown. Knowing that a force **P** of magnitude 275 N will just start the lever rotating clockwise, determine (*a*) the coefficient of static friction between the shaft and the lever, (*b*) the smallest force **P** for which the lever does not start rotating counterclockwise.

SOLUTION

(*a*) Impending motion ↻



$$W = (40\text{ kg})(9.81\text{ m/s}^2) = 392.4\text{ N}$$

$$+\circlearrowright \Sigma M_D = 0: \quad P(160 - r_f) - W(100 + r_f) = 0$$

$$r_f = \frac{160P - 100W}{P + W}$$

$$r_f = \frac{(160\text{ mm})(275\text{ N}) - (100\text{ mm})(392.4\text{ N})}{275\text{ N} + 392.4\text{ N}}$$

$$r_f = 7.132\text{ mm}$$

$$r_f = r\sin\phi_s = r\mu_s$$

$$\mu_s = \frac{r_f}{r} = \frac{7.132\text{ mm}}{30\text{ mm}} = 0.2377$$

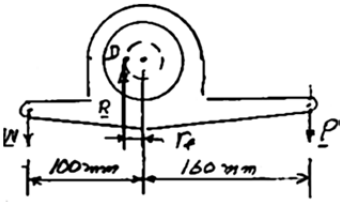
$\mu_s = 0.238$ ◀

(*b*) Impending motion ↺

$$r_f = r\sin\phi_s = r\mu_s$$

$$= (30\text{ mm})(0.2377)$$

$$r_f = 7.132\text{ mm}$$



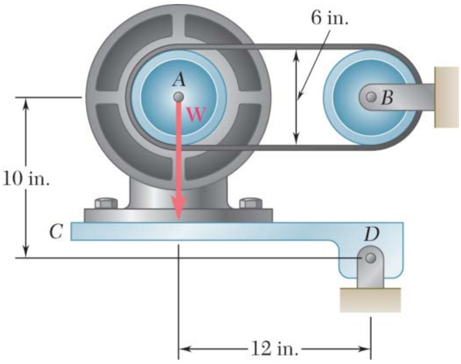
$$+\circlearrowright \Sigma M_D = 0: \quad P(160 + r_f) - W(100 - r_f) = 0$$

$$P = W\frac{100 - r_f}{160 + r_f}$$

$$P = (392.4\text{ N})\frac{100\text{ mm} - 7.132\text{ mm}}{160\text{ mm} + 7.132\text{ mm}}$$

$$P = 218.04\text{ N}$$

P = 218 N ↓ ◀

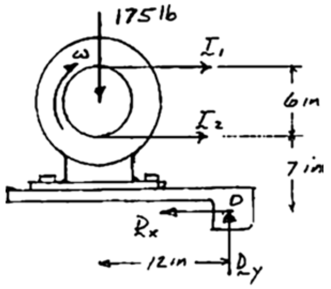


PROBLEM 8.145

In the pivoted motor mount shown the weight **W** of the 175-lb motor is used to maintain tension in the drive belt. Knowing that the coefficient of static friction between the flat belt and drums A and B is 0.40, and neglecting the weight of platform CD, determine the largest couple that can be transmitted to drum B when the drive drum A is rotating clockwise.

SOLUTION

FBD motor and mount:



Impending belt slip: cw rotation

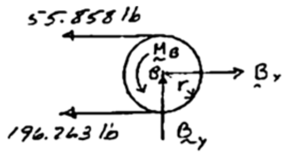
$$T_2 = T_1 e^{\mu_s \beta} = T_1 e^{0.40\pi} = 3.5136 T_1$$

$$\circlearrowleft \Sigma M_D = 0: \quad (12 \text{ in.})(175 \text{ lb}) - (7 \text{ in.})T_2 - (13 \text{ in.})T_1 = 0$$

$$2100 \text{ lb} = [(7 \text{ in.})(3.5136) + 13 \text{ in.}]T_1$$

$$T_1 = 55.858 \text{ lb}, \quad T_2 = 3.5136 T_1 = 196.263 \text{ lb}$$

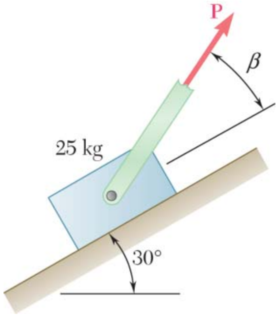
FBD drum at B:



$$\circlearrowleft \Sigma M_B = 0: \quad M_B - (3 \text{ in.})(196.263 \text{ lb} - 55.858 \text{ lb}) = 0$$

$r = 3$ in.

$M_B = 421$ lb · in. ◀

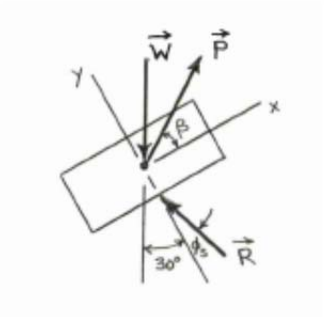


PROBLEM 8.F1

Knowing that the coefficient of friction between the 25-kg block and the incline is $\mu_s = 0.25$, draw the free-body diagram needed to determine both the smallest value of *P* required to start the block moving up the incline and the corresponding value of *β*.

SOLUTION

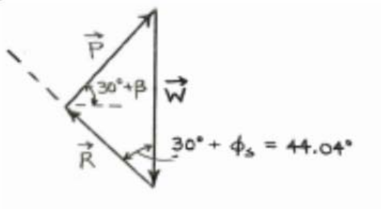
Free body: Block


$\vec{W}$
$\vec{P}$
y
x
β
30°
ϕ_s
$\vec{R}$

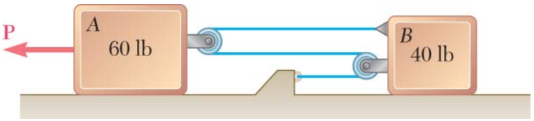
$$W = (25\text{ kg})(9.81\text{ m/s}^2) = 245.25\text{ N}$$

$$\phi_s = \tan^{-1}\mu_s = \tan^{-1}(0.25) = 14.04^\circ$$

Force Triangle:


$\vec{P}$
30°+β
$\vec{W}$
$\vec{R}$
30° + ϕ_s = 44.04°

Note: For minimum *P*, **P** must be perpendicular to **R**; thus, $\beta = \phi_s = 14.04^\circ$

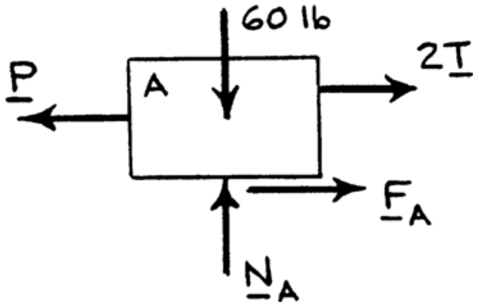


PROBLEM 8.F2

Two blocks *A* and *B* are connected by a cable as shown. Knowing that the coefficient of static friction at all surfaces of contact is 0.30 and neglecting the friction of the pulleys, draw the free-body diagrams needed to determine the smallest force **P** required to move the blocks.

SOLUTION

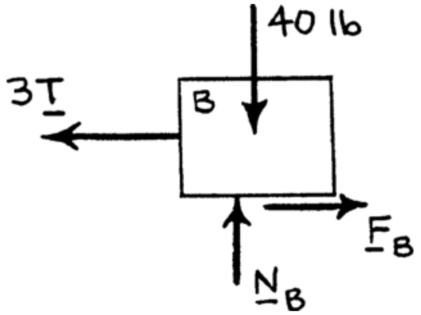
Free body: Block *A*



Motion impends to left, with

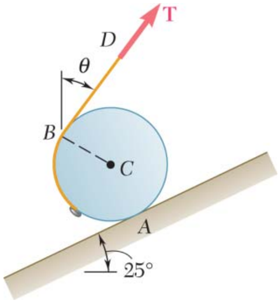
$F_A = \mu_S N_A = 0.30 N_A$

Free body: Block *B*



Motion impends to left, with

$F_B = \mu_s N_B = 0.30\ N_B$



PROBLEM 8.F3

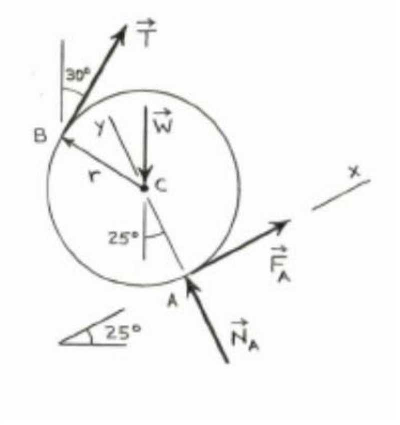
A cord is attached to and partially wound around a cylinder with a weightof *W* and radius *r* that rests on an incline as shown. Knowing that $\theta = 30°$, draw the free-body diagram needed to determine both the tension in the cord and the smallest allowable value of the coefficient of static friction between the cylinder and the incline for which equilibrium is maintained.

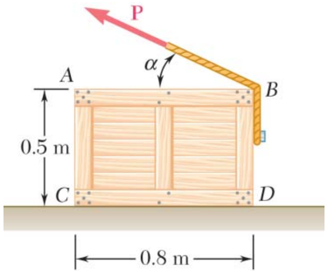
SOLUTION

Free body: Cylinder

For minimum μ_s , motion impends at *A* with

$F_A = \mu_s N_A$

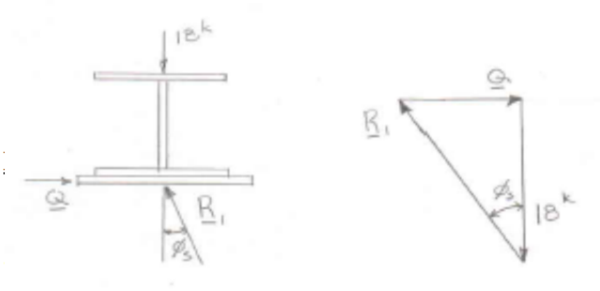




PROBLEM 8.F4

A 40-kg packing crate must be moved to the left along the floor without tipping. Knowing that the coefficient of static friction between the crate and the floor is 0.35, draw the free-body diagram needed to determine both the largest allowable value of α and the corresponding magnitude of the force **P**.

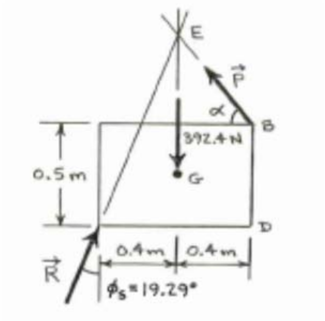
SOLUTION



$1\ \text{m/s}^2) = 392.4\ \text{N}$

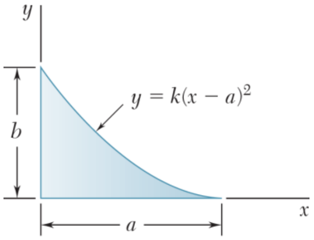
crate and ground is at C only, and the reaction **R** is de, **R** must form with the vertical an angle $\phi_s = \tan^{-1}\mu_s = \tan^{-1}(0.35) = 19.29°$.

Since the crate is a 3-force body, **P** must pass through point E where **R**and **W** intersect. Free body of the crate is then:



Note: After the crate starts moving, μ_s should be replaced with the lower value μ_k. This will yield a larger value of α.

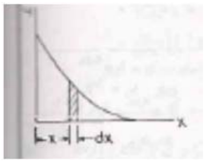
CHAPTER 9



PROBLEM 9.1

Determine by direct integration the moment of inertia of the shaded area with respect to the *y* axis.

SOLUTION



At $x = 0, \quad y = b: \quad b = k(0 - a)^2$

$k = \dfrac{b}{a^2}$ then

$$y = \frac{b}{a^2}(x-a)^2; \; dA = y dx$$

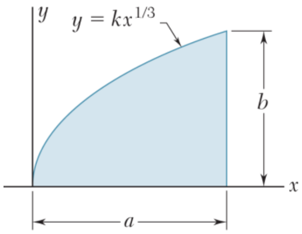
$$dI_y = x^2 dA = x^2 y\, dx$$

$$I_y = \int_0^a \left[\frac{b}{a^2} x^2 (x-a)^2\right] dx$$

$$= \int_0^a \left[\frac{b}{a^2}\left(x^4 - 2ax^3 + a^2x^2\right)\right] dx$$

$$= \frac{b}{a^2}\left[\frac{1}{5}x^5 - \frac{1}{2}ax^4 + \frac{1}{3}a^2x^3\right]_0^a$$

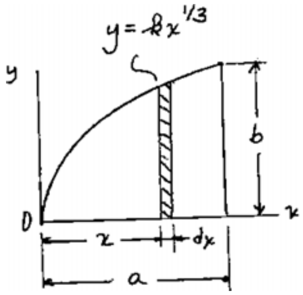
$$I_y = \frac{1}{30}a^3 b \blacktriangleleft$$



PROBLEM 9.2

Determine by direct integration the moment of inertia of the shaded area with respect to the *y* axis.

SOLUTION



$$y = kx^{1/3}$$

For $x = a$:

$$b = ka^{1/3}$$

$$k = b/a^{1/3}$$

Thus:

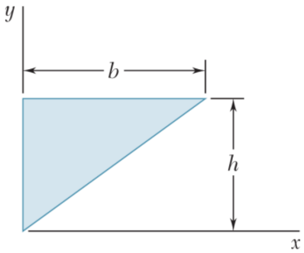
$$y = \frac{b}{a^{1/3}} x^{1/3}$$

$$dI_y = x^2 dA = x^2 y dx$$

$$dI_y = x^2 \frac{b}{a^{1/3}} x^{1/3} dx = \frac{b}{a^{1/3}} x^{7/3} dx$$

$$I_y = \int dI_y = \frac{b}{a^{1/3}} \int_0^a x^{7/3} dx = \frac{b}{a^{1/3}} \left(\frac{3}{10} a^{10/3} \right)$$

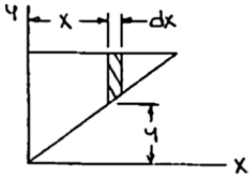
$$I_y = \frac{3}{10} a^3 b \blacktriangleleft$$



PROBLEM 9.3

Determine by direct integration the moment of inertia of the shaded area with respect to the *y* axis.

SOLUTION

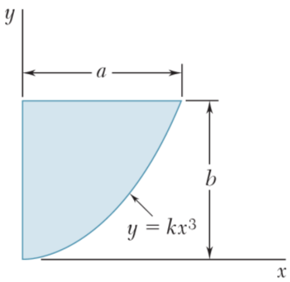


By observation
$$y = \frac{h}{b}x$$

Now
$$dI_y = x^2 \quad dA = x^2[(h-y)dx]$$
$$= hx^2\left(1-\frac{x}{b}\right)dx$$

Then
$$I_y = \int dI_y = \int_0^b hx^2\left(1-\frac{x}{b}\right)dx$$
$$= h\left[\frac{1}{3}x^3 - \frac{x^4}{4b}\right]_0^b$$

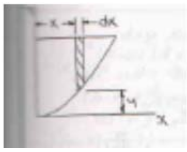
or $I_y = \frac{1}{12}b^3h$ ◀



PROBLEM 9.4

Determine by direct integration the moment of inertia of the shaded area with respect to the *y* axis.

SOLUTION



At $x = 0, \quad y = b: \quad b = ka^3$

$k = \dfrac{b}{a^3}$ then

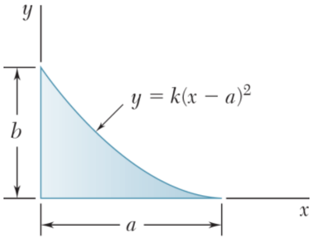
$$y = \frac{b}{a^3}\text{x}^3;\ dA = ydx$$

$$dI_y = x^2 dA = x^2\left(b - y\right)dx$$

$$I_y = \int_0^a \left[bx^2\left(1 - \frac{x^3}{a^3}\right)\right]dx$$

$$= b\left[\frac{1}{3}x^3 - \frac{x^6}{6a^3}\right]_0^a$$

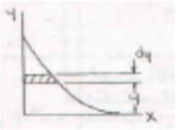
$$I_y = \frac{1}{6}a^3 b \blacktriangleleft$$



PROBLEM 9.5

Determine by direct integration the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION



At $x = 0, \quad y = b: \quad b = k(0-a)^2$

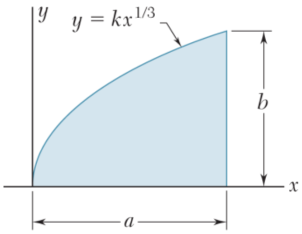
$k = \dfrac{b}{a^2}$ then $y = \dfrac{b}{a^2}(x-a)^2$ or $x = a\left(1 \pm \dfrac{\sqrt{y}}{\sqrt{b}}\right)$, take the negative root so that $y = b$ when $x = 0$.

$$dI_x = y^2 dA = y^2 x\, dy$$

$$I_x = \int_0^b \left[a\left(y^2 - \frac{1}{b^{1/2}} y^{5/2} \right) \right] dy$$

$$= a\left[\frac{1}{3} y^3 - \frac{2}{7}\left(\frac{1}{b^{1/2}}\right) y^{7/2} \right]_0^b$$

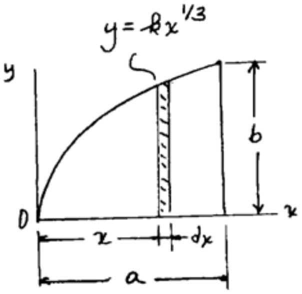
$$I_x = \frac{1}{21} ab^3 \quad ◀$$



PROBLEM 9.6

Determine by direct integration the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION

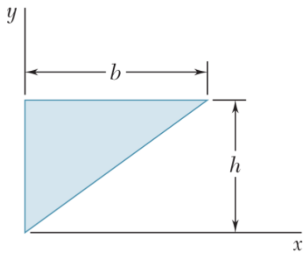


$$y = \frac{b}{a^{1/3}} x^{1/3}$$

$$dI_x = \frac{1}{3} y^3 dx = \frac{1}{3}\left(\frac{b}{a^{1/3}} x^{1/3}\right)^3 dx = \frac{1}{3}\frac{b^3}{a} x dx$$

$$I_x = \int dI_x = \int_0^a \frac{1}{3}\frac{b^3}{a} x dx = \frac{1}{3}\frac{b^3}{a}\frac{a^2}{2}$$

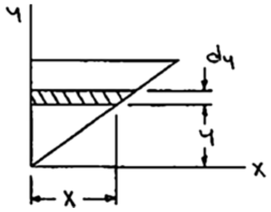
$$I_x = \frac{1}{6} ab^3 \blacktriangleleft$$



PROBLEM 9.7

Determine by direct integration the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION



By observation $y = \frac{h}{b}x$

or $x = \frac{b}{h}y$

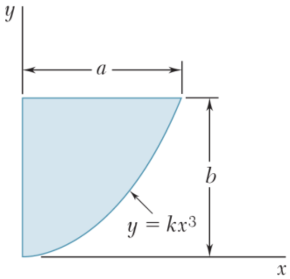
Now

$$dI_x = y^2 dA = y^2 (x\,dy)$$
$$= y^2\left(\frac{b}{h}y\,dy\right)$$
$$= \frac{b}{h}y^3 dy$$

Then

$$I_x = \int dI_x = \int_0^h \frac{b}{h}y^3 dy$$
$$= \frac{b}{h}\left[\frac{1}{4}y^4\right]_0^h$$

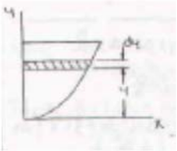
or $I_x = \frac{1}{4}bh^3$ ◀



PROBLEM 9.8

Determine by direct integration the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION



At $x = a, \quad y = b: \quad b = ka^3$

$k = \dfrac{b}{a^3}$ then

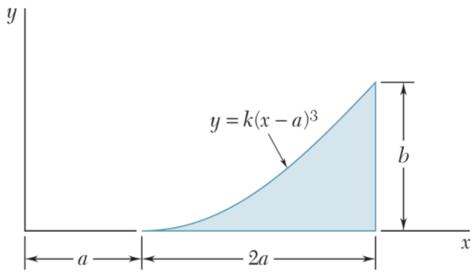
$$y = \frac{b}{a^3}x^3; \; x = \frac{a}{b^{1/3}}y^{1/3}$$

$$dI_x = y^2 dA = y^2\left(\frac{a}{b^{1/3}}y^{1/3}\right)dy$$

$$I_x = \int_0^b \left[\frac{a}{b^{1/3}}y^{7/3}\right]dy$$

$$= \left[\frac{3}{10}\left(\frac{a}{b^{1/3}}\right)y^{10/3}\right]_0^b$$

$$I_x = \frac{3}{10}ab^3 \blacktriangleleft$$

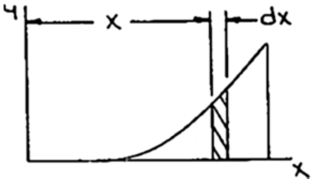


PROBLEM 9.9

Determine by direct integration the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION

At $x = 3a, \quad y = b$: $\quad b = k(3a-a)^3$



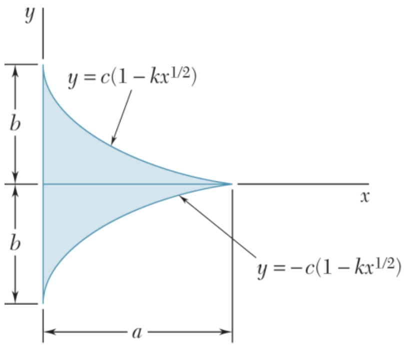
or $\quad k = \dfrac{b}{8a^3}$

Then $\quad y = \dfrac{b}{8a^3}(x-a)^3$

Now
$$dI_x = \frac{1}{3}y^3dx$$
$$= \frac{1}{3}\left[\frac{b}{8a^3}(x-a)^3\right]^3 dx$$
$$= \frac{b^3}{1536a^9}(x-a)^9\,dx$$

Then
$$I_x = \int dI_x = \int_a^{3a}\frac{b^3}{1536a^9}(x-a)^9dx = \frac{b^3}{1536a^9}\left[\frac{1}{10}(x-a)^{10}\right]_a^{3a}$$
$$= \frac{b^3}{15{,}360a^9}[(3a-a)^{10} - 0]$$

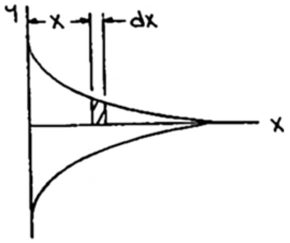
or $\quad I_x = \dfrac{1}{15}ab^3$ ◀



PROBLEM 9.10

Determine by direct integration the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION



At $x = 0, y = b$: $b = c(1-0)$ or $c = b$

At $x = a, y = 0$: $0 = b(1 - ka^{1/2})$ or $k = \dfrac{1}{a^{1/2}}$

Then $$y = b\left(1 - \frac{x^{1/2}}{a^{1/2}}\right)$$

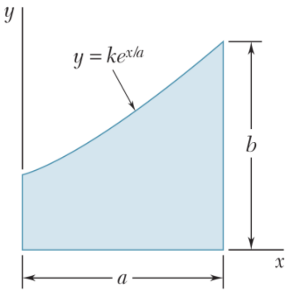
Now $$dI_x = \frac{1}{3}y^3dx = \frac{1}{3}\left[b\left(1 - \frac{x^{1/2}}{a^{1/2}}\right)\right]^3 dx$$

or $$dI_x = \frac{1}{3}b^3\left(1 - 3\frac{x^{1/2}}{a^{1/2}} + 3\frac{x}{a} - \frac{x^{3/2}}{a^{3/2}}\right)dx$$

Then $$I_x = \int dI_x = 2\int_0^a \frac{1}{3}b^3\left(1 - 3\frac{x^{1/2}}{a^{1/2}} + 3\frac{x}{a} - \frac{x^{3/2}}{a^{3/2}}\right)dx$$

$$= \frac{2}{3}b^3\left[x - 2\frac{x^{3/2}}{a^{1/2}} + \frac{3}{2}\frac{x^2}{a} - \frac{2}{5}\frac{x^{5/2}}{a^{3/2}}\right]_0^a$$

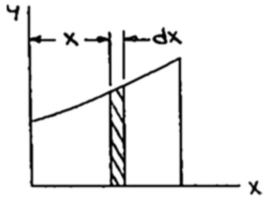
or $I_x = \dfrac{1}{15}ab^3$ ◀



PROBLEM 9.11

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION

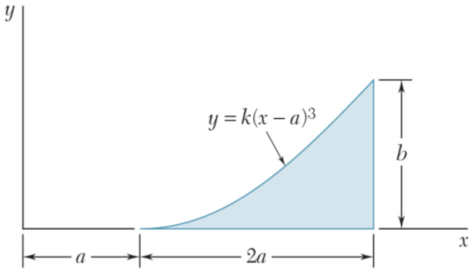


At $x = a,\ y = b$: $\quad b = ke^{a/a} \quad \text{or} \quad k = \frac{b}{e}$

Then $\quad y = \frac{b}{e}e^{x/a} = be^{x/a-1}$

Now
$$dI_x = \frac{1}{3}y^3dx = \frac{1}{3}(be^{x/a-1})^3\,dx$$
$$= \frac{1}{3}b^3e^{3(x/a-1)}dx$$

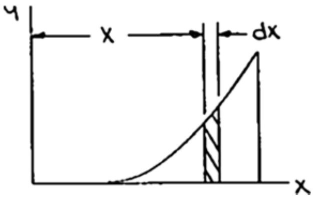
Then
$$I_x = \int dI_x = \int_0^a \frac{1}{3}b^3e^{3(x/a-1)}dx = \frac{b^3}{3}\left[\frac{a}{3}e^{3(x/a-1)}\right]_0^a$$
$$= \frac{1}{9}ab^3(1-e^{-3}) \qquad \text{or} \quad I_x = 0.1056ab^3 \blacktriangleleft$$



PROBLEM 9.12

Determine by direct integration the moment of inertia of the shaded area with respect to the *y* axis.

SOLUTION



At $x = 3a, \quad y = b$: $\quad b = k(3a-a)^3$

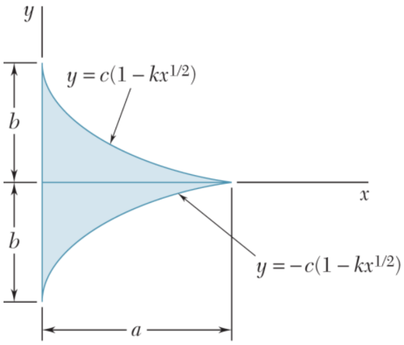
or $\quad k = \dfrac{b}{8a^3}$

Then $\quad y = \dfrac{b}{8a^3}(x-a)^3$

Now
$$dI_y = x^2 dA = x^2(y\,dx) = x^2\left[\frac{b}{8a^3}(x-a)^3\,dx\right]$$
$$= \frac{b}{8a^3}x^2(x^3 - 3x^2a + 3xa^2 - a^3)dx$$

Then
$$I_y = \int dI_y = \int_a^{3a} \frac{b}{8a^3}(x^5 - 3x^4a + 3x^3a^2 - a^3x^2)dx$$
$$= \frac{b}{8a^3}\left[\frac{1}{6}x^6 - \frac{3}{5}ax^5 + \frac{3}{4}a^2x^4 - \frac{1}{3}a^3x^3\right]_a^{3a}$$
$$= \frac{b}{8a^3}\left\{\left[\frac{1}{6}(3a)^6 - \frac{3}{5}a(3a)^5 + \frac{3}{4}a^2(3a)^4 - \frac{1}{3}a^3(3a)^3\right]\right.$$
$$\left. - \left[\frac{1}{6}(a)^6 - \frac{3}{5}a(a)^5 + \frac{3}{4}a^2(a)^4 - \frac{1}{3}a^3(a)^3\right]\right\}$$

or $\quad I_y = 3.43a^3b$ ◀



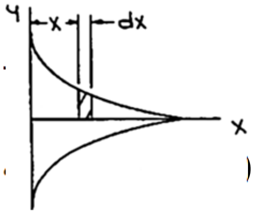
PROBLEM 9.13

Determine by direct integration the moment of inertia of the shaded area with respect to the *y* axis.

SOLUTION

At $x = 0, \quad y = b$: $\qquad b = c(1-0) \qquad$ or $\quad c = b$

$x = a, \quad y = 0$: $\qquad 0 = c(1 - ka^{1/2}) \qquad$ or $\quad k = \dfrac{1}{a^{1/2}}$

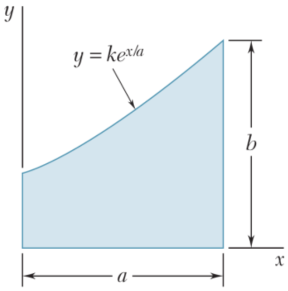


$$\Big) = x^2\left[b\left(1-\frac{x^{1/2}}{a^{1/2}}\right)dx\right]$$

$$I_y = \int dI_y = 2\int_0^a bx^2\left(1-\frac{x^{1/2}}{a^{1/2}}\right)dx$$

$$I_y = 2b\left[\frac{1}{3}x^3 - \frac{2}{7}\frac{x^{7/2}}{a^{1/2}}\right]_0^a$$

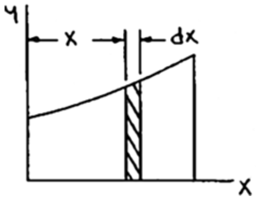
Then $\qquad I_y = \dfrac{2}{21}a^3b$ ◀



PROBLEM 9.14

Determine by direct integration the moment of inertia of the shaded area with respect to the *y* axis.

SOLUTION



At $x = a, \quad y = b$: $\qquad b = ke^{a/a}$

or $\qquad k = \dfrac{b}{e}$

Then $\qquad y = \dfrac{b}{e}e^{x/a} = be^{x/a-1}$

Now
$$dI_y = x^2 dA = x^2(y\,dx)$$
$$= x^2(be^{x/a-1}dx)$$

Then $\qquad I_y = \int dI_y = \int_0^a bx^2 e^{x/a-1}dx$

Now use integration by parts with

$$u = x^2 \qquad dv = e^{x/a-1}dx$$
$$du = 2x\,dx \qquad v = ae^{x/a-1}$$

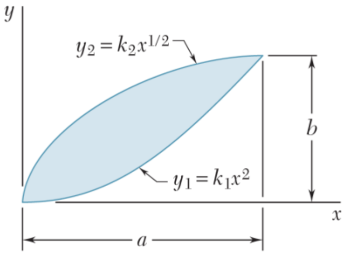
Then
$$\int_0^a x^2 e^{x/a-1}dx = \left[x^2 ae^{x/a-1}\right]_0^a - \int_0^a (ae^{x/a-1})2x\,dx$$
$$= a^3 - 2a\int_0^a xe^{x/a-1}dx$$

Using integration by parts with

$$u = x \qquad dv = e^{x/a-1}dx$$
$$du = dx \qquad v = ae^{x/a-1}$$

Then
$$I_y = b\left\{a^3 - 2a\left[(xae^{x/a-1})\big|_0^a - \int_0^a (ae^{x/a-1})dx\right]\right\}$$
$$= b\left\{a^3 - 2a\left[a^2 - (a^2e^{x/a-1})\big|_0^a\right]\right\}$$
$$= b\left\{a^3 - 2a\left[a^2 - (a^2 - a^2e^{-1})\right]\right\}$$

or $\quad I_y = 0.264a^3b$ ◀

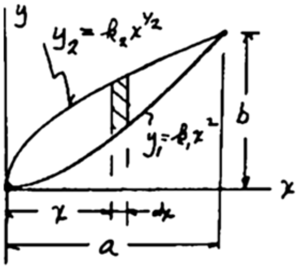


PROBLEM 9.15

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the *x* axis.

SOLUTION

$$y_1 = k_1x^2 \qquad y_2 = k_2x^{1/2}$$



For

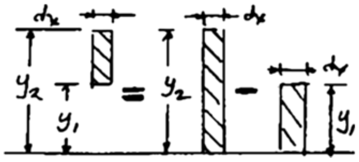
$$x = 0 \quad \text{and} \quad y_1 = y_2 = b$$

$$b = k_1a^2 \qquad b = k_2a^{1/2}$$

$$k_1 = \frac{b}{a^2} \qquad k_2 = \frac{b}{a^{1/2}}$$

Thus,

$$y_1 = \frac{b}{a^2}x^2 \qquad y_2 = \frac{b}{a^{1/2}}x^{1/2}$$



$$dA = (y_2 - y_1)dx$$

$$A = \int_0^a \left[\frac{b}{a^{1/2}}x^{1/2} - \frac{b}{a^2}x^2\right]dx$$

$$A = \frac{2}{3}\frac{ba^{3/2}}{a^{1/2}} - \frac{ba^3}{3a^2}$$

$$A = \frac{1}{3}ab$$

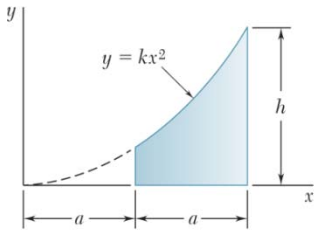
$$dI_x = \frac{1}{3}y_2^3dx - \frac{1}{3}y_1^3dx$$

$$= \frac{1}{3}\frac{b^3}{a^{3/2}}x^{3/2}dx - \frac{1}{3}\frac{b^3}{a^6}x^6dx$$

$$I_x = \int dI_x = \frac{b^3}{3a^{3/2}}\int_0^a x^{3/2}dx - \frac{b^3}{3a^6}\int_0^a x^6dx$$

$$= \frac{b^3}{3a^{3/2}}\frac{a^{5/2}}{\left(\frac{5}{2}\right)} - \frac{b^3}{3a^6}\frac{a^7}{7} = \left(\frac{2}{15} - \frac{1}{21}\right)ab^3 \qquad I_x = \frac{3}{35}ab^3 \blacktriangleleft$$

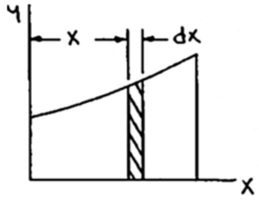
$$k_x^2 = \frac{I_x}{A} = \frac{\left(\frac{3}{35}ab^3\right)}{\frac{ab}{b}} \qquad k_x = b\sqrt{\frac{9}{35}} \blacktriangleleft$$



PROBLEM 9.16

Determine by direct integration the moment of inertia of the shaded area with respect to the *y* axis.

SOLUTION



At $x = 2a, \quad y = h$: $\quad h = k(2a)^2$

or $\quad k = \dfrac{h}{4a^2}$

Then $\quad y = \dfrac{h}{4a^2}x^2$

Now $\quad dA = ydx = \dfrac{h}{4a^2}x^2dx$

$$A = \int dA = \frac{h}{4a^2}\int_a^{2a} x^2dx$$

$$A = \left(\frac{h}{4a^2}\right)\left(\frac{x^3}{3}\right)_a^{2a} = \left(\frac{h}{4a^2}\right)\left(\frac{8a^3}{3} - \frac{a^3}{3}\right) = \frac{7}{12}ah$$

$$dI_x = \frac{1}{3}y^3dx = \frac{1}{3}\left(\frac{h}{64a^6}\right)x^6dx$$

Then $\quad I_x = \int dI_x = \dfrac{h^3}{192a^6}\displaystyle\int_0^{2a} x^6dx$

$$I_x = \left(\frac{h^3}{192a^6}\right)\frac{x^7}{7}\bigg|_a^{2a} = \frac{h^3}{192a^6}\left(\frac{128a^7 - a^7}{7}\right)$$

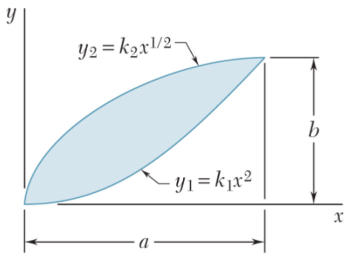
$$I_x = \frac{127}{(7)(192)}ah^3 = 0.094494ah^3$$

$$I_x = 0.0945ah^3 \blacktriangleleft$$

$$k_x{}^2 = \frac{I_x}{A} = \frac{0.094494ah^3}{\frac{7}{12}ah} = 0.161983h^2$$

$$k_x = 0.402h \blacktriangleleft$$

PROBLEM 9.17



Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION

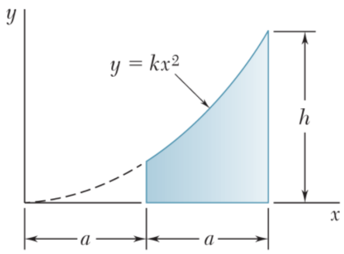
See figure of solution on Problem 9.15.

$$A = \frac{1}{3}ab \qquad dI_y = x^2 dA = x^2(y_2 - y_1)dx$$

$$I_y = \int_0^a x^2\left(\frac{b}{a^{1/2}}x^{1/2} - \frac{b}{a^2}x^2\right)dx = \frac{b}{a^{1/2}}\int_0^a x^{5/2}dx - \frac{b}{a^2}\int_0^a x^4 dx$$

$$I_y = \frac{b}{a^{1/2}} \cdot \frac{b^{7/2}}{\left(\frac{7}{2}\right)} - \frac{b}{a^2} \cdot \frac{a^5}{5} = \left(\frac{2}{7} - \frac{1}{5}\right)a^3 b \qquad I_y = \frac{3}{35}a^3 b \blacktriangleleft$$

$$k_y^2 = \frac{I_y}{A} = \frac{\left(\frac{3}{35}a^3 b\right)}{\frac{ab}{3}} \qquad k_y = a\sqrt{\frac{9}{35}} \blacktriangleleft$$



PROBLEM 9.18

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the *y* axis.

SOLUTION

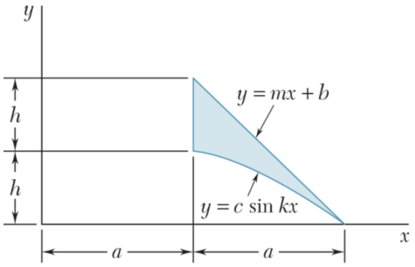
See figure of solution on Problem 9.16

$$y = \frac{h}{4a^2}x^2 \qquad A = \frac{7}{12}ah \qquad dA = ydx \qquad dI_y = x^2 dA = x^2\left(\frac{h}{4a^2}x^2\right)dx$$

$$I_y = \frac{h}{4a^2}\int_0^{2a} x^4 dx = \frac{h}{4a^2}\left(\frac{x^5}{5}\right)\Bigg|_a^{2a}$$

$$I_y = \frac{h}{4a^2}\left(\frac{32a^5}{5} - \frac{a^5}{5}\right) = \frac{31}{20}ha^3 \qquad I_y = \frac{31}{20}ha^3 \blacktriangleleft$$

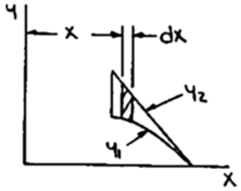
$$k_y^2 = \frac{I_y}{A} = \frac{\left(\frac{31}{20}ha^3\right)}{\frac{7}{12}ah} = \frac{93}{35}a^2 \qquad k_y = a\sqrt{\frac{93}{35}} \blacktriangleleft$$



PROBLEM 9.19

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the *x* axis.

SOLUTION



y_1: At $x = 2a$, $y = 0$: $\quad 0 = c\sin k(2a)$

$$2ak = \pi \quad \text{or} \quad k = \frac{\pi}{2a}$$

At $x = a$, $y = h$: $\quad h = c\sin\frac{\pi}{2a}(a) \quad \text{or} \quad c = h$

y_2: At $x = a$, $y = 2h$: $\quad 2h = ma - b$

At $x = 2a$, $y = 0$: $\quad 0 = m(2a) + b$

Solving yields $\quad m = -\frac{2h}{a}, \quad b = 4h$

Then $\quad y_1 = h\sin\frac{\pi}{2a}x \qquad y_2 = -\frac{2h}{a}x + 4h$

$$= \frac{2h}{a}(-x + 2a)$$

Now $\quad dA = (y_2 - y_1)dx = \left[\frac{2h}{a}(-x + 2a) - h\sin\frac{\pi}{2a}x\right]dx$

Then $\quad A = \int dA = \int_a^{2a} h\left[\frac{2}{a}(-x + 2a) - \sin\frac{\pi}{2a}x\right]dx$

$$= h\left[-\frac{1}{a}(-x + 2a)^2 + \frac{2a}{\pi}\cos\frac{\pi}{2a}x\right]_a^{2a}$$

$$= h\left[\left(-\frac{2a}{\pi}\right) + \frac{1}{a}(-a + 2a)^2\right] = ah\left(1 - \frac{2}{\pi}\right)$$

$$= 0.36338ah$$

PROBLEM 9.19 (Continued)

Find: I_x and k_x

We have
$$dI_x = \left(\frac{1}{3}y_2 - \frac{1}{3}y_1\right)dx = \frac{1}{3}\left\{\left[\frac{2h}{a}(-x+2a)\right]^3 - \left(h\sin\frac{\pi}{2a}x\right)^3\right\}dx$$
$$= \frac{h^3}{3}\left[\frac{8}{a^3}(-x+2a)^3 - \sin^3\frac{\pi}{2a}x\right]$$

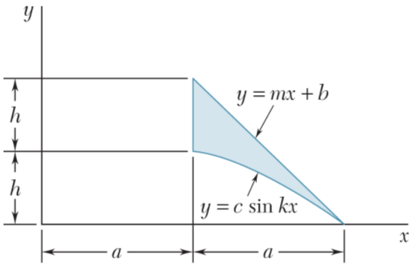
Then
$$I_x = \int dI_x = \int_a^{2a}\frac{h^3}{3}\left[\frac{8}{a^3}(-x+2a)^3 - \sin^3\frac{\pi}{2a}x\right]dx$$

Now
$$\sin^3\theta = \sin\theta(1-\cos^2\theta) = \sin\theta - \sin\theta\cos^2\theta$$

Then
$$I_x = \frac{h^3}{3}\int_a^{2a}\left[\frac{8}{a^3}(-x+2a)^3 - \left(\sin\frac{\pi}{2a}x - \sin\frac{\pi}{2a}x\cos^2\frac{\pi}{2a}x\right)\right]dx$$
$$= \frac{h^3}{3}\left[-\frac{2}{a^3}(-x+2a)^4 + \frac{2a}{\pi}\cos\frac{\pi}{2a}x - \frac{2a}{3\pi}\cos^3\frac{\pi}{2a}x\right]_a^{2a}$$
$$= \frac{h^3}{3}\left[\left(-\frac{2a}{\pi} + \frac{2a}{3\pi}\right) + \frac{2}{a^3}(-a+2a)^4\right]$$
$$= \frac{2}{3}ah^3\left(1 - \frac{2}{3\pi}\right)$$

$$I_x = 0.52520ah^3$$ or $I_x = 0.525ah^3$ ◀

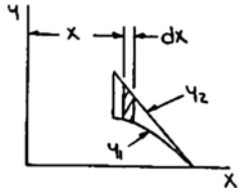
and
$$k_x^2 = \frac{I_x}{A} = \frac{0.52520ah^3}{0.36338ah}$$ or $k_x = 1.202h$ ◀



PROBLEM 9.20

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the *y* axis.

SOLUTION



y_1: At $x = 2a, \quad y = 0$: $\quad 0 = c \sin k(2a)$

$$2ak = \pi \quad \text{or} \quad k = \frac{\pi}{2a}$$

At $x = a, \quad y = h$: $\quad h = c \sin \frac{\pi}{2a}(a) \quad \text{or} \quad c = h$

y_2: At $x = a, \quad y = 2h$: $\quad 2h = ma - b$

At $x = 2a, \quad y = 0$: $\quad 0 = m(2a) + b$

Solving yields $\quad m = -\frac{2h}{a}, \quad b = 4h$

Then $\quad y_1 = h \sin \frac{\pi}{2a} x \qquad y_2 = -\frac{2h}{a} x + 4h$

$$= \frac{2h}{a}(-x + 2a)$$

Now $\quad dA = (y_2 - y_1)dx = \left[\frac{2h}{a}(-x + 2a) - h \sin \frac{\pi}{2a} x\right] dx$

Then $\quad A = \int dA = \int_a^{2a} h \left[\frac{2}{a}(-x + 2a) - \sin \frac{\pi}{2a} x\right] dx$

$$= h \left[-\frac{1}{a}(-x + 2a)^2 + \frac{2a}{\pi} \cos \frac{\pi}{2a} x\right]_a^{2a}$$

$$= h \left[\left(-\frac{2a}{\pi}\right) + \frac{1}{a}(-a + 2a)^2\right] = ah\left(1 - \frac{2}{\pi}\right)$$

$$= 0.36338ah$$

PROBLEM 9.20 (Continued)

Find: I_y and k_y

We have
$$dI_y = x^2 dA = x^2\left\{\left[\frac{2h}{a}(-x+2a) - h\sin\frac{\pi}{2a}x\right]dx\right\}$$
$$= h\left[\frac{2}{a}(-x^3+2ax^2) - x^2\sin\frac{\pi}{2a}x\right]dx$$

Then
$$I_y = \int dI_y = \int_a^{2a} h\left[\frac{2}{a}(-x^3+2ax^2) - x^2\sin\frac{\pi}{2a}x\right]dx$$

Now using integration by parts with

$$u = x^2 \qquad dv = \sin\frac{\pi}{2a}x\,dx$$

$$du = 2x\,dx \qquad v = -\frac{2a}{\pi}\cos\frac{\pi}{2a}x$$

Then
$$\int x^2\sin\frac{\pi}{2a}x\,dx = x^2\left(-\frac{2a}{\pi}\cos\frac{\pi}{2a}x\right) - \int\left(-\frac{2a}{\pi}\cos\frac{\pi}{2a}x\right)(2x\,dx)$$

Now let
$$u = x \qquad dv = \cos\frac{\pi}{2a}x\,dx$$

$$du = dx \qquad v = \frac{2a}{\pi}\sin\frac{\pi}{2a}x$$

Then
$$\int x^2\sin\frac{\pi}{2a}x\,dx = -\frac{2a}{\pi}x^2\cos\frac{\pi}{2a}x + \frac{4a}{\pi}\left[x\left(\frac{2a}{\pi}\sin\frac{\pi}{2a}x\right) - \int\left(\frac{2a}{\pi}\sin\frac{\pi}{2a}x\right)dx\right]$$

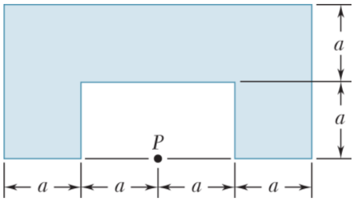
$$I_y = h\left[\frac{2}{a}\left(-\frac{1}{4}x^4 + \frac{2}{3}ax^3\right)\right.$$
$$\left.-\left(-\frac{2a}{\pi}x^2\cos\frac{\pi}{2a}x + \frac{8a^2}{\pi^2}x\sin\frac{\pi}{2a}x + \frac{16a^3}{\pi^3}\cos\frac{\pi}{2a}x\right)\right]_a^{2a}$$

$$= h\left\{\frac{2}{a}\left[-\frac{1}{4}(2a)^4 + \frac{2}{3}a(2a)^3\right] - \frac{2a}{\pi}(2a)^2 + \frac{16a^3}{\pi^3}\right\}$$
$$-h\left\{\frac{2}{a}\left[-\frac{1}{4}(a)^4 + \frac{2}{3}a(a)^3\right] - \frac{8a^2}{\pi^2}(a)\right\}$$
$$= 0.61345a^3h$$

or $I_y = 0.613a^3h$ ◀

and
$$k_y^2 = \frac{I_y}{A} = \frac{0.61345a^3h}{0.36338ah}$$

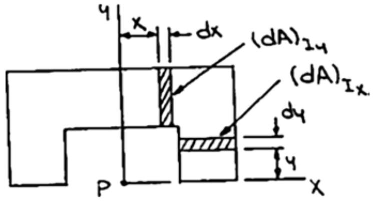
or $k_y = 1.299a$ ◀



PROBLEM 9.21

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point *P*.

SOLUTION



We have $$dI_x = y^2 dA = y^2[(x_2 - x_1)dy]$$

Then $$I_x = 2\left[\int_2^a y^2(2a-a)dy + \int_a^{2a} y^2(2a-0)dy\right]$$

$$= 2\left\{a\left[\frac{1}{3}y^3\right]_0^a + 2a\left[\frac{1}{3}y^3\right]_a^{2a}\right\}$$

$$= 2\left\{a\left[\frac{1}{3}(a)^3\right] + \frac{2}{3}a\left[(2a)^3 - (a)^3\right]\right\}$$

$$= 10a^4$$

Also $$dI_y = x^2 dA = x^2[(y_2 - y_1)dx]$$

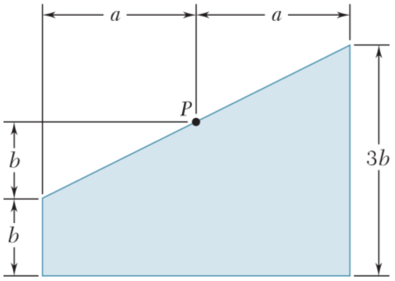
Then $$I_y = 2\left[\int_0^a x^2(2a-a)dx + \int_a^{2a} x^2(2a-0)dx\right]$$

$$= 10a^4$$

Now $$J_P = I_x + I_y = 10a^4 + 10a^4$$ or $J_P = 20a^4$ ◀

and $$k_P^2 = \frac{J_P}{A} = \frac{20a^4}{(4a)(2a)-(2a)(a)}$$

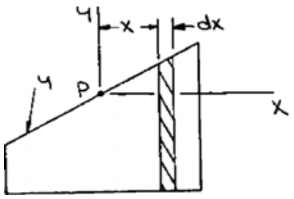
$$= \frac{10}{3}a^2$$ or $k_P = 1.826a$ ◀



PROBLEM 9.22

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point P.

SOLUTION



By observation
$$y = \frac{b}{a}x$$

First note
$$dA = (y + 2b)dx$$
$$= \frac{b}{a}(x + 2a)dx$$

Now
$$dI_x = \left(\frac{1}{3}y_{\text{top}}^3 - \frac{1}{3}y_{\text{bottom}}^3\right)dx$$
$$= \frac{1}{3}\left[\left(\frac{b}{a}x\right)^3 - (-2b)^3\right]dx$$
$$= \frac{1}{3}\frac{b^3}{a^3}(x^3 + 8a^3)dx$$

Then
$$I_x = \int dI_x = \int_{-a}^{a} \frac{1}{3}\frac{b^3}{a^3}(x^2 + 8a^3)dx$$
$$= \frac{1}{3}\frac{b^3}{a^3}\left[\frac{1}{4}x^4 + 8a^3x\right]_{-a}^{a}$$
$$= \frac{1}{3}\frac{b^3}{a^3}\left\{\left[\frac{1}{4}(a)^4 + 8a^3(a)\right] - \left[\frac{1}{4}(-a)^4 + 8a^3(-a)\right]\right\} = \frac{16}{3}ab^3$$

Also
$$dI_y = x^2dA = x^2\left[\frac{b}{a}(x + 2a)dx\right]$$

Then
$$I_y = \int dI_y = \int_{-a}^{a} \frac{b}{a}x^2(x + 2a)dx$$
$$= \frac{b}{a}\left[\frac{1}{4}x^4 + \frac{2}{3}ax^3\right]_{-a}^{a}$$
$$= \frac{b}{a}\left\{\left[\frac{1}{4}(a)^4 + \frac{2}{3}a(a)^3\right] - \left[\frac{1}{4}(-a)^4 + \frac{2}{3}a(-a)^3\right]\right\} = \frac{4}{3}a^3b$$

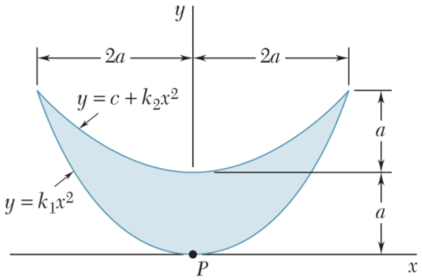
PROBLEM 9.22 (Continued)

Now $$J_P = I_x + I_y = \frac{16}{3}ab^3 + \frac{4}{3}a^3b$$

$$J_P = \frac{4}{3}ab(a^2 + 4b^2) \blacktriangleleft$$

and $$k_P^2 = \frac{J_P}{A} = \frac{\frac{4}{3}ab(a^2 + 4b^2)}{(2a)(3b) - \frac{1}{2}(2a)(2b)}$$

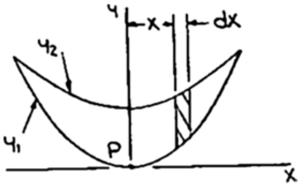
$$= \frac{1}{3}(a^2 + 4b^2) \text{ or } \qquad k_P = \sqrt{\frac{a^2 + 4b^2}{3}} \blacktriangleleft$$



PROBLEM 9.23

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point *P*.

SOLUTION



y_1: At $x = 2a$, $y = 2a$:

$$2a = k_1(2a)^2 \quad \text{or} \quad k_1 = \frac{1}{2a}$$

y_2: At $x = 0$, $y = a$: $a = c$

At $x = 2a$, $y = 2a$:

$$2a = a + k_2(2a)^2 \quad \text{or} \quad k_2 = \frac{1}{4a}$$

Then

$$y_1 = \frac{1}{2a}x^2 \qquad y_2 = a + \frac{1}{4a}x^2$$
$$= \frac{1}{4a}(4a^2 + x^2)$$

Now

$$dA = (y_2 - y_1)dx = \left[\frac{1}{4a}(4a^2 + x^2) - \frac{1}{2a}x^2\right]dx$$
$$= \frac{1}{4a}(4a^2 - x^2)dx$$

Then

$$A = \int dA = 2\int_0^{2a} \frac{1}{4a}(4a^2 - x^2)dx = \frac{1}{2a}\left[4a^2x - \frac{1}{3}x^3\right]_0^{2a} = \frac{8}{3}a^2$$

Now

$$dI_x = \left(\frac{1}{3}y_2^3 - \frac{1}{3}y_1^3\right)dx = \frac{1}{3}\left\{\left[\frac{1}{4a}(4a^2 + x^2)\right]^3 - \left[\frac{1}{2a}x^2\right]^3\right\}dx$$
$$= \frac{1}{3}\left[\frac{1}{64a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 + x^6) - \frac{1}{8a^3}x^6\right]dx$$
$$= \frac{1}{192a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 - 7x^6)dx$$

PROBLEM 9.23 (Continued)

Then $$I_x = \int dI_x = 2\int_0^{2a} \frac{1}{192a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 - 7x^6)dx$$

$$= \frac{1}{96a^3}\left[64a^6x + 16a^4x^3 + \frac{12}{5}a^2x^5 - x^7\right]_0^{2a}$$

$$= \frac{1}{96a^3}\left[64a^6(2a) + 16a^4(2a)^3 + \frac{12}{5}a^2(2a)^5 - (2a)^7\right]$$

$$= \frac{1}{96}a^4\left(128 + 128 + \frac{12}{5} \times 32 - 128\right) = \frac{32}{15}a^4$$

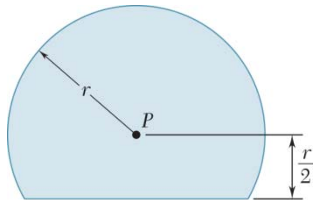
Also $$dI_y = x^2dA = x^2\left[\frac{1}{4a}(4a^2 - x^2)dx\right]$$

Then $$I_y = \int dI_y = 2\int_0^{2a} \frac{1}{4a}x^2(4a^2 - x^2)dx = \frac{1}{2a}\left[\frac{4}{3}a^2x^3 - \frac{1}{5}x^5\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{4}{3}a^2(2a)^3 - \frac{1}{5}(2a)^5\right] = \frac{32}{2}a^4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{32}{15}a^4$$

Now $$J_P = I_x + I_y = \frac{32}{15}a^4 + \frac{32}{15}a^4$$ or $J_P = \dfrac{64}{15}a^4$ ◀

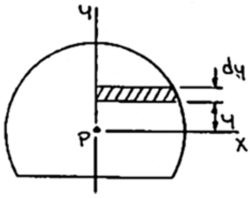
and $$k_P^2 = \frac{J_P}{A} = \frac{\frac{64}{15}a^4}{\frac{8}{3}a^2} = \frac{8}{5}a^2$$ or $k_P = 1.265a$ ◀



PROBLEM 9.24

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point *P*.

SOLUTION



The equation of the circle is

$$x^2 + y^2 = r^2$$

So that

$$x = \sqrt{r^2 - y^2}$$

Now

$$dA = xdy = \sqrt{r^2 - y^2}\,dy$$

Then

$$A = \int dA = 2\int_{-r/2}^{r} \sqrt{r^2 - y^2}\,dy$$

Let

$$y = r\sin\theta; \quad dy = r\cos\theta\,d\theta$$

Then

$$A = 2\int_{-\pi/6}^{\pi/2} \sqrt{r^2 - (r\sin\theta)^2}\,r\cos\,d\theta$$

$$= 2\int_{-\pi/6}^{\pi/2} r^2\cos^2\theta\,d\theta = 2r^2\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{-\pi/6}^{\pi/2}$$

$$= 2r^2\left[\frac{\frac{\pi}{2}}{2} - \left(\frac{-\frac{\pi}{6}}{2} + \frac{\sin -\frac{\pi}{3}}{4}\right)\right] = 2r^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right)$$

$$= 2.5274r^2$$

Now

$$dI_x = y^2 dA = y^2\left(\sqrt{r^2 - y^2}\,dy\right)$$

Then

$$I_x = \int dI_x = 2\int_{-r/2}^{r} y^2\sqrt{r^2 - y^2}\,dy$$

Let

$$y = r\sin\theta; \quad dy = r\cos\theta\,d\theta$$

Then

$$I_x = 2\int_{-\pi/6}^{\pi/2} (r\sin\theta)^2\sqrt{r^2 - (r\sin\theta)^2}\,r\cos\theta\,d\theta$$

$$= 2\int_{-\pi/6}^{\pi/2} r^2\sin^2\theta(r\cos\theta)r\cos\theta\,d\theta$$

Now

$$\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \sin^2\theta\cos^2\theta = \frac{1}{4}\sin 2\theta$$

PROBLEM 9.24 (Continued)

Then

$$I_x = 2\int_{-\pi/6}^{\pi/2} r^4\left(\frac{1}{4}\sin^2 2\theta\right)d\theta = \frac{r^4}{2}\left[\frac{\theta}{2}-\frac{\sin 4\theta}{8}\right]_{-\pi/6}^{\pi/2}$$

$$= \frac{r^4}{2}\left[\frac{\frac{\pi}{2}}{2} - \left(\frac{\frac{\pi}{6}}{2} - \frac{\sin - \frac{2\pi}{3}}{8}\right)\right]$$

$$= \frac{r^4}{2}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{16}\right)$$

Also

$$dI_y = \frac{1}{3}x^3 dy = \frac{1}{3}\left(\sqrt{r^2-y^2}\right)^3 dy$$

Then

$$I_y = \int dI_y = 2\int_{-r/2}^{r}\frac{1}{3}(r^2-y^2)^{3/2}dy$$

Let

$$y = r\sin\theta; \quad dy = r\cos\theta\, d\theta$$

Then

$$I_y = \frac{2}{3}\int_{-\pi/6}^{\pi/2}[r^2-(r\sin\theta)^2]^{3/2} r\cos\theta\, d\theta$$

$$I_y = \frac{2}{3}\int_{-\pi/6}^{\pi/2}(r^3\cos^3\theta) r\cos\theta\, d\theta$$

Now

$$\cos^4\theta = \cos^2\theta(1-\sin^2\theta) = \cos^2\theta - \frac{1}{4}\sin^2 2\theta$$

Then

$$I_y = \frac{2}{3}\int_{-\pi/6}^{\pi/2} r^4\left(\cos^2\theta - \frac{1}{4}\sin^2 2\theta\right)d\theta$$

$$= \frac{2}{3}r^4\left[\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)-\frac{1}{4}\left(\frac{\theta}{2}-\frac{\sin 4\theta}{8}\right)\right]_{-\pi/6}^{\pi/2}$$

$$= \frac{2}{3}r^4\left\{\left[\frac{\frac{\pi}{2}}{2}-\frac{1}{4}\left(\frac{\frac{\pi}{2}}{2}\right)\right]-\left[\frac{-\frac{\pi}{6}}{2}+\frac{\sin-\frac{\pi}{3}}{4}-\frac{1}{4}\left(\frac{-\frac{\pi}{6}}{2}-\frac{\sin-\frac{2\pi}{3}}{8}\right)\right]\right\}$$

$$= \frac{2}{3}r^4\left[\frac{\pi}{4}-\frac{\pi}{16}+\frac{\pi}{12}+\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)-\frac{\pi}{48}+\frac{1}{32}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{2}{3}r^4\left(\frac{\pi}{4}+\frac{9\sqrt{3}}{64}\right)$$

PROBLEM 9.24 (Continued)

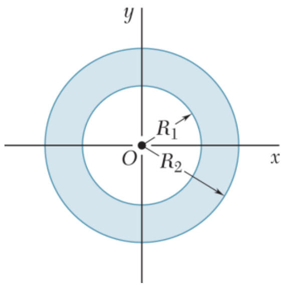
Now $$J_P = I_x + I_y = \frac{r^4}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{16}\right) + \frac{2}{3}r^4\left(\frac{\pi}{4} + \frac{9\sqrt{3}}{64}\right)$$

$$= r^4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{16}\right) = 1.15545r^4$$

or $J_P = 1.155r^4$ ◀

and $$k_P^2 = \frac{J_P}{A} = \frac{1.15545r^4}{2.5274r^2}$$

or $k_P = 0.676r$ ◀



PROBLEM 9.25

(*a*) Determine by direct integration the polar moment of inertia of the annular area shown with respect to Point *O*. (*b*) Using the result of Part *a*, determine the moment of inertia of the given area with respect to the *x* axis.

SOLUTION

(*a*)

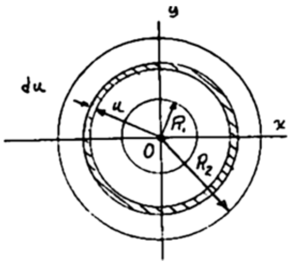
$$dA = 2\pi u\,du$$

$$dJ_O = u^2 dA = u^2(2\pi u\,du)$$

$$= 2\pi u^3\,du$$

$$J_O = \int dJ_O = 2\pi\int_{R_1}^{R_2} u^3 du$$

$$= 2\pi\left|\frac{1}{4}u^4\right|_{R_1}^{R_2}$$



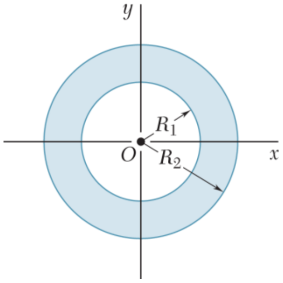
$$J_O = \frac{\pi}{2}\left[R_2^4 - R_1^4\right] \blacktriangleleft$$

(*b*) From Eq. (9.4): (*Note* by symmetry.) $I_x = I_y$

$$J_O = I_x + I_y = 2I_x$$

$$I_x = \frac{1}{2}J_O = \frac{\pi}{4}\left[R_2^4 - R_1^4\right]$$

$$I_x = \frac{\pi}{4}\left[R_2^4 - R_1^4\right] \blacktriangleleft$$

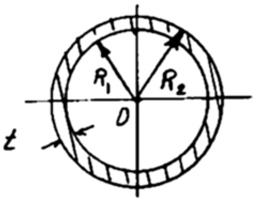


PROBLEM 9.26

(*a*) Show that the polar radius of gyration k_O of the annular area shown is approximately equal to the mean radius $R_m = (R_1 + R_2)/2$ for small values of the thickness $t = R_2 - R_1$. (*b*) Determine the percentage error introduced by using R_m in place of k_O for the following values of t/R_m: 1, $\frac{1}{2}$, and $\frac{1}{10}$.

SOLUTION

(*a*) From Problem 9.23:



$$J_O = \frac{\pi}{2}\left[R_2^4 - R_1^4\right]$$

$$k_O^2 = \frac{J_O}{A} = \frac{\frac{\pi}{2}\left[R_2^4 - R_1^4\right]}{\pi\left(R_2^2 - R_1^2\right)}$$

$$k_O^2 = \frac{1}{2}\left[R_2^2 + R_1^2\right]$$

Thickness: $t = R_2 - R_1$

Mean radius: $R_m = \frac{1}{2}(R_1 + R_2)$

Thus, $R_1 = R_m - \frac{1}{2}t$ and $R_2 = R_m + \frac{1}{2}t$

$$k_O^2 = \frac{1}{2}\left[\left(R_m + \frac{1}{2}t\right)^2 + \left(R_m - \frac{1}{2}t\right)^2\right] = R_m^2 + \frac{1}{4}t^2$$

For t small compared to R_m: $k_O^2 \approx R_m^2$ $\qquad k_O \approx R_m$ ◀

(*b*) Percentage error $= \dfrac{\text{(exact value)} - \text{(approximation value)}}{\text{(exact value)}}100$

$$\text{P.E.} = -\frac{\sqrt{R_m^2 + \frac{1}{4}t^2} - R_m}{\sqrt{R_m^2 + \frac{1}{4}t^2}}(100) = -\frac{\sqrt{1 + \frac{1}{4}\left(\frac{t}{R_m}\right)^2} - 1}{\sqrt{1 + \frac{1}{4}\left(\frac{t}{R_m}\right)^2}}100$$

PROBLEM 9.26 (Continued)

For $\dfrac{t}{R_m} = 1$:

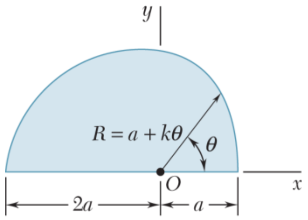
$$\text{P.E.} = \frac{\sqrt{1+\frac{1}{4}}-1}{\sqrt{1+\frac{1}{4}}}(100) = -10.56\%$$ ◀

For $\dfrac{t}{R_m} = \dfrac{1}{2}$:

$$\text{P.E.} = -\frac{\sqrt{1+\frac{1}{4}\left(\frac{1}{2}\right)^2}-1}{\sqrt{1+\frac{1}{4}\left(\frac{1}{2}\right)^2}}(100) = -2.99\%$$ ◀

For $\dfrac{t}{R_m} = \dfrac{1}{10}$:

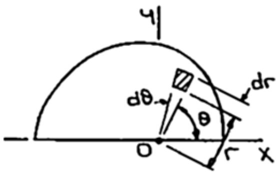
$$\text{P.E.} = \frac{\sqrt{1+\frac{1}{4}\left(\frac{1}{10}\right)^2}-1}{\sqrt{1+\frac{1}{4}\left(\frac{1}{10}\right)^2}}(100) = -0.1250\%$$ ◀



PROBLEM 9.27

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point *O*.

SOLUTION



At $\theta = \pi, \quad R = 2a$: $$2a = a + k(\pi)$$

or $$k = \frac{a}{\pi}$$

Then $$R = a + \frac{a}{\pi}\theta = a\left(1+\frac{\theta}{\pi}\right)$$

Now $$dA = (dr)(r\,d\theta)$$ $$= r dr\,d\theta$$

Then $$A = \int dA = \int_0^{\pi}\int_0^{a(1+\theta/\pi)} r dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}r^2\right]_0^{a(1+\theta/\pi)} d\theta$$

$$A = \int_0^{\pi}\frac{1}{2}a^2\left(1+\frac{\theta}{\pi}\right)^2 d\theta = \frac{1}{2}a^2\left[\frac{\pi}{3}\left(1+\frac{\theta}{\pi}\right)^3\right]_0^{\pi}$$

$$= \frac{1}{6}\pi a^2\left[\left(1+\frac{\pi}{\pi}\right)^3 - (1)^3\right] = \frac{7}{6}\pi a^2$$

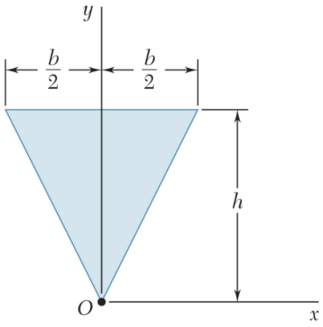
Now $$dJ_O = r^2 dA = r^2(r dr\,d\theta)$$

Then $$J_O = \int dJ_O = \int_0^{\pi}\int_0^{a(1+\theta/\pi)} r^3 dr\,d\theta$$

$$= \int_0^{\pi}\left[\frac{1}{4}r^4\right]_0^{a(1+\theta/\pi)} d\theta = \int_0^{\pi}\frac{1}{4}a^4\left(1+\frac{\theta}{\pi}\right)^4 d\theta$$

$$= \frac{1}{4}a^4\left[\frac{\pi}{5}\left(1+\frac{\theta}{\pi}\right)^5\right]_0^{\pi} = \frac{1}{20}\pi a^4\left[\left(1+\frac{\pi}{\pi}\right)^5 - (1)^5\right]$$ or $J_O = \frac{31}{20}\pi a^4$ ◀

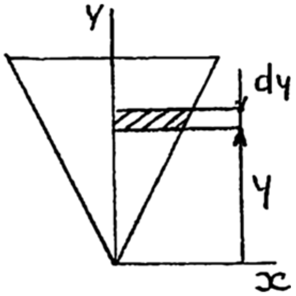
and $$k_O^2 = \frac{J_O}{A} = \frac{\frac{31}{20}\pi a^4}{\frac{7}{6}\pi a^2} = \frac{93}{70}a^2$$ or $k_O = 1.153a$ ◀



PROBLEM 9.28

Determine the polar moment of inertia and the polar radius of gyration of the isosceles triangle shown with respect to Point *O*.

SOLUTION

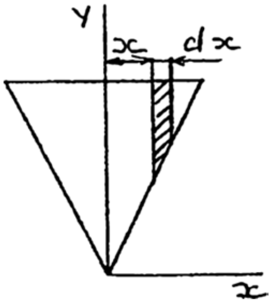


By observation: $$y = \frac{h}{\frac{b}{2}} x$$

or $$x = \frac{b}{2h} y$$

Now $$dA = x dy = \left(\frac{b}{2h} y \right) dy$$

and $$dI_x = y^2 dA = \frac{b}{2h} y^3 dy$$



Then $$I_x = \int dI_x = 2\int_0^h \frac{b}{2h} y^3 dy$$

$$= \frac{b}{h} \frac{y^4}{4} \bigg|_0^h = \frac{1}{4} bh^3$$

From above: $$y = \frac{2h}{b} x$$

Now $$dA = (h - y)dx = \left(h - \frac{2h}{b} x \right) dx$$

$$= \frac{h}{b}(b - 2x)dx$$

and $$dI_y = x^2 dA = x^2 \frac{h}{b}(b - 2x)dx$$

PROBLEM 9.28 (Continued)

Then

$$I_y = \int dI_y = 2\int_0^{b/2} \frac{h}{b} x^2 (b-2x)dx$$

$$= 2\frac{h}{b}\left[\frac{1}{3}bx^3 - \frac{1}{2}x^4\right]_0^{b/2}$$

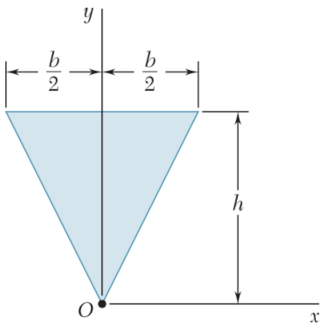
$$= 2\frac{h}{b}\left[\frac{b}{3}\left(\frac{b}{2}\right)^3 - \frac{1}{2}\left(\frac{b}{2}\right)^4\right] = \frac{1}{48}b^3h$$

Now

$$J_O = I_x + I_y = \frac{1}{4}bh^3 + \frac{1}{48}b^3h \qquad \text{or} \quad J_O = \frac{bh}{48}(12h^2 + b^2) \blacktriangleleft$$

and

$$k_O^2 = \frac{J_O}{A} = \frac{\frac{bh}{48}(12h^2 + b^2)}{\frac{1}{2}bh} = \frac{1}{24}(12h^2 + b^2) \quad \text{or} \quad k_O = \frac{\sqrt{12h^2 + b^2}}{24} \blacktriangleleft$$



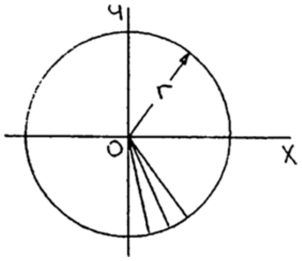
PROBLEM 9.29*

Using the polar moment of inertia of the isosceles triangle of Problem 9.28, show that the centroidal polar moment of inertia of a circular area of radius r is $\pi r^4/2$. (*Hint:* As a circular area is divided into an increasing number of equal circular sectors, what is the approximate shape of each circular sector?)

PROBLEM 9.28 Determine the polar moment of inertia and the polar radius of gyration of the isosceles triangle shown with respect to Point O.

SOLUTION

First the circular area is divided into an increasing number of identical circular sectors. The sectors can be approximated by isosceles triangles. For a large number of sectors the approximate dimensions of one of the isosceles triangles are as shown.



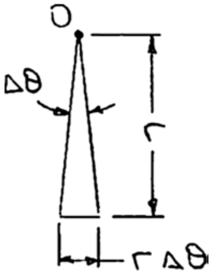
For an isosceles triangle (see Problem 9.28):

$$J_O = \frac{bh}{48}(12h^2 + b^2)$$

Then with $\quad b = r\Delta\theta \quad \text{and} \quad h = r$

$$(\Delta J_O)_{\text{sector}} \simeq \frac{1}{48}(r\Delta\theta)(r)[12r^2 + (r\Delta\theta)^2]$$

$$= \frac{1}{48}r^4\Delta\theta\left[(12 + \Delta\theta^2)\right]$$



Now $$\frac{dJ_{O\,\text{sector}}}{d\theta} = \lim_{\Delta\theta\to 0}\left(\frac{\Delta J_{O\,\text{sector}}}{\Delta\theta}\right) = \lim_{\Delta\theta\to 0}\left\{\frac{1}{48}r^4[12 + (\Delta\theta)^2]\right\}$$

$$= \frac{1}{4}r^4$$

Then $$(J_O)_{\text{circle}} = \int dJ_{O\,\text{sector}} = \int_0^{2\pi}\frac{1}{4}r^4 d\theta = \frac{1}{4}r^4\left[\theta\right]_0^{2\pi}$$

or $\quad (J_O)_{\text{circle}} = \dfrac{\pi}{2}r^4$ ◀

PROBLEM 9.30*

Prove that the centroidal polar moment of inertia of a given area A cannot be smaller than $A^2/2\pi$. (*Hint:* Compare the moment of inertia of the given area with the moment of inertia of a circle that has the same area and the same centroid.)

SOLUTION

From the solution to sample Problem 9.2, the centroidal polar moment of inertia of a circular area is

$$(J_C)_{\text{cir}} = \frac{\pi}{2} r^4$$

The area of the circle is

$$A_{\text{cir}} = \pi r^2$$

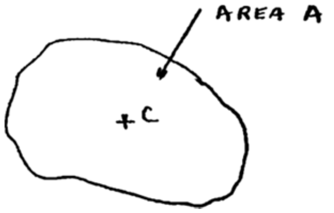
So that

$$[J_C(A)]_{\text{cir}} = \frac{A^2}{2\pi}$$

Two methods of solution will be presented. However, both methods depend upon the observation that as a given element of area dA is moved closer to some Point C. The value of J_C will be decreased $(J_C = \int r^2 dA;$ as r decreases, so must $J_C)$.

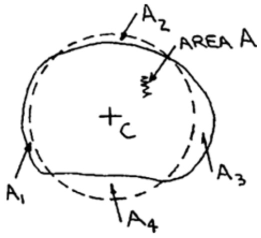
Solution 1

Imagine taking the area A and drawing it into a thin strip of negligible width and of sufficient length so that its area is equal to A. To minimize the value of $(J_C)_A$, the area would have to be distributed as closely as possible about C. This is accomplished by winding the strip into a tightly wound roll with C as its center; any voids in the roll would place the corresponding area farther from C than is necessary, thus increasing the value of $(J_C)_A$. (The process is analogous to rewinding a length of tape back into a roll.) Since the shape of the roll is circular, with the centroid of its area at C, it follows that



$$(J_C)_A \ge \frac{A^2}{2\pi} \quad \text{Q.E.D.} \blacktriangleleft$$

where the equality applies when the original area is circular.



PROBLEM 9.30* (Continued)

Solution 2

Consider an area *A* with its centroid at Point *C* and a circular area of area *A* with its center (and centroid) at Point *C*. Without loss of generality, assume that

$$A_1 = A_2 \qquad A_3 = A_4$$

It then follows that

$$(J_C)_A = (J_C)_{\text{cir}} + [J_C(A_1) - J_C(A_2) + J_C(A_3) - J_C(A_4)]$$

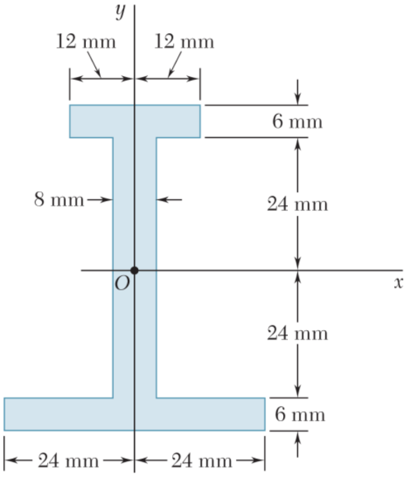
Now observe that

$$J_C(A_1) - J_C(A_2) \geq 0$$

$$J_C(A_3) - J_C(A_4) \geq 0$$

Since as a given area is moved farther away from *C*, its polar moment of inertia with respect to *C* must increase.

$$(J_C)_A \geq (J_C)_{\text{cir}} \qquad \text{or} \quad (J_C)_A \geq \frac{A^2}{2\pi} \quad \text{Q.E.D.} \blacktriangleleft$$



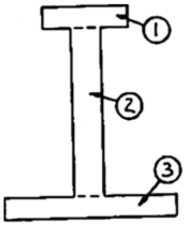
PROBLEM 9.31

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the *x* axis.

SOLUTION

First note that

$$A = A_1 + A_2 + A_3$$
$$= [(24)(6) + (8)(48) + (48)(6)]\ \text{mm}^2$$
$$= (144 + 384 + 288)\ \text{mm}^2$$
$$= 816\ \text{mm}^2$$



Now $$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$

where

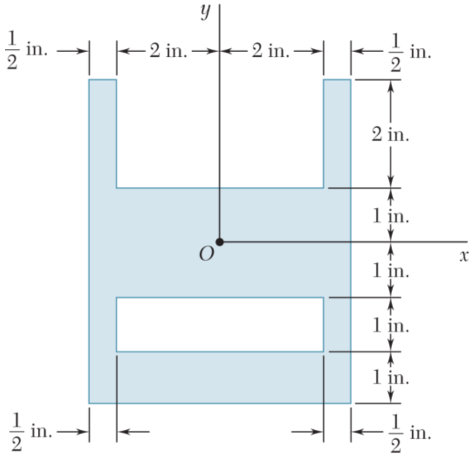
$$(I_x)_1 = \frac{1}{12}(24\ \text{mm})(6\ \text{mm})^3 + (144\ \text{mm}^2)(27\ \text{mm})^2$$
$$= (432 + 104{,}976)\ \text{mm}^4$$
$$= 105{,}408\ \text{mm}^4$$

$$(I_x)_2 = \frac{1}{12}(8\ \text{mm})(48\ \text{mm})^3 = 73{,}728\ \text{mm}^4$$

$$(I_x)_3 = \frac{1}{12}(48\ \text{mm})(6\ \text{mm})^3 + (288\ \text{mm}^2)(27\ \text{mm})^2$$
$$= (864 + 209{,}952)\ \text{mm}^4 = 210{,}816\ \text{mm}^4$$

Then $$I_x = (105{,}408 + 73{,}728 + 210{,}816)\ \text{mm}^4$$
$$= 389{,}952\ \text{mm}^4$$ or $I_x = 390 \times 10^3\ \text{mm}^4$ ◀

and $$k_x^2 = \frac{I_x}{A} = \frac{389{,}952\ \text{mm}^4}{816\ \text{mm}^2}$$ or $k_x = 21.9$ mm ◀



y
$\frac{1}{2}$ in.
2 in.
2 in.
$\frac{1}{2}$ in.
2 in.
1 in.
O
x
1 in.
1 in.
1 in.
$\frac{1}{2}$ in.
$\frac{1}{2}$ in.

PROBLEM 9.32

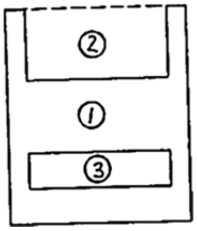
Determine the moment of inertia and the radius of gyration of the shaded area with respect to the *x* axis.

SOLUTION

First note that

$$
\begin{aligned}
A &= A_1 - A_2 - A_3 \\
&= [(5)(6) - (4)(2) - (4)(1)]\ \text{in}^2 \\
&= (30 - 8 - 4)\ \text{in}^2 \\
&= 18\ \text{in}^2
\end{aligned}
$$

Now $I_x = (I_x)_1 - (I_x)_2 - (I_x)_3$



where

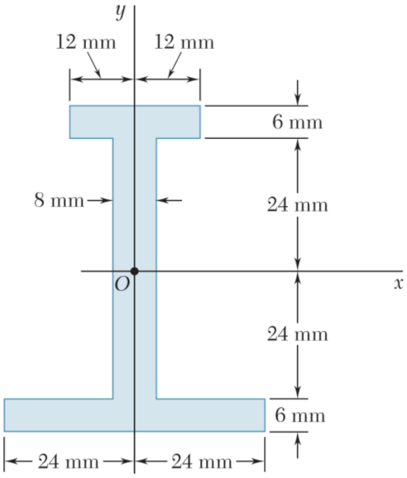
$$(I_x)_1 = \frac{1}{12}(5\ \text{in.})(6\ \text{in.})^3 = 90\ \text{in}^4$$

$$
\begin{aligned}
(I_x)_2 &= \frac{1}{12}(4\ \text{in.})(2\ \text{in.})^3 + (8\ \text{in}^2)(2\ \text{in.})^2 \\
&= 34\frac{2}{3}\ \text{in}^4
\end{aligned}
$$

$$
\begin{aligned}
(I_x)_3 &= \frac{1}{12}(4\ \text{in.})(1\ \text{in.})^3 + (4\ \text{in}^2)\left(\frac{3}{2}\ \text{in.}\right)^2 \\
&= 9\frac{1}{3}\ \text{in}^4
\end{aligned}
$$

Then $$I_x = \left(90 - 34\frac{2}{3} - 9\frac{1}{3}\right)\text{in}^4$$ or $I_x = 46.0\ \text{in}^4$ ◀

and $$k_x^2 = \frac{I_x}{A} = \frac{46.0\ \text{in}^4}{18\ \text{in}^4}$$ or $k_x = 1.599$ in. ◀



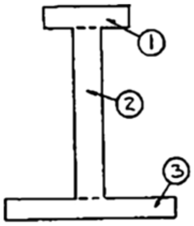
PROBLEM 9.33

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the *y* axis.

SOLUTION

First note that

$$A = A_1 + A_2 + A_3$$
$$= [(24 \times 6) + (8)(48) + (48)(6)]\ \text{mm}^2$$
$$= (144 + 384 + 288)\ \text{mm}^2$$
$$= 816\ \text{mm}^2$$



Now $I_y = (I_y)_1 + (I_y)_2 + (I_y)_3$

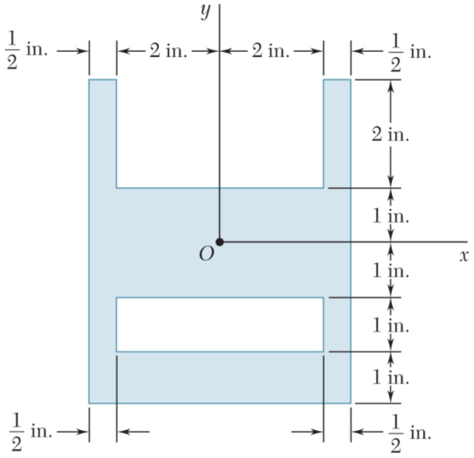
where

$$(I_y)_1 = \frac{1}{12}(6\ \text{mm})(24\ \text{mm})^3 = 6912\ \text{mm}^4$$
$$(I_y)_2 = \frac{1}{12}(48\ \text{mm})(8\ \text{mm})^3 = 2048\ \text{mm}^4$$
$$(I_y)_3 = \frac{1}{12}(6\ \text{mm})(48\ \text{mm})^3 = 55{,}296\ \text{mm}^4$$

Then $I_y = (6912 + 2048 + 55{,}296)\ \text{mm}^4 = 64{,}256\ \text{mm}^4$

or $I_y = 64.3 \times 10^3\ \text{mm}^4$ ◀

and $k_y^2 = \dfrac{I_y}{A} = \dfrac{64{,}256\ \text{mm}^4}{816\ \text{mm}^2}$ or $k_y = 8.87\ \text{mm}$ ◀



y
$\frac{1}{2}$ in.
2 in.
2 in.
$\frac{1}{2}$ in.
2 in.
1 in.
O
x
1 in.
1 in.
1 in.
$\frac{1}{2}$ in.
$\frac{1}{2}$ in.

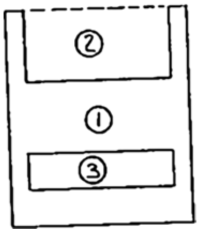
PROBLEM 9.34

Determine the moment of inertia and the radius of gyration of the shaded area with respect to the *y* axis.

SOLUTION

First note that

$$A = A_1 - A_2 - A_3$$
$$= [(5)(6) - (4)(2) - (4)(1)]\ \text{in}^2$$
$$= (30 - 8 - 4)\ \text{in}^2$$
$$= 18\ \text{in}^2$$



Now $$I_y = (I_y)_1 - (I_y)_2 - (I_y)_3$$

where

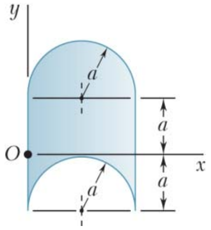
$$(I_y)_1 = \frac{1}{12}(6\text{ in.})(5\text{ in.})^3 = 62.5\text{ in}^4$$

$$(I_y)_2 = \frac{1}{12}(2\text{ in.})(4\text{ in.})^3 = 10\frac{2}{3}\ \text{in}^4$$

$$(I_y)_3 = \frac{1}{12}(1\text{ in.})(4\text{ in.})^3 = 5\frac{1}{3}\text{in}^4$$

Then $$I_y = \left(62.5 - 10\frac{2}{3} - 5\frac{1}{3}\right)\text{in}^4$$ or $I_y = 46.5\text{ in}^4$ ◀

and $$k_y^2 = \frac{I_y}{A} = \frac{46.5\text{ in}^4}{18\text{ in}^2}$$ or $k_y = 1.607$ in. ◀

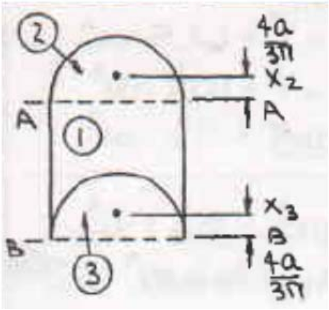


PROBLEM 9.35

Determine the moments of inertia of the shaded area shown with respect to the *x* and *y* axes.

SOLUTION We have $I_x = (I_x)_1 + (I_x)_2 - \left(I_x\right)_3$ where $(I_x)_1 = \frac{1}{12}(2a)(2\text{a})^3$

$$= \frac{4}{3}a^4$$



$$(I_{AA})_2 = \left(I_{BB}\right)_3 = \frac{\pi}{8}a^4$$

$$\left(I_{AA}\right)_2 = \left(\bar{I}_{x_2}\right)_2 + Ad^2$$

$$\left(\bar{I}_{x_2}\right)_2 = \left(\bar{I}_{x_3}\right)_3 = \frac{\pi}{8}a^4 - \left(\frac{\pi}{2}a^2\right)\left(\frac{4a}{3\pi}\right)^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)a^4$$

$$\left(I_x\right)_2 = \left(\bar{I}_{x_2}\right)_2 + Ad^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)a^4 + \left(\frac{\pi}{2}\right)a^2\left(a + \frac{4a}{3\pi}\right)^2 = \left(\frac{4}{3} + \frac{5\pi}{8}\right)a^4$$

$$\left(I_x\right)_3 = \left(\bar{I}_{x_3}\right)_3 + Ad^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)a^4 + \left(\frac{\pi}{2}\right)a^2\left(a - \frac{4a}{3\pi}\right)^2 = \left(-\frac{4}{3} + \frac{5\pi}{8}\right)a^4$$

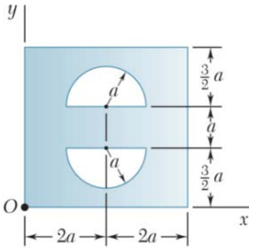
Finally, $I_x = \frac{4}{3}a^4 + \left(\frac{4}{3} + \frac{5\pi}{8}\right)a^4 - \left(-\frac{4}{3} + \frac{5\pi}{8}\right)a^4$ $I_x = 4a^4$ ◀

Also $I_y = (I_y)_1 + (I_y)_2 - \left(I_y\right)_3$

where $(I_y)_1 = \frac{1}{12}\left(2a\right)\left(2a\right)^3 + \left(2a\right)^2\left(a\right)^2 = \frac{16}{3}a^4$

but $(I_y)_2 = \left(I_y\right)_3$

Then $I_y = \frac{16}{3}a^4$ ◀



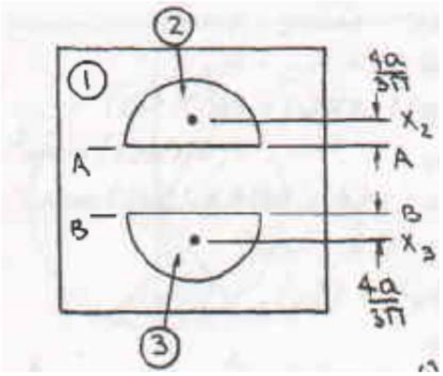
PROBLEM 9.36

Determine the moments of inertia of the shaded area shown with respect to the *x* and *y* axes.

SOLUTION

We have $$I_x = (I_x)_1 - (I_x)_2 - (I_x)_3$$

where $$(I_x)_1 = \frac{1}{12}(4a)(4a)^3 + (4a)^2(2a)^2$$
$$= \frac{256}{3}a^4$$



$$(I_{AA})_2 = (I_{BB})_3 = \frac{\pi}{8}a^4$$
$$(I_{AA})_2 = (\overline{I}_{x_2})_2 + Ad^2$$

$$(\overline{I}_{x_2})_2 = (\overline{I}_{x_3})_3 = \frac{\pi}{8}a^4 - \left(\frac{\pi}{2}a^2\right)\left(\frac{4a}{3\pi}\right)^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)a^4$$

$$(I_x)_2 = (\overline{I}_{x_2})_2 + Ad^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)a^4 + \left(\frac{\pi}{2}\right)a^2\left(\frac{5}{2}a + \frac{4a}{3\pi}\right)^2 = \left(\frac{10}{3} + \frac{13\pi}{4}\right)a^4$$

$$(I_x)_3 = (\overline{I}_{x_3})_3 + Ad^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)a^4 + \left(\frac{\pi}{2}\right)a^2\left(\frac{3}{2}a - \frac{4a}{3\pi}\right)^2 = \left(-2 + \frac{5\pi}{4}\right)a^4$$

Finally, $$I_x = \frac{256}{3}a^4 - \left(\frac{10}{3} + \frac{13\pi}{4}\right)a^4 - \left(-2 + \frac{5\pi}{4}\right)a^4 \quad I_x = 69.9a^4 \blacktriangleleft$$

Also $$I_y = (I_y)_1 - (I_y)_2 - (I_y)_3$$

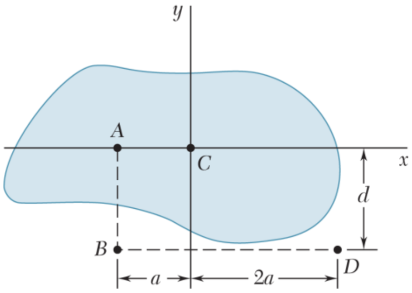
PROBLEM 9.36 (Continued)

$$(I_y)_1 = \frac{1}{12}(4a)(4a)^3 + (4a)^2(2a)^2 = \frac{256}{3}a^4$$

$$(I_y)_2 = \left(I_y\right)_3 = \frac{\pi}{8}a^4 + \left(\frac{\pi}{2}a^2\right)(2a)^2 = \frac{17}{8}\pi a^4$$

Then

$$I_y = \frac{256}{3}a^4 - 2\left(\frac{17}{8}\pi a^4\right) \qquad I_y = 72.0a^4 \blacktriangleleft$$



PROBLEM 9.37

The centroidal polar moment of inertia $\bar{J}_C$ of the 24-in^2 shaded area is 600 in^4. Determine the polar moments of inertia J_B and J_D of the shaded area knowing that $J_D = 2J_B$ and $d = 5$ in.

SOLUTION

Given: $A = 24\text{ in}^2 \quad J_D = 2J_B \quad J_C = 600\text{ in}^4$

$$J_B = \bar{J}_C + A(d_{CB})^2 \quad (1)$$

$$J_D = \bar{J}_C + A(d_{CD})^2 \quad (2)$$

$$\bar{J}_C + A\left(d_{CD}\right)^2 = 2\left(\bar{J}_C + A\left(d_{CB}\right)^2\right)$$

$$\left(d_{CB}\right)^2 = a^2 + d^2 \quad \text{and} \quad \left(d_{CD}\right)^2 = \left(2a\right)^2 + d^2$$

$$A\left(4a^2 + d^2\right) = \bar{J}_C + 2A\left(a^2 + d^2\right)$$

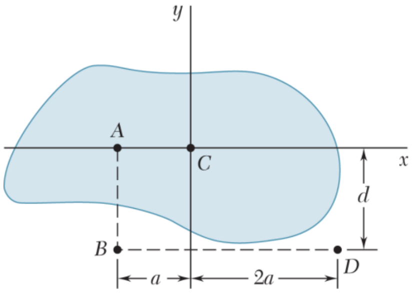
$$a^2 = \frac{1}{2}\left(\frac{\bar{J}_C}{A} + d^2\right)$$

$$a^2 = \frac{1}{2}\left(\frac{600}{24} + (5)^2\right)$$

$a = 5$ in.

Then from Eqn. (1) $\quad J_B = 600 + 24\left(5^2 + 5^2\right) \quad J_B = 1800\text{ in}^4$ ◀

Then from Eqn. (2) $\quad J_D = 600 + 24\left(10^2 + 5^2\right) \quad J_D = 3600\text{ in}^4$ ◀



PROBLEM 9.38

Determine the centroidal polar moment of inertia $\bar{J}_C$ of the 25-in^2 shaded area knowing that the polar moments of inertia of the area with respect to points *A*, *B*, and *D* are, respectively, $J_A = 281$ in^4, $J_B = 810$ in^4, and $J_D = 1578$ in^4.

SOLUTION

$$J_A = \bar{J}_C + A(d_{CA})^2 \qquad (d_{CA})^2 = a^2$$

$$J_B = \bar{J}_C + A(a)^2 \qquad (1)$$

$$J_B = \bar{J}_C + A(d_{CB})^2 \qquad (d_{CB})^2 = a^2 + d^2$$

$$J_B = \bar{J}_C + A(a^2 + d^2) \quad (2)$$

$$J_D = \bar{J}_C + A(d_{CD})^2 \qquad (d_{CD})^2 = 4a^2 + d^2$$

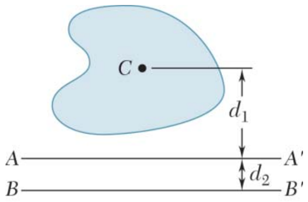
$$J_D = \bar{J}_C + A\left(4a^2 + d^2\right) \qquad (3)$$

Eqn. (3) – Eqn. (2): $$J_D - J_B = 3Aa^2 \qquad (4)$$

Eqn. (4) -3[Eqn. (1)] $$J_D - J_B - 3J_A = 3\bar{J}_C$$

$$\bar{J}_C = 281 - \frac{1}{3}(1578 - 810)$$

$\bar{J}_C = 25.0$ in^4 ◀



PROBLEM 9.39

Determine the shaded area and its moment of inertia with respect to the centroidal axis parallel to AA' knowing that d_1 = 25 mm and d_2 = 10 mm and that its moments of inertia with respect to AA' and BB' are 2.2 × 10^6 mm^4 and 4 × 10^6 mm^4, respectively.

SOLUTION

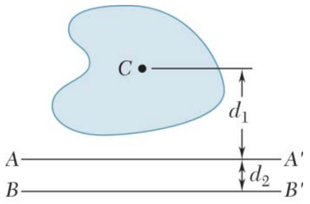
$$I_{AA'} = 2.2\times10^6 \text{ mm}^4 = \bar{I} + A(25\text{ mm})^2 \qquad (1)$$

$$I_{BB'} = 4\times10^6 \text{ mm}^4 = \bar{I} + A(35\text{ mm})^2$$

$$I_{BB'} - I_{AA'} = (4-2.2)\times10^6 = A(35^2 - 25^2)$$

$$1.8\times10^6 = A(600) \qquad A = 3000 \text{ mm}^2 \blacktriangleleft$$

Eq. (1): $$2.2\times10^6 = \bar{I} + (3000)(25)^2 \qquad \bar{I} = 325\times10^3 \text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.40

Knowing that the shaded area is equal to 6000 mm^2 and that its moment of inertia with respect to *AA′* is 18×10^6 mm^4, determine its moment of inertia with respect to *BB′* for $d_1 = 50$ mm and $d_2 = 10$ mm.

SOLUTION

Given: $A = 6000\text{ mm}^2$

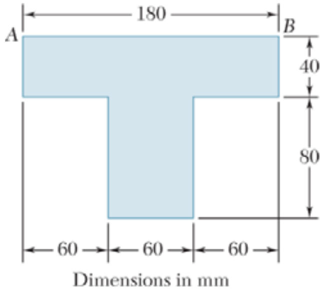
$$I_{AA'} = \bar{I} + Ad_1^2; \quad 18 \times 10^6\text{ mm}^2 = \bar{I} + (6000\text{ mm}^2)(50\text{ mm})^2$$

$$\bar{I} = 3 \times 10^6\text{ mm}^4$$

$$I_{BB'} = \bar{I} + Ad^2 = 3 \times 10^6\text{ mm}^4 + (6000\text{ mm}^2)(50\text{ mm} + 10\text{ mm})^2$$

$$= 3 \times 10^6 + 6000(60)^2 = 24.6 \times 10^6\text{ mm}^4$$

$$I_{BB'} = 24.6 \times 10^6\text{ mm}^4 \blacktriangleleft$$



PROBLEM 9.41

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side *AB*.

SOLUTION

First locate centroid *C* of the area.

Symmetry implies $\bar{X} = 0$

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
Flng.	$180 \times 40 = 7200$	20	144,000
Web	$80 \times 60 = 4800$	80	384,000
Σ	12,000		528,000

Then $\bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{X}(12{,}000 \text{ mm}^2) = 528{,}000 \text{ mm}^3$

or $\bar{Y} = 44.0$ mm (measured from top of flange)

Now $\bar{I}_x = (I_x)_F + (I_x)_W$

where

$$(I_x)_F = \frac{1}{12}(180 \text{ mm})(40 \text{ mm})^3 + \left(7200 \text{ mm}^2\right)\left(44 \text{ mm} - 20 \text{ mm}\right)^2 = 5.170 \times 10^6 \text{mm}^4$$

$$(I_x)_W = \left[\frac{1}{12}(60 \text{ mm})(80 \text{ mm})^3 + \left(4800 \text{ mm}^2\right)(80 \text{ mm} - 44 \text{ mm})^2\right] = 8.78 \times 10^6 \text{ mm}^4$$

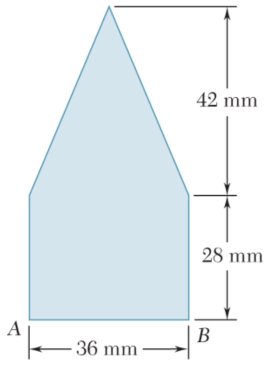
Then $\bar{I}_x = (5.107 + 8.78) \times 10^6 \text{mm}^4$

or $\bar{I}_x = 13.89 \times 10^6 \text{mm}^4$ ◀

Also $\bar{I}_y = (I_y)_F + (I_y)_W$

Then $\bar{I}_y = \left[\frac{1}{12}(40)(180)^3 + \frac{1}{12}(80)(60)^3\right] \text{mm}^4$

or $\bar{I}_y = 20.9 \times 10^6 \text{mm}^4$ ◀



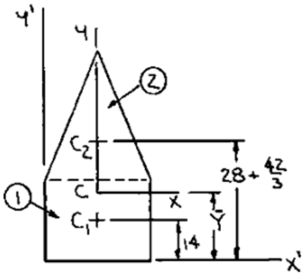
PROBLEM 9.42

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side *AB*.

SOLUTION

First locate *C* of the area.

Symmetry implies $\bar{X} = 18\text{ mm}$.



Dimensions in mm

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$36 \times 28 = 1008$	14	14,112
2	$\frac{1}{2} \times 36 \times 42 = 756$	42	31,752
Σ	1764		45,864

Then $\bar{Y}\Sigma A = \Sigma \bar{y}A$: $\bar{Y}(1764\text{ mm}^2) = 45{,}864\text{ mm}^3$

or $\bar{Y} = 26.0\text{ mm}$

Now $\bar{I}_x = (I_x)_1 + (I_x)_2$

where

$$(I_x)_1 = \frac{1}{12}(36\text{ mm})(28\text{ mm})^3 + (1008\text{ mm}^2)[(26-14)\text{ mm}]^2$$

$$= (65{,}856 + 145{,}152)\text{ mm}^4 = 211.008 \times 10^3\text{ mm}^4$$

$$(I_x)_2 = \frac{1}{36}(36\text{ mm})(42\text{ mm})^3 + (756\text{ mm}^2)[(42-26)\text{ mm}]^2$$

$$= (74{,}088 + 193{,}536)\text{ mm}^4 = 267.624 \times 10^3\text{ mm}^4$$

Then $\bar{I}_x = (211.008 + 267.624) \times 10^3\text{ mm}^4$

or $\bar{I}_x = 479 \times 10^3\text{ mm}^4$ ◀

PROBLEM 9.42 (Continued)

Also $$\bar{I}_y = (I_y)_1 + (I_y)_2$$

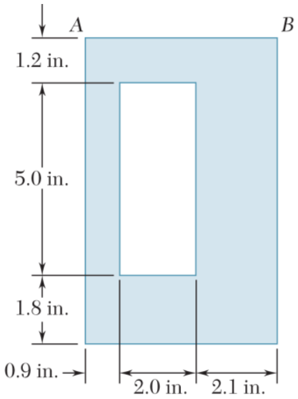
where $$(I_y)_1 = \frac{1}{12}(28\text{ mm})(36\text{ mm})^3 = 108.864\times 10^3\text{ mm}^4$$

$$(I_y)_2 = 2\left[\frac{1}{36}(42\text{ mm})(18\text{ mm})^3 + \left(\frac{1}{2}\times 18\text{ mm}\times 42\text{ mm}\right)(6\text{ mm})^2\right]$$

$$= 2(6804 + 13,608)\text{ mm}^4 = 40.824\times 10^3\text{ mm}^4$$

[$(I_y)_2$ is obtained by dividing A_2 into]

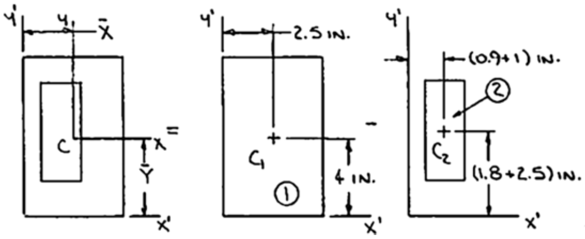
Then $$\bar{I}_y = (108.864 + 40.824\times 10^3\text{ mm}^4$$ or $$\bar{I}_y = 149.7\times 10^3\text{ mm}^4$$ ◀


A
B
1.2 in.
5.0 in.
1.8 in.
0.9 in.
2.0 in.
2.1 in.

PROBLEM 9.43

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side *AB*.

SOLUTION


y'
y
$\bar{X}$
C
x
$\bar{Y}$
x'
=
y'
2.5 in.
C_1
①
4 in.
x'
−
y'
(0.9+1) in.
②
C_2
(1.8+2.5) in.
x'

First locate centroid *C* of the area.

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$5\times8=40$	2.5	4	100	160
2	$-2\times5=-10$	1.9	4.3	–19	–43
Σ	30			81	117

Then $\bar{X}\Sigma A=\Sigma\bar{x}A$: $\quad\bar{X}(30\text{ in}^2)=81\text{ in}^3$

or $\bar{X}=2.70$ in.

and $\bar{Y}\Sigma A=\Sigma\bar{y}A$: $\quad\bar{Y}(30\text{ in}^2)=117\text{ in}^3$

or $\bar{Y}=3.90$ in.

Now $\bar{I}_x=(I_x)_1-(I_x)_2$

where

$$(I_x)_1=\frac{1}{12}(5\text{ in.})(8\text{ in.})^3+(40\text{ in}^2)[(4-3.9)\text{ in.}]^2$$

$$=(213.33+0.4)\text{ in}^4=213.73\text{ in}^4$$

$$(I_x)_2=\frac{1}{12}(2\text{ in.})(5\text{ in.})^3+(10\text{ in}^2)[(4.3-3.9)\text{ in.}]^2$$

$$=(20.83+1.60)=22.43\text{ in}^4$$

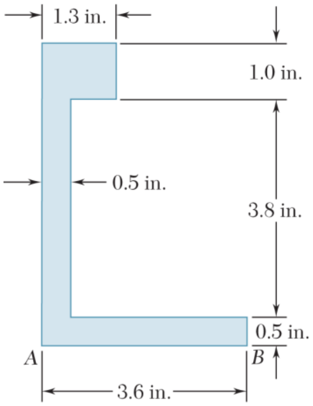
PROBLEM 9.43 (Continued)

Then $\bar{I}_x = (213.73 - 22.43)\text{ in}^4$ or $\bar{I}_x = 191.3\text{ in}^4$ ◀

Also $\bar{I}_y = (I_y)_1 - (I_y)_2$

where
$$(I_y)_1 = \frac{1}{12}(8\text{ in.})(5\text{ in.})^3 + (40\text{ in}^2)[(2.7 - 2.5)\text{ in.}]^2$$
$$= (83.333 + 1.6)\text{ in}^4 = 84.933\text{ in}^4$$
$$(I_y)_2 = \frac{1}{12}(5\text{ in.})(2\text{ in.})^3 + (10\text{ in}^2)[(2.7 - 1.9)\text{ in.}]^2$$
$$= (3.333 + 6.4)\text{ in}^4 = 9.733\text{ in}^4$$

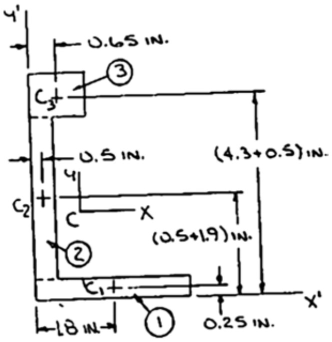
Then $\bar{I}_y = (84.933 - 9.733)\text{ in}^4$ or $\bar{I}_y = 75.2\text{ in}^4$ ◀


1.3 in.
1.0 in.
0.5 in.
3.8 in.
0.5 in.
A
B
3.6 in.

PROBLEM 9.44

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side *AB*.

SOLUTION


y'
0.65 IN.
③
C_3
0.5 IN.
(4.3+0.5) IN.
y
C_2
C
x
(0.5+1.9) IN.
②
C_1
x'
1.8 IN.
①
0.25 IN.

First locate centroid *C* of the area.

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$3.6 \times 0.5 = 1.8$	1.8	0.25	3.24	0.45
2	$0.5 \times 3.8 = 1.9$	0.25	2.4	0.475	4.56
3	$1.3 \times 1 = 1.3$	0.65	4.8	0.845	6.24
Σ	5.0			4.560	11.25

Then $\qquad \bar{X}\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(5\text{ in}^2) = 4.560\text{ in}^3$

or $\qquad \bar{X} = 0.912\text{ in.}$

and $\qquad \bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(5\text{ in}^2) = 11.25\text{ in}^3$

or $\qquad \bar{Y} = 2.25\text{ in.}$

PROBLEM 9.44 (Continued)

Now $$\bar{I}_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$

where $$(I_x)_1 = \frac{1}{12}(3.6\text{ in.})(0.5\text{ in.})^3 + (1.8\text{ in}^2)[(2.25 - 0.25)\text{ in.}]^2$$

$$= (0.0375 + 7.20)\text{ in}^4 = 7.2375\text{ in}^4$$

$$(I_x)_2 = \frac{1}{12}(0.5\text{ in.})(3.8\text{ in.})^3 + (1.9\text{ in}^2)[(2.4 - 2.25)\text{ in.}]^2$$

$$= (2.2863 + 0.0428)\text{ in}^4 = 2.3291\text{ in}^4$$

$$(I_x)_3 = \frac{1}{12}(1.3\text{ in.})(1\text{ in.})^3 + (1.3\text{ in}^2)[(4.8 - 2.25\text{ in.})]^2$$

$$= (0.1083 + 8.4533)\text{ in}^4 = 8.5616\text{ in}^4$$

Then $$\bar{I}_x = (7.2375 + 2.3291 + 8.5616)\text{ in}^4 = 18.1282\text{ in}^4$$

or $\bar{I}_x = 18.13\text{ in}^4$ ◀

Also $$\bar{I}_y = (I_y)_1 + (I_y)_2 + (I_y)_3$$

where $$(I_y)_1 = \frac{1}{12}(0.5\text{ in.})(3.6\text{ in.})^3 + (1.8\text{ in}^2)[(1.8 - 0.912)\text{ in.}]^2$$

$$= (1.9440 + 1.4194)\text{ in}^4 = 3.3634\text{ in}^4$$

$$(I_y)_2 = \frac{1}{12}(3.8\text{ in.})(0.5\text{ in.})^3 + (1.9\text{ in}^2)[(0.912 - 0.25)\text{ in.}]^2$$

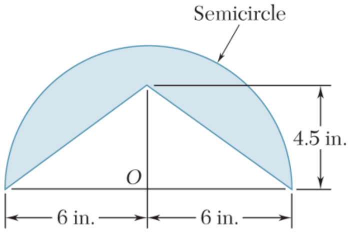
$$= (0.0396 + 0.8327)\text{ in}^4 = 0.8723\text{ in}^4$$

$$(I_y)_3 = \frac{1}{12}(1\text{ in.})(1.3\text{ in.})^3 + (1.3\text{ in}^2)[(0.912 - 0.65)\text{ in.}]^2$$

$$= (0.1831 + 0.0892)\text{ in}^4 = 0.2723\text{ in}^4$$

Then $$\bar{I}_y = (3.3634 + 0.8723 + 0.2723)\text{in}^4 = 4.5080\text{ in}^4$$

or $\bar{I}_y = 4.51\text{ in}^4$ ◀

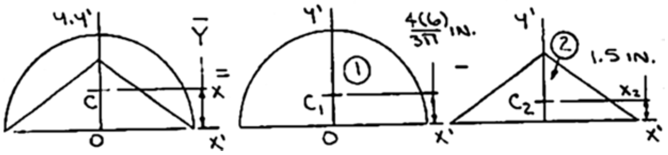

Semicircle
4.5 in.
O
6 in.
6 in.

PROBLEM 9.45

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

First locate centroid *C* of the figure.


y, y'
C
O
$\bar{Y}$
=
x'
y'
①
C₁
O
x'
$\frac{4(6)}{3\pi}$ IN.
−
y'
②
1.5 IN.
C₂
x₂
x'

	A, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
1	$\frac{\pi}{2}(6)^2 = 56.5487$	$\frac{8}{\pi} = 2.5465$	144
2	$-\frac{1}{2}(12)(4.5) = -27$	1.5	–40.5
Σ	29.5487		103.5

Then $$\bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(29.5487 \text{ in}^2) = 103.5 \text{ in}^3$$

or $$\bar{Y} = 3.5027 \text{ in.}$$

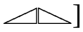
(*a*) $$J_O = (J_O)_1 - (J_O)_2$$

where $$(J_O)_1 = \frac{\pi}{4}(6 \text{ in.})^4 = 107.876 \text{ in}^4$$

$$(J_O)_2 = (I_{x'})_2 + (I_{y'})_2$$

Now $$(I_{x'})_2 = \frac{1}{12}(12 \text{ in.})(4.5 \text{ in.})^3 = 91.125 \text{ in}^4$$

and $$(I_{y'})_2 = 2\left[\frac{1}{12}(4.5 \text{ in.})(6 \text{ in.})^3\right] = 162.0 \text{ in}^4$$

[*Note:* $(I_{y'})_2$ is obtained using ]

Then $$(J_O)_2 = (91.125 + 162.0)\text{ in}^4$$
$$= 253.125\text{ in}^4$$

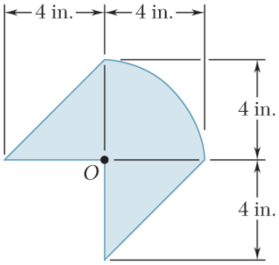
Finally $$J_O = (1017.876 - 253.125)\text{ in}^4 = 764.751\text{ in}^4$$

or $J_O = 765\text{ in}^4$ ◀

(*b*) $$J_O = \bar{J}_C + A\bar{Y}^2$$

or $$J_C = 764.751\text{ in.}^4 - (29.5487\text{ in}^2)(3.5027\text{ in.})^2$$

or $\bar{J}_C = 402\text{ in}^4$ ◀



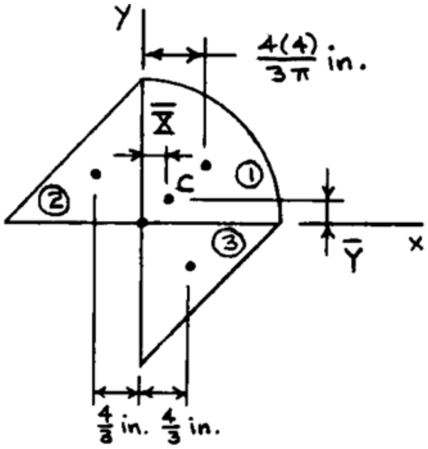
PROBLEM 9.46

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

Determination of centroid *C* of entire section:

Section	Area, in^2	$\bar{x}$, in.	$\bar{x}A$, in^3
1	$\frac{\pi}{4}(4)^2 = 4\pi$	$\frac{16}{3\pi}$	21.333
2	$\frac{1}{2}(4)(4) = 8$	$-\frac{4}{3}$	−10.6667
3	$\frac{1}{2}(4)(4) = 8$	$\frac{4}{3}$	10.6667
Σ	28.566		21.333



$$\bar{X}\Sigma A = \Sigma\bar{x}A: \quad \bar{X}(28.566\text{ in}^2) = 21.333\text{ in}^3$$

$$\bar{X} = 0.74680\text{ in.}; \quad \text{by symmetry,} \quad \bar{Y} = \bar{X} = 0.74680\text{ in.}$$

(*a*) For section ①: $I_x = \frac{1}{4}\left(\frac{1}{4}\pi r^4\right) = \frac{1}{16}\pi(4)^4 = 50.265\text{ in}^4$

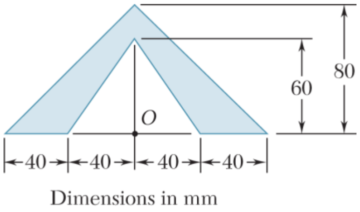
For sections ②,③: $I_x = \frac{1}{12}bh^3 = \frac{1}{12}(4)(4)^3 = 21.333\text{ in}^4$

For total area, $I_x = 50.265 + 2(21.333) = 92.931\text{ in}^4$

By symmetry, $I_y = I_x; \quad J_O = I_x + I_y = 185.862\text{ in}^4$ $\quad J_O = 185.9\text{ in}^4$ ◀

(*b*) Parallel axis theorem: $J_C = J_O - A(OC)^2 = J_O - A(\bar{X}^2 + \bar{Y}^2)$

$J_C = 185.862 - (28.566)(0.74680^2 + 0.74680^2)$ $\quad J_C = 154.0\text{ in}^4$ ◀

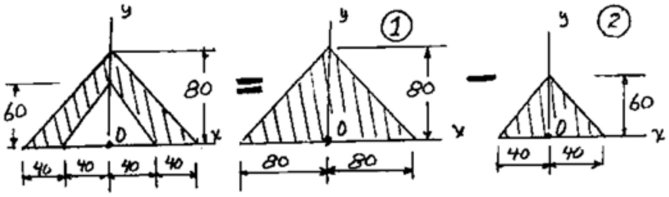

80
60
O
40
40
40
40
Dimensions in mm

PROBLEM 9.47

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

Dimensions in mm Symmetry: $\bar{X} = 0$


y
60
80
O
x
40
40
40
40
=
y
①
80
O
x
80
80
−
y
②
60
O
x
40
40

Determination of centroid *C* of entire section

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

$$\bar{Y}(4000\ \text{mm}^2) = 122.67 \times 10^3\ \text{mm}^3$$

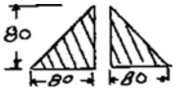
$$\bar{Y} = 30.667\ \text{mm}$$

	Area mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$\frac{1}{2}(160)(80) = 6400$	$\frac{80}{3}$	170.67×10^3
2	$-\frac{1}{2}(80)(60) = -2400$	20	-48×10^3
Σ	4000		122.67×10^3

(*a*) Polar moment of inertia J_O:

Section ①: $$I_x = \frac{1}{12}(160)(80)^3 = 6.8267 \times 10^6\ \text{mm}^4$$

For I_y consider the following two triangles



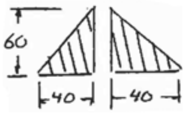
$$I_y = 2\left[\frac{1}{12}(80)(80)^3\right] = 6.8267 \times 10^6\ \text{mm}^4$$

$$J_O = I_x + I_y = (6.8267 + 6.8267)10^6 = 13.653 \times 10^6\ \text{mm}^4$$

PROBLEM 9.47 (Continued)

(*b*) Section ②: $I_x = \frac{1}{12}(80)(60)^3 = 1.44\times10^6 \text{ mm}^4$

For I_y consider the following two triangles



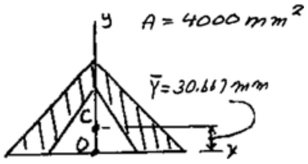
$$I_y = 2\left[\frac{1}{12}(60)(40)^3\right] = 0.640\times10^6 \text{ mm}^4$$

$$J_O = I_x + I_y = (1.44+0.640)10^6 = 2.08\times10^6 \text{ mm}^4$$

Entire section $J_O = (J_O)_1 - (J_O)_2 = 13.653\times10^6 - 2.08\times10^6$

$$J_O = 11.573\times10^6 \text{ mm}^4 \qquad J_O = 11.57\times10^6 \text{ mm}^4 \blacktriangleleft$$

(*c*) Polar moment of inertia J_O of intire area

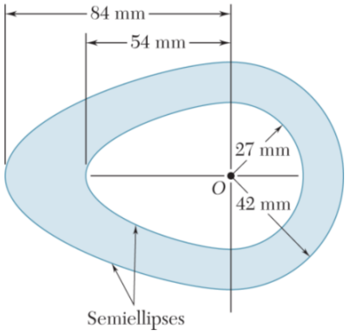


$$J_O = J_C + A\overline{Y}^2$$

$$11.573\times10^6 = J_C + (4000)(30.667)^2$$

$$J_C = 7.811\times10^6 \text{ mm}^4$$

$$J_C = 7.81\times10^6 \text{ mm}^4 \blacktriangleleft$$


84 mm
54 mm
27 mm
O
42 mm
Semiellipses

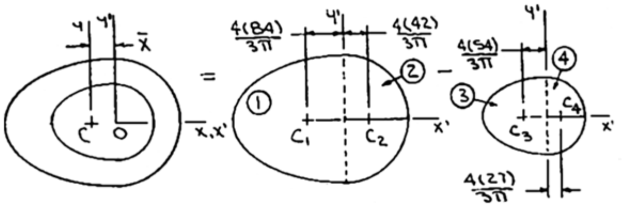
PROBLEM 9.48

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

First locate centroid *C* of the figure.

Note that symmetry implies $\bar{Y} = 0$.


4
4'
$\bar{X}$
C
O
x, x'
=
$\frac{4(84)}{3\pi}$
4'
$\frac{4(42)}{3\pi}$
①
②
C_1
C_2
x'
−
$\frac{4(54)}{3\pi}$
4'
④
③
C_4
C_3
x'
$\frac{4(27)}{3\pi}$

Dimensions in mm

	A, mm^2	$\bar{X}$, mm	$\bar{X}A$, mm^3
1	$\frac{\pi}{2}(84)(42) = 5541.77$	$-\frac{112}{\pi} = -35.6507$	$-197,568$
2	$\frac{\pi}{2}(42)^2 = 2770.88$	$\frac{56}{\pi} = 17.8254$	49,392
3	$-\frac{\pi}{2}(54)(27) = -2290.22$	$-\frac{72}{\pi} = -22.9183$	52,488
4	$-\frac{\pi}{2}(27)^2 = -1145.11$	$\frac{36}{\pi} = 11.4592$	$-13,122$
Σ	4877.32		$-108,810$

Then $\bar{X}\Sigma A = \Sigma \bar{x}A$: $\quad \bar{X}(4877.32 \text{ mm}^2) = -108,810 \text{ mm}^3$

or $\bar{X} = -22.3094$

(*a*) $J_O = (J_O)_1 + (J_O)_2 - (J_O)_3 - (J_O)_4$

where
$$(J_O)_1 = \frac{\pi}{8}(84 \text{ mm})(42 \text{ mm})[(84 \text{ mm})^2 + (42 \text{ mm})^2]$$
$$= 12.21960 \times 10^6 \text{ mm}^4$$

PROBLEM 9.48 (Continued)

$$(J_O)_2 = \frac{\pi}{4}(42\text{ mm})^4$$
$$= 2.44392\times 10^6\,\text{mm}^4$$

$$(J_O)_3 = \frac{\pi}{8}(54\text{ mm})(27\text{ mm})[(54\text{ mm})^2 + (27\text{ mm})^2]$$
$$= 2.08696\times 10^6\,\text{mm}^4$$

$$(J_O)_4 = \frac{\pi}{4}(27\text{ mm})^4$$
$$= 0.41739\times 10^6\,\text{mm}^4$$

Then $J_O = (12.21960 + 2.44392 - 2.08696 - 0.41739)\times 10^6\,\text{mm}^4$

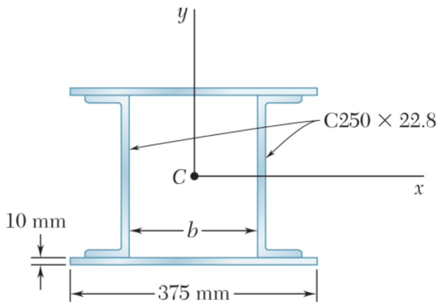
$$= 12.15917\times 10^6\,\text{mm}^4$$

or $J_O = 12.16\times 10^6\,\text{mm}^4$ ◀

(*b*) $J_O = \bar{J}_C + A\bar{X}^2$

or $\bar{J}_C = 12.15917\times 10^6\,\text{mm}^4 - (4877.32\text{ mm}^2)(-22.3094\text{ mm})^2$

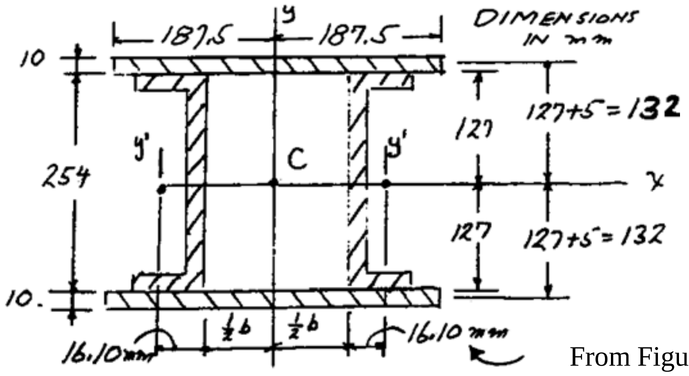
or $\bar{J}_C = 9.73\times 10^6\,\text{mm}^4$ ◀


y
C250 × 22.8
C
x
10 mm
b
375 mm

PROBLEM 9.49

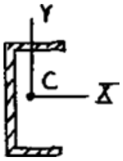
Two channels and two plates are used to form the column section shown. For $b = 200$ mm, determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal *x* and *y* axes.

SOLUTION


187.5
187.5
y
DIMENSIONS IN mm
10
127
127+5=132
254
y'
C
y'
x
127
127+5=132
10.
½b
½b
16.10 mm
16.10mm

From Figure 9.13B

For C250 × 22.8



$$A = 2890 \text{ mm}^2$$
$$I_x = 28.0 \times 10^6 \text{ mm}^4$$
$$I_y = 0.945 \times 10^6 \text{ mm}^4$$

Total area $\quad A = 2[2890 \text{ mm}^4 + (10 \text{ mm})(375 \text{ mm})] = 13.28 \times 10^3 \text{ mm}^2$

Given *b*= 200 mm:

$$\bar{I}_x = 2[28.0 \times 10^6 \text{ mm}^4] + 2\left[\frac{1}{12}(375 \text{ mm})(10 \text{ mm})^3 + (375 \text{ mm})(10 \text{ mm})(132 \text{ mm})^2\right]$$

$$= 186.743 \times 10^6 \qquad \bar{I}_x = 186.7 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\bar{k}_x^2 = \frac{\bar{I}_x}{A} = \frac{186.743 \times 10^6}{13.28 \times 10^3} = 14.0620 \times 10^3 \text{ mm}^2 \qquad \bar{k}_x = 118.6 \text{ mm} \blacktriangleleft$$

PROBLEM 9.49 (Continued)

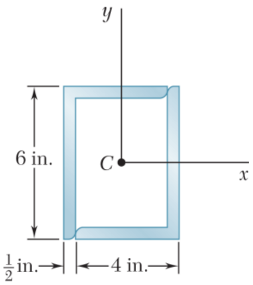
$$\bar{I}_y = \Sigma(\bar{I}_y + Ad^2) = 2\overset{\text{Channel}}{[\bar{I}_y + Ad^2]} + 2\overset{\text{Plate}}{\left[\frac{1}{12}(10\text{ mm})(375\text{ mm})^3\right]}$$

$$= 2\left[0.945\times10^6\text{ mm}^4 + (2890\text{ mm}^2)\left(\frac{200\text{ mm}}{2} + 16.10\text{ mm}\right)^2\right] + 87.891\times10^6\text{ mm}^4$$

$$= 2[0.945\times10^6 + 38.955\times10^6] + 87.891\times10^6$$

$$= 167.691\times10^6\text{ mm}^4 \qquad \bar{I}_y = 167.7\times10^6\text{ mm}^4 \blacktriangleleft$$

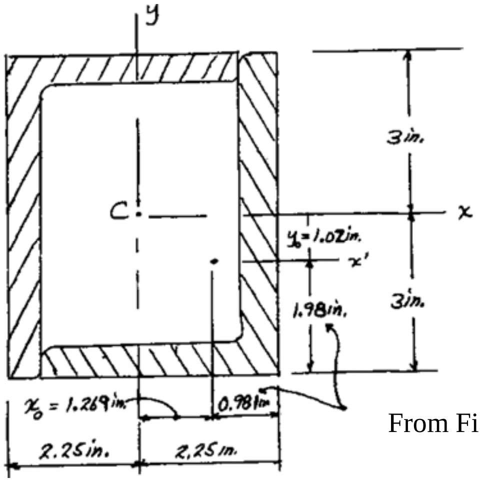
$$k_y^2 = \frac{I_y}{A} = \frac{167.691\times10^6}{13.28\times10^3} = 12.6273\times10^3 \qquad \bar{k}_y = 112.4\text{ mm} \blacktriangleleft$$



PROBLEM 9.50

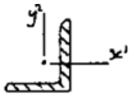
Two L6 × 4 × $\frac{1}{2}$ -in. angles are welded together to form the section shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal *x* and *y* axes.

SOLUTION



From Figure 9.13A

From Figure 9.13A: Area $= A = 4.75\ \text{in}^2$



$$I_{x'} = 17.3\ \text{in}^4$$

$$I_{y'} = 6.22\ \text{in}^4$$

$$\bar{I}_x = 2[\bar{I}_{x'} + Ay_O^2] = 2[17.3\ \text{in}^4 + (4.75\ \text{in}^2)(1.02\ \text{in.})^2] = 44.484\ \text{in}^4$$

$$\bar{I}_x = 44.5\ \text{in}^4 \blacktriangleleft$$

Total area = 2(4.75) = 9.50 in^2

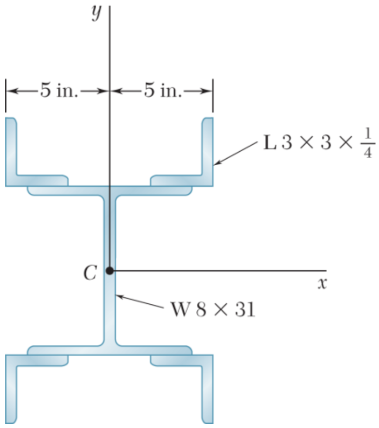
$$\bar{k}_x^2 = \frac{\bar{I}_x}{\text{area}} = \frac{44.484\ \text{in}^4}{9.50\ \text{in}^2} = 4.6825\ \text{in}^2 \qquad \bar{k}_x = 2.16\ \text{in.} \blacktriangleleft$$

$$\bar{I}_y = 2[\bar{I}_{y'} + Ax_O^2] = 2[6.22\ \text{in}^4 + (4.75\ \text{in}^2)(1.269\ \text{in.})^2]$$

$$= 27.738\ \text{in}^4 \qquad \bar{I}_y = 27.7\ \text{in}^4 \blacktriangleleft$$

Total area = 2(4.75) = 9.50 in^2

$$\bar{k}_y^2 = \frac{\bar{I}_y}{\text{area}} = \frac{27.738\ \text{in}^4}{9.50\ \text{in}^2} = 2.9198 \qquad \bar{k}_y = 1.709\ \text{in} \blacktriangleleft$$



PROBLEM 9.51

Two L6 × 4 × $\frac{1}{2}$ -in. angles are welded together to form the section shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal *x* and *y* axes.

SOLUTION

W section: $A = 9.12\text{ in}^2 \quad \bar{I}_x = 110\text{ in}^4 \quad \bar{I}_y = 37.1\text{ in}^4$

Angle: $A = 1.44\text{ in}^2 \quad \bar{I}_x = 1.23\text{ in}^4 \quad \bar{I}_y = 1.23\text{ in}^4$

$$A_{\text{total}} = A_{\text{W}} + 4A_{\text{A}}$$

$$= 9.12 + 4(1.44) = 14.880\text{ in}^2$$

Now $$\bar{I}_x = (\bar{I}_x)_{\text{W}} + 4(I_x)_{\text{A}}$$

where $$(I_x)_{\text{A}} = \bar{I}_{x_{\text{A}}} + Ad^2$$

$$= 1.23\text{ in}^4 + (1.44\text{ in}^2)(4.00\text{ in.} + 0.836\text{ in.})^2 = 34.907\text{ in}^4$$

Then $$\bar{I}_x = (110 + 4 \times 34.907)\text{ in}^4 = 249.63\text{ in}^4$$ $\bar{I}_x = 250\text{ in}^4$ ◀

and $$\bar{k}_x^2 = \frac{\bar{I}_x}{A_{\text{total}}} = \frac{249.63\text{ in}^4}{14.880\text{ in}^2}$$ $\bar{k}_x = 4.10\text{ in.}$ ◀

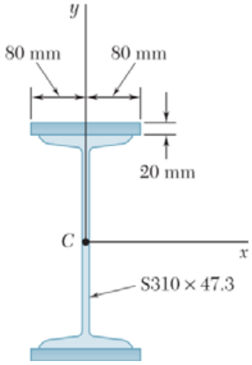
Also $$\bar{I}_y = (\bar{I}_y)_{\text{W}} + 4(\bar{I}_y)_{\text{A}}$$

where $$\left(\bar{I}_y\right)_{\text{A}} = \bar{I}_{y_{\text{A}}} + Ad^2$$

$$= \left(1.23 + 1.44\left(5 - 0.836\right)^2\right)\text{ in}^4 = 26.198\text{ in}^4$$

$$\bar{I}_y = (37.1 + 4 \times 26.198)\text{ in}^4 = 141.892\text{ in}^4$$ $\bar{I}_y = 141.9\text{ in}^4$ ◀

$$\bar{k}_y^2 = \frac{\bar{I}_y}{A_{\text{total}}} = \frac{141.892\text{ in}^4}{14.880\text{ in}^2}$$ $\bar{k}_y = 3.09\text{ in.}$ ◀



PROBLEM 9.52

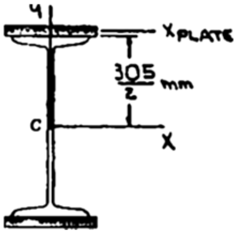
Two 20-mm steel plates are welded to a rolled S section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal *x* and *y* axes.

SOLUTION

S section: $A = 6010\text{ mm}^2$

$\bar{I}_x = 90.3 \times 10^6\text{ mm}^4$

$\bar{I}_y = 3.88 \times 10^6\text{ mm}^4$



Note:
$$A_{\text{total}} = A_S + 2A_{\text{plate}}$$
$$= 6010\text{ mm}^2 + 2(160\text{ mm})(20\text{ mm})$$
$$= 12,410\text{ mm}^2$$

Now
$$\bar{I}_x = (\bar{I}_x)_S + 2(I_x)_{\text{plate}}$$

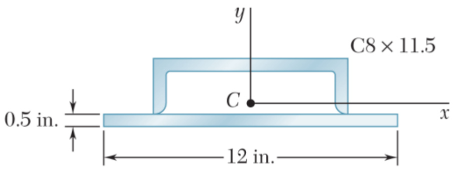
where
$$(I_x)_{\text{plate}} = \bar{I}_{x_{\text{plate}}} + Ad^2$$
$$= \frac{1}{12}(160\text{ mm})(20\text{ mm})^3 + (3200\text{ mm}^2)[(152.5+10)\text{ mm}]^2$$
$$= 84.6067 \times 10^6\text{ mm}^4$$

Then
$$\bar{I}_x = (90.3 + 2 \times 84.6067) \times 10^6\text{ mm}^4$$
$$= 259.5134 \times 10^6\text{ mm}^4 \qquad \text{or} \quad \bar{I}_x = 260 \times 10^6\text{ mm}^4 \blacktriangleleft$$

and
$$\bar{k}_x^2 = \frac{\bar{I}_x}{A_{\text{total}}} = \frac{259.5134 \times 10^6\text{ mm}^4}{12410\text{ mm}^2} \qquad \text{or} \quad \bar{k}_x = 144.6\text{ mm} \blacktriangleleft$$

Also
$$\bar{I}_y = (\bar{I}_y)_S + 2(\bar{I}_y)_{\text{plate}}$$
$$= 3.88 \times 10^6\text{ mm}^4 + 2\left[\frac{1}{12}(20\text{ mm})(160\text{ mm})^3\right]$$
$$= 17.5333 \times 10^6\text{ mm}^4 \qquad \text{or} \quad \bar{I}_y = 17.53 \times 10^6\text{ mm}^4 \blacktriangleleft$$

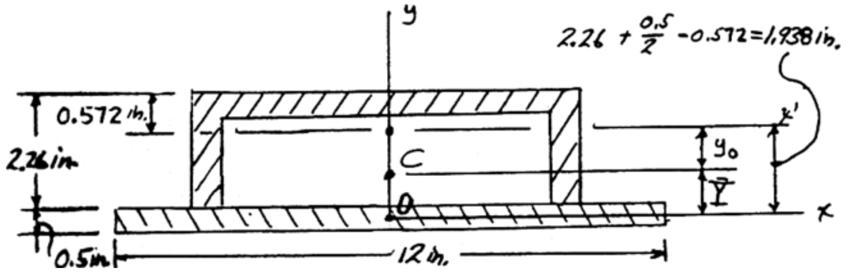
and
$$\bar{k}_y^2 = \frac{\bar{I}_y}{A_{\text{total}}} = \frac{17.5333 \times 10^6\text{ mm}^4}{12,410\text{ mm}^2} \qquad \text{or} \quad \bar{k}_y = 37.6\text{ mm} \blacktriangleleft$$

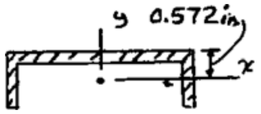


PROBLEM 9.53

A channel and a plate are welded together as shown to form a section that is symmetrical with respect to the *y* axis. Determine the moments of inertia of the combined section with respect to its centroidal *x* and *y* axes.

SOLUTION





From Figure 9.13B $\qquad B = 3.37\ \text{in}^2$

For C8 × 11.5: $\qquad \bar{I}_y = 1.31\ \text{in}^4$

(Note change of axes) $\qquad \bar{I}_y = 32.5\ \text{in}^4$

Location of centroid

$$\bar{Y}\Sigma A = \Sigma \bar{y} A: \qquad \bar{Y}[3.7\ \text{in}^2 + (12\ \text{in.})(0.5\ \text{in.})] = (3.37\ \text{in}^2)(1.938\ \text{in.})$$

$$\bar{Y} = 0.69702\ \text{in.}$$

Moment of inertia with respect to *x* axis.

Plate:
$$I_x = \bar{I}_x + A\bar{Y}^2 = \frac{1}{12}(12\ \text{in.})(0.5\ \text{in.})^3 + (12\ \text{in.})(0.5\ \text{in.})(0.69702\ \text{in.})^2$$
$$= 0.125 + 2.9150 = 3.0400\ \text{in}^4$$

Channel:
$$I_x = \bar{I}_{x'} + A\bar{Y}_0^2 = 1.31\ \text{in}^4 + (3.37\ \text{in.})(1.938\ \text{in.} - 0.69702\ \text{in.})^2$$
$$= 1.31 + 5.1899 = 6.4999\ \text{in}^4$$

Entire section:
$$I_x = 3.0400\ \text{in}^4 + 6.4999\ \text{in}^4 = 9.5399\ \text{in}^4$$

$\bar{I}_x = 9.54\ \text{in}^4$ ◀

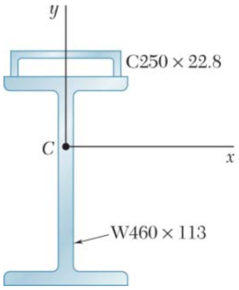
Moment of inertia with respect to *y* axis.

Plate:
$$\bar{I}_y = \frac{1}{12}(0.5\ \text{in.})(12\ \text{in.})^3 = 72\ \text{in}^4$$

Channel:
$$\bar{I}_y = 32.5\ \text{in}^4$$

Entire section:
$$\bar{I}_y = 72\ \text{in}^4 + 32.5\ \text{in}^4 = 104.5\ \text{in}^4$$

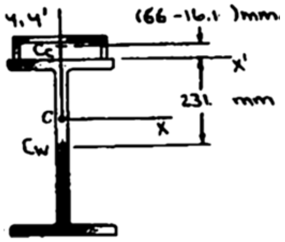
$\bar{I}_y = 104.5\ \text{in}^4$ ◀



PROBLEM 9.54

The strength of the rolled W section shown is increased by welding a channel to its upper flange. Determine the moments of inertia of the combined section with respect to its centroidal *x* and *y* axes.

SOLUTION



W section: $A = 14,400\text{ mm}^2$

$$\bar{I}_x = 554\times10^6\text{ mm}^4$$

$$\bar{I}_y = 63.3\times10^6\text{ mm}^4$$

Channel: $A = 2890\text{ mm}^2$

$$\bar{I}_x = 0.945\times10^6\text{ mm}^4$$

$$\bar{I}_y = 28.0\times10^6\text{ mm}^4$$

First locate centroid *C* of the section.

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
W Section	14,400	–231	–33,26,400
Channel	2,890	49.9	1,44,211
Σ	17,290		–31,82,189

Then $\bar{Y}\Sigma A = \Sigma\bar{y}A$: $\bar{Y}(17,290\text{ mm}^2) = -3,182,189\text{ mm}^3$

or $\bar{Y} = -184.047\text{ mm}$

Now $\bar{I}_x = (I_x)_\text{W} + (I_x)_\text{C}$

where

$$(I_x)_\text{W} = \bar{I}_x + Ad^2$$

$$= 554\times10^6\text{ mm}^4 + (14,400\text{ mm}^2)(231-184.047)^2\text{ mm}^2$$

$$= 585.75\times10^6\text{ mm}^4$$

$$(I_x)_\text{C} = \bar{I}_x - Ad^2$$

$$= 0.945\times10^6\text{ mm}^4 + (2,890\text{ mm}^2)(49.9+184.047)^2\text{ mm}^2$$

$$= 159.12\times10^6\text{ mm}^4$$

PROBLEM 9.54 (Continued)

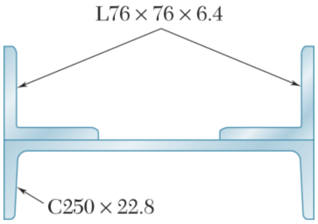
Then $\bar{I}_x = (585.75 + 159.12) \times 10^6 \text{ mm}^4$

or $\bar{I}_x = 745 \times 10^6 \text{ mm}^4$ ◀

also $\bar{I}_y = (I_y)_\text{W} + (I_y)_C$

$= (63.3 + 28.0) \times 10^6 \text{ mm}^4$

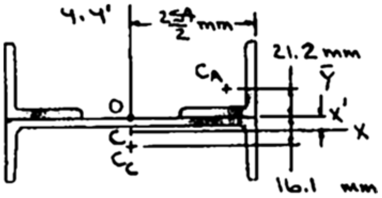
or $\bar{I}_y = 91.3 \times 10^6 \text{ mm}^4$ ◀



PROBLEM 9.55

Two L76 × 76 × 6.4-mm angles are welded to a C250 × 22.8 channel. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the web of the channel.

SOLUTION



Angle: $A = 929\ \text{mm}^2$

$\bar{I}_x = \bar{I}_y = 0.512 \times 10^6\ \text{mm}^4$

Channel: $A = 2890\ \text{mm}^2$

$\bar{I}_x = 0.945 \times 10^6\ \text{mm}^4$

$\bar{I}_y = 28.0 \times 10^6\ \text{mm}^4$

First locate centroid *C* of the section

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
Angle	$2(929) = 1858$	21.2	39,389.6
Channel	2890	−16.1	−46,529
Σ	4748		−7139.4

Then $\bar{Y}\Sigma A = \Sigma \bar{y} A$: $\quad \bar{Y}(4748\ \text{mm}^2) = -7139.4\ \text{mm}^3$

or $\bar{Y} = -1.50366\ \text{mm}$

Now $\bar{I}_x = 2(I_x)_L + (I_x)_C$

where

$$(I_x)_L = \bar{I}_x + Ad^2 = 0.512 \times 10^6\ \text{mm}^4 + (929\ \text{mm}^2)[(21.2 + 1.50366)\ \text{mm}]^2$$
$$= 0.990859 \times 10^6\ \text{mm}^4$$
$$(I_x)_C = \bar{I}_x + Ad^2 = 0.949 \times 10^6\ \text{mm}^4 + (2890\ \text{mm}^2)[(16.1 - 1.50366)\ \text{mm}]^2$$
$$= 1.56472 \times 10^6\ \text{mm}^4$$

Then $\bar{I}_x = [2(0.990859) + 1.56472 \times 10^6]\ \text{mm}^4$

or $\bar{I}_x = 3.55 \times 10^6\ \text{mm}^4$ ◀

PROBLEM 9.55 (Continued)

Also $$\bar{I}_y = 2(I_y)_L + (I_y)_C$$

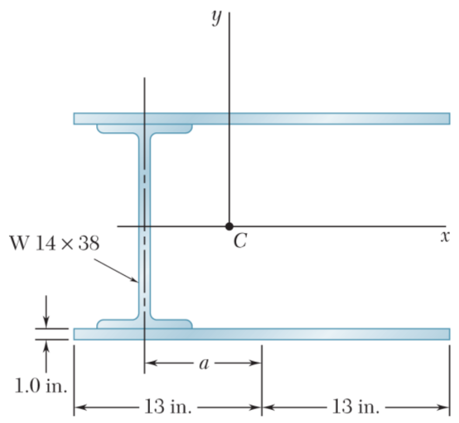
where $$(I_y)_L = \bar{I}_y + Ad^2 = 0.512\times10^6\text{ mm}^4 + (929\text{ mm}^2)[(127-21.2)\text{ mm}]^2$$

$$= 10.9109\times10^6\text{ mm}^4$$

$$(I_y)_C = \bar{I}_y$$

Then $$\bar{I}_y = [2(10.9109) + 28.0]\times10^6\text{ mm}^4$$

or $\bar{I}_y = 49.8\times10^6\text{ mm}^4$ ◀



PROBLEM 9.56

Two steel plates are welded to a rolled W section as indicated. Knowing that the centroidal moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the combined section are equal, determine (*a*) the distance *a*, (*b*) the moments of inertia with respect to the centroidal *x* and *y* axes.

SOLUTION

For W-section: $A = 11.2\text{ in}^2, \quad \bar{I}_x = 385\text{ in}^4, \quad \bar{I}_y = 26.7\text{ in}^4$

Section	Area, in^2	$\bar{x}$ in.	$\bar{x}A$, in^3
Plate	$2(1)(26) = 52$	a	$52a$
W-section	11.2	0	0
Σ	63.2		$52a$

$$\bar{X}\Sigma A = \Sigma\bar{x}A: \qquad \bar{X}\left(63.2\text{ in}^2\right) = 52a\text{ in}^3$$

$$\bar{x} = 0.82279a\text{ in.}$$

Entire section:

$$\bar{I}_x = (\bar{I}_{x_W} + 2I_{x_{PL}}); \quad I_{x_{PL}} = \bar{I}_x + Ad^2$$

$$= \frac{1}{12}(26\text{ in.})(1\text{ in.})^3 + \left(26\text{ in}^3\right)(7.05\text{ in.} + 0.5\text{ in.})^2$$

$$= 1484.232\text{ in}^4$$

$$\bar{I}_x = 385 + 2(1484.232) = 3353.46\text{ in}^4 \qquad \bar{I}_x = 3350\text{ in}^4 \blacktriangleleft$$

$$\bar{I}_x = \bar{I}_y \qquad \bar{I}_y = 3350\text{ in}^4 \blacktriangleleft$$

$$\bar{I}_y = (\bar{I}_{y_W} + 2I_{y_{PL}}); \quad I_{y_{PL}} = \bar{I}_y + Ad^2$$

$$= \frac{1}{12}(1\text{ in.})(26\text{ in.})^3 + \left(26\text{ in}^2\right)(a - 0.82279a\text{ in.})^2$$

$$= \left(1464.667 + 0.81649a^2\right)\text{ in}^4$$

PROBLEM 9.56 (Continued)

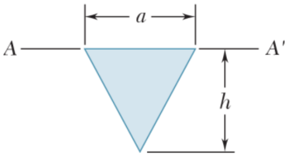
$$\bar{I}_{y_w} = \bar{I}_{y_w} + Ad^2$$

$$= 26.7\text{ in}^4 + \left(11.2\text{ in}^2\right)\left(0.82279a\text{ in.}\right)^2$$

$$= \left(26.7 + 7.5822a^2\right)\text{in}^4$$

$$\bar{I}_y = \left(26.7 + 7.5822a^2\right) + 2\left(1464.667 + 0.81649a^2\right) = 3353.46$$

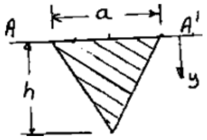
$a = 6.57$ in. ◀



PROBLEM 9.57

The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION



From section 9.2: $R = \gamma \int y dA, \quad M_{AA'} = \gamma \int y^2 dA$

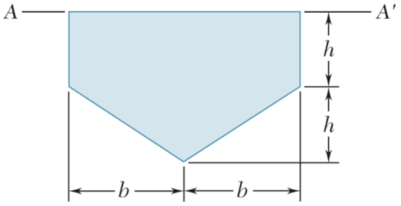
Let y_P= distance of center of pressure from AA'. We must have:

$$Ry_P = M_{AA'}: \quad y_P = \frac{M_{AA'}}{R} = \frac{\gamma \int y^2 dA}{\gamma \int y dA} = \frac{I_{AA'}}{\bar{y} A} \qquad (1)$$

For triangular panel:

$$I_{AA'} = \frac{1}{12} a h^3 \qquad \bar{y} = \frac{1}{3} h \qquad A = \frac{1}{2} a h$$

$$y_P = \frac{I_{AA'}}{\bar{y} A} = \frac{\frac{1}{12} a h^3}{\left(\frac{1}{3} h\right)\left(\frac{1}{2} a h\right)} \qquad y_P = \frac{1}{2} h \blacktriangleleft$$

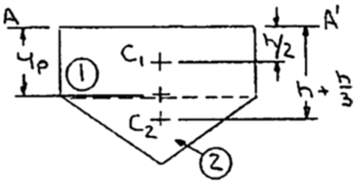


PROBLEM 9.58

The panel shown forms the end of a trough that is filled with water to the line *AA*′. Referring to Section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

Using the equation developed on page 506 of the text:



$$y_P = \frac{I_{AA'}}{\bar{y}A}$$

Now $\bar{Y}A = \Sigma \bar{y}A$

$$= \frac{h}{2}(2b \times h) + \frac{4}{3}h\left(\frac{1}{2} \times 2b \times h\right)$$

$$= \frac{7}{3}bh^2$$

and $I_{AA'} = (I_{AA'})_1 + (I_{AA'})_2$

where

$$(I_{AA'})_1 = \frac{1}{3}(2b)(h)^3 = \frac{2}{3}bh^3$$

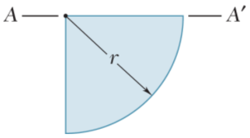
$$(I_{AA'})_2 = \bar{I}_x + Ad^2 = \frac{1}{36}(2b)(h)^3 + \left(\frac{1}{2} \times 2b \times h\right)\left(\frac{4}{3}h\right)^2$$

$$= \frac{11}{6}bh^3$$

Then $I_{AA'} = \frac{2}{3}bh^3 + \frac{11}{6}bh^3 = \frac{5}{2}bh^3$

Finally, $y_P = \dfrac{\frac{5}{2}bh^3}{\frac{7}{3}bh^2}$

or $y_P = \frac{15}{14}h$ ◀



PROBLEM 9.59

The panel shown forms the end of a trough that is filled with water to the line *AA*′. Referring to section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

Using the equation developed on page 506 of the text have:

$$y_P = \frac{I_{AA'}}{\overline{y}A}$$

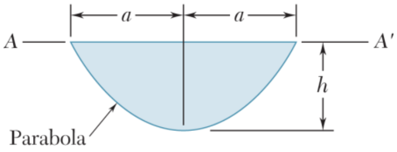
For a quartér circle:

$$I_{AA'} = \frac{\pi}{16} r^4$$

$$\overline{y} = \frac{4r}{3\pi} \qquad A = \frac{\pi}{4} r^2$$

Then $y_P = \dfrac{\frac{\pi}{16} r^4}{\frac{4r}{3\pi} \times \frac{\pi}{4} r^2}$ or

or $y_P = \dfrac{3\pi}{16} r$ ◀



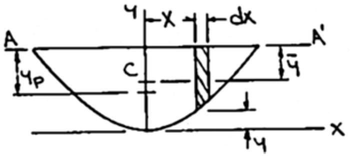
PROBLEM 9.60*

The panel shown forms the end of a trough that is filled with water to the line *AA′*. Referring to section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

Using the equation developed on page 506 of the text:

$$y_P = \frac{I_{AA'}}{\bar{y}A}$$



For a parabola:

$$\bar{y} = \frac{2}{5}h \qquad A = \frac{4}{3}ah$$

Now

$$dI_{AA'} = \frac{1}{3}(h-y)^3\,dx$$

By observation

$$y = \frac{h}{a^2}x^2$$

So that

$$dI_{AA'} = \frac{1}{3}\left(h - \frac{h}{a^2}x^2\right)^3 dx = \frac{1}{3}\frac{h^3}{a^6}(a^2 - x^2)^3\,dx$$

$$= \frac{1}{3}\frac{h}{a^6}(a^6 - 3a^4x^2 + 3a^2x^4 - x^6)dx$$

Then

$$I_{AA'} = 2\int_0^a \frac{1}{3}\frac{h^3}{a^6}(a^6 - 3a^4x^2 + 3a^2x^4 - x^6)dx$$

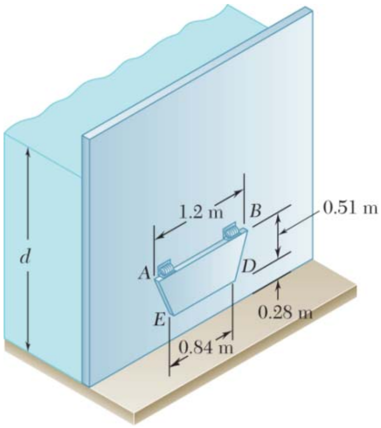
$$= \frac{2}{3}\frac{h^3}{a^6}\left[a^6x - a^4x^3 + \frac{3}{5}a^2x^5 - \frac{1}{7}x^7\right]_0^a$$

$$= \frac{32}{105}ah^3$$

Finally,

$$y_p = \frac{\frac{32}{105}ah^3}{\frac{2}{5}h \times \frac{4}{3}ah}$$

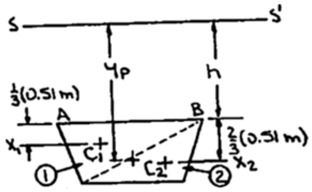
or $y_P = \frac{4}{7}h$ ◀



PROBLEM 9.61

A vertical trapezoidal gate that is used as an automatic valve is held shut by two springs attached to hinges located along edge *AB*. Knowing that each spring exerts a couple of magnitude 1470 N · m, determine the depth *d* of water for which the gate will open.

SOLUTION



From section 9.2:

$$R = \gamma \bar{y} A \qquad y_P = \frac{I_{SS'}}{\bar{y}A}$$

where *R* is the resultant of the hydrostatic forces acting on the gate and y_P is the depth to the point of application of *R*. Now

$$\bar{y}A = \Sigma \bar{y}A = [(h+0.17)\text{ m}]\left(\frac{1}{2}\times 1.2\text{ m}\times 0.51\text{ m}\right) + [(h+0.34)\text{ m}]\left(\frac{1}{2}\times 0.84\times 0.51\text{ m}\right)$$
$$= (0.5202h + 0.124848)\text{ m}^3$$

Recalling that $\gamma = py,$ we have

$$R = (10^3\text{ kg/m}^3 \times 9.81\text{ m/s}^2)(0.5202h + 0.124848)\text{ m}^3$$
$$= 5103.162(h + 0.24)\text{ N}$$

Also
$$I_{SS'} = (I_{SS'})_1 + (I_{SS'})_2$$

where
$$(I_{SS'})_1 = I_x + Ad^2$$
$$= \frac{1}{36}(1.2\text{ m})(0.51\text{ m})^3 + \left(\frac{1}{2}\times 1.2\text{ m}\times 0.51\text{ m}\right)[(h+0.17)\text{ m}]^2$$
$$= [0.0044217 + 0.306(h+0.17)^2]\text{ m}^4$$
$$= (0.306h^2 + 0.10404h + 0.0132651)\text{ m}^4$$

$$(I_{SS'})_2 = \bar{I}_{X_2} + Ad^2$$
$$= \frac{1}{36}(0.84\text{ m})(0.51\text{ m})^3 + \left(\frac{1}{2}\times 0.84\text{ m}\times 0.51\text{ m}\right)[(h+0.34)\text{ m}]^2$$
$$= [0.0030952 + 0.2142(h+0.34)^2]\text{ m}^4$$
$$= (0.2142h^2 + 0.145656h + 0.0278567)\text{ m}^4$$

PROBLEM 9.61 (Continued)

Then
$$I_{SS'} = (I_{SS'})_1 + (I_{SS'})_2$$
$$= (0.5202h^2 + 0.249696h + 0.0411218)\ \text{m}^4$$

and
$$y_P = \frac{(0.5202h^2 + 0.244696h + 0.0411218)\ \text{m}^4}{(0.5202h + 0.124848)\ \text{m}^3}$$
$$= \frac{h^2 + 0.48h + 0.07905}{h + 0.24}\ \text{m}$$

For the gate to open, require that

$$\Sigma M_{AB}: \quad M_{\text{open}} = (y_P - h)R$$

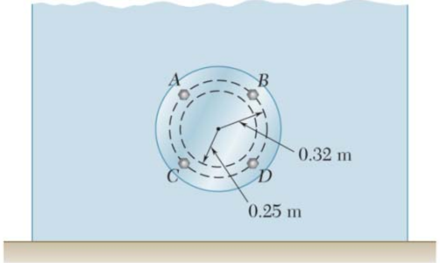
Substituting
$$2940\ \text{N}\cdot\text{m} = \left(\frac{h^2 + 0.48h + 0.07905}{h + 0.24} - h\right)\text{m} \times 5103.162(h + 0.24)\ \text{N}$$

or $5103.162(0.24h + 0.07905) = 2940$

or
$$h = 2.0711\ \text{m}$$

Then
$$d = (2.0711 + 0.79)\ \text{m}$$

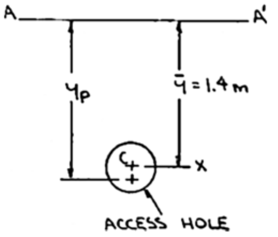
or $d = 2.86\ \text{m}$ ◀



PROBLEM 9.62

The cover for a 0.5-m-diameter access hole in a water storage tank is attached to the tank with four equally spaced bolts as shown. Determine the additional force on each bolt due to the water pressure when the center of the cover is located 1.4 m below the water surface.

SOLUTION



From section 9.2:

$$R = \gamma \bar{y} A \quad y_P = \frac{I_{AA'}}{\bar{y}A}$$

where R is the resultant of the hydrostatic forces acting on the cover and y_P is the depth to the point of application of R.

Recalling that $\gamma = p \cdot y,$ we have

$$R = (10^3 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2)(1.4 \text{ m})[\pi(0.25 \text{ m})^2]$$
$$= 2696.67 \text{ N}$$

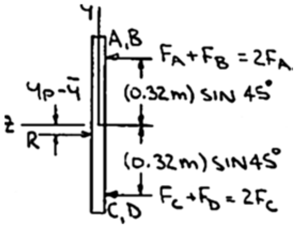
Also
$$I_{AA'} = \bar{I}_x + A\bar{y}^2 = \frac{\pi}{4}(0.25 \text{ m})^4 + [\pi(0.25 \text{ m})^2](1.4 \text{ m})^2$$
$$= 0.387913 \text{ m}^4$$

Then
$$y_P = \frac{0.387913 \text{ m}^4}{(1.4 \text{ m})[\pi(0.25 \text{ m})^2]} = 1.41116 \text{ m}$$

Now note that symmetry implies

$$F_A = F_B \qquad F_C = F_D$$

Next consider the free-body of the cover.



We have
$$+\circlearrowleft \Sigma M_{CD} = 0: \quad [2(0.32 \text{ m}) \sin 45°](2F_A)$$
$$-[0.32 \sin 45° - (1.41116 - 1.4)] \text{ m}$$
$$\times (2696.67 \text{ N}) = 0$$

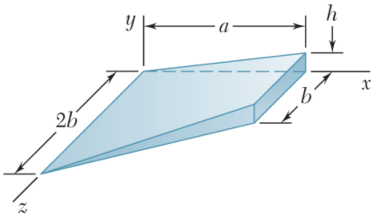
or
$$F_A = 640.92 \text{ N}$$

Then
$$\overset{+}{\longleftarrow} \Sigma F_z = 0: \quad 2(640.92 \text{ N}) + 2F_C - 2696.67 \text{ N} = 0$$

or
$$F_C = 707.42 \text{ N}$$

$$F_A = F_B = 641 \text{ N} \blacktriangleleft$$

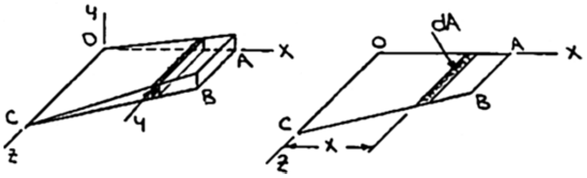
and
$$F_C = F_D = 707 \text{ N} \blacktriangleleft$$



PROBLEM 9.63*

Determine the x coordinate of the centroid of the volume shown. (*Hint:* The height y of the volume is proportional to the x coordinate; consider an analogy between this height and the water pressure on a submerged surface.)

SOLUTION



First note that

$$y = \frac{h}{a}x$$

Now

$$\bar{x}\int dV = \int \bar{x}_{EL}\,dV$$

where

$$\bar{x}_{EL} = x \quad dV = y\,dA = \left(\frac{h}{a}x\right)dA$$

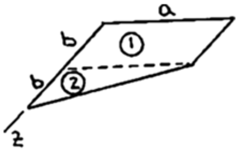
Then

$$\bar{x} = \frac{\int x\left(\frac{h}{a}x\,dA\right)}{\int \frac{h}{a}x\,dA} = \frac{\int x^2 dA}{\int x\,dA} = \frac{(I_z)_A}{(\bar{x}A)_A}$$

where $(I_z)_A$ and $(\bar{x}_A)_A$ pertain to area.

OABC: $(I_z)_A$ is the moment of inertia of the area with respect to the z axis, $\bar{x}_A$ is the x coordinate of the centroid of the area, and A is the area of *OABC*. Then

$$\begin{aligned}(I_z)_A &= (I_z)_{A_1} + (I_z)_{A_2}\\ &= \frac{1}{3}(b)(a)^3 + \frac{1}{12}(b)(a)^3\\ &= \frac{5}{12}a^3 b\end{aligned}$$



and

$$\begin{aligned}(\bar{x}A)_A &= \Sigma\bar{x}A\\ &= \left[\left(\frac{a}{2}\right)(a\times b)\right] + \left[\left(\frac{a}{3}\right)\left(\frac{1}{2}\times a\times b\right)\right]\\ &= \frac{2}{3}a^2 b\end{aligned}$$

PROBLEM 9.63* (Continued)

Finally,
$$\bar{x} = \frac{\frac{5}{12}a^3b}{\frac{2}{3}a^2b}$$

or $\bar{x} = \dfrac{5}{8}a$ ◀

Analogy with hydrostatic pressure on a submerged plate:

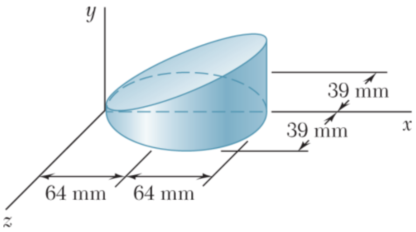
Recalling that $P = \gamma y$, it follows that the following analogies can be established.

Height $y \sim P$

$$dV = ydA \sim pdA = dF$$
$$xdV = x(ydA) \sim ydF = dM$$

Recalling that
$$y_P = \frac{M_x}{R}\left(= \frac{\int dM}{\int dF}\right)$$

It can then be concluded that $x \sim y_P$



PROBLEM 9.64*

Determine the x coordinate of the centroid of the volume shown; this volume was obtained by intersecting an elliptic cylinder with an oblique plane. (*Hint:* The height y of the volume is proportional to the x coordinate; consider an analogy between this height and the water pressure on a submerged surface.)

SOLUTION

Following the “Hint,” it can be shown that (see solution to Problem 9.63)

$$\bar{x} = \frac{(I_z)_A}{(\bar{x}A)_A} x$$

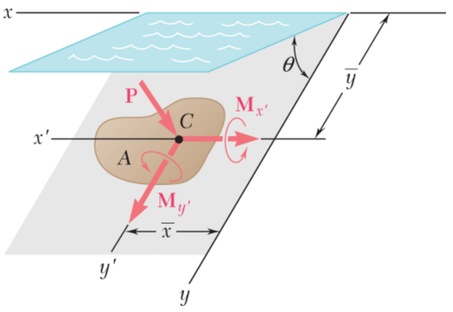
where $(I_z)_A$ and $(\bar{x}A)_A$ are the moment of inertia and the first moment of the area, respectively, of the elliptical area of the base of the volume with respect to the z axis. Then

$$\begin{aligned}(I_z)_A &= \bar{I}_z + Ad^2 \\ &= \frac{\pi}{4}(39 \text{ mm})(64 \text{ mm})^3 + [\pi(64 \text{ mm})(39 \text{ mm})](64 \text{ mm})^2 \\ &= 12.779520\pi \times 10^6 \text{ mm}^4\end{aligned}$$

$$\begin{aligned}(\bar{x}A)_A &= (64 \text{ mm})[\pi(64 \text{ mm})(39 \text{ mm})] \\ &= 0.159744\pi \times 10^6 \text{ mm}^3\end{aligned}$$

Finally $$\bar{x} = \frac{12.779520\pi \times 10^6 \text{ mm}^4}{0.159744\pi \times 10^6 \text{ mm}^4}$$

or $\bar{x} = 80.0$ mm ◀



PROBLEM 9.65*

Show that the system of hydrostatic forces acting on a submerged plane area A can be reduced to a force **P** at the centroid C of the area and two couples. The force **P** is perpendicular to the area and is of magnitude $P = \gamma A\bar{y}\sin\theta$, where γ is the specific weight of the liquid, and the couples are $\mathbf{M}_{x'} = (\gamma \bar{I}_{x'} \sin\theta)\mathbf{i}$ and $\mathbf{M}_{y'} = (\gamma \bar{I}_{x'y'} \sin\theta)\mathbf{j}$, where $\bar{I}_{x'y'} = \int x'y'\,dA$ (see section 9.8). Note that the couples are independent of the depth at which the area is submerged.

SOLUTION

The pressure p at an arbitrary depth $(y\sin\theta)$ is

$$p = \gamma(y\sin\theta)$$

so that the hydrostatic force dF exerted on an infinitesimal area dA is

$$dF = (\gamma y\sin\theta)dA$$

Equivalence of the force **P** and the system of infinitesimal forces dF requires

$$\Sigma F\text{:}\quad P = \int dF = \int \gamma y\sin\theta\,dA = \gamma\sin\theta\int y\,dA$$

or $$P = \gamma A\bar{y}\sin\theta \blacktriangleleft$$

Equivalence of the force and couple $(\mathbf{P},\ \mathbf{M}_{x'} + \mathbf{M}_{y'})$ and the system of infinitesimal hydrostatic forces requires

$$\Sigma M_x\text{:}\quad -\bar{y}P - M_{x'} = \int(-ydF)$$

Now $$-\int ydF = -\int y(\gamma y\sin\theta)dA = -\gamma\sin\theta\int y^2dA$$

$$= -(\gamma\sin\theta)I_x$$

Then $$-\bar{y}P - M_{x'} = -(\gamma\sin\theta)I_x$$

or $$M_{x'} = (\gamma\sin\theta)I_x - \bar{y}(\gamma A\bar{y}\sin\theta)$$

$$= \gamma\sin\theta(I_x - A\bar{y}^2)$$

or $$M_{x'} = \gamma\bar{I}_{x'}\sin\theta \blacktriangleleft$$

$$\Sigma M_y\text{:}\quad \bar{x}P + M_{y'} = \int x\,dF$$

Now $$\int x\,dF = \int x(\gamma y\sin\theta)dA = \gamma\sin\theta\int xy\,dA$$

$$= (\gamma\sin\theta)I_{xy} \qquad \text{(Equation 9.12)}$$

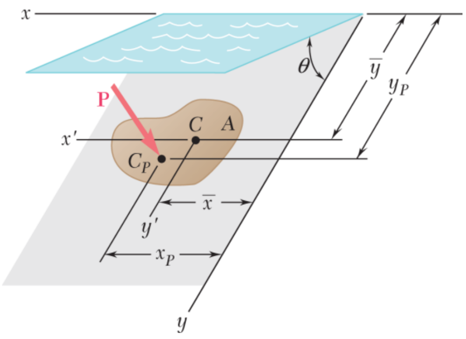
PROBLEM 9.65* (Continued)

Then $$\bar{x}P + M_{y'} = (\gamma \sin\theta) I_{xy}$$

or $$M_{y'} = (\gamma \sin\theta) I_{xy} - \bar{x}(\gamma A \bar{y} \sin\theta)$$

$$= \gamma \sin\theta (I_{xy} - A\bar{x}\,\bar{y})$$

or $$M_{y'} = \gamma \bar{I}_{x'y'} \sin\theta \blacktriangleleft$$



PROBLEM 9.66*

Show that the resultant of the hydrostatic forces acting on a submerged plane area A is a force **P** perpendicular to the area and of magnitude $P = \gamma A\bar{y}\sin\theta = \bar{p}A$, where γ is the specific weight of the liquid and $\bar{p}$ is the pressure at the centroid C of the area. Show that **P** is applied at a Point C_P, called the center of pressure, whose coordinates are $x_P = I_{xy}/A\bar{y}$ and $y_P = I_x/A\bar{y}$, where $\bar{I}_{xy} = \int xy\,dA$ (see section 9.8). Show also that the difference of ordinates $y_P - \bar{y}$ is equal to $\bar{k}_{x'}^2/\bar{y}$ and thus depends upon the depth at which the area is submerged.

SOLUTION

The pressure P at an arbitrary depth $(y\sin\theta)$ is

$$P = \gamma(y\sin\theta)$$

so that the hydrostatic force dP exerted on an infinitesimal area dA is

$$dP = (\gamma y\sin\theta)dA$$

The magnitude **P** of the resultant force acting on the plane area is then

$$P = \int dP = \int \gamma y\sin\theta\,dA = \gamma\sin\theta\int y\,dA$$

$$= \gamma\sin\theta(\bar{y}A)$$

Now $\bar{p} = \gamma\bar{y}\sin\theta$ $\quad P = \bar{p}A$ ◀

Next observe that the resultant **P** is equivalent to the system of infinitesimal forces d**P**. Equivalence then requires

$$\Sigma M_x:\quad -y_P P = -\int y\,dP$$

Now

$$\int y\,dP = \int y(\gamma y\sin\theta)dA = \gamma\sin\theta\int y^2\,dA$$

$$= (\gamma\sin\theta)I_x$$

Then

$$y_P P = (\gamma\sin\theta)I_x$$

or

$$y_P = \frac{(\gamma\sin\theta)I_x}{\gamma\sin\theta(\bar{y}A)}$$

or $\quad y_P = \dfrac{I_x}{A\bar{y}}$ ◀

PROBLEM 9.66* (Continued)

$$\Sigma M_y: \quad x_P P = \int x dP$$

Now $$\int x dP = \int x(\gamma y \sin\theta) dA = \gamma \sin\theta \int xy dA$$

$$= (\gamma \sin\theta) I_{xy} \qquad \text{(Equation 9.12)}$$

Then $$x_P P = (\gamma \sin\theta) I_{xy}$$

or $$x_P = \frac{(\gamma \sin\theta) I_{xy}}{\gamma \sin\theta(\bar{y}A)}$$

or $$x_P = \frac{I_{xy}}{A\bar{y}} \blacktriangleleft$$

Now $$I_x = \bar{I}_{x'} + A\bar{y}^2$$

From above $$I_x = (A\bar{y}) y_P$$

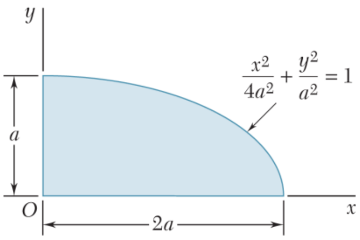
By definition $$\bar{I}_{x'} = \bar{k}_{x'}^2 A$$

Substituting $$(A\bar{y}) y_P = \bar{k}_{x'}^2 A + A\bar{y}^2$$

Rearranging yields $$y_P - \bar{y} = \frac{\bar{k}_{x'}^2}{\bar{y}} \blacktriangleleft$$

Although $\bar{k}_{x'}$ is not a function of the depth of the area (it depends only on the shape of A), $\bar{y}$ is dependent on the depth.

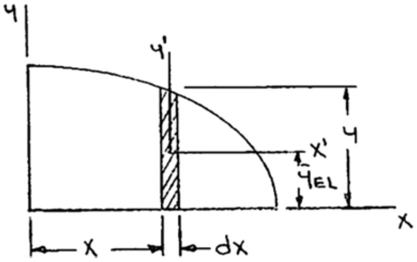
$$(y_P - \bar{y}) = f(\text{depth})$$



PROBLEM 9.67

Determine by direct integration the product of inertia of the given area with respect to the *x* and *y* axes.

SOLUTION



First note

$$y = a\sqrt{1-\frac{x^2}{4a^2}}$$

$$= \frac{1}{2}\sqrt{4a^2 - x^2}$$

We have

$$dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}\,dA$$

where

$$d\bar{I}_{x'y'} = 0 \quad \text{(symmetry)} \quad \bar{x}_{EL} = x$$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{1}{4}\sqrt{4a^2 - x^2}$$

$$dA = ydx = \frac{1}{2}\sqrt{4a^2 - x^2 dx}$$

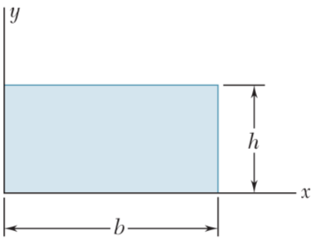
Then

$$I_{xy} = \int dI_{xy} = \int_0^{2a} x\left(\frac{1}{4}\sqrt{4a^2 - x^2}\right)\left(\frac{1}{2}\sqrt{4a^2 - x^2}\right)dx$$

$$= \frac{1}{8}\int_0^{2a}(4a^2x - x^3)dx = \frac{1}{8}\left[2a^2x^2 - \frac{1}{4}x^4\right]_0^{2a}$$

$$= \frac{a^4}{8}\left[2(2)^2 - \frac{1}{4}(2)^4\right]$$

or $I_{xy} = \frac{1}{2}a^4$ ◀



PROBLEM 9.68

Determine by direct integration the product of inertia of the given area with respect to the *x* and *y* axes.

SOLUTION

$$dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}\,dA$$

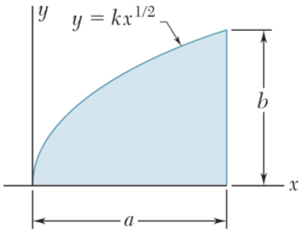
But $d\bar{I}_{x'y'} = 0$ (by symmetry) $\bar{x}_{EL} = x$ $\bar{y}_{EL} = \frac{1}{2}h$

$$dA = hd_x$$

$$I_{xy} = \int dI_{xy} = \int_0^b x\left(\frac{1}{2}h\right)hdx = \frac{1}{2}h^2\int_0^b xdx$$

$$= \frac{1}{2}h^2\frac{b^2}{2}$$

$$I_{xy} = \frac{1}{4}b^2h^2 \blacktriangleleft$$



PROBLEM 9.69

Determine by direct integration the product of inertia of the given area with respect to the *x* and *y* axes.

SOLUTION

For $$x = a, \quad b = ka^{1/2} \quad \text{or} \quad k = \frac{b}{a^{1/2}}$$

Thus, $$y = \frac{b}{a^{1/2}} x^{1/2}$$

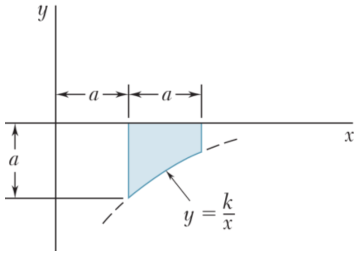
$$d\overline{I}_{xy} = d\overline{I}_{x'y'} + \overline{x}_{EL}\overline{y}_{EL}\, dA, \quad \text{But} \quad dI_{x'y'} = 0 \quad \text{(by symmetry)}$$

with $$\overline{x}_{EL} = x \quad \overline{y}_{EL} = \frac{y}{2} \quad dA = ydx$$

$$I_{xy} = \int dI_{xy} = \int_0^a x\left(\frac{y}{2}\right) ydx = \frac{1}{2}\int_0^a x\left(\frac{b}{a^{1/2}} x^{1/2}\right)^2 dx$$

$$= \frac{1}{2}\frac{b^2}{a}\int_0^a x^2 dx = \frac{b^2}{2a}\left[\frac{x^3}{3}\right]_0^a$$

$$I_{xy} = a^2b^2/6 \blacktriangleleft$$



PROBLEM 9.70

Determine by direct integration the product of inertia of the given area with respect to the *x* and *y* axes.

SOLUTION

For $$x = a, \quad y = -a; \quad -a = \frac{k}{a} \quad \text{or} \quad k = -a^2$$

Thus, $$y = -\frac{a^2}{x}$$

$$d\bar{I}_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}\,dA, \quad \text{But} \quad dI_{x'y'} = 0 \quad \text{(by symmetry)}$$

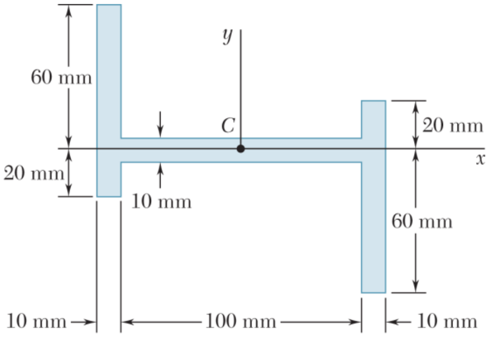
with $$\bar{x}_{EL} = x \quad \bar{y}_{EL} = \frac{y}{2} = -\frac{1}{2}\left(\frac{a^2}{x}\right) \quad dA = y\,dx$$

$$I_{xy} = \int dI_{xy} = \int_a^a x\left(\frac{1}{2}\frac{a^2}{x}\right)\left(\frac{a^2}{x}\right)dx = -\frac{a^4}{2}\int_a^{2a}\frac{1}{x}\,dx$$

$$= -\frac{a^4}{2}\left[\ln x\right]_a^{2a}$$

$$= -\frac{a^4}{2}\ln\frac{2a}{a}$$

$$I_{xy} = -0.347a^4 \blacktriangleleft$$



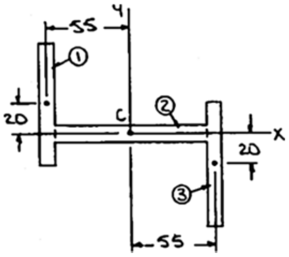
60 mm
y
20 mm
C
20 mm
x
10 mm
60 mm
10 mm
100 mm
10 mm

PROBLEM 9.71

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

Dimensions in mm



55
y
①
20
C
②
x
20
③
55

We have $\bar{I}_{xy} = (I_{xy})_1 + (\bar{I}_{xy})_2 + (I_{xy})_3$

Now symmetry implies

$$(\bar{I}_{xy})_2 = 0$$

and for the other rectangles

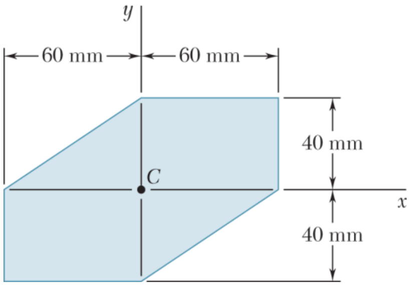
$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$$

where $I_{x'y'} = 0$ (symmetry)

Thus $\bar{I}_{xy} = (\bar{x}\,\bar{y}A)_1 + (\bar{x}\,\bar{y}A)_3$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\,\bar{y}A$, mm^4
1	$10 \times 80 = 800$	-55	20	$-880,000$
3	$10 \times 80 = 800$	55	-20	$-880,000$
Σ				$-1,760,000$

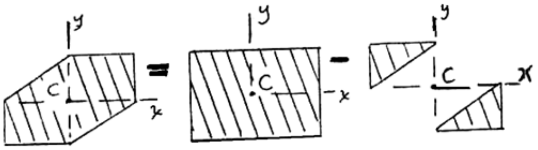
$\bar{I}_{xy} = -1.760 \times 10^6\ \text{mm}^4$ ◀


y
60 mm
60 mm
40 mm
C
x
40 mm

PROBLEM 9.72

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

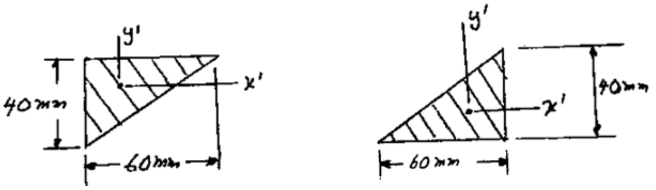
SOLUTION



Given area = Rectangle − (Two triangles)

For rectangle $I_{xy} = 0,$ by symmetry

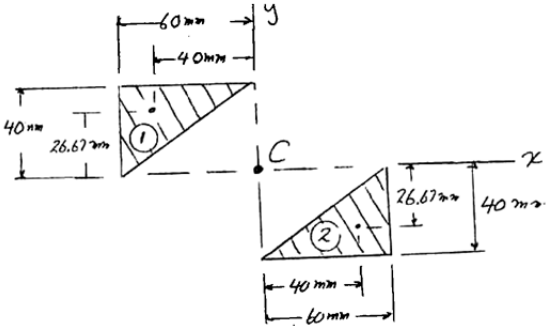
For each triangle:


y'
x'
40 mm
60 mm
y'
x'
40 mm
60 mm

Compare these triangles with the triangle of sample Problem 9.6, where $\bar{I}_{x'y'} = -\dfrac{1}{72}b^2h^2.$

For orientation of axes of this problem,

$$\bar{I}_{x'y'} = +\frac{1}{72}b^2h^2 = \frac{1}{72}(60\text{ mm})^2(40\text{ mm})^2 = +80\times 10^3\text{mm}^4$$


60 mm
y
40 mm
40 mm
26.67 mm
1
C
x
26.67 mm
40 mm
2
40 mm
60 mm

PROBLEM 9.72 (Continued)

	Area, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\,\bar{y}A$, mm^4
1	$\frac{1}{2}(60)(40) = 1200$	-40	$+26.67$	-1.280×10^6
2	$\frac{1}{2}(60)(40) = 1200$	$+40$	-26.67	-1.280×10^6

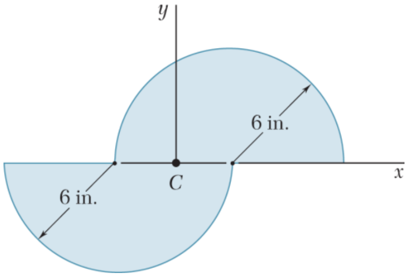
$$\Sigma\,\bar{x}\,\bar{y}\,A = -2.56 \times 10^6\,\text{mm}^4$$

For two triangles:

$$I_{xy} = \Sigma(\bar{I}_{x'y'} + \bar{x}\,\bar{y}\,A) = +2(80 \times 10^3) - 2.56 \times 10^6$$

$$= -2.40 \times 10^6\,\text{mm}^4$$

Since we must subtract triangles, $I_{xy} = +2.40 \times 10^6\,\text{mm}^4$ ◀



PROBLEM 9.73

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal *x* and *y* axes.

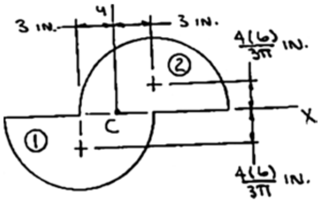
SOLUTION

We have $\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$

For each semicircle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

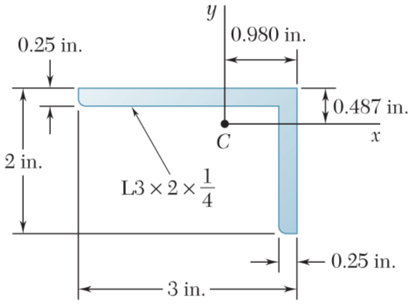
and $I_{x'y'} = 0$ (symmetry)

Thus $\bar{I}_{xy} = \Sigma \bar{x}\,\bar{y}A$



	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\bar{y}A$
1	$\frac{\pi}{2}(6)^2 = 18\pi$	-3	$-\frac{8}{\pi}$	432
2	$\frac{\pi}{2}(6)^2 = 18\pi$	3	$\frac{8}{\pi}$	432
Σ				864

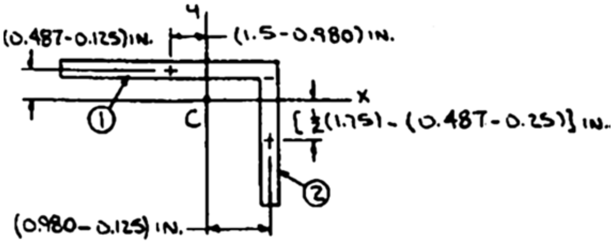
$\bar{I}_{xy} = 864\text{ in}^4$ ◀


y
0.980 in.
0.25 in.
0.487 in.
x
C
2 in.
L3 × 2 × 1/4
0.25 in.
3 in.

PROBLEM 9.74

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal *x* and *y* axes.

SOLUTION


(0.487-0.125) IN.
y
(1.5-0.980) IN.
x
①
C
[½(1.75) - (0.487-0.25)] IN.
②
(0.980-0.125) IN.

We have $\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$

For each rectangle

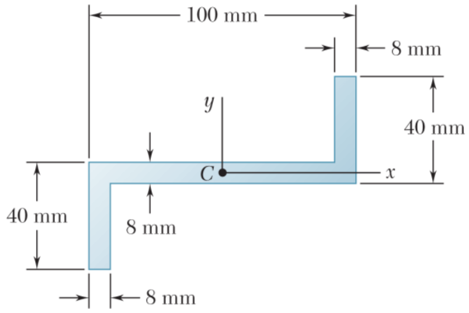
$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$$

and $\bar{I}_{x'y'} = 0 \quad \text{(symmetry)}$

Thus $\bar{I}_{xy} = \Sigma\bar{x}\,\bar{y}A$

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\,\bar{y}A$, in^4
1	$3 \times 0.25 = 0.75$	–0.520	0.362	–0.141180
2	$0.25 \times 1.75 = 0.4375$	0.855	–0.638	–0.238652
Σ				–0.379832

$\bar{I}_{xy} = -0.380 \text{ in}^4$ ◀

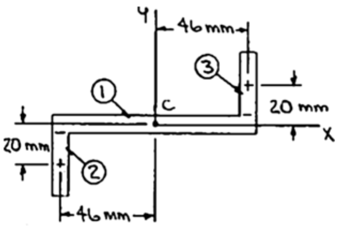

100 mm
8 mm
y
40 mm
C
x
40 mm
8 mm
8 mm

PROBLEM 9.75

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

We have $$I_{xy} = (\bar{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$$


y
46 mm
③
①
C
+
−
20 mm
x
20 mm
−
+
②
46 mm

Now symmetry implies $$(\bar{I}_{xy})_1 = 0$$

and for the other rectangles

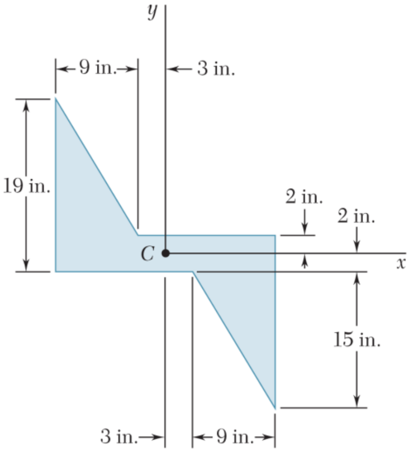
$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

where $$\bar{I}_{x'y'} = 0 \text{ (symmetry)}$$

Thus $$\bar{I}_{xy} = (\bar{x}\bar{y}A)_2 + (\bar{x}\bar{y}A)_3$$

	A, mm^2	$\bar{X}$, mm	$\bar{y}$, mm	$\bar{x}\bar{y}A$, mm^4
2	$8 \times 32 = 256$	–46	–20	235,520
3	$8 \times 32 = 256$	46	20	235,520
Σ				471,040

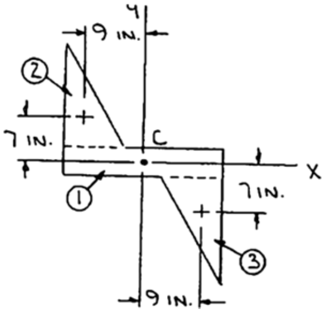
$\bar{I}_{xy} = 471 \times 10^3$ mm^4 ◀



PROBLEM 9.76

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal *x* and *y* axes.

SOLUTION



We have $\bar{I}_{xy} = (\bar{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$

Now, symmetry implies

$$(\bar{I}_{xy})_1 = 0$$

and for each triangle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

where, using the results of Sample Problem 9.6, $\bar{I}_{x'y'} = -\frac{1}{72}b^2h^2$ for both triangles. Note that the sign of $\bar{I}_{x'y'}$ is unchanged because the angles of rotation are $0°$ and $180°$ for triangles 2 and 3, respectively.

Now

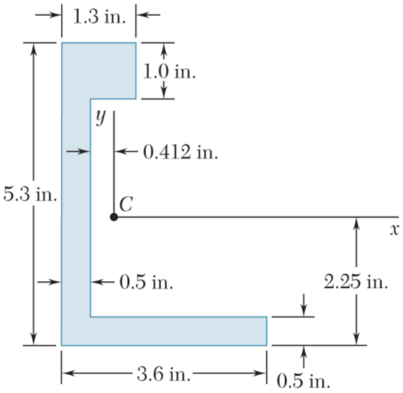
	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\,\bar{y}A$, in^4
2	$\frac{1}{2}(9)(15) = 67.5$	–9	7	–4252.5
3	$\frac{1}{2}(9)(15) = 67.5$	9	–7	–4252.5
Σ				–8505

Then

$$\bar{I}_{xy} = 2\left[-\frac{1}{72}(9\text{ in.})^2\,(15\text{ in.})^2\right] - 8505\text{ in}^4$$

$$= -9011.25\text{ in}^4$$

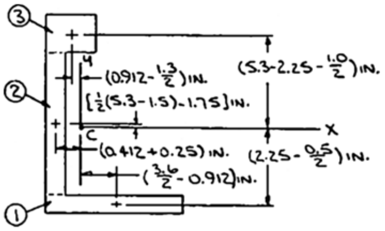
or $\bar{I}_{xy} = -9010\text{ in}^4$ ◀


1.3 in.
1.0 in.
y
0.412 in.
5.3 in.
C
x
0.5 in.
2.25 in.
3.6 in.
0.5 in.

PROBLEM 9.77

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal *x* and *y* axes.

SOLUTION


③
②
①
y
C
x
$(0.912-\frac{1.3}{2})$ in.
$[\frac{1}{2}(5.3-1.5)-1.75]$ in.
$(0.412+0.25)$ in.
$(\frac{3.6}{2}-0.912)$ in.
$(5.3-2.25-\frac{1.0}{2})$ in.
$(2.25-\frac{0.5}{2})$ in.

We have $\bar{I}_{xy} = (\bar{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$

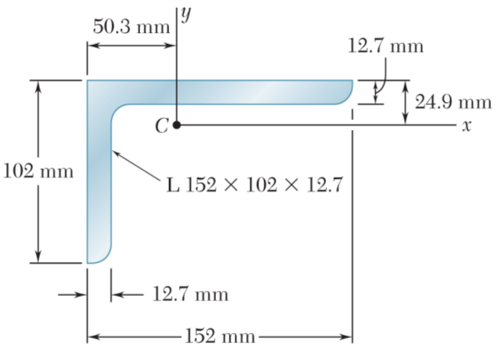
For each rectangle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

and $\bar{I}_{x'y'} = 0$ (symmetry)

Thus $I_{xy} = \Sigma\bar{x}\,\bar{y}A$

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\,\bar{y}A$, in^4
1	$3.6\times0.5=1.8$	0.888	–2.00	–3.196
2	$0.5\times3.8=1.9$	–0.662	0.15	–0.18867
3	$1.3\times1.0=1.3$	–0.262	2.55	–0.86853
Σ				–4.25320

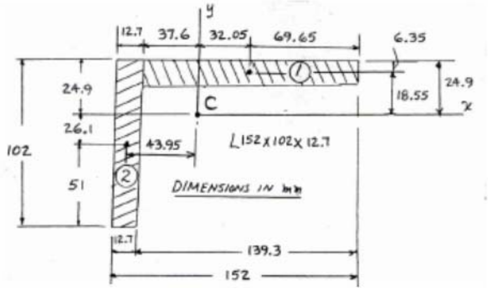
$\bar{I}_{xy} = -4.25\text{ in}^4$ ◀


50.3 mm
y
12.7 mm
24.9 mm
C
x
102 mm
L 152 × 102 × 12.7
12.7 mm
152 mm

PROBLEM 9.78

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION


y
12.7
37.6
32.05
69.65
6.35
24.9
18.55
24.9
x
1
C
26.1
43.95
L 152 x 102 x 12.7
102
51
2
DIMENSIONS IN mm
12.7
139.3
152

We have $\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$

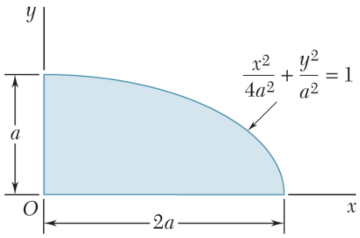
For each rectangle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$

and $\bar{I}_{x'y'} = 0$ (symmetry)

Thus $I_{xy} = \Sigma\bar{x}\bar{y}A$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\bar{y}A$, mm^4
1	$139.3 \times 12.7 = 1769.11$	+32.05	+18.55	$+1051.78 \times 10^3$
2	$12.7 \times (102) = 1295.4$	-43.95	-26.1	$+1485.95 \times 10^3$
Σ				2537.7×10^3

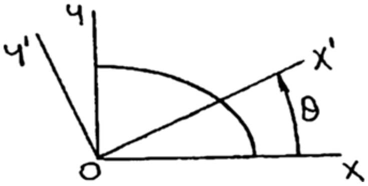
$\bar{I}_{xy} = 2.54 \times 10^6 \text{ mm}^4$ ◀



PROBLEM 9.79

Determine for the quarter ellipse of Problem 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the *x* and *y* axes about *O* (*a*) through 45° counterclockwise, (*b*) through 30° clockwise.

SOLUTION



From Figure 9.12:

$$I_x = \frac{\pi}{16}(2a)(a)^3 = \frac{\pi}{8}a^4$$

$$I_y = \frac{\pi}{16}(2a)^3(a) = \frac{\pi}{2}a^4$$

From Problem 9.67:

$$I_{xy} = \frac{1}{2}a^4$$

First note

$$\frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) = \frac{5}{16}\pi a^4$$

$$\frac{1}{2}(I_x - I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right) = -\frac{3}{16}\pi a^4$$

Now use Equations (9.18), (9.19), and (9.20).

Equation (9.18):

$$I_{x'} = \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y)\cos 2\theta - I_{xy}\sin 2\theta$$

$$= \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4\cos 2\theta - \frac{1}{2}a^4\sin 2\theta$$

Equation (9.19):

$$I_{y'} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\cos 2\theta + I_{xy}\sin 2\theta$$

$$= \frac{5}{16}\pi a^4 + \frac{3}{16}\pi a^4\cos 2\theta + \frac{1}{2}a^4\sin 2\theta$$

Equation (9.20):

$$I_{x'y'} = \frac{1}{2}(I_x - I_y)\sin 2\theta + I_{xy}\cos 2\theta$$

$$= -\frac{3}{16}\pi a^4\sin 2\theta + \frac{1}{2}a^4\cos 2\theta$$

PROBLEM 9.79 (Continued)

(*a*) $\theta = +45°$:

$$I_{x'} = \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos 90° - \frac{1}{2}a^4 \sin 90°$$

or $I_{x'} = 0.482a^4$ ◀

$$I_{y'} = \frac{5}{16}\pi + \frac{3}{16}\pi a^4 \cos 90° + \frac{1}{2}a^4$$

or $I_{y'} = 1.482a^4$ ◀

$$I_{x'y'} = -\frac{3}{16}\pi a^4 \sin 90° + \frac{1}{2}a^4 \cos 90°$$

or $I_{x'y'} = -0.589a^4$ ◀

(*b*) $\theta = -30°$:

$$I_{x'} = \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos(-60°) - \frac{1}{2}a^4 \sin(-60°)$$

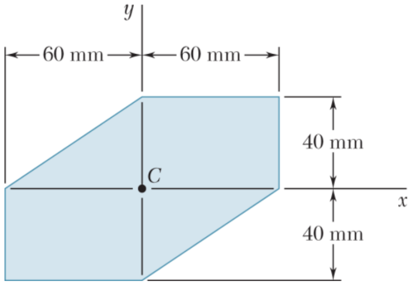
or $I_{x'} = 1.120a^4$ ◀

$$I_{y'} = \frac{5}{16}\pi a^4 + \frac{3}{16}\pi a^4 \cos(-60°) + \frac{1}{2}a^4 \sin(-60°)$$

or $I_{y'} = 0.843a^4$ ◀

$$I_{x'y'} = -\frac{3}{16}\pi a^4 \sin(-60°) + \frac{1}{2}a^4 \cos(-60°)$$

or $I_{x'y'} = 0.760a^4$ ◀

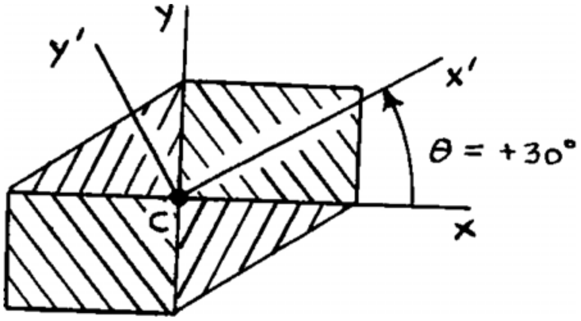


PROBLEM 9.80

Determine the moments of inertia and the product of inertia of the area of Problem 9.72 with respect to new centroidal axes obtained by rotating the *x* and *y* axes 30° counterclockwise.

SOLUTION

From Problem 9.72: $\bar{I}_{xy} = 2.40 \times 10^6 \text{ mm}^4$



Now, with two rectangles and two triangles:

$$\bar{I}_x = 2\left[\frac{1}{3}(60 \text{ mm})(40 \text{ mm})^3\right] + 2\left[\frac{1}{12}(60 \text{ mm})(40 \text{ mm})^3\right]$$

$$\bar{I}_x = 3.20 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 2\left[\frac{1}{3}(40 \text{ mm})(60 \text{ mm})^3\right] + 2\left[\frac{1}{12}(40 \text{ mm})(60 \text{ mm})^3\right] = 7.20 \times 10^6 \text{ mm}^4$$

Now
$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(3.20 + 7.20) \times 10^6 = 5.20 \times 10^6 \text{ mm}^4$$

$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(3.20 - 7.20) \times 10^6 = -2.00 \times 10^6 \text{ mm}^4$$

Using Eqs. (9.18), (9.19), (9.20):

Eq. (9.18):
$$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$

$$= [5.20 + (-2.00)\cos 60° - (2.40)\sin 60°] \times 10^6 \text{ mm}^4$$

or $\bar{I}_{x'} = 2.12 \times 10^6 \text{ mm}^4$ ◀

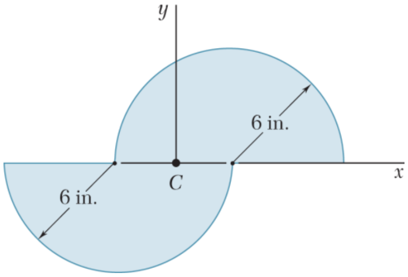
PROBLEM 9.80 (Continued)

Eq. (9.19):
$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$
$$= [5.20 - (-2.00)\cos 60° + (2.40)\sin 60°] \times 10^6 \text{ mm}^4$$

or $\bar{I}_{y'} = 8.28 \times 10^6 \text{ mm}^4$ ◀

Eq. (9.20):
$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\sin 2\theta$$
$$= [(-2.00)\sin 60° + (2.40)\cos 60°] \times 10^6 \text{ mm}^4$$

or $\bar{I}_{x'y'} = -0.532 \times 10^6 \text{ mm}^4$ ◀



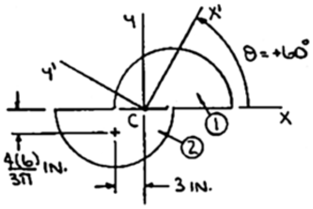
PROBLEM 9.81

Determine the moments of inertia and the product of inertia of the area of Problem 9.73 with respect to new centroidal axes obtained by rotating the x and y axes 60° counterclockwise.

SOLUTION

From Problem 9.73:

$$\bar{I}_{xy} = 864 \text{ in}^4$$



Now $$\bar{I}_x = (I_x)_1 + (I_x)_2$$

where $$(I_x)_1 = (I_x)_2 = \frac{\pi}{8}(6 \text{ in.})^4$$
$$= 162\pi \text{ in}^4$$

Then $$\bar{I}_x = 2(162\pi \text{ in}^4) = 324\pi \text{ in}^4$$

Also $$\bar{I}_y = (I_y)_1 + (I_y)_2$$

where $$(I_y)_1 = (I_y)_2 = \frac{\pi}{8}(6 \text{ in.})^4 + \left[\frac{\pi}{2}(6 \text{ in.})^2\right](3 \text{ in.})^2 = 324\pi \text{ in}^4$$

Then $$\bar{I}_y = 2(324\pi \text{ in}^4) = 648\pi \text{ in}^4$$

Now $$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(324\pi + 648\pi) = 486\pi \text{ in}^4$$
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(324\pi - 648\pi) = -162\pi \text{ in}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18): $$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$
$$= [486\pi + (-162\pi)\cos 120° - 864 \sin 120°] \text{ in}^4$$

or $\bar{I}_{x'} = 1033 \text{ in}^4$ ◀

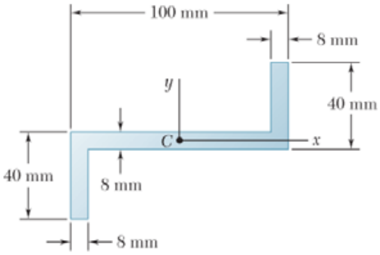
PROBLEM 9.81 (Continued)

Eq. (9.19):
$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$
$$= [486\pi - (-162\pi)\cos 120° + 864 \sin 120°]\text{ in}^4$$

or $\bar{I}_{y'} = 2020\text{ in}^4$ ◀

Eq. (9.20):
$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$
$$= [(-162\pi)\sin 120° + 864\cos 120°]\text{ in}^4$$

or $\bar{I}_{x'y'} = -873\text{ in}^4$ ◀

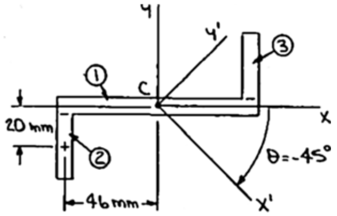


PROBLEM 9.82

Determine the moments of inertia and the product of inertia of the area of Problem 9.75 with respect to new centroidal axes obtained by rotating the *x* and *y* axes 45° clockwise.

SOLUTION

From Problem 9.75:



$$\bar{I}_{xy} = 471{,}040 \text{ mm}^4$$

Now $$\bar{I}_x = (\bar{I}_x)_1 + (I_x)_2 + (I_x)_3$$

where $$(\bar{I}_x)_1 = \frac{1}{12}(100 \text{ mm})(8 \text{ mm})^3$$
$$= 4266.67 \text{ mm}^4$$

$$(I_x)_2 = (I_x)_3 = \frac{1}{12}(8 \text{ mm})(32 \text{ mm})^3 + [(8 \text{ mm})(32 \text{ mm})](20 \text{ mm})^2$$
$$= 124{,}245.33$$

Then $$\bar{I}_x = [4266.67 + 2(124{,}245.33)] \text{ mm}^4 = 252{,}757 \text{ mm}^4$$

Also $$\bar{I}_y = (\bar{I}_y)_1 + (I_y)_2 + (I_y)_3$$

where $$(\bar{I}_y)_1 = \frac{1}{12}(8 \text{ mm})(100 \text{ mm})^3 = 666{,}666.7 \text{ mm}^4$$

$$(\bar{I}_y)_2 = (I_y)_3 = \frac{1}{12}(32 \text{ mm})(8 \text{ mm})^3 + [(8 \text{ mm})(32 \text{ mm})](46 \text{ mm})^2$$
$$= 543{,}061.3 \text{ mm}^4$$

Then $$\bar{I}_y = [666{,}666.7 + 2(543{,}061.3)] \text{ mm}^4 = 1{,}752{,}789 \text{ mm}^4$$

Now $$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(252{,}757 + 1{,}752{,}789) \text{ mm}^4 = 1{,}002{,}773 \text{ mm}^4$$
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(252{,}757 - 1{,}752{,}789) \text{ mm}^4 = -750{,}016 \text{ mm}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18): $$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$
$$= [1{,}002{,}773 + (-750{,}016)\cos(-90°) - 471{,}040\sin(-90°)]$$

or $\bar{I}_{x'} = 1.474 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.82 (Continued)

Eq. (9.19):

$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= [1{,}002{,}773 - (-750{,}016)\cos(-90°) + 471{,}040\sin(-90°)]$$

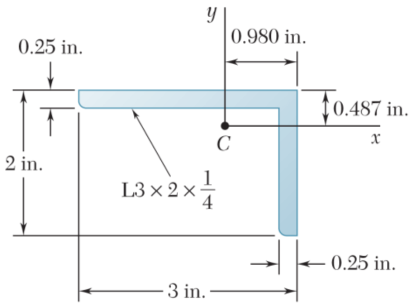
or $\bar{I}_{y'} = 0.532 \times 10^6\,\text{mm}^4$ ◀

Eq. (9.20):

$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$

$$= [(-750{,}016)\sin(-90°) + 471{,}040\cos(-90°)]$$

or $\bar{I}_{x'y'} = 0.750 \times 10^6\,\text{mm}^4$ ◀

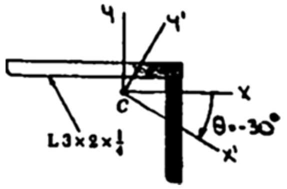


PROBLEM 9.83

Determine the moments of inertia and the product of inertia of the $L3\times 2\times \frac{1}{4}$-in. angle cross section of Problem 9.74 with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise.

SOLUTION

From Figure 9.13:



$$\bar{I}_x = 0.390 \text{ in}^4$$
$$\bar{I}_y = 1.09 \text{ in}^4$$

From Problem 9.74:

$$\bar{I}_{xy} = -0.37983 \text{ in}^4$$

Now

$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(0.390 + 1.09) \text{ in}^4 = 0.740 \text{ in}^4$$
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(0.390 - 1.09) \text{ in}^4 = -0.350 \text{ in}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18):

$$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$
$$= [0.740 + (-0.350)\cos(-60°) - (-0.37983)\sin(-60°)]$$

or $\bar{I}_{x'} = 0.236 \text{ in}^4$ ◀

Eq. (9.19):

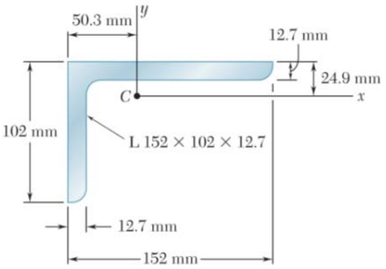
$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$
$$= [0.740 - (-0.350)\cos(-60°) + (-0.37983)\sin(-60°)]$$

or $\bar{I}_{y'} = 1.244 \text{ in}^4$ ◀

Eq. (9.20):

$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$
$$= [(-0.350)\sin(-60°) + (-0.37983)\cos(-60°)]$$

or $I_{x'y'} = 0.1132 \text{ in}^4$ ◀



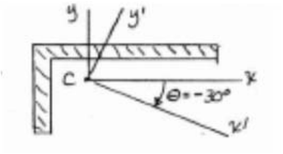
PROBLEM 9.84

Determine the moments of inertia and the product of inertia of the L152 × 102 × 12.7-mm angle cross section of Prob. 9.78 with respect to new centroidal axes obtained by rotating the *x* and *y* axes 30° clockwise.

SOLUTION

From Figure 9.13:

Note the axes are reversed in sketch.



$$\bar{I}_x = 2.59\times 10^6\,\text{mm}^4$$

$$\bar{I}_y = 7.20\times 10^6\,\text{mm}^4$$

From Problem 9.78:

$$\bar{I}_{xy} = 2.54\times 10^6\,\text{mm}^4$$

Now

$$\frac{1}{2}(\bar{I}_x+\bar{I}_y) = \frac{1}{2}(2.59+7.20)\times 10^6\,\text{mm}^4$$

$$= 4.895\times 10^6\,\text{mm}^4$$

$$\frac{1}{2}(\bar{I}_x-\bar{I}_y) = \frac{1}{2}(2.59-7.20)\times 10^6\,\text{mm}^4 = -2.305\times 10^6\,\text{mm}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18):

$$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x+\bar{I}_y)+\frac{1}{2}(\bar{I}_x-\bar{I}_y)\cos 2\theta-\bar{I}_{xy}\sin 2\theta$$

$$= [4.895-2.305\cos(-60°)-2.54\sin(-60°)]\times 10^6\,\text{mm}^4$$

or $\bar{I}_{x'} = 5.94\times 10^6\,\text{mm}^4$ ◀

Eq. (9.19):

$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x+\bar{I}_y)-\frac{1}{2}(\bar{I}_x-\bar{I}_y)\cos 2\theta+\bar{I}_{xy}\sin 2\theta$$

$$= [4.895+2.305\cos(-60°)+2.54\sin(-60°)]\times 10^6\,\text{mm}^4$$

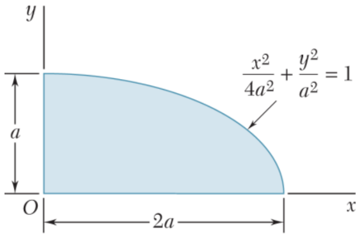
or $\bar{I}_{y'} = 3.85\times 10^6\,\text{mm}^4$ ◀

Eq. (9.20):

$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x-\bar{I}_y)\sin 2\theta+\bar{I}_{xy}\cos 2\theta$$

$$= [(-2.305\sin(-60°)+2.54\cos(-60°)]\times 10^6\,\text{mm}^4$$

or $\bar{I}_{x'y'} = 3.27\times 10^6\,\text{mm}^4$ ◀



PROBLEM 9.85

For the quarter ellipse of Problem 9.67, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

SOLUTION

From Problem 9.79: $I_x = \frac{\pi}{8}a^4 \qquad I_y = \frac{\pi}{2}a^4$

Problem 9.67: $I_{xy} = \frac{1}{2}a^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2\left(\frac{1}{2}a^4\right)}{\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4}$$

$$= \frac{8}{3\pi} = 0.84883$$

Then $2\theta_m = 40.326° \quad \text{and} \quad 220.326°$

or $\theta_m = 20.2° \text{ and } 110.2°$ ◀

Also Eq. (9.27):

$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$= \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right)$$

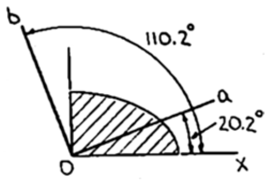
$$\pm \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right)\right]^2 + \left(\frac{1}{2}a^4\right)^2}$$

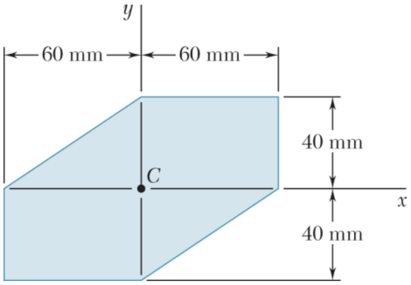
$$= (0.981{,}748 \pm 0.772{,}644)a^4$$

or $I_{\max} = 1.754a^4$ ◀

and $I_{\min} = 0.209a^4$ ◀

By inspection, the *a* axis corresponds to $I_{\min}$ and the *b* axis corresponds to $I_{\max}$.





PROBLEM 9.86

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.72.

SOLUTION

From Problem 9.80: $\bar{I}_x = 3.20 \times 10^6 \text{ mm}^4$

$\bar{I}_y = 7.20 \times 10^6 \text{ mm}^4$

From Problem 9.72: $\bar{I}_{xy} = 2.40 \times 10^6 \text{ mm}^4$

Now Eq. (9.25):
$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(2.40 \times 10^6)}{(3.20 - 7.20) \times 10^6} = 1.200$$

Then $2\theta_m = 50.194°$ and $230.194°$

or $\theta_m = 25.1°$ and $115.1°$ ◀

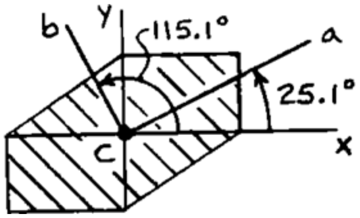
Also Eq. (9.27):
$$\bar{I}_{\max,\min} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + I_{xy}^2}$$

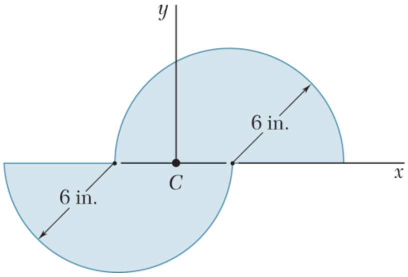
Then
$$\bar{I}_{\max,\min} = \left[\frac{3.20 + 7.20}{2} \pm \sqrt{\left(\frac{3.20 - 7.20}{2}\right)^2 + (2.40)^2}\right] \times 10^6 \text{ mm}^4$$
$$= (5.20 \pm 3.1241) \times 10^6 \text{ mm}^4$$

or $\bar{I}_{\max} = 8.32 \times 10^6 \text{ mm}^4$ ◀

and $\bar{I}_{\min} = 2.08 \times 10^6 \text{ mm}^4$ ◀

By inspection, the *a* axis corresponds to $\bar{I}_{\min}$ and the *b* axis corresponds to $\bar{I}_{\max}$.





PROBLEM 9.87

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.73.

SOLUTION

From Problem 9.81: $\bar{I}_x = 324\pi \text{ in}^4 \qquad \bar{I}_y = 648\pi \text{ in}^4$

Problem 9.73: $\bar{I}_{xy} = 864 \text{ in}^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(864)}{324\pi - 648\pi}$$
$$= 1.69765$$

Then $2\theta_m = 59.500° \quad \text{and} \quad 239.500°$

or $\theta_m = 29.7° \quad \text{and} \quad 119.7°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\max,\min} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

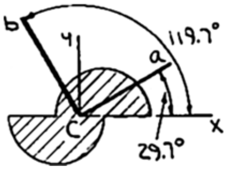
Then

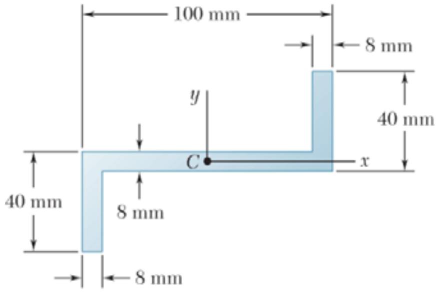
$$\bar{I}_{\max,\min} = \frac{324\pi + 648\pi}{2} \pm \sqrt{\left(\frac{324\pi - 648\pi}{2}\right)^2 + 864^2}$$
$$= (1526.81 \pm 1002.75) \text{ in}^4$$

or $\bar{I}_{\max} = 2530 \text{ in}^4$ ◀

and $\bar{I}_{\min} = 524 \text{ in}^4$ ◀

By inspection, the *a* axis corresponds to $\bar{I}_{\min}$ and the *b* axis corresponds to $\bar{I}_{\max}$.





PROBLEM 9.88

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.75.

SOLUTION

From Problem 9.82:

$$\bar{I}_x = 252,757 \text{ mm}^4$$

$$\bar{I}_y = 1,752,789 \text{ mm}^4$$

Problem 9.75:

$$\bar{I}_{xy} = 471,040 \text{ mm}^4$$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(471,040)}{252,757 - 1,752,789} = 0.62804$$

Then

$$2\theta_m = 32.130° \quad \text{and} \quad 212.130°$$

or $\theta_m = 16.07°$ and $106.1°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\text{max, min}} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

Then

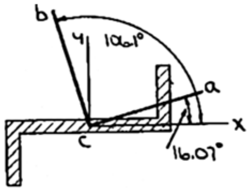
$$\bar{I}_{\text{max, min}} = \frac{252,757 + 1,752,789}{2} \pm \sqrt{\left(\frac{252,757 - 1,752,789}{2}\right)^2 + 471040^2}$$

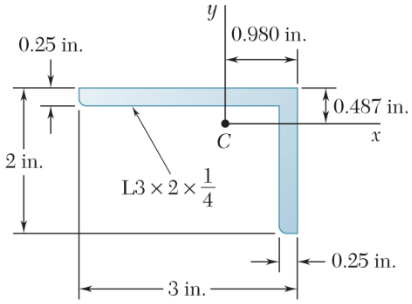
$$= (1,002,773 \pm 885,665) \text{ mm}^4$$

or $\bar{I}_{\max} = 1.888 \times 10^6 \text{ mm}^4$ ◀

and $\bar{I}_{\min} = 0.1171 \times 10^6 \text{ mm}^4$ ◀

By inspection, the *a* axis corresponds to $\bar{I}_{\min}$ and the *b* axis corresponds to $\bar{I}_{\max}$.





PROBLEM 9.89

For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

The $L3 \times 2 \times \frac{1}{4}$-in. angle cross section of Problem 9.74.

SOLUTION

From Problem 9.83: $\bar{I}_x = 0.390 \text{ in}^4 \qquad \bar{I}_y = 1.09 \text{ in}^4$

Problem 9.74: $\bar{I}_{xy} = -0.37983 \text{ in}^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(-0.37983)}{0.390 - 1.09}$$
$$= -1.08523$$

Then $2\theta_m = -47.341° \text{ and } 132.659°$

or $\theta_m = -23.7° \text{ and } 66.3°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\text{max, min}} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

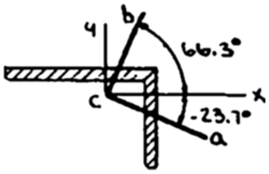
Then

$$\bar{I}_{\text{max, min}} = \frac{0.390 + 1.09}{2} \pm \sqrt{\left(\frac{0.390 - 1.09}{2}\right)^2 + (-0.37983)^2}$$
$$= (0.740 \pm 0.51650)^2 \text{ in}^4$$

or $\bar{I}_{\max} = 1.257 \text{ in}^4$ ◀

and $\bar{I}_{\min} = 0.224 \text{ in}^4$ ◀

By inspection, the *a* axis corresponds to $\bar{I}_{\min}$ and the *b* axis corresponds to $\bar{I}_{\max}$.



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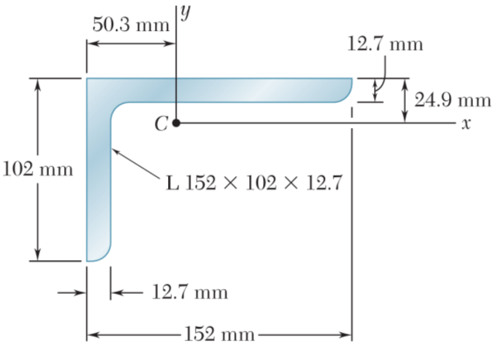


PROBLEM 9.90

For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

The $L152\times102\times12.7$-mm angle cross section of Problem 9.78.

SOLUTION

From Figure 9.13B: $\bar{I}_x = 2.59\times10^6\,\text{mm}^4$

$\bar{I}_y = 7.20\times10^6\,\text{mm}^4$

Problem 9.78: $\bar{I}_{xy} = 2.54\times10^6\,\text{mm}^4$

Now Eq. (9.25):
$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x-\bar{I}_y} = -\frac{2(2.54\times10^6)}{(2.59-7.20)\times10^6}$$
$$=1.10195$$

Then $2\theta_m = 47.777°$ and $227.777°$

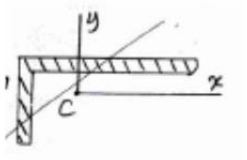
or $\theta_m = 23.9°$ and $113.9°$ ◀

Also Eq. (9.27):
$$\bar{I}_{\text{max, min}} = \frac{\bar{I}_x+\bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x-\bar{I}_y}{2}\right)^2+\bar{I}_{xy}^2}$$

Then
$$\bar{I}_{\text{max, min}} = \left[\frac{2.59+7.20}{2} \pm \sqrt{\left(\frac{2.59-7.20}{2}\right)^2+2.54^2}\right]\times10^6\,\text{mm}^4$$
$$=(4.895\pm3.4300)\times10^6\,\text{mm}^4$$

or $\bar{I}_{\text{max}} = 8.33\times10^6\,\text{mm}^4$ ◀

and $\bar{I}_{\text{min}} = 1.465\times10^6\,\text{mm}^4$ ◀



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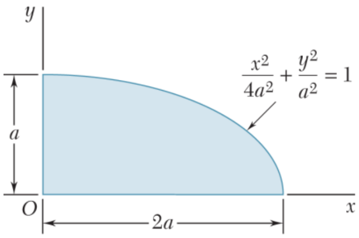
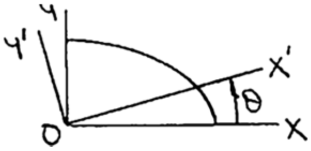

y
$\frac{x^2}{4a^2}+\frac{y^2}{a^2}=1$
a
O
2a
x

PROBLEM 9.91

Using Mohr’s circle, determine for the quarter ellipse of Problem 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the *x* and *y* axes about *O* (*a*) through 45° counterclockwise, (*b*) through 30° clockwise.

SOLUTION

From Problem 9.79:
$$I_x = \frac{\pi}{8}a^4$$



$$I_y = \frac{\pi}{2}a^4$$

Problem 9.67:
$$I_{xy} = \frac{1}{2}a^4$$

The Mohr’s circle is defined by the diameter *XY*, where

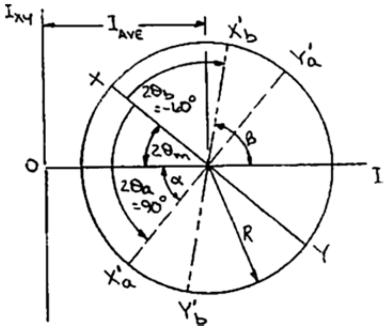
$$X\left(\frac{\pi}{8}a^4, \frac{1}{2}a^4\right) \quad \text{and} \quad Y\left(\frac{\pi}{2}a^4, -\frac{1}{2}a^4\right)$$

Now
$$I_{\text{ave}} = \frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) = \frac{5}{16}\pi a^4 = 0.98175a^4$$

and
$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} = \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right)\right]^2 + \left(\frac{1}{2}a^4\right)^2}$$

$$= 0.77264a^4$$

The Mohr’s circle is then drawn as shown.


I_{xy}
I_{AVE}
X'_b
Y'_a
X
$2\theta_b = -60°$
$2\theta_m$
β
O
I
$2\theta_a = 90°$
α
R
Y
X'_a
Y'_b

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$

$$= -\frac{2\left(\frac{1}{2}a^4\right)}{\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4}$$

$$= 0.84883$$

or
$$2\theta_m = 40.326°$$

Then

$$\alpha = 90° - 40.326° = 49.674°$$

$$\beta = 180° - (40.326° + 60°) = 79.674°$$

(*a*) $\theta = +45°$:

$$I_{x'} = I_{\text{ave}} - R\cos\alpha = 0.98175a^4 - 0.77264a^4 \cos 49.674°$$

or $I_{x'} = 0.482a^4$ ◀

$$I_{y'} = I_{\text{ave}} + R\cos\alpha = 0.98175a^4 + 0.77264a^4 \cos 49.674°$$

or $I_{y'} = 1.482a^4$ ◀

$$I_{x'y'} = -R\sin\alpha = -0.77264a^4 \sin 49.674°$$

or $I_{x'y'} = -0.589a^4$ ◀

(*b*) $\theta = -30°$:

$$I_{x'} = I_{\text{ave}} + R\cos\beta = 0.98175a^4 + 0.77264a^4 \cos 79.674°$$

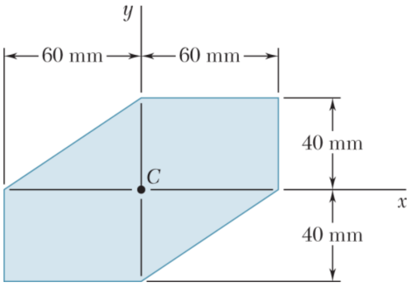
or $I_{x'} = 1.120a^4$ ◀

$$I_{y'} = I_{\text{ave}} - R\cos\beta = 0.98175a^4 - 0.77264a^4 \cos 79.674°$$

or $I_{y'} = 0.843a^4$ ◀

$$I_{x'y'} = R\sin\beta = 0.77264a^4 \sin 79.674°$$

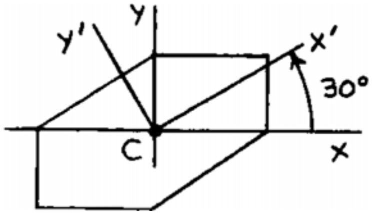
or $I_{x'y'} = 0.760a^4$ ◀



PROBLEM 9.92

Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Problem 9.72 with respect to new centroidal axes obtained by rotating the *x* and *y* axes 30° counterclockwise.

SOLUTION



From Problem 9.80: $\bar{I}_x = 3.20 \times 10^6 \text{ mm}^4$

$\bar{I}_y = 7.20 \times 10^6 \text{ mm}^4$

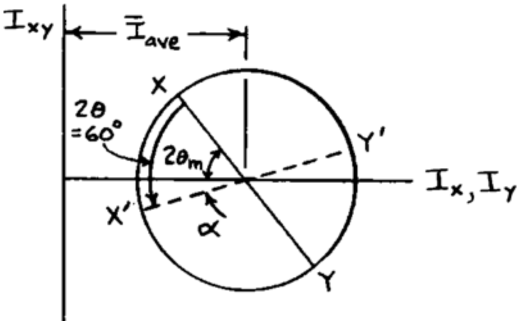
From Problem 9.72: $\bar{I}_{xy} = 2.40 \times 10^6 \text{ mm}^4$

The Mohr's circle is defined by the diameter *XY*, where $X(3.20 \times 10^6,\ 2.40 \times 10^6)$ and $Y(7.20 \times 10^6,\ -2.40 \times 10^6)$.

Now
$$I_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(3.20 + 7.20) \times 10^6 = 5.20 \times 10^6 \text{ mm}^4$$

and
$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2} = \left\{\sqrt{\left[\frac{1}{2}(3.20 - 7.20)\right]^2 + (2.40)^2}\right\} \times 10^6 \text{ mm}^4$$
$$= 3.1241 \times 10^6 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(2.40 \times 10^6)}{(3.20 - 7.20) \times 10^6}$$
$$= 1.200$$

or
$$2\theta_m = 50.1944°$$

Then
$$\alpha = 60° - 50.1944° = 9.8056°$$

PROBLEM 9.92 (Continued)

Then $$\bar{I}_{x'} = \bar{I}_{\text{ave}} - R\cos\alpha = (5.20 - 3.1241\cos 9.8056°)\times 10^6$$

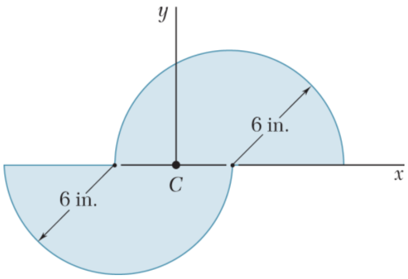
or $\bar{I}_{x'} = 2.12\times 10^6\,\text{mm}^4$ ◀

$$\bar{I}_{y'} = \bar{I}_{\text{ave}} + R\cos\alpha = (5.20 + 3.1241\cos 9.8056°)\times 10^6$$

or $\bar{I}_{y'} = 8.28\times 10^6\,\text{mm}^4$ ◀

$$\bar{I}_{x'y'} = -R\sin\alpha = -(3.1241\times 10^6)\sin 9.8056°$$

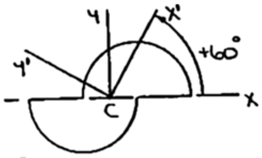
or $\bar{I}_{x'y'} = -0.532\times 10^6\,\text{mm}^4$ ◀



PROBLEM 9.93

Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Problem 9.73 with respect to new centroidal axes obtained by rotating the *x* and *y* axes 60° counterclockwise.

SOLUTION



From Problem 9.81: $\bar{I}_x = 324\pi \text{ in}^4$

$\bar{I}_y = 648\pi \text{ in}^4$

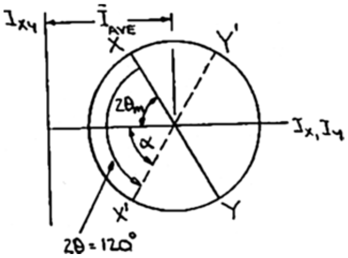
Problem 9.73: $\bar{I}_{xy} = 864 \text{ in}^4$

The Mohr's circle is defined by the diameter *XY*, where $X(324\pi,\ 864)$ and $Y(648\pi,\ -864)$.

Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(324\pi + 648\pi) = 1526.81 \text{ in}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2} = \sqrt{\left[\frac{1}{2}(324\pi + 648\pi)\right]^2 + 864^2}$$
$$= 1002.75 \text{ in}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(864)}{324\pi - 648\pi}$$
$$= 1.69765$$

or $$2\theta_m = 59.500°$$

Then $$\alpha = 120° - 59.500°$$
$$= 60.500°$$

PROBLEM 9.93 (Continued)

Then $$\bar{I}_{x'} = \bar{I}_{\text{ave}} - R\cos\alpha = 1526.81 - 1002.75\cos 60.500°$$

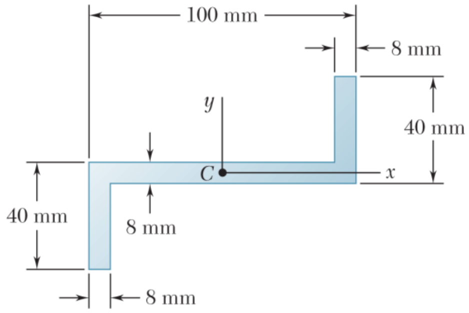
or $\bar{I}_{x'} = 1033 \text{ in}^4$ ◀

$$\bar{I}_{y} = \bar{I}_{\text{ave}} + R\cos\alpha = 1526.81 + 1002.75\cos 60.500°$$

or $\bar{I}_{y'} = 2020 \text{ in}^4$ ◀

$$\bar{I}_{x'y'} = -R\sin\alpha = -1002.75\sin 60.500°$$

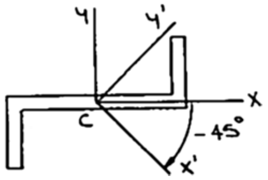
or $\bar{I}_{x'y'} = -873 \text{ in}^4$ ◀



PROBLEM 9.94

Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Problem 9.75 with respect to new centroidal axes obtained by rotating the *x* and *y* axes 45° clockwise.

SOLUTION



From Problem 9.82: $\bar{I}_x = 252{,}757 \text{ mm}^4$

$\bar{I}_y = 1{,}752{,}789 \text{ mm}^4$

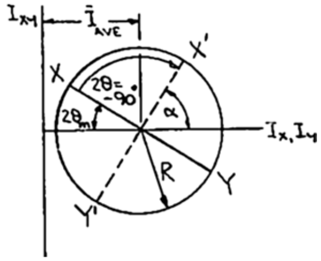
Problem 9.75: $\bar{I}_{xy} = 471{,}040 \text{ mm}^4$

The Mohr's circle is defined by the diameter *XY*, where *X* (252,757; 471,040) and *Y* (1,752,789; −471,040).

Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(252{,}757 + 1{,}752{,}789)$$
$$= 1{,}002{,}773 \text{ mm}^4$$

and
$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$
$$= \sqrt{\left[\frac{1}{2}(252{,}757 - 1{,}752{,}789)^2\right] + 471{,}040^2}$$
$$= 885{,}665 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(471{,}040)}{252{,}757 - 1{,}752{,}789}$$
$$= 0.62804$$

or $$2\theta_m = 32.130°$$

Then
$$\alpha = 180° - (32.130 + 90°)$$
$$= 57.870°$$

PROBLEM 9.94 (Continued)

Then $\bar{I}_{x'} = \bar{I}_{\text{ave}} + R\cos\alpha = 1{,}002{,}773 + 885{,}665\cos 57.870°$

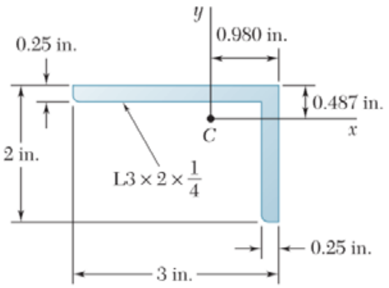
or $\bar{I}_{x'} = 1.474\times 10^6\,\text{mm}^4$ ◀

$\bar{I}_{y'} = \bar{I}_{\text{ave}} - R\cos\alpha = 1{,}002{,}773 - 885{,}665\cos 57.870°$

or $\bar{I}_{y'} = 0.532\times 10^6\,\text{mm}^4$ ◀

$\bar{I}_{x'y'} = R\sin\alpha = 885{,}665\sin 57.870°$

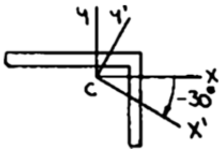
or $\bar{I}_{x'y'} = 0.750\times 10^6\,\text{mm}^4$ ◀


y
0.980 in.
0.25 in.
0.487 in.
C
x
2 in.
L3 × 2 × $\frac{1}{4}$
0.25 in.
3 in.

PROBLEM 9.95

Using Mohr's circle, determine the moments of inertia and the product of inertia of the L3×2×$\frac{1}{4}$-in. angle cross section of Problem 9.74 with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise.

SOLUTION



From Problem 9.83:

$$\bar{I}_x = 0.390 \text{ in}^4$$

$$\bar{I}_y = 1.09 \text{ in}^4$$

Problem 9.74:

$$\bar{I}_{xy} = -0.37983 \text{ in}^4$$

The Mohr's circle is defined by the diameter XY, where $X(0.390, -0.37983)$ and $Y(1.09,\ 0.37983)$.

Now

$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y)$$

$$= \frac{1}{2}(0.390 + 1.09)$$

$$= 0.740 \text{ in}^4$$

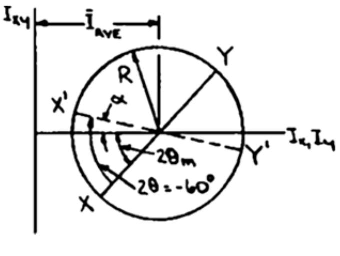
and

$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$

$$= \sqrt{\left[\frac{1}{2}(0.390 - 1.09)\right]^2 + (-0.37983)^2}$$

$$= 0.51650 \text{ in}^4$$

The Mohr's circle is then drawn as shown.


I_{xy}
$\bar{I}_{AVE}$
R
Y
X'
α
$2\theta_m$
Y'
I_x, I_y
$2\theta = -60°$
X

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$

$$= -\frac{2(-0.37983)}{0.390 - 1.09}$$

$$= -1.08523$$

or

$$2\theta_m = -47.341°$$

Then

$$\alpha = 60° - 47.341° = 12.659°$$

PROBLEM 9.95 (Continued)

Then $\bar{I}_{x'} = \bar{I}_{\text{ave}} - R\cos\alpha = 0.740 - 0.51650\cos 12.659°$

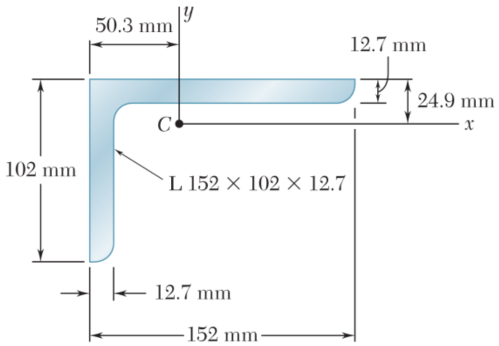
or $\bar{I}_{x'} = 0.236 \text{ in}^4$ ◀

$\bar{I}_{y'} = \bar{I}_{\text{ave}} + R\cos\alpha = 0.740 + 0.51650\cos 12.659°$

or $\bar{I}_{y} = 1.244 \text{ in}^4$ ◀

$\bar{I}_{x'y'} = R\sin\alpha = 0.51650\sin 12.659°$

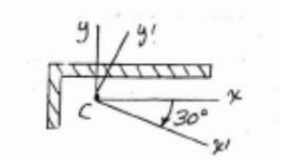
or $\bar{I}_{x'y'} = 0.1132 \text{ in}^4$ ◀


50.3 mm
y
12.7 mm
24.9 mm
x
C
102 mm
L 152 × 102 × 12.7
12.7 mm
152 mm

PROBLEM 9.96

Using Mohr's circle, determine the moments of inertia and the product of inertia of the $L152\times102\times12.7$-mm angle cross section of Problem 9.78 with respect to new centroidal axes obtained by rotating the *x* and *y* axes 45° counterclockwise.

SOLUTION



From Problem 9.84: $\bar{I}_x = 2.59\times10^6\,\text{mm}^4$

$\bar{I}_y = 7.20\times10^6\,\text{mm}^4$

Problem 9.78: $\bar{I}_{xy} = 2.54\times10^6\,\text{mm}^4$

The Mohr's circle is defined by the diameter *XY*, where $X(2.59\times10^6,\ 2.54\times10^6)$, $Y(7.20\times10^6,\ -2.54\times10^6)$.

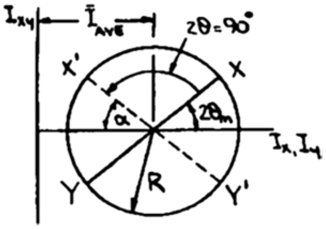
Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x+\bar{I}_y) = \frac{1}{2}(2.59+7.20)\times10^6 = 4.895\times10^6\,\text{mm}^4$$

and $$R = \sqrt{\left(\bar{I}_{avg}-\bar{I}_x\right)^2+\bar{I}_{xy}^2}$$

$$= \left\{\sqrt{\left(4.895-2.54\right)^2+2.54^2}\right\}\times10^6\,\text{mm}^4$$

$$= 3.43\times10^6\,\text{mm}^4$$

The Mohr's circle is then drawn as shown.


I_{xy}
$\bar{I}_{AVE}$
2θ=90°
X'
X
α
$2\theta_m$
I_x, I_y
Y
R
Y'

$$\tan\beta = \frac{DX}{DC}$$

$$= \frac{2.54\times10^6}{2.305\times10^6}$$

$$= 1.10195$$

or $\beta = 42.777°$

PROBLEM 9.96 (Continued)

Then $\gamma = 180° - 60° - 47.777°$

$= 72.223°$

and $\bar{I}_{x'} = \bar{I}_{\text{ave}} + R\cos\gamma = (4.895 + 3.43\cos 72.223°)\times 10^6$

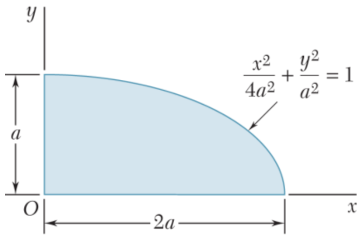
or $\bar{I}_{x'} = 5.94\times 10^6\,\text{mm}^4$ ◀

$\bar{I}_{y'} = \bar{I}_{\text{ave}} - R\cos\gamma = (4.895 - 3.43\cos 72.223°)\times 10^6$

or $\bar{I}_{y'} = 3.85\times 10^6\,\text{mm}^4$ ◀

$\bar{I}_{x'y'} = R\sin\gamma = (3.43\times 10^6)\sin 72.223°$

or $\bar{I}_{x'y'} = 3.27\times 10^6\,\text{mm}^4$ ◀



PROBLEM 9.97

For the quarter ellipse of Problem 9.67, use Mohr's circle to determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

SOLUTION

From Problem 9.79: $$I_x = \frac{\pi}{8}a^4 \qquad I_y = \frac{\pi}{2}a^4$$

Problem 9.67: $$I_{xy} = \frac{1}{2}a^4$$

The Mohr's circle is defined by the diameter *XY*, where

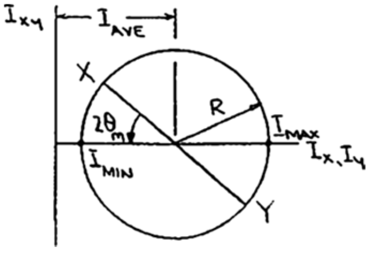
$$X\left(\frac{\pi}{8}a^4,\ \frac{1}{2}a^4\right) \quad \text{and} \quad Y\left(\frac{\pi}{2}a^4,\ -\frac{1}{2}a^4\right)$$

Now $$I_{\text{ave}} = \frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) = 0.98175a^4$$

and

$$R = \sqrt{\left[\frac{1}{2}(I_x - I_y)\right]^2 + I_{xy}^2}$$
$$= \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right)\right]^2 + \left(\frac{1}{2}a^4\right)^2}$$
$$= 0.77264a^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$
$$= -\frac{2\left(\frac{1}{2}a^4\right)}{\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4}$$
$$= 0.84883$$

or $$2\theta_m = 40.326°$$

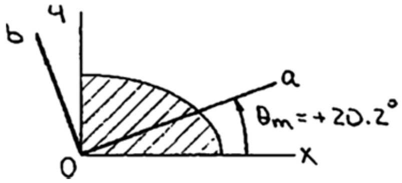
and $$\theta_m = 20.2°$$

PROBLEM 9.97 (Continued)

The principal axes are obtained by rotating the *xy* axes through

20.2° counterclockwise ◀

about *O*.

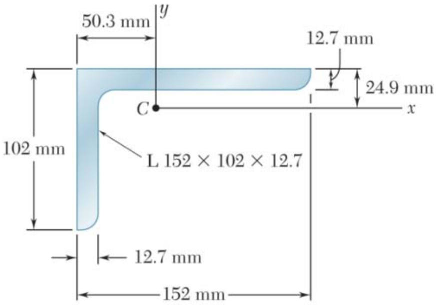


Now $$I_{\max,\min} = I_{\text{ave}} \pm R = 0.98175a^4 \pm 0.77264a^4$$

or $I_{\max} = 1.754a^4$ ◀

and $I_{\min} = 0.209a^4$ ◀

From the Mohr's circle it is seen that the *a* axis corresponds to $I_{\min}$ and the *b* axis corresponds to $I_{\max}$.


50.3 mm
y
12.7 mm
24.9 mm
C
x
102 mm
L 152 × 102 × 12.7
12.7 mm
152 mm

PROBLEM 9.98

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.78.

SOLUTION

From Fig. 9.13: $\bar{I}_x = 2.59 \times 10^6 \text{ mm}^4$

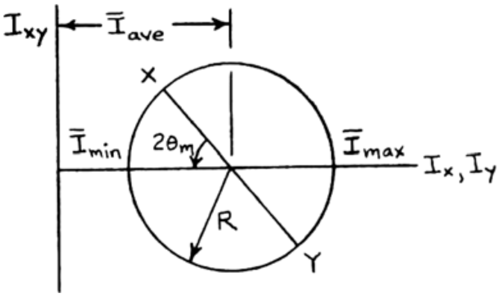
$\bar{I}_y = 7.20 \times 10^6 \text{ mm}^4$

From Problem 9.78: $\bar{I}_{xy} = 2.54 \times 10^6 \text{ mm}^4$

The Mohr's circle is defined by the diameter XY, where $X(2.59 \times 10^6, 2.54 \times 10^6)$ and $Y(7.20 \times 10^6, -2.54 \times 10^6)$.

Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y)$$
$$= \frac{1}{2}(2.59 + 7.20) \times 10^6 \text{ mm}^4$$
$$= 4.895 \times 10^6 \text{ mm}^4$$

and
$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$
$$= \left\{\sqrt{\left[\frac{1}{2}(2.59 - 7.20)\right]^2 + (2.54)^2}\right\} \times 10^6 \text{ mm}^4$$
$$= 3.430 \times 10^6 \text{ mm}^4$$


I_{xy}
$\bar{I}_{ave}$
X
$\bar{I}_{min}$
$2\theta_m$
$\bar{I}_{max}$
I_x, I_y
R
Y

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(2.54 \times 10^6)}{(2.59 - 7.20) \times 10^6} = 1.10195$$

or
$$2\theta_m = 47.777°$$

and
$$\theta_m = 23.9°$$

PROBLEM 9.98 (Continued)

The principal axes are obtained by rotating the *xy* axes through

23.9° counterclockwise ◀

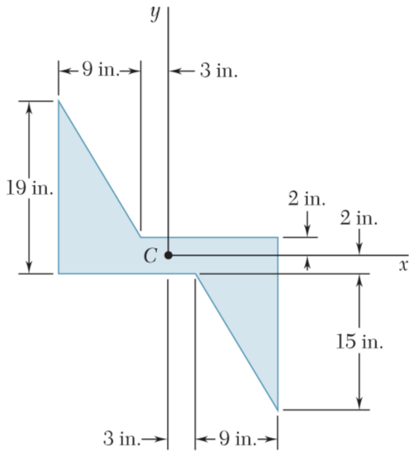
about *C*.

Now $$I_{\max,\min} = \bar{I}_{ave} \pm R = (4.895 \pm 3.43) \times 10^6$$

or $$I_{\max} = 8.33 \times 10^6 \text{ mm}^4$$ ◀

and $$I_{\min} = 1.465 \times 10^6 \text{ mm}^4$$ ◀

From the Mohr's circle it is seen that the x' axis corresponds to $\bar{I}_{\min}$ and the y' axis corresponds to $\bar{I}_{\max}$.

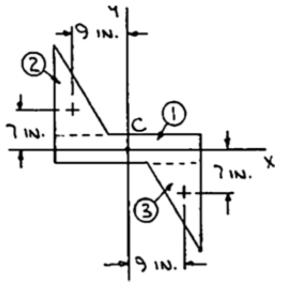


PROBLEM 9.99

Using Mohr’s circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.76.

SOLUTION



From Problem 9.76:

$$\bar{I}_{xy} = -9011.25 \text{ in}^4$$

Now

$$\bar{I}_x = (\bar{I}_x)_1 + (I_x)_2 + (I_x)_3$$

where

$$(\bar{I}_x)_1 = \frac{1}{12}(24 \text{ in.})(4 \text{ in.})^3 = 128 \text{ in}^4$$

$$(I_x)_2 = (I_x)_3 = \frac{1}{36}(9 \text{ in.})(15 \text{ in.})^3 + \left[\frac{1}{2}(9 \text{ in.})(15 \text{ in.})\right](7 \text{ in.})^2$$

$$= 4151.25 \text{ in}^4$$

Then

$$\bar{I}_x = [128 + 2(4151.25)] \text{ in}^4$$

$$= 8430.5 \text{ in}^4$$

Also

$$\bar{I}_y = (\bar{I}_y)_1 + (I_y)_2 + (I_y)_3$$

where

$$(\bar{I}_y)_1 = \frac{1}{12}(4 \text{ in.})(24 \text{ in.})^3 = 4608 \text{ in}^4$$

$$(I_y)_2 = (I_y)_3 = \frac{1}{36}(15 \text{ in.})(9 \text{ in.})^3 + \left[\frac{1}{2}(9 \text{ in.})(15 \text{ in.})\right](9 \text{ in.})^2$$

$$= 5771.25 \text{ in}^4$$

Then

$$\bar{I}_y = [4608 + 2(5771.25)] \text{ in}^4 = 16150.5 \text{ in}^4$$

PROBLEM 9.99 (Continued)

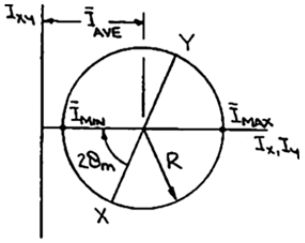
The Mohr's circle is defined by the diameter *XY*, where $X(8430.5,\ -9011.25)$ and $Y(16150.5,\ 9011.25)$.

Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(8430.5 + 16150.5) = 12290.5 \text{ in}^4$$

and
$$\begin{aligned} R &= \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2} \\ &= \sqrt{\left[\frac{1}{2}(8430.5 - 16150.5)\right]^2 + (-9011.25)^2} \\ &= 9803.17 \text{ in}^4 \end{aligned}$$

The Mohr's circle is then drawn as shown.

$$\begin{aligned} \tan 2\theta_m &= -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} \\ &= -\frac{2(-9011.25)}{8430.5 - 16150.5} \\ &= -2.33452 \end{aligned}$$



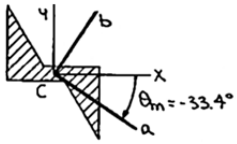
or
$$2\theta_m = -66.812°$$

and
$$\theta_m = -33.4°$$

The principal axes are obtained by rotating the *xy* axes through

33.4° clockwise ◀

about *C*.

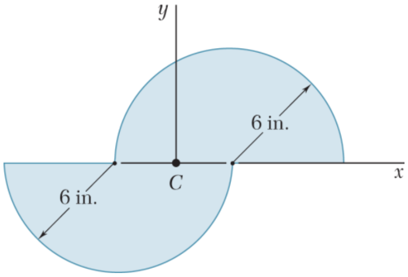


Now
$$\bar{I}_{\text{max, min}} = \bar{I}_{\text{ave}} \pm R = 12290.5 \pm 9803.17$$

or $\bar{I}_{\text{max}} = 22.1 \times 10^3 \text{in}^4$ ◀

and $\bar{I}_{\text{min}} = 2490 \text{ in}^4$ ◀

From the Mohr's circle it is seen that the *a* axis corresponds to $\bar{I}_{\text{min}}$ and the *b* axis corresponds to $\bar{I}_{\text{max}}$.



PROBLEM 9.100

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.73

SOLUTION

From Problem 9.81: $\bar{I}_x = 324\pi \text{ in}^4 \qquad \bar{I}_y = 648\pi \text{ in}^4$

Problem 9.73: $\bar{I}_{xy} = 864 \text{ in}^4$

The Mohr's circle is defined by the diameter *XY*, where $X(324\pi,\ 864)$ and $Y(648\pi,\ -864)$.

Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(324\pi + 648\pi) = 1526.81 \text{ in}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)^2\right] + I_{xy}^2}$$

$$= \sqrt{\left[\frac{1}{2}(324\pi - 648\pi)\right]^2 + 864^2}$$

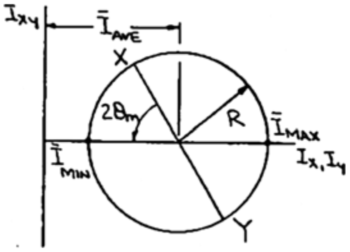
$$= 1002.75 \text{ in}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(864)}{324\pi - 648\pi}$$

$$= 1.69765$$



or $2\theta_m = 59.4998°$

and $\theta_m = 29.7°$

The principal axes are obtained by rotating the *xy* axes through

29.7° counterclockwise ◀

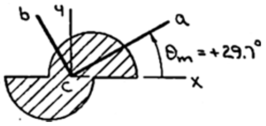
about *C*.

PROBLEM 9.100 (Continued)

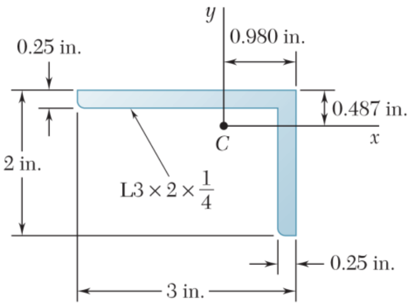
Now $$\bar{I}_{\max,\min} = \bar{I}_{\text{ave}} \pm R = 1526.81 \pm 1002.75$$

or $I_{\max} = 2530 \text{ in}^4$ ◀

and $\bar{I}_{\min} = 524 \text{ in}^4$ ◀



From the Mohr’s circle it is seen that the *a* axis corresponds to $I_{\min}$ and the *b* axis corresponds to $\bar{I}_{\max}$.



PROBLEM 9.101

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.74.

SOLUTION

From Problem 9.83: $\bar{I}_x = 0.390 \text{ in}^4$

$\bar{I}_y = 1.09 \text{ in}^4$

Problem 9.74: $\bar{I}_{xy} = -0.37983 \text{ in}^4$

The Mohr's circle is defined by the diameter XY, where $X(0.390, -0.37983)$ and $Y(1.09, 0.37983)$.

Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(0.390 + 1.09) = 0.740 \text{ in}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$

$$= \sqrt{\left[\frac{1}{2}(0.390 - 1.09)\right]^2 + (-0.37983)^2}$$

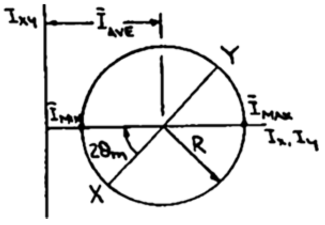
$$= 0.51650 \text{ in}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(-0.37983)}{0.390 - 1.09}$$

$$= -1.08523$$



Then $2\theta_m = -47.341°$

and $\theta_m = -23.7°$

The principal axes are obtained by rotating the xy axes through

23.7° clockwise ◀

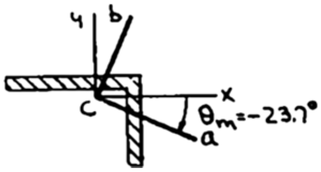
about C.

PROBLEM 9.101 (Continued)

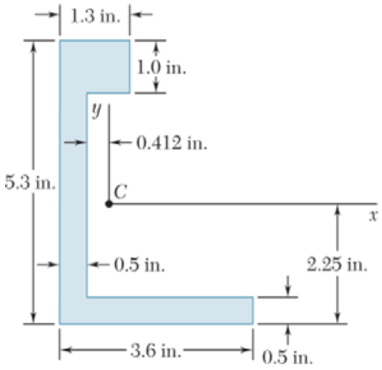
Now $$\bar{I}_{\text{max,min}} = \bar{I}_{\text{ave}} \pm R = 0.740 \pm 0.51650$$

or $\bar{I}_{\text{max}} = 1.257 \text{ in}^4$ ◀

and $\bar{I}_{\text{min}} = 0.224 \text{ in}^4$ ◀



From the Mohr's circle it is seen that the *a* axis corresponds to I_{min} and the *b* axis corresponds to I_{max}.



PROBLEM 9.102

Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

Area of Problem 9.77

(The moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area of Problem 9.102 were determined in Problem 9.44).

SOLUTION

From Problem 9.44:
$$\bar{I}_x = 18.1282 \text{ in}^4$$
$$\bar{I}_y = 4.5080 \text{ in}^4$$

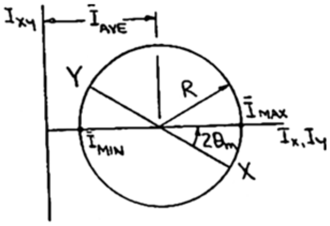
Problem 9.77:
$$\bar{I}_{xy} = -4.25320 \text{ in}^4$$

The Mohr's circle is defined by the diameter XY, where $X(18.1282, -4.25320)$ and $Y(4.5080, 4.25320)$.

Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(18.1282 + 4.5080) = 11.3181 \text{ in}^4$$

and
$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$
$$= \sqrt{\left[\frac{1}{2}(18.1282 - 4.5080)\right]^2 + (-4.25320)^2}$$
$$= 8.02915 \text{ in}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$
$$= -\frac{2(-4.25320)}{18.1282 - 4.5080}$$
$$= 0.62454$$

or
$$2\theta_m = 31.986°$$

and
$$\theta_m = 15.99°$$

The principal axes are obtained by rotating the xy axes through

15.99° counterclockwise ◀

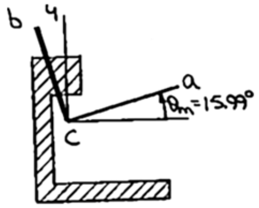
about C.

PROBLEM 9.102 (Continued)

Now $$\bar{I}_{\max,\min} = \bar{I}_{\text{ave}} \pm R = 11.3181 \pm 8.02915$$

or $$\bar{I}_{\max} = 19.35 \text{ in}^4 \blacktriangleleft$$

and $$\bar{I}_{\min} = 3.29 \text{ in}^4 \blacktriangleleft$$



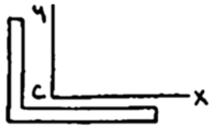
From the Mohr's circle it is seen that the *a* axis corresponds to $I_{\max}$ and the *b* axis corresponds to $I_{\min}$.

PROBLEM 9.103

The moments and product of inertia of an L4 × 3 × $\frac{1}{4}$-in. angle cross section with respect to two rectangular axes x and y through C are, respectively, $\bar{I}_x = 1.33$ in^4, $\bar{I}_y = 2.75$ in^4, and $\bar{I}_{xy} < 0$, with the minimum value of the moment of inertia of the area with respect to any axis through C being $\bar{I}_{\min} = 0.692$ in^4. Using Mohr's circle, determine (*a*) the product of inertia $\bar{I}_{xy}$ of the area, (*b*) the orientation of the principal axes, (*c*) the value of $\bar{I}_{\max}$.

SOLUTION

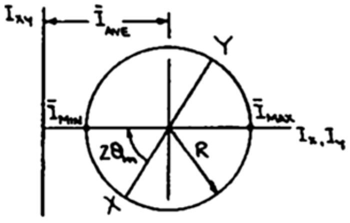
(*Note:* A review of a table of rolled-steel shapes reveals that the given values of $\bar{I}_x$ and $\bar{I}_y$ are obtained when the 4-in. leg of the angle is parallel to the x axis. Further, for $\bar{I}_{xy} < 0$, the angle must be oriented as shown.)



Now $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(1.33 + 2.75) = 2.040 \text{ in}^4$$

and $$\bar{I}_{\min} = \bar{I}_{\text{ave}} - R \quad \text{or} \quad R = 2.040 - 0.692$$
$$= 1.348 \text{ in}^4$$

Using $\bar{I}_{\text{ave}}$ and R, the Mohr's circle is then drawn as shown; note that for the diameter XY, $X(1.33,\ \bar{I}_{xy})$ and $Y(2.75,\ |\bar{I}_{xy}|)$.



(*a*) We have $$R^2 = \left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2$$

or $$\bar{I}_{xy}^2 = 1.348^2 - \left[\frac{1}{2}(1.33 - 2.75)\right]^2$$

Solving for $\bar{I}_{xy}$ and taking the negative root (since $\bar{I}_{xy} < 0$) yields $\bar{I}_{xy} = -1.14586$ in^4.

$\bar{I}_{xy} = -1.146$ in^4 ◀

(*b*) We have $$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(-1.14586)}{1.33 - 2.75}$$
$$= -1.61389$$

or $$2\theta_m = -58.217° \qquad \theta_m = -29.1°$$

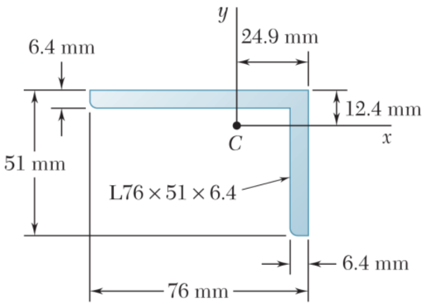
The principal axes are obtained by rotating the xy axes through

29.1° clockwise ◀

about C.

(*c*) We have $$\bar{I}_{\max} = \bar{I}_{\text{ave}} + R = 2.040 + 1.348$$

or $\bar{I}_{\max} = 3.39$ in^4 ◀


y
24.9 mm
6.4 mm
12.4 mm
x
C
51 mm
L76 × 51 × 6.4
6.4 mm
76 mm

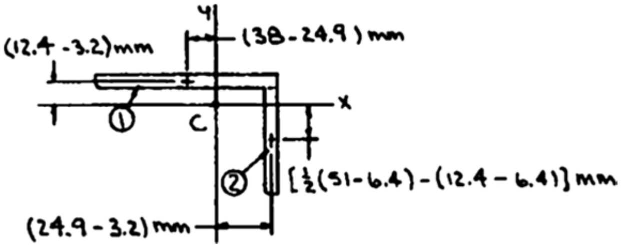
PROBLEM 9.104

Using Mohr's circle, determine for the cross section of the rolled-steel angle shown the orientation of the principal centroidal axes and the corresponding values of the moments of inertia. (Properties of the cross sections are given in Figure 9.13.)

SOLUTION

From Figure 9.13B: $\bar{I}_x = 0.162 \times 10^6 \text{ mm}^4$

$\bar{I}_y = 0.454 \times 10^6 \text{ mm}^4$


y
(12.4 - 3.2) mm
(38 - 24.9) mm
x
①
C
②
[½(51 - 6.4) - (12.4 - 6.4)] mm
(24.9 - 3.2) mm

We have $\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$

For each rectangle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

and $\bar{I}_{x'y'} = 0$ (symmetry) $I_{xy} = \Sigma\bar{x}\,\bar{y}A$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\,\bar{y}A$, mm^4
1	$76 \times 6.4 = 486.4$	–13.1	9.2	–58620.93
2	$6.4 \times (51 - 6.4) = 285.44$	21.7	–16.3	–100962.98
Σ				–159583.91

$\bar{I}_{xy} = -159584 \text{ mm}^4$

The Mohr's circle is defined by the diameter XY where $X(0.162 \times 10^6, -0.159584 \times 10^6)$ and $Y(0.454 \times 10^6, 0.159584 \times 10^6)$

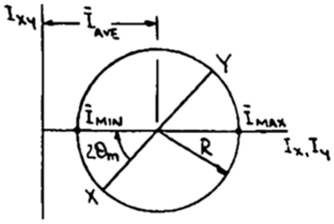
Now
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(0.162 + 0.454) \times 10^6$$
$$= 0.3080 \times 10^6 \text{ mm}^4$$

PROBLEM 9.104 (Continued)

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + I_{xy}^2} = \left\{\sqrt{\left[\frac{1}{2}(0.162 - 0.454)\right]^2 + (-0.159584)^2}\right\} \times 10^6$$

$$= 0.21629 \times 10^6 \text{ mm}^4$$

The Mohr's circle is then drawn as shown.



$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(-0.159584 \times 10^6)}{(0.162 - 0.454) \times 10^6}$$

$$= -1.09304$$

or $$2\theta_m = -47.545$$

and $$\theta_m = -23.8°$$

The principal axes are obtained by rotating the *xy* axes through

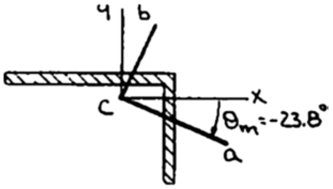
23.8° clockwise ◀

About *C*.

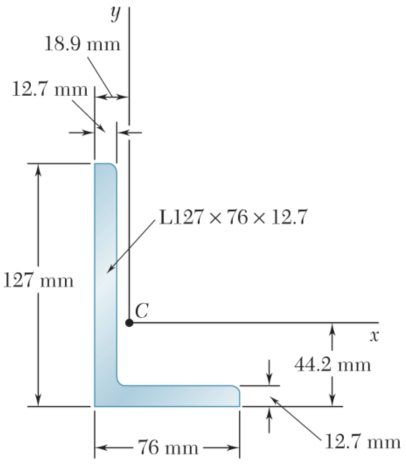
Now $$\bar{I}_{\text{max, min}} = \bar{I}_{\text{ave}} \pm R = (0.3080 \pm 0.21629) \times 10^6$$

or $$\bar{I}_{\text{max}} = 0.524 \times 10^6 \text{ mm}^4$$ ◀

and $$\bar{I}_{\text{min}} = 0.0917 \times 10^6 \text{ mm}^4$$ ◀



From the Mohr's circle it is seen that the *a* axis corresponds to $\bar{I}_{\text{min}}$ and the *b* axis corresponds to $\bar{I}_{\text{max}}$.


y
18.9 mm
12.7 mm
L127 × 76 × 12.7
127 mm
C
x
44.2 mm
76 mm
12.7 mm

PROBLEM 9.105

Using Mohr's circle, determine for the cross section of the rolled-steel angle shown the orientation of the principal centroidal axes and the corresponding values of the moments of inertia. (Properties of the cross sections are given in Figure 9.13.)

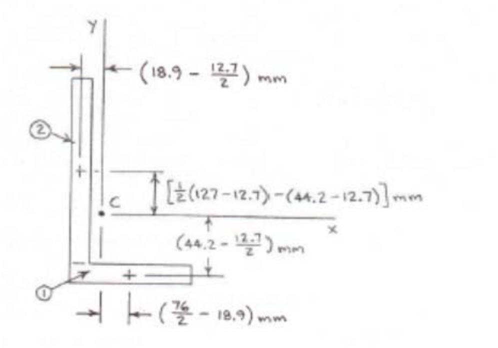
SOLUTION

From Figure 9.13B:

$$\bar{I}_x = 3.93 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 1.06 \times 10^6 \text{ mm}^4$$

Determine $\bar{I}_{xy}$:


y
$(18.9 - \frac{12.7}{2})$ mm
②
C
$[\frac{1}{2}(127-12.7)-(44.2-12.7)]$ mm
x
$(44.2 - \frac{12.7}{2})$ mm
①
$(\frac{76}{2} - 18.9)$ mm

$\bar{I}_{xy} = \left(I_{xy}\right)_1 + \left(I_{xy}\right)_2$ and for each rectangle,

$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$ and by symmetry $\bar{I}_{x'y'} = 0$ thus $\bar{I}_{xy} = \Sigma\bar{x}\bar{y}A$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\bar{y}A$, mm^4
1	$76 \times 12.7 = 965.2$	19.1	-37.85	-697,777
2	$12.7(127 - 12.7) = 1451.61$	-12.55	25.65	-467,284
Σ				-1,165,061

PROBLEM 9.105 (Continued)

$$\bar{I}_{xy} = -1.16506 \times 10^6 \text{ mm}^4$$

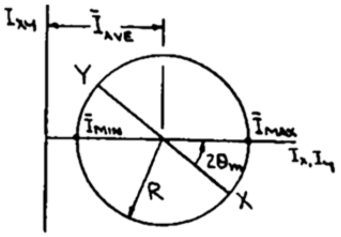
The Mohr's circle is defined by the diameter *XY*, where $X(3.93 \times 10^6, -1.165061 \times 10^6)$ and $Y(1.06 \times 10^6, 1.165061 \times 10^6)$.

Now $$I_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y)$$
$$= \frac{1}{2}(3.93 + 1.06) \times 10^6$$
$$= 2.495 \times 10^6 \text{mm}^4$$

and $$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2}$$
$$= \left\{\sqrt{\left[\frac{1}{2}(3.93 - 1.06)\right]^2 + (-1.165061)^2}\right\} \times 10^6 = 1.84840 \times 10^6 \text{mm}^4$$

The Mohr's circle is then drawn as shown.

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(-1.165061 \times 10^6)}{(3.93 - 1.06) \times 10^6}$$
$$= 0.81189$$



or $$2\theta_m = 39.073°$$

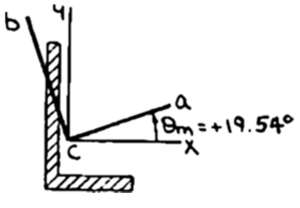
and $$\theta_m = 19.54°$$

The principal axes are obtained by rotating the *xy* axes through 19.54° counterclockwise ◀ about *C*.

Now $$\bar{I}_{\text{max, min}} = \bar{I}_{\text{ave}} \pm R = (2.495 \pm 1.84840) \times 10^6$$

or $\bar{I}_{\text{max}} = 4.34 \times 10^6 \text{mm}^4$ ◀

and $\bar{I}_{\text{min}} = 0.647 \times 10^6 \text{mm}^4$ ◀



From the Mohr's circle it is seen that the *a* axis corresponds to $\bar{I}_{\text{max}}$ and the *b* axis corresponds to $\bar{I}_{\text{min}}$.

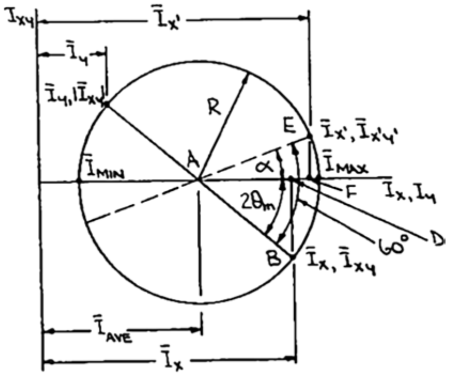
PROBLEM 9.106*

For a given area the moments of inertia with respect to two rectangular centroidal x and y axes are $\bar{I}_x = 1200\ \text{in}^4$ and $\bar{I}_y = 300\ \text{in}^4$, respectively. Knowing that after rotating the x and y axes about the centroid 30° counterclockwise, the moment of inertia relative to the rotated x axis is 1450 in^4, use Mohr's circle to determine (*a*) the orientation of the principal axes, (*b*) the principal centroidal moments of inertia.

SOLUTION

We have $$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(1200 + 300) = 750\ \text{in.}^4$$

Now observe that $\bar{I}_x > \bar{I}_{\text{ave}}$, $\bar{I}_{x'} > \bar{I}_x$, and $2\theta = +60°$. This is possible only if $\bar{I}_{xy} < 0$. Therefore, assume $\bar{I}_{xy} < 0$ and (for convenience) $\bar{I}_{x'y'} > 0$. Mohr's circle is then drawn as shown.



We have $$2\theta_m + \alpha = 60°$$

Now using ΔABD: $$R = \frac{\bar{I}_x - \bar{I}_{\text{ave}}}{\cos 2\theta_m} = \frac{1200 - 750}{\cos 2\theta_m}$$
$$= \frac{450}{\cos 2\theta_m} \quad (\text{in}^4)$$

Using ΔAEF: $$R = \frac{\bar{I}_{x'} - \bar{I}_{\text{ave}}}{\cos \alpha} = \frac{1450 - 750}{\cos \alpha}$$
$$= \frac{700}{\cos \alpha} \quad (\text{in}^4)$$

Then $$\frac{450}{\cos 2\theta_m} = \frac{700}{\cos \alpha} \qquad \alpha = 60° - 2\theta_m$$

or $$9\cos(60° - 2\theta_m) = 14\cos 2\theta_m$$

Expanding: $$9(\cos 60° \cos 2\theta_m + \sin 60° \sin 2\theta_m) = 14\cos 2\theta_m$$

or $$\tan 2\theta_m = \frac{14 - 9\cos 60°}{9\sin 60°} = 1.21885$$

or $$2\theta_m = 50.633° \quad \text{and} \quad \theta_m = 25.3°$$

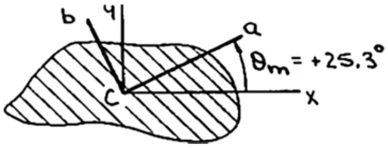
(*Note:* $2\theta_m < 60°$ implies assumption $\bar{I}_{x'y'} > 0$ is correct.)

Finally, $$R = \frac{450}{\cos 50.633°} = 709.46\ \text{in}^4$$

(*a*) From the Mohr's circle it is seen that the principal axes are obtained by rotating the given centroidal x and y axes through θ_m about the centroid C or

25.3° counterclockwise ◀

PROBLEM 9.106* (Continued)



(*b*) We have $\bar{I}_{\max,\min} = \bar{I}_{\text{ave}} \pm R = 750 \pm 709.46$

or $\bar{I}_{\max} = 1459 \text{ in}^4$ ◀

and $\bar{I}_{\min} = 40.5 \text{ in}^4$ ◀

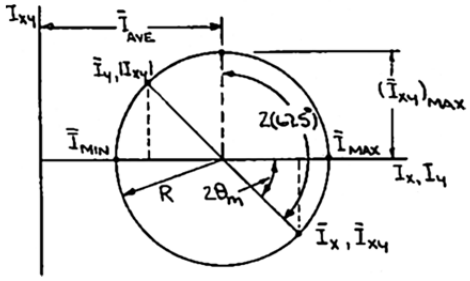
From the Mohr's circle it is seen that the *a* axis corresponds to $\bar{I}_{\max}$ and the *b* axis corresponds to $\bar{I}_{\min}$.

PROBLEM 9.107

It is known that for a given area $\bar{I}_y = 48 \times 10^6$ mm^4 and $\bar{I}_{xy} = -20 \times 10^6$ mm^4, where the *x* and *y* axes are rectangular centroidal axes. If the axis corresponding to the maximum product of inertia is obtained by rotating the *x* axis 67.5° counterclockwise about *C*, use Mohr's circle to determine (*a*) the moment of inertia $\bar{I}_x$ of the area, (*b*) the principal centroidal moments of inertia.

SOLUTION

First assume $\bar{I}_x > \bar{I}_y$ and then draw the Mohr's circle as shown. (*Note:* Assuming $\bar{I}_x < \bar{I}_y$ is not consistent with the requirement that the axis corresponding to $(\bar{I}_{xy})_{\max}$ is obtained after rotating the *x* axis through 67.5° CCW.)



From the Mohr's circle we have

$$2\theta_m = 2(67.5°) - 90° = 45°$$

(*a*) From the Mohr's circle we have

$$\bar{I}_x = \bar{I}_y + 2\frac{|I_{xy}|}{\tan 2\theta_m} = 48 \times 10^6 + 2\,\frac{20 \times 10^6}{\tan 45°}$$

or $\bar{I}_x = 88.0 \times 10^6 \text{ mm}^4$ ◀

(*b*) We have

$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(88.0 + 48) \times 10^6$$
$$= 68.0 \times 10^6 \text{ mm}^4$$

and

$$R = \frac{|I_{xy}|}{\sin 2\theta_m} = \frac{20 \times 10^6}{\sin 45°} = 28.284 \times 10^6 \text{ mm}^4$$

Now

$$\bar{I}_{\text{max, min}} = \bar{I}_{\text{ave}} \pm R = (68.0 \pm 28.284) \times 10^6$$

or $\bar{I}_{\max} = 96.3 \times 10^6 \text{ mm}^4$ ◀

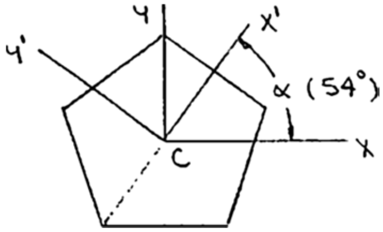
and $\bar{I}_{\min} = 39.7 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.108

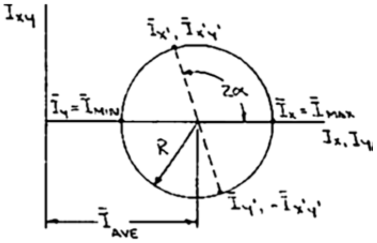
Using Mohr's circle, show that for any regular polygon (such as a pentagon) (*a*) the moment of inertia with respect to every axis through the centroid is the same, (*b*) the product of inertia with respect to every pair of rectangular axes through the centroid is zero.

SOLUTION

Consider the regular pentagon shown, with centroidal axes *x* and *y*.



Because the *y* axis is an axis of symmetry, it follows that $\bar{I}_{xy} = 0.$ Since $\bar{I}_{xy} = 0,$ the *x* and *y* axes must be principal axes. Assuming $\bar{I}_x = \bar{I}_{\text{max}}$ and $\bar{I}_y = \bar{I}_{\text{min}},$ the Mohr's circle is then drawn as shown.


I_{xy}
$\bar{I}_{x'}, \bar{I}_{x'y'}$
2α
$\bar{I}_y = \bar{I}_{MIN}$
$\bar{I}_x = \bar{I}_{MAX}$
I_x, I_y
R
$\bar{I}_{y'}, -\bar{I}_{x'y'}$
$\bar{I}_{AVE}$

Now rotate the coordinate axes through an angle α as shown; the resulting moments of inertia, $\bar{I}_{x'}$ and $\bar{I}_{y'}$, and product of inertia, $\bar{I}_{x'y'}$, are indicated on the Mohr's circle. However, the x' axis is an axis of symmetry, which implies $\bar{I}_{x'y'} = 0.$ For this to be possible on the Mohr's circle, the radius *R* must be equal to zero (thus, the circle degenerates into a point). With $R = 0,$ it immediately follows that

(*a*) $\bar{I}_x = \bar{I}_y = \bar{I}_{x'} = \bar{I}_{y'} = \bar{I}_{\text{ave}}$ (for all moments of inertia with respect to an axis through *C*) ◀

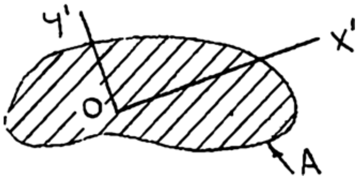
(*b*) $\bar{I}_{xy} = \bar{I}_{x'y'} = 0$ (for all products of inertia with respect to all pairs of rectangular axes with origin at *C*) ◀

PROBLEM 9.109

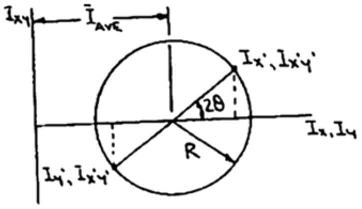
Using Mohr's circle, prove that the expression $I_{x'}I_{y'} - I_{x'y'}^2$ is independent of the orientation of the x' and y' axes, where $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ represent the moments and product of inertia, respectively, of a given area with respect to a pair of rectangular axes x' and y' through a given Point O. Also show that the given expression is equal to the square of the length of the tangent drawn from the origin of the coordinate system to Mohr's circle.

SOLUTION

First observe that for a given area A and origin O of a rectangular coordinate system, the values of I_{ave} and R are the same for all orientations of the coordinate axes. Shown below is a Mohr's circle, with the moments of inertia, $I_{x'}$ and $I_{y'}$, and the product of inertia, $I_{x'y'}$, having been computed for an arbitrary orientation of the $x'y'$ axes.



From the Mohr's circle



$$I_{x'} = I_{\text{ave}} + R\cos 2\theta$$

$$I_{y'} = I_{\text{ave}} - R\cos 2\theta$$

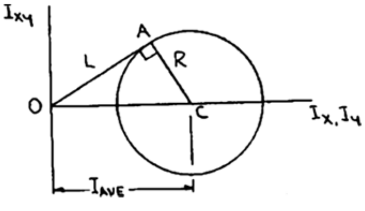
$$I_{x'y'} = R\sin 2\theta$$

Then, forming the expression $I_{x'}I_{y'} - I_{x'y'}^2$

$$\begin{aligned} I_{x'}I_{y'} - I_{x'y'}^2 &= (I_{\text{ave}} + R\cos 2\theta)(I_{\text{ave}} - R\cos 2\theta) - (R\sin 2\theta)^2 \\ &= \left(I_{\text{ave}}^2 - R^2\cos^2 2\theta\right) - (R^2\sin^2 2\theta) \\ &= I_{\text{ave}}^2 - R^2 \quad \text{which is a constant} \end{aligned}$$

$I_{x'}I_{y'} - I_{x'y'}^2$ is independent of the orientation of the coordinate axes Q.E.D. ◀

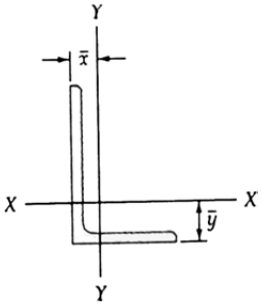
Shown is a Mohr's circle, with line $\overline{OA}$, of length L, the required tangent.



Noting that $\angle OAC$ is a right angle, it follows that

$$L^2 = I_{\text{ave}}^2 - R^2$$

or $L^2 = I_{x'}I_{y'} - I_{x'y'}^2$ Q.E.D. ◀



PROBLEM 9.110

Using the invariance property established in the preceding problem, express the product of inertia I_{xy} of an area A with respect to a pair of rectangular axes through O in terms of the moments of inertia I_x and I_y of A and the principal moments of inertia $I_{\min}$ and $I_{\max}$ of A about O. Use the formula obtained to calculate the product of inertia I_{xy} of the L3 × 2 × $\frac{1}{4}$ -in. angle cross section shown in Figure 9.13A, knowing that its maximum moment of inertia is 1.257 in^4.

SOLUTION

Consider the following two sets of moments and products of inertia, which correspond to two different orientations of the coordinate axes whole origin is at Point *O*.

Case 1: $I_{x'} = I_x, \quad I_{y'} = I_y, \quad I_{x'y'} = I_{xy}$

Case 2: $I_{x'} = I_{\max}, \quad I_{y'} = I_{\min}, \quad I_{x'y'} = 0$

The invariance property then requires

$$I_x I_y - I_{xy}^2 = I_{\max} I_{\min} \qquad \text{or} \qquad I_{xy} = \pm\sqrt{I_x I_y - I_{\max} I_{\min}} \blacktriangleleft$$

From Figure 9.13A: $\bar{I}_x = 1.09 \text{ in}^4$

$\bar{I}_y = 0.390 \text{ in}^4$

Using Eq. (9.21): $\bar{I}_x + \bar{I}_y = \bar{I}_{\max} + \bar{I}_{\min}$

Substituting $1.09 + 0.390 = 1.257 + \bar{I}_{\min}$

or $\bar{I}_{\min} = 0.223 \text{ in}^4$

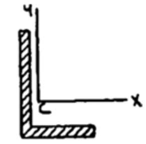
Then

$$\bar{I}_{xy} = \sqrt{(1.09)(0.390) - (1.257)(0.223)}$$

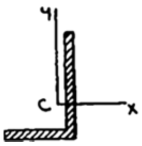
$$= \pm 0.381 \text{ in}^4$$

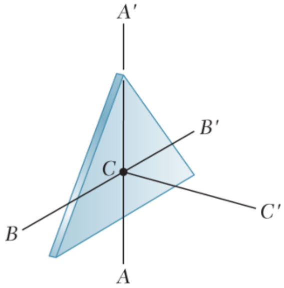
The two roots correspond to the following two orientations of the cross section.

For $\bar{I}_{xy} = -0.381 \text{ in}^4 \blacktriangleleft$



and for $\bar{I}_{xy} = 0.381 \text{ in}^4 \blacktriangleleft$



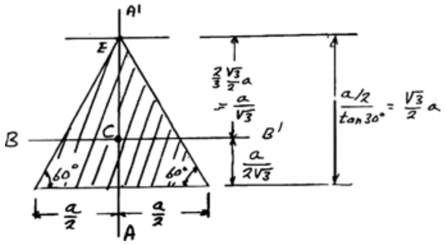


PROBLEM 9.111

A thin plate of mass m is cut in the shape of an equilateral triangle of side a. Determine the mass moment of inertia of the plate with respect to (*a*) the centroidal axes AA' and BB', (*b*) the centroidal axis CC' that is perpendicular to the plate.

SOLUTION

$$\text{Area} = \frac{1}{2}a\left(\frac{\sqrt{3}}{2}a\right) = \frac{\sqrt{3}}{4}a^2$$


A'
E
B
C
B'
60°
60°
$\frac{2}{3}\frac{\sqrt{3}}{2}a$
$= \frac{a}{\sqrt{3}}$
$\frac{a}{2\sqrt{3}}$
$\frac{a/2}{\tan 30°} = \frac{\sqrt{3}}{2}a$
$\frac{a}{2}$
$\frac{a}{2}$
A

$$\text{Mass} = m = \rho V = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

(*a*) Axis AA':

$$I_{AA',\text{area}} = 2\left[\frac{1}{12}\left(\frac{\sqrt{3}}{2}a\right)\left(\frac{a}{2}\right)^3\right] = \frac{\sqrt{3}}{96}a^4$$

$$I_{AA',\text{ mass}} = \frac{m}{A} I_{AA',\text{ area}} = \frac{\frac{m}{\sqrt{3}}}{4}a^2\left(\frac{\sqrt{3}}{96}a^4\right) = \frac{1}{24}ma^2 \quad ◀$$

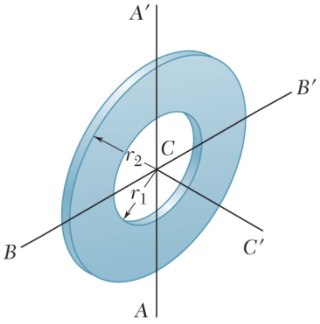
Axis BB'':

$$I_{BB',\text{ area}} = \frac{1}{36}a\left(\frac{\sqrt{3}}{2}a\right)^3 = \frac{\sqrt{3}}{96}a^4 \qquad (\text{we check that } I_{AA} = I_{BB})$$

$$I_{BB',\text{mass}} = \frac{m}{A} I_{BB',\text{ area}} = \frac{\frac{m}{\sqrt{3}}}{4}a^2\left(\frac{\sqrt{3}}{96}a^4\right) = \frac{1}{24}ma^2 \quad ◀$$

(b) Eq. (9.38):

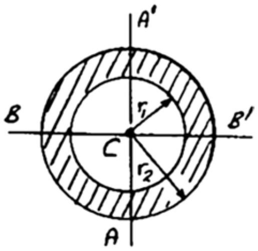
$$I_{CC'} = I_{AA'} + I_{BB'} = 2\left(\frac{1}{24}ma^2\right) \qquad I_{CC'} = \frac{1}{12}ma^2 \quad ◀$$



PROBLEM 9.112

Determine the mass moment of inertia of a ring of mass *m*, cut from a thin uniform plate, with respect to (*a*) the axis *AA*′, (*b*) the centroidal axis *CC*′ that is perpendicular to the plane of the ring.

SOLUTION



$$\text{Mass} = m = \rho V = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

We first determine

$$A = \pi r_2^2 - \pi r_1^2 = \pi\left(r_2^2 - r_1^2\right)$$

$$I_{AA',\text{area}} = \frac{\pi}{4} r_2^4 - \frac{\pi}{4} r_1^4 = \frac{\pi}{4}\left(r_2^4 - r_1^4\right)$$

(*a*)

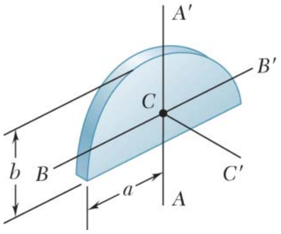
$$I_{AA',\text{ mass}} = \frac{m}{A} I_{AA',\text{ area}} = \frac{m}{\pi\left(r_2^2 - r_1^2\right)} \frac{\pi}{4}\left(r_2^4 - r_1^4\right) = \frac{1}{4} m\left(r_2^2 + r_1^2\right)$$

$$I_{AA'} = \frac{1}{4} m\left(r_1^2 + r_2^2\right) \blacktriangleleft$$

(*b*) By symmetry: $I_{BB'} = I_{AA'}$

Eq. (9.38): $I_{CC'} = I_{AA'} + I_{BB'} = 2I_{AA'}$

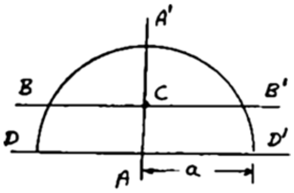
$$I_{CC'} = \frac{1}{2} m\left(r_1^2 + r_2^2\right) \blacktriangleleft$$



PROBLEM 9.113

A thin, semielliptical plate has a mass m. Determine the mass moment of inertia of the plate with respect to (a) the centroidal axis BB', (b) the centroidal axis CC' that is perpendicular to the plate.

SOLUTION



$$\text{mass} = m = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{2m}{\pi ab} I_{\text{area}}$$

Area: $$A = \frac{1}{2}\pi a^2$$

$$I_{x,\text{area}} = I_{BB',\text{area}} + A\bar{y}^2$$

$$I_{BB',\text{area}} = \frac{\pi}{8}ab^3 - \left(\frac{\pi}{2}ab\right)\left(\frac{4b}{3\pi}\right)^2$$

$$I_{BB',\text{area}} = \frac{\pi}{8}ab^3\left(1 - \frac{64}{9\pi^2}\right)$$

(a) $$I_{BB',\text{mass}} = \frac{2m}{\pi ab}\left(\frac{\pi}{8}ab^3\right)\left(1 - \frac{64}{9\pi^2}\right)$$

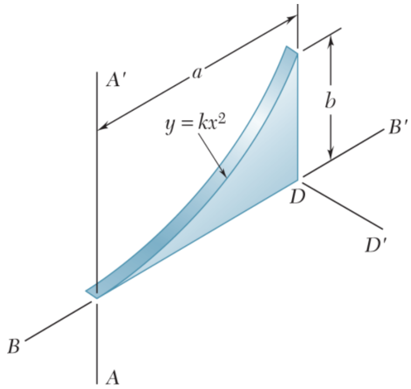
$$= \frac{1}{4}mb^2\left(1 - \frac{64}{9\pi^2}\right)$$

$$I_{BB'} = 0.0699mb^2 \blacktriangleleft$$

(b) Fig. (9.12): $I_{AA',area} = \frac{\pi}{8}a^3b$, then $I_{AA',mass} = \left(\frac{2m}{\pi ab}\right)\left(\frac{\pi}{8}a^3b\right) = \frac{1}{4}ma^2$

$$I_{CC',mass} = I_{AA',mass} + I_{BB',mass} = \frac{1}{4}ma^2 + \frac{1}{4}mb^2\left(1 - \frac{64}{9\pi^2}\right)$$

$$I_{CC'} = \frac{1}{4}\text{m}(a^2 + 0.279b^2) \blacktriangleleft$$



PROBLEM 9.114

The parabolic spandrel shown was cut from a thin, uniform plate. Denoting the mass of the spandrel by *m*, determine its mass moment of inertia with respect to (*a*) the axis *BB*′, (*b*) the axis *DD*′ that is perpendicular to the spandrel. (*Hint:* See Sample Problem 9.3.)

SOLUTION

First note

$$\text{mass} = m = \rho V = \rho t A$$

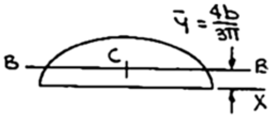
$$= \rho t\left(\frac{\pi}{2}ab\right)$$

Also

$$I_{\text{mass}} = \rho t I_{\text{area}}$$

$$= \frac{2m}{\pi ab} I_{\text{area}}$$

(*a*) We have



$$I_{x,\text{area}} = I_{BB',\text{area}} + A\bar{y}^2$$

or

$$I_{BB',\text{area}} = \frac{\pi}{8}ab^3 - \left(\frac{\pi}{2}ab\right)\left(\frac{4b}{3\pi}\right)^2$$

(*a*) From Sample Problem 9.3:

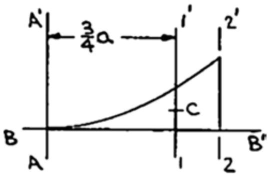
$$I_{BB',\text{area}} = \frac{1}{21}ab^3$$

Then

$$I_{BB',\text{mass}} = \frac{3m}{ab} \times \frac{1}{21}ab^3$$

or $\quad I_{BB'} = \frac{1}{7}mb^2$ ◀

(*b*) From Sample Problem 9.3:



$$I_{AA',\text{area}} = \frac{1}{5}a^3b$$

Now

$$I_{AA',\text{area}} = I_{11',\text{area}} + A\left(\frac{3}{4}a\right)^2$$

PROBLEM 9.114 (Continued)

and $$I_{22',\text{area}} = I_{11',\text{area}} + A\left(\frac{1}{4}a\right)^2$$

Then $$I_{22',\text{area}} = I_{AA',\text{area}} + A\left[\left(\frac{1}{4}a\right)^2 - \left(\frac{3}{4}a\right)^2\right]$$

$$= \frac{1}{5}a^3b + \frac{1}{3}ab\left(\frac{1}{16}a^2 - \frac{9}{16}a^2\right)$$

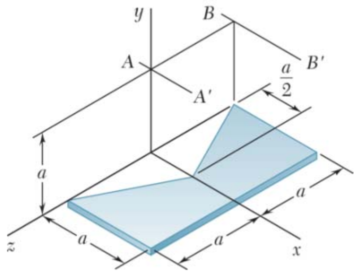
$$= \frac{1}{30}a^3b$$

and $$I_{22',\text{mass}} = \frac{3m}{ab} \times \frac{1}{30}a^3b$$

$$= \frac{1}{10}ma^2$$

Finally, $$I_{DD',\text{mass}} = I_{BB',\text{mass}} + I_{22',\text{mass}}$$

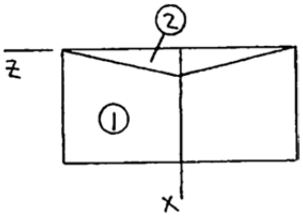
$$= \frac{1}{7}mb^2 + \frac{1}{10}ma^2$$ or $$I_{DD'} = \frac{1}{70}m(7a^2 + 10b^2)$$ ◀



PROBLEM 9.115

A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by *m*, determine its mass moment of inertia with respect to (*a*) the *x* axis, (*b*) the *y* axis.

SOLUTION



First note

$$\text{mass} = m = \rho V = \rho t A$$

$$= \rho t\left[(2a)(a) - \frac{1}{2}(2a)\left(\frac{a}{2}\right)\right]$$

$$= \frac{3}{2}\rho t a^2$$

Also

$$I_{\text{mass}} = \rho t I_{\text{area}}$$

$$= \frac{2m}{3a^2} I_{\text{area}}$$

(*a*) Now

$$\bar{I}_{x,\text{area}} = (I_x)_{1,\text{area}} - 2(I_x)_{2,\text{area}}$$

$$= \frac{1}{12}(a)(2a)^3 - 2\left[\frac{1}{12}\left(\frac{a}{2}\right)(a)^3\right]$$

$$= \frac{7}{12}a^4$$

Then

$$\bar{I}_{x,\text{mass}} = \frac{2m}{3a^2} \times \frac{7}{12}a^4$$

or $\bar{I}_x = \frac{7}{18}ma^2$ ◀

(*b*) We have

$$\bar{I}_{z,\text{area}} = (I_z)_{1,\text{area}} - 2(I_z)_{2,\text{area}}$$

$$= \frac{1}{3}(2a)(a)^3 - 2\left[\frac{1}{12}(a)\left(\frac{a}{2}\right)^3\right]$$

$$= \frac{31}{48}a^4$$

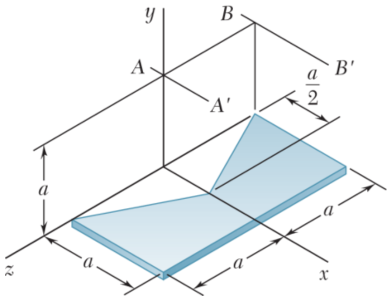
Then

$$I_{z,\text{mass}} = \frac{2m}{3a^2} \times \frac{31}{48}a^4$$

$$= \frac{31}{72}ma^2$$

Finally,
$$\begin{aligned} I_{y,\text{mass}} &= \bar{I}_{x,\text{mass}} + I_{z,\text{mass}} \\ &= \frac{7}{18}ma^2 + \frac{31}{72}ma^2 \\ &= \frac{59}{72}ma^2 \end{aligned}$$

or $I_y = 0.819ma^2$ ◀



PROBLEM 9.116

A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by m, determine its mass moment of inertia with respect to (a) the axis AA', (b) the axis BB', where the AA' and BB' axes are parallel to the x axis and lie in a plane parallel to and at a distance a above the xz plane.

SOLUTION

First note that the x axis is a centroidal axis so that

$$I = \bar{I}_{x,\text{mass}} + md^2$$

and that from the solution to Problem 9.115,

$$\bar{I}_{x,\text{mass}} = \frac{7}{18}ma^2$$

(a) We have

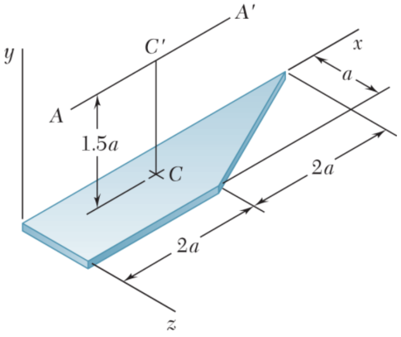
$$I_{AA',\text{mass}} = \frac{7}{18}ma^2 + m(a)^2$$

or $I_{AA'} = 1.389ma^2$ ◀

(b) We have

$$I_{BB',\text{mass}} = \frac{7}{18}ma^2 + m(a\sqrt{2})^2$$

or $I_{BB'} = 2.39ma^2$ ◀



PROBLEM 9.117

A thin plate of mass *m* has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (*a*) the *x* axis, (*b*) the *y* axis.

SOLUTION

First note

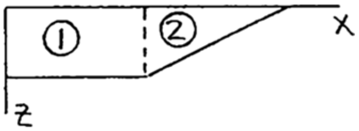
$$\text{mass} = m = \rho V = \rho t A$$

$$= \rho t \left[(2a)(a) + \frac{1}{2}(2a)(a) \right] = 3\rho t a^2$$

Also

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{3a^2} I_{\text{area}}$$

(*a*) Now



$$I_{x,\text{area}} = (I_x)_{1,\text{area}} + (I_x)_{2,\text{area}}$$

$$= \frac{1}{3}(2a)(a)^3 + \frac{1}{12}(2a)(a)^3$$

$$= \frac{5}{6}a^4$$

Then

$$I_{x,\text{mass}} = \frac{m}{3a^2} \times \frac{5}{6}a^4$$

or $I_{x,\text{ mass}} = \frac{5}{18}ma^2$ ◀

(*b*) We have

$$I_{z,\text{area}} = (I_z)_{1,\text{area}} + (I_z)_{2,\text{ area}}$$

$$= \left[\frac{1}{3}(a)(2a)^3 \right] + \left[\frac{1}{36}(a)(2a)^3 + \frac{1}{2}(2a)(a)\left(2a + \frac{1}{3} \times 2a \right)^2 \right] = 10a^4$$

Then

$$I_{x,\text{mass}} = \frac{m}{3a^2} \times 10a^4 = \frac{10}{3}ma^2$$

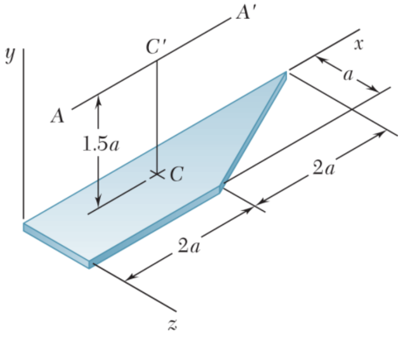
Finally,

$$I_{y,\text{mass}} = I_{x,\text{mass}} + I_{z,\text{mass}}$$

$$= \frac{5}{18}ma^2 + \frac{10}{3}ma^2$$

$$= \frac{65}{18}ma^2$$

or $I_{y,\text{mass}} = 3.61ma^2$ ◀



PROBLEM 9.118

A thin plate of mass m has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (a) the centroidal axis CC' that is perpendicular to the plate, (b) the axis AA' that is parallel to the x axis and is located at a distance $1.5a$ from the plate.

SOLUTION

First locate the centroid C.

$$\bar{X}\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(2a^2 + a^2) = a(2a^2) + \left(2a + \frac{1}{3}\times 2a\right)(a^2)$$

or
$$\bar{X} = \frac{14}{9}a$$

$$\bar{Z}\Sigma A = \Sigma \bar{z}A: \quad \bar{Z}(2a^2 + a^2) = \left(\frac{1}{2}a\right)(2a^2) + \left(\frac{1}{3}a\right)(a^2)$$

or
$$\bar{Z} = \frac{4}{9}a$$

(a) We have
$$I_{y,\,\text{mass}} = \bar{I}_{CC',\,\text{mass}} + m(\bar{X}^2 + \bar{Z}^2)$$

From the solution to Problem 9.117:

$$I_{y,\text{mass}} = \frac{65}{18}ma^2$$

Then
$$\bar{I}_{cc',\text{mass}} = \frac{65}{18}ma^2 - m\left[\left(\frac{14}{9}a\right)^2 + \left(\frac{4}{9}a\right)^2\right]$$

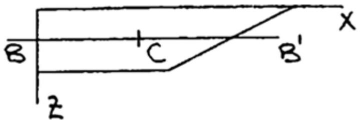
or $\bar{I}_{cc'} = 0.994ma^2$ ◀

(b) We have
$$I_{x,\text{mass}} = \bar{I}_{BB',\text{mass}} + m(\bar{Z})^2$$

and
$$I_{AA',\,\text{mass}} = \bar{I}_{BB',\,\text{mass}} + m(1.5a)^2$$

Then
$$I_{AA',\,\text{mass}} = I_{x,\,\text{mass}} + m\left[(1.5a)^2 - \left(\frac{4}{9}a\right)^2\right]$$

PROBLEM 9.118 (Continued)

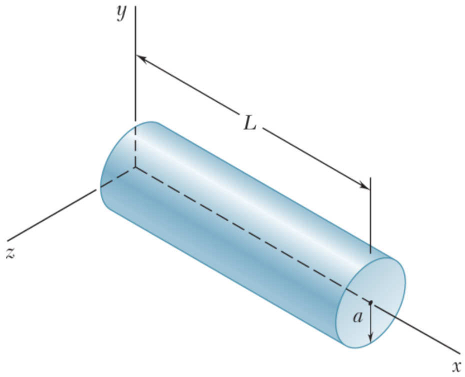


From the solution to Problem 9.193:

$$I_{x,\text{mass}} = \frac{5}{18}ma^2$$

Then
$$I_{AA',\text{mass}} = \frac{5}{18}ma^2 + m\left[(1.5a)^2 - \left(\frac{4}{9}a\right)^2\right]$$

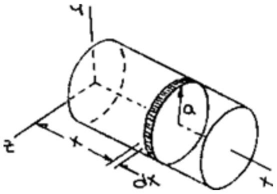
or $I_{AA'} = 2.33ma^2$ ◀



PROBLEM 9.119

Determine by direct integration the mass moment of inertia with respect to the z axis of the right circular cylinder shown, assuming that it has a uniform density and a mass m.

SOLUTION



For the cylinder: $m = \rho V = \rho\pi a^2 L$

For the element shown: $dm = \rho\pi a^2 dx$

$$= \frac{m}{L}dx$$

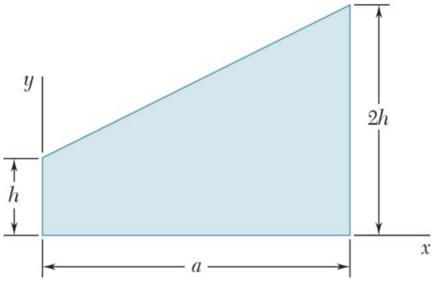
and $dI_z = dI_z + x^2 dm$

$$= \frac{1}{4}a^2 dm + x^2 dm$$

Then $I_z = \int dI_z = \int_0^L \left(\frac{1}{4}a^2 + x^2\right)\left(\frac{m}{L}dx\right)$

$$= \frac{m}{L}\left[\frac{1}{4}a^2 x + \frac{1}{3}x^3\right]_0^L$$

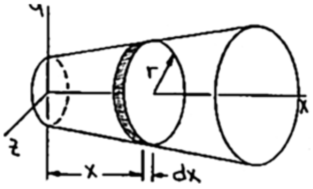
$$I_z = \frac{1}{12}m(3a^2 + 4L^2) \blacktriangleleft$$



PROBLEM 9.120

The area shown is revolved about the *x* axis to form a homogeneous solid of revolution of mass *m*. Using direct integration, express the mass moment of inertia of the solid with respect to the *x* axis in terms of *m* and *h*.

SOLUTION



We have $$y = \frac{2h - h}{a}x + h$$

so that $$r = \frac{h}{a}(x + a)$$

For the element shown:

$$dm = \rho\pi r^2 dx \quad dI_x = \frac{1}{2}r^2 dm$$

$$= \rho\pi\left[\frac{h}{a}(x + a)\right]^2 dx$$

Then $$m = \int dm = \int_0^a \rho\pi\frac{h^2}{a^2}(x + a)^2 dx = \frac{1}{3}\rho\pi\frac{h^2}{a^2}[(x + a)^3]_0^a$$

$$= \frac{1}{3}\rho\pi\frac{h^2}{a^2}(8a^3 - a^3) = \frac{7}{3}\rho\pi a h^2$$

Now $$I_x = \int dI_x = \int\frac{1}{2}r^2(\rho\pi r^2 dx) = \frac{1}{2}\rho\pi\int_0^a\left[\frac{h}{a}(x + a)\right]^4 dx$$

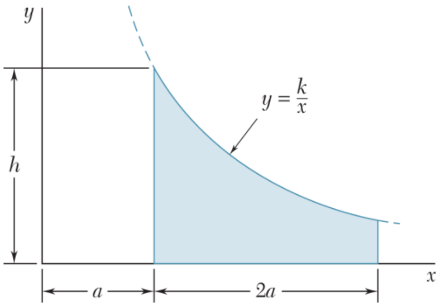
$$= \frac{1}{2}\rho\pi \times \frac{1}{5}\frac{h^4}{a^4}[(x + a)^5]_0^a = \frac{1}{10}\rho\pi\frac{h^4}{a^4}(32a^5 - a^5)$$

$$= \frac{31}{10}\rho\pi a h^4$$

From above: $$\rho\pi a h^2 = \frac{3}{7}m$$

Then $$I_x = \frac{31}{10}\left(\frac{3}{7}m\right)h^2 = \frac{93}{70}mh^2$$

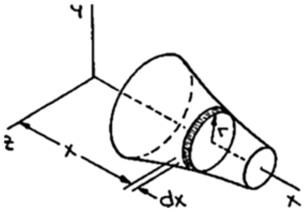
or $I_x = 1.329\, mh^2$ ◀



PROBLEM 9.121

The area shown is revolved about the *x* axis to form a homogeneous solid of revolution of mass *m*. Determine by direct integration the mass moment of inertia of the solid with respect to (*a*) the *x* axis, (*b*) the *y* axis. Express your answers in terms of *m* and the dimensions of the solid.

SOLUTION



We have at (a, h): $h = \frac{k}{a}$

or $k = ah$

For the element shown:

$$r = 4$$

$$dm = \rho\pi r^2 dx$$

$$= \rho\pi\left(\frac{ah}{x}\right)^2 dx$$

Then $$m = \int dm = \int_a^{3a} \rho\pi\left(\frac{ah}{x}\right)^2 dx$$

$$= \rho\pi a^2h^2\left[-\frac{1}{x}\right]_a^{3a}$$

$$= \rho\pi a^2h^2\left[-\frac{1}{3a} - \left(-\frac{1}{a}\right)\right] = \frac{2}{3}\rho\pi a h^2$$

(*a*) For the element: $dI_x = \frac{1}{2}r^2 dm = \frac{1}{2}\rho\pi r^4 dx$

Then $$I_x = \int dI_x = \int_a^{3a} \frac{1}{2}\rho\pi\left(\frac{ah}{x}\right)^4 dx = \frac{1}{2}\rho\pi a^4h^4\left[-\frac{1}{3}\frac{1}{x^3}\right]_a^{3a}$$

$$= -\frac{1}{6}\rho\pi a^4h^4\left[\left(\frac{1}{3a}\right)^3 - \left(\frac{1}{a}\right)^3\right] = \frac{1}{6}\times\frac{26}{27}\rho\pi a h^4$$

$$= \frac{1}{6}\times\frac{2}{3}\rho\pi a h^2 \times \frac{13}{9}h^2 = \frac{13}{54}mh^2$$

or $I_x = 0.241\,mh^2$ ◀

PROBLEM 9.121 (Continued)

(*b*) For the element:

$$dI_y = dI_y + x^2 dm$$

$$= \frac{1}{4} r^2 dm + x^2 dm$$

Then

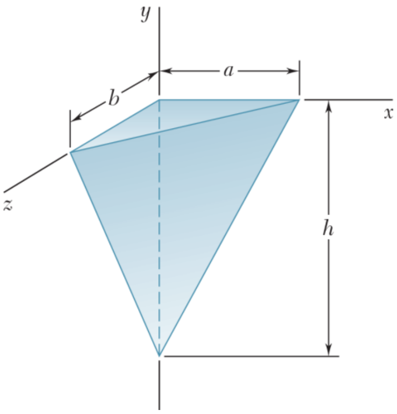
$$I_y = \int dI_y = \int_a^{3a} \left[\frac{1}{4} \left(\frac{ah}{x} \right)^2 + x^2 \right] \rho\pi \left(\frac{ah}{x} \right)^2 dx$$

$$= \rho\pi a^2 h^2 \int_a^{3a} \left(\frac{1}{4} \frac{a^2 h^2}{x^4} + 1 \right) dx = \rho\pi a^2 h^2 \left[-\frac{1}{12} \frac{a^2 h^2}{x^3} + x \right]_a^{3a}$$

$$= \left(\frac{3}{2} m \right) a \left\{ \left[-\frac{1}{12} \frac{a^2 h^2}{(3a)^3} + 3a \right] - \left[-\frac{1}{12} \frac{a^2 h^2}{(a)^3} + a \right] \right\}$$

$$= \frac{3}{2} ma \left(\frac{1}{12} \times \frac{26}{27} \frac{h^2}{a} + 2a \right) = m \left(\frac{13}{108} h^2 + 3a^2 \right)$$

or $I_y = m(3a^2 + 0.1204h^2)$ ◀



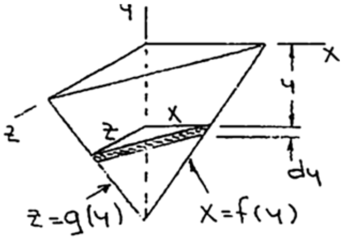
PROBLEM 9.122

Determine by direct integration the mass moment of inertia with respect to the *x* axis of the tetrahedron shown, assuming that it has a uniform density and a mass *m*.

SOLUTION

We have $$x = \frac{a}{h}y + a = a\left(1 + \frac{y}{h}\right)$$

and $$z = \frac{b}{h}y + b = b\left(1 + \frac{y}{h}\right)$$



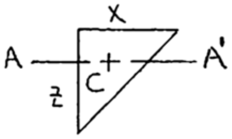
For the element shown: $$dm = \rho\left(\frac{1}{2}xz\,dy\right) = \frac{1}{2}\rho ab\left(1 + \frac{y}{h}\right)^2 dy$$

Then $$m = \int dm = \int_{-h}^{0} \frac{1}{2}\rho ab\left(1 + \frac{y}{h}\right)^2 dy$$

$$= \frac{1}{2}\rho ab \times \frac{h}{3}\left[\left(1 + \frac{y}{h}\right)^3\right]_{-h}^{0}$$

$$= \frac{1}{6}\rho abh\left[(1)^3 - (1-1)^3\right]$$

$$= \frac{1}{6}\rho abh$$



Now, for the element: $$I_{AA',\text{area}} = \frac{1}{36}xz^3 = \frac{1}{36}ab^3\left(1 + \frac{y}{h}\right)^4$$

Then $$dI_{AA',\text{mass}} = \rho t I_{AA',\text{area}} = \rho(dy)\left[\frac{1}{3}ab^3\left(1 + \frac{y}{h}\right)^4\right]$$

PROBLEM 9.122 (Continued)

Now

$$dI_x = dI_{AA',\text{mass}} + \left[y^2 + \left(\frac{1}{3}z\right)^2\right]dm$$

$$= \frac{1}{36}\rho ab^3\left(1+\frac{y}{h}\right)^4 dy$$

$$+\left\{y^2 + \left[\frac{1}{3}b\left(1+\frac{y}{h}\right)\right]^2\right\}\left[\frac{1}{2}\rho ab\left(1+\frac{y}{h}\right)^2 dy\right]$$

$$= \frac{1}{12}\rho ab^3\left(1+\frac{y}{h}\right)^4 dy + \frac{1}{2}\rho ab\left(y^2 + 2\frac{y^3}{h} + \frac{y^4}{h^2}\right)dy$$

Now $$m = \frac{1}{6}\rho abh$$

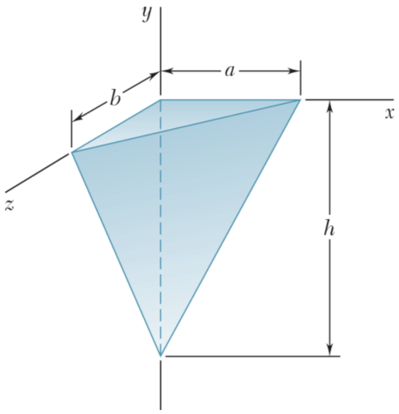
Then $$dI_x = \left[\frac{1}{2}m\frac{b^2}{h}\left(1+\frac{y}{h}\right)^4 + \frac{3m}{h}\left(y^2 + 2\frac{y^3}{h} + \frac{y^4}{h^2}\right)\right]dy$$

and $$I_x = \int dI_x = \int_{-h}^{0}\frac{m}{2h}\left[b^2\left(1+\frac{y}{h}\right)^4 + 6\left(y^2 + 2\frac{y^3}{h} + \frac{y^4}{h^2}\right)\right]dy$$

$$= \frac{m}{2h}\left[b^2 \times \frac{h}{5}\left(1+\frac{y}{h}\right)^5 + 6\left(\frac{1}{3}y^3 + \frac{1}{2}\frac{y^4}{h} + \frac{y^5}{5h^2}\right)\right]_{-h}^{0}$$

$$= \frac{m}{2h}\left\{\frac{1}{5}b^2h(1)^5 - 6\left[\frac{1}{3}(-h)^3 + \frac{1}{2h}(-h)^4 + \frac{1}{5h^2}(-h)^5\right]\right\}$$

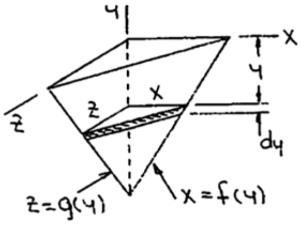
or $$I_x = \frac{1}{10}m(b^2 + h^2) \blacktriangleleft$$



PROBLEM 9.123

Determine by direct integration the mass moment of inertia with respect to the *y* axis of the tetrahedron shown, assuming that it has a uniform density and a mass *m*.

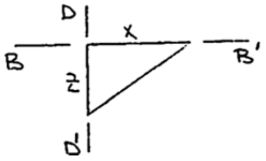
SOLUTION



We have $$x = \frac{a}{h}y + a = a\left(1 + \frac{y}{h}\right)$$

and $$z = \frac{b}{h}y + b = b\left(1 + \frac{y}{h}\right)$$

For the element shown: $$dm = \rho\left(\frac{1}{2}xz\,dy\right) = \frac{1}{2}\rho ab\left(1 + \frac{y}{h}\right)^2 dy$$



Then $$m = \int dm = \int_{-h}^{0} \frac{1}{2}\rho ab\left(1 + \frac{y}{h}\right)^2 dy$$

$$= \frac{1}{2}\rho ab \times \frac{h}{3}\left[\left(1 + \frac{y}{h}\right)^3\right]_{-h}^{0}$$

$$= \frac{1}{6}\rho abh\left[(1)^3 - (1-1)^3\right]$$

$$= \frac{1}{6}\rho abh$$

Also $$I_{BB',\text{area}} = \frac{1}{12}xz^3 \qquad I_{DD',\text{area}} = \frac{1}{12}zx^3$$

Then, using $$I_{\text{mass}} = \rho t I_{\text{area}}$$

we have $$dI_{BB',\text{mass}} = \rho(dy)\left(\frac{1}{12}xz^3\right) \qquad dI_{DD',\text{mass}} = \rho(dy)\left(\frac{1}{12}zx^3\right)$$

PROBLEM 9.123 (Continued)

Now

$$dI_y = dI_{BB',\,\text{mass}} + dI_{DD',\,\text{mass}}$$

$$= \frac{1}{12}\rho xz(x^2 + z^2)dy$$

$$= \frac{1}{12}\rho ab\left(1+\frac{y}{h}\right)^2\left[(a^2+b^2)\left(1+\frac{y}{h}\right)^2\right]dy$$

We have $m = \frac{1}{6}\rho abh \Rightarrow dI_y = \frac{m}{2h}(a^2+b^2)\left(1+\frac{y}{h}\right)^4 dy$

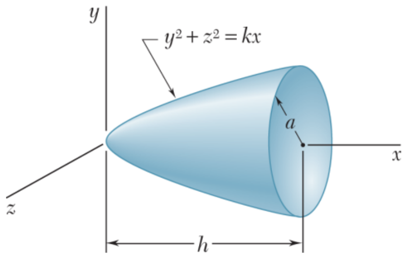
Then

$$I_y = \int dI_y = \int_{-h}^{0}\frac{m}{2h}(a^2+b^2)\left(1+\frac{y}{h}\right)^4 dy$$

$$= \frac{m}{2h}\left(a^2+b^2\right)\times\frac{h}{5}\left[\left(1+\frac{y}{h}\right)^5\right]_{-h}^{0}$$

$$= \frac{m}{10}\left(a^2+b^2\right)\left[(1)^5 - (1-1)^5\right]$$

or $I_y = \frac{1}{10}m(a^2+b^2)$ ◀



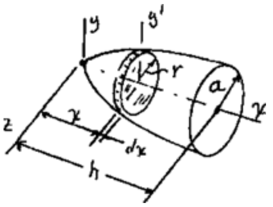
PROBLEM 9.124

Determine by direct integration the mass moment of inertia and the radius of gyration with respect to the x axis of the paraboloid shown, assuming that it has a uniform density and a mass m.

SOLUTION

$r^2 = y^2 + z^2 = kx$: at $x = h,\ r = a;\quad a^2 = kh;\quad k = \dfrac{a^2}{h}$

Thus: $r^2 = \dfrac{a^2}{h}x$



$$dm = \rho\pi r^2 dx = \rho\pi \frac{a^2}{h} xdy$$

$$m = \int_0^h dm = \rho\pi \frac{a^2}{h}\int_0^h xdx;\quad m = \frac{1}{2}\rho\pi a^2 h$$

$$dI_x = \left(\frac{1}{2}r^2\right)dm$$

$$I_x = \int_0^h dI_x = \frac{1}{2}\int_0^h \left(r^2\right)\left(\rho\pi r^2 dx\right) = \frac{1}{2}\rho\pi\int_0^h r^4 dx$$

$$= \frac{1}{2}\rho\pi\int_0^h \left(\frac{a^2}{h}x\right)^2 dx = \frac{\rho\pi a^4}{2h^2}\int_0^h x^2 dx$$

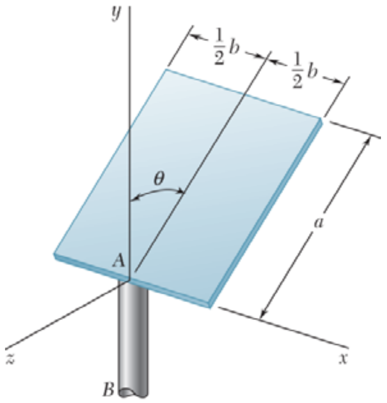
$$I_x = \frac{1}{6}\rho\pi a^4 h \ \triangleleft$$

Recall: $m = \dfrac{1}{2}\rho\pi a^2 h$; $\quad I_x = \left(\dfrac{1}{2}\rho\pi a^2 h\right)\left(\dfrac{1}{3}\right)a^2$

$$I_x = \frac{1}{3}ma^2 \ \blacktriangleleft$$

$$\left(k_x\right)^2 = \frac{I}{m} = \left(\frac{\frac{1}{3}ma^2}{m}\right) = \frac{1}{3}a^2$$

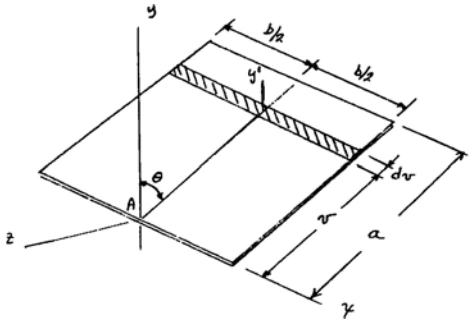
$$k_x = \frac{a}{\sqrt{3}} \ \blacktriangleleft$$



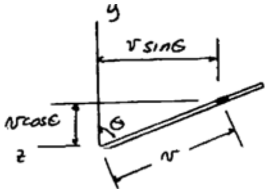
PROBLEM 9.125

A thin rectangular plate of mass *m* is welded to a vertical shaft *AB* as shown. Knowing that the plate forms an angle θ with the *y* axis, determine by direct integration the mass moment of inertia of the plate with respect to (*a*) the *y* axis, (*b*) the *z* axis.

SOLUTION



Projection on *yz* plane



Mass of plate: $m = \rho tab$

(*a*) For element shown: $dm = \rho btdv$

$$d\bar{I}_y = d\bar{I}_{y'} + (v\sin\theta)^2\,dm = \frac{1}{12}b^2dm + v^2\sin^2\theta dm$$

$$= \left(\frac{1}{12}b^2 + v^2\sin^2\theta\right)\rho btdv$$

$$I_y = \int dI_y = \rho bt\int_0^a\left(\frac{1}{12}b^2 + v^2\sin^2\theta\right)dv$$

$$= \rho bt\left|\frac{1}{12}b^2v + \frac{v^3}{3}\sin\theta\right|_0^a = \rho bt\left(\frac{1}{12}ab^2 + \frac{a^3}{3}\sin^2\theta\right)$$

$$= (\rho tab)\frac{1}{12}(b^2 + 4a^2\sin^2\theta)$$

PROBLEM 9.125 (Continued)

$$I_y = \frac{1}{12}m(b^2 + 4a^2 \sin^2 \theta) \blacktriangleleft$$

(b)

$$dI_z = d\bar{I}_{z'} + (v\cos\theta)^2 dm = \frac{1}{12}b^2 dm + v^2 \cos^2 \theta dm$$

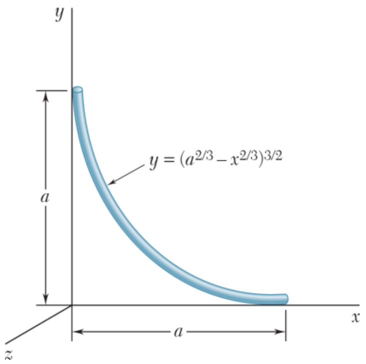
$$= \left(\frac{1}{12}b^2 + v^2 \cos^2 \theta\right)\rho btdv$$

$$I_z = \int dI_z = \rho bt\int_0^a \left(\frac{1}{12}b^2 + v^2 \cos^2 \theta\right)dv$$

$$= \rho bt\left|\frac{1}{12}b^2 v + \frac{v^3}{3}\cos^2 \theta\right|_0^a = \rho bt\left(\frac{1}{12}ab^2 + \frac{a^3}{3}\cos^2 \theta\right)$$

$$= (\rho tab)\frac{1}{12}(b^2 + 4a^2 \cos^2 \theta)$$

$$I_z = \frac{1}{12}m(b^2 + 4a^2 \cos^2 \theta) \blacktriangleleft$$



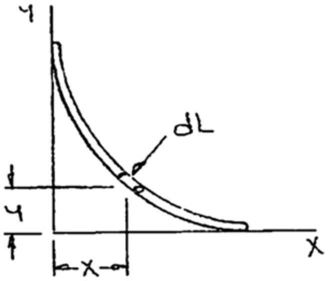
PROBLEM 9.126*

A thin steel wire is bent into the shape shown. Denoting the mass per unit length of the wire by m', determine by direct integration the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION

First note
$$\frac{dy}{dx} = -x^{-1/3}(a^{2/3} - x^{2/3})^{1/2}$$

Then
$$1+\left(\frac{dy}{dx}\right)^2 = 1 + x^{-2/3}(a^{2/3} - x^{2/3})$$
$$= \left(\frac{a}{x}\right)^{2/3}$$



For the element shown:
$$dm = m'dL = m'\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= m'\left(\frac{a}{x}\right)^{1/3} dx$$

Then
$$m = \int dm = \int_0^a m'\frac{a^{1/3}}{x^{1/3}}dx = \frac{3}{2}m'a^{1/3}\left[x^{2/3}\right]_0^a = \frac{3}{2}m'a$$

Now
$$I_x = \int y^2 dm = \int_0^a (a^{2/3} - x^{2/3})^3\left(m'\frac{a^{1/3}}{x^{1/3}}dx\right)$$
$$= m'a^{1/3}\int_0^a\left(\frac{a^2}{x^{1/3}} - 3a^{4/3}x^{1/3} + 3a^{2/3}x - x^{5/3}\right)dx$$
$$= m'a^{1/3}\left[\frac{3}{2}a^2x^{2/3} - \frac{9}{4}a^{4/3}x^{4/3} + \frac{3}{2}a^{2/3}x^2 - \frac{3}{8}x^{8/3}\right]_0^a$$
$$= m'a^3\left(\frac{3}{2} - \frac{9}{4} + \frac{3}{2} - \frac{3}{8}\right) = \frac{3}{8}m'a^3$$

or $I_x = \frac{1}{4}ma^2$ ◀

Symmetry implies $I_y = \frac{1}{4}ma^2$ ◀

PROBLEM 9.126* (Continued)

Alternative solution:

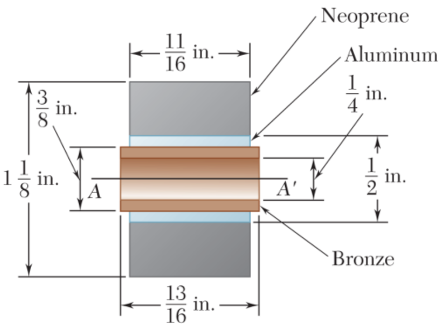
$$I_y = \int x^2 dm = \int_0^a x^2 \left(m' \frac{a^{1/3}}{x^{1/3}} dx \right) = m'a^{1/3} \int_0^a x^{5/3} dx$$

$$= m'a^{1/3} \times \frac{3}{8} \left[x^{8/3} \right]_0^a = \frac{3}{8} m'a^3$$

$$= \frac{1}{4} ma^2$$

Also $$I_z = \int (x^2 + y^2)\, dm = I_y + I_x$$

or $$I_z = \frac{1}{2} ma^2 \blacktriangleleft$$



PROBLEM 9.127

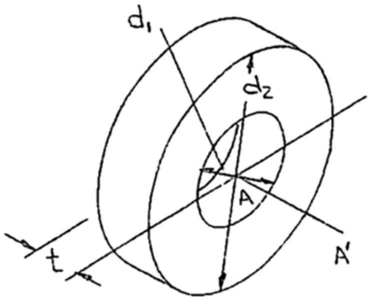
Shown is the cross section of an idler roller. Determine its mass moment of inertia and its radius of gyration with respect to the axis AA'. (The specific weight of bronze is 0.310 lb/in^3; of aluminum, 0.100 lb/in^3; and of neoprene, 0.0452 lb/in^3.)

SOLUTION

First note for the cylindrical ring shown that

$$m = \rho V = \rho t \times \frac{\pi}{4}\left(d_2^2 - d_1^2\right) = \frac{\pi}{4}\rho t\left(d_2^2 - d_1^2\right)$$

and, using Figure 9.28, that



$$\begin{aligned}
I_{AA'} &= \frac{1}{2}m_2\left(\frac{d_2}{2}\right)^2 - \frac{1}{2}m_1\left(\frac{d_1}{2}\right)^2 \\
&= \frac{1}{8}\left[\left(\rho t \times \frac{\pi}{4}d_2^2\right)d_2^2 - \left(\rho t \times \frac{\pi}{4}d_1^2\right)d_1^2\right] \\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^4 - d_1^4\right) \\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^2 - d_1^2\right)\left(d_2^2 + d_1^2\right) \\
&= \frac{1}{8}m\left(d_1^2 + d_2^2\right)
\end{aligned}$$

Now treat the roller as three concentric rings and, working from the bronze outward, we have

$$\begin{aligned}
m = \frac{\pi}{4} \times \frac{1}{32.2}\,\text{ft/s}^2 \Bigg\{ &(0.310\ \text{lb/in}^3)\left(\frac{13}{16}\text{in.}\right)\left[\left(\frac{3}{8}\right)^2 - \left(\frac{1}{4}\right)^2\right]\text{in}^2 \\
&+ (0.100\ \text{lb/in}^3)\left(\frac{11}{16}\ \text{in.}\right)\left[\left(\frac{1}{2}\right)^2 - \left(\frac{3}{8}\right)^2\right]\text{in}^2 \\
&+ (0.0452\ \text{lb/in}^3)\left(\frac{11}{16}\text{in.}\right)\left[\left(1\frac{1}{8}\right)^2 - \left(\frac{1}{2}\right)^2\right]\text{in}^2 \Bigg\} \\
= {} &(479.96 + 183.41 + 769.80) \times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft} \\
= {} &1.4332 \times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}
\end{aligned}$$

PROBLEM 9.127 (Continued)

and
$$I_{AA'} = \frac{1}{8}\left\{(479.96)\left[\left(\frac{1}{4}\right)^2 + \left(\frac{3}{8}\right)^2\right] + (183.41)\left[\left(\frac{3}{8}\right)^2 + \left(\frac{1}{2}\right)^2\right]\right.$$

$$\left. + (769.80)\left[\left(1\frac{1}{8}\right)^2 + \left(\frac{1}{2}\right)^2\right]\right\} \times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft} \times \text{in}^2 \times \frac{1\ \text{ft}^2}{144\ \text{in}^2}$$

$$= (84.628 + 62.191 + 1012.78) \times 10^{-9}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

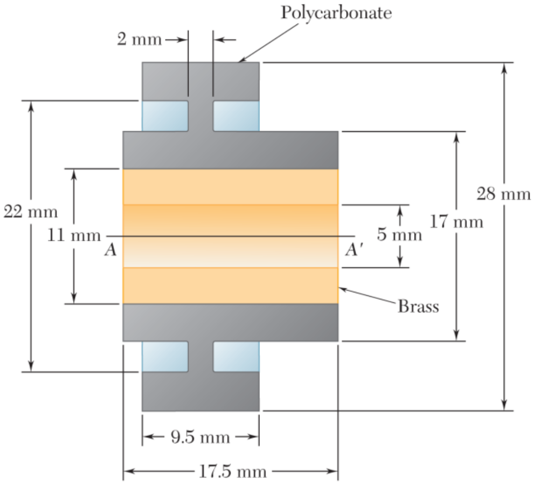
$$= 1.15960 \times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_{AA'} = 1.160 \times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

Now
$$k_{AA'}^2 = \frac{I_{AA'}}{m} = \frac{1.15960 \times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2}{1.4332 \times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}} = 809.09 \times 10^{-6}\ \text{ft}^2$$

Then
$$k_{AA'} = 28.445 \times 10^{-3}\ \text{ft}$$

or $k_{AA'} = 0.341\ \text{in.}$ ◀



PROBLEM 9.128

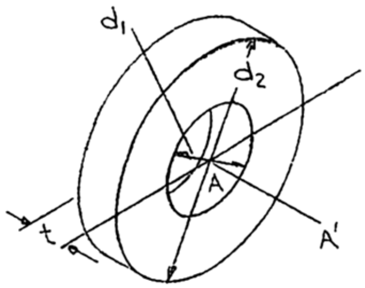
Shown is the cross section of a molded flat-belt pulley. Determine its mass moment of inertia and its radius of gyration with respect to the axis *AA*′. (The density of brass is 8650 kg/m^3 and the density of the fiber-reinforced polycarbonate used is 1250 kg/m^3.)

SOLUTION

First note for the cylindrical ring shown that

$$m = \rho V = \rho t \times \frac{\pi}{4}\left(d_2^2 - d_1^2\right)$$

and, using Figure 9.28, that



$$\begin{aligned}
I_{AA'} &= \frac{1}{2}m_2\left(\frac{d_2}{2}\right)^2 - \frac{1}{2}m_1\left(\frac{d_1}{2}\right)^2 \\
&= \frac{1}{8}\left[\left(\rho t \times \frac{\pi}{4}d_2^2\right)d_2^2 - \left(\rho t \times \frac{\pi}{4}d_1^2\right)d_1^2\right] \\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^4 - d_1^4\right) \\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^2 - d_1^2\right)\left(d_2^2 + d_1^2\right) \\
&= \frac{1}{8}m\left(d_1^2 + d_2^2\right)
\end{aligned}$$

Now treat the pulley as four concentric rings and, working from the brass outward, we have

$$\begin{aligned}
m = \frac{\pi}{4}\{&8650\text{ kg/m}^3 \times (0.0175\text{ m}) \times (0.011^2 - 0.005^2)\text{ m}^2 \\
&+ 1250\text{ kg/m}^3[(0.0175\text{ m}) \times (0.017^2 - 0.011^2)\text{ m}^2 \\
&+ (0.002\text{ m}) \times (0.022^2 - 0.017^2)\text{ m}^2 \\
&+ (0.0095\text{ m}) \times (0.028^2 - 0.022^2)\text{ m}^2]\} \\
= {}&(11.4134 + 2.8863 + 0.38288 + 2.7980) \times 10^{-3}\text{ kg} \\
= {}&17.4806 \times 10^{-3}\text{ kg}
\end{aligned}$$

PROBLEM 9.128 (Continued)

and

$$I_{AA'} = \frac{1}{8}[(11.4134)(0.005^2 + 0.011^2) + (2.8863)(0.011^2 + 0.017^2)$$

$$+ (0.38288)(0.017^2 + 0.022^2)$$

$$+ (2.7980)(0.022^2 + 0.028^2)] \times 10^{-3}\text{ kg} \times \text{m}^2$$

$$= (208.29 + 147.92 + 37.00 + 443.48) \times 10^{-9}\text{ kg}\cdot\text{m}^2$$

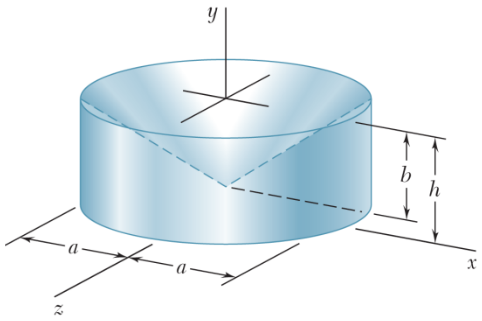
$$= 836.69 \times 10^{-9}\text{ kg}\cdot\text{m}^2$$

or $I_{AA'} = 837 \times 10^{-9}\text{ kg}\cdot\text{m}^2$ ◀

Now

$$k_{AA'}^2 = \frac{I_{AA'}}{m} = \frac{836.69 \times 10^{-9}\text{ kg}\cdot\text{m}^2}{17.4806 \times 10^{-3}\text{ kg}} = 47.864 \times 10^{-6}\text{ m}^2$$

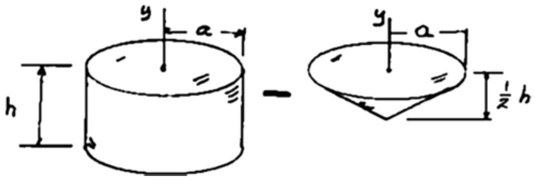
or $k_{AA'} = 6.92\text{ mm}$ ◀



PROBLEM 9.129

The machine part shown is formed by machining a conical surface into a circular cylinder. For $b = \frac{1}{2}h$, determine the mass moment of inertia and the radius of gyration of the machine part with respect to the y axis.

SOLUTION



Mass:

$$m_{\text{cyl}} = \rho\pi a^2 h \qquad m_{\text{cone}} = \frac{1}{3}\rho\pi a^2 \frac{h}{2} = \frac{1}{6}\rho\pi a^2 h$$

$$I_y: \quad I_{\text{cyl}} = \frac{1}{2}m_{\text{cyl}}a^2 \qquad I_{\text{cone}} = \frac{3}{10}m_{\text{cone}}a^2$$

$$= \frac{1}{2}\rho\pi a^4 h \qquad = \frac{1}{20}\rho\pi a^4 h$$

For entire machine part:

$$m = m_{\text{cyl}} - m_{\text{cone}} = \rho\pi a^2 h - \frac{1}{6}\rho\pi a^2 h = \frac{5}{6}\rho\pi a^2 h$$

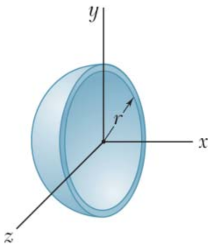
$$I_y = I_{\text{cyl}} - I_{\text{cone}} = \frac{1}{2}\rho\pi a^4 h - \frac{1}{20}\rho\pi a^4 h = \frac{9}{20}\rho\pi a^4 h$$

or

$$I_y = \left(\frac{5}{6}\rho\pi a^2 h\right)\left(\frac{6}{5}\right)\left(\frac{9}{20}\right)a^2 \qquad I_y = \frac{27}{50}ma^2 \blacktriangleleft$$

Then

$$k_y^2 = \frac{I}{m} = \frac{27}{50}a^2 \qquad k_y = 0.735a \blacktriangleleft$$



PROBLEM 9.130

Knowing that the thin hemispherical shell shown has a mass m and thickness t, determine the mass moment of inertia and the radius of gyration of the shell with respect to the x axis. (*Hint:* Consider the shell as formed by removing a hemisphere of radius r from a hemisphere of radius $r + t$; then neglect the terms containing t^2 and t^3 and keep those terms containing t.)

SOLUTION

Consider shell to be formed by removing hemisphere of radius r from hemisphere of radius $r+t$.

For a hemisphere,

$$I = \frac{2}{5}mr^2; \quad A = \frac{1}{2}\left(4\pi r^2\right) = 2\pi r^2$$

$$m = \rho V = \frac{1}{2}\rho\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\rho\pi r^3$$

Then
$$I = \frac{2}{5}\left(\frac{2}{3}\rho\pi r^3\right)r^2$$

$$= \frac{4}{15}\rho\pi r^5$$

Now following the hint, for a hemispherical shell,

$$I = \frac{4}{15}\rho\pi\left[\left(r+t\right)^5 - r^5\right]$$

$$I = \frac{4}{15}\rho\pi\left[r^5 + 5r^4t + 10r^3t^2 + \ldots - r^5\right]$$

Neglect terms with powers of $t > 1$. $I = \frac{4}{3}\rho\pi r^4 t = \left(2\rho\pi r^2\right)\frac{2}{3}r^2 t$

Mass of shell,

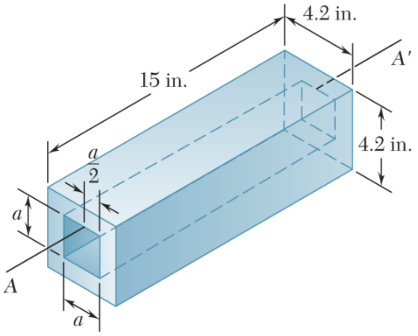
$$m = \rho V = \rho t A = \rho t\left(2\pi r^2\right) = 2\rho\pi r^2 t$$

Substituting with expression for m to simplify,

$$I = \frac{2}{3}mr^2 \blacktriangleleft$$

Radius of gyration, $k^2 = \frac{I}{m} = \frac{2}{3}r^2$

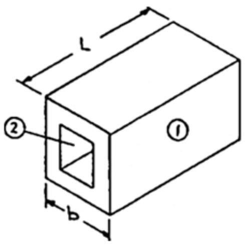
$$k = 0.816r \blacktriangleleft$$



PROBLEM 9.131

A square hole is centered in and extends through the aluminum machine component shown. Determine (*a*) the value of *a* for which the mass moment of inertia of the component with respect to the axis *AA*′, which bisects the top surface of the hole, is maximum, (*b*) the corresponding values of the mass moment of inertia and the radius of gyration with respect to the axis *AA*′. (The specific weight of aluminum is 0.100 lb/in^3.)

SOLUTION



First note $$m_1 = \rho V_1 = \rho b^2 L$$

and $$m_2 = \rho V_2 = \rho a^2 L$$

(*a*) Using Figure 9.28 and the parallel-axis theorem, we have

$$\begin{aligned}
I_{AA'} &= (I_{AA'})_1 - (I_{AA'})_2 \\
&= \left[\frac{1}{12}m_1(b^2+b^2) + m_1\left(\frac{a}{2}\right)^2\right] \\
&\quad - \left[\frac{1}{12}m_2(a^2+a^2) + m_2\left(\frac{a}{2}\right)^2\right] \\
&= (\rho b^2 L)\left(\frac{1}{6}b^2 + \frac{1}{4}a^2\right) - (\rho a^2 L)\left(\frac{5}{12}a^2\right) \\
&= \frac{\rho L}{12}(2b^4 + 3b^2a^2 - 5a^4)
\end{aligned}$$

Then $$\frac{dI_{AA'}}{da} = \frac{\rho L}{12}(6b^2 a - 20a^3) = 0$$

or $$a = 0 \qquad \text{and} \qquad a = b\sqrt{\frac{3}{10}}$$

Also $$\frac{d^2 I_{AA'}}{da^2} = \frac{\rho L}{12}(6b^2 - 60a^2) = \frac{1}{2}\rho L(b^2 - 10a^2)$$

Now for $a = 0$, $$\frac{d^2 I_{AA'}}{da^2} > 0$$

and for $a = b\sqrt{\frac{3}{10}}$, $$\frac{d^2 I_{AA'}}{da^2} < 0$$

$(I_{AA'})_{\text{max}}$ occurs when $a = b\sqrt{\frac{3}{10}}$

Then $a = (4.2\text{ in.})\sqrt{\frac{3}{10}}$

or $a = 2.30\text{ in.}$ ◀

(*b*) From part *a*:

$$(I_{AA'})_{\text{max}} = \frac{\rho L}{12}\left[2b^4 + 3b^2\left(b\sqrt{\frac{3}{10}}\right)^2 - 5\left(b\sqrt{\frac{3}{10}}\right)^4\right] = \frac{49}{240}\rho L b^4$$

$$= \frac{49}{240}\frac{\gamma_{AL}}{g}Lb^4 = \frac{49}{240}\times\frac{0.100\text{ lb/in}^3}{32.2\text{ ft/s}^2}\times(15\text{ in.})(4.2\text{ in.})^4\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

or $(I_{AA'})_{\text{max}} = 20.6\times10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

Now $k_{AA'}^2 = \frac{(I_{AA'})_{\text{max}}}{m}$

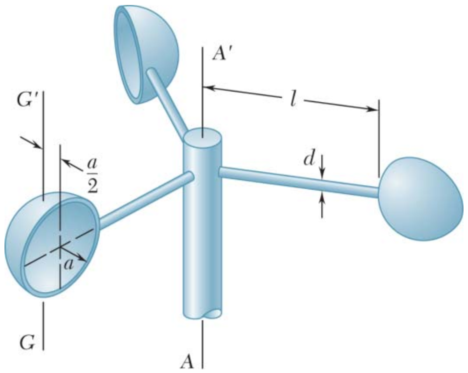
where $m = m_1 - m_2 = \rho L(b^2 - a^2)$

$$= \rho L\left[b^2 - \left(b\sqrt{\frac{3}{10}}\right)^2\right]$$

$$= \frac{7}{10}\rho L b^2$$

Then $k_{AA'}^2 = \frac{\frac{49}{240}\rho L b^4}{\frac{7}{10}\rho L b^2} = \frac{7}{24}b^2 = \frac{7}{24}(4.2\text{ in.})^2$

or $k_{AA'} = 2.27\text{ in.}$ ◀

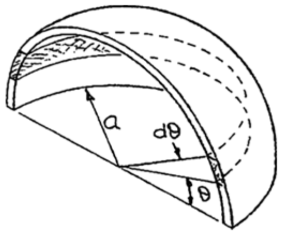


PROBLEM 9.132

The cups and the arms of an anemometer are fabricated from a material of density ρ. Knowing that the mass moment of inertia of a thin, hemispherical shell of mass m and thickness t with respect to its centroidal axis GG' is $5ma^2/12$, determine (*a*) the mass moment of inertia of the anemometer with respect to the axis AA', (*b*) the ratio of a to l for which the centroidal moment of inertia of the cups is equal to 1 percent of the moment of inertia of the cups with respect to the axis AA'.

SOLUTION

(*a*) First note
$$m_{\text{arm}} = \rho V_{\text{arm}} = \rho \times \frac{\pi}{4} d^2 l$$



and
$$dm_{\text{cup}} = \rho dV_{\text{cup}}$$
$$= \rho[(2\pi a \cos\theta)(t)(a d\theta)]$$

Then
$$m_{\text{cup}} = \int dm_{\text{cup}} = \int_0^{\pi/2} 2\pi\rho a^2 t \cos\theta d\theta$$
$$= 2\pi\rho a^2 t[\sin\theta]_0^{\pi/2}$$
$$= 2\pi\rho a^2 t$$

Now
$$(I_{AA'})_{\text{anem}} = (I_{AA'})_{\text{cups}} + (I_{AA'})_{\text{arms}}$$

Using the parallel-axis theorem and assuming the arms are slender rods, we have

$$(I_{AA'})_{\text{anem}} = 3\left[(I_{GG'})_{\text{cup}} + m_{\text{cup}} d_{AG}^2\right] + 3\left[\bar{I}_{\text{arm}} + m_{\text{arm}} d_{AG_{\text{arm}}}\right]$$
$$= 3\left\{\frac{5}{12} m_{\text{cup}} a^2 + m_{\text{cup}}\left[(l+a)^2 + \left(\frac{a}{2}\right)^2\right]\right\} + 3\left[\frac{1}{2} m_{\text{arm}} l^2 + m_{\text{arm}}\left(\frac{l}{2}\right)^2\right]$$
$$= 3m_{\text{cup}}\left(\frac{5}{3}a^2 + 2la + l^2\right) + m_{\text{arm}} l^2$$
$$= 3(2\pi\rho a^2 t)\left(\frac{5}{3}a^2 + 2la + l^2\right) + \left(\frac{\pi}{4}\rho d^2 l\right)(l^2)$$

or
$$(I_{AA'})_{\text{anem}} = \pi\rho l^2\left[6a^2 t\left(\frac{5}{3}\frac{a^2}{l^2} + 2\frac{a}{l} + 1\right) + \frac{d^2 l}{4}\right] \blacktriangleleft$$

(*b*) We have
$$\frac{(I_{GG'})_{\text{cup}}}{(I_{AA'})_{\text{cup}}} = 0.01$$

or
$$\frac{5}{12} m_{\text{cup}} a^2 = 0.01 m_{\text{cup}} \left(\frac{5}{3} a^2 + 2la + l^2 \right) \quad \text{(from part } a\text{)}$$

Now let $\zeta = \dfrac{a}{l}$.

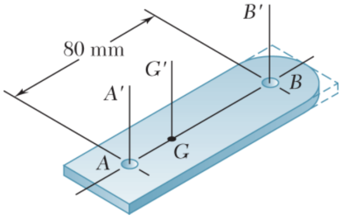
Then
$$5\zeta^2 = 0.12\left(\frac{5}{3}\zeta^2 + 2\zeta + 1 \right)$$

or
$$40\zeta^2 - 2\zeta - 1 = 0$$

Then
$$\zeta = \frac{2 \pm \sqrt{(-2)^2 - 4(40)(-1)}}{2(40)}$$

or
$$\zeta = 0.1851 \quad \text{and} \quad \zeta = -0.1351$$

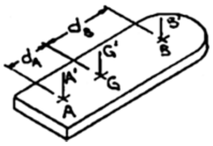
$$\frac{a}{l} = 0.1851 \blacktriangleleft$$



PROBLEM 9.133

After a period of use, one of the blades of a shredder has been worn to the shape shown and is of mass 0.18 kg. Knowing that the mass moments of inertia of the blade with respect to the *AA*′ and *BB*′ axes are $0.320\ \text{g}\cdot\text{m}^2$ and $0.680\ \text{g}\cdot\text{m}^2$, respectively, determine (*a*) the location of the centroidal axis *GG*′, (*b*) the radius of gyration with respect to axis *GG*′.

SOLUTION



(*a*) We have $d_B = (0.08 - d_A)\ \text{m}$

and, using the parallel axis

$$I_{AA'} = \overline{I}_{GG'} + md_A^2$$
$$I_{BB'} = \overline{I}_{GG'} + md_B^2$$

Then

$$\begin{aligned} I_{BB'} - I_{AA'} &= m\left(d_B^2 - d_A^2\right) \\ &= m\left[(0.08 - d_A)^2 - d_A^2\right] \\ &= m(0.0064 - 0.16d_A) \end{aligned}$$

Substituting:

$$(0.68 - 0.32)\times 10^{-3}\ \text{kg}\cdot\text{m}^2 = 0.18\ \text{kg}(0.0064 - 0.16d_A)\ \text{m}^2$$

or $d_A = 27.5\ \text{mm}$ ◀

to the right of *A*

(*b*) We have $I_{AA'} = I_{GG'} + md_A^2$

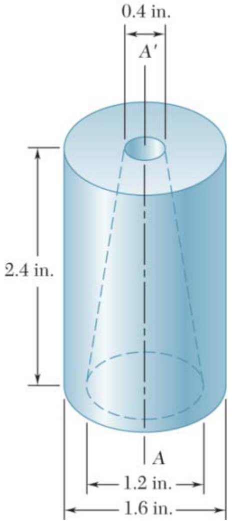
or

$$\begin{aligned} I_{GG'} &= 0.32\times 10^{-3}\ \text{kg}\cdot\text{m}^2 - 0.18\ \text{kg}\cdot(0.0275\ \text{m})^2 \\ &= 0.183875\times 10^{-3}\ \text{kg}\cdot\text{m}^2 \end{aligned}$$

Then

$$k_{GG'}^2 = \frac{I_{GG'}}{m} = \frac{0.183875\times 10^{-3}\ \text{kg}\cdot\text{m}^2}{0.18\ \text{kg}} = 1.02153\times 10^{-3}\ \text{m}^2$$

or $k_{GG'} = 32.0\ \text{mm}$ ◀

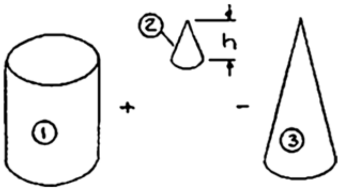


PROBLEM 9.134

Determine the mass moment of inertia of the 0.9-lb machine component shown with respect to the axis *AA*′.

SOLUTION

First note that the given shape can be formed adding a small cone to a cylinder and then removing a larger cone as indicated.



Now $$\frac{h}{0.4} = \frac{h + 2.4}{1.2} \quad \text{or} \quad h = 1.2 \text{ in.}$$

The weight of the body is given by

$$W = mg = g(m_1 + m_2 - m_3) = \rho g(V_1 + V_2 - V_3)$$

or $$0.9 \text{ lb} = \rho \times 32.2 \text{ ft/s}^2$$

$$\left[\pi(0.8)^2(2.4) + \frac{\pi}{3}(0.2)^2(1.2) - \frac{\pi}{3}(0.6)^2(3.6)\right] \text{in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$

$$= \rho \times 32.2 \text{ ft/s}^2 (2.79253 + 0.02909 - 0.78540) \times 10^{-3} \text{ ft}^3$$

or $$\rho = 13.7266 \text{ lb}\cdot\text{s}^2/\text{ft}^4$$

Then $$m_1 = (13.7266)(2.79253 \times 10^{-3}) = 0.038332 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = (13.7266)(0.02909 \times 10^{-3}) = 0.000399 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = (13.7266)(0.78540 \times 10^{-3}) = 0.010781 \text{ lb} \cdot \text{s}^2/\text{ft}$$

PROBLEM 9.134 (Continued)

Finally, using Figure 9.28, we have

$$\begin{aligned} I_{AA'} &= (I_{AA'})_1 + (I_{AA'})_2 - (I_{AA'})_3 \\ &= \frac{1}{2}m_1a_1^2 + \frac{3}{10}m_2a_2^2 - \frac{3}{10}m_3a_3^2 \\ &= \left[\frac{1}{2}(0.038332)(0.8)^2 + \frac{3}{10}(0.000399)(0.2)^2 - \frac{3}{10}(0.010781)(0.6)^2\right](\text{lb}\cdot\text{s}^2/\text{ft})\times\text{in}^2\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2 \\ &= (85.1822 + 0.0333 - 8.0858)\times 10^{-6}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2 \end{aligned}$$

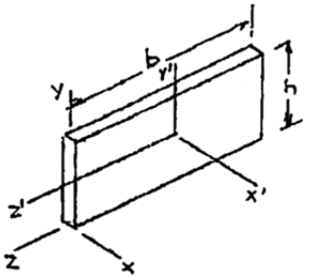
or $I_{AA'} = 77.1\times 10^{-6}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

To the instructor:

The following formulas for the mass moment of inertia of thin plates and a half cylindrical shell are derived at this time for use in the solutions of Problems 9.135 through 9.140.

Thin rectangular plate

$$\begin{aligned}(I_x)_m &= (\bar{I}_{x'})_m + md^2\\ &= \frac{1}{12}m(b^2+h^2) + m\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]\\ &= \frac{1}{3}m(b^2+h^2)\end{aligned}$$

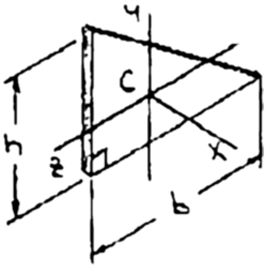


$$\begin{aligned}(I_y)_m &= (\bar{I}_{y'})_m + md^2\\ &= \frac{1}{12}mb^2 + m\left(\frac{b}{2}\right)^2 = \frac{1}{3}mb^2\end{aligned}$$

$$\begin{aligned}I_z &= (\bar{I}_{z'})_m + md^2\\ &= \frac{1}{12}mh^2 + m\left(\frac{h}{2}\right)^2 = \frac{1}{3}mh^2\end{aligned}$$

Thin triangular plate

We have $$m = \rho V = \rho\left(\frac{1}{2}bht\right)$$



and $$\bar{I}_{z,\text{area}} = \frac{1}{36}bh^3$$

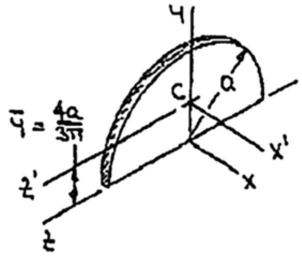
Then $$\begin{aligned}\bar{I}_{z,\text{mass}} &= \rho t I_{z,\text{area}}\\ &= \rho t \times \frac{1}{36}bh^3\\ &= \frac{1}{18}mh^2\end{aligned}$$

Similarly, $$\bar{I}_{y,\text{mass}} = \frac{1}{18}mb^2$$

Now $$\bar{I}_{x,\text{mass}} = \bar{I}_{y,\text{mass}} + \bar{I}_{z,\text{mass}} = \frac{1}{18}m(b^2+h^2)$$

Thin semicircular plate

We have $$m = \rho V = \rho\left(\frac{\pi}{2}a^2 t\right)$$



and $$\bar{I}_{y,\text{area}} = I_{z,\text{area}} = \frac{\pi}{8}a^4$$

(Continued)

Then
$$\bar{I}_{y,\text{mass}} = I_{z,\text{mass}} = \rho t \bar{I}_{y,\text{area}}$$
$$= \rho t \times \frac{\pi}{8} a^4$$
$$= \frac{1}{4} m a^2$$

Now
$$I_{x,\text{mass}} = \bar{I}_{y,\text{mass}} + I_{z,\text{mass}} = \frac{1}{2} m a^2$$

Also
$$I_{x,\text{mass}} = \bar{I}_{x',\text{mass}} + m\bar{y}^2 \quad \text{or} \quad \bar{I}_{x',\text{mass}} = m\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)a^2$$

and
$$I_{z,\text{mass}} = \bar{I}_{z',\text{mass}} + m\bar{y}^2 \quad \text{or} \quad \bar{I}_{z',\text{mass}} = m\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)a^2$$

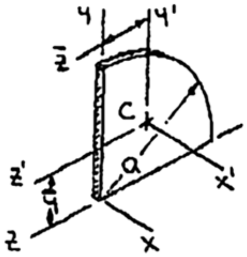
$$\bar{y} = \bar{z} = \frac{4a}{3\pi}$$

Thin Quarter-Circular Plate

We have
$$m = \rho V = \rho\left(\frac{\pi}{4} a^2 t\right)$$

and
$$I_{y,\text{area}} = I_{z,\text{area}} = \frac{\pi}{16} a^4$$

Then
$$I_{y,\text{mass}} = I_{z,\text{mass}} = \rho t I_{y,\text{area}}$$
$$= \rho t \times \frac{\pi}{16} a^4$$
$$= \frac{1}{4} m a^2$$



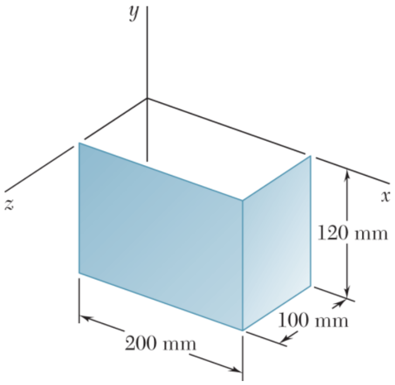
Now
$$I_{x,\text{mass}} = I_{y,\text{mass}} + I_{z,\text{mass}} = \frac{1}{2} m a^2$$

Also
$$I_{x,\text{mass}} = I_{x',\text{mass}} + m(\bar{y}^2 + \bar{z}^2)$$

or
$$\bar{I}_{x',\text{mass}} = m\left(\frac{1}{2} - \frac{32}{9\pi^2}\right)a^2$$

and
$$I_{y,\text{mass}} = I_{y',\text{mass}} + m\bar{z}^2$$

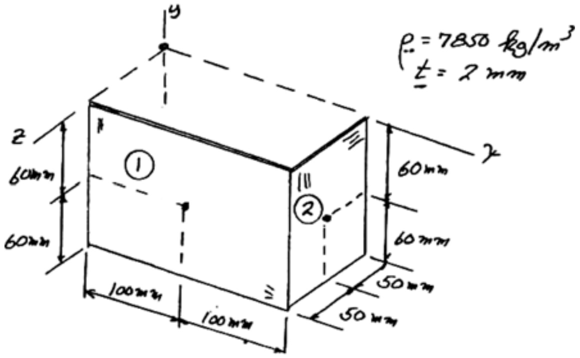
or
$$\bar{I}_{y',\text{mass}} = m\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)a^2$$



PROBLEM 9.135

A 2-mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION



$$m_1 = \rho V = 7850 \text{ kg/m}^3 (0.120 \text{ m})(0.200 \text{ m})(0.002 \text{ m}) = 0.3768 \text{ kg}$$

$$m_2 = \rho V = 7850 \text{ kg/m}^3 (0.120 \text{ m})(0.100 \text{ m})(0.002 \text{ m}) = 0.1884 \text{ kg}$$

Panel 1:

$$I_x = \bar{I}_x + md^2 = \frac{1}{12}(0.3768 \text{ kg})(0.12 \text{ m})^2 + (0.3768 \text{ kg})\left[(0.1 \text{ m})^2 + (0.06 \text{ m})^2\right]$$

$$I_x = 5.577\times10^{-3} \text{ kg}\cdot\text{m}^2$$

$$I_y = \bar{I}_y + md^2 = \frac{1}{12}(0.3768 \text{ kg})(0.2 \text{ m})^2 + (0.3768 \text{ kg})\left[(0.1 \text{ m})^2 + (0.1 \text{ m})^2\right]$$

$$I_y = 8.792\times10^{-3} \text{ kg}\cdot\text{m}^2$$

$$I_z = \bar{I}_z + md^2 = \frac{1}{12}(0.3768 \text{ kg})\left[(0.2 \text{ m})^2 + (0.12 \text{ m})^2\right] + (0.3768 \text{ kg})\left[(0.1 \text{ m})^2 + (0.06 \text{ m})^2\right]$$

$$I_z = 6.833\times10^{-3} \text{ kg}\cdot\text{m}^2$$

PROBLEM 9.135 (Continued)

Panel 2:

$$I_x = \bar{I}_x + md^2 = \frac{1}{12}(0.1884\text{ kg})\left[(0.12\text{ m})^2 + (0.1\text{ m})^2\right] + (0.1884\text{ kg})\left[(0.05\text{ m})^2 + (0.06\text{ m})^2\right]$$

$I_x = 1.532 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

$$I_y = \bar{I}_y + md^2 = \frac{1}{12}(0.1884\text{ kg})(0.1\text{ m})^2 + (0.1884\text{ kg})\left[(0.05\text{ m})^2 + (0.2\text{ m})^2\right]$$

$I_y = 8.164 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

$$I_z = \bar{I}_z + md^2 = \frac{1}{12}(0.1884\text{ kg})(0.12\text{ m})^2 + (0.1884\text{ kg})\left[(0.06\text{ m})^2 + (0.2\text{ m})^2\right]$$

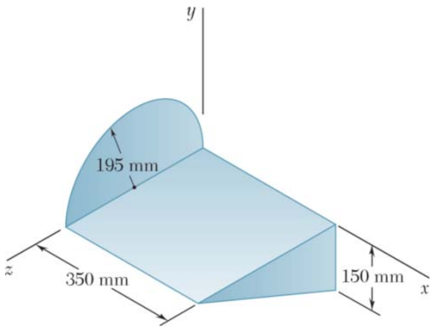
$I_z = 8.440 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

Total Mass Moments of Inertia: $I_T = I_1 + I_2$

$I_x = 7.11 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

$I_y = 16.96 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

$I_z = 15.27 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀



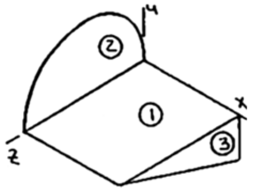
PROBLEM 9.136

A 2-mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \rho_{\text{ST}} tA$$



Then

$$m_1 = (7850\ \text{kg/m}^3)(0.002\ \text{m})(0.35 \times 0.39)\ \text{m}^2$$
$$= 2.14305\ \text{kg}$$

$$m_2 = (7850\ \text{kg/m}^3)(0.002\ \text{m})\left(\frac{\pi}{2} \times 0.195^2\right)\text{m}^2 = 0.93775\ \text{kg}$$

$$m_3 = (7850\ \text{kg/m}^3)(0.002\ \text{m})\left(\frac{1}{2} \times 0.39 \times 0.15\right)\text{m}^2 = 0.45923\ \text{kg}$$

Using Figure 9.28 for component 1 and the equations derived above for components 2 and 3, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$
$$= \left[\frac{1}{12}(2.14305\ \text{kg})(0.39\ \text{m})^2 + (2.14305\ \text{kg})\left(\frac{0.39}{2}\ \text{m}\right)^2\right]$$
$$+ \left\{\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(0.93775\ \text{kg})(0.195\ \text{m})^2 + (0.93775\ \text{kg})\left[\left(\frac{4 \times 0.195}{3\pi}\right)^2 + (0.195)^2\right]\text{m}^2\right\}$$
$$+ \left\{\frac{1}{18}(0.45923\ \text{kg})[(0.39)^2 + (0.15)^2]\ \text{m}^2 + (0.45923\ \text{kg})\left[\left(\frac{0.39}{3}\right)^2 + \left(\frac{0.15}{3}\right)^2\right]\text{m}^2\right\}$$
$$= [(0.027163 + 0.081489) + (0.011406 + 0.042081) + (0.004455 + 0.008909)]\ \text{kg}\cdot\text{m}^2$$
$$= (0.108652 + 0.053487 + 0.013364)\ \text{kg}\cdot\text{m}^2$$
$$= 0.175503\ \text{kg}\cdot\text{m}^2$$

or $I_x = 175.5 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3$$

$$= \left\{\frac{1}{12}(2.14305\text{ kg})[(0.35)^2 + (0.39)^2]\text{ m}^2 + (2.14305\text{ kg})\left[\left(\frac{0.35}{2}\right)^2 + \left(\frac{0.39}{2}\right)^2\right]\text{ m}^2\right\}$$

$$+\left[\frac{1}{4}(0.93775\text{ kg})(0.195\text{ m})^2 + (0.93775\text{ kg})(0.195\text{ m})^2\right]$$

$$+\left\{\frac{1}{18}(0.45923\text{ kg})(0.39\text{ m})^2 + (0.45923\text{ kg})\left[(0.35)^2 + \left(\frac{0.39}{3}\right)^2\right]\text{ m}^2\right\}$$

$$= [(0.049040 + 0.147120) + (0.008914 + 0.035658)$$

$$+ (0.003880 + 0.064017)]\text{ kg}\cdot\text{m}^2$$

$$= (0.196160 + 0.044572 + 0.067897)\text{ kg}\cdot\text{m}^2$$

$$= 0.308629\text{ kg}\cdot\text{m}^2$$

or $I_y = 309\times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3$$

$$= \left[\frac{1}{12}(2.14305\text{ kg})(0.35\text{ m})^2 + (2.14305\text{ kg})\left(\frac{0.35}{2}\text{ m}\right)^2\right]$$

$$+\left[\frac{1}{4}(0.93775\text{ kg})(0.195\text{ m})^2\right]$$

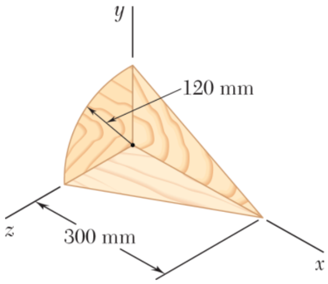
$$+\left\{\frac{1}{18}(0.45923\text{ kg})(0.15\text{ m})^2 + (0.45923\text{ kg})\left[(0.35)^2 + \left(\frac{0.15}{3}\right)^2\right]\text{ m}^2\right\}$$

$$= [(0.021877 + 0.065631) + 0.008914) + (0.000574 + 0.057404)]\text{ kg}\cdot\text{m}^2$$

$$= (0.087508 + 0.008914 + 0.057978)\text{ kg}\cdot\text{m}^2$$

$$= 0.154400\text{ kg}\cdot\text{m}^2$$

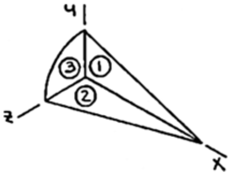
or $I_z = 154.4\times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀



PROBLEM 9.137

A subassembly for a model airplane is fabricated from three pieces of 1.5-mm plywood. Neglecting the mass of the adhesive used to assemble the three pieces, determine the mass moment of inertia of the subassembly with respect to each of the coordinate axes. (The density of the plywood is $780\ \text{kg/m}^3$.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho V = \rho t A$$

Then

$$m_1 = m_2 = (780\ \text{kg/m}^3)(0.0015\ \text{m})\left(\frac{1}{2}\times 0.3\times 0.12\right)\text{m}^2$$

$$= 21.0600\times 10^{-3}\ \text{kg}$$

$$m_3 = (780\ \text{kg/m}^3)(0.0015\ \text{m})\left(\frac{\pi}{4}\times 0.12^2\right)\text{m}^2 = 13.2324\times 10^{-3}\ \text{kg}$$

Using the equations derived above and the parallel-axis theorem, we have

$$(I_x)_1 = (I_x)_2$$

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$

$$= 2\left[\frac{1}{18}(21.0600\times 10^{-3}\ \text{kg})(0.12\ \text{m})^2 + (21.0600\times 10^{-3}\ \text{kg})\left(\frac{0.12}{3}\ \text{m}\right)^2\right]$$

$$+\left[\frac{1}{2}(13.2324\times 10^{-3}\ \text{kg})(0.12\ \text{m})^2\right]$$

$$= [2(16.8480 + 33.6960) + (95.2733)]\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$= [2(50.5440) + (95.2733)]\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

or $I_x = 196.4\times 10^{-6}\ \text{kg}\cdot\text{m}^2$ ◀

PROBLEM 9.137 (Continued)

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3$$

$$= \left[\frac{1}{18}(21.0600\times10^{-3}\text{ kg})(0.3\text{ m})^2 + (21.0600\text{ kg})\left(\frac{0.3}{3}\text{m}\right)^2\right]$$

$$+\left\{\frac{1}{18}(21.0600\times10^{-3}\text{ kg})[(0.3)^2+(0.12)^2]\,\text{m}^2\right.$$

$$\left.+(21.0600\times10^{-3}\text{ kg})\left[\left(\frac{0.3}{3}\right)^2+\left(\frac{0.12}{3}\right)^2\right]\text{m}^2\right\}$$

$$+\left[\frac{1}{4}(13.2324\times10^{-3}\text{ kg})(0.12\text{ m})^2\right]$$

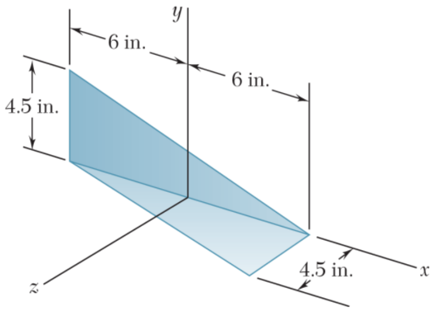
$$= [(105.300 + 210.600) + (122.148 + 244.296)$$
$$+ (47.637)]\times10^{-6}\text{ kg}\cdot\text{m}^2$$

$$= (315.900 + 366.444 + 47.637)\times10^{-6}\text{ kg}\cdot\text{m}^2$$

or $I_y = 730\times10^{-6}\text{ kg}\cdot\text{m}^2$ ◀

Symmetry implies $I_y = I_z$

$I_z = 730\times10^{-6}\text{ kg}\cdot\text{m}^2$ ◀



PROBLEM 9.138

A section of sheet steel 0.03 in. thick is cut and bent into the sheet metal machine component shown. Determine the mass moment of inertia of the component with respect to each of the coordinate axes. (The specific weight of steel is 490 lb/ft^3.)

SOLUTION

First consider one triangular plate with local axes *u*-*v*. We have

$$m = \rho V = \frac{\gamma}{g} tA = \left(\frac{490\text{ lb/ft}^3}{32.2\text{ ft/s}^2}\right)\frac{1}{2}(1\text{ ft})\left(\frac{4.5\text{ in.}}{12\text{ in./ft}}\right)\left(\frac{0.03\text{ in.}}{12\text{ in./ft}}\right) = 7.1332\times10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft}$$

$$I_{mass} = \rho t I_{area} = \frac{m}{A} I_{area}$$

Then $\bar{I}_u = \left(\frac{m}{\frac{1}{2}bh}\right)\left(\frac{1}{36}bh^3\right) = \frac{1}{18}mh^2,\qquad \bar{I}_v = \frac{1}{18}mb^2$

$$I_u = \frac{1}{18}(7.1332\times10^{-3})\left(\frac{4.5\text{ in.}}{12\text{ in./ft}}\right)^2 = 55.728\times10^{-6}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_v = \frac{1}{18}(7.1332\times10^{-3})(1\text{ ft})^2 = 396.29\times10^{-6}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

For composite area:

$$\bar{I}_x = 2\left[I_u + md^2\right] = 2\left[55.728\times10^{-6} + 7.1332\times10^{-3}\left(\frac{1.5\text{ in.}}{12\text{ in./ft}}\right)^2\right]$$

$$\bar{I}_x = 334\times10^{-6}\text{ lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$

For $\bar{I}_y$, consider plates separately.

Plate 1 in *x*-*y* plane:

$$\bar{I}_{y_1} = I_v + md^2 = 396.29\times10^{-6} + 7.1332\times10^{-3}\left(\frac{2\text{ in.}}{12\text{ in./ft}}\right)^2 = 594.43\times10^{-6}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

PROBLEM 9.138 (Continued)

Plate 2 in *x*-*z* plane:
Note: By Eqn. 9.38,

$$\bar{I}_{y'} = I_u + I_v = 55.728\times10^{-6} + 396.29\times10^{-6}$$

$$\bar{I}_{y'} = 452.02\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

Now applying the Parallel Axis Theorem,

$$\bar{I}_{y_2} = I_{y'} + md^2$$

$$\bar{I}_{y_2} = 452.02\times10^{-6} + \left(7.1332\times10^{-3}\right)\left[\left(\frac{2}{12}\right)^2 + \left(\frac{1.5}{12}\right)^2\right]$$

$$\bar{I}_{y_2} = 761.62\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

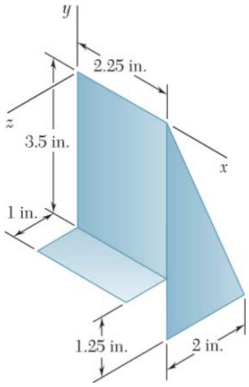
Finally, total for both plates 1 and 2 gives,

$$\bar{I}_y = \bar{I}_{y_1} + \bar{I}_{y_2} = 594.43\times10^{-6} + 761.62\times10^{-6}$$

$$I_y = 1.356\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$

By symmetry, $I_y = I_z$

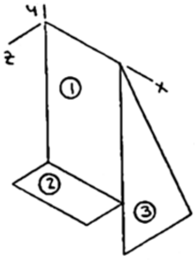
$$I_z = 1.356\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$



PROBLEM 9.139

A framing anchor is formed of 0.05-in.-thick galvanized steel. Determine the mass moment of inertia of the anchor with respect to each of the coordinate axes. (The specific weight of galvanized steel is 470 lb/ft^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho V = \frac{\gamma_{\text{G.S.}}}{g} tA$$

Then

$$m_1 = \frac{470 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times (2.25 \times 3.5) \text{ in}^2 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 3325.97 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{470 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times (2.25 \times 1) \text{ in}^2 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 950.28 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{470 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times \left(\frac{1}{2} \times 2 \times 4.75\right) \text{ in}^2 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 2006.14 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft}$$

Using Figure 9.28 for components 1 and 2 and the equations derived above for component 3, we have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 \\ &= \left[\frac{1}{12}(3325.97 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})(3.5 \text{ in.})^2 \right. \\ &\quad \left. + (3325.97 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})\left(\frac{3.5}{2} \text{ in}^2\right)\right]\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2 \\ &\quad + \left\{\frac{1}{12}(950.28 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})(1 \text{ in.})^2 \right. \\ &\quad \left. + (950.28 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})\left[(3.5)^2 + \left(\frac{1}{2}\right)^2\right] \text{ in}^2\right\}\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2 \end{aligned}$$

PROBLEM 9.139 (Continued)

$$+\left\{\frac{1}{18}(2006.14\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})[(4.75)^2+(2)^2]\ \text{in}^2\right.$$

$$\left.+(2006.14\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2}{3}\times4.75\right)^2+\left(\frac{1}{3}\times2\right)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$=[(23.578+70.735)+(0.550+82.490)$$

$$+(20.559+145.894)]\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$=(94.313+83.040+166.453)\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $\qquad I_x=344\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$I_y=(I_y)_1+(I_y)_2+(I_y)_3$$

$$=\left[\frac{1}{12}(3325.97\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})(2.25\ \text{in.})^2\right.$$

$$\left.+(3325.97\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left(\frac{2.25}{2}\ \text{in.}\right)^2\right]\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+\left\{\frac{1}{12}(950.28\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})[(2.25)^2+(1)^2]\ \text{in}^2\right.$$

$$\left.+(950.28\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2.25}{2}\right)^2+\left(\frac{1}{2}\right)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+\left\{\frac{1}{18}(2006.14\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})(2\ \text{in.})^2\right.$$

$$\left.+(2006.14\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[(2.25)^2+\left(\frac{1}{3}\times2\right)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$=[(9.744+29.232)+(3.334+10.002)$$

$$+(3.096+76.720)]\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$=(38.976+13.336+79.816)\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $\qquad I_y=132.1\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM 9.139 (Continued)

$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3$$

$$= \left\{ \frac{1}{12}(3325.97\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})[(2.25)^2 + (3.5)^2]\ \text{in}^2 \right.$$

$$\left. + (3325.97\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2.25}{2}\right)^2 + \left(\frac{3.5}{2}\right)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+ \left\{ \frac{1}{12}(950.28\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})(2.25\ \text{in.})^2 \right.$$

$$\left. + (950.28\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2.25}{2}\right)^2 + (3.5)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+ \left\{ \frac{1}{18}(2006.14\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})(4.75\ \text{in.})^2 \right.$$

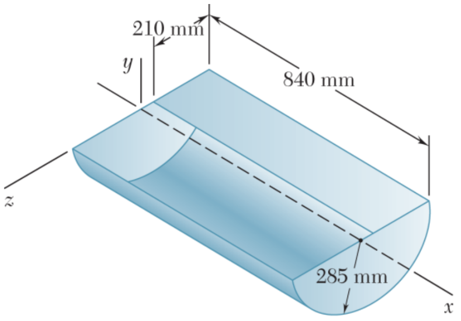
$$\left. + (2006.14\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[(2.25)^2 + \left(\frac{2}{3}\times4.75\right)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$= [(33.322 + 99.967) + (2.784 + 89.192)$$

$$+ (17.463 + 210.231)]\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= (133.289 + 91.976 + 227.694)\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

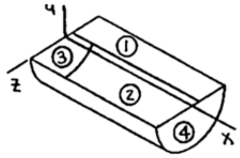
or $I_z = 453\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 9.140*

A farmer constructs a trough by welding a rectangular piece of 2-mm-thick sheet steel to half of a steel drum. Knowing that the density of steel is $7850\ \text{kg/m}^3$ and that the thickness of the walls of the drum is 1.8 mm, determine the mass moment of inertia of the trough with respect to each of the coordinate axes. Neglect the mass of the welds.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \rho_{\text{ST}} t A$$

Then

$$\begin{aligned} m_1 &= (7850\ \text{kg/m}^3)(0.002\ \text{m})(0.84 \times 0.21)\ \text{m}^2 \\ &= 2.76948\ \text{kg} \end{aligned}$$

$$\begin{aligned} m_2 &= (7850\ \text{kg/m}^3)(0.0018\ \text{m})(\pi \times 0.285 \times 0.84)\ \text{m}^2 \\ &= 10.62713\ \text{kg} \end{aligned}$$

$$\begin{aligned} m_3 = m_4 &= (7850\ \text{kg/m}^3)(0.0018\ \text{m})\left(\frac{\pi}{2} \times 0.285^2\right)\text{m}^2 \\ &= 1.80282\ \text{kg} \end{aligned}$$

Using Figure 9.28 for component 1 and the equations derived above for components 2 through 4, we have

$$\begin{aligned} (I_x)_3 &= (I_x)_4 \\ I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 \\ &= \left[\frac{1}{12}(2.76948\ \text{kg})(0.21\ \text{m})^2 + (2.76948\ \text{kg})\left(0.285 - \frac{0.21}{2}\right)^2 \text{m}^2\right] \\ &\quad + [(10.62713\ \text{kg})(0.285\ \text{m})^2] + 2\left[\frac{1}{2}(1.80282\ \text{kg})(0.285\ \text{m})^2\right] \\ &= [(0.01018 + 0.08973) + (0.86319) + 2(0.07322)]\ \text{kg}\cdot\text{m}^2 \\ &= [(0.09991 + 0.86319 + 2(0.07322)]\ \text{kg}\cdot\text{m}^2 \end{aligned}$$

or $I_x = 1.110\ \text{kg}\cdot\text{m}^2$ ◀

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4$$

$$= \left\{ \frac{1}{12}(2.76948\text{ kg})[(0.84)^2 + (0.21)^2]\text{ m}^2 \right.$$

$$\left. + (2.76948\text{ kg})\left[\left(\frac{0.84}{2}\right)^2 + \left(0.285 - \frac{0.21}{2}\right)^2\right]\text{m}^2 \right\}$$

$$+ \left\{ \frac{1}{12}(10.62713\text{ kg})[(0.84)^2 + 6(0.285)^2]\text{ m}^2 + (10.62713\text{ kg})\left(\frac{0.84}{2}\text{ m}\right)^2 \right\}$$

$$+ \left[\frac{1}{4}(1.80282\text{ kg})(0.285\text{ m})^2\right]$$

$$+ \left[\frac{1}{4}(1.80282\text{ kg})(0.285\text{ m})^2 + (1.80282\text{ kg})(0.84\text{ m})^2\right]$$

$$= [(0.17302 + 0.57827) + (1.05647 + 1.87463)$$

$$+ (0.03661) + (0.03661 + 1.27207)]\text{ kg}\cdot\text{m}^2$$

$$= (0.75129 + 2.93110 + 0.03661 + 1.30868)\text{ kg}\cdot\text{m}^2$$

or $I_y = 5.03\text{ kg}\cdot\text{m}^2$ ◀

$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3 + (I_z)_4$$

$$= \left[\frac{1}{12}(2.76948\text{ kg})(0.84\text{ m})^2 + (2.76948\text{ kg})\left(\frac{0.84}{2}\text{ m}\right)^2\right]$$

$$+ \left\{ \frac{1}{12}(10.62713\text{ kg})[(0.84)^2 + 6(0.285)^2]\text{ m}^2 + (10.62713\text{ kg})\left(\frac{0.84}{2}\text{ m}\right)^2 \right\}$$

$$+ \left[\frac{1}{4}(1.80282\text{ kg})(0.285\text{ m})^2\right]$$

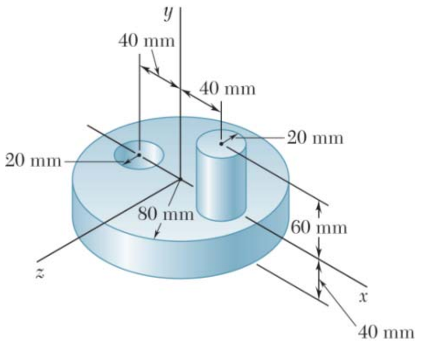
$$+ \left[\frac{1}{4}(1.80282\text{ kg})(0.285\text{ m})^2 + (1.80282\text{ kg})(0.84\text{ m})^2\right]$$

$$= [(0.16285 + 0.48854) + (1.05647 + 1.87463)$$

$$+ (0.03661) + (0.03661 + 1.27207)]\text{ kg}\cdot\text{m}^2$$

$$= (0.65139 + 2.93110 + 0.03661 + 1.30868)\text{ kg}\cdot\text{m}^2$$

or $I_z = 4.93\text{ kg}\cdot\text{m}^2$ ◀

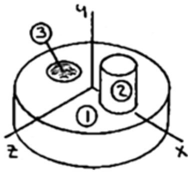


PROBLEM 9.141

The machine element shown is fabricated from steel. Determine the mass moment of inertia of the assembly with respect to (*a*) the x axis, (*b*) the y axis, (*c*) the z axis. (The density of steel is 7850 kg/m^3.)

SOLUTION

First compute the mass of each component. We have



$$m = \rho_{\text{ST}} V$$

Then

$$m_1 = (7850 \text{ kg/m}^3)(\pi(0.08 \text{ m})^2(0.04 \text{ m})]$$
$$= 6.31334 \text{ kg}$$

$$m_2 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.06 \text{ m})] = 0.59188 \text{ kg}$$

$$m_3 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.04 \text{ m})] = 0.39458 \text{ kg}$$

Using Figure 9.28 and the parallel-axis theorem, we have

(*a*)
$$I_x = (I_x)_1 + (I_x)_2 - (I_x)_3$$
$$= \left\{\frac{1}{12}(6.31334 \text{ kg})[3(0.08)^2 + (0.04)^2] \text{ m}^2 + (6.31334 \text{ kg})(0.02 \text{ m})^2\right\}$$
$$+\left\{\frac{1}{12}(0.59188 \text{ kg})[3(0.02)^2 + (0.06)^2] \text{ m}^2 + (0.59188 \text{ kg})(0.03 \text{ m})^2\right\}$$
$$-\left\{\frac{1}{12}(0.39458 \text{ kg})[3(0.02)^2 + (0.04)^2] \text{ m}^2 + (0.39458 \text{ kg})(0.02 \text{ m})^2\right\}$$
$$= [(10.94312 + 2.52534) + (0.23675 + 0.53269)$$
$$-(0.09207 + 0.15783)] \times 10^{-3} \text{ kg}\cdot\text{m}^2$$
$$= (13.46846 + 0.76944 - 0.24990) \times 10^{-3} \text{ kg}\cdot\text{m}^2$$
$$= 13.98800 \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

or $I_x = 13.99 \times 10^{-3} \text{ kg}\cdot\text{m}^2$ ◀

PROBLEM 9.141 (Continued)

(*b*)

$$I_y = (I_y)_1 + (I_y)_2 - (I_y)_3$$

$$= \left[\frac{1}{2}(6.31334\text{ kg})(0.08\text{ m})^2\right]$$

$$+ \left[\frac{1}{2}(0.59188\text{ kg})(0.02\text{ m})^2 + (0.59188\text{ kg})(0.04\text{ m})^2\right]$$

$$- \left[\frac{1}{2}(0.39458\text{ kg})(0.02\text{ m}^2) + (0.39458\text{ kg})(0.04\text{ m})^2\right]$$

$$= [(20.20269) + (0.11838 + 0.94701)$$

$$- (0.07892 + 0.63133)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (20.20269 + 1.06539 - 0.71025) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= 20.55783 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_y = 20.6 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

(*c*)

$$I_z = (I_z)_1 + (I_z)_2 - (I_z)_3$$

$$= \left\{\frac{1}{12}(6.31334\text{ kg})[3(0.08)^2 + (0.04)^2]\text{ m}^2 + (6.31334\text{ kg})(0.02\text{ m})^2\right\}$$

$$+ \left\{\frac{1}{12}(0.59188\text{ kg})[3(0.02)^2 + (0.06)^2]\text{ m}^2 + (0.59188\text{ kg})[(0.04)^2 + (0.03)^2]\text{ m}^2\right\}$$

$$- \left\{\frac{1}{12}(0.39458\text{ kg})[3(0.02)^2 + (0.04)^2]\text{ m}^2 + (0.03958\text{ kg})[(0.04)^2 + (0.02)^2]\text{ m}^2\right\}$$

$$= [(10.94312 + 2.52534) + (0.23675 + 1.47970)$$

$$- (0.09207 + 0.78916)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (13.46846 + 1.71645 - 0.88123) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

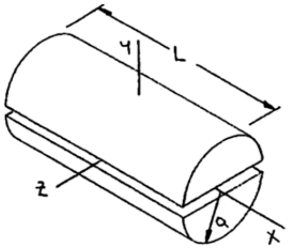
$$= 14.30368 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_z = 14.30 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

To the Instructor:

The following formulas for the mass of inertia of a semicylinder are derived at this time for use in the solutions of Problems 9.142 through 9.145.

From Figure 9.28:



Cylinder
$$(I_x)_{\text{cyl}} = \frac{1}{2} m_{\text{cyl}} a^2$$

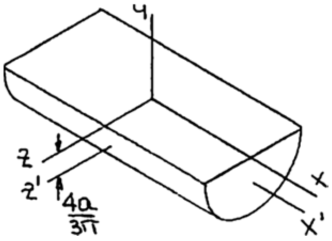
$$(I_y)_{\text{cyl}} = (I_z)_{\text{cyl}} = \frac{1}{12} m_{\text{cyl}} (3a^2 + L^2)$$

Symmetry and the definition of the mass moment of inertia $(I = \int r^2 dm)$ imply

$$(I)_{\text{semicylinder}} = \frac{1}{2}(I)_{\text{cylinder}}$$

$$(I_x)_{\text{sc}} = \frac{1}{2}\left(\frac{1}{2} m_{\text{cyl}} a^2\right)$$

and
$$(I_y)_{\text{sc}} = (I_z)_{\text{sc}} = \frac{1}{2}\left[\frac{1}{12} m_{\text{cyl}} (3a^2 + L^2)\right]$$



However,
$$m_{\text{sc}} = \frac{1}{2} m_{\text{cyl}}$$

Thus,
$$(I_x)_{\text{sc}} = \frac{1}{2} m_{\text{sc}} a^2$$

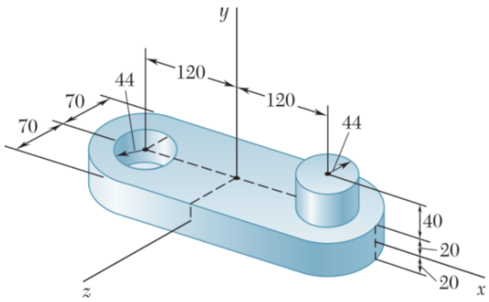
and
$$(I_y)_{\text{sc}} = (I_z)_{\text{sc}} = \frac{1}{12} m_{\text{sc}} (3a^2 + L^2)$$

Also, using the parallel axis theorem find

$$\bar{I}_{x'} = m_{\text{sc}}\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)a^2$$

$$\bar{I}_{z'} = m_{\text{sc}}\left[\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)a^2 + \frac{1}{12}L^2\right]$$

where x' and z' are centroidal axes.

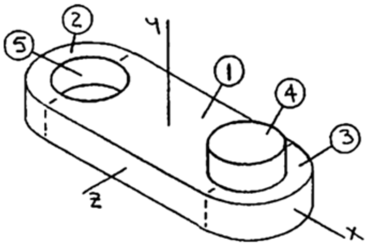


Dimensions in mm

PROBLEM 9.142

Determine the mass moments of inertia and the radii of gyration of the steel machine element shown with respect to the *x* and *y* axes. (The density of steel is 7850 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V$$

Then

$$m_1 = (7850\text{ kg/m}^3)(0.24 \times 0.04 \times 0.14)\text{ m}^3$$
$$= 10.5504\text{ kg}$$

$$m_2 = m_3 = (7850\text{ kg/m}^3)\left[\frac{\pi}{2}(0.07)^2 \times 0.04\right]\text{m}^3 = 2.41683\text{ kg}$$

$$m_4 = m_5 = (7850\text{ kg/m}^3)[\pi(0.044)^2 \times (0.04)]\text{ m}^3 = 1.90979\text{ kg}$$

Using Figure 9.28 for components 1, 4, and 5 and the equations derived above (before the solution to Problem 9.144) for a semicylinder, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 - (I_x)_5 \quad \text{where} \quad (I_x)_2 = (I_x)_3$$
$$= \left[\frac{1}{12}(10.5504\text{ kg})(0.04^2 + 0.14^2)\text{ m}^2\right] + 2\left\{\frac{1}{12}(2.41683\text{ kg})[3(0.07\text{ m})^2 + (0.04\text{ m})^2]\right\}$$
$$+ \left\{\frac{1}{12}(1.90979\text{ kg})[3(0.044\text{ m})^2 + (0.04\text{ m})^2] + (1.90979\text{ kg})(0.04\text{ m})^2\right\}$$
$$- \left\{\frac{1}{12}(1.90979\text{ kg})[3(0.044\text{ m})^2 + (0.04\text{ m})^2]\right\}$$
$$= [(0.0186390) + 2(0.0032829) + (0.0011790 + 0.0030557) - (0.0011790)]\text{ kg}\cdot\text{m}^2$$
$$= 0.0282605\text{ kg}\cdot\text{m}^2$$

or $I_x = 28.3 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

PROBLEM 9.142 (Continued)

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 - (I_y)_5$$

where $$(I_y)_2 = (I_y)_3 \quad (I_y)_4 = |(I_y)_5|$$

Then $$I_y = \left[\frac{1}{12}(10.5504\text{ kg})(0.24^2 + 0.14^2)\text{ m}^2\right]$$
$$+2\left[(2.41683\text{ kg})\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(0.07\text{ m}^2) + (2.41683\text{ kg})\left(0.12 + \frac{4\times 0.07}{3\pi}\right)^2\text{ m}^2\right]$$
$$= [(0.0678742) + 2(0.0037881 + 0.0541678)]\text{ kg}\cdot\text{m}^2$$
$$= 0.1837860\text{ kg}\cdot\text{m}^2$$

or $I_y = 183.8\times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

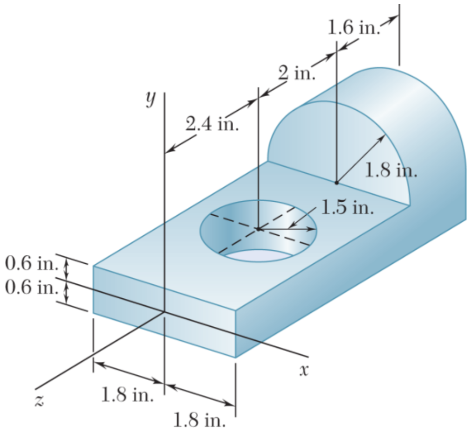
Also $$m = m_1 + m_2 + m_3 + m_4 - m_5 \text{ where } m_2 = m_3,\ m_4 = |m_5|$$
$$= (10.5504 + 2\times 2.41683)\text{ kg} = 15.38406\text{ kg}$$

Then $$k_x^2 = \frac{I_x}{m} = \frac{0.0282605\text{ kg}\cdot\text{m}^2}{15.38406\text{ kg}}$$

or $k_x = 42.9\text{ mm}$ ◀

and $$k_y^2 = \frac{I_y}{m} = \frac{0.1837860\text{ kg}\cdot\text{m}^2}{15.38406\text{ kg}}$$

or $k_y = 109.3\text{ mm}$ ◀

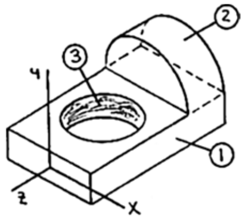


PROBLEM 9.143

Determine the mass moment of inertia of the steel machine element shown with respect to the *x* axis. (The density of steel is 490 lb/ft^3.)

SOLUTION

First compute the mass of each component. We have



$$\rho = \frac{\gamma}{g} = \frac{490\ \text{lb}\cdot\text{s}^2}{32.2\ \text{ft}}; \qquad m = \rho V$$

Then

$$m_1 = \rho\left(\frac{3.6\times 1.2\times 6}{12^3}\right)\text{ft}^3$$

$$= 15\times 10^{-3}\rho$$

$$m_2 = \rho\left[\frac{\pi}{2}\left(\frac{1.8}{12}\right)^2\left(\frac{1.6}{12}\right)\right]\text{ft}^3$$

$$= 4.7124\times 10^{-3}\rho$$

$$m_3 = \rho\left[\pi\left(\frac{1.5}{12}\right)^2\left(\frac{1.2}{12}\right)\right]\text{ft}^3 = 4.9087\times 10^{-3}\rho$$

Using Figure 9.28 for components 1 and 3 and the equation derived above (before the solution to Problem 9.142) for a semicylinder, we have

$$I_x = (I_x)_1 + (I_x)_2 - (I_x)_3$$

$$I_{x_1} = \left[\frac{1}{12}\left(15\times 10^{-3}\rho\right)\left(\left(\frac{1.2}{12}\right)^2 + \left(\frac{6}{12}\right)^2\right) + \left(15\times 10^{-3}\rho\right)\left(\frac{3}{12}\right)^2\right]$$

$$I_{x_1} = 1.26250\times 10^{-3}\rho$$

Half-cylinder, centroidal, $I_{x_{2''}} = \frac{m}{12}\left(3r^2 + L^2\right) = \frac{4.7124\times 10^{-3}\rho}{12}\left(3\left(\frac{1.8}{12}\right)^2 + \left(\frac{1.6}{12}\right)^2\right) = 3.3489\times 10^{-5}\rho$

PROBLEM 9.143 (Continued)

Half-cylinder about base,

$$I_{x_{2''}} = \bar{I}_{x_{2'}} + md^2$$

$$3.3489\times10^{-5}\rho = \bar{I}_{x_{2'}} + 4.7124\times10^{-3}\rho\left(\frac{0.7639}{12}\right)^2$$

$$\bar{I}_{x_{2'}} = 1.43925\times10^{-5}\rho$$

$$I_{x_2} = \bar{I}_{x_{2'}} + md^2$$

$$I_{x_2} = 1.43925\times10^{-5}\rho + 4.7124\times10^{-3}\rho\left[\left(\frac{5.2}{12}\right)^2 + \left(\frac{0.6+0.7639}{12}\right)^2\right]$$

$$I_{x_2} = 9.6015\times10^{-4}\rho$$

$$I_{x_3} = \frac{m}{12}\left(3r^2 + L^2\right) + md^2$$

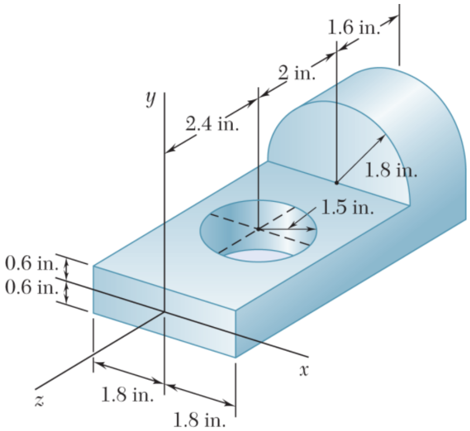
$$I_{x_3} = \frac{4.9087\times10^{-3}\rho}{12}\left[3\left(\frac{1.5}{12}\right)^2 + \left(\frac{1.2}{12}\right)^2\right] + 4.9087\times10^{-3}\rho\left(\frac{2.4}{12}\right)^2$$

$$I_{x_3} = 2.1961\times10^{-4}\rho$$

Now combining all components and substituting for ρ,

$$I_x = \frac{490}{32.2}\left(1.26250\times10^{-3} + 9.6015\times10^{-4} - 2.1961\times10^{-4}\right)$$

or $I_x = 30.5\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

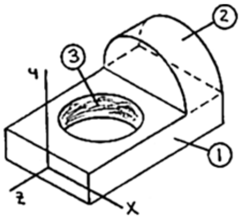


PROBLEM 9.144

Determine the mass moment of inertia of the steel machine element shown with respect to the *y* axis. (The density of steel is 490 lb/ft^3.)

SOLUTION

First compute the mass of each component. We have



$$\rho = \frac{\gamma}{g} = \frac{490 \text{ lb}\cdot\text{s}^2}{32.2 \text{ ft}}; \qquad m = \rho V$$

Then

$$m_1 = \rho\left(\frac{3.6\times 1.2\times 6}{12^3}\right)\text{ ft}^3$$

$$= 15\times 10^{-3}\rho$$

$$m_2 = \rho\left[\frac{\pi}{2}\left(\frac{1.8}{12}\right)^2\left(\frac{1.6}{12}\right)\right]\text{ ft}^3$$

$$= 4.7124\times 10^{-3}\rho$$

$$m_3 = \rho\left[\pi\left(\frac{1.5}{12}\right)^2\left(\frac{1.2}{12}\right)\right]\text{ ft}^3 = 4.9087\times 10^{-3}\rho$$

Using Figure 9.28 for components 1 and 3 and the equation derived above (before the solution to Problem 9.142) for a semicylinder, we have

$$I_y = (I_y)_1 + (I_y)_2 - (I_y)_3$$

$$I_{y_1} = \left[\frac{1}{12}(15\times 10^{-3}\rho)\left(\left(\frac{3.6}{12}\right)^2 + \left(\frac{6}{12}\right)^2\right) + (15\times 10^{-3}\rho)\left(\frac{3}{12}\right)^2\right]$$

$$I_{y_1} = 1.36250\times 10^{-3}\rho$$

PROBLEM 9.144 (Continued)

$$I_{y_2} = \bar{I}_{y_{2'}} + md^2$$

$$I_{y_2} = \frac{4.7124\times10^{-3}\rho}{12}\left(3\left(\frac{1.8}{12}\right)^2 + \left(\frac{1.6}{12}\right)^2\right) + 4.7124\times10^{-3}\rho\left(\frac{5.2}{12}\right)^2$$

$$I_{y_2} = 9.1837\times10^{-4}\rho$$

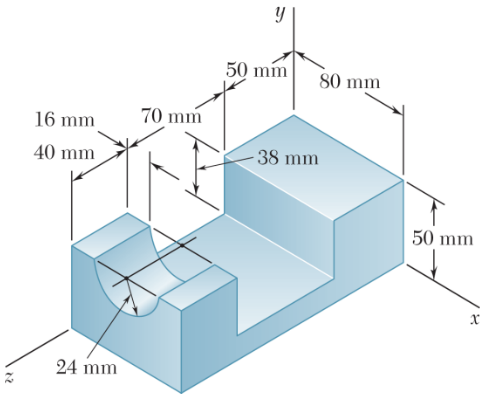
$$I_{y_3} = \frac{4.9087\times10^{-3}\rho}{2}\left(\frac{1.5}{12}\right)^2 + 4.9087\times10^{-3}\rho\left(\frac{2.4}{12}\right)^2$$

$$I_{y_3} = 2.3470\times10^{-4}\rho$$

Now combining all components and substituting for ρ,

$$I_y = \frac{490}{32.2}\left(1.36250\times10^{-3} + 9.1837\times10^{-4} - 2.3470\times10^{-4}\right)$$

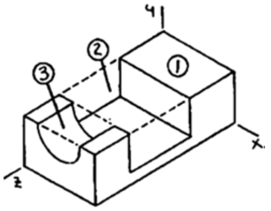
or $I_y = 31.1\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 9.145

Determine the mass moment of inertia of the steel fixture shown with respect to (*a*) the *x* axis, (*b*) the *y* axis, (*c*) the *z* axis. (The density of steel is 7850 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\mathrm{ST}} V$$

Then

$$m_1 = 7850\ \mathrm{kg/m^3} \times (0.08 \times 0.05 \times 0.160)\ \mathrm{m^3}$$
$$= 5.02400\ \mathrm{kg}$$

$$m_2 = 7850\ \mathrm{kg/m^3} \times (0.08 \times 0.038 \times 0.07)\ \mathrm{m^3} = 1.67048\ \mathrm{kg}$$

$$m_3 = 7850\ \mathrm{kg/m^3} \times \left(\frac{\pi}{2} \times 0.024^2 \times 0.04\right) \mathrm{m^3} = 0.28410\ \mathrm{kg}$$

Using Figure 9.28 for components 1 and 2 and the equations derived above for component 3, we have

(*a*) $I_x = (I_x)_1 - (I_x)_2 - (I_x)_3$

$$= \left\{\frac{1}{12}(5.02400\ \mathrm{kg})[(0.05)^2 + (0.16)^2]\ \mathrm{m^2} + (5.02400\ \mathrm{kg})\left[\left(\frac{0.05}{2}\right)^2 + \left(\frac{0.16}{2}\right)^2\right]\mathrm{m^2}\right\}$$

$$- \left\{\frac{1}{12}(1.67048\ \mathrm{kg})[(0.038)^2 + (0.07)^2]\ \mathrm{m^2} + (1.67048\ \mathrm{kg})\left[\left(0.05 - \frac{0.038}{2}\right)^2 + \left(0.05 + \frac{0.07}{2}\right)^2\right]\mathrm{m^2}\right\}$$

$$- \left\{(0.28410\ \mathrm{kg})\left[\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)(0.024)^2 + \frac{1}{12}(0.04)^2\right]\mathrm{m^2}\right.$$

$$\left. + (0.28410\ \mathrm{kg})\left[\left(0.05 - \frac{4 \times 0.024}{3\pi}\right)^2 + \left(0.16 - \frac{0.04}{2}\right)^2\right]\mathrm{m^2}\right\}$$

$$= [(11.7645 + 35.2936) - (0.8831 + 13.6745) - (0.0493 + 6.0187)] \times 10^{-3}\ \mathrm{kg \cdot m^2}$$

$$= (47.0581 - 14.5576 - 6.0680) \times 10^{-3}\ \mathrm{kg \cdot m^2}$$

$$= 26.4325 \times 10^{-3}\ \mathrm{kg \cdot m^2}$$

or $I_x = 26.4 \times 10^{-3}\ \mathrm{kg \cdot m^2}$ ◀

PROBLEM 9.145 (Continued)

(*b*) $$I_y = (I_y)_1 - (I_y)_2 - (I_y)_3$$

$$= \left\{\frac{1}{12}(5.02400\text{ kg})[(0.08)^2 + (0.16)^2]\text{ m}^2 + (5.02400\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(\frac{0.16}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(1.67048\text{ kg})[(0.08)^2 + (0.07)^2]\text{ m}^2 + (1.67048\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 + \frac{0.07}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(0.28410\text{ kg})[3(0.024)^2 + (0.04)^2]\text{ m}^2 + (0.28410\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.16 - \frac{0.04}{2}\right)^2\right]\text{m}^2\right\}$$

$$= [(13.3973 + 40.1920) - (1.5730 + 14.7420) - (0.0788 + 6.0229)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (53.5893 - 16.3150 - 6.1017) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= 31.1726 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_y = 31.2 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

(*c*) $$I_z = (I_z)_1 - (I_z)_2 - (I_z)_3$$

$$= \left\{\frac{1}{12}(5.02400\text{ kg})[(0.08)^2 + (0.05)^2]\text{ m}^2 + (5.02400\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(\frac{0.05}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(1.67048\text{ kg})[(0.08)^2 + (0.038)^2]\text{ m}^2 + (1.67048\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 - \frac{0.038}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{(0.28410\text{ kg})\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(0.024\text{ m})^2 + (0.28410\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 - \frac{4\times 0.024}{3\pi}\right)^2\right]\text{m}^2\right\}$$

$$= [(3.7261 + 11.1784) - (1.0919 + 4.2781) - (0.0523 + 0.9049)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (14.9045 - 5.3700 - 0.9572) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

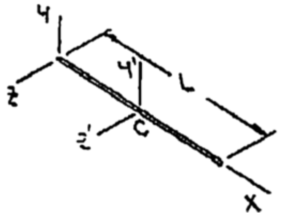
$$= 8.5773 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_z = 8.58 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

To the instructor:

The following formulas for the mass moment of inertia of wires are derived or summarized at this time for use in the solutions of Problems 9.146 through 9.148.

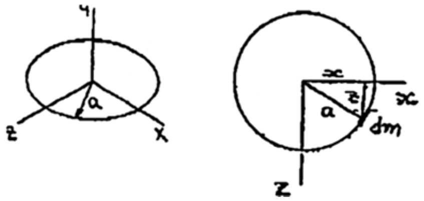
Slender Rod



$$I_x = 0 \qquad \bar{I}_{y'} = \bar{I}_{z'} = \frac{1}{12} mL^2 \text{ (Figure 9.28)}$$

$$I_y = I_z = \frac{1}{3} mL^2 \text{ (Sample Problem 9.9)}$$

Circle



We have $\bar{I}_y = \int r^2 dm = ma^2$

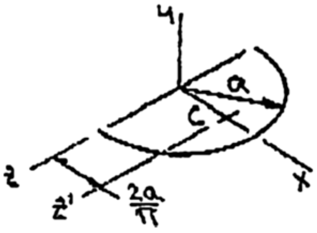
Now $\bar{I}_y = \bar{I}_x + \bar{I}_z$

And symmetry implies $\bar{I}_x = \bar{I}_z$

$$\bar{I}_x = \bar{I}_z = \frac{1}{2} ma^2$$

Semicircle

Following the above arguments for a circle, We have

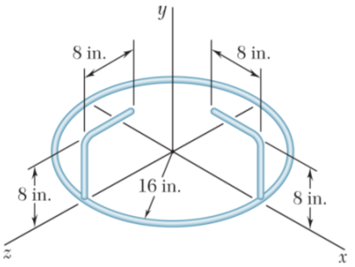


$$\bar{I}_x = I_z = \frac{1}{2} ma^2 \qquad I_y = ma^2$$

Using the parallel-axis theorem

$$I_z = \bar{I}_{z'} + m\bar{x}^2 \qquad x = \frac{2a}{\pi}$$

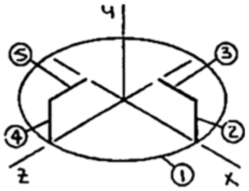
or $I_{z'} = m\left(\frac{1}{2} - \frac{4}{\pi^2}\right)a^2$



PROBLEM 9.146

Aluminum wire with a weight per unit length of 0.033 lb/ft is used to form the circle and the straight members of the figure shown. Determine the mass moment of inertia of the assembly with respect to each of the coordinate axes.

SOLUTION



First compute the mass of each component. We have

$$m=\frac{W}{g}=\frac{1}{g}\left(\frac{W}{L}\right)_{\text{AL}}L$$

Then

$$m_1=\frac{1}{32.2\ \text{ft/s}^2}\times 0.033\ \text{lb/ft}\times(2\pi\times 16\ \text{in.})\times\frac{1\ \text{ft}}{12\ \text{in.}}$$

$$=8.5857\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$m_2=m_3=m_4=m_5=\frac{1}{32.2\ \text{ft/s}^2}\times 0.033\ \text{lb/ft}\times 8\ \text{in.}\times\frac{1\ \text{ft}}{12\ \text{in.}}$$

$$=0.6832\ \text{lb}\cdot\text{s}^2/\text{ft}$$

Using the equations given above and the parallel-axis theorem, we have

$$I_x=(I_x)_1+(I_x)_2+(I_x)_3+(I_x)_4+(I_x)_5$$

$$=\left[\frac{1}{2}(8.5857\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(16\ \text{in.})^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+\left[\frac{1}{3}(0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(8\ \text{in.})^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+[0+(0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(8\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+\left[\frac{1}{12}(0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(8\ \text{in.})^2+(0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(4^2+16^2)\ \text{in}^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+\left[\frac{1}{12}(0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(8\ \text{in.})^2+(0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(8^2+12^2)\ \text{in}^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$=[(7.6315)+(0.1012)+(0.3036)+(0.0253+1.2905)+(0.0253+0.9868)]\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$=(7.6315+0.1012+0.3036)+1.3158+1.0121)\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$=10.3642\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_x=10.36\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM 9.146 (Continued)

$$(I_y)_2 = (I_y)_4, \qquad (I_y)_3 = (I_y)_5$$

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 + (I_y)_5$$

$$= [(8.5857\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(16\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+2[0 + (0.6832\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(16\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+2\left[\frac{1}{12}(0.6832\times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(8\ \text{in.})^2 + (0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(12\ \text{in.})^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

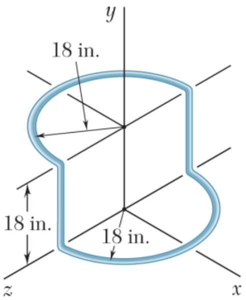
$$= [(15.2635) + 2(1.2146) + 2(0.0253 + 0.6832)]\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= [15.2635 + 2(1.2146) + 2(0.7085)]\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 19.1097\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_y = 19.11\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

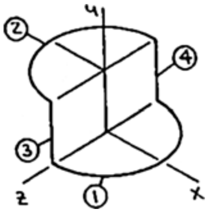
Symmetry implies $I_x = I_z$ $\qquad$ $I_z = 10.36\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 9.147

The figure shown is formed of $\frac{1}{8}$-in.-diameter steel wire. Knowing that the specific weight of the steel is 490 lb/ft^3, determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}}V = \frac{\gamma_{\text{ST}}}{g}AL$$

Then

$$m_1 = m_2 = \frac{490\text{ lb/ft}^3}{32.2\text{ ft/s}^2}\times\left[\frac{\pi}{4}\left(\frac{1}{8}\text{ in.}\right)^2\right]\times(\pi\times 18\text{ in.})\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^3$$

$$= 6.1112\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m_3 = m_4 = \frac{490\text{ lb/ft}^3}{32.2\text{ ft/s}^2} = \left[\frac{\pi}{4}\left(\frac{1}{8}\text{in.}\right)^2\right]\times 18\text{ in.}\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^3$$

$$= 1.9453\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft}$$

Using the equations given above and the parallel-axis theorem, we have

$$(I_x)_3 = (I_x)_4$$

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4$$

$$= \left[\frac{1}{2}(6.1112\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})(18\text{ in.})^2\right]\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$+\left[\frac{1}{2}(6.1112\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})(18\text{ in.})^2 + (6.1112\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})(18\text{ in.})^2\right]\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$+2\left[\frac{1}{12}(1.9453\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})(18\text{ in.})^2 + (6.1112\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})(9^2+18^2)\text{ in}^2\right]\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$= [(6.8751) + (6.8751 + 13.1502) + 2(0.3647 + 5.4712)]\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= [6.8751 + 20.6252 + 2(5.8359)]\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 39.1721\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_x = 0.0392\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM 9.147 (Continued)

$$(I_y)_1 = (I_y)_2, \quad (I_y)_3 = (I_y)_4$$

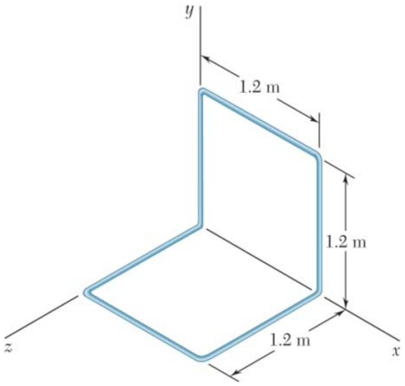
$$\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 \\
&= 2[(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&\quad + 2[(0+1.9453\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&= [2(13.7502)+2(4.3769)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \\
&= 36.2542\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2
\end{aligned}$$

or $I_y = 0.0363\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$(I_z)_3 = (I_z)_4$$

$$\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 + (I_z)_4 \\
&= \left[\frac{1}{2}(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&\quad + \left\{(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left(\frac{1}{2}-\frac{4}{\pi^2}\right)(18\ \text{in.})^2\right. \\
&\quad \left. +(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2\times18}{\pi}\right)^2+(18)^2\right]\text{in}^2\right\}\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&\quad + 2\left[\frac{1}{3}(1.9453\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&= [(6.8751)+(1.3024+19.3229)+2(1.4590)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \\
&= [6.8751+20.6253+2(1.4590)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \\
&= 30.4184\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2
\end{aligned}$$

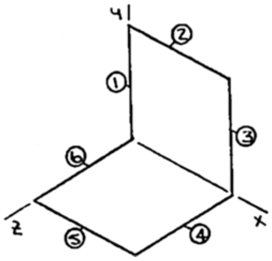
or $I_z = 0.0304\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 9.148

A homogeneous wire with a mass per unit length of 0.056 kg/m is used to form the figure shown. Determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION



First compute the mass m of each component. We have

$$m = (m/L)L$$
$$= 0.056 \text{ kg/m} \times 1.2 \text{ m}$$
$$= 0.0672 \text{ kg}$$

Using the equations given above and the parallel-axis theorem, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 + (I_x)_5 + (I_x)_6$$

Now $\quad (I_x)_1 = (I_x)_3 = (I_x)_4 = (I_x)_6 \quad \text{and} \quad (I_x)_2 = (I_x)_5$

Then

$$I_x = 4\left[\frac{1}{3}(0.0672 \text{ kg})(1.2 \text{ m})^2\right] + 2[0 + (0.0672 \text{ kg})(1.2 \text{ m})^2]$$
$$= [4(0.03226) + 2(0.09677)] \text{ kg}\cdot\text{m}^2$$
$$= 0.32258 \text{ kg}\cdot\text{m}^2 \qquad \text{or} \qquad I_x = 0.323 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

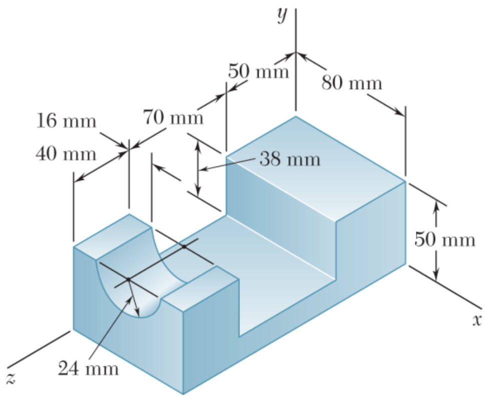
$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 + (I_y)_5 + (I_y)_6$$

Now $\quad (I_y)_1 = 0, \quad (I_y)_2 = (I_y)_6, \quad \text{and} \quad (I_y)_4 = (I_y)_5$

Then

$$I_y = 2\left[\frac{1}{3}(0.0672 \text{ kg})(1.2 \text{ m})^2\right] + [0 + (0.0672 \text{ kg})(1.2 \text{ m})^2]$$
$$+ 2\left[\frac{1}{12}(0.0672 \text{ kg})(1.2 \text{ m})^2 + (0.0672 \text{ kg})(1.2^2 + 0.6^2) \text{ m}^2\right]$$
$$= [2(0.03226) + (0.09677) + 2(0.00806 + 0.12096)] \text{ kg}\cdot\text{m}^2$$
$$= [2(0.03226) + (0.09677) + 2(0.12902)] \text{ kg}\cdot\text{m}^2$$
$$= 0.41933 \text{ kg}\cdot\text{m}^2 \qquad \text{or} \qquad I_y = 0.419 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

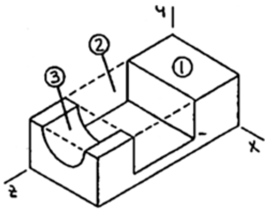
Symmetry implies $\quad I_y = I_z \qquad I_z = 0.419 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$



PROBLEM 9.149

Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the steel fixture shown. (The density of steel is 7850 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho V$$

Then

$$m_1 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.05 \times 0.16) \text{ m}^3 = 5.02400 \text{ kg}$$

$$m_2 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.038 \times 0.07) \text{ m}^3 = 1.67048 \text{ kg}$$

$$m_3 = 7850 \text{ kg/m}^3 \times \left(\frac{\pi}{2} \times 0.024^2 \times 0.04\right) \text{m}^3 = 0.28410 \text{ kg}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Now

$$\bar{y}_2 = \left(0.05 - \frac{0.038}{2}\right) \text{m} = 0.031 \text{ m}$$

$$\bar{y}_3 = \left(0.05 - \frac{4 \times 0.024}{3\pi}\right) \text{m} = 0.039814 \text{ m}$$

and then

	m, kg	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$m\bar{x}\bar{y}$, kg·m^2	$m\bar{y}\bar{z}$, kg·m^2	$m\bar{z}\bar{x}$, kg·m^2
1	5.02400	0.04	0.025	0.08	5.0240×10^{-3}	10.0480×10^{-3}	16.0768×10^{-3}
2	1.67048	0.04	0.031	0.085	2.0714×10^{-3}	4.4017×10^{-3}	5.6796×10^{-3}
3	0.28410	0.04	0.039814	0.14	0.4524×10^{-3}	1.5836×10^{-3}	1.5910×10^{-3}

Finally,

$$I_{xy} = (I_{xy})_1 - (I_{xy})_2 - (I_{xy})_3 = [(0 + 5.0240) - (0 + 2.0714) - (0 + 0.4524)] \times 10^{-3}$$

$$= 2.5002 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

or $I_{xy} = 2.50 \times 10^{-3}$ kg·m^2 ◀

$$I_{yz} = (I_{yz})_1 - (I_{yz})_2 - (I_{yz})_3 = [(0+10.0480) - (0+4.4017) - (0+1.5836)] \times 10^{-3}$$

$$= 4.0627 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

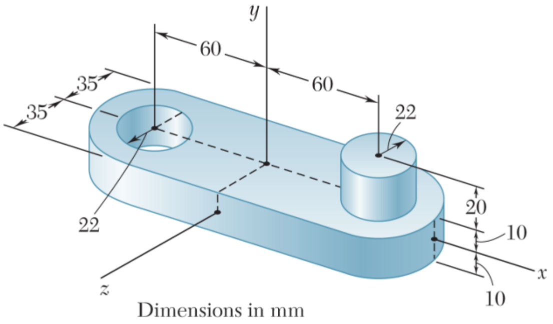
or $I_{yz} = 4.06 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ ◀

$$I_{zx} = (I_{zx})_1 - (I_{zx})_2 - (I_{zx})_3 = [(0+16.0768) - (0+5.6796) - (0+1.5910)] \times 10^{-3}$$

$$= 8.8062 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

or $I_{zx} = 8.81 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ ◀

PROBLEM 9.150


y
60
60
35
35
22
22
20
10
10
x
z
Dimensions in mm

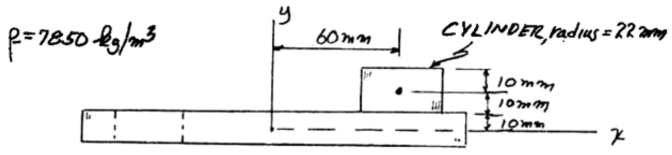
Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the steel machine element shown. (The density of steel is 7850 kg/m^3.)

SOLUTION

Since the machine element is symmetrical with respect to the *xy* plane, $I_{yz} = I_{zx} = 0$ ◀

Also, $I_{xy} = 0$

for all components except the cylinder, since these are symmetrical with respect to the *xz* plane.

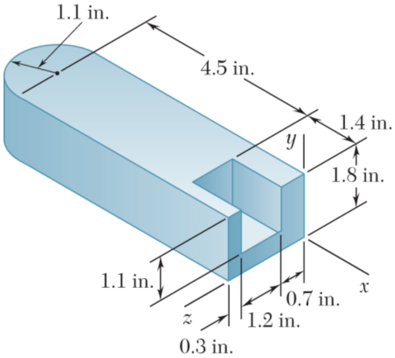

ρ=7850 kg/m³
y
60mm
CYLINDER, radius = 22 mm
10 mm
10mm
10mm
x

For the cylinder

$$m = \rho V = (7850\ \text{kg/m}^3)\pi(0.022\ \text{m})^2(0.02\ \text{m}) = 0.2387\ \text{kg}$$

$$I'_{xy} = \bar{I}_{x'y'} + m\bar{x}\,\bar{y} = 0 + (0.2387\ \text{kg})(0.06\ \text{m})(0.02\ \text{m}) = +286.4 \times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

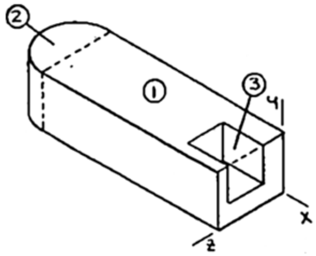
$$I_{xy} = 286 \times 10^{-6}\ \text{kg}\cdot\text{m}^2$$ ◀



PROBLEM 9.151

Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the cast aluminum machine component shown. (The specific weight of aluminum is 0.100 lb/in^3)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{AL}} V = \frac{\gamma_{\text{AL}}}{g} V$$

Then

$$m_1 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times (5.9 \times 1.8 \times 2.2) \text{ in}^3$$

$$= 72.5590 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times \left[\frac{\pi}{2}(1.1)^2 \times 1.8\right] \text{ in}^3$$

$$= 10.6248 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times (1.4 \times 1.1 \times 1.2) \text{ in}^3$$

$$= 5.7391 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Now

$$\bar{x}_2 = -\left(5.9 + \frac{4 \times 1.1}{3\pi}\right) \text{ in.}$$

$$= -6.36685 \text{ in.}$$

$$\bar{y}_3 = \left(1.8 - \frac{1.1}{2}\right) \text{ in.}$$

$$= 1.25 \text{ in.}$$

PROBLEM 9.151 (Continued)

and then

	m, $\text{lb}\cdot\text{s}^2/\text{ft}$	$\bar{x}$, ft	$\bar{y}$, ft	$\bar{z}$, ft	$m\bar{x}\,\bar{y}$, $\text{lb}\cdot\text{ft}\cdot\text{s}^2$	$m\bar{y}\,\bar{z}$, $\text{lb}\cdot\text{ft}\cdot\text{s}^2$	$m\bar{z}\,\bar{x}$, $\text{lb}\cdot\text{ft}\cdot\text{s}^2$
1	72.5590×10^{-3}	$-\frac{2.95}{12}$	$\frac{0.9}{12}$	$\frac{1.1}{12}$	-1.33781×10^{-3}	0.49884×10^{-3}	-1.63510×10^{-3}
2	10.6248×10^{-3}	$-\frac{6.36685}{12}$	$\frac{0.9}{12}$	$\frac{1.1}{12}$	-0.42279×10^{-3}	0.07305×10^{-3}	-0.51674×10^{-3}
3	5.7391×10^{-3}	$-\frac{0.7}{12}$	$\frac{1.25}{12}$	$\frac{1.3}{12}$	-0.03487×10^{-3}	0.06476×10^{-3}	-0.03627×10^{-3}

Finally,

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 - (I_{xy})_3$$
$$= [(0-1.33781)+(0-0.42279)-(0-0.03487)]\times10^{-3}$$

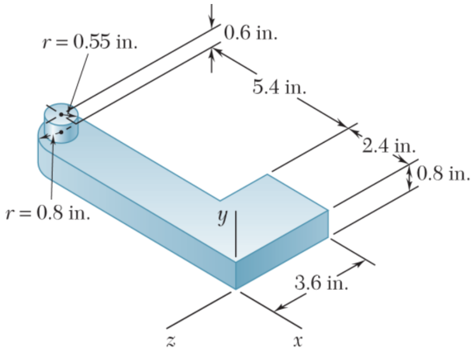
or $I_{xy} = -1.726\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$I_{yz} = (I_{yz})_1 + (I_{yz})_2 - (I_{yz})_3$$
$$= [(0+0.49884)+(0+0.07305)-(0+0.06476)]\times10^{-3}$$

or $I_{yz} = 0.507\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$I_{zx} = (I_{zx})_1 + (I_{zx})_2 - (I_{zx})_3$$
$$= [(0-1.63510)+(0-0.51674)-(0-0.3627)]\times10^{-3}$$

or $I_{zx} = -2.12\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



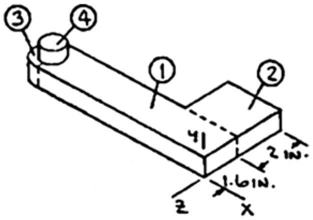
PROBLEM 9.152

Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the cast aluminum machine component shown. (The specific weight of aluminum is 0.100 lb/in^3)

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{\text{AL}} V = \frac{\gamma_{\text{AL}}}{g} V$$



Then

$$m_1 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times (7.8 \times 0.8 \times 1.6) \text{ in}^3$$

$$= 31.0062 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times (2.4 \times 0.8 \times 2) \text{ in}^3$$

$$= 11.9255 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times \left[\frac{\pi}{2}(0.8)^2 \times 0.8\right] \text{ in}^3 = 2.4977 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_4 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times [\pi(0.55)^2 \times 0.6] \text{ in}^3 = 1.7708 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Now

$$\bar{x}_3 = -\left(7.8 + \frac{4 \times 0.8}{3\pi}\right) \text{ in.} = -8.13953 \text{ in.}$$

PROBLEM 9.152 (Continued)

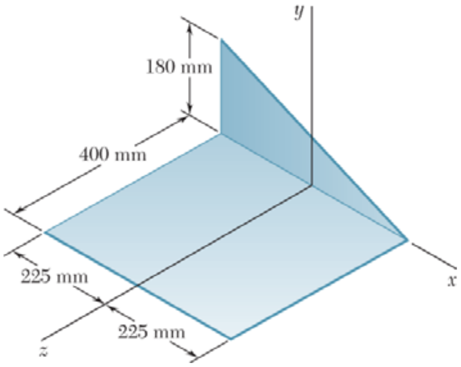
and then

	m, lb · s^2/ft	$\bar{x}$, ft	$\bar{y}$, ft	$\bar{z}$, ft	$m\bar{x}\,\bar{y}$, lb · ft · s^2	$m\bar{y}\,\bar{z}$, lb · ft · s^2	$m\bar{z}\,\bar{x}$, lb · ft · s^2
1	31.0062×10^{-3}	$-\frac{3.9}{12}$	$\frac{0.4}{12}$	$-\frac{0.8}{12}$	-335.901×10^{-6}	-68.903×10^{-6}	671.801×10^{-6}
2	11.9255×10^{-3}	$-\frac{1.2}{12}$	$\frac{0.4}{12}$	$-\frac{2.6}{12}$	-39.752×10^{-6}	-86.129×10^{-6}	258.386×10^{-6}
3	2.4977×10^{-3}	$-\frac{8.13953}{12}$	$\frac{0.4}{12}$	$-\frac{0.8}{12}$	-56.473×10^{-6}	-5.550×10^{-6}	112.945×10^{-6}
4	1.7708×10^{-3}	$-\frac{7.8}{12}$	$\frac{1.1}{12}$	$-\frac{0.8}{12}$	-105.511×10^{-6}	-10.822×10^{-6}	76.735×10^{-6}
Σ					-537.637×10^{-6}	-171.404×10^{-6}	1119.867×10^{-6}

Then $I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y})$ or $I_{xy} = -538\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z})$ or $I_{yz} = -171.4\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

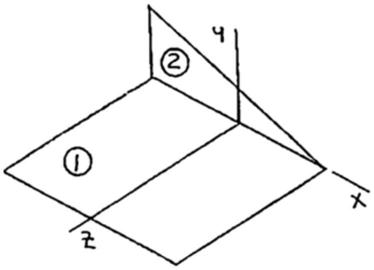
$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x})$ or $I_{zx} = 1120\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 9.153

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



First compute the mass of each component.

We have $$m = \rho_{\text{ST}} V = \rho_{\text{ST}} tA$$

Then $$m_1 = (7850\ \text{kg/m}^3)[(0.002)(0.4)(0.45)]\text{m}^3$$
$$= 2.8260\ \text{kg}$$
$$m_2 = 7850\ \text{kg/m}^3\left[(0.002)\left(\frac{1}{2}\times 0.45\times 0.18\right)\right]\text{m}^3$$
$$= 0.63585\ \text{kg}$$

Now observe that

$$(\bar{I}_{x'y'})_1 = (\bar{I}_{y'z'})_1 = (\bar{I}_{z'x'})_1 = 0$$
$$(\bar{I}_{y'z'})_2 = (\bar{I}_{z'x'})_2 = 0$$

From Sample Problem 9.6: $$(\bar{I}_{x'y'})_{2,\text{area}} = -\frac{1}{72}b_2^2h_2^2$$

Then $$(\bar{I}_{x'y'})_2 = \rho_{\text{ST}} t(I_{x'y'})_{2,\text{area}} = \rho_{\text{ST}} t\left(-\frac{1}{72}b_2^2h_2^2\right) = -\frac{1}{36}m_2b_2h_2$$

Also $$\bar{x}_1 = \bar{y}_1 = \bar{z}_2 = 0 \qquad \bar{x}_2 = \left(-0.225 + \frac{0.45}{3}\right)\text{m} = -0.075\ \text{m}$$

PROBLEM 9.153 (Continued)

Finally,
$$I_{xy} = \Sigma(\bar{I}_{xy} + m\bar{x}\,\bar{y}) = (0+0) + \left[-\frac{1}{36}(0.63585\text{ kg})(0.45\text{ m})(0.18\text{ m})\right.$$
$$\left. + (0.63585\text{ kg})(-0.075\text{ m})\left(\frac{0.18\text{ m}}{3}\right)\right]$$
$$= (-1.43066\times10^{-3} - 2.8613\times10^{-3})\text{ kg}\cdot\text{m}^2$$

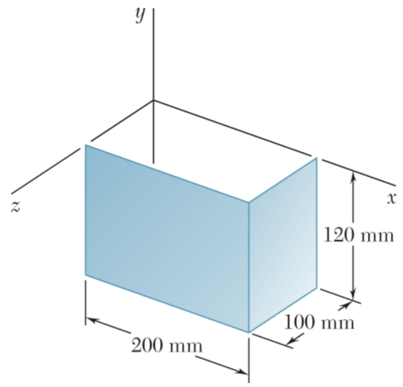
or $I_{xy} = -4.29\times10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

and
$$I_{yz} = \Sigma(\bar{I}_{y'z'} + m\bar{y}\,\bar{z}) = (0+0) + (0+0) = 0$$

or $I_{yz} = 0$ ◀

$$I_{zx} = \Sigma(\bar{I}_{z'x'} + m\bar{z}\,\bar{x}) = (0+0) + (0+0) = 0$$

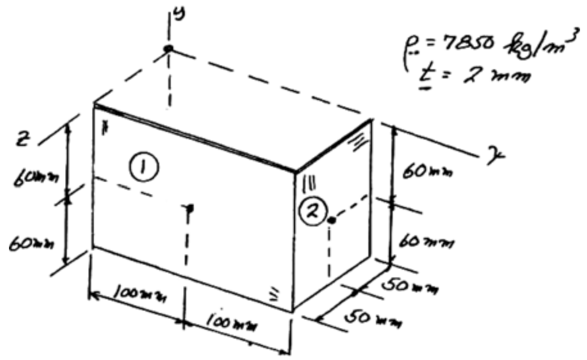
or $I_{zx} = 0$ ◀



PROBLEM 9.154

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



$$m_1 = \rho V = 7850\ \text{kg/m}^3\ (0.120\ \text{m})(0.200\ \text{m})(0.002\ \text{m}) = 0.3768\ \text{kg}$$

$$m_2 = \rho V = 7850\ \text{kg/m}^3\ (0.120\ \text{m})(0.100\ \text{m})(0.002\ \text{m}) = 0.1884\ \text{kg}$$

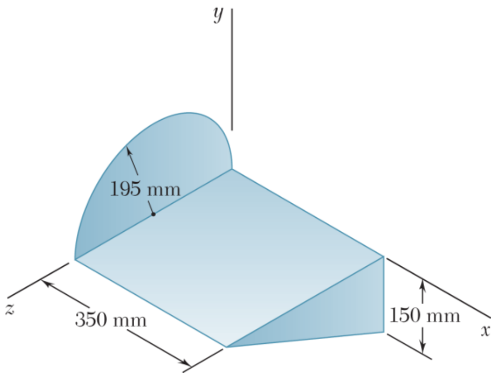
For each panel the centroidal product of inertia is zero with respect to each pair of coordinate axes.

	m, kg	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$m\bar{x}\bar{y}$	$m\bar{y}\bar{z}$	$m\bar{z}\bar{x}$
①	0.3768	0.1	–0.06	0.1	-2.261×10^{-3}	-2.261×10^{-3}	$+3.768\times10^{-3}$
②	0.1884	0.2	–0.06	0.05	-2.261×10^{-3}	-0.565×10^{-3}	$+1.884\times10^{-3}$
Σ					-4.522×10^{-3}	-2.826×10^{-3}	$+5.652\times10^{-3}$

$I_{xy} = -4.52\times10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

$I_{yz} = -2.83\times10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

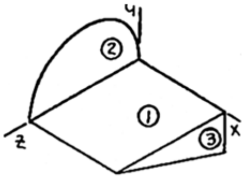
$I_{zx} = +5.65\times10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM 9.155

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\mathrm{ST}} V = \rho_{\mathrm{ST}} t A$$

Then
$$m_1 = (7850\text{ kg/m}^3)(0.002\text{ m})(0.35 \times 0.39)\text{ m}^2$$
$$= 2.14305\text{ kg}$$
$$m_2 = (7850\text{ kg/m}^3)(0.002\text{ m})\left(\frac{\pi}{2} \times 0.195^2\right)\text{ m}^2$$
$$= 0.93775\text{ kg}$$
$$m_3 = (7850\text{ kg/m}^3)(0.002\text{ m})\left(\frac{1}{2} \times 0.39 \times 0.15\right)\text{ m}^2$$
$$= 0.45923\text{ kg}$$

Now observe that because of symmetry the centroidal products of inertia of components 1 and 2 are zero and

$$(\bar{I}_{x'y'})_3 = (\bar{I}_{z'x'})_3 = 0$$

Also
$$(\bar{I}_{y'z'})_{3,\text{mass}} = \rho_{\mathrm{ST}} t (\bar{I}_{y'z'})_{3,\text{area}}$$

Using the results of Sample Problem 9.6 and noting that the orientation of the axes corresponds to a 90° rotation, we have

$$(\bar{I}_{y'z'})_{3,\text{area}} = \frac{1}{72} b_3^2 h_3^2$$

Then
$$(I_{y'z'})_3 = \rho_{\mathrm{ST}} t \left(\frac{1}{72} b_3^2 h_3^2\right) = \frac{1}{36} m_3 b_3 h_3$$

Also
$$\bar{y}_1 = \bar{x}_2 = 0$$
$$\bar{y}_2 = \frac{4 \times 0.195}{3\pi}\text{ m} = 0.082761\text{ m}$$

PROBLEM 9.155 (Continued)

Finally,

$$I_{xy} = \Sigma(\bar{I}_{x'y'} + m\bar{x}\,\bar{y})$$

$$= (0+0) + (0+0) + \left[0 + (0.45923\text{ kg})(0.35\text{ m})\left(-\frac{0.15}{3}\text{ m}\right)\right]$$

$$= -8.0365\times10^{-3}\text{ kg}\cdot\text{m}^2 \qquad \text{or } I_{xy} = -8.04\times10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

$$I_{yz} = \Sigma(\bar{I}_{y'z'} + m\bar{y}\,\bar{z})$$

$$= (0+0) + [0 + (0.93775\text{ kg})(0.082761\text{ m})(0.195\text{ m})]$$

$$+\left[\frac{1}{36}(0.45923\text{ kg})(0.39\text{ m})(0.15\text{ m}) + (0.45923\text{ kg})\left(-\frac{0.15}{3}\text{ m}\right)\left(\frac{0.39}{3}\text{ m}\right)\right]$$

$$= [(15.1338) + (0.7462 - 2.9850)]\times10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (15.1338 - 2.2388)\times10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= 12.8950\times10^{-3}\text{ kg}\cdot\text{m}^2 \qquad \text{or } I_{yz} = 12.90\times10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

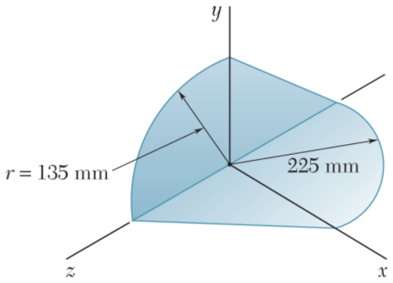
$$I_{zx} = \Sigma(\bar{I}_{z'x'} + m\bar{z}\,\bar{x})$$

$$= [0 + (2.14305\text{ kg})(0.175\text{ m})(0.195\text{ m})] + (0+0)$$

$$+\left[0 + (0.45923\text{ kg})\left(\frac{0.39}{3}\text{ m}\right)(0.35\text{ m})\right]$$

$$= (73.1316 + 20.8950)\times10^{-3}\text{ kg}\cdot\text{m}^2$$

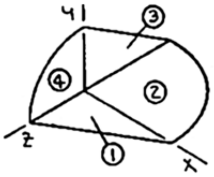
$$= 94.0266\times10^{-3}\text{ kg}\cdot\text{m}^2 \qquad \text{or } I_{zx} = 94.0\times10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$



PROBLEM 9.156

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\mathrm{ST}} V = \rho_{\mathrm{ST}} t A$$

Then

$$m_1 = m_3 = (7850\ \mathrm{kg/m^3})(0.002\ \mathrm{m})\left(\frac{1}{2} \times 0.225 \times 0.135\right)\mathrm{m^2} = 0.23844\ \mathrm{kg}$$

$$m_2 = (7850\ \mathrm{kg/m^3})(0.002\ \mathrm{m})\left(\frac{\pi}{4} \times 0.225^2\right)\mathrm{m^2} = 0.62424\ \mathrm{kg}$$

$$m_4 = (7850\ \mathrm{kg/m^3})(0.002\ \mathrm{m})\left(\frac{\pi}{4} \times 0.135^2\right)\mathrm{m^2} = 0.22473\ \mathrm{kg}$$

Now observe that the following centroidal products of inertia are zero because of symmetry.

$$(\bar{I}_{x'y'})_1 = (\bar{I}_{y'z'})_1 = 0 \qquad (\bar{I}_{x'y'})_2 = (\bar{I}_{y'z'})_2 = 0$$
$$(\bar{I}_{x'y'})_3 = (\bar{I}_{z'x'})_3 = 0 \qquad (\bar{I}_{x'y'})_4 = (\bar{I}_{z'x'}) = 0$$

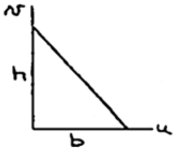
Also $\bar{y}_1 = \bar{y}_2 = 0 \qquad \bar{x}_3 = \bar{x}_4 = 0$

Now $I_{xy} = \Sigma(\bar{I}_{x'y'} + m\bar{x}\,\bar{y})$

so that $I_{xy} = 0$ ◀

Using the results of Sample Problem 9.6, we have

$$I_{uv,\mathrm{area}} = \frac{1}{24} b^2 h^2$$



Now

$$I_{uv,\mathrm{mass}} = \rho_{\mathrm{ST}} t\, I_{uv,\mathrm{area}}$$
$$= \rho_{\mathrm{ST}} t\left(\frac{1}{24} b^2 h^2\right)$$
$$= \frac{1}{12} mbh$$

Thus,

$$(I_{zx})_1 = \frac{1}{12} m_1 b_1 h_1$$

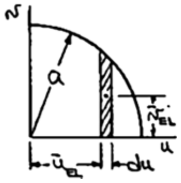
PROBLEM 9.156 (Continued)

While $$(I_{yz})_3 = -\frac{1}{12}m_3b_3h_3$$

because of a 90° rotation of the coordinate axes.

To determine I_{uv} for a quarter circle, we have

$$dI_{uv} = d\cancelto{0}{\bar{I}_{u'v'}} + u_{\text{EL}}v_{\text{EL}}\,dm$$



Where $$\bar{u}_{\text{EL}} = u \quad \bar{v}_{\text{EL}} = \frac{1}{2}v = \frac{1}{2}\sqrt{a^2 - u^2}$$

$$dm = \rho_{\text{ST}}t\,dA = \rho_{\text{ST}}tv\,du = \rho_{\text{ST}}t\sqrt{a^2 - u^2}\,du$$

Then $$I_{uv} = \int dI_{uv} = \int_0^a (u)\left(\frac{1}{2}\sqrt{a^2 - u^2}\right)\left(\rho_{\text{ST}}t\sqrt{a^2 - u^2}\,du\right)$$

$$= \frac{1}{2}\rho_{\text{ST}}t\int_0^a u(a^2 - u^2)\,du$$

$$= \frac{1}{2}\rho_{\text{ST}}t\left[\frac{1}{2}a^2u^2 - \frac{1}{4}u^4\right]_0^a = \frac{1}{8}\rho_{\text{ST}}t\;a^4 = \frac{1}{2\pi}ma^4$$

Thus $$(I_{zx})_2 = -\frac{1}{2\pi}m_2a_2^2$$

because of a 90° rotation of the coordinate axes. Also

$$(I_{yz})_4 = \frac{1}{2\pi}m_4a_4^2$$

Finally,

$$I_{yz} = \Sigma(I_{yz}) = [(\cancelto{0}{\bar{I}_{y'z'}}) + m_1\bar{y}_1\bar{z}_1] + [(\cancelto{0}{\bar{I}_{y'z'}}) + m_2\bar{y}_2\bar{z}_2] + (I_{yz})_3 + (I_{yz})_4$$

$$= \left[-\frac{1}{12}(0.23844\text{ kg})(0.225\text{ m})(0.135\text{ m})\right] + \left[\frac{1}{2\pi}(0.22473\text{ kg})(0.135\text{ m})^2\right]$$

$$= (-0.60355 + 0.65185)\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

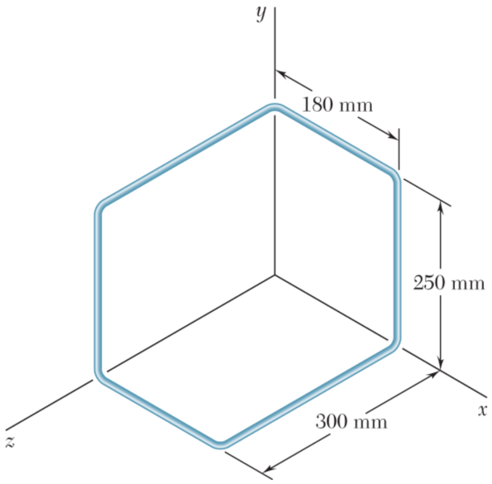
or $I_{yz} = 48.3\times 10^{-6}\text{ kg}\cdot\text{m}^2$ ◀

$$I_{zx} = \Sigma(I_{zx}) = (I_{zx})_1 + (I_{zx})_2 + [(\cancelto{0}{\bar{I}_{z'x'}})_3 + m_3\bar{z}_3\bar{x}_3] + [(\cancelto{0}{\bar{I}_{z'x'}})_4 + m_4\bar{z}_4\bar{x}_4]$$

$$= \left[\frac{1}{12}(0.23844\text{ kg})(0.225\text{ m})(0.135\text{ m})\right] + \left[-\frac{1}{2\pi}(0.62424\text{ kg})(0.225\text{ m})^2\right]$$

$$= (0.60355 - 5.02964)\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

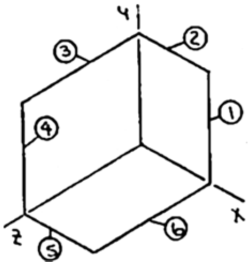
or $I_{zx} = -4.43\times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀



PROBLEM 9.157

The figure shown is formed of 1.5-mm-diameter aluminum wire. Knowing that the density of aluminum is 2800 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{AL}}V = \rho_{\text{AL}}AL$$

Then

$$m_1 = m_4 = (2800\text{ kg/m}^3)\left[\frac{\pi}{4}(0.0015\text{ m})^2\right](0.25\text{ m})$$

$$= 1.23700\times 10^{-3}\text{ kg}$$

$$m_2 = m_5 = (2800\text{ kg/m}^3)\left[\frac{\pi}{4}(0.0015\text{ m})^2\right](0.18\text{ m})$$

$$= 0.89064\times 10^{-3}\text{ kg}$$

$$m_3 = m_6 = (2800\text{ kg/m}^3)\left[\frac{\pi}{4}(0.0015\text{ m})^2\right](0.3\text{ m})$$

$$= 1.48440\times 10^{-3}\text{ kg}$$

PROBLEM 9.157 (Continued)

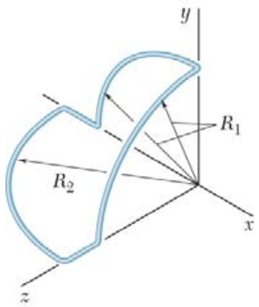
Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

	m, kg	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$m\bar{x}\,\bar{y}$, $\text{kg}\cdot\text{m}^2$	$m\bar{y}\,\bar{z}$, $\text{kg}\cdot\text{m}^2$	$m\bar{z}\,\bar{x}$, $\text{kg}\cdot\text{m}^2$
1	1.23700×10^{-3}	0.18	0.125	0	27.8325×10^{-6}	0	0
2	0.89064×10^{-3}	0.09	0.25	0	20.0394×10^{-6}	0	0
3	1.48440×10^{-3}	0	0.25	0.15	0	55.6650×10^{-6}	0
4	1.23700×10^{-3}	0	0.125	0.3	0	46.3875×10^{-6}	0
5	0.89064×10^{-3}	0.09	0	0.3	0	0	24.0473×10^{-6}
6	1.48440×10^{-3}	0.18	0	0.15	0	0	40.0788×10^{-6}
Σ					47.8719×10^{-6}	102.0525×10^{-6}	64.1261×10^{-6}

Then $I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y})$ or $I_{xy} = 47.9\times10^{-6}\ \text{kg}\cdot\text{m}^2$ ◀

$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z})$ or $I_{yz} = 102.1\times10^{-6}\ \text{kg}\cdot\text{m}^2$ ◀

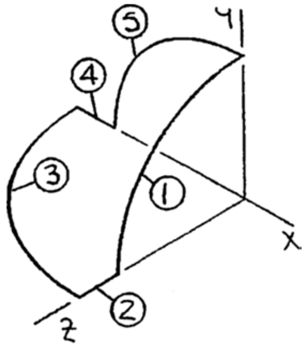
$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x})$ or $I_{zx} = 64.1\times10^{-6}\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM 9.158

Thin aluminum wire of uniform diameter is used to form the figure shown. Denoting by m' the mass per unit length of the wire, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION



First compute the mass of each component. We have

$$m = \left(\frac{m}{L}\right)L = m'L$$

Then

$$m_1 = m_5 = m'\left(\frac{\pi}{2}R_1\right) = \frac{\pi}{2}m'R_1$$

$$m_2 = m_4 = m'(R_2 - R_1)$$

$$m_3 = m'\left(\frac{\pi}{2}R_2\right) = \frac{\pi}{2}m'R_2$$

Now observe that because of symmetry the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'},$ and $\bar{I}_{z'x'},$ of components 2 and 4 are zero and

$$(\bar{I}_{x'y'})_1 = (\bar{I}_{z'x'})_1 = 0 \qquad (\bar{I}_{x'y'})_3 = (\bar{I}_{y'z'})_3 = 0$$

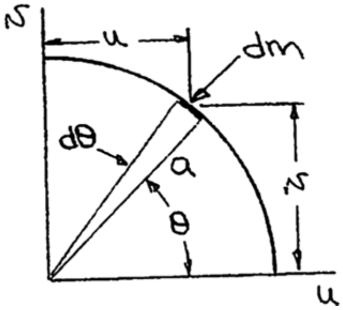
$$(\bar{I}_{y'z'})_5 = (\bar{I}_{z'x'})_5 = 0$$

Also

$$\bar{x}_1 = \bar{x}_2 = 0 \qquad \bar{y}_2 = \bar{y}_3 = \bar{y}_4 = 0 \qquad \bar{z}_4 = \bar{z}_5 = 0$$

Using the parallel-axis theorem [Equations (9.47)], it follows that $I_{xy} = I_{yz} = I_{zx}$ for components 2 and 4.

To determine I_{uv} for one quarter of a circular arc, we have $dI_{uv} = uv\,dm$



where

$$u = a\cos\theta \qquad v = a\sin\theta$$

and

$$dm = \rho dV = \rho[A(ad\theta)]$$

where A is the cross-sectional area of the wire. Now

$$m = m'\left(\frac{\pi}{2}a\right) = \rho A\left(\frac{\pi}{2}a\right)$$

so that

$$dm = m'ad\theta$$

and

$$dI_{uv} = (a\cos\theta)(a\sin\theta)(m'ad\theta)$$

$$= m'a^3 \sin\theta\cos\theta d\theta$$

PROBLEM 9.158 (Continued)

Then
$$I_{uv} = \int dI_{uv} = \int_0^{\pi/2} m'a^3 \sin\theta\cos\theta d\theta$$
$$= m'a^3\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} = \frac{1}{2}m'a^3$$

Thus,
$$(I_{yz})_1 = \frac{1}{2}m'R_1^3$$

and
$$(I_{zx})_3 = -\frac{1}{2}m'R_2^3 \quad (I_{xy})_5 = -\frac{1}{2}m'R_1^3$$

because of 90° rotations of the coordinate axes. Finally,

$$I_{xy} = \Sigma(I_{xy}) = [(\cancelto{0}{\bar{I}_{x'y'}})_1 + m_1\bar{x}_1\bar{y}_1] + [(\cancelto{0}{\bar{I}_{x'y'}})_3 + m_3\bar{x}_3\bar{y}_3] + (I_{xy})_5$$

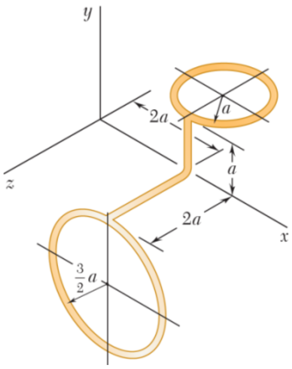
or $I_{xy} = -\frac{1}{2}m'R_1^3$ ◀

$$I_{yz} = \Sigma(I_{yz}) = (I_{yz})_1 + [(\cancelto{0}{\bar{I}_{y'z'}})_3 + m_3\bar{y}_3\bar{z}_3] + [(\cancelto{0}{\bar{I}_{y'z'}})_5 + m_5\bar{y}_5\bar{z}_5]$$

or $I_{yz} = \frac{1}{2}m'R_1^3$ ◀

$$I_{zx} = \Sigma(I_{zx}) = [(\cancelto{0}{\bar{I}_{z'x'}})_1 + m_1\bar{z}_1\bar{x}_1] + (I_{zx})_3 + [(\cancelto{0}{\bar{I}_{z'x'}})_5 + m_5\bar{z}_5\bar{x}_5]$$

or $I_{zx} = -\frac{1}{2}m'R_2^3$ ◀



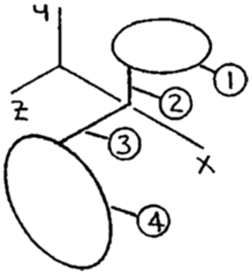
PROBLEM 9.159

Brass wire with a weight per unit length w is used to form the figure shown. Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION

First compute the mass of each component. We have

$$m = \frac{W}{g} = \frac{1}{g}wL$$



Then

$$m_1 = \frac{W}{g}(2\pi \times a) = 2\pi \frac{w}{g}a$$

$$m_2 = \frac{w}{g}(a) = \frac{w}{g}a$$

$$m_3 = \frac{w}{g}(2a) = 2\frac{w}{g}a$$

$$m_4 = \frac{w}{g}\left(2\pi \times \frac{3}{2}a\right) = 3\pi\frac{w}{g}a$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

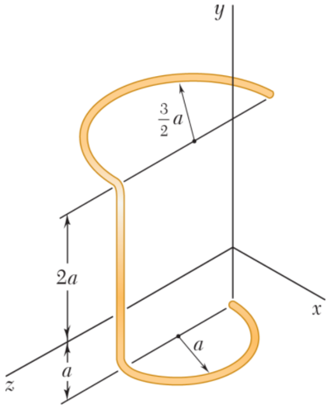
	m	$\bar{x}$	$\bar{y}$	$\bar{z}$	$m\bar{x}\,\bar{y}$	$m\bar{y}\,\bar{z}$	$m\bar{z}\,\bar{x}$
1	$2\pi\frac{w}{g}a$	$2a$	a	$-a$	$4\pi\frac{w}{g}a^3$	$-2\pi\frac{w}{g}a^3$	$-4\pi\frac{w}{g}a^3$
2	$\frac{w}{g}a$	$2a$	$\frac{1}{2}a$	0	$\frac{w}{g}a^3$	0	0
3	$2\frac{w}{g}a$	$2a$	0	a	0	0	$4\frac{w}{g}a^3$
4	$3\pi\frac{w}{g}a$	$2a$	$-\frac{3}{2}a$	$2a$	$-9\pi\frac{w}{g}a^3$	$-9\pi\frac{w}{g}a^3$	$12\pi\frac{w}{g}a^3$
Σ					$\frac{w}{g}(1-5\pi)a^3$	$-11\pi\frac{w}{g}a^3$	$4\frac{w}{g}(1+2\pi)a^3$

PROBLEM 9.159 (Continued)

Then $$I_{xy} = \Sigma(\overset{0}{\cancel{\bar{I}_{x'y'}}} + m\bar{x}\,\bar{y})$$ or $$I_{xy} = \frac{w}{g}a^3(1-5\pi) \blacktriangleleft$$

$$I_{yz} = \Sigma(\overset{0}{\cancel{\bar{I}_{y'z'}}} + m\bar{y}\,\bar{z})$$ or $$I_{yz} = -11\pi\frac{w}{g}a^3 \blacktriangleleft$$

$$I_{zx} = \Sigma(\overset{0}{\cancel{\bar{I}_{z'x'}}} + m\bar{z}\,\bar{x})$$ or $$I_{zx} = 4\frac{w}{g}a^3(1+2\pi) \blacktriangleleft$$

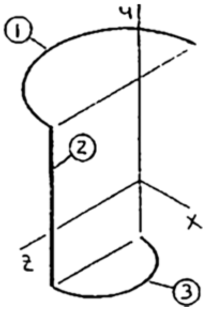


PROBLEM 9.160

Brass wire with a weight per unit length w is used to form the figure shown. Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION

First compute the mass of each component. We have



$$m = \frac{W}{g} = \frac{1}{g} wL$$

Then

$$m_1 = \frac{W}{g}\left(\pi \times \frac{3}{2}a\right) = \frac{3}{2}\pi \frac{w}{g} a$$

$$m_2 = \frac{w}{g}(3a) = 3\frac{w}{g} a$$

$$m_3 = \frac{w}{g}(\pi \times a) = \pi \frac{w}{g} a$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

	m	$\bar{x}$	$\bar{y}$	$\bar{z}$	$m\bar{x}\,\bar{y}$	$m\bar{y}\,\bar{z}$	$m\bar{z}\,\bar{x}$
1	$\frac{3}{2}\pi\frac{w}{g}a$	$-\frac{2}{\pi}\left(\frac{3}{2}a\right)$	$2a$	$\frac{1}{2}a$	$-9\frac{w}{g}a^3$	$\frac{3}{2}\pi\frac{w}{g}a^3$	$-\frac{9}{4}\frac{w}{g}a^3$
2	$3\frac{w}{g}a$	0	$\frac{1}{2}a$	$2a$	0	$3\frac{w}{g}a^3$	0
3	$\pi\frac{w}{g}a$	$\frac{2}{\pi}(a)$	$-a$	a	$-2\frac{w}{g}a^3$	$-\pi\frac{w}{g}a^3$	$2\frac{w}{g}a^3$
Σ					$-11\frac{w}{g}a^3$	$\frac{w}{g}\left(\frac{\pi}{2}+3\right)a^3$	$-\frac{1}{4}\frac{w}{g}a^3$

PROBLEM 9.160 (Continued)

Then $I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y})$ or $I_{xy} = -11\dfrac{w}{g}a^3$ ◀

$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z})$ or $I_{yz} = \dfrac{1}{2}\dfrac{w}{g}a^3(\pi + 6)$ ◀

$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x})$ or $I_{zx} = -\dfrac{1}{4}\dfrac{w}{g}a^3$ ◀

PROBLEM 9.161

Complete the derivation of Eqs. (9.47), which express the parallel-axis theorem for mass products of inertia.

SOLUTION

We have
$$I_{xy} = \int xydm \quad I_{yz} = \int yzdm \quad I_{zx} = \int zxdm \tag{9.45}$$

and
$$x = x' + \overline{x} \quad y = y' + \overline{y} \quad z = z' + \overline{z} \tag{9.31}$$

Consider
$$I_{xy} = \int xydm$$

Substituting for *x* and for *y*

$$I_{xy} = \int (x' + \overline{x})(y' + \overline{y})\, dm$$
$$= \int x'y'dm + \overline{y}\int x'dm + \overline{x}\int y'dm + \overline{x}\,\overline{y}\int dm$$

By definition
$$\overline{I}_{x'y'} = \int x'y'dm$$

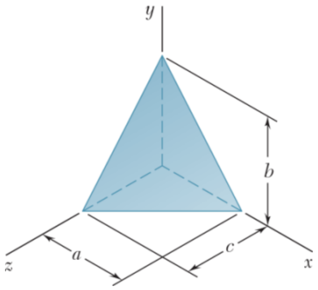
and
$$\int x'dm = m\overline{x}'$$
$$\int y'dm = m\overline{y}'$$

However, the origin of the primed coordinate system coincides with the mass center *G*, so that

$$\overline{x}' = \overline{y}' = 0$$

$$I_{xy} = \overline{I}_{x'y'} + m\overline{x}\,\overline{y} \quad \text{Q.E.D.} \blacktriangleleft$$

The expressions for I_{yz} and I_{zx} are obtained in a similar manner.

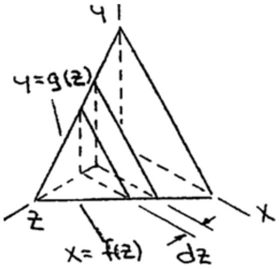


PROBLEM 9.162

For the homogeneous tetrahedron of mass *m* shown, (*a*) determine by direct integration the mass product of inertia I_{zx}, (*b*) deduce I_{yz} and I_{xy} from the result obtained in part *a*.

SOLUTION

(*a*) First divide the tetrahedron into a series of thin vertical slices of thickness *dz* as shown.



Now $$x = -\frac{a}{c}z + a = a\left(1 - \frac{z}{c}\right)$$

and $$y = -\frac{b}{c}z + b = b\left(1 - \frac{z}{c}\right)$$

The mass *dm* of the slab is

$$dm = \rho dV = \rho\left(\frac{1}{2}xydz\right) = \frac{1}{2}\rho ab\left(1 - \frac{z}{c}\right)^2 dz$$

Then $$m = \int dm = \int_0^c \frac{1}{2}\rho ab\left(1 - \frac{z}{c}\right)^2 dz$$

$$= \frac{1}{2}\rho ab\left[\left(-\frac{c}{3}\right)\left(1 - \frac{z}{c}\right)^3\right]_0^c = \frac{1}{6}\rho abc$$

Now $$dI_{zx} = d\bar{I}_{z'x'} + \bar{z}_{\text{EL}}\bar{x}_{\text{EL}}dm$$

where $$d\bar{I}_{z'x'} = 0 \quad \text{(symmetry)}$$

and $$\bar{z}_{\text{EL}} = z \qquad \bar{x}_{\text{EL}} = \frac{1}{3}x = \frac{1}{3}a\left(1 - \frac{z}{c}\right)$$

Then $$I_{zx} = \int dI_{zx} = \int_0^c z\left[\frac{1}{3}a\left(1 - \frac{z}{c}\right)\right]\left[\frac{1}{2}\rho ab\left(1 - \frac{z}{c}\right)^2 dz\right]$$

$$= \frac{1}{6}\rho a^2 b\int_0^c \left(z - 3\frac{z^2}{c} + 3\frac{z^3}{c^2} - \frac{z^4}{c^3}\right)dz$$

$$= \frac{m}{c}a\left[\frac{1}{2}z^2 - \frac{z^3}{c} + \frac{3}{4}\frac{z^4}{c^2} - \frac{1}{5}\frac{z^5}{c^3}\right]_0^c$$

or $$I_{zx} = \frac{1}{20}mac \blacktriangleleft$$

PROBLEM 9.162 (Continued)

(*b*) Because of the symmetry of the body, I_{xy} and I_{yz} can be deduced by considering the circular permutation of (x, y, z) and (a, b, c).

Thus,

$$I_{xy} = \frac{1}{20} mab \blacktriangleleft$$

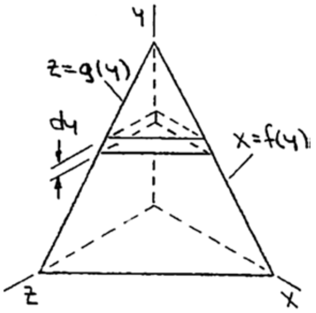
$$I_{yz} = \frac{1}{20} mbc \blacktriangleleft$$

Alternative solution for part *a*:

First divide the tetrahedron into a series of thin horizontal slices of thickness dy as shown.

Now $$x = -\frac{a}{b} y + a = a\left(1 - \frac{y}{b}\right)$$

and $$z = -\frac{c}{b} y + c = c\left(1 - \frac{y}{b}\right)$$



The mass dm of the slab is

$$dm = \rho dV = \rho\left(\frac{1}{2} xzdy\right) = \frac{1}{2}\rho ac\left(1 - \frac{y}{b}\right)^2 dy$$

Now $$dI_{zx} = \rho t dI_{zx,\text{area}}$$

where $$t = dy$$

and $dI_{zx,\text{area}} = \frac{1}{24} x^2 z^2$ from the results of Sample Problem 9.6.

Then $$dIzx = \rho(dy)\left\{\frac{1}{24}\left[a\left(1 - \frac{y}{b}\right)^2\right]\left[c\left(1 - \frac{y}{b}\right)\right]^2\right\}$$

$$= \frac{1}{24}\rho a^2 c^2 \left(1 - \frac{y}{b}\right)^4 dy = \frac{1}{4}\frac{m}{b} ac\left(1 - \frac{y}{b}\right)^4 dy$$

Finally, $$I_{zx} = \int dI_{zx} = \int_0^b \frac{1}{4}\frac{m}{b} ac\left(1 - \frac{y}{b}\right)^4 dy$$

$$= \frac{1}{4}\frac{m}{b} ac\left[\left(-\frac{b}{5}\right)\left(1 - \frac{y}{b}\right)^5\right]_0^b$$

or $$I_{zx} = \frac{1}{20} mac \blacktriangleleft$$

PROBLEM 9.162 (Continued)

Alternative solution for part *a*:

The equation of the included face of the tetrahedron is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

so that $$y=b\left(1-\frac{x}{a}-\frac{z}{c}\right)$$

For an infinitesimal element of sides $dx, dy,$ and dz:

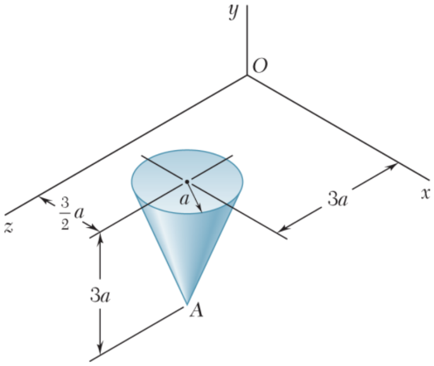
$$dm=\rho dV=\rho dydxdz$$

From part *a* $$x=a\left(1-\frac{z}{c}\right)$$

Now

$$\begin{aligned}
I_{zx}=\int zxdm &= \int_0^c\int_0^{a(1-z/c)}\int_0^{b(1-x/a-z/c)} zx(\rho dydxdz)\\
&= \rho\int_0^c\int_0^{a(1-z/c)} zx\left[b\left(1-\tfrac{x}{a}-\tfrac{z}{c}\right)\right]dxdz\\
&= \rho b\int_0^c z\left[\frac{1}{2}x^2-\frac{1}{3}\frac{x^3}{a}-\frac{1}{2}\frac{z}{c}x^2\right]_0^{a(1-z/c)} dz\\
&= \rho b\int_0^c z\left[\frac{1}{2}a^2\left(1-\frac{z}{c}\right)^2-\frac{1}{3a}a^3\left(1-\frac{z}{c}\right)^3-\frac{1}{2}\frac{z}{c}a^2\left(1-\frac{z}{c}\right)^2\right]dz\\
&= \rho b\int_0^c \frac{1}{6}a^2z\left(1-\frac{z}{c}\right)^3 dz\\
&= \frac{1}{6}\rho a^2b\int_0^c\left(z-3\frac{z^2}{c}+3\frac{z^3}{c^2}-\frac{z^4}{c^3}\right)dz\\
&= \frac{m}{c}a\left[\frac{1}{2}z^2-\frac{z^3}{c}+\frac{3}{4}\frac{z^4}{c^2}-\frac{1}{5}\frac{z^5}{c^3}\right]_0^c
\end{aligned}$$

or $I_{zx}=\dfrac{1}{20}mac$ ◀



PROBLEM 9.163

The homogeneous circular cone shown has a mass m. Determine the mass moment of inertia of the cone with respect to the line joining the origin O and Point A.

SOLUTION

First note that

$$d_{OA} = \sqrt{\left(\frac{3}{2}a\right)^2 + (-3a)^2 + (3a)^2} = \frac{9}{2}a$$

Then

$$\lambda_{OA} = \frac{1}{\frac{9}{2}a}\left(\frac{3}{2}a\mathbf{i} - 3a\mathbf{j} + 3a\mathbf{k}\right) = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

For a rectangular coordinate system with origin at Point A and axes aligned with the given x, y, z axes, we have (using Figure 9.28)

$$I_x = I_z = \frac{3}{5}m\left[\frac{1}{4}a^2 + (3a)^2\right] \qquad I_y = \frac{3}{10}ma^2$$

$$= \frac{111}{20}ma^2$$

Also, symmetry implies $\quad I_{xy} = I_{yz} = I_{zx} = 0$

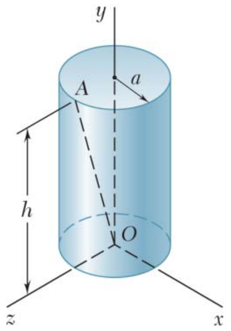
With the mass products of inertia equal to zero, Equation (9.46) reduces to

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2$$

$$= \frac{111}{20}ma^2\left(\frac{1}{3}\right)^2 + \frac{3}{10}ma^2\left(-\frac{2}{3}\right)^2 + \frac{111}{20}ma^2\left(\frac{2}{3}\right)^2$$

$$= \frac{193}{60}ma^2$$

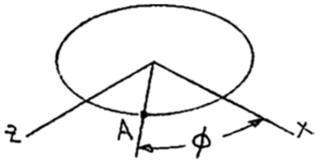
or $\quad I_{OA} = 3.22ma^2$ ◀



PROBLEM 9.164

The homogeneous circular cylinder shown has a mass *m*. Determine the mass moment of inertia of the cylinder with respect to the line joining the origin *O* and Point *A* that is located on the perimeter of the top surface of the cylinder.

SOLUTION



From Figure 9.28: $I_y = \frac{1}{2}ma^2$

and using the parallel-axis theorem

$$I_x = I_z = \frac{1}{12}m(3a^2 + h^2) + m\left(\frac{h}{2}\right)^2 = \frac{1}{12}m(3a^2 + 4h^2)$$

Symmetry implies $I_{xy} = I_{yz} = I_{zx} = 0$

For convenience, let Point *A* lie in the *yz* plane. Then

$$\lambda_{OA} = \frac{1}{\sqrt{h^2 + a^2}}(h\mathbf{j} + a\mathbf{k})$$

With the mass products of inertia equal to zero, Equation (9.46) reduces to

$$I_{OA} = \cancelto{0}{I_x\lambda_x^2} + I_y\lambda_y^2 + I_z\lambda_z^2$$

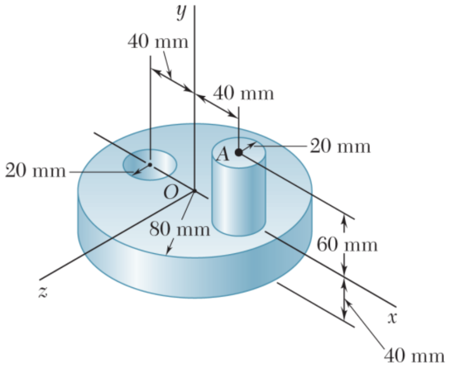
$$= \frac{1}{2}ma^2\left(\frac{h}{\sqrt{h^2 + a^2}}\right)^2 + \frac{1}{12}m(3a^2 + 4h^2)\left(\frac{a}{\sqrt{h^2 + a^2}}\right)^2$$

or $I_{OA} = \frac{1}{12}ma^2\frac{10h^2 + 3a^2}{h^2 + a^2}$ ◀

Note: For Point *A* located at an arbitrary point on the perimeter of the top surface, λ_{OA} is given by

$$\lambda_{OA} = \frac{1}{\sqrt{h^2 + a^2}}(a\cos\phi\mathbf{i} + h\mathbf{j} + a\sin\phi\mathbf{k})$$

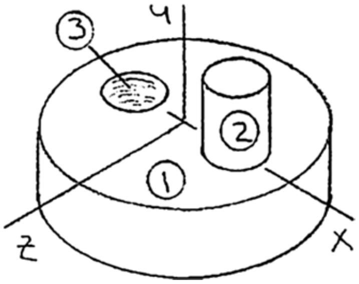
which results in the same expression for I_{OA}.



PROBLEM 9.165

Shown is the machine element of Problem 9.141. Determine its mass moment of inertia with respect to the line joining the origin *O* and Point *A*.

SOLUTION



First compute the mass of each component.

We have $$m = \rho_{\mathrm{ST}} V = \frac{0.284 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} V = (0.008819 \text{ lb} \cdot \text{s}^2/\text{ft} \cdot \text{in}^3)V$$

Then

$$m_1 = (7850 \text{ kg/m}^3)[\pi(0.08 \text{ m})^2(0.04 \text{ m})] = 6.31334 \text{ kg}$$
$$m_2 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.06 \text{ m})] = 0.59188 \text{ kg}$$
$$m_3 = (7850 \text{ kg/m}^3)[\pi(0.02 \text{ m})^2(0.04 \text{ m})] = 0.39458 \text{ kg}$$

Symmetry implies $$I_{yz} = I_{zx} = 0 \qquad (I_{xy})_1 = 0$$

and $$(\bar{I}_{x'y'})_2 = (\bar{I}_{x'y'})_3 = 0$$

Now
$$\begin{aligned} I_{xy} &= \Sigma(\bar{I}_{x'y'} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 - m_3\bar{x}_3\bar{y}_3 \\ &= [0.59188 \text{ kg } (0.04 \text{ m})(0.03 \text{ m})] - [0.39458 \text{ kg } (-0.04 \text{ m})(-0.02 \text{ m})] \\ &= (0.71026 - 0.31566) \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\ &= 0.39460 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

PROBLEM 9.165 (Continued)

From the solution to Problem 9.141, we have

$$I_x = 13.98800 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_y = 20.55783 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_z = 14.30368 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

By observation
$$\boldsymbol{\lambda}_{OA} = \frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{j})$$

Substituting into Eq. (9.46)

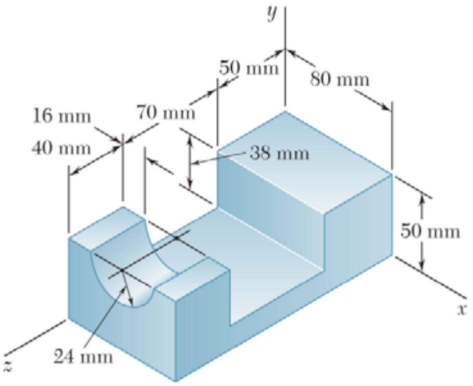
$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + \cancelto{0}{I_z\lambda_z^2} - 2I_{xy}\lambda_x\lambda_y - 2\cancelto{0}{I_{yz}}\lambda_y\lambda_z - 2\cancelto{0}{I_{zx}}\lambda_z\lambda_x$$

$$= \left[(13.98800)\left(\frac{2}{\sqrt{13}}\right)^2 + (20.55783)\left(\frac{3}{\sqrt{13}}\right)^2\right.$$

$$\left. - 2(0.39460)\left(\frac{2}{\sqrt{13}}\right)\left(\frac{2}{\sqrt{13}}\right)\right] \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$= (4.30400 + 14.23234 - 0.36425) \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

or $\quad I_{OA} = 18.17 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM 9.166

Determine the mass moment of inertia of the steel fixture of Problems 9.145 and 9.149 with respect to the axis through the origin that forms equal angles with the *x*, *y*, and *z* axes.

SOLUTION

From the solutions to Problems 9.145 and 9.149, we have

Problem 9.145: $I_x = 26.4325 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

$I_y = 31.1726 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

$I_z = 8.5773 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

Problem 9.149: $I_{xy} = 2.5002 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

$I_{yz} = 4.0627 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

$I_{zx} = 8.8062 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

From the problem statement it follows that

$$\lambda_x = \lambda_y = \lambda_z$$

Now $\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \Rightarrow 3\lambda_x^2 = 1$

or $\lambda_x = \lambda_y = \lambda_z = \dfrac{1}{\sqrt{3}}$

Substituting into Eq. (9.46)

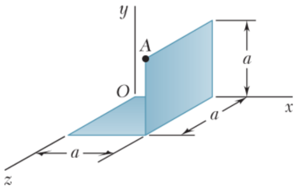
$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

Noting that $\lambda_x^2 = \lambda_y^2 = \lambda_z^2 = \lambda_x\lambda_y = \lambda_y\lambda_z = \lambda_z\lambda_x = \dfrac{1}{3}$

We have $I_{OL} = \dfrac{1}{3}[26.4325 + 31.1726 + 8.5773$

$-2(2.5002 + 4.0627 + 8.8062)] \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

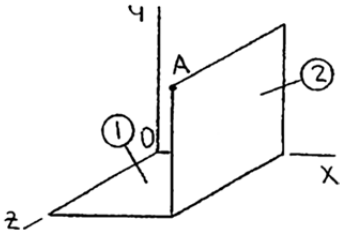
or $I_{OL} = 11.81 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM 9.167

The thin bent plate shown is of uniform density and weight W. Determine its mass moment of inertia with respect to the line joining the origin O and Point A.

SOLUTION



First note that

$$m_1 = m_2 = \frac{1}{2}\frac{W}{g}$$

and that

$$\lambda_{OA} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Using Figure 9.28 and the parallel-axis theorem, we have

$$I_x = (I_x)_1 + (I_x)_2$$

$$= \left[\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)^2\right]$$

$$+ \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)(a^2 + a^2) + \frac{1}{2}\frac{W}{g}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right]\right\}$$

$$= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{12} + \frac{1}{4}\right)a^2 + \left(\frac{1}{6} + \frac{1}{2}\right)a^2\right] = \frac{1}{2}\frac{W}{g}a^2$$

$$I_y = (I_y)_1 + (I_y)_2$$

$$= \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)(a^2 + a^2) + \frac{1}{2}\frac{W}{g}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right]\right\}$$

$$+ \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left[(a)^2 + \left(\frac{a}{2}\right)^2\right]\right\}$$

$$= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{6} + \frac{1}{2}\right)a^2 + \left(\frac{1}{12} + \frac{5}{4}\right)a^2\right] = \frac{W}{g}a^2$$

$$I_z = (I_z)_1 + (I_z)_2$$

$$= \left[\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)^2\right]$$

$$+ \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left[(a)^2 + \left(\frac{a}{2}\right)^2\right]\right\}$$

$$= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{12} + \frac{1}{4}\right)a^2 + \left(\frac{1}{12} + \frac{5}{4}\right)a^2\right] = \frac{5}{6}\frac{W}{g}a^2$$

PROBLEM 9.167 (Continued)

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of both components are zero because of symmetry. Also, $\bar{y}_1 = 0$

Then

$$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 = \frac{1}{2}\frac{W}{g}(a)\left(\frac{a}{2}\right) = \frac{1}{4}\frac{W}{g}a^2$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) = m_2\bar{y}_2\bar{z}_2 = \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) = \frac{1}{8}\frac{W}{g}a^2$$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) = m_1\bar{z}_1\bar{x}_1 + m_2\bar{z}_2\bar{x}_2$$

$$= \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)(a) = \frac{3}{8}\frac{W}{g}a^2$$

Substituting into Equation (9.46)

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

Noting that

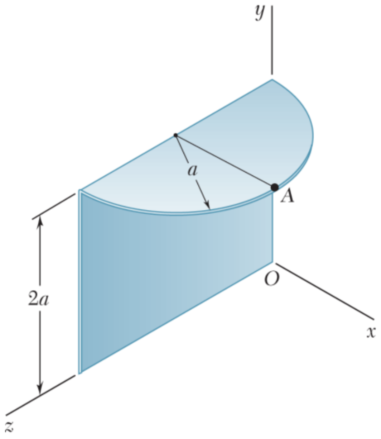
$$\lambda_x^2 = \lambda_y^2 = \lambda_z^2 = \lambda_x\lambda_y = \lambda_y\lambda_z = \lambda_z\lambda_x = \frac{1}{3}$$

We have

$$I_{OA} = \frac{1}{3}\left[\frac{1}{2}\frac{W}{g}a^2 + \frac{W}{g}a^2 + \frac{5}{6}\frac{W}{g}a^2 - 2\left(\frac{1}{4}\frac{W}{g}a^2 + \frac{1}{8}\frac{W}{g}a^2 + \frac{3}{8}\frac{W}{g}a^2\right)\right]$$

$$= \frac{1}{3}\left[\frac{14}{6} - 2\left(\frac{3}{4}\right)\right]\frac{W}{g}a^2$$

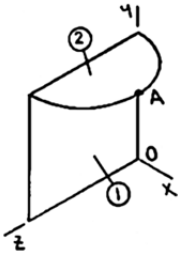
or $I_{OA} = \dfrac{5}{18}\dfrac{W}{g}a^2$ ◀



PROBLEM 9.168

A piece of sheet steel of thickness t and specific weight γ is cut and bent into the machine component shown. Determine the mass moment of inertia of the component with respect to the joining the origin O and Point A.

SOLUTION



First note that

$$\boldsymbol{\lambda}_{OA} = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

Next compute the mass of each component. We have

$$m = \rho V = \frac{\gamma}{g}(tA)$$

Then

$$m_1 = \frac{\gamma}{g}t(2a \times 2a) = 4\frac{\gamma t}{g}a^2$$

$$m_2 = \frac{\gamma}{g}t\left(\frac{\pi}{2} \times a^2\right) = \frac{\pi}{2}\frac{\gamma t}{g}a^2$$

Using Figure 9.28 for component 1 and the equations derived above (following the solution to Problem 9.134) for a semicircular plate for component 2, we have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 \\ &= \left\{\frac{1}{12}\left(4\frac{\gamma t}{g}a^2\right)[(2a)^2 + (2a)^2] + 4\frac{\gamma t}{g}a^2(a^2 + a^2)\right\} \\ &\quad + \left\{\frac{1}{4}\left(\frac{\pi}{2}\frac{\gamma t}{g}a^2\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2[(2a)^2 + (a)^2]\right\} \\ &= 4\frac{\gamma t}{g}a^2\left(\frac{2}{3} + 2\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{4} + 5\right)a^2 \\ &= 18.91335\frac{\gamma t}{g}a^4 \end{aligned}$$

PROBLEM 9.168 (Continued)

$$
\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 \\
&= \left[\frac{1}{12}\left(4\frac{\gamma t}{g}a^2\right)(2a)^2 + 4\frac{\gamma t}{g}a^2(a^2)\right] \\
&\quad + \left\{\frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{2}-\frac{16}{9\pi^2}\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left[\left(\frac{4a}{3\pi}\right)^2 + (a)^2\right]\Bigg]\right\} \\
&= 4\frac{\gamma t}{g}a^2\left(\frac{1}{3}+1\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{2}-\frac{16}{9\pi^2}+\frac{16}{9\pi^2}+1\right)a^2 \\
&= 7.68953\frac{\gamma t}{g}a^4
\end{aligned}
$$

$$
\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 \\
&= \left[\frac{1}{12}\left(4\frac{\gamma t}{g}a^2\right)(2a)^2 + 4\frac{\gamma t}{g}a^2(a^2)\right] \\
&\quad + \left\{\frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{4}-\frac{16}{9\pi^2}\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left[\left(\frac{4a}{3\pi}\right)^2 + (2a)^2\right]\Bigg]\right\} \\
&= 4\frac{\gamma t}{g}a^2\left(\frac{1}{3}+1\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{4}-\frac{16}{9\pi^2}+\frac{16}{9\pi^2}+4\right)a^2 \\
&= 12.00922\frac{\gamma t}{g}a^4
\end{aligned}
$$

Now observe that the centroidal products of inertia, $\overline{I}_{x'y'}, \overline{I}_{y'z'}$, and $\overline{I}_{z'x'}$, of both components are zero because of symmetry. Also $\overline{x}_1 = 0$.

Then

$$
\begin{aligned}
I_{xy} &= \Sigma(\cancelto{0}{\overline{I}_{x'y'}} + m\overline{x}\,\overline{y}) = m_2\overline{x}_2\overline{y}_2 = \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{4a}{3\pi}\right)(2a) \\
&= 1.33333\frac{\gamma t}{g}a^4
\end{aligned}
$$

$$
\begin{aligned}
I_{yz} &= \Sigma(\cancelto{0}{\overline{I}_{y'z'}} + m\overline{y}\,\overline{z}) = m_1\overline{y}_1\overline{z}_1 + m_2\overline{y}_2\overline{z}_2 \\
&= 4\frac{\gamma t}{g}a^2(a)(a) + \frac{\pi}{2}\frac{\gamma t}{g}a^2(2a)(a) \\
&= 7.14159\frac{\gamma t}{g}a^4
\end{aligned}
$$

$$
\begin{aligned}
I_{zx} &= \Sigma(\cancelto{0}{\overline{I}_{z'x'}} + m\overline{z}\,\overline{x}) = m_2\overline{z}_2\overline{x}_2 = \frac{\pi}{2}\frac{\gamma t}{g}a^2(a)\left(\frac{4a}{3\pi}\right) \\
&= 0.66667\frac{\gamma t}{g}a^4
\end{aligned}
$$

PROBLEM 9.168 (Continued)

Substituting into Eq. (9.46)

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

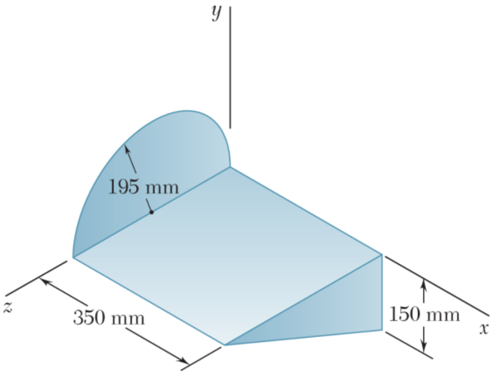
$$= 18.91335\frac{\gamma t}{g}a^4\left(\frac{1}{\sqrt{6}}\right)^2 + 7.68953\frac{\gamma t}{g}a^4\left(\frac{2}{\sqrt{6}}\right)^2$$

$$+12.00922\frac{\gamma t}{g}a^4\left(\frac{1}{\sqrt{6}}\right)^2 - 2\left(1.33333\frac{\gamma t}{g}a^4\right)\left(\frac{1}{\sqrt{6}}\right)\left(\frac{2}{\sqrt{6}}\right)$$

$$-2\left(7.14159\frac{\gamma t}{g}a^4\right)\left(\frac{2}{\sqrt{6}}\right)\left(\frac{2}{\sqrt{6}}\right) - 2\left(0.66667\frac{\gamma t}{g}a^4\right)\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right)$$

$$= (3.15223 + 5.12635 + 2.00154 - 0.88889 - 4.76106 - 0.22222)\frac{\gamma t}{g}a^4$$

or $I_{OA} = 4.41\frac{\gamma t}{g}a^4$ ◀



PROBLEM 9.169

Determine the mass moment of inertia of the machine component of Problems 9.136 and 9.155 with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-4\mathbf{i} + 8\mathbf{j} + \mathbf{k})/9$.

SOLUTION

From the solutions to Problems 9.136 and 9.155. We have

Problem 9.136:
$$I_x = 175.503 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$I_y = 308.629 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$I_z = 154.400 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

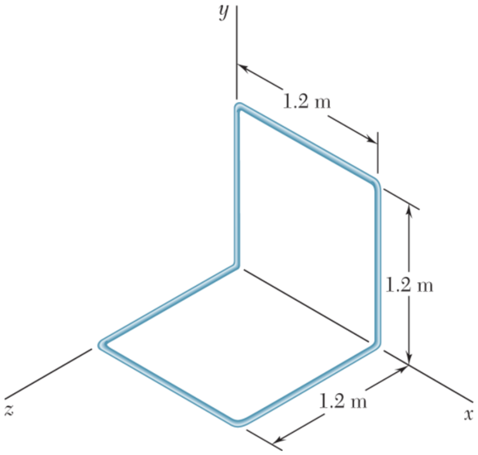
Problem 9.155:
$$I_{xy} = -8.0365 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$I_{yz} = 12.8950 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$I_{zx} = 94.0266 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

Substituting into Eq. (9.46)

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

$$= \left[175.503\left(-\frac{4}{9}\right)^2 + 308.629\left(\frac{8}{9}\right)^2 + 154.400\left(\frac{1}{9}\right)^2 \right.$$
$$-2(-8.0365)\left(-\frac{4}{9}\right)\left(\frac{8}{9}\right) - 2(12.8950)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right)$$
$$\left. -2(94.0266)\left(\frac{1}{9}\right)\left(-\frac{4}{9}\right)\right] \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$= (34.6673 + 243.855 + 1.906 - 6.350$$
$$-2.547 + 9.287) \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

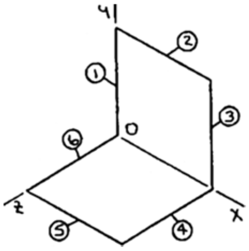
or $I_{OL} = 281 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM 9.170

For the wire figure of Problem 9.148, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION



First compute the mass of each component. We have

$$m = \left(\frac{m}{L}\right)L = 0.056 \text{ kg/m} \times 1.2 \text{ m}$$
$$= 0.0672 \text{ kg}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, for each component are zero because of symmetry.

Also

$$\bar{x}_1 = \bar{x}_6 = 0 \qquad \bar{y}_4 = \bar{y}_5 = \bar{y}_6 = 0 \qquad \bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$$

Then
$$I_{xy} = \Sigma(\overset{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 + m_3\bar{x}_3\bar{y}_3$$
$$= (0.0672 \text{ kg})(0.6 \text{ m})(1.2 \text{ m}) + (0.0672 \text{ kg})(1.2 \text{ m})(0.6 \text{ m})$$
$$= 0.096768 \text{ kg}\cdot\text{m}^2$$

$$I_{yz} = \Sigma(\overset{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) = 0$$

$$I_{zx} = \Sigma(\overset{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) = m_4\bar{z}_4\bar{x}_4 + m_5\bar{z}_5\bar{x}_5$$
$$= (0.0672 \text{ kg})(0.6 \text{ m})(1.2 \text{ m}) + (0.0672 \text{ kg})(1.2 \text{ m})(0.6 \text{ m})$$
$$= 0.096768 \text{ kg}\cdot\text{m}^2$$

From the solution to Problem 9.148, we have

$$I_x = 0.32258 \text{ kg}\cdot\text{m}^2$$
$$I_y = I_z = 0.41933 \text{ kg}\cdot\text{m}^2$$

PROBLEM 9.170 (Continued)

Substituting into Eq. (9.46)

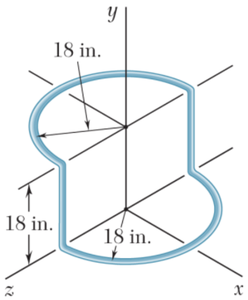
$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2\cancelto{0}{I_{yz}}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

$$= \left[0.32258\left(-\frac{3}{7}\right)^2 + 0.41933\left(-\frac{6}{7}\right)^2 + 0.41933\left(\frac{2}{7}\right)^2\right.$$

$$\left. - 2(0.096768)\left(-\frac{3}{7}\right)\left(-\frac{6}{7}\right) - 2(0.096768)\left(\frac{2}{7}\right)\left(-\frac{3}{7}\right)\right] \text{kg} \cdot \text{m}^2$$

$$I_{OL} = (0.059249 + 0.30808 + 0.034231 - 0.071095 + 0.023698)\ \text{kg} \cdot \text{m}^2$$

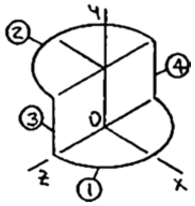
or $I_{OL} = 0.354\ \text{kg} \cdot \text{m}^2$ ◀



PROBLEM 9.171

For the wire figure of Problem 9.147, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \frac{\gamma_{\text{ST}}}{g} AL$$

Then

$$m_1 = m_2 = \frac{490\ \text{lb/ft}^3}{32.2\ \text{ft/s}^2} \times \left[\frac{\pi}{4}\left(\frac{1}{8}\ \text{in.}\right)^2\right] \times (\pi \times 18\ \text{in.}) \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^3$$

$$= 6.1112 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = m_4 = \frac{490\ \text{lb/ft}^3}{32.2\ \text{ft/s}^2} \times \left[\frac{\pi}{4}\left(\frac{1}{8}\ \text{in.}\right)^2\right] \times 18\ \text{in.} \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^3$$

$$= 1.9453\ \text{lb} \cdot \text{s}^2/\text{ft}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, for each component are zero because of symmetry.

Also $\qquad \bar{x}_3 = \bar{x}_4 = 0 \qquad \bar{y}_1 = 0 \qquad \bar{z}_1 = \bar{z}_2 = 0$

Then $\qquad I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2$

$$= (6.1112 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft})\left(-\frac{2 \times 18}{\pi}\ \text{in.}\right)(18\ \text{in.})\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$= -8.75480 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) = m_3\bar{y}_3\bar{z}_3 + m_4\bar{y}_4\bar{z}_4$$

Now $\qquad m_3 = m_4, \qquad \bar{y}_3 = \bar{y}_4, \qquad \bar{z}_4 = -\bar{z}_3 \qquad I_{yz} = 0$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) \quad \text{or} \quad I_{zx} = 0$$

PROBLEM 9.171 (Continued)

From the solution to Problem 9.147, we have

$$I_x = 39.1721 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

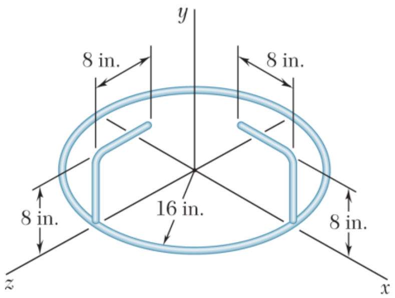
$$I_y = 36.2542 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_z = 30.4184 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

Substituting into Eq. (9.46)

$$\begin{aligned}
I_{OL} &= I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2\cancelto{0}{I_{yz}}\lambda_y\lambda_z - 2\cancelto{0}{I_{zx}}\lambda_z\lambda_x \\
&= \left[39.1721\left(-\frac{3}{7}\right)^2 + 36.2542\left(-\frac{6}{7}\right)^2 + 30.4184\left(\frac{2}{7}\right)^2 \right. \\
&\quad \left. - 2(-8.75480)\left(-\frac{3}{7}\right)\left(-\frac{6}{7}\right)\right] \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2 \\
&= (7.19488 + 26.6357 + 2.48313 + 6.43210) \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2
\end{aligned}$$

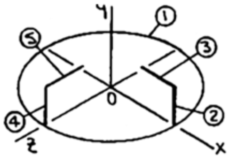
or $I_{OL} = 0.0427\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$ ◀



PROBLEM 9.172

For the wire figure of Problem 9.146, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION



First compute the mass of each component. We have

$$m = \frac{W}{g} = \frac{1}{g}(W/L)_{\text{AL}}\,L$$

Then

$$m_1 = \frac{1}{32.2\ \text{ft/s}^2}(0.033\ \text{lb/ft})(2\pi \times 16\ \text{in.}) \times \frac{1\ \text{ft}}{12\ \text{in.}}$$

$$= 8.5857 \times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$m_2 = m_3 = m_4 = m_5 = \frac{1}{32.2\ \text{ft/s}^2}(0.033\ \text{lb/ft})(8\ \text{in.}) \times \frac{1\ \text{ft}}{12\ \text{in.}}$$

$$= 0.6832 \times 10^{-3}\ \text{lb}\cdot\text{ft/s}^2$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Also

$$\bar{x}_1 = \bar{x}_4 = \bar{x}_5 = 0 \qquad \bar{y}_1 = 0 \qquad \bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$$

Then

$$I_{xy} = \Sigma(\overset{0}{\cancel{\bar{I}_{x'y'}}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 + m_3\bar{x}_3\bar{y}_3$$

$$= 0.6832 \times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}\left(\frac{16}{12}\ \text{ft}\right)\left(\frac{4}{12}\ \text{ft}\right)$$

$$+ 0.6832 \times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}\left(\frac{12}{12}\ \text{ft}\right)\left(\frac{8}{12}\ \text{ft}\right)$$

$$= (0.30364 + 0.45547) \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 0.75911 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

Symmetry implies

$$I_{yz} = I_{xy} \qquad I_{vz} = 0.75911 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_{zx} = \Sigma(\overset{0}{\cancel{\bar{I}_{z'x'}}} + m\bar{z}\,\bar{x}) = 0$$

PROBLEM 9.172 (Continued)

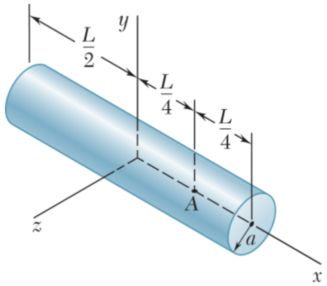
From the solution to Problem 9.146, we have

$$I_x = I_z = 10.3642 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$
$$I_y = 19.1097 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Substituting into Eq. (9.46)

$$\begin{aligned} I_{OL} &= I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2\cancelto{0}{I_{zx}}\lambda_z\lambda_x \\ &= \left[10.3642\left(-\frac{3}{7}\right)^2 + 19.1097\left(-\frac{6}{7}\right)^2 + 10.3642\left(\frac{2}{7}\right)^2 \right. \\ &\quad \left. - 2(0.75911)\left(-\frac{3}{7}\right)\left(-\frac{6}{7}\right) - 2(0.75911)\left(-\frac{6}{7}\right)\left(\frac{2}{7}\right) \right] \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \\ &= (1.90663 + 14.03978 + 0.84606 - 0.55771 + 0.37181) \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \end{aligned}$$

or $I_{OL} = 16.61 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ◀

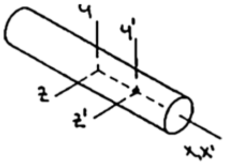


PROBLEM 9.173

For the homogeneous circular cylinder shown, of radius a and length L, determine the value of the ratio a/L for which the ellipsoid of inertia of the cylinder is a sphere when computed (*a*) at the centroid of the cylinder, (*b*) at Point A.

SOLUTION

(*a*) From Figure 9.28:



$$\bar{I}_x = \frac{1}{2}ma^2$$

$$\bar{I}_y = \bar{I}_z = \frac{1}{12}m(3a^2 + L^2)$$

Now observe that symmetry implies $I_{xy} = I_{yz} = I_{zx} = 0$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_x x^2 + I_y y^2 + I_z z^2 = 1: \quad \frac{1}{2}ma^2x^2 + \frac{1}{12}m(3a^2 + L^2)y^2 + \frac{1}{12}m(3a^2 + L^2) = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore

$$\frac{1}{2}ma^2 = \frac{1}{12}m(3a^2 + L^2) \qquad \text{or} \qquad \frac{a}{L} = \frac{1}{\sqrt{3}} \blacktriangleleft$$

(*b*) Using Figure 9.28 and the parallel-axis theorem, we have

$$I_{x'} = \frac{1}{2}ma^2$$

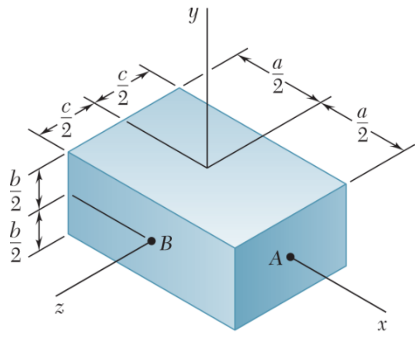
$$I_{y'} = I_{z'} = \frac{1}{12}m(3a^2 + L^2) + m\left(\frac{L}{4}\right)^2 = m\left(\frac{1}{4}a^2 + \frac{7}{48}L^2\right)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

From Part *a* it then immediately follows that

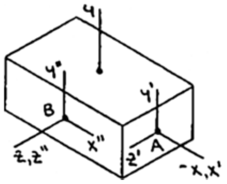
$$\frac{1}{2}ma^2 = m\left(\frac{1}{4}a^2 + \frac{7}{48}L^2\right) \qquad \text{or} \qquad \frac{a}{L} = \sqrt{\frac{7}{12}} \blacktriangleleft$$



PROBLEM 9.174

For the rectangular prism shown, determine the values of the ratios *b*/*a* and *c*/*a* so that the ellipsoid of inertia of the prism is a sphere when computed (*a*) at Point *A*, (*b*) at Point *B*.

SOLUTION



(*a*) Using Figure 9.28 and the parallel-axis theorem, we have at Point *A*

$$I_{x'} = \frac{1}{12}m(b^2 + c^2)$$

$$I_{y'} = \frac{1}{12}m(a^2 + c^2) + m\left(\frac{a}{2}\right)^2$$

$$= \frac{1}{12}m(4a^2 + c^2)$$

$$I_{z'} = \frac{1}{12}m(a^2 + b^2) + m\left(\frac{a}{2}\right)^2 = \frac{1}{12}m(4a^2 + b^2)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_{x'}x^2 + I_{y'}y^2 + I_{z'}z^2 = 1: \quad \frac{1}{12}m(b^2 + c^2)x^2 + \frac{1}{12}m(4a^2 + c^2)y^2 + \frac{1}{12}m(4a^2 + b^2)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore

$$\frac{1}{12}m(b^2 + c^2) = \frac{1}{12}m(4a^2 + c^2)$$

$$= \frac{1}{12}m(4a^2 + b^2)$$

Then $\quad b^2 + c^2 = 4a^2 + c^2 \quad$ or $\quad \frac{b}{a} = 2$ ◀

and $\quad b^2 + c^2 = 4a^2 + b^2 \quad$ or $\quad \frac{c}{a} = 2$ ◀

PROBLEM 9.174 (Continued)

(*b*) Using Figure 9.28 and the parallel-axis theorem, we have at Point *B*

$$I_{x''} = \frac{1}{12}m(b^2 + c^2) + m\left(\frac{c}{2}\right)^2 = \frac{1}{12}m(b^2 + 4c^2)$$

$$I_{y''} = \frac{1}{12}m(a^2 + c^2) + m\left(\frac{c}{2}\right)^2 = \frac{1}{12}m(a^2 + 4c^2)$$

$$I_{z''} = \frac{1}{12}m(a^2 + b^2)$$

Now observe that symmetry implies

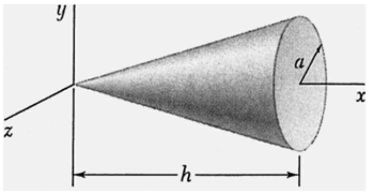
$$I_{x''y''} = I_{y''z''} = I_{z''x''} = 0$$

From part *a* it then immediately follows that

$$\frac{1}{12}m(b^2 + 4c^2) = \frac{1}{12}m(a^2 + 4c^2) = \frac{1}{12}m(a^2 + b^2)$$

Then $b^2 + 4c^2 = a^2 + 4c^2$ or $\frac{b}{a} = 1$ ◀

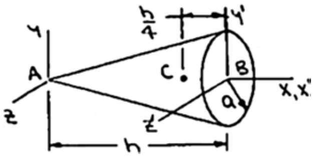
and $b^2 + 4c^2 = a^2 + b^2$ or $\frac{c}{a} = \frac{1}{2}$ ◀



PROBLEM 9.175

For the right circular cone of Sample Problem 9.11, determine the value of the ratio *a/h* for which the ellipsoid of inertia of the cone is a sphere when computed (*a*) at the apex of the cone, (*b*) at the center of the base of the cone.

SOLUTION



(*a*) From Sample Problem 9.11, we have at the apex *A*

$$I_x = \frac{3}{10}ma^2$$

$$I_y = I_z = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)$$

Now observe that symmetry implies $I_{xy} = I_{yz} = I_{zx} = 0$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_x x^2 + I_y y^2 + I_z z^2 = 1: \quad \frac{3}{10}ma^2x^2 + \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)y^2 + \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore,

$$\frac{3}{10}ma^2 = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right) \qquad \text{or} \qquad \frac{a}{h} = 2 \blacktriangleleft$$

(*b*) From Sample Problem 9.11, we have

$$I_{x'} = \frac{3}{10}ma^2$$

and at the centroid *C* $\quad I_{y''} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right)$

Then $\quad I_{y'} = I_{z'} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right) + m\left(\frac{h}{4}\right)^2 = \frac{1}{20}m(3a^2 + 2h^2)$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

From Part *a* it then immediately follows that

$$\frac{3}{10}ma^2 = \frac{1}{20}m(3a^2 + 2h^2) \qquad \text{or} \qquad \frac{a}{h} = \sqrt{\frac{2}{3}} \blacktriangleleft$$

PROBLEM 9.176

Given an arbitrary body and three rectangular axes x, y, and z, prove that the mass moment of inertia of the body with respect to any one of the three axes cannot be larger than the sum of the mass moments of inertia of the body with respect to the other two axes. That is, prove that the inequality $I_x \le I_y + I_z$ and the two similar inequalities are satisfied. Further, prove that $I_y \ge \frac{1}{2} I_x$ if the body is a homogeneous solid of revolution, where x is the axis of revolution and y is a transverse axis.

SOLUTION

(i) To prove $I_y + I_z \ge I_x$

By definition

$$I_y = \int (z^2 + x^2)dm$$

$$I_z = \int (x^2 + y^2)dm$$

Then

$$I_y + I_z = \int (z^2 + x^2)dm + \int (x^2 + y^2)dm$$

$$= \int (y^2 + z^2)dm + 2\int x^2 dm$$

Now $\int (y^2 + z^2)dm = I_x$ and $\int x^2 dm \ge 0$

$$I_y + I_z \ge I_x \qquad \text{Q.E.D.}$$

The proofs of the other two inequalities follow similar steps.

(ii) If the x axis is the axis of revolution, then

$$I_y = I_z$$

and from part (i) $I_y + I_z \ge I_x$

or $2I_y \ge I_x$

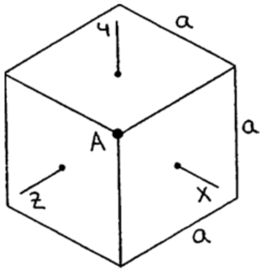
or

$$I_y \ge \frac{1}{2} I_x \qquad \text{Q.E.D.}$$

PROBLEM 9.177

Consider a cube of mass m and side a. (a) Show that the ellipsoid of inertia at the center of the cube is a sphere, and use this property to determine the moment of inertia of the cube with respect to one of its diagonals. (b) Show that the ellipsoid of inertia at one of the corners of the cube is an ellipsoid of revolution, and determine the principal moments of inertia of the cube at that point.

SOLUTION



(a) At the center of the cube have (using Figure 9.28)

$$I_x = I_y = I_z = \frac{1}{12}m(a^2 + a^2) = \frac{1}{6}ma^2$$

Now observe that symmetry implies

$$I_{xy} = I_{yz} = I_{zx} = 0$$

Using Equation (9.48), the equation of the ellipsoid of inertia is

$$\left(\frac{1}{6}ma^2\right)x^2 + \left(\frac{1}{6}ma^2\right)y^2 + \left(\frac{1}{6}ma^2\right)z^2 = 1$$

$$\text{or} \quad x^2 + y^2 + z^2 = \frac{6}{ma^2} \ (= R^2) \blacktriangleleft$$

which is the equation of a sphere.

Since the ellipsoid of inertia is a sphere, the moment of inertia with respect to any axis OL through the center O of the cube must always

be the same $\left(R = \dfrac{1}{\sqrt{I_{OL}}}\right)$.

$$I_{OL} = \frac{1}{6}ma^2 \blacktriangleleft$$

(b) The above sketch of the cube is the view seen if the line of sight is along the diagonal that passes through corner A. For a rectangular coordinate system at A and with one of the coordinate axes aligned with the diagonal, an ellipsoid of inertia at A could be constructed. If the cube is then rotated 120° about the diagonal, the mass distribution will remain unchanged. Thus, the ellipsoid will also remain unchanged after it is rotated. As noted at the end of Section 9.17, this is possible only if the ellipsoid is an ellipsoid of revolution, where the diagonal is both the axis of revolution and a principal axis.

It then follows that

$$I_{x'} = I_{OL} = \frac{1}{6}ma^2 \blacktriangleleft$$

In addition, for an ellipsoid of revolution, the two transverse principal moments of inertia are equal and any axis perpendicular to the axis of revolution is a principal axis. Then, applying the parallel-axis theorem between the center of the cube and corner A for any perpendicular axis

$$I_{y'} = I_{z'} = \frac{1}{6}ma^2 + m\left(\frac{\sqrt{3}}{2}a\right)^2$$

$$\text{or} \quad I_{y'} = I_{z'} = \frac{11}{12}ma^2 \blacktriangleleft$$

(*Note:* Part b can also be solved using the method of Section 9.18.)

PROBLEM 9.177 (Continued)

First note that at corner *A*

$$I_x = I_y = I_z = \frac{2}{3}ma^2$$

$$I_{xy} = I_{yz} = I_{zx} = \frac{1}{4}ma^2$$

Substituting into Equation (9.56) yields

$$k^3 - 2ma^2k^2 + \frac{55}{48}m^2a^6k - \frac{121}{864}m^3a^9 = 0$$

For which the roots are

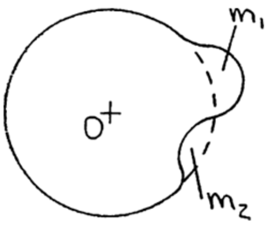
$$k_1 = \frac{1}{6}ma^2$$

$$k_2 = k_3 = \frac{11}{12}ma^2$$

PROBLEM 9.178

Given a homogeneous body of mass m and of arbitrary shape and three rectangular axes x, y, and z with origin at O, prove that the sum $I_x + I_y + I_z$ of the mass moments of inertia of the body cannot be smaller than the similar sum computed for a sphere of the same mass and the same material centered at O. Further, using the result of Problem 9.176, prove that if the body is a solid of revolution, where x is the axis of revolution, its mass moment of inertia I_y about a transverse axis y cannot be smaller than $3ma^2/10$, where a is the radius of the sphere of the same mass and the same material.

SOLUTION



(i) Using Equation (9.30), we have

$$I_x + I_y + I_z = \int (y^2 + z^2)dm + \int (z^2 + x^2)\,dm + \int (x^2 + y^2)dm$$

$$= 2\int (x^2 + y^2 + z^2)dm$$

$$= 2\int r^2 dm$$

where r is the distance from the origin O to the element of mass dm. Now assume that the given body can be formed by adding and subtracting appropriate volumes V_1 and V_2 from a sphere of mass m and radius a which is centered at O; it then follows that

$$m_1 = m_2 \ (m_{\text{body}} = m_{\text{sphere}} = m)$$

Then

$$(I_x + I_y + I_z)_{\text{body}} = (I_x + I_y + I_z)_{\text{sphere}} + (I_x + I_y + I_z)_{V_1} - (I_x + I_y + I_z)_{V_2}$$

or

$$(I_x + I_y + I_z)_{\text{body}} = (I_x + I_y + I_z)_{\text{sphere}} + 2\int_{m_1} r^2 dm - 2\int_{m_2} r^2 dm$$

Now, $m_1 = m_2$ and $r_1 \geq r_2$ for all elements of mass dm in volumes 1 and 2.

$$\int_{m_1} r^2 dm - \int_{m_2} r^2 dm \geq 0$$

so that

$$(I_x + I_y + I_z)_{\text{body}} \geq (I_x + I_y + I_z)_{\text{sphere}} \qquad \text{Q.E.D.}$$

PROBLEM 9.178 (Continued)

(ii) First note from Figure 9.28 that for a sphere

$$I_x = I_y = I_z = \frac{2}{5}ma^2$$

Thus $$(I_x + I_y + I_z)_{\text{sphere}} = \frac{6}{5}ma^2$$

For a solid of revolution, where the *x* axis is the axis of revolution, we have

$$I_y = I_z$$

Then, using the results of part (i)

$$(I_x + 2I_y)_{\text{body}} \geq \frac{6}{5}ma^2$$

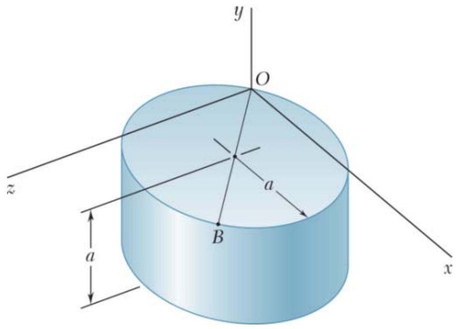
From Problem 9.178 we have $$I_y \geq \frac{1}{2}I_x$$

or $$(2I_y - I_x)_{\text{body}} \geq 0$$

Adding the last two inequalities yields

$$(4I_y)_{\text{body}} \geq \frac{6}{5}ma^2$$

or $$(I_y)_{\text{body}} \geq \frac{3}{10}ma^2 \quad \text{Q.E.D.}$$

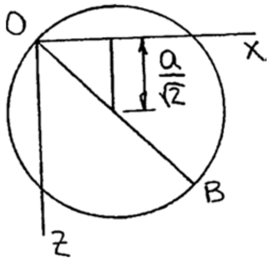


PROBLEM 9.179*

The homogeneous circular cylinder shown has a mass m, and the diameter OB of its top surface forms 45° angles with the x and z axes. (a) Determine the principal mass moments of inertia of the cylinder at the origin O. (b) Compute the angles that the principal axes of inertia at O form with the coordinate axes. (c) Sketch the cylinder, and show the orientation of the principal axes of inertia relative to the x, y, and z axes.

SOLUTION

(a) First compute the moments of inertia using Figure 9.28 and the parallel-axis theorem.



$$I_x = I_z = \frac{1}{12}m(3a^2 + a^2) + m\left[\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{2}\right)^2\right] = \frac{13}{12}ma^2$$

$$I_y = \frac{1}{2}ma^2 + m(a)^2 = \frac{3}{2}ma^2$$

Next observe that the centroidal products of inertia are zero because of symmetry. Then

$$I_{xy} = \cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y} = m\left(\frac{a}{\sqrt{2}}\right)\left(-\frac{a}{2}\right) = -\frac{1}{2\sqrt{2}}ma^2$$

$$I_{yz} = \cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z} = m\left(-\frac{a}{2}\right)\left(\frac{a}{\sqrt{2}}\right) = -\frac{1}{2\sqrt{2}}ma^2$$

$$I_{zx} = \cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x} = m\left(\frac{a}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}}\right) = \frac{1}{2}ma^2$$

Substituting into Equation (9.56)

$$K^3 - \left(\frac{13}{12} + \frac{3}{2} + \frac{13}{12}\right)ma^2K^2$$

$$+\left[\left(\frac{13}{12}\times\frac{3}{2}\right) + \left(\frac{3}{2}\times\frac{13}{12}\right) + \left(\frac{13}{12}\times\frac{13}{12}\right) - \left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{1}{2}\right)^2\right](ma^2)^2K$$

$$-\left[\left(\frac{13}{12}\times\frac{3}{2}\times\frac{13}{12}\right) - \left(\frac{13}{12}\right)\left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2\right.$$

$$\left.-\left(\frac{13}{12}\right)\left(-\frac{1}{2\sqrt{2}}\right)^2 - 2\left(-\frac{1}{2\sqrt{2}}\right)\left(-\frac{1}{2\sqrt{2}}\right)\left(\frac{1}{2}\right)\right](ma^2)^3 = 0$$

Simplifying and letting $K = ma^2\zeta$ yields

$$\zeta^3 - \frac{11}{3}\zeta^2 + \frac{565}{144}\zeta - \frac{95}{96} = 0$$

Solving yields

$$\zeta_1 = 0.363383 \qquad \zeta_2 = \frac{19}{12} \qquad \zeta_3 = 1.71995$$

The principal moments of inertia are then

$K_1 = 0.363ma^2$ ◀

$K_2 = 1.583ma^2$ ◀

$K_3 = 1.720ma^2$ ◀

(*b*) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, we use two of the equations of Equations (9.54) and (9.57).

Thus,

$$(I_x - K)\lambda_x - I_{xy}\lambda_y - I_{zx}\lambda_z = 0 \tag{9.54a}$$

$$-I_{zx}\lambda_x - I_{yz}\lambda_y + (I_z - K)\lambda_z = 0 \tag{9.54c}$$

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \tag{9.57}$$

(*Note:* Since $I_{xy} = I_{yz}$, Equations (9.54a) and (9.54c) were chosen to simplify the "elimination" of λ_y during the solution process.)

Substituting for the moments and products of inertia in Equations (9.54a) and (9.54c)

$$\left(\frac{13}{12}ma^2 - K\right)\lambda_x - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_y - \left(\frac{1}{2}ma^2\right)\lambda_z = 0$$

$$-\left(\frac{1}{2}ma^2\right)\lambda_x - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_y + \left(\frac{13}{12}ma^2 - K\right)\lambda_z = 0$$

or

$$\left(\frac{13}{12} - \zeta\right)\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y - \frac{1}{2}\lambda_z = 0 \tag{i}$$

and

$$-\frac{1}{2}\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y + \left(\frac{13}{12} - \zeta\right)\lambda_z = 0 \tag{ii}$$

Observe that these equations will be identical, so that one will need to be replaced, if

$$\frac{13}{12} - \zeta = -\frac{1}{2} \qquad \text{or} \qquad \zeta = \frac{19}{12}$$

Thus, a third independent equation will be needed when the direction cosines associated with K_2 are determined. Then for K_1 and K_3

Eq. (i) through Eq. (ii): $\left[\frac{13}{12}-\zeta-\left(-\frac{1}{2}\right)\right]\lambda_x+\left[-\frac{1}{2}-\left(\frac{13}{12}-\zeta\right)\right]\lambda_z=0$

or $\lambda_z=\lambda_x$

Substituting into Eq. (i): $\left(\frac{13}{12}-\zeta\right)\lambda_x+\frac{1}{2\sqrt{2}}\lambda_y-\frac{1}{2}\lambda_x=0$

or $\lambda_y=2\sqrt{2}\left(\zeta-\frac{7}{12}\right)\lambda_x$

Substituting into Equation (9.57):

$$\lambda_x^2+\left[2\sqrt{2}\left(\zeta-\frac{7}{12}\right)\lambda_x\right]^2+(\lambda_x)^2=1$$

or $$\left[2+8\left(\zeta-\frac{7}{12}\right)^2\right]\lambda_x^2=1 \qquad \text{(iii)}$$

$\mathbf{K}_1$: Substituting the value of ζ_1 into Eq. (iii):

$$\left[2+8\left(0.363383-\frac{7}{12}\right)^2\right](\lambda_x)_1^2=1$$

or $(\lambda_x)_1=(\lambda_z)_1=0.647249$

and then $(\lambda_y)_1=2\sqrt{2}\left(0.363383-\frac{7}{12}\right)(0.647249)$

$=-0.402662$

$(\theta_x)_1=(\theta_z)_1=49.7°\quad(\theta_y)_1=113.7°$ ◀

$\mathbf{K}_3$: Substituting the value of ζ_3 into Eq. (iii):

$$\left[2+8\left(1.71995-\frac{7}{12}\right)^2\right](\lambda_x)_3^2=1$$

or $(\lambda_x)_3=(\lambda_z)_3=0.284726$

and then $(\lambda_y)_3=2\sqrt{2}\left(1.71995-\frac{7}{12}\right)(0.284726)$

$=0.915348$

$(\theta_x)_3=(\theta_z)_3=73.5°\quad(\theta_y)_3=23.7°$ ◀

K$_2$: For this case, the set of equations to be solved consists of Equations (9.54a), (9.54b), and (9.57).

Now

$$-I_{xy}\lambda_x + (I_y - K)\lambda_y - I_{yz}\lambda_z = 0 \qquad (9.54b)$$

Substituting for the moments and products of inertia.

$$-\left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_x + \left(\frac{3}{2}ma^2 - K\right)\lambda_y - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_z = 0$$

or

$$\frac{1}{2\sqrt{2}}\lambda_x + \left(\frac{3}{2} - \xi\right)\lambda_y + \frac{1}{2\sqrt{2}}\lambda_z = 0 \qquad (iv)$$

Substituting the value of ξ_2 into Eqs. (i) and (iv):

$$\left(\frac{13}{12} - \frac{19}{12}\right)(\lambda_x)_2 + \frac{1}{2\sqrt{2}}(\lambda_y)_2 - \frac{1}{2}(\lambda_z)_2 = 0$$

$$\frac{1}{2\sqrt{2}}(\lambda_x)_2 + \left(\frac{3}{2} - \frac{19}{12}\right)(\lambda_y)_2 + \frac{1}{2\sqrt{2}}(\lambda_z)_2 = 0$$

or

$$-(\lambda_x)_2 + \frac{1}{\sqrt{2}}(\lambda_y)_2 - (\lambda_z)_2 = 0$$

and

$$(\lambda_x)_2 - \frac{\sqrt{2}}{6}(\lambda_y)_2 + (\lambda_z)_2 = 0$$

Adding yields

$$(\lambda_y)_2 = 0$$

and then

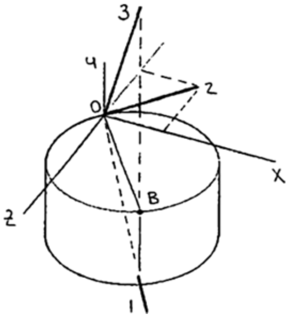
$$(\lambda_y)_2 = -(\lambda_x)_2$$

Substituting into Equation (9.57)

$$(\lambda_x)_2^2 + \overset{0}{\cancel{(\lambda_y)_2^2}} + (-\lambda_x)_2^2 = 1$$

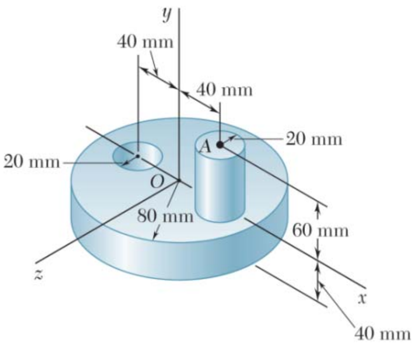
or

$$(\lambda_x)_2 = \frac{1}{\sqrt{2}} \quad \text{and} \quad (\lambda_z)_2 = -\frac{1}{\sqrt{2}}$$



$$(\theta_x)_2 = 45.0° \quad (\theta_y)_2 = 90.0° \quad (\theta_z)_2 = 135.0° \blacktriangleleft$$

(*c*) Principal axes 1 and 3 lie in the vertical plane of symmetry passing through Points *O* and *B*. Principal axis 2 lies in the *xz* plane.



PROBLEM 9.180

For the component described in Problem 9.165, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solutions to Problems 9.141 and 9.165 we have

Problem 9.141:

$$I_x = 13.98800 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$I_y = 20.55783 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$
$$I_z = 14.30368 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

Problem 9.165:

$$I_{yz} = I_{zx} = 0$$
$$I_{xy} = 0.39460 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

Eq. (9.55) then becomes

$$\begin{vmatrix} I_x - K & -I_{xy} & 0 \\ -I_{xy} & I_y - K & 0 \\ 0 & 0 & I_z - K \end{vmatrix} = 0 \quad \text{or} \quad (I_x - K)(I_y - K)(I_z - K) - (I_z - K)I_{xy}^2 = 0$$

Thus $\qquad I_z - K = 0 \qquad I_x I_y - (I_x + I_y)K + K^2 - I_{xy}^2 = 0$

Substituting: $\qquad K_1 = 14.30368 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

or $\qquad K_1 = 14.30 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

and

$$(13.98800 \times 10^{-3})(20.55783 \times 10^{-3}) - (13.98800 + 20.55783)(10^{-3})K + K^2 - (0.39460 \times 10^{-3})^2 = 0$$

or

$$K^2 - (34.54583 \times 10^{-3})K + 287.4072 \times 10^{-6} = 0$$

Solving yields $\qquad K_2 = 13.96438 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$

or $\qquad K_2 = 13.96 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

and $\qquad K_3 = 20.58145 \times 10^{-3}\ \text{kg}\cdot\text{m}^2 \qquad$ or $\qquad K_3 = 20.6 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

PROBLEM 9.180 (Continued)

(*b*) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, use two of the equations of Eq. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*) with $I_{yz} = I_{zx} = 0.$

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 = 0$$
$$-I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 = 0$$

Substituting:

$$[(13.98800 - 14.30368)\times 10^{-3}](\lambda_x)_1 - (0.39460\times 10^{-3})(\lambda_y)_1 = 0$$
$$-(0.39460\times 10^{-3})(\lambda_x)_1 + [(20.55783 - 14.30368)\times 10^{-3}](\lambda_y)_1 = 0$$

Adding yields $(\lambda_x)_1 = (\lambda_y)_1 = 0$

Then using Eq. (9.57) $\cancelto{0}{(\lambda_x)_1^2} + \cancelto{0}{(\lambda_y)_1^2} + (\lambda_z)_1^2 = 1$

or $(\lambda_z)_1 = 1$

$(\theta_x)_1 = 90.0° \quad (\theta_y)_1 = 90.0° \quad (\theta_z)_1 = 0°$ ◀

K_2: Begin with Eqs. (9.54*b*) and (9.54*c*) with $I_{yz} = I_{zx} = 0.$

$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 = 0 \qquad \text{(i)}$$
$$(I_z - K_2)(\lambda_z)_2 = 0$$

Now $I_z \neq K_2 \Rightarrow (\lambda_z)_2 = 0$

Substituting into Eq. (i):

$$-(0.39460\times 10^{-3})(\lambda_x)_z + [(20.55783 - 13.96438)\times 10^{-3}](\lambda_y)_2 = 0$$

or $(\lambda_y)_2 = 0.059847(\lambda_x)_2$

Using Eq. (9.57): $(\lambda_x)_2^2 + [0.05984](\lambda_x)_2]^2 + \cancelto{0}{(\lambda_z)_2^2} = 1$

or $(\lambda_x)_2 = 0.998214$

and $(\lambda_y)_2 = 0.059740$

$(\theta_x)_2 = 3.4° \qquad (\theta_y)_2 = 86.6° \qquad (\theta_z)_2 = 90.0°$ ◀

K_3: Begin with Eqs. (9.54*b*) and (9.54*c*) with $I_{yz} = I_{zx} = 0.$

$$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 = 0 \qquad \text{(ii)}$$
$$(I_z - K_3)(\lambda_z)_3 = 0$$

PROBLEM 9.180 (Continued)

Now $$I_z \neq K_3 \Rightarrow (\lambda_z)_3 = 0$$

Substituting into Eq. (ii):

$$-(0.39460\times10^{-3})(\lambda_x)_3 + [(20.55783-20.58145)\times10^{-3}](\lambda_y)_3 = 0$$

or $$(\lambda_y)_3 = -16.70618(\lambda_x)_3$$

Using Eq. (9.57): $$(\lambda_x)_3^2 + [-16.70618(\lambda_x)_3]^2 + \cancelto{0}{(\lambda_z)_3^2} = 1$$

or $$(\lambda_x)_3 = -0.059751 \quad \text{(axes right-handed set } \Rightarrow \text{"\_")}$$

and $$(\lambda_y)_3 = 0.998211$$

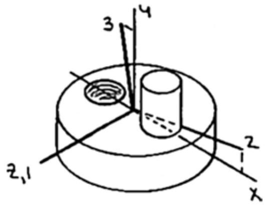
$$(\theta_x)_3 = 93.4^\circ \qquad (\theta_y)_3 = 3.43^\circ \qquad (\theta_z)_3 = 90.0^\circ \blacktriangleleft$$

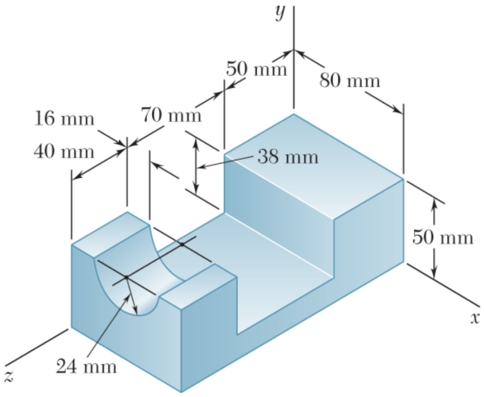
Note: Since the principal axes are orthogonal and $(\theta_z)_2 = (\theta_z)_3 = 90^\circ$, it follows that

$$|(\lambda_x)_2| = |(\lambda_y)_3| \qquad |(\lambda_y)_2| = |(\lambda_z)_3|$$

The differences in the above values are due to round-off errors.

(*c*) Principal axis 1 coincides with the *z* axis, while principal axes 2 and 3 lie in the *xy* plane.





PROBLEM 9.181*

For the component described in Problems 9.145 and 9.149, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solutions to Problems 9.145 and 9.149, we have

Problem 9.145: $I_x = 26.4325\times10^{-3}\ \text{kg}\cdot\text{m}^2$ Problem 9.149: $I_{xy} = 2.5002\times10^{-3}\ \text{kg}\cdot\text{m}^2$

$I_y = 31.1726\times10^{-3}\ \text{kg}\cdot\text{m}^2$ $I_{yz} = 4.0627\times10^{-3}\ \text{kg}\cdot\text{m}^2$

$I_z = 8.5773\times10^{-3}\ \text{kg}\cdot\text{m}^2$ $I_{zx} = 8.8062\times10^{-3}\ \text{kg}\cdot\text{m}^2$

Substituting into Eq. (9.56):

$$\begin{aligned}
&K^3 - [(26.4325 + 31.1726 + 8.5773)(10^{-3})]K^2\\
&\quad + [(26.4325)(31.1726) + (31.1726)(8.5773) + (8.5773)(26.4325)\\
&\quad - (2.5002)^2 - (4.0627)^2 - (8.8062)^2](10^{-6})K\\
&\quad - [(26.4325)(31.1726)(8.5773) - (26.4325)(4.0627)^2\\
&\quad - (31.1726)(8.8062)^2 - (8.5773)(2.5002)^2\\
&\quad - 2(2.5002)(4.0627)(8.8062)](10^{-9}) = 0
\end{aligned}$$

or $$K^3 - (66.1824\times10^{-3})K^2 + (1217.76\times10^{-6})K - (3981.23\times10^{-9}) = 0$$

Solving yields

$K_1 = 4.1443\times10^{-3}\ \text{kg}\cdot\text{m}^2$ or $K_1 = 4.14\times10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

$K_2 = 29.7840\times10^{-3}\ \text{kg}\cdot\text{m}^2$ or $K_2 = 29.8\times10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

$K_3 = 32.2541\times10^{-3}\ \text{kg}\cdot\text{m}^2$ or $K_3 = 32.3\times10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

PROBLEM 9.181* (Continued)

(*b*) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, use two of the equations of Eqs. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$
$$I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting:

$$[(26.4325 - 4.1443)(10^{-3})](\lambda_x)_1 - (2.5002 \times 10^{-3})(\lambda_y)_1 - (8.8062 \times 10^{-3})(\lambda_z)_1 = 0$$

$$-(2.5002 \times 10^{-3})(\lambda_x)_1 + [(31.1726 - 4.1443)(10^{-3})](\lambda_y)_1 - (4.0627 \times 10^{-3})(\lambda_z)_1 = 0$$

Simplifying

$$8.9146(\lambda_x)_1 - (\lambda_y)_1 - 3.5222(\lambda_z)_1 = 0$$
$$-0.0925(\lambda_x)_1 + (\lambda_y)_1 - 0.1503(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$:

$$(\lambda_z)_1 = 2.4022(\lambda_x)_1$$

and then

$$(\lambda_y)_1 = [8.9146 - 3.5222(2.4022)](\lambda_x)_1$$
$$= 0.45357(\lambda_x)_1$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_1^2 + [0.45357(\lambda_x)_1]^2 + [2.4022(\lambda_x)_1]^2 = 1$$

or $(\lambda_x)_1 = 0.37861$

and $(\lambda_y)_1 = 0.17173$ $(\lambda_z)_1 = 0.90950$

$$(\theta_x)_1 = 67.8° \quad (\theta_y)_1 = 80.1° \quad (\theta_z)_1 = 24.6° \blacktriangleleft$$

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

Substituting:

$$[(26.4325 - 29.7840)(10^{-3})](\lambda_x)_2 - (2.5002 \times 10^{-3})(\lambda_y)_2 - (8.8062 \times 10^{-3})(\lambda_z)_2 = 0$$

$$-(2.5002 \times 10^{-3})(\lambda_x)_2 + [(31.1726 - 29.7840)(10^{-3})](\lambda_y)_2 - (4.0627 \times 10^{-3})(\lambda_z)_2 = 0$$

PROBLEM 9.181* (Continued)

Simplifying

$$-1.3405(\lambda_x)_2 - (\lambda_y)_2 - 3.5222(\lambda_z)_2 = 0$$
$$-1.8005(\lambda_x)_2 + (\lambda_y)_2 - 2.9258(\lambda_z)_2 = 0$$

Adding and solving for $(\lambda_z)_2$:

$$(\lambda_z)_2 = -0.48713(\lambda_x)_2$$

and then

$$(\lambda_y)_2 = [-1.3405 - 3.5222(-0.48713)](\lambda_x)_2$$
$$= 0.37527(\lambda_x)_2$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2 + [0.37527(\lambda_x)_2]^2 + [-0.48713(\lambda_x)_2]^2 = 1$$

or $(\lambda_x)_2 = 0.85184$

and $(\lambda_y)_2 = 0.31967$ $(\lambda_z)_2 = -0.41496$

$$(\theta_x)_1 = 31.6° \quad (\theta_y)_2 = 71.4° \quad (\theta_z)_2 = 114.5° \blacktriangleleft$$

K_3: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_3)(\lambda_x)_3 - I_{xy}(\lambda_y)_3 - I_{zx}(\lambda_z)_3 = 0$$
$$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 - I_{yz}(\lambda_z)_3 = 0$$

Substituting:

$$[(26.4325 - 32.2541)(10^{-3})](\lambda_x)_3 - (2.5002 \times 10^{-3})(\lambda_y)_3 - (8.8062 \times 10^{-3})(\lambda_z)_3 = 0$$

$$-(2.5002 \times 10^{-3})(\lambda_x)_3 + [(31.1726 - 32.2541)(10^{-3})](\lambda_y)_3 - (4.0627 \times 10^{-3})(\lambda_z)_3 = 0$$

Simplifying

$$-2.3285(\lambda_x)_3 - (\lambda_y)_3 - 3.5222(\lambda_z)_3 = 0$$
$$2.3118(\lambda_x)_3 + (\lambda_y)_3 + 3.7565(\lambda_z)_3 = 0$$

Adding and solving for $(\lambda_z)_3$:

$$(\lambda_z)_3 = 0.071276(\lambda_x)_3$$

and then

$$(\lambda_y)_3 = [-2.3285 - 3.5222(0.071276)](\lambda_x)_3$$
$$= -2.5795(\lambda_x)_3$$

PROBLEM 9.181* (Continued)

Now substitute into Eq. (9.57):

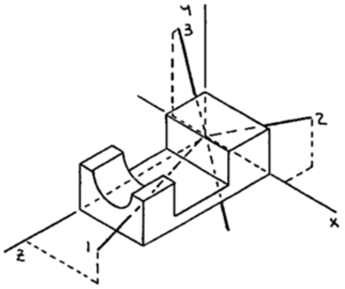
$$(\lambda_x)_3^2 + [-2.5795(\lambda_x)_3]^2 + [0.071276(\lambda_x)_3]^2 = 1 \qquad \text{(i)}$$

or $$(\lambda_x)_3 = 0.36134$$

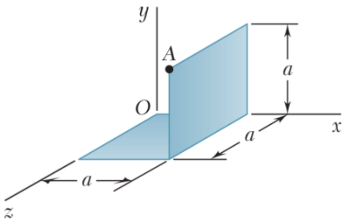
and $(\lambda_y)_3 = 0.93208$ $\qquad (\lambda_z)_3 = 0.025755$

$$(\theta_x)_3 = 68.8° \quad (\theta_y)_3 = 158.8° \quad (\theta_z)_3 = 88.5° \blacktriangleleft$$

(*c*) *Note:* Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.36134$.



Then $$(\theta_x)_3 = 111.2° \quad (\theta_y)_3 = 21.2° \quad (\theta_z)_3 = 91.5° \blacktriangleleft$$



PROBLEM 9.182*

For the component described in Problem 9.167, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solution of Problem 9.167, we have

$$I_x = \frac{1}{2}\frac{W}{g}a^2 \qquad I_{xy} = \frac{1}{4}\frac{W}{g}a^2$$

$$I_y = \frac{W}{g}a^2 \qquad I_{yz} = \frac{1}{8}\frac{W}{g}a^2$$

$$I_z = \frac{5}{6}\frac{W}{g}a^2 \qquad I_{zx} = \frac{3}{8}\frac{W}{g}a^2$$

Substituting into Eq. (9.56):

$$K^3 - \left[\left(\frac{1}{2}+1+\frac{5}{6}\right)\left(\frac{W}{g}a^2\right)\right]K^2$$

$$+\left[\left(\frac{1}{2}\right)(1)+(1)\left(\frac{5}{6}\right)+\left(\frac{5}{6}\right)\left(\frac{1}{2}\right)-\left(\frac{1}{4}\right)^2-\left(\frac{1}{8}\right)^2-\left(\frac{3}{8}\right)^2\right]\left(\frac{W}{g}a^2\right)^2 K$$

$$-\left[\left(\frac{1}{2}\right)(1)\left(\frac{5}{6}\right)-\left(\frac{1}{2}\right)\left(\frac{1}{8}\right)^2-(1)\left(\frac{3}{8}\right)^2-\left(\frac{5}{6}\right)\left(\frac{1}{4}\right)^2-2\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)\left(\frac{3}{8}\right)\right]\left(\frac{W}{g}a^2\right)^3=0$$

Simplifying and letting $K = \frac{W}{g}a^2K$ yields

$$K^3 - 2.33333K^2 + 1.53125K - 0.192708 = 0$$

Solving yields

$$K_1 = 0.163917 \quad K_2 = 1.05402 \quad K_3 = 1.11539$$

The principal moments of inertia are then $K_1 = 0.1639\frac{W}{g}a^2$ ◀

$K_2 = 1.054\frac{W}{g}a^2$ ◀

$K_3 = 1.115\frac{W}{g}a^2$ ◀

PROBLEM 9.182* (Continued)

(*b*) To determine the direction cosines $\lambda_x,\ \lambda_y,\ \lambda_z$ of each principal axis, use two of the equations of Eq. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$
$$-I_{xy}(\lambda_x)_1 + (I_y - K_2)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting

$$\left[\left(\frac{1}{2} - 0.163917\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_1 - \left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_1 - \left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_1 = 0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_1 + \left[(1 - 0.163917)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_1 - \left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_1 = 0$$

Simplifying

$$1.34433(\lambda_x)_1 - (\lambda_y)_1 - 1.5(\lambda_z)_1 = 0$$

$$-0.299013(\lambda_x)_1 + (\lambda_y)_1 - 0.149507(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$:

$$(\lambda_z)_1 = 0.633715(\lambda_x)_1$$

and then
$$(\lambda_y)_1 = [1.34433 - 1.5(0.633715)](\lambda_x)_1$$
$$= 0.393758(\lambda_x)_1$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_1^2 + [0.393758(\lambda_x)_1]^2 + (0.633715(\lambda_x)_1]^2 = 1$$

or $(\lambda_x)_1 = 0.801504$

and $(\lambda_y)_1 = 0.315599$ $(\lambda_z)_1 = 0.507925$

$(\theta_x)_1 = 36.7°$ $(\theta_y)_1 = 71.6°$ $(\theta_z)_1 = 59.5°$ ◀

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - k_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - k_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

PROBLEM 9.182* (Continued)

Substituting

$$\left[\left(\frac{1}{2}-1.05402\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_2-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_2-\left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_2=0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_2+\left[(1-1.05402)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_2-\left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_2=0$$

Simplifying

$$-2.21608(\lambda_x)_2-(\lambda_y)_2-1.5(\lambda_z)_2=0$$

$$4.62792(\lambda_x)_2+(\lambda_y)_2+2.31396(\lambda_z)_2=0$$

Adding and solving for $(\lambda_z)_2$

$$(\lambda_z)_2=-2.96309(\lambda_x)_2$$

and then

$$(\lambda_y)_2=[-2.21608-1.5(-2.96309)](\lambda_x)_2$$

$$=2.22856(\lambda_x)_2$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2+[2.22856(\lambda_x)_2]^2+[-2.96309(\lambda_x)_2]^2=1$$

or $(\lambda_x)_2=0.260410$

and $(\lambda_y)_2=0.580339$ $(\lambda_z)_2=-0.771618$

$$(\theta_x)_2=74.9° \quad (\theta_y)_2=54.5° \quad (\theta_z)_2=140.5° \blacktriangleleft$$

K_3: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x-K_3)(\lambda_x)_3-I_{xy}(\lambda_y)_3-I_{zx}(\lambda_z)_3=0$$

$$-I_{xy}(\lambda_x)_3+(I_y-K_3)(\lambda_y)_3-I_{yz}(\lambda_z)_3=0$$

Substituting

$$\left[\left(\frac{1}{2}-1.11539\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_3-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_3-\left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_3=0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_3+\left[(1-1.11539)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_3-\left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_3=0$$

Simplifying

$$-2.46156(\lambda_x)_3-(\lambda_y)_3-1.5(\lambda_z)_3=0$$

$$2.16657(\lambda_x)_3+(\lambda_y)_3+1.08328(\lambda_z)_3=0$$

PROBLEM 9.182* (Continued)

Adding and solving for $(\lambda_z)_3$

$$(\lambda_z)_3 = -0.707885(\lambda_x)_3$$

and then

$$\begin{aligned}(\lambda_y)_3 &= [-2.46156 - 1.5(-0.707885)](\lambda_x)_3 \\ &= -1.39973(\lambda_x)_3\end{aligned}$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_3^2 + [-1.39973(\lambda_x)_3]^2 + [-0.707885(\lambda_x)_3]^2 = 1 \qquad \text{(i)}$$

or

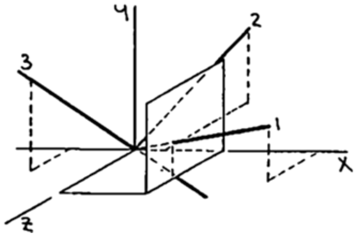
$$(\lambda_x)_3 = 0.537577$$

and

$$(\lambda_y)_3 = -0.752463 \qquad (\lambda_z)_3 = -0.380543$$

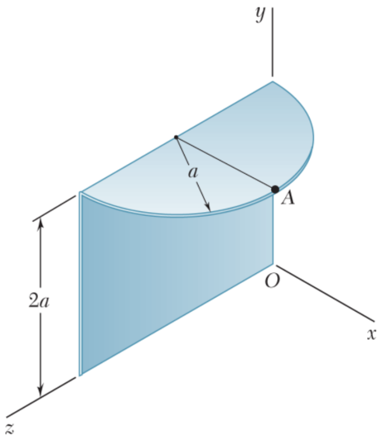
$$(\theta_x)_3 = 57.5° \quad (\theta_y)_3 = 138.8° \quad (\theta_z)_3 = 112.4° \blacktriangleleft$$

(*c*) *Note:* Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.537577$.



Then

$$(\theta_x)_3 = 122.5° \quad (\theta_y)_3 = 41.2° \quad (\theta_z)_3 = 67.6° \blacktriangleleft$$



PROBLEM 9.183*

For the component described in Problem 9.168, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solution to Problem 9.168, we have

$$I_x = 18.91335\frac{\gamma t}{g}a^4 \qquad I_{xy} = 1.33333\frac{\gamma t}{g}a^4$$

$$I_y = 7.68953\frac{\gamma t}{g}a^4 \qquad I_{yz} = 7.14159\frac{\gamma t}{g}a^4$$

$$I_z = 12.00922\frac{\gamma t}{g}a^4 \qquad I_{zx} = 0.66667\frac{\gamma t}{g}a^4$$

Substituting into Eq. (9.56):

$$K^3 - \left[(18.91335 + 7.68953 + 12.00922)\left(\frac{\gamma t}{g}a^4\right)\right]K^2$$

$$+[(18.91335)(7.68953) + (7.68953)(12.00922) + (12.00922)(18.91335)$$

$$-(1.33333)^2 - (7.14159)^2 + (0.66667)^2]\left(\frac{\gamma t}{g}a^4\right)^2 K$$

$$-[(18.91335)(7.68953)(12.00922) - (18.91335)(7.14159)^2$$

$$-(7.68953)(0.66667)^2 - (12.00922)(1.33333)^2$$

$$-2(1.33333)(7.14159)(0.66667)]\left(\frac{\gamma t}{g}a^4\right)^3 = 0$$

Simplifying and letting

$$K = \frac{\gamma t}{g}a^4 K \quad \text{yields}$$

$$K^3 - 38.61210K^2 + 411.69009K - 744.47027 = 0$$

PROBLEM 9.183* (Continued)

Solving yields

$$K_1 = 2.25890 \quad K_2 = 17.27274 \quad K_3 = 19.08046$$

The principal moments of inertia are then $K_1 = 2.26\dfrac{\gamma t}{g}a^4$ ◀

$$K_2 = 17.27\frac{\gamma t}{g}a^4 \blacktriangleleft$$

$$K_3 = 19.08\frac{\gamma t}{g}a^4 \blacktriangleleft$$

(*b*) To determine the direction cosines $\lambda_x,\ \lambda_y,\ \lambda_z$ of each principal axis, use two of the equations of Eq. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$
$$-I_{xy}(\lambda_x)_1 + (I_y - K_2)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting

$$\left[(18.91335 - 2.25890)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_x)_1 - \left(1.33333\frac{\gamma t}{a}a^4\right)(\lambda_y)_1 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_1 = 0$$

$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_1 + \left[(7.68953 - 2.25890)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_1 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_1 = 0$$

Simplifying

$$12.49087(\lambda_x)_1 - (\lambda_y)_1 - 0.5(\lambda_z)_1 = 0$$
$$-0.24552(\lambda_x)_1 + (\lambda_y)_1 - 1.31506(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$

$$(\lambda_z)_1 = 6.74653(\lambda_x)_1$$

and then
$$(\lambda_y)_1 = [12.49087 - (0.5)(6.74653)](\lambda_x)_1$$
$$= 9.11761(\lambda_x)_1$$

Now substitute into Eq. (9.57): $(\lambda_x)_1^2 + [9.11761(\lambda_x)_1]^2 + [6.74653(\lambda_x)_1]^2 = 1$

or $(\lambda_x)_1 = 0.087825$

and $(\lambda_y)_1 = 0.80075 \qquad (\lambda_z)_1 = 0.59251$

$$(\theta_x)_1 = 85.0° \quad (\theta_y)_1 = 36.8° \quad (\theta_z)_1 = 53.7° \blacktriangleleft$$

PROBLEM 9.183* (Continued)

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

Substituting

$$[(18.91335 - 17.27274)\left(\frac{\gamma t}{g}a^4\right)(\lambda_x)_2 - \left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_y)_2 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_2 = 0$$
$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_2 + \left[(7.68953 - 17.27274)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_2 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_2 = 0$$

Simplifying

$$1.23046(\lambda_x)_2 - (\lambda_y)_2 - 0.5(\lambda_z)_2 = 0$$
$$0.13913(\lambda_x)_2 + (\lambda_y)_2 + 0.74522(\lambda_z)_2 = 0$$

Adding and solving for $(\lambda_z)_2$

$$(\lambda_z)_2 = -5.58515(\lambda_x)_2$$

and then $(\lambda_y)_2 = [1.23046 - (0.5)(-5.58515)](\lambda_x)_2$

$$= 4.02304\,(\lambda_x)_2$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2 + [4.02304(\lambda_x)_2]^2 + [-5.58515(\lambda_x)_2]^2 = 1$$

or $(\lambda_x)_2 = 0.14377$

and $(\lambda_y)_2 = 0.57839 \qquad (\lambda_z)_2 = -0.80298$

$$(\theta_x)_2 = 81.7° \quad (\theta_y)_2 = 54.7° \quad (\theta_z)_2 = 143.4° \blacktriangleleft$$

K_3: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - K_3)(\lambda_x)_3 - I_{xy}(\lambda_y)_3 - I_{zx}(\lambda_z)_3 = 0$$
$$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 - I_{yz}(\lambda_y)_3 = 0$$

Substituting

$$[(18.91335 - 19.08046)\left(\frac{\gamma t}{g}a^4\right)(\lambda_x)_3 - \left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_y)_3 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_3 = 0$$
$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_3 + \left[(7.68953 - 19.08046)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_3 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_3 = 0$$

PROBLEM 9.183* (Continued)

Simplifying

$$-0.12533(\lambda_x)_3 - (\lambda_y)_3 - 0.5(\lambda_z)_3 = 0$$
$$0.11705(\lambda_x)_3 + (\lambda_y)_3 + 0.62695(\lambda_z)_3 = 0$$

Adding and solving for $(\lambda_z)_3$

$$(\lambda_z)_3 = 0.06522(\lambda_x)_3$$

and then
$$(\lambda_y)_3 = [-0.12533 - (0.5)(0.06522)](\lambda_x)_3$$
$$= -0.15794(\lambda_x)_3$$

Now substitute into Eq. (9.57):

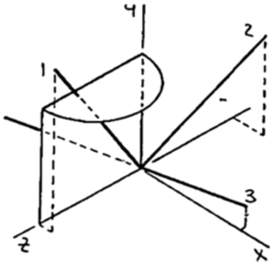
$$(\lambda_x)_3^2 + [-0.15794(\lambda_x)_3]^2 + [0.06522(\lambda_x]_3]^2 = 1 \qquad \text{(i)}$$

or
$$(\lambda_x)_3 = 0.98571$$

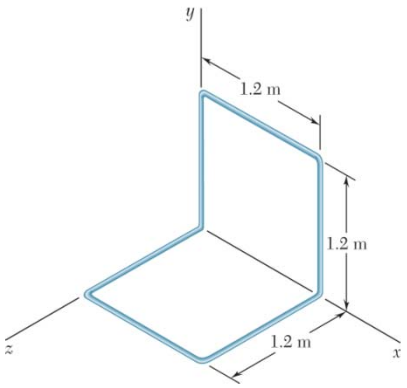
and
$$(\lambda_y)_3 = -0.15568 \qquad (\lambda_z)_3 = 0.06429$$

$$(\theta_x)_3 = 9.7° \quad (\theta_y)_3 = 99.0° \quad (\theta_z)_3 = 86.3° ◀$$

(*c*) *Note:* Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.98571$.



Then
$$(\theta_x)_3 = 170.3° \quad (\theta_y)_3 = 81.0° \quad (\theta_z)_3 = 93.7° ◀$$



PROBLEM 9.184*

For the component described in Problems 9.148 and 9.170, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solutions to Problems 9.148 and 9.170. We have

$$I_x = 0.32258\ \text{kg}\cdot\text{m}^2 \qquad I_y = I_z = 0.41933\ \text{kg}\cdot\text{m}^2$$

$$I_{xy} = I_{zx} = 0.096768\ \text{kg}\cdot\text{m}^2 \qquad I_{yz} = 0$$

Substituting into Eq. (9.56) and using

$$I_y = I_z \qquad I_{xy} = I_{zx} \qquad I_{yz} = 0$$

$$\begin{aligned}K^3 &- [0.32258 + 2(0.41933)]K^2\\ &+ [2(0.32258)(0.41933) + (0.41933)^2 - 2(0.096768)^2]K\\ &- [(0.32258)(0.41933)^2 - 2(0.41933)(0.096768)^2] = 0\end{aligned}$$

Simplifying

$$K^3 - 1.16124K^2 + 0.42764K - 0.048869 = 0$$

Solving yields

$$K_1 = 0.22583\ \text{kg}\cdot\text{m}^2 \qquad \text{or} \qquad K_1 = 0.226\ \text{kg}\cdot\text{m}^2 \blacktriangleleft$$

$$K_2 = 0.41920\ \text{kg}\cdot\text{m}^2 \qquad \text{or} \qquad K_2 = 0.419\ \text{kg}\cdot\text{m}^2 \blacktriangleleft$$

$$K_3 = 0.51621\ \text{kg}\cdot\text{m}^2 \qquad \text{or} \qquad K_3 = 0.516\ \text{kg}\cdot\text{m}^2 \blacktriangleleft$$

(*b*) To determine the direction cosines λ_x, λ_y, λ_z of each principal axis, use two of the equations of Eqs. (9.54) and (9.57). Then

K_1: Begin with Eqs. (9.54*b*) and (9.54*c*):

$$-I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 - \cancelto{0}{I_{yz}}(\lambda_z)_3 = 0$$

$$-I_{zx}(\lambda_x)_1 - \cancelto{0}{I_{yz}}(\lambda_y)_1 + (I_z - K_1)(\lambda_z)_3 = 0$$

PROBLEM 9.184* (Continued)

Substituting

$$-(0.096768)(\lambda_x)_1 + (0.41933 - 0.22583)(\lambda_y)_1 = 0$$
$$-(0.096768)(\lambda_x)_1 + (0.41933 - 0.22583)(\lambda_z)_1 = 0$$

Simplifying yields

$$(\lambda_y)_1 = (\lambda_z)_1 = 0.50009(\lambda_x)_1$$

Now substitute into Eq. (9.54):

$$(\lambda_x)^2 + 2[0.50009(\lambda_x)_1]^2 = 1$$

or $$(\lambda_x)_1 = 0.81645$$

and $$(\lambda_y)_1 = (\lambda_z)_1 = 0.40830$$

$$(\theta_x)_1 = 35.3° \quad (\theta_y)_1 = (\theta_z)_1 = 65.9° \blacktriangleleft$$

$K_2:$ Begin with Eqs. (9.54*a*) and (9.54*b*)

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - \cancelto{0}{I_{yz}}(\lambda_z)_2 = 0$$

Substituting

$$(0.32258 - 0.41920)(\lambda_x)_2 - (0.096768)(\lambda_y)_2 - (0.096768)(\lambda_z)_2 = 0 \qquad \text{(i)}$$
$$-(0.96768)(\lambda_x)_2 + (0.41933 - 0.41920)(\lambda_y)_2 = 0 \qquad \text{(ii)}$$

Eq. (ii) $\Rightarrow (\lambda_x)_2 = 0$

and then Eq. (i) $\Rightarrow (\lambda_z)_2 = -(\lambda_y)_2$

Now substitute into Eq. (9.57):

$$\cancelto{0}{(\lambda_x)_2^2} + (\lambda_y)_2^2 + [-(\lambda_y)_2]^2 = 1$$

or $$(\lambda_y)_2 = \frac{1}{\sqrt{2}}$$

and $$(\lambda_z)_2 = -\frac{1}{\sqrt{2}}$$

$$(\theta_x)_2 = 90.0° \quad (\theta_y)_2 = 45.0° \quad (\theta_z)_2 = 135.0° \blacktriangleleft$$

PROBLEM 9.184* (Continued)

K_3: Begin with Eqs. (9.54*b*) and (9.54*c*):

$$I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 + \cancelto{0}{I_{yz}}(\lambda_z)_3 = 0$$

$$-I_{zx}(\lambda_x)_3 - \cancelto{0}{I_{yz}}(\lambda_y)_3 + (I_z - K_3)(\lambda_z)_3 = 0$$

Substituting

$$-(0.096768)(\lambda_x)_3 + (0.41933 - 0.51621)(\lambda_y)_3 = 0$$

$$-(0.096768)(\lambda_x)_3 + (0.41933 - 0.51621)(\lambda_z)_3 = 0$$

Simplifying yields

$$(\lambda_y)_3 = (\lambda_z)_3 = -(\lambda_x)_3$$

Now substitute into Eq. (9.57):

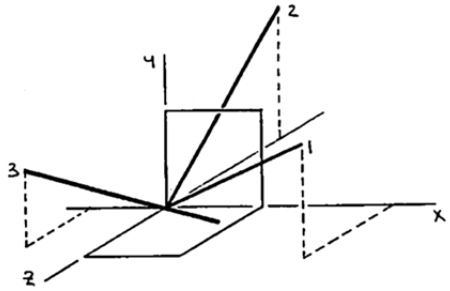
$$(\lambda_x)_3^2 + 2[-(\lambda_x)_3]^2 = 1 \tag{i}$$

or
$$(\lambda_x)_3 = \frac{1}{\sqrt{3}}$$

and
$$(\lambda_y)_3 = (\lambda_z) = -\frac{1}{\sqrt{3}}$$

$$(\theta_x)_3 = 54.7° \quad (\theta_y)_3 = (\theta_z)_3 = 125.3° \blacktriangleleft$$

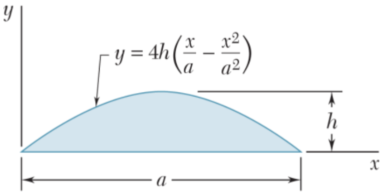
(*c*)



Note: Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen;

That is,
$$(\lambda_x)_3 = -\frac{1}{\sqrt{3}}$$

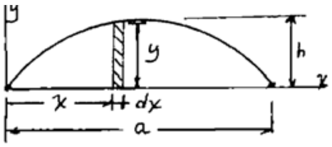
Then
$$(\theta_x)_3 = 125.3° \qquad (\theta_y)_3 = (\theta_z)_3 = 54.7° \blacktriangleleft$$



PROBLEM 9.185

Determine by direct integration the moments of inertia of the shaded area with respect to the *x* and *y* axes.

SOLUTION



$$y = 4h\left(\frac{x}{a} - \frac{x^2}{a^2}\right)$$

$$dI_x = \frac{1}{3}y^3dx = \frac{1}{3}\left[4h\left(\frac{x}{a} - \frac{x^2}{a^2}\right)\right]^3 dx$$

$$I_x = \int dI_x = \frac{64h^3}{3}\int_0^a \left(\frac{x^3}{a^3} - 3\frac{x^4}{a^4} + 3\frac{x^5}{a^5} - \frac{x^6}{a^6}\right)dx$$

$$= \frac{64h^3}{3}\left|\frac{1}{4}\frac{x^4}{a^3} - \frac{3}{5}\frac{x^5}{a^4} + \frac{1}{2}\frac{x^6}{a^5} - \frac{1}{7}\frac{x^7}{a^6}\right|_0^a$$

$$= \frac{64h^3}{3}a\left(\frac{1}{4} - \frac{3}{5} + \frac{1}{2} - \frac{1}{7}\right) = \frac{64h^3}{3}a\left(\frac{3}{420}\right)$$

$$I_x = \frac{16}{105}ah^3 \blacktriangleleft$$

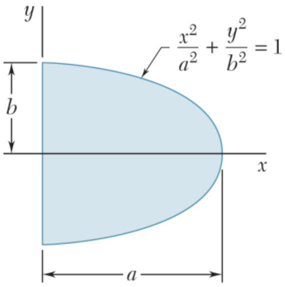
$$y = 4h\left(\frac{x}{a} - \frac{x^2}{a^2}\right)$$

$$dA = ydx$$

$$dI_y = x^2dA = 4hx^2\left(\frac{x}{a} - \frac{x^2}{a^2}\right)dx$$

$$I_y = 4h\int_0^a \left(\frac{x^3}{a} - \frac{x^4}{a^2}\right)dx$$

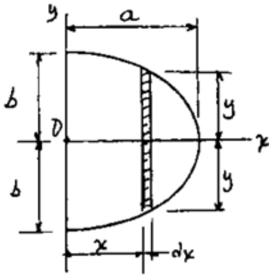
$$I_y = 4h\left[\frac{x^4}{4a} - \frac{x^5}{5a^2}\right]_0^a = 4h\left(\frac{a^3}{4} - \frac{a^3}{5}\right) \qquad I_y = \frac{1}{5}ha^3 \blacktriangleleft$$



PROBLEM 9.186

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$y=b\sqrt{1-\frac{x^2}{a^2}}$$

$$dA=2y\,dx$$

$$dI_y=x^2dA=2x^2y\,dx$$

$$I_y=\int dI_y=\int_0^a 2x^2y\,dx=2b\int_0^a x^2\sqrt{1-\frac{x^2}{a^2}}\,dx$$

Set: $x=a\sin\theta \qquad dx=a\cos\theta\,d\theta$

$$I_y=2b\int_0^{\pi/2}a^2\sin^2\theta\sqrt{1-\sin^2\theta}\quad a\cos\theta\,d\theta$$

$$=2a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta=2a^3b\int_0^{\pi/2}\frac{1}{4}\sin^2 2\theta\,d\theta$$

$$=\frac{1}{2}a^3b\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)d\theta=\frac{1}{4}a^3b\left|\theta-\frac{1}{4}\sin 4\theta\right|_0^{\pi/2}$$

$$=\frac{1}{4}a^3b\left[\frac{\pi}{2}-0\right]=\frac{\pi}{8}a^3b$$

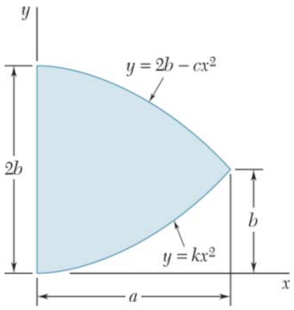
$$I_y=\frac{1}{8}\pi a^3b \blacktriangleleft$$

From solution of Problem 9.16: $A=\frac{1}{2}\pi ab$

Thus:

$$I_y=k_y^2A \quad k_y^2=\frac{I_y}{A}=\frac{\frac{1}{8}\pi a^3b}{\frac{1}{2}\pi ab}=\frac{1}{4}a^2$$

$$k_y=\frac{1}{2}a \blacktriangleleft$$



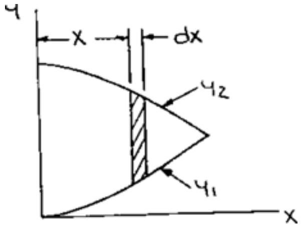
PROBLEM 9.187

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the *x* axis.

SOLUTION

At $x = a_1,\ y_1 = y_2 = b$: $\quad y_1:\quad b = ka^2 \quad \text{or} \quad k = \dfrac{b}{a^2}$

$$y_2:\quad b = 2b - ca^2 \quad \text{or} \quad c = \frac{b}{a^2}$$



Then $$y_1 = \frac{b}{a^2}x^2 \quad y_2 = b\left(2 - \frac{x^2}{a^2}\right)$$

Now $$dA = (y_2 - y_1)dx = \left[b\left(2 - \frac{x^2}{a^2}\right) - \frac{b}{a^2}x^2\right]dx = \frac{2b}{a^2}(a^2 - x^2)dx$$

Then $$A = \int dA = \int_0^a \frac{2b}{a^2}(a^2 - x^2)dx = \frac{2b}{a^2}\left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}ab$$

Now $$dI_x = \left(\frac{1}{3}y_2^3 - \frac{1}{3}y_1^3\right)dx = \frac{1}{3}\left\{\left[b\left(2 - \frac{x^2}{a^2}\right)\right]^3 - \left[\frac{b}{a^2}x^2\right]^3\right\}dx$$

$$= \frac{1}{3}\frac{b^3}{a^6}(8a^6 - 12a^4x^2 + 6a^2x^4 - x^6 - x^6)dx$$

$$= \frac{2}{3}\frac{b^3}{a^6}(4a^6 - 6a^4x^2 + 3a^2x^4 - x^6)dx$$

Then $$I_x = \int dI_x = \int_0^a \frac{2}{3}\frac{b^3}{a^6}(4a^6 - 6a^4x^2 + 3a^2x^4 - x^6)dx$$

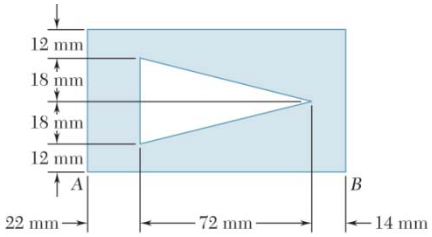
$$= \frac{2}{3}\frac{b^3}{a^6}\left[4a^6x - 2a^4x^3 + \frac{3}{5}a^6x^5 - \frac{1}{7}x^7\right]_0^a$$

$$= \frac{172}{105}ab^3$$

or $I_x = 1.638ab^3$ ◀

and $$k_x^2 = \frac{I_x}{A} = \frac{\frac{172}{105}ab^3}{\frac{4}{3}ab} = \frac{43}{35}b^2$$

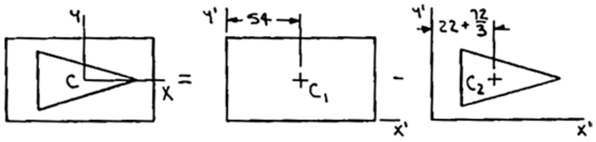
or $k_x = 1.108b$ ◀



PROBLEM 9.188

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side *AB*.

SOLUTION



Dimensions in mm

First locate centroid *C* of the area.

Symmetry implies $\bar{Y} = 30$ mm.

	A, mm^2	$\bar{x}$, mm	$\bar{x}A$, mm^3
1	$108 \times 60 = 6480$	54	349,920
2	$-\frac{1}{2} \times 72 \times 36 = -1296$	46	–59,616
Σ	5184		290,304

Then $$\bar{X}\,\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(5184 \text{ mm}^2) = 290{,}304 \text{ mm}^3$$

or $$\bar{X} = 56.0 \text{ mm}$$

Now $$\bar{I}_x = (I_x)_1 - (I_x)_2$$

where $$(I_x)_1 = \frac{1}{12}(108 \text{ mm})(60 \text{ mm})^3 = 1.944 \times 10^6 \text{ mm}^4$$

$$(I_x)_2 = 2\left[\frac{1}{36}(72 \text{ mm})(18 \text{ mm})^3 + \left(\frac{1}{2} \times 72 \text{ mm} \times 18 \text{ mm}\right)(6 \text{ mm})^2\right]$$

$$= 2(11{,}664 + 23{,}328) \text{ mm}^4 = 69.984 \times 10^3 \text{ mm}^4$$

[$(I_x)_2$ is obtained by dividing A_2 into]

Then $$\bar{I}_x = (1.944 - 0.069984) \times 10^6 \text{ mm}^4$$

or $$\bar{I}_x = 1.874 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

PROBLEM 9.188 (Continued)

Also $$\bar{I}_y = (I_y)_1 - (I_y)_2$$

where $$(I_y)_1 = \frac{1}{12}(60\text{ mm})(108\text{ mm})^3 + (6480\text{ mm}^2)[(56.54)\text{ mm}]^2$$

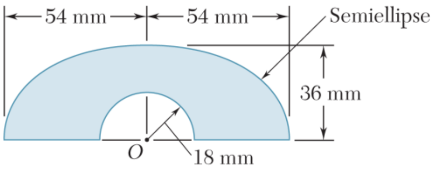
$$= (6,298,560 + 25,920)\text{ mm}^4 = 6.324\times 10^6\text{ mm}^4$$

$$(I_y)_2 = \frac{1}{36}(36\text{ mm})(72\text{ mm})^3 + (1296\text{ mm}^2)[(56-46)\text{ mm}]^2$$

$$= (373,248 + 129,600)\text{ mm}^4 = 0.502\times 10^6\text{ mm}^4$$

Then $$\bar{I}_y = (6.324 - 0.502)10^6\text{ mm}^4$$

or $\bar{I}_y = 5.82\times 10^6\text{ mm}^4$ ◀

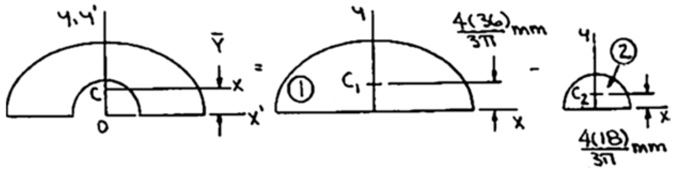

54 mm
54 mm
Semiellipse
36 mm
O
18 mm

PROBLEM 9.189

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

First locate centroid *C* of the area.


y, y'
$\bar{Y}$
C
x
x'
O
=
y
$\frac{4(36)}{3\pi}$ mm
①
C_1
x
−
y
②
C_2
x
$\frac{4(18)}{3\pi}$ mm

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$\frac{\pi}{2}(54)(36) = 3053.6$	$\frac{48}{\pi} = 15.2789$	46,656
2	$-\frac{\pi}{2}(18)^2 = -508.9$	$\frac{24}{\pi} = 7.6394$	−3888
Σ	2544.7		42,768

Then $\bar{Y}\Sigma A = \Sigma \bar{y} A: \quad \bar{Y}(2544.7 \text{ mm}^2) = 42,768 \text{ mm}^3$

or $\bar{Y} = 16.8067 \text{ mm}$ ◁

(*a*)

$$J_O = (J_O)_1 - (J_O)_2$$

$$= \frac{\pi}{8}(54 \text{ mm})(36 \text{ mm})[(54 \text{ mm})^2 + (36 \text{ mm})^2] - \frac{\pi}{4}(18 \text{ mm})^4$$

$$= (3.2155\times10^6 - 0.0824\times10^6) \text{ mm}^4$$

$$= 3.1331\times10^6 \text{ mm}^4$$

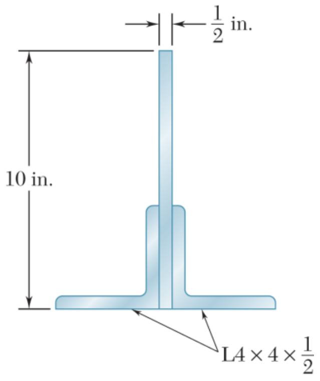
or $J_O = 3.13\times10^6 \text{ mm}^4$ ◀

(*b*)

$$J_O = \bar{J}_C + A(\bar{Y})^2$$

or $\bar{J}_C = 3.1331\times10^6 \text{ mm}^4 - (2544.7 \text{ mm}^2)(16.8067 \text{ mm})^2$

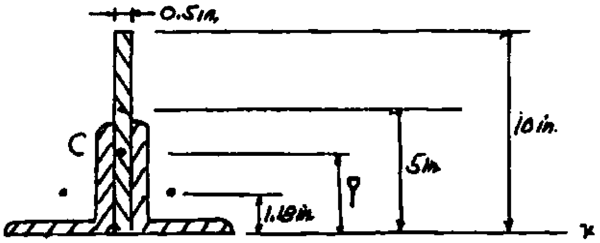
or $\bar{J}_C = 2.41\times10^6 \text{ mm}^4$ ◀


$\frac{1}{2}$ in.
10 in.
L4 × 4 × $\frac{1}{2}$

PROBLEM 9.190

Two L4 × 4 × $\frac{1}{2}$ -in. angles are welded to a steel plate as shown. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the plate.

SOLUTION


0.5in.
10 in.
5in
C
$\bar{Y}$
1.18in
x

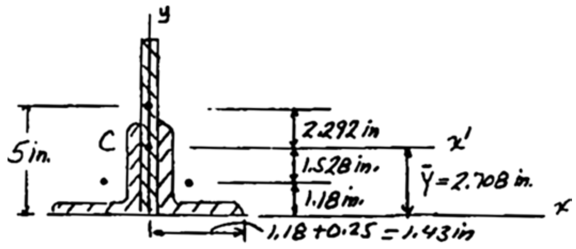
For $4 \times 4 \times \frac{1}{2}$-in. angle: $A = 3.75\ \text{in}^2, \quad \bar{I}_x = \bar{I}_y = 5.52\ \text{in}^4$

$$\bar{Y}A = \Sigma \bar{y}\, A$$

$$\bar{Y}(12.5\ \text{in}^2) = 33.85\ \text{in}^3$$

$$\bar{Y} = 2.708\ \text{in.}$$

Section	Area, in^2	$\bar{y}$ in.	$\bar{y}A$, in^3
Plate	$(0.5)(10) = 5$	5	25
Two angles	$2(3.75) = 7.5$	1.18	8.85
Σ	12.5		33.85


y
5 in.
C
2.292 in
1.528 in.
1.18 in.
x'
$\bar{Y}$ = 2.708 in.
x
1.18 + 0.25 = 1.43 in

PROBLEM 9.190 (Continued)

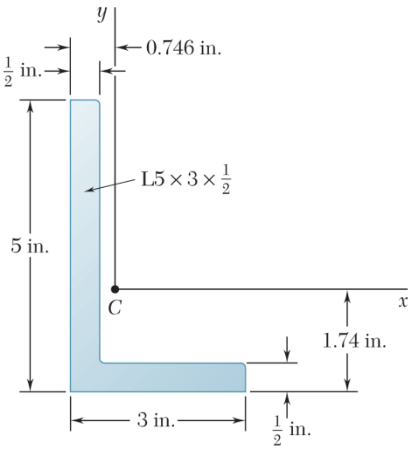
Entire section:

$$\bar{I}_x = \Sigma(\bar{I}_{x'} + Ad^2) = \left[\frac{1}{12}(0.5)(10)^3 + (0.5)(10)(2.292)^2\right] + 2[5.52 + (3.75)(1.528)^2]$$

$$= 41.667 + 26.266 + 1604 + 17.511 = 96.48 \text{ in}^4 \qquad \bar{I}_x = 96.5 \text{ in}^4 \blacktriangleleft$$

$$\bar{I}_y = \frac{1}{12}(10)(0.5)^3 + 2[5.52 + (3.75)(1.43)^2]$$

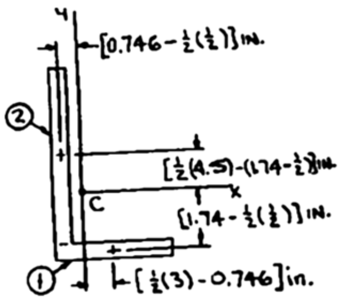
$$= 0.104 + 11.04 + 15.367 = 26.51 \text{ in}^4 \qquad \bar{I}_y = 26.5 \text{ in}^4 \blacktriangleleft$$


y
0.746 in.
$\frac{1}{2}$ in.
L5 × 3 × $\frac{1}{2}$
5 in.
C
x
1.74 in.
3 in.
$\frac{1}{2}$ in.

PROBLEM 9.191

Using the parallel-axis theorem, determine the product of inertia of the L5 × 3 × $\frac{1}{2}$ -in. angle cross section shown with respect to the centroidal x and y axes.

SOLUTION


y
[0.746 − ½(½)] in.
②
[½(4.5)−(1.74−½)] in.
C
x
[1.74 − ½(½)] in.
①
[½(3) − 0.746] in.

We have

$$\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$$

For each rectangle:

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

and

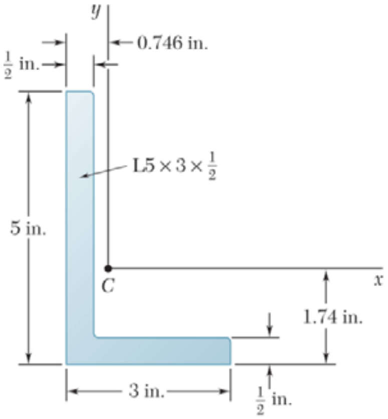
$$\bar{I}_{x'y'} = 0 \ \text{(symmetry)}$$

Thus

$$I_{xy} = \Sigma \bar{x}\bar{y}A$$

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\bar{y}A$, in^4
1	$3 \times \frac{1}{2} = 1.5$	0.754	−1.49	−1.68519
2	$4.5 \times \frac{1}{2} = 2.25$	−0.496	1.01	−1.12716
Σ				−2.81235

$$\bar{I}_{xy} = -2.81\,\text{in}^4 \blacktriangleleft$$

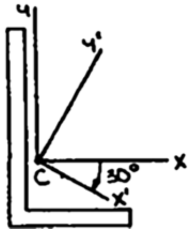

y
0.746 in.
$\frac{1}{2}$ in.
L5 × 3 × $\frac{1}{2}$
5 in.
C
x
1.74 in.
3 in.
$\frac{1}{2}$ in.

PROBLEM 9.192

For the L5 × 3 × $\frac{1}{2}$-in. angle cross section shown, use Mohr's circle to determine (*a*) the moments of inertia and the product of inertia with respect to new centroidal axes obtained by rotating the *x* and *y* axes 30° clockwise, (*b*) the orientation of the principal axes through the centroid and the corresponding values of the moments of inertia.

SOLUTION

From Figure 9.13*a*:



$$\bar{I}_x = 9.43\text{ in}^4 \qquad \bar{I}_y = 2.55\text{ in}^4$$

From the solution to Problem 9.191

$$\bar{I}_{xy} = -2.81235\text{ in}^4$$

The Mohr's circle is defined by the diameter *XY*, where

$$X(9.43 - 2.81235), \qquad Y(2.55,\ 2.81235)$$

Now

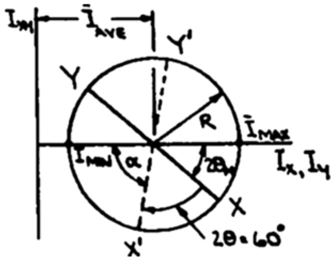
$$\bar{I}_{\text{ave}} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(9.43 + 2.55) = 5.99\text{ in}^4$$

and

$$R = \sqrt{\left[\frac{1}{2}(\bar{I}_x - \bar{I}_y)\right]^2 + \bar{I}_{xy}^2} = \sqrt{\left[\frac{1}{2}(9.43 - 2.55)\right]^2 + (-2.81235)^2}$$

$$= 4.4433\text{ in}^4$$

The Mohr's circle is then drawn as shown.


I_xy
Ī_AVE
Y'
Y
R
Ī_MAX
I_MIN
α
2θ_M
I_x, I_y
X
X'
2θ = 60°

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(-2.81235)}{9.43 - 2.55}$$

$$= 0.81754$$

or

$$2\theta_m = 39.267°$$

and

$$\theta_m = 19.6335°$$

Then

$$\alpha = 180° - (39.267° + 60°)$$

$$= 80.733°$$

PROBLEM 9.192 (Continued)

(*a*) We have $\bar{I}_{x'} = \bar{I}_{\text{ave}} - R\cos\alpha = 5.99 - 4.4433\cos 80.733°$

or $\bar{I}_{x'} = 5.27\text{ in}^4$ ◀

$\bar{I}_{y'} = \bar{I}_{\text{ave}} + R\cos\alpha = 5.99 + 4.4433\cos 80.733°$

or $\bar{I}_{y'} = 6.71\text{ in}^4$ ◀

$\bar{I}_{x'y'} = -R\sin\alpha = -4.4433\sin 80.733°$

or $\bar{I}_{x'y'} = -4.39\text{ in}^4$ ◀

(*b*) First observe that the principal axes are obtained by rotating the *xy* axes through

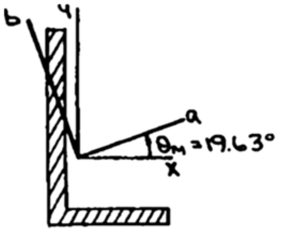
19.63° counterclockwise ◀

about *C*.

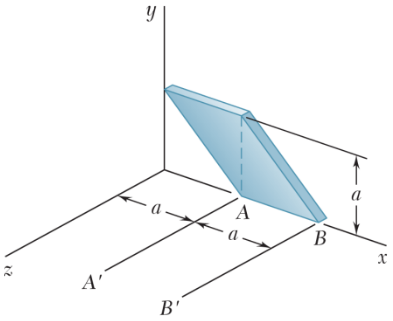
Now $\bar{I}_{\text{max, min}} = \bar{I}_{\text{ave}} \pm R = 5.99 \pm 4.4433$

or $\bar{I}_{\text{max}} = 10.43\text{ in}^4$ ◀

$\bar{I}_{\text{min}} = 1.547\text{ in}^4$ ◀



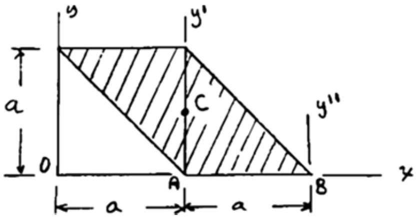
From the Mohr's circle it is seen that the *a* axis corresponds to $\bar{I}_{\text{max}}$ and the *b* axis corresponds to $\bar{I}_{\text{min}}$.



PROBLEM 9.193

A thin plate of mass m was cut in the shape of a parallelogram as shown. Determine the mass moment of inertia of the plate with respect to (*a*) the x axis, (*b*) the axis BB', which is perpendicular to the plate.

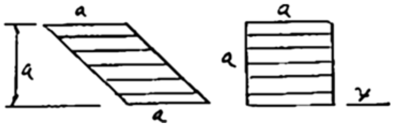
SOLUTION



$$\text{mass} = m = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

(*a*) Consider parallelogram as made of horizontal strips and slide strips to form a square since distance from each strip to x axis is unchanged.



$$I_{x,\text{area}} = \frac{1}{3}a^4$$

$$I_{x,\text{mass}} = \frac{m}{A} I_{x,\text{area}} = \frac{m}{a^2}\left(\frac{1}{3}a^4\right) \qquad I_x = \frac{1}{3}ma^2 \blacktriangleleft$$

(*b*) For centroidal axis y':

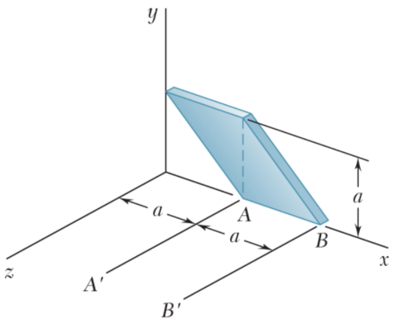
$$\bar{I}_{y',\text{area}} = 2\left[\frac{1}{12}a^4\right] = \frac{1}{6}a^4$$

$$\bar{I}_{y',\text{mass}} = \frac{m}{A} I_{y',\text{area}} = \frac{m}{a^2}\left(\frac{1}{6}a^4\right) = \frac{1}{6}ma^2$$

$$I_{y''} = \bar{I}_{y'} + ma^2 = \frac{1}{6}ma^2 + ma^2 = \frac{7}{6}ma^2$$

For axis $BB' \perp$ to plate, Eq. (9.38):

$$I_{BB'} = I_x + I_{y''} = \frac{1}{3}ma^2 + \frac{7}{6}ma^2 \qquad I_{BB'} = \frac{3}{2}ma^2 \blacktriangleleft$$



PROBLEM 9.194

A thin plate of mass *m* was cut in the shape of a parallelogram as shown. Determine the mass moment of inertia of the plate with respect to (*a*) the *y* axis, (*b*) the axis *AA*′, which is perpendicular to the plate.

SOLUTION

See sketch of solution of Problem 9.117.

(*a*) From part *b* of solution of Problem 9.117:

$$\bar{I}_{y'} = \frac{1}{6}ma^2$$

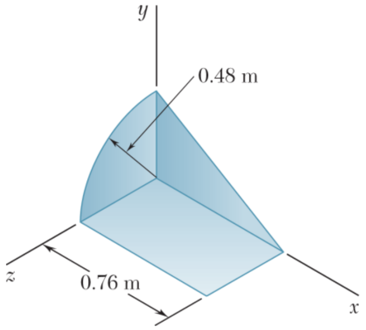
$$I_y = \bar{I}_{y'} + ma^2 = \frac{1}{6}ma^2 + ma^2 \qquad I_y = \frac{7}{6}ma^2 \blacktriangleleft$$

(*b*) From solution of Problem 9.115:

$$\bar{I}_{y'} = \frac{1}{6}ma^2 \quad \text{and} \quad I_x = \frac{1}{3}ma^2$$

Eq. (9.38): $I_{AA'} = \bar{I}_{y'} + I_x$

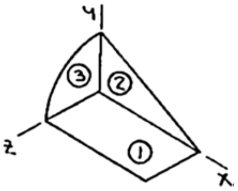
$$= \frac{1}{6}ma^2 + \frac{1}{3}ma^2 \qquad I_{AA'} = \frac{1}{2}ma^2 \blacktriangleleft$$



PROBLEM 9.195

A 2-mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{ST} V = \rho_{ST} t A$$

Then

$$m_1 = (7850\text{ kg/m}^3)(0.002\text{ m})(0.76 \times 0.48)\text{ m}^2$$
$$= 5.72736\text{ kg}$$

$$m_2 = (7850\text{ kg/m}^3)(0.002\text{ m})\left(\frac{1}{2} \times 0.76 \times 0.48\right)\text{m}^2$$
$$= 2.86368\text{ kg}$$

$$m_3 = (7850\text{ kg/m}^3)(0.002\text{ m})\left(\frac{\pi}{4} \times 0.48^2\right)\text{m}^2$$
$$= 2.84101\text{ kg}$$

Using Figure 9.28 for component 1 and the equations derived above for components 2 and 3, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$
$$= \left[\frac{1}{12}(5.72736\text{ kg})(0.48\text{ m})^2 + (5.72736\text{ kg})\left(\frac{0.48}{2}\text{ m}\right)^2\right]$$
$$+ \left[\frac{1}{18}(2.86368\text{ kg})(0.48\text{ m})^2 + (2.86368\text{ kg})\left(\frac{0.48}{3}\text{ m}\right)^2\right] + \left[\frac{1}{2}(2.84101\text{ kg})(0.48\text{ m})^2\right]$$
$$= [(0.109965 + 0.329896) + (0.036655 + 0.073310) + (0.327284)]\text{ kg} \cdot \text{m}^2$$
$$= (0.439861 + 0.109965 + 0.327284)\text{ kg} \cdot \text{m}^2$$

or $I_x = 0.877\text{ kg} \cdot \text{m}^2$ ◀

PROBLEM 9.195 (Continued)

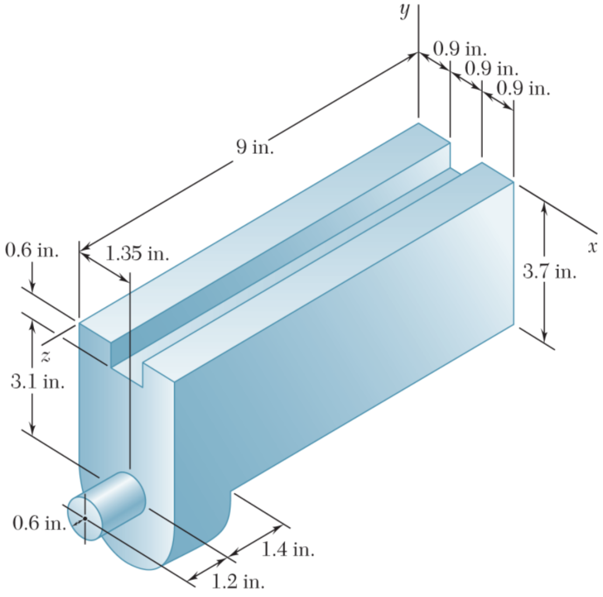
$$
\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 \\
&= \left\{\frac{1}{12}(5.72736\text{ kg})(0.76^2 + 0.48^2)\text{ m}^2 + (5.72736\text{ kg})\left[\left(\frac{0.76}{2}\right)^2 + \left(\frac{0.48}{2}\right)^2\right]\text{m}^2\right\} \\
&\quad + \left[\frac{1}{18}(2.86368\text{ kg})(0.76\text{ m})^2 + (2.86368\text{ kg})\left(\frac{0.76}{3}\text{ m}\right)^2\right] + \left[\frac{1}{4}(2.84101\text{ kg})(0.48\text{ m})^2\right] \\
&= [(0.385642 + 1.156927) + (0.091892 + 0.183785) + (0.163642)]\text{ kg}\cdot\text{m}^2 \\
&= (1.542590 + 0.275677 + 0.163642)\text{ kg}\cdot\text{m}^2
\end{aligned}
$$

or $I_y = 1.982\text{ kg}\cdot\text{m}^2$ ◀

$$
\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 \\
&= \left[\frac{1}{12}(5.72736\text{ kg})(0.76\text{ m})^2 + (5.72736\text{ kg})\left(\frac{0.76}{2}\text{ m}\right)^2\right] \\
&\quad + \left\{\frac{1}{18}(2.86368\text{ kg})(0.76^2 + 0.48^2)\text{ m}^2 + (2.86368\text{ kg})\left[\left(\frac{0.76}{3}\right)^2 + \left(\frac{0.48}{3}\right)^2\right]\text{m}^2\right\} \\
&\quad + \left[\frac{1}{4}(2.84101\text{ kg})(0.48\text{ m})^2\right]
\end{aligned}
$$

$$
\begin{aligned}
I_z &= [(0.275677 + 0.827031) + (0.128548 + 0.257095) \\
&\quad + (0.163642)]\text{ kg}\cdot\text{m}^2 \\
&= (1.102708 + 0.385643 + 0.163642)\text{ kg}\cdot\text{m}^2
\end{aligned}
$$

or $I_z = 1.652\text{ kg}\cdot\text{m}^2$ ◀



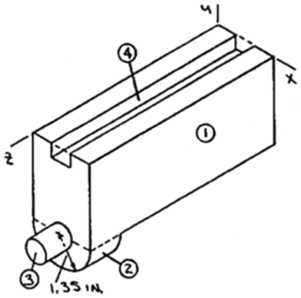
PROBLEM 9.196

Determine the mass moment of inertia of the steel machine element shown with respect to the *z* axis. (The specific weight of steel is $490\ \text{lb/ft}^3$.)

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \frac{\gamma_{\text{ST}}}{g} V$$



Then

$$m_1 = \frac{490\ \text{lb/ft}^3}{32.2\ \text{ft/s}^2} \times (2.7 \times 3.7 \times 9)\ \text{in}^3 \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^3$$
$$= 791.780 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{490\ \text{lb/ft}^3}{32.2\ \text{ft/s}^2} \times \left(\frac{\pi}{2} \times 1.35^2 \times 1.4\right)\ \text{in}^3 \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^3$$
$$= 35.295 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{490\ \text{lb/ft}^3}{32.2\ \text{ft/s}^2} \times (\pi \times 0.6^2 \times 1.2)\ \text{in}^3 \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^3$$
$$= 11.952 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$

$$m_4 = \frac{490\ \text{lb/ft}^3}{32.2\ \text{ft/s}^2} \times (0.9 \times 0.6 \times 9)\ \text{in}^3 \times \left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^3 = 42.799 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$

The mass moments of inertia are now computed using Figure 9.28 (components 1, 3, and 4) and the equations derived above (component 2).

PROBLEM 9.196 (Continued)

Find: I_z

We have

$$I_z = (I_z)_1 + (I_z)_2 + (I_z)_3 - (I_z)_4$$

$$= \left\{ \frac{1}{12}(791.780\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})[(2.7)^2 + (3.7)^2]\text{ in}^2 \right.$$

$$\left. + (791.780\text{ lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2.7}{2}\right)^2 + \left(\frac{3.7}{2}\right)^2\right]\text{ in}^2 \right\}\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$+ \left\{ (35.295\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(1.35\text{ in.})^2 \right.$$

$$\left. +(35.295\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})\left[(1.35)^2 + \left(3.7 + \frac{4\times 1.35}{3\pi}\right)^2\right]\text{ in}^2 \right\}\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$+ \left\{ \frac{1}{12}(11.952\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})(0.6\text{ in.})^2 \right.$$

$$\left. +(11.952\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})[(1.35)^2 + (3.7)^2]\text{ in}^2 \right\}\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$- \left\{ \frac{1}{12}(42.799\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})[(0.9)^2 + (0.6)^2]\text{ in}^2 \right.$$

$$\left. +(42.799\times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})\left[(1.35)^2 + \left(\frac{0.6}{2}\right)^2\right]\text{ in}^2 \right\}\times\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

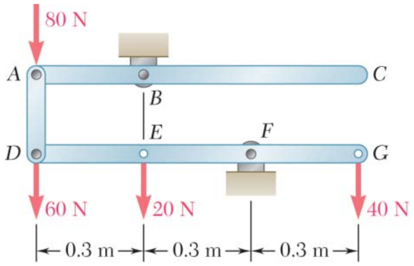
$$I_z = [(9.6132 + 28.8395) + (0.1429 + 4.9219)$$

$$+ (0.0149 + 1.2875) - (0.0290 + 0.5684)]\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= (38.4527 + 5.0648 + 1.3024 - 0.5974)\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_z = 0.0442\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

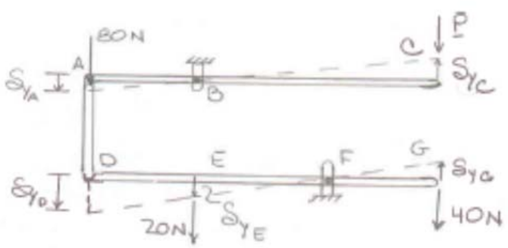
CHAPTER 10


80 N
A
B
C
E
F
D
G
60 N
20 N
40 N
0.3 m
0.3 m
0.3 m

PROBLEM 10.1

Determine the vertical force **P** that must be applied at *C* to maintain the equilibrium of the linkage.

SOLUTION


80N
A
B
C
P
δy_A
δy_C
D
E
F
G
δy_D
δy_G
δy_E
20N
40N
L
2

Assume $\delta y_A \downarrow$

$$\delta y_C = 2\delta y_A \uparrow$$

$$\delta y_D = \delta y_A \downarrow$$

$$\delta y_E = \frac{1}{2}\delta y_D = \frac{1}{2}\delta y_A \downarrow$$

$$\delta y_G = \delta y_E = \frac{1}{2}\delta y_A \uparrow$$

Also $\delta\theta = \dfrac{\delta y_A}{a} \circlearrowleft \qquad a = 0.3 \text{ m}$

Virtual Work: We apply both *P* and *M* to member *ABC*.

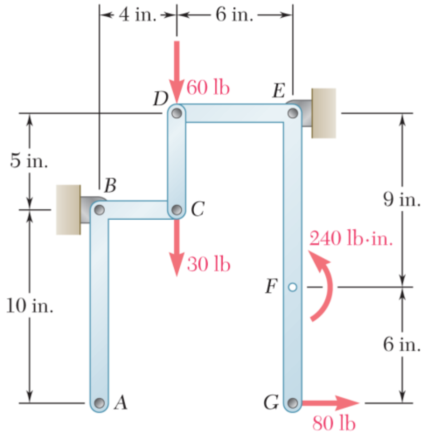
$$\delta U = 80\delta y_A - P\delta y_C - M\delta\theta + 60\delta y_D + 20\delta y_E - 40\delta y_G = 0$$

$$80\delta y_A - P(2\delta y_A) - M\left(\frac{\delta y_A}{a}\right) + 60\delta y_A + 20\left(\frac{1}{2}\delta y_A\right) - 40\left(\frac{1}{2}\delta y_A\right) = 0$$

$$80 - 2P - \frac{M}{a} + 60 + 10 - 20 = 0$$

$$2P + \frac{M}{a} = 130 \text{ N} \qquad (1)$$

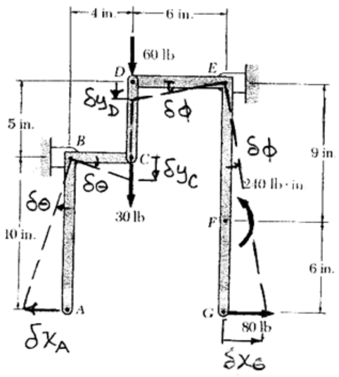
Now from Eq. (1) for $M = 0$ $\qquad 2P = 130 \text{ N} \qquad$ $\mathbf{P} = 65.0 \text{ N} \downarrow$ ◀


4 in.
6 in.
D
60 lb
E
5 in.
B
C
9 in.
240 lb·in.
30 lb
F
10 in.
6 in.
A
G
80 lb

PROBLEM 10.2

Determine the horizontal force **P** which must be applied at *A* so that the linkage is in equilibrium for the position shown.

SOLUTION


4 in.
6 in.
60 lb
D
E
δy_D
$\delta\phi$
5 in.
$\delta\phi$
B
C
δy_C
9 in
$\delta\theta$
$\delta\theta$
240 lb·in
30 lb
F
10 in.
6 in.
A
G
80 lb
δx_A
δx_G

Assume $\delta\theta \circlearrowright$

$$\delta x_A = 10\,\delta\theta \text{ in.} \longleftarrow$$

$$\delta y_C = 4\,\delta\theta \text{ in.} \downarrow$$

$$\delta y_D = \delta y_C = 4\,\delta\theta \text{ in.} \downarrow$$

$$\delta\phi = \frac{\delta y_D}{6} = \frac{2}{3}\delta\theta \circlearrowright$$

$$\delta x_G = 15\,\delta\phi = 15\left(\frac{2}{3}\delta\theta\right) = 10\,\delta\theta \text{ in.} \longrightarrow$$

Virtual Work: Assume that force **P** is applied at *A*.

$\delta U = 0$: $\delta U = -P\delta x_A + 30\,\delta y_C + 60\,\delta y_D + 240\,\delta\phi + 80\,\delta x_G = 0$

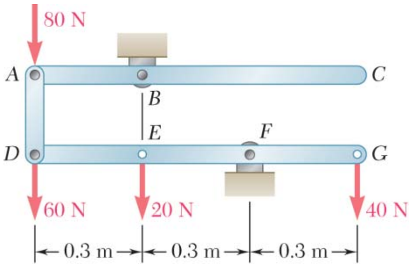
$$-P(10\ \delta\theta \text{ in.}) + (30 \text{ lb})(4\delta\theta \text{ in.}) + (60 \text{ lb})(4\ \delta\theta) + (240 \text{ lb}\cdot\text{in.})\left(\frac{2}{3}\delta\theta\right)$$

$$+(80 \text{ lb})(10\ \delta\theta \text{ in.}) = 0$$

$$-10P + 120 + 240 + 160 + 800 = 0$$

$$10P = 1320$$

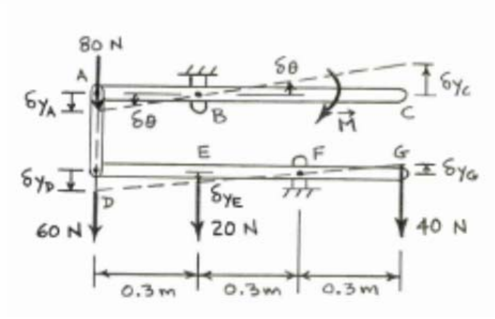
$\mathbf{P} = 132.0 \text{ lb} \longrightarrow$ ◀


80 N
A
B
C
E
F
D
G
60 N
20 N
40 N
0.3 m
0.3 m
0.3 m

PROBLEM 10.3

Determine the couple **M** that must be applied to member *ABC* to maintain the equilibrium of the linkage.

SOLUTION


80 N
A
δy_A
δθ
δθ
B
M
C
δy_C
E
F
G
δy_G
δy_D
D
δy_E
60 N
20 N
40 N
0.3m
0.3m
0.3m

Assume $\delta y_A \downarrow$

$$\delta y_D = \delta y_A \downarrow$$

$$\delta y_E = \frac{1}{2}\delta y_D = \frac{1}{2}\delta y_A \downarrow$$

$$\delta y_G = \delta y_E = \frac{1}{2}\delta y_A \uparrow$$

Also $$\delta\theta = \frac{\delta y_A}{0.3} \circlearrowleft$$

Virtual Work: We apply both *P* and *M* to member *ABC*.

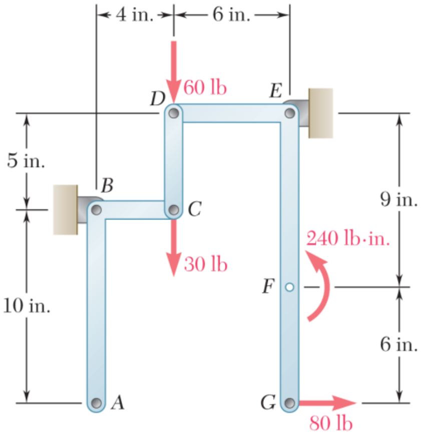
$$\delta U = 80\delta y_A - P\delta y_C - M\delta\theta + 60\delta y_D + 20\delta y_E - 40\delta y_G = 0$$

$$80\delta y_A - P(2\delta y_A) - M\left(\frac{\delta y_A}{0.3\text{ m}}\right) + 60\delta y_A + 20\left(\frac{1}{2}\delta y_A\right) - 40\left(\frac{1}{2}\delta y_A\right) = 0$$

$$80 - 2P - \frac{M}{0.3\text{ m}} + 60 + 10 - 20 = 0$$

$$2P + \frac{M}{0.3\text{ m}} = 130\text{ N} \qquad (1)$$

Now from Eq. (1) for $P = 0$ $$\frac{M}{0.3\text{ m}} = 130\text{ N}$$ $\mathbf{M} = 39.0\ \text{N}\cdot\text{m} \circlearrowright$ ◀


4 in.
6 in.
D
60 lb
E
5 in.
B
C
9 in.
240 lb·in.
30 lb
F
10 in.
6 in.
A
G
80 lb

PROBLEM 10.4

Determine the couple **M** that must be applied to member *ABC* to maintain the equilibrium of the linkage.

SOLUTION

Assume $\delta\theta$ ↻

$$\delta x_A = 10\delta\theta \leftarrow$$

$$\delta y_C = 4\delta\theta \downarrow$$

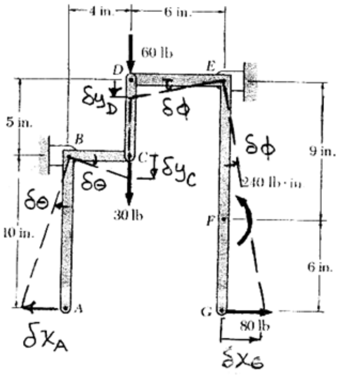
$$\delta y_D = \delta y_C = 4\delta\theta \downarrow$$

$$\delta\phi = \frac{\delta y_D}{6} = \frac{2}{3}\delta\theta \circlearrowleft$$

$$\delta x_G = 15\delta\phi$$

$$= 15\left(\frac{2}{3}\delta\theta\right)$$

$$= 10\delta\theta \rightarrow$$


4 in.
6 in.
60 lb
D
E
δy_D
$\delta\phi$
5 in.
B
C
δy_C
$\delta\phi$
9 in.
$\delta\theta$
240 lb·in
30 lb
F
10 in.
6 in.
A
G
80 lb
δx_A
δx_G

Virtual Work: We shall assume that a force **P** and a couple **M** are applied to member *ABC* as shown.

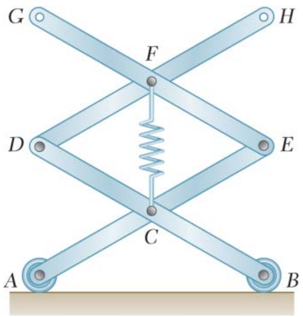
$$\delta U = -P\delta x_A - M\,\delta\theta + 30\delta y_C + 40\delta y_D + 180\delta\phi + 80\delta x_G = 0$$

$$-P(10\delta\theta) - M\,\delta\theta + 30(4\delta\theta) + 60(4\delta\theta) + 240\left(\frac{2}{3}\delta\theta\right) + 80(10\delta\theta) = 0$$

$$-10P - M + 120 + 240 + 160 + 800 = 0$$

$$(10\text{ in.})P + M = 1320\text{ lb}\cdot\text{in.} \qquad (1)$$

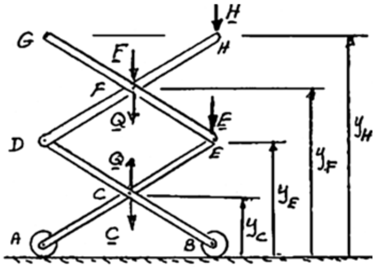
Now from Eq. (1) for $P = 0$ $\mathbf{M} = 1320\text{ lb}\cdot\text{in.}$ ↺ ◀



PROBLEM 10.5

A spring of constant 15 kN/m connects Points *C* and *F* of the linkage shown. Neglecting the weight of the spring and linkage, determine the force in the spring and the vertical motion of Point *G* when a vertical downward 120-N force is applied (*a*) at Point *C*, (*b*) at Points *C* and *H*.

SOLUTION



$$y_G = 4y_C$$

$$y_H = 4y_C \qquad \delta y_H = 4\delta y_C$$

$$y_F = 3y_C \qquad \delta y_F = 3\delta y_C$$

$$y_E = 2y_C \qquad \delta y_E = 2\delta y_C$$

For spring:

$$\Delta = y_F - y_C$$

Q = Force in spring (assumed in tension)

$$Q = +k\Delta = k(y_F - y_C) = k(3y_C - y_C) = 2ky_C \qquad (1)$$

(*a*) $\mathbf{C} = 120 \text{ N}, \quad \mathbf{E} = \mathbf{F} = \mathbf{H} = 0$

Virtual Work:

$$\delta U = 0: \quad -(120 \text{ N})\delta y_C + Q\delta y_C - Q\delta y_F = 0$$

$$-120\delta y_C + Q\delta y_C - Q(3\delta y_C) = 0$$

$Q = -60$ N $\qquad$ $Q = 60.0$ N C ◀

Eq. (1): $Q = 2ky_C, \quad -60 \text{ N} = 2(15 \text{ kN/m})y_C, \quad y_C = -2 \text{ mm}$

At Point *G*: $y_G = 4y_C = 4(-2 \text{ mm}) = -8 \text{ mm}$ $\qquad$ $\mathbf{y}_G = 8.00 \text{ mm}\downarrow$ ◀

(*b*) $\mathbf{C} = \mathbf{H} = 120 \text{ N}, \mathbf{E} = \mathbf{F} = 0$

Virtual Work:

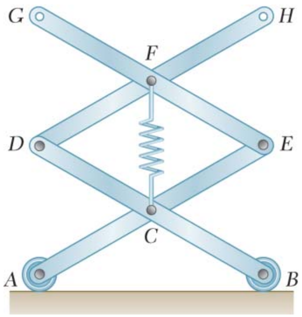
$$\delta U = 0: \quad -(120 \text{ N})\delta y_C - (120 \text{ N})y_H + Q\delta y_C - Q\delta y_F = 0$$

$$-120\delta y_C - 120(4\delta y_C) + Q\delta y_C - Q(3\delta y_C) = 0$$

$Q = -300$ N $\quad$ $Q = 300$ N C ◀

Eq. (1): $Q = 2ky_C \quad -300 \text{ N} = 2(15 \text{ kN/m})y_C, \quad y_C = -10 \text{ mm}$

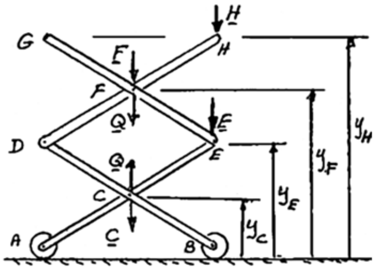
At Point *G*: $y_G = 4y_C = 4(-10 \text{ mm}) = -40 \text{ mm}$ $\qquad$ $\mathbf{y}_G = 40.0 \text{ mm}\downarrow$ ◀



PROBLEM 10.6

A spring of constant 15 kN/m connects Points *C* and *F* of the linkage shown. Neglecting the weight of the spring and linkage, determine the force in the spring and the vertical motion of Point *G* when a vertical downward 120-N force is applied (*a*) at Point *E*, (*b*) at Points *E* and *F*.

SOLUTION



$$y_G = 4y_C$$

$$y_H = 4y_C \quad \delta y_H = 4\delta y_C$$

$$y_F = 3y_C \quad \delta y_F = 3\delta y_C$$

$$y_E = 2y_C \quad \delta y_E = 2\delta y_C$$

For spring: $\Delta = y_F - y_C$

Q = Force in spring (assumed in tension)

$$\mathbf{Q} = +k\Delta = k(y_F - y_C) = k(3y_C - y_C) = 2ky_C \qquad (1)$$

(*a*) $\mathbf{E} = 120\text{ N}, \quad \mathbf{C} = \mathbf{F} = \mathbf{H} = 0$

Virtual Work: $\delta U = 0\text{:} \quad -(120\text{ N})\delta y_E + Q\delta y_C - Q\delta y_F = 0$

$$-120(2\delta y_C) + Q\delta y_C - Q(3\delta y_C) = 0$$

$Q = -120\text{ N} \quad Q = 120.0\text{ N} \quad \text{C}$ ◀

Eq. (1): $Q = 2ky_C, \quad -120\text{ N} = 2(15\text{ kN/m})y_C, \quad y_C = -4\text{ mm}$

At Point *G*: $y_G = 4y_C = 4(-4\text{ mm}) = 16\text{ mm}$ $\quad \mathbf{y}_G = 16.00\text{ mm}\downarrow$ ◀

(*b*) $\mathbf{E} = \mathbf{F} = 120\text{ N}, \mathbf{C} = \mathbf{H} = 0$

Virtual Work:

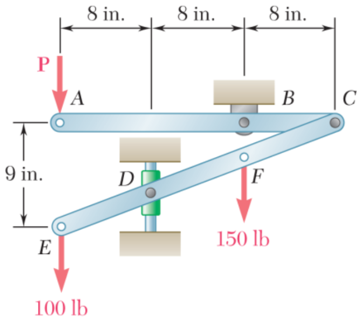
$$\delta U = 0\text{:} \quad -(120\text{ N})\delta y_E - (120\text{ N})\delta y_F + Q\delta y_C - Q\delta y_F = 0$$

$$-120(2\delta y_C) - 120(3\delta y_C) + Q\delta y_C - Q(3\delta y_C) = 0$$

$Q = -300\text{ N} \qquad Q = 300\text{ N} \quad \text{C}$ ◀

Eq. (1): $Q = 2ky_C, \quad -300\text{ N} = 2(15\text{ kN/m})y_C, \quad y_C = -10\text{ mm}$

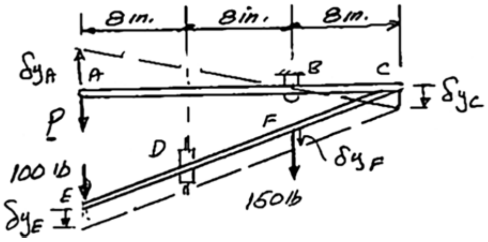
At Point *G*: $y_G = 4y_C = 4(-10\text{ mm}) = -40\text{ mm}$ $\quad \mathbf{y}_G = 40.0\text{ mm}\downarrow$ ◀


8 in.
8 in.
8 in.
P
A
B
C
9 in.
D
F
E
150 lb
100 lb

PROBLEM 10.7

The two-bar linkage shown is supported by a pin and bracket at *B* and a collar at *D* that slides freely on a vertical rod. Determine the force **P** required to maintain the equilibrium of the linkage.

SOLUTION


8 in.
8 in.
8 in.
δy_A
A
B
C
δy_C
P
F
D
δy_F
100 lb
E
δy_E
150 lb

Assume $\delta y_A \uparrow$:

$$\delta y_C = \frac{8\text{ in.}}{16\text{ in.}}\delta y_A; \quad \delta y_C = \frac{1}{2}\delta y_A \downarrow$$

Since bar *CD* move in translation

$$\delta y_E = \delta y_F = \delta y_C$$

or

$$\delta y_E = \delta y_F = \frac{1}{2} y_A \downarrow$$

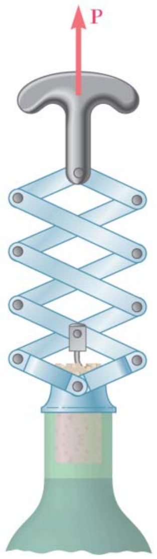
Virtual Work:

$$\delta U = 0: \quad -P\delta y_A + (100\text{ lb})\delta y_E + (150\text{ lb})\delta y_F = 0$$

$$-P\delta y_A + 100\left(\frac{1}{2}\delta y_A\right) + 150\left(\frac{1}{2}\delta y_A\right) = 0$$

$$P = 125\text{ lb}$$

$$\mathbf{P} = 125.0\text{ lb} \downarrow \blacktriangleleft$$



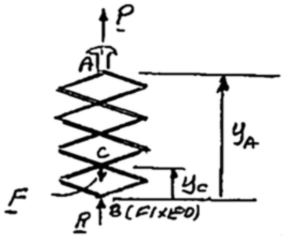
PROBLEM 10.8

Knowing that the maximum friction force exerted by the bottle on the cork is 60 lb, determine (*a*) the force **P** that must be applied to the corkscrew to open the bottle, (*b*) the maximum force exerted by the base of the corkscrew on the top of the bottle.

SOLUTION

From sketch $y_A = 4y_C$

Thus $\delta y_A = 4\delta y_C$



(*a*) Virtual Work:

$$\delta U = 0: \quad P\delta y_A - F\delta y_C = 0$$

$$P(4\delta y_C) - F\delta y_C = 0$$

$$P = \frac{1}{4}F$$

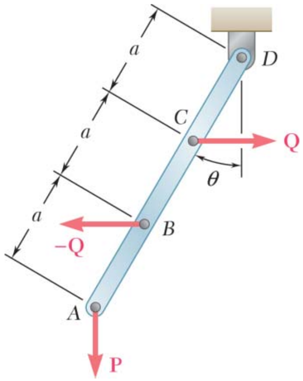
$$F = 60 \text{ lb}: \quad P = \frac{1}{4}(60 \text{ lb}) = 15 \text{ lb}$$

$$\mathbf{P} = 15.00 \text{ lb} \uparrow \blacktriangleleft$$

(*b*) Free body: Corkscrew

$$+\!\uparrow \Sigma F_y = 0 \quad R + P - F = 0; \quad R + 15 \text{ lb} - 60 \text{ lb} = 0$$

On corkscrew: $\mathbf{R} = 45 \text{ lb} \uparrow$, On bottle: $\mathbf{R} = 45.0 \text{ lb} \downarrow \blacktriangleleft$



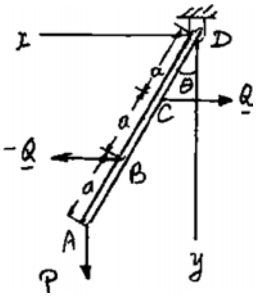
PROBLEM 10.9

Rod *AD* is acted upon by a vertical force **P** at end *A*, and by two equal and opposite horizontal forces of magnitude *Q* at points *B* and *C*. Derive an expression for the magnitude *Q* of the horizontal forces required for equilibrium.

SOLUTION

We have

$$x_C = a\sin\theta$$
$$\delta x_C = a\cos\theta\delta\theta$$
$$x_B = 2a\sin\theta$$
$$\delta x_B = 2a\cos\theta\delta\theta$$
$$y_A = 3a\cos\theta$$
$$\delta y_A = -3a\sin\theta\delta\theta$$

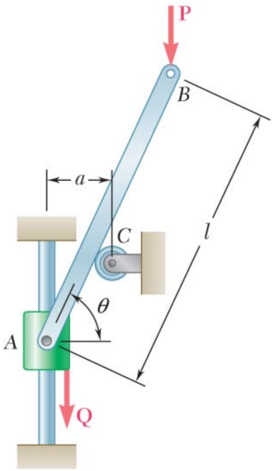


Virtual Work: We note that **P** tends to increase y_A and −**Q** tends to increase x_B, while **Q** tends to decrease x_C. Therefore

$$\delta U = P\delta y_A + Q\delta x_B - Q\delta x_C = 0$$
$$= P(-3a\sin\theta\delta\theta) + Q(2a\cos\theta\delta\theta) - Q(a\cos\theta\delta\theta) = 0$$

$$Q\cos\theta = 3P\sin\theta'$$

$$Q = 3P\tan\theta \blacktriangleleft$$

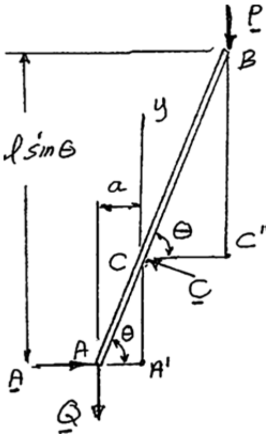


PROBLEM 10.10

The slender rod *AB* is attached to a collar *A* and rests on a small wheel at *C*. Neglecting the radius of the wheel and the effect of friction, derive an expression for the magnitude of the force **Q** required to maintain the equilibrium of the rod.

SOLUTION

For $\Delta AA'C$:



$$A'C = a\tan\theta$$

$$y_A = -(A'C) = -a\tan\theta$$

$$\delta y_A = -\frac{a}{\cos^2\theta}\delta\theta$$

For $\Delta CC'B$:

$$BC' = l\sin\theta - A'C$$

$$= l\sin\theta - a\tan\theta$$

$$y_B = BC' = l\sin\theta - a\tan\theta$$

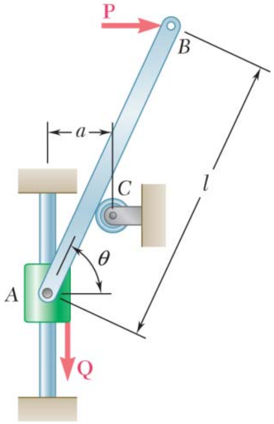
$$\delta y_B = l\cos\theta\delta\theta - \frac{a}{\cos^2\theta}\delta\theta$$

Virtual Work: $\delta U = 0$: $Q\delta y_A - P\delta y_B = 0$

$$-Q\left(-\frac{a}{\cos^2\theta}\right)\delta\theta - P\left(l\cos\theta - \frac{a}{\cos^2\theta}\right)\delta\theta = 0$$

$$Q\left(\frac{a}{\cos^2\theta}\right) = P\left(l\cos\theta - \frac{a}{\cos^2\theta}\right)$$

$$Q = P\left(\frac{l}{a}\cos^3\theta - 1\right) \blacktriangleleft$$

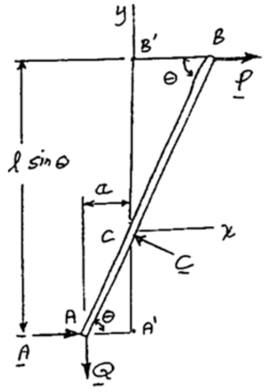


PROBLEM 10.11

The slender rod *AB* is attached to a collar *A* and rests on a small wheel at *C*. Neglecting the radius of the wheel and the effect of friction, derive an expression for the magnitude of the force **Q** required to maintain the equilibrium of the rod.

SOLUTION

For $\Delta AA'C$:



$$A'C = a\tan\theta$$

$$y_A = -(A'C) = -a\tan\theta$$

$$\delta y_A = -\frac{a}{\cos^2\theta}\delta\theta$$

For $\Delta BB'C$:

$$B'C = l\sin\theta - A'C$$

$$= l\sin\theta - a\tan\theta$$

$$BB' = \frac{B'C}{\tan\theta} = \frac{l\sin\theta - a\tan\theta}{\tan\theta}$$

$$x_B = BB' = l\cos\theta - a$$

$$\delta x_B = -l\sin\theta\delta\theta$$

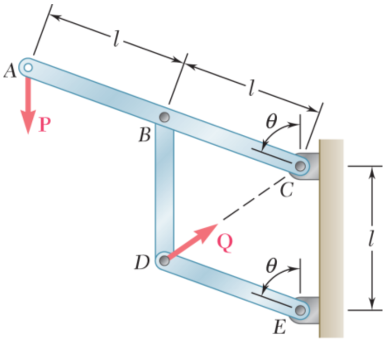
Virtual Work:

$$\delta U = 0: \quad P\delta x_B - Q\delta y_A = 0$$

$$P(-l\sin\theta\delta\theta) - Q\left(-\frac{a}{\cos^2\theta}\delta\theta\right) = 0$$

or

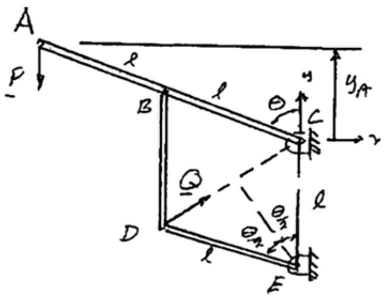
$$Pl\sin\theta\cos^2\theta = Qa \qquad Q = P\frac{l}{a}\sin\theta\cos^2\theta \blacktriangleleft$$


l
l
A
P
B
θ
C
Q
D
θ
E
l

PROBLEM 10.12

Knowing that the line of action of the force **Q** passes through Point *C*, derive an expression for the magnitude of **Q** required to maintain equilibrium.

SOLUTION


A
P
l
B
l
θ
y
y_A
C
x
Q
l
$\frac{\theta}{2}$
$\frac{\theta}{2}$
D
l
E

We have

$$y_A = 2l\cos\theta; \quad \delta y_A = -2l\sin\theta\,\delta\theta$$

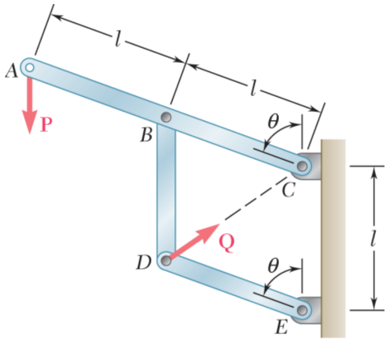
$$CD = 2l\sin\frac{\theta}{2}; \quad \delta(CD) = l\cos\frac{\theta}{2}\delta\theta$$

Virtual Work:

$$\delta U = 0: \quad -P\delta y_A - Q\delta(CD) = 0$$

$$-P(-2l\sin\theta\,\delta\theta) - Q\left(l\cos\frac{\theta}{2}\delta\theta\right) = 0$$

$$Q = 2P\frac{\sin\theta}{\cos(\theta/2)} \blacktriangleleft$$

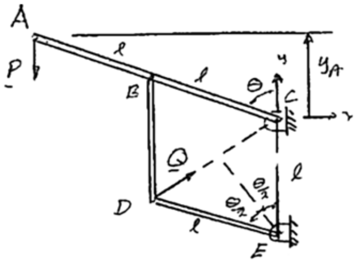


PROBLEM 10.13

Solve Problem 10.12 assuming that the force **P** applied at Point *A* acts horizontally to the left.

PROBLEM 10.12 Knowing that the line of action of the force **Q** passes through Point *C*, derive an expression for the magnitude of **Q** required to maintain equilibrium.

SOLUTION



We have

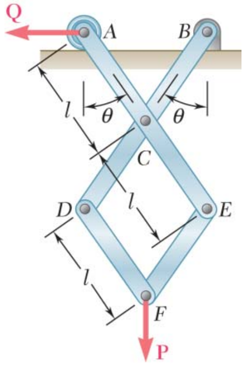
$$x_A = 2l\cos\theta; \quad \delta x_A = 2l\cos\theta\delta\theta$$

$$CD = 2l\sin\frac{\theta}{2}; \quad \delta(CD) = l\cos\frac{\theta}{2}\delta\theta$$

Virtual Work:

$$\delta U = 0: \quad P\delta x_A - Q\delta(CD) = 0$$

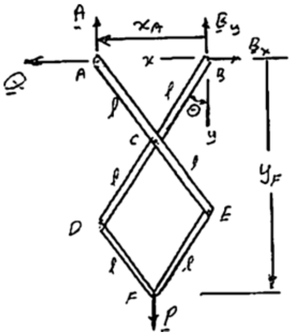
$$P(2l\cos\theta\delta\theta) - Q\left(l\cos\frac{\theta}{2}\delta\theta\right) = 0 \qquad Q = 2P\frac{\cos\theta}{\cos(\theta/2)} \blacktriangleleft$$


Q
A
B
l
θ
θ
C
D
E
l
l
F
P

PROBLEM 10.14

The mechanism shown is acted upon by the force **P**; derive an expression for the magnitude of the force **Q** required to maintain equilibrium.

SOLUTION


A
x_A
B_y
B_x
Q
A
x
B
l
θ
l
y
C
l
y_F
l
D
E
l
l
F
P

Virtual Work:

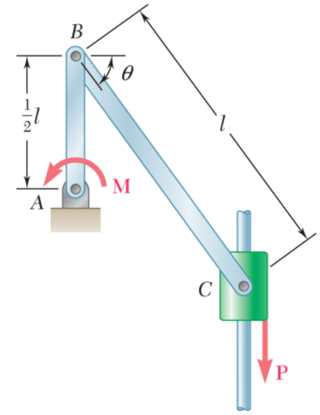
We have
$$x_A = 2l\sin\theta$$
$$\delta x_A = 2l\cos\theta\delta\theta$$

and
$$y_F = 3l\cos\theta$$
$$\delta y_F = -3l\sin\theta\delta\theta$$

Virtual Work:
$$\delta U = 0: \quad Q\delta x_A + P\delta y_F = 0$$

$$Q(2l\cos\theta\delta\theta) + P(-3l\sin\theta\delta\theta) = 0$$

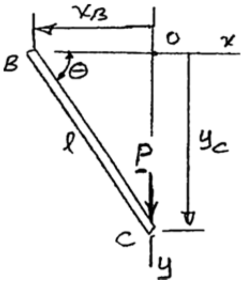
$$Q = \frac{3}{2}P\tan\theta \blacktriangleleft$$


B
θ
$\frac{1}{2}l$
l
M
A
C
P

PROBLEM 10.15

Derive an expression for the magnitude of the couple **M** required to maintain the equilibrium of the linkage shown.

SOLUTION


x_B
O
x
B
θ
ℓ
P
y_C
C
y

We have $x_B = l\cos\theta$

$$\delta x_B = -l\sin\theta\,\delta\theta \qquad (1)$$

$$y_C = l\sin\theta$$

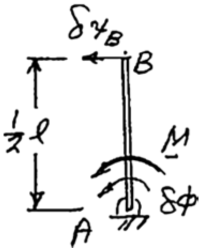
$$\delta y_C = l\cos\theta\,\delta\theta$$

Now $\delta x_B = \frac{1}{2}l\,\delta\theta$

Substituting from Equation (1)

$$-l\sin\theta\,\delta\theta = \frac{1}{2}l\,\delta\phi$$

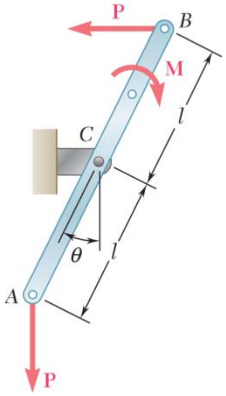
or $\delta\phi = -2\sin\theta\,\delta\theta \circlearrowleft$


δx_B
B
$\frac{1}{2}\ell$
M
$\delta\phi$
A

Virtual Work: $\delta U = 0$: $M\,\delta\varphi + P\,\delta y_C = 0$

$$M(-2\sin\theta\,\delta\theta) + P(l\cos\theta\,\delta\theta) = 0$$

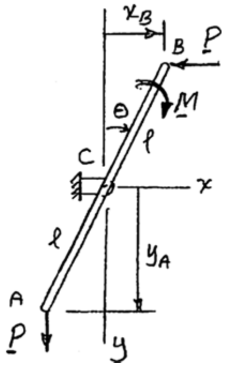
or $M = \frac{1}{2}Pl\frac{\cos\theta}{\sin\theta}$ $M = \frac{Pl}{2\tan\theta}$ ◀



PROBLEM 10.16

Derive an expression for the magnitude of the couple **M** required to maintain the equilibrium of the linkage shown.

SOLUTION



We have

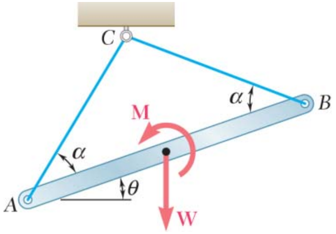
$$x_B = l\sin\theta$$
$$\delta x_B = l\cos\theta\delta\theta$$
$$y_A = l\cos\theta$$
$$\delta y_A = -l\sin\theta\delta\theta$$

Virtual Work:

$$\delta U = 0: \quad M\delta\theta - P\delta x_B + P\delta y_A = 0$$

$$M\delta\theta - P(l\cos\theta\delta\theta) + P(-l\sin\theta\delta\theta) = 0$$

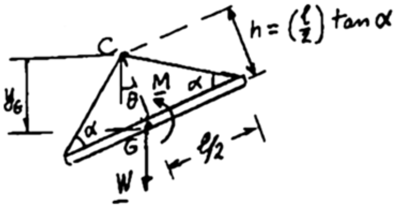
$$M = Pl(\sin\theta + \cos\theta) \blacktriangleleft$$



PROBLEM 10.17

A uniform rod *AB* of length *l* and weight *W* is suspended from two cords *AC* and *BC* of equal length. Derive an expression for the magnitude of the couple **M** required to maintain equilibrium of the rod in the position shown.

SOLUTION



$$y_G = h\cos\theta = \frac{1}{2}l\tan\alpha\cos\theta$$

$$\delta y_G = -\frac{1}{2}l\tan\alpha\,\sin\theta\delta\theta$$

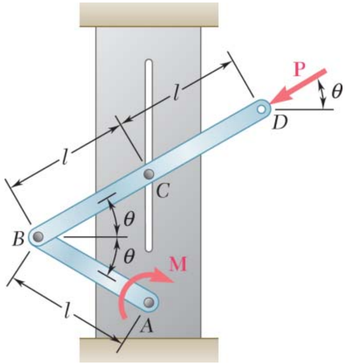
Virtual Work:

$$\delta U = W\delta y_G + M\delta\theta = 0$$

$$W\left(-\frac{1}{2}l\tan\alpha\sin\theta\delta\theta\right) + M\delta\theta = 0$$

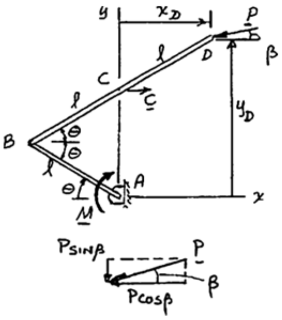
$$M = \frac{1}{2}Wl\tan\alpha\sin\theta \quad ◀$$

PROBLEM 10.18



The pin at C is attached to member BCD and can slide along a slot cut in the fixed plate shown. Neglecting the effect of friction, derive an expression for the magnitude of the couple **M** required to maintain equilibrium when the force **P** that acts at D is directed (a) as shown, (b) vertically downward, (c) horizontally to the right.

SOLUTION



We have
$$x_D = l\cos\theta$$
$$\delta x_D = -l\sin\theta\delta\theta$$
$$y_D = 3l\sin\theta$$
$$\delta y_D = 3l\cos\theta\delta\theta$$

Virtual Work: $\delta U = 0$: $M\delta\theta - (P\cos\beta)\delta x_D - (P\sin\beta)\delta y_D = 0$

$$M\delta\theta - (P\cos\beta)(-l\sin\theta\delta\theta) - (P\sin\beta)(3l\cos\theta\delta\theta) = 0$$

$$M = Pl(3\sin\beta\cos\theta - \cos\beta\sin\theta) \qquad (1)$$

(a) For **P** directed along BCD, $\beta = \theta$

Equation (1): $M = Pl(3\sin\theta\cos\theta - \cos\theta\sin\theta)$

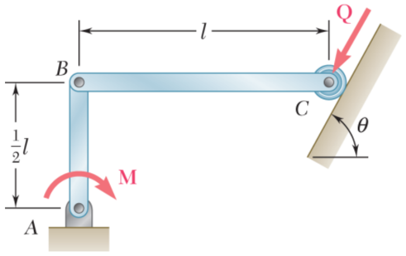
$M = Pl(2\sin\theta\cos\theta)$ $\qquad M = Pl\sin 2\theta$ ◀

(b) For **P** directed ↓, $\beta = 90°$

Equation (1): $M = Pl(3\sin 90°\cos\theta - \cos 90°\sin\theta)$ $\qquad M = 3Pl\cos\theta$ ◀

(c) For **P** directed →, $\beta = 180°$

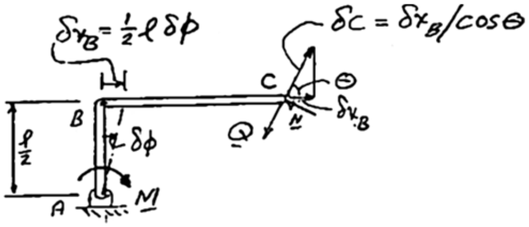
Equation (1): $M = Pl(3\sin 180°\cos\theta - \cos 180°\sin\theta)$ $\qquad M = Pl\sin\theta$ ◀


l
Q
B
C
θ
$\frac{1}{2}l$
M
A

PROBLEM 10.19

For the linkage shown, determine the couple **M** required for equilibrium when $l = 1.8$ ft, $Q = 40$ lb, and $\theta = 65°$.

SOLUTION


$\delta x_B = \frac{1}{2}\ell\,\delta\phi$
$\delta C = \delta x_B/\cos\theta$
C
θ
δx_B
B
$\frac{\ell}{2}$
$\delta\phi$
Q
M
A

$$\delta C = \frac{\frac{1}{2} l \delta\phi}{\cos\theta}$$

Virtual Work:

$$\delta U = 0: \quad M\delta\phi - Q\delta C = 0$$

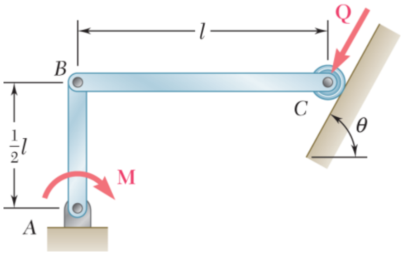
$$M\delta\phi - Q\left(\frac{1}{2}\frac{l}{\cos\theta}\right)\delta\phi = 0$$

$$M = \frac{1}{2}\frac{Ql}{\cos\theta}$$

Data:

$$M = \frac{1}{2}\frac{(40\text{ lb})(1.8\text{ ft})}{\cos 65°} = 85.18\text{ lb}\cdot\text{ft}$$

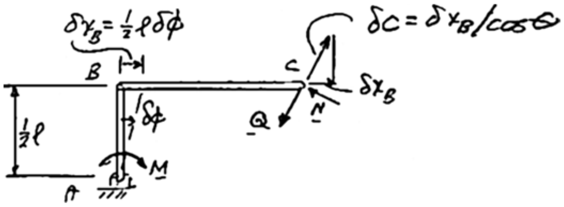
$\mathbf{M} = 85.2\text{ lb}\cdot\text{ft} \circlearrowright$ ◀


Q
l
B
C
θ
$\frac{1}{2}l$
M
A

PROBLEM 10.20

For the linkage shown, determine the force **Q** required for equilibrium when $l = 18$ in., $M = 600$ lb · in., and $\theta = 70°$.

SOLUTION


$\delta x_B = \frac{1}{2}l\delta\phi$
$\delta C = \delta x_B / \cos\theta$
B
C
δx_B
$l\delta\phi$
$\frac{1}{2}l$
Q
N
M
A

$$\delta C = \frac{1}{2}\frac{l\delta\phi}{\cos\theta}$$

Virtual Work:

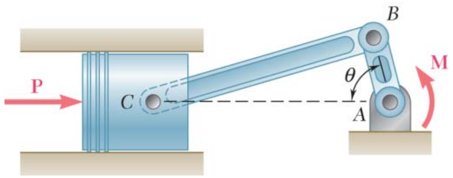
$$\delta U = 0: \qquad M\,\delta\phi - Q\delta C = 0$$

$$M\,\delta\phi - Q\left(\frac{1}{2}\frac{l}{\cos\theta}\right)\delta\phi = 0$$

$$Q = \frac{2M\cos\theta}{l}$$

Data:
$$Q = \frac{2(600\text{ lb}\cdot\text{in.})\cos 70°}{18\text{ in.}} = 22.801\text{ lb}$$

Q = 22.8 lb ∠70.0° ◀

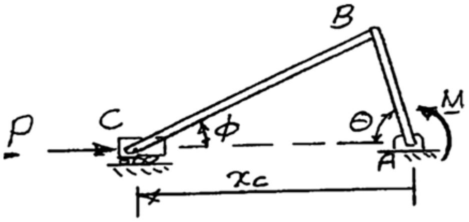


PROBLEM 10.21

A 4-kN force **P** is applied as shown to the piston of the engine system. Knowing that $AB = 50$ mm and $BC = 200$ mm, determine the couple **M** required to maintain the equilibrium of the system when (*a*) $\theta = 30°$, (*b*) $\theta = 150°$.

SOLUTION

Analysis of the geometry:



Law of sines

$$\frac{\sin\phi}{AB} = \frac{\sin\theta}{BC}$$

$$\sin\phi = \frac{AB}{BC}\sin\theta \qquad (1)$$

Now

$$x_C = AB\cos\theta + BC\cos\phi$$

$$\delta x_C = -AB\sin\theta\delta\theta - BC\sin\phi\delta\phi \qquad (2)$$

Now, from Equation (1) $$\cos\phi\delta\phi = \frac{AB}{BC}\cos\theta\delta\theta$$

or $$\delta\phi = \frac{AB}{BC}\frac{\cos\theta}{\cos\phi}\delta\theta \qquad (3)$$

From Equation (2)

$$\delta x_C = -AB\sin\theta\delta\theta - BC\sin\phi\left(\frac{AB}{BC}\frac{\cos\theta}{\cos\phi}\delta\theta\right)$$

or $$\delta x_C = -\frac{AB}{\cos\phi}(\sin\theta\cos\phi + \sin\phi\cos\theta)\delta\theta$$

Then $$\delta x_C = -\frac{AB\sin(\theta+\phi)}{\cos\phi}\delta\theta$$

PROBLEM 10.21 (Continued)

Virtual Work:

$$\delta U = 0{:}\quad -P\delta x_C - M\,\delta\theta = 0$$

$$-P\left[-\frac{AB\sin(\theta+\phi)}{\cos\phi}\delta\theta\right] - M\,\delta\theta = 0$$

Thus,

$$M = AB\frac{\sin(\theta+\phi)}{\cos\phi}P \qquad (4)$$

(*a*) $P = 4\text{ kN},\quad \theta = 30°$

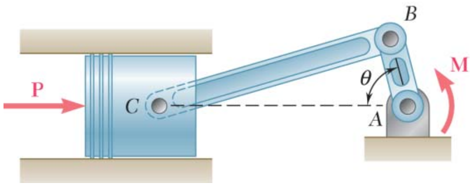
Eq. (1): $\sin\phi = \dfrac{50\text{ mm}}{200\text{ mm}}\sin 30° \qquad \phi = 7.181°$

Eq. (4): $M = (0.05\text{ m})\dfrac{\sin(30°+7.181°)}{\cos 7.818°}(4\text{ kN})$ $\qquad$ $\mathbf{M} = 121.8\text{ N}\cdot\text{m}$ ↺ ◀

(*b*) $P = 4\text{ kN},\quad \theta = 150°$

Eq. (1): $\sin\phi = \dfrac{50\text{ mm}}{200\text{ mm}}\sin 160° \qquad \phi = 7.181°$

Eq. (4): $M = (0.05\text{ m})\dfrac{\sin(150°+7.181°)}{\cos 7.181°}(4\text{ kN})$ $\qquad$ $\mathbf{M} = 78.2\text{ N}\cdot\text{m}$ ↺ ◀

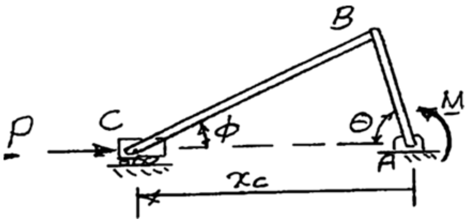


PROBLEM 10.22

A couple **M** of magnitude 100 N · m is applied as shown to the crank of the engine system. Knowing that $AB = 50$ mm and $BC = 200$ mm, determine the force **P** required to maintain the equilibrium of the system when (*a*) $\theta = 60°$, (*b*) $\theta = 120°$.

SOLUTION

Analysis of the geometry:



Law of sines

$$\frac{\sin\phi}{AB} = \frac{\sin\theta}{BC}$$

$$\sin\phi = \frac{AB}{BC}\sin\theta \qquad (1)$$

Now

$$x_C = AB\cos\theta + BC\cos\phi$$

$$\delta x_C = -AB\sin\theta\delta\theta - BC\sin\phi\delta\phi \qquad (2)$$

Now, from Equation (1) $$\cos\phi\delta\phi = \frac{AB}{BC}\cos\theta\delta\theta$$

or $$\delta\phi = \frac{AB}{BC}\frac{\cos\theta}{\cos\phi}\delta\theta \qquad (3)$$

From Equation (2)

$$\delta x_C = -AB\sin\theta\delta\theta - BC\sin\phi\left(\frac{AB}{BC}\frac{\cos\theta}{\cos\phi}\delta\theta\right)$$

or $$\delta x_C = -\frac{AB}{\cos\phi}(\sin\theta\cos\phi + \sin\phi\cos\theta)\delta\theta$$

Then $$\delta x_C = -\frac{AB\sin(\theta+\phi)}{\cos\phi}\delta\theta$$

PROBLEM 10.22 (Continued)

Virtual Work: $\delta U = 0$: $-P\delta x_C - M\,\delta\theta = 0$

$$-P\left[-\frac{AB\sin(\theta+\phi)}{\cos\phi}\delta\theta\right] - M\,\delta\theta = 0$$

Thus, $$M = AB\frac{\sin(\theta+\phi)}{\cos\phi}P \qquad (4)$$

(*a*) $M = 100\ \text{N}\cdot\text{m}, \quad \theta = 60°$

Eq. (1): $$\sin\phi = \frac{50\ \text{mm}}{200\ \text{mm}}\sin 60° \qquad \phi = 12.504°$$

Eq. (4): $$100\ \text{N}\cdot\text{m} = (0.05\ \text{m})\frac{\sin\,(60° + 12.504°)}{\cos 12.504°}P$$

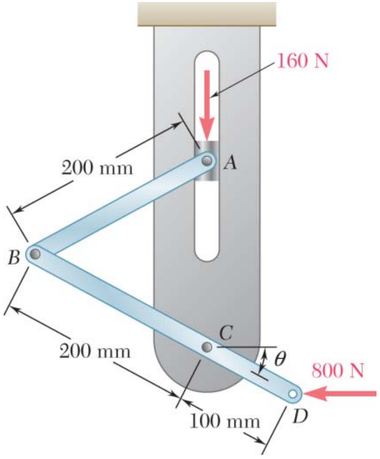
$$P = 2047\ \text{N} \qquad \mathbf{P} = 2.05\ \text{kN} \longrightarrow \blacktriangleleft$$

(*b*) $M = 100\ \text{N}\cdot\text{m}, \quad \theta = 120°$

Eq. (1): $$\sin\phi = \frac{50\ \text{mm}}{200\ \text{mm}}\sin 120° \qquad \phi = 12.504°$$

Eq. (4): $$100\ \text{N}\cdot\text{m} = (0.05\ \text{m})\frac{\sin\,(120° + 12.504°)}{\cos 12.504°}P$$

$$P = 2649\ \text{N} \qquad \mathbf{P} = 2.65\ \text{kN} \longrightarrow \blacktriangleleft$$

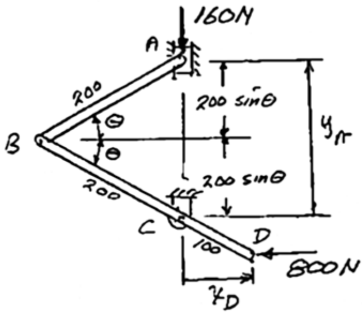

160 N
A
200 mm
B
200 mm
C
θ
800 N
100 mm
D

PROBLEM 10.23

Rod *AB* is attached to a block at *A* that can slide freely in the vertical slot shown. Neglecting the effect of friction and the weights of the rods, determine the value of θ corresponding to equilibrium.

SOLUTION

Dimensions in mm.


160N
A
200
200 sinθ
y_A
B
θ
θ
200 sinθ
200
C
D
100
800N
x_D

$$y_A = 400\sin\theta$$

$$\delta y_A = 400\cos\theta\,\delta\theta$$

$$x_D = 100\cos\theta$$

$$\delta x_D = -100\sin\theta\,\delta\theta$$

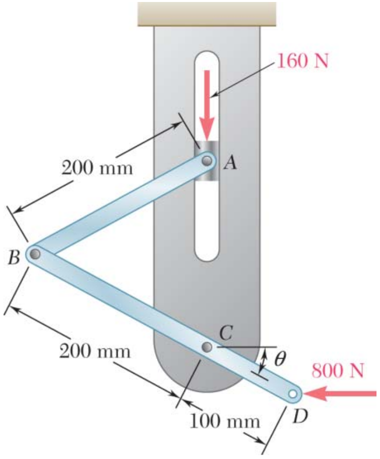
Virtual Work:

$$\delta U = -(160\text{ N})\delta y_A - (800\text{ N})\delta x_D = 0$$

$$-(160)(400\cos\theta\,\delta\theta) - (800)(-100\sin\theta\,\delta\theta) = 0$$

$$\frac{\sin\theta}{\cos\theta} = 0.8; \quad \tan\theta = 0.8$$

$$\theta = 38.7° \blacktriangleleft$$

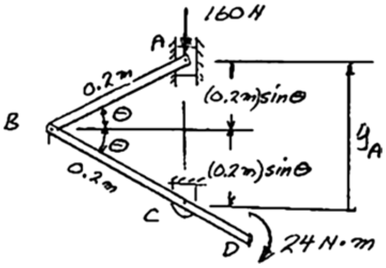

160 N
200 mm
A
B
200 mm
C
θ
800 N
100 mm
D

PROBLEM 10.24

Solve Problem 10.23 assuming that the 800-N force is replaced by a $24\text{-N} \cdot \text{m}$ clockwise couple applied at *D*.

PROBLEM 10.23 Rod *AB* is attached to a block at *A* that can slide freely in the vertical slot shown. Neglecting the effect of friction and the weights of the rods, determine the value of θ corresponding to equilibrium.

SOLUTION


160 N
A
0.2 m
(0.2 m) sin θ
B
θ
θ
y_A
0.2 m
(0.2 m) sin θ
C
D
24 N·m

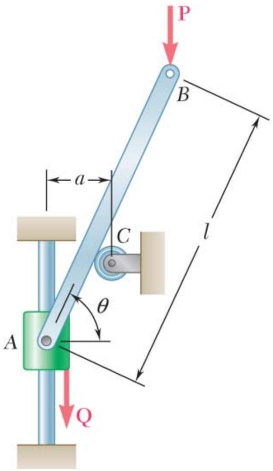
$$y_A = (0.4 \text{ m}) \sin\theta$$

$$\delta y_A = (0.4 \text{ m}) \cos\theta \, \delta\theta$$

Virtual Work: $$\delta U = -(160 \text{ N})\delta y_A + (24 \text{ N} \cdot \text{m})\delta\theta = 0$$

$$-(160 \text{ N})(0.4 \text{ m})\cos\theta \, \delta\theta + (24 \text{ N} \cdot \text{m})\delta\theta = 0$$

$$\cos\theta = 0.375 \qquad \theta = 68.0° \blacktriangleleft$$

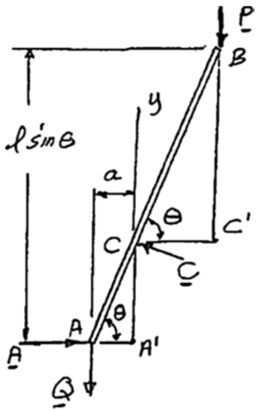


PROBLEM 10.25

Determine the value of θ corresponding to the equilibrium position of the rod of Problem 10.10 when $l = 30$ in., $a = 5$ in., $P = 25$ lb, and $Q = 40$ lb.

PROBLEM 10.10 The slender rod *AB* is attached to a collar *A* and rests on a small wheel at *C*. Neglecting the radius of the wheel and the effect of friction, derive an expression for the magnitude of the force **Q** required to maintain the equilibrium of the rod.

SOLUTION



For triangle $AA'C$:

$$A'C = a\tan\theta$$

$$y_A = -(A'C) = -a\tan\theta$$

$$\delta y_A = -\frac{a}{\cos^2\theta}\delta\theta$$

For triangle $CC'B$:

$$BC' = l\sin\theta - A'C$$

$$= l\sin\theta - a\tan\theta$$

$$y_B = BC' = l\sin\theta - a\tan\theta$$

$$\delta y_B = l\cos\theta\delta\theta - \frac{a}{\cos^2\theta}\delta\theta$$

Virtual Work:

$$\delta U = 0: \quad -Q\delta y_A - P\delta y_B = 0$$

$$-Q\left(-\frac{a}{\cos^2\theta}\right)\delta\theta - P\left(l\cos\theta - \frac{a}{\cos^2\theta}\right)\delta\theta = 0$$

$$Q\left(\frac{a}{\cos^2\theta}\right) = P\left(l\cos\theta - \frac{a}{\cos^2\theta}\right)$$

or

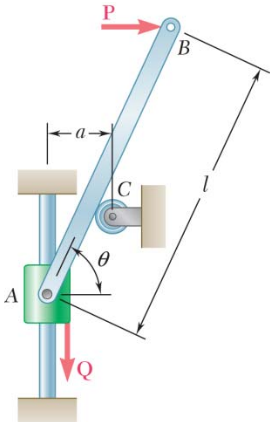
$$Q = P\left(\frac{l}{a}\cos^3\theta - 1\right)$$

PROBLEM 10.25 (Continued)

with $l = 30 \text{ in.}, a = 5 \text{ in.}, P = 25 \text{ lb, and } Q = 40 \text{ lb}$

$$(40 \text{ lb}) = (25 \text{ lb})\left(\frac{30 \text{ in.}}{5 \text{ in.}}\cos^3\theta - 1\right)$$

or $\cos^3\theta = 0.4333$ $\theta = 40.8° \blacktriangleleft$

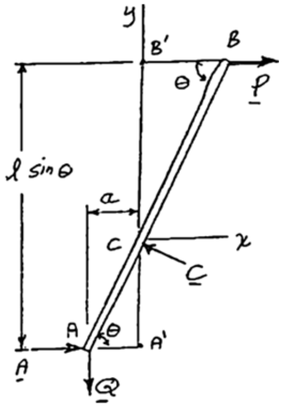


PROBLEM 10.26

Determine the values of θ corresponding to the equilibrium position of the rod of Prob. 10.11 when $l = 24$ in., $a = 4$ in., $P = 10$ lb, and $Q = 18$ lb.

PROBLEM 10.11 The slender rod *AB* is attached to a collar *A* and rests on a small wheel at *C*. Neglecting the radius of the wheel and the effect of friction, derive an expression for the magnitude of the force **Q** required to maintain the equilibrium of the rod.

SOLUTION



For triangle $AA'C$:

$$A'C = a\tan\theta$$

$$y_A = -(A'C) = -a\tan\theta$$

$$\delta y_A = -\frac{a}{\cos^2\theta}\delta\theta$$

For triangle $BB'C$:

$$B'C = l\sin\theta - A'C$$

$$= l\sin\theta - a\tan\theta$$

$$BB' = \frac{B'C}{\tan\theta} = \frac{l\sin\theta - a\tan\theta}{\tan\theta}$$

$$x_B = BB' = l\cos\theta - a$$

$$\delta x_B = -l\sin\theta\delta\theta$$

Virtual Work:

$$\delta U = 0: \quad P\delta x_B - Q\delta y_A = 0$$

$$P(-l\sin\theta\delta\theta) - Q\left(-\frac{a}{\cos^2\theta}\delta\theta\right) = 0$$

$$Pl\sin\theta\cos^2\theta = Qa$$

or

$$Q = P\frac{l}{a}\sin\theta\cos^2\theta$$

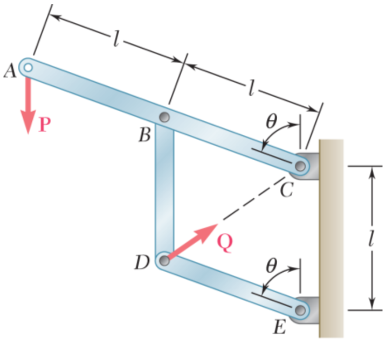
PROBLEM 10.26 (Continued)

with $l = 24 \text{ in.}, a = 4 \text{ in.}, P = 10 \text{ lb, and } Q = 18 \text{ lb}$

$$18 \text{ lb} = (10 \text{ lb})\frac{24 \text{ in.}}{4 \text{ in.}}\sin\theta\cos^2\theta$$

or $\sin\theta\cos^2\theta = 0.300$

Solving numerically $\theta = 19.81° \text{ and } 51.9°$ ◀



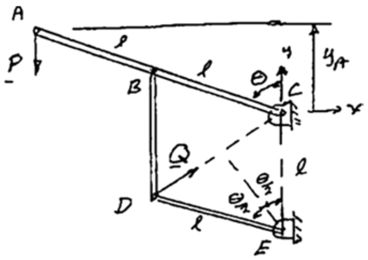
PROBLEM 10.27

Determine the value of θ corresponding to the equilibrium position of the mechanism of Problem 10.12 when $P = 80$ N and $Q = 100$ N.

PROBLEM 10.12 Knowing that the line of action of the force **Q** passes through Point C, derive an expression for the magnitude of **Q** required to maintain equilibrium.

SOLUTION

From geometry



$$y_A = 2l\cos\theta, \quad \delta y_A = -2l\sin\theta\delta\theta$$

$$CD = 2l\sin\frac{\theta}{2}, \quad \delta(CD) = l\cos\frac{\theta}{2}\delta\theta$$

Virtual Work:

$$\delta U = 0: \quad -P\delta y_A - Q\delta(CD) = 0$$

$$-P(-2l\sin\theta\delta\theta) - Q\left(l\cos\frac{\theta}{2}\delta\theta\right) = 0$$

or

$$Q = 2P\frac{\sin\theta}{\cos\left(\frac{\theta}{2}\right)}$$

with

$$P = 80\text{ N}, \quad Q = 100\text{ N}$$

$$(100\text{ N}) = 2(80\text{ N})\frac{\sin\theta}{\cos\left(\frac{\theta}{2}\right)}$$

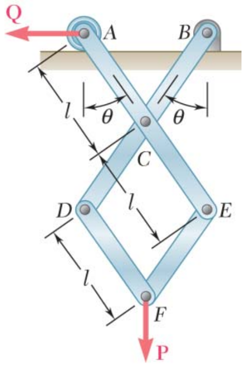
$$\frac{\sin\theta}{\cos\left(\frac{\theta}{2}\right)} = 0.625$$

or

$$\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = 0.625$$

$$\theta = 36.42° \qquad \theta = 36.4° \blacktriangleleft$$

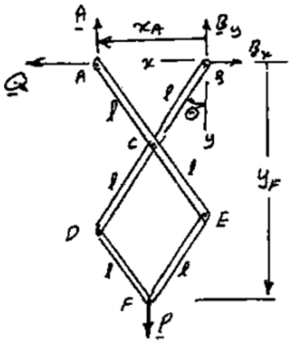
(Additional solutions discarded as not applicable are $\theta = \pm 180°$)



PROBLEM 10.28

Determine the value of θ corresponding to the equilibrium position of the mechanism of Prob. 10.14 when $P = 270$ N and $Q = 960$ N.

SOLUTION



Virtual Work:

$$x_A = 2l\sin\theta, \quad \delta x_A = 2l\cos\theta\delta\theta$$

$$y_F = 3l\cos\theta, \quad \delta y_F = -3l\sin\theta\delta\theta$$

$$\delta U = 0: \qquad Q\delta x_A + P\delta y_F = 0$$

$$\delta U = 0: \qquad Q\delta\theta x_A + P\delta y_F = 0$$

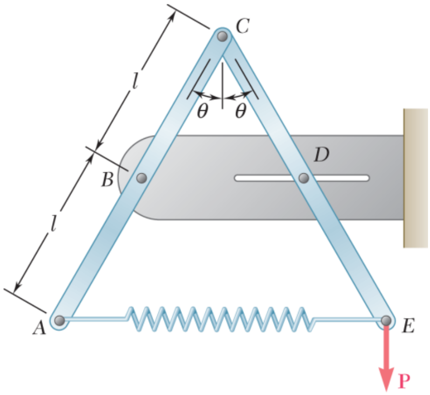
$$Q(2l\cos\theta\delta\theta) + P(-3l\sin\theta\delta\theta) = 0$$

$$Q = \frac{3}{2}P\tan\theta$$

Data: $P = 270$ N, $Q = 960$ N

$$(960\text{ N}) = \frac{3}{2}(270\text{ N})\tan\theta$$

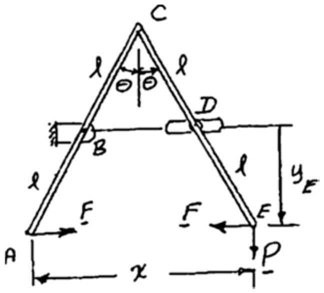
$\theta = 67.1°$ ◀



PROBLEM 10.29

Two rods *AC* and *CE* are connected by a pin at *C* and by a spring *AE*. The constant of the spring is *k*, and the spring is unstretched when $\theta = 30°$. For the loading shown, derive an equation in *P*, *θ*, *l*, and *k* that must be satisfied when the system is in equilibrium.

SOLUTION



$$y_E = l\cos\theta$$

$$\delta y_E = -l\sin\theta\,\delta\theta$$

Spring:

Unstretched length = $2l$

$$x = 2(2l\sin\theta) = 4l\sin\theta$$

$$\delta x = 4l\cos\theta\delta\theta$$

$$F = k(x - 2l)$$

$$F = k(4l\sin\theta - 2l)$$

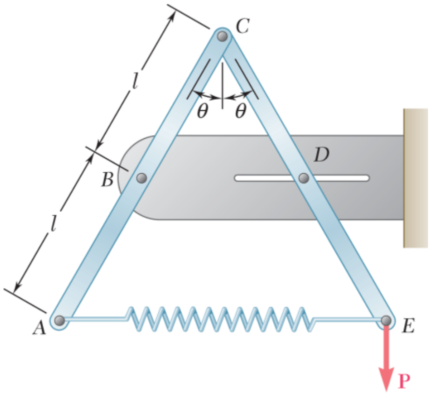
Virtual Work:

$$\delta U = 0: \quad P\delta y_E - F\delta x = 0$$

$$P(-l\sin\theta\,\delta\theta) - k(4l\sin\theta - 2l)(4l\cos\theta\,\delta\theta) = 0$$

$$-P\sin\theta - 8kl(2\sin\theta - 1)\cos\theta = 0$$

or $$\frac{P}{8kl} = (1 - 2\sin\theta)\frac{\cos\theta}{\sin\theta} \qquad \frac{P}{8kl} = \frac{1 - 2\sin\theta}{\tan\theta} \blacktriangleleft$$



PROBLEM 10.30

Two rods *AC* and *CE* are connected by a pin at *C* and by a spring *AE*. The constant of the spring is 1.5 lb/in., and the spring is unstretched when $\theta = 30°$. Knowing that $l = 10$ in. and neglecting the weight of the rods, determine the value of θ corresponding to equilibrium when $P = 40$ lb.

SOLUTION

From the analysis of Problem 10.29

$$\frac{P}{8kl} = \frac{1-2\sin\theta}{\tan\theta}$$

Then with

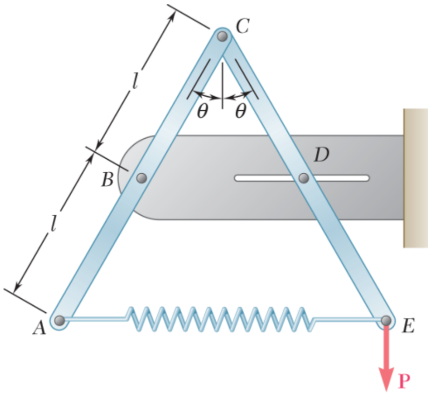
$$P = 160 \text{ N}, \; l = 0.2 \text{ m, and } k = 300 \text{ N/m}$$

$$\frac{160 \text{ N}}{8(300 \text{ N/m})(0.2 \text{ m})} = \frac{1-2\sin\theta}{\tan\theta}$$

or

$$\frac{1-2\sin\theta}{\tan\theta} = \frac{1}{3} = 0.3333$$

Solving numerically $\theta = 25.0°$ ◀

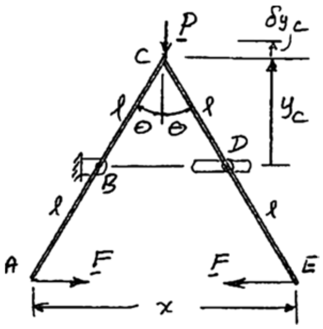


PROBLEM 10.31

Solve Problem 10.30 assuming that force **P** is moved to *C* and acts vertically downward.

PROBLEM 10.30 Two rods *AC* and *CE* are connected by a pin at *C* and by a spring *AE*. The constant of the spring is 1.5 lb/in., and the spring is unstretched when $\theta = 30°$. Knowing that $l = 10$ in. and neglecting the weight of the rods, determine the value of θ corresponding to equilibrium when $P = 40$ lb.

SOLUTION



$$y_C = l\cos\theta, \quad \delta y_C = -l\sin\theta\delta\theta$$

Spring:

$$\text{Unstretched length} = 2l$$

$$x = 2(2l\sin\theta) = 4l\sin\theta$$

$$\delta x = 4l\cos\theta\delta\theta$$

$$F = k(x - 2l)$$

$$F = k(4l\sin\theta - 2l)$$

Virtual Work:

$$\delta U = 0: \quad -P\delta y_C - F\delta x$$

$$-P(-l\sin\theta\delta\theta) - k(4l\sin\theta - 2l)(4l\cos\theta\delta\theta) = 0$$

$$P\sin\theta - 8kl(2\sin\theta - 1)\cos\theta = 0$$

or

$$\frac{P}{8kl} = (2\sin\theta - 1)\frac{\cos\theta}{\sin\theta}$$

with

$$l = 200 \text{ mm}, \ k = 300 \text{ N/m, and } P = 160 \text{ N}$$

$$\frac{(160 \text{ N})}{8(300 \text{ N/m})(0.2)} = (2\sin\theta - 1)\frac{\cos\theta}{\sin\theta}$$

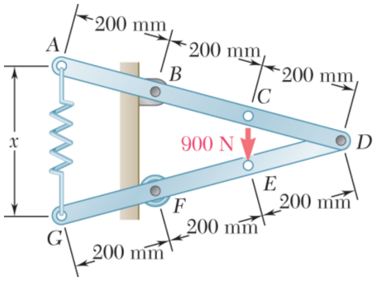
or

$$(2\sin\theta - 1)\frac{\cos\theta}{\sin\theta} = \frac{1}{3}$$

Solving numerically $\theta = 39.65°$

and $\theta = 68.96°$ $\qquad \theta = 39.7°$ ◀

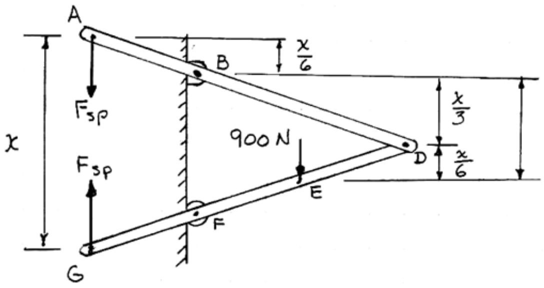
and $\qquad \theta = 69.0°$ ◀


200 mm
A
200 mm
B
200 mm
C
x
900 N
D
E
200 mm
F
200 mm
G
200 mm

PROBLEM 10.32

Two bars *AD* and *DG* are connected by a pin at *D* and by a spring *AG*. Knowing that the spring is 300 mm long when unstretched and that the constant of the spring is 5 kN/m, determine the value of *x* corresponding to equilibrium when a 900-N load is applied at *E* as shown.

SOLUTION


A
B
x/6
F_{SP}
x
F_{SP}
900 N
x/3
D
x/6
E
F
G

$$y_E = \frac{x}{3} + \frac{x}{6} = \frac{x}{2} \qquad \therefore\ \delta y_E = \frac{1}{2}\delta x$$

Linear Spring: $F_{SP} = ks = (5000\text{ N/m})(x - 0.30\text{ m})$

Virtual Work: $\delta U = 0$

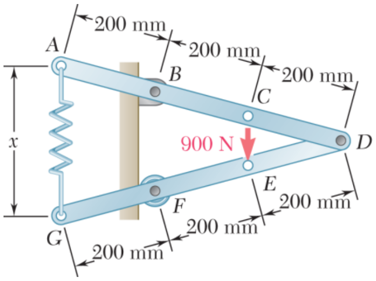
$$\delta U = -F_{SP}\delta x + P\delta_E = 0$$

$$-(5000\text{ N/m})(x - 0.30\text{ m})\delta x + (900\text{ N})\left(\frac{1}{2}\delta x\right) = 0$$

$$-5000x + 1500 + 450 = 0$$

$x = 0.390$ m

or $x = 390$ mm ◀



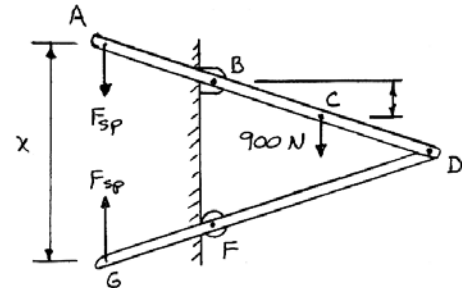
200 mm
200 mm
200 mm
A
B
C
x
900 N
D
E
F
200 mm
G
200 mm
200 mm

PROBLEM 10.33

Solve Prob. 10.32 assuming that the 900-N load is applied at *C* instead of *E*.

Problem 10.32: Two bars *AD* and *DG* are connected by a pin at *D* and by a spring *AG*. Knowing that the spring is 300 mm long when unstretched and that the constant of the spring is 5 kN/m, determine the value of *x* corresponding to equilibrium when 900-N load is applied at *E* as shown.

SOLUTION



A
B
C
x
F_SP
900 N
D
F_SP
F
G

$$y_C = \frac{x}{6} \qquad \therefore\ \delta y_C = \frac{1}{6}\delta x$$

Linear Spring: $F_{SP} = ks = (5000\text{ N/m})(x - 0.30\text{ m})$

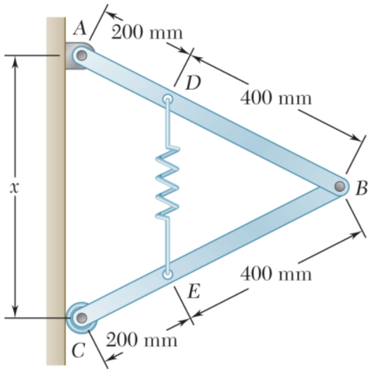
Virtual Work: $\delta U = 0$

$$\delta U = -F_{SP}\delta x + P\delta y_C = 0$$

$$-(5000\text{ N/m})(x - 0.30\text{ m})\delta x + (900\text{ N})\left(\frac{1}{6}\delta x\right) = 0$$

$$-5000x + 1500 + 150 = 0$$

$x = 0.330$ m $\qquad$ or $x = 330$ mm ◀

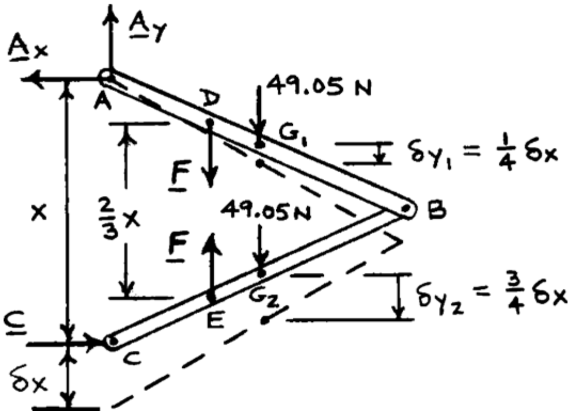


PROBLEM 10.34

Two 5-kg bars *AB* and *BC* are connected by a pin at *B* and by a spring *DE*. Knowing that the spring is 150 mm long when unstretched and that the constant of the spring is 1 kN/m, determine the value of *x* corresponding to equilibrium.

SOLUTION

First note: $W_{\text{bar}} = (5\text{ kg})(9.81\text{ m/s}^2) = 49.05\text{ N}$



During the virtual displacement, Points *D* and *E* move apart a distance $\delta(DE) = \frac{2}{3}\delta x$ and the total work done by the forces exerted at *D* and *E* is $-F\left(\frac{2}{3}\delta x\right)$

$$\delta U = 0: \quad -F\left(\frac{2}{3}\delta x\right) + 49.05\text{ N}\left(\frac{1}{4}\delta x\right) + 49.05\text{ N}\left(\frac{3}{4}\delta x\right) = 0$$

$$F = 73.575\text{ N}$$

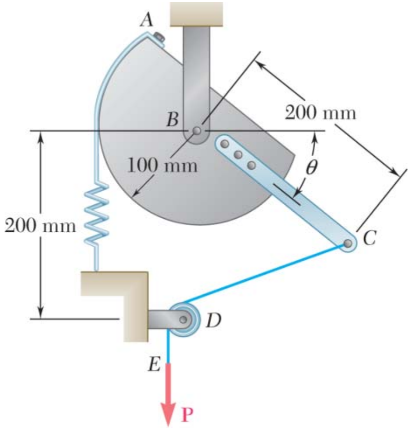
For $F = 73.575$ N, elongation of spring is

$$\frac{F}{k} = \frac{73.575\text{ N}}{1000\text{ N/m}} = 73.575\times 10^{-3}\text{m}$$

$$= 73.575\text{ mm}$$

Since undeformed length of spring is 150 mm, total length is

$$DE = \frac{2}{3}x = 150\text{ mm} + 73.575\text{ mm} \qquad x = 335\text{ mm} \blacktriangleleft$$



PROBLEM 10.35

A vertical force **P** of magnitude 150 N is applied to end *E* of cable *CDE*, which passes over a small pulley *D* and is attached to the mechanism at *C*. The constant of the spring is $k = 4$ kN/m, and the spring is unstretched when $\theta = 0$. Neglecting the weight of the mechanism and the radius of the pulley, determine the value of θ corresponding to equilibrium.

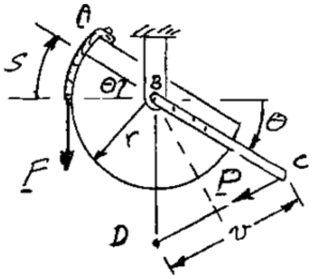
SOLUTION

$$l = BC = 0.2 \text{ m}$$

$$r = 0.1 \text{ m}$$

$< CBD = 90° - \theta$

$$v = 2l\sin\left(45° - \frac{\theta}{2}\right)$$



$$\delta v = -l\cos\left(45° - \frac{\theta}{2}\right)\delta\theta$$

$$s = r\theta \quad \delta s = r\delta\theta$$

$$F = ks = kr\theta$$

Virtual Work: $\delta U = 0: \quad -P\delta v - F\delta s = 0$

$$-P\left[-l\cos\left(45° - \frac{\theta}{2}\right)\right]\delta\theta - kr\theta(r\delta\theta) = 0$$

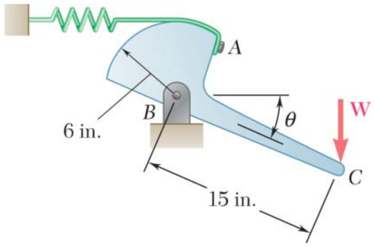
$$\frac{Pl}{kr^2} = \frac{\theta}{\cos\left(45° - \frac{\theta}{2}\right)}$$

$$\frac{Pl}{kr^2} = \frac{(150 \text{ N})(0.2 \text{ m})}{(4000 \text{ N/m})(0.1 \text{ m})^2} = 0.75$$

$$0.75 = \frac{\theta}{\cos\left(45° - \frac{\theta}{2}\right)}$$

$$\theta = 0.67623 \text{ rad} = 38.745°$$

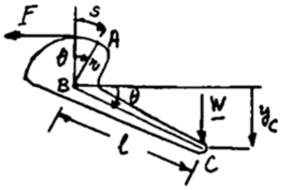
$\theta = 38.7°$ ◀



PROBLEM 10.36

A load **W** of magnitude 72 lb is applied to the mechanism at *C*. Neglecting the weight of the mechanism, determine the value of θ corresponding to equilibrium. The constant of the spring is $k = 20$ lb/in., and the spring is unstretched when $\theta = 0$.

SOLUTION



$$s = r\theta$$

$$\delta s = r\delta\theta$$

$$F = ks = kr\theta$$

$$y_C = l\sin\theta$$

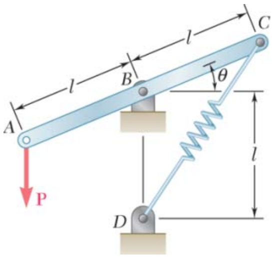
$$\delta y_C = l\cos\theta\delta\theta$$

Virtual Work: $\delta U = -F\delta s + W\delta y_C = 0$

$$-kr\theta(r\delta\theta) + W(l\cos\theta\delta\theta) = 0$$

$$\frac{\theta_C}{\cos\theta} = \frac{Wl}{kr^2} = \frac{(72\text{ lb})(15\text{ in.})}{(20\text{ lb/in.})(6\text{ in.})^2} = 1.500$$

Solving by trial and error: $\theta = 0.91486$ rad $\qquad \theta = 52.4° ◀$

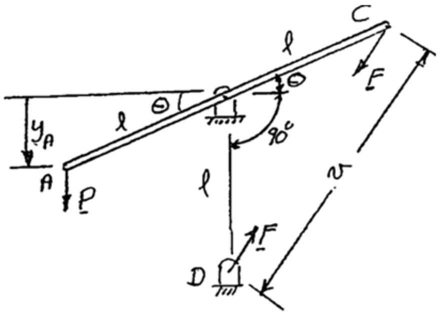


PROBLEM 10.37

Knowing that the constant of spring *CD* is *k* and that the spring is unstretched when rod *ABC* is horizontal, determine the value of *θ* corresponding to equilibrium for the data indicated.

$$P = 300 \text{ N}, \quad l = 400 \text{ mm}, \quad k = 5 \text{ kN/m}.$$

SOLUTION



$$y_A = l\sin\theta$$

$$\delta y_A = l\cos\theta\delta\theta$$

Spring: $v = CD$

Unstretched when $\theta = 0$

so that $v_0 = \sqrt{2}l$

For θ: $v = 2l\sin\left(\frac{90^\circ + \theta}{2}\right)$

$$\delta v = l\cos\left(45^\circ + \frac{\theta}{2}\right)\delta\theta$$

Stretched length: $s = v - v_0 = 2l\sin\left(45^\circ + \frac{\theta}{2}\right) - \sqrt{2}l$

Then $F = ks = kl\left[2\sin\left(45^\circ + \frac{\theta}{2}\right) - \sqrt{2}\right]$

Virtual Work:

$$\delta U = 0: \quad P\delta y_A - F\delta v = 0$$

$$Pl\cos\theta\delta\theta - kl\left[2\sin\left(45^\circ + \frac{\theta}{2}\right) - \sqrt{2}\right]l\cos\left(45^\circ + \frac{\theta}{2}\right)\delta\theta = 0$$

or $\frac{P}{kl} = \frac{1}{\cos\theta}\left[2\sin\left(45^\circ + \frac{\theta}{2}\right)\cos\left(45^\circ + \frac{\theta}{2}\right) - \sqrt{2}\cos\left(45^\circ + \frac{\theta}{2}\right)\right]$

$$= \frac{1}{\cos\theta}\left[2\sin\left(45^\circ + \frac{\theta}{2}\right)\cos\left(45^\circ + \frac{\theta}{2}\right)\cos\theta - \sqrt{2}\cos\left(45^\circ + \frac{\theta}{2}\right)\right]$$

$$= 1 - \sqrt{2}\frac{\cos\left(45^\circ \frac{\theta}{2}\right)}{\cos\theta}$$

PROBLEM 10.37 (Continued)

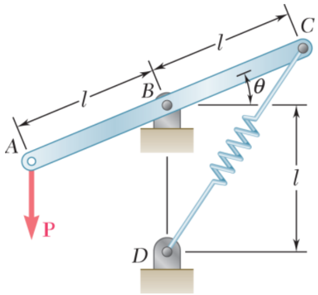
Now, with $P = 300\text{ N}, \quad l = 400\text{ mm}, \quad \text{and} \quad k = 5\text{ kN/m}$

$$\frac{(300\text{ N})}{(5000\text{ N/m})(0.4\text{ m})} = 1 - \sqrt{2}\,\frac{\cos\left(45° + \frac{\theta}{2}\right)}{\cos\theta}$$

or

$$\frac{\cos\left(45° + \frac{\theta}{2}\right)}{\cos\theta} = 0.60104$$

Solving numerically $\theta = 22.6°$ ◀



PROBLEM 10.38

Knowing that the constant of spring *CD* is k and that the spring is unstretched when rod *ABC* is horizontal, determine the value of θ corresponding to equilibrium for the data indicated.

$$P = 75 \text{ lb}, \quad l = 15 \text{ in.}, \quad k = 20 \text{ lb/in.}$$

SOLUTION

From the analysis of Problem 10.37, we have

$$\frac{P}{kl} = 1 - \sqrt{2}\frac{\cos\left(45^\circ + \frac{\theta}{2}\right)}{\cos\theta}$$

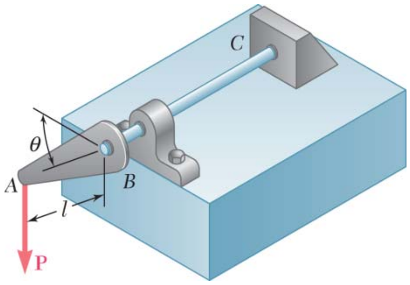
with $P = 75 \text{ lb}, \quad l = 15 \text{ in.} \quad \text{and} \quad k = 20 \text{ lb/in.}$

$$\frac{(75 \text{ lb})}{(20 \text{ lb/in.})(15 \text{ in.})} = 1 - \sqrt{2}\frac{\cos\left(45^\circ + \frac{\theta}{2}\right)}{\cos\theta}$$

or

$$\frac{\cos\left(45^\circ + \frac{\theta}{2}\right)}{\cos\theta} = 0.53033$$

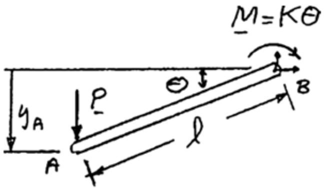
Solving numerically $\theta = 51.1^\circ$ ◀



PROBLEM 10.39

The lever *AB* is attached to the horizontal shaft *BC* that passes through a bearing and is welded to a fixed support at *C*. The torsional spring constant of the shaft *BC* is *K*; that is, a couple of magnitude *K* is required to rotate end *B* through 1 rad. Knowing that the shaft is untwisted when *AB* is horizontal, determine the value of θ corresponding to the position of equilibrium when $P = 100$ N, $l = 250$ mm, and $K = 12.5$ N · m/rad.

SOLUTION



We have

$$y_A = l\sin\theta$$

$$\delta y_A = l\cos\theta\delta\theta$$

Virtual Work:

$$\delta U = 0: \quad P\delta y_A - M\delta\theta = 0$$

$$Pl\cos\theta\delta\theta - K\theta\delta\theta = 0$$

or

$$\frac{\theta}{\cos\theta} = \frac{Pl}{K} \qquad (1)$$

with

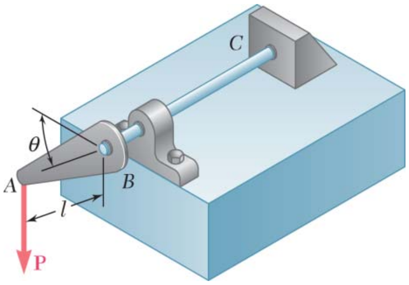
$$P = 100\text{ N}, \quad l = 250\text{ mm} \quad \text{and} \quad K = 12.5\text{ N}\cdot\text{m/rad}$$

$$\frac{\theta}{\cos\theta} = \frac{(100\text{ N})(0.250\text{ m})}{12.5\text{ N}\cdot\text{m/rad}}$$

or

$$\frac{\theta}{\cos\theta} = 2.0000$$

Solving numerically $\theta = 59.0°$ ◀



PROBLEM 10.40

Solve Problem 10.39 assuming that $P = 350$ N, $l = 250$ mm, and $K = 12.5$ N · m/rad. Obtain answers in each of the following quadrants: $0 < \theta < 90°$, $270° < \theta < 360°$, $360° < \theta < 450°$.

PROBLEM 10.39 The lever *AB* is attached to the horizontal shaft *BC* that passes through a bearing and is welded to a fixed support at *C*. The torsional spring constant of the shaft *BC* is *K*; that is, a couple of magnitude *K* is required to rotate end *B* through 1 rad. Knowing that the shaft is untwisted when *AB* is horizontal, determine the value of θ corresponding to the position of equilibrium when $P = 100$ N, $l = 250$ mm, and $K = 12.5$ N · m/rad.

SOLUTION

Using Equation (1) of Problem 10.39 and

$$P = 350 \text{ N}, \quad l = 250 \text{ mm} \quad \text{and} \quad K = 12.5 \text{ N} \cdot \text{m/rad}$$

We have

$$\frac{\theta}{\cos\theta} = \frac{(350 \text{ N})(0.250 \text{ m})}{12.5 \text{ N} \cdot \text{m/rad}}$$

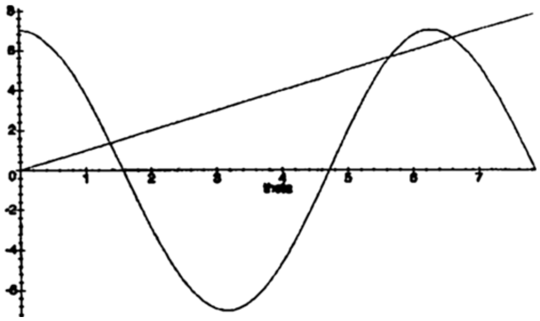
or

$$\frac{\theta}{\cos\theta} = 7 \quad \text{or} \quad \theta = 7\cos\theta \qquad (1)$$

The solutions to this equation can be shown graphically using any appropriate graphing tool, such as Maple, with the command: plot ({theta, 7 * cos(theta)}, t = 0.5 * Pi/2);

Thus, we plot $y = \theta$ and $y = 7\cos\theta$ in the range

$$0 \le \theta \le \frac{5\pi}{2}$$



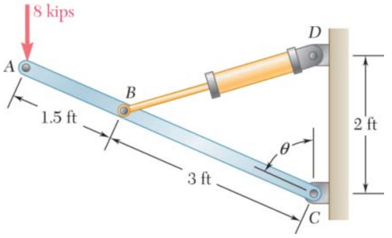
PROBLEM 10.40 (Continued)

We observe that there are three points of intersection, which implies that Equation (1) has three roots in the specified range of θ.

$$0 \le \theta \le 90°\left(\frac{\pi}{2}\right); \quad \theta = 1.37333 \text{ rad}, \quad \theta = 78.69° \qquad \theta = 78.7° \blacktriangleleft$$

$$270 \le \theta \le 360°\left(\frac{3\pi}{2} \le \theta \le 2\pi\right); \quad \theta = 5.65222 \text{ rad}, \quad \theta = 323.85° \qquad \theta = 324° \blacktriangleleft$$

$$360 \le \theta \le 450°\left(2\pi \le \theta \le \frac{5\pi}{2}\right); \quad \theta = 6.61597 \text{ rad}, \quad \theta = 379.07 \qquad \theta = 379° \blacktriangleleft$$

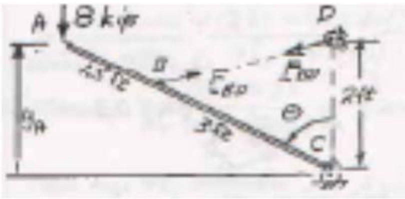


PROBLEM 10.41

The position of boom *ABC* is controlled by the hydraulic cylinder *BD*. For the loading shown, determine the force exerted by the hydraulic cylinder on pin *B* when $\theta = 70°$.

SOLUTION

We have



$$y_A = (4.5\,\text{ft})\cos\theta \qquad \delta y_A = -4.5\sin\theta\delta\theta$$

$$\begin{aligned}(BD)^2 &= (BC)^2 + (CD)^2 - 2(BC)(CD)\cos\theta \\ &= (3)^2 + (2)^2 - 2(3)(2)\cos\theta\end{aligned}$$

$$(BD)^2 = 13 - 12\cos\theta \tag{1}$$

Differentiating: $2(BD)\delta(BD) = 12\sin\theta\delta\theta$

$$\delta(BD) = \frac{6}{BD}\sin\theta\delta\theta \tag{2}$$

Virtual Work: Noting that **P** tends to decrease y_A and $\mathbf{F}_{BD}$ tends to increase *BD*, write

$$\delta U = -P\delta y_A + F_{BD}\delta(BD) = 0$$

$$-8\text{ kips}(-4.5\sin\theta\delta\theta) + F_{BD}\left(-\frac{6}{BD}\sin\theta\delta\theta\right) = 0$$

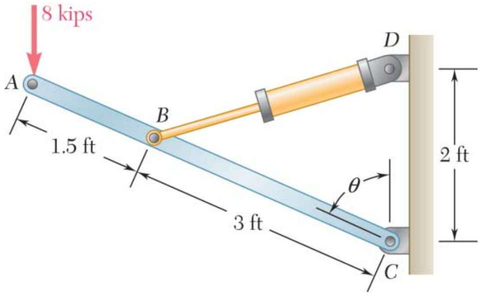
$$F_{BD} = \frac{36(BD)\sin\theta}{6\sin\theta}$$

or, $$F_{BD} = 6\text{ kips}(BD) \tag{3}$$

Making $\theta = 70°$ in Eq. (1), we have

$$(BD)^2 = 13 - 12\cos 70° = 8.8958$$
$$BD = 2.9826\text{ ft}$$

Carrying into Eq. (3): $F_{BD} = 6(2.9826) = 17.8956$ kips $\qquad F_{BD} = 17.90$ kips ◀



PROBLEM 10.42

The position of boom *ABC* is controlled by the hydraulic cylinder *BD*. For the loading shown, determine the largest allowable value of the angle θ if the maximum force that the cylinder can exert on pin *B* is 25 kips.

SOLUTION

(*a*) See solution of Problem 10.41 for the derivation of Eq. (3):

$$F_{BD} = (6\text{ kips})(BD) \blacktriangleleft$$

(*b*) For $(F_{BD})_{\max} = 25\text{ kips},$ we have

$$25\text{ kips} = (6\text{ kips})(BD)$$

$$BD = 4.1667\text{ ft}$$

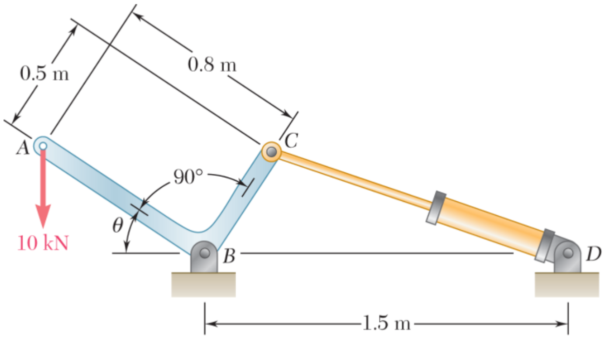
Carrying this value into Eq. (1) of Problem 10.41, write

$$(BD)^2 = 13 - 12\cos\theta$$

$$(4.1667)^2 = 13 - 12\cos\theta$$

$$\cos\theta = -0.36343$$

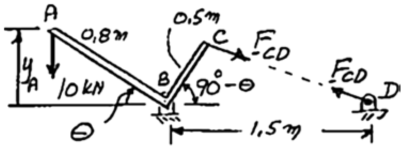
$$\theta = 111.3° \blacktriangleleft$$



PROBLEM 10.43

The position of member *ABC* is controlled by the hydraulic cylinder *CD*. For the loading shown, determine the force exerted by the hydraulic cylinder on pin *C* when $\theta = 55°$.

SOLUTION



$$y_A = (0.8\text{ m})\sin\theta$$

$$\delta y_A = 0.8\cos\theta\delta\theta$$

$$CD^2 = BC^2 + BD^2 - 2(BC)(BD)\cos(90° - \theta)$$

$$CD^2 = 0.5^2 + 1.5^2 - 2(0.5)(1.5)\sin\theta$$

$$CD^2 = 2.5 - 1.5\sin\theta \qquad (1)$$

$$2(CD)(\delta_{CD}) = -1.5\cos\theta\delta\theta \qquad \delta_{CD} = -\frac{3\cos\theta}{4CD}\delta\theta$$

Virtual Work: $\delta U = 0\text{:} \quad -(10\text{ kN})\delta y_A - F_{CD}\delta_{CD} = 0$

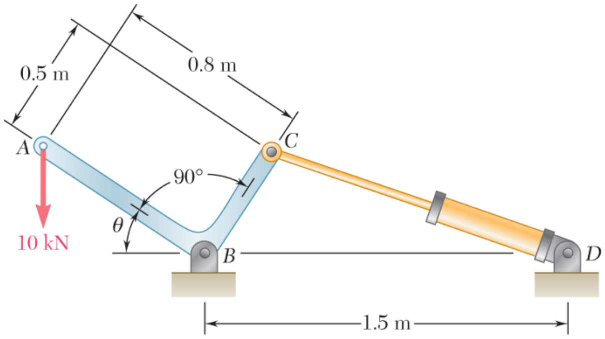
$$-10(0.8\cos\theta\delta\theta) - F_{CD}\left(-\frac{3\cos\theta}{4CD}\delta\theta\right) = 0$$

$$F_{CD} = \frac{32}{3}CD \qquad (2)$$

For $\theta = 55°$:

Eq. (1): $CD^2 = 2.5 - 1.5\sin 55° = 1.2713; \quad CD = 1.1275\text{ m}$

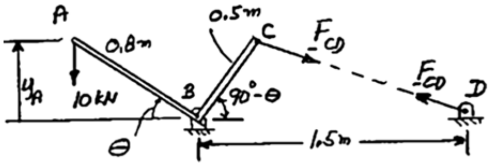
Eq. (2): $F_{CD} = \frac{32}{3}CD = \frac{32}{3}(1.1275) = 12.027\text{ kN}$ $\qquad \mathbf{F}_{CD} = 12.03\text{ kN}$ ↘ ◀



PROBLEM 10.44

The position of member *ABC* is controlled by the hydraulic cylinder *CD*. Determine the angle θ knowing that the hydraulic cylinder exerts a 15-kN force on pin *C*.

SOLUTION



$$y_A = (0.8\text{ m})\sin\theta$$

$$\delta y_A = 0.8\cos\theta\delta\theta$$

$$CD^2 = BC^2 + BD^2 - 2(BC)(BD)\cos(90° - \theta)$$

$$CD^2 = 0.5^2 + 1.5^2 - 2(0.5)(1.5)\sin\theta$$

$$CD^2 = 2.5 - 1.5\sin\theta \qquad (1)$$

$$2(CD)(\delta_{CD}) = -1.5\cos\theta\delta\theta; \qquad \delta_{CD} = -\frac{3\cos\theta}{4CD}\delta\theta$$

Virtual Work:

$$\delta U = 0: \quad -(10\text{ kN})\delta y_A - F_{CD}\delta_{CD} = 0$$

$$-10(0.8\cos\theta\delta\theta) - F_{CD}\left(-\frac{3\cos\theta}{4CD}\delta\theta\right) = 0$$

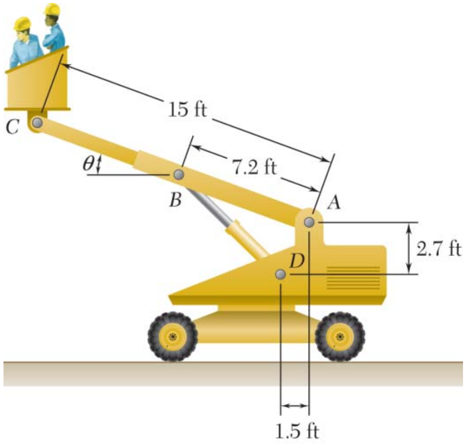
$$F_{CD} = \frac{32}{3}CD \qquad (2)$$

For $F_{CD} = 15\text{ kN}$:

Eq. (2): $15\text{ kN} = \frac{32}{3}CD$: $\qquad CD = \frac{45}{32} = 1.40625\text{ m}$

Eq. (1): $(1.40625)^2 = 2.5 - 1.5\sin\theta$: $\qquad \sin\theta = 0.34831$

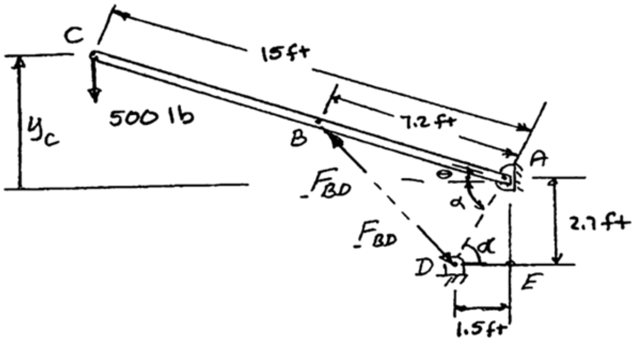
$\theta = 20.38°$ $\qquad \theta = 20.4°$ ◀



PROBLEM 10.45

The telescoping arm *ABC* is used to provide an elevated platform for construction workers. The workers and the platform together weigh 500 lb and their combined center of gravity is located directly above *C*. For the position when $\theta = 20°$, determine the force exerted on pin *B* by the single hydraulic cylinder *BD*.

SOLUTION



In ΔADE:

$$\tan\alpha = \frac{AE}{DE} = \frac{2.7 \text{ ft}}{1.5 \text{ ft}}$$

$$\alpha = 60.945°$$

$$AD = \frac{2.7 \text{ ft}}{\sin 60.945°} = 3.0887 \text{ m}$$

From the geometry:

$$y_C = (15 \text{ ft})\sin\theta$$

$$\delta y_C = (15 \text{ ft})\cos\theta\delta\theta$$

Then, in triangle *BAD*: Angle $BAD = \alpha + \theta$

Law of cosines:

$$BD^2 = AB^2 + AD^2 - 2(AB)(AD)\cos(\alpha + \theta)$$

or

$$BD^2 = (7.2 \text{ ft})^2 + (3.0887 \text{ ft})^2 - 2(7.2 \text{ ft})(3.0887 \text{ ft})\cos(\alpha + \theta)$$

$$BD^2 = 61.380 \text{ ft}^2 - (44.477\cos(\alpha + \theta)) \text{ ft}^2 \quad (1)$$

PROBLEM 10.45 (Continued)

And then

$$2(BD)(\delta BD) = (44.477\sin(\alpha+\theta))\delta\theta$$

$$\delta BD = \frac{44.477\sin(\alpha+\theta)}{2(BD)}\delta\theta$$

Virtual Work: $\delta U = 0{:}\quad -P\delta y_C + F_{BD}\delta BD = 0$

Substituting $-(500\text{ lb})(15\text{ ft})\cos\theta\delta\theta + F_{BD}\left[\frac{(44.477\text{ ft}^2)\sin(\alpha+\theta)}{2(BD)}\delta\theta\right] = 0$

or $F_{BD} = \left[337.25\frac{\cos\theta}{\sin(\alpha+\theta)}BD\right]\text{ lb/ft}$ (2)

Now, with $\theta = 20°$ and $\alpha = 60.945°$

Equation (1):

$$BD^2 = 61.380 - 44.477\cos(60.945° + 20°)$$

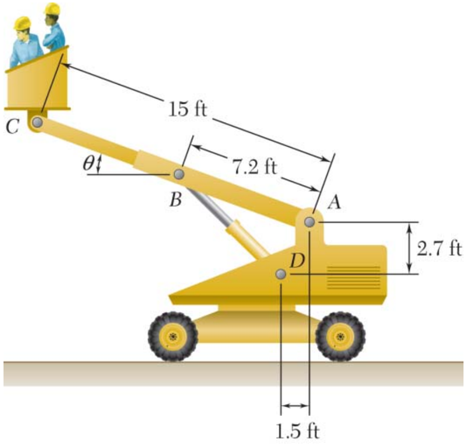
$$BD^2 = 54.380$$

$$BD = 7.3743\text{ ft}$$

Equation (2):

$$F_{BD} = \left[337.25\frac{\cos 20°}{\sin(60.945° + 20°)}(7.3743\text{ ft})\right]\text{ lb/ft}$$

or $F_{BD} = 2366\text{ lb}$ $\mathbf{F}_{BD} = 2370\text{ lb}$ ↖ ◀



PROBLEM 10.46

Solve Problem 10.45 assuming that the workers are lowered to a point near the ground so that $\theta = -20°$.

PROBLEM 10.45 The telescoping arm *ABC* is used to provide an elevated platform for construction workers. The workers and the platform together weigh 500 lb and their combined center of gravity is located directly above *C*. For the position when $\theta = 20°$, determine the force exerted on pin *B* by the single hydraulic cylinder *BD*.

SOLUTION

Using the figure and analysis of Problem 10.45, including Equations (1) and (2), and with $\theta = -20°$, we have

Equation (1):

$$BD^2 = 61.380 - 44.477\cos(60.945° - 20°)$$

$$BD^2 = 27.785$$

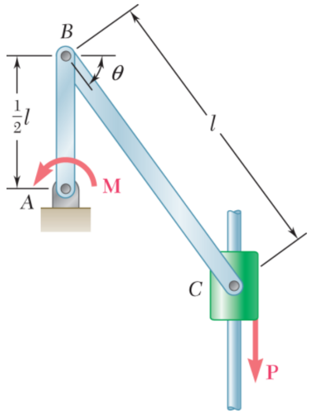
$$BD = 5.2711 \text{ ft}$$

Equation (2):

$$F_{BD} = 337.25\frac{\cos(-20°)}{\sin(60.945° - 20°)}(5.2711)$$

$$F_{BD} = 2549 \text{ lb}$$

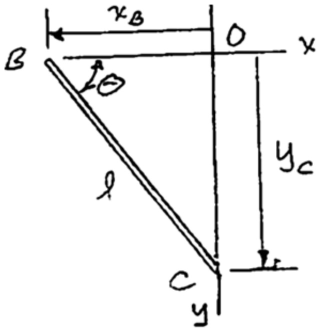
or $\mathbf{F}_{BD} = 2550 \text{ lb} \nwarrow$ ◀


B
θ
$\frac{1}{2}l$
l
M
A
C
P

PROBLEM 10.47

Denoting by μ_s the coefficient of static friction between collar *C* and the vertical rod, derive an expression for the magnitude of the largest couple **M** for which equilibrium is maintained in the position shown. Explain what happens if $\mu_s \geq \tan\theta$.

SOLUTION


x_B
O
x
B
θ
y_C
l
C
y

Member *BC*: We have

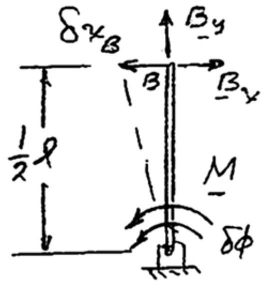
$$x_B = l\cos\theta$$

$$\delta x_B = -l\sin\theta\delta\theta \qquad (1)$$

and

$$y_C = l\sin\theta$$

$$\delta y_C = l\cos\theta\delta\theta \qquad (2)$$


δx_B
B_y
B
B_x
$\frac{1}{2}l$
M
$\delta\phi$

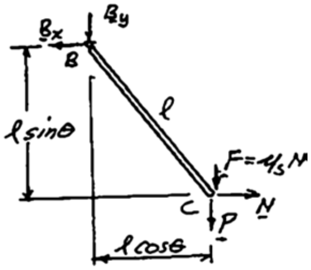
Member *AB*: We have

$$\delta x_B = \frac{1}{2}l\delta\phi$$

Substituting from Equation (1),

$$-l\sin\theta\delta\theta = \frac{1}{2}l\delta\phi$$

or $$\delta\phi = -2\sin\theta\delta\theta \qquad (3)$$


B_x
B_y
B
l
$l\sin\theta$
$F=\mu_s N$
C
P
N
$l\cos\theta$

Free body of rod *BC*

For $M_{\max}$, motion of collar *C* impends upward

$$+\circlearrowright \Sigma M_B = 0: \quad Nl\sin\theta - (P + \mu_s N)(l\cos\theta) = 0$$

$$N\tan\theta - \mu_s N = P$$

$$N = \frac{P}{\tan\theta - \mu_s}$$

PROBLEM 10.47 (Continued)

Virtual Work:

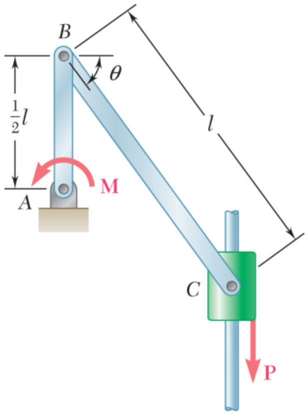
$$\delta U = 0: \quad M\,\delta\phi + (P + \mu_s N)\delta y_C = 0$$

$$M(-2\sin\theta\delta\theta) + (P + \mu_s N)l\cos\theta\delta\theta = 0$$

$$M_{\max} = \frac{(P + \mu_s N)}{2\tan\theta}l = \frac{P + \mu_s \frac{P}{\tan\theta - \mu_s}}{2\tan\theta}l$$

$$\text{or} \quad M_{\max} = \frac{Pl}{2(\tan\theta - \mu_s)} \quad \blacktriangleleft$$

If $\mu_s = \tan\theta,\ M = \infty,$ system becomes *self-locking*.



PROBLEM 10.48

Knowing that the coefficient of static friction between collar C and the vertical rod is 0.40, determine the magnitude of the largest and smallest couple **M** for which equilibrium is maintained in the position shown, when $\theta = 35°$, $l = 600$ mm, and $P = 300$ N.

SOLUTION

From the analysis of Problem 10.50, we have

$$M_{\max} = \frac{Pl}{2(\tan\theta + \mu_s)}$$

With $\theta = 35°, \quad l = 0.6\text{ m}, \quad P = 300\text{ N}$

$$M_{\max} = \frac{(300\text{ N})(0.6\text{ m})}{2(\tan 35° - 0.4)}$$

$$= 299.80\text{ N}\cdot\text{m} \qquad M_{\max} = 300\text{ N}\cdot\text{m} \blacktriangleleft$$

For $M_{\min}$, motion of C impends downward and F acts upward. The equations of Problem 10.50 can still be used if we replace μ_s by $-\mu_s$. Then

$$M_{\min} = \frac{Pl}{2(\tan\theta + \mu_s)}$$

Substituting,

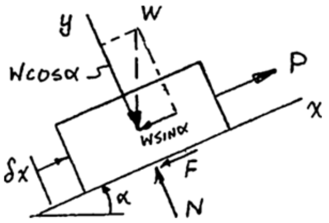
$$M_{\min} = \frac{(300\text{ N})(0.6\text{ m})}{2(\tan 35° + 0.4)}$$

$$= 81.803\text{ N}\cdot\text{m} \qquad M_{\min} = 81.8\text{ N}\cdot\text{m} \blacktriangleleft$$

PROBLEM 10.49

A block of weight W is pulled up a plane forming an angle α with the horizontal by a force **P** directed along the plane. If μ is the coefficient of friction between the block and the plane, derive an expression for the mechanical efficiency of the system. Show that the mechanical efficiency cannot exceed $\frac{1}{2}$ if the block is to remain in place when the force **P** is removed.

SOLUTION



$$\text{Input work} = P\delta x$$

$$\text{Output work} = (W\sin\alpha)\delta x$$

Efficiency:

$$\eta = \frac{W\sin\alpha\delta x}{P\delta x} \quad \text{or} \quad \eta = \frac{W\sin\alpha}{P} \qquad (1)$$

$$+\nearrow \Sigma F_x = 0\text{:} \quad P - F - W\sin\alpha = 0 \quad \text{or} \quad P = W\sin\alpha + F \qquad (2)$$

$$+\nwarrow \Sigma F_y = 0\text{:} \quad N - W\cos\alpha = 0 \quad \text{or} \quad N = W\cos\alpha$$

$$F = \mu N = \mu W\cos\alpha$$

Equation (2): $$P = W\sin\alpha + \mu W\cos\alpha = W(\sin\alpha + \mu\cos\alpha)$$

Equation (1): $$\eta = \frac{W\sin\alpha}{W(\sin\alpha + \mu\cos\alpha)} \quad \text{or} \quad \eta = \frac{1}{1 + \mu\cot\alpha}$$

If block is to remain in place when $P = 0$, we know (see Chapter 8) that $\phi_s \geq \alpha$ or, since

$$\mu = \tan\phi_s, \quad \mu \geq \tan\alpha$$

Multiply by $\cot\alpha$: $$\mu\cot\alpha \geq \tan\alpha\cot\alpha = 1$$

Add 1 to each side: $$1 + \mu\cot\alpha \geq 2$$

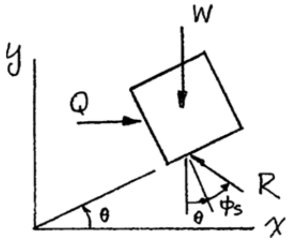
Recalling the expression for η, we find $$\eta \geq \frac{1}{2} \blacktriangleleft$$

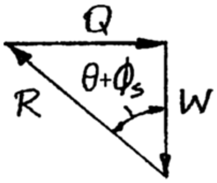
PROBLEM 10.50

Derive an expression for the mechanical efficiency of the jack discussed in Section 8.6. Show that if the jack is to be self-locking, the mechanical efficiency cannot exceed $\frac{1}{2}$.

SOLUTION

Recall Figure 8.9*a*. Draw force triangle





$$Q = W\tan(\theta + \phi_s)$$

$$y = x\tan\theta \text{ so that } \delta y = \delta x\tan\theta$$

$$\text{Input work} = Q\delta x = W\tan(\theta + \phi_s)\delta x$$

$$\text{Output work} = W\delta y = W(\delta x)\tan\theta$$

Efficiency:
$$\eta = \frac{W\tan\theta\delta x}{W\tan(\theta + \phi_s)\delta x}; \qquad \eta = \frac{\tan\theta}{\tan(\theta + \phi_s)} \blacktriangleleft$$

From Page 432, we know the jack is self-locking if

$$\phi \geq \theta$$

Then
$$\theta + \phi_s \geq 2\theta$$

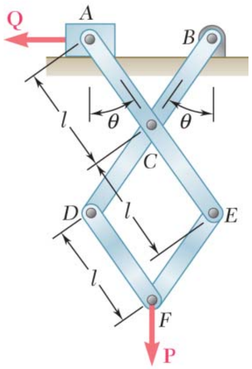
so that
$$\tan(\theta + \phi_s) \geq \tan 2\theta$$

From above
$$\eta = \frac{\tan\theta}{\tan(\theta + \phi_s)}$$

It then follows that
$$\eta \leq \frac{\tan\theta}{\tan 2\theta}$$

But
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Then
$$\eta \leq \frac{\tan\theta(1 - \tan^2\theta)}{2\tan\theta} = \frac{1 - \tan^2\theta}{2} \qquad \eta \leq \frac{1}{2} \blacktriangleleft$$



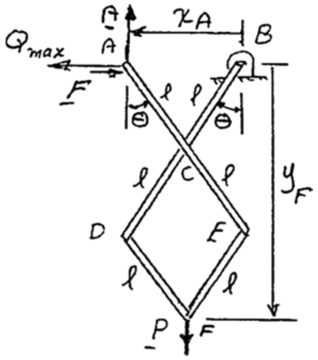
PROBLEM 10.51

Denoting by μ_s the coefficient of static friction between the block attached to rod *ACE* and the horizontal surface, derive expressions in terms of P, μ_s, and θ for the largest and smallest magnitude of the force **Q** for which equilibrium is maintained.

SOLUTION

For the linkage:

$$+\circlearrowleft \Sigma M_B = 0: \quad -x_A + \frac{x_A}{2}P = 0 \quad \text{or} \quad \mathbf{A} = \frac{P}{2}\uparrow$$



Then:

$$F = \mu_s A = \mu_s \frac{P}{2} = \frac{1}{2}\mu_s P$$

Now

$$x_A = 2l\sin\theta$$
$$\delta x_A = 2l\cos\theta\delta\theta$$

and

$$y_F = 3l\cos\theta$$
$$\delta y_F = -3l\sin\theta\delta\theta$$

Virtual Work:

$$\delta U = 0: \quad (Q_{\max} - F)\delta x_A + P\delta y_F = 0$$

$$\left(Q_{\max} - \frac{1}{2}\mu_s P\right)(2l\cos\theta\delta\theta) + P(-3l\sin\theta\delta\theta) = 0$$

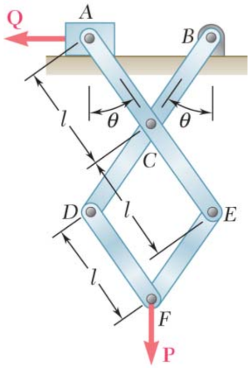
or

$$Q_{\max} = \frac{3}{2}P\tan\theta + \frac{1}{2}\mu_s P$$

$$Q_{\max} = \frac{P}{2}(3\tan\theta + \mu_s) \blacktriangleleft$$

For $Q_{\min}$, motion of *A* impends to the right and **F** acts to the left. We change μ_s to $-\mu_s$ and find

$$Q_{\min} = \frac{P}{2}(3\tan\theta - \mu_s) \blacktriangleleft$$



PROBLEM 10.52

Knowing that the coefficient of static friction between the block attached to rod *ACE* and the horizontal surface is 0.15, determine the magnitude of the largest and smallest force **Q** for which equilibrium is maintained when $\theta = 30°$, $l = 0.2$ m, and $P = 40$ N.

SOLUTION

Using the results of Problem 10.48 with

$$\theta = 30°$$
$$l = 0.2 \text{ m}$$
$$P = 40 \text{ N}, \quad \text{and} \quad \mu_s = 0.15$$

We have

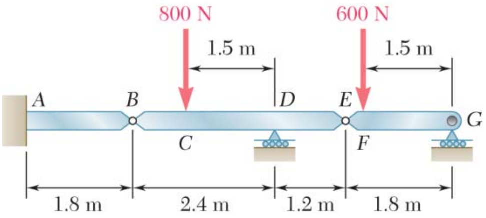
$$Q_{\max} = \frac{P}{2}(3\tan\theta + \mu_s)$$
$$= \frac{(40 \text{ N})}{2}(3\tan 30° + 0.15)$$
$$= 37.64 \text{ N}$$

$Q_{\max} = 37.6$ N ◀

and

$$Q_{\min} = \frac{P}{2}(3\tan\theta - \mu_s)$$
$$= \frac{(40 \text{ N})}{2}(3\tan 30° - 0.15)$$
$$= 31.64 \text{ N}$$

$Q_{\min} = 31.6$ N ◀

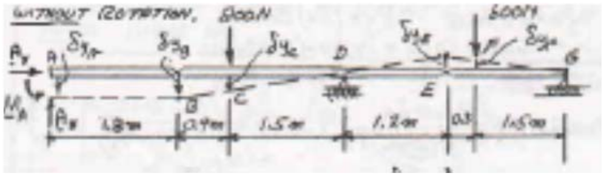

800 N
600 N
1.5 m
1.5 m
A
B
D
E
G
C
F
1.8 m
2.4 m
1.2 m
1.8 m

PROBLEM 10.53

Using the method of virtual work, determine separately the force and couple representing the reaction at *A*.

SOLUTION

Force at *A*. We give a vertical virtual displacement δy_A to Point *A*, keeping member *AB* horizontal.


WITHOUT ROTATION, 800N
600N
δy_A
δy_B
δy_C
D
δy_E
F
δy_F
G
A_x
A
M_A
B
C
E
A_y
1.8m
0.9m
1.5m
1.2m
0.3
1.5m

From the geometry of the diagram:

$$\delta y_A = \delta y_B$$

$$\delta y_C = \frac{5}{8}\delta y_B;\ \delta y_E = \frac{1}{2}\delta y_B$$

$$\delta y_F = \frac{5}{6}\delta y_E = \frac{5}{6}\left(\frac{1}{2}\delta y_B\right) = \frac{5}{12}\delta y_B$$

Virtual Work:

$$\delta U = 0;$$

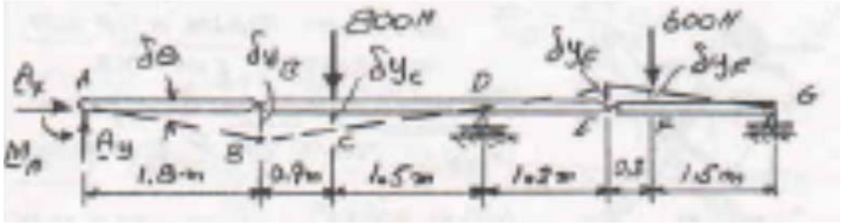
$$-A\delta y_A + (800\text{ N})\delta y_C - (600\text{ N})\delta y_F = 0$$

$$-A\delta y_B + (800)\left(\frac{5}{8}\right)\delta y_B - (600)\left(\frac{5}{12}\delta y_B\right) = 0$$

$$A = +250\text{ N}$$

A = 250 N↑ ◀

Couple at *A*. We rotate beam *AB* through $\delta\theta$ about Point *A*.


800N
600N
A_x
A
$\delta\theta$
δy_B
δy_C
D
δy_E
δy_F
G
M_A
A_y
B
C
E
F
1.8m
0.9m
1.5m
1.2m
0.3
1.5m

PROBLEM 10.53 (Continued)

From the geometry of the diagram:

$$\delta y_B = 1.8\delta\theta$$

$$\delta y_E = \frac{1}{5}\delta y_B = \frac{1}{5}(1.8\delta\theta) = 0.9\delta\theta$$

$$\delta y_C = \frac{5}{8}\delta y_B = \frac{5}{8}(1.8\delta\theta) = 1.125\delta\theta$$

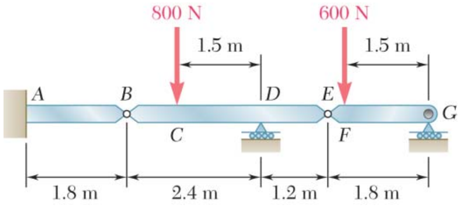
$$\delta y_F = \frac{5}{6}\delta y_E = \frac{5}{6}(0.9\delta\theta) = 0.75\delta\theta$$

Virtual Work:

$$\delta U = 0; \quad -M_A\delta\theta + (800\text{ N})\delta y_C - (600\text{ N})\delta y_F = 0$$

$$-M_A\delta\theta 2 + 800(1.125\delta\theta) - 600(0.75\delta\theta) = 0$$

$M_A = 450\text{ N}\cdot\text{m}$ $\qquad\qquad$ $\mathbf{M}_A = 450\text{ N}\cdot\text{m}$ ↻ ◀

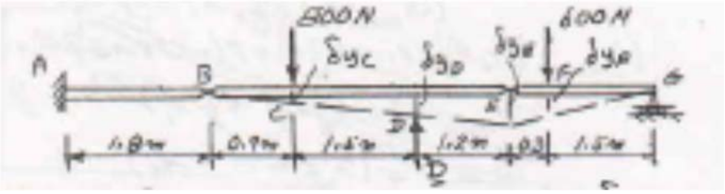

800 N
600 N
1.5 m
1.5 m
A
B
D
E
G
C
F
1.8 m
2.4 m
1.2 m
1.8 m

PROBLEM 10.54

Using the method of virtual work, determine the reaction at *D*.

SOLUTION

We release the support at *D* and assume a virtual displacement δy_D for Point *D*.


800 N
600 N
δyc
δyD
δyE
δyF
A
B
C
D
E
F
G
1.8m
0.9m
1.5m
1.2m
0.3
1.5m
D

From similar triangles:

$$\delta y_C = \frac{3}{8}\delta y_D$$

$$\delta y_E = \frac{3}{2}\delta y_D$$

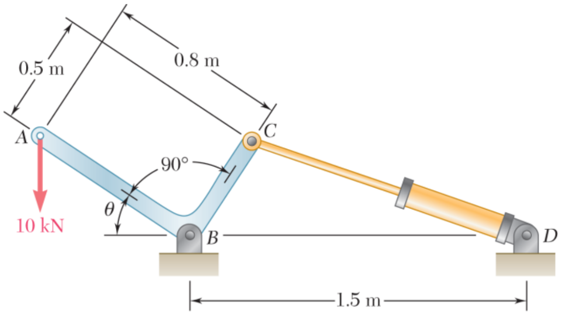
$$\delta y_F = \frac{5}{6}\delta y_E = \frac{5}{6}\left(\frac{3}{2}\delta y_D\right) = \frac{5}{4}\delta y_D$$

Virtual Work:

$$\delta U = 0; \quad -D\delta \mathrm{y}_D + (800\text{ N})\delta y_C + (600\text{ N})\delta y_F = 0$$

$$-D\delta y_D + 800\left(\frac{3}{8}\delta y_D\right) + 600\left(\frac{5}{4}\delta y_D\right) = 0$$

$D = +1050$ N $\qquad$ $\mathbf{D} = 1050\text{ N}\uparrow$ ◀

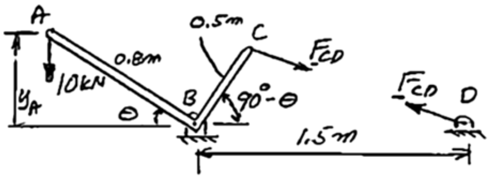


PROBLEM 10.55

Referring to Problem 10.43 and using the value found for the force exerted by the hydraulic cylinder *CD*, determine the change in the length of *CD* required to raise the 10-kN load by 15 mm.

PROBLEM 10.43 The position of member *ABC* is controlled by the hydraulic cylinder *CD*. For the loading shown, determine the force exerted by the hydraulic cylinder on pin *C* when $\theta = 55°$.

SOLUTION



Virtual Work: Assume both δy_A and δ_{CD} increase

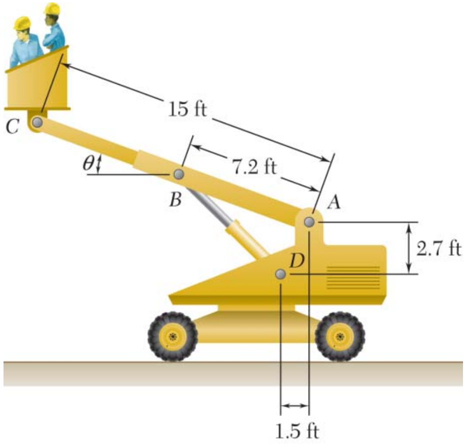
$$\delta U = 0: \quad -(10 \text{ kN})\delta y_A - F_{CD}\delta_{CD} = 0$$

Substitute: $\delta y_A = 15 \text{ mm} \quad \text{and} \quad F_{CD} = 12.03 \text{ kN}$

$$-(10 \text{ kN})(15 \text{ mm}) - (12.03 \text{ kN})\delta_{CD} = 0$$

$$\delta_{CD} = -12.47 \text{ mm}$$

The negative sign indicates that *CD* shortened $\delta_{CD} = 12.47$ mm shorter ◀

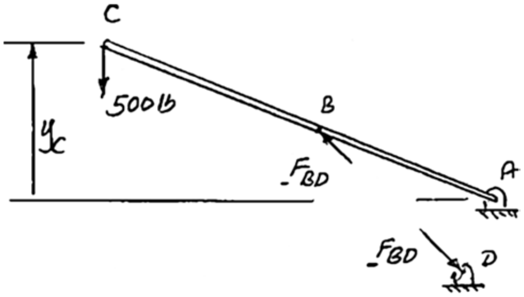


PROBLEM 10.56

Referring to Problem 10.45 and using the value found for the force exerted by the hydraulic cylinder *BD*, determine the change in the length of *BD* required to raise the platform attached at *C* by 2.5 in.

PROBLEM 10.45 The telescoping arm *ABC* is used to provide an elevated platform for construction workers. The workers and the platform together weigh 500 lb and their combined center of gravity is located directly above *C*. For the position when $\theta = 20°$, determine the force exerted on pin *B* by the single hydraulic cylinder *BD*.

SOLUTION



Virtual Work: Assume both δy_C and δ_{BD} increase

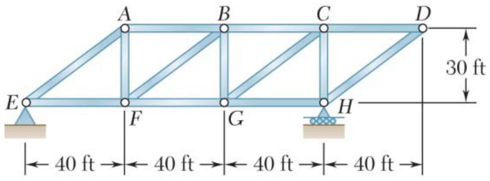
$$\delta U = 0: \quad -(500\text{ lb})\delta y_C + F_{BD}\delta_{BD} = 0$$

Substitute: $\delta y_C = 2.5\text{ in.}$ and $F_{BD} = 2370\text{ lb}$

$$-(500\text{ lb})(2.5\text{ in.}) + (2370\text{ lb})\delta_{BD} = 0$$

$$\delta_{BD} = +0.527\text{ in.}$$

The positive sign indicates that cylinder *BD* increases in length $\delta_{BD} = 0.527$ in. longer ◀

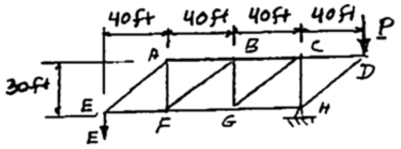


PROBLEM 10.57

Determine the vertical movement of joint D if the length of member BF is increased by 1.5 in. (*Hint:* Apply a vertical load at joint D, and, using the methods of Chapter 6, compute the force exerted by member BF on joints B and F. Then apply the method of virtual work for a virtual displacement resulting in the specified increase in length of member BF. This method should be used only for small changes in the lengths of members.)

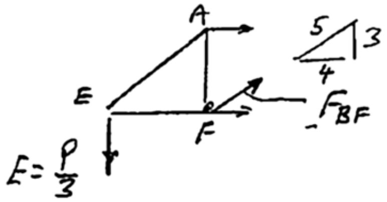
SOLUTION

Apply vertical load P at D.



$$+\circlearrowleft \Sigma M_H = 0: \quad -P(40 \text{ ft}) + E(120 \text{ ft}) = 0$$

$$\mathbf{E} = \frac{P}{3}\downarrow$$



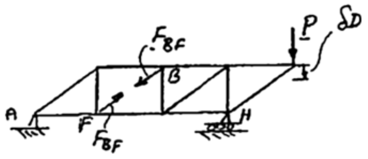
$$+\uparrow \Sigma F_y = 0: \quad \frac{3}{5}F_{BF} - \frac{P}{3} = 0$$

$$F_{BF} = \frac{5}{9}P$$

Virtual Work:

We remove member BF and replace it with forces $\mathbf{F}_{BF}$ and $-\mathbf{F}_{BF}$ at pins F and B, respectively. Denoting the virtual displacements of Points B and F as $\delta\mathbf{r}_B$ and $\delta\mathbf{r}_F$, respectively, and noting that $\mathbf{P}$ and $\overrightarrow{\delta D}$ have the same direction, we have

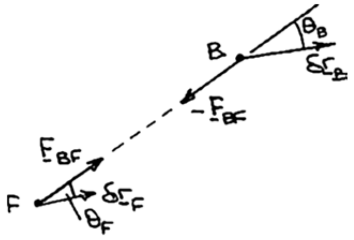
Virtual Work:



$$\delta U = 0: \qquad P\delta D + \mathbf{F}_{BF} \cdot \delta\mathbf{r}_F + (-\mathbf{F}_{BF}) \cdot \delta\mathbf{r}_B = 0$$

$$P\delta D + F_{BF}\delta r_F \cos\theta_F - F_{BF}\delta r_B \cos\theta_B = 0$$

$$P\delta D - F_{BF}(\delta r_B \cos\theta_B - \delta r_F \cos\theta_F) = 0$$



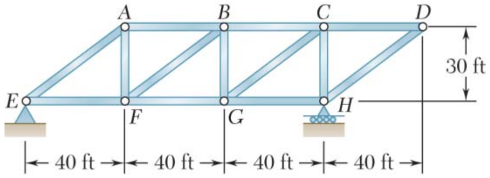
where $(\delta r_B \cos\theta_B - \delta r_F \cos\theta_F) = \delta_{BF}$, which is the change in length of member BF. Thus,

$$P\delta D - F_{BF}\delta_{BF} = 0$$

$$P\delta D - \left(\frac{5}{9}P\right)(1.5 \text{ in.}) = 0$$

$$\delta D = 0.833 \text{ in.}$$

$\delta D = 0.833$ in.↓ ◀

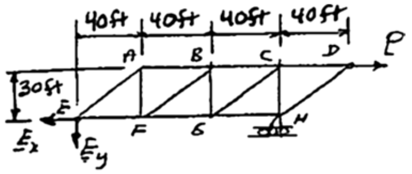

A
B
C
D
E
F
G
H
30 ft
40 ft
40 ft
40 ft
40 ft

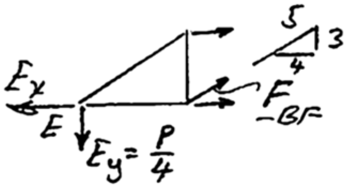
PROBLEM 10.58

Determine the horizontal movement of joint *D* if the length of member *BF* is increased by 1.5 in. (See the hint for Problem 10.57.)

SOLUTION

Apply horizontal load *P* at *D*.


40ft
40ft
40ft
40ft
A
B
C
D
P
30ft
E
F
G
H
Ex
Ey


5
3
4
Ex
E
Ey = P/4
F_BF

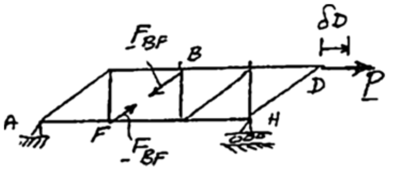
$$+\circlearrowleft \Sigma M_H = 0: \quad P(30\text{ ft}) - E_y(120\text{ ft}) = 0$$

$$\mathbf{E}_y = \frac{P}{4}\downarrow$$

$$+\uparrow \Sigma F_y = 0: \quad \frac{3}{5}F_{BF} - \frac{P}{4} = 0$$

$$F_{BF} = \frac{5}{12}P$$

We remove member *BF* and replace it with forces $\mathbf{F}_{BF}$ and $-\mathbf{F}_{BF}$ at pins *F* and *B*, respectively. Denoting the virtual displacements of Points *B* and *F* as $\delta\mathbf{r}_B$ and $\delta\mathbf{r}_F$, respectively, and noting that **P** and $\overrightarrow{\delta D}$ have the same direction, we have


δD
F_BF
B
D
P
A
F
H
-F_BF

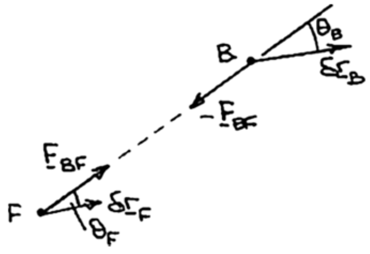
Virtual Work:

$$\delta U = 0: \qquad P\delta D + \mathbf{F}_{BF}\cdot\delta\mathbf{r}_F + (-\mathbf{F}_{BF})\cdot\delta\mathbf{r}_B = 0$$

$$P\delta D + F_{BF}\delta r_F\cos\theta_F - F_{BF}\delta r_B\cos\theta_B = 0$$

$$P\delta D - F_{BF}(\delta r_B\cos\theta_B - \delta r_F\cos\theta_F) = 0$$

where $(\delta r_B\cos\theta_B - \delta r_F\cos\theta_F) = \delta_{BF},$ which is the change in length of member *BF*. Thus,

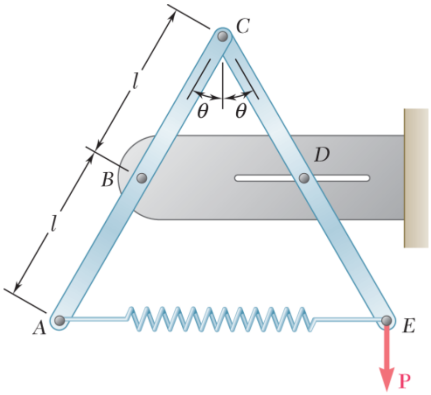

θB
B
δrB
-F_BF
F_BF
F
δrF
θF

$$P\delta D - F_{BF}\delta_{BF} = 0$$

$$P\delta D - \left(\frac{5}{9}P\right)(1.5\text{ in.}) = 0$$

$$\delta D = 0.625\text{ in.}$$

$$\delta D = 0.625\text{ in.}\longrightarrow$$ ◀

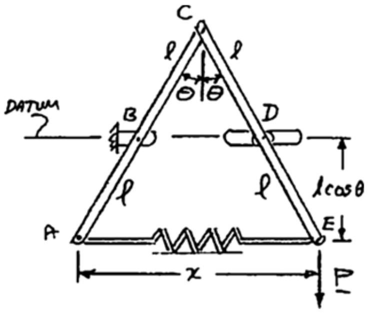


PROBLEM 10.59

Using the method of Section 10.8, solve Problem 10.29.

PROBLEM 10.29 Two rods *AC* and *CE* are connected by a pin at *C* and by a spring *AE*. The constant of the spring is *k*, and the spring is unstretched when $\theta = 30°$. For the loading shown, derive an equation in *P*, *θ*, *l*, and *k* that must be satisfied when the system is in equilibrium.

SOLUTION



Spring: $AE = x = 2(2l\sin\theta) = 4l\sin\theta$

Unstretched length: $x_0 = 4l\sin 30° = 2l$

Deflection of spring

$$s = x - x_0$$

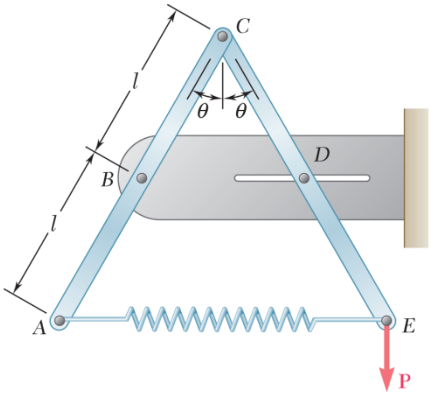
$$s = 2l(2\sin\theta - 1)$$

$$V = \frac{1}{2}ks^2 + Py_E$$

$$V = \frac{1}{2}k\left[2l(2\sin\theta - 1)\right]^2 + P(-l\cos\theta)$$

$$\frac{dV}{d\theta} = 4kl^2(2\sin\theta - 1)2\cos\theta + Pl\sin\theta = 0$$

$$(2\sin\theta - 1)\frac{\cos\theta}{\sin\theta} + \frac{P}{8kl} = 0 \qquad \frac{P}{8kl} = \frac{1 - 2\sin\theta}{\tan\theta} \blacktriangleleft$$



PROBLEM 10.60

Using the method of Section 10.8, solve Problem 10.30.

PROBLEM 10.30 Two rods *AC* and *CE* are connected by a pin at *C* and by a spring *AE*. The constant of the spring is 1.5 lb/in., and the spring is unstretched when $\theta = 30°$. Knowing that $l = 10$ in. and neglecting the weight of the rods, determine the value of θ corresponding to equilibrium when $P = 40$ lb.

SOLUTION

Using the result of Problem 10.59, with

$$P = 40 \text{ lb}$$

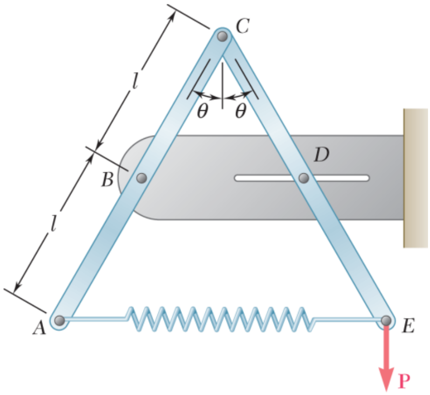
$$l = 10 \text{ in.} \quad \text{and} \quad k = 1.5 \text{ lb/in.}$$

$$\frac{P}{8kl} = \frac{1 - 2\sin\theta}{\tan\theta}$$

or
$$\frac{1 - 2\sin\theta}{\tan\theta} = \frac{40 \text{ lb}}{8(1.5 \text{ lb/in.})(10 \text{ in.})}$$

$$= \frac{1}{3}$$

Solving numerically, $\theta = 25.0°$ ◀

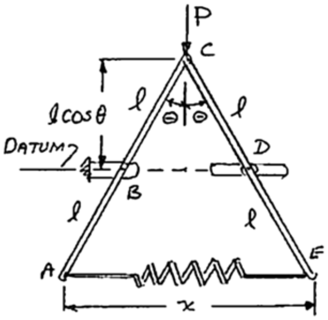


PROBLEM 10.61

Using the method of Section 10.8, solve Problem 10.31.

PROBLEM 10.31 Solve Problem 10.30 assuming that force **P** is moved to *C* and acts vertically downward.

SOLUTION



Spring: $AE = x = 2(2l\sin\theta) = 4l\sin\theta$

Unstretched length: $x_0 = 4l\sin 30° = 2l$

Deflection of spring

$$s = x - x_0$$

$$s = 2l(2\sin\theta - 1)$$

$$V = \frac{1}{2}ks^2 + Py_C$$

$$= \frac{1}{2}k[2l(2\sin\theta - 1)]^2 + P(l\cos\theta)$$

$$V = 2kl^2(2\sin\theta - 1)^2 + Pl\cos\theta$$

$$\frac{dV}{d\theta} = 4kl^2(2\sin\theta - 1)2\cos\theta - Pl\sin\theta = 0$$

$$(1 - 2\sin\theta)\frac{\cos\theta}{\sin\theta} + \frac{P}{8kl} = 0$$

$$\frac{P}{8kl} = \frac{2\sin\theta - 1}{\tan\theta}$$

with $P = 40\text{ lb}, \quad l = 10\text{ in.} \quad \text{and} \quad k = 1.5\text{ lb/in.}$

We have

$$\frac{(40\text{ lb})}{8(1.5\text{ lb/in.})(10\text{ in.})} = \frac{2\sin\theta - 1}{\tan\theta}$$

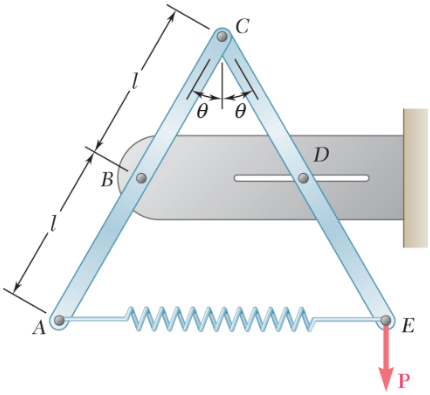
or

$$\frac{2\sin - 1}{\tan\theta} = \frac{1}{3}$$

Solving numerically $\theta = 39.65°$ and $68.96°$

$\theta = 39.7°$

$\theta = 69.0°$ ◀

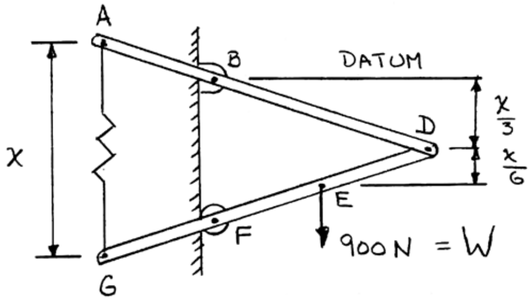


PROBLEM 10.62

Using the method of Sec. 10.8, solve Prob. 10.30.

Problem 10.30: Two bars *AD* and *DG* are connected by a pin at *D* and by a spring *AG*. Knowing that the spring is 300 mm long when unstretched and that the constant of the spring is 5 kN/m, determine the value of *x* corresponding to equilibrium when a 900-N load is applied at *E* as shown.

SOLUTION



Spring: $s = x - 0.3$ m

$$y_E = -\frac{x}{3} - \frac{x}{6} = -\frac{x}{2}$$

Potential Energy:

$$V = \frac{1}{2}ks^2 + Wy_E$$

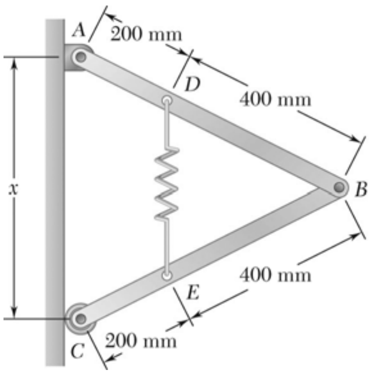
$$= \frac{1}{2}k(x-0.3)^2 - W\left(\frac{x}{2}\right)$$

For equilibrium:

$$\frac{dV}{dx} = k(x-0.3) - \frac{1}{2}W = 0$$

$$(5000\text{ N/m})(x-0.3)\text{m} - \frac{1}{2}(900\text{ N}) = 0$$

Solving $x = 0.390$ m $x = 390$ mm ◀



A
200 mm
D
400 mm
x
B
400 mm
E
C
200 mm

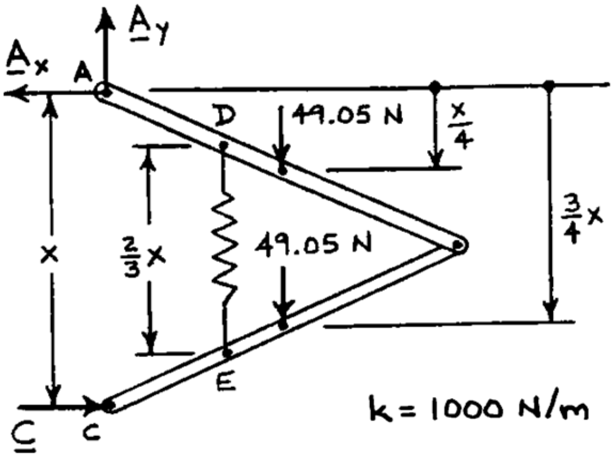
PROBLEM 10.63

Using the method of Section 10.8, solve Problem 10.33.

PROBLEM 10.33 Two 5-kg bars *AB* and *BC* are connected by a pin at *B* and by a spring *DE*. Knowing that the spring is 150 mm long when unstretched and that the constant of the spring is 1 kN/m, determine the value of *x* corresponding to equilibrium.

SOLUTION

First note: $W_{\text{bar}} = (5\text{ kg})(9.81\text{ m/s}^2) = 49.05\text{ N}$



A_y
A_x
A
D
49.05 N
x/4
3/4 x
x
2/3 x
49.05 N
E
C
C
k = 1000 N/m

Since unstretched length of spring is 150 mm, or 0.15 m, we have

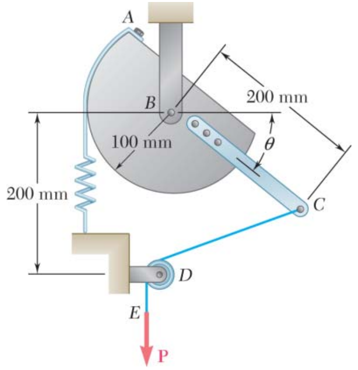
$$\Delta_S = \frac{2}{3}x - 0.15$$

$$\mathbf{V} = \frac{1}{2}k\Delta_S^2 - (49.05\text{ N})\frac{x}{4} - (49.05\text{ N})\frac{3x}{4}$$

$$V = \frac{1}{2}(1000\text{ N/m})\left(\frac{2}{3}x - 0.15\right)^2 - 12.2625x - 36.7875x$$

$$\frac{dV}{dx} = 1000\left(\frac{2}{3}x - 0.15\right)\frac{2}{3} - 12.2625 - 36.7875 = 0$$

$x = 0.335\text{ m}$ $\qquad$ $x = 335\text{ mm}$ ◀

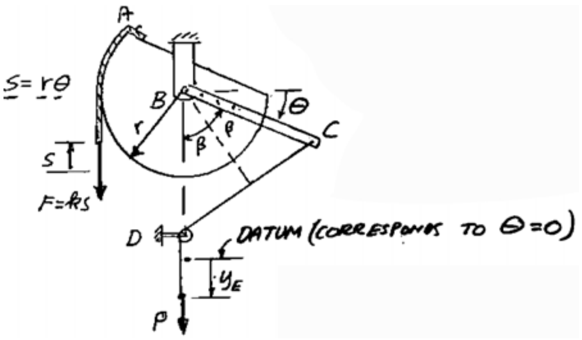


PROBLEM 10.64

Using the method of Section 10.8, solve Problem 10.35.

PROBLEM 10.35 A vertical force **P** of magnitude 150 N is applied to end *E* of cable *CDE*, which passes over a small pulley *D* and is attached to the mechanism at *C*. The constant of the spring is *k* = 4 kN/m, and the spring is unstretched when $\theta = 0$. Neglecting the weight of the mechanism and the radius of the pulley, determine the value of θ corresponding to equilibrium.

SOLUTION



$$\beta = \frac{1}{2}(90° - \theta) = 45° - \frac{\theta}{2}$$

$$BC = BD = l$$

$$CD = 2l\sin\beta = 2l\sin\left(45 - \frac{\theta}{2}\right)$$

For $\theta = 0$: $(CD)_0 = 2l\sin 45 = \sqrt{2}\,l$

$$y_E = (CD)_0 - CD$$

$$= \sqrt{2}\,l - 2l\sin\left(45° - \frac{\theta}{2}\right)$$

Potential energy:

$$V = \frac{1}{2}ks^2 - Py_E$$

$$V = \frac{1}{2}k(r\theta)^2 - P\left[\sqrt{2}l - 2l\sin\left(45° - \frac{\theta}{2}\right)\right]$$

$$\frac{dV}{d\theta} = kr^2\theta + 2Pl\cos\left(45° - \frac{\theta}{2}\right)\left(-\frac{1}{2}\right) = 0$$

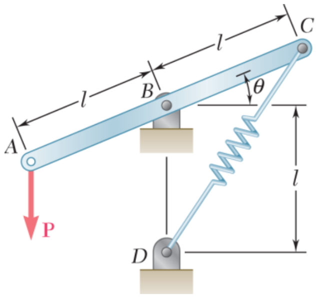
$$\frac{Pl}{kr^2} = \frac{\theta}{\cos\left(45° - \frac{\theta}{2}\right)}$$

$$\frac{Pl}{kr^2} = \frac{(150\text{ N})(0.2\text{ m})}{(4000\text{ N/m})(0.1\text{ m})^2} = 0.75$$

$$0.75 = \frac{\theta}{\cos\left(45° - \frac{\theta}{2}\right)}$$

Solve by trial and error: $\theta = 0.67623$ rad

$\theta = 38.745°$ $\theta = 38.7°$ ◀



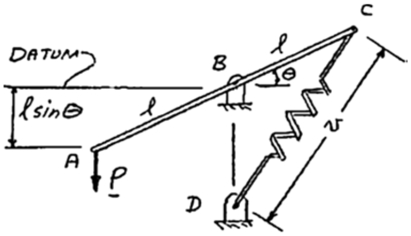
PROBLEM 10.65

Using the method of Section 10.8, solve Problem 10.37.

PROBLEM 10.37 and 10.38 Knowing that the constant of spring *CD* is *k* and that the spring is unstretched when rod *ABC* is horizontal, determine the value of θ corresponding to equilibrium for the data indicated.

PROBLEM 10.37 $P = 300$ N, $l = 400$ mm, $k = 5$ kN/m.

SOLUTION



Spring

$$v = 2l\sin\left(\frac{90° + \theta}{2}\right)$$

$$v = 2l\sin\left(45° + \frac{\theta}{2}\right)$$

Unstretched $(\theta = 0)$

$$v_0 = 2l\sin 45° = \sqrt{2}l$$

Deflection of spring

$$s = v - v_0 = 2l\sin\left(45° + \frac{\theta}{2}\right) - \sqrt{2}l$$

$$V = \frac{1}{2}ks^2 + Py_A = \frac{1}{2}kl^2\left[2\sin\left(45° + \frac{\theta}{2}\right) - \sqrt{2}\right]^2 + P(-l\sin\theta)$$

$$\frac{dV}{d\theta} = kl^2\left[2\sin\left(45° + \frac{\theta}{2}\right) - \sqrt{2}\right]\cos\left(45° + \frac{\theta}{2}\right) - Pl\cos\theta = 0$$

$$\left[2\sin\left(45° + \frac{\theta}{2}\right)\cos\left(45° + \frac{\theta}{2}\right) - \sqrt{2}\cos\left(45° + \frac{\theta}{2}\right)\right] = \frac{P}{kl}\cos\theta$$

$$\cos\theta - \sqrt{2}\cos\left(45° + \frac{\theta}{2}\right) = \frac{P}{kl}\cos\theta$$

Divide each member by cos θ

$$1 - \sqrt{2}\frac{\cos\left(45° + \frac{\theta}{2}\right)}{\cos\theta} = \frac{P}{kl}$$

Then with $P = 300$ N, $l = 0.4$ m and $k = 5000$ N/m

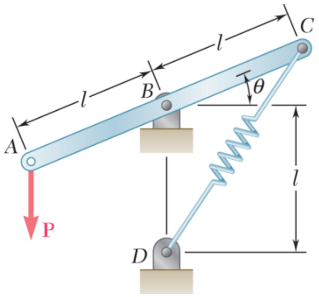
$$1 - \sqrt{2}\frac{\cos\left(45° + \frac{\theta}{2}\right)}{\cos\theta} = \frac{300 \text{ N}}{(5000 \text{ N/m})(0.4 \text{ m})}$$

$$= 0.15$$

or

$$\frac{\cos\left(45° + \frac{\theta}{2}\right)}{\cos\theta} = 0.60104$$

Solving numerically $\theta = 22.6°$ ◀



PROBLEM 10.66

Using the method of Section 10.8, solve Problem 10.38.

PROBLEM 10.37 and 10.38 Knowing that the constant of spring *CD* is *k* and that the spring is unstretched when rod *ABC* is horizontal, determine the value of θ corresponding to equilibrium for the data indicated.

PROBLEM 10.38 $P = 75$ lb, $l = 15$ in., $k = 20$ lb/in.

SOLUTION

Using the results of Problem 10.65 with $P = 75$ lb, $l = 15$ in. and $k = 20$ lb/in., we have

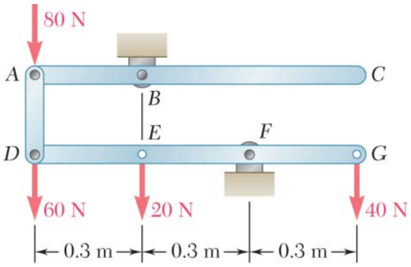
$$1-\sqrt{2}\frac{\cos\left(45^\circ+\frac{\theta}{2}\right)}{\cos\theta}=\frac{P}{kl}$$

$$=\frac{75\text{ lb}}{(20\text{ lb/in.})(15\text{ in.})}$$

$$=0.25$$

or
$$\frac{\cos\left(45^\circ+\frac{\theta}{2}\right)}{\cos\theta}=0.53033$$

Solving numerically $\theta = 51.058^\circ$ $\theta = 51.1^\circ$ ◀

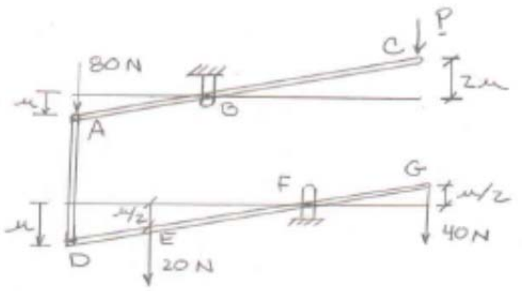

80 N
A
C
B
E
F
D
G
60 N
20 N
40 N
0.3 m
0.3 m
0.3 m

PROBLEM 10.67

Show that equilibrium is neutral in Problem 10.1.

PROBLEM 10.1 Determine the vertical force **P** that must be applied at C to maintain the equilibrium of the linkage.

SOLUTION


P
C
80 N
2u
u
A
B
F
G
u/2
u/2
u
E
D
20 N
40 N

Designate vertical distance between link AC and DG as b.

We have
$$y_A = b - u, \quad y_C = b + 2u, \quad y_D = -u, \quad y_E = -\frac{1}{2}u, \quad y_G = +\frac{1}{2}u$$

$$V = (80\text{ N})y_A + P(y_C) + (60\text{ N})y_D + (20\text{ N})y_E + (40\text{ N})y_G$$

$$V = 80(\text{b}-u) + P(b+2u) + 60(-u) + 20\left(-\frac{1}{2}u\right) + 40\left(\frac{1}{2}u\right)$$

$$\frac{dV}{du} = -80 + 2P - 60 - 10 + 20 = 0$$

$$P = 65\text{ N}$$

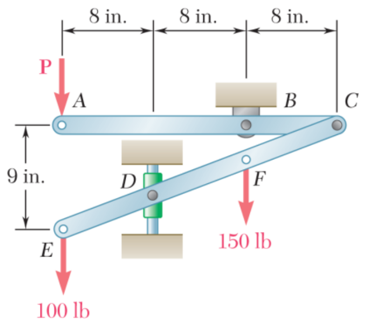
Substituting $P = 270\text{ N}$ in the expression for V, we have

$$V = (80 + 65)b + (-80 + 130 - 60 - 10 + 20)u$$

$$V = 145b$$

Thus V is constant

and equilibrium is neutral ◀

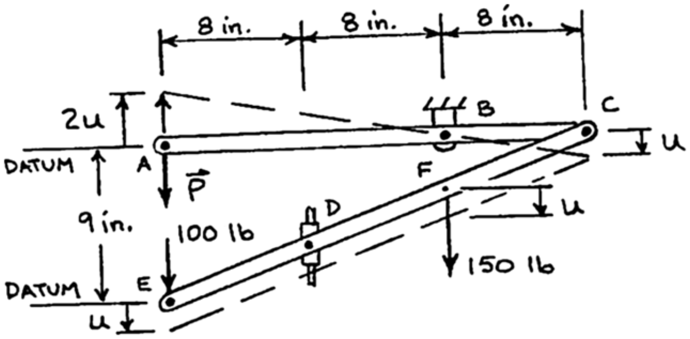

8 in.
8 in.
8 in.
P
A
B
C
9 in.
D
F
E
150 lb
100 lb

PROBLEM 10.68

Show that equilibrium is neutral in Problem 10.7.

PROBLEM 10.7 The two-bar linkage shown is supported by a pin and bracket at *B* and a collar at *D* that slides freely on a vertical rod. Determine the force **P** required to maintain the equilibrium of the linkage.

SOLUTION


8 in.
8 in.
8 in.
2u
B
C
u
DATUM
A
$\vec{P}$
F
D
u
9 in.
100 lb
150 lb
DATUM
E
u

$$y_A = 2u$$

$$y_E = -u$$

$$y_F = -u$$

$$V = Py_A + (100 \text{ lb})y_E + (150 \text{ lb})y_F$$

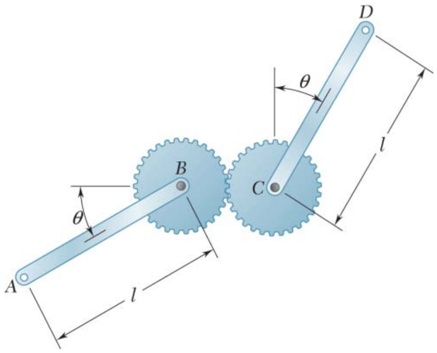
$$V = P(2u) + (100 \text{ lb})(-u) + (150 \text{ lb})(-u)$$

$$\frac{dV}{du} = 2P - 100 - 150 = 0$$

$$P = 125 \text{ lb}$$

Now, substitute $P = 125$ lb in expression for *V*, making $V = 0$. Thus, *V* is constant and

equilibrium is neutral. ◀



PROBLEM 10.69

Two uniform rods, each of mass *m*, are attached to gears of equal radii as shown. Determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

SOLUTION

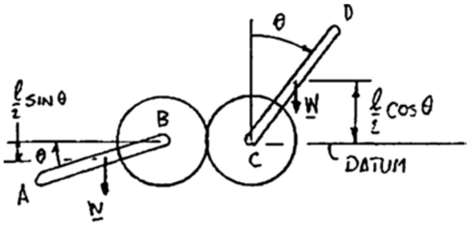
Potential energy

$$V = W\left(-\frac{l}{2}\sin\theta\right) + W\left(\frac{l}{2}\cos\theta\right) \quad W = mg$$

$$= W\frac{l}{2}(\cos\theta - \sin\theta)$$

$$\frac{dV}{d\theta} = \frac{Wl}{2}(-\sin\theta - \cos\theta)$$

$$\frac{d^2V}{d\theta^2} = \frac{Wl}{2}(\sin\theta - \cos\theta)$$



For equilibrium: $\frac{dV}{d\theta} = 0: \quad \sin\theta = -\cos\theta$

or $\tan\theta = -1$

Thus $\theta = -45.0°$ and $\theta = 135.0°$

Stability:

At $\theta = -45.0°$:

$$\frac{d^2V}{d\theta^2} = \frac{Wl}{2}[\sin(-45°) - \cos 45°]$$

$$= \frac{Wl}{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) < 0$$

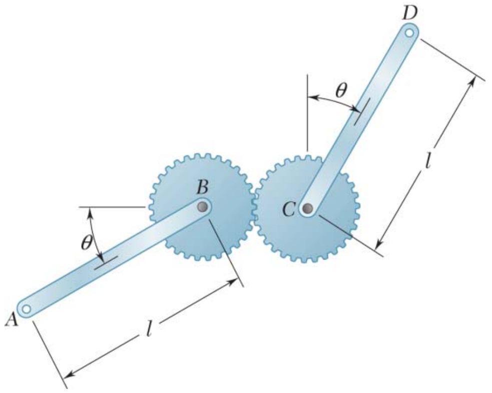
$\theta = -45.0°$, Unstable ◀

At $\theta = 135.0°$:

$$\frac{d^2V}{d\theta^2} = \frac{Wl}{2}(\sin 135° - \cos 135°)$$

$$= \frac{Wl}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) > 0$$

$\theta = 135.0°$, Stable ◀



PROBLEM 10.70

Two uniform rods, *AB* and *CD*, are attached to gears of equal radii as shown. Knowing that $W_{AB} = 8\text{ lb}$ and $W_{CD} = 4\text{ lb}$, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

SOLUTION

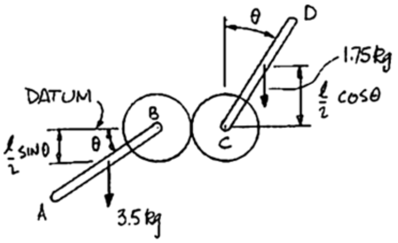
Potential energy

$$V = (3.5\text{ kg}\times 9.81\text{ m/s}^2)\left(-\frac{l}{2}\sin\theta\right) + (1.75\text{ kg}\times 9.81\text{ m/s}^2)\left(\frac{l}{2}\cos\theta\right)$$

$$= (8.5838\text{ N})l(-2\sin\theta + \cos\theta)$$

$$\frac{dV}{d\theta} = (8.5838\text{ N})l(-2\cos\theta - \sin\theta)$$

$$\frac{d^2V}{d\theta^2} = (8.5838\text{ N})l(2\sin\theta - \cos\theta)$$



Equilibrium: $\frac{dV}{d\theta} = 0$: $\quad -2\cos\theta - \sin\theta = 0$

or $\tan\theta = -2$

Thus $\theta = -63.4°$ and $116.6°$

Stability:

At $\theta = -63.4°$:

$$\frac{d^2V}{d\theta^2} = (8.5838\text{ N})l[2\sin(-63.4°) - \cos(-63.4°)]$$

$$= (8.5838\text{ N})l(-1.788 - 0.448) < 0$$

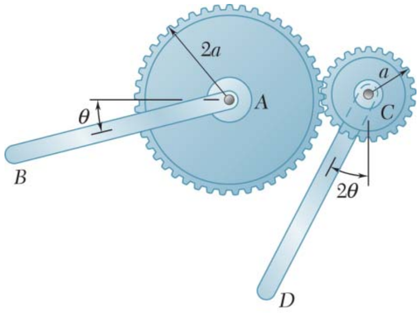
$\theta = -63.4°$, Unstable ◀

At $\theta = 116.6°$:

$$\frac{d^2V}{d\theta^2} = (8.5838\text{ N})l[2\sin(116.6°) - \cos(116.6°)]$$

$$= (8.5838\text{ N})l(1.788 + 0.447) > 0$$

$\theta = 116.6°$, Stable ◀



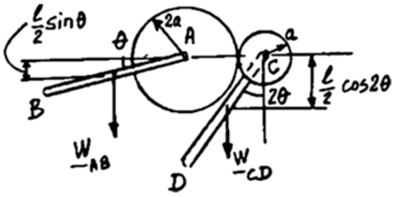
PROBLEM 10.71

Two uniform rods *AB* and *CD*, of the same length *l*, are attached to gears as shown. Knowing that rod *AB* weighs 3 lb and that rod *CD* weighs 2 lb, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

SOLUTION

$$V = -W_{AB}\left(\frac{l}{2}\sin\theta\right) - W_{CD}\left(\frac{l}{2}\cos 2\theta\right)$$

$$\frac{dV}{d\theta} = -\frac{1}{2}W_{AB}l\cos\theta + W_{CD}l\sin 2\theta$$



Equilibrium:

$$\frac{dV}{d\theta} = 0$$

$$-\frac{1}{2}W_{AB}\,l\cos\theta + W_{CD}l\sin 2\theta = 0$$

$$-W_{AB}\cos\theta + 4W_{CD}\sin\theta\cos\theta = 0$$

$$-W_{AB}\cos\theta\left(1 - \frac{W_{CD}}{W_{AB}}\sin\theta\right) = 0$$

$$\cos\theta = 0 \quad \text{and} \quad \sin\theta = \frac{W_{AB}}{4W_{CD}}$$

$$\theta = 90°, \quad \theta = 270°, \quad \theta = \sin^{-1}\frac{W_{AB}}{4W_{CD}} \qquad (1)$$

For $W_{AB} = 3\text{ lb} \quad \text{and} \quad W_{CD} = 2\text{ lb},$ We have

$$\theta = 90°, \quad \theta = 270°, \quad \theta = \sin^{-1}\frac{3}{8}$$

$$\theta = 90°, \quad \theta = 270°, \quad \theta = 22.0°, \quad \theta = 158.0°$$ ◀

Stability:

$$\frac{d^2V}{d\theta^2} = \left(\frac{1}{2}W_{AB}\sin\theta + 2W_{CD}\cos 2\theta)\right)l \qquad (2)$$

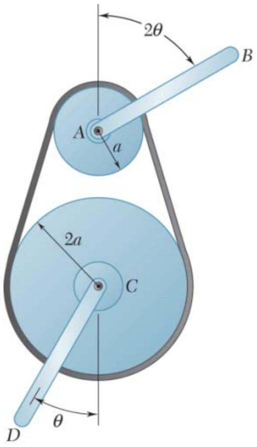
$$\theta = 270°: \quad \frac{d^2V}{d\theta^2} = \left[\frac{1}{2}(3)\sin(270°) + 2(2)\cos(540°)\right]l = -5.5l < 0, \text{ unstable}$$ ◀

$$\theta = 22.0°: \quad \frac{d^2V}{d\theta^2} = \left[\frac{1}{2}(3)\sin 220° + 2(2)\cos 44.0°\right]l > 0 \quad \text{Stable}$$ ◀

PROBLEM 10.71 (Continued)

$$\theta = 90°: \quad \frac{d^2V}{d\theta^2} = \left[\frac{1}{2}(3)\sin 90° + 2(2)\cos 180°\right] l = -2.5l < 0, \quad \text{Unstable} \quad ◀$$

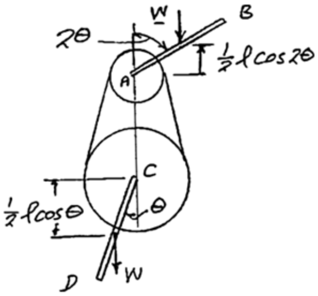
$$\theta = 158.0°: \quad \frac{d^2V}{d\theta^2} = \left[\frac{1}{2}(3)\sin 158° + 2(2)\cos 316°\right] l = 3.44l > 0, \quad \text{Stable} \quad ◀$$



PROBLEM 10.72

Two uniform rods, each of mass *m* and length *l*, are attached to drums that are connected by a belt as shown. Assuming that no slipping occurs between the belt and the drums, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

SOLUTION



$$W = mg$$

$$V = W\left(\frac{l}{2}\cos 2\theta\right) - W\left(\frac{l}{2}\cos\theta\right)$$

$$\frac{dV}{d\theta} = W\frac{l}{2}(-2\sin 2 + \sin\theta)$$

$$\frac{d^2V}{d\theta^2} = W\frac{1}{2}(-4\cos 2\theta - \cos\theta)$$

Equilibrium: $\frac{dV}{d\theta} = 0$: $\frac{Wl}{2}(-2\sin 2\theta + \sin\theta) = 0$

or $\sin\theta(-4\cos\theta + 1) = 0$

Solving $\theta = 0,\ 75.5°, 180°, \text{and } 284°$

Stability: $\frac{d^2V}{d\theta^2} = W\frac{l}{2}(-4\cos 2\theta - \cos\theta)$

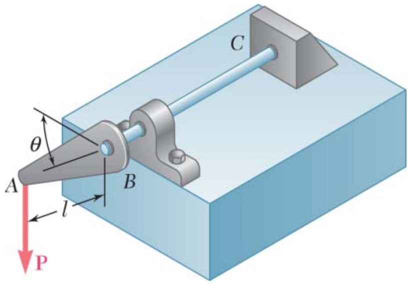
At $\theta = 0$: $\frac{d^2V}{d\theta^2} = W\frac{l}{2}(-4 - 1) < 0$ $\theta = 0,$ Unstable ◀

At $\theta = 75.5°$: $\frac{d^2V}{d\theta^2} = W\frac{l}{2}(-4(-.874) - .25) > 0$ $\theta = 75.5°,$ Stable ◀

At $\theta = 180°$: $\frac{d^2V}{d\theta^2} = W\frac{l}{2}(-4 + 1) < 0$ $\theta = 180.0°,$ Unstable ◀

At $\theta = 284°$: $\frac{d^2V}{d\theta^2} = W\frac{l}{2}(-4(-.874) - .25) > 0$ $\theta = 284°,$ Stable ◀

PROBLEM 10.73



Using the method of Section 10.8, solve Problem 10.39. Determine whether the equilibrium is stable, unstable, or neutral. (*Hint:* The potential energy corresponding to the couple exerted by a torsion spring is $\frac{1}{2}K\theta^2$, where K is the torsional spring constant and θ is the angle of twist.)

PROBLEM 10.39 The lever AB is attached to the horizontal shaft BC that passes through a bearing and is welded to a fixed support at C. The torsional spring constant of the shaft BC is K; that is, a couple of magnitude K is required to rotate end B through 1 rad. Knowing that the shaft is untwisted when AB is horizontal, determine the value of θ corresponding to the position of equilibrium when $P = 100$ N, $l = 250$ mm, and $K = 12.5 \text{ N}\cdot\text{m/rad}$.

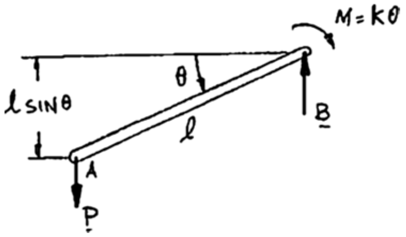
SOLUTION

Potential energy

$$V = \frac{1}{2}K\theta^2 - Pl\sin\theta$$

$$\frac{dV}{d\theta} = K\theta - Pl\cos\theta$$

$$\frac{d^2V}{d\theta^2} = K + Pl\sin\theta$$



Equilibrium:

$$\frac{dV}{d\theta} = 0: \quad \cos\theta = \frac{K}{Pl}\theta$$

For

$$P = 100 \text{ N}, \quad l = 0.25 \text{ m.}, \quad K = 12.5 \text{ N}\cdot\text{m/rad}$$

$$\cos\theta = \frac{12.5 \text{ N}\cdot\text{m/rad}}{(100)(0.25 \text{ m})}\theta$$

$$= 0.500\theta$$

Solving numerically, we obtain

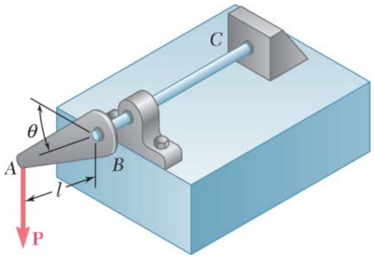
$$\theta = 1.02967 \text{ rad} = 59.000°$$

$\theta = 59.0°$ ◀

Stability

$$\frac{d^2V}{d\theta^2} = (12.5 \text{ N}\cdot\text{m/rad}) + (100 \text{ N})(0.25 \text{ m})\sin 59.0° > 0$$

Stable ◀

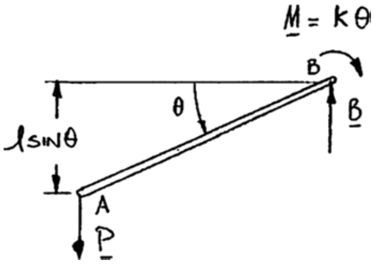


PROBLEM 10.74

In Problem 10.40, determine whether each of the positions of equilibrium is stable, unstable, or neutral. (See hint for Problem 10.73.)

PROBLEM 10.40 Solve Problem 10.39 assuming that $P = 350$ N, $l = 250$ mm, and $K = 12.5\ \text{N}\cdot\text{m/rad}$. Obtain answers in each of the following quadrants: $0 < \theta < 90°$, $270° < \theta < 360°$, $360° < \theta < 450°$.

SOLUTION



Potential energy $$V = \frac{1}{2}K\theta^2 - Pl\sin\theta$$

$$\frac{dV}{d\theta} = K\theta - Pl\cos\theta$$

$$\frac{d^2V}{d\theta^2} = K + Pl\sin\theta$$

Equilibrium $$\frac{dV}{d\theta} = 0: \quad \cos\theta = \frac{K}{Pl}\theta$$

For $P = 350\ \text{N}, \quad l = 0.250\ \text{m} \quad \text{and} \quad K = 12.5\ \text{N}\cdot\text{m/rad}$

$$\cos\theta = \frac{12.5\ \text{N}\cdot\text{m/rad}}{(350\ \text{N})(0.250\ \text{m})}\theta$$

or $$\cos\theta = \frac{\theta}{7}$$

Solving numerically $\theta = 1.37333$ rad, 5.652 rad, and 6.616 rad

or $\theta = 78.7°$, 323.8°, 379.1°

Stability at $\theta = 78.7°$: $$\frac{d^2V}{d\theta^2} = (12.5\ \text{N}\cdot\text{m/rad}) + (350\ \text{N})(0.250\ \text{m})\sin 78.7°$$

$$= 98.304 > 0$$

$\theta = 78.7°$, Stable ◀

At $\theta = 323.8°$: $$\frac{d^2V}{d\theta^2} = (12.5\ \text{N}\cdot\text{m/rad}) + (350\ \text{N})(0.250\ \text{m})\sin 323.8°$$

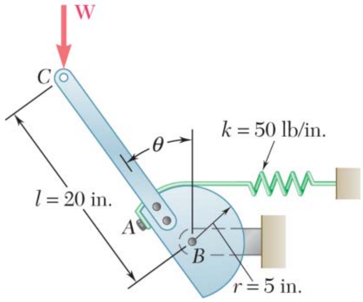
$$= -39.178\ \text{N}\cdot\text{m} < 0$$

$\theta = 324°$, Unstable ◀

At At $\theta = 379.1°$: $$\frac{d^2V}{d\theta^2} = (12.5\ \text{N}\cdot\text{m/rad}) + (350\ \text{N})(0.250\ \text{m})\sin 379.1°$$

$$= 44.132\ \text{N}\cdot\text{m} > 0$$

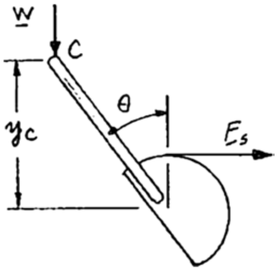
$\theta = 379°$, Stable ◀



PROBLEM 10.75

A load **W** of magnitude 100 lb is applied to the mechanism at *C*. Knowing that the spring is unstretched when $\theta = 15°$, determine that value of θ corresponding to equilibrium and check that the equilibrium is stable.

SOLUTION



We have $$y_C = l\cos\theta$$

$$V = \frac{1}{2}k[r(\theta-\theta_0)]^2 + Wy_C \quad \theta_0 = 15° = \frac{\pi}{12}\text{rad}$$

$$= \frac{1}{2}kr^2(\theta-\theta_0)^2 + Wl\cos\theta$$

$$\frac{dV}{d\theta} = kr^2(\theta-\theta_0) - Wl\sin\theta$$

Equilibrium $$\frac{dV}{d\theta} = 0: \quad kr^2(\theta-\theta_0) - wl\sin\theta = 0 \tag{1}$$

with $W = 100$ lb, $R = 50$ lb./in., $l = 20$ in., and $r = 5$ in.

$$(50\text{ lb./in.})(25\text{ in.}^2)\left(\theta - \frac{\pi}{12}\right) - (100\text{ lb})(20\text{ in.})\sin\,\theta = 0$$

or $$0.625\theta - \sin\theta = 0.16362$$

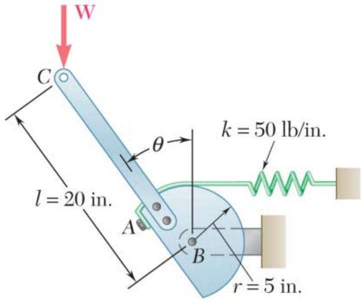
Solving numerically $$\theta = 1.8145\text{ rad} = 103.97°$$

$\theta = 104.0°$ ◀

Stability $$\frac{d^2V}{d\theta^2} = kr^2 - Wl\cos\theta \tag{2}$$

or $$= 1250 - 2000\cos\theta$$

For $\theta = 104.0°$: $= 1734\text{ in.}\cdot\text{lb} > 0$ Stable ◀



PROBLEM 10.76

A load **W** of magnitude 100 lb is applied to the mechanism at *C*. Knowing that the spring is unstretched when $\theta = 30°$, determine that value of θ corresponding to equilibrium and check that the equilibrium is stable.

SOLUTION

Using the solution of Problem 10.75, particularly Equation (1), with 15° replaced by $30°\left(\frac{\pi}{6}\text{ rad}\right)$:

For equilibrium $$kr^2\left(\theta - \frac{\pi}{6}\right) - Wl\sin\theta = 0$$

With $k = 50$ lb/in., $W = 100$ lb, $r = 5$ in., and $l = 20$ in.

$$(50\text{ lb/in.})(25\text{ in.}^2)\left(\theta - \frac{\pi}{6}\right) - (100\text{ lb})(20\text{ in.})\sin\theta = 0$$

or $$1250\theta - 654.5 - 200\sin\theta = 0$$

Solving numerically, $$\theta = 1.9870\text{ rad } = 113.8°$$

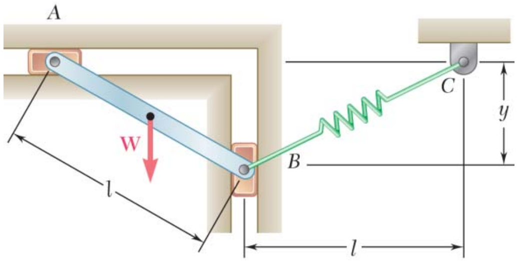
$\theta = 113.8°$ ◀

Stability: Equation (2), Problem 75:

$$\frac{d^2V}{d\theta^2} = kr^2 - Wl\cos\theta$$

or $$= 1250 - 2000\cos\theta$$

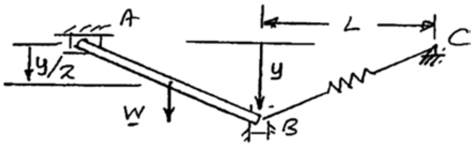
For $\theta = 113.8°$: $$= 2057\text{ in.}\cdot\text{lb} > 0$$ Stable ◀



PROBLEM 10.77

A slender rod *AB*, of weight *W*, is attached to two blocks *A* and *B* that can move freely in the guides shown. Knowing that the spring is unstretched when $y = 0$, determine the value of y corresponding to equilibrium when $W = 80$ N, $l = 500$ mm, and $k = 600$ N/m.

SOLUTION



Deflection of spring = *s*, where

$$s = \sqrt{l^2 + y^2} - l$$

$$\frac{ds}{dy} = \frac{y}{\sqrt{l^2 - y^2}}$$

Potential energy:

$$V = \frac{1}{2}ks^2 - W\frac{y}{2}$$

$$\frac{dV}{dy} = ks\frac{ds}{dy} - \frac{1}{2}W$$

$$\frac{dV}{dy} = k\left(\sqrt{l^2 + y^2} - l\right)\frac{y}{\sqrt{l^2 + y^2}} - \frac{1}{2}W$$

$$= k\left(1 - \frac{l}{\sqrt{l^2 + y^2}}\right)y - \frac{1}{2}W$$

Equilibrium

$$\frac{dV}{dy} = 0: \quad \left(1 - \frac{l}{\sqrt{l^2 + y^2}}\right)y = \frac{1}{2}\frac{W}{k}$$

Now $W = 80$ N, $l = 0.500$ m, and $k = 600$ N/m

Then

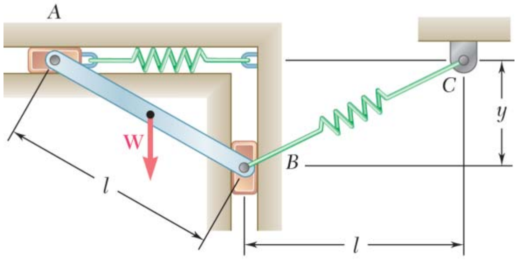
$$\left(1 - \frac{0.500 \text{ m}}{\sqrt{(0.500)^2 + y^2}}\right)y = \frac{1}{2}\frac{(80 \text{ N})}{(600 \text{ N/m})}$$

or

$$\left(1 - \frac{0.500}{\sqrt{0.25 + y^2}}\right)y = 0.066667$$

Solving numerically, $y = 0.357$ m

$y = 357$ mm ◀

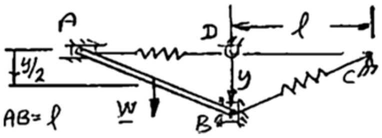


PROBLEM 10.78

A slender rod AB, of weight W, is attached to two blocks A and B that can move freely in the guides shown. Knowing that both springs are unstretched when $y = 0$, determine the value of y corresponding to equilibrium when $W = 80$ N, $l = 550$ mm, and $k = 600$ N/m.

SOLUTION

Spring deflections



$$S_{AD} = l - \sqrt{l^2 - y^2}$$

$$S_{BC} = \sqrt{l^2 + y^2} - l$$

$$V = \frac{1}{2}kS_{AD}^2 + \frac{1}{2}kS_{BC}^2 - W\frac{y}{2}$$

$$V = \frac{1}{2}k\left(l - \sqrt{l^2 - y^2}\right)^2 + \frac{1}{2}k\left(\sqrt{l^2 + y^2} - l\right)^2 - W\frac{y}{2}$$

$$\frac{dV}{dy} = k\left(l - \sqrt{l^2 - y^2}\right)\left(\frac{y}{\sqrt{l^2 - y^2}}\right) + k\left(\sqrt{l^2 + y^2} - l\right)\left(\frac{y}{\sqrt{l^2 + y^2}}\right) - \frac{W}{2}$$

$$\frac{dV}{dy} = 0: \quad \left[\left(\frac{l}{\sqrt{l^2 - y^2}} - 1\right) + \left(1 - \frac{l}{\sqrt{l^2 + y^2}}\right)\right]y = \frac{W}{2k}$$

Data: $W = 80$ N, $l = 0.5$ m, $k = 600$ N/m

$$\left[\frac{0.5}{\sqrt{(0.5)^2 - y^2}} - \frac{0.5}{\sqrt{(0.5)^2 + y^2}}\right]y = \frac{80}{2(1200)} = 0.066667$$

Solve by trial and error: $y = 0.252$ m $\qquad y = 252$ mm ◀

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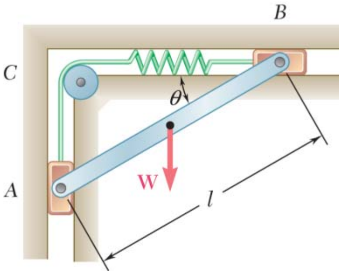
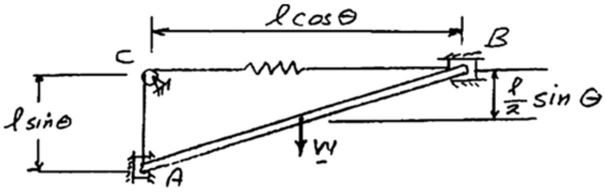


PROBLEM 10.79

A slender rod *AB*, of weight *W*, is attached to two blocks *A* and *B* that can move freely in the guides shown. The constant of the spring is *k*, and the spring is unstretched when *AB* is horizontal. Neglecting the weight of the blocks, derive an equation in *θ*, *W*, *l*, and *k* that must be satisfied when the rod is in equilibrium.

SOLUTION



Elongation of spring:

$$s = l\sin\theta + l\cos\theta - l$$

$$s = l(\sin\theta + \cos\theta - 1)$$

Potential energy:

$$V = \frac{1}{2}ks^2 - W\frac{1}{2}\sin\theta \quad W = mg$$

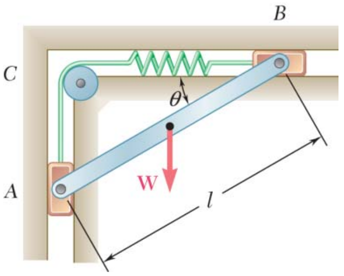
$$= \frac{1}{2}kl^2(\sin\theta + \cos\theta - 1)^2 - mg\frac{l}{2}\sin\theta$$

$$\frac{dV}{d\theta} = kl^2(\sin\theta + \cos\theta - 1)(\cos\theta - \sin\theta) - \frac{1}{2}mgl\cos\theta \tag{1}$$

Equilibrium:

$$\frac{dV}{d\theta} = 0: \quad (\sin\theta + \cos\theta - 1)(\cos\theta - \sin\theta) - \frac{mg}{2kl}\cos\theta = 0$$

or $\cos\theta\left[(\sin\theta + \cos\theta - 1)(1 - \tan\theta) - \dfrac{mg}{2kl}\right] = 0$ ◀



PROBLEM 10.80

A slender rod AB, of weight W, is attached to two blocks A and B that can move freely in the guides shown. Knowing that the spring is unstretched when AB is horizontal, determine three values of θ corresponding to equilibrium when $W = 300$ lb, $l = 16$ in., and $k = 75$ lb/in. State in each case whether the equilibrium is stable, unstable, or neutral.

SOLUTION

Using the results of Problem 10.79, particularly the condition of equilibrium

$$\cos\theta\left[(\sin\theta+\cos\theta-1)(1-\tan\theta)-\frac{mg}{2kl}\right]=0$$

Now, with $W = 300$ lb, $l = 16$ in., and $k = 75$ lb/in.

$$\frac{W}{2kl}=\frac{300\text{ lb}}{(16\text{ in.})(75\text{ lb/in.})}=0.25$$

Thus: $$\cos\theta\left[(\sin\theta+\cos\theta-1)(1-\tan\theta)-0.25\right]=0$$

$$\cos\theta=0\quad\text{and}\quad(\sin\theta+\cos\theta-1)(1-\tan\theta)=0.25$$

First equation yields $\theta = 90°$. Solving the second equation by trial, we find $\theta = 9.39°$ and $34.16°$
Values of θ for equilibrium are

$$\theta=9.39°,\ 34.2°,\text{ and }90.0°$$

Stability: we differentiate Eq. (1).

$$\frac{d^2y}{ds^2}=kl^2[(\cos\theta-\sin\theta)(\cos\theta-\sin\theta)+(\sin\theta+\cos\theta-1)(-\sin\theta-\cos\theta)]+\frac{1}{2}wl\sin\theta$$

$$=kl^2\left[\cos^2\theta+\sin^2\theta-2\cos\theta\sin\theta-\sin^2\theta-\cos^2\theta-2\cos\theta\sin\theta+\sin\theta+\cos\theta+\frac{W}{2kl}\sin\theta\right]$$

$$=kl^2\left[\left(1+\frac{W}{2kl}\right)\sin\theta+\cos\theta-2\sin2\theta\right]$$

$$\frac{d^2V}{d\theta^2}=kl^2(1.25\sin\theta+\cos\theta-2\sin2\theta)$$

$\theta = 9.39°$: $$\frac{d^2V}{d\theta^2}=kl^2(1.25\sin9.4+\cos9.4-2\sin18.8)$$

$$=kl^2(+0.55)<0$$ Stable ◀

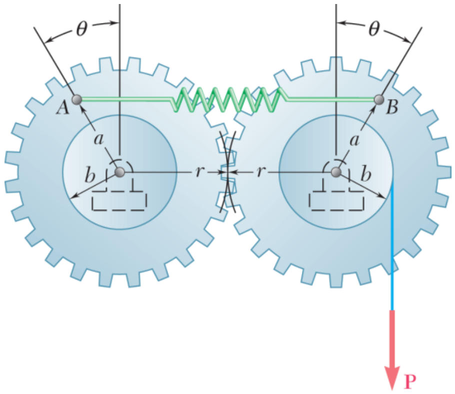
PROBLEM 10.80 (Continued)

$\theta = 34.2°$: $$\frac{d^2v}{d\theta^2} = kl^2(1.25\sin 34.2° + \cos 34.3° - 2\sin 68.4°)$$

$$= kl^2(-0.33) < 0$$ Unstable ◀

$\theta = 90.0°$: $$\frac{d^2V}{d\theta^2} = kl^2(1.25\sin 90° + \cos 90° - 2\sin 180°)$$

$$= kl^2(1.25) > 0$$ Stable ◀

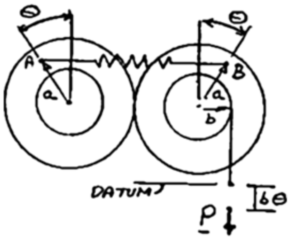


PROBLEM 10.81

A spring *AB* of constant *k* is attached to two identical gears as shown. Knowing that the spring is undeformed when $\theta = 0$, determine two values of the angle θ corresponding to equilibrium when $P = 30$ lb, $a = 4$ in., $b = 3$ in., $r = 6$ in., and $k = 5$ lb/in. State in each case whether the equilibrium is stable, unstable, or neutral.

SOLUTION

Elongation of spring



$$s = 2(a\sin\theta) = 2a\sin\theta$$

$$V = \frac{1}{2}ks^2 - Pb\theta$$

$$= \frac{1}{2}k(2a\sin\theta)^2 - Pb\theta$$

$$\frac{dV}{d\theta} = 4ka^2\sin\theta\cos\theta - Pb$$

$$= 2ka^2\sin 2\theta - Pb \qquad (1)$$

Equilibrium $$\frac{dV}{d\theta} = 0: \quad \sin 2\theta = \frac{Pb}{2ka^2}$$

$$\sin 2\theta = \frac{(30\text{ lb})(3\text{ in.})}{2(5\text{ lb/in.})(4\text{ in.})^2}; \quad \sin 2\theta = 0.5625$$

$$2\theta = 34.229° \quad \text{and} \quad 145.771°$$

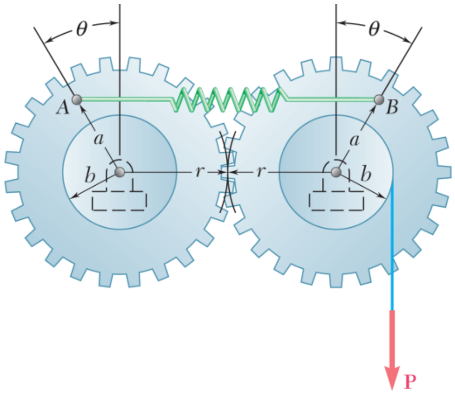
$$\theta = 17.11° \quad \text{and} \quad 72.9° \blacktriangleleft$$

Stability: We differentiate Eq. (1)

$$\frac{d^2V}{d\theta^2} = 4ka^2\cos 2\theta$$

$$\theta = 17.11°: \quad \frac{d^2v}{d\theta^2} = 4ka^2\cos 34.2° = 4ka^2(0.83) > 0 \qquad \text{Stable} \blacktriangleleft$$

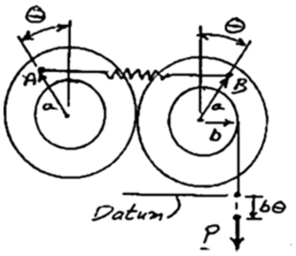
$$\theta = 72.9°: \quad \frac{d^2v}{d\theta^2} = 4ka^2\cos 145.8° = 4ka^2(-0.83) < 0 \qquad \text{Unstable} \blacktriangleleft$$



PROBLEM 10.82

A spring *AB* of constant *k* is attached to two identical gears as shown. Knowing that the spring is undeformed when $\theta = 0$, and given that $a = 60$ mm, $b = 45$ mm, $r = 90$ mm, and $k = 6$ kN/m, determine (*a*) the range of values of *P* for which a position of equilibrium exists, (*b*) two values of θ corresponding to equilibrium if the value of *P* is equal to half the upper limit of the range found in part *a*.

SOLUTION



Elongation of spring

$$s = 2(a\sin\theta) = 2a\sin\theta$$

Potential energy

$$V = \frac{1}{2}ks^2 - Pb\theta = \frac{1}{2}k(2a\sin\theta)^2 - Pb\theta$$

$$V = 2ka^2\sin^2\theta - Pb\theta$$

Equilibrium

$$\frac{dV}{d\theta} = 0: \quad \frac{dV}{d\theta} = 4ka^2\sin\theta\cos\theta - Pb \qquad (1)$$

$$= 2ka^2\sin 2\theta - Pb = 0$$

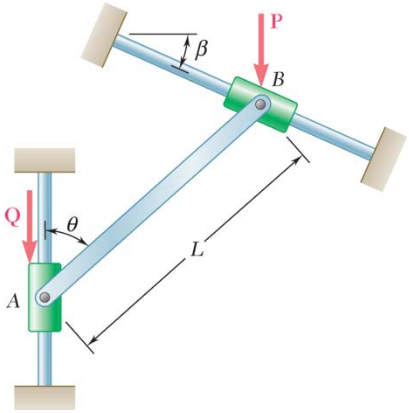
$$\sin 2\theta = \frac{Pb}{2ka^2}; \quad \text{For} \quad P_{\max}; \ \frac{P_{\max}b}{2ka^2} = 1$$

(*a*) $$\frac{P_{\max}(0.045\text{ m})}{2(6000\text{ N/m})(0.06\text{ m})^2} = 1$$

$P_{\max} = 960$ N ◀

(*b*) For $P = \frac{1}{2}P_{\max}$, $$\sin 2\theta = \frac{1}{2}; \ 2\theta = 30° \quad \text{and} \quad 150°$$

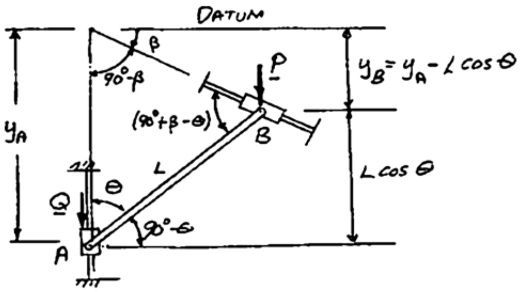
$\theta = 15.00°$ and $75.0°$ ◀



PROBLEM 10.83

A slender rod AB is attached to two collars A and B that can move freely along the guide rods shown. Knowing that $\beta = 30°$ and $P = Q = 400$ N, determine the value of the angle θ corresponding to equilibrium.

SOLUTION



Law of Sines

$$\frac{y_A}{\sin(90° + \beta - \theta)} = \frac{L}{\sin(90 - \beta)}$$

$$\frac{y_A}{\cos(\theta - \beta)} = \frac{L}{\cos\beta}$$

or

$$y_A = L\frac{\cos(\theta - \beta)}{\cos\beta}$$

From the figure:

$$y_B = L\frac{\cos(\theta - \beta)}{\cos\beta} - L\cos\theta$$

Potential Energy:

$$V = -Py_B - Qy_A = -P\left[L\frac{\cos(\theta - \beta)}{\cos\beta} - L\cos\theta\right] - QL\frac{\cos(\theta - \beta)}{\cos\beta}$$

$$\frac{dV}{d\theta} = -PL\left[-\frac{\sin(\theta - \beta)}{\cos\beta} + \sin\theta\right] + QL\frac{\sin(\theta - \beta)}{\cos\beta}$$

$$= L(P + Q)\frac{\sin(\theta - \beta)}{\cos\beta} - PL\sin\theta$$

PROBLEM 10.83 (Continued)

Equilibrium
$$\frac{dV}{d\theta}=0: \quad L(P+Q)\frac{\sin(\theta-\beta)}{\cos\beta}-PL\sin\theta=0$$

or
$$(P+Q)\sin(\theta-\beta)=P\sin\theta\cos\beta$$

$$(P+Q)(\sin\theta\cos\beta-\cos\theta\sin\beta)=P\sin\theta\cos\beta$$

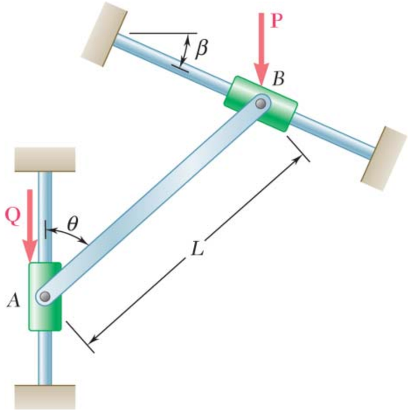
or
$$-(P+Q)\cos\theta\sin\beta+Q\sin\theta\cos\beta=0$$

$$-\frac{P+Q}{Q}\frac{\sin\beta}{\cos\beta}+\frac{\sin\theta}{\cos\theta}=0$$

$$\tan\theta=\frac{P+Q}{Q}\tan\beta \qquad (2)$$

With
$$P=Q=400\text{ N}, \quad \beta=30°$$

$$\tan\theta=\frac{800\text{ N}}{400\text{ N}}\tan 30°=1.1547 \qquad \theta=49.1° \blacktriangleleft$$



PROBLEM 10.84

A slender rod *AB* is attached to two collars *A* and *B* that can move freely along the guide rods shown. Knowing that $\beta = 30°$, $P = 100\text{ N}$, and $Q = 25\text{ N}$, determine the value of the angle θ corresponding to equilibrium.

SOLUTION

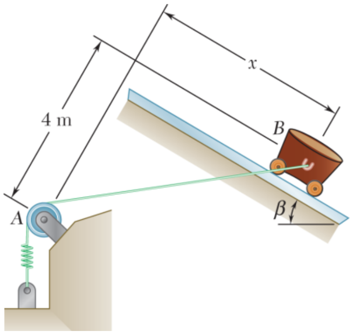
Using Equation (2) of Problem 10.83, with $P = 100$ N, $Q = 25$ N, and $\beta = 30°$, we have

$$\tan\theta = \frac{(100\text{ N})(25\text{ N})}{(25\text{ N})}\tan 30°$$

$$= 57.735$$

$$\theta = 89.007°$$

$\theta = 89.0°$ ◀

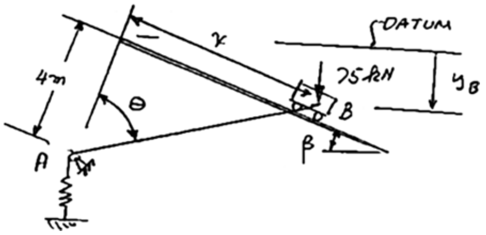


PROBLEM 10.85

Cart *B*, which weighs 75 kN, rolls along a sloping track that forms an angle β with the horizontal. The spring constant is 5 kN/m, and the spring is unstretched when $x = 0$. Determine the distance x corresponding to equilibrium for the angle β indicated.

Angle $\beta = 30°$.

SOLUTION



$$x = (4\text{ m})\tan\theta \qquad (1)$$

$$y_B = x\sin\beta = 4\tan\theta\sin\beta$$

$$AC = (4\text{ m})\cos\theta$$

For $x = 0$, $$(AC)_0 = 4\text{ m}$$

Stretch of spring. $$s = AC - (AC)_0 = \frac{4}{\cos\theta} - 4 = 4\left(\frac{1}{\cos\theta} - 1\right)$$

$$V = \frac{1}{2}ks^2 - (75\text{ kN})y_B$$

$$= \frac{1}{2}(5\text{ kN/m})16\left(\frac{1}{\cos\theta} - 1\right)^2 - (75\text{ kN})4\tan\theta\sin\beta$$

$$\frac{dV}{d\theta} = 80\left(\frac{1}{\cos\theta} - 1\right)\frac{\sin\theta}{\cos^2\theta} - 300\frac{\sin\beta}{\cos^2\theta}$$

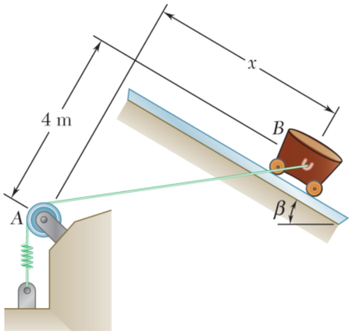
Equilibrium $$\frac{dV}{d\theta} = 0: \quad \left(\frac{1}{\cos\theta} - 1\right)\sin\theta = 3.75\sin\beta \qquad (2)$$

Given: $$\beta = 30°, \quad \sin\theta = 0.5$$

Eq. (2): $$\left(\frac{1}{\cos\theta} - 1\right)\sin\theta = 3.75(0.5) = 1.875$$

Solve by trial and error: $$\theta = 70.46°$$

Eq. (1): $$x = (4\text{ m})\tan 70.46° \qquad x = 11.27\text{ m} \blacktriangleleft$$

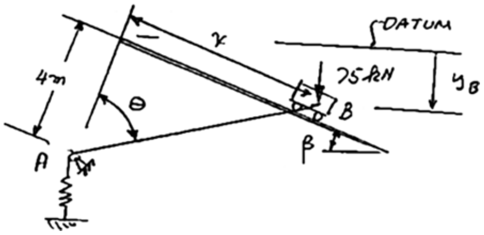


PROBLEM 10.86

Cart *B*, which weighs 75 kN, rolls along a sloping track that forms an angle β with the horizontal. The spring constant is 5 kN/m, and the spring is unstretched when $x = 0$. Determine the distance *x* corresponding to equilibrium for the angle β indicated.

Angle $\beta = 60°$.

SOLUTION



$$x = (4\text{ m})\tan\theta \qquad (1)$$

$$y_B = x\sin\beta = 4\tan\theta\sin\beta$$

$$AC = (4\text{ m})\cos\theta$$

For $x = 0$, $(AC)_0 = 4\text{ m}$

Stretch of spring: $s = AC - (AC)_0 = \dfrac{4}{\cos\theta} - 4 = 4\left(\dfrac{1}{\cos\theta} - 1\right)$

$$V = \frac{1}{2}ks^2 - (75\text{ kN})y_B$$

$$= \frac{1}{2}(5\text{ kN/m})16\left(\frac{1}{\cos\theta} - 1\right)^2 - (75\text{ kN})4\tan\theta\sin\beta$$

$$\frac{dV}{d\theta} = 80\left(\frac{1}{\cos\theta} - 1\right)\frac{\sin\theta}{\cos^2\theta} - 300\frac{\sin\beta}{\cos^2\theta}$$

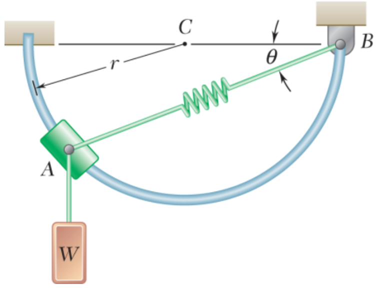
Equilibrium $\dfrac{dV}{d\theta} = 0\text{:} \quad \left(\dfrac{1}{\cos\theta} - 1\right)\sin\theta = 3.75\sin\beta \qquad (2)$

Given: $\beta = 60°, \sin\theta = 0.86603$

Eq. (2): $\left(\dfrac{1}{\cos\theta} - 1\right)\sin\theta = 3.75(0.86603) = 3.2476$

Solve by trial and error: $\theta = 76.67°$

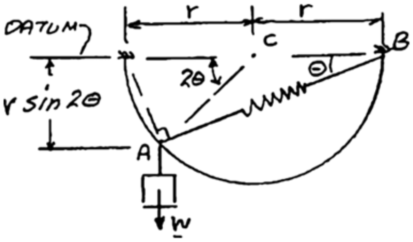
Eq. (1): $x = (4\text{ m})\tan 26.67°$ $\quad x = 16.88\text{ m}$ ◀



PROBLEM 10.87

Collar *A* can slide freely on the semicircular rod shown. Knowing that the constant of the spring is *k* and that the unstretched length of the spring is equal to the radius *r*, determine the value of θ corresponding to equilibrium when $W = 50$ lb, $r = 9$ in., and $k = 15$ lb/in.

SOLUTION



Stretch of spring
$$s = AB - r$$
$$s = 2(r\cos\theta) - r$$
$$s = r(2\cos\theta - 1)$$

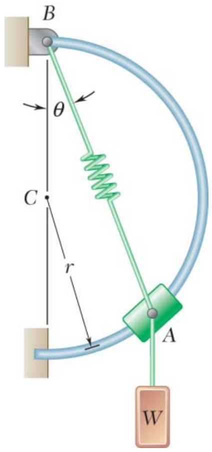
Potential energy:
$$V = \frac{1}{2}ks^2 - Wr\sin 2\theta \qquad W = mg$$
$$V = \frac{1}{2}kr^2(2\cos\theta - 1)^2 - Wr\sin 2\theta$$
$$\frac{dV}{d\theta} = -kr^2(2\cos\theta - 1)2\sin\theta - 2Wr\cos 2\theta$$

Equilibrium
$$\frac{dV}{d\theta} = 0: \quad -kr^2(2\cos\theta - 1)\sin\theta - Wr\cos 2\theta = 0$$
$$\frac{(2\cos\theta - 1)\sin\theta}{\cos 2\theta} = -\frac{W}{kr}$$

Now
$$\frac{W}{kr} = \frac{(50\text{ lb})}{(15\text{ lb/in.})(9\text{ in.})} = 0.37037$$

Then
$$\frac{(2\cos\theta - 1)\sin\theta}{\cos 2\theta} = -0.37037$$

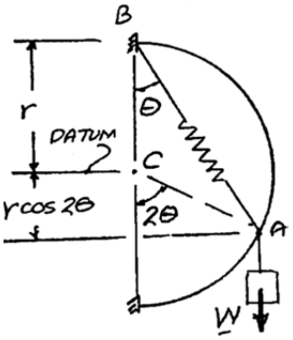
Solving numerically, $\theta = 0.95637\text{ rad} = 54.8°$ $\qquad \theta = 54.8°$ ◀



PROBLEM 10.88

Collar *A* can slide freely on the semicircular rod shown. Knowing that the constant of the spring is *k* and that the unstretched length of the spring is equal to the radius *r*, determine the value of θ corresponding to equilibrium when $W = 50$ lb, $r = 9$ in., and $k = 15$ lb/in.

SOLUTION



Stretch of spring

$$s = AB - r = 2(r\cos\theta) - r$$

$$s = r(2\cos\theta - 1)$$

$$V = \frac{1}{2}ks^2 - Wr\cos 2\theta$$

$$= \frac{1}{2}kr^2(2\cos\theta - 1)^2 - Wr\cos 2\theta$$

$$\frac{dV}{d\theta} = -kr^2(2\cos\theta - 1)2\sin\theta + 2Wr\sin 2\theta$$

Equilibrium

$$\frac{dV}{d\theta} = 0: \quad -kr^2(2\cos\theta - 1)\sin\theta + Wr\sin 2\theta = 0$$

$$-kr^2(2\cos\theta - 1)\sin\theta + Wr(2\sin\theta\cos\theta) = 0$$

or

$$\frac{(2\cos\theta - 1)\sin\theta}{2\cos\theta} = \frac{W}{kr}$$

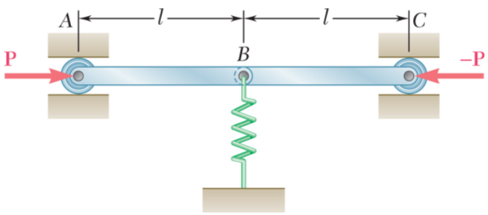
Now

$$\frac{W}{kr} = \frac{(50\text{ lb})}{(15\text{ lb/in.})(9\text{ in.})} = 0.37037$$

Then

$$\frac{2\cos\theta - 1}{2\cos\theta} = 0.37037$$

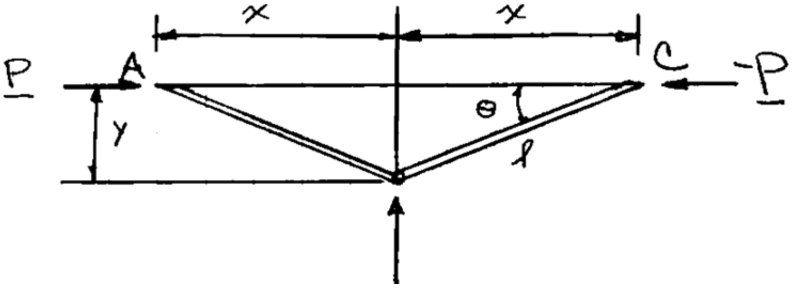
Solving $\theta = 37.4°$ ◀



PROBLEM 10.89

Two bars *AB* and *BC* of negligible weight are attached to a single spring of constant *k* that is unstretched when the bars are horizontal. Determine the range of values of the magnitude *P* of two equal and opposite forces **P** and –**P** for which the equilibrium of the system is stable in the position shown.

SOLUTION



$$x = l\cos\theta$$
$$y = l\sin\theta$$

$$V = 2Px + \frac{1}{2}ky^2$$

$$v = 2Pl\cos\theta + \frac{1}{2}kl^2\sin^2\theta$$

$$\frac{dV}{d\theta} = -2Pl\sin\theta + kl^2\sin\theta\cos\theta$$

$$= -2Pl\sin\theta + \frac{1}{2}kl^2\sin 2\theta$$

$$\frac{d^2V}{d\theta^2} = -2Pl\cos\theta + kl^2\cos 2\theta \qquad (1)$$

For equilibrium position $\theta = 0$ to be stable

$$\frac{d^2V}{d\theta^2} = -2Pl + kl^2 > 0 \qquad (2)$$

$$P < \frac{1}{2}kl \blacktriangleleft$$

PROBLEM 10.89 (Continued)

Note: For $P = \frac{1}{2}kl,$ we have $\frac{d^2V}{d\theta^2} = 0$ and we must determine which is the first derivative to be $\neq 0.$

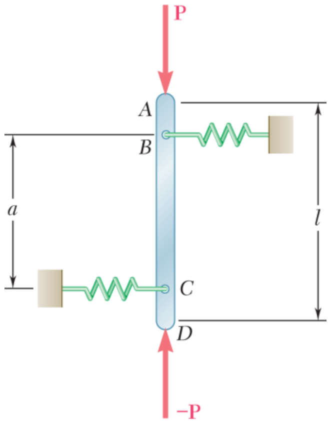
Differentiating Eq. (1):

$$\frac{d^3V}{d\theta^3} = +2Pl\sin\theta - 2kl^2\sin 2\theta = 0 \text{ for } \theta = 0$$

$$\frac{d^4V}{d\theta^4} = 2Pl\cos\theta - 4kl^2\cos 2\theta = 2Pl - 4kl^2 \text{ for } \theta = 0$$

But $P = \frac{1}{2}kl.$ Thus $\frac{d^4V}{d\theta^4} = kl^2 - 4kl^2 < 0$ and we conclude that the equilibrium is unstable for $P = \frac{1}{2}kl.$

The sign < in Eq. (2) is thus correct.



PROBLEM 10.90

A vertical bar *AD* is attached to two springs of constant *k* and is in equilibrium in the position shown. Determine the range of values of the magnitude *P* of two equal and opposite vertical forces **P** and –**P** for which the equilibrium position is stable if (*a*) $AB = CD$, (*b*) $AB = 2CD$.

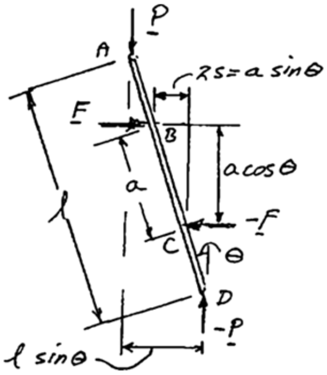
SOLUTION

For both (*a*) and (*b*): Since **P** and –**P** are vertical, they form a couple of moment

$$M_P = +Pl\sin\theta$$

The forces **F** and –**F** exerted by springs must, therefore, also form a couple, with moment

$$M_F = -Fa\cos\theta$$



We have $$dU = M_P d\theta + M_F d\theta$$
$$= (Pl\sin\theta - Fa\cos\theta)d\theta$$

but $$F = ks = k\left(\frac{1}{2}a\sin\theta\right)$$

Thus, $$dU = \left(Pl\sin\theta - \frac{1}{2}ka^2\sin\theta\cos\theta\right)d\theta$$

From Equation (10.19), page 580, we have

$$dV = -dU = -Pl\sin\theta d\theta + \frac{1}{4}ka^2\sin 2\theta d\theta$$

or $$\frac{dV}{d\theta} = -Pl\sin\theta + \frac{1}{4}ka^2\sin 2\theta$$

and $$\frac{d^2V}{d\theta^2} = -Pl\cos\theta + \frac{1}{2}ka^2\cos 2\theta \qquad (1)$$

For $\theta = 0$: $$\frac{d^2V}{d\theta^2} = -Pl + \frac{1}{2}ka^2$$

PROBLEM 10.90 (Continued)

For Stability: $\dfrac{d^2V}{d\theta^2} > 0, \quad -Pl + \dfrac{1}{2}ka^2 > 0$

or (for Parts *a* and *b*) $P < \dfrac{ka^2}{2l}$ ◀

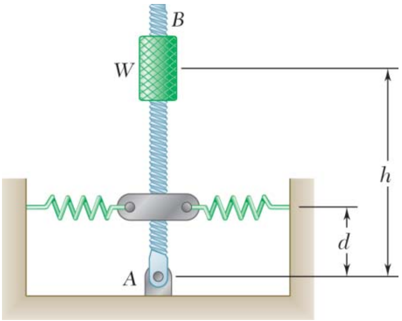
Note: To check that equilibrium is unstable for $P = \frac{ka^2}{2l}$, we differentiate (1) twice:

$$\frac{d^3V}{d\theta^3} = +Pl\sin\theta - ka^2\sin 2\theta = 0, \quad \text{for} \quad \theta = 0,$$

$$\frac{d^4V}{d\theta^4} = Pl\cos\theta - 2ka^2\cos 2\theta$$

For $\theta = 0$ $\dfrac{d^4V}{d\theta^4} = Pl - 2ka^2 = \dfrac{ka^2}{2} - 2ka^2 < 0$

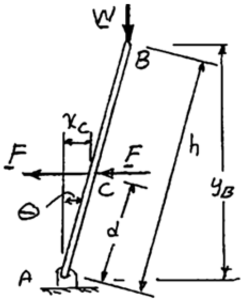
Thus, equilibrium is unstable when $P = \dfrac{ka^2}{2l}$



PROBLEM 10.91

Rod AB is attached to a hinge at A and to two springs, each of constant k. If $h = 25$ in., $d = 12$ in., and $W = 80$ lb, determine the range of values of k for which the equilibrium of the rod is stable in the position shown. Each spring can act in either tension or compression.

SOLUTION



We have $$x_C = d\sin\theta \qquad y_B = h\cos\theta$$

Potential Energy: $$V = 2\left(\frac{1}{2}kx_C^2 + Wy_B\right)$$

$$= kd^2\sin^2\theta + Wh\cos\theta$$

Then $$\frac{dV}{d\theta} = 2kd^2\sin\theta\cos\theta - Wh\sin\theta$$

$$= kd^2\sin 2\theta - Wh\sin\theta$$

and $$\frac{d^2V}{d\theta^2} = 2kd^2\cos 2\theta - Wh\cos\theta \tag{1}$$

For equilibrium position $\theta = 0$ to be stable, we must have

$$\frac{d^2V}{d\theta^2} = 2kd^2 - Wh > 0$$

or $$kd^2 > \frac{1}{2}Wh \tag{2}$$

Note: For $kd^2 = \frac{1}{2}Wh,$ we have $\frac{d^2V}{d\theta^2} = 0,$ so that we must determine which is the first derivative that is not equal to zero. Differentiating Equation (1), we write

$$\frac{d^3V}{d\theta^3} = -4kd^2\sin 2\theta + Wh\sin\theta = 0 \qquad \text{for } \theta = 0$$

$$\frac{d^4V}{d\theta^2} = -8kd^2\cos 2\theta + Wh\cos\theta$$

For $\theta = 0$: $$\frac{d^4V}{d\theta^4} = -8kd^2 + Wh$$

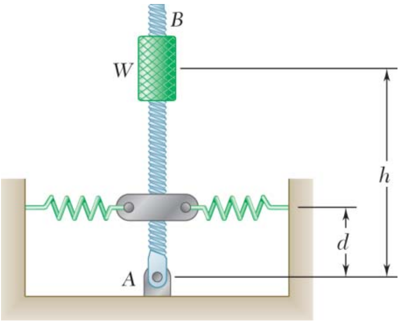
PROBLEM 10.91 (Continued)

Since $kd^2 = \frac{1}{2}Wh$, $\frac{d^4V}{d\theta^4} = -4Wh + Wh < 0$, we conclude that the equilibrium is unstable for $kd^2 = \frac{1}{2}Wh$ and the $>$ sign in Equation (2) is correct.

With $W = 80$ lb, $h = 25$ in., and $d = 12$ in.

Equation (2) gives $$k(12\text{ in.})^2 > \frac{1}{2}(80\text{ lb})(25\text{ in.})$$

or $k > 6.944$ lb/in. $k > 6.94$ lb/in. ◀



PROBLEM 10.92

Rod *AB* is attached to a hinge at *A* and to two springs, each of constant *k*. If $h = 45$ in., $k = 6$ lb/in., and $W = 60$ lb, determine the smallest distance *d* for which the equilibrium of the rod is stable in the position shown. Each spring can act in either tension or compression.

SOLUTION

Using Equation (2) of Problem 10.91 with

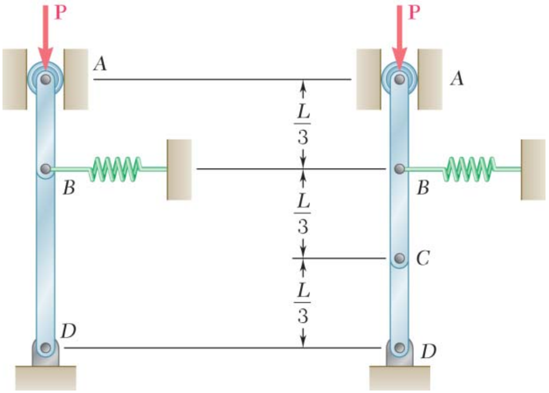
$$h = 45 \text{ in.}, \quad k = 6 \text{ lb/in.}, \quad \text{and} \quad W = 60 \text{ lb}$$

$$(6 \text{ lb/in.})d^2 > \frac{1}{2}(60 \text{ lb})(45 \text{ in.})$$

or

$$d^2 > 225 \text{ in.}^2$$

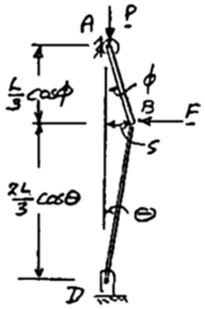
$$d > 15.0000 \text{ in.} \qquad \text{smallest } d = 15.00 \text{ in.} \blacktriangleleft$$



PROBLEM 10.93

Two bars are attached to a single spring of constant k that is unstretched when the bars are vertical. Determine the range of values of P for which the equilibrium of the system is stable in the position shown.

SOLUTION



$$s = \frac{L}{3}\sin\phi = \frac{2L}{3}\sin\theta$$

For small values of ϕ and θ

$$\phi = 2\theta$$

$$V = P\left(\frac{L}{3}\cos\phi + \frac{2L}{3}\cos\theta\right) + \frac{1}{2}ks^2$$

$$V = \frac{PL}{3}(\cos 2\theta + 2\cos\theta) + \frac{1}{2}k\left(\frac{2L}{3}\sin\theta\right)^2$$

$$\frac{dV}{d\theta} = \frac{PL}{3}(-2\sin 2\theta - 2\sin\theta) + \frac{2}{9}kL^2\sin\theta\cos\theta$$

$$= -\frac{PL}{3}(2\sin 2\theta + 2\sin\theta) + \frac{2}{9}kL^2\sin 2\theta$$

$$\frac{d^2V}{d\theta^2} = -\frac{PL}{3}(4\cos 2\theta + 2\cos\theta) + \frac{4}{9}kL^2\cos 2\theta$$

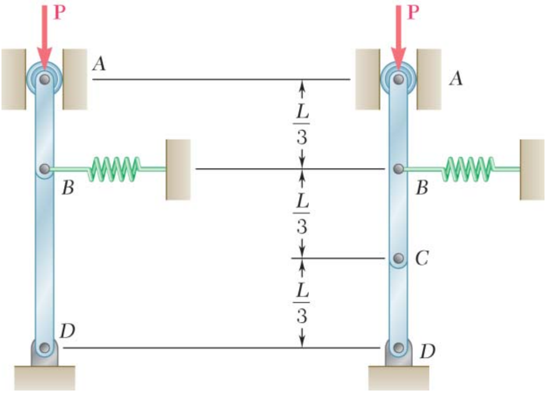
when $\theta = 0$:

$$\frac{d^2V}{d\theta^2} = -\frac{6PL}{3} + \frac{4}{9}kL^2$$

For stability:

$$\frac{d^2V}{d\theta^2} > 0, \quad -2PL + \frac{4}{9}kL^2 > 0$$

$$P < \frac{2}{9}kL \blacktriangleleft$$

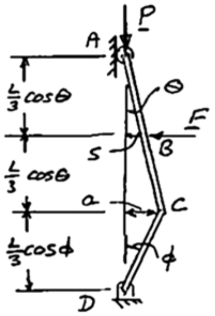

P
P
A
A
$\frac{L}{3}$
$\frac{L}{3}$
$\frac{L}{3}$
B
B
C
D
D

PROBLEM 10.94

Two bars are attached to a single spring of constant k that is unstretched when the bars are vertical. Determine the range of values of P for which the equilibrium of the system is stable in the position shown.

SOLUTION

$$a = \frac{2L}{3}\sin\theta = \frac{L}{3}\sin\phi$$


P
A
$\frac{L}{3}\cos\theta$
θ
F
s
B
$\frac{L}{3}\cos\theta$
a
C
$\frac{L}{3}\cos\phi$
ϕ
D

For small values of ϕ and θ

$$\phi = 2\theta$$

$$s = \frac{L}{3}\sin\theta$$

$$V = P\left(\frac{2L}{3}\cos\theta + \frac{L}{3}\cos\phi\right) + \frac{1}{2}ks^2$$

$$= \frac{PL}{3}(2\cos\theta + \cos 2\theta) + \frac{1}{2}k\left(\frac{L}{3}\sin\theta\right)^2$$

$$\frac{dV}{d\theta} = \frac{PL}{3}(-2\sin\theta - 2\sin 2\theta) + \frac{kL^2}{9}\sin\theta\cos\theta$$

$$\frac{dV}{d\theta} = -\frac{2PL}{3}(\sin\theta + \sin 2\theta) + \frac{kL^2}{18}\sin 2\theta$$

$$\frac{d^2V}{d\theta^2} = -\frac{2PL}{3}(\cos\theta + 2\cos 2\theta) + \frac{kL^2}{9}\cos 2\theta$$

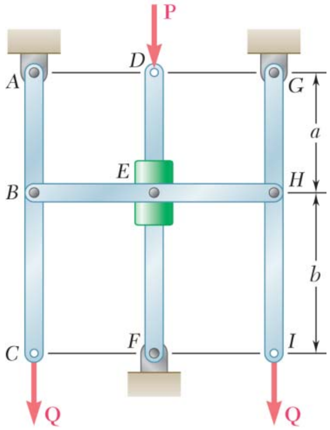
when $\theta = 0$:

$$\frac{dV^2}{d\theta^2} = -2PL + \frac{kL^2}{9}$$

For stability:

$$\frac{d^2V}{d\theta^2} > 0, \quad -2PL + \frac{kL^2}{9} > 0$$

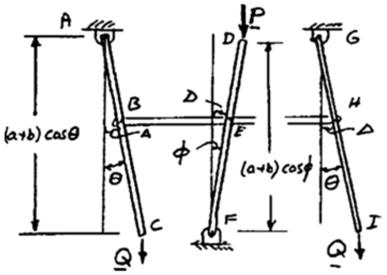
$$P < \frac{1}{18}kL \blacktriangleleft$$



PROBLEM 10.95

The horizontal bar *BEH* is connected to three vertical bars. The collar at *E* can slide freely on bar *DF*. Determine the range of values of *Q* for which the equilibrium of the system is stable in the position shown when $a = 24$ in., $b = 20$ in., and $P = 150$ lb.

SOLUTION



First note $\Delta = a\sin\theta = b\sin\phi$

For small values of θ and ϕ: $a\theta = b\phi$

or

$$\phi = \frac{a}{b}\theta$$

$$V = P(a+b)\cos\phi - 2Q(a+b)\cos\theta$$

$$= (a+b)\left[P\cos\left(\frac{a}{b}\theta\right) - 2Q\cos\theta\right]$$

$$\frac{dV}{d\theta} = (a+b)\left[-\frac{a}{b}P\sin\left(\frac{a}{b}\theta\right) + 2Q\sin\theta\right]$$

$$\frac{d^2V}{d\theta^2} = (a+b)\left[-\frac{a^2}{b^2}P\cos\left(\frac{a}{b}\theta\right) + 2Q\cos\theta\right]$$

when $\theta = 0$:

$$\frac{d^2V}{d\theta^2} = (a+b)\left(-\frac{a^2}{b^2}P + 2Q\right)$$

PROBLEM 10.95 (Continued)

Stability: $$\frac{d^2V}{d\theta^2} > 0: \quad -\frac{a^2}{b^2}P + 2Q > 0$$

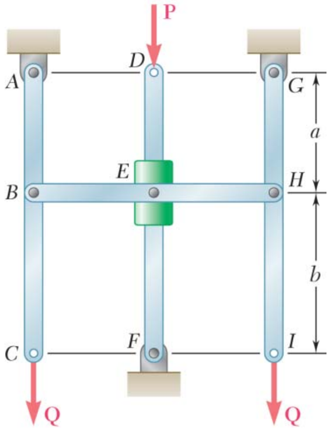
$$P < 2\frac{b^2}{a^2}Q \qquad (1)$$

or $$Q > \frac{a^2}{2b^2}P \qquad (2)$$

with $P = 150$ lb, $a = 24$ in., and $b = 20$ in.

Equation (1): $$Q > \frac{(24 \text{ in.})^2}{2(20 \text{ in.})^2}(150 \text{ lb}) = 108.000 \text{ lb}$$

For stability $Q > 108.0$ lb ◀



PROBLEM 10.96

The horizontal bar *BEH* is connected to three vertical bars. The collar at *E* can slide freely on bar *DF*. Determine the range of values of *P* for which the equilibrium of the system is stable in the position shown when $a = 150$ mm, $b = 200$ mm, and $Q = 45$ N.

SOLUTION

Using Equation (2) of Problem 10.95 with

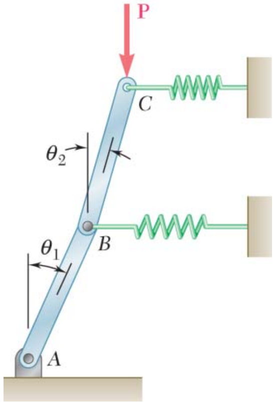
$$Q = 45\text{ N},\quad a = 150\text{ mm},\quad \text{and}\quad b = 200\text{ mm}$$

Equation (2)

$$P < 2\frac{(200\text{ mm})^2}{(150\text{ mm})^2}(45\text{ N})$$

$$= 160.000\text{ N}$$

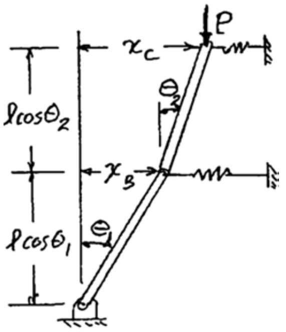
For stability $P < 160.0\text{ N}$ ◀



PROBLEM 10.97*

Bars *AB* and *BC*, each of length *l* and of negligible weight, are attached to two springs, each of constant *k*. The springs are undeformed, and the system is in equilibrium when $\theta_1 = \theta_2 = 0$. Determine the range of values of *P* for which the equilibrium position is stable.

SOLUTION



We have

$$x_B = l\sin\theta$$

$$x_C = l\sin\theta_1 + l\sin\theta_2$$

$$y_C = l\cos\theta_1 + l\cos\theta_2$$

$$V = Py_C + \frac{1}{2}kx_B^2 + \frac{1}{2}kx_C^2$$

or

$$V = Pl(\cos\theta_1 + \cos\theta_2) + \frac{1}{2}kl^2\left[\sin^2\theta_1 + (\sin\theta_1 + \sin\theta_2)^2\right]$$

For small values of θ_1 and θ_2:

$$\sin\theta_1 \approx \theta_1, \quad \sin\theta_2 \approx \theta_2, \quad \cos\theta_1 \approx 1 - \frac{1}{2}\theta_1^2, \quad \cos\theta_2 \approx 1 - \frac{1}{2}\theta_2^2$$

Then

$$V = Pl\left(1 - \frac{\theta_1^2}{2} + 1 - \frac{\theta_2^2}{2}\right) + \frac{1}{2}kl^2\left[\theta_1^2 + \left(\theta_1 + \theta_2\right)^2\right]$$

and

$$\frac{\partial V}{\partial\theta_1} = -Pl\theta_1 + kl^2[\theta_1 + (\theta_1 + \theta_2)]$$

$$\frac{\partial V}{\partial\theta_2} = -Pl\theta_2 + kl^2(\theta_1 + \theta_2)$$

$$\frac{\partial^2 V}{\partial\theta_1^2} = -Pl + 2kl^2 \qquad \frac{\partial^2 V}{\partial\theta_2^2} = -Pl + kl^2$$

$$\frac{\partial^2 V}{\partial\theta_1\partial\theta_2} = kl^2$$

Stability: Conditions for stability (see Page 583).

PROBLEM 10.97* (Continued)

For $$\theta_1 = \theta_2 = 0: \quad \frac{\partial V}{\partial \theta_1} = \frac{\partial V}{\partial \theta_2} = 0 \quad \text{(condition satisfied)}$$

$$\left(\frac{\partial^2 V}{\partial \theta_1 \partial \theta_2}\right)^2 - \frac{\partial^2 V}{\partial \theta_1^2}\frac{\partial^2 V}{\partial \theta_2^2} < 0$$

Substituting $$(kl^2)^2 - (-Pl + 2kl^2)(-Pl + kl) < 0$$

$$k^2 l^4 - P^2 l^2 + 3Pkl^3 - 2k^2 l^4 < 0$$

$$P^2 - 3klP + k^2 l^2 > 0$$

Solving $$P < \frac{3-\sqrt{5}}{2}kl \quad \text{or} \quad P > \frac{3+\sqrt{5}}{2}kl$$

or $$P < 0.382kl \quad \text{or} \quad P > 2.62kl$$

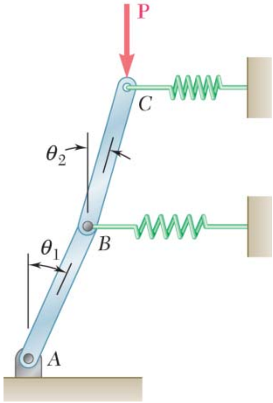
$$\frac{\partial^2 V}{\partial \theta_1^2} > 0: \quad -Pl + 2kl^2 > 0$$

or $$P < \frac{1}{2}kl$$

$$\frac{\partial^2 V}{\partial \theta_2^2} > 0: \quad -Pl + kl^2 > 0$$

or $$P < kl$$

Therefore, all conditions for stable equilibrium are satisfied when $0 \le P < 0.382kl$ ◀



PROBLEM 10.98*

Solve Problem 10.97 knowing that $l = 800$ mm and $k = 2.5$ kN/m.

PROBLEM 10.97* Bars *AB* and *BC*, each of length *l* and of negligible weight, are attached to two springs, each of constant *k*. The springs are undeformed, and the system is in equilibrium when $\theta_1 = \theta_2 = 0$. Determine the range of values of *P* for which the equilibrium position is stable.

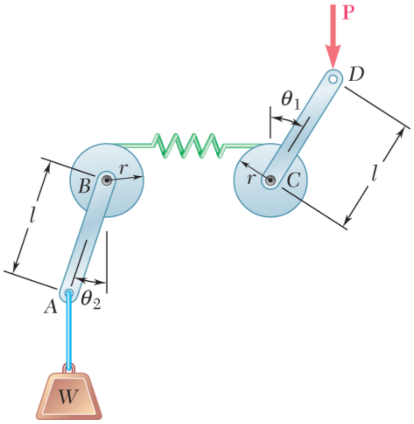
SOLUTION

From the analysis of Problem 10.97 with

$$l = 800 \text{ mm} \quad \text{and} \quad k = 2.5 \text{ kN/m}$$

$$P < 0.382kl = 0.382(2500 \text{ N/m})(0.8 \text{ m}) = 764 \text{ N}$$

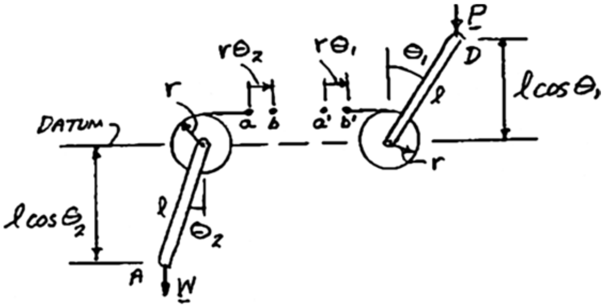
$P < 764$ N ◀


P
D
θ_1
B
r
r
C
l
l
A
θ_2
W

PROBLEM 10.99*

Two rods of negligible weight are attached to drums of radius r that are connected by a belt and spring of constant k. Knowing that the spring is undeformed when the rods are vertical, determine the range of values of P for which the equilibrium position $\theta_1 = \theta_2 = 0$ is stable.

SOLUTION


P
$r\theta_2$
$r\theta_1$
θ_1
D
r
l
$l\cos\theta_1$
DATUM
a
b
a'
b'
r
$l\cos\theta_2$
l
θ_2
A
W

Left end of spring moves from a to b. Right end of spring moves from a' to b'. Elongation of spring

$$s = a'b' - ab = r\theta_1 - r\theta_2 = r(\theta_1 - \theta_2)$$

$$V = \frac{1}{2}ks^2 + pl\cos\theta_1 - wl\cos\theta_2$$

$$= \frac{1}{2}kr^2(\theta_1 - \theta_2)^2 + pl\cos\theta_1 - wl\cos\theta_2$$

$$\frac{\partial v}{\partial\theta_1} = kr^2(\theta_1 - \theta_2) - pl\sin\theta_1$$

$$\frac{\partial v}{\partial\theta_2} = -kr^2(\theta_1 - \theta_2) + wl\sin\theta_2$$

$$\frac{\partial^2 v}{\partial\theta_1^2} = kr^2 - pl\cos\theta_1$$

$$\frac{\partial^2 v}{\partial\theta_2^2} = kr^2 + wl\cos\theta_2$$

$$\frac{\partial^2 v}{\partial\theta_1\partial\theta_2} = -kr^2$$

PROBLEM 10.99* (Continued)

For $\theta_1 = \theta_2 = 0$: $\dfrac{\partial^2 v}{\partial \theta_1^2} = kr^2 - pl, \quad \dfrac{\partial^2 v}{\partial \theta_2^2} = +kr^2 + wl, \quad \dfrac{r^2 v}{\partial \theta, \partial \theta_2} = -kr^2$

Conditions for stability (see Page 583)

$$\left(\frac{\partial^2 v}{\partial \theta_1 \partial \theta_2}\right)^2 - \frac{\partial^2 v}{\partial \theta_1^2} \cdot \frac{\partial^2 v}{\partial \theta_2^2} < 0$$

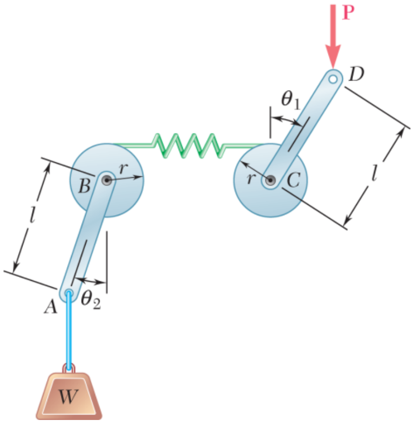
$$(kr^2)^2 - (kr^2 - pl)(kr^2 + wl) < 0$$

$$pl(kr^2 + wl) - kr^2 wl < 0$$

$$P < \frac{wkr^2}{kr^2 + wl}; \quad P < \frac{kr^2}{l}\left(\frac{W}{\frac{kr^2}{l} + w}\right) \quad \triangleleft$$

$$\frac{\partial^2 v}{\partial \theta_1^2} >: \quad kr^2 - pl > 0; \quad P < \frac{kr^2}{l} \quad \triangleleft$$

We choose: $P < \dfrac{kr^2}{l}\left(\dfrac{W}{\frac{kr^2}{l} + W}\right)$ ◀



PROBLEM 10.100*

Solve Problem 10.99 knowing that k = 20 lb/in., r = 3 in., l = 6 in., and (*a*) $W = 15$ lb, (*b*) $W = 60$ lb.

PROBLEM 10.99* Two rods of negligible weight are attached to drums of radius r that are connected by a belt and spring of constant k. Knowing that the spring is undeformed when the rods are vertical, determine the range of values of P for which the equilibrium position $\theta_1 = \theta_2 = 0$ is stable.

SOLUTION

$$k = 20 \text{ lb/in.}$$

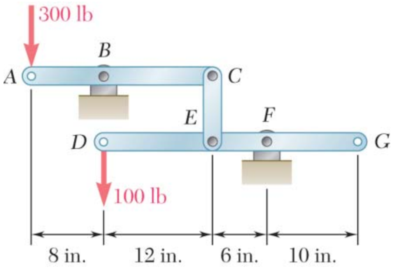
$$r = 3 \text{ in.}$$

$$l = 6 \text{ in.}$$

$$\frac{kr^2}{l} = \frac{(20 \text{ lb/in.})(3 \text{ in.})^2}{6 \text{ in.}} = 30 \text{ lb}$$

(*a*) $W = 15 \text{ lb}: \quad P < (30 \text{ lb})\dfrac{15 \text{ lb}}{(30 \text{ lb}) + (15 \text{ lb})}$ $\qquad P < 10.00 \text{ lb}$ ◀

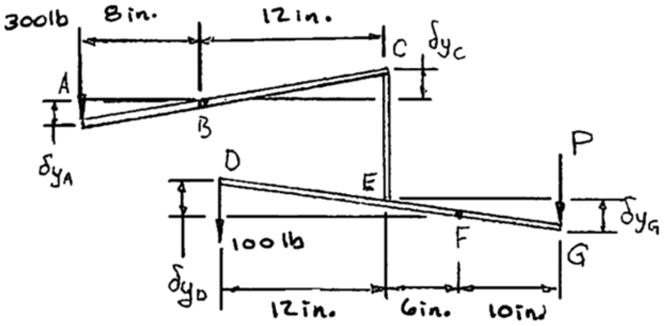
(*b*) $W = 60 \text{ lb}: \quad P < (30 \text{ lb})\dfrac{60 \text{ lb}}{(30 \text{ lb}) + (60 \text{ lb})}$ $\qquad P < 20.0 \text{ lb}$ ◀


300 lb
B
A
C
E
F
D
G
100 lb
8 in.
12 in.
6 in.
10 in.

PROBLEM 10.101

Determine the vertical force **P** that must be applied at *G* to maintain the equilibrium of the linkage.

SOLUTION


300lb
8in.
12in.
C
δy_C
A
B
δy_A
P
D
E
δy_G
F
100 lb
G
δy_D
12in.
6in.
10in.

Assuming $\delta y_A \downarrow$

it follows

$$\delta y_C = \frac{12}{8}\delta y_A = 1.5\delta y_A \uparrow$$

$$\delta y_E = \delta y_C = 1.5\delta y_A \uparrow$$

$$\delta y_D = \frac{18}{6}\delta y_A = 3(1.5\delta y_A) = 4.5\delta y_A \uparrow$$

$$\delta y_G = \frac{10}{6}\delta y_A = \frac{10}{6}(1.5\delta y_A) = 2.5\delta y_A \downarrow$$

Then, by virtual work

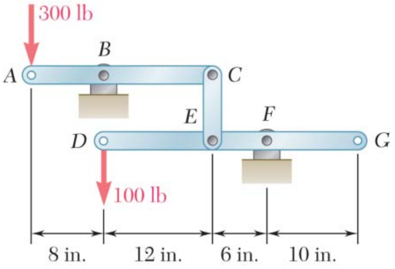
$$\delta U = 0: \quad (300 \text{ lb})\delta y_A - (100 \text{ lb})\delta y_D + P\delta y_G = 0$$

$$300\delta y_A - 100(4.5\delta y_A) + P(2.5\delta y_A) = 0$$

$$300 - 450 + 2.5P = 0$$

$$P = 60.0 \text{ lb}$$

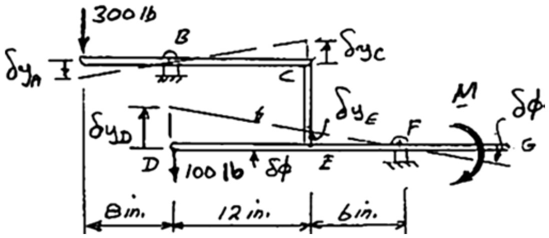
$\mathbf{P} = 60.0 \text{ lb} \downarrow$ ◀


300 lb
B
A
C
E
F
D
G
100 lb
8 in.
12 in.
6 in.
10 in.

PROBLEM 10.102

Determine the couple **M** that must be applied to member *DEFG* to maintain the equilibrium of the linkage.

SOLUTION


300 lb
B
δy_A
δy_C
C
δy_E
M
$\delta\phi$
F
G
δy_D
D
100 lb
$\delta\phi$
E
8 in.
12 in.
6 in.

Assume $\delta y_A \downarrow$:

$$\delta y_C = \frac{12}{8}\delta y_A = 1.5\delta y_A \uparrow, \qquad \delta y_E = \delta y_C = 1.5\delta y_A \uparrow$$

$$\delta y_D = \frac{18}{6}\delta y_E = 3(1.5\delta y_A) = 4.5\delta y_A \uparrow$$

$$\delta\phi = \frac{\delta y_E}{6} = \frac{1.5\delta y_A}{6} = \frac{1}{4}\delta y_A \curvearrowright$$

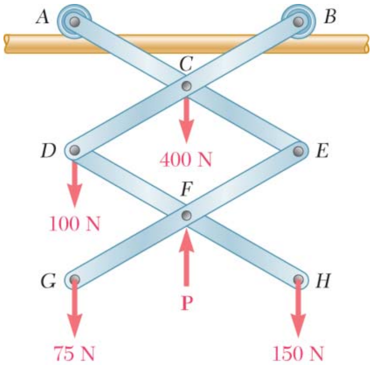
Virtual Work:

$$\delta U = 0: \qquad (300\text{ lb})\delta y_A - (100\text{ lb})\delta y_D + M\,\delta\phi = 0$$

$$300\delta y_A - 100(1.5\delta y_A) + M\left(\frac{1}{4}\delta y_A\right) = 0$$

$$300 - 450 + \frac{1}{4}M = 0$$

$$M = +600\text{ lb}\cdot\text{in.} \qquad \mathbf{M} = 600\text{ lb}\cdot\text{in.} \curvearrowright \blacktriangleleft$$


A
B
C
D
400 N
E
F
100 N
G
H
P
75 N
150 N

PROBLEM 10.103

Determine the force **P** required to maintain the equilibrium of the linkage shown. All members are of the same length and the wheels at *A* and *B* roll freely on the horizontal rod.

SOLUTION

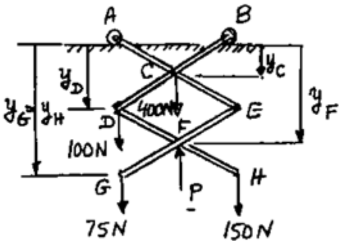
Using y_C as independent variable:

$$y_D = 2y_C \qquad \delta y_D = 2\delta y_C$$

$$y_F = 3y_C \qquad \delta y_F = 3\delta y_C$$

$$y_G = y_H = 4y_C$$

$$\delta y_G = \delta y_H = 4\delta y_C$$


A
B
C
y_C
y_D
y_G
y_H
400N
D
E
F
y_F
100N
G
P
H
75N
150N

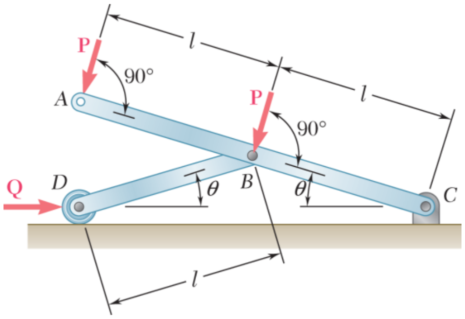
Virtual Work:

$$\delta U = (400\text{ N})\delta y_C + (100\text{ N})\delta y_D - P\delta y_F + (75\text{ N})\delta y_G + (150\text{ N})\delta y_H = 0$$

$$400\delta y_C + 100(2\delta y_C) - P(3\delta y_C) + (75+150)(4\delta y_C) = 0$$

$$3P = 400 + 200 + 900 \qquad P = +500\text{ N}$$

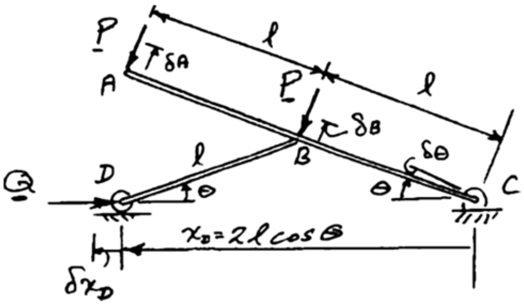
$$\mathbf{P} = 500\text{ N}\uparrow \blacktriangleleft$$


P
90°
A
P
90°
l
l
Q
D
θ
B
θ
C
l

PROBLEM 10.104

Derive an expression for the magnitude of the force **Q** required to maintain the equilibrium of the mechanism shown.

SOLUTION


P
δA
A
l
P
l
δB
B
δθ
Q
D
l
θ
θ
C
$x_D = 2l\cos\theta$
δx_D

We have $x_D = 2l\cos\theta$ so that $\delta x_D = -2l\sin\theta\,\delta\theta$

$$\delta A = 2l\delta\theta$$

$$\delta B = l\delta\theta$$

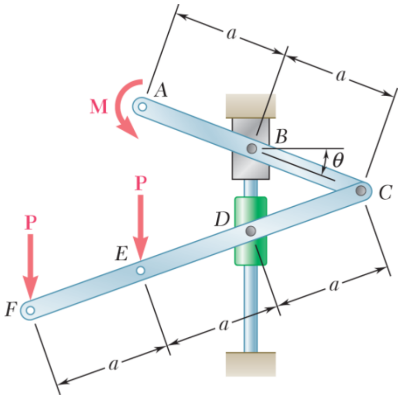
Virtual Work:

$$\delta U = 0: \quad -Q\delta x_D - P\delta A - P\delta B = 0$$

$$-Q(-2l\sin\theta\delta\theta) - P(2l\delta\theta) - P(l\delta\theta) = 0$$

$$2Ql\sin\theta - 3Pl = 0$$

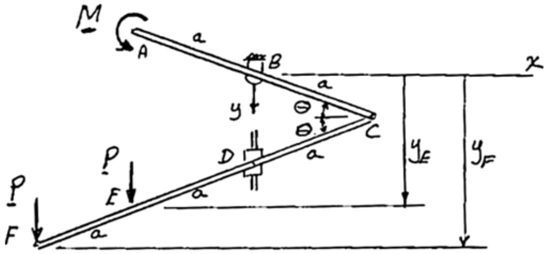
$$Q = \frac{3}{2}\frac{P}{\sin\theta} \blacktriangleleft$$


a
a
M
A
B
θ
C
P
P
D
E
F
a
a
a

PROBLEM 10.105

Derive an expression for the magnitude of the couple **M** required to maintain the equilibrium of the linkage shown.

SOLUTION


M
a
A
B
x
a
y
θ
θ
C
P
D
a
y_E
y_F
P
E
a
F
a

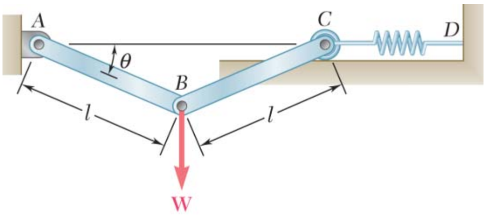
$$y_E = 3a\sin\theta \quad \delta y_E = 3a\cos\theta\delta\theta$$
$$y_F = 4a\sin\theta \quad \delta y_F = 4a\cos\theta\delta\theta$$

Virtual Work:

$$\delta U = 0: \quad -M\,\delta\theta + P\delta y_E + P\delta y_F = 0$$

$$-M\,\delta\theta + P(3a\cos\theta\delta\theta) + P(4a\cos\theta\delta\theta) = 0$$

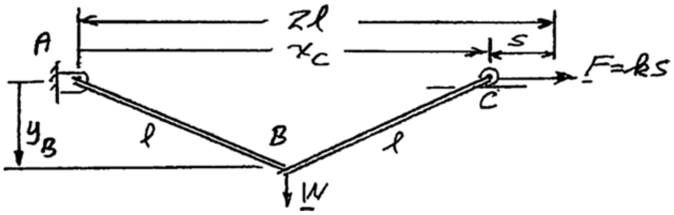
$M = 7Pa\cos\theta$ ◀



PROBLEM 10.106

A vertical load **W** is applied to the linkage at *B*. The constant of the spring is *k*, and the spring is unstretched when *AB* and *BC* are horizontal. Neglecting the weight of the linkage, derive an equation in *θ*, *W*, *l*, and *k* that must be satisfied when the linkage is in equilibrium.

SOLUTION



$$x_C = 2l\cos\theta \quad \delta x_C = -2l\sin\theta\delta\theta$$
$$y_B = l\sin\theta \quad \delta y_B = l\cos\theta\delta\theta$$
$$F = ks = k(2l - x_C) = 2kl(1-\cos\theta)$$

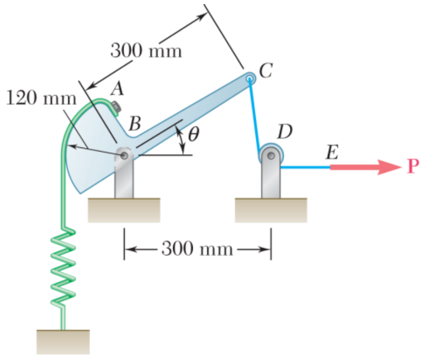
Virtual Work: $\delta U = 0$: $\quad F\delta x_C + W\delta y_B = 0$

$$2kl(1-\cos\theta)(-2l\sin\theta\delta\theta) + W(l\cos\theta\delta\theta) = 0$$

$$4kl^2(1-\cos\theta)\sin\theta = Wl\cos\theta$$

or $$(1-\cos\theta)\tan\theta = \frac{W}{4kl}$$

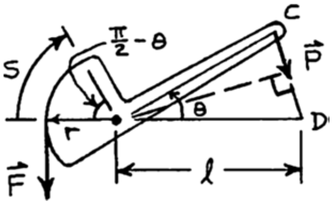
From above $$(1-\cos\theta)\tan\theta = \frac{W}{4kl} \blacktriangleleft$$



PROBLEM 10.107

A force **P** of magnitude 240 N is applied to end *E* of cable *CDE*, which passes under pulley *D* and is attached to the mechanism at *C*. Neglecting the weight of the mechanism and the radius of the pulley, determine the value of θ corresponding to equilibrium. The constant of the spring is $k = 4$ kN/m, and the spring is unstretched when $\theta = 90°$.

SOLUTION



$$s = r\left(\frac{\pi}{2} - \theta\right)$$

$$\delta s = -r\delta\theta$$

$$F = ks = kr\left(\frac{\pi}{2} - \theta\right)$$

$$CD = 2l\sin\frac{\theta}{2}$$

$$\delta(CD) = 2l\cos\frac{\theta}{2}\left(\frac{1}{2}\delta\theta\right)$$

$$= l\cos\frac{\theta}{2}\delta\theta$$

Virtual Work:

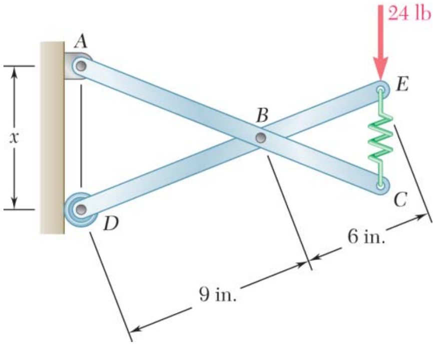
Since **F** tends to decrease *s* and **P** tends to decrease *CD*, we have

$$\delta U = -F\delta s - P\delta(CD) = 0$$

$$-kr\left(\frac{\pi}{2} - \theta\right)(-r\delta\theta) - P\left(l\cos\frac{\theta}{2}\delta\theta\right) = 0$$

$$\frac{\frac{\pi}{2} - \theta}{\cos\frac{\theta}{2}} = \frac{pl}{kr^2} = \frac{(240\text{ N})(0.3\text{ m})}{(4000\text{ N/m})(0.12\text{ m})^2} = 1.25$$

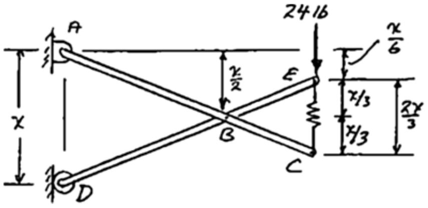
Solving by trial and error: $\theta = 0.33868$ rad $\theta = 19.40°$ ◀



PROBLEM 10.108

Two identical rods *ABC* and *DBE* are connected by a pin at *B* and by a spring *CE*. Knowing that the spring is 4 in. long when unstretched and that the constant of the spring is 8 lb/in., determine the distance *x* corresponding to equilibrium when a 24-lb load is applied at *E* as shown.

SOLUTION



Deformation of spring

$$s = EC - 4\text{ in.} = \frac{2x}{3} - 4$$

$$V = \frac{1}{2}ks^2 - (24\text{ lb})\frac{x}{6} = \frac{1}{2}(8\text{ lb/in.})\left(\frac{2x}{3} - 4\right)^2 - 4x$$

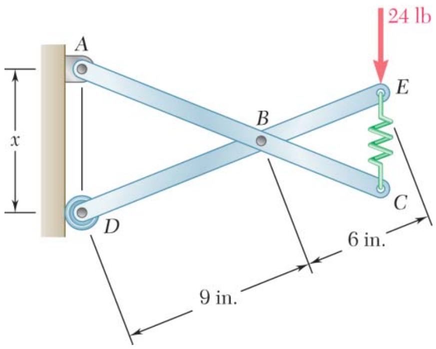
$$\frac{dV}{dx} = 8\left(\frac{2x}{3} - 4\right)\frac{2}{3} - 4$$

Equilibrium: $\frac{dV}{dx} = 0 \qquad \frac{16}{3}\left(\frac{2x}{3} - 4\right) - 4 = 0$

$$\frac{2x}{3} - 4 = 4\left(\frac{3}{16}\right)$$

$$\frac{2x}{3} = 4 + \frac{3}{4}$$

$x = 7.13\text{ in.}$ ◀

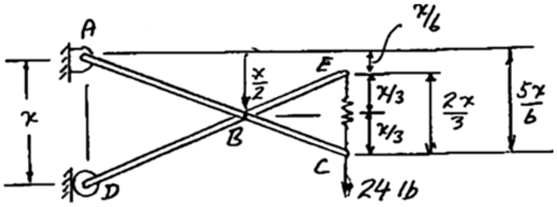


PROBLEM 10.109

Solve Problem 10.108 assuming that the 24-lb load is applied at *C* instead of *E*.

PROBLEM 10.108 Two identical rods *ABC* and *DBE* are connected by a pin at *B* and by a spring *CE*. Knowing that the spring is 4 in. long when unstretched and that the constant of the spring is 8 lb/in., determine the distance *x* corresponding to equilibrium when a 24-lb load is applied at *E* as shown.

SOLUTION



Deformation of spring

$$s = EC - 4\text{ in.} = \frac{2x}{3} - 4$$

$$V = \frac{1}{2}ks^2 - (24\text{ lb})\frac{5x}{6} = \frac{1}{2}(8\text{ lb/in.})\left(\frac{2x}{3} - 4\right)^2 - 20x$$

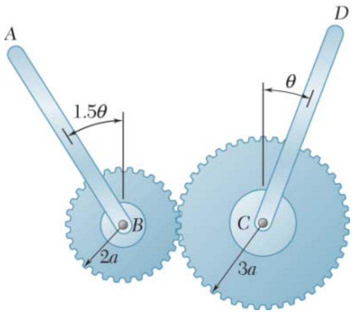
$$\frac{dV}{dx} = 8\left(\frac{2x}{3} - 4\right)\frac{2}{3} - 20$$

Equilibrium: $\frac{dV}{dx} = 0 \qquad \frac{16}{3}\left(\frac{2x}{3} - 4\right) - 20 = 0$

$$\frac{2x}{3} - 4 = 20\left(\frac{3}{16}\right)$$

$$\frac{2x}{3} = 4 + 3.75$$

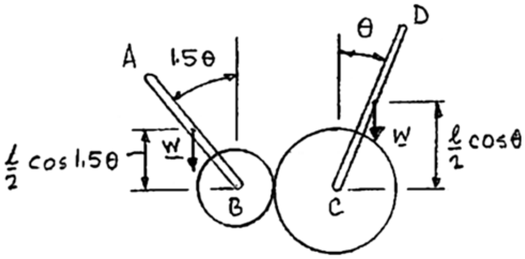
$x = 11.63$ in. ◀



PROBLEM 10.110

Two uniform rods, each of mass *m* and length *l*, are attached to gears as shown. For the range $0 \le \theta \le 180°$, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

SOLUTION



Potential energy
$$V = W\left(\frac{l}{2}\cos 1.5\theta\right) + W\left(\frac{l}{2}\cos\theta\right) \quad W = mg$$

$$\frac{dV}{d\theta} = \frac{Wl}{2}(-1.5\sin 1.5\theta) + \frac{Wl}{2}(-\sin\theta)$$
$$= -\frac{Wl}{2}(1.5\sin 1.5\theta + \sin\theta)$$
$$\frac{d^2V}{d\theta^2} = -\frac{Wl}{2}(2.25\cos 1.5\theta + \cos\theta)$$

For equilibrium
$$\frac{dV}{d\theta} = 0: \quad 1.5\sin 1.5\theta + \sin\theta = 0$$

Solutions: One solution, by inspection, is $\theta = 0$, and a second angle less than 180° can be found numerically:

$$\theta = 2.4042 \text{ rad} = 137.8°$$

Now
$$\frac{d^2V}{d\theta^2} = -\frac{Wl}{2}(2.25\cos 1.5\theta + \cos\theta)$$

At $\theta = 0$:
$$\frac{d^2V}{d\theta^2} = -\frac{Wl}{2}(2.25\cos 0° + \cos 0°)$$
$$= -\frac{Wl}{2}(3.25)(< 0)$$

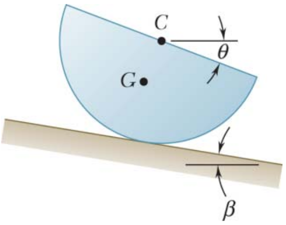
$\theta = 0$, Unstable ◀

PROBLEM 10.110 (Continued)

At $\theta = 137.8°$: $$\frac{d^2V}{d\theta^2} = -\frac{Wl}{2}[2.25\cos(1.5\times 137.8°) + \cos 137.8°]$$

$$= \frac{Wl}{2}(2.75)(> 0)$$

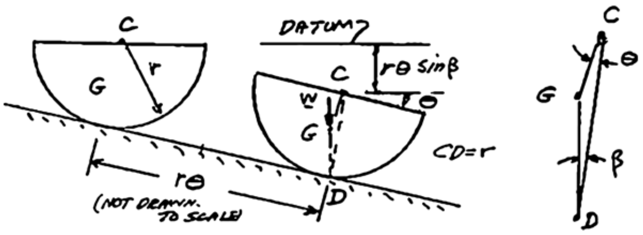
$\theta = 137.8°$, Stable ◀



PROBLEM 10.111

A homogeneous hemisphere of radius r is placed on an incline as shown. Assuming that friction is sufficient to prevent slipping between the hemisphere and the incline, determine the angle θ corresponding to equilibrium when $\beta = 10°$.

SOLUTION



Detail

$$CG = \frac{3}{8}r$$

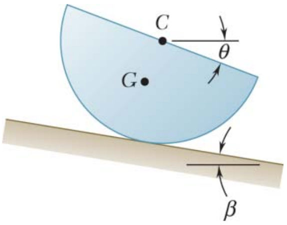
$$V = W(-r\theta\sin\beta - (CG)\cos\theta)$$

$$\frac{dV}{d\theta} = W\left(-r\sin\beta + \frac{3}{8}r\sin\theta\right)$$

Equilibrium: $\dfrac{dV}{d\theta} = 0, \quad -\sin\beta + \dfrac{3}{8}\sin\theta = 0$

$$\sin\beta = \frac{3}{8}\sin\theta \qquad (1)$$

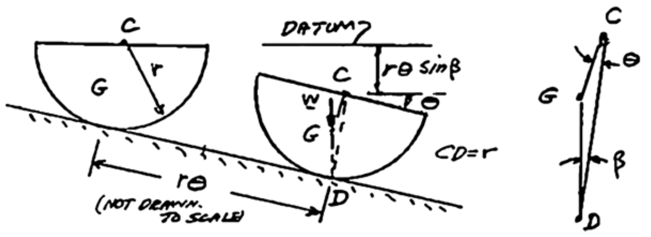
For $\beta = 10°$ $\quad \sin 10° = \dfrac{3}{8}\sin\theta \quad \sin\theta = 0.46306, \quad \theta = 27.6°$ ◀



PROBLEM 10.112

A homogeneous hemisphere of radius r is placed on an incline as shown. Assuming that friction is sufficient to prevent slipping between the hemisphere and the incline, determine (a) the largest angle β for which a position of equilibrium exists, (b) the angle θ corresponding to equilibrium when the angle β is equal to half the value found in part a.

SOLUTION



Detail

$$CG = \frac{3}{8}r$$

$$V = W(-r\theta\sin\beta - (CG)\cos\theta)$$

$$\frac{dV}{d\theta} = W\left(-r\sin\beta + \frac{3}{8}r\sin\theta\right)$$

Equilibrium: $$\frac{dV}{d\theta} = 0, \quad -\sin\beta + \frac{3}{8}\sin\theta = 0$$

$$\sin\beta = \frac{3}{8}\sin\theta \qquad (1)$$

(a) For $\beta_{\max}$, $\theta = 90°$

Eq. (1) $\quad \sin\beta_{\max} = \frac{3}{8}\sin 90°, \quad \sin\beta_{\max} = \frac{3}{8} = 22.02°$ $\qquad \beta_{\max} = 22.0°$ ◀

(b) When $\beta = \frac{1}{2}\beta_{\max} = 11.01°$

Eq. (1) $\quad \sin 11.01° = \frac{3}{8}\sin\theta; \quad \sin\theta = 0.5093$ $\qquad \theta = 30.6°$ ◀

Note: We can also use ΔCGD and law of sines to derive Eq. (1).

$$\frac{\sin\beta}{CG} = \frac{\sin\theta}{CD}; \quad \sin\beta = \frac{CG}{CD}\sin\theta; \quad \sin\beta = \frac{3}{8}\sin\theta$$

Instructor's and Solutions Manual
to accompany

Vector Mechanics for Engineers, Dynamics

Eleventh Edition

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University of Connecticut

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TO THE INSTRUCTOR

As indicated in its preface, *Vector Mechanics for Engineers: Dynamics* is designed for a first course in dynamics. New concepts have, therefore, been presented in simple terms and every step has been explained in detail. However, because of the large number of optional sections that have been included, this text can also be used to teach a course that will challenge the more advanced student.

The text has been divided into units, each corresponding to a well-defined topic and consisting of one or several theory sections, one or several Sample Problems, a section entitled *Solving Problems on Your Own*, and a large number of problems to be assigned. To assist instructors in making up schedules of assignments that will best fit their classes, the various topics covered in the text have been listed in Table I and a suggested number of periods to be spent on each topic has been indicated. Both a minimum and a maximum number of periods have been suggested, and the topics which form the standard basic course in dynamics have been separated from those that are optional. The total number of periods required to teach the basic material varies from 27 to 48, while covering the entire text would require from 40 to 67 periods. In most instances, of course, the instructor will want to include some, but not all, of the additional material presented in the text. If allowance is made for the time spent for review and exams, it is seen that this text is equally suitable for teaching the standard basic dynamics course in 40 to 45 periods and for teaching a more complete dynamics course to advanced students. In addition, it should be noted that *Statics* and *Dynamics* can be used together to teach a combined 4- or 5-credit-hour course covering all the essential topics in dynamics as well as those sections of statics that are prerequisites to the study of dynamics.

The problems have been grouped according to the portions of material they illustrate and have been arranged in order of increasing difficulty, with problems requiring special attention indicated by asterisks. We note that, in most cases, problems have been arranged in groups of six or more, all problems of the same group being closely related. This means that instructors will easily find additional problems to amplify a particular point they may have brought up in discussing a problem assigned for homework.

Educational research has shown that students can often choose appropriate equations and solve algorithmic problems without having a strong conceptual understanding of mechanics principles. To help assess and develop student conceptual understanding, we have included Concept Questions, which are multiple choice problems that require few, if any, calculations. Each possible incorrect answer typically represents a common misconception (e.g., students often think that a vehicle moving in a curved path at constant speed has zero acceleration). Students are encouraged to solve these problems using the principles and techniques discussed in the text and to use these principles to help them develop their intuition. Mastery and discussion of these Concept Questions will deepen students' conceptual understanding and help them to solve dynamics problems.

Drawing diagrams correctly is a critical step in solving kinetics problems in dynamics. A new type of problem has been added to the text to emphasize the importance of drawing these diagrams. In Chapters 12 and 16 the Free-Body Practice Problems require students to draw a free-body diagram (FBD) showing the applied forces and an equivalent diagram called a “kinetic diagram” (KD) showing $m\mathbf{a}$ (or its components) and $\bar{I}\alpha$. These diagrams provide students with a pictorial representation of Newton’s second law and are critical in helping students to correctly solve kinetic problems. In Chapters 13 and 17 the Impulse-Momentum Practice problems require students to draw diagrams showing the momenta of the bodies before impact, the impulses exerted on the body during impact, and the final momenta of the bodies.

A group of problems designed to be solved with computational software can be found through Connect. Also on Connect are solutions for these problems, including analyses of the problems and problem solutions and output for the most widely used computational programs.

To assist in the preparation of homework assignments, Table II provides a brief description of all groups of problems and a classification of the problems in each group according to the units used. It should also be noted that the answers to all problems are given at the end of the text, except for those with a number in italic. Because of the large number of problems available in both systems of units, the instructor has the choice of assigning problems using SI units and problems using U.S. customary units in whatever proportion is found to be desirable. To illustrate this point, sample lesson schedules are shown in Tables III and IV, together with various alternative lists of assigned problems. Half of the problems in each of the six lists suggested in Table III are stated in SI units and half in U.S. customary units. On the other hand, 75% of the problems in the four lists suggested in Table IV are stated in SI units and 25% in U.S. customary units.

Because the approach used in this text differs in a number of respects from the approach used in other books, instructors will be well advised to read the preface to *Vector Mechanics for Engineers,* in which the authors have outlined their general philosophy. In addition, instructors will find in the following pages a description, chapter by chapter, of the more significant features of this text. It is hoped that this material will help instructors in organizing their courses to best fit the needs of their students.

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Brian P. Self

DESCRIPTION OF THE MATERIAL CONTAINED IN *VECTOR MECHANICS FOR ENGINEERS: DYNAMICS*, Tenth Edition

Chapter 11
Kinematics of Particles

In this chapter, the motion of bodies is studied without regard to their size; all bodies are assumed to reduce to single particles. The analysis of the effect of the size of a body and the study of the relative motion of the various particles forming a given body are postponed until Chap. 15. In order to present the simpler topics first, Chap. 11 has been divided into two parts: rectilinear motion of particles, and curvilinear motion of particles.

In Sec. 11.1A, position, velocity, and acceleration are defined for a particle in rectilinear motion. They are defined as quantities which may be either positive or negative and students should be warned not to confuse position coordinate and distance traveled, or velocity and speed. The significance of positive and negative acceleration should be stressed. Negative acceleration may indicate a loss in speed in the positive direction or a gain in speed in the negative direction.

As they begin the study of dynamics, many students are under the belief that the motion of a particle must be either uniform or uniformly accelerated. To destroy this misconception, the motion of a particle is first described under very general conditions, assuming a variable acceleration which may depend upon the time, the position, or the velocity of the particle (Sec. 11.1B). To facilitate the handling of the initial conditions, definite integrals, rather than indefinite integrals, are used in the integration of the equations of motion.

The special equations related to uniform and uniformly accelerated motion are derived in Secs. 11.2A and 11.2B. Before using these equations, students should be warned to check carefully that the motion under consideration is actually a uniform or a uniformly accelerated motion.

Two important concepts are introduced in Sec. 11.2C: (1) the concept of relative motion, which will be developed further in Secs. 11.4D, 15.2B and (2) the concept of dependent motions and degrees of freedom.

The first part of Chap. 11 ends with the presentation of several graphical methods of solution of rectilinear-motion problems (Sec. 11.3). This material is optional and may be omitted. Several problems in which the data are given in graphical form have been included.

The second part of the chapter begins with the introduction of the vectors defining the position, velocity and acceleration of a particle in curvilinear motion. The derivative of a vector function is defined and introduced at this point (Sec. 11.4B). The motion of a particle is first studied in terms of rectangular components (Sec. 11.4C); it is shown that in many cases (for example, projectiles) the study of curvilinear motion can be reduced to that of two independent rectilinear motions. The concept of fixed and moving frames of reference is introduced in Sec. 11.4D and is immediately used to treat the relative motion of particles.

The use of tangential and normal components, and of radial and transverse components is discussed in Secs. 11.5. Each system of components is first introduced in two dimensions and then extended to include three-dimensional space.

Chapter 12
Kinetics of Particles: Newton's Second Law

As indicated earlier, this chapter and the following two are concerned only with the kinetics of particles and systems of particles. They neglect the effect of the size of the bodies considered and ignore the rotation of the bodies about their mass center. The effect of size will be taken into account in Chaps. 16 through 18, which deal with the kinetics of rigid bodies.

Sec. 12.1 presents Newton's second law of motion and introduces the concept of a newtonian frame of reference. In Sec. 12.1B the concept of linear momentum of a particle is introduced, and Newton's second law is expressed in its alternative form, which states that the resultant of the forces acting on a particle is equal to the rate of change of the linear momentum of the particle. Section 12.1C reviews the two systems of units used in this text, the SI metric units and the U.S. customary units which were previously discussed in Sec. 1.3. This section also emphasizes the difference between an absolute and a gravitational system of units.

A number of problems with two degrees of freedom have been included (Problems 12.28 through 12.35), some of which require a careful analysis of the accelerations involved (see Sample Problem 12.5).

Section 12.1D applies Newton's second law to the study of the motion of a particle in terms of rectangular components, tangential and normal components, and radial and transverse components. The term *inertia vector* is used in preference to inertia force or reversed effective force to avoid any possible confusion with actual forces.

In Sec. 12.2 the concept of angular momentum of a particle is introduced, and Newton's second law is used to show that the sum of the moments about a point *O* of the forces acting on a particle is equal to the rate of change of the angular momentum of the particle about *O*. Section 12.2B considers the particular case of the motion of a particle under a central force. The early introduction of the concept of angular momentum greatly facilitates the discussion of this motion. Section 12.2C presents Newton's law of gravitation and its application to the study of the motion of earth satellites.

Section 12.3 is optional. Section 12.3A derives the differential equation of the trajectory of a particle under a central force, while Sec. 12.3B discusses the trajectories of satellites and other space vehicles under the gravitational attraction of the earth. While the general equation of orbital motion is derived (Eq. 12.36), its application is restricted to launchings in which the velocity at burnout is parallel to the surface of the earth. (Oblique launchings are considered in Sec. 13.2D.) The periodic time is found directly from the fundamental definition of areal velocity rather than by formulas requiring a previous knowledge of the properties of conic sections.

Instructors may omit Section 12.3 and yet assign a number of interesting space mechanics problems to their students after they have reached Sec. 13.2D.

Chapter 13
Kinetics of Particles: Energy and Momentum Methods

After a brief introduction designed to give to students some motivation for the study of this chapter, the concept of work of a force is introduced in Sec. 13.1A. The term work is always used in connection with a well-defined force. Three examples considered are the work of a weight (i.e., the work of the force exerted by the earth on a given body), the work of the force exerted by a spring on a given body, and the work of a gravitational force. Confusing statements such as the work done on a body, or the work done on a spring are avoided.

The concept of kinetic energy is introduced in Sec. 13.1B and the principle of work and energy is derived by integration of Newton's equation of motion. In applying the principle of work and energy, students should be encouraged to draw separate sketches representing the initial and final positions of the body (Sec. 13.1C). Section 13.1D introduces the concepts of power and efficiency.

Section 13.2 is devoted to the concepts of conservative forces and potential energy and to the principle of conservation of energy. Potential energy should always be associated with a given conservative force acting on a body. By avoiding statements such as "the energy contained in a spring" a clearer presentation of the subject is obtained, which will not conflict with the more advanced concepts that students may encounter in later courses. In applying the principle of conservation of energy, students should again be encouraged to draw separate sketches representing the initial and final positions of the body considered.

In Sec. 13.2D, the principles of conservation of energy and conservation of angular momentum are applied jointly to the solution of problems involving conservative central forces. A large number of problems of this type, dealing with the motion of satellites and other space vehicles, are available for homework assignment. As noted earlier, these problems (except the last two, Probs. 13.117 and 13.118) can be solved even if Section 12.3 has been omitted.

The second part of Chap. 13 is devoted to the principle of impulse and momentum and to its application to the study of the motion of a particle. Section 13.3 introduces the concept of linear impulse and derives the principle of impulse and momentum from Newton's second law. The instructor should emphasize the fact that impulses and momenta are vector quantities. Students should be encouraged to draw three separate sketches when applying the principle of impulse and momentum and to show clearly the vectors representing the initial momentum, the impulses, and the final momentum. It is only after the concept of impulsive force has been presented that students will begin to appreciate the effectiveness of the method of impulse and momentum (Sec. 13.3B).

Direct central impact and oblique central impact are studied in Section 13.4. Note that the coefficient of restitution is defined as the ratio of the impulses during the period of restitution and the period of deformation. This more basic approach will make it possible in Chapter 17 to extend the results obtained here for central impact to the case of eccentric impact. Emphasis should be placed on the fact that, except for perfectly elastic impact, energy is not conserved. Note that the discussion of oblique central impact in Sec. 13.4B has been expanded to cover the case when one or both of the colliding bodies are constrained in their motions.

Section 13.4C shows how to select from the three fundamental methods studied in Chaps. 12 and 13 the one best suited for the solution of a given problem. It also shows how several methods can be combined to solve a given problem. Note that problems have been included (Probs. 13.177 through 13.189) which require the use of both the method of energy and the method of momentum in their solutions.

Chapter 14
Systems of Particles

Chapter 14 is devoted to the study of the motion of systems of particles. Sections 14.1A and 14.1B derive the fundamental equations (14.10) and (14.11) relating, respectively, the resultant and the moment resultant of the external forces to the rate of change of the linear and angular momentum of a system of particles. Sections 14.1C and 14.1D are devoted, respectively, to the motion of the mass center of a system and to the motion of the system about its mass center. Section 14.1E discusses the conditions under which the linear momentum and the angular momentum of a system of particles are conserved. Sections 14.2A and 14.2B deal with the application of the work-energy principle to a system of particles, and in Sec. 14.2C the application of the impulse-momentum principle is discussed.

A number of challenging problems have been provided to illustrate the application of the principles discussed in this chapter. The first group of problems (Probs. 14.1 through 14.30) deal chiefly with the conservation of the linear momentum of a system of particles and with the motion of the mass center of the system, while the second group of problems (Probs. 14.31 through 14.58) involve the combined use of the principles of conservation of energy, linear momentum, and angular momentum. However, the main purpose of these sections is to lay the proper foundation for the later study of the kinetics of rigid bodies (Chaps. 16 through 18). Depending upon the preparation and interest of the students, a greater or lesser emphasis may be placed on this part of the course. It is essential, however, that the significance of Eqs. (14.16) and (14.23) be pointed out to students, in view of the role played by these equations in the study of the motion of rigid bodies.

The instructor should note the distinction made in Sec. 14.1A between *equivalent* systems of forces (i.e., systems of forces which have the same effect) and *equipollent* systems of forces (i.e., systems of forces which have the same resultant and the same moment resultant). The equivalence of two systems of forces has been indicated in diagrams by *red* equals signs, and their equipollence by *blue* equals signs.

Section 14.3 is optional. It is devoted to the study of variable systems of particles, with applications to the determination of the forces exerted by deflected streams and the thrust of propellers, jet engines, and rockets. Since Newton's second law $\mathbf{F} = m\mathbf{a}$ was stated for a particle with a constant mass and does not apply, in general, to a system with a variable mass, the derivations given in Sec. 14.3A for a steady stream of particles and in Sec. 14.3B for a system gaining or losing mass are based on the consideration of an auxiliary system consisting of unchanging particles. This approach should give students a basic understanding of the subject and lead them to more advanced courses in mechanics of fluids.

Chapter 15
Kinematics of Rigid Bodies

With this chapter we start the study of the dynamics of rigid bodies. After an introduction in which the fundamental types of plane motion are defined (Sec. 15.1A), the relations defining the velocity and the acceleration of any given particle of a rigid body are established for two particular cases: translation (Sec. 15.1A) and rotation about a fixed axis (Secs. 15.1B and 15.1C).

In Sec. 15.2 it is shown that the most general plane motion can always be considered as the sum of a translation and a rotation. This property is established by considering the relative motion of two particles of the rigid body and is immediately applied in Sec. 15.2B to the determination of velocities.

Section 15.3 introduces the concept of the instantaneous center of rotation. The instructor should stress the fact that, while the instantaneous-center method simplifies the solution of many problems involving velocities, it cannot be used to determine accelerations.

In Sec. 15.4 the concept of relative motion is used again, this time to determine accelerations in plane motion. Students should be warned against any unwarranted assumptions concerning the direction of unknown accelerations. Section 15.4B is optional. It presents an analytical method for the determination of velocities and accelerations based on the use of a parameter.

Section 15.5A discusses the rate of change of a vector with respect to a rotating frame, and Sec. 15.5B applies the results obtained to the determination of Coriolis acceleration in plane motion. It should be kept in mind, however, that Coriolis acceleration does not depend upon the existence of an actual slab or rod on which the particle moves. It can appear whenever *rotating* axes are used. There lies the fundamental difference between the approach of Sec. 15.5 and that of Secs. 11.4D and 15.4A where moving axes *of fixed direction* were used.

The remaining sections of Chap. 15 are devoted to the kinematics of rigid bodies in three dimensions and are optional. However, they should be included in a course covering the kinetics of rigid bodies in three dimensions (Chap. 18) and should be taught either at this point or immediately before Chap. 18.

In Sec. 15.6A the motion of a rigid body about a fixed point is presented. Students should note that while the finite rotations of a rigid body have magnitude and direction, the rotations are not vectors; on the other hand, both angular velocity and angular acceleration are vector quantities. In Sec. 15.6B, the general motion of a rigid body is analyzed. It should be emphasized that in this section the moving frame of reference is of fixed orientation and does not rotate.

Section 15.7 considers the motion of particles and rigid bodies with respect to rotating frames of reference and extend the concept of Coriolis acceleration to three-dimensional motion.

Chapter 16
Plane Motion of Rigid Bodies: Forces and Accelerations

This chapter is devoted to the plane motion of rigid bodies which consist of plane slabs or which are symmetrical with respect to the reference plane. Cases involving the plane motion of nonsymmetrical bodies and, more generally, the motion of rigid bodies in three dimensions are considered in Chap. 18. If the determination of mass moments of inertia has not been covered in the previous statics course, the instructor should include material from Appendix B at this point.

In Sec. 16.1 the fundamental relations derived in Chap. 14 for a system of particles are used to show that the external forces acting on a rigid body are equipollent to the vector $m\overline{\mathbf{a}}$ attached at the mass center G of the body and the couple of moment $\dot{\mathbf{H}}_G$. This result, which is illustrated in Fig. 16.3, is valid in the most general case of motion of a rigid body (three-dimensional as well as plane motion).

It is shown in Sec. 16.1B that in the case of the plane motion of a slab or symmetrical body, the angular momentum $\mathbf{H}_G$ reduces to $\overline{I}\omega$ and its rate of change to $\overline{I}\alpha$. It is shown that the external forces acting on a rigid body are actually *equivalent* to the inertial terms represented by the vector $m\overline{\mathbf{a}}$ and the couple $\overline{I}\alpha$. As noted in Sec. 16.5, this result is obtained independently of the principle of transmissibility and can be used to *derive* this principle from the other axioms of mechanics.

At this point students will have reached the climax of their study of rigid-body motion in two dimensions. Indeed, they can solve any problem by drawing two sketches—one showing the external forces, the other the vector $m\overline{\mathbf{a}}$ and the couple $\overline{I}\alpha$—and then expressing that the two systems of vectors shown are equivalent.

The various types of plane-motion problems have been grouped according to their kinematic characteristics. Translation, centroidal rotation, and plane motion consisting of a translation and an unrelated centroidal rotation are considered first, since they are the simplest ones to analyze. They are followed by plane motions with various kinematic constraints: non-centroidal rotation, rolling motion, and other types of plane motion. Problems involving systems of rigid bodies have been included at the end of this chapter, with either one degree of freedom (Probs. 16.126 through 16.134) or two degrees of freedom (Probs. 16.135 through 16.141). The instructor should stress the fact that, in spite of the different kinematic characteristics of these various motions, the approach to the kinetics of the motion is consistently the same: all problems are solved by drawing two sketches—one showing the external forces, the other the vector $m\overline{\mathbf{a}}$ and the couple $\overline{I}\alpha$—and then expressing that the two systems of vectors are equivalent.

Since the approach used in this text differs from others in the emphasis placed on the direct application of Newton’s second Law, rather than on specialized formulas, it might be appropriate at this point to summarize the advantages derived from this approach.

(1) A single method is used, which applies to all cases of plane motion, regardless of their kinematic characteristics, and which can be used safely under any conditions. This is in contrast to using the equation $\Sigma M = I\alpha$ which is limited in its applications.

(2) By stressing the use of the free-body diagram and kinetic diagram, this method provides a better understanding of the kinetics of the motion. There will be little danger, for example, in the solution of a problem of non-centroidal rotation, that students will forget the effect of the acceleration of the mass center on the reaction at the fixed point, a mistake which occurs frequently when the specialized formula $\Sigma M_O = I_O\alpha$ is used.

(3) The method used divides the solution of a problem into two main parts, one in which the kinematic and kinetic characteristics of the problem are considered (separately if necessary), and the other in which the methods of statics are used. In this way the techniques of each separate field can be used most efficiently. For example, moment equations can be written to eliminate unwanted reactions, just as it was done in statics; this can be done independently of the kinetic characteristics of the problem.

(4) By resolving every plane motion (even a non-centroidal rotation) into a translation and a centroidal rotation, a unified approach is obtained, which will also be used in Chap. 17 with the method of work and energy and with the method of impulse and momentum, and which will be extended in Chap. 18 to the study of the three-dimensional motion of a rigid body. This approach is a basic one, which can be applied effectively throughout the study of mechanics in advanced courses as well as in elementary ones.

Chapter 17
Plane Motion of Rigid Bodies: Energy and Momentum Methods

The first portion of the chapter extends the method of work and energy, already used in Chap. 13, to the study of the plane motion of rigid bodies. The expressions for the work of a couple and for the kinetic energy of a rigid body are derived in Secs. 17.1B and 17.1C. Using the results obtained in Sec. 14.2A, the kinetic energy of a rigid body is separated into a translational part and a rotational part (about the mass center). The authors believe that, while it may lead to slightly longer solutions, this method is more fundamental and should be used in preference to special formulas. Indeed, it follows the basic idea of resolving every plane motion into a translation and a centroidal rotation.

It is shown in Sec. 17.1D that the method of work and energy is especially effective in the case of systems of rigid bodies connected by pins, inextensible cords, and meshed gears. In Sec. 17.1E the principle of conservation of energy is used to analyze the plane motion of rigid bodies.

In the second part of Chap. 17, the method of impulse and momentum is extended to the study of the motion of rigid bodies. The approach used is different from that of most elementary textbooks. Ready-to-use formulas are avoided; instead, students are taught to express the general principle of impulse and momentum by means of impulse-momentum diagrams and to write the equations most appropriate to the solution of the problem considered.

The results obtained in Sec. 14.2C for a system of particles are directly applicable to the system of particles forming a rigid body and can be used to analyze the plane motion of rigid bodies. It is shown in Sec. 17.2A that the momenta of the various particles forming a rigid body reduce to a vector $m\overline{\mathbf{v}}$ and a couple $\overline{I}\omega$ in the most general case of plane motion.

While the principle of conservation of angular momentum is discussed in Sec. 17.2C because of its physical and historical significance, it is not actually used in the solution of problems. To solve any problem, regardless of the type of motion, and whether it involves constant forces of finite magnitude applied for a finite time or impulsive forces applied for a very short time interval, students are told to draw three separate sketches showing, respectively, the initial momenta, the impulses of the external forces, and the final momenta. The momenta of a rigid body are represented in the most general case by a *momentum vector* $m\overline{\mathbf{v}}$ attached at the mass center and a *momentum couple* $\overline{I}\omega$. If students then consider the components of the vectors involved, they obtain relations between linear impulses and linear momenta. If they consider the moments of the same vectors, they obtain angular impulses and angular momenta. If, by equating moments about a point such as a pivot, they obtain an equation which does not involve any of the external forces, they will have automatically established conservation of angular momentum about that point.

The advantages derived from this approach can be summarized as follows:

(1) Students learn only one method of solution, a method based directly on a fundamental principle and which can be used safely under any conditions. This is in contrast with the equation $\sum M_0 \Delta t = I_0(\omega_2 - \omega_1)$, which is limited in its applications.

(2) The method stresses the use of impulse and momentum diagrams and thus provides a better understanding of the kinetics of the motion. It is unlikely, for example, that students will forget an impulsive reaction at a fixed support.

(3) Students use the basic tools they learned in statics: reduction of a system of vectors to a vector and a couple and equations relating the components or the moments of these vectors.

(4) Again, the same unified approach is used: every plane motion is resolved into a translation and a centroidal rotation. In Chap. 18 this approach will be extended to the solution of problems involving the three-dimensional motion of rigid bodies.

Some teachers may fear that the inclusion of momentum vectors and momentum couples in the same diagrams may lead to confusion. This will not be the case, however, if students are instructed to write separate equations involving either components or moments, as they did in statics. The first equations will contain linear impulses and linear momenta expressed in N•s or lb•s, and the latter angular impulses and angular momenta expressed in N•m•s or lb•ft•s.

Section 17.3 is devoted to the eccentric impact of two rigid bodies, a topic seldom included in an elementary text. No special difficulty will be encountered, however, if separate sketches are used as indicated above.

Chapter 18
Kinetics of Rigid Bodies in Three Dimensions

In this chapter the restrictions imposed in preceding chapters (e.g., plane motion, symmetrical bodies) are lifted and students proceed to the analysis of more general (and more difficult) problems, such as the rotation of nonsymmetrical bodies about fixed axes and the motion of gyroscopes.

In Sec. 18.1, the general result obtained in Sec. 16.1 is recalled, namely, that the external forces acting on a rigid body are equipollent to the vector $m\overline{\mathbf{a}}$ attached at the mass center G of the body and the couple of moment $\dot{\mathbf{H}}_G$. It is also pointed out that the main feature of the impulse-momentum method, namely, the reduction of the momenta of the particles of a rigid body to a linear momentum vector $m\overline{\mathbf{v}}$ attached at G and an angular momentum, remains valid and that the work-energy principle and the principle of conservation of energy still apply in the case of the motion of a rigid body in three dimensions. The difficulties encountered in the study of the three dimensional motion of a rigid body are related to the determination of the angular momentum $\mathbf{H}_G$, of its rate of change $\dot{\mathbf{H}}_G$, and of the kinetic energy of the body.

The determination of the angular momentum $\mathbf{H}_G$ of a rigid body from its angular velocity ω is discussed in Sec. 18.1A. Since this requires the use of mass products of inertia, as well as the use of mass moments of inertia, the instructor should cover the material from Appendix B (or from the second part of Chap. 9) if this material has not been included in the previous statics course.

Section 18.1B is devoted to the application of the impulse-momentum principle to the three-dimensional motion of a rigid body, and Sec. 18.1C to the determination of its kinetic energy.

In Secs. 18.2A and 18.2B, the rate of change of the angular momentum $\mathbf{H}_G$ is computed and the equations of motion for a rigid body in three dimensions are derived. Newton's second law is extended to the case of three-dimensional motion by showing that the external forces are actually *equivalent* to the effective forces represented by the vector $m\overline{\mathbf{a}}$ and the couple $\dot{\mathbf{H}}_G$. Sections 18.2C and 18.2D are devoted to the particular cases of the motion of a rigid body about a fixed point and the rotation of a rigid body about a fixed axis, with applications to the balancing of rotating shafts.

While Euler's equations of motion have been derived, it should be noted that the more fundamental vector relations represented by Equations (18.22), (18.23), and (18.28) are used in the solution of problems.

The remaining portion of this chapter (Secs. 18.2D through 18.3C) is designed for advanced students and, in general, should be omitted for ordinary classes. In Secs. 18.3A and 18.3B the motion of a gyroscope is considered. At this point Eulerian angles are introduced. It should be carefully noted that the rotating system of axes $Oxyz$ is attached to the inner gimbal; these axes are principal axes of inertia and they follow the precession and nutation of the gyroscope; they do not, however, spin with the gyroscope. The special case of steady precession is considered in Sec. 18.3B. Several problems dealing with the steady precession of a top and other axisymmetrical bodies have been included and, in one of them (Prob. 18.110), it is shown that the formula usually given in introductory texts is only approximate.

The motion of an axisymmetrical body under no force (Sec. 18.3C) introduces students to one of the most interesting aspects of classical dynamics—an aspect which has gained widespread attention in recent years due to the interest in space vehicles and artificial satellites. In this connection, it should be pointed out that Poinsot's theory of the motion of a nonsymmetrical body under no force may be covered by assigning Probs. 18.143 through 18.145. An additional problem (Prob. 18.146) relates to the stability of the rotation of a nonsymmetrical body about a principal axis.

Chapter 19
Mechanical Vibrations

This chapter provides an introduction to the study of mechanical vibrations. While only one-degree-of-freedom systems are included, all the basic principles are presented. The various topics covered are as follows:

(a) Free, undamped vibrations of a particle (Sec. 19.1A). The differential equation characterizing simple harmonic motion is derived and all basic terms, such as period, natural frequency, and amplitude, are defined. Both the analytical and the geometrical methods of solution are described. It is shown in Sec. 19.1B that the motion of a simple pendulum can be approximated by a simple harmonic motion. Section 19.1C, which is optional, shows how an exact solution can be obtained for the period of oscillations of a simple pendulum.

(b) Free, undamped vibrations of a rigid body. The principle of equivalence of the systems of applied forces and inertial terms is first used to determine the natural frequency and the period of oscillations of a rigid body (Sec. 19.2). Note that the same positive sense is assumed for the angular acceleration and displacement The principle of conservation of energy is then used to solve the same type of problems (Sec. 19.3).

(*c*) *Forced, undamped vibrations of a particle* (Sec. 19.4). This section introduces students to the concepts of forced frequency, transient and steady-state vibrations, and resonance. While all students will be able to understand this section, those with a knowledge of elementary differential equations will derive a greater benefit from it since it provides a direct application of the solution of linear nonhomogeneous equations with constant coefficients.

(d) Free, damped vibrations of a particle (Sec. 19.5A), and

(e) Forced, damped vibrations of a particle (Sec. 19.5B). These two sections take into account the effect of friction and thus provide a more rigorous analysis of the vibrations of a particle. They are not recommended, however, to students who do not possess a basic knowledge of elementary differential equations.

In Sec. 19.5C the electrical analogue for a vibrating mechanical system is discussed; this section is optional and should not be assigned unless Secs. 19.5A and 19.5B have been covered.

TABLE I. LIST OF TOPICS COVERED IN *VECTOR MECHANICS FOR ENGINEERS: DYNAMICS*

		Suggested Number of Periods	
Sections	**Topics**	**Basic Course**	**Additional Topics**
11. KINEMATICS OF PARTICLES			
11.1	Rectilinear Motion	1	
11.2	Special Cases and Relative Motion	1	
*11.3	Graphical Methods		1
11.4	Curvilinear Motion (Rectangular Components)	1-2	
11.5	Tangential and Normal Components	0.5-1	
11.5B	Radial and Transverse Components	0.5-1	
12. KINETICS OF PARTICLES: NEWTON'S SECOND LAW			
12.1	Newton’s Second Law and Linear Momentum		1-2
12.2	Angular Momentum and Orbital Mechanics	1-2	
*12.3	Applications of Central-Force Motion		1-2
13. KINETICS OF PARTICLES: ENERGY AND MOMENTUM METHODS			
13.1	Work and Energy	1-2	
13.2	Conservation of Energy; Central Forces	1-2	
13.3	Impulse and Momentum; Impulsive Motion	1-2	
13.4	Impact; Mixed Problems	1-2	
14. SYSTEMS OF PARTICLES			
14.1	Motion of Systems of Particles	0.5-1	
14.2	Energy and Momentum Methods for Systems of Particles	0.5-1	
*14.3	Variable Systems of Particles		1-2
15. KINEMATICS OF RIGID BODIES			
15.1	Translation; Rotation About a Fixed Axis	1-1.5	
15.2	Velocities in Plane Motion	1-1.5	
15.3	Instantaneous Center	1	
15.4	Accelerations in Plane Motion	1-2	0.5
15.5	Motion With Respect to a Rotating Frame	1-2	
*15.6	Motion of a Rigid Body in Space		1-2
*15.7	Motion Relative to a Moving Reference Frame		1-2
16. PLANE MOTION OF RIGID BODIES: FORCES AND ACCELERATIONS			
16.1	Kinetics of a Rigid Body in Plane Motion	2-3	
16.2	Constrained Plane Motion	2-4	
17. PLANE MOTION OF RIGID BODIES: ENERGY AND MOMENTUM METHODS			
17.1	Energy Methods for a Rigid Body	1-2	
17.2	Momentum Methods for a Rigid Body	1-2	
17.3	Eccentric Impact	1-2	
18. KINETICS OF RIGID BODIES IN THREE DIMENSIONS			
*18.1	Energy and Momentum in Three Dimensions		2
*18.2	Motion of a Rigid Body in Three Dimensions		1-2
*18.3	Motion of a Gyroscope		2
19. MECHANICAL VIBRATIONS			
19.1	Vibrations of Particles without Damping		1-2
*19.1C	Exact Solution of a Simple Pendulum		0.5
19.2	Free Vibrations of Rigid Bodies	1-2	
19.3	Energy Methods		1-1.5
19.4	Forced Vibrations	1-1.5	
*19.5	Damped Vibrations		1-2
*19.5C	Electrical Analogues		1
	Total Number of Periods	27-48	13-19

TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
		CHAPTER 11: KINEMATICS OF PARTICLES
		RECTILINEAR MOTION OF PARTICLES
		Analyze the motion of a particle given
11.3,4	11.1,2	position = *f(t)*
11.5,7		
	11.*6,8*	also find total distance traveled
11.11,12	11.9,10	acceleration = *f(t)*
11.*13,14*		
11.15,18	11.16,17	acceleration = *f(x)*
11.*19,20*		
11.21,24	11.22,23	acceleration = *f(v)*
11.25,26		
11.27	11.28	velocity = *f(x)*
	11.*29,30*	acceleration $\propto 1/r^2$
11.31	11.32	Simple harmonic motion
		Uniformly accelerated motion
11.33,34	11.35,36	motion of one particle
11.*37,38*		
11.39,40	11.41,42	motion of two particles
11.44,*45*	11.43,*46*	
		Relative motion of particles
11.47,49	11.48,50	systems with one degree of freedom
11.51,52	11.*53,54*	
11.57,58	11.55,56	systems with two degrees of freedom
11.*59,60*		
		Graphical methods
	11.*61,62*	analyze motion using a given motion curve
11.63,64		
11.66	11.65	motion of one particle
11.69,70	11.*67,68*	
11.71,75	11.72,73	motion of two particles
11.*76*	11.74	
11.78	11.*77*,79 80	problems involving rate of change of acceleration
11.*81,82*	11.83	problems solved by numerical integration
	11.84	problem solved using a *v*-*x* curve
11.*87,88*	11.85,*86*	problems solved using the moment-area method
		CURVILINEAR MOTION OF PARTICLES
		Analyze the motion of a particle given
11.89,90	11.91,92	*x* and *y* as functions of *t*
11.*93*	11.*94*	the position vector: two dimensions
11.95	*11.96	the position vector: three dimensions

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Answers are not given to problems with the numbers set in italic type.

TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
		Projectiles
11.99,99	11.97,100	with horizontal initial velocity
11.102,103	11.*101,104*	with initial velocity of a given magnitude and direction
11.106,108	11.105,107	for a given direction, find magnitude of initial velocity
11.*109,110*		
11.111,114	11.112,113	for a given magnitude, find direction of initial velocity
11.*115*,*116		(Solution of a quadratic equation is required.)
		Relative motion of particles with
11.118,120	11.117,119	constant velocities
11.*121,122*		
	11.123, 124	given velocities and accelerations
11.125,126		given relative acceleration
	11.127,128	constant accelerations
11.129,131	11.*130,132*	two given observations of relative motion
		Normal and tangential components
11. 133,134,135	11. 136	zero tangential acceleration
11.138,139	11.137,141	constant tangential acceleration
11.*140,142*		
11.144,145	11.143,146	find radii of curvature of trajectory
11.147,149	11.*148,150*	
*11.151	*11.152	find radius of curvature of a curve in three dimensions
11.153,154	11.156,157	problems involving a satellite in a circular orbit
11.158,159	11.*157,160*	
		Radial and transverse components
11.163,164	11.161,162	given r and θ as functions of time
11.165,166		
11.*167,168*	11.169,*170	find relations among v, a, $\dot{\theta}$, and $\ddot{\theta}$
11.171	11.172	find approximate velocity from two observations of r and θ
		for motion along a spiral
11.*174*	11.*173*	find velocity
11.175	11.176	find acceleration
11.*177*	11.*179*	motion of a particle in three dimensions: cylindrical coordinates
	11.178	motion of a particle along a hyperbola or a circle
*11.180	*11.181	find direction of the binormal
11.182,183	11.*186*,187	Review problems
11.*184*,185	11.189,190	
11.188,*192*	11.191,*193*	
		CHAPTER 12: KINETICS OF PARTICLES
		NEWTON'S SECOND LAW
12.3,4	12.1,2	Definition of mass and weight
		Rectilinear motion
12.7,8	12.5,6	single particles
12.*9,10*		

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
12.11,12	12.*13*,*14*	several connected bodies
12. 17,18	12. 15,16	
12.19		
12.20,*21*		two bodies with impending relative motion
12.*22*,23		
12. 24,25	12.*26*,27	problems with variable acceleration
12.28,30	12.29,31	systems with two degrees of freedom
12.*32*,*33*		
12.34,35		
		Curvilinear motion: normal and tangential components
12.36,37		circular paths contained in a horizontal plane
12.38,39		
12.40,*41*	*12.42,*43	motion defined by two criteria
12.46,47	12.44,45	circular paths contained in a vertical plane
12.48,49		
12.50,51		
	12.*52*	cars traveling on banked curves
	12,53,*54*	
12.55,56		impending relative motion of a particle with respect to a moving support
12.57,58	12.*60*	
12,*59*	12.62,63	
12.64	12.65	cathode-ray tubes
		Curvilinear motion: radial and transverse components
12.68,69	12.66,67	find acceleration of and/or forces acting on a particle for a given motion
12.71,72	12.*70*,*73*	
12.74,*75*	12.76,*77*	motion under a central force: theoretical problems
12.79,81	12.78,80	motion defined by Newton's Law of Gravitation
12.82,85	12.*83*,*84*	
		Conservation of angular momentum
12.86,87	12.88,89	orbital motion of spacecraft
12.90,*93*	12.91,*92*	mechanical devices
		Trajectory of a particle under a central force
12.96,97	12.94,95	determination of the central force for a given trajectory
12.98,100	12.*99*	analysis of spacecraft orbits
12.101		
12.102,104	12.103,105	analysis of more advanced orbital problems
	12.*106*,107	
12.108	12.109	determination of the time in orbit
12.*110*,*111*		
12.112,113		
12.115,*116*	12.114,*117*	determination of the point of impact of a descending spacecraft
12.119,120	12.118,121	derivation of classical formulas
12.*123*,126	12.*122*,124	Review problems
12.127,128	12.125,130	
12.*129*,*131*	12.131,132	

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

CHAPTER 13: KINETICS OF PARTICLES
ENERGY AND MOMENTUM METHODS

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
13.1,*4*	13.2,*3*	Determine kinetic energy
13.7,8	13.5,6	Method of work and energy applied to problems involving
13.9,*10*		a single body
13.11,12	13.*13*,*14*	
13.15,16	13.17,18	two connected bodies
13.21,22	13.19,20	
13.23		
13.24	13.25	a body which drops out of the system
13.26,27	13.28.29	work of springs
13.*30*,*31*		
13.32,*35*	13.33,34	work of non-linear forces
13.37,38	13.36	work of a gravitational force
13.39,40	13.41,*42*	use of $F = ma$
13.44	13.*43*,45	
13.46,47		Introductory problems involving power and efficiency
13.48,51	13.*49*,*50*	Power in uniform and uniformly accelerated motion
13.52,*53*	13.54	
		Method of conservation of energy applied to problems involving
13.55,*56*	13.58,59	springs in a horizontal plane
13.57		
13.*60*,*61*		
13.64,65	13.62,*63*	springs and the effect of gravity
13.68,69	13.66,*67*	
13.70,71	13.72,*73*	use of $F = ma$
13.76,77	13.74,*75*	
	*13.78	
13.*79,80	13.*81,*82	Theory problems on conservative forces
13.*_83_,*_84_		
13.85,86	13.87,88	Space mechanics problems involving gravitational potential energy
13.89,90	13.*91*,*92*	
13.93,94	13.95,96	Conservation of energy and conservation of angular momentum
13.*98*,*99*	13.97	
		Applications to space mechanics
13.100,103	13.101,102	velocity perpendicular to the radius vector
13.*104*,*105*		
13.106,107	13.110,*111	velocity oblique to the radius vector
13.108,109	13.*112*,*113*	
13.*114,115	13.116,*117	derivations and theoretical problems
13.*118		
		Method of impulse and momentum applied to problems involving
13.119,120	13.121	a single body
13.*122*,123	13.124,125	
13.126,*128*	13.*127*	

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
13.131,132	13.129,130	two bodies
13.*133*		
13.*135*	13.134	a force defined by a graph
13.136	13.*137*,138	
13.141,142	13.139,140	impulsive motion of a single body
13.*143*,*144*		
13.146,147	13.145,148	impulsive motion of two bodies
	13.149	
13.151,152	13.150	impulsive motion with different supports
	13.*153*,*154*	
13.155,156	13.157,158	Direct central impact
13.*159*,*160*	13,161	
13.*162*,163		
13.166,167	13.164,165	Oblique central impact
13.168		
13.169	13.*170*,*171*	Impact against fixed surfaces
13.172,173		
13.175,177	13.174,176	Problems involving energy and momentum in direct impact
13.179,180	13.*178*,*181*	
13.182,183		
13.185,186	13.184,188	Problems involving energy and momentum in oblique impact
13.*187*,*189*		
13.193,195	13.190,191	Review problems
13.197,198	13.192,194	
13.*199*,200	13.*196*,*201*	

CHAPTER 14: SYSTEMS OF PARTICLES

14.1,2	14.3,4	Conservation of linear momentum (one dimension)
14.7,8	14.*5*,*6*	
14.9,10	14.11,12	Conservation of angular momentum
14.*13*,*14*		
14.15,16	14.19,20	Motion of mass center
14.*17*,*18*		
14.24,25	14.21,22	Conservation of linear momentum (2 or 3 dimensions)
14.26	14.*23*	
14.*27*,28	14.29,30	Derivations and proofs
14.31,34	14.32,33	Conservation of linear momentum and computation of change in kinetic energy
14.*34*,*36*		

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number* **SI Units**	 **U.S. Units**	 **Problem Description**
14.37,38	14.39,40	Conservation of linear momentum and conservation of energy
14.43,44	14._41 ,42_	
14.45,48	14.46,47	Conservation of linear and angular momentum
	14._49,50_	
14.51,52	14.55,56	Conservation of linear momentum, angular momentum and energy
14._53,54_		
14.57,68	14.59,60	Thrust caused by a diverted flow
14.61,62		
14.64,66	14._63,65_	Reactions at supports of vanes or conveyor belts
14.67,68		
14.69,71	14.70,72	Thrust developed by a jet engine, a propeller, or a fan
14._73,75_	14.74	
14.76		
14.77,78		Power and efficiency
14.79	14.80	
14. _81,83_	14._82_,*_84_	Theory problems from fluid mechanics
14.*_85_		
14.86,87	14._88_	Motion of chains
	14.90,91	Motion of a toy car
14.91,94	14._92,93_	Thrust of rockets
14.95,96	14._97,98_	
14.101	14.99,100	
	14.102	
	14.103,104	Efficiency formulas for jet engine and rocket
14.106,107	14._105_,108	Review problems
14._109_,110	14.112,_113_	
14._111_,115	14.114,116	

CHAPTER 15: KINEMATICS OF RIGID BODIES

		Rotation about a fixed axis
15.1,2	15.3,4	solutions using equations
15.5,6	15.7,8	
15.9		
15.10,11	15.12,13	rotation about a skew axis
15._14,15_		
15.17	15.16	motion of the earth
15.18,19		motion in two dimensions
15._20,21_	15.22,23	
	15.24	
15.26,27	15.25	rolling contact with no slipping
15.29,30	15.28,31	linear and angular motion
15._34,35_	15.32,33	disks brought into contact
*15.36	*15.37	special problems

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
		General plane motion: velocities
15.40,41	15.38,39	motion of s single rigid body
15._42_,_43_		
15.44,45		application of vector algebra to motion of a plate
15._46_,47		
15.49,50	15.48,_52_	planetary gear systems
15.51	15.53,_54_	
		motion of rod connected to
15.55,56	15.57,58	a crank and a sliding block
15.61,62	15. _59_,_60_	
15.63,64		two cranks
15.65	15._66_,_67_	
15.68,69	15.70,72	motion involving a rolling wheel
15.71		
		Instantaneous center of rotation
15.76,77	15._73_,74	problems involving parallel velocities
15.78,79	15.75,_81_	
15.80		
15._84_,_85_	15.82,83	single rod: angle between controlling velocities = 90^{o}
15.86,87		
15.88,90	15.89	single rod angle between controlling velocities $\neq 90^{o}$
15._91_,_92_		
15._93_	15.94	
15.96,97	15.95,98	problems involving two instantaneous centers
15.100	15.99	space and body centrodes
15.102,103	15._101_,_104_	previous problems solved using instantaneous center
		General plane motion: accelerations
15.105,106		find relations among accelerations ($\omega = 0$)
15.107,108	15.109,110	
15.111	15.112,113	rolling motion
15.115,_116_	15.114	
15._117_,_119_	15.118	
		motion of a body controlled by
15.120,121	15._123_,124	one crank and a sliding block
15.122	15.125,_126_	
15.127,128	15.129,130	two cranks
15.132,134	15.131,133	
15.136	15.135	
15.137		special problems
*15.138,*139	*15.140,*141	analysis of plane motion in terms of a parameter
*15._142_,*143	*15._144_,147	
*15._145_,_*146_	*15._148_,*149	

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
		Plane motion of a particle relative to a rotating frame
15.150,151	15.152,155	velocity of a particle with respect to a rotating frame
15.153,154	15.*156,157*	
		Coriolis acceleration
15.160,162	15.*158,159*	acceleration of a particle moving at constant speed with respect to frame rotating with constant velocity
15.163,164	15.161	
15.165,166		
15.167,168		
15.169	15.170,171	
15.*172,173*		
15.174,175		acceleration of a particle with respect to a frame rotating with an angular acceleration
15.176,177		
15.178	15.*179,180*	
*15.183	15.*181,*182	
		Motion about a fixed point. General Motion
15.184,185		relation between linear and angular velocity for a rigid body with a fixed point
15.186	15.187	
15.190,*191*	15.188,189	angular acceleration of a body rotating simultaneously about two axes
15.*192*,193	15.*194*,195	
15.197,198	15.196	velocity and/or acceleration of a body rotating about a fixed point
15.199,200	15.*201,202*	
15.205	15.203,204	rod attached to moving collars: find velocities of collars (angular velocity is indeterminate)
15.*208,209*	15.206,207	
	15.210,211	universal joints
15.213	15.112	rod with clevis attached at one end
15.*214*	15.*215*,216	rod attached to moving collars: find acceleration of collars (angular velocities and angular accelerations are indeterminate)
15.218,219	15.117	
		Three dimensional motion of a particle relative to a rotating frame. Coriolis Acceleration
15.222,223	15.220,221	$\dot{\Omega} = 0$ and $\dot{u} = 0$
15.224	15.*225,226*	
15.227,228		
15.*229*	15.230	Ω and u not constant
15.231,233	15.232	previous problems to be solved using rotating frames
15.234,235	15.236	axis of Ω in plane of mechanism
15.*237,238*	14,239	
15.244,245	15.240,241	frame of reference in general motion
15.*246,247*	15.242,243	
15.248,*251*	15.249,*250*	Review problems
15.252,*254*	15.253,*257*	
15.255,256	15.258,259	

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
		CHAPTER 16: PLANE MOTION OF RIGID BODIES FORCES AND ACCELERATIONS
		Rectilinear translation
16.4,5	16.1,2	find reactions or accelerations
16.6	16.3	
16.9,10	16.*7*,*8*	determine whether body tips or slides
16.11,*12*		
		Curvilinear translation
16.*13*,14		bodies with zero velocity ($a_n = 0$)
16.15,*16*		
16.*17*		
	16.18	bodies with velocities ($a_n \neq 0$)
	16.19,20	
16.*_22_	16.*_21_	draw shear and bending-moment diagrams for accelerated rod
		Centroidal rotation
16.23	16.24	proofs that effective forces reduce to $m\bar{\mathbf{a}}$ and $\bar{I}\alpha$
16.25,29	16.26,27	single body
16.30,31	16.28	
16.*32*,35	16.33,34	connected pulleys and blocks
16.*36*	16.*37*,*38*	
16.41,42	16.39,40	friction disks
16.43,44	16.45	
		General plane motion
16.46	16.47	proofs concerning system of effective forces
16.50,51	16.48,49	find accelerations for given loading
16.*53*,*54*	16.52	
16.55,56	16.57,60	vertical motion controlled by applied forces and gravity
16.58,59	16.*61*,*62*	
16.63	16.64,65	bars suspended from springs and suddenly released
16.66,67	16.*68*	
16.69,70	16.71,72	spheres, disks, or hoops rotating and sliding on horizontal surface
16.*73*,*74*		
		Non-centroidal rotation
16.75		derivation involving center of percussion
16.76,77	16.78,79	find acceleration and reactions for a suspended body
		find reactions for a body rotating with
16.80,83	16.*81*,*82*	constant angular velocity
16.84	16.85,86	angular acceleration and velocity
16.87,*90*	16.88,*89*	two connected bodies
16.*91*		
		Rolling motion
16.92	16.93	derivations involving use of $\Sigma M_C = I_C\alpha$
16.*96*,*97*	16.94,95	single body rolling on an incline under gravity

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
16.98,99	16.102,103	single body rolling, or rotating and sliding
16.*100*,*101*	16.104,105	
16.108,109	16.*106*.107	connected bodies
	16.*110*	
16.113,114	16.111,112	unbalanced bodies
	16.115,116	
		General plane motion
		single rigid body, find accelerations and reactions
16.119,120	16.117,118	problems in which the velocity is zero
16.121,*122*		
16.*123*		
16.124,127	16.125,126	problems in which the velocity is not zero
16.128,129		
16.*130*,131		
	16.*132*,*133*	problems involving an integration
		connected rigid bodies
16.135,136	16.134	systems with one degree of freedom
16.137,138	16.139,140	
16.*142*	16.*141*	needs rotating axis
16.*143,*145	16.*144	systems with two degrees of freedom
16.*146,	16.*147,*148	
16.*149,*150		
16.*150	16.*151,*152	draw shear and bending-moment diagrams for accelerated rod
16.*155*,156	16.153,154	Review problems
16.*159*,160	16.157,158	
16.163,164	16.*161*,*164*	

CHAPTER 17: PLANE MOTION OF RIGID BODIES
ENERGY AND MOMENTUM METHODS

ENERGY METHODS

SI Units	U.S. Units	Problem Description
		Centroidal rotation
17.1,4	17.2,3	single body with applied couple
17.6	17.5	
17.11,12	17.7,*8*	connected bodies
17.13,*14*	17.*9*,10	
17.*15*		
		Noncentroidal rotation
17.17	17.16,18	
17.21,22	17.19,20	
17.*23*,*24*		

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
		Rolling motion
17.25,26	17.27	single body
17.*28*,29	17.30	
17.31		
17.34,*35*	17.32,33	connected bodies
		General plane motion
17.36,37	17.39,40	single body
17.38	17.*41*	
17.42,43		connected bodies
17.44,45		
17.46,47		
17.49,50	17.48,51	Power
		MOMENTUM METHODS
		Centroidal rotation
17.52,54	17.53,55	single body
17.*56*,*57*	17.58,59	
17.*60*,*61*	17.63,64	connected bodies
17.62		
17.65,66	17.67,68	Derivations involving noncentroidal rotation (center of percussion)
		Rolling motion
17.70,71	17.69	single body
17.74,75	17.72,*73*	connected bodies
	17.*76*	
17.77	17.78	sliding and rotating sphere
		Conservation of angular momentum
17.79	17.80	problems involving changes in the configuration of a rotating assembly
17.82,83	17.81	problems involving relative motion between components of a rotating assembly
17.86,87	17.*84*,*85*	
17.88,89	17.90,91	problems involving conservation of energy
17.*92*,*93*	17.95	
17.94		
		Eccentric impact
17.96	17.97,98	projectile becomes embedded in a body suspended from a fixed point
17.101,102	17.99,100	
17.102,103	17. 104,105	single body, rotation about a fixed point
17.106,107	17.108,109	
17.*110*,*111*	17.112	
17.*113*		
17.*114*,115	17.118,119	problems involving method of work and energy
17.116,117	17.120	
17.121,122	17.123,124	two bodies, general plane motion
17.*126*	17.*125*	
17.127,128	*17.130*	
17.*129*		
17.131,*134	17.132,133	special problems

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TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
17.*138*,139	17.135,136	Review problems
17.140,*141*	17.137,142	
17.143,145	17.*144*,*146*	

CHAPTER 18: KINETICS OF RIGID BODIES IN THREE DIMENSIONS

SI Units	U.S. Units	Problem Description
18.1,5	18.2,3	Computation of angular momentum (no product of inertia)
18.7,8	18.*4*,*6*	
18.10,12	18.9,11	Use of equation $\mathbf{H}_O = \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + \mathbf{H}_G$
18.*13*,*14*		
18.15,16	18.17,18	Computation of angular momentum (using products of inertia)
18.*19*,*20*		
18.21,22	18.*23*,*24*	Impulse and momentum in three dimensions
18.27,28	18.25,26	
18.*29*,*30*	18.31,32	impact problems
18.35,36	18.33,34	
18.37	18.38	
18.39,43	18.40,41	Computation of kinetic energy
18.44,45	18.42,*46*	
18.47,49	18.*48*	
18.50		
18.*51*,*52*	18.53,54	
18.55,59	18.56,57	Computation of derivative of angular momentum
18.60	18.58,*61*	
18.*63*,64	18.*62*	
		Dynamic reactions on rotating shaft
18.65,68	18.66,67	for ω = constant
18.*69*,*70*		
18.71,74	18.72,73	for $\omega = 0$ and $\alpha \neq 0$
18.*77*,*78*	18.75,76	
18.79,81	18.*80*,*83*	Simple problems on gyroscopic effect
18.82,84		
18.86,87	18.85,88	Relative equilibrium under rotation about a fixed axis
18.90,91	18.*89*,*92*	
		Dynamic reactions for combined rotations
18.95,96	18.93,94	with constant rates
18.99,100	18.*97*,*98*	
18.103,104	18.101,102	with variable rates
18.*105*,*106 (starred)		
18.109,*110*	18.107,*108*	Steady precession. Simple problems
	18.111,112	

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* Problems which do not involve any specific system of units have been indicated by underlining their numbers.
Answers are not given to problems with the numbers set in italic type.

TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
18.115,116	18,113.114	Steady precession. More advanced problems
18.*118*	18.117	
18.119,120	18.121,122	Precession under no force (theory)
18.*123*,124	18.*125*,128	Precession under no force (applications)
18.126,127	18.*129*,130	
		Conservation of energy and angular momentum
18.131,*134*	18.132,133	for bodies with precession and nutation
18.135,136		
18.*139,*140	18.*137,*138	for bodies with precession, nutation, and spin
	18.**141*,**142*	
18.*143,144	18.*145,*146	Poinsot analysis
18.148,151	18.*147*,*149*	Review problems
18.*152*,153	18.150,154	
18.155,157	18.156,*158*	
		CHAPTER 19: MECHANICAL VIBRATIONS
		VIBRATIONS WITHOUT DAMPING
		Free vibrations of a particle
19.1,4	19.2,3	problems involving amplitude, frequency, maximum velocity and acceleration
19.5,7	19.6,*9*	
19.8	19.*10*	
19.14,*15*	19.11,12	problems involving velocity and acceleration at a given instant
19.*16*	19.13	
19.17,20	19.18,18	springs in parallel or in series
19.*21*,*22*		
19.24,26	19.23,25	change mass or stiffness of the system
19.28,30	19.*27*,29	special problems
19.31	19.*32*	
19.*34,*36	19.*33,*35	simple pendulum with large amplitude
		Free vibrations of rigid bodies
19.37,39	19.38,41	systems with elastic restoring forces
19.*40*,*43*	19.42	
19.44,45	19.*46*,*47*	systems with restoring forces due to gravity
19.48,49	19.50,51	
19.52,53	19.54	derivations for motion of compound pendulum
19.55,56	19.57,*58*	systems with elastic and gravity restoring forces
19.60,62	19.59,*61*	determine velocity and acceleration
19.63,64	19.67,68	torsional vibrations
19.*65.66*		

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Answers are not given to problems with the numbers set in italic type.

TABLE II: CLASSIFICATION AND DESCRIPTION OF PROBLEMS (CONTINUED)

Problem Number*		
SI Units	**U.S. Units**	**Problem Description**
		Conservation of energy used to analyze the motion of
19.69,72	19.70,71	particles
19.75,76	19.*73,74*	single rigid body; pendulum
19.77,80	19.78,79	single rigid body; elastic restoring forces
19.*81,82*		
19.83,*84*	19.85,86	connected rigid bodies
19.87,*88*		
19.89,92	19.90,91	special problems
19.95,96	19.93,94	
19.*98	19.*97	
		Forced vibrations
19.99,100	19.101,102	problems involving periodic applied force
19.*103,104*		
19.106,107	19.105,*108*	problems involving periodic support motion
19.109,112	19.110,*111*	
19.113,114	19.115,116	problems involving rotating machinery
19.117,119	19.*118,120*	
19.121,122	19.123,125	special problems
19.*124,126*		
		DAMPED VIBRATIONS
		Damped free vibrations
19.127	19.128	proofs concerning damped free vibration
19. 129,130	19.131	proofs involving the logarithmic decrement
19.133	19.132	problems that use logarithmic decrement
19.136	19.134,135	maximum deflection of damped free vibration
19.137	19.138	damped free vibration of a rigid body
19.141,142	19.139,140	Damped forced vibrations
19.144	19.143,145	problems involving rotating machinery
	19.146	
19.147,148	19.149,	forced transmission with damping
	19.*150	energy dissipation
19.151*,152*		write the differential equation
19.153,154	19.155,156	Electrical analogues
19.157,158		
19.159,162	19. 160,161	Review problems
19.*166,167*	19. *163,164*	
19.168,170	19. 165,169	

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* Problems which do not involve any specific system of units have been indicated by underlining their numbers.
Answers are not given to problems with the numbers set in italic type.

TABLE III: SAMPLE ASSIGNMENT SCHEDULE FOR A COURSE IN DYNAMICS

This schedule includes all of the material of *VECTOR MECHANICS FOR ENGINEERS: DYNAMICS* with the exception of Sections 11.7, 11.8, 12.11–12.13, 18.9–18.11 and 19.8–19.10.

50% OF THE PROBLEMS IN EACH OF THE FOLLOWING LISTS USE SI UNITS AND 50% U.S. CUSTOMARY UNITS

			ANSWERS TO ALL OF THESE PROBLEMS ARE GIVEN IN THE BACK OF THE BOOK				NO ANSWERS ARE GIVEN IN THE BOOK FOR ANY OF THESE PROBLEMS	
Period	Sections	Topics	List 1	List 2	List 3	List 4	List 5	List 6
1	11.1	Introduction						
2	11.1	Rectilinear Motion	11.3,10,15,23	11.4,9,18,22	11.1,12,16,24	11.2,11,17,21	11.6,14,19,30	11.8,13,20,29
3	11.2	Uniformly Accelerated Motion	11.35,40,48,55	11.36,39,50,57	11.33,42,47,56	11.34,41,49,55	11.37,46,53,60	11.38,43,54,59
4	11.4	Curvilinear Motion (Rect. Comps)	11.98,106,118,128	11.99,105,120,127	11.97,108,117,126	11.100,107,119,123	11.101,110,121,132	11.104,109,122,130
5	11.5	Curvilinear Motion (Other Comp.)	11.137,146,155,166	11.141,143,156,165	11.138,145,153,170	11.139,144,154,169	11.140,150,157,168	11.142,148,160,167
6	12.1	Newton's Second Law and Linear Momentum	12.7,16,28,47	12.8,15,30,46	12.5,23,29,45	12.6,20,31,44	12.13,22,32,54	12.14,21,33,52
7	12.2	Angular Momentum and Orbital Mechanics	12.57,69,78,88	12.58,6880,87	12.55,67,79,91	12.56,66,81,90	12.59,73,83,93	12.60,70,84,92,
8		Review						
9		EXAM NUMBER ONE						
10	13.1	Work and Energy, Power	13.7,18,26,45	13.8,17,27,41	13.5,16,28,40	13.4,15,29,39	13.13,22,30,50	13.14,21,35,49
11	13.2	Conservation of Energy	13.58,65,72,80	13.59,64,74,79	13.55,67,70,82	13.57,62,71,81	13.60,63,73,84	13.61,66,75,83
12	13.2D	Application to Space Mechanics	13.85,96,100,111	13.86,95,103,110	13.87,94,101,107	13.88,93,102,106	13.91,99,104,114	13.92,98,105,113
13	13.3	Impulse and Momentum	13.119,130,141,150	13.120,129,142,148	13.121,132,139,146	13.124,131,140,145	13.125,137,143,154	13.127,133,144,153
14	13.4	Impact	13.157,168,174,186	13.158,166,176,185	13.155,165,175,188	13.156,164,177,184	13.160,171,178,189	13.162,170,181,187
15	14.1	Systems of Particles	14.1,12,15,22	14.2,11,16,21	14.3,10,19,25	14.4,9,20,24	14.5,14,17,27	14.6,13,18,23
16	14.2	Systems of Particles	14.32,38,46,52	14.33,37,47,51	14.31,40,45,56	14.35,39,48,55	14.34,42,49,54	14.36,41,50,53
17	14.3	Variable Systems of Particles	14.61,72,77,87	14.62,70,78,86	14.59,71,80,94	14.60,69,83,94	14.63,75,92,98	14.65,73,93,97
18		EXAM NUMBER TWO						
19	15.1	Translation, Rotation	15.1,11,18,29	15.2,10,19,28	15.3,13,23,31	15.4,12,24,30	15.7,15,20,35	15.8,14,21,34
20	15.2	General Plane Motion	15.38,50,57,70	15.39,49,58,64	15.40,53,55,69	15.41,48,56,68	15.42,54,59,67	15.43,52,60,66
21	15.3	Instantaneous Center	15.76,83,88,98	15.77,82,90,95	15.74,87,89,97	15.75,86,94,96	15.73,85,92,104	15.81,84,93,101
22	15.4	Acceleration in Plane Motion	15.112,121,127,139	15.113,120,128,138	15.111,125,129,141	15.115,124,130,140	15.116,126,131,146	15.117,123,135,145
23	15.5	Motion With Respect to a Rotating Frame	15.150,163,168,178	15.151,162,169,177	15.152,161,166,175	15.155,160,167,174	15.156,159,172,180	15.157,158,173,179
24	15.6	Motion of a Rigid Body in Space	15.187,198,203,213	15.188,197,204,213	15.184,196,205,211	15.185,195,205,210	15.191,202,208,215	15.192,201,209,214
25	15.7	Moving Reference Frames in Three Dimensions	15.222,232,234,243	15.223,230,235,242	15.220,233,236,241	15.221,231,239,240	15.225,229,237,247	15.226,229,238,246
26	16.1	Plane Motion of Rigid Bodies	16.4,19,25,40	16.5,18,30,39	16.1,15,27,42	16.2,14,28,41	16.7,17,32,45	15.8,16,35,45
27	16.1	Plane Motion of Rigid Bodies	16.48,56,64,70	16.48,55,65,69	16.50,58,63,72	16.51,57,63,71	16.53,62,68,74	16.54,61,68,73
28		EXAM NUMBER THREE						
29	16.2	Constrained Plane Motion	16.76,86,98,116	16.77,85,99,107	16.78,87,102,109	16.79,84,103,108	16.81,91,100,110	16.82,90,101,106
30	16.2	Constrained Plane Motion	16.117,125,134,143	16.118,124,139,144	16.121,126,135,147	16.121,125,136,146	16.122,133,141,150	16.123,132,142,149
31	17.1	Work and Energy	17.1,19,25,40	17.6,18,26,39	17.2,23,27,37	17.7,17,30,36	17.14,22,28,42	17.15,21,34,41
32	17.2	Impulse and Momentum	17.55,71,80,89	17.59,70,81,88	17.52,72,79,95	17.54,69,82,90	17.56,75,84,93	17.57,73,85,92
33	17.3	Eccentric Impact	17.96,108,116,124	17.101,105,117,123	17.97, 107,119,122	17.98,109,120,121	17.99,111,113,130	17.100,110,114,125
34	18.1	Momentum and Energy in Three Dimensions	18.1,11,15,26	18.5,9,16,25	18.2,12,17,22	18.3,10,18,21	18.4,14,19,24	18.6,13,20,23
35	18.1	Momentum and Energy in Three Dimensions	18.30,41,50	18.29,42,49	18.32,39,54	18.31,40,53	18.36,45,52	18.35,46,51
36	18.2	Motion in Three Dimensions	18.55,68,71,78	18.56,67,72,77	18.57,66,75,80	18.58,65,76,79	18.60,70,73,82	18.64,69,74,81
37	18.2	Motion in Three Dimensions	18.86,95,102	18.85,96.101	18.88,93,104	18.87,94,103	18.92,97,106	18.91,98,105
38		EXAM NUMBER FOUR						
39	19.1	Free Vibrations of Particles	19.1,14,17,25	19.4,8,20,23	19.3,12,18,26	19.5,11,19,24	19.9,16,21,32	19.10,15,22,27
40	19.2	Free Vibrations of Rigid Bodies	19.38,45,57,64	19.41,44,59,63	19.37,51,55,68	19.41,50,56,67	19.40,47,58,66	19.43,46,61,65
41	19.3	Energy Methods	19.69,79,83,91	19.72,78,87,90	19.70,80,85,92	19.71,77,86,89	19.73,2,84,94	19.74,81,88,93
42	19.4	Forced Vibrations	19.101,107,115,122	19.102,106,116,121	19.99,110,113,125	19.100,105,114,123	19.103,111,118126	19.104,108,120,124

TABLE IV: SAMPLE ASSIGNMENT SCHEDULE FOR A COURSE IN DYNAMICS

This schedule includes all of the material of *VECTOR MECHANICS FOR ENGINEERS: DYNAMICS* with the exception of Sections 11.7, 11.8, 12.11–12.13, 18.9–18.11 and 19.8–19.10.

75% OF THE PROBLEMS IN EACH OF THE FOLLOWING LISTS USE SI UNITS AND 25% U.S. CUSTOMARY UNITS

ANSWERS TO ALL OF THESE PROBLEMS ARE GIVEN IN THE BACK OF THE BOOK

Period	Sections	Topics	List 1a	List 2a	List 3a	List 4a
1	11.1	Introduction				
2	11.1	Rectilinear Motion	11.3,10,15,27	11.4,9,18,26	11.5,12,16,24	11.7,11,17,21
3	11.2	Uniformly Accelerated Motion	11.35,40,51,55	11.36,39,52,57	11.33,45,47,56	11.34,44,49,55
4	11.4	Curvilinear Motion (Rect. Comps)	11.98,106,118,131	11.99,105,120,129	11.102,108,117,126	11.103,107,119,123
5	11.5	Curvilinear Motion (Other Comp.)	11.137,146,158,166	11.141,143,159,165	11.138,149,153,170	11.139,147,154,169
6	12.1	Newton’s Second Law and Linear Momentum	12.7,16,28,49	12.8,15,30,48	12.9,23,29,45	12.10,20,31,44
7	12.2	Angular Momentum and Orbital Mechanics	12.57,69,82,88	12.58,68,85,87	12.55,72,79,91	12.56,71,81,90
8		Review				
9		EXAM NUMBER ONE				
10	13.1	Work and Energy, Power	13.7,18,26,52	13.8,17,27,51	13.11,16,28,40	13.12,15,29,39
11	13.2	Conservation of Energy	13.58,65,76,80	13.59,64,77,79	13.55,69,70,82	13.57,68,71,81
12	13.2D	Application to Space Mechanics	13.85,96,100,109	13.86,95,103,108	13.89,94,101,107	13.90,93,102,106
13	13.3	Impulse and Momentum	13.119,130,141,152	13.120,129,142,151	13.126,132,139,146	13.128,131,140,145
14	13.4	Impact	13.157,168,179,186	13.158,166,180,185	13.155,172,175,188	13.156,169,177,184
15	14.1	Systems of Particles	14.1,12,15,28	14.2,11,16,26	14.7,10,19,25	14.8,9,20,24
16	14.2	Systems of Particles	14.32,38,51,52	14.33,37,51,52	14.31,44,45,56	14.35,43,48,55
17	14.3	Variable Systems of Particles	14.61,72,77,96	14.62,70,78,95	14.64,71,80,94	14.66,69,83,94
18		EXAM NUMBER TWO				
19	15.1	Translation, Rotation	15.1,11,18,33	15.2,10,19,32	15.5,13,23,31	15.6,12,24,30
20	15.2	General Plane Motion	15.38,50,61,70	15.39,49,62,64	15.40,51,55,69	15.41,50,56,68
21	15.3	Instantaneous Center	15.76,83,88,103	15.77,82,90,102	15.78,87,89,97	15.79,86,94,96
22	15.4	Acceleration in Plane Motion	15.112,121,132,139	15.113,120,133,138	15.111,122,129,141	15.115,122,130,140
23	15.5	Motion With Respect to a Rotating Frame	15.150,163,168,176	15.151,162,169,181	15.153,161,166,175	15.154,160,167,174
24	15.6	Motion of a Rigid Body in Space	15.187,198,206,213	15.188,197,207,213	15.184,199,205,211	15.185,200,205,210
25	15.7	Moving Reference Frames in Three Dimensions	15.222,232,234,245	15.223,230,235,244	15.224,233,236,241	15.227,231,239,240
26	16.1	Plane Motion of Rigid Bodies	16.4,19,25,44	16.5,18,30,43	16.9,15,27,42	16.10,14,28,41
27	16.1	Plane Motion of Rigid Bodies	16.48,56,66,70	16.48,55,67,69	16.50,60,63,72	16.51,59,63,71
28		EXAM NUMBER THREE				
29	16.2	Constrained Plane Motion	16.76,86,98,114	16.77,85,99,113	16.80,87,102,109	16.83,84,103,108
30	16.2	Constrained Plane Motion	16.117,125,137,143	16.118,124,138,144	16.121,129,135,147	16.121,128,136,146
31	17.1	Work and Energy	17.1,19,25,46	17.6,18,26,45	17.11,23,27,37	17.12,17,30,36
32	17.3	Impulse and Momentum	17.55,71,83,89	17.59,70,86,88	17.52,75,79,95	17.54,74,82,90
33	17.3	Eccentric Impact	17.96,108,116,128	17.101,105,117,127	17.102,107,119,122	17.103,109,120,121
34	18.1	Momentum and Energy in Three Dimensions	18.1,11,15,28	18.5,9,16,27	18.7,12,17,22	18.8,10,18,21
35	18.1	Momentum and Energy in Three Dimensions	18.30,44,50	18.29,45,49	18.36,39,54	18.35,40,53
36	18.2	Motion in Three Dimensions	18.55,68,71,84	18.56,67,72,82	18.60,66,75,80	18.64,65,76,79
37	18.2	Motion in Three Dimensions	18.86,99,102	18.85,100.101	18.91,93,104	18.90,94,103
38		EXAM NUMBER FOUR				
39	19.1	Free Vibrations of Particles	19.1,14,17,31	19.4,8,20,28	19.5,12,18,26	19.7,11,19,24
40	19.2	Free Vibrations of Rigid Bodies	19.38,45,60,64	19.41,44,62,63	19.37,49,55,68	19.41,48,56,67
41	19.3	Energy Methods	19.69,79,83,96	19.72,78,87,95	19.75,80,85,92	19.76,77,86,89
42	19.4	Forced Vibrations	19.101,107,117,122	19.102,106,119,121	19.99,112,113,125	19.100,109,114,123

CHAPTER 11

PROBLEM 11.1

A snowboarder starts from rest at the top of a double black diamond hill. As he rides down the slope, GPS coordinates are used to determine his displacement as a function of time: $x = 0.5t^3 + t^2 + 2t$ where x and t are expressed in ft and seconds, respectively. Determine the position, velocity, and acceleration of the boarder when t = 5 seconds.

SOLUTION

Position: $x = 0.5t^3 + t^2 + 2t$

Velocity: $v = \dfrac{dx}{dt} = 1.5t^2 + 2t + 2$

Acceleration: $a = \dfrac{dv}{dt} = 3t + 2$

At $t = 5$ s, $x = 0.5(5)^3 + 5^2 + 2(5)$ $\qquad x = 97.5$ ft ◀

$v = 1.5(5)^2 + 2(5) + 2$ $\qquad v = 49.5$ ft/s ◀

$a = 3(5) + 2$ $\qquad a = 17$ ft/s^2 ◀

PROBLEM 11.2

The motion of a particle is defined by the relation $x = 2t^3 - 9t^2 + 12t + 10,$ where x and t are expressed in feet and seconds, respectively. Determine the time, the position, and the acceleration of the particle when $v = 0$.

SOLUTION

$$x = 2t^3 - 9t^2 + 12t + 10$$

Differentiating,

$$v = \frac{dx}{dt} = 6t^2 - 18t + 12 = 6(t^2 - 3t + 2)$$

$$= 6(t-2)(t-1)$$

$$a = \frac{dv}{dt} = 12t - 18$$

So $v = 0$ at $t = 1\text{ s}$ and $t = 2\text{ s}$.

At $t = 1\text{ s}$,

$$x_1 = 2 - 9 + 12 + 10 = 15$$

$t = 1.000\text{ s}$ ◀

$$a_1 = 12 - 18 = -6$$

$x_1 = 15.00\text{ ft}$ ◀

$a_1 = -6.00\text{ ft/s}^2$ ◀

At $t = 2\text{ s}$,

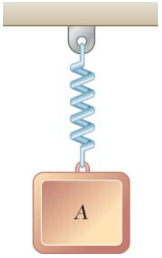
$$x_2 = 2(2)^3 - 9(2)^2 + 12(2) + 10 = 14$$

$t = 2.00\text{ s}$ ◀

$x_2 = 14.00\text{ ft}$ ◀

$$a_2 = (12)(2) - 18 = 6$$

$a_2 = 6.00\text{ ft/s}^2$ ◀



PROBLEM 11.3

The vertical motion of mass A is defined by the relation $x = 10 \sin 2t + 15 \cos 2t + 100$, where x and t are expressed in mm and seconds, respectively. Determine (a) the position, velocity and acceleration of A when $t = 1$ s, (b) the maximum velocity and acceleration of A.

SOLUTION

$$x = 10 \sin 2t + 15 \cos 2t + 100$$

$$v = \frac{dx}{dt} = 20 \cos 2t - 30 \sin 2t$$

$$a = \frac{dv}{dt} = -40 \sin 2t - 60 \cos 2t$$

For trigonometric functions set calculator to radians:

(a) At $t = 1$ s.

$$x_1 = 10 \sin 2 + 15 \cos 2 + 100 = 102.9 \qquad x_1 = 102.9 \text{ mm} \blacktriangleleft$$

$$v_1 = 20 \cos 2 - 30 \sin 2 = -35.6 \qquad v_1 = -35.6 \text{ mm/s} \blacktriangleleft$$

$$a_1 = -40 \sin 2 - 60 \cos 2 = -11.40 \qquad a_1 = -11.40 \text{ mm/s}^2 \blacktriangleleft$$

(b) Maximum velocity occurs when $a = 0$.

$$-40 \sin 2t - 60 \cos 2t = 0$$

$$\tan 2t = -\frac{60}{40} = -1.5$$

$$2t = \tan^{-1}(-1.5) = -0.9828 \text{ and } -0.9828 + \pi$$

Reject the negative value. $2t = 2.1588$

$$t = 1.0794 \text{ s}$$

$$t = 1.0794 \text{ s for } v_{\max}$$

so

$$v_{\max} = 20 \cos(2.1588) - 30 \sin(2.1588)$$
$$= -36.056 \qquad v_{\max} = -36.1 \text{ mm/s} \blacktriangleleft$$

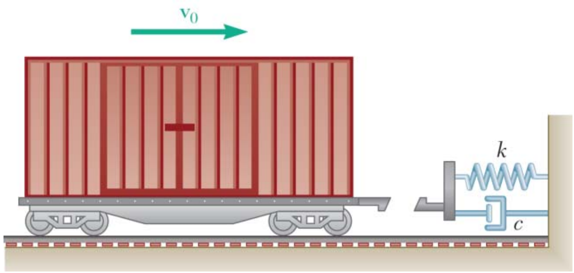
Note that we could have also used

$$v_{\max} = \sqrt{20^2 + 30^2} = 36.056$$

by combining the sine and cosine terms.

For $a_{\max}$ we can take the derivative and set equal to zero or just combine the sine and cosine terms.

$$a_{\max} = \sqrt{40^2 + 60^2} = 72.1 \text{ mm/s}^2 \qquad a_{\max} = 72.1 \text{ mm/s}^2 \blacktriangleleft$$



PROBLEM 11.4

A loaded railroad car is rolling at a constant velocity when it couples with a spring and dashpot bumper system. After the coupling, the motion of the car is defined by the relation $x = 60e^{-4.8t}\sin 16t$ where x and t are expressed in mm and seconds, respectively. Determine the position, the velocity and the acceleration of the railroad car when (*a*) $t = 0$, (*b*) $t = 0.3$ s.

SOLUTION

$$x = 60e^{-4.8t}\sin 16t$$

$$v = \frac{dx}{dt} = 60(-4.8)e^{-4.8t}\sin 16t + 60(16)e^{-4.8t}\cos 16t$$

$$v = -288e^{-4.8t}\sin 16t + 960e^{-4.8t}\cos 16t$$

$$a = \frac{dv}{dt} = 1382.4e^{-4.8t}\sin 16t - 4608e^{-4.8t}\cos 16t$$

$$-4608e^{-4.8t}\cos 16t - 15360e^{-4.8t}\sin 16t$$

$$a = -13977.6e^{-4.8t}\sin 16t - 9216e^{-4.8}\cos 16t$$

(*a*) At $t = 0$,

$x_0 = 0$ $\qquad$ $x_0 = 0$ mm ◀

$v_0 = 960$ mm/s $\qquad$ $v_0 = 960$ mm/s $\longrightarrow$ ◀

$a_0 = -9216$ mm/s^2 $\qquad$ $a_0 = 9220$ mm/s^2 $\longleftarrow$ ◀

(*b*) At $t = 0.3$ s,

$$e^{-4.8t} = e^{-1.44} = 0.23692$$

$$\sin 16t = \sin 4.8 = -0.99616$$

$$\cos 16t = \cos 4.8 = 0.08750$$

$x_{0.3} = (60)(0.23692)(-0.99616) = -14.16$ $\qquad$ $x_{0.3} = 14.16$ mm $\longleftarrow$ ◀

$$v_{0.3} = -(288)(0.23692)(-0.99616)$$

$+(960)(0.23692)(0.08750) = 87.9$ $\qquad$ $v_{0.3} = 87.9$ mm/s $\longrightarrow$ ◀

$$a_{0.3} = -(13977.6)(0.23692)(-0.99616)$$

$-(9216)(0.23692)(0.08750) = 3108$ $\qquad$ $a_{0.3} = 3110$ mm/s^2 ◀

or 3.11 m/s^2 $\longrightarrow$ ◀

PROBLEM 11.5

The motion of a particle is defined by the relation $x = 6t^4 - 2t^3 - 12t^2 + 3t + 3$, where x and t are expressed in meters and seconds, respectively. Determine the time, the position, and the velocity when $a = 0$.

SOLUTION

We have $$x = 6t^4 - 2t^3 - 12t^2 + 3t + 3$$

Then $$v = \frac{dx}{dt} = 24t^3 - 6t^2 - 24t + 3$$

and $$a = \frac{dv}{dt} = 72t^2 - 12t - 24$$

When $a = 0$: $$72t^2 - 12t - 24 = 12(6t^2 - t - 2) = 0$$

or $$(3t - 2)(2t + 1) = 0$$

or $$t = \frac{2}{3}\text{ s} \quad \text{and} \quad t = -\frac{1}{2}\text{ s} \quad \text{(Reject)}$$

$t = 0.667$ s ◀

At $t = \frac{2}{3}$ s: $$x_{2/3} = 6\left(\frac{2}{3}\right)^4 - 2\left(\frac{2}{3}\right)^3 - 12\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right) + 3$$

or $x_{2/3} = 0.259$ m ◀

$$v_{2/3} = 24\left(\frac{2}{3}\right)^3 - 6\left(\frac{2}{3}\right)^2 - 24\left(\frac{2}{3}\right) + 3$$

or $v_{2/3} = -8.56$ m/s ◀

PROBLEM 11.6

The motion of a particle is defined by the relation $x = t^3 - 9t^2 + 24t - 8$, where x and t are expressed in inches and seconds, respectively. Determine (a) when the velocity is zero, (b) the position and the total distance traveled when the acceleration is zero.

SOLUTION

We have $$x = t^3 - 9t^2 + 24t - 8$$

Then $$v = \frac{dx}{dt} = 3t^2 - 18t + 24$$

and $$a = \frac{dv}{dt} = 6t - 18$$

(a) When $v = 0$: $$3t^2 - 18t + 24 = 3(t^2 - 6t + 8) = 0$$

$$(t-2)(t-4) = 0$$

$t = 2.00$ s and $t = 4.00$ s ◀

(b) When $a = 0$: $$6t - 18 = 0 \quad \text{or} \quad t = 3 \text{ s}$$

At $t = 3$ s: $$x_3 = (3)^3 - 9(3)^2 + 24(3) - 8$$ or $x_3 = 10.00$ in. ◀

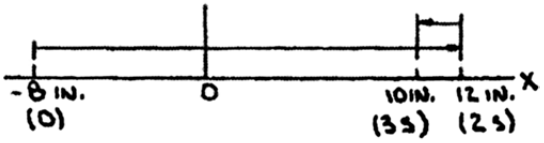
First observe that $0 \le t < 2$ s: $v > 0$

$2 \text{ s} < t \le 3$ s: $v < 0$

Now

At $t = 0$: $x_0 = -8$ in.

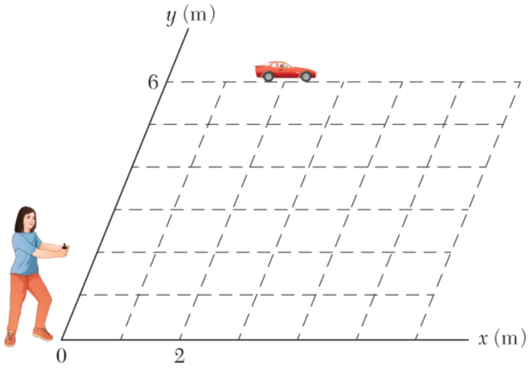
At $t = 2$ s: $x_2 = (2)^3 - 9(2)^2 + 24(2) - 8 = 12$ in.



Then $$x_2 - x_0 = 12 - (-8) = 20 \text{ in.}$$

$$|x_3 - x_2| = |10 - 12| = 2 \text{ in.}$$

Total distance traveled $= (20 + 2)$ in. Total distance $= 22.0$ in. ◀



PROBLEM 11.7

A girl operates a radio-controlled model car in a vacant parking lot. The girl's position is at the origin of the *xy* coordinate axes, and the surface of the parking lot lies in the x-y plane. She drives the car in a straight line so that the *x* coordinate is defined by the relation $x(t) = 0.5t^3 - 3t^2 + 3t + 2$, where *x* and *t* are expressed in meters and seconds, respectively. Determine (*a*) when the velocity is zero, (*b*) the position and total distance travelled when the acceleration is zero.

SOLUTION

Position: $x(t) = 0.5t^3 - 3t^2 + 3t + 2$

Velocity: $v(t) = \dfrac{dx}{dt}$

$$v(t) = 1.5t^2 - 6t + 3$$

(*a*) Time when v=0

$$0 = 1.5t^2 - 6t + 3$$

$$t = \frac{6 \pm \sqrt{6^2 - 4(1.5)(3)}}{2 * 1.5}$$

$t = 0.586\text{ s}$ and $t = 3.414\text{ s}$ ◀

Acceleration: $a(t) = \dfrac{dv}{dt}$

$$a(t) = 3t - 6$$

Time when a=0

$$0 = 3t - 6$$

$$t = 2\text{ s}$$

(*b*) Position at t=2 s

$$x(2) = 0.5(2)^3 - 3(2)^2 + 3 * 2 + 2$$

$x(2) = 0\text{ m}$ ◀

To find total distance note that car changes direction at t=0.586 s

Position at t=0 s

$$x(0) = 0.5(0)^3 - 3(0)^2 + 3 * 0 + 2$$

$$x(0) = 2$$

Position at t=0.586 s

$$x(0.586) = 0.5(0.586)^3 - 3(0.586)^2 + 3 * 0.586 + 2$$

$$x(0.586) = 2.828\text{ m}$$

Distances traveled:

From t=0 to t=0.586 s: $|x(0.586) - x(0)| = 0.828\text{ m}$

From t=0.586 to t=2 s: $|x(2) - x(0.586)| = 2.828\text{ m}$

Total distance traveled $= 0.828\text{ m} + 2.828\text{ m}$

Total distance $= 3.656\text{ m}$ ◀

PROBLEM 11.8

The motion of a particle is defined by the relation $x = t^2 - (t-2)^3$, where *x* and *t* are expressed in feet and seconds, respectively. Determine (*a*) the two positions at which the velocity is zero, (*b*) the total distance traveled by the particle from *t* = 0 to *t* = 4 s..

SOLUTION

Position: $x(t) = t^2 - (t-2)^3$

Velocity: $v(t) = \frac{dx}{dt}$

$$v(t) = 2t - 3(t-2)^2$$

$$v(t) = 2t - 3(t^2 - 4t + 4)$$

$$= -3t^2 + 14t - 12$$

Time when v(t)=0 $\quad 0 = -3t^2 + 14t - 12$

$$t = \frac{-14 \pm \sqrt{14^2 - 4(-3)(-12)}}{2(-3)}$$

$t = 1.131$ s and $t = 3.535$ s

(a) Position at t=1.131 s $\quad x(1.131) = (1.131)^2 - (1.1.31 - 2)^3$

$x(1.131) = 1.935$ ft. ◀

Position at t=3.535 s $\quad x(3.535) = (3.535)^2 - (3.535 - 2)^3$

$x(3.531) = 8.879$ ft. ◀

To find total distance traveled note that the particle changes direction at t=1.131 s and again at t=3.535 s.

Position at t=0 s $\quad x(0) = (0)^2 - (0 - 2)^3$

$x(0) = 8$ ft

Position at t=4 s $\quad x(4) = (4)^2 - (4 - 2)^3$

$x(4) = 8$ ft

(b) Distances traveled:

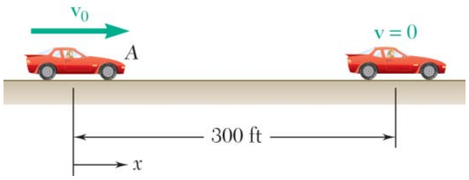
From t=0 to t=1.131 s: $\quad |x(1.131) - x(0)| = 6.065$ ft.

From t=1.131 to t=3.531 s: $\quad |x(3.535) - x(1.131)| = 6.944$ ft.

From t=3.531 to t=4 s: $\quad |x(4) - x(3.535)| = 0.879$ ft.

Total distance traveled $= 6.065$ ft $+ 6.944$ ft $+ 0.879$ ft

Total distance $= 13.888$ ft. ◀



PROBLEM 11.9

The brakes of a car are applied, causing it to slow down at a rate of 10 ft/s^2. Knowing that the car stops in 100 ft, determine (*a*) how fast the car was traveling immediately before the brakes were applied, (*b*) the time required for the car to stop.

SOLUTION

$$a = -10 \text{ ft/s}^2$$

(*a*) Velocity at $x = 0$.

$$v\frac{dv}{dx} = a = -10$$

$$\int_{v_0}^{0} v\,dv = -\int_0^{x_f} (-10)dx$$

$$0 - \frac{v_0^2}{2} = -10x_f = -(10)(300)$$

$$v_0^2 = 6000 \qquad v_0 = 77.5 \text{ ft/s}^2 \blacktriangleleft$$

(*b*) Time to stop.

$$\frac{dv}{dx} = a = -10$$

$$\int_{v_0}^{0} dv = -\int_0^{t_f} -10dt$$

$$0 - v_0 = -10t_f$$

$$t_f = \frac{v_0}{10} = \frac{77.5}{10} \qquad t_f = 7.75 \text{ s} \blacktriangleleft$$

PROBLEM 11.10

The acceleration of a particle is defined by the relation $a = 3e^{-0.2t}$, where a and t are expressed in ft/s^2 and seconds, respectively. Knowing that $x = 0$ and $v = 0$ at $t = 0$, determine the velocity and position of the particle when $t = 0.5$ s.

SOLUTION

Acceleration: $a = 3e^{-0.2t}$ ft/s^2

Given: $v_0 = 0$ ft/s, $x_0 = 0$ ft

Velocity: $a = \frac{dv}{dt} \Rightarrow dv = adt$

$$\int_{v_0}^{v} dv = \int_0^t adt$$

$$v - v_o = \int_0^t 3e^{-0.2t}dt \Rightarrow v - 0 = -15e^{-0.2t}\Big|_0^t$$

$$v = 15\left(1 - e^{-0.2t}\right) \text{ ft/s}$$

Position: $v = \frac{dx}{dt} \Rightarrow dx = vdt$

$$\int_{x_0}^{x} dx = \int_0^t vdt$$

$$x - x_o = \int_0^t 15\left(1 - e^{-0.2t}\right)dt \Rightarrow \text{x} - 0 = 15\left(t + 5e^{-0.2t}\right)\Big|_0^t$$

$$x = 15\left(t + 5e^{-0.2t}\right) - 75 \text{ ft}$$

Velocity at t=0.5 s $v = 15\left(1 - e^{-0.2*0.5}\right)$ ft/s

$$v(0.5) = 1.427 \text{ ft/s} \blacktriangleleft$$

Position at t=0.5 s $x = 15\left(0.5 + 5e^{-0.2*0.5}\right) - 75$ ft

$$x(0.5) = 0.363 \text{ ft.} \blacktriangleleft$$

PROBLEM 11.11

The acceleration of a particle is directly proportional to the square of the time t. When $t = 0$, the particle is at $x = 24$ m. Knowing that at $t = 6$ s, $x = 96$ m and $v = 18$ m/s, express x and v in terms of t.

SOLUTION

We have $$a = kt^2 \quad k = \text{constant}$$

Now $$\frac{dv}{dt} = a = kt^2$$

At $t = 6$ s, $v = 18$ m/s: $$\int_{18}^{v} dv = \int_{6}^{t} kt^2 dt$$

or $$v - 18 = \frac{1}{3}k(t^3 - 216)$$

or $$v = 18 + \frac{1}{3}k(t^3 - 216)(\text{m/s})$$

Also $$\frac{dx}{dt} = v = 18 + \frac{1}{3}k(t^3 - 216)$$

At $t = 0$, $x = 24$ m: $$\int_{24}^{x} dx = \int_{0}^{t}\left[18 + \frac{1}{3}k(t^3 - 216)\right]dt$$

or $$x - 24 = 18t + \frac{1}{3}k\left(\frac{1}{4}t^4 - 216t\right)$$

Now

At $t = 6$ s, $x = 96$ m: $$96 - 24 = 18(6) + \frac{1}{3}k\left[\frac{1}{4}(6)^4 - 216(6)\right]$$

or $$k = \frac{1}{9} \text{ m/s}^4$$

Then $$x - 24 = 18t + \frac{1}{3}\left(\frac{1}{9}\right)\left(\frac{1}{4}t^4 - 216t\right)$$

or $$x(t) = \frac{1}{108}t^4 + 10t + 24 \quad ◀$$

and $$v = 18 + \frac{1}{3}\left(\frac{1}{9}\right)(t^3 - 216)$$

or $$v(t) = \frac{1}{27}t^3 + 10 \quad ◀$$

PROBLEM 11.12

The acceleration of a particle is defined by the relation $a = kt^2$. (*a*) Knowing that $v = -8$ m/s when $t = 0$ and that $v = +8$ m/s when $t = 2$ s, determine the constant *k*. (*b*) Write the equations of motion, knowing also that $x = 0$ when $t = 2$ s.

SOLUTION

$$a = kt^2 \qquad (1)$$

$$\frac{dv}{dt} = a = kt^2$$

$t = 0,\ v = -8$ m/s and $t = 2$ s, $v = +8$ ft/s

(*a*)

$$\int_{-8}^{8} dv = \int_0^2 kt^2\, dt$$

$$8 - (-8) = \frac{1}{3}k(2)^3$$

$k = 6.00\ \text{m/s}^4$ ◀

(*b*) Substituting $k = 6\ \text{m/s}^4$ into (1)

$$\frac{dv}{dt} = a = 6t^2$$

$a = 6t^2$ ◀

$t = 0, v = -8$ m/s:

$$\int_{-8}^{v} dv = \int_0^t 6t^2\, dt$$

$$v - (-8) = \frac{1}{3}6(t)^3$$

$v = 2t^3 - 8$ ◀

$$\frac{dx}{dt} = v = 2t^3 - 8$$

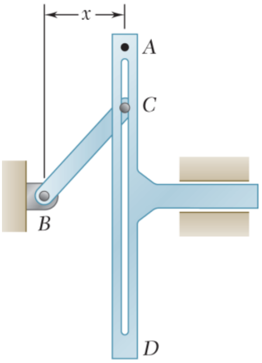
$t = 2$ s, $x = 0$:

$$\int_0^x dx = \int_2^t (2t^3 - 8)dt; \quad x = \left|\frac{1}{2}t^4 - 8t\right|_2^t$$

$$x = \left[\frac{1}{2}t^4 - 8t\right] - \left[\frac{1}{2}(2)^4 - 8(2)\right]$$

$$x = \frac{1}{2}t^4 - 8t - 8 + 16$$

$x = \frac{1}{2}t^4 - 8t + 8$ ◀



PROBLEM 11.13

A Scotch yoke is a mechanism that transforms the circular motion of a crank into the reciprocating motion of a shaft (or vice versa). It has been used in a number of different internal combustion engines and in control valves. In the Scotch yoke shown, the acceleration of Point A is defined by the relation $a = -1.8\sin kt$, where a and t are expressed in m/s^2 and seconds, respectively, and $k = 3$ rad/s. Knowing that $x = 0$ and $v = 0.6$ m/s when $t = 0$, determine the velocity and position of Point A when $t = 0.5$ s.

SOLUTION

Acceleration: $a = -1.8\sin kt \text{ m/s}^2$

Given: $v_0 = 0.6 \text{ m/s}, \quad x_0 = 0, \quad k = 3 \text{ rad/s}$

Velocity:

$$a = \frac{dv}{dt} \Rightarrow \quad dv = adt \quad \Rightarrow \int_{v_0}^{v} dv = \int_0^t adt$$

$$v - v_0 = \int_0^t a\,dt = -1.8\int_0^t \sin kt\,dt = \frac{1.8}{k}\cos kt\Big|_0^t$$

$$v - 0.6 = \frac{1.8}{3}(\cos kt - 1) = 0.6\cos kt - 0.6$$

$$v = 0.6\cos kt \text{ m/s}$$

Position:

$$v = \frac{dx}{dt} \Rightarrow \quad dx = vdt \quad \Rightarrow \int_{x_0}^{x} dx = \int_0^t vdt$$

$$x - x_0 = \int_0^t v\,dt = 0.6\int_0^t \cos kt\,dt = \frac{0.6}{k}\sin kt\Big|_0^t$$

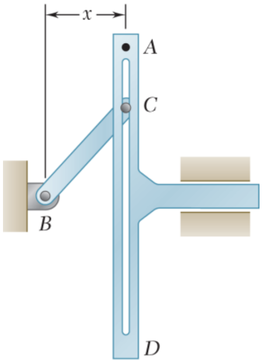
$$x - 0 = \frac{0.6}{3}(\sin kt - 0) = 0.2\sin kt$$

$$x = 0.2\sin kt \text{ m}$$

When t =0.5 s, $kt = (3)(0.5) = 1.5 \text{ rad}$

$v = 0.6\cos 1.5 = 0.0424 \text{ m/s}$ $\quad v = 42.4 \text{ mm/s}$ ◀

$x = 0.2\sin 1.5 = 0.1995 \text{ m}$ $\quad x = 199.5 \text{ mm}$ ◀



PROBLEM 11.14

For the scotch yoke mechanism shown, the acceleration of Point A is defined by the relation $a = -1.08\sin kt - 1.44\cos kt$, where a and t are expressed in m/s^2 and seconds, respectively, and $k = 3$ rad/s. Knowing that $x = 0.16$ m and $v = 0.36$ m/s when $t = 0$, determine the velocity and position of Point A when $t = 0.5$ s.

SOLUTION

Acceleration: $a = -1.08\sin kt - 1.44\cos kt \text{ m/s}^2$

Given: $v_0 = 0.36 \text{ m/s}, \quad x_0 = 0.16, \quad k = 3 \text{ rad/s}$

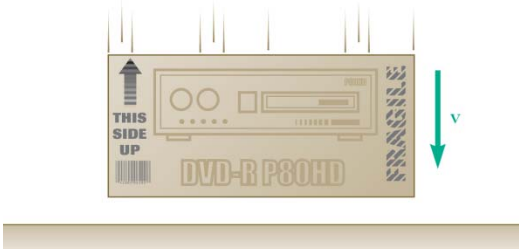
Velocity:
$$a = \frac{dv}{dt} \Rightarrow \quad dv = a\,dt \quad \Rightarrow \int_{v_0}^{v} dv = \int_0^t a\,dt$$

Integrate:
$$v - v_0 = -1.08 \int_0^t \sin kt\,dt - 1.44 \int_0^t \cos kt\,dt$$
$$v - 0.36 = \frac{1.08}{k}\cos kt\Big|_0^t - \frac{1.44}{k}\sin kt\Big|_0^t$$
$$= \frac{1.08}{3}(\cos 3t - 1) - \frac{1.44}{3}(\sin 3t - 0)$$
$$= 0.36\cos 3t - 0.36 - 0.48\sin 3t$$
$$v = 0.36\cos 3t - 0.48\sin 3t \text{ m/s}$$

Evaluate at $t = 0.5$ s
$$= 0.36\cos 1.5 - 0.36 - 0.48\sin 1.5$$
$v = -453$ mm/s ◀

Position:
$$v = \frac{dx}{dt} \Rightarrow \quad dx = v\,dt \quad \Rightarrow \int_{x_0}^{x} dx = \int_0^t v\,dt$$
$$x - x_0 = \int_0^t v\,dt = 0.36 \int_0^t \cos kt\,dt - 0.48 \int_0^t \sin kt\,dt$$
$$x - 0.16 = \frac{0.36}{k}\sin kt\Big|_0^t + \frac{0.48}{k}\cos kt\Big|_0^t$$
$$= \frac{0.36}{3}(\sin 3t - 0) + \frac{0.48}{3}(\cos 3t - 1)$$
$$= 0.12\sin 3t + 0.16\cos 3t - 0.16$$
$$x = 0.12\sin 3t + 0.16\cos 3t \text{ m}$$

Evaluate at $t = 0.5$ s
$$x = 0.12\sin 1.5 + 0.16\cos 1.5 = 0.1310 \text{ m}$$
$x = 131.0$ mm ◀



PROBLEM 11.15

A piece of electronic equipment that is surrounded by packing material is dropped so that it hits the ground with a speed of 4 m/s. After contact the equipment experiences an acceleration of $a = -kx$, where k is a constant and x is the compression of the packing material. If the packing material experiences a maximum compression of 20 mm, determine the maximum acceleration of the equipment.

SOLUTION

$$a = \frac{vdv}{dx} = -kx$$

Separate and integrate.

$$\int_{v_0}^{v_f} vdv = -\int_0^{x_f} kx\,dx$$

$$\frac{1}{2}v_f^2 - \frac{1}{2}v_0^2 = -\frac{1}{2}kx^2\bigg|_0^{x_f} = -\frac{1}{2}kx_f^2$$

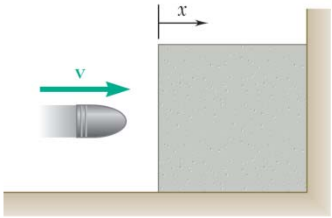
Use $v_0 = 4$ m/s, $x_f = 0.02$ m, and $v_f = 0$. Solve for k.

$$0 - \frac{1}{2}(4)^2 = -\frac{1}{2}k(0.02)^2 \qquad k = 40{,}000 \text{ s}^{-2}$$

Maximum acceleration.

$$a_{\max} = -kx_{\max}: \quad (-40{,}000)(0.02) = -800 \text{ m/s}^2$$

$a = 800 \text{ m/s}^2 \uparrow$ ◀



PROBLEM 11.16

A projectile enters a resisting medium at $x = 0$ with an initial velocity $v_0 = 900$ ft/s and travels 4 in. before coming to rest. Assuming that the velocity of the projectile is defined by the relation $v = v_0 - kx$, where v is expressed in ft/s and x is in feet, determine (*a*) the initial acceleration of the projectile, (*b*) the time required for the projectile to penetrate 3.9 in. into the resisting medium.

SOLUTION

First note

When $x = \frac{4}{12}$ ft, $v = 0$: $\quad 0 = (900 \text{ ft/s}) - k\left(\frac{4}{12} \text{ ft}\right)$

or $\quad k = 2700 \frac{1}{\text{s}}$

(*a*) We have $\quad v = v_0 - kx$

Then $\quad a = \frac{dv}{dt} = \frac{d}{dt}(v_0 - kx) = -kv$

or $\quad a = -k(v_0 - kx)$

At $t = 0$: $\quad a = 2700 \frac{1}{\text{s}}(900 \text{ ft/s} - 0)$

or $\quad a_0 = -2.43 \times 10^6 \text{ ft/s}^2$ ◀

(*b*) We have $\quad \frac{dx}{dt} = v = v_0 - kx$

At $t = 0$, $x = 0$: $\quad \int_0^x \frac{dx}{v_0 - kx} = \int_0^t dt$

or $\quad -\frac{1}{k}[\ln(v_0 - kx)]_0^x = t$

or $\quad t = \frac{1}{k}\ln\left(\frac{v_0}{v_0 - kx}\right) = \frac{1}{k}\ln\left(\frac{1}{1 - \frac{k}{v_0}x}\right)$

When $x = 3.9$ in.: $\quad t = \frac{1}{2700 \frac{1}{\text{s}}}\ln\left[\frac{1}{1 - \frac{2700 \text{ 1/s}}{900 \text{ ft/s}}\left(\frac{3.9}{12} \text{ ft}\right)}\right]$

or $\quad t = 1.366 \times 10^{-3}$ s ◀

PROBLEM 11.17

The acceleration of a particle is defined by the relation $a = -k/x$. It has been experimentally determined that $v = 15$ ft/s when $x = 0.6$ ft and that $v = 9$ ft/s when $x = 1.2$ ft. Determine (*a*) the velocity of the particle when $x = 1.5$ ft, (*b*) the position of the particle at which its velocity is zero.

SOLUTION

$$a = \frac{v\,dv}{dx} = \frac{-k}{x}$$

Separate and integrate using $x = 0.6$ ft, $v = 15$ ft/s.

$$\int_{15}^{v} v\,dv = -k\int_{0.6}^{x} \frac{dx}{x}$$

$$\frac{1}{2}v^2\Big|_{15}^{v} = -k\ln x\Big|_{0.6}^{x}$$

$$\frac{1}{2}v^2 - \frac{1}{2}(15)^2 = -k\ln\left(\frac{x}{0.6}\right) \qquad (1)$$

When $v = 9$ ft/s, $x = 1.2$ ft

$$\frac{1}{2}(9)^2 - \frac{1}{2}(15)^2 = -k\ln\left(\frac{1.2}{0.6}\right)$$

Solve for *k*.

$$k = 103.874 \text{ ft}^2/\text{s}^2$$

(*a*) Velocity when $x = 65$ ft.

Substitute $k = 103.874 \text{ ft}^2/\text{s}^2$ and $x = 1.5$ ft into (1).

$$\frac{1}{2}v^2 - \frac{1}{2}(15)^2 = -103.874\ln\left(\frac{1.5}{0.6}\right)$$

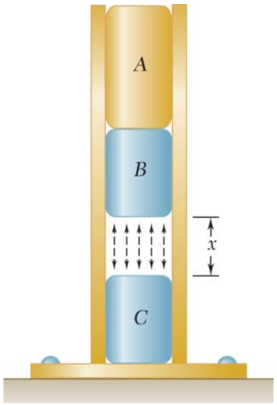
$v = 5.89$ ft/s ◀

(*b*) Position when for $v = 0$,

$$0 - \frac{1}{2}(15)^2 = -103.874\ln\left(\frac{x}{0.6}\right)$$

$$\ln\left(\frac{x}{0.6}\right) = 1.083$$

$x = 1.772$ ft ◀



PROBLEM 11.18

A brass (nonmagnetic) block *A* and a steel magnet *B* are in equilibrium in a brass tube under the magnetic repelling force of another steel magnet *C* located at a distance $x = 0.004$ m from *B*. The force is inversely proportional to the square of the distance between *B* and *C*. If block *A* is suddenly removed, the acceleration of block *B* is $a = -9.81 + k/x^2$, where a and x are expressed in m/s^2 and m, respectively, and $k = 4 \times 10^{-4}\ \text{m}^3/\text{s}^2$. Determine the maximum velocity and acceleration of *B*.

SOLUTION

The maximum velocity occurs when $a = 0$. $\qquad 0 = -9.81 + \dfrac{k}{x_m^2}$

$$x_m^2 = \frac{k}{9.81} = \frac{4 \times 10^{-4}}{9.81} = 40.775 \times 10^{-6}\ \text{m}^2 \qquad x_m = 0.0063855\ \text{m}$$

The acceleration is given as a function of *x*.

$$v\frac{dv}{dx} = a = -9.81 + \frac{k}{x^2}$$

Separate variables and integrate:

$$vdv = -9.81dx + \frac{k\,dx}{x^2}$$

$$\int_0^v vdv = -9.81\int_{x_0}^{x} dx + k\int_{x_0}^{x} \frac{dx}{x^2}$$

$$\frac{1}{2}v^2 = -9.81(x - x_0) - k\left(\frac{1}{x} - \frac{1}{x_0}\right)$$

$$\frac{1}{2}v_m^2 = -9.81(x_m - x_0) - k\left(\frac{1}{x_m} - \frac{1}{x_0}\right)$$

$$= -9.81(0.0063855 - 0.004) - (4 \times 10^{-4})\left(\frac{1}{0.0063855} - \frac{1}{0.004}\right)$$

$$= -0.023402 + 0.037358 = 0.013956\ \text{m}^2/\text{s}^2$$

Maximum velocity: $\qquad v_m = 0.1671\ \text{m/s} \qquad\qquad v_m = 167.1\ \text{mm/s}$ ↑ ◀

The maximum acceleration occurs when *x* is smallest, that is, $x = 0.004$ m.

$$a_m = -9.81 + \frac{4 \times 10^{-4}}{(0.004)^2} \qquad\qquad a_m = 15.19\ \text{m/s}^2$$ ↑ ◀

PROBLEM 11.19

Based on experimental observations, the acceleration of a particle is defined by the relation $a = -(0.1 + \sin x/b)$, where a and x are expressed in m/s^2 and meters, respectively. Knowing that $b = 0.8$ m and that $v = 1$ m/s when $x = 0$, determine (*a*) the velocity of the particle when $x = -1$ m, (*b*) the position where the velocity is maximum, (*c*) the maximum velocity.

SOLUTION

We have $$v\frac{dv}{dx} = a = -\left(0.1 + \sin\frac{x}{0.8}\right)$$

When $x = 0,\ v = 1$ m/s: $$\int_1^v v\,dv = \int_0^x -\left(0.1 + \sin\frac{x}{0.8}\right)dx$$

or $$\frac{1}{2}(v^2 - 1) = -\left[0.1x - 0.8\cos\frac{x}{0.8}\right]_0^x$$

or $$\frac{1}{2}v^2 = -0.1x + 0.8\cos\frac{x}{0.8} - 0.3$$

(*a*) When $x = -1$ m: $$\frac{1}{2}v^2 = -0.1(-1) + 0.8\cos\frac{-1}{0.8} - 0.3$$

or $v = \pm 0.323$ m/s ◀

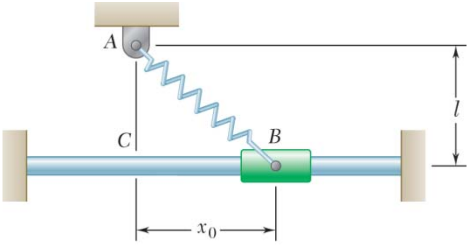
(*b*) When $v = v_{\text{max}},\ a = 0$: $$-\left(0.1 + \sin\frac{x}{0.8}\right) = 0$$

or $x = -0.080\,134$ m $x = -0.0801$ m ◀

(*c*) When $x = -0.080\,134$ m:

$$\frac{1}{2}v_{\text{max}}^2 = -0.1(-0.080\,134) + 0.8\cos\frac{-0.080\,134}{0.8} - 0.3$$
$$= 0.504\ \text{m}^2/\text{s}^2$$

or $v_{\text{max}} = 1.004$ m/s ◀



PROBLEM 11.20

A spring AB is attached to a support at A and to a collar. The unstretched length of the spring is l. Knowing that the collar is released from rest at $x = x_0$ and has an acceleration defined by the relation $a = -100(x - lx/\sqrt{l^2 + x^2})$, determine the velocity of the collar as it passes through Point C.

SOLUTION

Since a is function of x,

$$a = v\frac{dv}{dx} = -100\left(x - \frac{lx}{\sqrt{l^2 + x^2}}\right)$$

Separate variables and integrate:

$$\int_{v_0}^{v_f} v\,dv = -100\int_{x_0}^{0}\left(x - \frac{lx}{\sqrt{l^2 + x^2}}\right)dx$$

$$\frac{1}{2}v_f^2 - \frac{1}{2}v_0^2 = -100\left(\frac{x^2}{2} - l\sqrt{l^2 + x^2}\right)\Bigg|_{x_0}^{0}$$

$$\frac{1}{2}v_f^2 - 0 = -100\left(-\frac{x_0^2}{2} - l^2 + l\sqrt{l^2 + x_0^2}\right)$$

$$\frac{1}{2}v_f^2 = \frac{100}{2}(-l^2 + x_0^2 - l^2 - 2l\sqrt{l^2 + x_0^2})$$

$$= \frac{100}{2}(\sqrt{l^2 + x_0^2} - l)^2$$

$$v_f = 10(\sqrt{l^2 + x_0^2} - l) \blacktriangleleft$$

PROBLEM 11.21

The acceleration of a particle is defined by the relation $a = k\left(1 - e^{-x}\right)$, where k is a constant. Knowing that the velocity of the particle is $v = +9$ m/s when $x = -3$ m and that the particle comes to rest at the origin, determine (*a*) the value of k, (*b*) the velocity of the particle when $x = -2$ m.

SOLUTION

Acceleration: $a = k\left(1 - e^{-x}\right)$

Given: at $x = -3$ m, $v = 9$ m/s

at $x = 0$ m, $v = 0$ m/s

$$adx = vdv$$

$$\mathrm{k}\left(1 - e^{-x}\right)dx = vdv$$

Integrate using x=-3 m and v=9 m/s as the lower limits of the integrals

$$\int_{-3}^{x} \mathrm{k}\left(1 - e^{-x}\right)dx = \int_{9}^{v} vdv$$

$$k\left(x + e^{-x}\right)\Big|_{-3}^{x} = \frac{1}{2}v^2\Big|_{9}^{v}$$

Velocity: $k\left(x + e^{-x} - \left(-3 + e^{3}\right)\right) = \frac{1}{2}v^2 - \frac{1}{2}(9)^2$ (1)

(a) Now substitute v=0 m/s and x=0 m into (1) and solve for k

$$k\left(0 + e^{-0} - \left(-3 + e^{3}\right)\right) = \frac{1}{2}0^2 - \frac{1}{2}(9)^2$$

$k = 2.52\ \mathrm{m}^2/\mathrm{s}^2$ ◀

(b) Find velocity when x= -2 m using the equation (1) and the value of k

$$2.518\left(-2 + e^{2} - \left(-3 + e^{3}\right)\right) = \frac{1}{2}v^2 - \frac{1}{2}(9)^2$$

$v = 4.70$ m/s ◀

PROBLEM 11.22

Starting from $x = 0$ with no initial velocity, a particle is given an acceleration $a = 0.1\sqrt{v^2 + 16}$, where a and v are expressed in ft/s^2 and ft/s, respectively. Determine (*a*) the position of the particle when $v =$ 3ft/s, (*b*) the speed and acceleration of the particle when $x = 4$ ft.

SOLUTION

$$a = \frac{vdv}{dx} = 0.1(v^2 + 16)^{1/2} \tag{1}$$

Separate and integrate.

$$\int_0^v \frac{vdv}{\sqrt{v^2 + 16}} = \int_0^x 0.1\, dx$$

$$(v^2 + 16)^{1/2}\Big|_0^v = 0.1x$$

$$(v^2 + 16)^{1/2} - 4 = 0.1x$$

$$x = 10[(v^2 + 16)^{1/2} - 4] \tag{2}$$

(*a*) $v = 3$ ft/s.

$$x = 10[(3^2 + 16)^{1/2} - 4] \qquad x = 10.00 \text{ ft} \blacktriangleleft$$

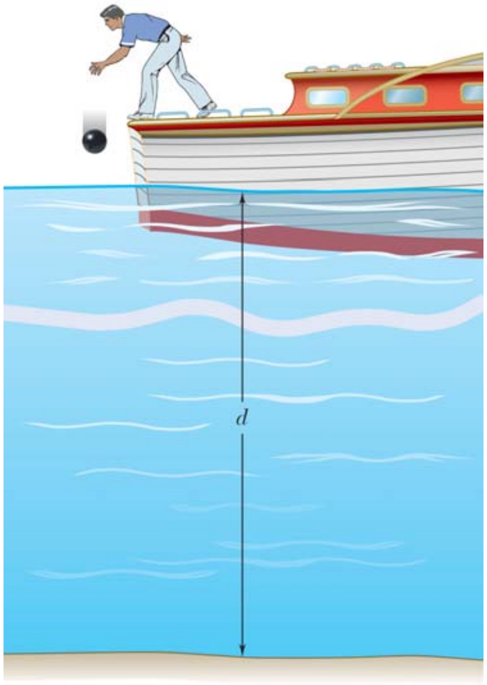
(*b*) $x = 4$ ft.

From (2), $$(v^2 + 16)^{1/2} = 4 + 0.1x = 4 + (0.1)(4) = 4.4$$

$$v^2 + 16 = 19.36$$

$$v^2 = 3.36 \text{ ft}^2/\text{s}^2 \qquad v = 1.833 \text{ ft/s} \blacktriangleleft$$

From (1), $$a = 0.1(1.833^2 + 16)^{1/2} \qquad a = 0.440 \text{ ft/s}^2 \blacktriangleleft$$



PROBLEM 11.23

A ball is dropped from a boat so that it strikes the surface of a lake with a speed of 16.5 ft/s. While in the water the ball experiences an acceleration of $a = 10 - 0.8v$, where a and v are expressed in ft/s^2 and ft/s, respectively. Knowing the ball takes 3 s to reach the bottom of the lake, determine (*a*) the depth of the lake, (*b*) the speed of the ball when it hits the bottom of the lake.

SOLUTION

$$a = \frac{dv}{dt} = 10 - 0.8v$$

Separate and integrate:

$$\int_{v_0}^{v} \frac{dv}{10 - 0.8v} = \int_0^t dt$$

$$-\frac{1}{0.8}\ln(10 - 0.8v)\bigg|_{v_0}^{v} = t$$

$$\ln\left(\frac{10 - 0.8v}{10 - 0.8v_0}\right) = -0.8t$$

$$10 - 0.8v = (10 - 0.8v_0)e^{-0.8t}$$

or $$0.8v = 10 - (10 - 0.8v_0)e^{-0.8t}$$

$$v = 12.5 - (12.5 - v_0)e^{-0.8t}$$

With $v_0 = 16.5\,\text{ft/s}$ $$v = 12.5 + 4e^{-0.8t}$$

Integrate to determine x as a function of t.

$$v = \frac{dx}{dt} = 12.5 + 4e^{-0.8t}$$

$$\int_0^x dx = \int_0^t (12.5 + 4e^{-0.8t})dt$$

PROBLEM 11.23 (Continued)

$$x = 12.5t - 5e^{-0.8t}\Big|_0^t = 12.5t - 5e^{-0.8t} + 5$$

(*a*) At $t = 35$ s,

$x = 12.5(3) - 5e^{-2.4} + 5 = 42.046$ ft $\qquad x = 42.0$ ft ◀

(*b*) $v = 12.5 + 4e^{-2.4} = 12.863$ ft/s $\qquad v = 12.86$ ft/s ◀

PROBLEM 11.24

The acceleration of a particle is defined by the relation $a = -k\sqrt{v}$, where k is a constant. Knowing that $x = 0$ and $v = 81$ m/s at $t = 0$ and that $v = 36$ m/s when $x = 18$ m, determine (*a*) the velocity of the particle when $x = 20$ m, (*b*) the time required for the particle to come to rest.

SOLUTION

(*a*) We have $$v\frac{dv}{dx} = a = -k\sqrt{v}$$

so that $$\sqrt{v}\, dv = -k\,dx$$

When $x = 0, v = 81\text{ m/s}$: $$\int_{81}^{v} \sqrt{v}\, dv = \int_{0}^{x} -k\,dx$$

or $$\frac{2}{3}[v^{3/2}]_{81}^{v} = -kx$$

or $$\frac{2}{3}[v^{3/2} - 729] = -kx$$

When $x = 18\text{ m}, v = 36\text{ m/s}$: $$\frac{2}{3}(36^{3/2} - 729) = -k(18)$$

or $$k = 19\sqrt{\text{m/s}^2}$$

Finally

When $x = 20\text{ m}$: $$\frac{2}{3}(v^{3/2} - 729) = -19(20)$$

or $$v^{3/2} = 159$$ $v = 29.3$ m/s ◀

(*b*) We have $$\frac{dv}{dt} = a = -19\sqrt{v}$$

At $t = 0, v = 81\text{ m/s}$: $$\int_{81}^{v} \frac{dv}{\sqrt{v}} = \int_{0}^{t} -19dt$$

or $$2[\sqrt{v}]_{81}^{v} = -19t$$

or $$2(\sqrt{v} - 9) = -19t$$

When $v = 0$: $$2(-9) = -19t$$

or $t = 0.947$ s ◀

PROBLEM 11.25

The acceleration of a particle is defined by the relation $a = -kv^{2.5}$, where k is a constant. The particle starts at $x = 0$ with a velocity of 16 mm/s, and when $x = 6$ mm the velocity is observed to be 4 mm/s. Determine (*a*) the velocity of the particle when $x = 5$ mm, (*b*) the time at which the velocity of the particle is 9 mm/s.

SOLUTION

Acceleration: $a = -kv^{2.5}$

Given: at t=0, $x = 0$ mm, $v = 16$ mm/s

at $x = 6$ mm, $v = 4$ mm/s

$$adx = vdv$$

$$-kv^{2.5}dx = vdv$$

Separate variables $-kdx = v^{-3/2}dv$

Integrate using x=-0 m and v=16 mm/s as the lower limits of the integrals

$$\int_0^x \text{-k}dx = \int_{16}^v v^{-3/2}dv$$

$$-kx\Big|_0^x = -2v^{-1/2}\Big|_{16}^v$$

Velocity and position: $kx = 2v^{-1/2} - \frac{1}{2}$ (1)

Now substitute v=4 mm/s and x=6 mm into (1) and solve for k

$$k(6) = 4^{-1/2} - \frac{1}{2}$$

$$k = 0.0833 \text{ mm}^{-3/2}s^{1/2}$$

(a) Find velocity when x= 5 mm using the equation (1) and the value of k

$$k(5) = 2v^{-1/2} - \frac{1}{2}$$

$v = 4.76$ mm/s◀

(b) $a = \frac{dv}{dt}$ or $adt = dv$

$a = \frac{dv}{dt}$ or $adt = dv$

Separate variables $-kdt = v^{-2.5}dv$

PROBLEM 11.25 (Continued)

Integrate using t=0 and v=16 mm/s as the lower limits of the integrals

$$\int_0^t \text{-k}dt = \int_{16}^v v^{-5/2}dv$$

$$-kt\Big|_0^t = -\frac{2}{3}v^{-3/2}\Big|_{16}^v$$

Velocity and time: $$kt = \frac{2}{3}v^{-3/2} - \frac{1}{96} \qquad (2)$$

Find time when v= 9 mm/s using the equation (2) and the value of k

$$kt = \frac{2}{3}(9)^{-3/2} - \frac{1}{96}$$

$t = 0.171$ s◀

Problem 11.26

A human powered vehicle (HPV) team wants to model the acceleration during the 260 *m* sprint race (the first 60 *m* is called a flying start) using $a = A - Cv^2$, where a is acceleration in m/s^2 and v is the velocity in m/s. From wind tunnel testing, they found that $C = 0.0012\ \text{m}^{-1}$. Knowing that the cyclist is going 100 km/h at the 260 meter mark, what is the value of A?

SOLUTION

Acceleration: $a = A - Cv^2\ \text{m/s}^2$

Given: $C = 0.0012\ \text{m}^{-1}$, $v_f = 100$ km/hr when $x_f = 260$ m

Note: 100 km/hr = 27.78 m/s

$$adx = vdv$$

$$\left(A - Cv^2\right)dx = vdv$$

Separate variables $$dx = \frac{vdv}{A - Cv^2}$$

Integrate starting from rest and traveling a distance x_f with a final velocity v_f.

$$\int_0^{x_f} dx = \int_0^{v_f} \frac{vdv}{A - Cv^2}$$

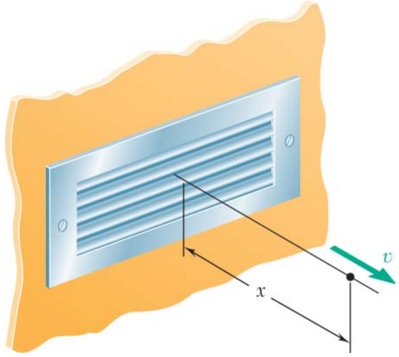
$$x\Big|_0^{x_f} = -\frac{1}{2C}\ln\left(A - Cv^2\right)\Big|_0^{v_f}$$

$$x_f = -\frac{1}{2C}\ln\left(A - v_f^2\right) + \frac{1}{2C}\ln\left(A\right)$$

$$x_f = \frac{1}{2C}\ln\left(\frac{A}{A - Cv_f^2}\right)$$

Next solve for A $$A = -\frac{Cv_f^2 e^{2Cx_f}}{1 - e^{2Cx_f}}$$

Now substitute values of C, x_f and v_f and solve for A $\quad A = 1.995\ \text{m/s}^2$ ◀



PROBLEM 11.27

Experimental data indicate that in a region downstream of a given louvered supply vent the velocity of the emitted air is defined by $v = 0.18v_0/x$, where v and x are expressed in m/s and meters, respectively, and v_0 is the initial discharge velocity of the air. For $v_0 = 3.6$ m/s, determine (*a*) the acceleration of the air at $x = 2$ m, (*b*) the time required for the air to flow from $x = 1$ to $x = 3$ m.

SOLUTION

(*a*) We have $$a = v\frac{dv}{dx}$$

$$= \frac{0.18v_0}{x}\frac{d}{dx}\left(\frac{0.18v_0}{x}\right)$$

$$= -\frac{0.0324v_0^2}{x^3}$$

When $x = 2$ m: $$a = -\frac{0.0324(3.6)^2}{(2)^3}$$

or $$a = -0.0525 \text{ m/s}^2 \blacktriangleleft$$

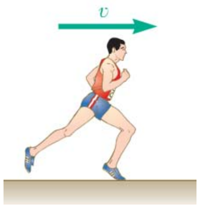
(*b*) We have $$\frac{dx}{dt} = v = \frac{0.18v_0}{x}$$

From $x = 1$ m to $x = 3$ m: $$\int_1^3 x\,dx = \int_{t_1}^{t_3} 0.18v_0\,dt$$

or $$\left[\frac{1}{2}x^2\right]_1^3 = 0.18v_0(t_3 - t_1)$$

or $$(t_3 - t_1) = \frac{\frac{1}{2}(9-1)}{0.18(3.6)}$$

or $$t_3 - t_1 = 6.17 \text{ s} \blacktriangleleft$$



PROBLEM 11.28

Based on observations, the speed of a jogger can be approximated by the relation $v = 7.5(1-0.04x)^{0.3}$, where v and x are expressed in mi/h and miles, respectively. Knowing that $x = 0$ at $t = 0$, determine (*a*) the distance the jogger has run when $t = 1$ h, (*b*) the jogger's acceleration in ft/s^2 at $t = 0$, (*c*) the time required for the jogger to run 6 mi.

SOLUTION

(*a*) We have

$$\frac{dx}{dt} = v = 7.5(1-0.04x)^{0.3}$$

At $t = 0,\ x = 0$:

$$\int_0^x \frac{dx}{(1-0.04x)^{0.3}} = \int_0^t 7.5dt$$

or

$$\frac{1}{0.7}\left(-\frac{1}{0.04}\right)[(1-0.04x)^{0.7}]_0^x = 7.5t$$

or

$$1-(1-0.04x)^{0.7} = 0.21t \qquad (1)$$

or

$$x = \frac{1}{0.04}[1-(1-0.21t)^{1/0.7}]$$

At $t = 1$ h:

$$x = \frac{1}{0.04}\{1-[1-0.21(1)]^{1/0.7}\}$$

or

$$x = 7.15 \text{ mi} \blacktriangleleft$$

(*b*) We have

$$\begin{aligned} a &= v\frac{dv}{dx} \\ &= 7.5(1-0.04x)^{0.3}\frac{d}{dx}[7.5(1-0.04x)^{0.3}] \\ &= 7.5^2(1-0.04x)^{0.3}[(0.3)(-0.04)(1-0.04x)^{-0.7}] \\ &= -0.675(1-0.04x)^{-0.4} \end{aligned}$$

At $t = 0,\ x = 0$:

$$a_0 = -0.675 \text{ mi/h}^2 \times \frac{5280 \text{ ft}}{1 \text{ mi}} \times \left(\frac{1 \text{ h}}{3600 \text{ s}}\right)^2$$

or

$$a_0 = -275 \times 10^{-6} \text{ ft/s}^2 \blacktriangleleft$$

(*c*) From Eq. (1)

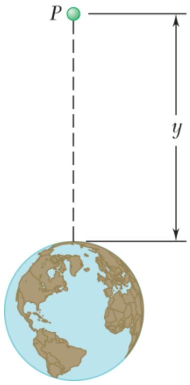
$$t = \frac{1}{0.21}[1-(1-0.04x)^{0.7}]$$

When $x = 6$ mi:

$$\begin{aligned} t &= \frac{1}{0.21}\{1-[1-0.04(6)]^{0.7}\} \\ &= 0.83229 \text{ h} \end{aligned}$$

or

$$t = 49.9 \text{ min} \blacktriangleleft$$



PROBLEM 11.29

The acceleration due to gravity at an altitude y above the surface of the earth can be expressed as

$$a = \frac{-32.2}{[1 + (y/20.9 \times 10^6)]^2}$$

where a and y are expressed in ft/s^2 and feet, respectively. Using this expression, compute the height reached by a projectile fired vertically upward from the surface of the earth if its initial velocity is (*a*) 1800 ft/s, (*b*) 3000 ft/s, (*c*) 36,700 ft/s.

SOLUTION

We have
$$v\frac{dv}{dy} = a = -\frac{32.2}{\left(1 + \frac{y}{20.9 \times 10^6}\right)^2}$$

When
$$y = 0, \qquad v = v_0$$

provided that v does reduce to zero,
$$y = y_{\max}, \quad v = 0$$

Then
$$\int_{v_0}^{0} v\,dv = \int_0^{y_{\max}} \frac{-32.2}{\left(1 + \frac{y}{20.9 \times 10^6}\right)^2} dy$$

or
$$-\frac{1}{2}v_0^2 = -32.2\left[-20.9 \times 10^6 \frac{1}{1 + \frac{y}{20.9 \times 10^6}}\right]_0^{y_{\max}}$$

or
$$v_0^2 = 1345.96 \times 10^6 \left(1 - \frac{1}{1 + \frac{y_{\max}}{20.9 \times 10^6}}\right)$$

or
$$y_{\max} = \frac{v_0^2}{64.4 - \frac{v_0^2}{20.9 \times 10^6}}$$

(*a*) $v_0 = 1800$ ft/s:
$$y_{\max} = \frac{(1800)^2}{64.4 - \frac{(1800)^2}{20.9 \times 10^6}}$$

or $$y_{\max} = 50.4 \times 10^3 \text{ ft} \blacktriangleleft$$

(*b*) $v_0 = 3000$ ft/s:
$$y_{\max} = \frac{(3000)^2}{64.4 - \frac{(3000)^2}{20.9 \times 10^6}}$$

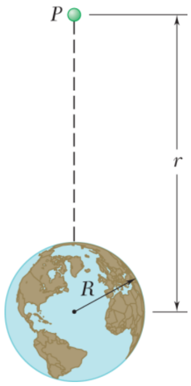
or $$y_{\max} = 140.7 \times 10^3 \text{ ft} \blacktriangleleft$$

PROBLEM 11.29 (Continued)

(*c*) $v_0 = 36,700$ ft/s: $$y_{\max} = \frac{(36,700)^2}{64.4 - \frac{(36,700)^2}{20.9\times 10^6}} = -3.03\times 10^{10} \text{ ft}$$

This solution is invalid since the velocity does not reduce to zero. The velocity 36,700 ft/s is above the escape velocity v_R from the earth. For v_R and above.

$y_{\max} \longrightarrow \infty$ ◀



PROBLEM 11.30

The acceleration due to gravity of a particle falling toward the earth is $a = -gR^2/r^2$, where r is the distance from the *center* of the earth to the particle, R is the radius of the earth, and g is the acceleration due to gravity at the surface of the earth. If $R = 3960$ mi, calculate the *escape velocity,* that is, the minimum velocity with which a particle must be projected vertically upward from the surface of the earth if it is not to return to the earth. (*Hint:* $v = 0$ for $r = \infty$.)

SOLUTION

We have $$v\frac{dv}{dr} = a = -\frac{gR^2}{r^2}$$

When $$r = R, \quad v = v_e$$
$$r = \infty, \quad v = 0$$

then $$\int_{v_e}^{0} v\,dv = \int_R^\infty -\frac{gR^2}{r^2}\,dr$$

or $$-\frac{1}{2}v_e^2 = gR^2\left[\frac{1}{r}\right]_R^\infty$$

or $$v_e = \sqrt{2gR}$$
$$= \left(2 \times 32.2\ \text{ft/s}^2 \times 3960\ \text{mi} \times \frac{5280\ \text{ft}}{1\ \text{mi}}\right)^{1/2}$$

or $$v_e = 36.7 \times 10^3\ \text{ft/s}$$ ◀

PROBLEM 11.31

The velocity of a particle is $v = v_0[1 - \sin(\pi t/T)]$. Knowing that the particle starts from the origin with an initial velocity v_0, determine (*a*) its position and its acceleration at $t = 3T$, (*b*) its average velocity during the interval $t = 0$ to $t = T$.

SOLUTION

(*a*) We have

$$\frac{dx}{dt} = v = v_0\left[1 - \sin\left(\frac{\pi t}{T}\right)\right]$$

At $t = 0, x = 0$:

$$\int_0^x dx = \int_0^t v_0\left[1 - \sin\left(\frac{\pi t}{T}\right)\right]dt$$

$$x = v_0\left[t + \frac{T}{\pi}\cos\left(\frac{\pi t}{T}\right)\right]_0^t = v_0\left[t + \frac{T}{\pi}\cos\left(\frac{\pi t}{T}\right) - \frac{T}{\pi}\right] \qquad (1)$$

At $t = 3T$:

$$x_{3T} = v_0\left[3T + \frac{T}{\pi}\cos\left(\frac{\pi \times 3T}{T}\right) - \frac{T}{\pi}\right] = v_0\left(3T - \frac{2T}{\pi}\right) \qquad x_{3T} = 2.36\, v_0 T \blacktriangleleft$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left\{v_0\left[1 - \sin\left(\frac{\pi t}{T}\right)\right]\right\} = -v_0\frac{\pi}{T}\cos\frac{\pi t}{T}$$

At $t = 3T$:

$$a_{3T} = -v_0\frac{\pi}{T}\cos\frac{\pi \times 3T}{T} \qquad a_{3T} = \frac{\pi v_0}{T} \blacktriangleleft$$

(*b*) Using Eq. (1)

At $t = 0$:

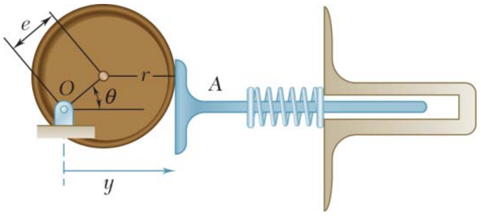
$$x_0 = v_0\left[0 + \frac{T}{\pi}\cos(0) - \frac{T}{\pi}\right] = 0$$

At $t = T$:

$$x_T = v_0\left[T + \frac{T}{\pi}\cos\left(\frac{\pi T}{T}\right) - \frac{T}{\pi}\right] = v_0\left(T - \frac{2T}{\pi}\right) = 0.363 v_0 T$$

Now

$$v_{\text{ave}} = \frac{x_T - x_0}{\Delta t} = \frac{0.363 v_0 T - 0}{T - 0} \qquad v_{\text{ave}} = 0.363 v_0 \blacktriangleleft$$



Problem 11.32

An eccentric circular cam, which serves a similar function as the Scotch yoke mechanism in Problem 11.13, is used in conjunction with a flat face follower to control motion in pumps and in steam engine valves. Knowing that the eccentricity is denoted by e, the maximum range of the displacement of the follower is $d_{\max}$, and the maximum velocity of the follower is $v_{\max}$, determine the displacement, velocity, and acceleration of the follower.

SOLUTION

Constraint:

$$y = r + e\cos\theta \tag{1}$$

Differentiate:

$$\dot{y} = -e\dot{\theta}\sin\theta \tag{2}$$

Differentiate again:

$$\ddot{y} = -e\ddot{\theta}\sin\theta - e\dot{\theta}^2\cos\theta \tag{3}$$

$y_{\max}$ occurs when cosθ =1 and $y_{\min}$ occurs when cosθ = -1

$$d_{\max} = y_{\max} - y_{\min}$$

$$d_{\max} = r + e - (r - e)$$

$$d_{\max} = 2e$$

$$e = \frac{d_{\max}}{2} \tag{4}$$

Substitute (4) into (1) to get Position $\quad y = r + \dfrac{d_{\max}}{2}\cos\theta$ ◀

Max Velocity occurs when $\sin\theta = \pm 1$

$$v_{\max} = \mp e\dot{\theta}$$

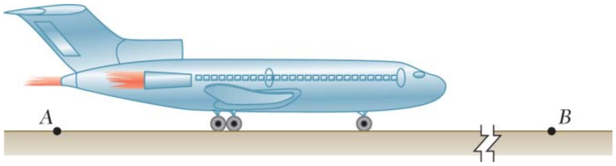
$$\dot{\theta} = \frac{v_{\max}}{\mp e} \tag{5}$$

Substitute (5) into (2) to get velocity and assume cw rotation. $\quad \dot{y} = -v_{\max}\sin\theta$

Substitute (4) and (5) into (3)

$$\ddot{y} = -\frac{d_{\max}}{2}\ddot{\theta}\sin\theta - \frac{d_{\max}}{2}\left(\frac{4v_{\max}^2}{d_{\max}^2}\right)\cos\theta$$

Acceleration: $\quad \ddot{y} = -\dfrac{d_{\max}}{2}\ddot{\theta}\sin\theta - \dfrac{2v_{\max}^2}{d_{\max}}\cos\theta$ ◀



Problem 11.33

An airplane begins its take-off run at *A* with zero velocity and a constant acceleration *a*. Knowing that it becomes airborne 30 s later at *B* and that the distance *AB* is 900 m, determine (*a*) the acceleration *a*, (*b*) the take-off velocity v_B.

SOLUTION

Given: $x_A = 0,\ x_B = 900\text{ m},\ v_A = 0,\ t = 30\text{ s}$

Uniform Acceleration: $x_B = x_A + v_A t + \frac{1}{2}at^2$

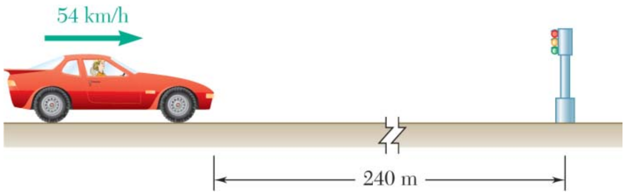
Substitute known values: $900\text{ m} = 0 + 0 + \frac{1}{2}a(30\text{ s})^2$

(a) Solve for acceleration $a = 2.0\text{ m/s}^2$ ◀

Uniform Acceleration: $v_B = v_A + at$

Substitute known values: $v_B = 0 + (2\text{ m/s}^2)(30\text{ s})$

(b) Velocity at takeoff (point B) $v_B = 60.0\text{ m/s}$ ◀



PROBLEM 11.34

A motorist is traveling at 54 km/h when she observes that a traffic light 240 m ahead of her turns red. The traffic light is timed to stay red for 24 s. If the motorist wishes to pass the light without stopping just as it turns green again, determine (*a*) the required uniform deceleration of the car, (*b*) the speed of the car as it passes the light.

SOLUTION

Uniformly accelerated motion:

$$x_0 = 0 \quad v_0 = 54 \text{ km/h} = 15 \text{ m/s}$$

(a) $x = x_0 + v_0 t + \frac{1}{2} a t^2$

when $t = 24\text{ s}$, $x = 240$ m:

$$240 \text{ m} = 0 + (15 \text{ m/s})(24 \text{ s}) + \frac{1}{2} a (24 \text{ s})^2$$

$$a = -0.4167 \text{ m/s}^2$$

$a = -0.417 \text{ m/s}^2$ ◀

(*b*) $v = v_0 + a t$

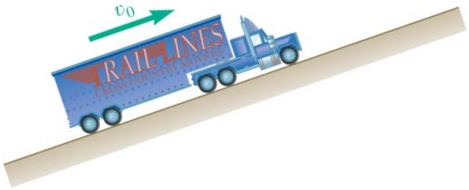
when $t = 24\text{ s}$:

$$v = (15 \text{ m/s}) + (-0.4167 \text{ m/s})(24 \text{ s})$$

$$v = 5.00 \text{ m/s}$$

$$v = 18.00 \text{ km/h}$$

$v = 18.00$ km/h ◀



Problem 11.35

Steep safety ramps are built beside mountain highways to enable vehicles with defective brakes to stop safely. A truck enters a 750-ft ramp at a high speed v_0 and travels 540 ft in 6 s at constant deceleration before its speed is reduced to $v_0/2$. Assuming the same constant deceleration, determine (*a*) the additional time required for the truck to stop, (*b*) the additional distance traveled by the truck.

SOLUTION

Given: $x_o = 0,\ x_A = 540\text{ m},\ t_A = 6\text{ s},\ v_A = \frac{1}{2}v_o,\ L_{ramp} = 750\,ft$

Uniform Acceleration: $v_A = v_o + at_A$

Substitute known values: $\frac{1}{2}v_o = v_o + (a)(6)$

$$a = -\frac{1}{12}v_o$$

Uniform Acceleration: $x_A = x_o + v_o t_A + \frac{1}{2}at_A^2$

Substitute known values: $540 = 0 + v_o(6) - \frac{1}{24}v_o(6)^2$

$$v_o = 120\text{ ft/s and } a = -10\text{ ft/s}^2$$

(a) $v_B = v_o + at_B$

Substitute known values: $0 = 120 - 10t_B$

Solve for t_B: $t_B = 12\text{ s}$

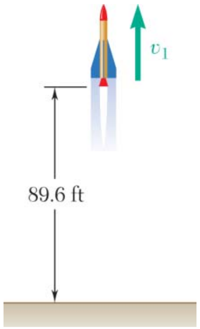
Additional time to stop $t_B - t_A = 6.0\text{ s}$ ◀

(b) $x_B = x_o + v_o t_B + \frac{1}{2}at_B^2$

Substitute known values: $x_B = 0 + 120(12) - \frac{1}{2}10(12)^2$

Solve for x_B: $x_B = 720\text{ ft}$

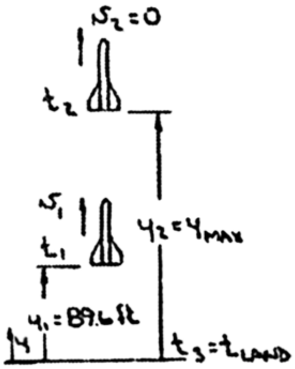
Additional time to stop $x_B - x_A = 180.0\text{ ft}$ ◀



PROBLEM 11.36

A group of students launches a model rocket in the vertical direction. Based on tracking data, they determine that the altitude of the rocket was 89.6 ft at the end of the powered portion of the flight and that the rocket landed 16 s later. Knowing that the descent parachute failed to deploy so that the rocket fell freely to the ground after reaching its maximum altitude and assuming that $g = 32.2\ \text{ft/s}^2$, determine (*a*) the speed v_1 of the rocket at the end of powered flight, (*b*) the maximum altitude reached by the rocket.

SOLUTION



(*a*) We have $y = y_1 + v_1 t + \frac{1}{2}at^2$

At t_{land}, $y = 0$

Then $0 = 89.6\ \text{ft} + v_1(16\ \text{s}) + \frac{1}{2}(-32.2\ \text{ft/s}^2)(16\ \text{s})^2$

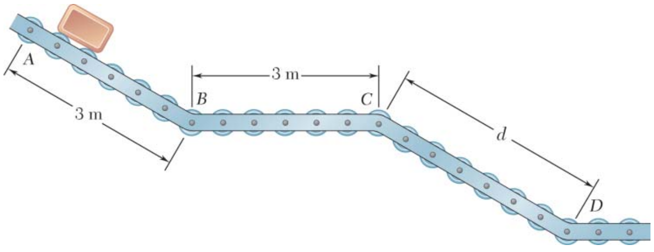
or $v_1 = 252\ \text{ft/s}$ ◀

(*b*) We have $v^2 = v_1^2 + 2a(y - y_1)$

At $y = y_{\text{max}}, \quad v = 0$

Then $0 = (252\ \text{ft/s})^2 + 2(-32.2\ \text{ft/s}^2)(y_{\text{max}} - 89.6)\ \text{ft}$

or $y_{\text{max}} = 1076\ \text{ft}$ ◀



PROBLEM 11.37

A small package is released from rest at A and moves along the skate wheel conveyor $ABCD$. The package has a uniform acceleration of 4.8 m/s^2 as it moves down sections AB and CD, and its velocity is constant between B and C. If the velocity of the package at D is 7.2 m/s, determine (a) the distance d between C and D, (b) the time required for the package to reach D.

SOLUTION

(a) For $A \longrightarrow B$ and $C \longrightarrow D$ we have

$$v^2 = v_0^2 + 2a(x - x_0)$$

Then, at B $\quad v_{BC}^2 = 0 + 2(4.8\text{ m/s}^2)(3-0)\text{ m}$

$$= 28.8\text{ m}^2/\text{s}^2 \qquad (v_{BC} = 5.3666\text{ m/s})$$

and at D $\quad v_D^2 = v_{BC}^2 + 2a_{CD}(x_D - x_C) \qquad d = x_D - x_C$

or $\quad (7.2\text{ m/s})^2 = (28.8\text{ m}^2/\text{s}^2) + 2(4.8\text{ m/s}^2)d$

or $\quad d = 2.40\text{ m}$ ◀

(b) For $A \longrightarrow B$ and $C \longrightarrow D$ we have

$$v = v_0 + at$$

Then $A \longrightarrow B$ $\quad 5.3666\text{ m/s} = 0 + (4.8\text{ m/s}^2)t_{AB}$

or $\quad t_{AB} = 1.11804\text{ s}$

and $C \longrightarrow D$ $\quad 7.2\text{ m/s} = 5.3666\text{ m/s} + (4.8\text{ m/s}^2)t_{CD}$

or $\quad t_{CD} = 0.38196\text{ s}$

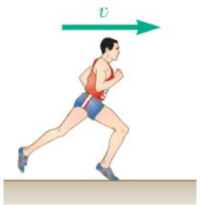
Now, for $B \longrightarrow C$, we have $\quad x_C = x_B + v_{BC}t_{BC}$

or $\quad 3\text{ m} = (5.3666\text{ m/s})t_{BC}$

or $\quad t_{BC} = 0.55901\text{ s}$

Finally, $\quad t_D = t_{AB} + t_{BC} + t_{CD} = (1.11804 + 0.55901 + 0.38196)\text{ s}$

or $\quad t_D = 2.06\text{ s}$ ◀



PROBLEM 11.38

A sprinter in a 100-m race accelerates uniformly for the first 35 m and then runs with constant velocity. If the sprinter's time for the first 35 m is 5.4 s, determine (*a*) his acceleration, (*b*) his final velocity, (*c*) his time for the race.

SOLUTION

Given: $0 \le x \le 35\text{ m}, \quad a = \text{constant}$

$35\text{ m} < x \le 100\text{ m}, \quad v = \text{constant}$

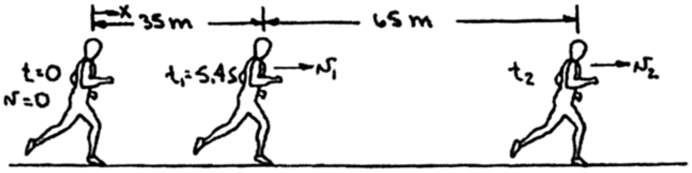
At $t = 0, \quad v = 0 \quad$ when $\quad x = 35\text{ m}, \quad t = 5.4\text{ s}$

Find:

(*a*) a

(*b*) v when $x = 100\text{ m}$

(*c*) t when $x = 100\text{ m}$



(*a*) We have $$x = 0 + 0t + \frac{1}{2}at^2 \quad \text{for} \quad 0 \le x \le 35\text{ m}$$

At $t = 5.4$ s: $$35\text{ m} = \frac{1}{2}a(5.4\text{ s})^2$$

or $$a = 2.4005\text{ m/s}^2$$

$a = 2.40\text{ m/s}^2$ ◀

(*b*) First note that $v = v_{\max}$ for $35\text{ m} \le x \le 100\text{ m}$.

Now $$v^2 = 0 + 2a(x - 0) \quad \text{for} \quad 0 \le x \le 35\text{ m}$$

When $x = 35$ m: $$v_{\max}^2 = 2(2.4005\text{ m/s}^2)(35\text{ m})$$

or $$v_{\max} = 12.9628\text{ m/s}$$

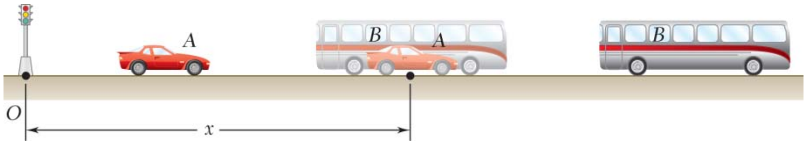
$v_{\max} = 12.96\text{ m/s}$ ◀

(*c*) We have $$x = x_1 + v_0(t - t_1) \quad \text{for} \quad 35\text{ m} < x \le 100\text{ m}$$

When $x = 100$ m: $$100\text{ m} = 35\text{ m} + (12.9628\text{ m/s})(t_2 - 5.4)\text{ s}$$

or

$t_2 = 10.41\text{ s}$ ◀



PROBLEM 11.39

Automobile *A* starts from *O* and accelerates at the constant rate of 0.75 m/s^2. A short time later it is passed by bus *B* which is traveling in the opposite direction at a constant speed of 6 m/s. Knowing that bus *B* passes point *O* 20 s after automobile *A* started from there, determine when and where the vehicles passed each other.

SOLUTION

Place origin at 0.

Motion of auto. $(x_A)_0 = 0, \quad (v_A)_0 = 0, \quad a_A = 0.75 \text{ m/s}^2$

$$x_A = (x_A)_0 + (v_A)_0 t + \frac{1}{2}a_A t^2 = 0 + 0 + \left(\frac{1}{2}\right)(0.75)t^2$$

$$x_A = 0.375t^2 \text{ m}$$

Motion of bus. $(x_B)_0 = ?, \quad (v_B)_0 = -6 \text{ m/s}, \quad a_B = 0$

$$x_B = (x_B)_0 - (v_B)_0 t = (x_B)_0 - 6t \text{ m}$$

At $t = 20 s, \quad x_B = 0.$

$$0 = (x_B)_0 - (6)(20) \qquad (x_B)_0 = 120 \text{ m}$$

Hence, $x_B = 120 - 6t$

When the vehicles pass each other, $x_B = x_A$.

$$120 - 6t = 0.375\, t^2$$

$$0.375\, t^2 + 6t - 120 = 0$$

$$t = \frac{-6 \pm \sqrt{(6)^2 - (4)(0.375)(-120)}}{(2)(0.375)}$$

$$t = \frac{-6 \pm 14.697}{0.75} = 11.596 \text{ s} \quad \text{and} \quad -27.6 \text{ s}$$

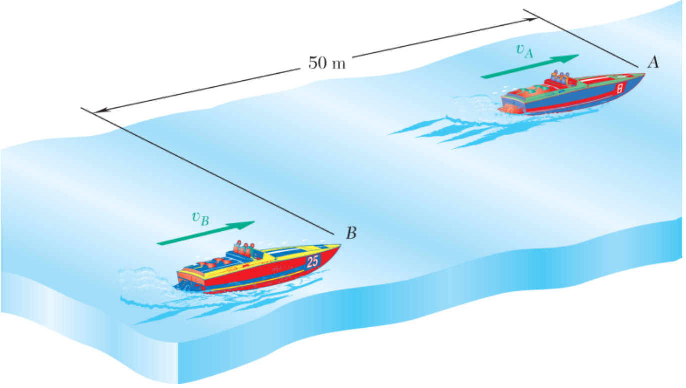
Reject the negative root. $t = 11.60 \text{ s}$ ◀

Corresponding values of x_A and x_B.

$$x_A = (0.375)(11.596)^2 = 50.4 \text{ m}$$

$$x_B = 120 - (6)(11.596) = 50.4 \text{ m}$$

$x = 50.4 \text{ m}$ ◀



PROBLEM 11.40

In a boat race, boat *A* is leading boat *B* by 50 m and both boats are traveling at a constant speed of 180 km/h. At $t = 0$, the boats accelerate at constant rates. Knowing that when *B* passes *A*, $t = 8$ s and $v_A = 225$ km/h, determine (*a*) the acceleration of *A*, (*b*) the acceleration of *B*.

SOLUTION

(*a*) We have $$v_A = (v_A)_0 + a_A t$$

$$(v_A)_0 = 180 \text{ km/h} = 50 \text{ m/s}$$

At $t = 8$ s: $$v_A = 225 \text{ km/h} = 62.5 \text{ m/s}$$

Then $$62.5 \text{ m/s} = 50 \text{ m/s} + a_A (8 \text{ s})$$

or $$a_A = 1.563 \text{ m/s}^2 \blacktriangleleft$$

(*b*) We have

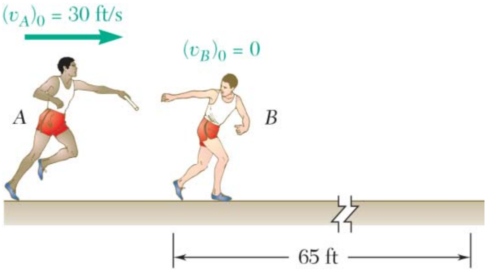
$$x_A = (x_A)_0 + (v_A)_0 t + \frac{1}{2} a_A t^2 = 50 \text{ m} + (50 \text{ m/s})(8 \text{ s}) + \frac{1}{2}(1.5625 \text{ m/s}^2)(8 \text{ s})^2 = 500 \text{ m}$$

and $$x_B = 0 + (v_B)_0 t + \frac{1}{2} a_B t^2 \qquad (v_B)_0 = 50 \text{ m/s}$$

At $t = 8$ s: $$x_A = x_B$$

$$500 \text{ m} = (50 \text{ m/s})(8 \text{ s}) + \frac{1}{2} a_B (8 \text{ s})^2$$

or $$a_B = 3.13 \text{ m/s}^2 \blacktriangleleft$$



PROBLEM 11.41

As relay runner *A* enters the 65-ft-long exchange zone with a speed of 30 ft/s, he begins to slow down. He hands the baton to runner *B* 2.5 s later as they leave the exchange zone with the same velocity. Determine (*a*) the uniform acceleration of each of the runners, (*b*) when runner *B* should begin to run.

SOLUTION

Let $x = 0$ at the start of the exchange zone, and $t = 0$ as runner *A* enters the exchange zone.

Then, $(v_A)_0 = 30$ ft/s.

Motion of runner *A*: $v_A = (v_A)_0 + a_A t$

$$x_A = (v_A)_0 t + \frac{1}{2} a_A t^2$$

(*a*) Solving for a_A, and noting that $x_A = 65$ ft when $t = 2.5$ s

$$a_A = \frac{2\left[x_A - (v_A)_0 t\right]}{t^2} = \frac{2\left[65 - (30)(2.5)\right]}{2.5^2}$$

$a_A = -3.20$ ft/s^2 ◀

At $t = 2.5$ s, $(v_A)_f = 30 + (-3.20)(2.5) = 22.0$ ft/s

Motion of runner *B*: $(v_B)_0 = 0$ and $x_B = 0$ at the starting time t_B of runner *B*.

Then, $v_B^2 - (v_B)_0^2 = 2a_B x_B$

Solving for a_B, and noting that $v_B = (v_A)_f = 22.0$ ft/s

When $x_B = 65$ ft, $$a_B = \frac{(v_B)^2 - (v_B)_0^2}{2x_B} = \frac{22.0^2 - 0}{(2)(65)}$$

$a_B = 3.723$ ft/s^2 ◀

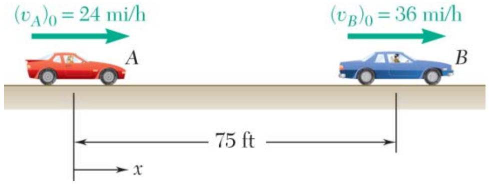
$$v_B = (v_B)_0 + a_B(t - t_B) = a_B(t - t_B)$$

$$t - t_B = \frac{v_B}{a_B}$$

(*b*) Solving for t_B, and noting that $v_B = 22.0$ ft/s when $t = 2.5$ s

$$t_B = t - \frac{v_B}{a_B} = 2.5 - \frac{22.0}{3.723} = -3.41 \text{ s}$$

Runner *B* should begin 3.41 s before runner *A* reaches the exchange zone. ◀



PROBLEM 11.42

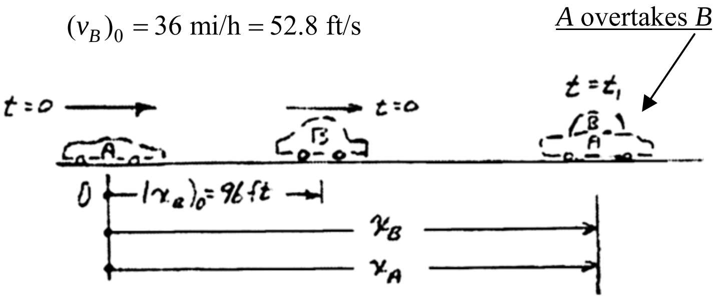
Automobiles A and B are traveling in adjacent highway lanes and at $t = 0$ have the positions and speeds shown. Knowing that automobile A has a constant acceleration of 1.8 ft/s^2 and that B has a constant deceleration of 1.2 ft/s^2, determine (*a*) when and where A will overtake B, (*b*) the speed of each automobile at that time.

SOLUTION

$$a_A = +1.8 \text{ ft/s}^2 \quad a_B = -1.2 \text{ ft/s}^2$$

$$(v_A)_0 = 24 \text{ mi/h} = 35.2 \text{ ft/s}$$

$$(v_B)_0 = 36 \text{ mi/h} = 52.8 \text{ ft/s}$$



Motion of auto A:

$$v_A = (v_A)_0 + a_A t = 35.2 + 1.8t \qquad (1)$$

$$x_A = (x_A)_0 + (v_A)_0 t + \frac{1}{2} a_A t^2 = 0 + 35.2t + \frac{1}{2}(1.8)t^2 \qquad (2)$$

Motion of auto B:

$$v_B = (v_B)_0 + a_B t = 52.8 - 1.2t \qquad (3)$$

$$x_B = (x_B)_0 + (v_B)_0 t + \frac{1}{2} a_B t^2 = 75 + 52.8t + \frac{1}{2}(-1.2)t^2 \qquad (4)$$

(*a*) A overtakes B at $t = t_1$.

$$x_A = x_B: \quad 35.2t + 0.9t_1^2 = 75 + 52.8t_1 - 0.6t_1^2$$

$$1.5t_1^2 - 17.6t_1 - 75 = 0$$

$$t_1 = -3.22 \text{ s} \quad \text{and} \quad t_1 = 15.0546 \qquad t_1 = 15.05 \text{ s} ◀$$

Eq. (2): $$x_A = 35.2(15.05) + 0.9(15.05)^2 \qquad x_A = 734 \text{ ft} ◀$$

PROBLEM 11.42 (Continued)

(*b*) Velocities when $t_1 = 15.05 \text{ s}$

Eq. (1): $v_A = 35.2 + 1.8(15.05)$

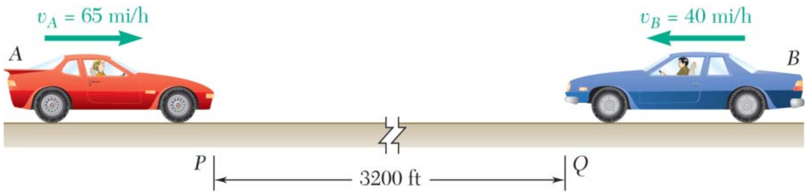
$v_A = 62.29 \text{ ft/s}$ $\qquad v_A = 42.5 \text{ mi/h} \longrightarrow$ ◀

Eq. (3): $v_B = 52.8 - 1.2(15.05)$

$v_B = 34.74 \text{ ft/s}$ $\qquad v_B = 23.7 \text{ mi/h} \longrightarrow$ ◀

PROBLEM 11.43

Two automobiles *A* and *B* are approaching each other in adjacent highway lanes. At $t = 0$, *A* and *B* are 3200 ft apart, their speeds are $v_A = 65$ mi/h and $v_B = 40$ mi/h, and they are at Points *P* and *Q*, respectively. Knowing that *A* passes Point *Q* 40 s after *B* was there and that *B* passes Point *P* 42 s after *A* was there, determine (*a*) the uniform accelerations of *A* and *B*, (*b*) when the vehicles pass each other, (*c*) the speed of *B* at that time.



SOLUTION

(*a*) We have $x_A = 0 + (v_A)_0 t + \frac{1}{2} a_A t^2 \quad (v_A)_0 = 65 \text{ mi/h} = 95.33 \text{ ft/s}$

(x is positive $\longrightarrow$; origin at *P*.)

At $t = 40$ s: $3200 \text{ m} = (95.333 \text{ m/s})(40 \text{ s}) + \frac{1}{2} a_A (40 \text{ s})^2$ $\quad a_A = -0.767 \text{ ft/s}^2$ ◀

Also, $x_B = 0 + (v_B)_0 t + \frac{1}{2} a_B t^2 \quad (v_B)_0 = 40 \text{ mi/h} = 58.667 \text{ ft/s}$

(x_B is positive $\longleftarrow$; origin at *Q*.)

At $t = 42$ s: $3200 \text{ ft} = (58.667 \text{ ft/s})(42 \text{ s}) + \frac{1}{2} a_B (42 \text{ s})^2$

or $a_B = 0.83447 \text{ ft/s}^2$ $\quad a_B = 0.834 \text{ ft/s}^2$ ◀

(*b*) When the cars pass each other $x_A + x_B = 3200$ ft

Then $(95.333 \text{ ft/s}) t_{AB} + \frac{1}{2}(-0.76667 \text{ ft/s}) t_{AB}^2 + (58.667 \text{ ft/s}) t_{AB} + \frac{1}{2}(0.83447 \text{ ft/s}^2) t_{AB}^2 = 3200 \text{ ft}$

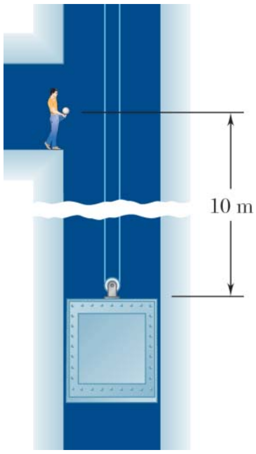
or $0.03390 t_{AB}^2 + 154 t_{AB} - 3200 = 0$

Solving $t = 20.685 \text{ s}$ and $t = -4563 \text{ s}$ $\quad t > 0 \Rightarrow \quad t_{AB} = 20.7 \text{ s}$ ◀

(*c*) We have $v_B = (v_B)_0 + a_B t$

At $t = t_{AB}$: $v_B = 58.667 \text{ ft/s} + (0.83447 \text{ ft/s}^2)(20.685 \text{ s})$

$= 75.927 \text{ ft/s}$ $\quad v_B = 51.8 \text{ mi/h}$ ◀



PROBLEM 11.44

An elevator is moving upward at a constant speed of 4 m/s. A man standing 10 m above the top of the elevator throws a ball upward with a speed of 3 m/s. Determine (*a*) when the ball will hit the elevator, (*b*) where the ball will hit the elevator with respect to the location of the man.

SOLUTION

Place the origin of the position coordinate at the level of the standing man, the positive direction being up. The ball undergoes uniformly accelerated motion.

$$y_B = (y_B)_0 + (v_B)_0 t - \frac{1}{2} g t^2$$

with $(y_B)_0 = 0, (v_B)_0 = 3$ m/s, and $g = 9.81\text{ m/s}^2$.

$$y_B = 3t - 4.905t^2$$

The elevator undergoes uniform motion.

$$y_E = (y_E)_0 + v_E t$$

with $(y_E)_0 = -10$ m and $v_E = 4$ m/s.

(*a*) Time of impact. Set $y_B = y_E$

$$3t - 4.905t^2 = -10 + 4t$$

$$4.905t^2 + t - 10 = 0$$

$t = 1.3295$ and -1.5334

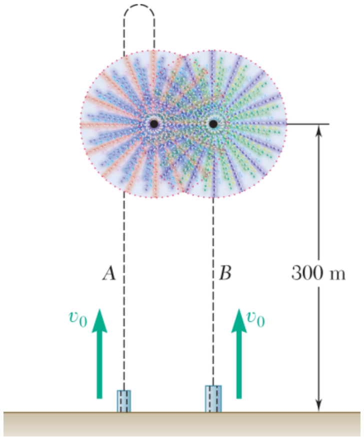
$t = 1.330\text{ s}$ ◀

(*b*) Location of impact.

$$y_B = (3)(1.3295) - (4.905)(1.3295)^2 = -4.68\text{ m}$$

$$y_E = -10 + (4)(1.3295) = -4.68\text{ m} \qquad \text{(checks)}$$

4.68 m below the man ◀



PROBLEM 11.45

Two rockets are launched at a fireworks display. Rocket A is launched with an initial velocity $v_0 = 100$ m/s and rocket B is launched t_1 seconds later with the same initial velocity. The two rockets are timed to explode simultaneously at a height of 300 m as A is falling and B is rising. Assuming a constant acceleration $g = 9.81\ \text{m/s}^2$, determine (a) the time t_1, (b) the velocity of B relative to A at the time of the explosion.

SOLUTION

Place origin at ground level. The motion of rockets A and B is

Rocket A:

$$v_A = (v_A)_0 - gt = 100 - 9.81t \tag{1}$$

$$y_A = (y_A)_0 + (v_A)_0 t - \frac{1}{2}gt^2 = 100t - 4.905t^2 \tag{2}$$

Rocket B:

$$v_B = (v_B)_0 - g(t - t_1) = 100 - 9.81(t - t_1) \tag{3}$$

$$\begin{aligned} y_B &= (y_B)_0 + (v_B)_0(t - t_1) - \frac{1}{2}g(t - t_1)^2 \\ &= 100(t - t_1) - 4.905(t - t_1)^2 \end{aligned} \tag{4}$$

Time of explosion of rockets A and B. $y_A = y_B = 300$ ft

From (2),

$$300 = 100t - 4.905t^2$$

$$4.905t^2 - 100t + 300 = 0$$

$$t = 16.732\ \text{s} \quad \text{and} \quad 3.655\ \text{s}$$

From (4),

$$300 = 100(t - t_1) - 4.905(t - t_1^2)$$

$$t - t_1 = 16.732\ \text{s} \quad \text{and} \quad 3.655\ \text{s}$$

Since rocket A is falling, $t = 16.732$ s

Since rocket B is rising, $t - t_1 = 3.655$ s

(a) Time t_1: $t_1 = t - (t - t_1)$ $\qquad t_1 = 13.08$ s ◀

(b) Relative velocity at explosion.

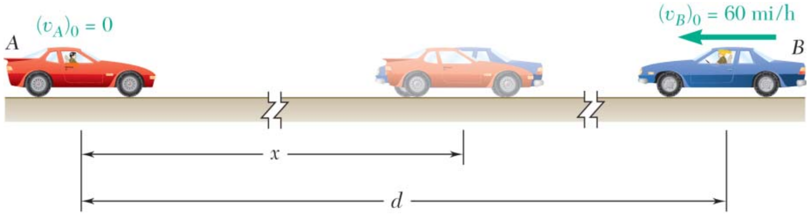
From (1), $v_A = 100 - (9.81)(16.732) = -64.15$ m/s

From (3), $v_B = 100 - (9.81)(16.732 - 13.08) = 64.15$ m/s

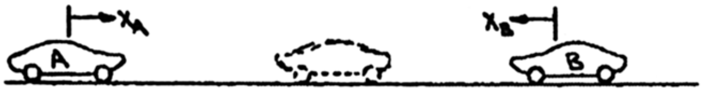
Relative velocity: $v_{B/A} = v_B - v_A$ $\qquad v_{B/A} = 128.3$ m/s ↑ ◀

PROBLEM 11.46

Car *A* is parked along the northbound lane of a highway, and car *B* is traveling in the southbound lane at a constant speed of 60 mi/h. At $t = 0,$ *A* starts and accelerates at a constant rate $a_A,$ while at $t = 5$ s, *B* begins to slow down with a constant deceleration of magnitude $a_A/6.$ Knowing that when the cars pass each other $x = 294$ ft and $v_A = v_B,$ determine (*a*) the acceleration $a_A,$ (*b*) when the vehicles pass each other, (*c*) the distance *d* between the vehicles at $t = 0.$



SOLUTION



For $t \geq 0$: $v_A = 0 + a_A t$

$$x_A = 0 + 0 + \frac{1}{2}a_A t^2$$

$0 \leq t < 5$ s: $x_B = 0 + (v_B)_0 t \quad (v_B)_0 = 60 \text{ mi/h} = 88 \text{ ft/s}$

At $t = 5$ s: $x_B = (88 \text{ ft/s})(5 \text{ s}) = 440 \text{ ft}$

For $t \geq 5$ s: $v_B = (v_B)_0 + a_B(t-5) \qquad a_B = -\frac{1}{6}a_A$

$$x_B = (x_B)_5 + (v_B)_0(t-5) + \frac{1}{2}a_B(t-5)^2$$

Assume $t > 5$ s when the cars pass each other.

At that time (t_{AB}),

$v_A = v_B$: $a_A t_{AB} = (88 \text{ ft/s}) - \frac{a_A}{6}(t_{AB} - 5)$

$x_A = 294$ ft: $294 \text{ ft} = \frac{1}{2}a_A t_{AB}^2$

Then $$\frac{a_A\left(\frac{7}{6}t_{AB} - \frac{5}{6}\right)}{\frac{1}{2}a_A t_{AB}^2} = \frac{88}{294}$$

or $$44t_{AB}^2 - 343t_{AB} + 245 = 0$$

PROBLEM 11.46 (Continued)

Solving $t_{AB} = 0.795\text{ s} \quad \text{and} \quad t_{AB} = 7.00\text{ s}$

(*a*) With $t_{AB} > 5\text{ s}$, $294\text{ ft} = \frac{1}{2}a_A(7.00\text{ s})^2$

or $a_A = 12.00\text{ ft/s}^2$ ◀

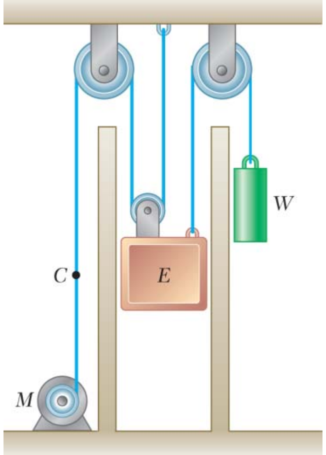
(*b*) From above $t_{AB} = 7.00\text{ s}$ ◀

Note: An acceptable solution cannot be found if it is assumed that $t_{AB} \le 5\text{ s}$.

(*c*) We have

$$\begin{aligned} d &= x + (x_B)_{t_{AB}} \\ &= 294\text{ ft} + 440\text{ ft} + (88\text{ ft/s})(2.00\text{ s}) \\ &\quad + \frac{1}{2}\left(-\frac{1}{6}\times 12.00\text{ ft/s}^2\right)(2.00\text{ s})^2 \end{aligned}$$

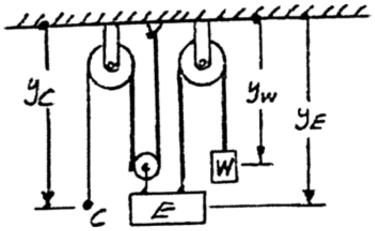
or $d = 906\text{ ft}$ ◀



PROBLEM 11.47

The elevator shown in the figure moves downward with a constant velocity of 4 m/s. Determine (*a*) the velocity of the cable *C*, (*b*) the velocity of the counterweight *W*, (*c*) the relative velocity of the cable *C* with respect to the elevator, (*d*) the relative velocity of the counterweight *W* with respect to the elevator.

SOLUTION



Choose the positive direction downward.

(*a*) Velocity of cable *C*.

$$y_C + 2y_E = \text{constant}$$

$$v_C + 2v_E = 0$$

But, $v_E = 4\ \text{m/s}$

or $v_C = -2v_E = -8\ \text{m/s}$

$\mathbf{v}_C = 8.00\ \text{m/s}\uparrow$ ◀

(*b*) Velocity of counterweight *W*.

$$y_W + y_E = \text{constant}$$

$$v_W + v_E = 0 \quad v_W = -v_E = -4\ \text{m/s}$$

$\mathbf{v}_W = 4.00\ \text{m/s}\uparrow$ ◀

(*c*) Relative velocity of *C* with respect to *E*.

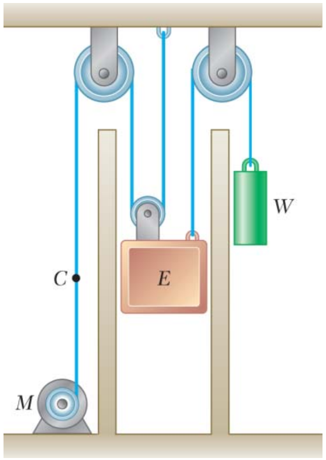
$$v_{C/E} = v_C - v_E = (-8\ \text{m/s}) - (+4\ \text{m/s}) = -12\ \text{m/s}$$

$\mathbf{v}_{C/E} = 12.00\ \text{m/s}\uparrow$ ◀

(*d*) Relative velocity of *W* with respect to *E*.

$$v_{W/E} = v_W - v_E = (-4\ \text{m/s}) - (4\ \text{m/s}) = -8\ \text{m/s}$$

$\mathbf{v}_{W/E} = 8.00\ \text{m/s}\uparrow$ ◀

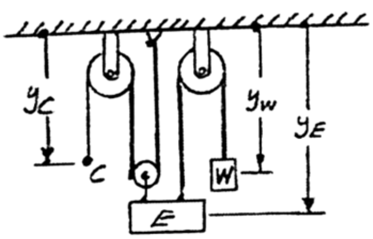


PROBLEM 11.48

The elevator shown starts from rest and moves upward with a constant acceleration. If the counterweight W moves through 30 ft in 5 s, determine (a) the acceleration of the elevator and the cable C, (b) the velocity of the elevator after 5 s.

SOLUTION

We choose positive direction downward for motion of counterweight.



$$y_W = \frac{1}{2} a_W t^2$$

At $t = 5$ s,

$$y_W = 30 \text{ ft}$$

$$30 \text{ ft} = \frac{1}{2} a_W (5 \text{ s})^2$$

$$a_W = 2.4 \text{ ft/s}^2 \qquad \mathbf{a}_W = 2.4 \text{ ft/s}^2 \downarrow$$

(a) Accelerations of E and C.

Since $\quad y_W + y_E = \text{constant} \quad v_W + v_E = 0, \quad \text{and} \quad a_W + a_E = 0$

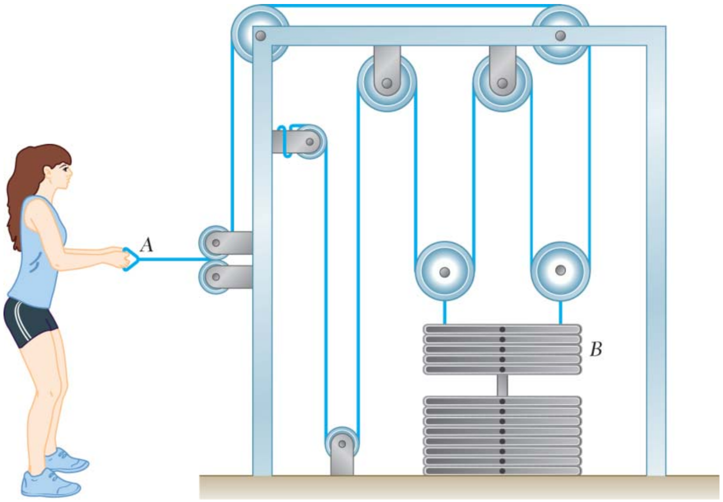
Thus: $\quad a_E = -a_W = -(2.4 \text{ ft/s}^2), \qquad \mathbf{a}_E = 2.40 \text{ ft/s}^2 \uparrow$ ◀

Also, $\quad y_C + 2y_E = \text{ constant}, \quad v_C + 2v_E = 0, \quad \text{and} \quad a_C + 2a_E = 0$

Thus: $\quad a_C = -2a_E = -2(-2.4 \text{ ft/s}^2) = +4.8 \text{ ft/s}^2, \qquad \mathbf{a}_C = 4.80 \text{ ft/s}^2 \downarrow$ ◀

(b) Velocity of elevator after 5 s.

$$v_E = (v_E)_0 + a_E t = 0 + (-2.4 \text{ ft/s}^2)(5 \text{ s}) = -12 \text{ ft/s} \qquad (\mathbf{v}_E)_5 = 12.00 \text{ ft/s} \uparrow$$ ◀

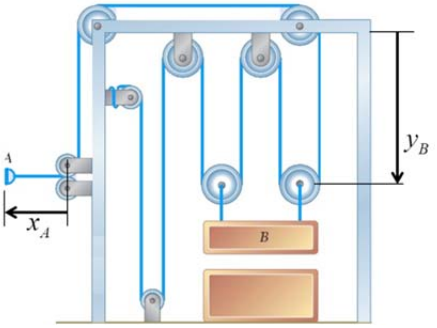


Problem 11.49

An athlete pulls handle *A* to the left with a constant velocity of 0.5 m/s. Determine (*a*) the velocity of the weight *B*, (*b*) the relative velocity of weight *B* with respect to the handle *A*.

SOLUTION

From the diagram: $x_A + 4y_B = \text{constant}$ Sketch:



(a) Differentiate $v_A + 4v_B = 0$ (1)

$x_A = 0.5 \text{ m/s}$

Solve (1) for v_B $v_B = -0.125 \text{ m/s}$

$v_B = 0.125 \text{ m/s} \uparrow$ ◀

(b) Finding the relative velocity:

$$\vec{v}_B = 0.125 \uparrow \text{ m/s}$$

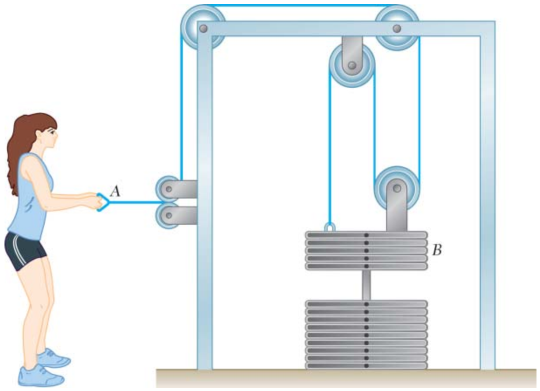
$$\vec{v}_A = 0.5 \leftarrow \text{ m/s}$$

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$$

$$= 0.125 \uparrow - (0.5 \leftarrow) \text{ m/s}$$

or $\mathbf{v}_{B/A} = 0.5 \rightarrow + 0.125 \uparrow \text{ m/s}$

$\mathbf{v}_{B/A} = 0.5154 \text{ m/s} \measuredangle 14°$ ◀



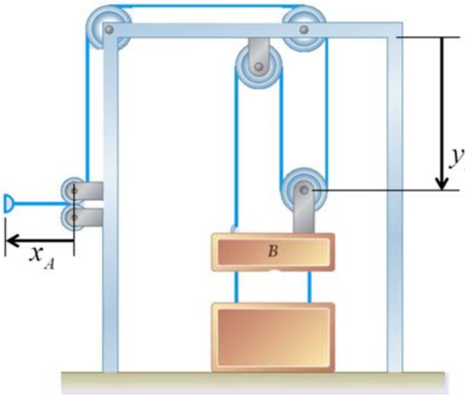
Problem 11.50

An athlete pulls handle A to the left with a constant acceleration. Knowing that after the weight B has been lifted 4 in. its velocity is 2 ft/s, determine (*a*) the accelerations of handle A and weight B (*b*) the velocity and change in position of handle A after 0.5 sec.

SOLUTION

(*a*) From the diagram:

Sketch:



$$x_A + 3y_B = \text{constant}$$

$$v_A + 3v_B = 0 \qquad (1)$$

$$a_A + 3a_B = 0 \qquad (2)$$

Uniform Acceleration of weight B

$$\mathrm{v}_B{}^2 = (\mathrm{v}_B)_o^2 + 2a_B\left(y_B - (y_B)_o\right)$$

Substitute in known values

$$(-2\text{ ft/s})^2 = 0^2 + 2a_B\left(-\frac{4}{12}\text{ ft} - 0\right)$$

$$a_B = -6\text{ ft/s}^2$$

Using (2)

$$a_A + 3(-6\text{ ft/s}^2) = 0$$

$$a_A = 18\text{ ft/s}^2 \leftarrow ◀$$

$$a_B = 6\text{ ft/s}^2 \uparrow ◀$$

(*b*) Uniform Acceleration of Handle A

$$v_A = (v_A)_o + a_A t$$

$$v_A = 0 + \left(18\text{ ft/s}^2\right)(0.5\text{ s})$$

or

$$v_A = 9\text{ ft/s} \leftarrow ◀$$

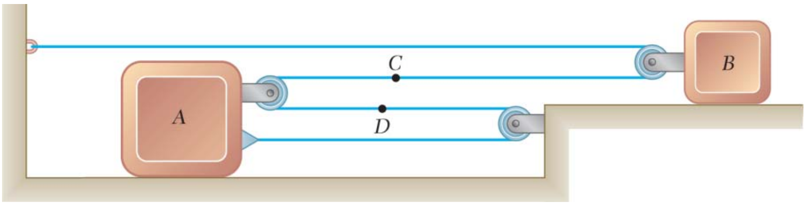
Also

$$y_A = (y_A)_o + (v_A)_o t + \frac{1}{2}a_A t^2$$

$$y_A - (y_A)_o = 0 + \frac{1}{2}\left(18\text{ ft/s}^2\right)(0.5\text{ s})^2$$

or

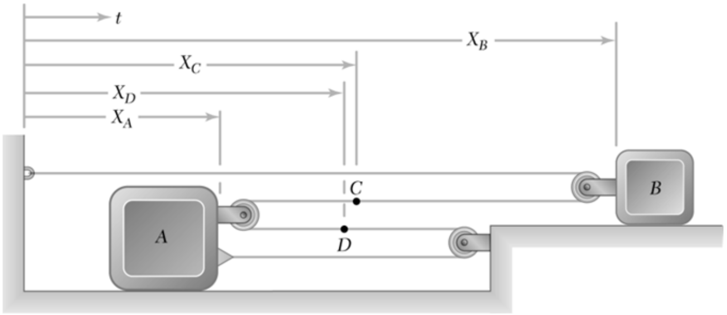
$$\Delta y_A = 2.25\text{ ft} \leftarrow ◀$$



PROBLEM 11.51

Slider block *B* moves to the right with a constant velocity of 300 mm/s. Determine (*a*) the velocity of slider block *A*, (*b*) the velocity of portion *C* of the cable, (*c*) the velocity of portion *D* of the cable, (*d*) the relative velocity of portion *C* of the cable with respect to slider block *A*.

SOLUTION



From the diagram $$x_B + (x_B - x_A) - 2x_A = \text{constant}$$

Then $$2v_B - 3v_A = 0 \tag{1}$$

and $$2a_B - 3a_A = 0 \tag{2}$$

Also, we have $$-x_D - x_A = \text{constant}$$

Then $$v_D + v_A = 0 \tag{3}$$

(*a*) Substituting into Eq. (1) $$2(300\text{ mm/s}) - 3v_A = 0$$

or $$\mathbf{v}_A = 200\text{ mm/s} \longrightarrow \blacktriangleleft$$

(*b*) From the diagram $$x_B + (x_B - x_C) = \text{constant}$$

Then $$2v_B - v_C = 0$$

Substituting $$2(300\text{ mm/s}) - v_C = 0$$

or $$\mathbf{v}_C = 600\text{ mm/s} \longrightarrow \blacktriangleleft$$

(*c*) From the diagram $$(x_C - x_A) + (x_D - x_A) = \text{constant}$$

Then $$v_C - 2v_A + v_D = 0$$

Substituting $$600\text{ mm/s} - 2(200\text{ mm/s}) + v_D = 0$$

or $$\mathbf{v}_D = 200\text{ mm/s} \longleftarrow \blacktriangleleft$$

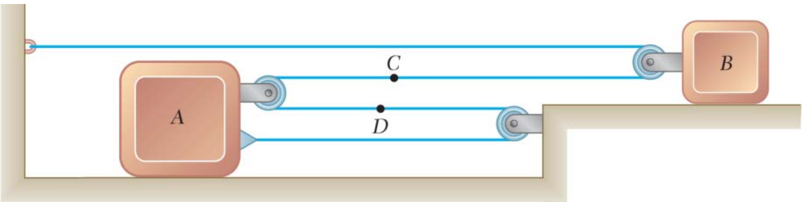
PROBLEM 11.51 (Continued)

(*d*) We have

$$v_{C/A} = v_C - v_A$$
$$= 600 \text{ mm/s} - 200 \text{ mm/s}$$

or

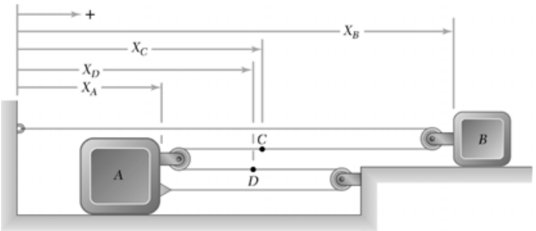
$$\mathbf{v}_{C/A} = 400 \text{ mm/s} \longrightarrow \blacktriangleleft$$



PROBLEM 11.52

At the instant shown, slider block *B* is moving with a constant acceleration, and its speed is 150 mm/s. Knowing that after slider block *A* has moved 240 mm to the right its velocity is 60 mm/s, determine (*a*) the accelerations of *A* and *B*, (*b*) the acceleration of portion *D* of the cable, (*c*) the velocity and change in position of slider block *B* after 4 s.

SOLUTION



From the diagram $\quad x_B + (x_B - x_A) - 2x_A = \text{constant}$

Then $\quad 2v_B - 3v_A = 0 \qquad (1)$

and $\quad 2a_B - 3a_A = 0 \qquad (2)$

(*a*) First observe that if block *A* moves to the right, $\mathbf{v}_A \rightarrow$ and Eq. (1) $\Rightarrow \mathbf{v}_B \rightarrow$. Then, using Eq. (1) at $t = 0$

$$2(150\ \text{mm/s}) - 3(v_A)_0 = 0$$

or $\quad (v_A)_0 = 100\ \text{mm/s}$

Also, Eq. (2) and $a_B = \text{constant} \Rightarrow a_A = \text{constant}$

Then $\quad v_A^2 = (v_A)_0^2 + 2a_A[x_A - (x_A)_0]$

When $x_A - (x_A)_0 = 240$ mm:

$$(60\ \text{mm/s})^2 = (100\ \text{mm/s})^2 + 2a_A(240\ \text{mm})$$

or $\quad a_A = -\dfrac{40}{3}\ \text{mm/s}^2$

or $\quad \mathbf{a}_A = 13.33\ \text{mm/s}^2 \longleftarrow$ ◀

PROBLEM 11.52 (Continued)

Then, substituting into Eq. (2)

$$2a_B - 3\left(-\frac{40}{3}\text{ mm/s}^2\right) = 0$$

or $$a_B = -20\text{ mm/s}^2 \qquad \mathbf{a}_B = 20.0\text{ mm/s}^2 \longleftarrow \blacktriangleleft$$

(*b*) From the diagram, $-x_D - x_A = \text{constant}$

$$v_D + v_A = 0$$

Then $$a_D + a_A = 0$$

Substituting $$a_D + \left(-\frac{40}{3}\text{ mm/s}^2\right) = 0$$

or $$\mathbf{a}_D = 13.33\text{ mm/s}^2 \longrightarrow \blacktriangleleft$$

(*c*) We have $$v_B = (v_B)_0 + a_B t$$

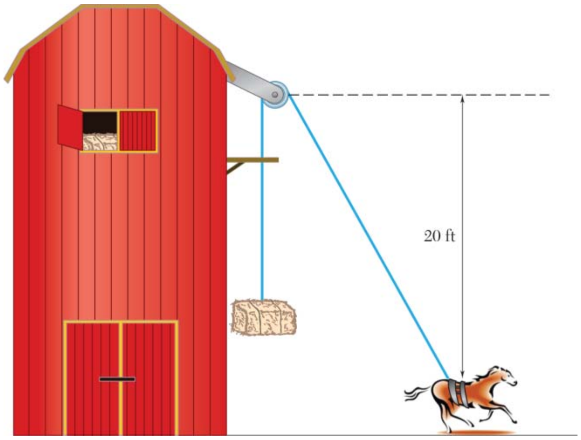
At $t = 4$ s: $$v_B = 150\text{ mm/s} + (-20.0\text{ mm/s}^2)(4\text{ s})$$

or $$\mathbf{v}_B = 70.0\text{ mm/s} \longrightarrow \blacktriangleleft$$

Also $$x_B = (x_B)_0 + (v_B)_0 t + \frac{1}{2}a_B t^2$$

At $t = 4$ s: $$x_B - (x_B)_0 = (150\text{ mm/s})(4\text{ s}) + \frac{1}{2}(-20.0\text{ mm/s}^2)(4\text{ s})^2$$

or $$\mathbf{x}_B - (\mathbf{x}_B)_0 = 440\text{ mm} \longrightarrow \blacktriangleleft$$



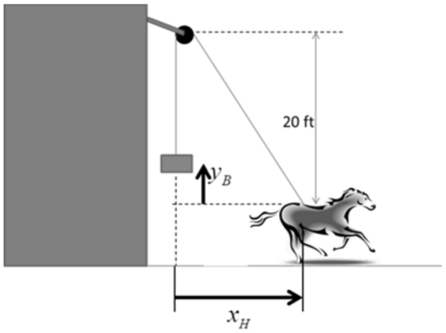
PROBLEM 11.53

A farmer wants to get his hay bales into the top loft of his barn by walking his horse forward with a constant velocity of 1 ft/s. Determine the velocity and acceleration of the hay bale when the horse is 10 ft away from the barn.

SOLUTION

From the diagram, we can write an expression for the total length of rope:

$$L = 20 - y_B + \sqrt{x_H^2 + (20)^2}$$



Differentiate
$$\frac{dL}{dt} = 0$$

$$0 = -\dot{y}_B + \frac{x_H \dot{x}_H}{\sqrt{x_H^2 + (20)^2}}$$

Velocities
$$v_B = \dot{y}_B \text{ and } v_H = \dot{x}_H$$

$$v_B = \frac{x_H v_H}{\sqrt{x_H^2 + (20)^2}}$$

Differentiate again:
$$a_B = \frac{dv_B}{dt}$$

$$a_B = \frac{\dot{x}_H v_H}{\sqrt{x_H^2 + (20)^2}} + \frac{x_H \dot{v}_H}{\sqrt{x_H^2 + (20)^2}} + \frac{x_H v_H (-\tfrac{1}{2})(2x_H)\dot{x}_H}{\left(x_H^2 + (20)^2\right)^{3/2}}$$

or
$$a_B = \frac{v_H^2}{\sqrt{x_H^2 + (20)^2}} + \frac{x_H a_H}{\sqrt{x_H^2 + (20)^2}} - \frac{x_H^2 v_H^2}{\left(x_H^2 + (20)^2\right)^{3/2}}$$

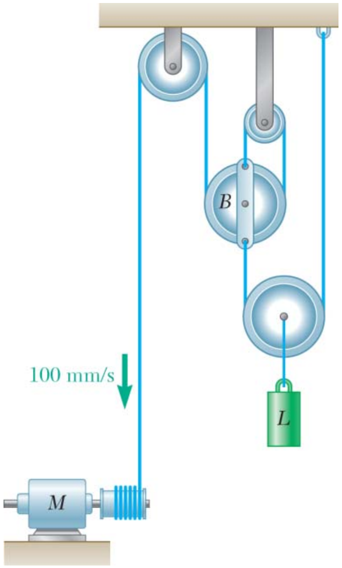
at $x_H = 10$
$$v_B = \frac{10(1)}{\sqrt{(10)^2 + (20)^2}}$$

$v_B = 0.447$ ft/s ◀

and
$$a_B = \frac{(1)^2}{\sqrt{(10)^2 + (20)^2}} + \frac{(10)(0)}{\sqrt{(10)^2 + (20)^2}} - \frac{(10)^2 (1)^2}{\left((10)^2 + (20)^2\right)^{3/2}}$$

$$a_B = 0.0447 + 0 - 0.000894$$

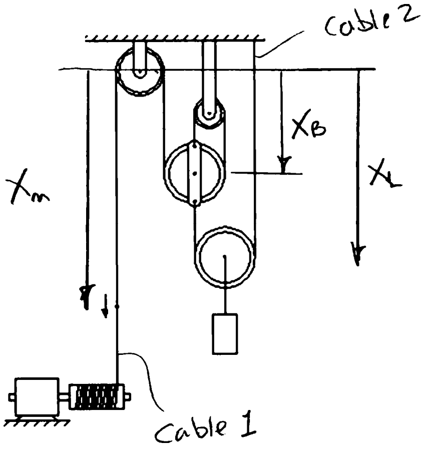
$a_B = 0.0358$ ft/s^2 ◀



PROBLEM 11.54

The motor M reels in the cable at a constant rate of 100 mm/s. Determine (a) the velocity of load L, (b) the velocity of pulley B with respect to load L.

SOLUTION



Let x_B and x_L be the positions, respectively, of pulley B and load L measured downward from a fixed elevation above both. Let x_M be the position of a point on the cable about to enter the reel driven by the motor. Then, considering the lengths of the two cables,

$$x_M + 3x_B = \text{constant} \qquad v_M + 3v_B = 0$$

$$x_L + (x_L - x_B) = \text{constant} \qquad 2v_L + v_B = 0$$

with

$$v_M = 100 \text{ mm/s}$$

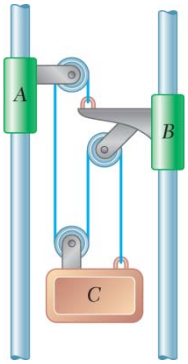
$$v_B = -\frac{v_M}{3} = -33.333 \text{ m/s}$$

$$v_L = \frac{v_B}{2} = -16.667 \text{ mm/s}$$

(a) Velocity of load L. $\mathbf{v}_L = 16.67 \text{ mm/s}\uparrow$ ◀

(b) Velocity of pulley B with respect to load L. $v_{B/L} = v_B - v_L = -33.333 - (-16.667) = -16.667$

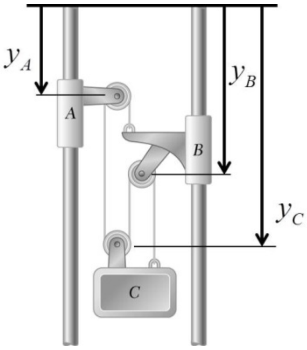
$\mathbf{v}_{B/L} = 16.67 \text{ mm/s}\uparrow$ ◀



PROBLEM 11.55

Collar A starts from rest at $t = 0$ and moves upward with a constant acceleration of 3.6 in./s^2. Knowing that collar B moves downward with a constant velocity of 18 in./s, determine (*a*) the time at which the velocity of block C is zero, (*b*) the corresponding position of block C.

SOLUTION



Define positions as positive downward from a fixed level.

Constraint of cable. $(y_B - y_A) + (y_C - y_A) + 2(y_C - y_B) = \text{constant}$

$3y_C - y_B - 2y_A = \text{constant}$

Differentiate $3v_C - v_B - 2v_A = 0$ (1)

Differentiate again $3a_C - a_B - 2a_A = 0$ (2)

Motion of block C:

$$(v_A)_0 = 0, \;\; a_A = -3.6 \text{ in./s}^2, \;\; v_B = (v_B)_0 = 18 \text{ in./s}, \;\; a_B = 0$$

Using (1) $(v_C)_0 = \frac{1}{3}\left[(v_B)_0 + 2(v_A)_0\right] = 6 \text{ in./s}$

Using (2) $a_C = \frac{1}{3}(a_B + 2a_A) = \frac{1}{3}\left[0 + (2)(-3.6)\right] = -2.4 \text{ in./s}^2$

Constant acceleration

$$\begin{aligned} v_C &= (v_C)_0 + a_C t \\ &= 6 - 1.2t \end{aligned} \qquad (3)$$

Constant acceleration

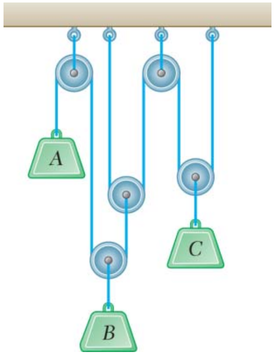
$$\begin{aligned} x_C - (x_C)_0 &= (v_C)_0 t + \frac{1}{2}a_C t^2 \\ &= 6t - 0.6t^2 \end{aligned} \qquad (4)$$

(a) Time at $v_C = 0$:

Using (3) $0 = 6 - 2.4t$ $t = 2.5 \text{ s}$ ◀

(b) Corresponding position of block C:

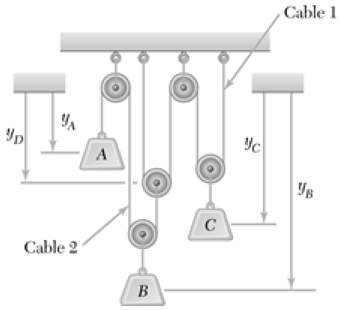
Using (4) $x_C - (x_C)_0 = (6)(2.5) + \left(\frac{1}{2}\right)(-2.4)(2.5)^2$ $x_C - (x_C)_0 = 7.5 \text{ in.} \downarrow$ ◀



PROBLEM 11.56

Block A starts from rest at $t = 0$ and moves downward with a constant acceleration of 6 in./s^2. Knowing that block B moves up with a constant velocity of 3 in./s, determine (a) the time when the velocity of block C is zero, (b) the corresponding position of block C.

SOLUTION



The cable lengths are constant.

$$L_1 = 2y_C + 2y_D + \text{constant}$$
$$L_2 = y_A + y_B + (y_B - y_D) + \text{constant}$$

Eliminate y_D.

$$L_1 + 2L_2 = 2y_C + 2y_D + 2y_A + 2y_B + 2(y_B - y_D) + \text{constant}$$
$$2(y_C + y_A + 2y_B) = \text{constant}$$

Differentiate to obtain relationships for velocities and accelerations, positive downward.

$$v_C + v_A + 2v_B = 0 \qquad (1)$$
$$a_C + a_A + 2a_B = 0 \qquad (2)$$

Use units of inches and seconds.

Motion of block A:

$$v_A = a_A t + 6t$$
$$\Delta y_A = \frac{1}{2}a_A t^2 = \frac{1}{2}(6)t^2 = 3t^2$$

Motion of block B:

$$\mathbf{v}_B = 3 \text{ in./s} \uparrow \qquad v_B = -3 \text{ in./s}$$
$$\Delta y_B = v_B t = -3t$$

Motion of block C: From (1),

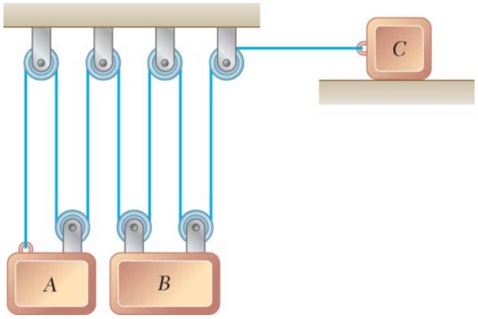
$$v_C = -v_A - 2v_B = -6t - 2(-3) = 6 - 6t$$
$$\Delta y_C = \int_0^t v_C dt = 6t - 3t^2$$

(a) Time when v_C is zero. $6 - 6t = 0$ $\quad t = 1.000 \text{ s}$ ◀

(b) Corresponding position.

$$\Delta y_C = (6)(1) - (3)(1)^2 = 3 \text{ in.}$$

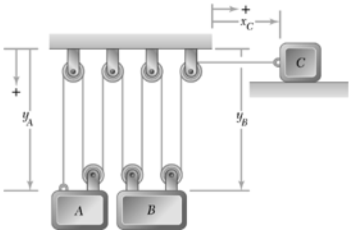
$\Delta y_C = 3.00 \text{ in.} \downarrow$ ◀



PROBLEM 11.57

Block B starts from rest, block A moves with a constant acceleration, and slider block C moves to the right with a constant acceleration of $75\ \text{mm/s}^2$. Knowing that at $t = 2$ s the velocities of B and C are 480 mm/s downward and 280 mm/s to the right, respectively, determine (*a*) the accelerations of A and B, (*b*) the initial velocities of A and C, (*c*) the change in position of slider block C after 3 s.

SOLUTION



From the diagram

$$3y_A + 4y_B + x_C = \text{constant}$$

Then $$3v_A + 4v_B + v_C = 0 \qquad (1)$$

and $$3a_A + 4a_B + a_C = 0 \qquad (2)$$

Given: $$(v_B) = 0,$$

$$a_A = \text{constent}$$

$$(\mathbf{a}_C) = 75\ \text{mm/s}^2 \longrightarrow$$

At $t = 2$ s, $$\mathbf{v}_B = 480\ \text{mm/s} \downarrow$$

$$\mathbf{v}_C = 280\ \text{mm/s} \longrightarrow$$

(*a*) Eq. (2) and $a_A = \text{constant}$ and $a_C = \text{constant} \Rightarrow a_B = \text{constant}$

Then $$v_B = 0 + a_B t$$

At $t = 2$ s: $$480\ \text{mm/s} = a_B(2\ \text{s})$$

$$a_B = 240\ \text{mm/s}^2 \qquad \text{or} \qquad \mathbf{a}_B = 240\ \text{mm/s}^2 \downarrow$$ ◀

Substituting into Eq. (2)

$$3a_A + 4(240\ \text{mm/s}^2) + (75\ \text{mm/s}^2) = 0$$

$$a_A = -345\ \text{mm/s} \qquad \text{or} \qquad \mathbf{a}_A = 345\ \text{mm/s}^2 \uparrow$$ ◀

PROBLEM 11.57 (Continued)

(*b*) We have $v_C = (v_C)_0 + a_C t$

At $t = 2$ s: $280 \text{ mm/s} = (v_C)_0 + (75 \text{ mm/s})(2 \text{ s})$

$v_C = 130 \text{ mm/s}$ or $(\mathbf{v}_C)_0 = 130.0 \text{ mm/s} \longrightarrow$ ◀

Then, substituting into Eq. (1) at $t = 0$

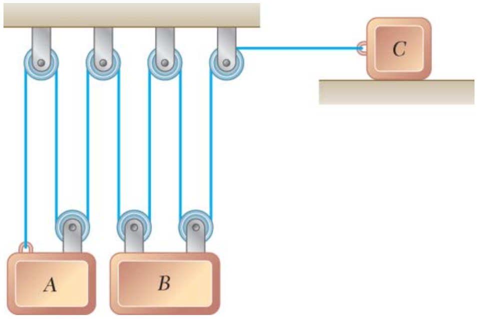
$$3(v_A)_0 + 4(0) + (130 \text{ mm/s}) = 0$$

$v_A = -43.3 \text{ mm/s}$ or $(\mathbf{v}_A)_0 = 43.3 \text{ mm/s} \uparrow$ ◀

(*c*) We have $x_C = (x_C)_0 + (v_C)_0 t + \frac{1}{2} a_C t^2$

At $t = 3$ s: $x_C - (x_C)_0 = (130 \text{ mm/s})(3 \text{ s}) + \frac{1}{2}(75 \text{ mm/s}^2)(3 \text{ s})^2$

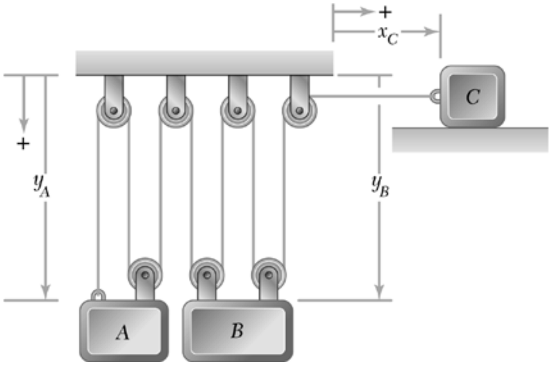
$= 728 \text{ mm}$ or $\mathbf{x}_C - (\mathbf{x}_C)_0 = 728 \text{ mm} \longrightarrow$ ◀



PROBLEM 11.58

Block B moves downward with a constant velocity of 20 mm/s. At $t = 0$, block A is moving upward with a constant acceleration, and its velocity is 30 mm/s. Knowing that at $t = 3$ s slider block C has moved 57 mm to the right, determine (a) the velocity of slider block C at $t = 0$, (b) the accelerations of A and C, (c) the change in position of block A after 5 s.

SOLUTION



From the diagram

$$3y_A + 4y_B + x_C = \text{constant}$$

Then
$$3v_A + 4v_B + v_C = 0 \qquad (1)$$

and
$$3a_A + 4a_B + a_C = 0 \qquad (2)$$

Given:
$$\mathbf{v}_B = 20 \text{ mm/s}\downarrow;$$
$$(\mathbf{v}_A)_0 = 30 \text{ mm/s}\uparrow$$

(a) Substituting into Eq. (1) at $t = 0$

$$3(-30 \text{ mm/s}) + 4(20 \text{ mm/s}) + (v_C)_0 = 0$$

$$(v_C)_0 = 10 \text{ mm/s} \qquad \text{or} \qquad (\mathbf{v}_C)_0 = 10.00 \text{ mm/s} \longrightarrow \blacktriangleleft$$

(b) We have
$$x_C = (x_C)_0 + (v_C)_0 t + \frac{1}{2}a_C t^2$$

At $t = 3$ s:
$$57 \text{ mm} = (10 \text{ mm/s})(3 \text{ s}) + \frac{1}{2}a_C(3 \text{ s})^2$$

$$a_C = 6 \text{ mm/s}^2 \qquad \text{or} \qquad \mathbf{a}_C = 6.00 \text{ mm/s}^2 \longrightarrow \blacktriangleleft$$

Now
$$\mathbf{v}_B = \text{constant} \rightarrow a_B = 0$$

Then, substituting into Eq. (2)

$$3a_A + 4(0) + (6\text{ mm/s}^2) = 0$$

$$a_A = -2\text{ mm/s}^2 \qquad \text{or} \qquad \mathbf{a}_A = 2.00\text{ mm/s}^2 \uparrow \blacktriangleleft$$

(*c*) We have

$$y_A = (y_A)_0 + (v_A)_0 t + \frac{1}{2}a_A t^2$$

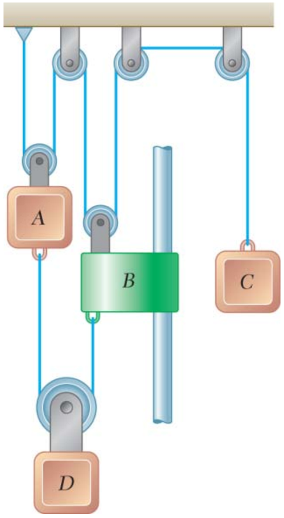
At $t = 5$ s:

$$y_A - (y_A)_0 = (-30\text{ mm/s})(5\text{ s}) + \frac{1}{2}(-2\text{ mm/s}^2)(5\text{ s})^2$$

$$= -175\text{ mm}$$

or

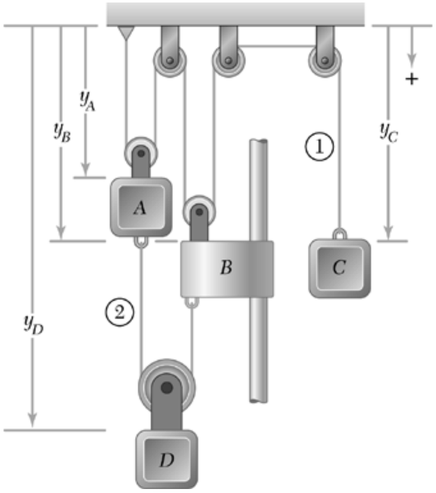
$$\mathbf{y}_A - (\mathbf{y}_A)_0 = 175.0\text{ mm} \uparrow \blacktriangleleft$$



PROBLEM 11.59

The system shown starts from rest, and each component moves with a constant acceleration. If the relative acceleration of block *C* with respect to collar *B* is 60 mm/s^2 upward and the relative acceleration of block *D* with respect to block *A* is 110 mm/s^2 downward, determine (*a*) the velocity of block *C* after 3 s, (*b*) the change in position of block *D* after 5 s.

SOLUTION



From the diagram

Cable 1: $2y_A + 2y_B + y_C = \text{constant}$

Then $2v_A + 2v_B + v_C = 0$ (1)

and $2a_A + 2a_B + a_C = 0$ (2)

Cable 2: $(y_D - y_A) + (y_D - y_B) = \text{constant}$

Then $-v_A - v_B + 2v_D = 0$ (3)

and $-a_A - a_B + 2a_D = 0$ (4)

Given: At $t = 0$, $v = 0$; all accelerations constant;
$a_{C/B} = 60\ \text{mm/s}^2 \uparrow$, $a_{D/A} = 110\ \text{mm/s}^2 \downarrow$

(a) We have $a_{C/B} = a_C - a_B = -60$ or $a_B = a_C + 60$

and $a_{D/A} = a_D - a_A = 110$ or $a_A = a_D - 110$

Substituting into Eqs. (2) and (4)

Eq. (2): $2(a_D - 110) + 2(a_C + 60) + a_C = 0$

or $3a_C + 2a_D = 100$ (5)

Eq. (4): $-(a_D - 110) - (a_C + 60) + 2a_D = 0$

or $-a_C + a_D = -50$ (6)

PROBLEM 11.59 (Continued)

Solving Eqs. (5) and (6) for a_C and a_D

$$a_C = 40 \text{ mm/s}^2$$

$$a_D = -10 \text{ mm/s}^2$$

Now $$v_C = 0 + a_C t$$

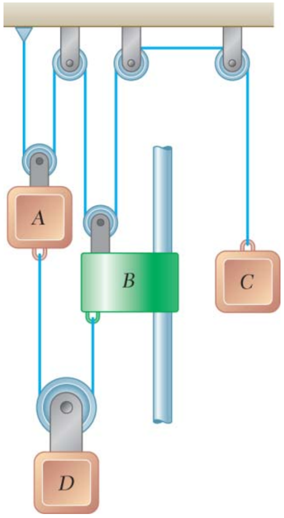
At $t = 3$ s: $$v_C = (40 \text{ mm/s}^2)(3 \text{ s})$$

or $$\mathbf{v}_C = 120.0 \text{ mm/s} \downarrow$$ ◀

(*b*) We have $$y_D = (y_D)_0 + (0)t + \frac{1}{2}a_D t^2$$

At $t = 5$ s: $$y_D - (y_D)_0 = \frac{1}{2}(-10 \text{ mm/s}^2)(5 \text{ s})^2$$

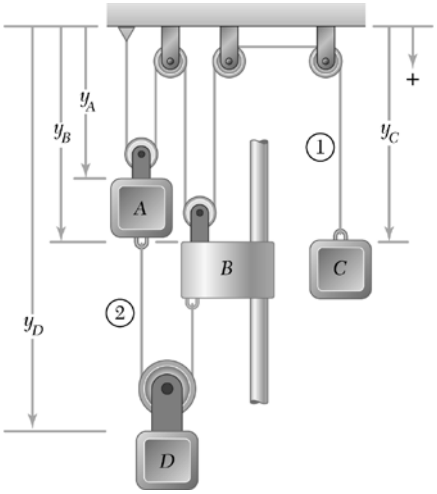
or $$\mathbf{y}_D - (\mathbf{y}_D)_0 = 125.0 \text{ mm} \uparrow$$ ◀



PROBLEM 11.60*

The system shown starts from rest, and the length of the upper cord is adjusted so that *A*, *B*, and *C* are initially at the same level. Each component moves with a constant acceleration, and after 2 s the relative change in position of block *C* with respect to block *A* is 280 mm upward. Knowing that when the relative velocity of collar *B* with respect to block *A* is 80 mm/s downward, the displacements of *A* and *B* are 160 mm downward and 320 mm downward, respectively, determine (*a*) the accelerations of *A* and *B* if $a_B > 10\text{ mm/s}^2$, (*b*) the change in position of block *D* when the velocity of block *C* is 600 mm/s upward.

SOLUTION



From the diagram

Cable 1: $2y_A + 2y_B + y_C = \text{constant}$

Then $$2v_A + 2v_B + v_C = 0 \tag{1}$$

and $$2a_A + 2a_B + a_C = 0 \tag{2}$$

Cable 2: $(y_D - y_A) + (y_D - y_B) = \text{constant}$

Then $$-v_A - v_B - 2v_D = 0 \tag{3}$$

and $$-a_A - a_B + 2a_D = 0 \tag{4}$$

Given: At $t = 0$

$$v = 0$$

$$(y_A)_0 = (y_B)_0 = (y_C)_0$$

All accelerations constant.

At $t = 2\text{ s}$

$$y_{C/A} = 280\text{ mm}\uparrow$$

When $v_{B/A} = 80\text{ mm/s}\downarrow$

$$y_A - (y_A)_0 = 160\text{ mm}\uparrow$$

$$y_B - (y_B)_0 = 320\text{ mm}\downarrow$$

$$a_B > 10\text{ mm/s}^2$$

(*a*) We have $y_A = (y_A)_0 + (0)t + \frac{1}{2}a_A t^2$

and $y_C = (y_C)_0 + (0)t + \frac{1}{2}a_C t^2$

Then $y_{C/A} = y_C - y_A = \frac{1}{2}(a_C - a_A)t^2$

At $t = 2$ s, $y_{C/A} = -280$ mm:

$$-280 \text{ mm} = \frac{1}{2}(a_C - a_A)(2 \text{ s})^2$$

or $a_C = a_A - 140$ (5)

Substituting into Eq. (2)

$$2a_A + 2a_B + (a_A - 140) = 0$$

or $a_A = \frac{1}{3}(140 - 2a_B)$ (6)

Now $v_B = 0 + a_B t$

$v_A = 0 + a_A t$

$v_{B/A} = v_B - v_A = (a_B - a_A)t$

Also $y_B = (y_B)_0 + (0)t + \frac{1}{2}a_B t^2$

When $\mathbf{v}_{B/A} = 80 \text{ mm/s}\downarrow$: $80 = (a_B - a_A)t$ (7)

$\Delta y_A = 160 \text{ mm}\downarrow$: $160 = \frac{1}{2}a_A t^2$

$\Delta y_B = 320 \text{ mm}\downarrow$: $320 = \frac{1}{2}a_B t^2$

Then $160 = \frac{1}{2}(a_B - a_A)t^2$

Using Eq. (7) $320 = (80)t$ or $t = 4$ s

Then $160 = \frac{1}{2}a_A(4)^2$ or $\mathbf{a}_A = 20.0 \text{ mm/s}^2\downarrow$ ◀

and $320 = \frac{1}{2}a_B(4)^2$ or $\mathbf{a}_B = 40.0 \text{ mm/s}^2\downarrow$ ◀

Note that Eq. (6) is not used; thus, the problem is over-determined.

PROBLEM 11.60* (Continued)

(*b*) Substituting into Eq. (5)

$$a_C = 20 - 140 = -120 \text{ mm/s}^2$$

and into Eq. (4)

$$-(20 \text{ mm/s}^2) - (40 \text{ mm/s}^2) + 2a_D = 0$$

or $$a_D = 30 \text{ mm/s}^2$$

Now $$v_C = 0 + a_C t$$

When $v_C = -600$ mm/s: $$-600 \text{ mm/s} = (-120 \text{ mm/s}^2)t$$

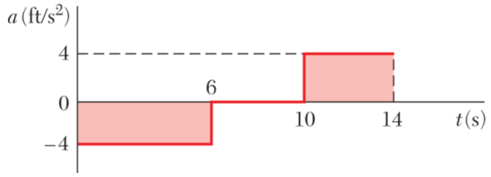
or $$t = 5 \text{ s}$$

Also $$y_D = (y_D)_0 + (0)t + \frac{1}{2}a_D t^2$$

At $t = 5$ s: $$y_D - (y_D)_0 = \frac{1}{2}(30 \text{ mm/s}^2)(5 \text{ s})^2$$

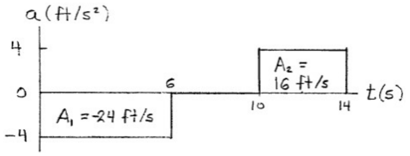
or $$\mathbf{y}_D - (\mathbf{y}_D)_0 = 375 \text{ mm}\downarrow \blacktriangleleft$$

PROBLEM 11.61



A particle moves in a straight line with a constant acceleration of −4 ft/s^2 for 6 s, zero acceleration for the next 4 s, and a constant acceleration of +4 ft/s^2 for the next 4 s. Knowing that the particle starts from the origin and that its velocity is −8 ft/s during the zero acceleration time interval, (*a*) construct the $v-t$ and $x-t$ curves for $0 \le t \le 14$ s, (*b*) determine the position and the velocity of the particle and the total distance traveled when $t = 14$ s.

SOLUTION



From the $a-t$ curve

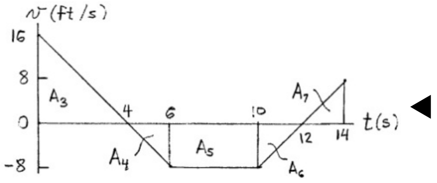
$$A_1 = -24 \text{ ft/s}, A_2 = 16 \text{ ft/s}$$

(*a*) Construct the $v-t$ curve: $v_6 = -8$ ft/s

$$v_0 = v_6 - A_1$$
$$= -8 - (-24)$$
$$= 16 \text{ ft/s}$$

$$v_{10} = -8 \text{ ft/s}$$

$$v_{14} = v_{10} + A_2$$
$$= -8 + 16$$
$$= 8 \text{ ft/s}$$



From the $v-t$ curve $A_3 = 32$ ft, $A_4 = -8$ ft

$$A_5 = -32 \text{ ft}, A_6 = -8 \text{ ft}, A_7 = 8 \text{ ft}$$

(b) Construct the $x-t$ curve $x_0 = 0$

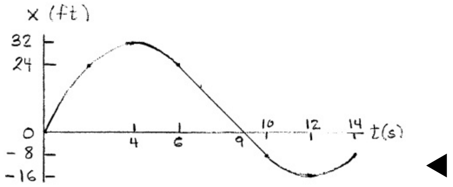
$$x_4 = x_0 + A_3 = 32 \text{ ft}$$

$$x_6 = x_4 + A_4 = 24 \text{ ft}$$

$$x_{10} = x_6 + A_5 = -8 \text{ ft}$$

$$x_{12} = x_{10} + A_6 = -16 \text{ ft}$$

$$x_{14} = x_{12} + A_7 = -8 \text{ ft}$$



Total Distance Traveled

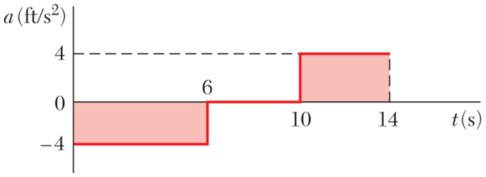
$0 \le t \le 4$ s, $d_1 = |32 - 0| = 32$ ft

$4 \text{ s} \le t \le 12$ s, $d_2 = |-16 - 32| = 48$ ft

$12 \text{ s} \le t \le 14$ s, $d_3 = |-8 - (-16)| = 8$ ft

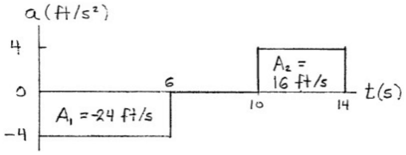
$d = 32 + 48 + 8$ $d = 88$ ft ◀

Problem 11.62



A particle moves in a straight line with a constant acceleration of −4 ft/s^2 for 6 s, zero acceleration for the next 4 s, and a constant acceleration of +4 ft/s^2 for the next 4 s. Knowing that the particle starts from the origin with $v_0 = 16$ ft/s, (*a*) construct the $v-t$ and $x-t$ curves for $0 \le t \le 14$ s, (*b*) determine the amount of time during which the particle is further than 16 ft from the origin.

SOLUTION



From the $a-t$ curve

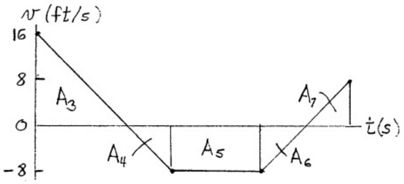
$$A_1 = -24 \text{ ft/s}, A_2 = 16 \text{ ft/s}$$

(b) Construct the $v-t$ curve : $v_0 = 16$ ft/s

$$v_6 = v_0 + A_1$$
$$= 16 + (-24)$$
$$= -8 \text{ ft/s}$$

$$v_{10} = v_6 = -8 \text{ ft/s}$$

$$v_{14} = v_{10} + A_2$$
$$= -8 + 16$$
$$= 8 \text{ ft/s}$$



◀

From the $v-t$ curve $A_3 = 32$ ft, $A_4 = -8$ ft

$$A_5 = -32 \text{ ft}, A_6 = -8 \text{ ft}, A_7 = 8 \text{ ft}$$

Construct the $x-t$ curve $x_0 = 0$

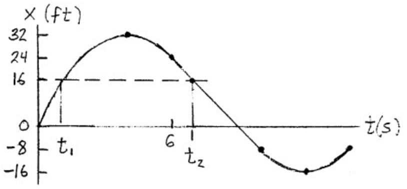
$$x_4 = x_0 + A_3 = 32 \text{ ft}$$

$$x_6 = x_4 + A_4 = 24 \text{ ft}$$

$$x_{10} = x_6 + A_5 = -8 \text{ ft}$$

$$x_{12} = x_{10} + A_6 = -16 \text{ ft}$$

$$x_{14} = x_{12} + A_7 = -8 \text{ ft}$$



◀

(b) Time for $|x| > 16$ ft.

From the $x-t$ diagram, this is time interval t_1 to t_2.

From $0 < t < 6$ s, $\dfrac{dx}{dt} = v = 16 - 4t$

Integrating, using limits $x = 0$ when $t = 0$ and $x = 16$ ft when $t = t_1$

$$[x]_0^{16} = \left[16t - 2t^2\right]_0^{t} \quad \text{or} \quad 16 = 16t_1 - 2t_1^2$$

PROBLEM 11.62 (Continued)

or $t_1^2 - 8t_1 + 8 = 0$

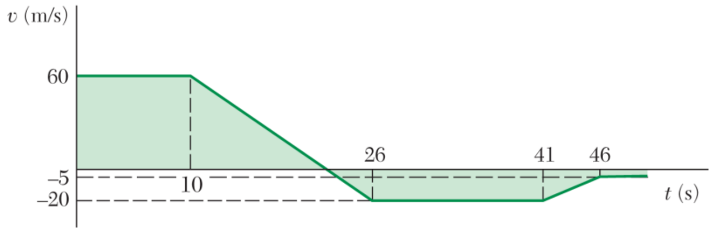
Using the quadratic formula: $t_1 = \dfrac{8 \pm \sqrt{(8)^2 - (4)(1)(8)}}{(2)(1)} = 4 \pm 2.828 = 1.172\text{ s}$ and 6.828 s

The larger root is out of range, thus $t_1 = 1.172\text{ s}$

From $6 < t < 10$, $x = 24 - 8(t - 6) = 72 - 8t$

Setting $x = 16$, $16 = 72 - 8t_2$ or $t_2 = 7\text{ s}$

Time Interval: $\Delta t = t_2 - t_1$ $\Delta t = 5.83\text{ s}$◀



PROBLEM 11.63

A particle moves in a straight line with the velocity shown in the figure. Knowing that $x = -540$ m at $t = 0$, (*a*) construct the a–t and x–t curves for $0 < t < 50$ s, and determine (*b*) the total distance traveled by the particle when $t = 50$ s, (*c*) the two times at which $x = 0$.

SOLUTION

(*a*) a_t = slope of v–t curve at time t

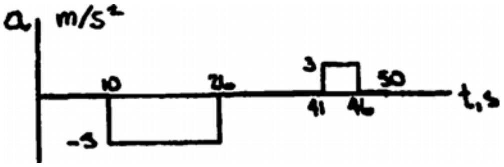
From $t = 0$ to $t = 10$ s: $v = \text{constant} \Rightarrow a = 0$

$t = 10$ s to $t = 26$ s: $$a = \frac{-20 - 60}{26 - 10} = -5 \text{ m/s}^2$$

$t = 26$ s to $t = 41$ s: $v = \text{constant} \Rightarrow a = 0$

$t = 41$ s to $t = 46$ s: $$a = \frac{-5 - (-20)}{46 - 41} = 3 \text{ m/s}^2$$

$t = 46$ s: $v = \text{constant} \Rightarrow a = 0$



$x_2 = x_1 +$ (area under v–t curve from t_1 to t_2)

At $t = 10$ s: $x_{10} = -540 + 10(60) + 60$ m

Next, find time at which $v = 0$. Using similar triangles

$$\frac{t_{v=0} - 10}{60} = \frac{26 - 10}{80} \quad \text{or} \quad t_{v=0} = 22 \text{ s}$$

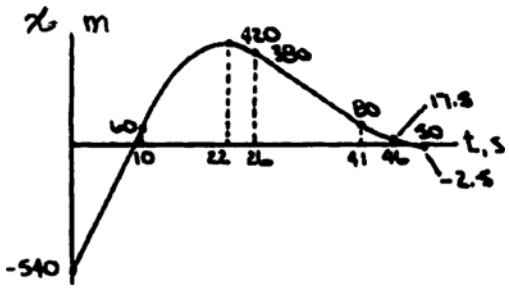
At $t = 22$ s: $x_{22} = 60 + \frac{1}{2}(12)(60) = 420$ m

$t = 26$ s: $x_{26} = 420 - \frac{1}{2}(4)(20) = 380$ m

$t = 41$ s: $x_{41} = 380 - 15(20) = 80$ m

$t = 46$ s: $x_{46} = 80 - 5\left(\frac{20 + 5}{2}\right) = 17.5$ m

$t = 50$ s: $x_{50} = 17.5 - 4(5) = -2.5$ m



PROBLEM 11.63 (Continued)

(*b*) From $t = 0$ to $t = 22\text{ s}$: Distance traveled $= 420 - (-540)$

$$= 960\text{ m}$$

$t = 22\text{ s}$ to $t = 50\text{ s}$: Distance traveled $= |-2.5 - 420|$

$$= 422.5\text{ m}$$

Total distance traveled $= (960 + 422.5)\text{ ft} = 1382.5\text{ m}$

Total distance traveled $= 1383\text{ m}$ ◀

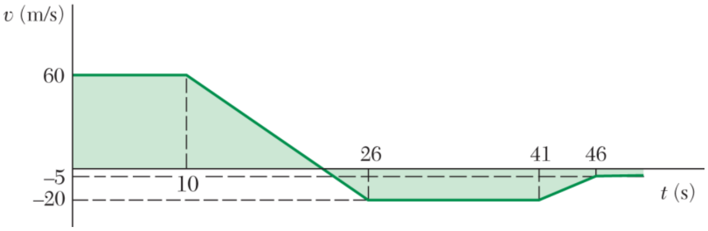
(*c*) Using similar triangles

Between 0 and 10 s:
$$\frac{(t_{x=0})_1 - 0}{540} = \frac{10}{600}$$

$(t_{x=0})_1 = 9.00\text{ s}$ ◀

Between 46 s and 50 s:
$$\frac{(t_{x=0})_2 - 46}{17.5} = \frac{4}{20}$$

$(t_{x=0})_2 = 49.5\text{ s}$ ◀



PROBLEM 11.64

A particle moves in a straight line with the velocity shown in the figure. Knowing that $x = -540$ m at $t = 0$, (a) construct the a–t and x–t curves for $0 < t < 50$ s, and determine (b) the maximum value of the position coordinate of the particle, (c) the values of t for which the particle is at $x = 100$ m.

SOLUTION

(a) a_t = slope of v–t curve at time t

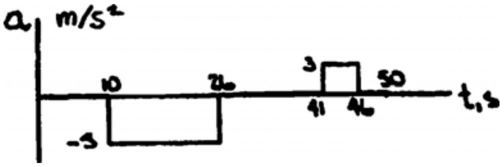
From $t = 0$ to $t = 10$ s: $v = \text{constant} \Rightarrow a = 0$

$t = 10$ s to $t = 26$ s: $a = \dfrac{-20-60}{26-10} = -5 \text{ m/s}^2$

$t = 26$ s to $t = 41$ s: $v = \text{constant} \Rightarrow a = 0$

$t = 41$ s to $t = 46$ s: $a = \dfrac{-5-(-20)}{46-41} = 3 \text{ m/s}^2$

$t = 46$ s: $v = \text{constant} \Rightarrow a = 0$



$x_2 = x_1 +$ (area under v–t curve from t_1 to t_2)

At $t = 10$ s: $x_{10} = -540 + 10(60) = 60$ m

Next, find time at which $v = 0$. Using similar triangles

$$\frac{t_{v=0} - 10}{60} = \frac{26-10}{80} \quad \text{or} \quad t_{v=0} = 22 \text{ s}$$

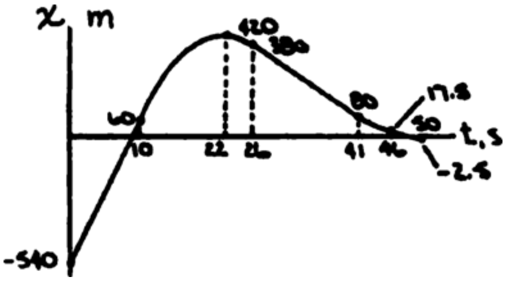
At $t = 22$ s: $x_{22} = 60 + \dfrac{1}{2}(12)(60) = 420$ m

$t = 26$ s: $x_{26} = 420 - \dfrac{1}{2}(4)(20) = 380$ m

$t = 41$ s: $x_{41} = 380 - 15(20) = 80$ m

$t = 46$ s: $x_{46} = 80 - 5\left(\dfrac{20+5}{2}\right) = 17.5$ m

$t = 50$ s: $x_{50} = 17.5 - 4(5) = -2.5$ m

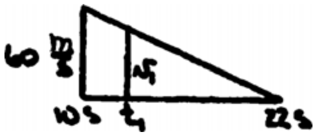


PROBLEM 11.64 (Continued)

(*b*) Reading from the x–t curve $x_{\max} = 420 \text{ m}$ ◀

(*c*) Between 10 s and 22 s

$100 \text{ m} = 420 \text{ m} - (\text{area under } v\text{–}t \text{ curve from } t, \text{ to } 22 \text{ s}) \text{ m}$



$$100 = 420 - \frac{1}{2}(22 - t_1)(v_1)$$

$$(22 - t_1)(v_1) = 640$$

Using similar triangles

$$\frac{v_1}{22 - t_1} = \frac{60}{12} \qquad \text{or} \qquad v_1 = 5(22 - t_1)$$

Then

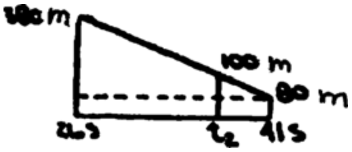
$$(22 - t_1)[5(22 - t_1)] = 640$$

$$t_1 = 10.69 \text{ s} \quad \text{and} \quad t_1 = 33.3 \text{ s}$$

We have $10 \text{ s} < t_1 < 22 \text{ s} \Rightarrow$ $t_1 = 10.69 \text{ s}$ ◀

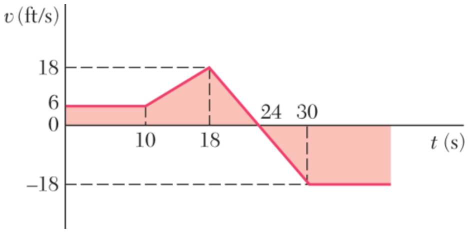
Between 26 s and 41 s:

Using similar triangles



$$\frac{41 - t_2}{20} = \frac{15}{300}$$

$t_2 = 40.0 \text{ s}$ ◀



Problem 11.65

A particle moves in a straight line with the velocity shown in the figure. Knowing that $x = -48$ ft at $t = 0$, draw the a–t and x–t curves for $0 < t < 40$ s and determine (*a*) the maximum value of the position coordinate of the particle, (*b*) the values of t for which the particle is at a distance of 108 ft from the origin.

SOLUTION

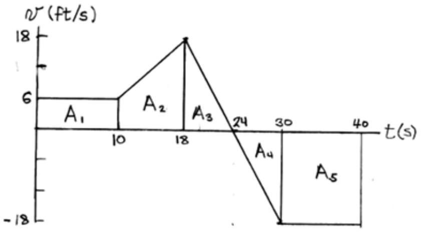
The $a - t$ curve is the slope of the $v - t$ curve

$0 < t < 10$ s, $\quad a = 0$

$10 \text{ s} < t < 18$ s, $\quad a = \dfrac{18 - 6}{18 - 10} = 1.5 \text{ ft/s}^2$

$18 \text{ s} < t < 30$ s, $\quad a = \dfrac{-18 - 18}{30 - 18} = -3 \text{ ft/s}^2$

$30 \text{ s} < t < 40$ s $\quad a = 0$



◀

Points on the x–t curve may be calculated using areas of the v–t curve.

$$A_1 = (10)(6) = 60 \text{ ft}$$

$$A_2 = \frac{1}{2}(6 + 18)(18 - 10) = 96 \text{ ft}$$

$$A_3 = \frac{1}{2}(18)(24 - 18) = 54 \text{ ft}$$

$$A_4 = \frac{1}{2}(-18)(30 - 24) = -54 \text{ ft}$$

$$A_5 = (-18)(40 - 30) = -180 \text{ ft}$$

Value of x at specific times:

$$x_0 = -48 \text{ ft}$$

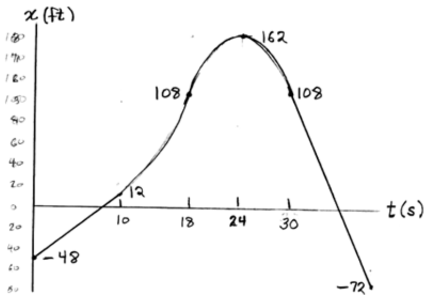
$$x_{10} = x_0 + A_1 = 12 \text{ ft}$$

$$x_{18} = x_{10} + A_2 = 108 \text{ ft}$$

$$x_{24} = x_{18} + A_3 = 162 \text{ ft}$$

$$x_{30} = x_{24} + A_4 = 108 \text{ ft}$$

$$x_{40} = x_{30} + A_5 = -72 \text{ ft}$$



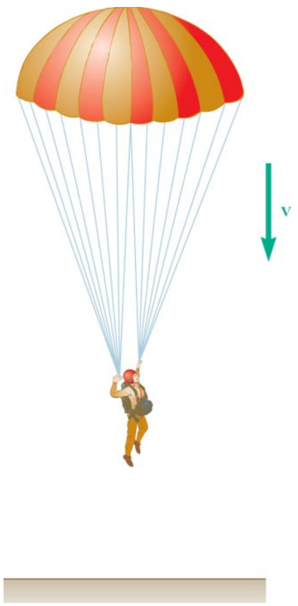
◀

PROBLEM 11.65 (Continued)

(a) Maximum value of x occurs when: $v = 0$, i.e. $t = 24$ s. $x_{\max} = 162$ ft ◀

(b) Times when $x = 108$ ft.

From the $x - t$ curve $t = 18$ s and $t = 30$ s ◀



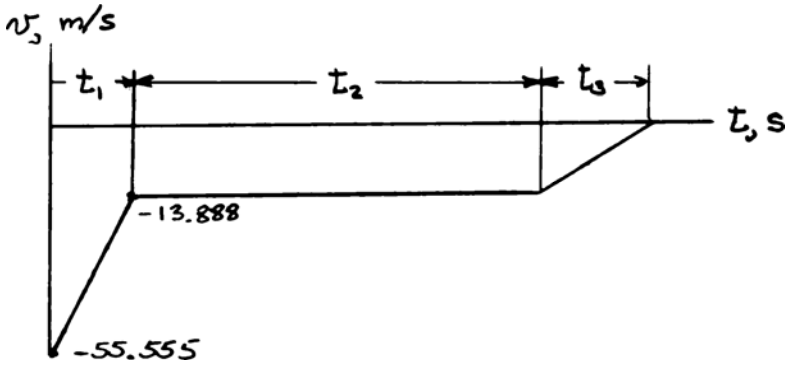
PROBLEM 11.66

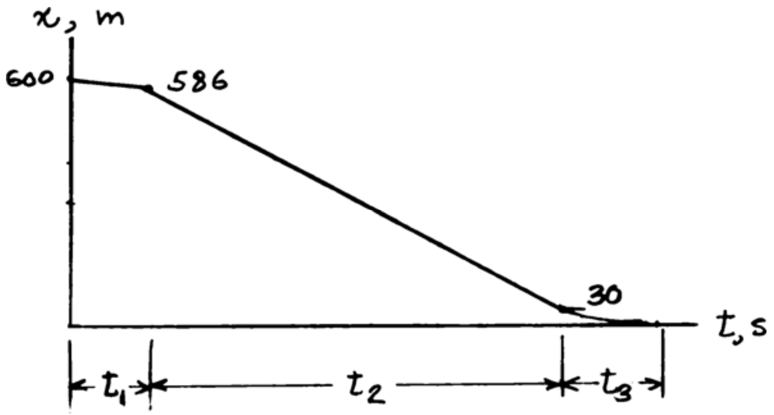
A parachutist is in free fall at a rate of 200 km/h when he opens his parachute at an altitude of 600 m. Following a rapid and constant deceleration, he then descends at a constant rate of 50 km/h from 586 m to 30 m, where he maneuvers the parachute into the wind to further slow his descent. Knowing that the parachutist lands with a negligible downward velocity, determine (*a*) the time required for the parachutist to land after opening his parachute, (*b*) the initial deceleration.

SOLUTION

Assume second deceleration is constant. Also, note that

$$200 \text{ km/h} = 55.555 \text{ m/s},$$
$$50 \text{ km/h} = 13.888 \text{ m/s}$$





(*a*) Now $\Delta x =$ area under $v{-}t$ curve for given time interval

Then

$$(586 - 600)\text{ m} = -t_1\left(\frac{55.555 + 13.888}{2}\right)\text{ m/s}$$

$$t_1 = 0.4032\text{ s}$$

$$(30 - 586)\text{ m} = -t_2(13.888\text{ m/s})$$

$$t_2 = 40.0346\text{ s}$$

$$(0 - 30)\text{ m} = -\frac{1}{2}(t_3)(13.888\text{ m/s})$$

$$t_3 = 4.3203\text{ s}$$

$$t_{\text{total}} = (0.4032 + 40.0346 + 4.3203)\text{ s}$$

$t_{\text{total}} = 44.8\text{ s}$ ◀

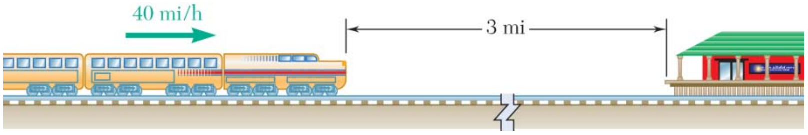
(*b*) We have

$$a_{\text{initial}} = \frac{\Delta v_{\text{initial}}}{t_1}$$

$$= \frac{[-13.888 - (-55.555)]\text{ m/s}}{0.4032\text{ s}}$$

$$= 103.3\text{ m/s}^2$$

$\mathbf{a}_{\text{initial}} = 103.3\text{ m/s}^2 \uparrow$ ◀



PROBLEM 11.67

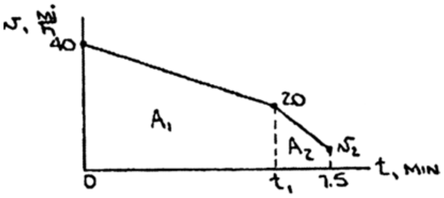
A commuter train traveling at 40 mi/h is 3 mi from a station. The train then decelerates so that its speed is 20 mi/h when it is 0.5 mi from the station. Knowing that the train arrives at the station 7.5 min after beginning to decelerate and assuming constant decelerations, determine (*a*) the time required for the train to travel the first 2.5 mi, (*b*) the speed of the train as it arrives at the station, (*c*) the final constant deceleration of the train.

SOLUTION

Given: At $t = 0$, $v = 40$ mi/h, $x = 0$; when $x = 2.5$ mi, $v = 20$ mi/h;

at $t = 7.5$ min, $x = 3$ mi; constant decelerations.

The $v-t$ curve is first drawn as shown.



(*a*) We have $A_1 = 2.5$ mi

$$(t_1 \text{ min})\left(\frac{40+20}{2}\right)\text{ mi/h} \times \frac{1\text{ h}}{60\text{ min}} = 2.5\text{ mi}$$

$t_1 = 5.00$ min ◀

(*b*) We have $A_2 = 0.5$ mi

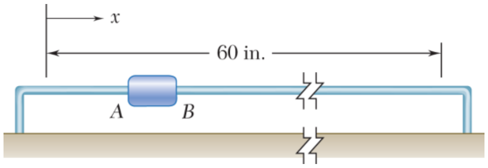
$$(7.5-5)\text{ min} \times \left(\frac{20+v_2}{2}\right)\text{mi/h} \times \frac{1\text{ h}}{60\text{ min}} = 0.5\text{ mi}$$

$v_2 = 4.00$ mi/h ◀

(*c*) We have $a_{\text{final}} = a_{12}$

$$= \frac{(4-20)\text{ mi/h}}{(7.5-5)\text{ min}} \times \frac{5280\text{ ft}}{\text{mi}} \times \frac{1\text{ min}}{60\text{ s}} \times \frac{1\text{ h}}{3600\text{ s}}$$

$a_{\text{final}} = -0.1564\text{ ft/s}^2$ ◀



PROBLEM 11.68

A temperature sensor is attached to slider *AB* which moves back and forth through 60 in. The maximum velocities of the slider are 12 in./s to the right and 30 in./s to the left. When the slider is moving to the right, it accelerates and decelerates at a constant rate of 6 in./s^2; when moving to the left, the slider accelerates and decelerates at a constant rate of 20 in./s^2. Determine the time required for the slider to complete a full cycle, and construct the $v-t$ and $x-t$ curves of its motion.

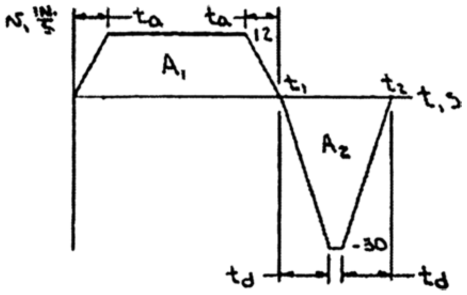
SOLUTION

The $v-t$ curve is first drawn as shown. Then

$$t_a = \frac{v_{\text{right}}}{a_{\text{right}}} = \frac{12 \text{ in./s}}{6 \text{ in./s}^2} = 2 \text{ s}$$

$$t_d = \frac{v_{\text{left}}}{a_{\text{left}}} = \frac{30 \text{ in./s}}{20 \text{ in./s}}$$

$$= 1.5 \text{ s}$$



Now $A_1 = 60$ in.

or $[(t_1 - 2) \text{ s}](12 \text{ in./s}) = 60$ in.

or $t_1 = 7$ s

and $A_2 = 60$ in.

or $\{[(t_2 - 7) - 1.5] \text{ s}\}(30 \text{ in./s}) = 60$ in.

or $t_2 = 10.5$ s

Now $t_{\text{cycle}} = t_2$

$t_{\text{cycle}} = 10.5$ s ◀

We have $x_{ii} = x_i + (\text{area under } v-t \text{ curve from } t_i \text{ to } t_{ii})$

At $t = 2$ s: $x_2 = \frac{1}{2}(2)(12) = 12$ in.

$t = 5$ s: $x_5 = 12 + (5-2)(12)$

$= 48$ in.

$t = 7$ s: $x_7 = 60$ in.

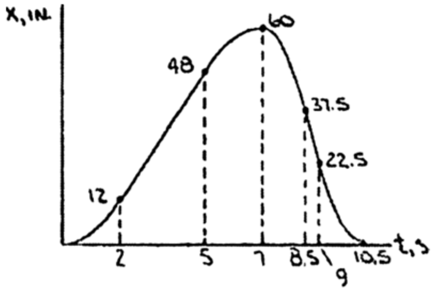
$t = 8.5$ s: $x_{8.5} = 60 - \frac{1}{2}(1.5)(30)$

$= 37.5$ in.

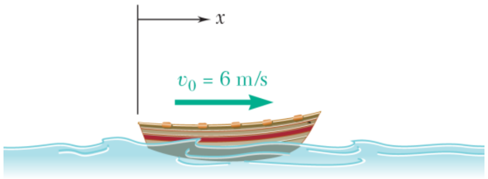
$t = 9$ s: $x_9 = 37.5 - (0.5)(30)$

$= 22.5$ in.

$t = 10.5$ s: $x_{10.5} = 0$



PROBLEM 11.69



In a water-tank test involving the launching of a small model boat, the model's initial horizontal velocity is 6 m/s, and its horizontal acceleration varies linearly from $-12\ \text{m/s}^2$ at $t = 0$ to $-2\ \text{m/s}^2$ at $t = t_1$ and then remains equal to $-2\ \text{m/s}^2$ until $t = 1.4$ s. Knowing that $v = 1.8$ m/s when $t = t_1$, determine (*a*) the value of t_1, (*b*) the velocity and the position of the model at $t = 1.4$ s.

SOLUTION

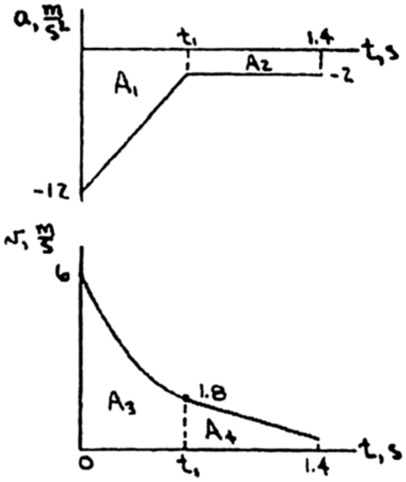
Given: $v_0 = 6$ m/s; for $0 < t < t_1$,

for $t_1 < t < 1.4\ \text{s} \quad a = -2\ \text{m/s}^2$;

at $t = 0 \quad a = -12\ \text{m/s}^2$;

at $t = t_1$ $a = -2\ \text{m/s}^2,\ v = 1.8\ \text{m/s}^2$

The a–t and v–t curves are first drawn as shown. The time axis is not drawn to scale.



(*a*) We have $v_{t_1} = v_0 + A_1$

$$1.8\ \text{m/s} = 6\ \text{m/s} - (t_1\ \text{s})\left(\frac{12+2}{2}\right)\text{m/s}^2$$

$$t_1 = 0.6\ \text{s} \quad ◀$$

(*b*) We have $v_{1.4} = v_{t_1} + A_2$

$$v_{1.4} = 1.8\ \text{m/s} - (1.4 - 0.6)\ \text{s} \times 2\ \text{m/s}^2$$

$$v_{1.4} = 0.20\ \text{m/s} \quad ◀$$

Now $x_{1.4} = A_3 + A_4$, where A_3 is most easily determined using integration. Thus,

for $0 < t < t_1$: $$a = \frac{-2-(-12)}{0.6}t - 12 = \frac{50}{3}t - 12$$

Now $$\frac{dv}{dt} = a = \frac{50}{3}t - 12$$

At $t = 0$, $v = 6$ m/s: $$\int_6^v dv = \int_0^t \left(\frac{50}{3}t - 12\right)dt$$

or $$v = 6 + \frac{25}{3}t^2 - 12t$$

We have $$\frac{dx}{dt} = v = 6 - 12t + \frac{25}{3}t^2$$

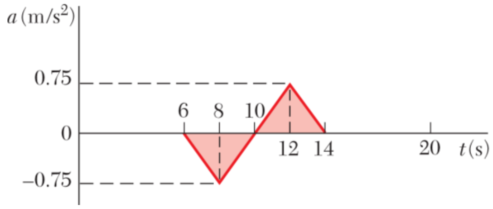
Then $$A_3 = \int_0^{x_{t_1}} dx = \int_0^{0.6} (6 - 12t + \frac{25}{3}t^2)dt$$

$$= \left[6t - 6t^2 + \frac{25}{9}t^3\right]_0^{0.6} = 2.04 \text{ m}$$

Also $$A_4 = (1.4 - 0.6)\left(\frac{1.8 + 0.2}{2}\right) = 0.8 \text{ m}$$

Then $$x_{1.4} = (2.04 + 0.8) \text{ m}$$

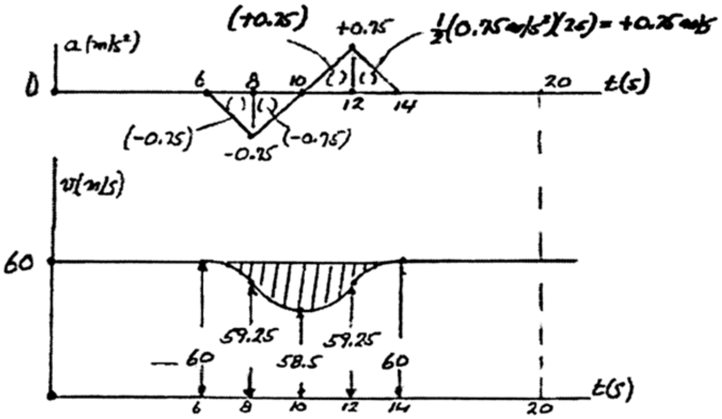
or $$x_{1.4} = 2.84 \text{ m} \blacktriangleleft$$



PROBLEM 11.70

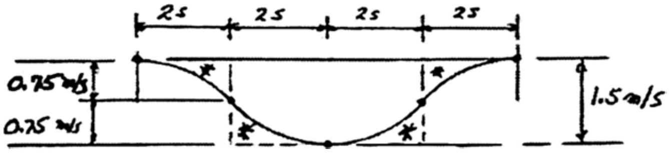
The acceleration record shown was obtained for a small airplane traveling along a straight course. Knowing that $x = 0$ and $v = 60$ m/s when $t = 0$, determine (a) the velocity and position of the plane at $t = 20$ s, (b) its average velocity during the interval $6\text{ s} < t < 14\text{ s}$.

SOLUTION



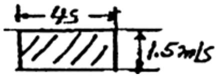
a (m/s²)
(+0.75)
+0.75
½(0.75 m/s²)(2s) = +0.75 m/s
0
6
8
10
12
14
20
t(s)
(−0.75)
−0.75
(−0.75)
v(m/s)
60
59.25
58.5
59.25
60
60
6
8
10
12
14
20
t(s)

Geometry of "bell-shaped" portion of v–t curve



2s
2s
2s
2s
0.75 m/s
0.75 m/s
1.5 m/s
*
*
*
*

The parabolic spandrels marked by * are of equal area. Thus, total area of shaded portion of v–t diagram is:



$= \Delta x = 6$ m

(a) When $t = 20$ s:

$v_{20} = 60$ m/s ◀

$$x_{20} = (60\text{ m/s})(20\text{ s}) - (\text{shaded area})$$
$$= 1200\text{ m} - 6\text{ m}$$

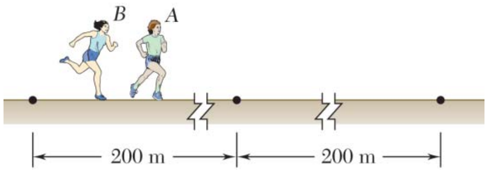
$x_{20} = 1194$ m ◀

(b) From $t = 6$ s to $t = 14$ s: $\Delta t = 8$ s

$$\Delta x = (60\text{ m/s})(14\text{ s} - 6\text{ s}) - (\text{shaded area})$$
$$= (60\text{ m/s})(8\text{ s}) - 6\text{ m} = 480\text{ m} - 6\text{ m} = 474\text{ m}$$

$$v_{\text{average}} = \frac{\Delta x}{\Delta t} = \frac{474\text{ m}}{8\text{ s}}$$

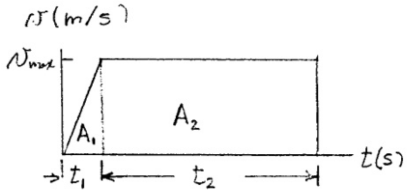
$v_{\text{average}} = 59.25$ m/s ◀

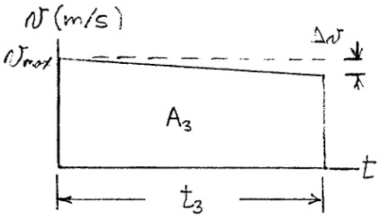


PROBLEM 11.71

In a 400-m race, runner A reaches her maximum velocity v_A in 4 s with constant acceleration and maintains that velocity until she reaches the half-way point with a split time of 25 s. Runner B reaches her maximum velocity v_B in 5 s with constant acceleration and maintains that velocity until she reaches the half-way point with a split time of 25.2 s. Both runners then run the second half of the race with the same constant deceleration of 0.1 m/s^2. Determine (*a*) the race times for both runners, (*b*) the position of the winner relative to the loser when the winner reaches the finish line.

SOLUTION





Sketch $v-t$ curves for first 200 m.

Runner A: $t_1 = 4\text{ s}, t_2 = 25 - 4 = 21\text{ s}$

$$A_1 = \frac{1}{2}(4)(v_A)_{\max} = 2(v_A)_{\max}$$

$$A_2 = 21(v_A)_{\max}$$

$$A_1 + A_2 = \Delta x = 200\text{ m}$$

$$23(v_A)_{\max} = 200 \qquad \text{or} \qquad (v_A)_{\max} = 8.6957\text{ m/s}$$

Runner B: $t_1 = 5\text{ s}, \qquad t_2 = 25.2 - 5 = 20.2\text{ s}$

$$A_1 = \frac{1}{2}(5)(v_B)_{\max} = 2.5(v_B)_{\max}$$

$$A_2 = 20.2(v_B)_{\max}$$

$$A_1 + A_2 = \Delta x = 200\text{ m}$$

$$22.7(v_B)_{\max} = 200 \qquad \text{or} \qquad (v_B)_{\max} = 8.8106\text{ m/s}$$

Sketch $v-t$ curve for second 200 m. $\Delta v = |a|t_3 = 0.1t_3$

$$A_3 = v_{\max}t_3 - \frac{1}{2}\Delta v t_3 = 200 \qquad \text{or} \qquad 0.05t_3^2 - v_{\max}t_3 + 200 = 0$$

$$t_3 = \frac{v_{\max} \pm \sqrt{(v_{\max})^2 - (4)(0.05)(200)}}{(2)(0.05)} = 10\left(v_{\max} \pm \sqrt{(v_{\max})^2 - 40}\right)$$

Runner A: $(v_{\max})_A = 8.6957, \qquad (t_3)_A = 146.64\text{ s} \qquad \text{and} \qquad 27.279\text{ s}$

Reject the larger root. Then total time $t_A = 25 + 27.279 = 52.279\text{ s}$ $\qquad t_A = 52.2\text{ s}$ ◀

PROBLEM 11.71 (Continued)

Runner B: $(v_{\max})_B = 8.8106$, $(t_3)_B = 149.45$ s and 26.765 s

Reject the larger root. Then total time $t_B = 25.2 + 26.765 = 51.965$ s

$t_B = 52.0$ s ◀

Velocity of A at $t = 51.965$ s:

$$v_1 = 8.6957 - (0.1)(51.965 - 25) = 5.999 \text{ m/s}$$

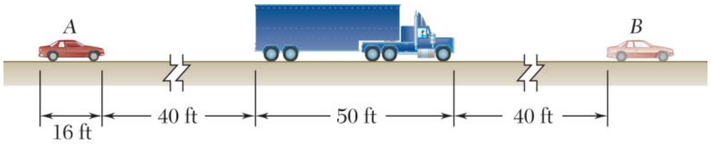
Velocity of A at $t = 51.279$ s:

$$v_2 = 8.6957 - (0.1)(52.279 - 25) = 5.968 \text{ m/s}$$

Over $51.965 \text{ s} \le t \le 52.965$ s, runner A covers a distance Δx

$$\Delta x = v_{\text{ave}}(\Delta t) = \frac{1}{2}(5.999 + 5.968)(52.279 - 51.965)$$

$\Delta x = 1.879$ m ◀



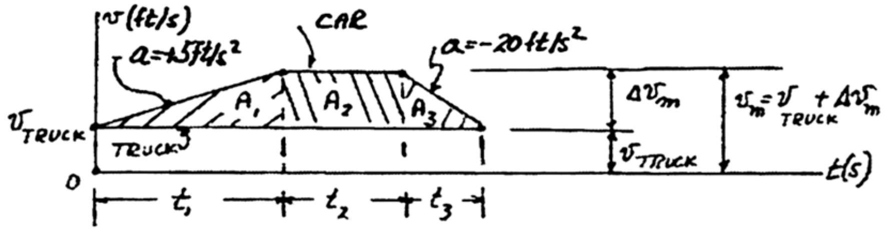
PROBLEM 11.72

A car and a truck are both traveling at the constant speed of 35 mi/h; the car is 40 ft behind the truck. The driver of the car wants to pass the truck, i.e., he wishes to place his car at *B*, 40 ft in front of the truck, and then resume the speed of 35 mi/h. The maximum acceleration of the car is $5\ \text{ft/s}^2$ and the maximum deceleration obtained by applying the brakes is $20\ \text{ft/s}^2$. What is the shortest time in which the driver of the car can complete the passing operation if he does not at any time exceed a speed of 50 mi/h? Draw the $v{-}t$ curve.

SOLUTION

Relative to truck, car must move a distance: $\Delta x = 16 + 40 + 50 + 40 = 146\ \text{ft}$

Allowable increase in speed: $\Delta v_m = 50 - 35 = 15\ \text{mi/h} = 22\ \text{ft/s}$



Acceleration Phase: $t_1 = 22/5 = 4.4\ \text{s}$ $A_1 = \frac{1}{2}(22)(4.4) = 48.4\ \text{ft}$

Deceleration Phase: $t_3 = 22/20 = 1.1\ \text{s}$ $A_3 = \frac{1}{2}(22)(1.1) = 12.1\ \text{ft}$

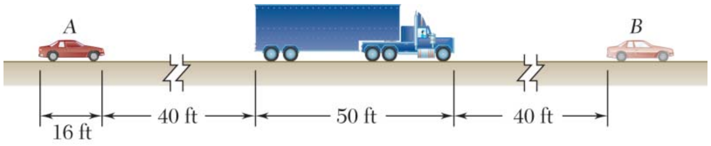
But: $\Delta x = A_1 + A_2 + A_3$: $146\ \text{ft} = 48.4 + (22)t_2 + 12.1$ $t_2 = 3.89\ \text{s}$

$t_{\text{total}} = t_1 + t_2 + t_3 = 4.4\ \text{s} + 3.89\ \text{s} + 1.1\ \text{s} = 9.39\ \text{s}$ $t_B = 9.39\ \text{s}$ ◀

PROBLEM 11.73

Solve Problem 11.72, assuming that the driver of the car does not pay any attention to the speed limit while passing and concentrates on reaching position *B* and resuming a speed of 35 mi/h in the shortest possible time. What is the maximum speed reached? Draw the v–t curve.

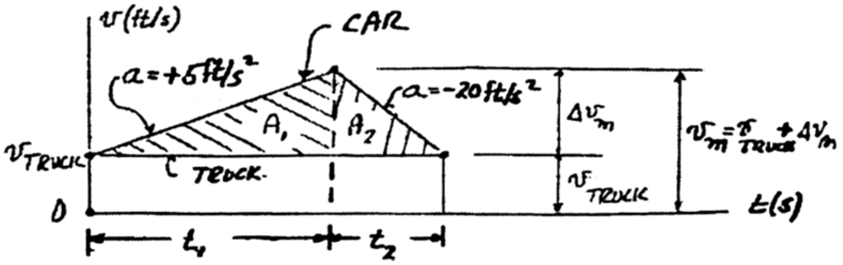
PROBLEM 11.72 A car and a truck are both traveling at the constant speed of 35 mi/h; the car is 40 ft behind the truck. The driver of the car wants to pass the truck, i.e., he wishes to place his car at *B*, 40 ft in front of the truck, and then resume the speed of 35 mi/h. The maximum acceleration of the car is 5 ft/s^2 and the maximum deceleration obtained by applying the brakes is 20 ft/s^2. What is the shortest time in which the driver of the car can complete the passing operation if he does not at any time exceed a speed of 50 mi/h? Draw the v–t curve.



SOLUTION

Relative to truck, car must move a distance:

$$\Delta x = 16 + 40 + 50 + 40 = 146 \text{ ft}$$



$$\Delta v_m = 5t_1 = 20t_2; \qquad t_2 = \frac{1}{4}t_1$$

$\Delta x = A_1 + A_2$:

$$146 \text{ ft} = \frac{1}{2}(\Delta v_m)(t_1 + t_2)$$

$$146 \text{ ft} = \frac{1}{2}(5t_1)\left(t_1 + \frac{1}{4}t_1\right)$$

$$t_1^2 = 46.72 \qquad t_1 = 6.835 \text{ s} \qquad t_2 = \frac{1}{4}t_1 = 1.709$$

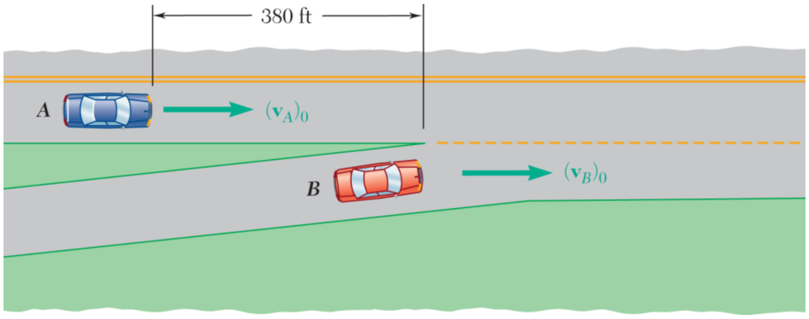
$$t_{\text{total}} = t_1 + t_2 = 6.835 + 1.709$$

$t_B = 8.54$ s ◀

$$\Delta v_m = 5t_1 = 5(6.835) = 34.18 \text{ ft/s} = 23.3 \text{ mi/h}$$

Speed $v_{\text{total}} = 35$ mi/h, $v_m = 35 \text{ mi/h} + 23.3 \text{ mi/h}$

$v_m = 58.3$ mi/h ◀



PROBLEM 11.74

Car A is traveling on a highway at a constant speed $(v_A)_0 = 60$ mi/h, and is 380 ft from the entrance of an access ramp when car B enters the acceleration lane at that point at a speed $(v_B)_0 = 15$ mi/h. Car B accelerates uniformly and enters the main traffic lane after traveling 200 ft in 5 s. It then continues to accelerate at the same rate until it reaches a speed of 60 mi/h, which it then maintains. Determine the final distance between the two cars.

SOLUTION

Given: $(v_A)_0 = 60$ mi/h, $(v_B)_0 = 1.5$ mi/h; at $t = 0$,

$(x_A)_0 = -380$ ft, $(x_B)_0 = 0$; at $t = 5$ s,

$x_B = 200$ ft; for 15 mi/h $< v_B \leq 60$ mi/h,

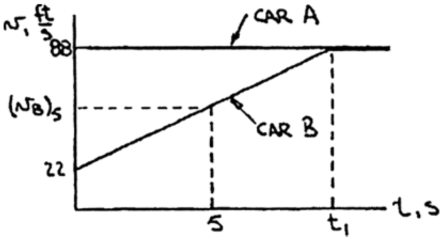
$a_B =$ constant; for $v_B = 60$ mi/h,

$a_B = 0$

First note 60 mi/h = 88 ft/s

15 mi/h = 22 ft/s

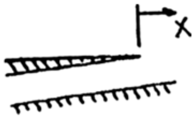
The v–t curves of the two cars are then drawn as shown.



Using the coordinate system shown, we have

at $t = 5$ s, $x_B = 200$ ft: $$(5\text{ s})\left[\frac{22+(v_B)_5}{2}\right]\text{ ft/s} = 200\text{ ft}$$

or $$(v_B)_5 = 58\text{ ft/s}$$



Then, using similar triangles, we have

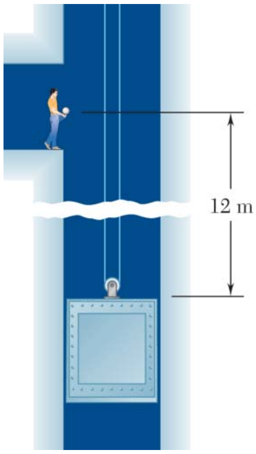
$$\frac{(88-22)\text{ ft/s}}{t_1} = \frac{(58-22)\text{ ft/s}}{5\text{ s}} (= a_B)$$

or $$t_1 = 9.1667\text{ s}$$

Finally, at $t = t_1$

$$x_{B/A} = x_B - x_A = \left[(9.1667\text{ s})\left(\frac{22+88}{2}\right)\text{ft/s}\right] - [-380\text{ ft} + (9.1667\text{ s})(88\text{ ft/s})]$$

or $$x_{B/A} = 77.5\text{ ft} ◀$$



PROBLEM 11.75

An elevator starts from rest and moves upward, accelerating at a rate of 1.2 m/s^2 until it reaches a speed of 7.8 m/s, which it then maintains. Two seconds after the elevator begins to move, a man standing 12 m above the initial position of the top of the elevator throws a ball upward with an initial velocity of 20 m/s. Determine when the ball will hit the elevator.

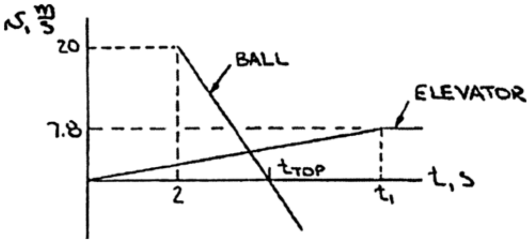
SOLUTION

Given: At $t = 0 \quad v_E = 0$; For $0 < v_E \leq 7.8$ m/s, $a_E = 1.2 \text{ m/s}^2 \uparrow$;

For $v_E = 7.8$ m/s, $a_E = 0$;

At $t = 2$ s, $v_B = 20 \text{m/s} \uparrow$

The v–t curves of the ball and the elevator are first drawn as shown. Note that the initial slope of the curve for the elevator is 1.2 m/s^2, while the slope of the curve for the ball is $-g\,(-9.81 \text{ m/s}^2)$.



The time t_1 is the time when v_E reaches 7.8 m/s.

Thus, $$v_E = (0) + a_E t$$

or $$7.8 \text{ m/s} = (1.2 \text{ m/s}^2)t_1$$

or $$t_1 = 6.5 \text{ s}$$

The time t_{top} is the time at which the ball reaches the top of its trajectory.

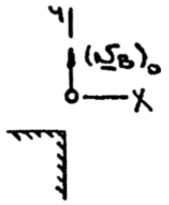
Thus, $$v_B = (v_B)_0 - g(t-2)$$

or $$0 = 20 \text{ m/s} - (9.81 \text{ m/s}^2)(t_{\text{top}} - 2) \text{ s}$$

or $$t_{\text{top}} = 4.0387 \text{ s}$$

PROBLEM 11.75 (Continued)

Using the coordinate system shown, we have



$0 < t < t_1$: $$y_E = -12 \text{ m} + \left(\frac{1}{2}a_E t^2\right) \text{ m}$$

At $t = t_{\text{top}}$: $$y_B = \frac{1}{2}(4.0387 - 2) \text{ s} \times (20 \text{ m/s})$$
$$= 20.387 \text{ m}$$

and $$y_E = -12 \text{ m} + \frac{1}{2}(1.2 \text{ m/s}^2)(4.0387 \text{ s})^2$$
$$= -2.213 \text{ m}$$

At $$t = [2 + 2(4.0387 - 2)] \text{ s} = 6.0774 \text{ s}, \ \ y_B = 0$$

and at $t = t_1$, $$y_E = -12 \text{ m} + \frac{1}{2}(6.5 \text{ s})(7.8 \text{ m/s}) = 13.35 \text{ m}$$

The ball hits the elevator $(y_B = y_E)$ when $t_{\text{top}} \le t \le t_1$.

For $t \ge t_{\text{top}}$: $$y_B = 20.387 \text{ m} - \left[\frac{1}{2} \text{ g } (t - t_{\text{top}})^2\right] \text{ m}$$

Then,

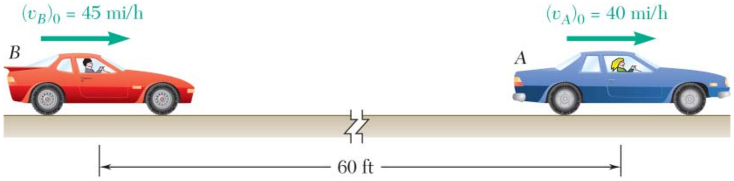
when $$y_B = y_E$$

$$20.387 \text{ m} - \frac{1}{2}(9.81 \text{ m/s}^2)(t - 4.0387)^2$$
$$= -12 \text{ m} + \frac{1}{2}(1.2 \text{ m/s}^2)(t \text{ s})^2$$

or $$5.505t^2 - 39.6196t + 47.619 = 0$$

Solving $$t = 1.525 \text{ s} \quad \text{and} \quad t = 5.67 \text{ s}$$

Since 1.525 s is less than 2 s, $t = 5.67$ s ◀



PROBLEM 11.76

Car *A* is traveling at 40 mi/h when it enters a 30 mi/h speed zone. The driver of car *A* decelerates at a rate of 16 ft/s^2 until reaching a speed of 30 mi/h, which she then maintains. When car *B*, which was initially 60 ft behind car *A* and traveling at a constant speed of 45 mi/h, enters the speed zone, its driver decelerates at a rate of 20 ft/s^2 until reaching a speed of 28 mi/h. Knowing that the driver of car *B* maintains a speed of 28 mi/h, determine (*a*) the closest that car *B* comes to car *A*, (*b*) the time at which car *A* is 70 ft in front of car *B*.

SOLUTION

Given: $(v_A)_0 = 40$ mi/h; For $30 \text{ mi/h} < v_A \leq 40$ mi/h, $a_A = -16 \text{ ft/s}^2$; For $v_A = 30$ mi/h, $a_A = 0$;

$(x_{A/B})_0 = 60$ ft; $(v_B)_0 = 45$ mi/h;

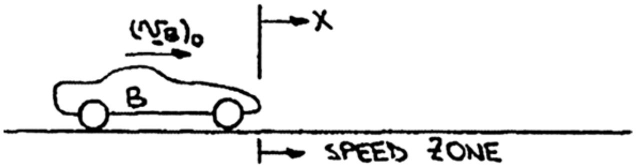
When $x_B = 0$, $a_B = -20 \text{ ft/s}^2$;

For $v_B = 28$ mi/h, $a_B = 0$

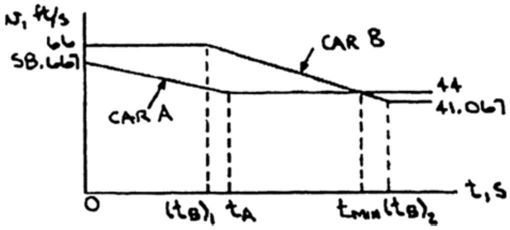
First note $40 \text{ mi/h} = 58.667 \text{ ft/s}$ $30 \text{ mi/h} = 44 \text{ ft/s}$

$45 \text{ mi/h} = 66 \text{ ft/s}$ $28 \text{ mi/h} = 41.067 \text{ ft/s}$

At $t = 0$



The v–t curves of the two cars are as shown.



At $t = 0$: Car *A* enters the speed zone.

$t = (t_B)_1$: Car *B* enters the speed zone.

$t = t_A$: Car *A* reaches its final speed.

$t = t_{\min}$: $v_A = v_B$

$t = (t_B)_2$: Car *B* reaches its final speed.

PROBLEM 11.76 (Continued)

(*a*) We have $$a_A = \frac{(v_A)_{\text{final}} - (v_A)_0}{t_A}$$

or $$-16\text{ ft/s}^2 = \frac{(44 - 58.667)\text{ ft/s}}{t_A}$$

or $$t_A = 0.91669\text{ s}$$

Also $$60\text{ ft} = (t_B)_1(v_B)_0$$

or $$60\text{ ft} = (t_B)_1(66\text{ ft/s}) \qquad \text{or} \qquad (t_B)_1 = 0.90909\text{ s}$$

and $$a_B = \frac{(v_B)_{\text{final}} - (v_B)_0}{(t_B)_2 - (t_B)_1}$$

or $$-20\text{ ft/s}^2 = \frac{(41.067 - 66)\text{ ft/s}}{[(t_B)_2 - 0.90909]\text{ s}}$$

Car *B* will continue to overtake car *A* while $v_B > v_A$. Therefore, $(x_{A/B})_{\min}$ will occur when $v_A = v_B$, which occurs for

$$(t_B)_1 < t_{\min} < (t_B)_2$$

For this time interval

$$v_A = 44\text{ ft/s}$$
$$v_B = (v_B)_0 + a_B[t - (t_B)_1]$$

Then at $t = t_{\min}$: $$44\text{ ft/s} = 66\text{ ft/s} + (-20\text{ ft/s}^2)(t_{\min} - 0.90909)\text{ s}$$

or $$t_{\min} = 2.00909\text{ s}$$

Finally $$(x_{A/B})_{\min} = (x_A)_{t_{\min}} - (x_B)_{t_{\min}}$$

$$= \left\{ t_A \left[\frac{(v_A)_0 + (v_A)_{\text{final}}}{2} \right] + (t_{\min} - t_A)(v_A)_{\text{final}} \right\}$$

$$- \left\{ (x_B)_0 + (t_B)_1(v_B)_0 + [t_{\min} - (t_B)_1] \left[\frac{(v_B)_0 + (v_A)_{\text{final}}}{2} \right] \right\}$$

$$= \left[(0.91669\text{ s}) \left(\frac{58.667 + 44}{2} \right) \text{ft/s} + (2.00909 - 0.91669)\text{ s} \times (44\text{ ft/s}) \right]$$

$$- \left[-60\text{ ft} + (0.90909\text{ s})(66\text{ ft/s}) + (2.00909 - 0.90909)\text{ s} \times \left(\frac{66 + 44}{2} \right) \text{ft/s} \right]$$

$$= (47.057 + 48.066)\text{ ft} - (-60 + 60.000 + 60.500)\text{ ft}$$

$$= 34.623\text{ ft}$$

or $(x_{A/B})_{\min} = 34.6\text{ ft}$ ◀

PROBLEM 11.76 (Continued)

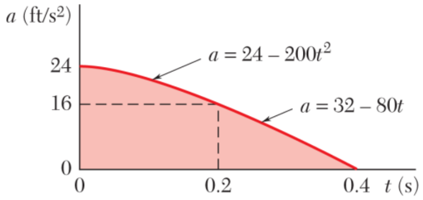
(*b*) Since $(x_{A/B}) \le 60$ ft for $t \le t_{\min}$, it follows that $x_{A/B} = 70$ ft for $t > (t_B)_2$

[Note $(t_B)_2 \simeq t_{\min}$]. Then, for $t > (t_B)_2$

$$x_{A/B} = (x_{A/B})_{\min} + [(t - t_{\min})(v_A)_{\text{final}}]$$
$$- \left\{[(t_B)_2 - (t_{\min})]\left[\frac{(v_A)_{\text{final}} + (v_B)_{\text{final}}}{2}\right] + [t - (t_B)_2](v_B)_{\text{final}}\right\}$$

or
$$70 \text{ ft} = 34.623 \text{ ft} + [(t - 2.00909) \text{ s} \times (44 \text{ ft/s})]$$
$$- \left[(2.15574 - 2.00909) \text{ s} \times \left(\frac{44 + 41.06}{2}\right) \text{ft/s} + (t - 2.15574) \text{ s} \times (41.067) \text{ ft/s}\right]$$

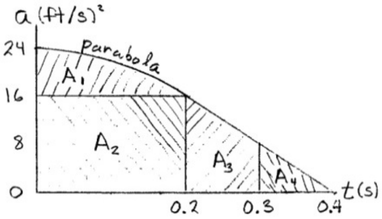
or $t = 14.14$ s ◀

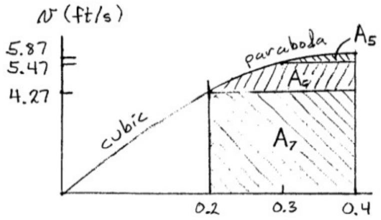


PROBLEM 11.77

An accelerometer record for the motion of a given part of a mechanism is approximated by an arc of a parabola for 0.2 s and a straight line for the next 0.2 s as shown in the figure. Knowing that $v = 0$ when $t = 0$ and $x = 0.8$ ft when $t = 0.4$ s, (*a*) construct the $v-t$ curve for $0 \le t \le 0.4$ s, (*b*) determine the position of the part at $t = 0.3$ s and $t = 0.2$ s.

SOLUTION





Divide the area of the $a-t$ curve into the four areas A_1, A_2, A_3 and A_4.

$$A_1 = \frac{2}{3}(8)(0.2) = 1.0667 \text{ ft/s}$$

$$A_2 = (16)(0.2) = 3.2 \text{ ft/s}$$

$$A_3 = \frac{1}{2}(16 + 8)(0.1) = 1.2 \text{ ft/s}$$

$$A_4 = \frac{1}{2}(8)(0.1) = 0.4 \text{ ft/s}$$

Velocities: $v_0 = 0$

$v_{0.2} = v_0 + A_1 + A_2$ $\quad v_{0.2} = 4.27$ ft/s ◀

$v_{0.3} = v_{0.2} + A_3$ $\quad v_{0.3} = 5.47$ ft/s ◀

$v_{0.4} = v_{0.3} + A_4$ $\quad v_{0.4} = 5.87$ ft/s ◀

Sketch the $v-t$ curve and divide its area into A_5, A_6, and A_7 as shown.

$$\int_x^{0.8} dx = 0.8 - x = \int_t^{0.4} v\,dt \qquad \text{or} \qquad x = 0.8 - \int_t^{0.4} v\,dt$$

At $t = 0.3$ s, $\quad x_{0.3} = 0.8 - A_5 - (5.47)(0.1)$

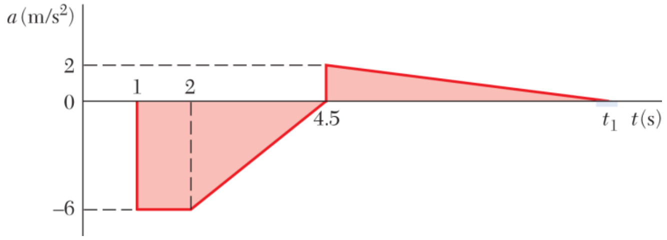
With $A_5 = \frac{2}{3}(0.4)(0.1) = 0.0267$ ft, $\quad x_{0.3} = 0.227$ ft ◀

At $t = 0.2$ s, $\quad x_{0.2} = 0.8 - (A_5 + A_6) - A_7$

With $A_5 + A_6 = \frac{2}{3}(1.6)(0.2) = 0.2133$ ft,

and $\quad A_7 = (4.27)(0.2) = 0.8533$ ft

$x_{0.2} = 0.8 - 0.2133 - 0.8533$ $\quad x_{0.2} = -0.267$ ft ◀



PROBLEM 11.78

A car is traveling at a constant speed of 54 km/h when its driver sees a child run into the road. The driver applies her brakes until the child returns to the sidewalk and then accelerates to resume her original speed of 54 km/h; the acceleration record of the car is shown in the figure. Assuming $x = 0$ when $t = 0$, determine (*a*) the time t_1 at which the velocity is again 54 km/h, (*b*) the position of the car at that time, (*c*) the average velocity of the car during the interval $1\text{ s} \le t \le t_1$.

SOLUTION

Given: At $t = 0, \quad x = 0, \quad v = 54\text{ km/h};$

For $t = t_1, \quad v = 54\text{ km/h}$

First note $54\text{ km/h} = 15\text{ m/s}$

(*a*) We have $v_b = v_a + (\text{area under } a\text{–}t \text{ curve from } t_a \text{ to } t_b)$

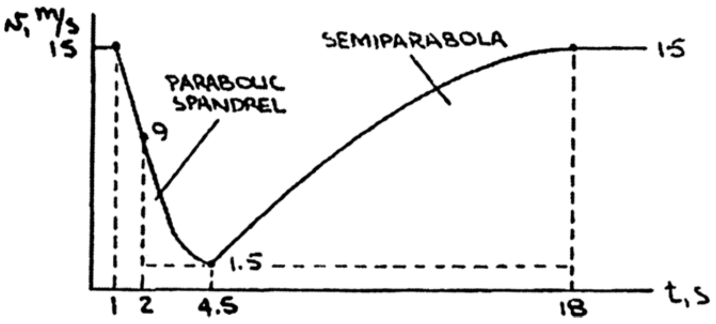
Then at $t = 2\text{ s}: \quad v = 15 - (1)(6) = 9\text{ m/s}$

$$t = 4.5\text{ s}: \quad v = 9 - \frac{1}{2}(2.5)(6) = 1.5\text{ m/s}$$

$$t = t_1: \quad 15 = 1.5 + \frac{1}{2}(t_1 - 4.5)(2)$$

or $t_1 = 18.00\text{ s}$ ◀

(*b*) Using the above values of the velocities, the v–t curve is drawn as shown.



Now x at $t = 18$ s

$$x_{18} = 0 + \Sigma\,(\text{area under the } v\text{–}t \text{ curve from } t = 0 \text{ to } t = 18\text{ s})$$

$$= (1\text{ s})(15\text{ m/s}) + (1\text{ s})\left(\frac{15+9}{2}\right)\text{m/s}$$

$$+\left[(2.5\text{ s})(1.5\text{ m/s}) + \frac{1}{3}(2.5\text{ s})(7.5\text{ m/s})\right]$$

$$+\left[(13.5\text{ s})(1.5\text{ m/s}) + \frac{2}{3}(13.5\text{ s})(13.5\text{ m/s})\right]$$

$$= [15 + 12 + (3.75 + 6.25) + (20.25 + 121.50)]\text{ m}$$

$$= 178.75\text{ m}$$

or $x_{18} = 178.8$ m ◀

(*c*) First note $x_1 = 15$ m

$$x_{18} = 178.75\text{ m}$$

Now $$v_{\text{ave}} = \frac{\Delta x}{\Delta t} = \frac{(178.75 - 15)\text{ m}}{(18-1)\text{ s}} = 9.6324\text{ m/s}$$

or $v_{\text{ave}} = 34.7$ km/h ◀

PROBLEM 11.79

An airport shuttle train travels between two terminals that are 1.6 mi apart. To maintain passenger comfort, the acceleration of the train is limited to ±4 ft/s^2, and the jerk, or rate of change of acceleration, is limited to ±0.8 ft/s^2 per second. If the shuttle has a maximum speed of 20 mi/h, determine (*a*) the shortest time for the shuttle to travel between the two terminals, (*b*) the corresponding average velocity of the shuttle.

SOLUTION

Given: $x_{\max} = 1.6 \text{ mi}; \quad |a_{\max}| = 4 \text{ ft/s}^2$

$$\left|\left(\frac{da}{dt}\right)_{\max}\right| = 0.8 \text{ ft/s}^2\text{/s}; \quad v_{\max} = 20 \text{ mi/h}$$

First note $20 \text{ mi/h} = 29.333 \text{ ft/s}$

$$1.6 \text{ mi} = 8448 \text{ ft}$$

(*a*) To obtain $t_{\min}$, the train must accelerate and decelerate at the maximum rate to maximize the time for which $v = v_{\max}$. The time Δt required for the train to have an acceleration of 4 ft/s^2 is found from

$$\left(\frac{da}{dt}\right)_{\max} = \frac{a_{\max}}{\Delta t}$$

or $$\Delta t = \frac{4 \text{ ft/s}^2}{0.8 \text{ ft/s}^2\text{/s}}$$

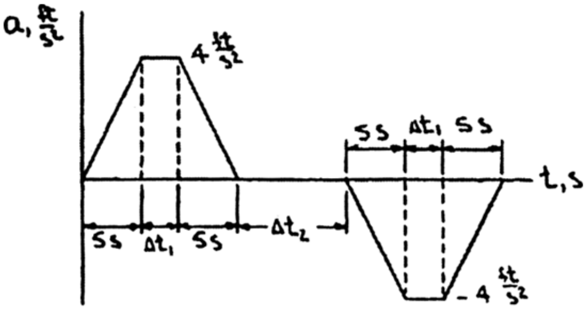
or $$\Delta t = 5 \text{ s}$$

Now,

after 5 s, the speed of the train is $v_5 = \frac{1}{2}(\Delta t)(a_{\max}) \quad \left(\text{since } \frac{da}{dt} = \text{constant}\right)$

or $$v_5 = \frac{1}{2}(5 \text{ s})(4 \text{ ft/s}^2) = 10 \text{ ft/s}$$

Then, since $v_5 < v_{\max}$, the train will continue to accelerate at 4 ft/s^2 until $v = v_{\max}$. The a–t curve must then have the shape shown. Note that the magnitude of the slope of each inclined portion of the curve is 0.8 ft/s^2/s.



Now at $t = (10 + \Delta t_1)$ s, $v = v_{\max}$:

$$2\left[\frac{1}{2}(5\text{ s})(4\text{ ft/s}^2)\right] + (\Delta t_1)(4\text{ ft/s}^2) = 29.333\text{ ft/s}$$

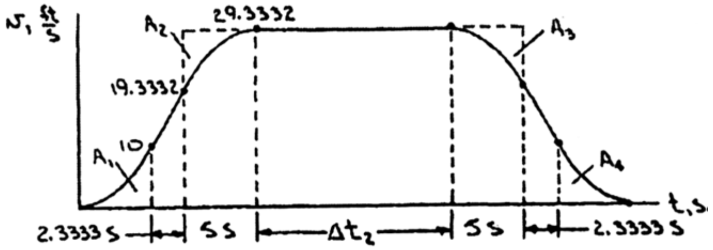
or $\Delta t_1 = 2.3333$ s

Then at $t = 5$ s: $v = 0 + \frac{1}{2}(5)(4) = 10$ ft/s

$t = 7.3333$ s: $v = 10 + (2.3333)(4) = 19.3332$ ft/s

$t = 12.3333$ s: $v = 19.3332 + \frac{1}{2}(5)(4) = 29.3332$ ft/s

Using symmetry, the v–t curve is then drawn as shown.



Noting that $A_1 = A_2 = A_3 = A_4$ and that the area under the v–t curve is equal to $x_{\max}$, we have

$$2\left[(2.3333\text{ s})\left(\frac{10 + 19.3332}{2}\right)\text{ft/s}\right] + (10 + \Delta t_2)\text{ s} \times (29.3332\text{ ft/s}) = 8448\text{ ft}$$

or $\Delta t_2 = 275.67$ s

Then $t_{\min} = 4(5\text{ s}) + 2(2.3333\text{ s}) + 275.67\text{ s}$

$= 300.34$ s

or $t_{\min} = 5.01$ min ◀

(*b*) We have $v_{\text{ave}} = \dfrac{\Delta x}{\Delta t} = \dfrac{1.6\text{ mi}}{300.34\text{ s}} \times \dfrac{3600\text{ s}}{1\text{ h}}$

or $v_{\text{ave}} = 19.18$ mi/h ◀

PROBLEM 11.80

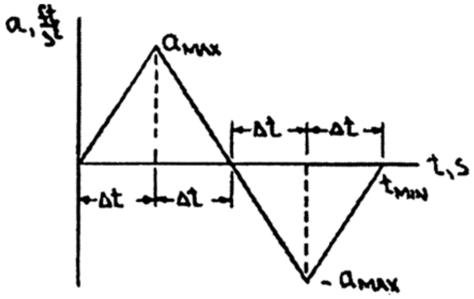
During a manufacturing process, a conveyor belt starts from rest and travels a total of 1.2 ft before temporarily coming to rest. Knowing that the jerk, or rate of change of acceleration, is limited to ±4.8 ft/s^2 per second, determine (*a*) the shortest time required for the belt to move 1.2 ft, (*b*) the maximum and average values of the velocity of the belt during that time.

SOLUTION

Given: At $t = 0, \quad x = 0, \quad v = 0; \quad x_{\max} = 1.2 \text{ ft};$

when $x = x_{\max}, \quad v = 0; \quad \left| \left(\frac{da}{dt} \right)_{\max} \right| = 4.8 \text{ ft/s}^2$

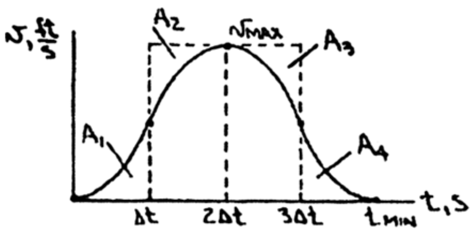
(*a*) Observing that $v_{\max}$ must occur at $t = \frac{1}{2} t_{\min}$, the a–t curve must have the shape shown. Note that the magnitude of the slope of each portion of the curve is 4.8 ft/s^2/s.



We have at $t = \Delta t$: $v = 0 + \frac{1}{2}(\Delta t)(a_{\max}) = \frac{1}{2} a_{\max} \Delta t$

$t = 2\Delta t$: $v_{\max} = \frac{1}{2} a_{\max} \Delta t + \frac{1}{2}(\Delta t)(a_{\max}) = a_{\max} \Delta t$

Using symmetry, the v–t is then drawn as shown.



Noting that $A_1 = A_2 = A_3 = A_4$ and that the area under the v–t curve is equal to $x_{\max}$, we have

$$(2\Delta t)(v_{\max}) = x_{\max}$$

$$v_{\max} = a_{\max} \Delta t \Rightarrow 2 a_{\max} \Delta t^2 = x_{\max}$$

PROBLEM 11.80 (Continued)

Now $\frac{a_{\max}}{\Delta t} = 4.8 \text{ ft/s}^2/\text{s}$ so that

$$2(4.8\Delta t \text{ ft/s}^3)\Delta t^2 = 1.2 \text{ ft}$$

or $\Delta t = 0.5 \text{ s}$

Then $t_{\min} = 4\Delta t$

or $t_{\min} = 2.00 \text{ s}$ ◀

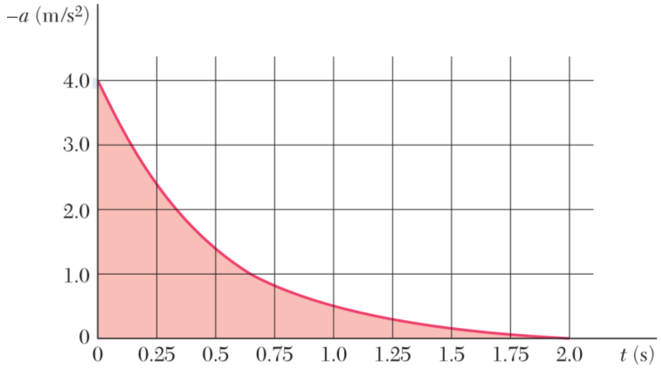
(*b*) We have

$$\begin{aligned} v_{\max} &= a_{\max}\Delta t \\ &= (4.8 \text{ ft/s}^2/\text{s} \times \Delta\text{t})\Delta t \\ &= 4.8 \text{ ft/s}^2/\text{s} \times (0.5 \text{ s})^2 \end{aligned}$$

or $v_{\max} = 1.2 \text{ ft/s}$ ◀

Also $v_{\text{ave}} = \frac{\Delta x}{\Delta t_{\text{total}}} = \frac{1.2 \text{ ft}}{2.00 \text{ s}}$

or $v_{\text{ave}} = 0.6 \text{ ft/s}$ ◀



PROBLEM 11.81

Two seconds are required to bring the piston rod of an air cylinder to rest; the acceleration record of the piston rod during the 2 s is as shown. Determine by approximate means (*a*) the initial velocity of the piston rod, (*b*) the distance traveled by the piston rod as it is brought to rest.

SOLUTION

Given: a–t curve; at $t = 2$ s, $v = 0$

1. The a–t curve is first approximated with a series of rectangles, each of width $\Delta t = 0.25$ s. The area $(\Delta t)(a_{\text{ave}})$ of each rectangle is approximately equal to the change in velocity Δv for the specified interval of time. Thus,

$$\Delta v \cong a_{\text{ave}}\,\Delta t$$

where the values of a_{ave} and Δv are given in columns 1 and 2, respectively, of the following table.

2. Now

$$v(2) = v_0 + \int_0^2 a\,dt = 0$$

and approximating the area $\int_0^2 a\,dt$ under the a–t curve by $\Sigma a_{\text{ave}}\Delta t \approx \Sigma \Delta v$, the initial velocity is then equal to

$$v_0 = -\Sigma \Delta v$$

Finally, using

$$v_2 = v_1 + \Delta v_{12}$$

where Δv_{12} is the change in velocity between times t_1 and t_2, the velocity at the end of each 0.25 interval can be computed; see column 3 of the table and the v–t curve.

3. The v–t curve is then approximated with a series of rectangles, each of width 0.25 s. The area $(\Delta t)(v_{\text{ave}})$ of each rectangle is approximately equal to the change in position Δx for the specified interval of time. Thus

$$\Delta x \approx v_{\text{ave}}\Delta t$$

where v_{ave} and Δx are given in columns 4 and 5, respectively, of the table.

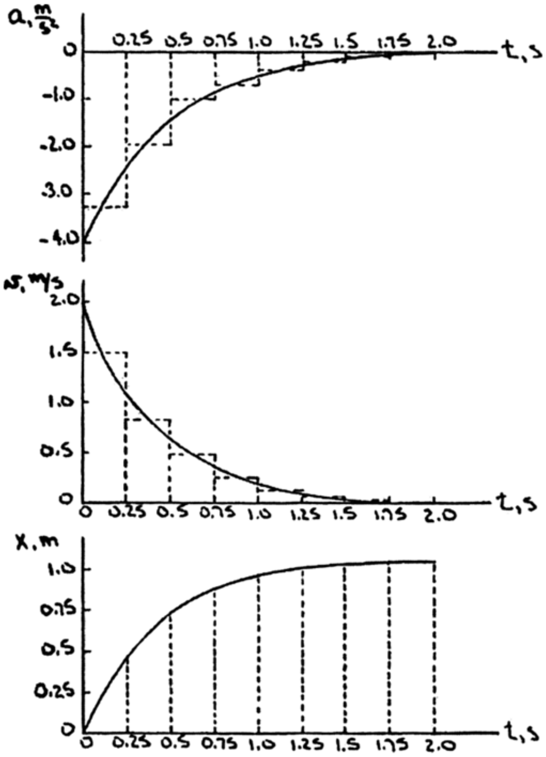
PROBLEM 11.81 (Continued)

4. With $x_0 = 0$ and noting that

$$x_2 = x_1 + \Delta x_{12}$$

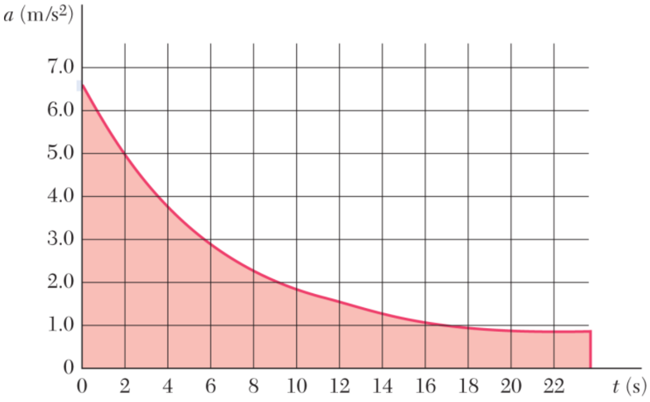
where Δx_{12} is the change in position between times t_1 and t_2, the position at the end of each 0.25 s interval can be computed; see column 6 of the table and the x–t curve.

		1	2	3	4	5	6
t, s	a, m/s^2	a$_{AVE}$, m/s^2	Δv, m/s	v, m/s	v$_{AVE}$, m/s	Δx, m	x, m
0	- 4.00	▨	▨	1.914	▨	▨	0
		-3.215	-0.804		1.512	0.378	
0.25	- 2.43			1.110			0.378
		-1.915	-0.479		0.871	0.218	
0.50	-1.40			0.631			0.596
		-1.125	-0.281		0.491	0.123	
0.75	-0.85			0.350			0.719
		-0.675	-0.169		0.266	0.067	
1.00	-0.50			0.181			0.786
		-0.390	-0.098		0.132	0.033	
1.25	-0.28			0.083			0.819
		-0.205	-0.051		0.058	0.015	
1.50	- 0.13			0.032			0.834
		-0.095	-0.024		0.020	0.005	
1.75	- 0.06			0.008			0.839
		-0.030	-0.008		0.004	0.001	
2.00	0	▨	▨	0	▨	▨	0.840



(*a*) We had found $v_0 = 1.914$ m/s ◀

(*b*) At $t = 2$ s $x = 0.840$ m ◀



PROBLEM 11.82

The acceleration record shown was obtained during the speed trials of a sports car. Knowing that the car starts from rest, determine by approximate means (*a*) the velocity of the car at $t = 8$ s, (*b*) the distance the car has traveled at $t = 20$ s.

SOLUTION

Given: a–t curve; at $t = 0, \ x = 0, \ v = 0$

1. The a–t curve is first approximated with a series of rectangles, each of width $\Delta t = 2$ s. The area $(\Delta t)(a_{\text{ave}})$ of each rectangle is approximately equal to the change in velocity Δv for the specified interval of time. Thus,

$$\Delta v \cong a_{\text{ave}} \ \Delta t$$

where the values of a_{ave} and Δv are given in columns 1 and 2, respectively, of the following table.

2. Noting that $v_0 = 0$ and that

$$v_2 = v_1 + \Delta v_{12}$$

where Δv_{12} is the change in velocity between times t_1 and t_2, the velocity at the end of each 2 s interval can be computed; see column 3 of the table and the v–t curve.

3. The v–t curve is next approximated with a series of rectangles, each of width $\Delta t = 2$ s. The area $(\Delta t)(v_{\text{ave}})$ of each rectangle is approximately equal to the change in position Δx for the specified interval of time.

Thus, $\Delta x \cong v_{\text{ave}} \ \Delta t$

where v_{ave} and Δx are given in columns 4 and 5, respectively, of the table.

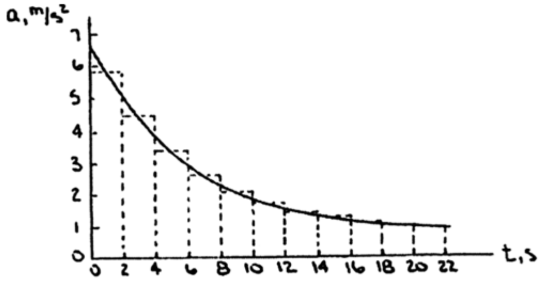
4. With $x_0 = 0$ and noting that

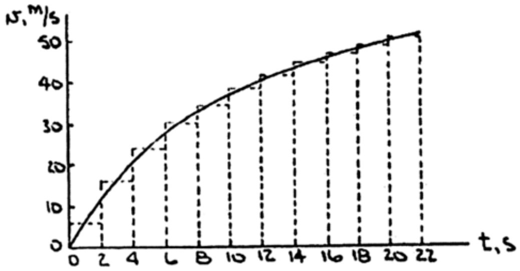
$$x_2 = x_1 + \Delta x_{12}$$

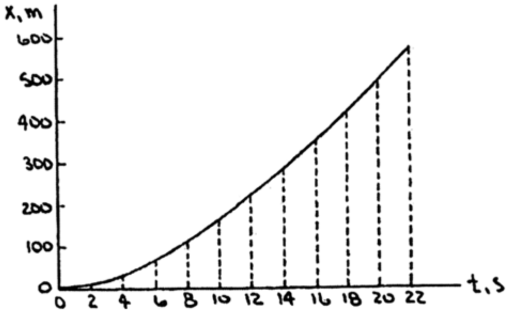
where Δx_{12} is the change in position between times t_1 and t_2, the position at the end of each 2 s interval can be computed; see column 6 of the table and the x–t curve.

PROBLEM 11.82 (Continued)

		1	2	3	4	5	6
t, s	a, m/s²	a_{AVE}, m/s²	Δv, m/s	v, m/s	v_{AVE}, m/s	Δx, m	x, m
0	6.63	▨	▨	0	▨	▨	0
		5.86	11.72		5.86	11.72	
2	5.08			11.72			11.72
		4.47	8.94		16.19	32.38	
4	3.86			20.66			44.10
		3.38	6.76		24.04	48.08	
6	2.90			27.42			92.18
		2.58	5.16		30.00	60.00	
8	2.25			32.58			152.18
		2.06	4.12		34.64	69.28	
10	1.87			36.70			221.46
		1.71	3.42		38.41	76.82	
12	1.54			40.12			298.28
		1.42	2.84		41.54	83.08	
14	1.29			42.96			381.36
		1.23	2.46		44.19	88.38	
16	1.16			45.42			469.74
		1.10	2.20		46.52	93.04	
18	1.03			47.62			562.78
		1.00	2.00		48.62	97.24	
20	0.97			49.62			660.02
		0.94	1.88		50.56	101.12	
22	0.90	▨	▨	51.50	▨	▨	761.14

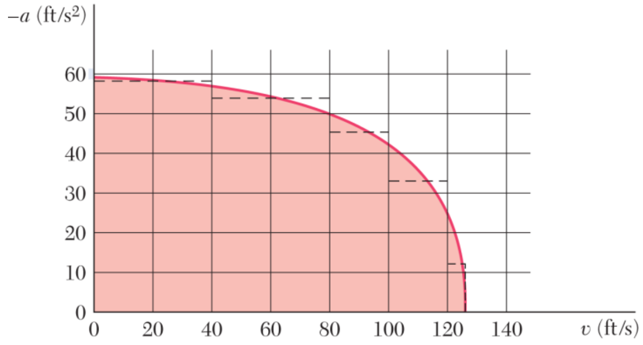






(*a*) At $t = 8$ s, $v = 32.58$ m/s or $v = 117.3$ km/h ◀

(*b*) At $t = 20$ s $x = 660$ m ◀



PROBLEM 11.83

A training airplane has a velocity of 126 ft/s when it lands on an aircraft carrier. As the arresting gear of the carrier brings the airplane to rest, the velocity and the acceleration of the airplane are recorded; the results are shown (solid curve) in the figure. Determine by approximate means (*a*) the time required for the airplane to come to rest, (*b*) the distance traveled in that time.

SOLUTION

Given: $a\!-\!v$ curve:

$$v_0 = 126 \text{ ft/s}$$

The given curve is approximated by a series of uniformly accelerated motions (the horizontal dashed lines on the figure).

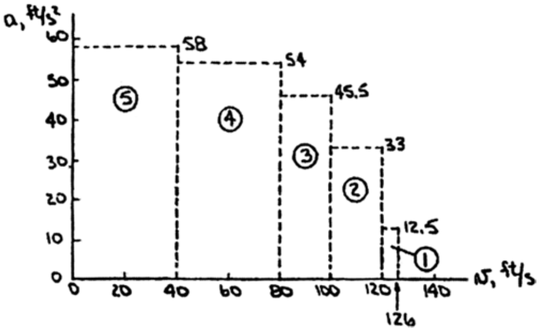
For uniformly accelerated motion

$$v_2^2 = v_1^2 + 2a(x_2 - x_1)$$
$$v_2 = v_1 + a(t_2 - t_1)$$

or

$$\Delta x = \frac{v_2^2 - v_1^2}{2a}$$

$$\Delta t = \frac{v_2 - v_1}{a}$$

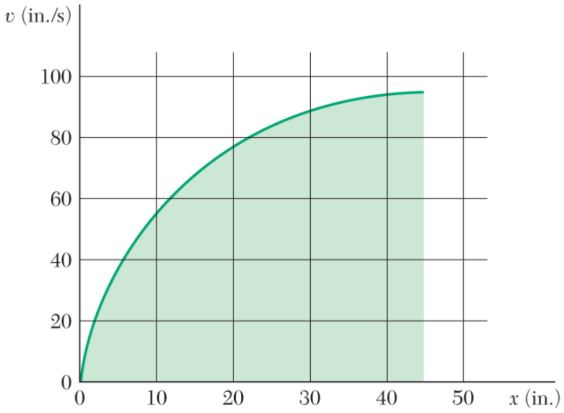


For the five regions shown above, we have

Region	v_1, ft/s	v_2, ft/s	a, ft/s^2	Δx, ft	Δt, s
1	126	120	−12.5	59.0	0.480
2	120	100	−33	66.7	0.606
3	100	80	−45.5	39.6	0.440
4	80	40	−54	44.4	0.741
5	40	0	−58	13.8	0.690
Σ				223.5	2.957

(*a*) From the table, when $v = 0$ $t = 2.96$ s ◀

(*b*) From the table and assuming $x_0 = 0$, when $v = 0$ $x = 224$ ft ◀

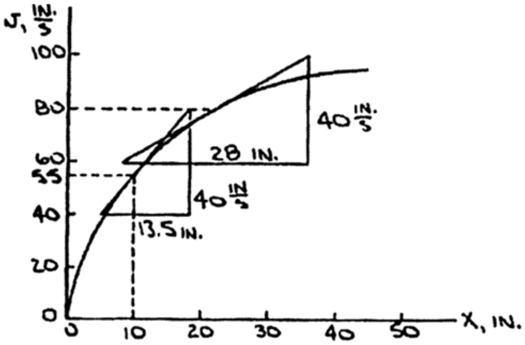


PROBLEM 11.84

Shown in the figure is a portion of the experimentally determined $v-x$ curve for a shuttle cart. Determine by approximate means the acceleration of the cart (*a*) when $x = 10$ in., (*b*) when $v = 80$ in./s.

SOLUTION

Given: $v-x$ curve



First note that the slope of the above curve is $\frac{dv}{dx}$. Now

$$a = v\frac{dv}{dx}$$

(*a*) When $x = 10$ in., $v = 55$ in./s

Then

$$a = 55 \text{ in./s}\left(\frac{40 \text{ in./s}}{13.5 \text{ in.}}\right)$$

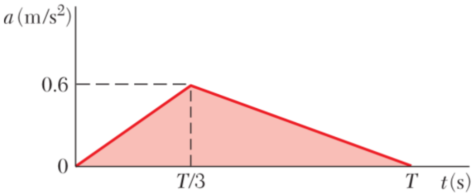
or $a = 163.0 \text{ in./s}^2$ ◀

(*b*) When $v = 80$ in./s, we have

$$a = 80 \text{ in./s}\left(\frac{40 \text{ in./s}}{28 \text{ in.}}\right)$$

or $a = 114.3 \text{ in./s}^2$ ◀

Note: To use the method of measuring the subnormal outlined at the end of Section 11.8, it is necessary that the same scale be used for the *x* and *v* axes (e.g., 1 in. = 50 in., 1 in. = 50 in./s). In the above solution, Δv and Δx were measured directly, so different scales could be used.



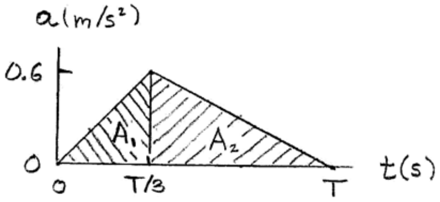
PROBLEM 11.85

An elevator starts from rest and rises 40 m to its maximum velocity in T s with the acceleration record shown in the figure. Determine (*a*) the required time T, (*b*) the maximum velocity, (*c*) the velocity and position of the elevator at $t = T/2$.

SOLUTION

Find the position of the elevator from the $a-t$ curve using the moment-area method.

First, find the areas A_1 and A_2 on the $a-t$ curve



$$A_1 = \frac{1}{2}(0.6)\frac{T}{3} = 0.1T \text{ m/s}$$

$$A_2 = \frac{1}{2}(0.6)\frac{2T}{3} = 0.2T \text{ m/s}$$

(a) Apply the moment-area formula: $x - x_0 = v_0 t + \int_0^T (T-t)\,a dt$

$$x = v_0 t + (A_1)\left(T - \frac{2}{3}\left(\frac{T}{3}\right)\right) + A_2\left(T - \left(\frac{T}{3} + \frac{1}{3}\left(\frac{2T}{3}\right)\right)\right)$$

$$40 = 0 + \frac{7}{90}T^2 + \frac{8}{90}T^2$$

$$= \frac{15}{90}T^2$$

$$= \frac{1}{6}T^2$$

$$T^2 = (40)(6)$$

$$= 240 \text{ s}^2$$

$T = 15.49$ s ◀

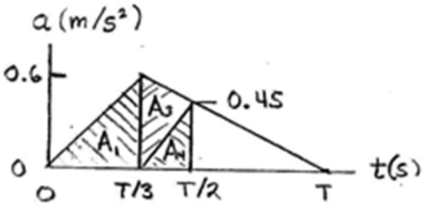
(b) $v_{\max} = v_0 + A_1 + A_2 = 0 + 0.1T + 0.2T = 0.3T$

$v_{\max} = 4.65$ m/s ◀

(c) To find the position at t=T/2, first find the areas A_1 and A_2 on the $a-t$ curve

$$A_1 = 0.1T \qquad A_3 = \frac{1}{2}(0.6)\frac{T}{6} = 0.05T$$

$$A_4 = \frac{1}{2}(0.45)\frac{T}{6} = 0.0375T$$



Apply the moment-area formula: $x - x_0 = v_0 t + \int_0^{T/2}\left(\frac{T}{2} - t\right) a dt$

PROBLEM 11.85 (Continued)

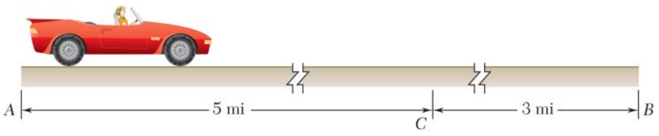
$$x = v_0\frac{T}{2} + A_1\left(\frac{T}{2} - \frac{2T}{9}\right) + A_3\left(\frac{2}{3}\cdot\frac{T}{6}\right) + A_4\left(\frac{1}{3}\cdot\frac{T}{6}\right)$$

$$= 0 + (0.1T)\left(\frac{5T}{18}\right) + (0.05T)\left(\frac{T}{9}\right) + (0.0375T)\frac{T}{18} = 0.035417T^2$$

$$= (0.035417)(15.49)^2$$

$x = 8.50 \text{ m}$ ◀

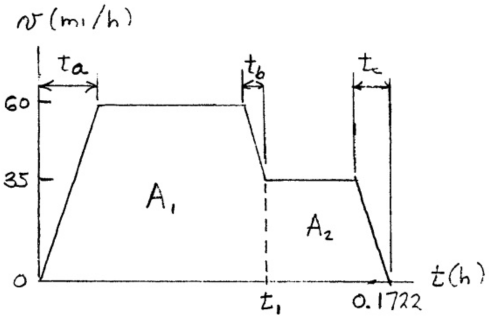
$v = v_0 + A_1 + A_3 + A_4 = 0.1875T$ $\quad$ $v = 2.90 \text{ m/s}$ ◀



PROBLEM 11.86

Two road rally checkpoints *A* and *B* are located on the same highway and are 8 mi apart. The speed limits for the first 5 mi and the last 3 mi are 60 mi/h and 35 mi/h, respectively. Drivers must stop at each checkpoint, and the specified time between points *A* and *B* is 10 min 20 s. Knowing that a driver accelerates and decelerates at the same constant rate, determine the magnitude of her acceleration if she travels at the speed limit as much as possible.

SOLUTION



Given

$$10 \text{ min } 20 \text{ s} = \frac{10}{60} + \frac{20}{3600}$$

$$= 0.1722 \text{ h}$$

Sketch the $v - t$ curve and note $t_a = \frac{60}{a}$, $t_b = \frac{25}{a}$, $t_c = \frac{35}{a}$

Find Areas A_1 and A_2

$$A_1 = 60t_1 - \frac{1}{2}(60)(t_a) - \frac{1}{2}(25)t_b$$

$$= 60\,t_1 - 1800\frac{1}{a} - 312.5\frac{1}{a} \qquad (1)$$

$$A_2 = 35(0.1722 - t_1) - \frac{1}{2}(35)t_c$$

$$= 6.0278 - 35t_1 - 612.5\frac{1}{a} \qquad (2)$$

Also given $\qquad A_1 = 5 \text{ mi and } A_2 = 3 \text{ mi}$

Setting (1) equal to 5 mi: $\qquad 60t_1 - 2112.5\frac{1}{a} = 5 \qquad (3)$

Setting (2) equal to 3 mi: $\qquad 35t_1 + 612.5\frac{1}{a} = 3.0278 \qquad (4)$

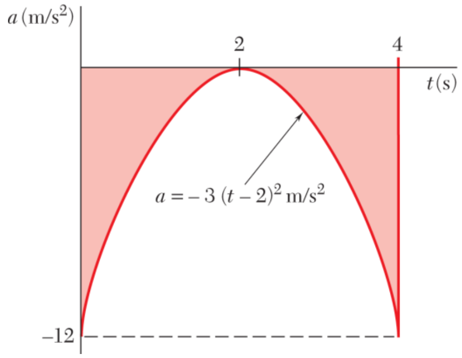
Solving equations (3) and (4) for t_1 and $\frac{1}{a}$:

$$t_1 = 85.45 \times 10^{-3} \text{ h} = 5.13 \text{ min}$$

$$\frac{1}{a} = 60.23 \times 10^{-6} \text{ h}^2/\text{mi}$$

$$a = 16.616 \times 10^3 \text{ mi/h}^2 = \frac{(16.616 \times 10^3)(5280)}{(3600)^2}$$

$a = 6.77 \text{ ft/s}^2$ ◀



PROBLEM 11.87

As shown in the figure, from $t = 0$ to $t = 4$ s, the acceleration of a given particle is represented by a parabola. Knowing that $x = 0$ and $v = 8$ m/s when $t = 0$, (*a*) construct the v–t and x–t curves for $0 < t < 4$ s, (*b*) determine the position of the particle at $t = 3$ s. (*Hint:* Use table inside the front cover.)

SOLUTION

Given At $t = 0, \quad x = 0, \quad v = 8\ \mathrm{m/s}$

(*a*) We have $v_2 = v_1 + (\text{area under } a\text{–}t \text{ curve from } t_1 \text{ to } t_2)$

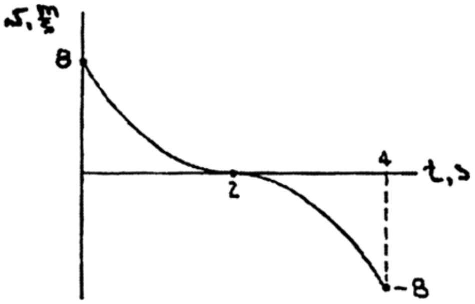
and $x_2 = x_1 + (\text{area under } v\text{–}t \text{ curve from } t_1 \text{ to } t_2)$

Then, using the formula for the area of a parabolic spandrel, we have

$$\text{at } t = 2\ \mathrm{s}: \quad v = 8 - \frac{1}{3}(2)(12) = 0$$

$$t = 4\ \mathrm{s}: \quad v = 0 - \frac{1}{3}(2)(12) = -8\ \mathrm{m/s}$$

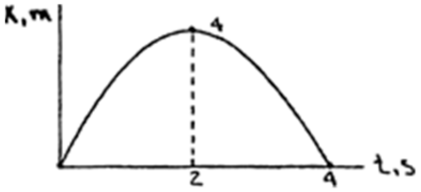
The v–t curve is then drawn as shown.



Note: The area under each portion of the curve is a spandrel of Order no. 3.

$$\text{Now} \quad \text{at } t = 2\ \mathrm{s}: \quad x = 0 + \frac{(2)(8)}{3+1} = 4\ \mathrm{m}$$

$$t = 4\ \mathrm{s}: \quad x = 4 - \frac{(2)(8)}{3+1} = 0$$



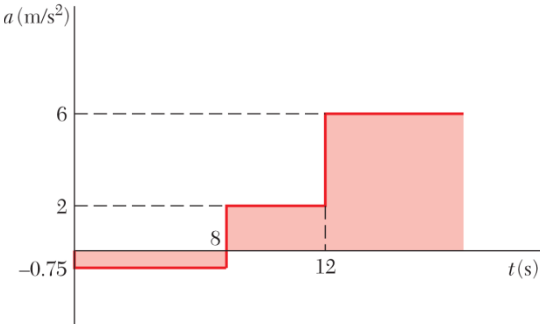
The x–t curve is then drawn as shown. ◀

(*b*) We have At $t = 3$ s: $a = -3(3-2)^2 = -3\ \mathrm{m/s^2}$

$$v = 0 - \frac{1}{3}(1)(3) = -1\ \mathrm{m/s}$$

$$x = 4 - \frac{(1)(1)}{3+1}$$

or $x_3 = 3.75\ \mathrm{m}$ ◀

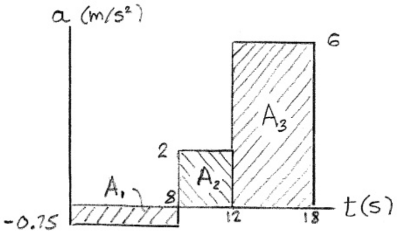


PROBLEM 11.88

A particle moves in a straight line with the acceleration shown in the figure. Knowing that the particle starts from the origin with $v_0 = -2$ m/s, (*a*) construct the $v - t$ and $x - t$ curves for $0 < t < 18$ s, (*b*) determine the position and the velocity of the particle and the total distance traveled when $t = 18$ s.

SOLUTION

Find the indicated areas under the a–t curve:



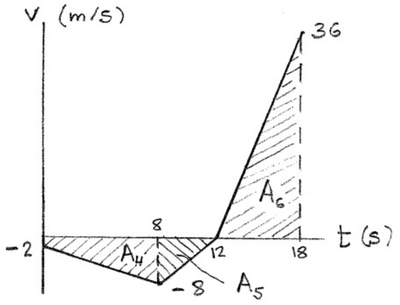
$$A_1 = (-0.75)(8) = -6 \text{ m/s}$$
$$A_2 = (2)(4) = 8 \text{ m/s}$$
$$A_3 = (6)(6) = 36 \text{ m/s}$$

Find the values of velocity at times indicated:

$$v_0 = -2 \text{ m/s}$$
$$v_8 = v_0 + A_1 = -8 \text{ m/s}$$
$$v_{12} = v_8 + A_2 = 0$$
$$v_{18} = v_{12} + A_3 = 36 \text{ m/s}$$

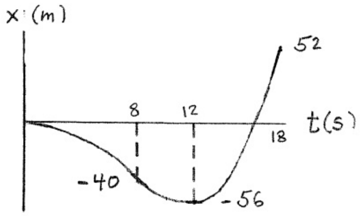
(a) Sketch $v - t$ curve using straight line portions over the constant acceleration periods.

Find the indicated areas under the v–t curve:



$$A_4 = \frac{1}{2}(-2 - 8)(8) = -40 \text{ m}$$
$$A_5 = \frac{1}{2}(-8)(4) = -16 \text{ m}$$
$$A_6 = \frac{1}{2}(36)(6) = 108 \text{ m}$$

Find the values of position at times indicated:



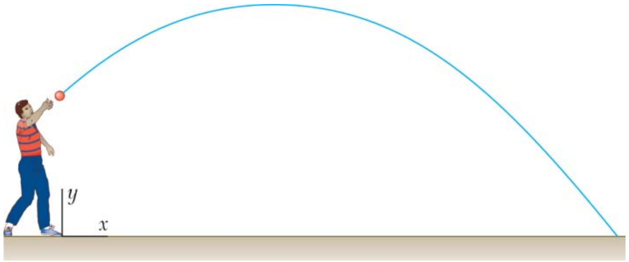
$$x_0 = 0$$
$$x_8 = x_0 + A_4 = -40 \text{ m}$$
$$x_{12} = x_8 + A_5 = -56 \text{ m}$$
$$x_{18} = x_{12} + A_6 = 52 \text{ m}$$

Sketch the $x - t$ curve using the positions and times calculated:

(b) Position at t=18s $x_{18} = 52$ m ◀

Velocity at t-18s $v_{18} = 36$ m/s ◀

Total Distance Traveled: = 56 + 108 $d = 164$ m ◀



PROBLEM 11.89

A ball is thrown so that the motion is defined by the equations $x = 5t$ and $y = 2 + 6t - 4.9t^2$, where x and y are expressed in meters and t is expressed in seconds. Determine (*a*) the velocity at $t = 1$ s, (*b*) the horizontal distance the ball travels before hitting the ground.

SOLUTION

Units are meters and seconds.

Horizontal motion: $$v_x = \frac{dx}{dt} = 5$$

Vertical motion: $$v_y = \frac{dy}{dt} = 6 - 9.8t$$

(*a*) Velocity at $t = 1$ s.

$$v_x = 5$$
$$v_y = 6 - 9.8 = -3.8$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{5^2 + 3.8^2} = 6.28 \text{ m/s}$$

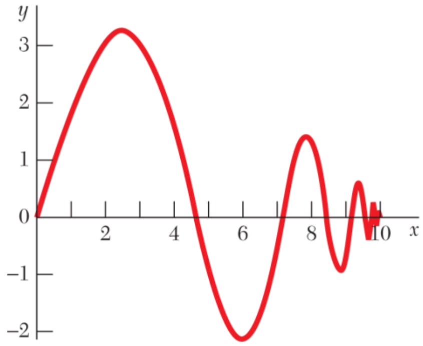
$$\tan\theta = \frac{v_y}{v_x} = \frac{-3.8}{5} \qquad \theta = -37.2°$$

$v = 6.28$ m/s $\searrow$ 37.2° ◀

(*b*) Horizontal distance: $(y = 0)$

$$y = 2 + 6t - 4.9t^2$$
$$t = 1.4971 \text{ s}$$
$$x = (5)(1.4971) = 7.4856 \text{ m}$$

$x = 7.49$ m ◀



PROBLEM 11.90

The motion of a vibrating particle is defined by the position vector $\mathbf{r} = 10(1 - e^{-3t})\mathbf{i} + (4e^{-2t}\sin 15t)\mathbf{j}$, where $\mathbf{r}$ and t are expressed in millimeters and seconds, respectively. Determine the velocity and acceleration when (*a*) $t = 0$, (*b*) $t = 0.5$ s.

SOLUTION

$$\mathbf{r} = 10(1 - e^{-3t})\mathbf{i} + (4e^{-2t}\sin 15t)\mathbf{j}$$

Then $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 30e^{-3t}\mathbf{i} + [60e^{-2t}\cos 15t - 8e^{-2t}\sin 15t]\mathbf{j}$$

and $$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -90e^{-3t}\mathbf{i} + [-120e^{-2t}\cos 15t - 900e^{-2t}\sin 15t - 120e^{-2t}\cos 15t + 16e^{-2t}\sin 15t]\mathbf{j}$$

$$= -90e^{-3t}\mathbf{i} + [-240e^{-2t}\cos 15t - 884e^{-2t}\sin 15t]\mathbf{j}$$

(*a*) When $t = 0$:

$\mathbf{v} = 30\mathbf{i} + 60\mathbf{j}$ mm/s — $\mathbf{v} = 67.1$ mm/s ∠ 63.4° ◀

$\mathbf{a} = -90\mathbf{i} - 240\mathbf{j}$ mm/s^2 — $\mathbf{a} = 256$ mm/s^2 ∠ 69.4° ◀

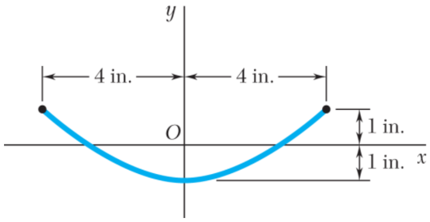
When $t = 0.5$ s:

$$\mathbf{v} = 30e^{-1.5}\mathbf{i} + [60e^{-1}\cos 7.5 - 8e^{-1}\sin 7.5]$$
$$= 6.694\mathbf{i} + 4.8906\mathbf{j} \text{ mm/s}$$

$\mathbf{v} = 8.29$ mm/s ∠ 36.2° ◀

$$\mathbf{a} = 90e^{-1.5}\mathbf{i} + [-240e^{-1}\cos 7.5 - 884e^{-1}\sin 7.5\mathbf{j}]$$
$$= -20.08\mathbf{i} - 335.65\mathbf{j} \text{ mm/s}^2$$

$\mathbf{a} = 336$ mm/s^2 ∠ 86.6° ◀



PROBLEM 11.91

The motion of a vibrating particle is defined by the position vector $\mathbf{r} = (4\sin \pi t)\mathbf{i} - (\cos 2\pi t)\mathbf{j}$, where r is expressed in inches and t in seconds. (*a*) Determine the velocity and acceleration when $t = 1$ s. (*b*) Show that the path of the particle is parabolic.

SOLUTION

$$\mathbf{r} = (4\sin \pi t)\mathbf{i} - (\cos 2\pi t)\mathbf{j}$$
$$\mathbf{v} = (4\pi \cos \pi t)\mathbf{i} + (2\pi \sin 2\pi t)\mathbf{j}$$
$$\mathbf{a} = -(4\pi^2 \sin \pi t)\mathbf{i} + (4\pi^2 \cos 2\pi t)\mathbf{j}$$

(*a*) When $t = 1$ s:

$$\mathbf{v} = (4\pi \cos \pi)\mathbf{i} + (2\pi \sin 2\pi)\mathbf{j} \qquad \mathbf{v} = -(4\pi \text{ in/s})\mathbf{i} \blacktriangleleft$$

$$\mathbf{a} = -(4\pi^2 \sin \pi)\mathbf{i} - (4\pi^2 \cos \pi)\mathbf{j} \qquad \mathbf{a} = -(4\pi^2 \text{in/s}^2)\mathbf{j} \blacktriangleleft$$

(*b*) Path of particle:

Since $\mathbf{r} = x\mathbf{i} + y\,\mathbf{j}$; $x = 4\sin \pi t,$ $y = -\cos 2\pi t$

Recall that $\cos 2\theta = 1 - 2\sin^2 \theta$ and write

$$y = -\cos 2\pi t = -(1 - 2\sin^2 \pi t) \qquad (1)$$

But since $x = 4\sin \pi t$ or $\sin \pi t = \frac{1}{4}x,$ Eq.(1) yields

$$y = -\left[1 - 2\left(\frac{1}{4}x\right)^2\right] \qquad y = \frac{1}{8}x^2 - 1 \text{ (Parabola)} \blacktriangleleft$$

PROBLEM 11.92

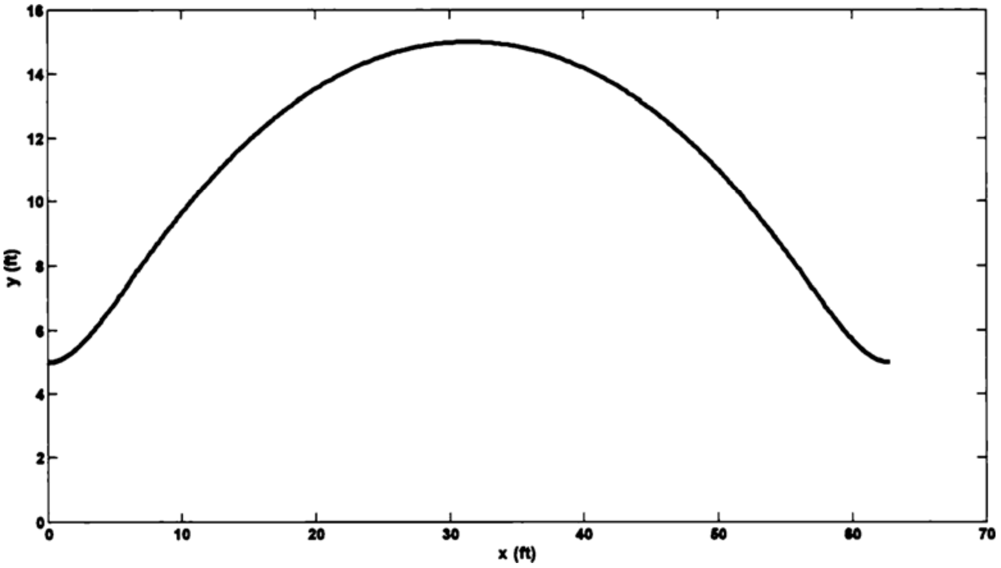
The motion of a particle is defined by the equations $x = 10t - 5\sin t$ and $y = 10 - 5\cos t$, where x and y are expresed in feet and t is expressed in seconds. Sketch the path of the particle for the time interval $0 \le t \le 2\pi$, and determine (*a*) the magnitudes of the smallest and largest velocities reached by the particle, (*b*) the corresponding times, positions, and directions of the velocities.

SOLUTION

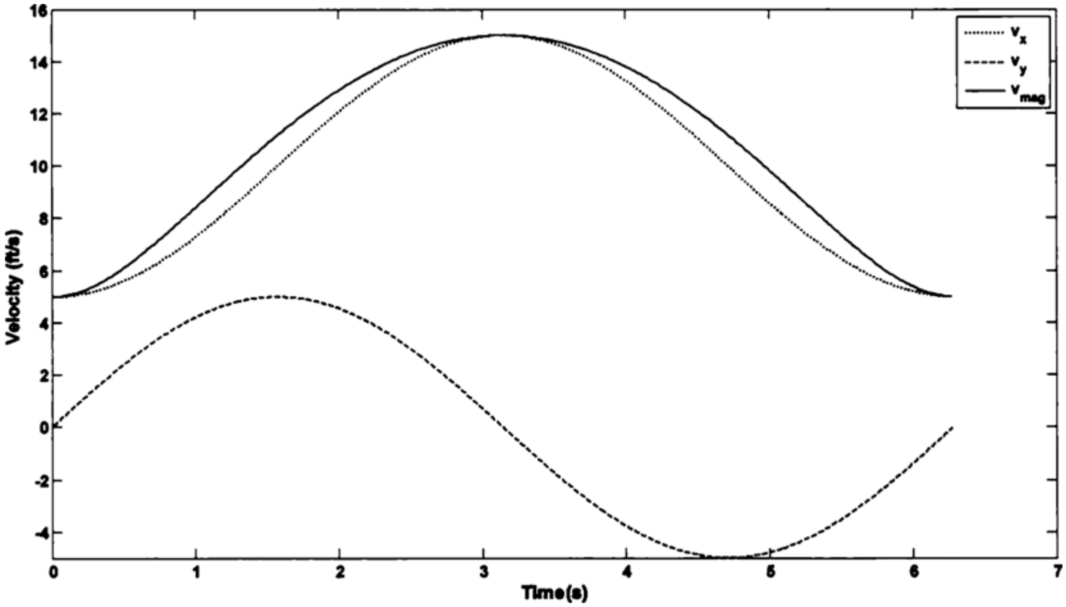
Sketch the path of the particle, i.e., plot of *y* versus *x*.

Using $x = 10t - 5\sin t$, and $y = 10 - 5\cos t$ obtain the values in the table below. Plot as shown.

$t(s)$	x(ft)	y(ft)
0	0.00	5
$\frac{\pi}{2}$	10.71	10
π	31.41	15
$3\frac{\pi}{2}$	52.12	10
2π	62.83	5



PROBLEM 11.92 (Continued)



(*a*) Differentiate with respect to *t* to obtain velocity components.

$$v_x = \frac{dx}{dt} = 10 - 5\cos t \quad \text{and} \quad v_y = 5\sin t$$

$$v^2 = v_x^2 + v_y^2 = (10 - 5\cos t)^2 + 25\sin^2 t = 125 - 100\cos t$$

$$\frac{d(v)^2}{dt} = 100\sin t = 0 \quad t = 0.\ \pm\pi.\ \pm 2\pi \ldots \pm N\pi$$

When $t = 2N\pi.$ $\quad \cos t = 1.$ $\quad$ and $\quad v^2$ is minimum.

When $t = (2N+1)\pi.$ $\quad \cos t = -1.$ $\quad$ and $\quad v^2$ is maximum.

$$(v^2)_{\min} = 125 - 100 = 25(\text{ft/s})^2 \quad v_{\min} = 5 \text{ ft/s} \blacktriangleleft$$

$$(v^2)_{\max} = 125 + 100 = 225(\text{ft/s})^2 \qquad v_{\max} = 15 \text{ ft/s} \blacktriangleleft$$

(*b*) When $v = v_{\min}.$

When $N = 0, 1, 2, \ldots$

$$x = 10(2\pi N) - 5\sin(2\pi N) \qquad x = 20\pi N \text{ ft} \blacktriangleleft$$

$$y = 10 - 5\cos(2\pi N) \qquad y = 5 \text{ ft} \blacktriangleleft$$

$$v_x = 10 - 5\cos(2\pi N) \qquad v_x = 5 \text{ ft/s} \blacktriangleleft$$

$$v_y = 5\sin(2\pi N) \qquad v_y = 0 \blacktriangleleft$$

$$\tan\theta = \frac{v_y}{v_x} = 0, \qquad \theta = 0 \blacktriangleleft$$

PROBLEM 11.92 (Continued)

When $v = v_{\max}$. $t = (2N+1)\pi\text{ s}$ ◀

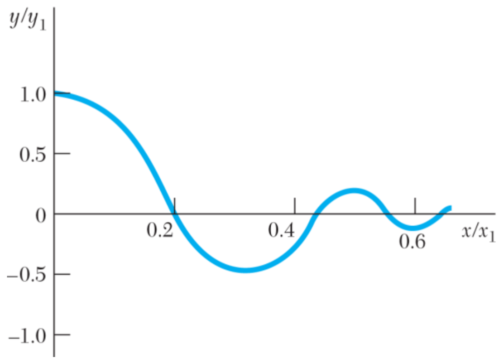
$x = 10[2\pi(N-1)] - 5\sin[2\pi(N+1)]$ $x = 20\pi(N+1)\text{ ft}$ ◀

$y = 10 - 5\cos[2\pi(N+1)]$ $y = 15\text{ ft}$ ◀

$v_x = 10 - 5\cos[2\pi(N+1)]$ $v_x = 15\text{ ft/s}$ ◀

$v_y = 5\sin[2\pi(N+1)]$ $v_y = 0$ ◀

$\tan\theta = \dfrac{v_y}{v_x} = 0,$ $\theta = 0$ ◀



PROBLEM 11.93

The damped motion of a vibrating particle is defined by the position vector $\mathbf{r} = x_1[1-1/(t+1)]\mathbf{i} + (y_1 e^{-\pi t/2}\cos 2\pi t)\mathbf{j}$, where t is expressed in seconds. For $x_1 = 30$ mm and $y_1 = 20$ mm, determine the position, the velocity, and the acceleration of the particle when (*a*) $t = 0$, (*b*) $t = 1.5$ s.

SOLUTION

We have $$\mathbf{r} = 30\left(1-\frac{1}{t+1}\right)\mathbf{i} + 20(e^{-\pi t/2}\cos 2\pi t)\mathbf{j}$$

Then $$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$= 30\frac{1}{(t+1)^2}\mathbf{i} + 20\left(-\frac{\pi}{2}e^{-\pi t/2}\cos 2\pi t - 2\pi e^{-\pi t/2}\sin 2\pi t\right)\mathbf{j}$$

$$= 30\frac{1}{(t+1)^2}\mathbf{i} - 20\pi\left[e^{-\pi t/2}\left(\frac{1}{2}\cos 2\pi t + 2\sin 2\pi t\right)\right]\mathbf{j}$$

and $$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$= -30\frac{2}{(t+1)^3}\mathbf{i} - 20\pi\left[-\frac{\pi}{2}e^{-\pi t/2}\left(\frac{1}{2}\cos 2\pi t + 2\sin 2\pi t\right) + e^{-\pi t/2}(-\pi\sin 2\pi t + 4\cos 2\pi t)\right]\mathbf{j}$$

$$= -\frac{60}{(t+1)^3}\mathbf{i} + 10\pi^2 e^{-\pi t/2}(4\sin 2\pi t - 7.5\cos 2\pi t)\mathbf{j}$$

(*a*) At $t = 0$: $$\mathbf{r} = 30\left(1-\frac{1}{1}\right)\mathbf{i} + 20(1)\mathbf{j}$$

or $\mathbf{r} = 20\text{ mm}$ ↑ ◀

$$\mathbf{v} = 30\left(\frac{1}{1}\right)\mathbf{i} - 20\pi\left[(1)\left(\frac{1}{2}+0\right)\right]\mathbf{j}$$

or $\mathbf{v} = 43.4\text{ mm/s}$ ↘ 46.3° ◀

$$\mathbf{a} = -\frac{60}{(1)}\mathbf{i} + 10\pi^2(1)(0-7.5)\mathbf{j}$$

or $\mathbf{a} = 743\text{ mm/s}^2$ ↙ 85.4° ◀

PROBLEM 11.93 (Continued)

(*b*) At $t = 1.5$ s: $\quad \mathbf{r} = 30\left(1 - \frac{1}{2.5}\right)\mathbf{i} + 20e^{-0.75\pi}(\cos 3\pi)\mathbf{j}$

$$= (18 \text{ mm})\mathbf{i} + (-1.8956 \text{ mm})\mathbf{j}$$

or $\quad \mathbf{r} = 18.10 \text{ mm}$ ↘ 6.01° ◀

$$\mathbf{v} = \frac{30}{(2.5)^2}\mathbf{i} - 20\pi e^{-0.75\pi}\left(\frac{1}{2}\cos 3\pi + 0\right)\mathbf{j}$$

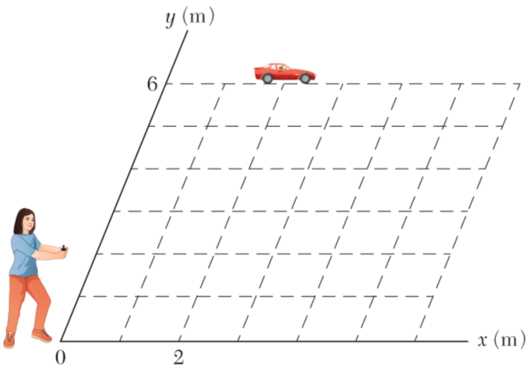
$$= (4.80 \text{ mm/s})\mathbf{i} + (2.9778 \text{ mm/s})\mathbf{j}$$

or $\quad \mathbf{v} = 5.65 \text{ mm/s}$ ↗ 31.8° ◀

$$\mathbf{a} = -\frac{60}{(2.5)^3}\mathbf{i} + 10\pi^2 e^{-0.75\pi}(0 - 7.5\cos 3\pi)\mathbf{j}$$

$$= (-3.84 \text{ mm/s}^2)\mathbf{i} + (70.1582 \text{ mm/s}^2)\mathbf{j}$$

or $\quad \mathbf{a} = 70.3 \text{ mm/s}^2$ ↖ 86.9° ◀



Problem 11.94

A girl operates a radio-controlled model car in a vacant parking lot. The girl's position is at the origin of the *xy* coordinate axes, and the surface of the parking lot lies in the *x*-*y* plane. The motion of the car is defined by the position vector $\vec{r} = (2+2t^2)\hat{i} + (6+t^3)\hat{j}$ where *r* and *t* are expressed in meters and seconds, respectively. Determine (a) the distance between the car and the girl when $t = 2$ s, (*b*) the distance the car traveled in the interval from $t = 0$ to $t = 2$ s, (*c*) the speed and direction of the car's velocity at $t = 2$ s, (d) the magnitude of the car's acceleration at $t = 2$ s.

SOLUTION

Given: $\mathbf{r} = (2+2t^2)\mathbf{i} + (6+t^3)\mathbf{j}$

(a) At t=2s $\quad \mathbf{r}(2) = 10\mathbf{i} + 14\mathbf{j}$ m

$|\mathbf{r}(2)| = \sqrt{10^2 + 14^2}$ m $\qquad |\mathbf{r}(2)| = 17.20$ m ◀

(b) The car is traveling on a curved path. The distance that the car travels during any infinitesimal interval is given by: $\quad ds = \sqrt{dx^2 + dy^2}$

where: $\quad dx = 4t\,dt$ and $dy = 3t^2 dt$

Substituting $\quad ds = \sqrt{16t^2 + 9t^4}\,dt$

Integrating: $\quad \int_0^s ds = \int_0^2 t\sqrt{16 + 9t^2}\,dt$

$s = \frac{1}{27}\left(16 + 9t^2\right)^{3/2}\Big|_0^2 \qquad s = 11.52$ m ◀

(c) Velocity can be found by $\quad \mathbf{v} = \frac{d\mathbf{r}}{dt}$

$\mathbf{v} = 4t\mathbf{i} + 3t^2\mathbf{j}$ m/s

At t=2 s $\quad \mathbf{v} = 8\mathbf{i} + 12\mathbf{j}$ m/s $\qquad |\mathbf{v}| = 14.42$ m/s ∡ 56.31° ◀

(d) Acceleration can be found by $\mathbf{a} = \frac{d\mathbf{v}}{dt}$

$\mathbf{a} = 4\mathbf{i} + 6t\mathbf{j}$ m/s^2

At $t = 2$ s $\quad \mathbf{a} = 4\mathbf{i} + 12\mathbf{j}$ m/s^2 $\qquad |\mathbf{a}| = 12.65$ m/s^2 ◀

PROBLEM 11.95

The three-dimensional motion of a particle is defined by the position vector **r** = (*Rt* cos $\omega_n t$)**i** + *ct***j** + (*Rt* sin $\omega_n t$)**k**. Determine the magnitudes of the velocity and acceleration of the particle. (The space curve described by the particle is a conic helix.)

SOLUTION

We have
$$\mathbf{r} = (Rt\cos\omega_n t)\mathbf{i} + ct\mathbf{j} + (Rt\sin\omega_n t)\mathbf{k}$$

Then
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = R(\cos\omega_n t - \omega_n t\sin\omega_n t)\mathbf{i} + c\mathbf{j} + R(\sin\omega_n t + \omega_n t\cos\omega_n t)\mathbf{k}$$

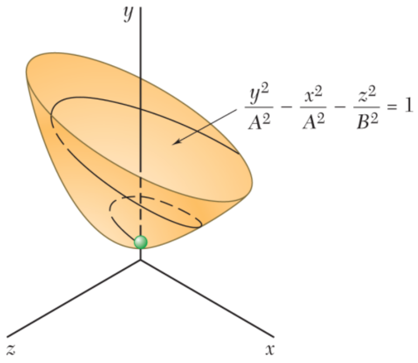
and
$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt}\\ &= R(-\omega_n\sin\omega_n t - \omega_n\sin\omega_n t - \omega_n^2 t\cos\omega_n t)\mathbf{i}\\ &\quad + R(\omega_n\cos\omega_n t + \omega_n\cos\omega_n t - \omega_n^2 t\sin\omega_n t)\mathbf{k}\\ &= R(-2\omega_n\sin\omega_n t - \omega_n^2 t\cos\omega_n t)\mathbf{i} + R(2\omega_n\cos\omega_n t - \omega_n^2 t\sin\omega_n t)\mathbf{k}\end{aligned}$$

Now
$$\begin{aligned}v^2 &= v_x^2 + v_y^2 + v_z^2\\ &= [R(\cos\omega_n t - \omega_n t\sin\omega_n t)]^2 + (c)^2 + [R(\sin\omega_n t + \omega_n t\cos\omega_n t)]^2\\ &= R^2\Big[\left(\cos^2\omega_n t - 2\omega_n t\sin\omega_n t\cos\omega_n t + \omega_n^2 t^2\sin^2\omega_n t\right)\\ &\quad + \left(\sin^2\omega_n t + 2\omega_n t\sin\omega_n t\cos\omega_n t + \omega_n^2 t^2\cos^2\omega_n t\right)\Big] + c^2\\ &= R^2\left(1 + \omega_n^2 t^2\right) + c^2\end{aligned}$$

or
$$v = \sqrt{R^2\left(1 + \omega_n^2 t^2\right) + c^2} \quad ◀$$

Also,
$$\begin{aligned}a^2 &= a_x^2 + a_y^2 + a_z^2\\ &= \left[R\left(-2\omega_n\sin\omega_n t - \omega_n^2 t\cos\omega_n t\right)\right]^2 + (0)^2\\ &\quad + \left[R\left(2\omega_n\cos\omega_n t - \omega_n^2 t\sin\omega_n t\right)\right]^2\\ &= R^2\Big[\left(4\omega_n^2\sin^2\omega_n t + 4\omega_n^3 t\sin\omega_n t\cos\omega_n t + \omega_n^4 t^2\cos^2\omega_n t\right)\\ &\quad + \left(4\omega_n^2\cos^2\omega_n t - 4\omega_n^3 t\sin\omega_n t\cos\omega_n t + \omega_n^4 t^2\sin^2\omega_n t\right)\Big]\\ &= R^2\left(4\omega_n^2 + \omega_n^4 t^2\right)\end{aligned}$$

or
$$a = R\omega_n\sqrt{4 + \omega_n^2 t^2} \quad ◀$$



PROBLEM 11.96

The three-dimensional motion of a particle is defined by the position vector $\mathbf{r} = (At\cos t)\mathbf{i} + (A\sqrt{t^2+1})\mathbf{j} + (Bt\sin t)\mathbf{k}$, where r and t are expressed in feet and seconds, respectively. Show that the curve described by the particle lies on the hyperboloid $(y/A)^2 - (x/A)^2 - (z/B)^2 = 1$. For $A = 3$ and $B = 1$, determine (*a*) the magnitudes of the velocity and acceleration when $t = 0$, (*b*) the smallest nonzero value of t for which the position vector and the velocity are perpendicular to each other.

SOLUTION

We have
$$\mathbf{r} = (At\cos t)\mathbf{i} + (A\sqrt{t^2+1})\mathbf{j} + (Bt\sin t)\mathbf{k}$$

or
$$x = At\cos t \quad y = A\sqrt{t^2+1} \quad z = Bt\sin t$$

Then
$$\cos t = \frac{x}{At} \qquad \sin t = \frac{z}{Bt} \qquad t^2 = \left(\frac{y}{A}\right)^2 - 1$$

Now
$$\cos^2 t + \sin^2 t = 1 \quad \Rightarrow \quad \left(\frac{x}{At}\right)^2 + \left(\frac{z}{Bt}\right)^2 = 1$$

or
$$t^2 = \left(\frac{x}{A}\right)^2 + \left(\frac{z}{B}\right)^2$$

Then
$$\left(\frac{y}{A}\right)^2 - 1 = \left(\frac{x}{A}\right)^2 + \left(\frac{z}{B}\right)^2$$

or
$$\left(\frac{y}{A}\right)^2 - \left(\frac{x}{A}\right)^2 - \left(\frac{z}{B}\right)^2 = 1 \qquad \text{Q.E.D.} \qquad \blacktriangleleft$$

(*a*) With $A = 3$ and $B = 1$, we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 3(\cos t - t\sin t)\mathbf{i} + 3\frac{t}{\sqrt{t^2+1}}\mathbf{j} + (\sin t + t\cos t)\mathbf{k}$$

and
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 3(-\sin t - \sin t - t\cos t)\mathbf{i} + 3\frac{\sqrt{t^2+1} - t\left(\frac{t}{\sqrt{t^2+1}}\right)}{(t^2+1)}\mathbf{j}$$
$$+ (\cos t + \cos t - t\sin t)\mathbf{k}$$
$$= -3(2\sin t + t\cos t)\mathbf{i} + 3\frac{1}{(t^2+1)^{3/2}}\mathbf{j} + (2\cos t - t\sin t)\mathbf{k}$$

At $t = 0$: $\mathbf{v} = 3(1-0)\mathbf{i} + (0)\mathbf{j} + (0)\mathbf{k}$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

or $v = 3 \text{ ft/s}$ ◀

and $\mathbf{a} = -3(0)\mathbf{i} + 3(1)\mathbf{j} + (2-0)\mathbf{k}$

Then $a^2 = (0)^2 + (3)^2 + (2)^2 = 13$

or $a = 3.61 \text{ ft/s}^2$ ◀

(*b*) If **r** and **v** are perpendicular, $\mathbf{r} \cdot \mathbf{v} = 0$

$$[(3t \cos t)\mathbf{i} + (3\sqrt{t^2+1})\mathbf{j} + (t \sin t)\mathbf{k}] \cdot [3(\cos t - t \sin t)\mathbf{i} + \left(3\frac{t}{\sqrt{t^2+1}}\right)\mathbf{j} + (\sin t + t \cos t)\mathbf{k}] = 0$$

or
$$(3t \cos t)[3(\cos t - t \sin t)] + (3\sqrt{t^2+1})\left(3\frac{t}{\sqrt{t^2+1}}\right) + (t \sin t)(\sin t + t \cos t) = 0$$

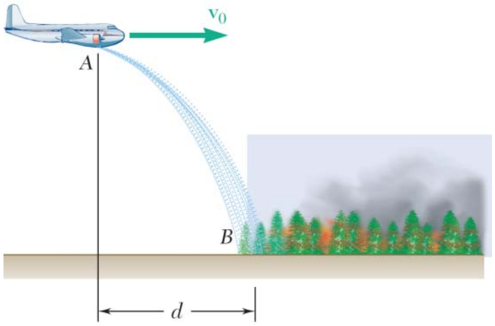
Expanding
$$(9t \cos^2 t - 9t^2 \sin t \cos t) + (9t) + (t \sin^2 t + t^2 \sin t \cos t) = 0$$

or (with $t \neq 0$)
$$10 + 8\cos^2 t - 8t \sin t \cos t = 0$$

or
$$7 + 2\cos 2t - 2t \sin 2t = 0$$

Using "trial and error" or numerical methods, the smallest root is $t = 3.82 \text{ s}$ ◀

Note: The next root is $t = 4.38$ s.



PROBLEM 11.97

An airplane used to drop water on brushfires is flying horizontally in a straight line at 180 mi/h at an altitude of 300 ft. Determine the distance d at which the pilot should release the water so that it will hit the fire at B.

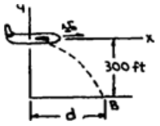
SOLUTION

First note $\quad v_0 = 180 \text{ km/h} = 264 \text{ ft/s}$

Place origin of coordinates at Point A.

Vertical motion. (Uniformly accelerated motion)

$$y = 0 + (0)t - \frac{1}{2}gt^2$$



At B: $\quad -300 \text{ ft} = -\frac{1}{2}(32.2 \text{ ft/s}^2)t^2$

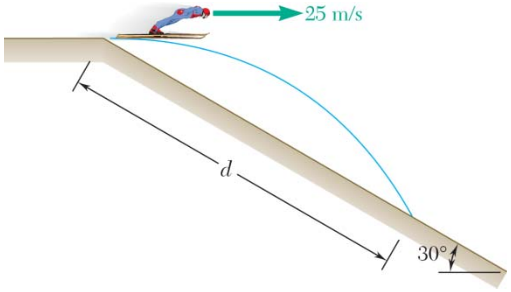
or $\quad t_B = 4.31666 \text{ s}$

Horizontal motion. (Uniform)

$$x = 0 + (v_x)_0 t$$

At B: $\quad d = (264 \text{ ft/s})(4.31666 \text{ s})$

or $\quad d = 1140 \text{ ft}$ ◀



PROBLEM 11.98

A ski jumper starts with a horizontal take-off velocity of 25 m/s and lands on a straight landing hill inclined at 30°. Determine (*a*) the time between take-off and landing, (*b*) the length *d* of the jump, (*c*) the maximum vertical distance between the jumper and the landing hill.

SOLUTION

(*a*) At the landing point, $y = -x\tan 30°$

Horizontal motion: $x = x_0 + (v_x)_0 t = v_0 t$

Vertical motion: $y = y_0 + (v_y)_0 t - \frac{1}{2}gt^2 = -\frac{1}{2}gt^2$

from which $t^2 = -\frac{2y}{g} = \frac{2x\tan 30°}{g} = \frac{2v_0 t\tan 30°}{g}$

Rejecting the $t = 0$ solution gives $t = \frac{2v_0\tan 30°}{g} = \frac{(2)(25)\tan 30°}{9.81}$ $t = 2.94\text{ s}$ ◀

(*b*) Landing distance: $d = \frac{x}{\cos 30°} = \frac{v_0 t}{\cos 30°} = \frac{(25)(2.94)}{\cos 30°}$ $d = 84.9\text{ m}$ ◀

(*c*) Vertical distance: $h = x\tan 30° + y$

or $h = v_0 t\tan 30° - \frac{1}{2}gt^2$

Differentiating and setting equal to zero,

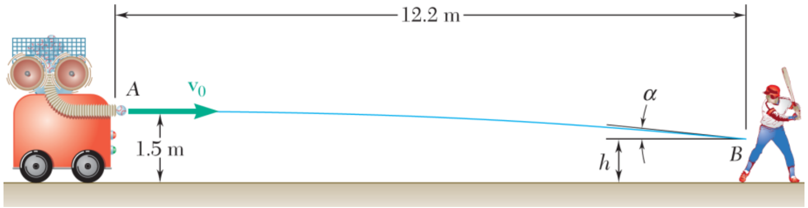
$$\frac{dh}{dt} = v_0\tan 30° - gt = 0 \qquad \text{or} \qquad t = \frac{v_o\tan 30°}{g}$$

Then,

$$h_{\max} = \frac{(v_0)(v_0\tan 30°)\tan 30°}{g} - \frac{1}{2}g\left(\frac{v_0\tan 30°}{g}\right)^2$$

$$= \frac{v_0^2\tan^2 30°}{2g} = \frac{(25)^2(\tan 30°)^2}{(2)(9.81)}$$

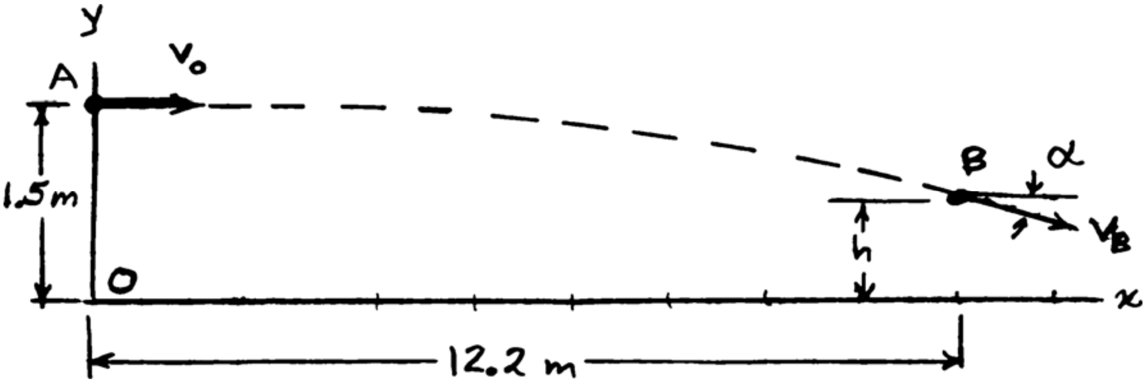
$h_{\max} = 10.62\text{ m}$ ◀



PROBLEM 11.99

A baseball pitching machine "throws" baseballs with a horizontal velocity $\mathbf{v}_0$. Knowing that height h varies between 788 mm and 1068 mm, determine (*a*) the range of values of v_0, (*b*) the values of α corresponding to $h = 788$ mm and $h = 1068$ mm.

SOLUTION



(*a*) Vertical motion: $y_0 = 1.5 \text{ m}, \ (v_y)_0 = 0$

$$y = y_0 + (v_y)_0 t - \frac{1}{2}gt^2 \quad \text{or} \quad t = \sqrt{\frac{2(y_0 - y)}{g}}$$

At Point B, $y = h$ or $t_B = \sqrt{\dfrac{2(y_0 - h)}{g}}$

When $h = 788 \text{ mm} = 0.788 \text{ m}$, $t_B = \sqrt{\dfrac{(2)(1.5 - 0.788)}{9.81}} = 0.3810 \text{ s}$

When $h = 1068 \text{ mm} = 1.068 \text{ m}$, $t_B = \sqrt{\dfrac{(2)(1.5 - 1.068)}{9.81}} = 0.2968 \text{ s}$

Horizontal motion: $x_0 = 0, \ (v_x)_0 = v_0,$

$$x = v_0 t \quad \text{or} \quad v_0 = \frac{x}{t} = \frac{x_B}{t_B}$$

PROBLEM 11.99 (Continued)

With $x_B = 12.2$ m, we get $v_0 = \dfrac{12.2}{0.3810} = 32.02$ m/s

and $v_0 = \dfrac{12.2}{0.2968} = 41.11$ m/s

$32.02 \text{ m/s} \le v_0 \le 41.11 \text{ m/s}$ or $115.3 \text{ km/h} \le v_0 \le 148.0 \text{ km/h}$ ◀

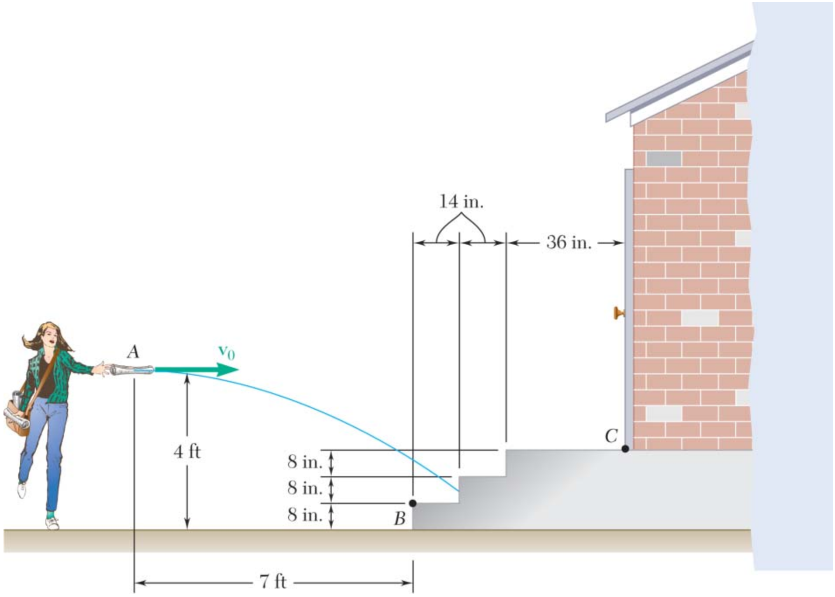
(*b*) Vertical motion: $v_y = (v_y)_0 - gt = -gt$

Horizontal motion: $v_x = v_0$

$$\tan\alpha = -\frac{dy}{dx} = -\frac{(v_y)_B}{(v_x)_B} = \frac{gt_B}{v_0}$$

For $h = 0.788$ m, $\tan\alpha = \dfrac{(9.81)(0.3810)}{32.02} = 0.11673,$ $\alpha = 6.66°$ ◀

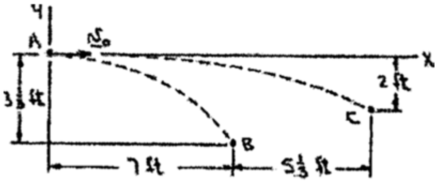
For $h = 1.068$ m, $\tan\alpha = \dfrac{(9.81)(0.2968)}{41.11} = 0.07082,$ $\alpha = 4.05°$ ◀


14 in.
36 in.
A
v_0
4 ft
8 in.
8 in.
8 in.
B
C
7 ft

PROBLEM 11.100

While delivering newspapers, a girl throws a newspaper with a horizontal velocity $\mathbf{v}_0$. Determine the range of values of v_0 if the newspaper is to land between Points *B* and *C*.

SOLUTION


y
A
v_0
x
2 ft
C
$3\frac{1}{3}$ ft
B
7 ft
$5\frac{1}{3}$ ft

Vertical motion. (Uniformly accelerated motion)

$$y = 0 + (0)t - \frac{1}{2}gt^2$$

Horizontal motion. (Uniform)

$$x = 0 + (v_x)_0 t = v_0 t$$

At *B*: y: $-3\frac{1}{3}\text{ ft} = -\frac{1}{2}(32.2\text{ ft/s}^2)t^2$

or $t_B = 0.455016\text{ s}$

Then x: $7\text{ ft} = (v_0)_B(0.455016\text{ s})$

or $(v_0)_B = 15.38\text{ ft/s}$

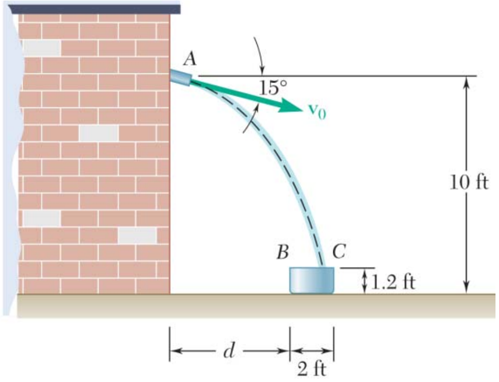
At *C*: y: $-2\text{ ft} = -\frac{1}{2}(32.2\text{ ft/s}^2)t^2$

or $t_C = 0.352454\text{ s}$

Then x: $12\frac{1}{3}\text{ ft} = (v_0)_C(0.352454\text{ s})$

or $(v_0)_C = 35.0\text{ ft/s}$

$15.38\text{ ft/s} < v_0 < 35.0\text{ ft/s}$ ◀



PROBLEM 11.101

Water flows from a drain spout with an initial velocity of 2.5 ft/s at an angle of 15° with the horizontal. Determine the range of values of the distance *d* for which the water will enter the trough *BC*.

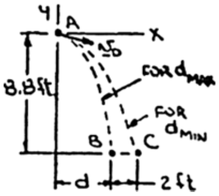
SOLUTION

First note

$$(v_x)_0 = (2.5\text{ ft/s})\cos 15° = 2.4148\text{ ft/s}$$

$$(v_y)_0 = -(2.5\text{ ft/s})\sin 15° = -0.64705\text{ ft/s}$$

Vertical motion. (Uniformly accelerated motion)



$$y = 0 + (v_y)_0 t - \frac{1}{2}gt^2$$

At the top of the trough

$$-8.8\text{ ft} = (-0.64705\text{ ft/s})t - \frac{1}{2}(32.2\text{ ft/s}^2)t^2$$

or $t_{BC} = 0.719491\text{ s}$ (the other root is negative)

Horizontal motion. (Uniform)

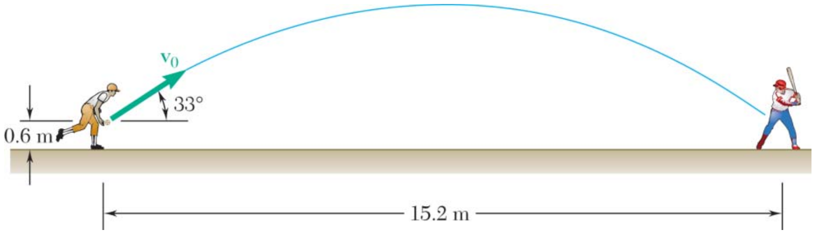
$$x = 0 + (v_x)_0 t$$

In time t_{BC} $\quad x_{BC} = (2.4148\text{ ft/s})(0.719491\text{ s}) = 1.737\text{ ft}$

Thus, the trough must be placed so that

$$x_B < 1.737\text{ ft or } x_C \geq 1.737\text{ ft}$$

Since the trough is 2 ft wide, it then follows that $0 < d < 1.737\text{ ft}$ ◀



PROBLEM 11.102

In slow pitch softball the underhand pitch must reach a maximum height of between 1.8 m and 3.7 m above the ground. A pitch is made with an initial velocity $\mathbf{v}_0$ of magnitude 13 m/s at an angle of 33° with the horizontal. Determine (*a*) if the pitch meets the maximum height requirement, (*b*) the height of the ball as it reaches the batter.

SOLUTION

$$v_0 = 13 \text{ m/s}, \ \alpha = 33°, \ x_0 = 0, \ y_0 = 0.6 \text{ m}$$

Vertical motion:

$$v_y = v_0 \sin\alpha - gt$$

$$y = y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

At maximum height,

$$v_y = 0 \qquad \text{or} \qquad t = \frac{v_0 \sin\alpha}{g}$$

(*a*)

$$t = \frac{13\sin 33°}{9.81} = 0.7217 \text{ s}$$

$$y_{\max} = 0.6 + (13\sin 33°)(0.7217) - \frac{1}{2}(9.81)(0.7217)^2$$

$$y_{\max} = 3.16 \text{ m} \blacktriangleleft$$

$$1.8 \text{ m} < 3.16 \text{ m} < 3.7 \text{ m}$$

yes ◀

Horizontal motion:

$$x = x_0 + (v_0 \cos\alpha)t \qquad \text{or} \qquad t = \frac{x - x_0}{v_0 \cos\alpha}$$

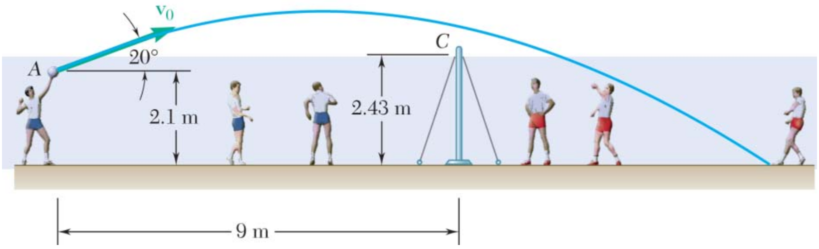
At $x = 15.2$ m,

$$t = \frac{15.2 - 0}{13\cos 33°} = 1.3941 \text{ s}$$

(*b*) Corresponding value of y:

$$y = 0.6 + (13\sin 33°)(1.3941) - \frac{1}{2}(9.81)(1.3941)^2$$

$$y = 0.937 \text{ m} \blacktriangleleft$$



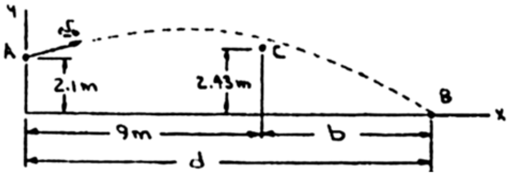
PROBLEM 11.103

A volleyball player serves the ball with an initial velocity $\mathbf{v}_0$ of magnitude 13.40 m/s at an angle of 20° with the horizontal. Determine (*a*) if the ball will clear the top of the net, (*b*) how far from the net the ball will land.

SOLUTION

First note

$$(v_x)_0 = (13.40 \text{ m/s}) \cos 20° = 12.5919 \text{ m/s}$$
$$(v_y)_0 = (13.40 \text{ m/s}) \sin 20° = 4.5831 \text{ m/s}$$



(*a*) Horizontal motion. (Uniform)

$$x = 0 + (v_x)_0 t$$

At *C*

$$9 \text{ m} = (12.5919 \text{ m/s})t \quad \text{or} \quad t_C = 0.71475 \text{ s}$$

Vertical motion. (Uniformly accelerated motion)

$$y = y_0 + (v_y)_0 t - \frac{1}{2} g t^2$$

At *C*:

$$\begin{aligned} y_C &= 2.1 \text{ m} + (4.5831 \text{ m/s})(0.71475 \text{ s}) \\ &\quad - \frac{1}{2}(9.81 \text{ m/s}^2)(0.71475 \text{ s})^2 \\ &= 2.87 \text{ m} \end{aligned}$$

$y_C > 2.43$ m (height of net) ⇒ ball clears net ◀

(*b*) At *B*, $y = 0$:

$$0 = 2.1 \text{ m} + (4.5831 \text{ m/s})t - \frac{1}{2}(9.81 \text{ m/s}^2)t^2$$

Solving

$$t_B = 1.271175 \text{ s} \quad \text{(the other root is negative)}$$

Then

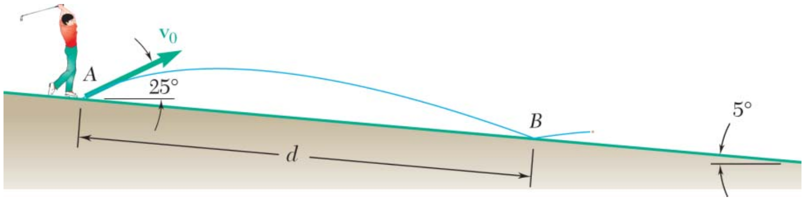
$$\begin{aligned} d &= (v_x)_0 t_B = (12.5919 \text{ m/s})(1.271175 \text{ s}) \\ &= 16.01 \text{ m} \end{aligned}$$

The ball lands

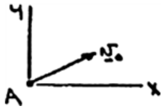
$b = (16.01 - 9.00)$ m $= 7.01$ m from the net ◀

PROBLEM 11.104

A golfer hits a golf ball with an initial velocity of 160 ft/s at an angle of 25° with the horizontal. Knowing that the fairway slopes downward at an average angle of 5°, determine the distance *d* between the golfer and Point *B* where the ball first lands.



SOLUTION



First note $$(v_x)_0 = (160 \text{ ft/s}) \cos 25°$$ $$(v_y)_0 = (160 \text{ ft/s}) \sin 25°$$

and at *B* $$x_B = d \cos 5° \quad y_B = -d \sin 5°$$

Now Horizontal motion. (Uniform)

$$x = 0 + (v_x)_0 t$$

At *B* $$d \cos 5° = (160 \cos 25°)t \quad \text{or} \quad t_B = \frac{\cos 5°}{160 \cos 25°} d$$

Vertical motion. (Uniformly accelerated motion)

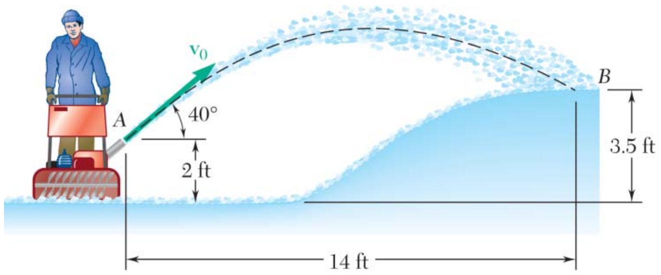
$$y = 0 + (v_y)_0 t - \frac{1}{2} g t^2 \quad (g = 32.2 \text{ ft/s}^2)$$

At *B*: $$-d \sin 5° = (160 \sin 25°) t_B - \frac{1}{2} g t_B^2$$

Substituting for t_B $$-d \sin 5° = (160 \sin 25°)\left(\frac{\cos 5°}{160 \cos 25°}\right) d - \frac{1}{2} g \left(\frac{\cos 5°}{160 \cos 25°}\right)^2 d^2$$

or $$d = \frac{2}{32.2 \cos 5°}(160 \cos 25°)^2 (\tan 5° + \tan 25°)$$ $$= 726.06 \text{ ft}$$

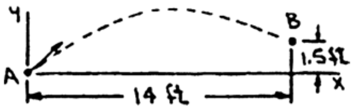
or $d = 242$ yd ◀



PROBLEM 11.105

A homeowner uses a snowblower to clear his driveway. Knowing that the snow is discharged at an average angle of 40° with the horizontal, determine the initial velocity v_0 of the snow.

SOLUTION



First note $(v_x)_0 = v_0 \cos 40°$

$(v_y)_0 = v_0 \sin 40°$

Horizontal motion. (Uniform)

$$x = 0 + (v_x)_0 t$$

At B: $14 = (v_0 \cos 40°)t \quad \text{or} \quad t_B = \dfrac{14}{v_0 \cos 40°}$

Vertical motion. (Uniformly accelerated motion)

$$y = 0 + (v_y)_0 t - \frac{1}{2}gt^2 \qquad (g = 32.2 \text{ ft/s}^2)$$

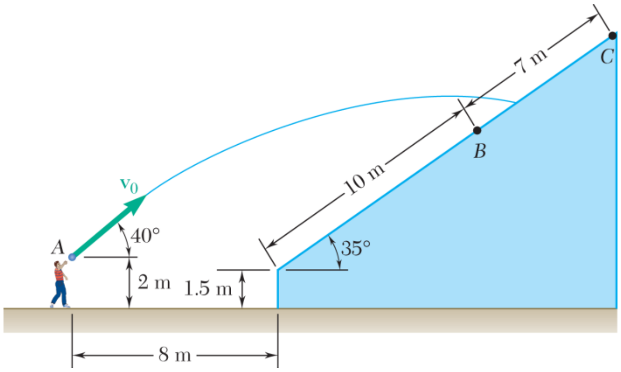
At B: $1.5 = (v_0 \sin 40°)t_B - \dfrac{1}{2}gt_B^2$

Substituting for t_B

$$1.5 = (v_0 \sin 40°)\left(\frac{14}{v_0 \cos 40°}\right) - \frac{1}{2}g\left(\frac{14}{v_0 \cos 40°}\right)^2$$

or $v_0^2 = \dfrac{\frac{1}{2}(32.2)(196)/\cos^2 40°}{-1.5 + 14 \tan 40°}$

or $v_0 = 22.9 \text{ ft/s}$ ◀



PROBLEM 11.106

At halftime of a football game souvenir balls are thrown to the spectators with a velocity $\mathbf{v}_0$. Determine the range of values of v_0 if the balls are to land between Points B and C.

SOLUTION

The motion is projectile motion. Place the origin of the xy-coordinate system at ground level just below Point A. The coordinates of Point A are $x_0 = 0,\ y_0 = 2\text{m}.$ The components of initial velocity are $(v_x)_0 = v_0 \cos 40°$ m/s and $(v_y)_0 = v_0 \sin 40°.$

Horizontal motion: $$x = x_0 + (v_x)_0 t = (v_0 \cos 40°)t \qquad (1)$$

Vertical motion: $$y = y_0 + (v_y)_0 t = \frac{1}{2} g t^2$$

$$= 2 + (v_0 \sin 40°) = -\frac{1}{2}(9.81)t^2 \qquad (2)$$

From (1), $$v_0 t = \frac{x}{\cos 40°} \qquad (3)$$

Then $$y = 2 + x \tan 40° - 4.905 t^2$$

$$t^2 = \frac{2 + x \tan 40° - y}{4.905} \qquad (4)$$

Point B: $$x = 8 + 10 \cos 35° = 16.1915 \text{ m}$$

$$y = 1.5 + 10 \sin 35° = 7.2358 \text{ m}$$

$$v_0 t = \frac{16.1915}{\cos 40°} = 21.1365 \text{ m}$$

$$t^2 = \frac{2 + 16.1915 \tan 40° - 7.2358}{4.905} \qquad t = 1.3048 \text{ s}$$

$$v_0 = \frac{21.1365}{1.3048} \qquad v_0 = 16.199 \text{ m/s}$$

PROBLEM 11.106 (Continued)

Point C:

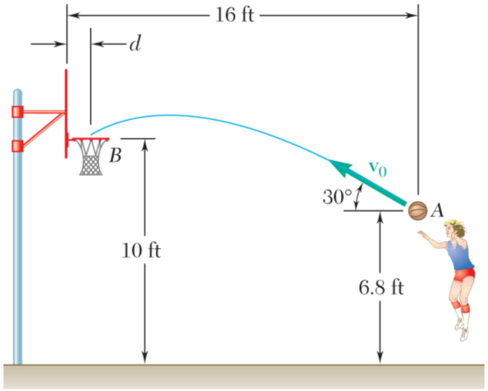
$$x = 8 + (10 + 7)\cos 35° = 21.9256 \text{ m}$$
$$y = 1.5 + (10 + 7)\sin 35° = 11.2508 \text{ m}$$

$$v_0 t = \frac{21.9256}{\cos 40°} = 28.622 \text{ m}$$

$$t^2 = \frac{2 + 21.9256 \tan 40° - 11.2508}{4.905} \qquad t = 1.3656 \text{ s}$$

$$v_0 = \frac{28.622}{1.3656} \qquad v_0 = 20.96 \text{ m/s}$$

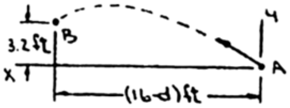
Range of values of v_0. $16.20 \text{ m/s} < v_0 < 21.0 \text{ m/s}$ ◀



PROBLEM 11.107

A basketball player shoots when she is 16 ft from the backboard. Knowing that the ball has an initial velocity $\mathbf{v}_0$ at an angle of 30° with the horizontal, determine the value of v_0 when d is equal to (*a*) 9 in., (*b*) 17 in.

SOLUTION



First note $\quad (v_x)_0 = v_0 \cos 30° \qquad (v_y)_0 = v_0 \sin 30°$

Horizontal motion. (Uniform) $\quad x = 0 + (v_x)_0 t$

At *B*: $\quad (16 - d) = (v_0 \cos 30°)t \quad \text{or} \quad t_B = \dfrac{16 - d}{v_0 \cos 30°}$

Vertical motion. (Uniformly accelerated motion) $\quad y = 0 + (v_y)_0 t - \dfrac{1}{2}gt^2 \quad (g = 32.2 \text{ ft/s}^2)$

At *B*: $\quad 3.2 = (v_0 \sin 30°)t_B - \dfrac{1}{2}gt_B^2$

Substituting for t_B $\quad 3.2 = (v_0 \sin 30°)\left(\dfrac{16 - d}{v_0 \cos 30°}\right) - \dfrac{1}{2}g\left(\dfrac{16 - d}{v_0 \cos 30°}\right)^2$

or

$$v_0^2 = \frac{2g(16 - d)^2}{3\left[\frac{1}{\sqrt{3}}(16 - d) - 3.2\right]}$$

(*a*) $\quad d = 9$ in.:

$$v_0^2 = \frac{2(32.2)\left(16 - \frac{9}{12}\right)^2}{3\left[\frac{1}{\sqrt{3}}\left(16 - \frac{9}{12}\right) - 3.2\right]}$$

$v_0 = 29.8$ ft/s ◀

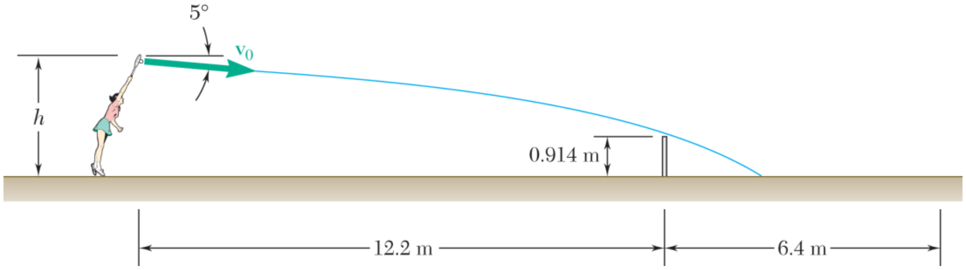
(*b*) $\quad d = 17$ in.:

$$v_0^2 = \frac{2(32.2)\left(16 - \frac{17}{12}\right)^2}{3\left[\frac{1}{\sqrt{3}}\left(16 - \frac{17}{12}\right) - 3.2\right]}$$

$v_0 = 29.6$ ft/s ◀

PROBLEM 11.108

A tennis player serves the ball at a height h = 2.5 m with an initial velocity of $\mathbf{v}_0$ at an angle of 5° with the horizontal. Determine the range for which of v_0 for which the ball will land in the service area which extends to 6.4 m beyond the net.



SOLUTION

The motion is projectile motion. Place the origin of the xy-coordinate system at ground level just below the point where the racket impacts the ball. The coordinates of this impact point are $x_0 = 0,\ y_0 = h = 2.5\text{ m}$. The components of initial velocity are $(v_x)_0 = v_0 \cos 5°$ and $(v_y)_0 = v_0 \sin 5°$.

Horizontal motion:

$$x = x_0 + (v_x)_0 t = (v_0 \cos 5°)t \qquad (1)$$

Vertical motion:

$$y = y_0 + (v_y)_0 t = \frac{1}{2} g t^2$$

$$= 2.5 - (v_0 \sin 5°)t = -\frac{1}{2}(9.81)t^2 \qquad (2)$$

From (1),

$$v_0 t = \frac{x}{\cos 5°} \qquad (3)$$

Then

$$y = 2.5 - x \tan 5° - 4.905 t^2$$

$$t^2 = \frac{2.5 - x \tan 5° - y}{4.905} \qquad (4)$$

At the minimum speed the ball just clears the net.

$$x = 12.2\text{ m}, \quad y = 0.914\text{ m}$$

$$v_0 t = \frac{12.2}{\cos 5°} = 12.2466\text{ m}$$

$$t^2 = \frac{2.5 - 12.2 \tan 5° - 0.914}{4.905} \qquad t = 0.32517\text{ s}$$

$$v_0 = \frac{12.2466}{0.32517} \qquad v_0 = 37.66\text{ m/s}$$

PROBLEM 11.108 (Continued)

At the maximum speed the ball lands 6.4 m beyond the net.

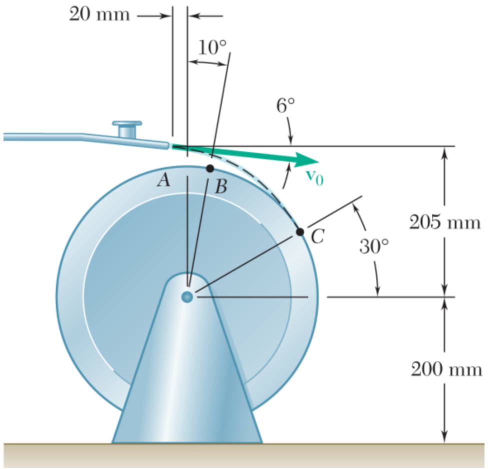
$$x = 12.2 + 6.4 = 18.6 \text{ m} \qquad y = 0$$

$$v_0 t = \frac{18.6}{\cos 5^\circ} = 18.6710 \text{ m}$$

$$t^2 = \frac{2.5 - 18.6 \tan 5^\circ - 0}{4.905} \qquad t = 0.42181 \text{ s}$$

$$v_0 = \frac{18.6710}{0.42181} \qquad v_0 = 44.26 \text{ m/s}$$

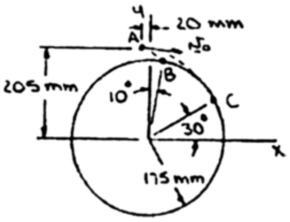
Range for v_0. $37.7 \text{ m/s} < v_0 < 44.3 \text{ m/s}$ ◀



PROBLEM 11.109

The nozzle at *A* discharges cooling water with an initial velocity $\mathbf{v}_0$ at an angle of 6° with the horizontal onto a grinding wheel 350 mm in diameter. Determine the range of values of the initial velocity for which the water will land on the grinding wheel between Points *B* and *C*.

SOLUTION



First note

$$(v_x)_0 = v_0 \cos 6°$$
$$(v_y)_0 = -v_0 \sin 6°$$

Horizontal motion. (Uniform)

$$x = x_0 + (v_x)_0 t$$

Vertical motion. (Uniformly accelerated motion)

$$y = y_0 + (v_y)_0 t - \frac{1}{2}gt^2 \quad (g = 9.81 \text{ m/s}^2)$$

At Point *B*:

$$x = (0.175 \text{ m}) \sin 10°$$
$$y = (0.175 \text{ m}) \cos 10°$$

$$x: \quad 0.175 \sin 10° = -0.020 + (v_0 \cos 6°)t$$

or

$$t_B = \frac{0.050388}{v_0 \cos 6°}$$

$$y: \quad 0.175 \cos 10° = 0.205 + (-v_0 \sin 6°)t_B - \frac{1}{2}gt_B^2$$

Substituting for t_B

$$-0.032659 = (-v_0 \sin 6°)\left(\frac{0.050388}{v_0 \cos 6°}\right) - \frac{1}{2}(9.81)\left(\frac{0.050388}{v_0 \cos 6°}\right)^2$$

PROBLEM 11.109 (Continued)

or $$v_0^2 = \frac{\frac{1}{2}(9.81)(0.050388)^2}{\cos^2 6°(0.032659 - 0.050388 \tan 6°)}$$

or $$(v_0)_B = 0.678 \text{ m/s}$$

At Point C: $$x = (0.175 \text{ m}) \cos 30°$$
$$y = (0.175 \text{ m}) \sin 30°$$

$$x: \quad 0.175 \cos 30° = -0.020 + (v_0 \cos 6°)t$$

or $$t_C = \frac{0.171554}{v_0 \cos 6°}$$

$$y: \quad 0.175 \sin 30° = 0.205 + (-v_0 \sin 6°)t_C - \frac{1}{2}gt_C^2$$

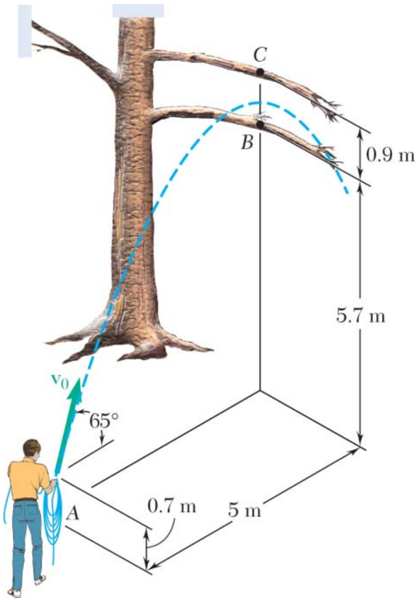
Substituting for t_C

$$-0.117500 = (-v_0 \sin 6°)\left(\frac{0.171554}{v_0 \cos 6°}\right) - \frac{1}{2}(9.81)\left(\frac{0.171554}{v_0 \cos 6°}\right)^2$$

or $$v_0^2 = \frac{\frac{1}{2}(9.81)(0.171554)^2}{\cos^2 6°(0.117500 - 0.171554 \tan 6°)}$$

or $$(v_0)_C = 1.211 \text{ m/s}$$

$$0.678 \text{ m/s} < v_0 < 1.211 \text{ m/s} \blacktriangleleft$$



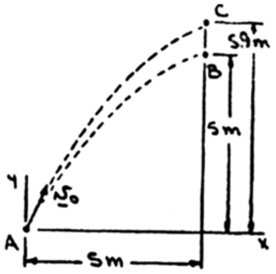
PROBLEM 11.110

While holding one of its ends, a worker lobs a coil of rope over the lowest limb of a tree. If he throws the rope with an initial velocity $\mathbf{v}_0$ at an angle of 65° with the horizontal, determine the range of values of v_0 for which the rope will go over only the lowest limb.

SOLUTION

First note

$$(v_x)_0 = v_0 \cos 65°$$
$$(v_y)_0 = v_0 \sin 65°$$



Horizontal motion. (Uniform)

$$x = 0 + (v_x)_0 t$$

At either B or C, $x = 5$ m

$$s = (v_0 \cos 65°)t_{B,C}$$

or
$$t_{B,C} = \frac{5}{(v_0 \cos 65°)}$$

Vertical motion. (Uniformly accelerated motion)

$$y = 0 + (v_y)_0 t - \frac{1}{2}gt^2 \qquad (g = 9.81 \text{ m/s}^2)$$

At the tree limbs, $t = t_{B,C}$

$$y_{B,C} = (v_0 \sin 65°)\left(\frac{5}{v_0 \cos 65°}\right) - \frac{1}{2}g\left(\frac{5}{v_0 \cos 65°}\right)^2$$

PROBLEM 11.110 (Continued)

or
$$v_0^2 = \frac{\frac{1}{2}(9.81)(25)}{\cos^2 65°(5 \tan 65° - y_{B,C})}$$
$$= \frac{686.566}{5 \tan 65° - y_{B,C}}$$

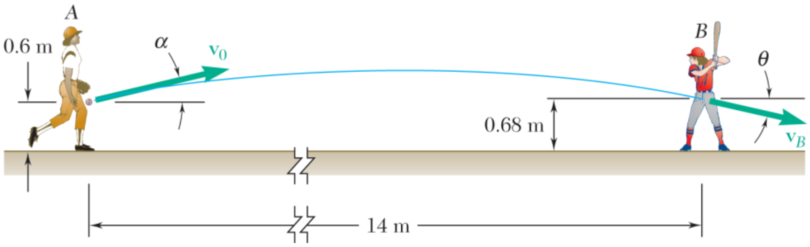
At Point *B*: $$v_0^2 = \frac{686.566}{5 \tan 65° - 5} \qquad \text{or} \qquad (v_0)_B = 10.95 \text{ m/s}$$

At Point *C*: $$v_0^2 = \frac{686.566}{5 \tan 65° - 5.9} \qquad \text{or} \qquad (v_0)_C = 11.93 \text{ m/s}$$

$10.95 \text{ m/s} < v_0 < 11.93 \text{ m/s}$ ◀

PROBLEM 11.111

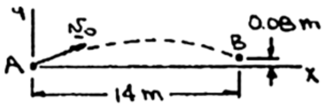
The pitcher in a softball game throws a ball with an initial velocity $\mathbf{v}_0$ of 72 km/h at an angle α with the horizontal. If the height of the ball at Point B is 0.68 m, determine (*a*) the angle α, (*b*) the angle θ that the velocity of the ball at Point B forms with the horizontal.



SOLUTION

First note

$$v_0 = 72 \text{ km/h} = 20 \text{ m/s}$$



and

$$(v_x)_0 = v_0 \cos\alpha = (20 \text{ m/s}) \cos\alpha$$
$$(v_y)_0 = v_0 \sin\alpha = (20 \text{ m/s}) \sin\alpha$$

(*a*) Horizontal motion. (Uniform)

$$x = 0 + (v_x)_0 t = (20 \cos\alpha)t$$

At Point B:

$$14 = (20\cos\alpha)t \quad \text{or} \quad t_B = \frac{7}{10\cos\alpha}$$

Vertical motion. (Uniformly accelerated motion)

$$y = 0 + (v_y)_0 t - \frac{1}{2}gt^2 = (20\sin\alpha)t - \frac{1}{2}gt^2 \quad (g = 9.81 \text{ m/s}^2)$$

At Point B:

$$0.08 = (20\sin\alpha)t_B - \frac{1}{2}gt_B^2$$

Substituting for t_B

$$0.08 = (20\sin\alpha)\left(\frac{7}{10\cos\alpha}\right) - \frac{1}{2}g\left(\frac{7}{10\cos\alpha}\right)^2$$

or

$$8 = 1400\tan\alpha - \frac{1}{2}g\frac{49}{\cos^2\alpha}$$

Now

$$\frac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$$

PROBLEM 11.111 (Continued)

Then $$8 = 1400\tan\alpha - 24.5g(1+\tan^2\alpha)$$

or $$240.345\tan^2\alpha - 1400\tan\alpha + 248.345 = 0$$

Solving $$\alpha = 10.3786° \quad \text{and} \quad \alpha = 79.949°$$

Rejecting the second root because it is not physically reasonable, we have

$\alpha = 10.38°$ ◀

(*b*) We have $$v_x = (v_x)_0 = 20\cos\alpha$$

and $$v_y = (v_y)_0 - gt = 20\sin\alpha - gt$$

At Point *B*: $$(v_y)_B = 20\sin\alpha - gt_B$$

$$= 20\sin\alpha - \frac{7g}{10\cos\alpha}$$

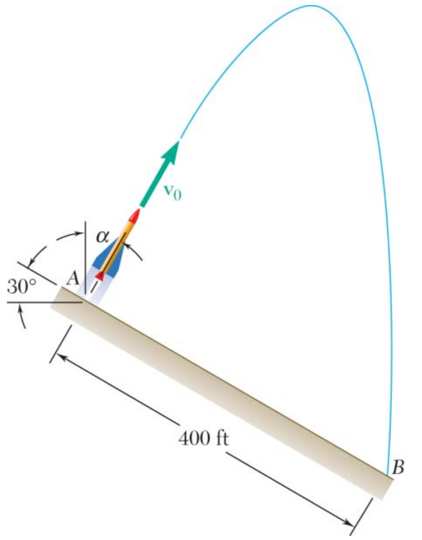
Noting that at Point *B*, $v_y < 0$, we have

$$\tan\theta = \frac{|(v_y)_B|}{v_x}$$

$$= \frac{\frac{7g}{10\cos\alpha} - 20\sin\alpha}{20\cos\alpha}$$

$$= \frac{\frac{7}{200}\frac{9.81}{\cos 10.3786°} - \sin 10.3786°}{\cos 10.3786°}$$

or $\theta = 9.74°$ ◀



PROBLEM 11.112

A model rocket is launched from Point A with an initial velocity $\mathbf{v}_0$ of 75 m/s. If the rocket's descent parachute does not deploy and the rocket lands a distance $d = 100$ m from A, determine (*a*) the angle α that $\mathbf{v}_0$ forms with the vertical, (*b*) the maximum height above Point A reached by the rocket, and (*c*) the duration of the flight.

SOLUTION

Set the origin at Point A. $\quad x_0 = 0, \quad y_0 = 0$

Horizontal motion: $\quad x = v_0 t \sin\alpha \qquad \sin\alpha = \dfrac{x}{v_0 t} \qquad (1)$

Vertical motion: $\quad y = v_0 t \cos\alpha - \dfrac{1}{2}gt^2$

$$\cos\alpha = \frac{1}{v_0 t}\left(y + \frac{1}{2}gt^2\right) \qquad (2)$$

$$\sin^2\alpha + \cos^2\alpha = \frac{1}{(v_0 t)^2}\left[x^2 + \left(y + \frac{1}{2}gt^2\right)^2\right] = 1$$

$$x^2 + y^2 + gyt^2 + \frac{1}{4}g^2t^4 = v_0^2 t^2$$

$$\frac{1}{4}g^2t^4 - \left(v_0^2 - gy\right)t^2 + (x^2 + y^2) = 0 \qquad (3)$$

At Point B, $\quad \sqrt{x^2 + y^2} = 100\text{ m}, \quad x = 100\cos 30°\text{ m}$

$$y = -100\sin 30° = -50\text{ m}$$

$$\frac{1}{4}(9.81)^2t^4 - [75^2 - (9.81)(-50)]t^2 + 100^2 = 0$$

$$24.0590\,t^4 - 6115.5\,t^2 + 10000 = 0$$

$$t^2 = 252.54\text{ s}^2 \quad \text{and} \quad 1.6458\text{ s}^2$$

$$t = 15.8916\text{ s} \quad \text{and} \quad 1.2829\text{ s}$$

PROBLEM 11.112 (Continued)

Restriction on α: $0 < \alpha < 120°$

$$\tan\alpha = \frac{x}{y + \frac{1}{2}gt^2} = \frac{100\cos 30°}{-50 + (4.905)(15.8916)^2} = 0.0729$$

$$\alpha = 4.1669°$$

and $$\frac{100\cos 30°}{-50 + (4.905)(1.2829)^2} = -2.0655$$

$$\alpha = 115.8331°$$

Use $\alpha = 4.1669°$ corresponding to the steeper possible trajectory.

(*a*) Angle α. $\alpha = 4.17°$ ◀

(*b*) Maximum height. $v_y = 0 \quad \text{at} \quad y = y_{\max}$

$$v_y = v_0\cos\alpha - gt = 0$$

$$t = \frac{v_0\cos\alpha}{g}$$

$$y_{\max} = v_0 t\cos\alpha - \frac{1}{2}gt = \frac{v_0^2\cos^2\alpha}{2g}$$

$$= \frac{(75)^2\cos^2 4.1669°}{(2)(9.81)}$$

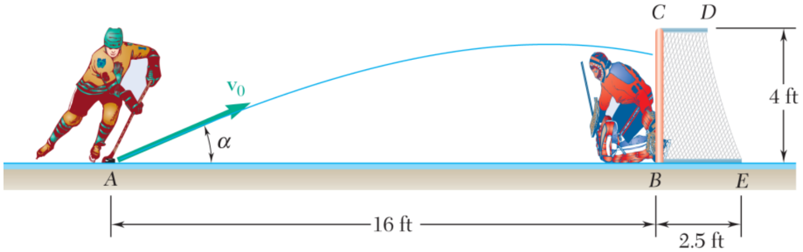
$y_{\max} = 285\text{ m}$ ◀

(*c*) Duration of the flight. (time to reach *B*)

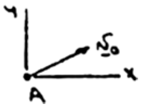
$t = 15.89\text{ s}$ ◀

PROBLEM 11.113

The initial velocity $\mathbf{v}_0$ of a hockey puck is 105 mi/h. Determine (*a*) the largest value (less than 45°) of the angle α for which the puck will enter the net, (*b*) the corresponding time required for the puck to reach the net.



SOLUTION



First note $\quad v_0 = 105 \text{ mi/h} = 154 \text{ ft/s}$

and

$$(v_x)_0 = v_0 \cos\alpha = (154 \text{ ft/s}) \cos\alpha$$

$$(v_y)_0 = v_0 \sin\alpha = (154 \text{ ft/s}) \sin\alpha$$

(*a*) Horizontal motion. (Uniform)

$$x = 0 + (v_x)_0 t = (154 \cos\alpha)t$$

At the front of the net, $\quad x = 16 \text{ ft}$

Then $\quad 16 = (154 \cos\alpha)t$

or

$$t_{\text{enter}} = \frac{8}{77 \cos\alpha}$$

Vertical motion. (Uniformly accelerated motion)

$$y = 0 + (v_y)_0 t - \frac{1}{2}gt^2$$

$$= (154 \sin\alpha)t - \frac{1}{2}gt^2 \quad (g = 32.2 \text{ ft/s}^2)$$

At the front of the net,

$$y_{\text{front}} = (154 \sin\alpha)t_{\text{enter}} - \frac{1}{2}gt_{\text{enter}}^2$$

$$= (154 \sin\alpha)\left(\frac{8}{77 \cos\alpha}\right) - \frac{1}{2}g\left(\frac{8}{77 \cos\alpha}\right)^2$$

$$= 16 \tan\alpha - \frac{32g}{5929 \cos^2\alpha}$$

Now $$\frac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$$

Then $$y_{\text{front}} = 16\tan\alpha - \frac{32g}{5929}(1 + \tan^2\alpha)$$

or $$\tan^2\alpha - \frac{5929}{2g}\tan\alpha + \left(1 + \frac{5929}{32g}y_{\text{front}}\right) = 0$$

Then $$\tan\alpha = \frac{\frac{5929}{2g} \pm \left[\left(-\frac{5929}{2g}\right)^2 - 4\left(1 + \frac{5929}{32g}y_{\text{front}}\right)\right]^{1/2}}{2}$$

or $$\tan\alpha = \frac{5929}{4\times 32.2} \pm \left[\left(-\frac{5929}{4\times 32.2}\right)^2 - \left(1 + \frac{5929}{32\times 32.2}y_{\text{front}}\right)\right]^{1/2}$$

or $$\tan\alpha = 46.0326 \pm [(46.0326)^2 - (1 + 5.7541\, y_{\text{front}})]^{1/2}$$

Now $0 < y_{\text{front}} < 4$ ft so that the positive root will yield values of $\alpha > 45°$ for all values of y_{front}. When the negative root is selected, α increases as y_{front} is increased. Therefore, for $\alpha_{\max}$, set

$$y_{\text{front}} = y_C = 4 \text{ ft}$$

Then $$\tan\alpha = 46.0326 - [(46.0326)^2 - (1 + 5.7541 + 4)]^{1/2}$$

or $$\alpha_{\max} = 14.6604°$$

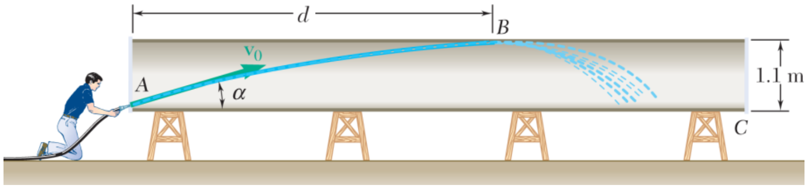
$\alpha_{\max} = 14.66°$ ◀

(*b*) We had found $$t_{\text{enter}} = \frac{8}{77\cos\alpha} = \frac{8}{77\cos 14.6604°}$$

or $t_{\text{enter}} = 0.1074 \text{ s}$ ◀

PROBLEM 11.114

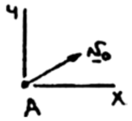
A worker uses high-pressure water to clean the inside of a long drainpipe. If the water is discharged with an initial velocity $\mathbf{v}_0$ of 11.5 m/s, determine (*a*) the distance *d* to the farthest Point *B* on the top of the pipe that the worker can wash from his position at *A*, (*b*) the corresponding angle α.



SOLUTION

First note

$$(v_x)_0 = v_0 \cos\alpha = (11.5 \text{ m/s})\cos\alpha$$
$$(v_y)_0 = v_0 \sin\alpha = (11.5 \text{ m/s})\sin\alpha$$



By observation, $d_{\max}$ occurs when $y_{\max} = 1.1$ m.

Vertical motion. (Uniformly accelerated motion)

$$v_y = (v_y)_0 - gt \qquad y = 0 + (v_y)_0 t - \frac{1}{2}gt^2$$
$$= (11.5 \sin\alpha) - gt \qquad = (11.5 \sin\alpha)t - \frac{1}{2}gt^2$$

When $y = y_{\max}$ at B, $(v_y)_B = 0$

Then $(v_y)_B = 0 = (11.5 \sin\alpha) - gt$

or $t_B = \dfrac{11.5 \sin\alpha}{g} \quad (g = 9.81 \text{ m/s}^2)$

and $y_B = (11.5 \sin\alpha)t_B - \dfrac{1}{2}gt_B^2$

Substituting for t_B and noting $y_B = 1.1$ m

$$1.1 = (11.5 \sin\alpha)\left(\frac{11.5 \sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{11.5 \sin\alpha}{g}\right)^2$$
$$= \frac{1}{2g}(11.5)^2 \sin^2\alpha$$

or $\sin^2\alpha = \dfrac{2.2 \times 9.81}{11.5^2} \qquad \alpha = 23.8265°$

PROBLEM 11.114 (Continued)

(*a*) Horizontal motion. (Uniform)

$$x = 0 + (v_x)_0 t = (11.5 \cos \alpha) t$$

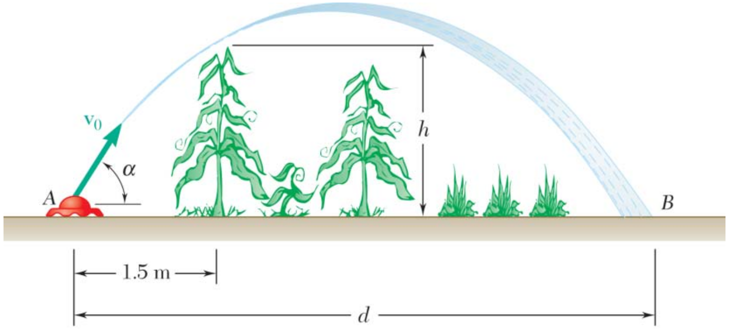
At Point *B*: $x = d_{\max} \quad \text{and} \quad t = t_B$

where $t_B = \dfrac{11.5}{9.81} \sin 23.8265° = 0.47356 \text{ s}$

Then $d_{\max} = (11.5)(\cos 23.8265°)(0.47356)$

or $d_{\max} = 4.98 \text{ m}$ ◀

(*b*) From above $\alpha = 23.8°$ ◀



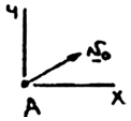
PROBLEM 11.115

An oscillating garden sprinkler which discharges water with an initial velocity $\mathbf{v}_0$ of 8 m/s is used to water a vegetable garden. Determine the distance d to the farthest Point B that will be watered and the corresponding angle α when (*a*) the vegetables are just beginning to grow, (*b*) the height h of the corn is 1.8 m.

SOLUTION

First note $(v_x)_0 = v_0 \cos\alpha = (8 \text{ m/s}) \cos\alpha$

$(v_y)_0 = v_0 \sin\alpha = (8 \text{ m/s}) \sin\alpha$



Horizontal motion. (Uniform)

$$x = 0 + (v_x)_0 t = (8\cos\alpha)t$$

At Point B: $x = d: \quad d = (8\cos\alpha)t$

or $t_B = \dfrac{d}{8\cos\alpha}$

Vertical motion. (Uniformly accelerated motion)

$$y = 0 + (v_y)_0 t - \frac{1}{2}gt^2$$

$$= (8\sin\alpha)t - \frac{1}{2}gt^2 \quad (g = 9.81 \text{ m/s}^2)$$

At Point B: $0 = (8\sin\alpha)t_B - \dfrac{1}{2}gt_B^2$

Simplifying and substituting for t_B

$$0 = 8\sin\alpha - \frac{1}{2}g\left(\frac{d}{8\cos\alpha}\right)$$

or $d = \dfrac{64}{g}\sin 2\alpha$ (1)

(*a*) When $h = 0$, the water can follow any physically possible trajectory. It then follows from Eq. (1) that d is maximum when $2\alpha = 90°$

or $\alpha = 45°$ ◀

Then $d = \dfrac{64}{9.81}\sin(2\times 45°)$

or $d_{\max} = 6.52 \text{ m}$ ◀

(*b*) Based on Eq. (1) and the results of Part *a*, it can be concluded that *d* increases in value as α increases in value from 0 to 45° and then *d* decreases as α is further increased. Thus, $d_{\max}$ occurs for the value of α closest to 45° and for which the water just passes over the first row of corn plants. At this row, $x_{\text{corn}} = 1.5$ m

so that

$$t_{\text{corn}} = \frac{1.5}{8\cos\alpha}$$

Also, with $y_{\text{corn}} = h,$ we have

$$h = (8\sin\alpha)t_{\text{corn}} - \frac{1}{2}gt_{\text{corn}}^2$$

Substituting for t_{corn} and noting $h = 1.8$ m,

$$1.8 = (8\sin\alpha)\left(\frac{1.5}{8\cos\alpha}\right) - \frac{1}{2}g\left(\frac{1.5}{8\cos\alpha}\right)^2$$

or

$$1.8 = 1.5\tan\alpha - \frac{2.25g}{128\cos^2\alpha}$$

Now

$$\frac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$$

Then

$$1.8 = 1.5\tan\alpha - \frac{2.25(9.81)}{128}(1 + \tan^2\alpha)$$

or

$$0.172441\tan^2\alpha - 1.5\tan\alpha + 1.972441 = 0$$

Solving $\alpha = 58.229°$ and $\alpha = 81.965°$

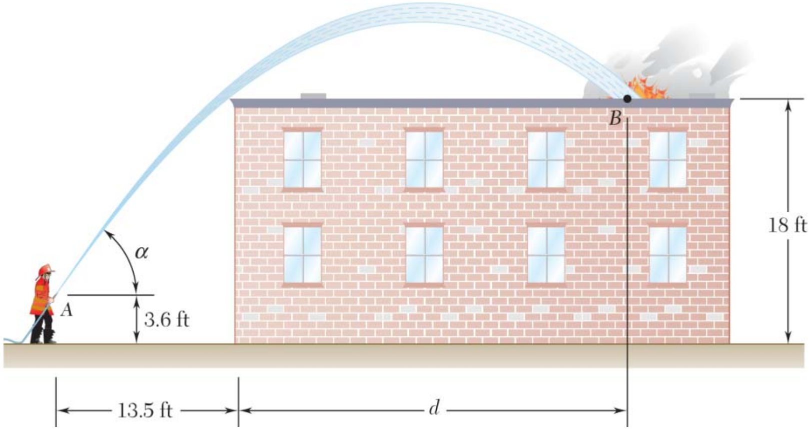
From the above discussion, it follows that $d = d_{\max}$ when

$\alpha = 58.2°$ ◀

Finally, using Eq. (1)

$$d = \frac{64}{9.81}\sin(2 \times 58.229°)$$

or $d_{\max} = 5.84$ m ◀



PROBLEM 11.116*

A nozzle at A discharges water with an initial velocity of 36 ft/s at an angle α with the horizontal. Determine (a) the distance d to the farthest point B on the roof that the water can reach, (b) the corresponding angle α. Check that the stream will clear the edge of the roof.

SOLUTION

Horizontal motion: $x = (v_0 \cos\alpha)t$ or $t = \dfrac{x}{v_0\cos\alpha}$

Vertical motion: $y = y_0 + (v_0 \sin\alpha)t - \dfrac{1}{2}gt^2 = y_0 + x\tan\alpha - \dfrac{1}{2}\dfrac{gx^2}{(v_0\cos\alpha)^2}$

$$y = y_0 + x\tan\alpha - \frac{gx^2}{2v_0^2}\left(1 + \tan^2\alpha\right)$$

Let $u = x\tan\alpha$ so that $y = y_0 + u - \dfrac{g}{2v_0^2}\left(x^2 + u^2\right)$

Solving for x^2: $x^2 = \dfrac{2v_0^2}{g}(u + y_0 - y) - u^2$

The maximum value of x^2 is required: $\dfrac{d\left(x^2\right)}{du} = 0.$

$$\frac{d\left(x^2\right)}{du} = \frac{2v_0^2}{g} - 2u = 0 \quad \text{or} \quad u = \frac{v_0^2}{g}$$

Data: $v_0 = 36$ ft/s, $y_0 = 3.6$ ft, $y_B = 18$ ft $u = \dfrac{(36)^2}{32.2} = 40.2484$ ft

$$(x_{\max})^2 = \frac{(2)(36)^2}{32.2}(40.2484 + 3.6 - 18) - (40.2484)^2 = 460.78 \text{ ft}^2 \qquad x_{\max} = 21.466 \text{ ft}$$

(a) Maximum distance: $d = x_{\max} - 13.5$ $d = 7.97$ ft ◀

PROBLEM 11.116* (Continued)

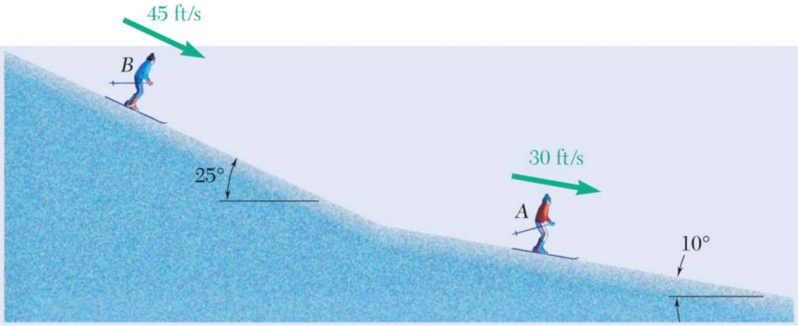
(*b*) Angle α. $$\tan\alpha = \frac{x_{\max}\tan\alpha}{x_{\max}} = \frac{u}{x_{\max}} = \frac{40.2484}{21.466} = 1.875$$

$$\alpha = 61.93°$$ $\alpha = 61.9°$ ◀

Check the edge. $$y = y_0 + x\tan\alpha + \frac{gx^2}{2(v_0\cos\alpha)^2}$$

$$y = 3.6 + (13.5)(1.875) - \frac{(32.2)(13.5)^2}{2[36\cos 61.93°]^2}$$ $y = 18.69$ ft ◀

Since $y > 18$ ft, the stream clears the edge.



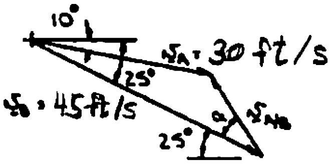
PROBLEM 11.117

The velocities of skiers *A* and *B* are as shown. Determine the velocity of *A* with respect to *B*.

SOLUTION

We have $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$

The graphical representation of this equation is then as shown.



Then $\mathbf{v}_{A/B}^2 = 30^2 + 45^2 - 2(30)(45)\cos 15°$

or $v_{A/B} = 17.80450$ ft/s

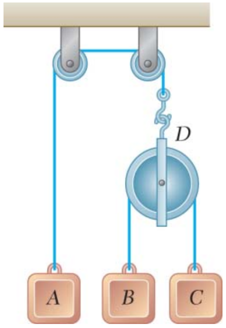
and $\dfrac{30}{\sin\alpha} = \dfrac{17.80450}{\sin 15°}$

or $\alpha = 25.8554°$

$\alpha + 25° = 50.8554°$ $\mathbf{v}_{A/B} = 17.8 \text{ ft/s} \measuredangle 50.9°$ ◀

Alternative solution.

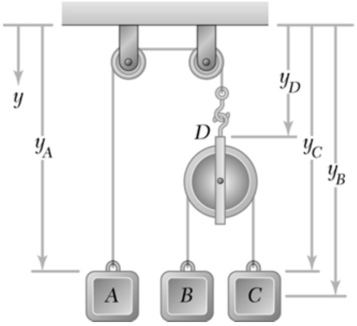
$$\begin{aligned}\mathbf{v}_{A/B} &= \mathbf{v}_A - \mathbf{v}_B\\ &= 30\cos 10°\,\mathbf{i} - 30\sin 10°\mathbf{j} - (45\cos 25°\mathbf{i} - 45°\sin 25°\mathbf{j})\\ &= 11.2396\mathbf{i} + 13.8084\,\mathbf{j}\\ &= 5.05 \text{ m/s} = 17.8 \text{ ft/s} \measuredangle 50.9°\end{aligned}$$



PROBLEM 11.118

The three blocks shown move with constant velocities. Find the velocity of each block, knowing that the relative velocity of *A* with respect to *C* is 300 mm/s upward and that the relative velocity of *B* with respect to *A* is 200 mm/s downward.

SOLUTION



From the diagram

Cable 1: $y_A + y_D = \text{constant}$

Then $v_A + v_D = 0$ (1)

Cable 2: $(y_B - y_D) + (y_C - y_D) = \text{constant}$

Then $v_B + v_C - 2v_D = 0$ (2)

Combining Eqs. (1) and (2) to eliminate v_D,

$$2v_A + v_B + v_C = 0 \quad (3)$$

Now $v_{A/C} = v_A - v_C = -300 \text{ mm/s}$ (4)

and $v_{B/A} = v_B - v_A = 200 \text{ mm/s}$ (5)

Then $(3) + (4) - (5) \Rightarrow$

$$(2v_A + v_B + v_C) + (v_A - v_C) - (v_B - v_A) = (-300) - (200)$$

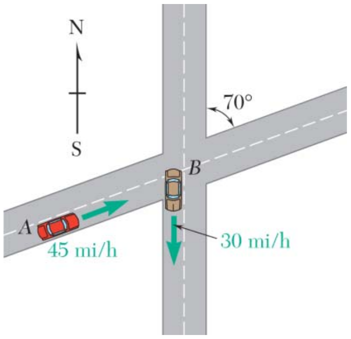
or $\mathbf{v}_A = 125 \text{ mm/s}\uparrow$ ◀

and using Eq. (5) $v_B - (-125) = 200$

or $\mathbf{v}_B = 75 \text{ mm/s}\downarrow$ ◀

Eq. (4) $-125 - v_C = -300$

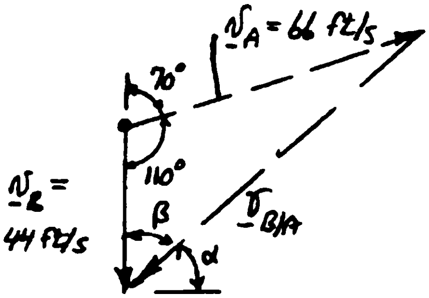
or $\mathbf{v}_C = 175 \text{ mm/s}\downarrow$ ◀



PROBLEM 11.119

Three seconds after automobile *B* passes through the intersection shown, automobile *A* passes through the same intersection. Knowing that the speed of each automobile is constant, determine (*a*) the relative velocity of *B* with respect to *A*, (*b*) the change in position of *B* with respect to *A* during a 4-s interval, (*c*) the distance between the two automobiles 2 s after *A* has passed through the intersection.

SOLUTION



$$v_A = 45 \text{ mi/h} = 66 \text{ ft/s}$$
$$v_B = 30 \text{ mi/h} = 44 \text{ ft/s}$$

Law of cosines

$$v_{B/A}^2 = 66^2 + 44^2 - 2(66)(44)\cos 110°$$
$$v_{B/A} = 90.99 \text{ ft/s}$$

$$v_B = v_A + v_{B/A}$$

Law of sines

$$\frac{\sin\beta}{66} = \frac{\sin 110°}{90.99} \qquad \beta = 42.97°$$
$$\alpha = 90° - \beta = 90° - 42.97° = 47.03°$$

(*a*) Relative velocity: $\mathbf{v}_{B/A} = 91.0$ ft/s ↙ 47.0° ◀

(*b*) Change in position for $\Delta t = 4$ s.

$$\Delta r_{B/A} = v_{B/A}\Delta t = (91.0 \text{ ft/s})(4 \text{ s})$$

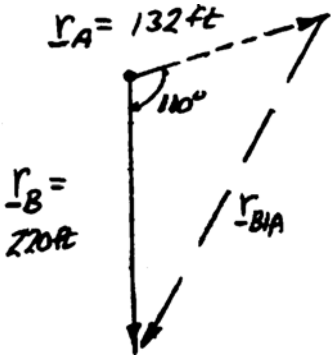
$\mathbf{r}_{B/A} = 364$ ft ↙ 47.0° ◀

(*c*) Distance between autos 2 seconds after auto *A* has passed intersection.

Auto *A* travels for 2 s.

$$v_A = 66 \text{ ft/s} \measuredangle 20°$$
$$r_A = v_A t = (66 \text{ ft/s})(2 \text{ s}) = 132 \text{ ft}$$
$$\mathbf{r}_A = 132 \text{ ft} \measuredangle 20°$$

PROBLEM 11.119 (Continued)


$\underline{r}_A = 132$ ft
110°
$\underline{r}_B =$ 220ft
$\underline{r}_{B/A}$

Auto *B*

$$\mathbf{v}_B = 44 \text{ ft/s}\downarrow$$

$$\mathbf{r}_B = \mathbf{v}_B t = (44 \text{ ft/s})(5 \text{ s}) = 220 \text{ ft}\downarrow$$

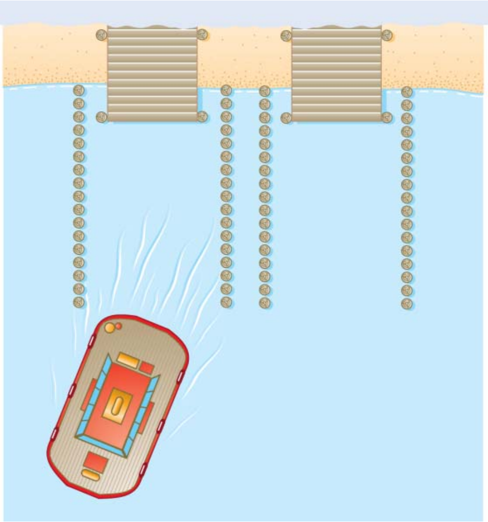
$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

Law of cosines

$$r_{B/A}^2 = (132)^2 + (220)^2 - 2(132)(220)\cos 110°$$

$$\mathbf{r}_{B/A} = 292.7 \text{ ft}$$

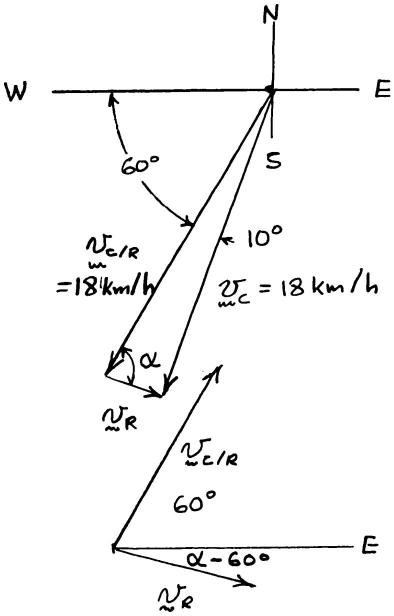
Distance between autos = 293 ft ◀



PROBLEM 11.120

Shore-based radar indicates that a ferry leaves its slip with a velocity $\mathbf{v} = 18$ km/h ↙ 70°, while instruments aboard the ferry indicate a speed of 18.4 km/h and a heading of 30° west of south relative to the river. Determine the velocity of the river.

SOLUTION



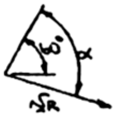
We have $\quad \mathbf{v}_F = \mathbf{v}_R + \mathbf{v}_{F/R} \quad$ or $\quad \mathbf{v}_F = \mathbf{v}_{F/R} + \mathbf{v}_R$

The graphical representation of the second equation is then as shown.

We have $\quad v_R^2 = 18^2 + 18.4^2 - 2(18)(18.4)\cos 10°$

or $\quad v_R = 3.1974$ km/h

and $\quad \dfrac{18}{\sin\alpha} = \dfrac{3.1974}{\sin 10°}$

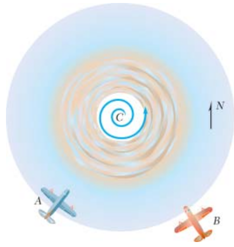


or $\quad \alpha = 77.84°$

Noting that

$\mathbf{v}_R = 3.20$ km/h ↘ 17.8° ◀

Alternatively one could use vector algebra.



PROBLEM 11.121

Airplanes A and B are flying at the same altitude and are tracking the eye of hurricane C. The relative velocity of C with respect to A is $\mathbf{v}_{C/A} = 350$ km/h ↙ 75°, and the relative velocity of C with respect to B is $\mathbf{v}_{C/B} = 400$ km/h ↘ 40°. Determine (a) the relative velocity of B with respect to A, (b) the velocity of A if ground-based radar indicates that the hurricane is moving at a speed of 30 km/h due north, (c) the change in position of C with respect to B during a 15-min interval.

SOLUTION

(a) We have $\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{C/A}$

and $\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{C/B}$

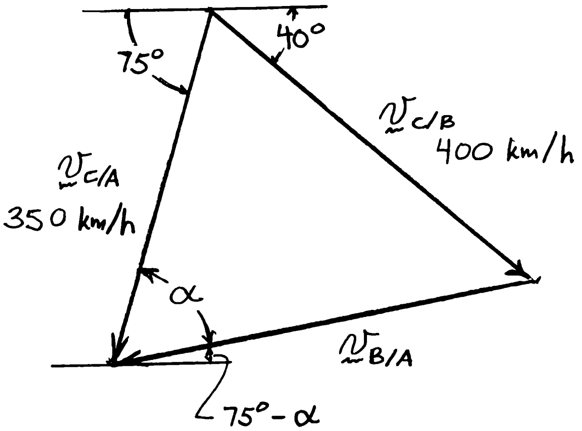
Then $\mathbf{v}_A + \mathbf{v}_{C/A} = \mathbf{v}_B + \mathbf{v}_{C/B}$

or $\mathbf{v}_B - \mathbf{v}_A = \mathbf{v}_{C/A} - \mathbf{v}_{C/B}$

Now $\mathbf{v}_B - \mathbf{v}_A = \mathbf{v}_{B/A}$

so that $\mathbf{v}_{B/A} = \mathbf{v}_{C/A} - \mathbf{v}_{C/B}$

or $\mathbf{v}_{C/A} = \mathbf{v}_{C/B} + \mathbf{v}_{B/A}$



The graphical representation of the last equation is then as shown.

We have $v_{B/A}^2 = 350^2 + 400^2 - 2(350)(400)\cos 65°$

or $v_{B/A} = 405.175$ km/h

and $\dfrac{400}{\sin\alpha} = \dfrac{405.175}{\sin 65°}$

or $\alpha = 63.474°$

$75° - \alpha = 11.526°$

$\mathbf{v}_{B/A} = 405$ km/h ↙ 11.53° ◀

PROBLEM 11.121 (Continued)

(*b*) We have $\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{C/A}$

or $\mathbf{v}_A = (30\text{ km/h})\mathbf{j} - (350\text{ km/h})(-\cos 75°\mathbf{i} - \sin 75°\mathbf{j})$

$$\mathbf{v}_A = (90.587\text{ km/h})\mathbf{i} + (368.07\text{ km/h})\mathbf{j}$$

or $\mathbf{v}_A = 379$ km/h ↗ 76.17° ◀

(*c*) Noting that the velocities of *B* and *C* are constant, we have

$$\mathbf{r}_B = (\mathbf{r}_B)_0 + \mathbf{v}_B t \qquad \mathbf{r}_C = (\mathbf{r}_C)_0 + \mathbf{v}_C t$$

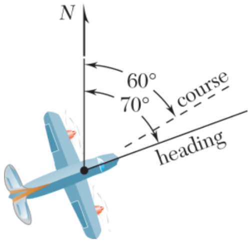
Now $\mathbf{r}_{C/B} = \mathbf{r}_C - \mathbf{r}_B = [(\mathbf{r}_C)_0 - (\mathbf{r}_B)_0] + (\mathbf{v}_C - \mathbf{v}_B)t$

$$= [(\mathbf{r}_C)_0 - (\mathbf{r}_B)_0] + \mathbf{v}_{C/B}t$$

Then $\Delta\mathbf{r}_{C/B} = (\mathbf{r}_{C/B})_{t_2} - (\mathbf{r}_{C/B})_{t_1} = \mathbf{v}_{C/B}(t_2 - t_1) = \mathbf{v}_{C/B}\Delta t$

For $\Delta t = 15$ min: $\Delta r_{C/B} = (400\text{ km/h})\left(\frac{1}{4}\text{h}\right) = 100\text{ km}$

$\Delta\mathbf{r}_{C/B} = 100$ km ↘ 40° ◀



PROBLEM 11.122

Instruments in an airplane which is in level flight indicate that the velocity relative to the air (airspeed) is 120 km/h and the direction of the relative velocity vector (heading) is 70° east of north. Instruments on the ground indicate that the velocity of the airplane (ground speed) is 110 km/h and the direction of flight (course) is 60° east of north. Determine the wind speed and direction.

SOLUTION

Let **i** and **j** be unit vectors in directions east and north respectively.

Velocity of plane relative to air.

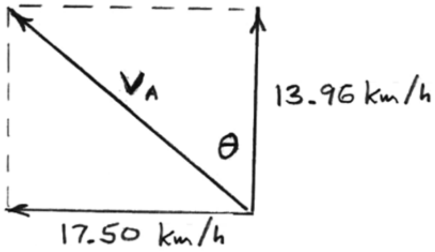
$$\mathbf{v}_{P/A} = (120 \text{ km/h})(\cos 20°\,\mathbf{i} + \sin 20°\,\mathbf{j})$$

Velocity of plane.

$$\mathbf{v}_P = (110 \text{ km/h})(\cos 30°\,\mathbf{i} + \sin 30°\,\mathbf{j})$$

But $\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{P/A}$

Velocity of air. $\mathbf{v}_A = \mathbf{v}_P - \mathbf{v}_{P/A}$



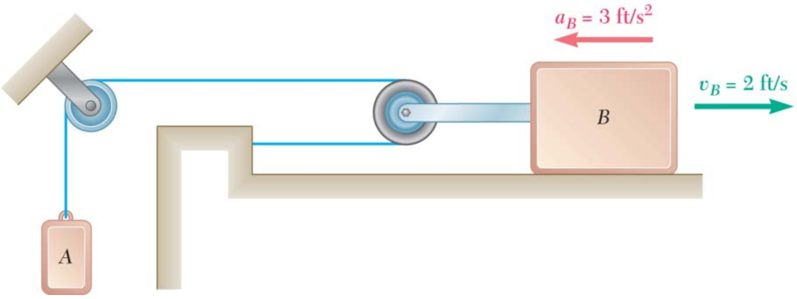
$$v_A = (110\cos 30° - 120\cos 20°)\mathbf{i} + (110\sin 30° - 120\sin 20°)\,\mathbf{j}$$

$$= (-17.50 \text{ km/h})\mathbf{i} + (13.96 \text{ km/h})\,\mathbf{j}$$

$$v_A = \sqrt{(17.50)^2 + (13.96)^2} = 22.4 \text{ km/h}$$

$$\tan\theta = \frac{17.50}{13.96} \qquad \theta = 51.4°$$

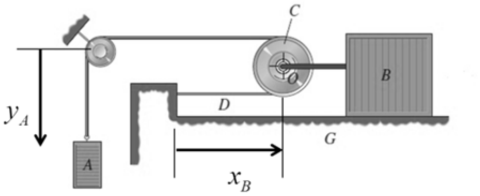
$\mathbf{v}_A = 22.4$ km/h at 51.4° west of north ◀



Problem 11.123

Knowing that at the instant shown block *B* has a velocity of 2 ft/s to the right and an acceleration of 3 ft/s^2 to the left, determine (*a*) the velocity of block *A*, (*b*) the acceleration of block A.

SOLUTION



Given: $v_B = 2 \text{ ft/s}, a_B = -3 \text{ ft/s}$

Length of Cable $L = y_A + 2x_B + \text{ constants}$

Differentiate Length Equation

$$0 = v_A + 2v_B \qquad (1)$$

Differentiate again $0 = a_A + 2a_B \qquad (2)$

(*e*) Rearrange (1)

$$v_A = -2v_B$$
$$= -2(2) \text{ ft/s}$$

$$v_A = -4 \text{ ft/s}$$

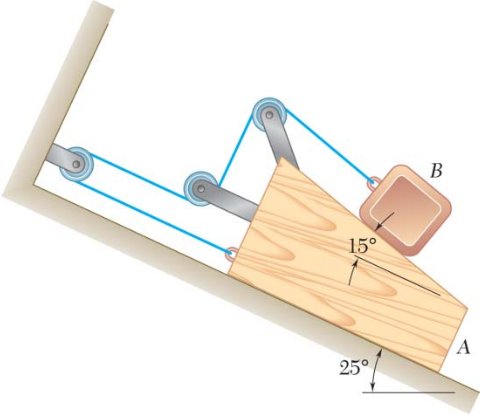
$$v_A = 4 \text{ ft/s} \uparrow \blacktriangleleft$$

(*f*) Rearrange (2)

$$a_A = -2a_B$$
$$= -2(-3) \text{ ft/s}^2$$

$$a_A = 6 \text{ ft/s}^2$$

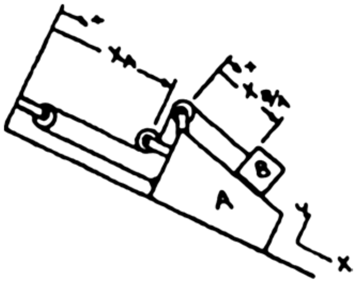
$$a_A = 6 \text{ ft/s}^2 \downarrow \blacktriangleleft$$



PROBLEM 11.124

Knowing that at the instant shown block A has a velocity of 8 in./s and an acceleration of 6 in./s^2 both directed down the incline, determine (a) the velocity of block B, (b) the acceleration of block B.

SOLUTION



From the diagram $\qquad 2x_A + x_{B/A} = \text{constant}$

Then $\qquad 2v_A + v_{B/A} = 0$

or $\qquad |v_{B/A}| = 16 \text{ in./s}$

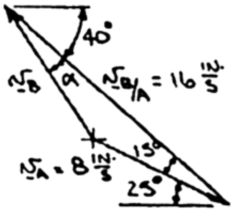
and $\qquad 2a_A + a_{B/A} = 0$

or $\qquad |a_{B/A}| = 12 \text{ in./s}^2$

Note that $\mathbf{v}_{B/A}$ and $\mathbf{a}_{B/A}$ must be parallel to the top surface of block A.

(a) We have $\qquad \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

The graphical representation of this equation is then as shown. Note that because A is moving downward, B must be moving upward relative to A.


40°
v_B
α
$v_{B/A} = 16 \frac{\text{in.}}{\text{s}}$
$v_A = 8 \frac{\text{in.}}{\text{s}}$
15°
25°

We have $\qquad v_B^2 = 8^2 + 16^2 - 2(8)(16)\cos 15°$

or $\qquad v_B = 8.5278 \text{ in./s}$

and $\qquad \dfrac{8}{\sin\alpha} = \dfrac{8.5278}{\sin 15°}$

or $\qquad \alpha = 14.05°$

$\mathbf{v}_B = 8.53$ in./s ∠ 54.1° ◀

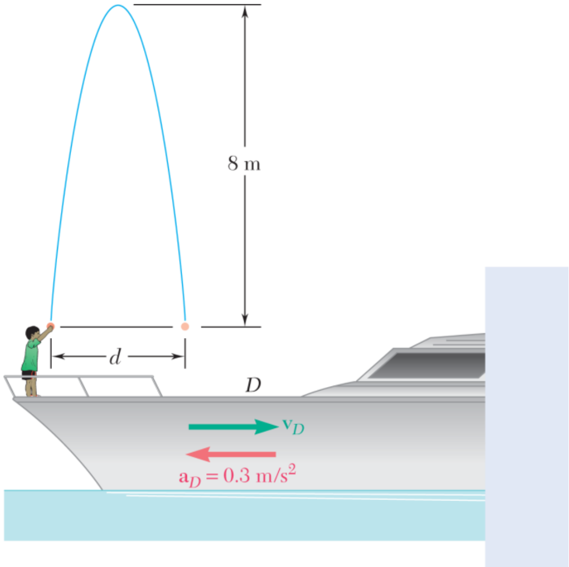
(b) The same technique that was used to determine $\mathbf{v}_B$ can be used to determine $\mathbf{a}_B$. An alternative method is as follows.

We have

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$
$$= (6\mathbf{i}) + 12(-\cos 15°\mathbf{i} + \sin 15°\mathbf{j})^*$$
$$= -(5.5911 \text{ in./s}^2)\mathbf{i} + (3.1058 \text{ in./s}^2)\mathbf{j}$$

or $\qquad \mathbf{a}_B = 6.40$ in./s^2 ∠ 54.1° ◀

* Note the orientation of the coordinate axes on the sketch of the system.



PROBLEM 11.125

A boat is moving to the right with a constant deceleration of 0.3 m/s^2 when a boy standing on the deck *D* throws a ball with an initial velocity relative to the deck which is vertical. The ball rises to a maximum height of 8 m above the release point and the boy must step forward a distance *d* to catch it at the same height as the release point. Determine (*a*) the distance *d*, (*b*) the relative velocity of the ball with respect to the deck when the ball is caught.

SOLUTION

Horizontal motion of the ball: $v_x = (v_x)_0, \quad x_{\text{ball}} = (v_x)_0 t$

Vertical motion of the ball: $v_y = (v_y)_0 - gt$

$$y_B = (v_y)_0 t - \frac{1}{2}gt^2, \quad (v_y)^2 - (v_y)_0^2 = -2gy$$

At maximum height, $v_y = 0$ and $y = y_{\max}$

$$(v_y)^2 = 2gy_{\max} = (2)(9.81)(8) = 156.96 \text{ m}^2/\text{s}^2$$

$$(v_y)_0 = 12.528 \text{ m/s}$$

At time of catch, $y = 0 = 12.528 - \frac{1}{2}(9.81)t^2$

or $t_{\text{catch}} = 2.554$ s and $v_y = 12.528$ m/s ↓

Motion of the deck: $v_x = (v_x)_0 + a_D t, \quad x_{\text{deck}} = (v_x)_0 t + \frac{1}{2}a_D t^2$

Motion of the ball relative to the deck:

$$(v_{B/D})_x = (v_x)_0 - [(v_x)_0 + a_D t] = -a_D t$$

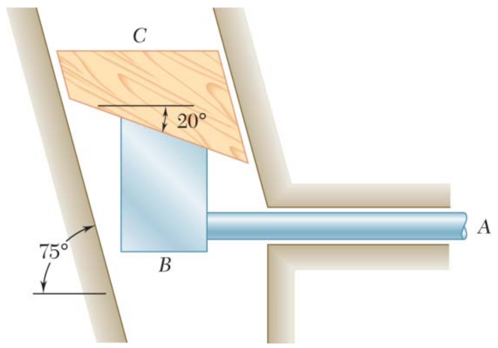
$$x_{B/D} = (v_x)_0 t - \left[(v_x)_0 t + \frac{1}{2}a_D t^2\right] = -\frac{1}{2}a_D t^2$$

$$(v_{B/D})_y = (v_y)_0 - gt, \quad y_{B/D} = y_B$$

(*a*) At time of catch, $d = x_{D/B} = -\frac{1}{2}(-0.3)(2.554)^2$ $d = 0.979$ m ◀

(*b*) $(v_{B/D})_x = -(-0.3)(2.554) = +0.766$ m/s or 0.766 m/s →

$(v_{B/D})_y = 12.528$ m/s ↓ $\mathbf{v}_{B/D} = 12.55$ m/s ↘ 86.5° ◀



PROBLEM 11.126

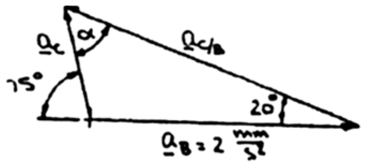
The assembly of rod *A* and wedge *B* starts from rest and moves to the right with a constant acceleration of 2 mm/s^2. Determine (*a*) the acceleration of wedge *C*, (*b*) the velocity of wedge *C* when $t = 10$ s.

SOLUTION

(*a*) We have

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}$$

The graphical representation of this equation is then as shown.



First note

$$\alpha = 180° - (20° + 105°)$$
$$= 55°$$

Then

$$\frac{a_C}{\sin 20°} = \frac{2}{\sin 55°}$$

$$a_C = 0.83506\ \text{mm/s}^2$$

$\mathbf{a}_C = 0.835\ \text{mm/s}^2$ ↘ 75° ◀

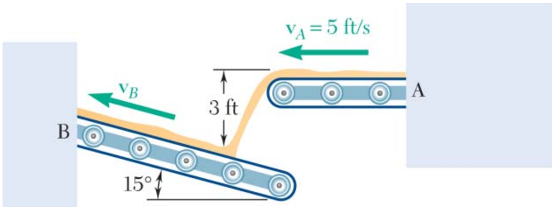
(*b*) For uniformly accelerated motion

$$v_C = 0 + a_C t$$

At $t = 10$ s:

$$v_C = (0.83506\ \text{mm/s}^2)(10\ \text{s})$$
$$= 8.3506\ \text{mm/s}$$

or $\mathbf{v}_C = 8.35\ \text{mm/s}$ ↘ 75° ◀



PROBLEM 11.127

Determine the required velocity of the belt *B* if the relative velocity with which the sand hits belt *B* is to be (*a*) vertical, (*b*) as small as possible.

SOLUTION

A grain of sand will undergo projectile motion.

$$v_{s_x} = v_{s_{x_0}} = \text{constant} = -5 \text{ ft/s}$$

y-direction. $$v_{s_y} = \sqrt{2gh} = \sqrt{(2)(32.2 \text{ ft/s}^2)(3 \text{ ft})} = 13.90 \text{ ft/s} \downarrow$$

Relative velocity. $$\mathbf{v}_{S/B} = \mathbf{v}_S - \mathbf{v}_B \qquad (1)$$

(*a*) If $v_{S/B}$ is vertical,

$$\begin{aligned} -v_{S/B}\mathbf{j} &= -5\mathbf{i} - 13.9\mathbf{j} - (-v_B \cos 15°\mathbf{i} + v_B \sin 15°\mathbf{j}) \\ &= -5\mathbf{i} - 13.9\mathbf{j} + v_B \cos 15°\mathbf{i} - v_B \sin 15°\mathbf{j} \end{aligned}$$

Equate components. $\mathbf{i}$: $0 = -5 + v_B \cos 15° \qquad v_B = \dfrac{5}{\cos 15°} = 5.176 \text{ ft/s}$

$$\mathbf{v}_B = 5.18 \text{ ft/s} \nwarrow 15° \blacktriangleleft$$

(*b*) $v_{S/C}$ is as small as possible, so make $v_{S/B} \perp$ to v_B into (1).

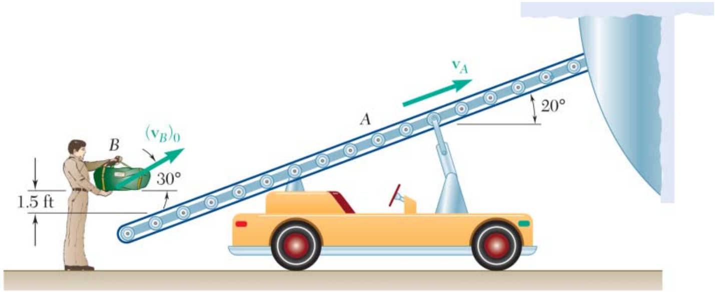
$$-v_{S/B} \sin 15°\mathbf{i} - v_{S/B} \cos 15°\mathbf{j} = -5\mathbf{i} - 13.9\mathbf{j} + v_B \cos 15°\mathbf{i} - v_B \sin 15°\mathbf{j}$$

Equate components and transpose terms.

$$(\sin 15°)\, v_{S/B} + (\cos 15°)\, v_B = 5$$
$$(\cos 15°)\, v_{S/B} - (\sin 15°)\, v_B = 13.90$$

Solving, $\qquad v_{S/B} = 14.72 \text{ ft/s} \qquad v_B = 1.232 \text{ ft/s}$

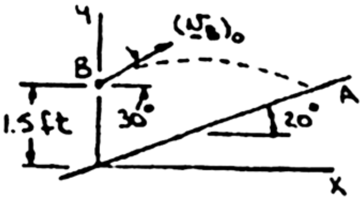
$$\mathbf{v}_B = 1.232 \text{ ft/s} \nwarrow 15° \blacktriangleleft$$



PROBLEM 11.128

Conveyor belt *A*, which forms a 20° angle with the horizontal, moves at a constant speed of 4 ft/s and is used to load an airplane. Knowing that a worker tosses duffel bag *B* with an initial velocity of 2.5 ft/s at an angle of 30° with the horizontal, determine the velocity of the bag relative to the belt as it lands on the belt.

SOLUTION



First determine the velocity of the bag as it lands on the belt. Now

$$[(v_B)_x]_0 = (v_B)_0 \cos 30^\circ$$
$$= (2.5 \text{ ft/s})\cos 30^\circ$$
$$[(v_B)_y]_0 = (v_B)_0 \sin 30^\circ$$
$$= (2.5 \text{ ft/s})\sin 30^\circ$$

Horizontal motion. (Uniform)

$$x = 0 + [(v_B)_x]_0\, t \qquad (v_B)_x = [(v_B)_x]_0$$
$$= (2.5 \cos 30^\circ)t \qquad = 2.5 \cos 30^\circ$$

Vertical motion. (Uniformly accelerated motion)

$$y = y_0 + [(v_B)_y]_0\, t - \frac{1}{2}gt^2 \qquad (v_B)_y = [(v_B)_y]_0 - gt$$
$$= 1.5 + (2.5 \sin 30^\circ)t - \frac{1}{2}gt^2 \qquad = 2.5 \sin 30^\circ - gt$$

The equation of the line collinear with the top surface of the belt is

$$y = x \tan 20^\circ$$

Thus, when the bag reaches the belt

$$1.5 + (2.5 \sin 30^\circ)t - \frac{1}{2}gt^2 = [(2.5 \cos 30^\circ)t] \tan 20^\circ$$

or
$$\frac{1}{2}(32.2)t^2 + 2.5(\cos 30^\circ \tan 20^\circ - \sin 30^\circ)t - 1.5 = 0$$

or
$$16.1t^2 - 0.46198t - 1.5 = 0$$

Solving $t = 0.31992$ s and $t = -0.29122$ s (Reject)

PROBLEM 11.128 (Continued)

The velocity $\mathbf{v}_B$ of the bag as it lands on the belt is then

$$\begin{aligned}\mathbf{v}_B &= (2.5\cos 30°)\mathbf{i} + [2.5\sin 30° - 32.2(0.319\,92)]\mathbf{j}\\ &= (2.1651\text{ ft/s})\mathbf{i} - (9.0514\text{ ft/s})\mathbf{j}\end{aligned}$$

Finally $$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

or $$\begin{aligned}\mathbf{v}_{B/A} &= (2.1651\mathbf{i} - 9.0514\mathbf{j}) - 4(\cos 20°\mathbf{i} + \sin 20°\mathbf{j})\\ &= -(1.59367\text{ ft/s})\mathbf{i} - (10.4195\text{ ft/s})\mathbf{j}\end{aligned}$$

or $$\mathbf{v}_{B/A} = 10.54\text{ ft/s} \swarrow 81.3° \blacktriangleleft$$

PROBLEM 11.129

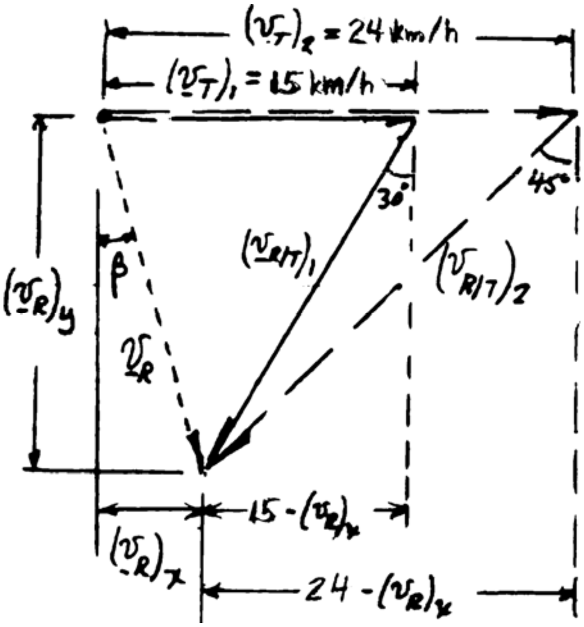
During a rainstorm the paths of the raindrops appear to form an angle of 30° with the vertical and to be directed to the left when observed from a side window of a train moving at a speed of 15 km/h. A short time later, after the speed of the train has increased to 24 km/h, the angle between the vertical and the paths of the drops appears to be 45°. If the train were stopped, at what angle and with what velocity would the drops be observed to fall?

SOLUTION

$$v_{\text{rain}} = v_{\text{train}} + v_{\text{rain/train}}$$

Case ①: $v_T = 15\text{ km/h} \longrightarrow; \quad v_{R/T} \swarrow 30^\circ$

Case ②: $v_T = 24\text{ km/h} \longrightarrow; \quad v_{R/T} \swarrow 45^\circ$



Case ①: $(v_R)_y \tan 30 = 15 - (v_R)_x$ (1)

Case ②: $(v_R)_y \tan 45^\circ = 24 - (v_R)_x$ (2)

Substract (1) from (2) $(v_R)_y (\tan 45^\circ - \tan 30^\circ) = 9$

$$(v_R)_y = 21.294\text{ km/h}$$

Eq. (2): $21.294 \tan 45^\circ = 25 - (v_R)_x$

$$(v_R)_x = 2.706\text{ km/h}$$

$$\tan\beta = \frac{3.706}{21.294}$$

$$\beta = 7.24^\circ$$

$$v_R = \frac{21.294}{\cos 7.24^\circ} = 21.47 \text{ km/h} = 5.96 \text{ m/s}$$

$v_R = 5.96 \text{ m/s} \searrow 82.8^\circ$ ◀

Alternate solution

Alternate, vector equation $\mathbf{v}_R = \mathbf{v}_T + \mathbf{v}_{R/T}$

For first case, $\mathbf{v}_R = 15\mathbf{i} + v_{R/T-1}(-\sin 30^\circ\mathbf{i} - \cos 30^\circ\mathbf{j})$

For second case, $\mathbf{v}_R = 24\mathbf{i} + v_{R/T-2}(-\sin 45^\circ\mathbf{i} - \cos 45^\circ\mathbf{j})$

Set equal

$$15\mathbf{i} + v_{R/T-1}(-\sin 30^\circ\mathbf{i} - \cos 30^\circ\mathbf{j}) = 24\mathbf{i} + v_{R/T-2}(-\sin 45^\circ\mathbf{i} - \cos 45^\circ\mathbf{j})$$

Separate into components:

i: $15 - v_{R/T-1}\sin 30^\circ = 24 - v_{R/T-2}\sin 45^\circ$

$$-v_{R/T-1}\sin 30^\circ + v_{R/T-2}\sin 45^\circ = 9 \qquad (3)$$

j: $-v_{R/T-1}\cos 30^\circ = -v_{R/T-2}\cos 45^\circ$

$$v_{R/T-1}\cos 30^\circ + v_{R/T-2}\cos 45^\circ = 0 \qquad (4)$$

Solving Eqs. (3) and (4) simultaneously,

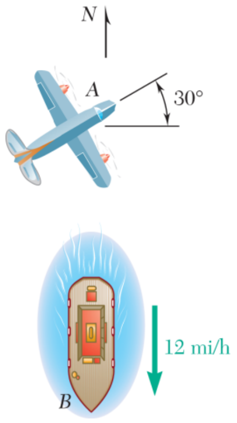
$v_{R/T-1} = 24.5885 \text{ km/h}$ $\qquad v_{R/T-2} = 30.1146 \text{ km/h}$

Substitute $\mathbf{v}_{R/T-2}$ back into equation for $\mathbf{v}_R$.

$$\mathbf{v}_R = 24\mathbf{i} + 30.1146(-\sin 45^\circ\mathbf{i} - \cos 45^\circ\mathbf{j})$$

$\mathbf{v}_R = 2.71\mathbf{i} - 21.29\mathbf{j}$ $\qquad \mathbf{v}_R = 21.4654 \text{ km/hr} = 5.96 \text{ m/s}$

$\theta = \tan^{-1}\left(\dfrac{-21.29}{2.71}\right) = -82.7585^\circ$ $\qquad \mathbf{v}_R = 5.96 \text{ m/s} \searrow 82.8^\circ$ ◀



PROBLEM 11.130

Instruments in airplane A indicate that with respect to the air the plane is headed 30° north of east with an airspeed of 300 mi/h. At the same time radar on ship B indicates that the relative velocity of the plane with respect to the ship is 280 mi/h in the direction 33° north of east. Knowing that the ship is steaming due south at 12 mi/h, determine (*a*) the velocity of the airplane, (*b*) the wind speed and direction.

SOLUTION

Let the x-axis be directed east, and the y-axis be directed north.

Airspeed: $\mathbf{v}_{A/W} = 300 \text{ mi/h} \measuredangle 30^\circ = 300(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) \text{ mi/h}$

Plane relative to ship: $\mathbf{v}_{A/B} = 280(\cos 33^\circ \mathbf{i} + \sin 33^\circ \mathbf{j}) \text{ mi/h}$

Ship: $\mathbf{v}_B = 12 \text{ mi/h} \downarrow = -12\mathbf{j}$

(*a*) Velocity of airplane.

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} = -12\mathbf{j} + 280(\cos 33^\circ \mathbf{i} + \sin 33^\circ \mathbf{j})$$

$$= 234.83\mathbf{i} + 140.50\mathbf{j}$$

$\mathbf{v}_A = 274 \text{ mi/h} \measuredangle 30.9^\circ$ ◀

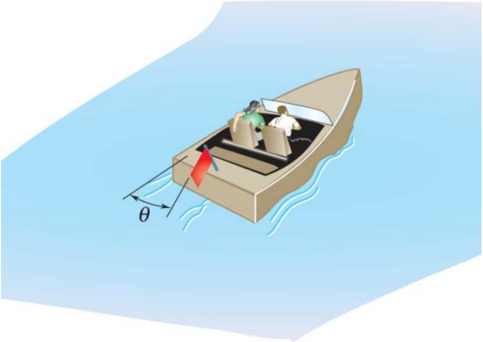
(*b*) Wind velocity.

$$\mathbf{v}_W = \mathbf{v}_A + \mathbf{v}_{W/A} = \mathbf{v}_A - \mathbf{v}_{A/W}$$

$$= 234.83\mathbf{i} + 140.50\mathbf{j} - 300(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$$

$$= -24.98\mathbf{i} - 9.50\mathbf{j}$$

$\mathbf{v}_W = 26.7 \text{ mi/h} \swarrow 20.8^\circ$ ◀



PROBLEM 11.131

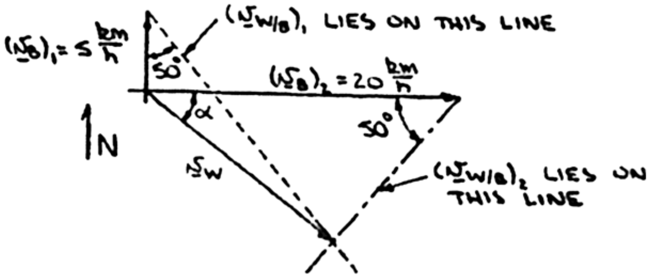
When a small boat travels north at 5 km/h, a flag mounted on its stern forms an angle $\theta = 50°$ with the centerline of the boat as shown. A short time later, when the boat travels east at 20 km/h, angle θ is again 50°. Determine the speed and the direction of the wind.

SOLUTION

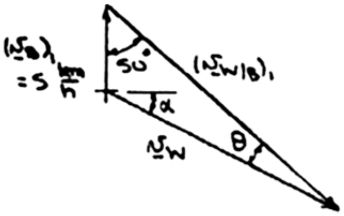
We have

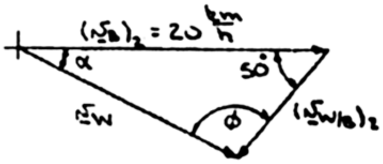
$$\mathbf{v}_W = \mathbf{v}_B + \mathbf{v}_{W/B}$$

Using this equation, the two cases are then graphically represented as shown.


(v_B)_1 = 5 km/h
(v_W/B)_1 LIES ON THIS LINE
50°
(v_B)_2 = 20 km/h
N
α
50°
v_W
(v_W/B)_2 LIES ON THIS LINE

With $\mathbf{v}_W$ now defined, the above diagram is redrawn for the two cases for clarity.


(v_B)_1 = 5 km/h
50°
(v_W/B)_1
α
v_W
θ


(v_B)_2 = 20 km/h
α
50°
v_W
φ
(v_W/B)_2

Noting that

$$\theta = 180° - (50° + 90° + \alpha) \qquad \phi = 180° - (50° + \alpha)$$
$$= 40° - \alpha \qquad\qquad = 130° - \alpha$$

We have

$$\frac{v_W}{\sin 50°} = \frac{5}{\sin(40° - \alpha)} \qquad \frac{v_W}{\sin 50°} = \frac{20}{\sin(130° - \alpha)}$$

PROBLEM 11.131 (Continued)

Therefore
$$\frac{5}{\sin(40^\circ - \alpha)} = \frac{20}{\sin(130^\circ - \alpha)}$$

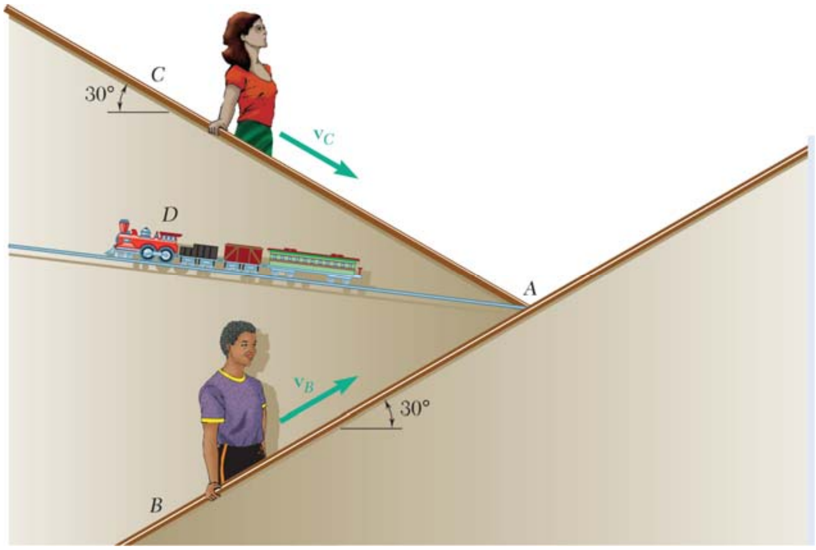
or
$$\sin 130^\circ \cos \alpha - \cos 130^\circ \sin \alpha = 4(\sin 40^\circ \cos \alpha - \cos 40^\circ \sin \alpha)$$

or
$$\tan \alpha = \frac{\sin 130^\circ - 4 \sin 40^\circ}{\cos 130^\circ - 4 \cos 40^\circ}$$

or
$$\alpha = 25.964^\circ$$

Then
$$v_W = \frac{5 \sin 50^\circ}{\sin(40^\circ - 25.964^\circ)} = 15.79 \text{ km/h}$$

$\mathbf{v}_W = 15.79$ km/h $\searrow$ 26.0° ◀


C
30°
$\mathbf{v}_C$
D
A
$\mathbf{v}_B$
30°
B

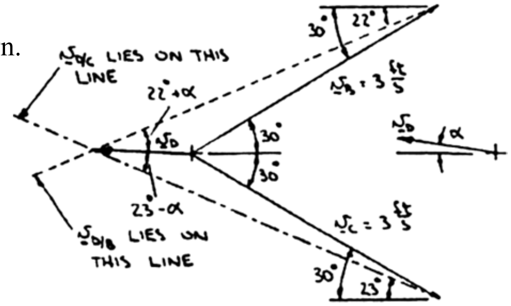
PROBLEM 11.132

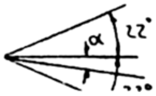
As part of a department store display, a model train *D* runs on a slight incline between the store's up and down escalators. When the train and shoppers pass Point *A*, the train appears to a shopper on the up escalator *B* to move downward at an angle of 22° with the horizontal, and to a shopper on the down escalator *C* to move upward at an angle of 23° with the horizontal and to travel to the left. Knowing that the speed of the escalators is 3 ft/s, determine the speed and the direction of the train.

SOLUTION

We have
$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$
$$\mathbf{v}_D = \mathbf{v}_C + \mathbf{v}_{D/C}$$

The graphical representations of these equations are then as shown.


$\mathbf{v}_{D/C}$ LIES ON THIS LINE
22°+α
$\mathbf{v}_D$
30°
22°
$\mathbf{v}_B = 3\frac{ft}{s}$
30°
30°
$\mathbf{v}_D$
α
23°-α
$\mathbf{v}_{D/B}$ LIES ON THIS LINE
$\mathbf{v}_C = 3\frac{ft}{s}$
30°
23°



Then $\dfrac{v_D}{\sin 8^\circ} = \dfrac{3}{\sin(22^\circ + \alpha)}$ $\qquad \dfrac{v_D}{\sin 7^\circ} = \dfrac{3}{\sin(23^\circ - \alpha)}$

Equating the expressions for $\dfrac{v_D}{3}$

$$\frac{\sin 8^\circ}{\sin(22^\circ + \alpha)} = \frac{\sin 7^\circ}{\sin(23^\circ - \alpha)}$$

or $\sin 8^\circ(\sin 23^\circ \cos\alpha - \cos 23^\circ \sin\alpha)$
$$= \sin 7^\circ(\sin 22^\circ \cos\alpha + \cos 22^\circ \sin\alpha)$$

or
$$\tan\alpha = \frac{\sin 8^\circ \sin 23^\circ - \sin 7^\circ \sin 22^\circ}{\sin 8^\circ \cos 23^\circ + \sin 7^\circ \cos 22^\circ}$$

or
$$\alpha = 2.0728^\circ$$

Then
$$v_D = \frac{3 \sin 8^\circ}{\sin(22^\circ + 2.0728^\circ)} = 1.024 \text{ ft/s}$$

$\mathbf{v}_D = 1.024$ ft/s ↘ 2.07° ◀

Alternate solution using components.

$$\mathbf{v}_B = (3\text{ ft/s}) \nearrow 30^\circ = (2.5981\text{ ft/s})\mathbf{i} + (1.5\text{ ft/s})\mathbf{j}$$

$$\mathbf{v}_C = (3\text{ ft/s}) \searrow 30^\circ = (2.5981\text{ ft/s})\mathbf{i} - (1.5\text{ ft/s})\mathbf{j}$$

$$\mathbf{v}_{D/B} = u_1 \swarrow 22^\circ = -(u_1 \cos 22^\circ)\mathbf{i} - (u_1 \sin 22^\circ)\mathbf{j}$$

$$\mathbf{v}_{D/C} = u_2 \nwarrow 23^\circ = -(u_2 \cos 23^\circ)\mathbf{i} + (u_2 \sin 23^\circ)\mathbf{j}$$

$$\mathbf{v}_D = v_D \nwarrow \alpha = -(v_D \cos \alpha)\mathbf{i} + (v_D \sin \alpha)\mathbf{j}$$

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B} = \mathbf{v}_C + \mathbf{v}_{D/C}$$

$$2.5981\mathbf{i} + 1.5\mathbf{j} - (u_1 \cos 22^\circ)\mathbf{i} - (u_1 \sin 22^\circ)\mathbf{j} = 2.5981\mathbf{i} - 1.5\mathbf{j} - (u_2 \cos 23^\circ)\mathbf{i} + (u_2 \sin 23^\circ)\mathbf{j}$$

Separate into components, transpose, and change signs.

$$u_1 \cos 22^\circ - u_2 \cos 23^\circ = 0$$

$$u_1 \sin 22^\circ + u_1 \sin 23^\circ = 3$$

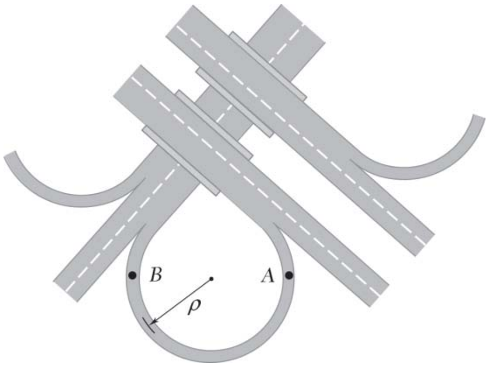
Solving for u_1 and u_2, $\quad u_1 = 3.9054\text{ ft/s} \qquad u_2 = 3.9337\text{ ft/s}$

$$\begin{aligned}\mathbf{v}_D &= 2.5981\mathbf{i} + 1.5\mathbf{j} - (3.9054 \cos 22^\circ)\mathbf{i} - (3.9054 \sin 22^\circ)\mathbf{j} \\ &= -(1.0229\text{ ft/s})\mathbf{i} + (0.0370\text{ ft/s})\mathbf{j}\end{aligned}$$

or

$$\begin{aligned}\mathbf{v}_D &= 2.5981\mathbf{i} - 1.5\mathbf{j} - (3.9337 \cos 23^\circ)\mathbf{i} + (3.9337 \sin 23^\circ)\mathbf{j} \\ &= -(1.0229\text{ ft/s})\mathbf{i} + (0.0370\text{ ft/s})\mathbf{j}\end{aligned}$$

$$\mathbf{v}_D = 1.024\text{ ft/s} \nwarrow 2.07^\circ \blacktriangleleft$$



PROBLEM 11.133

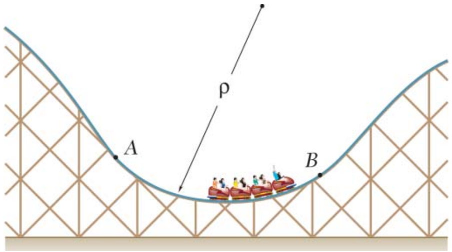
Determine the smallest radius that should be used for a highway if the normal component of the acceleration of a car traveling at 72 km/h is not to exceed 0.8 m/s^2.

SOLUTION

$$a_n = \frac{v^2}{\rho} \qquad a_n = 0.8 \text{ m/s}^2$$

$$v = 72 \text{ km/h} = 20 \text{ m/s}$$

$$0.8 \text{ m/s}^2 = \frac{(20 \text{ m/s})^2}{\rho} \qquad \rho = 500 \text{ m} \blacktriangleleft$$



PROBLEM 11.134

Determine the maximum speed that the cars of the roller-coaster can reach along the circular portion *AB* of the track if ρ is 25 m and the normal component of their acceleration cannot exceed 3 g.

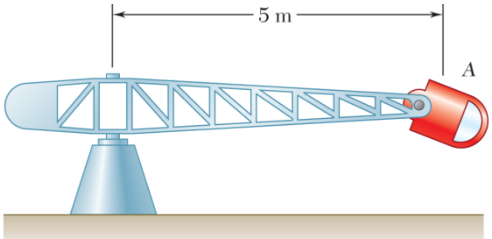
SOLUTION

We have $$a_n = \frac{v^2}{\rho}$$

Then $$(v_{\max})^2_{AB} = (3 \times 9.81\ \text{m/s}^2)(25\ \text{m})$$

or $$(v_{\max})_{AB} = 27.124\ \text{m/s}$$

or $$(v_{\max})_{AB} = 97.6\ \text{km/h} \blacktriangleleft$$



Problem 11.135

Human centrifuges are often used to simulate different acceleration levels for pilots and astronauts. Space shuttle pilots typically face inwards towards the center of the gondola in order to experience a simulated 3*g* forward acceleration. Knowing that the astronaut sits 5 m from the axis of rotation and experiences 3 g's inward, determine her velocity.

SOLUTION

Given: $a_n = 3g,\ \rho = 5 \text{ m}$

$$a_n = \frac{v^2}{\rho}$$

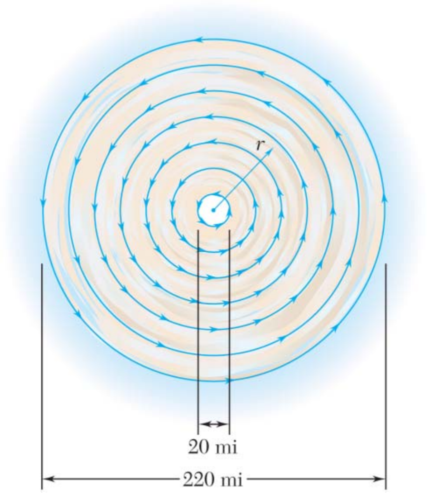
Rearrange $v = \sqrt{\rho a_n}$

$$v = \sqrt{\rho 3g}$$

Substitute in known values $v = \sqrt{5*3*9.81} \text{ m/s}$

$$v_A = 12.13 \text{ m/s} \blacktriangleleft$$

PROBLEM 11.136



The diameter of the eye of a stationary hurricane is 20 mi and the maximum wind speed is 100 mi/h at the eye wall, $r = 10$ mi. Assuming that the wind speed is constant for constant r and decreases uniformly with increasing r to 40 mi/h at $r = 110$ mi, determine the magnitude of the acceleration of the air at (*a*) $r = 10$ mi, (*b*) $r = 60$ mi, (*c*) $r = 110$ mi.

SOLUTION

(*a*) $$r = 10 \text{ mi} = 52800 \text{ ft}, \quad v = 100 \text{ mi/h} = 146.67 \text{ ft/s}$$

$$a = \frac{v^2}{r} = \frac{(146.67)^2}{52800} \qquad a = 0.407 \text{ ft/s}^2 \blacktriangleleft$$

(*b*) $$r = 60 \text{ mi} = 316800 \text{ ft}, \quad v = 100 - \frac{40-100}{110-10}(60-10) = 70 \text{ mi/h} = 102.67 \text{ ft/s}$$

$$a = \frac{v^2}{r} = \frac{(102.67)^2}{316800} \qquad a = 0.0333 \text{ ft/s}^2 \blacktriangleleft$$

(*c*) $$r = 110 \text{ mi} = 580800 \text{ ft}, \quad v = 40 \text{ mi/h} = 58.67 \text{ ft/s}$$

$$a = \frac{v^2}{r} = \frac{(58.67)^2}{580800} \qquad a = 0.00593 \text{ ft/s}^2 \blacktriangleleft$$

PROBLEM 11.137

The peripheral speed of the tooth of a 10-in.-diameter circular saw blade is 150 ft/s when the power to the saw is turned off. The speed of the tooth decreases at a constant rate, and the blade comes to rest in 9 s. Determine the time at which the total acceleration of the tooth is 130 ft/s^2.

SOLUTION

For uniformly decelerated motion: $v = v_0 + a_t t$

At $t = 9$ s, $0 = 150 - a_t(9)$, or $a_t = -16.667$ ft/s^2

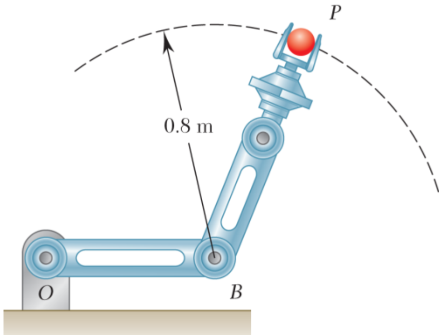
Total acceleration: $a^2 = a_t^2 + a_n^2$

$$a_n = \left[a^2 - a_t^2\right]^{1/2} = \left[(130)^2 - (-16.667)^2\right]^{1/2} = 128.93 \text{ ft/s}^2$$

Normal acceleration: $a_n = \dfrac{v^2}{\rho}$, where $\rho = \dfrac{1}{2}$ diameter $= \dfrac{5}{12}$ ft

$$v^2 = \rho a_n = \left(\frac{5}{12}\right)(128.93) = 53.72 \text{ ft}^2/\text{s}^2, \quad v = 7.329 \text{ ft/s}$$

Time: $t = \dfrac{v - v_0}{a_t} = \dfrac{7.329 - 150}{-16.667}$ $t = 8.56$ s ◀



PROBLEM 11.138

A robot arm moves so that P travels in a circle about Point B, which is not moving. Knowing that P starts from rest, and its speed increases at a constant rate of 10 mm/s^2, determine (*a*) the magnitude of the acceleration when $t = 4$ s, (*b*) the time for the magnitude of the acceleration to be 80 mm/s^2.

SOLUTION

Tangential acceleration: $a_t = 10 \text{ mm/s}^2$

Speed: $v = a_t t$

Normal acceleration: $a_n = \dfrac{v^2}{\rho} = \dfrac{a_t^2 t^2}{\rho}$

where $\rho = 0.8 \text{ m} = 800 \text{ mm}$

(*a*) When $t = 4$ s $\quad v = (10)(4) = 40 \text{ mm/s}$

$$a_n = \frac{(40)^2}{800} = 2 \text{ mm/s}^2$$

Acceleration: $a = \sqrt{a_t^2 + a_n^2} = \sqrt{(10)^2 + (2)^2}$

$a = 10.20 \text{ mm/s}^2$ ◀

(*b*) Time when $a = 80 \text{ mm/s}^2$

$$a^2 = a_n^2 + a_t^2$$

$$(80)^2 = \left[\frac{(10)^2 t^2}{800}\right]^2 + 10^2 \qquad t^4 = 403200 \text{ s}^4$$

$t = 25.2$ s ◀

PROBLEM 11.139

A monorail train starts from rest on a curve of radius 400 m and accelerates at the constant rate a_t. If the maximum total acceleration of the train must not exceed 1.5 m/s^2, determine (*a*) the shortest distance in which the train can reach a speed of 72 km/h, (*b*) the corresponding constant rate of acceleration a_t.

SOLUTION

When $v = 72\text{ km/h} = 20\text{ m/s}$ and $\rho = 400\text{ m}$,

$$a_n = \frac{v^2}{\rho} = \frac{(20)^2}{400} = 1.000\text{ m/s}^2$$

But $$a = \sqrt{a_n^2 + a_t^2}$$

$$a_t = \sqrt{a^2 - a_n^2} = \sqrt{(1.5)^2 - (1.000)^2} = \pm 1.11803\text{ m/s}^2$$

Since the train is accelerating, reject the negative value.

(*a*) Distance to reach the speed.

$$v_0 = 0$$

Let $$x_0 = 0$$

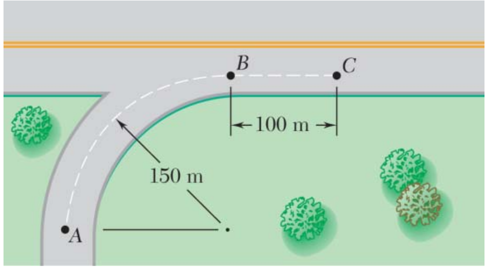
$$v_1^2 = v_0^2 + 2a_t(x_1 - x_0) = 2a_t x_1$$

$$x_1 = \frac{v_1^2}{2a_t} = \frac{(20)^2}{(2)(1.11803)}$$

$x_1 = 178.9\text{ m}$ ◀

(*b*) Corresponding tangential acceleration.

$a_t = 1.118\text{ m/s}^2$ ◀



PROBLEM 11.140

A motorist starts from rest at Point *A* on a circular entrance ramp when $t = 0$, increases the speed of her automobile at a constant rate and enters the highway at Point *B*. Knowing that her speed continues to increase at the same rate until it reaches 100 km/h at Point *C*, determine (*a*) the speed at Point *B*, (*b*) the magnitude of the total acceleration when $t = 20$ s.

SOLUTION

Speeds: $v_0 = 0 \qquad v_1 = 100 \text{ km/h} = 27.78 \text{ m/s}$

Distance: $s = \dfrac{\pi}{2}(150) + 100 = 335.6 \text{ m}$

Tangential component of acceleration: $v_1^2 = v_0^2 + 2a_t s$

$$a_t = \frac{v_1^2 - v_0^2}{2s} = \frac{(27.78)^2 - 0}{(2)(335.6)} = 1.1495 \text{ m/s}^2$$

At Point *B*, $v_B^2 = v_0^2 + 2a_t s_B$ where $s_B = \dfrac{\pi}{2}(150) = 235.6 \text{ m}$

$$v_B^2 = 0 + (2)(1.1495)(235.6) = 541.69 \text{ m}^2/\text{s}^2$$

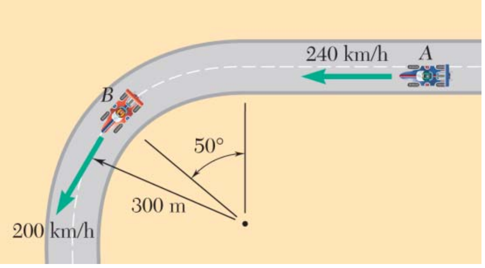
$v_B = 23.27 \text{ m/s}$ $\qquad v_B = 83.8 \text{ km/h}$ ◀

(*a*) At $t = 20$ s, $v = v_0 + a_t t = 0 + (1.1495)(20) = 22.99 \text{ m/s}$

Since $v < v_B$, the car is still on the curve. $\rho = 150 \text{ m}$

Normal component of acceleration: $a_n = \dfrac{v^2}{\rho} = \dfrac{(22.99)^2}{150} = 3.524 \text{ m/s}^2$

(*b*) Magnitude of total acceleration: $|a| = \sqrt{a_t^2 + a_n^2} = \sqrt{(1.1495)^2 + (3.524)^2}$ $\qquad |a| = 3.71 \text{ m/s}^2$ ◀



PROBLEM 11.141

Racecar A is traveling on a straight portion of the track while racecar B is traveling on a circular portion of the track. At the instant shown, the speed of A is increasing at the rate of 10 m/s^2, and the speed of B is decreasing at the rate of 6 m/s^2. For the position shown, determine (*a*) the velocity of B relative to A, (*b*) the acceleration of B relative to A.

SOLUTION

Speeds:

$$v_A = 240 \text{ km/h} = 66.67 \text{ m/s}$$
$$v_B = 200 \text{ km/h} = 55.56 \text{ m/s}$$

Velocities:

$$\mathbf{v}_A = 66.67 \text{ m/s} \longleftarrow$$
$$\mathbf{v}_B = 55.56 \text{ m/s} \swarrow 50^\circ$$

(*a*) Relative velocity:

$$\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$$

$$\mathbf{v}_{B/A} = (55.56\cos 50^\circ) \leftarrow + \; 55.56\sin 50^\circ \downarrow + \; 66.67 \longrightarrow$$

$$= 30.96 \rightarrow + \; 42.56 \downarrow$$

$$= 52.63 \text{ m/s} \searrow 53.96^\circ$$

$$\mathbf{v}_{B/A} = 189.5 \text{ km/h} \searrow 54.0^\circ \blacktriangleleft$$

Tangential accelerations:

$$(\mathbf{a}_A)_t = 10 \text{ m/s}^2 \longleftarrow$$
$$(\mathbf{a}_B)_t = 6 \text{ m/s}^2 \nearrow 50^\circ$$

Normal accelerations:

$$a_n = \frac{v^2}{\rho}$$

Car A: $(\rho = \infty)$ $\quad (\mathbf{a}_A)_n = 0$

Car B: $(\rho = 300 \text{ m})$

$$(\mathbf{a}_B)_n = \frac{(55.56)^2}{300} = 10.288 \qquad (\mathbf{a}_B)_n = 10.288 \text{ m/s}^2 \searrow 40^\circ$$

(*b*) Acceleration of B relative to A:

$$\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$$

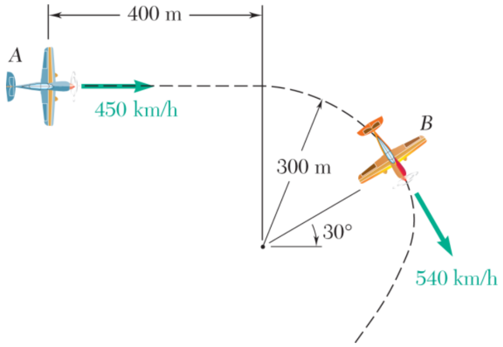
$$\mathbf{a}_{B/A} = (\mathbf{a}_B)_t + (\mathbf{a}_B)_n - (\mathbf{a}_A)_t - (\mathbf{a}_A)_n$$

$$= 6 \nearrow 50^\circ + 10.288 \searrow 40^\circ + 10 \rightarrow + \; 0$$

$$= (6\cos 50^\circ + 10.288\cos 40^\circ + 10) \longrightarrow$$

$$+ \; (6\sin 50^\circ - 10.288\sin 40^\circ) \uparrow$$

$$= 21.738 \rightarrow + \; 2.017 \downarrow \qquad \mathbf{a}_{B/A} = 21.8 \text{ m/s}^2 \searrow 5.3^\circ \blacktriangleleft$$



PROBLEM 11.142

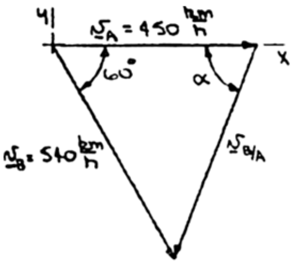
At a given instant in an airplane race, airplane *A* is flying horizontally in a straight line, and its speed is being increased at the rate of 8 m/s^2. Airplane *B* is flying at the same altitude as airplane *A* and, as it rounds a pylon, is following a circular path of 300-m radius. Knowing that at the given instant the speed of *B* is being decreased at the rate of 3 m/s^2, determine, for the positions shown, (*a*) the velocity of *B* relative to *A*, (*b*) the acceleration of *B* relative to *A*.

SOLUTION

First note $v_A = 450\ \text{km/h} \quad v_B = 540\ \text{km/h} = 150\ \text{m/s}$

(*a*) We have $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

The graphical representation of this equation is then as shown.


y
$v_A = 450\ \frac{\text{km}}{\text{h}}$
x
60°
α
$v_B = 540\ \frac{\text{km}}{\text{h}}$
$v_{B/A}$

We have $v_{B/A}^2 = 450^2 + 540^2 - 2(450)(540)\cos 60°$

$$v_{B/A} = 501.10\ \text{km/h}$$

and $\dfrac{540}{\sin\alpha} = \dfrac{501.10}{\sin 60°}$

$$\alpha = 68.9°$$

$$\mathbf{v}_{B/A} = 501\ \text{km/h} \swarrow 68.9° \blacktriangleleft$$

(*b*) First note $\mathbf{a}_A = 8\ \text{m/s}^2 \longrightarrow \qquad (\mathbf{a}_B)_t = 3\ \text{m/s}^2 \nwarrow 60°$

Now $(a_B)_n = \dfrac{v_B^2}{\rho_B} = \dfrac{(150\ \text{m/s})^2}{300\ \text{m}}$

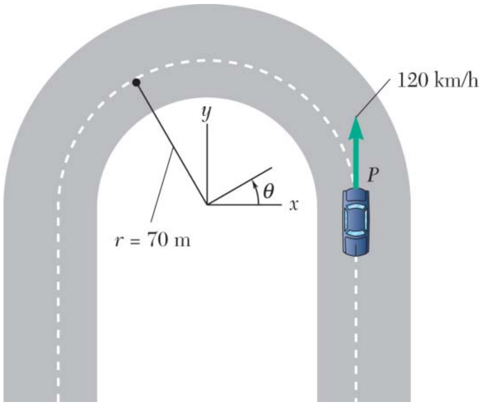
$$(\mathbf{a}_B)_n = 75\ \text{m/s}^2 \swarrow 30°$$

Then
$$\begin{aligned}\mathbf{a}_B &= (\mathbf{a}_B)_t + (\mathbf{a}_B)_n\\ &= 3(-\cos 60°\ \mathbf{i} + \sin 60°\mathbf{j}) + 75(-\cos 30°\,\mathbf{i} - \sin 30°\mathbf{j})\\ &= -(66.452\ \text{m/s}^2)\mathbf{i} - (34.902\ \text{m/s}^2)\mathbf{j}\end{aligned}$$

Finally
$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

$$\begin{aligned}\mathbf{a}_{B/A} &= (-66.452\mathbf{i} - 34.902\mathbf{j}) - (8\mathbf{i})\\ &= -(74.452\ \text{m/s}^2)\mathbf{i} - (34.902\ \text{m/s}^2)\mathbf{j}\end{aligned}$$

$$\mathbf{a}_{B/A} = 82.2\ \text{m/s}^2 \swarrow 25.1° \blacktriangleleft$$



PROBLEM 11.143

A race car enters the circular portion of a track that has a radius of 70 m. When the car enters the curve at point P, it is travelling with a speed of 120 km/h that is increasing at 5 m/s^2. Three seconds later, determine (a) the total acceleration of the car in xy components, (b) the linear velocity of the car in xy components.

SOLUTION

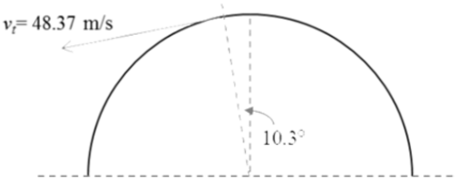
Speeds: $v_{t_o} = 120 \text{ km/h} = 33.33 \text{ m/s}$

$$v_t = v_{t_o} + a_t t$$

$$v_t = 33.33 + 5(3) = 48.33 \text{ m/s}$$

Distance: $s = s_o + v_{t_o} t + \frac{1}{2} a_t t^2$

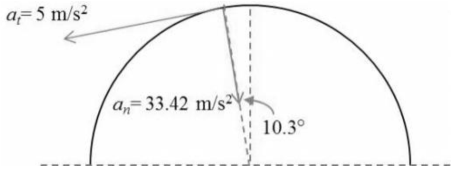
$$s = 33.33(3) + \frac{1}{2}(5)(3)^2 = 122.5 \text{ m}$$



Location: $s = R\theta$

$$\theta = \frac{s}{R} = \frac{122.5}{70} = 1.75 \text{ rad}$$

$$\theta = 100.3°$$



Tangential Acceleration: $a_t = 5 \text{ m/s}^2$

Normal Acceleration: $a_n = \dfrac{v^2}{\rho}$

$$a_n = \frac{48.33^2}{70} = 33.37 \text{ m/s}^2$$

(a) Accelerations: $a_x = -5\cos(10.3°) + 33.37\sin(10.3°)$

$$a_x = 1.047 \text{ m/s}^2 \blacktriangleleft$$

$$a_y = -5\sin(10.3°) - 33.37\cos(10.3°)$$

$$a_y = -33.726 \text{ m/s}^2 \blacktriangleleft$$

(b) Velocities: $v_x = -48.33\cos(10.3°)$

$$v_x = -47.55 \text{ m/s} \blacktriangleleft$$

$$v_y = -48.33\sin(10.3°)$$

$$v_y = -8.64 \text{ m/s} \blacktriangleleft$$

PROBLEM 11.144

An airplane flying at a constant speed of 240 m/s makes a banked horizontal turn. What is the minimum allowable radius of the turn if the structural specifications require that the acceleration of the airplane shall never exceed 4*g*?

SOLUTION

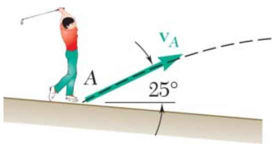
Given: $v = 240\text{ m/s},\ a_{\max} = 4g$

Acceleration: $a_n = \dfrac{v^2}{\rho}$

Rearrange: $\rho = \dfrac{v^2}{a_n}$

Substitute in known values: $\rho = \dfrac{240^2}{4*9.81}m$

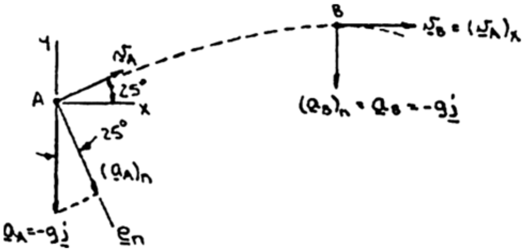
$\rho = 1467.9\text{ m}$ ◀


$\mathbf{v}_A$
A
25°

PROBLEM 11.145

A golfer hits a golf ball from Point *A* with an initial velocity of 50 m/s at an angle of 25° with the horizontal. Determine the radius of curvature of the trajectory described by the ball (*a*) at Point *A*, (*b*) at the highest point of the trajectory.

SOLUTION


y
A
x
$\underline{v}_A$
25°
25°
$(\underline{a}_A)_n$
$\underline{a}_A = -g\underline{j}$
$\underline{e}_n$
B
$\underline{v}_B = (\underline{v}_A)_x$
$(\underline{a}_B)_n = \underline{a}_B = -g\underline{j}$

(*a*) We have $(a_A)_n = \dfrac{v_A^2}{\rho_A}$

or $\rho_A = \dfrac{(50\text{ m/s})^2}{(9.81\text{ m/s}^2)\cos 25°}$

or $\rho_A = 281\text{ m}$ ◀

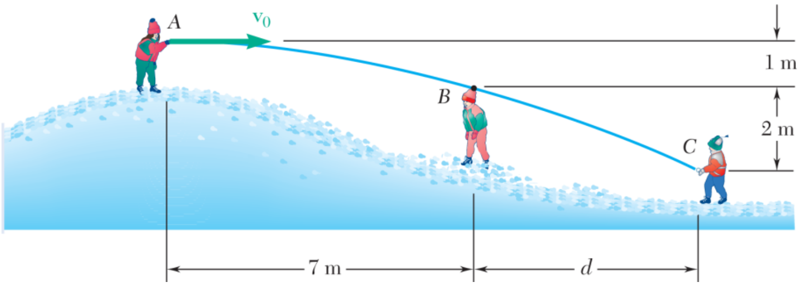
(*b*) We have $(a_B)_n = \dfrac{v_B^2}{\rho_B}$

where Point *B* is the highest point of the trajectory, so that

$$v_B = (v_A)_x = v_A \cos 25°$$

Then $\rho_B = \dfrac{[(50\text{ m/s})\cos 25°]^2}{9.81\text{ m/s}^2}$

or $\rho_B = 209\text{ m}$ ◀



PROBLEM 11.146

Three children are throwing snowballs at each other. Child *A* throws a snowball with a horizontal velocity $\mathbf{v}_0$. If the snowball just passes over the head of child *B* and hits child C, determine the radius of curvature of the trajectory described by the snowball (*a*) at Point *B*, (*b*) at Point *C*.

SOLUTION

The motion is projectile motion. Place the origin at Point *A*.

Horizontal motion: $\mathbf{v}_x = v_0 \qquad x = v_0 t$

Vertical motion: $y_0 = 0, \qquad (v_y) = 0$

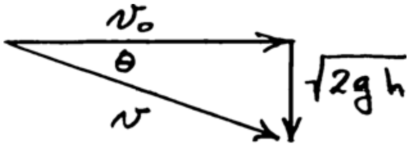
$$v_y = -gt \qquad y = -\frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2h}{g}}, \qquad \text{where } h \text{ is the vertical distance fallen.}$$

$$|v_y| = \sqrt{2gh}$$

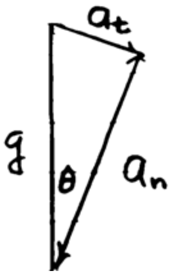
Speed: $v^2 = v_x^2 + v_y^2 = v_0^2 + 2gh$

Direction of velocity.



$$\cos\theta = \frac{v_0}{v}$$

Direction of normal acceleration.



$$a_n = g\cos\theta = \frac{gv_0}{v} = \frac{v^2}{\rho}$$

Radius of curvature: $\rho = \dfrac{v^3}{gv_0}$

At Point *B*, $h_B = 1 \text{ m}; \quad x_B = 7 \text{ m}$

$$t_B = \sqrt{\frac{(2)(1 \text{ m})}{9.81 \text{ m/s}^2}} = 0.45152 \text{ s}$$

$$x_B = v_0 t_B \qquad v_0 = \frac{x_B}{t_B} = \frac{7 \text{ m}}{0.45152 \text{ s}} = 15.504 \text{ m/s}$$

$$v_B^2 = (15.504)^2 + (2)(9.81)(1) = 259.97 \text{ m}^2/\text{s}^2$$

(*a*) Radius of curvature at Point *B*.

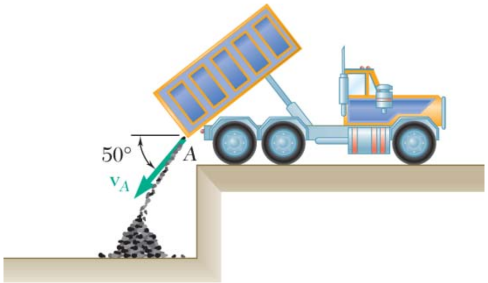
$$\rho_B = \frac{(259.97\ \text{m}^2/\text{s}^2)^{3/2}}{(9.81\ \text{m/s}^2)(15.504\ \text{m/s})} \qquad \rho_B = 27.6\ \text{m} \blacktriangleleft$$

At Point *C* $h_C = 1\ \text{m} + 2\ \text{m} = 3\ \text{m}$

$$v_C^2 = (15.504)^2 + (2)(9.81)(3) = 299.23\ \text{m}^2/\text{s}^2$$

(*b*) Radius of curvature at Point *C*.

$$\rho_C = \frac{(299.23\ \text{m}^2/\text{s}^2)^{3/2}}{(9.81\ \text{m/s}^2)(15.504\ \text{m/s})} \qquad \rho_C = 34.0\ \text{m} \blacktriangleleft$$



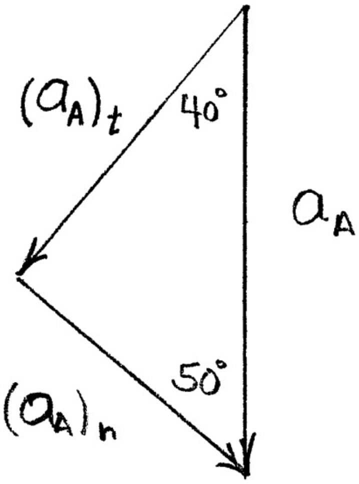
PROBLEM 11.147

Coal is discharged from the tailgate A of a dump truck with an initial velocity $\mathbf{v}_A = 2\text{ m/s} \nearrow 50°$. Determine the radius of curvature of the trajectory described by the coal (a) at Point A, (b) at the point of the trajectory 1 m below Point A.

SOLUTION

(a) At Point A. $a_A = g\downarrow = 9.81\text{ m/s}^2\downarrow$

Sketch tangential and normal components of acceleration at A.



$$(a_A)_n = g\cos 50°$$

$$\rho_A = \frac{v_A^2}{(a_A)_n} = \frac{(2)^2}{9.81\cos 50°}$$

$\rho_A = 0.634\text{ m}$ ◀

(b) At Point B, 1 meter below Point A.

Horizontal motion: $(v_B)_x = (v_A)_x = 2\cos 50° = 1.286\text{ m/s} \leftarrow$

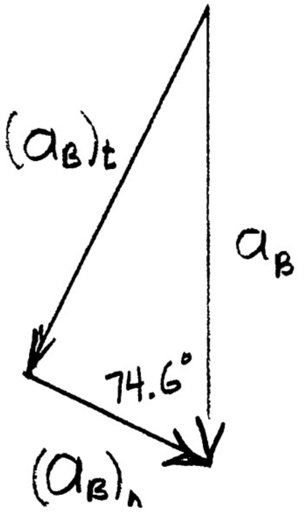
Vertical motion: $(v_B)_y^2 = (v_A)_y^2 + 2a_y(y_B - y_A)$

$$= (2\cos 40°)^2 + (2)(-9.81)(-1)$$

$$= 21.97\text{ m}^2/\text{s}^2$$

$$(v_B)_y = 4.687\text{ m/s}\downarrow$$

$$\tan\theta = \frac{(v_B)_y}{(v_B)_x} = \frac{4.687}{1.286}, \quad \text{or} \quad \theta = 74.6°$$

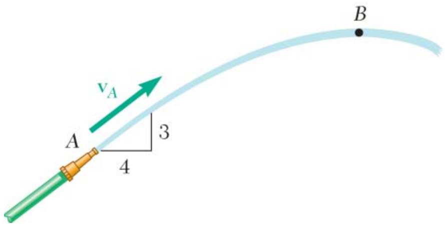


$$a_B = g\cos 74.6°$$

$$\rho_B = \frac{v_B^2}{(a_B)_n} = \frac{(v_B)_x^2 + (v_B)_y^2}{g\cos 74.6°}$$

$$= \frac{(1.286)^2 + 21.97}{9.81\cos 74.6°}$$

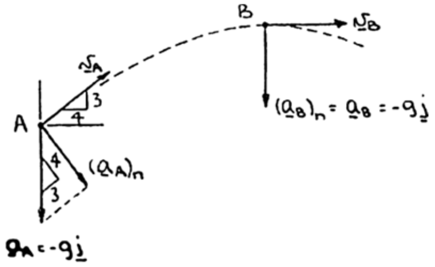
$\rho_B = 9.07\text{ m}$ ◀


B
$\mathbf{v}_A$
A
3
4

PROBLEM 11.148

From measurements of a photograph, it has been found that as the stream of water shown left the nozzle at *A*, it had a radius of curvature of 25 m. Determine (*a*) the initial velocity $\mathbf{v}_A$ of the stream, (*b*) the radius of curvature of the stream as it reaches its maximum height at *B*.

SOLUTION


B
$\underline{v}_B$
$\underline{v}_A$
3
4
A
$(\underline{a}_B)_n = \underline{a}_B = -g\underline{j}$
4
3
$(\underline{a}_A)_n$
$\underline{a}_A = -g\underline{j}$

(*a*) We have $(a_A)_n = \dfrac{v_A^2}{\rho_A}$

or $v_A^2 = \left[\dfrac{4}{5}(9.81\text{ m/s}^2)\right](25\text{ m})$

or $v_A = 14.0071\text{ m/s}$

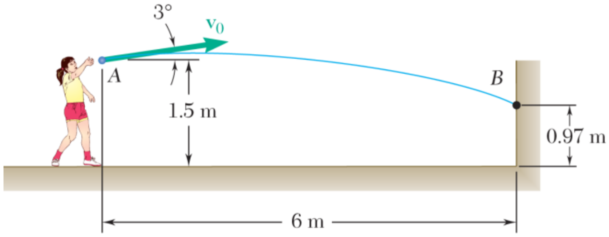
$\mathbf{v}_A = 14.01\text{ m/s} \measuredangle 36.9^\circ$ ◀

(*b*) We have $(a_B)_n = \dfrac{v_B^2}{\rho_B}$

Where $v_B = (v_A)_x = \dfrac{4}{5}v_A$

Then $\rho_B = \dfrac{\left(\frac{4}{5} \times 14.0071\text{ m/s}\right)^2}{9.81\text{ m/s}^2}$

or $\rho_B = 12.80\text{ m}$ ◀



PROBLEM 11.149

A child throws a ball from point *A* with an initial velocity $\mathbf{v}_0$ at an angle of 3° with the horizontal. Knowing that the ball hits a wall at point *B*, determine (*a*) the magnitude of the initial velocity, (*b*) the minimum radius of curvature of the trajectory.

SOLUTION

Horizontal motion. $v_x = v_0 \cos\alpha \qquad x = v_0 t \cos\alpha$

Vertical motion. $v_y = v_0 \sin\alpha - gt$

$$y = y_0 + v_0 t \sin\alpha - \frac{1}{2}gt^2$$

Eliminate *t*.

$$y = y_0 + x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} \qquad (1)$$

Solving (1) for v_0 and applying result at point *B*

$$v_0 = \sqrt{\frac{gx^2}{2(y_0 + x\tan\alpha - y)\cos^2\alpha}} = \sqrt{\frac{(9.81)(6)^2}{(2)(1.5 + 6\tan 3° - 0.97)(\cos^2 3°)}}$$

(*a*) Magnitude of initial velocity. $v_0 = 14.48 \text{ m/s}$ ◀

(*b*) Minimum radius of curvature of trajectory.

$$a_n = g = \frac{v^2}{\rho} \qquad \rho = \frac{v^2}{a_n} = \frac{v^2}{g\cos\theta} \qquad (2)$$

where θ is the slope angle of the trajectory.

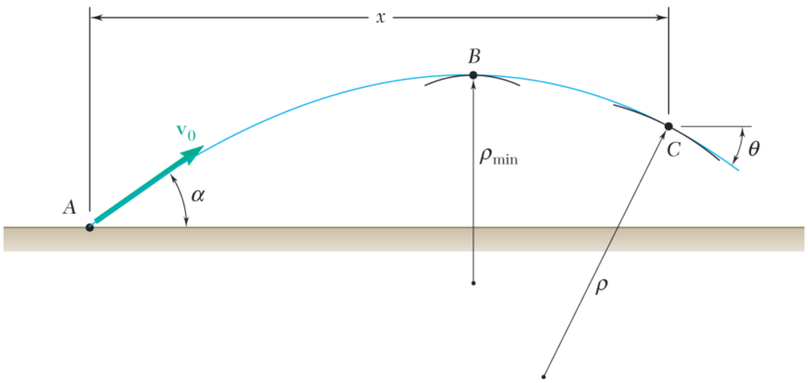
The minimum value of ρ occurs at the highest point of the trajectory where $\cos\theta = 1$

and $v = v_x = v_0\cos\alpha$

Then

$$\rho_{\min} = \frac{v_0^2\cos^2\alpha}{g} = \frac{(14.48)^2\cos^2 3°}{9.81}$$

$\rho_{\min} = 21.3 \text{ m}$ ◀

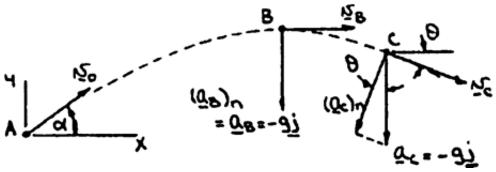


PROBLEM 11.150

A projectile is fired from Point A with an initial velocity $\mathbf{v}_0$. (a) Show that the radius of curvature of the trajectory of the projectile reaches its minimum value at the highest Point B of the trajectory. (b) Denoting by θ the angle formed by the trajectory and the horizontal at a given Point C, show that the radius of curvature of the trajectory at C is $\rho = \rho_{\min}/\cos^3\theta$.

SOLUTION

For the arbitrary Point C, we have



$$(a_C)_n = \frac{v_C^2}{\rho_C}$$

or $$\rho_C = \frac{v_C^2}{g\cos\theta}$$

Noting that the horizontal motion is uniform, we have

$$(v_A)_x = (v_C)_x$$

where $$(v_A)_x = v_0\cos\alpha \qquad (v_C)_x = v_C\cos\theta$$

Then $$v_0\cos\alpha = v_C\cos\theta$$

or $$v_C = \frac{\cos\alpha}{\cos\theta}v_0$$

so that $$\rho_C = \frac{1}{g\cos\theta}\left(\frac{\cos\alpha}{\cos\theta}v_0\right)^2 = \frac{v_0^2\cos^2\alpha}{g\cos^3\theta}$$

(a) In the expression for ρ_C, v_0, α, and g are constants, so that ρ_C is minimum where $\cos\theta$ is maximum. By observation, this occurs at Point B where $\theta = 0$.

$$\rho_{\min} = \rho_B = \frac{v_0^2\cos^2\alpha}{g}$$ Q.E.D. ◀

(b) $$\rho_C = \frac{1}{\cos^3\theta}\left(\frac{v_0^2\cos^2\alpha}{g}\right)$$

$$\rho_C = \frac{\rho_{\min}}{\cos^3\theta}$$ Q.E.D. ◀

PROBLEM 11.151*

Determine the radius of curvature of the path described by the particle of Problem 11.95 when $t = 0$.

PROBLEM 11.95 The three-dimensional motion of a particle is defined by the position vector $\mathbf{r} = (Rt \cos \omega_n t)\mathbf{i} + ct\mathbf{j} + (Rt \sin \omega_n t)\mathbf{k}$. Determine the magnitudes of the velocity and acceleration of the particle. (The space curve described by the particle is a conic helix.)

SOLUTION

We have $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = R(\cos\omega_n t - \omega_n t \sin\omega_n t)\mathbf{i} + c\mathbf{j} + R(\sin\omega_n t + \omega_n t \cos\omega_n t)\mathbf{k}$$

and $$\mathbf{a} = \frac{d\mathbf{v}}{dt} = R\left(-\omega_n \sin \omega_n t - \omega_n \sin \omega_n t - \omega_n^2 t \cos \omega_n t\right)\mathbf{i}$$
$$+ R\left(\omega_n \cos \omega_n t + \omega_n \cos \omega_n t - \omega_n^2 t \sin \omega_n t\right)\mathbf{k}$$

or $$\mathbf{a} = \omega_n R[-(2 \sin \omega_n t + \omega_n t \cos \omega_n t)\mathbf{i} + (2 \cos \omega_n t - \omega_n t \sin \omega_n t)\mathbf{k}]$$

Now $$v^2 = R^2(\cos \omega_n t - \omega_n t \sin \omega_n t)^2 + c^2 + R^2(\sin \omega_n t + \omega_n t \cos \omega_n t)^2$$
$$= R^2\left(1 + \omega_n^2 t^2\right) + c^2$$

Then $$v = \left[R^2\left(1 + \omega_n^2 t^2\right) + c^2\right]^{1/2}$$

and $$\frac{dv}{dt} = \frac{R^2 \omega_n^2 t}{\left[R^2\left(1 + \omega_n^2 t^2\right) + c^2\right]^{1/2}}$$

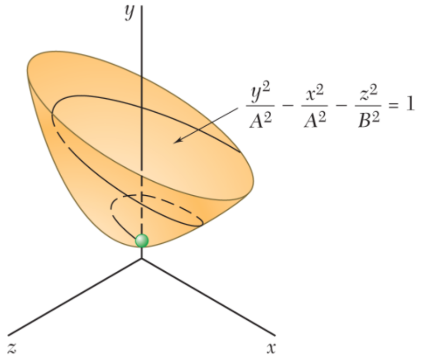
Now $$a^2 = a_t^2 + a_n^2 = \left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2$$

At $t = 0$: $$\frac{dv}{dt} = 0$$
$$\mathbf{a} = \omega_n R(2\mathbf{k}) \quad \text{or} \quad a = 2\omega_n R$$
$$v^2 = R^2 + c^2$$

Then, with $$\frac{dv}{dt} = 0,$$

we have $$a = \frac{v^2}{\rho}$$

or $$2\omega_n R = \frac{R^2 + c^2}{\rho}$$ $$\rho = \frac{R^2 + c^2}{2\omega_n R} \blacktriangleleft$$



PROBLEM 11.152*

Determine the radius of curvature of the path described by the particle of Problem 11.96 when $t = 0$, $A = 3$, and $B = 1$.

PROBLEM 11.96

The three-dimensional motion of a particle is defined by the position vector $\mathbf{r} = (At\cos t)\mathbf{i} + (A\sqrt{t^2+1})\mathbf{j} + (Bt\sin t)\mathbf{k}$, where r and t are expressed in feet and seconds, respectively. Show that the curve described by the particle lies on the hyperboloid $(y/A)^2 - (x/A)^2 - (z/B)^2 = 1$. For $A = 3$ and $B = 1$, determine (*a*) the magnitudes of the velocity and acceleration when $t = 0$, (*b*) the smallest nonzero value of t for which the position vector and the velocity are perpendicular to each other.

SOLUTION

With $$A = 3, \quad B = 1$$

we have $$\mathbf{r} = (3t\cos t)\mathbf{i} + \left(3\sqrt{t^2+1}\right)\mathbf{j} + (t\sin t)\mathbf{k}$$

Now $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 3(\cos t - t\sin t)\mathbf{i} + \left(\frac{3t}{\sqrt{t^2+1}}\right)\mathbf{j} + (\sin t + t\cos t)\mathbf{k}$$

and $$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 3(-\sin t - \sin t - t\cos t)\mathbf{i} + 3\left[\frac{\sqrt{t^2+1} - t\left(\frac{t}{\sqrt{t^2+1}}\right)}{t^2+1}\right]\mathbf{j}$$

$$+ (\cos t + \cos t - t\sin t)\mathbf{k}$$

$$= -3(2\sin t + t\cos t)\mathbf{i} + 3\frac{1}{(t^2+1)^{1/2}}\mathbf{j}$$

$$+ (2\cos t - t\sin t)\mathbf{k}$$

Then $$v^2 = 9(\cos t - t\sin t)^2 + 9\frac{t^2}{t^2+1} + (\sin t + t\cos t)^2$$

Expanding and simplifying yields

$$v^2 = t^4 + 19t^2 + 1 + 8(\cos^2 t + t^4\sin^2 t) - 8(t^3 + t)\sin 2t$$

Then $$v = [t^4 + 19t^2 + 1 + 8(\cos^2 t + t^4\sin^2 t) - 8(t^3 + t)\sin 2t]^{1/2}$$

and $$\frac{dv}{dt} = \frac{4t^3 + 38t + 8(-2\cos t\sin t + 4t^3\sin^2 t + 2t^4\sin t\cos t) - 8[(3t^2+1)\sin 2t + 2(t^3+t)\cos 2t]}{2[t^4 + 19t^2 + 1 + 8(\cos^2 t + t^4\sin^2 t) - 8(t^3+t)\sin 2t]^{1/2}}$$

PROBLEM 11.152* (Continued)

Now $$a^2 = a_t^2 + a_n^2 = \left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2$$

At $t = 0$: $$\mathbf{a} = 3\mathbf{j} + 2\mathbf{k}$$

or $$a = \sqrt{13}\ \text{ft/s}^2$$

$$\frac{dv}{dt} = 0$$

$$v^2 = 9\ (\text{ft/s})^2$$

Then, with $$\frac{dv}{dt} = 0,$$

we have $$a = \frac{v^2}{\rho}$$

or $$\rho = \frac{9\ \text{ft}^2/\text{s}^2}{\sqrt{13}\ \text{ft/s}^2}$$ $\rho = 2.50$ ft ◀

PROBLEM 11.153

A satellite will travel indefinitely in a circular orbit around a planet if the normal component of the acceleration of the satellite is equal to $g(R/r)^2$, where g is the acceleration of gravity at the surface of the planet, R is the radius of the planet, and r is the distance from the center of the planet to the satellite. Knowing that the diameter of the sun is 1.39 Gm and that the acceleration of gravity at its surface is 274 m/s^2, determine the radius of the orbit of the indicated planet around the sun assuming that the orbit is circular.

Earth: $(v_{\text{mean}})_{\text{orbit}} = 107 \text{ Mm/h}$.

SOLUTION

For the sun, $g = 274 \text{ m/s}^2$,

and $$R = \frac{1}{2}D = \left(\frac{1}{2}\right)(1.39 \times 10^9) = 0.695 \times 10^9 \text{ m}$$

Given that $a_n = \dfrac{gR^2}{r^2}$ and that for a circular orbit $a_n = \dfrac{v^2}{r}$

Eliminating a_n and solving for r, $$r = \frac{gR^2}{v^2}$$

For the planet Earth, $$v = 107 \times 10^6 \text{m/h} = 29.72 \times 10^3 \text{ m/s}$$

Then $$r = \frac{(274)(0.695 \times 10^9)^2}{(29.72 \times 10^3)^2} = 149.8 \times 10^9 \text{ m}$$

$r = 149.8 \text{ Gm}$ ◀

PROBLEM 11.154

A satellite will travel indefinitely in a circular orbit around a planet if the normal component of the acceleration of the satellite is equal to $g(R/r)^2$, where g is the acceleration of gravity at the surface of the planet, R is the radius of the planet, and r is the distance from the center of the planet to the satellite. Knowing that the diameter of the sun is 1.39 Gm and that the acceleration of gravity at its surface is 274 m/s^2, determine the radius of the orbit of the indicated planet around the sun assuming that the orbit is circular.

Saturn: $(\upsilon_{\text{mean}})_{\text{orbit}} = 34.7\text{ Mm/h}$.

SOLUTION

For the sun, $$g = 274\text{ m/s}^2$$

and $$R = \frac{1}{2}D = \left(\frac{1}{2}\right)(1.39\times 10^9) = 0.695\times 10^9\text{ m}$$

Given that $a_n = \dfrac{gR^2}{r^2}$ .and that for a circular orbit: $a_n = \dfrac{v^2}{r}$

Eliminating a_n and solving for r, $$r = \frac{gR^2}{v^2}$$

For the planet Saturn, $$v = 34.7\times 10^6\text{m/h} = 9.639\times 10^3\text{m/s}$$

Then, $$r = \frac{(274)(0.695\times 10^9)^2}{(9.639\times 10^3)^2} = 1.425\times 10^{12}\text{ m}$$ $$r = 1425\text{ Gm} \blacktriangleleft$$

PROBLEM 11.155

Determine the speed of a satellite relative to the indicated planet if the satellite is to travel indefinitely in a circular orbit 100 mi above the surface of the planet. (See information given in Problems 11.153–11.154).

Venus: $g = 29.20\text{ ft/s}^2, \ R = 3761\text{ mi.}$

SOLUTION

From Problems 11.153 and 11.154, $a_n = \dfrac{gR^2}{r^2}$

For a circular orbit, $a_n = \dfrac{v^2}{r}$

Eliminating a_n and solving for v, $v = R\sqrt{\dfrac{g}{r}}$

For Venus, $g = 29.20\text{ ft/s}^2$

$$R = 3761\text{ mi} = 19.858 \times 10^6\text{ ft.}$$

$$r = 3761 + 100 = 3861\text{ mi} = 20.386 \times 10^6\text{ ft}$$

Then, $$v = 19.858 \times 10^6\sqrt{\frac{29.20}{20.386 \times 10^6}} = 23.766 \times 10^3\text{ ft/s}$$

$v = 16200\text{ mi/h}$ ◀

PROBLEM 11.156

Determine the speed of a satellite relative to the indicated planet if the satellite is to travel indefinitely in a circular orbit 100 mi above the surface of the planet. (See information given in Problems 11.153–11.154).

Mars: $g = 12.17 \text{ ft/s}^2, \; R = 2102 \text{ mi}.$

SOLUTION

From Problems 11.153 and 11.154, $$a_n = \frac{gR^2}{r^2}$$

For a circular orbit, $$a_n = \frac{v^2}{r}$$

Eliminating a_n and solving for v, $$v = R\sqrt{\frac{g}{r}}$$

For Mars, $g = 12.17 \text{ ft/s}^2$

$R = 2102 \text{ mi} = 11.0986 \times 10^6 \text{ ft}$

$r = 2102 + 100 = 2202 \text{ mi} = 11.6266 \times 10^6 \text{ ft}$

Then, $$v = 11.0986 \times 10^6 \sqrt{\frac{12.17}{11.6266 \times 10^6}} = 11.35 \times 10^3 \text{ ft/s}$$

$v = 7740 \text{ mi/h}$ ◀

PROBLEM 11.157

Determine the speed of a satellite relative to the indicated planet if the satellite is to travel indefinitely in a circular orbit 100 mi above the surface of the planet. (See information given in Problems 11.153–11.154).

Jupiter: $g = 75.35 \text{ ft/s}^2$, $R = 44,432$ mi.

SOLUTION

From Problems 11.153 and 11.154,

$$a_n = \frac{gR^2}{r^2}$$

For a circular orbit,

$$a_n = \frac{v^2}{r}$$

Eliminating a_n and solving for v,

$$v = R\sqrt{\frac{g}{r}}$$

For Jupiter,

$$g = 75.35 \text{ ft/s}^2$$

$$R = 44432 \text{ mi} = 234.60 \times 10^6 \text{ ft}$$

$$r = 44432 + 100 = 44532 \text{ mi} = 235.13 \times 10^6 \text{ ft}$$

Then,

$$v = (234.60 \times 10^6)\sqrt{\frac{75.35}{235.13 \times 10^6}} = 132.8 \times 10^3 \text{ ft/s}$$

$$v = 90600 \text{ mi/h} \blacktriangleleft$$

PROBLEM 11.158

A satellite will travel indefinitely in a circular orbit around the earth if the normal component of its acceleration is equal to $g\left(R / r\right)^2$, where $g = 9.81\text{ m/s}^2$, R = radius of the earth = 6370 km, and r = distance from the center of the earth to the satellite. Assuming that the orbit of the moon is a circle of radius $384 \times 10^3\text{ km}$, determine the speed of the moon relative to the earth.

SOLUTION

Normal acceleration: $$a_n = \frac{gR^2}{r^2} \quad \text{and} \quad a_n = \frac{v^2}{\rho} = \frac{v^2}{r}$$

Solve for v^2: $$v^2 = ra_n = \frac{gR^2}{r}$$

Data: $$g = 9.81\text{ m/s}^2, \qquad R = 6370\text{ km} = 6.370 \times 10^6\text{ m}$$

$$r = 384 \times 10^3\text{ km} = 384 \times 10^6\text{ m}$$

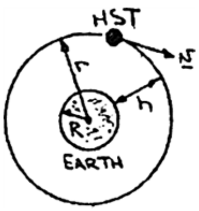
$$v^2 = \frac{(9.81)\left(6.370 \times 10^6\right)^2}{384 \times 10^6} = 1.0366 \times 10^6\text{ m}^2/\text{s}^2$$

$v = 1.018\text{ m/s}$ $\qquad v = 3670\text{ km/h}$ ◀

PROBLEM 11.159

Knowing that the radius of the earth is 6370 km, determine the time of one orbit of the Hubble Space Telescope, knowing that the telescope travels in a circular orbit 590 km above the surface of the earth. (See information given in Problems 11.153–11.155.)

SOLUTION



We have $$a_n = g\frac{R^2}{r^2} \quad \text{and} \quad a_n = \frac{v^2}{r}$$

Then $$g\frac{R^2}{r^2} = \frac{v^2}{r}$$

or $$v = R\sqrt{\frac{g}{r}} \quad \text{where} \quad r = R + h$$

The circumference s of the circular orbit is equal to

$$s = 2\pi r$$

Assuming that the speed of the telescope is constant, we have

$$s = vt_{\text{orbit}}$$

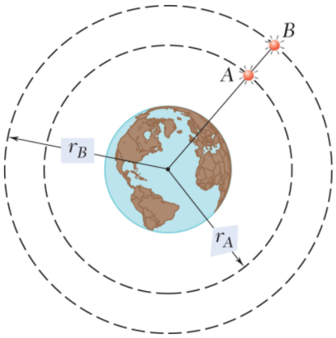
Substituting for s and v

$$2\pi r = R\sqrt{\frac{g}{r}}\,t_{\text{orbit}}$$

or $$t_{\text{orbit}} = \frac{2\pi}{R}\frac{r^{3/2}}{\sqrt{g}}$$

$$= \frac{2\pi}{6370\text{ km}}\frac{[(6370+590)\text{ km}]^{3/2}}{[9.81\times 10^{-3}\text{ km/s}^2]^{1/2}} \times \frac{1\text{ h}}{3600\text{ s}}$$

or $$t_{\text{orbit}} = 1.606\text{ h} \blacktriangleleft$$



PROBLEM 11.160

Satellites *A* and *B* are traveling in the same plane in circular orbits around the earth at altitudes of 120 and 200 mi, respectively. If at $t = 0$ the satellites are aligned as shown and knowing that the radius of the earth is $R = 3960$ mi, determine when the satellites will next be radially aligned. (See information given in Problems 11.153–11.155.)

SOLUTION

We have $$a_n = g\frac{R^2}{r^2} \quad \text{and} \quad a_n = \frac{v^2}{r}$$

Then $$g\frac{R^2}{r^2} = \frac{v^2}{r} \qquad \text{or} \qquad v = R\sqrt{\frac{g}{r}}$$

where $$r = R + h$$

The circumference *s* of a circular orbit is equal to $$s = 2\pi r$$

Assuming that the speeds of the satellites are constant, we have

$$s = vT$$

Substituting for *s* and *v*

$$2\pi r = R\sqrt{\frac{g}{r}}T$$

or $$T = \frac{2\pi}{R}\frac{r^{3/2}}{\sqrt{g}} = \frac{2\pi}{R}\frac{(R+h)^{3/2}}{\sqrt{g}}$$

Now $$h_B > h_A \Rightarrow (T)_B > (T)_A$$

Next let time T_C be the time at which the satellites are next radially aligned. Then, if in time T_C satellite *B* completes *N* orbits, satellite *A* must complete $(N+1)$ orbits.

Thus,

$$T_C = N(T)_B = (N+1)(T)_A$$

or $$N\left[\frac{2\pi}{R}\frac{(R+h_B)^{3/2}}{\sqrt{g}}\right] = (N+1)\left[\frac{2\pi}{R}\frac{(R+h_A)^{3/2}}{\sqrt{g}}\right]$$

or
$$N = \frac{(R+h_A)^{3/2}}{(R+h_B)^{3/2}-(R+h_A)^{3/2}} = \frac{1}{\left(\frac{R+h_B}{R+h_A}\right)^{3/2}-1}$$

$$= \frac{1}{\left(\frac{3960+200}{3960+120}\right)^{3/2}-1} = 33.835 \text{ orbits}$$

Then
$$T_C = N(T)_B = N\frac{2\pi}{R}\frac{(R+h_B)^{3/2}}{\sqrt{g}}$$

$$= 33.835\frac{2\pi}{3960 \text{ mi}}\frac{[(3960+200)\text{ mi}]^{3/2}}{\left(32.2 \text{ ft/s}^2 \times \frac{1 \text{ mi}}{5280 \text{ ft}}\right)^{1/2}} \times \frac{1 \text{ h}}{3600 \text{s}}$$

or $T_C = 51.2 \text{ h}$ ◀

Alternative solution

From above, we have $(T)_B > (T)_A$. Thus, when the satellites are next radially aligned, the angles θ_A and θ_B swept out by radial lines drawn to the satellites must differ by 2π. That is,

$$\theta_A = \theta_B + 2\pi$$

For a circular orbit $s = r\theta$

From above $s = vt$ and $v = R\sqrt{\frac{g}{r}}$

Then
$$\theta = \frac{s}{r} = \frac{vt}{r} = \frac{1}{r}\left(R\sqrt{\frac{g}{r}}\right)t = \frac{R\sqrt{g}}{r^{3/2}}t = \frac{R\sqrt{g}}{(R+h)^{3/2}}t$$

At time T_C:
$$\frac{R\sqrt{g}}{(R+h_A)^{3/2}}T_C = \frac{R\sqrt{g}}{(R+h_B)^{3/2}}T_C + 2\pi$$

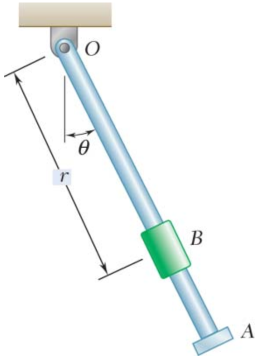
or
$$T_C = \frac{2\pi}{R\sqrt{g}\left[\frac{1}{(R+h_A)^{3/2}} - \frac{1}{(R+h_B)^{3/2}}\right]}$$

$$= \frac{2\pi}{(3960 \text{ mi})\left(32.2 \text{ ft/s}^2 \times \frac{1 \text{ mi}}{5280 \text{ ft}}\right)^{1/2}}$$

$$\times \frac{1}{\frac{1}{[(3960+120)\text{ mi}]^{3/2}} - \frac{1}{[(3960+200)\text{ mi}]^{3/2}}}$$

$$\times \frac{1 \text{ h}}{3600 \text{ s}}$$

or $T_C = 51.2 \text{ h}$ ◀



PROBLEM 11.161

The oscillation of rod OA about O is defined by the relation $\theta = (3/\pi)(\sin \pi t)$, where θ and t are expressed in radians and seconds, respectively. Collar B slides along the rod so that its distance from O is $r = 6(1 - e^{-2t})$ where r and t are expressed in inches and seconds, respectively. When $t = 1$ s, determine (*a*) the velocity of the collar, (*b*) the acceleration of the collar, (*c*) the acceleration of the collar relative to the rod.

SOLUTION

Calculate the derivatives with respect to time.

$$r = 6 - 6e^{-2t}\text{in.} \qquad \theta = \frac{3}{\pi}\sin \pi t \text{ rad}$$

$$\dot{r} = 12e^{-2t}\text{in/s} \qquad \dot{\theta} = 3\cos \pi t \text{ rad/s}$$

$$\ddot{r} = -24e^{-2t}\text{in/s}^2 \qquad \ddot{\theta} = -3\pi \sin \pi t \text{ rad/s}^2$$

At $t = 1$ s,

$$r = 6 - 6e^{-2} = 5.1880 \text{ in.} \qquad \theta = \frac{3}{\pi}\sin \pi = 0$$

$$\dot{r} = 12e^{-2} = 1.6240 \text{ in/s} \qquad \dot{\theta} = 3\cos \pi = -3 \text{ rad/s}$$

$$\ddot{r} = -24e^{-2} = -3.2480 \text{ in/s}^2 \qquad \ddot{\theta} = -3\pi \sin \pi = 0$$

(*a*) *Velocity of the collar*.

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta = 1.6240\,\mathbf{e}_r + (5.1880)(-3)\mathbf{e}_\theta$$

$$\mathbf{v} = (1.624 \text{ in/s})\mathbf{e}_r + (15.56 \text{ in/s})\mathbf{e}_\theta \blacktriangleleft$$

(*b*) *Acceleration of the collar*.

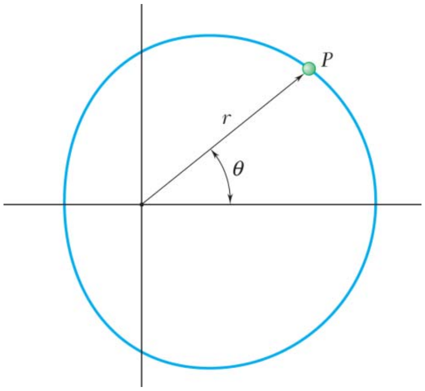
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

$$= [-3.2480 - (5.1880)(-3)^2]\mathbf{e}_r + (5.1880)(0) + (2)(1.6240)(-3)]\mathbf{e}_\theta$$

$$(-49.9 \text{ in/s}^2)\mathbf{e}_r + (-9.74 \text{ in/s}^2)\mathbf{e}_\theta \blacktriangleleft$$

(*c*) *Acceleration of the collar relative to the rod*.

$$\mathbf{a}_{B/OA} = \ddot{r}\mathbf{e}_r \qquad \mathbf{a}_{B/OA} = (-3.25 \text{ in/s}^2)\mathbf{e}_r \blacktriangleleft$$



PROBLEM 11.162

The path of a particle P is a limaçon. The motion of the particle is defined by the relations $r = b(2 + \cos \pi t)$ and $\theta = \pi t$, where t and θ are expressed in seconds and radians, respectively. Determine (a) the velocity and the acceleration of the particle when $t = 2$ s, (b) the value of θ for which the magnitude of the velocity is maximum.

SOLUTION

Differentiate the expressions for r and θ with respect to time.

$$r = b(2 + \cos \pi t),$$

$$\dot{r} = -\pi b \sin \pi t,$$

$$\ddot{r} = -\pi^2 b \cos \pi t$$

$$\theta = \pi t,$$

$$\dot{\theta} = \pi,$$

$$\ddot{\theta} = 0$$

(a) At $t = 2$ s, $\sin \pi t = 0, \quad \cos \pi t = 1$

$$r = 3b, \quad \dot{r} = 0, \quad \ddot{r} = -\pi^2 b, \quad \theta = 2\pi \text{ rad}, \quad \dot{\theta} = \pi \text{ rad/s}$$

$$v_r = \dot{r} = 0, \quad v_\theta = r\dot{\theta} = 3\pi b, \qquad \mathbf{v} = 3\pi b\mathbf{e}_\theta \blacktriangleleft$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = -\pi^2 b - (3b)\pi^2 = -4\pi^2 b$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0, \qquad \mathbf{a} = -4\pi^2 b\mathbf{e}_r \blacktriangleleft$$

(b) Values of θ for which v is maximum.

$$v_r = \dot{r} = -\pi b \sin \pi t$$

$$v_\theta = r\dot{\theta} = -b(2 + \cos \pi t)\pi$$

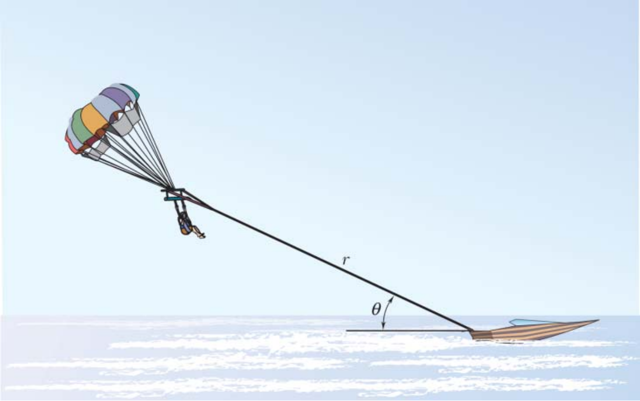
$$v^2 = v_r^2 + v_\theta^2 = \pi^2 b^2 \left[\sin^2 \pi t + (2 + \cos \pi t)^2\right]$$

$$= \pi^2 b^2 \left[\sin^2 \pi t + 4 + 4\cos \pi t + \cos^2 \pi t\right]$$

$$= \pi^2 b^2 (5 + 4\cos \pi t)$$

is maximum when $\cos \pi t = 1$ or $\pi t = 0, \ 2\pi, \ 4\pi, \ 6\pi,$ etc

$\theta = \pi t$, hence $\theta = 2N\pi, \ N = 0, 1, 2, \ldots \blacktriangleleft$



PROBLEM 11.163

During a parasailing ride, the boat is traveling at a constant 30 km/hr with a 200 m long tow line. At the instant shown, the angle between the line and the water is 30° and is increasing at a constant rate of 2°/s. Determine the velocity and acceleration of the parasailer at this instant.

SOLUTION

Given:

$\mathbf{v}_B = 30\mathbf{i}$ km/hr = $8.333\mathbf{i}$ m/s

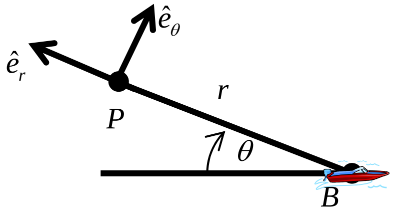
$\mathbf{a}_B = 0$

$r = 200$ m, $\dot{r} = 0$, $\ddot{r} = 0$

$\theta = 30°$

$\dot{\theta} = 2°/s = 0.0349$ rad/s

$\ddot{\theta} = 0$



Relative Motion relations:

$$\mathbf{v}_P = \mathbf{v}_B + \mathbf{v}_{P/B}$$

$$\mathbf{a}_P = \mathbf{a}_B + \mathbf{a}_{P/B}$$

Using Radial and Transverse components:

$$\mathbf{v}_{P/B} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\mathbf{a}_{P/B} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta$$

Substitute in known values:

$$\mathbf{v}_{P/B} = 6.981\mathbf{e}_\theta \text{ m/s}$$

$$\mathbf{a}_{P/B} = -0.2437\mathbf{e}_r \text{ m/s}^2$$

Change to rectangular coordinates:

$$\mathbf{v}_{P/B} = 6.981 * \sin 30°\mathbf{i} + 6.981 * \cos 30°\mathbf{j} \text{ m/s}$$

$$= 3.491\mathbf{i} + 6.046\mathbf{j} \text{ m/s}$$

$$\mathbf{a}_{P/B} = -0.2437 * \left(-\cos 30°\right)\mathbf{i} - 0.2437 * \sin 30°\mathbf{j} \text{ m/s}^2$$

$$= 0.2111\mathbf{i} - 0.1219\mathbf{j} \text{ m/s}^2$$

Substitute into Relative Motion relations:

$$\mathbf{v}_P = 8.33\mathbf{i} + 3.491\mathbf{i} + 6.046\mathbf{j} \text{ m/s}$$

$$= 11.824\mathbf{i} + 6.046\mathbf{j} \text{ m/s}$$

$$\mathbf{a}_P = 0 + 0.2111\mathbf{i} - 0.1219\mathbf{j} \text{ m/s}^2$$

$$= 0.2111\mathbf{i} - 0.1219\mathbf{j} \text{ m/s}^2$$

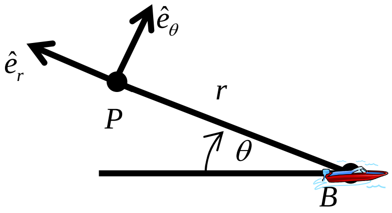
Velocity: $v_P = 13.280 \text{ m/s} \measuredangle 27.08°$ ◀

Acceleration: $a_P = 0.2437 \text{ m/s}^2 \measuredangle 30.00°$ ◀

PROBLEM 11.164

Some parasailing systems use a winch to pull the rider back to the boat. During the interval when θ is between 20^o and 40^o, (where t = 0 at $\theta = 20^\mathrm{o}$) the angle increases at the constant rate of 2 $^\mathrm{o}$/s. During this time, the length of the rope is defined by the relationship$r = 600 - \frac{1}{8}t^{5/2}$, where r and t are expressed in ft and s, respectively. Knowing that the boat is travelling at a constant rate of 15 knots (where 1 knot = 1.15 mi/h), (*a*) plot the magnitude of the velocity of the parasailer as a function of time (b) determine the magnitude of the acceleration of the parasailer when t = 5 s.

SOLUTION



Given:

$$\mathbf{v}_B = 15\mathbf{i}\ \text{knots}*\frac{1.15\text{mph}}{\text{knot}}*\frac{5280\,ft/mi}{3600s/hr} = 25.30\mathbf{i}\ \text{ft/s}$$

$$\mathbf{a}_B = 0$$

$$r = 600 - \frac{1}{8}t^{5/2}\ \text{m}$$

$$20° \le \theta \le 40°$$

$$\dot{\theta} = 2°/s = 0.0349\ \text{rad/s}$$

$$\ddot{\theta} = 0$$

Take Derivatives of r(t):

$$\dot{r} = -\frac{5}{16}t^{3/2}\ \text{m/s}$$

$$\ddot{r} = -\frac{15}{32}t^{1/2}\ \text{m/s}^2$$

Relative Motion relations:

$$\mathbf{v}_P = \mathbf{v}_B + \mathbf{v}_{P/B}$$

$$\mathbf{a}_P = \mathbf{a}_B + \mathbf{a}_{P/B}$$

Using Radial and Transverse components:

$$\mathbf{v}_{P/B} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\mathbf{a}_{P/B} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta$$

Change velocity to rectangular coordinates:

$$\mathbf{v}_{P/B} = \dot{r}(-\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + r\dot{\theta}\left(\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right)$$

Substitute into Relative Motion relations:

$$\mathbf{v}_P = \left(v_B - \dot{r}\cos\theta + r\dot{\theta}\sin\theta\right)\mathbf{i} + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\mathbf{j}$$

$$\mathbf{a}_P = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta$$

Note that:

$$\mathbf{v}_P = v_{Px}\mathbf{i} + v_{Py}\mathbf{j}$$

$$\mathbf{a}_P = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta$$

PROBLEM 11.164 (Continued)

Where:

$$v_{Px} = v_B - \dot{r}\cos\theta + r\dot{\theta}\sin\theta$$

$$v_{Py} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$

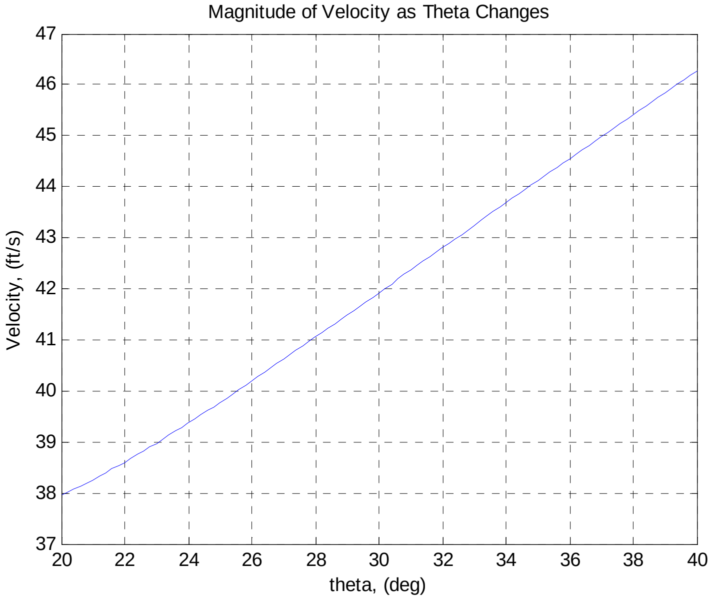
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

Magnitude of velocity:

$$\mathbf{v}_P = \sqrt{v_{Px}^2 + v_{Py}^2}$$

$$\mathbf{a}_P = \sqrt{a_r^2 + a_\theta^2}$$

(a) Plot of Velocity as a function of theta:



(b) Magnitude of Acceleration at t=5s:

$$r = 600 - \frac{1}{8}(5)^{5/2} = \ 593.012 \text{ m}$$

$$\dot{r} = -\frac{5}{16}(5)^{3/2} = \text{-}3.494 \text{ m/s}$$

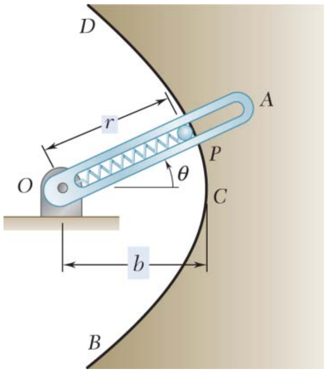
$$\ddot{r} = -\frac{15}{32}(5)^{1/2} = \text{-}1.048 \text{ m/s}^2$$

$$a_r = \left(\ddot{r} - r\dot{\theta}^2\right) = -1.771 \text{ m/s}^2$$

$$a_\theta = \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = -0.2439 \text{ m/s}^2$$

$$\mathbf{a}_P = \sqrt{-1.771^2 + -0.2439^2}$$

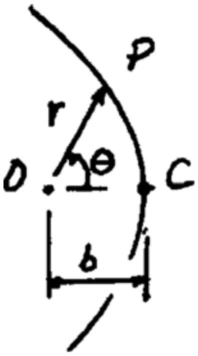
$$a_P = 1.787 \text{ m/s}^2 \blacktriangleleft$$



PROBLEM 11.165

As rod *OA* rotates, pin *P* moves along the parabola *BCD*. Knowing that the equation of this parabola is $r = 2b/(1+\cos\theta)$ and that $\theta = kt$, determine the velocity and acceleration of *P* when (*a*) $\theta = 0$, (*b*) $\theta = 90°$.

SOLUTION



$$r = \frac{2b}{1+\cos kt} \qquad \theta = kt$$

$$\dot{r} = \frac{2bk\sin kt}{(1+\cos kt)^2} \qquad \dot{\theta} = k \qquad \ddot{\theta} = 0$$

$$\ddot{r} = \frac{2bk}{(1+\cos kt)^4}[(1+\cos kt)^2 k\cos kt + (\sin kt)2(1+\cos kt)(k\sin kt)]$$

(*a*) When $\theta = kt = 0$:

$$r = b \qquad \dot{r} = 0 \qquad \ddot{r} = \frac{2bk}{(2)^4}[(2)^2 k(1)+0] = \frac{1}{2}bk^2$$

$$\theta = 0 \qquad \dot{\theta} = k \qquad \ddot{\theta} = 0$$

$$v_r = \dot{r} = 0 \qquad v_\theta = r\dot{\theta} = bk$$

$\mathbf{v} = bk\,\mathbf{e}_\theta$ ◀

$$\left.\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 = \frac{1}{2}bk^2 - bk^2 = -\frac{1}{2}bk^2 \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} = b(0) + 2(0) = 0 \end{aligned}\right\}$$

$\mathbf{a} = -\frac{1}{2}bk^2\mathbf{e}_r$ ◀

(*b*) When $\theta = kt = 90°$:

$$r = 2b \qquad \dot{r} = 2bk \qquad \ddot{r} = \frac{2bk}{19}[0+2k] = 4bk^2$$

$$\theta = 90° \qquad \dot{\theta} = k \qquad \ddot{\theta} = 0$$

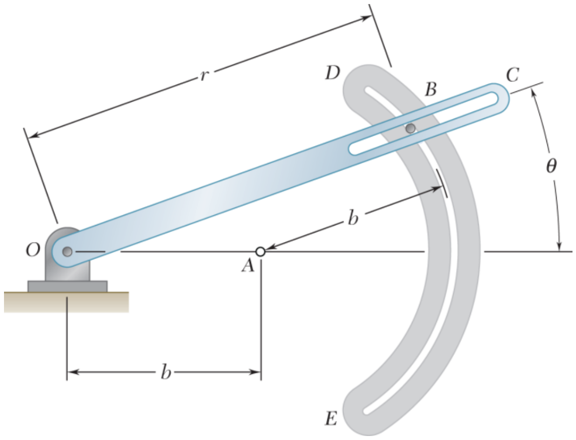
$$v_r = \dot{r} = 2bk \qquad v_\theta = r\dot{\theta} = 2bk$$

$\mathbf{v} = 2bk\,\mathbf{e}_r + 2bk\,\mathbf{e}_\theta$ ◀

$$a_r = \ddot{r} - r\dot{\theta}^2 = 4bk^2 - 2bk^2 = 2bk^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 2b(0) + 2(2bk)k = 4bk^2$$

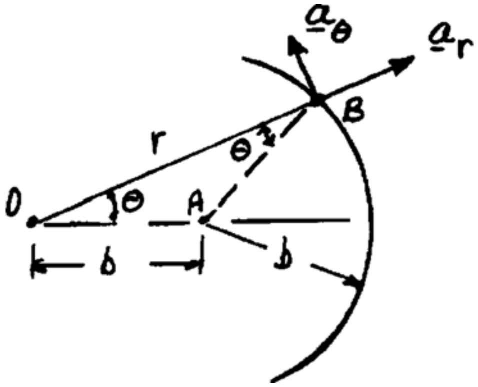
$\mathbf{a} = 2bk^2\mathbf{e}_r + 4bk^2\mathbf{e}_\theta$ ◀



PROBLEM 11.166

The pin at B is free to slide along the circular slot DE and along the rotating rod OC. Assuming that the rod OC rotates at a constant rate $\dot{\theta}$, (*a*) show that the acceleration of pin B is of constant magnitude, (*b*) determine the direction of the acceleration of pin B.

SOLUTION



From the sketch:

$$r = 2b\cos\theta$$

$$\dot{r} = -2b\sin\theta\,\dot{\theta}$$

Since $\dot{\theta} = \text{constant}$, $\quad \ddot{\theta} = 0$

$$\ddot{r} = -2b\cos\theta\,\dot{\theta}^2$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = -2b\cos\theta\,\dot{\theta}^2 - (2b\cos\theta)\dot{\theta}^2$$

$$a_r = -4b\cos\theta\,\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (2b\cos\theta)(0) + 2(-2b\sin\theta)\dot{\theta}^2$$

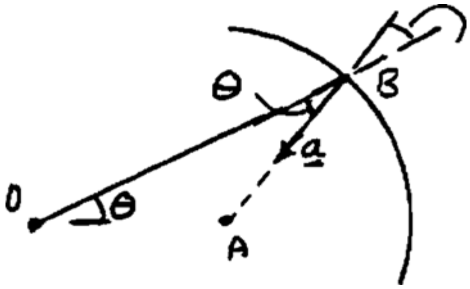
$$a_\theta = -4b\sin\theta\,\dot{\theta}^2$$

$$a = \sqrt{a_r^2 + a_\theta^2} = 4b\dot{\theta}^2\sqrt{(-\cos\theta)^2 + (-\sin\theta)^2}$$

$$a = 4b\dot{\theta}^2$$

Since both b and $\dot{\theta}$ are constant, we find that

$a = \text{constant}$ ◀

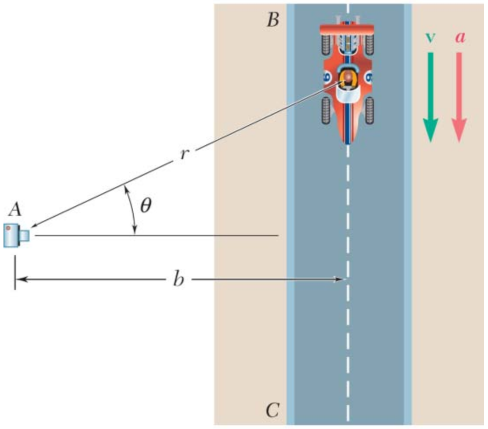


$$\gamma = \tan^{-1}\frac{a_\theta}{a_r} = \tan^{-1}\left(\frac{-4b\sin\theta\,\dot{\theta}^2}{-4b\cos\theta\,\dot{\theta}^2}\right)$$

$$\gamma = \tan^{-1}(\tan\theta)$$

$$\gamma = \theta$$

Thus, **a** is directed toward A ◀

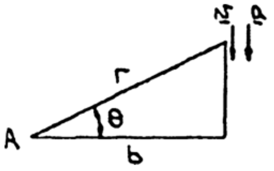


PROBLEM 11.167

To study the performance of a racecar, a high-speed camera is positioned at Point *A*. The camera is mounted on a mechanism which permits it to record the motion of the car as the car travels on straightway *BC*. Determine (*a*) the speed of the car in terms of *b*, θ, and $\dot{\theta}$, (*b*) the magnitude of the acceleration in terms of *b*, θ, $\dot{\theta}$, and $\ddot{\theta}$.

SOLUTION

(*a*) We have $$r = \frac{b}{\cos\theta}$$



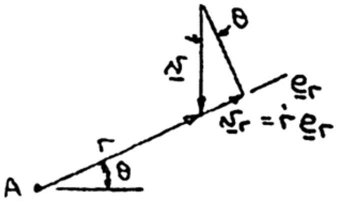
Then $$\dot{r} = \frac{b\dot{\theta}\sin\theta}{\cos^2\theta}$$

We have
$$v^2 = v_r^2 + v_\theta^2 = (\dot{r})^2 + (r\dot{\theta})^2$$
$$= \left(\frac{b\dot{\theta}\sin\theta}{\cos^2\theta}\right)^2 + \left(\frac{b\dot{\theta}}{\cos\theta}\right)^2$$
$$= \frac{b^2\dot{\theta}^2}{\cos^2\theta}\left(\frac{\sin^2\theta}{\cos^2\theta} + 1\right) = \frac{b^2\dot{\theta}^2}{\cos^4\theta}$$

or $$v = \pm\frac{b\dot{\theta}}{\cos^2\theta}$$

For the position of the car shown, θ is decreasing; thus, the negative root is chosen. $$v = -\frac{b\dot{\theta}}{\cos^2\theta} \blacktriangleleft$$

Alternative solution.



From the diagram $$\dot{r} = -v\sin\theta$$

or $$\frac{b\dot{\theta}\sin\theta}{\cos^2\theta} = -v\sin\theta$$

or $$v = -\frac{b\dot{\theta}}{\cos^2\theta} \blacktriangleleft$$

PROBLEM 11.167 (Continued)

(*b*) For rectilinear motion $a = \dfrac{dv}{dt}$

Using the answer from Part *a*

$$v = -\frac{b\dot{\theta}}{\cos^2\theta}$$

Then
$$a = \frac{d}{dt}\left(-\frac{b\dot{\theta}}{\cos^2\theta}\right)$$
$$= -b\frac{\ddot{\theta}\cos^2\theta - \dot{\theta}(-2\dot{\theta}\cos\theta\sin\theta)}{\cos^4\theta}$$

or
$$a = -\frac{b}{\cos^2\theta}(\ddot{\theta} + 2\dot{\theta}^2\tan\theta) \;◀$$

Alternative solution

From above
$$r = \frac{b}{\cos\theta} \qquad \dot{r} = \frac{b\dot{\theta}\sin\theta}{\cos^2\theta}$$

Then
$$\ddot{r} = b\frac{(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)(\cos^2\theta) - (\dot{\theta}\sin\theta)(-2\dot{\theta}\cos\theta\sin\theta)}{\cos^4\theta}$$
$$= b\left[\frac{\ddot{\theta}\sin\theta}{\cos^2\theta} + \frac{\dot{\theta}^2(1+\sin^2\theta)}{\cos^3\theta}\right]$$

Now
$$a^2 = a_r^2 + a_\theta^2$$

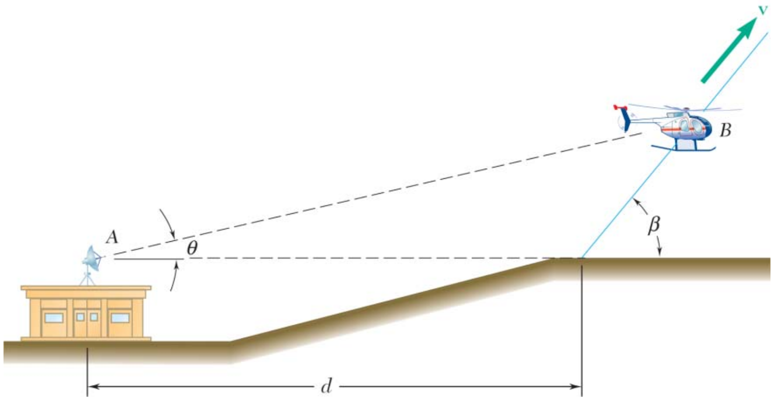
where
$$a_r = \ddot{r} - r\dot{\theta}^2 = b\left[\frac{\ddot{\theta}\sin\theta}{\cos^2\theta} + \frac{\dot{\theta}^2(1+\sin^2\theta)}{\cos^2\theta}\right] - \frac{b\dot{\theta}^2}{\cos\theta}$$
$$= \frac{b}{\cos^2\theta}\left(\ddot{\theta}\sin\theta + \frac{2\dot{\theta}^2\sin^2\theta}{\cos\theta}\right)$$
$$a_r = \frac{b\sin\theta}{\cos^2\theta}(\ddot{\theta} + 2\dot{\theta}^2\tan\theta)$$

and
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{b\ddot{\theta}}{\cos\theta} + 2\frac{b\dot{\theta}^2\sin\theta}{\cos^2\theta}$$
$$= \frac{b\cos\theta}{\cos^2\theta}(\ddot{\theta} + 2\dot{\theta}\tan\theta)$$

Then
$$a = \pm\frac{b}{\cos^2\theta}(\ddot{\theta} + 2\dot{\theta}^2\tan\theta)[(\sin\theta)^2 + (\cos\theta)^2]^{1/2}$$

For the position of the car shown, $\ddot{\theta}$ is negative; for *a* to be positive, the negative root is chosen.

$$a = -\frac{b}{\cos^2\theta}(\ddot{\theta} + 2\dot{\theta}^2\tan\theta) \;◀$$



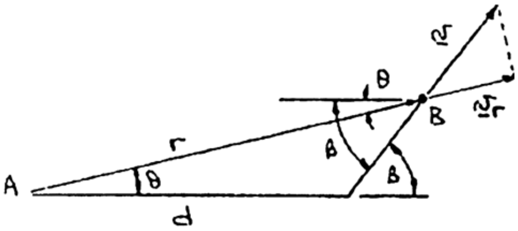
PROBLEM 11.168

After taking off, a helicopter climbs in a straight line at a constant angle β. Its flight is tracked by radar from Point A. Determine the speed of the helicopter in terms of d, β, θ, and $\dot{\theta}$.

SOLUTION

From the diagram

$$\frac{r}{\sin(180^\circ - \beta)} = \frac{d}{\sin(\beta - \theta)}$$



or $$d \sin\beta = r(\sin\beta\cos\theta - \cos\beta\sin\theta)$$

or $$r = d\,\frac{\tan\beta}{\tan\beta\cos\theta - \sin\theta}$$

Then $$\dot{r} = d\tan\beta\,\frac{-(-\tan\beta\sin\theta - \cos\theta)}{(\tan\beta\cos\theta - \sin\theta)^2}\,\dot{\theta}$$

$$= d\dot{\theta}\tan\beta\,\frac{\tan\beta\sin\theta + \cos\theta}{(\tan\beta\cos\theta - \sin\theta)^2}$$

From the diagram

$$v_r = v\cos(\beta - \theta) \qquad \text{where} \qquad v_r = \dot{r}$$

Then

$$d\dot{\theta}\tan\beta\,\frac{\tan\beta\sin\theta + \cos\theta}{(\tan\beta\cos\theta - \sin\theta)^2} = v(\cos\beta\cos\theta + \sin\beta\sin\theta)$$

$$= v\cos\beta\,(\tan\beta\sin\theta + \cos\theta)$$

or $$v = \frac{d\dot{\theta}\tan\beta\sec\beta}{(\tan\beta\cos\theta - \sin\theta)^2} \quad ◀$$

Alternative solution.

We have $$v^2 = v_r^2 + v_\theta^2 = (\dot{r})^2 + (r\dot{\theta})^2$$

PROBLEM 11.168 (Continued)

Using the expressions for r and $\dot{r}$ from above

$$v = \left[d\dot{\theta} \tan\beta \frac{\tan\beta \sin\theta + \cos\theta}{(\tan\beta \cos\theta - \sin\theta)^2} \right]^2$$

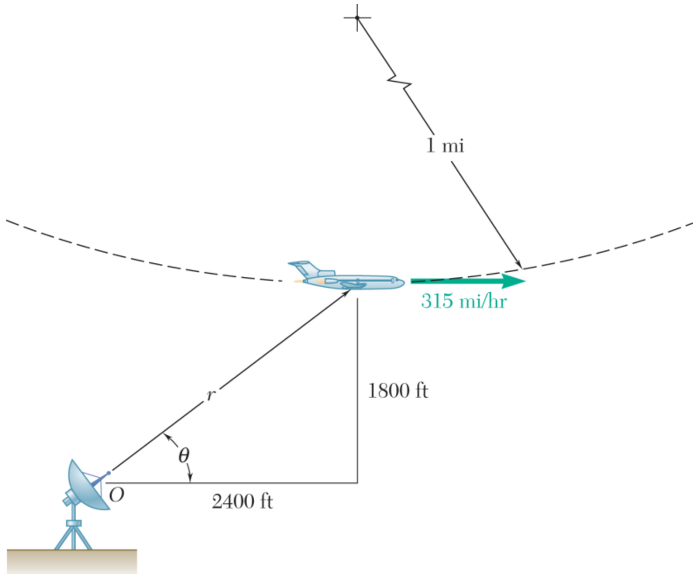
or

$$v = \pm \frac{d\dot{\theta} \tan\beta}{(\tan\beta \cos\theta - \sin\theta)} \left[\frac{(\tan\beta \sin\theta + \cos\theta)^2}{(\tan\beta \cos\theta - \sin\theta)^2} + 1 \right]^{1/2}$$

$$= \pm \frac{d\dot{\theta} \tan\beta}{(\tan\beta \cos\theta - \sin\theta)} \left[\frac{\tan^2\beta + 1}{(\tan\beta \cos\theta - \sin\theta)^2} \right]^{1/2}$$

Note that as θ increases, the helicopter moves in the indicated direction. Thus, the positive root is chosen.

$$v = \frac{d\dot{\theta} \tan\beta \sec\beta}{(\tan\beta \cos\theta - \sin\theta)^2} \blacktriangleleft$$



PROBLEM 11.169

At the bottom of a loop in the vertical plane, an airplane has a horizontal velocity of 315 mi/h and is speeding up at a rate of 10 ft/s^2. The radius of curvature of the loop is 1 mi. The plane is being tracked by radar at *O*. What are the recorded values of $\dot{r}$, $\ddot{r}$, $\dot{\theta}$ and $\ddot{\theta}$ for this instant?

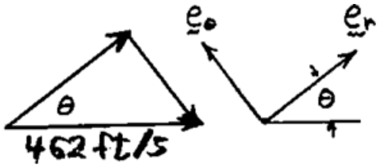
SOLUTION

Geometry. The polar coordinates are

$$r = \sqrt{(2400)^2 + (1800)^2} = 3000 \text{ ft} \qquad \theta = \tan^{-1}\left(\frac{1800}{2400}\right) = 36.87°$$

Velocity Analysis.

$$\mathbf{v} = 315 \text{ mi/h} = 462 \text{ ft/s} \longrightarrow$$



$$v_r = 462\cos\theta = 369.6 \text{ ft/s}$$

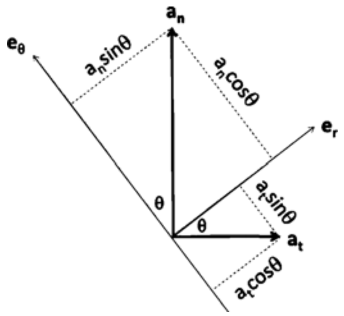
$$v_\theta = -462\sin\theta = -277.2 \text{ ft/s}$$

$$v_r = \dot{r} \qquad \dot{r} = 370 \text{ ft/s} \blacktriangleleft$$

$$v_\theta = r\dot{\theta} \qquad \dot{\theta} = \frac{v_\theta}{r} = -\frac{277.2}{3000}$$

$$\dot{\theta} = -0.0924 \text{ rad/s} \blacktriangleleft$$

Acceleration analysis.



$$a_t = 10 \text{ ft/s}^2$$

$$a_n = \frac{v^2}{\rho} = \frac{(462)^2}{5280} = 40.425 \text{ ft/s}^2$$

PROBLEM 11.169 (Continued)

$$a_r = a_t \cos\theta + a_n \sin\theta = 10 \cos 36.87° + 40.425 \sin 36.87° = 32.255 \text{ ft/s}^2$$

$$a_\theta = -a_t \sin\theta + a_n \cos\theta = -10 \sin 36.87° + 40.425 \cos 36.87° = 26.34 \text{ ft/s}^2$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad \ddot{r} = a_r + r\dot{\theta}^2$$

$$\ddot{r} = 32.255 + (3000)(-0.0924)^2$$

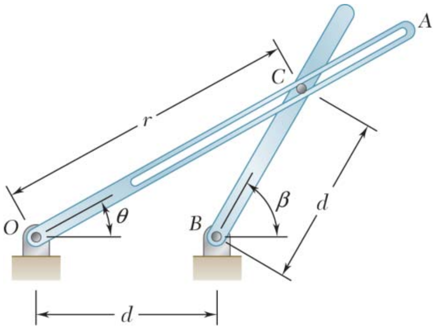
$\ddot{r} = 57.9 \text{ ft/s}^2$ ◀

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$\ddot{\theta} = \frac{a_\theta}{r} - \frac{2\dot{r}\dot{\theta}}{r}$$

$$= \frac{26.34}{3000} - \frac{(2)(369.6)(-0.0924)}{3000}$$

$\ddot{\theta} = 0.0315 \text{ rad/s}^2$ ◀



PROBLEM 11.170

Pin *C* is attached to rod *BC* and slides freely in the slot of rod *OA* which rotates at the constant rate ω. At the instant when $\beta = 60°$, determine (*a*) $\dot{r}$ and $\dot{\theta}$, (*b*) $\ddot{r}$ and $\ddot{\theta}$. Express your answers in terms of *d* and ω.

SOLUTION

Looking at *d* and β as polar coordinates with $\dot{d} = 0$,

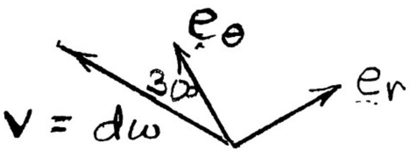
$$v_\beta = d\dot{\beta} = d\omega, \qquad v_d = \dot{d} = 0$$

$$a_\beta = d\ddot{\beta} + 2\dot{d}\dot{\beta} = 0, \qquad a_d = \ddot{d} - d\dot{\beta}^2 = -d\omega^2$$

Geometry analysis: $r = d\sqrt{3}$ for angles shown.

(*a*) Velocity analysis:

Sketch the directions of $\mathbf{v}$, $\mathbf{e}_r$ and $\mathbf{e}_\theta$.



$$v_r = \dot{r} = \mathbf{v}\cdot\mathbf{e}_r = d\omega\cos 120°$$

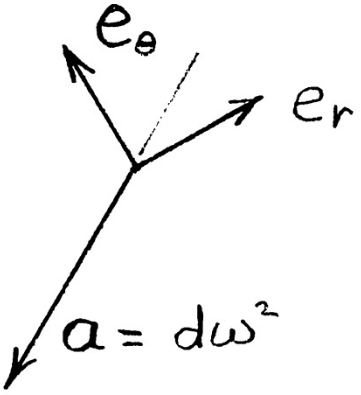
$$\dot{r} = -\frac{1}{2}d\omega \blacktriangleleft$$

$$v_\theta = r\dot{\theta} = \mathbf{v}\cdot\mathbf{e}_\theta = d\omega\cos 30°$$

$$\dot{\theta} = \frac{d\omega\cos 30°}{r} = \frac{d\omega\frac{\sqrt{3}}{2}}{d\sqrt{3}} \qquad \dot{\theta} = \frac{1}{2}\omega \blacktriangleleft$$

(*b*) Acceleration analysis:

Sketch the directions of $\mathbf{a}$, $\mathbf{e}_r$ and $\mathbf{e}_\theta$.



$$a_r = \mathbf{a}\cdot\mathbf{e}_r = a\cos 150° = -\frac{\sqrt{3}}{2}d\omega^2$$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\sqrt{3}}{2}d\omega^2$$

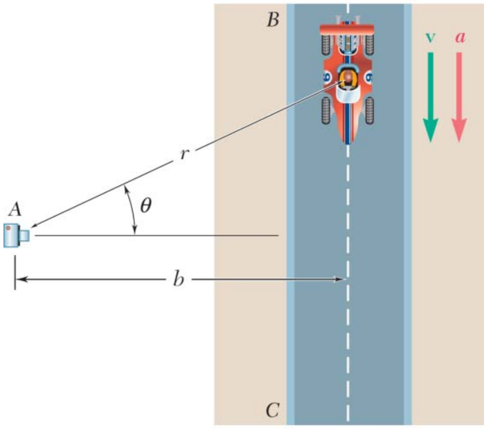
$$\ddot{r} = -\frac{\sqrt{3}}{2}d\omega^2 + r\dot{\theta}^2 = -\frac{\sqrt{3}}{2}d\omega^2 + d\sqrt{3}\left(\frac{1}{2}\omega\right)^2$$

$$\ddot{r} = -\frac{\sqrt{3}}{4}d\omega^2 \blacktriangleleft$$

$$a_\theta = \mathbf{a}\cdot\mathbf{e}_\theta = d\omega^2\cos 120° = -\frac{1}{2}d\omega^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$\ddot{\theta} = \frac{1}{r}(a_\theta - 2\dot{r}\dot{\theta}) = \frac{1}{\sqrt{3}d}\left[-\frac{1}{2}d\omega^2 - (2)\left(-\frac{1}{2}d\omega\right)\left(\frac{1}{2}\omega\right)\right] \qquad \ddot{\theta} = 0 \blacktriangleleft$$



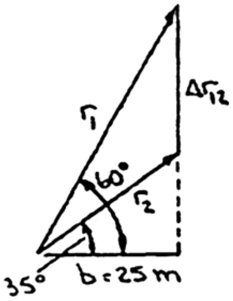
PROBLEM 11.171

For the racecar of Problem 11.167, it was found that it took 0.5 s for the car to travel from the position $\theta = 60°$ to the position $\theta = 35°$. Knowing that $b = 25$ m, determine the average speed of the car during the 0.5-s interval.

PROBLEM 11.167 To study the performance of a racecar, a high-speed camera is positioned at Point *A*. The camera is mounted on a mechanism which permits it to record the motion of the car as the car travels on straightway *BC*. Determine (*a*) the speed of the car in terms of b, θ, and $\dot{\theta}$, (*b*) the magnitude of the acceleration in terms of b, θ, $\dot{\theta}$, and $\ddot{\theta}$.

SOLUTION

From the diagram:



$$\Delta r_{12} = 25 \tan 60° - 25 \tan 35°$$

$$= 25.796 \text{ m}$$

Now

$$v_{\text{ave}} = \frac{\Delta r_{12}}{\Delta t_{12}}$$

$$= \frac{25.796 \text{ m}}{0.5 \text{ s}}$$

$$= 51.592 \text{ m/s}$$

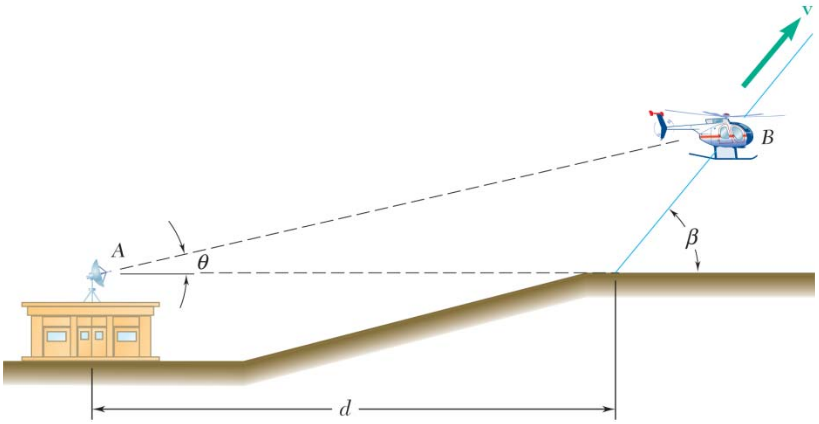
or

$$v_{\text{ave}} = 185.7 \text{ km/h} \blacktriangleleft$$

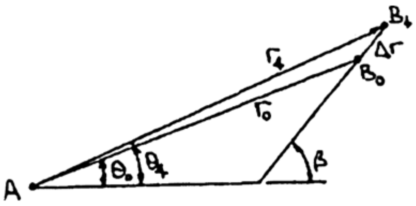
PROBLEM 11.172

For the helicopter of Problem 11.168, it was found that when the helicopter was at *B*, the distance and the angle of elevation of the helicopter were $r = 3000$ ft and $\theta = 20°$, respectively. Four seconds later, the radar station sighted the helicopter at $r = 3320$ ft and $\theta = 23.1°$. Determine the average speed and the angle of climb β of the helicopter during the 4-s interval.

PROBLEM 11.168 After taking off, a helicopter climbs in a straight line at a constant angle β. Its flight is tracked by radar from Point *A*. Determine the speed of the helicopter in terms of d, β, θ, and $\dot{\theta}$.



SOLUTION



We have $r_0 = 3000$ ft $\theta_0 = 20°$

$r_4 = 3320$ ft $\theta_4 = 23.1°$

From the diagram:

$$\Delta r^2 = 3000^2 + 3320^2 - 2(3000)(3320)\cos(23.1° - 20°)$$

or $\Delta r = 362.70$ ft

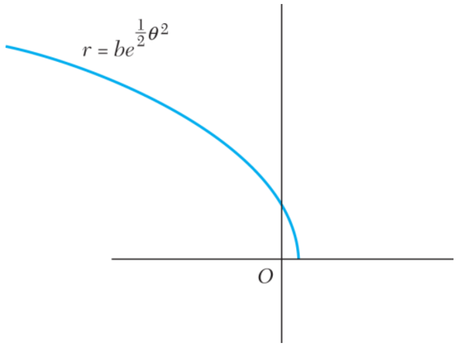
Now
$$v_{\text{ave}} = \frac{\Delta r}{\Delta t} = \frac{362.70 \text{ ft}}{4 \text{ s}} = 90.675 \text{ ft/s}$$

or $v_{\text{ave}} = 61.8$ mi/h ◀

Also, $\Delta r \cos\beta = r_4 \cos\theta_4 - r_0 \cos\theta_0$

or
$$\cos\beta = \frac{3320 \cos 23.1° - 3000 \cos 20°}{362.70}$$

or $\beta = 49.7°$ ◀



PROBLEM 11.173

A particle moves along the spiral shown. Determine the magnitude of the velocity of the particle in terms of $b,\ \theta,$ and $\dot{\theta}.$

SOLUTION

Given:

$$r = be^{\frac{1}{2}\theta^2}$$

Take time derivative

$$\dot{r} = be^{\frac{1}{2}\theta^2}\theta\dot{\theta}$$

Radial and Transverse Components

$$v_r = \dot{r} = be^{\frac{1}{2}\theta^2}\theta\dot{\theta}$$

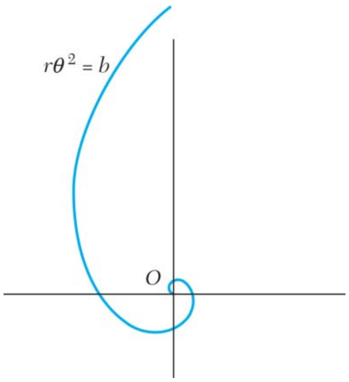
$$v_\theta = r\dot{\theta} = be^{\frac{1}{2}\theta^2}\dot{\theta}$$

Magnitude of Velocity

$$v^2 = v_r^2 + v_\theta^2$$

$$= \left(be^{\frac{1}{2}\theta^2}\right)^2\left(\theta^2 + 1\right)\dot{\theta}^2$$

$$v = \left|be^{\frac{1}{2}\theta^2}\left(\theta^2 + 1\right)^{1/2}\dot{\theta}\right| ◀$$



PROBLEM 11.174

A particle moves along the spiral shown. Determine the magnitude of the velocity of the particle in terms of b, θ, and $\dot{\theta}$.

SOLUTION

Given: $r = \dfrac{b}{\theta^2}$

Take time derivative $\dot{r} = -\dfrac{2b}{\theta^3}\dot{\theta}$

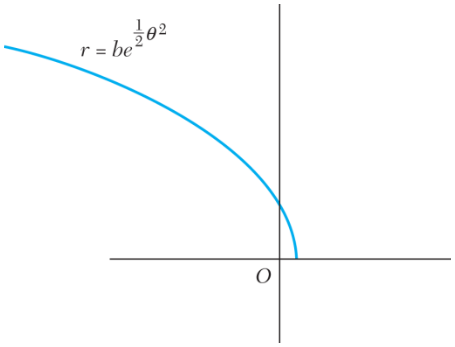
Radial and Transverse Components $v_r = \dot{r} = -\dfrac{2b}{\theta^3}\dot{\theta}$

$$v_\theta = r\dot{\theta} = \frac{b}{\theta^2}\dot{\theta}$$

Magnitude of Velocity

$$\begin{aligned} v^2 &= v_r^2 + v_\theta^2 \\ &= \frac{4b^2}{\theta^6}\dot{\theta}^2 + \frac{b^2}{\theta^4}\dot{\theta}^2 \\ &= \frac{b^2}{\theta^6}\left(4 + \theta^2\right)\dot{\theta}^2 \end{aligned}$$

$$v = \left|\frac{b}{\theta^3}\left(4 + \theta^2\right)^{1/2}\dot{\theta}\right| \blacktriangleleft$$



PROBLEM 11.175

A particle moves along the spiral shown. Knowing that $\dot{\theta}$ is constant and denoting this constant by ω, determine the magnitude of the acceleration of the particle in terms of b, θ, and $\dot{\theta}$.

SOLUTION

Given:

$$r = be^{\frac{1}{2}\theta^2}$$

$$\dot{\theta} = \omega \qquad \text{and} \qquad \ddot{\theta} = 0$$

Take time derivatives

$$\dot{r} = be^{\frac{1}{2}\theta^2}\theta\dot{\theta}$$

$$\ddot{r} = be^{\frac{1}{2}\theta^2}\left[\left(\theta\dot{\theta}\right)^2 + \dot{\theta}^2 + \theta\ddot{\theta}\right]$$

Radial Component of Acceleration

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$= be^{\frac{1}{2}\theta^2}\left[\left(\theta\dot{\theta}\right)^2 + \dot{\theta}^2 + \theta\ddot{\theta} - \dot{\theta}^2\right]$$

$$= be^{\frac{1}{2}\theta^2}\left[\left(\theta\dot{\theta}\right)^2 + \theta\ddot{\theta}\right]$$

Transverse Component of Acceleration

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$= be^{\frac{1}{2}\theta^2}\ddot{\theta} + 2be^{\frac{1}{2}\theta^2}\theta\dot{\theta}^2$$

$$= be^{\frac{1}{2}\theta^2}\left[\ddot{\theta} + 2\theta\dot{\theta}^2\right]$$

Substitute Given Values

$$\dot{\theta} = \omega \qquad \text{and} \qquad \ddot{\theta} = 0$$

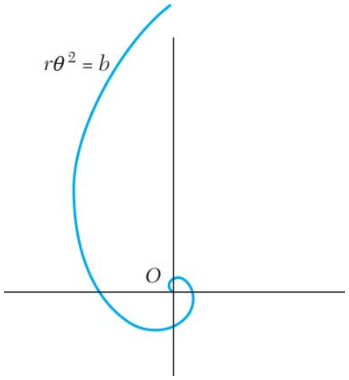
$$a_r = be^{\frac{1}{2}\theta^2}\left(\theta\omega\right)^2 \qquad \text{and} \qquad a_\theta = be^{\frac{1}{2}\theta^2}\left(2\theta\omega^2\right)$$

Magnitude of Acceleration

$$a^2 = a_r^2 + a_\theta^2$$

$$= \left(be^{\frac{1}{2}\theta^2}\right)^2\left(\theta^4 + 4\theta^2\right)\omega^4$$

$$a = be^{\frac{1}{2}\theta^2}\theta\left(\theta^2 + 4\right)^{\frac{1}{2}}\omega^2 \blacktriangleleft$$



PROBLEM 11.176

A particle moves along the spiral shown. Knowing that $\dot{\theta}$ is constant and denoting this constant by ω, determine the magnitude of the acceleration of the particle in terms of b, θ, and $\dot{\theta}$.

SOLUTION

Given:

$$r = \frac{b}{\theta^2}, \quad \dot{\theta} = \omega \text{ and } \ddot{\theta} = 0$$

Take time derivatives

$$\dot{r} = -\frac{2b}{\theta^3}\dot{\theta}$$

$$\ddot{r} = -\frac{2b}{\theta^3}\ddot{\theta} + \frac{6b}{\theta^4}\dot{\theta}^2$$

Radial Component of Acceleration

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ &= -\frac{2b}{\theta^3}\ddot{\theta} + \frac{6b}{\theta^4}\dot{\theta}^2 - \frac{b}{\theta^2}\dot{\theta}^2 \\ &= \frac{b}{\theta^4}\left(-2\theta\ddot{\theta} + 6\dot{\theta}^2 - \theta^2\dot{\theta}^2\right) \end{aligned}$$

Transverse Component of Acceleration

$$\begin{aligned} a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \\ &= \frac{b}{\theta^2}\ddot{\theta} + (2)\left(-\frac{2b}{\theta^3}\right)\dot{\theta}^2 \\ &= \frac{b}{\theta^3}\left(\theta\ddot{\theta} - 4\dot{\theta}^2\right) \end{aligned}$$

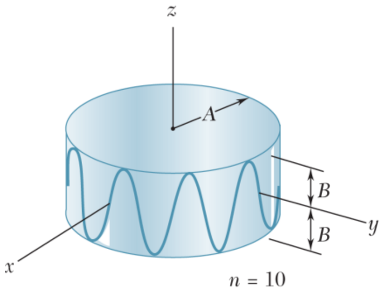
Substitute Given Values

$$a_r = \frac{b}{\theta^4}\left(6 - \theta^2\right)\omega^2 \qquad \text{and} \qquad a_\theta = -\frac{4b}{\theta^3}\omega^2$$

Magnitude of Acceleration

$$\begin{aligned} a^2 &= a_r^2 + a_\theta^2 \\ &= \frac{b^2}{\theta^8}\left(36 - 12\theta^2 + \theta^4\right)\omega^4 + \frac{16b^2}{\theta^6}\omega^4 \\ &= \frac{b^2}{\theta^8}\left(36 + 4\theta^2 + \theta^4\right)\omega^4 \end{aligned}$$

$$a = \frac{b}{\theta^4}\left(36 + 4\theta^2 + \theta^4\right)^{1/2}\omega^2 \blacktriangleleft$$



PROBLEM 11.177

The motion of a particle on the surface of a right circular cylinder is defined by the relations $R = A$, $\theta = 2\pi t$, and $z = B\sin 2\pi nt$, where A and B are constants and n is an integer. Determine the magnitudes of the velocity and acceleration of the particle at any time t.

SOLUTION

$$R = A \qquad \theta = 2\pi t \qquad z = B\sin 2\pi nt$$

$$\dot{R} = 0 \qquad \dot{\theta} = 2\pi \qquad \dot{z} = 2\pi n\, B\cos 2\pi nt$$

$$\ddot{R} = 0 \qquad \ddot{\theta} = 0 \qquad \ddot{z} = -4\pi^2 n^2\, B\sin 2\pi nt$$

Velocity (Eq. 11.49)

$$\mathbf{v} = \dot{R}\mathbf{e}_R + R\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{k}$$

$$\mathbf{v} = \quad + A(2\pi)\mathbf{e}_\theta + 2\pi n\, B\cos 2\pi nt\,\mathbf{k}$$

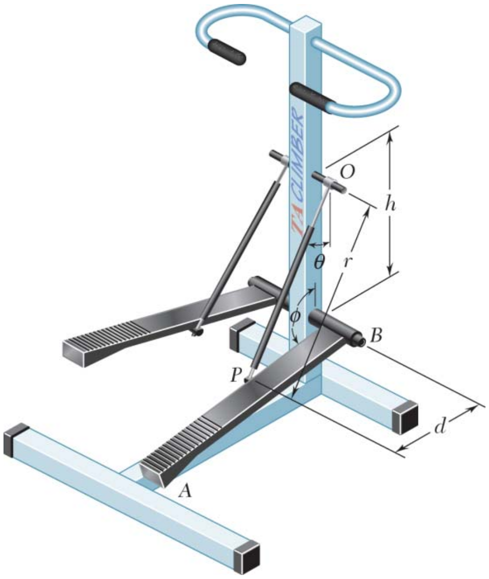
$$v = 2\pi\sqrt{A^2 + n^2B^2\cos^2 2\pi nt} \blacktriangleleft$$

Acceleration (Eq. 11.50)

$$\mathbf{a} = (\ddot{R} - R\dot{\theta}^2)\mathbf{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\mathbf{e}_\theta + \ddot{z}\mathbf{k}$$

$$\mathbf{a} = -4\pi^2 A\mathbf{e}_k - 4\pi^2 n^2\, B\sin 2\pi nt\,\mathbf{k}$$

$$a = 4\pi^2\sqrt{A^2 + n^4B^2\sin^2 2\pi nt} \blacktriangleleft$$



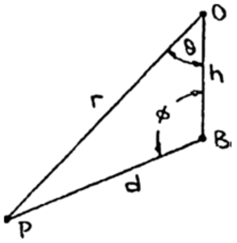
PROBLEM 11.178

Show that $\dot{r} = h\dot{\phi}\sin\theta$ knowing that at the instant shown, step *AB* of the step exerciser is rotating counterclockwise at a constant rate $\dot{\phi}$.

SOLUTION

From the diagram

$$r^2 = d^2 + h^2 - 2dh\cos\phi$$



Then $$2r\dot{r} = 2dh\dot{\phi}\sin\phi$$

Now $$\frac{r}{\sin\phi} = \frac{d}{\sin\theta}$$

or $$r = \frac{d\sin\phi}{\sin\theta}$$

Substituting for r in the expression for $\dot{r}$

$$\left(\frac{d\sin\phi}{\sin\theta}\right)\dot{r} = dh\dot{\phi}\sin\phi$$

or $$\dot{r} = h\dot{\phi}\sin\theta \qquad \text{Q.E.D.}$$ ◀

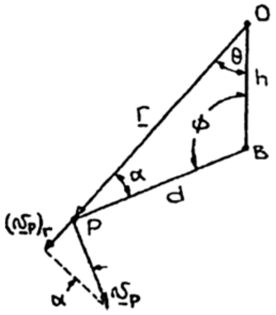
Alternative solution.

First note $$\alpha = 180^\circ - (\phi + \theta)$$

Now $$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_\theta = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

With B as the origin

$$v_P = d\dot{\phi} \qquad (d = \text{constant} \Rightarrow \dot{d} = 0)$$



With O as the origin $(v_P)_r = \dot{r}$

where $(v_P)_r = v_P \sin\alpha$

Then $\dot{r} = d\dot{\phi}\sin\alpha$

Now $\dfrac{h}{\sin\alpha} = \dfrac{d}{\sin\theta}$

or $d\sin\alpha = h\sin\theta$

substituting $\dot{r} = h\dot{\phi}\sin\theta$ Q.E.D.

PROBLEM 11.179

The three-dimensional motion of a particle is defined by the relations $R = A(1-e^{-t}), \theta = 2\pi t,$ and $z = B(1-e^{-t})$. Determine the magnitudes of the velocity and acceleration when (*a*) $t = 0$, (*b*) $t = \infty$.

SOLUTION

$$R = A(1-e^{-t}) \qquad \theta = 2\pi t \qquad z = B(1-e^{-t})$$

$$\dot{R} = Ae^{-t} \qquad \dot{\theta} = 2\pi \qquad \dot{z} = Be^{-t}$$

$$\ddot{R} = -Ae^{-t} \qquad \ddot{\theta} = 0 \qquad \ddot{z} = -Be^{-t}$$

Velocity (Eq. 11.49)

$$\mathbf{v} = \dot{R}\mathbf{e}_R + R\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{k}$$

$$\mathbf{v} = Ae^{-t}\mathbf{e}_R + 2\pi A(1-e^{-t})\mathbf{e}_\theta + Be^{-t}\mathbf{k}$$

(*a*) When $t = 0$: $e^{-t} = e^0 = 1$; $\mathbf{v} = A\mathbf{e}_R + B\mathbf{k}$ $\qquad v = \sqrt{A^2 + B^2}$ ◀

(*b*) When $t = \infty$: $e^{-t} = e^{-\infty} = 0$ $\quad \mathbf{v} = 2\pi A\mathbf{e}_\theta$ $\qquad v = 2\pi A$ ◀

Acceleration (Eq. 11.50)

$$\mathbf{a} = (\ddot{R} - R\dot{\theta}^1)\mathbf{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\mathbf{e}_\theta + \ddot{z}\,\mathbf{k}$$

$$= [-Ae^{-t} - A(1-e^{-t})4\pi^2]\mathbf{e}_R + [0 + 2Ae^{-t}(2\pi)]\mathbf{e}_\theta - Be^{-t}\mathbf{k}$$

(*a*) When $t = 0$: $e^{-t} = e^0 = 1$

$$\mathbf{a} = -A\mathbf{e}_R + 4\pi A\mathbf{e}_\theta - B\,\mathbf{k}$$

$$a = \sqrt{A^2 + (4\pi A)^2 + B^2} \qquad a = \sqrt{(1+16\pi^2)A^2 + B^2}$$ ◀

(*b*) When $t = \infty$: $e^{-t} = e^{-\infty} = 0$

$$\mathbf{a} = -4\pi^2 A\mathbf{e}_R \qquad a = 4\pi^2 A$$ ◀

PROBLEM 11.180*

For the conic helix of Problem 11.95, determine the angle that the osculating plane forms with the *y* axis.

PROBLEM 11.95 The three-dimensional motion of a particle is defined by the position vector $\mathbf{r} = (Rt\cos\omega_n t)\mathbf{i} + ct\mathbf{j} + (Rt\sin\omega_n t)\mathbf{k}$. Determine the magnitudes of the velocity and acceleration of the particle. (The space curve described by the particle is a conic helix.)

SOLUTION

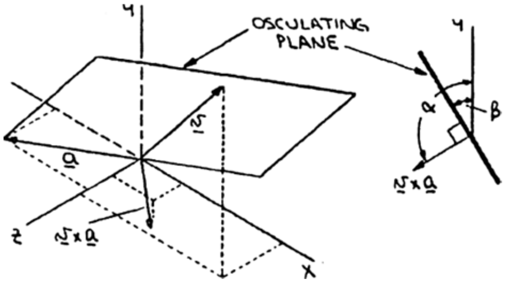
First note that the vectors **v** and **a** lie in the osculating plane.

Now $$\mathbf{r} = (Rt\cos\omega_n t)\mathbf{i} + ct\mathbf{j} + (Rt\sin\omega_n t)\mathbf{k}$$

Then $$\mathbf{v} = \frac{dr}{dt} = R(\cos\omega_n t - \omega_n t\sin\omega_n t)\mathbf{i} + c\mathbf{j} + R(\sin\omega_n t + \omega_n t\cos\omega_n t)\mathbf{k}$$

and $$\begin{aligned}\mathbf{a} &= \frac{dv}{dt}\\ &= R\left(-\omega_n\sin\omega_n t - \omega_n\sin\omega_n t - \omega_n^2 t\cos\omega_n t\right)\mathbf{i}\\ &\quad + R\left(\omega_n\cos\omega_n t + \omega_n\cos\omega_n t - \omega_n^2 t\sin\omega_n t\right)\mathbf{k}\\ &= \omega_n R[-(2\sin\omega_n t + \omega_n t\cos\omega_n t)\mathbf{i} + (2\cos\omega_n t - \omega_n t\sin\omega_n t)\mathbf{k}]\end{aligned}$$

It then follows that the vector $(\mathbf{v}\times\mathbf{a})$ is perpendicular to the osculating plane.



$$(\mathbf{v}\times\mathbf{a}) = \omega_n R\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ R(\cos\omega_n t - \omega_n t\sin\omega_n t) & c & R(\sin\omega_n t + \omega_n t\cos\omega_n t)\\ -(2\sin\omega_n t + \omega_n t\cos\omega_n t) & 0 & (2\cos\omega_n t - \omega_n t\sin\omega_n t)\end{vmatrix}$$

$$\begin{aligned}&= \omega_n R\{c(2\cos\omega_n t - \omega_n t\sin\omega_n t)\mathbf{i} + R[-(\sin\omega_n t + \omega_n t\cos\omega_n t)(2\sin\omega_n t + \omega_n t\cos\omega_n t)\\ &\quad - (\cos\omega_n t - \omega_n t\sin\omega_n t)(2\cos\omega_n t - \omega_n t\sin\omega_n t)]\mathbf{j} + c(2\sin\omega_n t + \omega_n t\cos\omega_n t)\mathbf{k}\\ &= \omega_n R\left[c(2\cos\omega_n t - \omega_n t\sin\omega_n t)\mathbf{i} - R\left(2 + \omega_n^2 t^2\right)\mathbf{j} + c(2\sin\omega_n t + \omega_n t\cos\omega_n t)\mathbf{k}\right]\end{aligned}$$

PROBLEM 11.180* (Continued)

The angle α formed by the vector $(\mathbf{v}\times\mathbf{a})$ and the y axis is found from

$$\cos\alpha = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{j}}{|(\mathbf{v}\times\mathbf{a})||\mathbf{j}|}$$

Where $$|\mathbf{j}| = 1$$

$$(\mathbf{v}\times\mathbf{a})\cdot\mathbf{j} = -\omega_n R^2\left(2+\omega_n^2 t^2\right)$$

$$|(\mathbf{v}\times\mathbf{a})| = \omega_n R\left[c^2(2\cos\omega_n t - \omega_n t\sin\omega_n t)^2 + R^2\left(2+\omega_n^2 t^2\right)^2 + c^2(2\sin\omega_n t + \omega_n t\cos\omega_n t)^2\right]^{1/2}$$

$$= \omega_n R\left[c^2\left(4+\omega_n^2 t^2\right) + R^2\left(2+\omega_n^2 t^2\right)^2\right]^{1/2}$$

Then $$\cos\alpha = \frac{-\omega_n R^2\left(2+\omega_n^2 t^2\right)}{\omega_n R\left[c^2\left(4+\omega_n^2 t^2\right) + R^2\left(2+\omega_n^2 t^2\right)^2\right]^{1/2}}$$

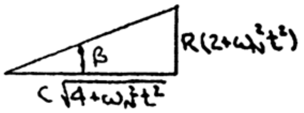
$$= \frac{-R\left(2+\omega_n^2 t^2\right)}{\left[c^2\left(4+\omega_n^2 t^2\right) + R^2\left(2+\omega_n^2 t^2\right)^2\right]^{1/2}}$$

The angle β that the osculating plane forms with y axis (see the above diagram) is equal to

$$\beta = \alpha - 90°$$

Then $$\cos\alpha = \cos(\beta + 90°) = -\sin\beta$$

$$-\sin\beta = \frac{-R\left(2+\omega_n^2 t^2\right)}{\left[c^2\left(4+\omega_n^2 t^2\right) + R^2\left(2+\omega_n^2 t^2\right)^2\right]^{1/2}}$$



Then $$\tan\beta = \frac{R\left(2+\omega_n^2 t^2\right)}{c\sqrt{4+\omega_n^2 t^2}}$$

or $$\beta = \tan^{-1}\left[\frac{R\left(2+\omega_n^2 t^2\right)}{c\sqrt{4+\omega_n^2 t^2}}\right] \blacktriangleleft$$

PROBLEM 11.181*

Determine the direction of the binormal of the path described by the particle of Problem 11.96 when (*a*) $t = 0$, (*b*) $t = \pi/2$ s.

SOLUTION

Given:
$$\mathbf{r} = (At\cos t)\mathbf{i} + \left(A\sqrt{t^2+1}\right)\mathbf{j} + (Bt\sin t)\mathbf{k}$$

$$r - \text{ft}, \quad t - \text{s}; \quad A = 3, \quad B - 1$$

First note that $\mathbf{e}_b$ is given by

$$\mathbf{e}_b = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$$

Now
$$\mathbf{r} = (3t\cos t)\mathbf{i} + \left(3\sqrt{t^2+1}\right)\mathbf{j} + (t\sin t)\mathbf{k}$$

Then
$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$
$$= 3(\cos t - t\sin t)\mathbf{i} + \frac{3t}{\sqrt{t^2+1}}\mathbf{j} + (\sin t + t\cos t)\mathbf{k}$$

and
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 3(-\sin t - \sin t - t\cos t)\mathbf{i} + 3\frac{\sqrt{t^2+1} - t\left(\frac{t}{\sqrt{t^2+1}}\right)}{t^2+1}\mathbf{j}$$
$$+ (\cos t + \cos t - t\sin t)\mathbf{k}$$
$$= -3(2\sin t + t\cos t)\mathbf{i} + \frac{3}{(t^2+1)^{3/2}}\mathbf{j} + (2\cos t - t\sin t)\mathbf{k}$$

(*a*) At $t = 0$:
$$\mathbf{v} = (3 \text{ ft/s})\mathbf{i}$$
$$\mathbf{a} = (3 \text{ ft/s}^2)\mathbf{j} + (2 \text{ ft/s}^2)\mathbf{k}$$

Then
$$\mathbf{v}\times\mathbf{a} = 3\mathbf{i}\times(3\mathbf{j} + 2\mathbf{k})$$
$$= 3(-2\mathbf{j} + 3\mathbf{k})$$

and
$$|\mathbf{v}\times\mathbf{a}| = 3\sqrt{(-2)^2 + (3)^2} = 3\sqrt{13}$$

Then
$$\mathbf{e}_b = \frac{3(-2\mathbf{j} + 3\mathbf{k})}{3\sqrt{13}} = \frac{1}{\sqrt{13}}(-2\mathbf{j} + 3\mathbf{k})$$

$$\cos\theta_x = 0 \qquad \cos\theta_y = -\frac{2}{\sqrt{13}} \qquad \cos\theta_z = \frac{3}{\sqrt{13}}$$

or
$$\theta_x = 90° \qquad \theta_y = 123.7° \qquad \theta_z = 33.7°$$ ◀

(*b*) At $t = \frac{\pi}{2}$ s:

$$\mathbf{v} = -\left(\frac{3\pi}{2}\ \text{ft/s}\right)\mathbf{i} + \left(\frac{3\pi}{\sqrt{\pi^2+4}}\ \text{ft/s}\right)\mathbf{j} + (1\ \text{ft/s})\mathbf{k}$$

$$\mathbf{a} = -(6\ \text{ft/s}^2)\mathbf{i} + \left[\frac{24}{(\pi^2+4)^{3/2}}\ \text{ft/s}^2\right]\mathbf{j} - \left(\frac{\pi}{2}\ \text{ft/s}^2\right)\mathbf{k}$$

Then

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{3\pi}{2} & \frac{3\pi}{(\pi^2+4)^{1/2}} & 1 \\ -6 & \frac{24}{(\pi^2+4)^{3/2}} & -\frac{\pi}{2} \end{vmatrix}$$

$$= -\left[\frac{3\pi^2}{2(\pi^2+4)^{1/2}} + \frac{24}{(\pi^2+4)^{3/2}}\right]\mathbf{i} - \left(6 + \frac{3\pi^2}{4}\right)\mathbf{j}$$

$$+\left[-\frac{36\pi}{(\pi^2+4)^{3/2}} + \frac{18\pi}{(\pi^2+4)^{1/2}}\right]\mathbf{k}$$

$$= -4.43984\mathbf{i} - 13.40220\mathbf{j} + 12.99459\mathbf{k}$$

and

$$|\mathbf{v}\times\mathbf{a}| = [(-4.43984)^2 + (-13.40220)^2 + (12.99459)^2]^{1/2}$$

$$= 19.18829$$

Then

$$\mathbf{e}_b = \frac{1}{19.1829}(-4.43984\mathbf{i} - 13.40220\mathbf{j} + 12.99459\mathbf{k})$$

$$\cos\theta_x = -\frac{4.43984}{19.18829} \quad \cos\theta_y = -\frac{13.40220}{19.18829} \quad \cos\theta_z = \frac{12.99459}{19.18829}$$

or $\theta_x = 103.4°$ $\theta_y = 134.3°$ $\theta_z = 47.4°$ ◀

PROBLEM 11.182

The motion of a particle is defined by the relation $x = 2t^3 - 15t^2 + 24t + 4$, where x and t are expressed in meters and seconds, respectively. Determine (*a*) when the velocity is zero, (*b*) the position and the total distance traveled when the acceleration is zero.

SOLUTION

$$x = 2t^3 - 15t^2 + 24t + 4$$

so $$v = \frac{dx}{dt} = 6t^2 - 30t + 24$$

$$a = \frac{dv}{dt} = 12t - 30$$

(*a*) Times when $v = 0$. $\quad 0 = 6t^2 - 30t + 24 = 6\,(t^2 - 5t + 4)$

$$(t-4)(t-1) = 0$$

$t = 1.00\text{ s}, \quad t = 4.00\text{ s}$ ◀

(*b*) Position and distance traveled when $a = 0$.

$$a = 12t - 30 = 0 \qquad t = 2.5\text{ s}$$

so $$x_2 = 2(2.5)^3 - 15(2.5)^2 + 24(2.5) + 4$$

Final position $\quad x = 1.50\text{ m}$ ◀

For $\quad 0 \le t \le 1\text{ s}, \quad v > 0.$

For $\quad 1\text{ s} \le t \le 2.5\text{ s}, \quad v \le 0.$

At $\quad t = 0, \quad x_0 = 4\text{ m}.$

At $\quad t = 1\text{ s}, \quad x_1 = (2)(1)^3 - (15)(1)^2 + (24)(1) + 4 = 15\text{ m}$

Distance traveled over interval: $x_1 - x_0 = 11\text{ m}$

For $\quad 1\text{ s} \le t \le 2.5\text{ s}, \qquad v \le 0$

Distance traveled over interval

$$|x_2 - x_1| = |1.5 - 15| = 13.5\text{ m}$$

Total distance: $\quad d = 11 + 13.5 \quad\quad d = 24.5\text{ m}$ ◀

PROBLEM 11.183

A drag car starts from rest and starts down the racetrack with and acceleration defined by $a = 50 - 10\,t$, where a and t are in m/s^2 and s, respectively. After reaching a speed of 125 m/s, a parachute is deployed to help slow down the dragster. Knowing that this deceleration is defined by the relationship $a = -0.02v^2$, where v is the velocity in m/s, determine (*a*) the total time from the beginning of the race until the car slows back down to 10 m/s, (*b*) the total distance the car travels during this time.

SOLUTION

Given:

$$a_{1-2} = 50 - 10t \text{ m/s}^2 \text{ for } 0 \le v \le 125 \text{ m/s}$$

$$a_{2-3} = -0.02v^2 \text{ after } v = 125 \text{ m/s}$$

Find v(t) as car speeds up

$$a = \frac{dv}{dt} \Rightarrow dv = adt$$

$$\int_0^{v(t)} dv = \int_0^t (50 - 10t)dt$$

$$v(t) = 50t - 5t^2$$

Find time to reach v=125 m/s

Rearrange and solve for t:

$$125 = 50t - 5t^2$$

$$t_2^2 - 10t_2 + 25 = 0$$

$$(t_2 - 5)^2 = 0$$

$$t_2 = 5 \text{ s}$$

Find distance to reach-125 m/s:

$$dx = vdt$$

$$\int_0^{x_2} dx = \int_0^{t_2} (50t - 5t^2)dt$$

$$x_2 = 25t^2 - \frac{5}{3}t^3\Big|_0^5$$

$$x_2 = 416.7 \text{ m}$$

(a) Find total time to slow down 10 m/s:

$$a = \frac{dv}{dt} \Rightarrow dt = \frac{dv}{a}$$

$$\int_5^{t_3} dt = -50\int_{125}^{10} \frac{dv}{v^2}$$

$$t_3 - 5 = \frac{50}{v}\Big|_{125}^{10}$$

$t_3 = 9.6$ s ◀

PROBLEM 11.183 (Continued)

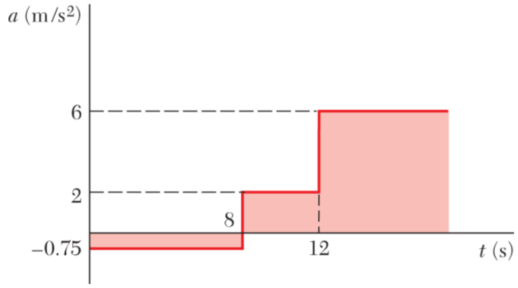
(b) Find total distance to slow down to 10 m/s

$$a\,dx = v\,dv \Rightarrow dx = \frac{v\,dv}{a}$$

$$\int_{x_2}^{x_3} dx = -50\int_{v_2}^{v_3} \frac{dv}{v}$$

$$x_3 - 416.7 = -50\ln v\Big|_{125}^{10}$$

$x_3 = 543.0$ m ◀

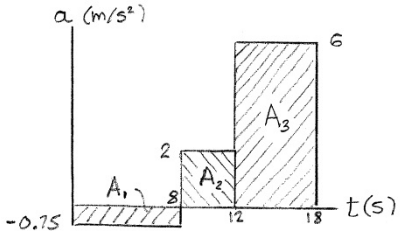


PROBLEM 11.184

A particle moves in a straight line with the acceleration shown in the figure. Knowing that the particle starts from the origin with $v_0 = -2$ m/s, (*a*) construct the $v-t$ and $x-t$ curves for $0 < t < 18$ s, (*b*) determine the position and the velocity of the particle and the total distance traveled when $t = 18$ s.

SOLUTION

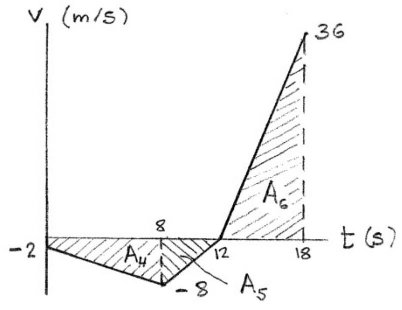
Compute areas under $a-t$ curve.



$$A_1 = (-0.75)(8) = -6 \text{ m/s}$$
$$A_2 = (2)(4) = 8 \text{ m/s}$$
$$A_3 = (6)(6) = 36 \text{ m/s}$$

$$v_0 = -2 \text{ m/s}$$
$$v_8 = v_0 + A_1 = -8 \text{ m/s}$$
$$v_{12} = v_8 + A_2 = 0$$
$$v_{18} = v_{12} + A_3$$

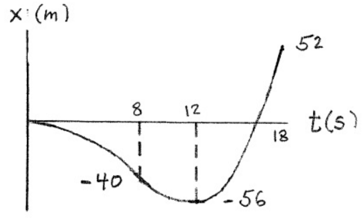
$v_{18} = 36$ m/s ◀



Sketch $v-t$ curve using straight line portions over the constant acceleration periods.

Compute areas under the $v-t$ curve.

$$A_4 = \frac{1}{2}(-2-8)(8) = -40 \text{ m}$$
$$A_5 = \frac{1}{2}(-8)(4) = -16 \text{ m}$$
$$A_6 = \frac{1}{2}(36)(6) = 108 \text{ m}$$

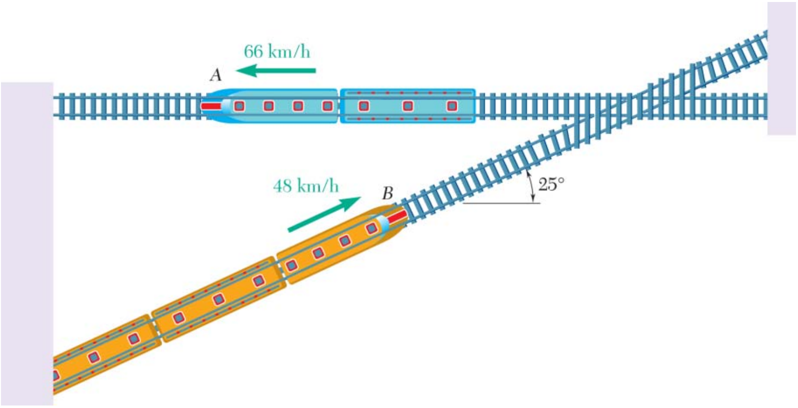


$$x_0 = 0$$
$$x_8 = x_0 + A_4 = -40 \text{ m}$$
$$x_{12} = x_8 + A_5 = -56 \text{ m}$$
$$x_{18} = x_{12} + A_6$$

$x_{18} = 52$ m ◀

Total distance traveled $= 56 + 108$

$d = 164$ m ◀



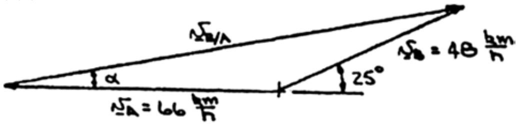
PROBLEM 11.185

The velocities of commuter trains *A* and *B* are as shown. Knowing that the speed of each train is constant and that *B* reaches the crossing 10 min after *A* passed through the same crossing, determine (*a*) the relative velocity of *B* with respect to *A*, (*b*) the distance between the fronts of the engines 3 min after *A* passed through the crossing.

SOLUTION

(*a*) We have $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

The graphical representation of this equation is then as shown.


$v_{B/A}$
α
$v_A = 66\ \frac{\text{km}}{\text{h}}$
25°
$v_B = 48\ \frac{\text{km}}{\text{h}}$

Then $$v_{B/A}^2 = 66^2 + 48^2 - 2(66)(48)\cos 155°$$

or $$v_{B/A} = 111.366 \text{ km/h}$$

and $$\frac{48}{\sin\alpha} = \frac{111.366}{\sin 155°}$$

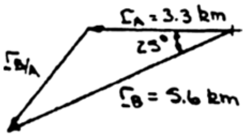
or $$\alpha = 10.50°$$

$$\mathbf{v}_{B/A} = 111.4 \text{ km/h} \measuredangle\ 10.50° \blacktriangleleft$$

(*b*) First note that

at $t = 3$ min, *A* is $(66 \text{ km/h})\left(\frac{3}{60}\right) = 3.3$ km west of the crossing.

at $t = 3$ min, *B* is $(48 \text{ km/h})\left(\frac{7}{60}\right) = 5.6$ km southwest of the crossing.

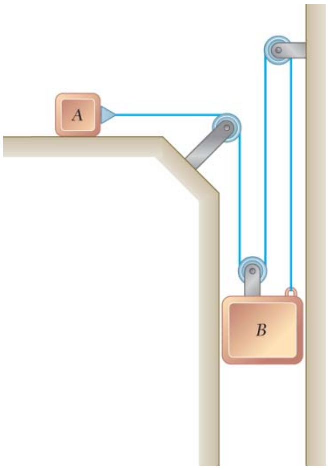


Now $$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

Then at $t = 3$ min, we have

$$r_{B/A}^2 = 3.3^2 + 5.6^2 - 2(3.3)(5.6)\cos 25°$$

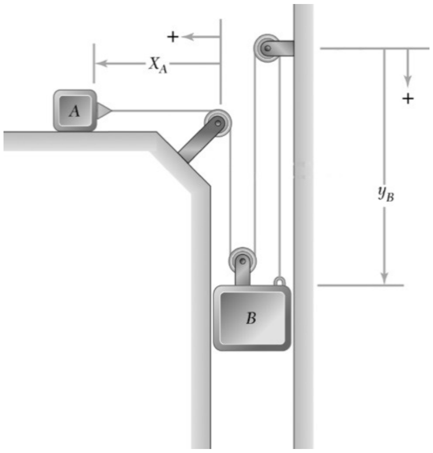
or $$r_{B/A} = 2.96 \text{ km} \blacktriangleleft$$



PROBLEM 11.186

Knowing that slider block *A* starts from rest and moves to the left with a constant acceleration of 1 ft/s^2, determine (*a*) the relative acceleration of block *A* with respect to block *B*, (*b*) the velocity of block *B* after 2 s.

SOLUTION



Given: $a_A = 1\text{ ft/s}^2$

From the diagram can write an expression for the length of the cable

$$L = x_A + 3y_B + \text{ constants}$$

Differentiate:

$$0 = \dot{x}_A + 3\dot{y}_B$$
$$0 = v_A + 3v_B$$

Differentiate again:

$$0 = a_A + 3a_B$$

Find the acceleration of B:

$$a_B = -\frac{1}{3}a_A$$
$$= -\frac{1}{3}\text{ft/s}^2$$

Write accelerations as vectors:

$$\mathbf{a}_A = -1\mathbf{i}\text{ ft/s}^2$$
$$\mathbf{a}_B = \frac{1}{3}\mathbf{j}\text{ ft/s}^2$$

(c) Find relative accelerations:

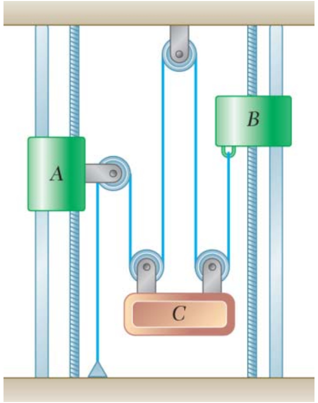
$$\mathbf{a}_{A/B} = \mathbf{a}_A - \mathbf{a}_B$$
$$= -1\mathbf{i} - \frac{1}{3}\mathbf{j}\text{ ft/s}^2$$

$$\text{a}_{A/B} = 1.054\text{ ft/s}^2 \swarrow 18.43° \blacktriangleleft$$

(d) Find speed at t=2 s:

$$v_B = (v_B)_o + a_Bt$$
$$= 0 + \frac{1}{3}(2)$$
$$= \frac{2}{3}\text{ m/s}$$

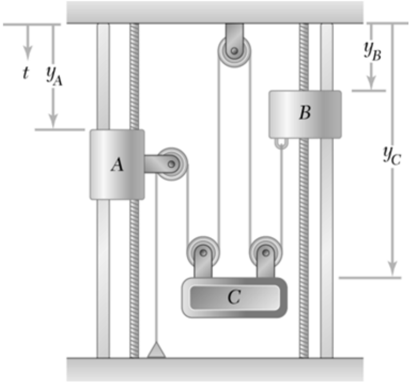
$$v_b = 0.667\text{ m/s}\uparrow \blacktriangleleft$$



PROBLEM 11.187

Collar A starts from rest at $t = 0$ and moves downward with a constant acceleration of 7 in./s^2. Collar B moves upward with a constant acceleration, and its initial velocity is 8 in./s. Knowing that collar B moves through 20 in. between $t = 0$ and $t = 2$ s, determine (*a*) the accelerations of collar B and block C, (*b*) the time at which the velocity of block C is zero, (*c*) the distance through which block C will have moved at that time.

SOLUTION



From the diagram

$$-y_A + (y_C - y_A) + 2y_C + (y_C - y_B) = \text{constant}$$

Then $$-2v_A - v_B + 4v_C = 0 \qquad (1)$$

and $$-2a_A - a_B + 4a_C = 0 \qquad (2)$$

Given: $$(v_A)_0 = 0$$

$$(\mathbf{a}_A) = 7\text{ in./s}^2\downarrow$$

$$(\mathbf{v}_B)_0 = 8\text{ in./s}\uparrow$$

$$\mathbf{a}_B = \text{constant}\uparrow$$

At $t = 2$ s $$y - (y_B)_0 = 20\text{ in.}\uparrow$$

(*a*) We have $$y_B = (y_B)_0 + (v_B)_0 t + \frac{1}{2}a_B t^2$$

At $t = 2$ s: $$-20\text{ in.} = (-8\text{ in./s})(2\text{ s}) + \frac{1}{2}a_B(2\text{ s})^2$$

$$a_B = -4\text{ in./s}^2 \quad \text{or} \quad \mathbf{a}_B = 2\text{ in./s}^2\uparrow \blacktriangleleft$$

Then, substituting into Eq. (2)

$$-2(7\text{ in./s}^2) - (-2\text{ in./s}^2) + 4a_C = 0$$

$$a_C = 3\text{ in./s}^2 \quad \text{or} \quad \mathbf{a}_C = 3\text{ in./s}^2\downarrow \blacktriangleleft$$

PROBLEM 11.187 (Continued)

(*b*) Substituting into Eq. (1) at $t = 0$

$$-2(0) - (-8 \text{ in./s}) + 4(v_C)_0 = 0 \quad \text{or} \quad (v_C)_0 = -2 \text{ in./s}$$

Now $$v_C = (v_C)_0 + a_C t$$

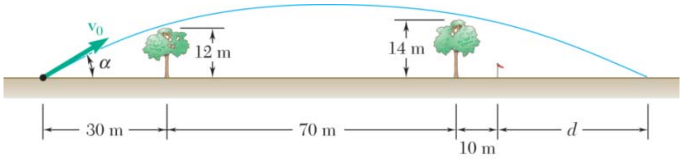
When $v_C = 0$: $$0 = (-2 \text{ in./s}) + (3 \text{ in./s}^2)t$$

or $$t = \frac{2}{3} \text{ s}$$ $t = 0.667$ s ◀

(*c*) We have $$y_C = (y_C)_0 + (v_C)_0 t + \frac{1}{2} a_C t^2$$

At $t = \frac{2}{3}$ s: $$y_C - (y_C)_0 = (-2 \text{ in./s})\left(\frac{2}{3} \text{ s}\right) + \frac{1}{2}(3 \text{ in./s}^2)\left(\frac{2}{3} \text{ s}\right)^2$$

$$= -0.667 \text{ in.}$$ or $\mathbf{y}_C - (\mathbf{y}_C)_0 = 0.667 \text{ in.} \uparrow$ ◀



PROBLEM 11.188

A golfer hits a ball with an initial velocity of magnitude v_0 at an angle α with the horizontal. Knowing that the ball must clear the tops of two trees and land as close as possible to the flag, determine v_0 and the distance d when the golfer uses (*a*) a six-iron with $\alpha = 31°$, (*b*) a five-iron with $\alpha = 27°$.

SOLUTION

The horizontal and vertical motions are

$$x = (v_0 \cos\alpha)t \qquad \text{or} \qquad v_0 = \frac{x}{t\cos\alpha} \tag{1}$$

$$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 = x\tan\alpha - \frac{1}{2}gt^2$$

or

$$t^2 = \frac{2(x\tan\alpha - y)}{g} \tag{2}$$

At the landing Point C:

$$y_C = 0, \qquad t = \frac{2v_0 \sin\alpha}{g}$$

And

$$x_C = (v_0\cos\alpha)t = \frac{2v_0^2 \sin\alpha\cos\alpha}{g} \tag{3}$$

(*a*) $\alpha = 31°$

To clear tree A:

$$x_A = 30 \text{ m}, \; y_A = 12 \text{ m}$$

From (2),

$$t_A^2 = \frac{2(30\tan 31° - 12)}{9.81} = 1.22851 \text{ s}^2, \qquad t_A = 1.1084 \text{ s}$$

From (1),

$$(v_0)_A = \frac{30}{1.1084\cos 31°} = 31.58 \text{ m/s}$$

To clear tree B:

$$x_B = 100 \text{ m}, \quad y_B = 14 \text{ m}$$

From (2),

$$(t_B)^2 = \frac{2(100\tan 31° - 14)}{9.81} = 9.3957 \text{ s}^2, \qquad t_B = 3.0652 \text{ s}$$

From (1),

$$(v_0)_B = \frac{100}{3.0652\cos 31°} = 38.06 \text{ m/s}$$

The larger value governs,

$$v_0 = 38.06 \text{ m/s}$$

$v_0 = 38.1$ m/s ◀

From (3),

$$x_C = \frac{(2)(38.06)^2 \sin 31° \cos 31°}{9.81} = 130.38 \text{ m}$$

$$d = x_C - 110$$

$d = 20.4$ m ◀

PROBLEM 11.188 (Continued)

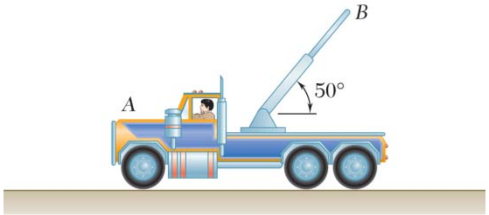
(*b*) $\alpha = 27°$

By a similar calculation, $t_A = 0.81846 \text{ s}, \quad (v_0)_A = 41.138 \text{ m/s},$

$t_B = 2.7447 \text{ s}, \quad (v_0)_B = 40.890 \text{ m/s},$

$v_0 = 41.138 \text{ m/s}$ $\qquad v_0 = 41.1 \text{ m/s}$ ◀

$x_C = 139.56 \text{ m},$ $\qquad d = 29.6 \text{ m}$ ◀



PROBLEM 11.189

As the truck shown begins to back up with a constant acceleration of 4 ft/s^2, the outer section B of its boom starts to retract with a constant acceleration of 1.6 ft/s^2 relative to the truck. Determine (*a*) the acceleration of section B, (*b*) the velocity of section B when $t = 2$ s.

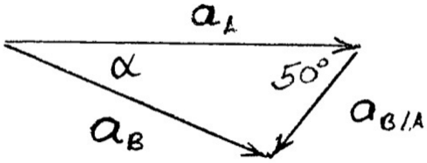
SOLUTION

For the truck, $\mathbf{a}_A = 4 \text{ ft/s}^2 \longrightarrow$

For the boom, $\mathbf{a}_{B/A} = 1.6 \text{ ft/s}^2 \swarrow 50°$

(*a*) $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$

Sketch the vector addition.



By law of cosines:

$$a_B^2 = a_A^2 + a_{B/A}^2 - 2a_A a_{B/A} \cos 50°$$

$$= 4^2 + 1.6^2 - 2(4)(1.6) \cos 50°$$

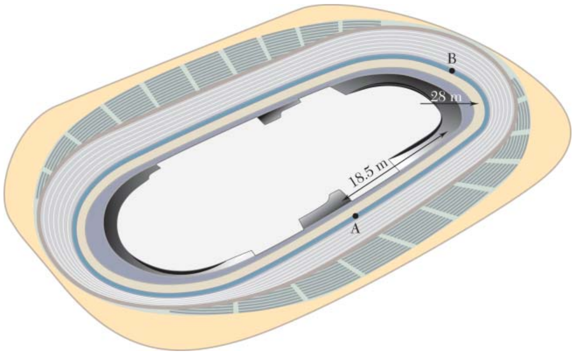
$$a_B = 3.214 \text{ ft/s}^2$$

Law of sines: $\sin\alpha = \dfrac{a_{B/A} \sin 50°}{a_B} = \dfrac{1.6 \sin 50°}{3.214} = 0.38131$

$\alpha = 22.4°$, $\mathbf{a}_B = 3.21 \text{ ft/s}^2 \searrow 22.4°$ ◀

(*b*) $\mathbf{v}_B = (v_B)_0 + a_B t = 0 + (3.214)(2)$

$\mathbf{v}_B = 6.43 \text{ ft/s}^2 \searrow 22.4°$ ◀



PROBLEM 11.190

A velodrome is a specially designed track used in bicycle racing. If a rider starts from rest and accelerates between points *A* and *B* according to the relationship $a_t = (11.46 - 0.01878\, v^2)$ m/s^2, what is her acceleration at point *B*?

SOLUTION

Given:

$$a_t = \left(11.46 \ - \ 0.01878v^2\right) \ \text{m/s}^2$$

Distance traveled between points A and B:

$$\begin{aligned} s_B &= 18.5 + r\theta \ \text{m} \\ &= 18.5 + 28 * \frac{\pi}{2} \ \text{m} \\ &= 62.48 \ \text{m} \end{aligned}$$

Find velocity at point B:

$$ads = vdv \Rightarrow ds = \frac{vdv}{a}$$

$$\int_0^{62.48} ds = \int_0^{v_B} \frac{vdv}{11.46 \ - \ 0.01878v^2}$$

$$62.48 = \frac{1}{-(2)0.01878} \ln\left(11.46 - 0.01878v^2\right)\Big|_0^{v_B}$$

$$-2.347 = \ln\left(11.46 - 0.01878v_B^2\right) - \ln\left(11.46\right)$$

$$e^{0.09211} = 11.46 - 0.01878v_B^2$$

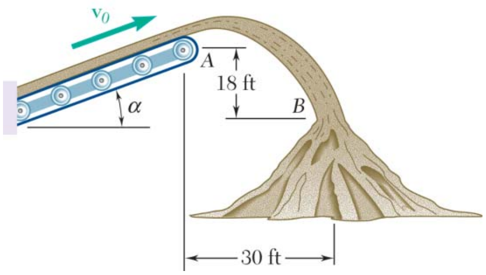
$$v_B = 23.49 \ \text{m/s}$$

Acceleration:

$$\mathbf{a}_B = a_t\mathbf{e}_t + \frac{v_B^2}{\rho}\mathbf{e}_n$$

$$= \left(11.46 \ - \ 0.01878v_B^2\right)\mathbf{e}_t + \frac{(23.49)^2}{28}\mathbf{e}_n$$

$$\mathbf{a}_B = 1.097\mathbf{e}_t + 19.71\mathbf{e}_n \ \text{m/s}^2 \blacktriangleleft$$



PROBLEM 11.191

Sand is discharged at A from a conveyor belt and falls onto the top of a stockpile at B. Knowing that the conveyor belt forms an angle $\alpha = 25°$ with the horizontal, determine (*a*) the speed v_0 of the belt, (*b*) the radius of curvature of the trajectory described by the sand at Point B.

SOLUTION

The motion is projectile motion. Place the origin at Point A. Then $x_0 = 0$ and $y_0 = 0$.
The coordinates of Point B are $x_B = 30$ ft and $y_B = -18$ ft.

Horizontal motion: $$v_x = v_0 \cos 25° \qquad (1)$$

$$x = v_0 t \cos 25° \qquad (2)$$

Vertical motion: $$v_y = v_0 \sin 25° - gt \qquad (3)$$

$$y = v_0 t \sin 25° - \frac{1}{2} g t^2 \qquad (4)$$

At Point B, Eq. (2) gives

$$v_0 t_B = \frac{x_B}{\cos 25°} = \frac{30}{\cos 25°} = 33.101 \text{ ft}$$

Substituting into Eq. (4),

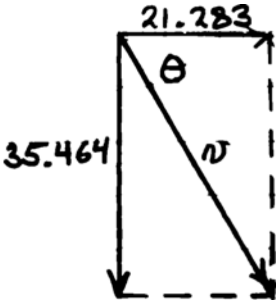
$$-18 = (33.101)(\sin 25°) - \frac{1}{2}(32.2) t_B^2$$

$$t_B = 1.40958 \text{ s}$$

(*a*) Speed of the belt. $$v_0 = \frac{v_0 t_B}{t_B} = \frac{33.101}{1.40958} = 23.483$$

$v_0 = 23.4$ ft/s ◀

Eqs. (1) and (3) give



$$v_x = 23.483 \cos 25° = 21.283 \text{ ft/s}$$

$$v_y = (23.483)\sin 25° - (32.2)(1.40958) = -35.464 \text{ ft/s}$$

$$\tan\theta \frac{-v_y}{v_x} = 1.66632 \qquad \theta = 59.03°$$

$$v = 41.36 \text{ ft/s}$$

PROBLEM 11.191 (Continued)

Components of acceleration.

$$\mathbf{a} = 32.2 \text{ ft/s}^2 \downarrow \qquad a_t = 32.2\sin\theta$$

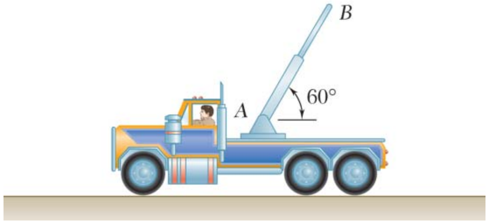
$$a_n = 32.2\cos\theta = 32.2\cos 59.03° = 16.57 \text{ ft/s}^2$$

(*b*) Radius of curvature at *B*.

$$a_n = \frac{v^2}{\rho}$$

$$\rho = \frac{v^2}{a_n} = \frac{(41.36)^2}{16.57}$$

$\rho = 103.2$ ft ◀



PROBLEM 11.192

The end Point B of a boom is originally 5 m from fixed Point A when the driver starts to retract the boom with a constant radial acceleration of $\ddot{r} = -1.0\text{ m/s}^2$ and lower it with a constant angular acceleration $\ddot{\theta} = -0.5\text{ rad/s}^2$. At $t = 2$ s, determine (*a*) the velocity of Point B, (*b*) the acceleration of Point B, (*c*) the radius of curvature of the path.

SOLUTION

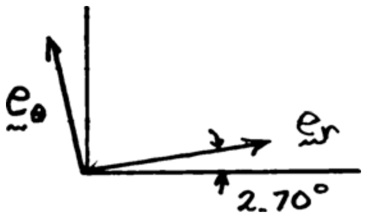
Radial motion.
$$r_0 = 5\text{ m}, \quad \dot{r}_0 = 0, \quad \ddot{r} = -1.0\text{ m/s}^2$$
$$r = r_0 + \dot{r}_0 t + \frac{1}{2}\ddot{r}t^2 = 5 + 0 - 0.5t^2$$
$$\dot{r} = \dot{r}_0 + \ddot{r}t = 0 - 1.0t$$

At $t = 2$ s,
$$r = 5 - (0.5)(2)^2 = 3\text{ m}$$
$$\dot{r} = (-1.0)(2) = -2\text{ m/s}$$

Angular motion.
$$\theta_0 = 60° = \frac{\pi}{3}\text{rad}, \quad \dot{\theta}_0 = 0, \quad \ddot{\theta} = -0.5\text{ rad/s}^2$$
$$\theta = \theta_0 + \dot{\theta}_0 + \frac{1}{2}\ddot{\theta}t^2 = \frac{\pi}{3} + 0 - 0.25t^2$$
$$\dot{\theta} = \dot{\theta}_0 + \ddot{\theta}t = 0 - 0.5t$$

At $t = 2$ s,
$$\theta = \frac{\pi}{3} + 0 - (0.25)(2)^2 = 0.047198\text{ rad} = 2.70°$$
$$\dot{\theta} = -(0.5)(2) = -1.0\text{ rad/s}$$

Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$.



(*a*) Velocity of Point B at $t = 2$ s.
$$\mathbf{v}_B = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$
$$= -(2\text{ m/s})\mathbf{e}_r + (3\text{ m})(-1.0\text{ rad/s})\mathbf{e}_\theta$$

$$\mathbf{v}_B = (-2.00\text{ m/s})\mathbf{e}_r + (-3.00\text{ m/s})\mathbf{e}_\theta \blacktriangleleft$$

$$\tan\alpha = \frac{v_\theta}{v_r} = \frac{-3.0}{-2.0} = 1.5 \qquad \alpha = 56.31°$$
$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{(-2)^2 + (-3)^2} = 3.6055\text{ m/s}$$

PROBLEM 11.192 (Continued)

Direction of velocity.

$$\mathbf{e}_t = \frac{\mathbf{v}}{v} = \frac{-2\mathbf{e}_r - 3\mathbf{e}_\theta}{3.6055} = -0.55470\mathbf{e}_r - 0.83205\mathbf{e}_\theta$$

$$\theta + \alpha = 2.70 + 56.31° = 59.01°$$

$$\mathbf{v}_B = 3.61 \text{ m/s} \measuredangle 59.0° \blacktriangleleft$$

(*b*) Acceleration of Point *B* at *t* = 2 s.

$$\mathbf{a}_B = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

$$= [-1.0 - (3)(-1)^2]\mathbf{e}_r + [(3)(-0.5) + (2)(-1.0)(-0.5)]\mathbf{e}_\theta$$

$$\mathbf{a}_B = (-4.00 \text{ m/s}^2)\mathbf{e}_r + (2.50 \text{ m/s}^2)\mathbf{e}_\theta \blacktriangleleft$$

$$\tan\beta = \frac{a_\theta}{a_r} = \frac{2.50}{-4.00} = -0.625 \qquad \beta = -32.00°$$

$$a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(-4)^2 + (2.5)^2} = 4.7170 \text{ m/s}^2$$

$$\theta + \beta = 2.70° - 32.00° = -29.30°$$

$$\mathbf{a}_B = 4.72 \text{ m/s}^2 \measuredangle 29.3° \blacktriangleleft$$

Tangential component: $\mathbf{a}_t = (\mathbf{a} \cdot \mathbf{e}_t)\mathbf{e}_t$

$$\mathbf{a}_t = (-4\mathbf{e}_r + 2.5\mathbf{e}_\theta) \cdot (-0.55470\mathbf{e}_r - 0.83205\mathbf{e}_\theta)\mathbf{e}_t$$

$$= [(-4)(-0.55470) + (2.5)(-0.83205)]\mathbf{e}_t$$

$$= (0.138675 \text{ m/s}^2)\mathbf{e}_t = 0.1389 \text{ m/s}^2 \measuredangle 59.0°$$

Normal component: $\mathbf{a}_n = \mathbf{a} - \mathbf{a}_t$

$$\mathbf{a}_n = -4\mathbf{e}_r + 2.5\mathbf{e}_\theta - (0.138675)(-0.55470\mathbf{e}_r - 0.83205\mathbf{e}_\theta)$$

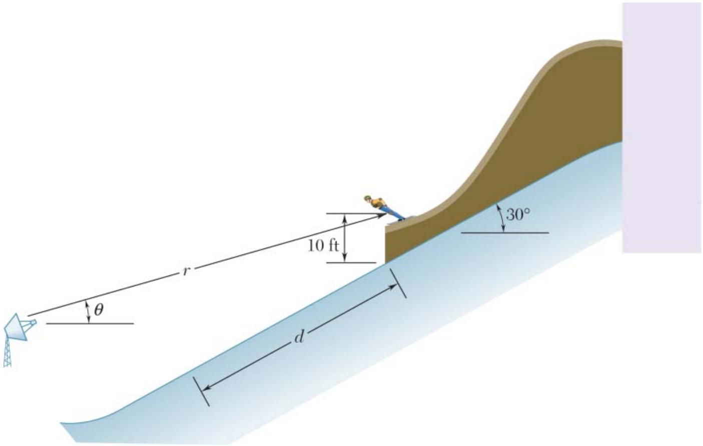
$$= (-3.9231 \text{ m/s}^2)\mathbf{e}_r + (2.6154 \text{ m/s}^2)\mathbf{e}_\theta$$

$$a_n = \sqrt{(3.9231)^2 + (2.6154)^2} = 4.7149 \text{ m/s}^2$$

(*c*) Radius of curvature of the path.

$$a_n = \frac{v^2}{\rho}$$

$$\rho = \frac{v^2}{a_n} = \frac{(3.6055 \text{ m/s})^2}{4.7149 \text{ m/s}} \qquad \rho = 2.76 \text{ m} \blacktriangleleft$$



PROBLEM 11.193

A telemetry system is used to quantify kinematic values of a ski jumper immediately before she leaves the ramp. According to the system $r = 500$ ft, $\dot{r} = -105$ ft/s, $\ddot{r} = -10$ ft/s^2, $\theta = 25°$, $\dot{\theta} = 0.07$ rad/s, $\ddot{\theta} = 0.06$ rad/s^2. Determine (*a*) the velocity of the skier immediately before she leaves the jump, (*b*) the acceleration of the skier at this instant, (*c*) the distance of the jump *d* neglecting lift and air resistance.

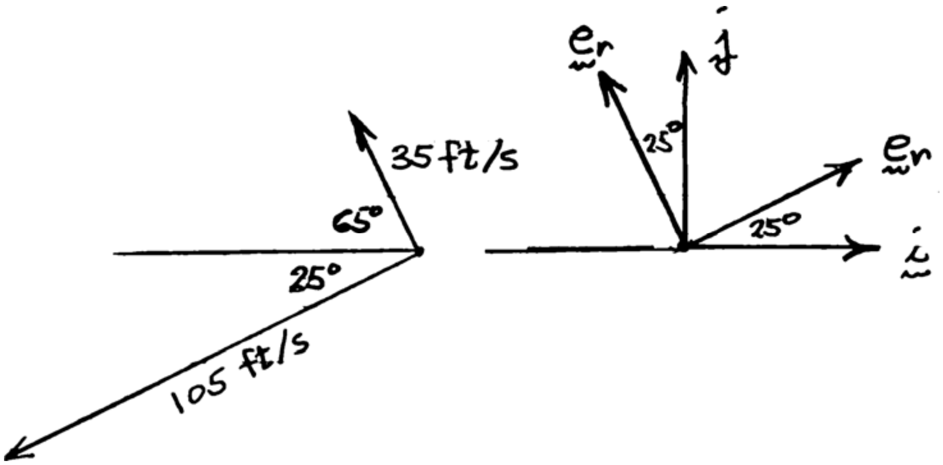
SOLUTION

(*a*) Velocity of the skier. $(r = 500 \text{ ft}, \quad \theta = 25°)$

$$\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$
$$= (-105 \text{ ft/s})\mathbf{e}_r + (500 \text{ ft})(0.07 \text{ rad/s})\mathbf{e}_\theta$$

$$\mathbf{v} = (-105 \text{ ft/s})\mathbf{e}_r + (35 \text{ ft/s})\mathbf{e}_\theta \quad ◀$$

Direction of velocity:



$$\mathbf{v} = (-105\cos 25° - 35\cos 65°)\mathbf{i} + (35\sin 65° - 105\sin 25°)\mathbf{j}$$
$$= (-109.95 \text{ ft/s})\mathbf{i} + (-12.654 \text{ ft/s})\mathbf{j}$$

$$\tan\alpha = \frac{v_y}{v_x} = \frac{-12.654}{-109.95} \qquad \alpha = 6.565°$$

$$v = \sqrt{(105)^2 + (35)^2} = 110.68 \text{ ft/s}$$

$$v = 110.7 \text{ ft/s} \measuredangle 6.57° \quad ◀$$

PROBLEM 11.193 (Continued)

(*b*) Acceleration of the skier.

$$\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$
$$a_r = -10 - (500)(0.07)^2 = -12.45 \text{ ft/s}^2$$
$$a_\theta = (500)(0.06) + (2)(-105)(0.07) = 15.30 \text{ ft/s}^2$$

$\mathbf{a} = (-12.45 \text{ ft/s}^2)\mathbf{e}_r + (15.30 \text{ ft/s}^2)\mathbf{e}_\theta$ ◀

$$\mathbf{a} = (-12.45)(\mathbf{i}\cos 25° + \mathbf{j}\sin 25°) + (15.30)(-\mathbf{i}\cos 65° + \mathbf{j}\sin 65°)$$
$$= (-17.750 \text{ ft/s}^2)\mathbf{i} + (8.6049 \text{ ft/s}^2)\mathbf{j}$$
$$\tan\beta = \frac{a_y}{a_x} = \frac{8.6049}{-17.750} \qquad \beta = -25.9°$$
$$a = \sqrt{(12.45)^2 + (15.30)^2} = 19.725 \text{ ft/s}^2$$

$a = 19.73 \text{ ft/s}^2$ ↙ 25.9° ◀

(*c*) Distance of the jump *d*.

Projectile motion. Place the origin of the *xy*-coordinate system at the end of the ramp with the *x*-coordinate horizontal and positive to the left and the *y*-coordinate vertical and positive downward.

Horizontal motion: (Uniform motion)

$$x_0 = 0$$
$$\dot{x}_0 = 109.95 \text{ ft/s} \qquad \text{(from Part } a\text{)}$$
$$x = x_0 + \dot{x}_0 t = 109.95t$$

Vertical motion: (Uniformly accelerated motion)

$$y_0 = 0$$
$$\dot{y}_0 = 12.654 \text{ ft/s} \qquad \text{(from Part } a\text{)}$$
$$\ddot{y} = 32.2 \text{ ft/s}^2$$
$$y = y_0 + \dot{y}_0 t + \frac{1}{2}\ddot{y}t^2 = 12.654t - 16.1t^2$$

At the landing point,

$$x = d\cos 30° \qquad (1)$$
$$y = 10 + d\sin 30° \quad \text{or} \quad y - 10 = d\sin 30° \qquad (2)$$

Multiply Eq. (1) by sin 30° and Eq. (2) by cos 30° and subtract

$$x\sin 30° - (y - 10)\cos 30° = 0$$

$$(109.95t)\sin 30° - (12.654t + 16.1t^2 - 10)\cos 30° = 0$$

$$-13.943t^2 + 44.016t + 8.6603 = 0$$

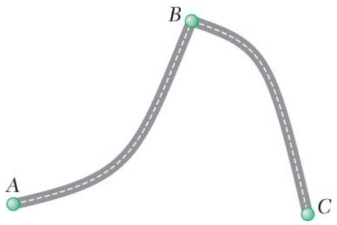
$$t = -0.1858 \text{ s} \quad \text{and} \quad 3.3427 \text{ s}$$

Reject the negative root.

$$x = (109.95 \text{ ft/s})(3.3427 \text{ s}) = 367.53 \text{ ft}$$

$$d = \frac{x}{\cos 30°}$$

$d = 424$ ft ◀



PROBLEM 11.CQ1

A bus travels the 100 miles between *A* and *B* at 50 mi/h and then another 100 miles between *B* and *C* at 70 mi/h. The average speed of the bus for the entire 200-mile trip is:

(*a*) more than 60 mi/h

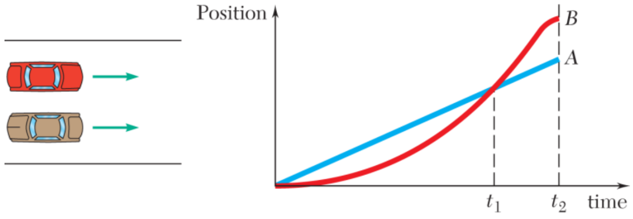
(*b*) equal to 60 mi/h

(*c*) less than 60 mi/h

SOLUTION

The time required for the bus to travel from *A* to *B* is 2 h and from *B* to *C* is 100/70 = 1.43 h, so the total time is 3.43 h and the average speed is 200/3.43 = 58 mph.

Answer: (*c*) ◀



PROBLEM 11CQ2

Two cars A and B race each other down a straight road. The position of each car as a function of time is shown. Which of the following statements are true (more than one answer can be correct)?

(*a*) At time t_2 both cars have traveled the same distance

(*b*) At time t_1 both cars have the same speed

(*c*) Both cars have the same speed at some time $t < t_1$

(*d*) Both cars have the same acceleration at some time $t < t_1$

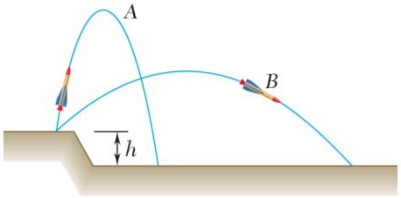
(*e*) Both cars have the same acceleration at some time $t_1 < t < t_2$

SOLUTION

The speed is the slope of the curve, so answer c) is true.

The acceleration is the second derivative of the position. Since A's position increases linearly the second derivative will always be zero. The second derivative of curve B is zero at the pont of inflection which occurs between t_1 and t_2.

Answers: (*c*) and (*e*) ◀



PROBLEM 11.CQ3

Two model rockets are fired simultaneously from a ledge and follow the trajectories shown. Neglecting air resistance, which of the rockets will hit the ground first?

(*a*) *A*

(*b*) *B*

(*c*) They hit at the same time.

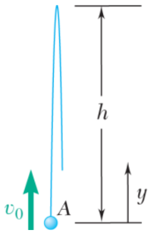
(*d*) The answer depends on *h*.

SOLUTION

The motion in the vertical direction depends on the initial velocity in the *y*-direction. Since *A* has a larger initial velocity in this direction it will take longer to hit the ground.

Answer: (*b*) ◀

PROBLEM 11.CQ4



Ball *A* is thrown straight up. Which of the following statements about the ball are true at the highest point in its path?

(*a*) The velocity and acceleration are both zero.

(*b*) The velocity is zero, but the acceleration is not zero.

(*c*) The velocity is not zero, but the acceleration is zero.

(*d*) Neither the velocity nor the acceleration are zero.

SOLUTION

At the highest point the velocity is zero. The acceleration is never zero.

Answer: (*b*) ◀

PROBLEM 11.CQ5

Ball *A* is thrown straight up with an initial speed v_0 and reaches a maximum elevation *h* before falling back down. When *A* reaches its maximum elevation, a second ball is thrown straight upward with the same initial speed v_0. At what height, *y*, will the balls cross paths?

(*a*) $y = h$

(*b*) $y > h/2$

(*c*) $y = h/2$

(*d*) $y < h/2$

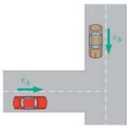
(*e*) $y = 0$

SOLUTION

When the ball is thrown up in the air it will be constantly slowing down until it reaches its apex, at which point it will have a speed of zero. So, the time it will take to travel the last half of the distance to the apex will be longer than the time it takes for the first half. This same argument can be made for the ball falling from the maximum elevation. It will be speeding up, so the first half of the distance will take longer than the second half. Therefore, the balls should cross above the half-way point.

Answer: (*b*) ◀

PROBLEM 11.CQ6

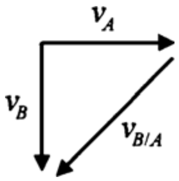


Two cars are approaching an intersection at constant speeds as shown. What velocity will car *B* appear to have to an observer in car *A*?

(*a*) → (*b*) ↘ (*c*) ↖ (*d*) ↗ (*e*) ↙

SOLUTION

Since $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ we can draw the vector triangle and see



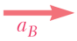
$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

Answer: (*e*) ◀

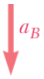
PROBLEM 11.CQ7

Blocks *A* and *B* are released from rest in the positions shown. Neglecting friction between all surfaces, which figure below best indicates the direction α of the acceleration of block *B*?

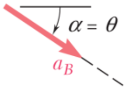
a.



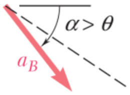
b.



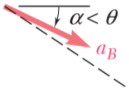
c.

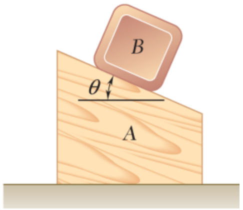

$\alpha = \theta$
a_B

d.


$\alpha > \theta$
a_B

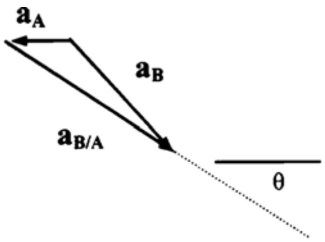
e.


$\alpha < \theta$
a_B

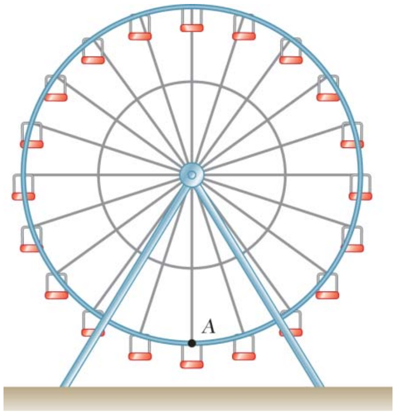


SOLUTION

Since $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ we get


$\mathbf{a_A}$
$\mathbf{a_B}$
$\mathbf{a_{B/A}}$
θ

Answer: (*d*) ◀



PROBLEM 11.CQ8

The Ferris wheel is rotating with a constant angular velocity ω. What is the direction of the acceleration of Point A?

(*a*) →

(*b*) ↑

(*c*) ↓

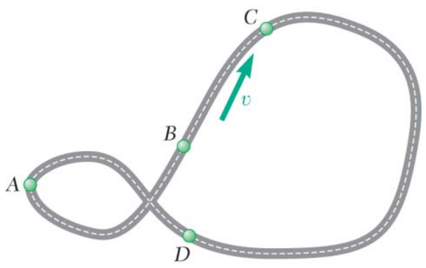
(*d*) ←

(*e*) The acceleration is zero.

SOLUTION

The tangential acceleration is zero since the speed is constant, so there will only be normal acceleration pointed upwards.

Answer: (*b*) ◀



PROBLEM 11.CQ9

A racecar travels around the track shown at a constant speed. At which point will the racecar have the largest acceleration?

(*a*) A

(*b*) B

(*c*) C

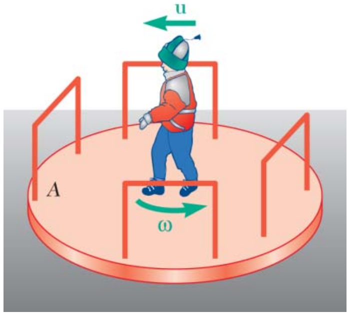
(*d*) The acceleration will be zero at all the points.

SOLUTION

The tangential acceleration is zero since the speed is constant, so there will only be normal acceleration. The normal acceleration will be maximum where the radius of curvature is a minimum that is at Point A.

Answer: (*a*) ◀

PROBLEM 11.CQ10



A child walks across merry-go-round *A* with a constant speed *u* relative to *A*. The merry-go-round undergoes fixed axis rotation about its center with a constant angular velocity ω counterclockwise. When the child is at the center of *A*, as shown, what is the direction of his acceleration when viewed from above.

(*a*) →

(*b*) ←

(*c*) ↑

(*d*) ↓

(*e*) The acceleration is zero.

SOLUTION

Polar coordinates are most natural for this problem, that is,

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \qquad (1)$$

From the information given, we know $\ddot{r} = 0,\ \ddot{\theta} = 0,\ r = 0,\ \dot{\theta} = \omega,\ \dot{r} = -u.$ When we substitute these values into (1), we will only have a term in the $-\theta$ direction.

Answer: (*d*) ◀

CHAPTER 12

PROBLEM 12.1

Astronauts who landed on the moon during the Apollo 15, 16 and 17 missions brought back a large collection of rocks to the earth. Knowing the rocks weighed 139 lb when they were on the moon, determine (*a*) the weight of the rocks on the earth, (*b*) the mass of the rocks in slugs. The acceleration due to gravity on the moon is 5.30 ft/s^2.

SOLUTION

Since the rocks weighed 139 lb on the moon, their mass is

$$m = \frac{W_{\text{moon}}}{g_{\text{moon}}} = \frac{139 \text{ lb}}{5.30 \text{ ft/s}^2} = 26.226 \text{ lb} \cdot \text{s}^2/\text{ft}$$

(*a*) On the earth, $W_{\text{earth}} = mg_{\text{earth}}$

$$w = (26.226 \text{ lb} \cdot \text{s}^2/\text{ft})(32.2 \text{ ft/s}^2) \qquad w = 844 \text{ lb} \blacktriangleleft$$

(*b*) Since $1 \text{ slug} = 1 \text{ lb} \cdot \text{s}^2/\text{ft}$, $\qquad m = 26.2 \text{ slugs} \blacktriangleleft$

PROBLEM 12.2

The value of g at any latitude ϕ may be obtained from the formula

$$g = 32.09(1 + 0.0053\ \sin^2\phi)\ \text{ft/s}^2$$

which takes into account the effect of the rotation of the earth, as well as the fact that the earth is not truly spherical. Knowing that the weight of a silver bar has been officially designated as 5 lb, determine to four significant figures (*a*) the mass is slugs, (*b*) the weight in pounds at the latitudes of 0°, 45°, and 60°.

SOLUTION

$$g = 32.09(1 + 0.0053\ \sin^2\phi)\ \text{ft/s}^2$$

$\phi = 0°$: $\quad g = 32.09\ \text{ft/s}^2$

$\phi = 45°$: $\quad g = 32.175\ \text{ft/s}^2$

$\phi = 90°$: $\quad g = 32.26\ \text{ft/s}^2$

(a) Mass at all latitudes: $\quad m = \dfrac{5.00\ \text{lb}}{32.175\ \text{ft/s}^2}$ $\quad m = 0.1554\ \text{lb}\cdot\text{s}^2/\text{ft}$ ◀

(*b*) Weight: $\quad W = mg$

$\phi = 0°$: $\quad W = (0.1554\ \text{lb}\cdot\text{s}^2/\text{ft})(32.09\ \text{ft/s}^2) = 4.987\ \text{lb}$ ◀

$\phi = 45°$: $\quad W = (0.1554\ \text{lb}\cdot\text{s}^2/\text{ft})(32.175\ \text{ft/s}^2) = 5.000\ \text{lb}$ ◀

$\phi = 90°$: $\quad W = (0.1554\ \text{lb}\cdot\text{s}^2/\text{ft})(32.26\ \text{ft/s}^2) = 5.013\ \text{lb}$ ◀

PROBLEM 12.3

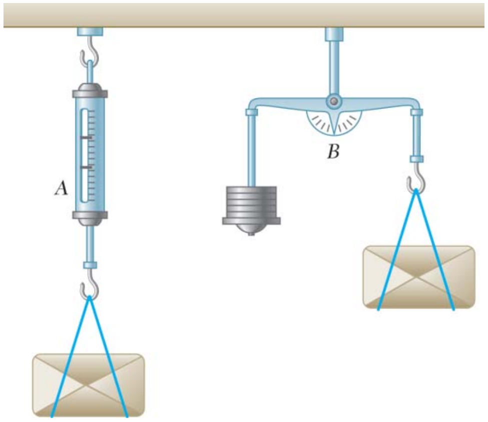
A 400-kg satellite has been placed in a circular orbit 1500 km above the surface of the earth. The acceleration of gravity at this elevation is 6.43 m/s^2. Determine the linear momentum of the satellite, knowing that its orbital speed is 25.6×10^3 km/h.

SOLUTION

Mass of satellite is independent of gravity: $m = 400$ kg

$$\begin{aligned} v &= 25.6\times10^3 \text{ km/h} \\ &= (25.6\times10^6 \text{ m/h})\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 7.111\times10^3 \text{ m/s} \end{aligned}$$

$$L = mv = (400 \text{ kg})(7.111\times10^3 \text{ m/s}) \qquad L = 2.84\times10^6 \text{ kg}\cdot\text{m/s} \blacktriangleleft$$



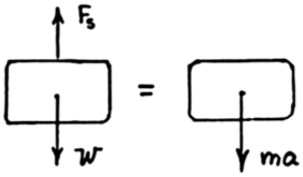
PROBLEM 12.4

A spring scale *A* and a lever scale *B* having equal lever arms are fastened to the roof of an elevator, and identical packages are attached to the scales as shown. Knowing that when the elevator moves downward with an acceleration of $1\ \text{m/s}^2$ the spring scale indicates a load of 60 N, determine (*a*) the weight of the packages, (*b*) the load indicated by the spring scale and the mass needed to balance the lever scale when the elevator moves upward with an acceleration of $1\ \text{m/s}^2$.

SOLUTION

Assume $$g = 9.81\ \text{m/s}^2$$

$$m = \frac{W}{g}$$



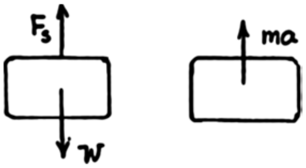
$$+\uparrow\ \Sigma F = ma:\quad F_s - W = -\frac{W}{g}a$$

$$W\left(1 - \frac{a}{g}\right) = F_s$$

or $$W = \frac{F_s}{1 - \frac{a}{g}} = \frac{60}{1 - \frac{1}{9.81}}$$

$$W = 66.8\ \text{N} \blacktriangleleft$$

(*b*)



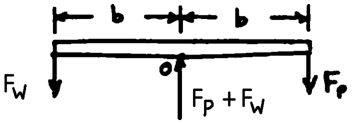
$$\Sigma F = ma:\quad F_s - W = \frac{W}{g}a$$

$$F_s = W\left(1 + \frac{a}{g}\right)$$

$$= 66.81\left(1 + \frac{1}{9.81}\right)$$

$$F_s = 73.6\ \text{N} \blacktriangleleft$$

For the balance system *B*,



$$\Sigma M_0 = 0:\quad bF_w - bF_p = 0$$

$$F_w = F_p$$

PROBLEM 12.4 (Continued)

But $$F_w = W_w\left(1+\frac{a}{g}\right)$$

and $$F_p = W_p\left(1+\frac{a}{g}\right)$$

so that $$W_w = W_p$$

and $$m_w = \frac{W_p}{g} = \frac{66.81}{9.81}$$

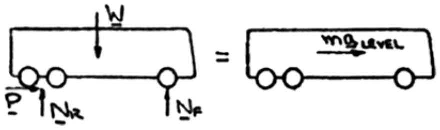
$m_w = 6.81\text{ kg}$ ◀

PROBLEM 12.5

In anticipation of a long 7° upgrade, a bus driver accelerates at a constant rate of 3 ft/s^2 while still on a level section of the highway. Knowing that the speed of the bus is 60 mi/h as it begins to climb the grade and that the driver does not change the setting of his throttle or shift gears, determine the distance traveled by the bus up the grade when its speed has decreased to 50 mi/h.

SOLUTION

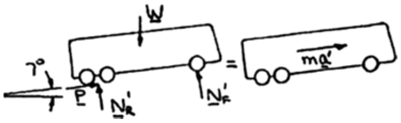
First consider when the bus is on the level section of the highway.



$$a_{\text{level}} = 3\text{ ft/s}^2$$

We have $\xrightarrow{+} \Sigma F_x = ma\text{:} \quad P = \dfrac{W}{g} a_{\text{level}}$

Now consider when the bus is on the upgrade.



We have $+\nearrow \Sigma F_x = ma\text{:} \quad P - W\sin 7° = \dfrac{W}{g} a'$

Substituting for P $\quad \dfrac{W}{g} a_{\text{level}} - W\sin 7° = \dfrac{W}{g} a'$

or

$$\begin{aligned} a' &= a_{\text{level}} - g\sin 7° \\ &= (3 - 32.2\sin 7°)\text{ ft/s}^2 \\ &= -0.92419\text{ ft/s}^2 \end{aligned}$$

For the uniformly decelerated motion

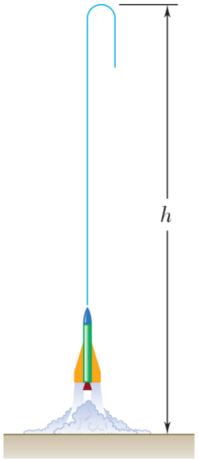
$$v^2 = (v_0)^2_{\text{upgrade}} + 2a'(x_{\text{upgrade}} - 0)$$

Noting that $60\text{ mi/h} = 88\text{ ft/s},$ then when $v = 50\text{ mi/h}\left(= \dfrac{5}{6} v_0\right),$ we have

$$\left(\frac{5}{6} \times 88\text{ ft/s}\right)^2 = (88\text{ ft/s})^2 + 2(-0.92419\text{ ft/s}^2)x_{\text{upgrade}}$$

or $\quad x_{\text{upgrade}} = 1280.16\text{ ft}$

or $\quad x_{\text{upgrade}} = 0.242\text{ mi}$ ◀



PROBLEM 12.6

A 0.2-lb model rocket is launched vertically from rest at time $t = 0$ with a constant thrust of 2 lb for one second and no thrust for $t > 1$ s. Neglecting air resistance and the decrease in mass of the rocket, determine (*a*) the maximum height *h* reached by the rocket, (*b*) the time required to reach this maximum height.

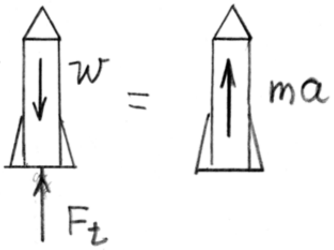
SOLUTION

Given: $W = 0.2$ lbs

For $0 \le t \le 1$ s $\quad F_t = 2$ lbs

For $t > 1$ s $\quad F_t = 0$ lbs

Free Body Diagram:



For the thrust phase, $0 \le t \le 1$ s: $+\!\uparrow \; \Sigma F = ma: \quad F_t - W = ma = \dfrac{W}{g}a$

$$a = g\left(\frac{F_t}{W} - 1\right) = (32.2)\left(\frac{2}{0.2} - 1\right) = 289.8 \text{ ft/s}^2$$

At $t = 1$ s: $\quad v = at = (289.8)(1) = 289.8$ ft/s

$$y = \frac{1}{2}at^2 = \frac{1}{2}(289.8)(1)^2 = 144.9 \text{ ft}$$

For free flight phase, $t > 1$ s: $\quad a = -g = -32.2$ ft/s

$$v = v_1 + a(t-1) = 289.8 + (-32.2)(t-1)$$

(*a*) At maximum height: $\quad v = 0, \quad t - 1 = \dfrac{289.8}{32.2} = 9.00 \text{ s}, \quad t = 10.00 \text{ s}$

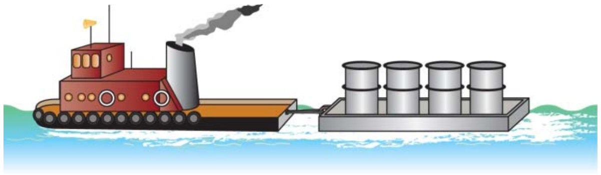
$$v^2 - v_1^2 = 2a(y - y_1) = -2g(y - y_1)$$

$$y - y_1 = -\frac{v^2 - v_1^2}{2g} = -\frac{0 - (289.8)^2}{(2)(32.2)} = 1304.1 \text{ ft}$$

$$y_{\max} = h = 1304.1 + 144.9$$

$h = 1449$ ft ◀

(*b*) Time to reach max height: $\quad t = 10.00$ s ◀



PROBLEM 12.7

A tugboat pulls a small barge through a harbor. The propeller thrust minus the drag produces a net thrust that varies linearly with speed. Knowing that the combined weight of the tug and barge is 3600 kN, determine (*a*) the time required to increase the speed from an initial value v_1 = 1.0 m/s to a final value v_2 = 2.5 m/s, (*b*) the distance traveled during this time interval.

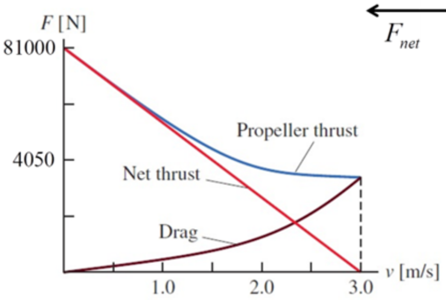
SOLUTION

Given:

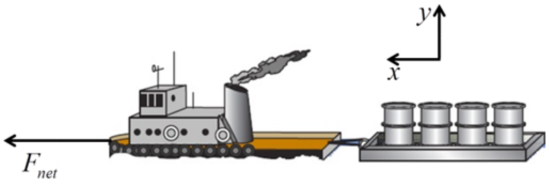
$$W = 3600 \text{ kN}$$

$$m = \frac{W}{g} \Rightarrow m = 366972.5 \text{ kg}$$

$$v_1 = 1.0 \text{ m/s}, \; v_2 = 2.5 \text{ m/s}$$



Free Body Diagram:



From Graph:

$$F_{net} = 81000 - \frac{81000}{3}v = 81000 - 27000v \text{ N}$$

From FBD:

$$\Sigma F_x = ma_x \Rightarrow \quad F_{net} = ma_x \Rightarrow a_x = \frac{F_{net}}{m}$$

$$a_x = \frac{81000 - 27000v}{m}$$

$$= \frac{27000}{m}(3 - v) \text{ m/s}^2$$

(*a*) From Kinematics:

$$a_x = \frac{dv}{dt} \Rightarrow dt = \frac{dv}{a_x}$$

$$\int_0^t dt = \frac{m}{27000}\int_{v_1}^{v_2} \frac{dv}{3 - v}$$

$$t = -\frac{m}{27000}\ln(3 - v)\Big|_1^{2.5}$$

$$t = 18.84 \text{ s} \blacktriangleleft$$

PROBLEM 12.7 (Continued)

(b) From Kinematics: $a_x dx = v dv \Rightarrow dx = \dfrac{v dv}{a_x}$

$$\int_0^x dx = \frac{m}{27000}\int_{v_1}^{v_2}\frac{v dv}{3-v}$$

$$x = \frac{m}{27000}\left(3 - v - 3\ln\left|3-v\right|\right)\Big|_1^{2.5}$$

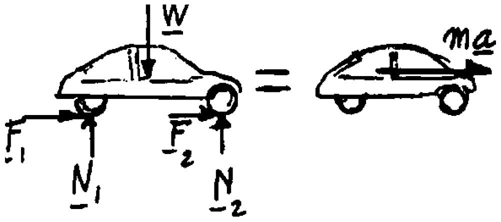
$x = 36.14$ m ◀

PROBLEM 12.8

Determine the maximum theoretical speed that may be achieved over a distance of 60 m by a car starting from rest, knowing that the coefficient of static friction is 0.80 between the tires and the pavement and that 60 percent of the weight of the car is distributed over its front wheels and 40 percent over its rear wheels. Assume (*a*) four-wheel drive, (*b*) front-wheel drive, (*c*) rear-wheel drive.

SOLUTION

(*a*) Four-wheel drive



$$+\uparrow \Sigma F_y = 0: \quad N_1 + N_2 - W = 0$$

$$F_1 + F_2 = \mu N_1 + \mu N_2$$
$$= \mu(N_1 + N_2) = \mu w$$

$$\xrightarrow{+} \Sigma F_x = ma: \quad F_1 + F_2 = ma$$

$$\mu w = ma$$

$$a = \frac{\mu W}{m} = \mu \frac{mg}{m} = \mu g = 0.80(9.81)$$

$$a = 7.848 \text{ m/s}^2$$

$$v^2 = 2ax = 2(7.848 \text{ m/s}^2)(60 \text{ m}) = 941.76 \text{ m}^2/\text{s}^2$$

$v = 30.69$ m/s $\qquad v = 110.5$ km/h ◀

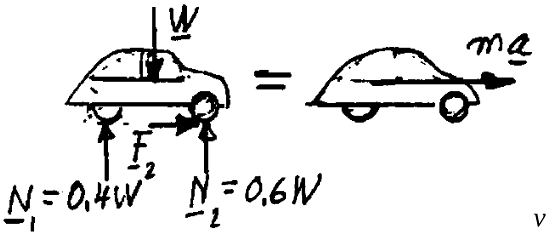
(*b*) Front-wheel drive

$$F_2 = ma$$

$$\mu(0.6\ W) = ma$$

$$a = \frac{0.6\mu W}{m} = \frac{0.6\mu mg}{m}$$
$$= 0.6\mu g = 0.6(0.80)(9.81)$$

$$a = 4.709 \text{ m/s}^2$$

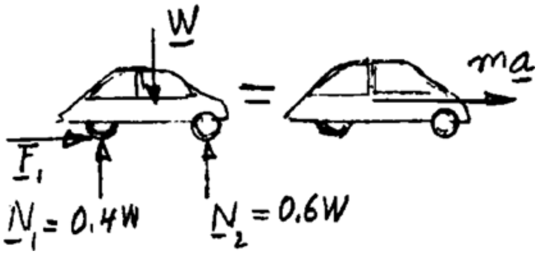


$$v^2 = 2ax = 2(4.709 \text{ m/s}^2)(60 \text{ m}) = 565.1 \text{ m}^2/\text{s}^2$$

$v = 23.77$ m/s $\qquad v = 85.6$ km/h ◀

PROBLEM 12.8 (Continued)

(*c*) Rear-wheel drive



$$F_1 = ma$$

$$\mu(0.4\ W) = ma$$

$$a = \frac{0.4\mu W}{m} = \frac{0.4\mu mg}{m}$$

$$= 0.4\mu g = 0.4(0.80)(9.81)$$

$$a = 3.139 \text{ m/s}^2$$

$$v^2 = 2ax = 2(3.139 \text{ m/s}^2)(60 \text{ m}) = 376.7 \text{ m}^2/\text{s}^2$$

$$v = 19.41 \text{ m/s} \qquad v = 69.9 \text{ km/h} \blacktriangleleft$$

PROBLEM 12.9

If an automobile's braking distance from 90 km/h is 45 m on level pavement, determine the automobile's braking distance from 90 km/h when it is (*a*) going up a 5° incline, (*b*) going down a 3-percent incline. Assume the braking force is independent of grade.

SOLUTION

Assume uniformly decelerated motion in all cases.

For braking on the level surface,

$$v_0 = 90 \text{ km/h} = 25 \text{ m/s}, \quad v_f = 0$$

$$x_f - x_0 = 45 \text{ m}$$

$$v_f^2 = v_0^2 + 2a(x_f - x_0)$$

$$a = \frac{v_f^2 - v_0^2}{2(x_f - x_0)} = \frac{0 - (25)^2}{(2)(45)} = -6.9444 \text{ m/s}^2$$

Braking force.

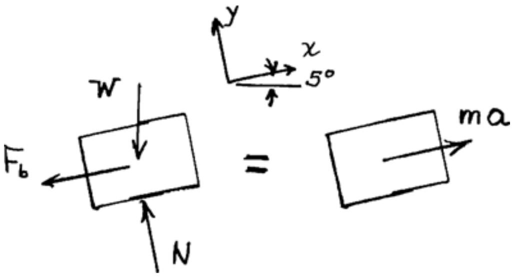
$$F_b = ma = \frac{W}{g}a = -\frac{6.944}{9.81}W = -0.70789W$$

(*a*) Going up a 5° incline.

$$\nearrow \Sigma F = ma$$

$$-F_b - W\sin 5° = \frac{W}{g}a$$

$$a = -\frac{F_b + W \sin 5°}{W}g = -(0.70789 + \sin 5°)(9.81) = -7.79944 \text{ m/s}^2$$

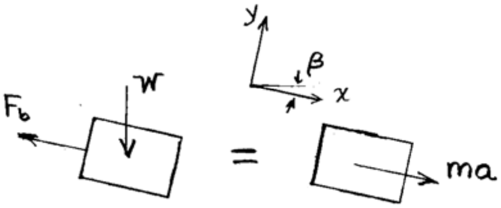


$$x_f - x_0 = \frac{v_f^2 - v_0^2}{2a} = \frac{0 - (25)^2}{(2)(-7.79944)}$$

$$x_f - x_0 = 40.1 \text{ m} \blacktriangleleft$$

PROBLEM 12.9 (Continued)

(*b*) Going down a 3 percent incline.



$$\tan\beta = \frac{3}{100} \qquad \beta = 1.71835°$$

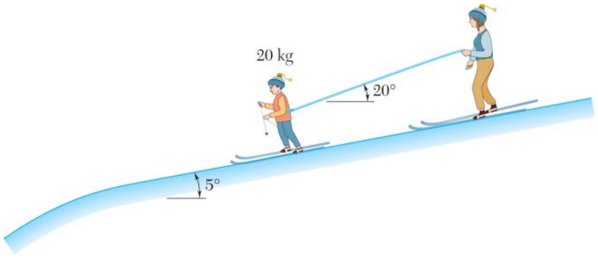
$$-F_b + W\sin\beta = \frac{W}{g}a$$

$$a = -(0.70789 - \sin\beta)(9.81)$$

$$= -6.65028 \text{ m/s}$$

$$x_f = x_0 = \frac{0-(25)^2}{(2)(-6.65028)}$$

$x_f - x_0 = 47.0$ m ◀

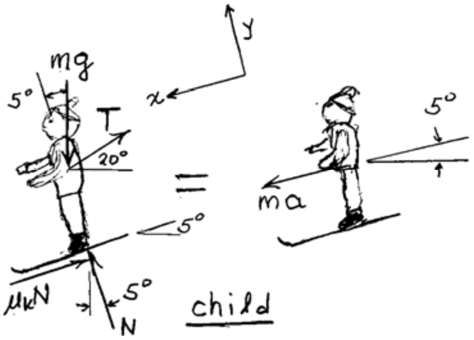


PROBLEM 12.10

A mother and her child are skiing together, and the mother is holding the end of a rope tied to the child's waist. They are moving at a speed of 7.2 km/h on a gently sloping portion of the ski slope when the mother observes that they are approaching a steep descent. She pulls on the rope with an average force of 7 N. Knowing the coefficient of friction between the child and the ground is 0.1 and the angle of the rope does not change, determine (*a*) the time required for the child's speed to be cut in half, (*b*) the distance traveled in this time.

SOLUTION

Draw free body diagram of child.



$\Sigma \mathbf{F} = m\mathbf{a}$:

x-direction: $mg\sin 5° - \mu_k N - T\cos 15° = ma$

y-direction: $N - mg\cos 5° + T\sin 15° = 0$

From y-direction,

$$N = mg\cos 5° - T\sin 15° = (20\text{ kg})(9.81\text{ m/s}^2)\cos 5° - (7\text{ N})\sin 15° = 193.64\text{ N}$$

From x-direction,

$$a = g\sin 5° - \frac{\mu_k N}{m} - \frac{T\cos 15°}{m}$$

$$= (9.81\text{ m/s}^2)\sin 5° - \frac{(0.1)(193.64\text{ N})}{20\text{ kg}} - \frac{(7\text{ N})\cos 15°}{20\text{ kg}}$$

$$= -0.45128\text{ m/s}^2 \qquad \text{(in } x\text{-direction.)}$$

$$v_0 = 7.2\text{ km/h} = 2\text{ m/s} \qquad x_0 = 0$$

$$v_f = \frac{1}{2}v_0 = 1\text{ m/s}$$

$$v_f = v_0 + at \qquad t = \frac{v_f - v_0}{a} = \frac{-1\text{ m/s}}{-0.45128\text{ m/s}^2} = 2.2159\text{ s}$$

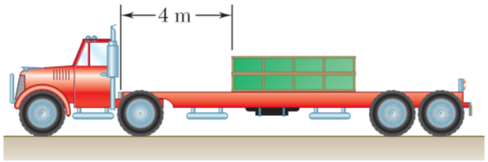
(*a*) *Time elapsed.* $t = 2.22\text{ s}$ ◀

(*b*) *Corresponding distance.*

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

$$= 0 + (2\text{ m/s})(2.2159\text{ s}) + \frac{1}{2}(-0.45128\text{ m/s}^2)(2.2159\text{ s})^2$$

$x = 3.32\text{ m}$ ◀

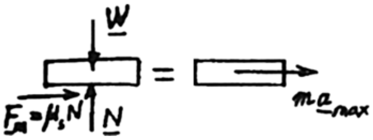


PROBLEM 12.11

The coefficients of friction between the load and the flat-bed trailer shown are $\mu_s = 0.40$ and $\mu_k = 0.30$. Knowing that the speed of the rig is 72 km/h, determine the shortest distance in which the rig can be brought to a stop if the load is not to shift.

SOLUTION

Load: We assume that sliding of load relative to trailer is impending:



$$F = F_m$$
$$= \mu_s N$$

Deceleration of load is same as deceleration of trailer, which is the maximum allowable deceleration $\mathbf{a}_{\max}$.

$$+\!\uparrow \Sigma F_y = 0: \quad N - W = 0 \quad N = W$$

$$F_m = \mu_s N = 0.40\,W$$

$$\xrightarrow{+} \Sigma F_x = ma: \quad F_m = ma_{\max}$$

$$0.40\ W = \frac{W}{g} a_{\max} \qquad a_{\max} = 3.924 \text{ m/s}^2$$

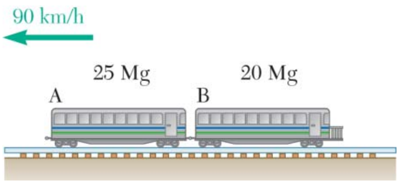
$$\mathbf{a}_{\max} = 3.92 \text{ m/s}^2 \longrightarrow$$

Uniformly accelerated motion.

$$v^2 = v_0^2 + 2ax \quad \text{with} \quad v = 0 \qquad\qquad v_0 = 72 \text{ km/h} = 20 \text{ m/s}$$

$$a = -a_{\max} = 3.924 \text{ m/s}^2$$

$$0 = (20)^2 + 2(-3.924)x \qquad\qquad x = 51.0 \text{ m} \blacktriangleleft$$


90 km/h
25 Mg
20 Mg
A
B

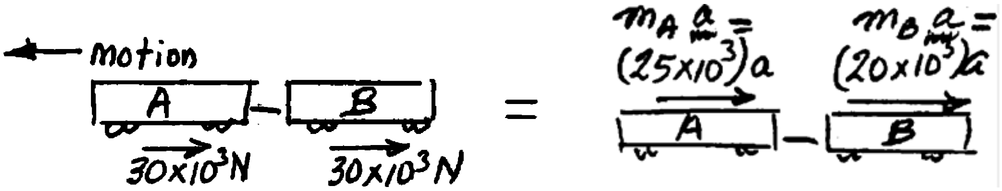
PROBLEM 12.12

A light train made up of two cars is traveling at 90 km/h when the brakes are applied to both cars. Knowing that car *A* has a mass of 25 Mg and car *B* a mass of 20 Mg, and that the braking force is 30 kN on each car, determine (*a*) the distance traveled by the train before it comes to a stop, (*b*) the force in the coupling between the cars while the train is showing down.

SOLUTION

$v_0 = 90 \text{ km/h} = 90/3.6 = 25 \text{ m/s}$

(*a*) Both cars:

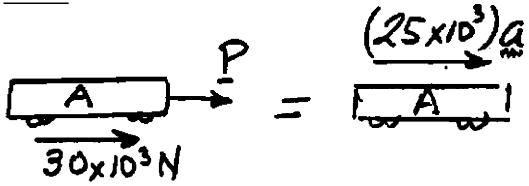

motion
A
B
30x10³N
30x10³N
=
m_A a =
(25x10³)a
m_B a =
(20x10³)a
A
B

$\xrightarrow{+} \Sigma F_x = \Sigma ma$: $60 \times 10^3 \text{ N} = (45 \times 10^3 \text{ kg})a$ $\qquad a = 1.333 \text{ m/s}^2 \longrightarrow$

$v^2 = v_0^1 + 2ax$: $0 = (25)^2 + 2(-1.333)x$

Stopping distance: $\qquad x = 234 \text{ m}$ ◀

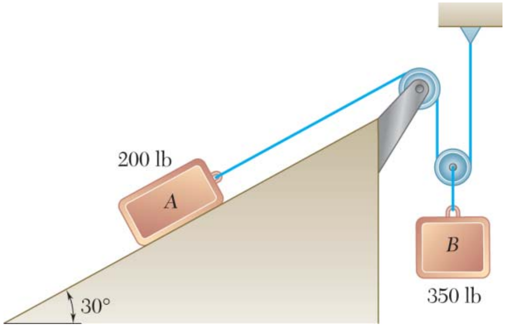
(*b*) Car *A*:


P
A
30x10³N
=
(25x10³)a
A

$\xrightarrow{+} \Sigma F_x = ma$: $30 \times 10^3 + P = (25 \times 10^3)a$

$P = (25 \times 10^3)(1.333) - 30 \times 10^3$

Coupling force: $\qquad P = +3332 \text{ N} \qquad P = 3.33 \text{ kN (tension)}$ ◀



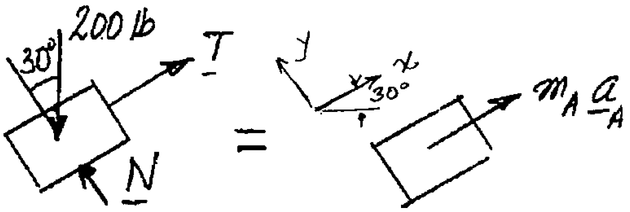
PROBLEM 12.13

The two blocks shown are originally at rest. Neglecting the masses of the pulleys and the effect of friction in the pulleys and between block *A* and the incline, determine (*a*) the acceleration of each block, (*b*) the tension in the cable.

SOLUTION

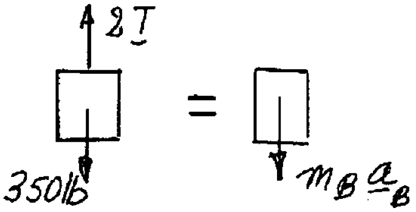
(*a*) We note that $a_B = \frac{1}{2}a_A$.

Block *A*



$$\nearrow\!+\ \Sigma F_x = m_A a_A: \quad T - (200\text{ lb})\sin 30° = \frac{200}{32.2}a_A \tag{1}$$

Block *B*



$$+\!\downarrow \Sigma F_y = m_B a_B: \quad 350\text{ lb} - 2T = \frac{350}{32.2}\left(\frac{1}{2}a_A\right) \tag{2}$$

(*a*) Multiply Eq. (1) by 2 and add Eq. (2) in order to eliminate *T*:

$$-2(200)\sin 30° + 350 = 2\frac{200}{32.2}a_A + \frac{350}{32.2}\left(\frac{1}{2}a_A\right)$$

$$150 = \frac{575}{32.2}a_A$$

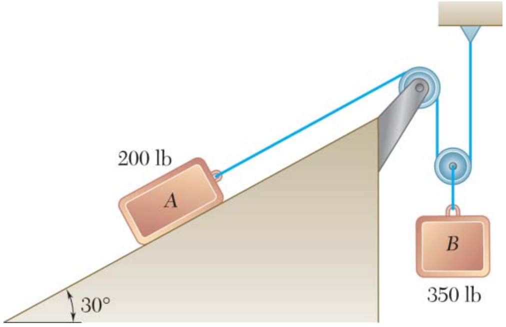
$\mathbf{a}_A = 8.40\text{ ft/s}^2 \measuredangle 30°$ ◀

$$a_B = \frac{1}{2}a_A = \frac{1}{2}(8.40\text{ ft/s}^2),$$

$\mathbf{a}_B = 4.20\text{ ft/s}^2\downarrow$ ◀

(*b*) From Eq. (1), $T - (200)\sin 30° = \dfrac{200}{32.2}(8.40)$

$T = 152.2\text{ lb}$ ◀



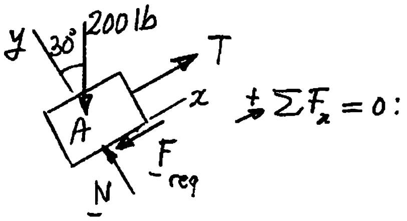
PROBLEM 12.14

Solve Problem 12.13, assuming that the coefficients of friction between block *A* and the incline are $\mu_s = 0.25$ and $\mu_k = 0.20$.

PROBLEM 12.13 The two blocks shown are originally at rest. Neglecting the masses of the pulleys and the effect of friction in the pulleys and between block *A* and the incline, determine (*a*) the acceleration of each block, (*b*) the tension in the cable.

SOLUTION

We first determine whether the blocks move by computing the friction force required to maintain block A in equilibrium. *T* = 175 lb. When *B* in equilibrium,



$$+\nearrow \Sigma F_x = 0: \quad 175 - 200\sin 30° - F_{\text{req}} = 0$$

$$F_{\text{req}} = 75.0 \text{ lb}$$

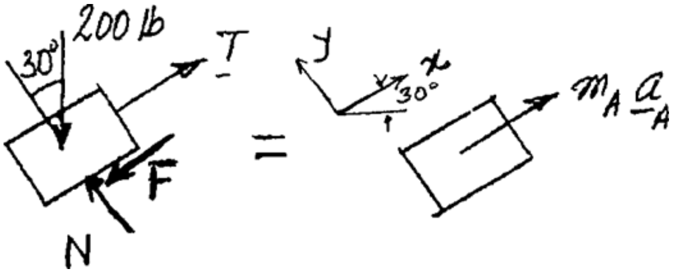
$$+\nwarrow \Sigma F_y = 0: \quad N - 200\cos 30° = 0 \quad N = 173.2 \text{ lb}$$

$$F_M = \mu_s N = 0.25(173.2 \text{ lb}) = 43.3 \text{ lb}$$

Since $F_{\text{req}} > \mathbf{F}_m$, blocks will move (*A* up and *B* down).

We note that $a_B = \frac{1}{2} a_A$.

Block *A*

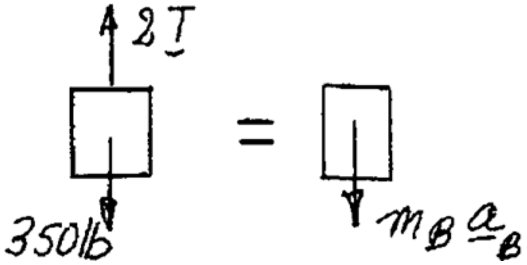


$$F = \mu_k N = (0.20)(173.2) = 34.64 \text{ lb.}$$

$$+\nearrow \Sigma F_x = m_A 0_A: \quad -200\sin 30° - 34.64 + T = \frac{200}{32.2} a_A \qquad (1)$$

PROBLEM 12.14 (Continued)

Block *B*



$$+\downarrow \Sigma F_y = m_B a_B: \quad 350\,\text{lb} - 2T = \frac{350}{32.2}\left(\frac{1}{2}a_A\right) \qquad (2)$$

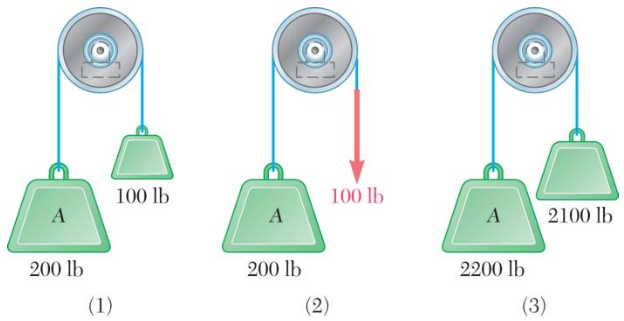
(*a*) Multiply Eq. (1) by 2 and add Eq. (2) in order to eliminate *T*:

$$-2(200)\sin 30^\circ - 2(34.64) + 350 = 2\frac{200}{32.2}a_A + \frac{350}{32.2}\left(\frac{1}{2}a_A\right)$$

$81.32 = \frac{575}{32.2}a_A$ $\qquad$ $a_A = 4.55\ \text{ft/s}^2 \measuredangle 30^\circ$ ◀

$a_B = \frac{1}{2}a_A = \frac{1}{2}(4.52\ \text{ft/s}^2),$ $\qquad$ $a_B = 2.28\ \text{ft/s}^2 \downarrow$ ◀

(*b*) From Eq. (1), $T - (200)\sin 30^\circ - 34.64 = \frac{200}{32.2}(4.52)$ $\qquad$ $T = 162.9$ lb ◀



PROBLEM 12.15

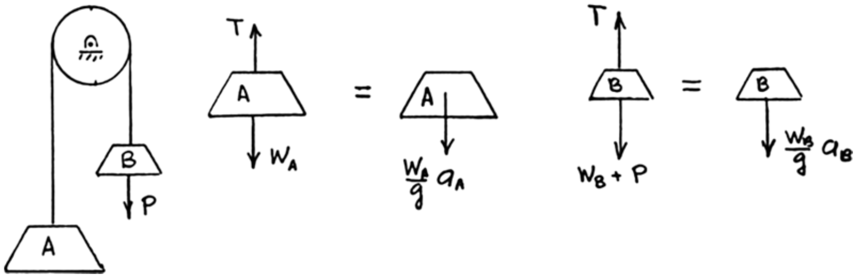
Each of the systems shown is initially at rest. Neglecting axle friction and the masses of the pulleys, determine for each system (*a*) the acceleration of block *A*, (*b*) the velocity of block *A* after it has moved through 10 ft, (*c*) the time required for block *A* to reach a velocity of 20 ft/s.

SOLUTION

Let y be positive downward for both blocks.

Constraint of cable: $y_A + y_B = \text{constant}$

$$a_A + a_B = 0 \qquad \text{or} \qquad a_B = -a_A$$



For blocks *A* and *B*, $+\!\downarrow \Sigma F = ma$:

Block *A*: $$W_A - T = \frac{W_A}{g}a_A \qquad \text{or} \qquad T = W_A - \frac{W_A}{g}a_A$$

Block *B*: $$P + W_B - T = \frac{W_B}{g}a_B = -\frac{W_B}{g}a_A$$

$$P + W_B - W_A + \frac{W_A}{g}a_A = -\frac{W_B}{g}a_A$$

Solving for a_A, $$a_A = \frac{W_A - W_B - P}{W_A + W_B}g \qquad (1)$$

$$v_A^2 - (v_A)_0^2 = 2a_A[y_A - (y_A)_0] \quad \text{with} \quad (v_A)_0 = 0$$

$$v_A = \sqrt{2a_A[y_A - (y_A)_0]} \qquad (2)$$

$$v_A - (v_A)_0 = a_A t \quad \text{with} \quad (v_A)_0 = 0$$

$$t = \frac{v_A}{a_A} \qquad (3)$$

PROBLEM 12.15 (Continued)

(*a*) Acceleration of block *A*.

System (1): $W_A = 200 \text{ lb}, \quad W_B = 100 \text{ lb}, \quad P = 0$

By formula (1), $(a_A)_1 = \dfrac{200-100}{200+100}(32.2)$ $\quad (\mathbf{a}_A)_1 = 10.73 \text{ ft/s}^2 \downarrow$ ◀

System (2): $W_A = 200 \text{ lb}, \quad W_B = 0, \quad P = 50 \text{ lb}$

By formula (1), $(a_A)_2 = \dfrac{200-100}{200}(32.2)$ $\quad (\mathbf{a}_A)_2 = 16.10 \text{ ft/s}^2 \downarrow$ ◀

System (3): $W_A = 2200 \text{ lb}, \quad W_B = 2100 \text{ lb}, \quad P = 0$

By formula (1), $(a_A)_3 = \dfrac{2200-2100}{2200+2100}(32.2)$ $\quad (\mathbf{a}_A)_3 = 0.749 \text{ ft/s}^2 \downarrow$ ◀

(*b*) v_A at $y_A - (y_A)_0 = 10$ ft. Use formula (2).

System (1): $(v_A)_1 = \sqrt{(2)(10.73)(10)}$ $\quad (v_A)_1 = 14.65 \text{ ft/s} \downarrow$ ◀

System (2): $(v_A)_2 = \sqrt{(2)(16.10)(10)}$ $\quad (v_A)_2 = 17.94 \text{ ft/s} \downarrow$ ◀

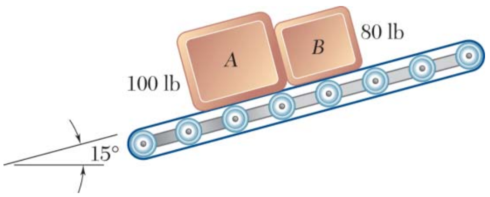
System (3): $(v_A)_3 = \sqrt{(2)(0.749)(10)}$ $\quad (v_A)_3 = 3.87 \text{ ft/s} \downarrow$ ◀

(*c*) Time at $v_A = 20$ ft/s. Use formula (3).

System (1): $t_1 = \dfrac{20}{10.73}$ $\quad t_1 = 1.864 \text{ s}$ ◀

System (2): $t_2 = \dfrac{20}{16.10}$ $\quad t_2 = 1.242 \text{ s}$ ◀

System (3): $t_3 = \dfrac{20}{0.749}$ $\quad t_3 = 26.7 \text{ s}$ ◀



PROBLEM 12.16

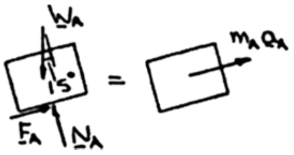
Boxes *A* and *B* are at rest on a conveyor belt that is initially at rest. The belt is suddenly started in an upward direction so that slipping occurs between the belt and the boxes. Knowing that the coefficients of kinetic friction between the belt and the boxes are $(\mu_k)_A = 0.30$ and $(\mu_k)_B = 0.32$, determine the initial acceleration of each box.

SOLUTION

Assume that $a_B > a_A$ so that the normal force N_{AB} between the boxes is zero.

A: $+\nwarrow \Sigma F_y = 0: \quad N_A - W_A \cos 15° = 0$

A:



or $N_A = W_A \cos 15°$

Slipping: $F_A = (\mu_k)_A N_A$

$= 0.3 W_A \cos 15°$

$+\nearrow \Sigma F_x = m_A a_A: \quad F_A - W_A \sin 15° = m_A a_A$

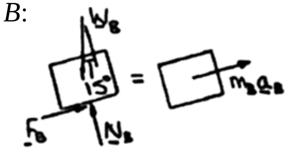
or $0.3 W_A \cos 15° - W_A \sin 15° = \dfrac{W_A}{g} a_A$

or $a_A = (32.2\ \text{ft/s}^2)(0.3 \cos 15° - \sin 15°)$

$= 0.997\ \text{ft/s}^2$

B: $+\nwarrow \Sigma F_y = 0: \quad N_B - W_B \cos 15° = 0$

B:



or $N_B = W_B \cos 15°$

Slipping: $F_B = (\mu_k)_B N_B$

$= 0.32 W_B \cos 15°$

$+\nearrow \Sigma F_x = m_B a_B: \quad F_B - W_B \sin 15° = m_B a_B$

or $0.32 W_B \cos 15° - W_B \sin 15° = \dfrac{W_B}{g} a_B$

or $a_B = (32.2\ \text{ft/s}^2)(0.32 \cos 15° - \sin 15°) = 1.619\ \text{ft/s}^2$

$a_B > a_A \Rightarrow$ assumption is correct

$\mathbf{a}_A = 0.997\ \text{ft/s}^2 \nearrow 15°$ ◀

$\mathbf{a}_B = 1.619\ \text{ft/s}^2 \nearrow 15°$ ◀

PROBLEM 12.16 (Continued)

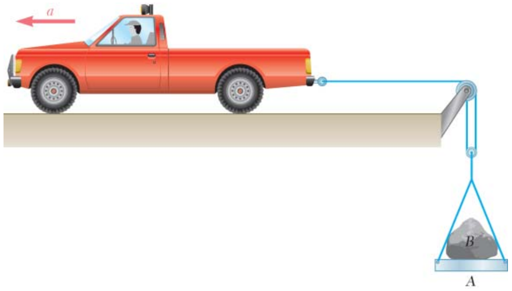
Note: If it is assumed that the boxes remain in contact $(N_{AB} \neq 0)$, then assuming N_{AB} to be compression,

$$a_A = a_B \quad \text{and find} \quad (\Sigma F_x = ma) \text{ for each box.}$$

A: $$0.3W_A \cos 15° - W_A \sin 15° - N_{AB} = \frac{W_A}{g}a$$

B: $$0.32W_B \cos 15° - W_B \sin 15° + N_{AB} = \frac{W_B}{g}a$$

Solving yields $a = 1.273 \text{ ft/s}^2$ and $N_{AB} = -0.859$ lb, which contradicts the assumption.



PROBLEM 12.17

A 5000-lb truck is being used to lift a 1000 lb boulder *B* that is on a 200 lb pallet *A*. Knowing the acceleration of the truck is $1\ \text{ft/s}^2$, determine (*a*) the horizontal force between the tires and the ground, (*b*) the force between the boulder and the pallet.

SOLUTION

Kinematics:

$$\mathbf{a}_T = 1\ \text{m/s}^2 \leftarrow$$

$$\mathbf{a}_A = \mathbf{a}_B = 0.5\ \text{m/s}^2 \uparrow$$

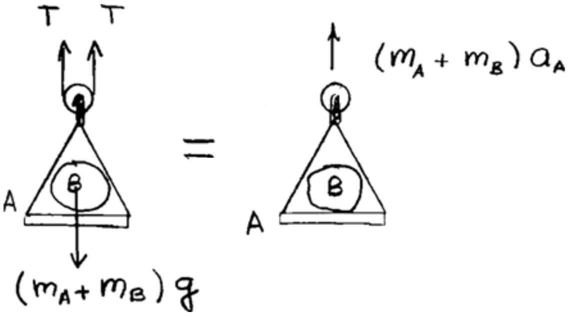
Masses:

$$m_T = \frac{5000}{32.2} = 155.28\ \text{slugs}$$

$$m_A = \frac{200}{32.2} = 6.211\ \text{slugs}$$

$$m_B = \frac{1000}{32.2} = 31.056\ \text{slugs}$$

Let *T* be the tension in the cable. Apply Newton's second law to the lower pulley, pallet and boulder.



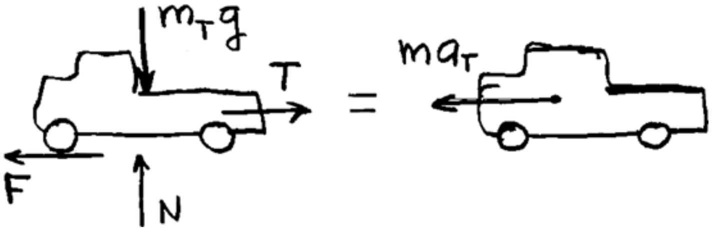
Vertical components $+\uparrow$:

$$2T - (m_A + m_B)g = (m_A + m_B)a_A$$

$$2T - (37.267)(32.2) = (37.267)(0.5)$$

$$T = 609.32\ \text{lb}$$

Apply Newton's second law to the truck.



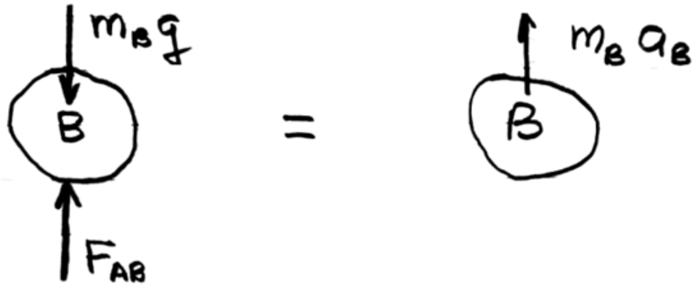
PROBLEM 12.17 (Continued)

Horizontal components $\overset{+}{\leftarrow}$: $F - T = m_T a_T$

(*a*) Horizontal friction force between tires and ground.

$$F = T + m_T a_T = 609.32 + (155.28)(1.0) \qquad F = 765 \text{ lb} \blacktriangleleft$$

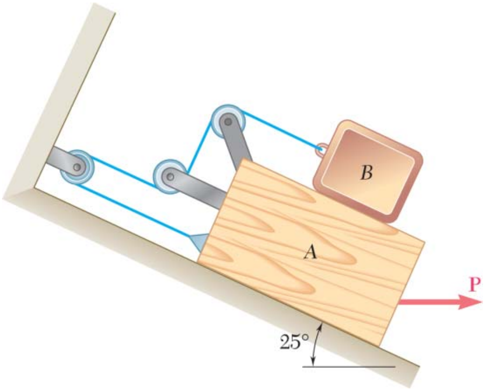
Apply Newton's second law to the boulder.



Vertical components $+\uparrow$: $F_{AB} - m_B g = m_B a_B$

$$F_{AB} = m_B(g + a) = 31.056(32.2 + 0.5)$$

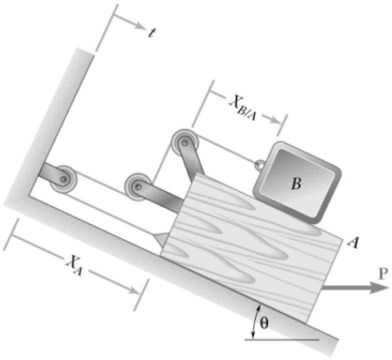
(*b*) Contact force between boulder and pallet: $F_{AB} = 1016 \text{ lb} \blacktriangleleft$



PROBLEM 12.18

Block A has a mass of 40 kg, and block B has a mass of 8 kg. The coefficients of friction between all surfaces of contact are $\mu_s = 0.20$ and $\mu_k = 0.15$. If $P = 0$, determine (*a*) the acceleration of block B, (*b*) the tension in the cord.

SOLUTION



From the constraint of the cord:

$$2x_A + x_{B/A} = \text{constant}$$

Then $$2v_A + v_{B/A} = 0$$

and $$2a_A + a_{B/A} = 0$$

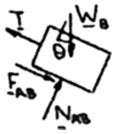
Now $$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

Then $$a_B = a_A + (-2a_A)$$

or $$a_B = -a_A \qquad (1)$$

First we determine if the blocks will move for the given value of θ. Thus, we seek the value of θ for which the blocks are in impending motion, with the impending motion of A down the incline.

B:



B: $$+\nearrow \Sigma F_y = 0: \quad N_{AB} - W_B \cos\theta = 0$$

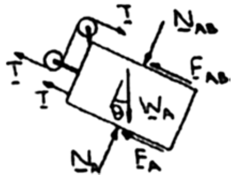
or $$N_{AB} = m_B g \cos\theta$$

Now $$F_{AB} = \mu_s N_{AB}$$
$$= 0.2 m_B g \cos\theta$$

$$\overset{+}{\searrow} \Sigma F_x = 0: \quad -T + F_{AB} + W_B \sin\theta = 0$$

or $$T = m_B g (0.2\cos\theta + \sin\theta)$$

A:



A: $$+\nearrow \Sigma F_y = 0: \quad N_A - N_{AB} - W_A \cos\theta = 0$$

or $$N_A = (m_A + m_B) g \cos\theta$$

Now $$F_A = \mu_s N_A$$
$$= 0.2(m_A + m_B) g \cos\theta$$

$$\overset{+}{\searrow} \Sigma F_x = 0: \quad -T - F_A - F_{AB} + W_A \sin\theta = 0$$

PROBLEM 12.18 (Continued)

or $$T = m_A g\sin\theta - 0.2(m_A + m_B)g\cos\theta - 0.2 m_B g\cos\theta$$
$$= g[m_A\sin\theta - 0.2(m_A + 2m_B)\cos\theta]$$

Equating the two expressions for T

$$m_B g(0.2\cos\theta + \sin\theta) = g[m_A\sin\theta - 0.2(m_A + 2m_B)\cos\theta]$$

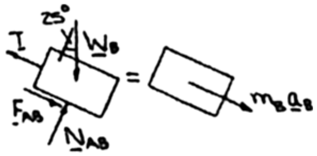
or $$8(0.2 + \tan\theta) = [40\tan\theta - 0.2(40 + 2\times 8)]$$

or $$\tan\theta = 0.4$$

or $\theta = 21.8^\circ$ for impending motion. Since $\theta < 25^\circ$, the blocks will move. Now consider the motion of the blocks.

(*a*) $+\nearrow\ \Sigma F_y = 0:\quad N_{AB} - W_B\cos 25^\circ = 0$

B:



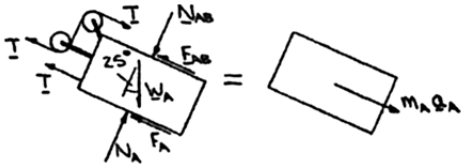
or $$N_{AB} = m_B g\cos 25^\circ$$

Sliding: $$F_{AB} = \mu_k N_{AB} = 0.15 m_B g\cos 25^\circ$$

$$+\searrow\ \Sigma F_x = m_B a_B:\quad -T + F_{AB} + W_B\sin 25^\circ = m_B a_B$$

or $$T = m_B[g(0.15\cos 25^\circ + \sin 25^\circ) - a_B]$$
$$= 8[9.81(0.15\cos 25^\circ + \sin 25^\circ) - a_B]$$
$$= 8(5.47952 - a_B) \qquad \text{(N)}$$

A:



$$+\nearrow\ \Sigma F_y = 0:\quad N_A - N_{AB} - W_A\cos 25^\circ = 0$$

or $$N_A = (m_A + m_B)g\cos 25^\circ$$

Sliding: $$F_A = \mu_k N_A = 0.15(m_A + m_B)g\cos 25^\circ$$

$$+\searrow\ \Sigma F_x = m_A a_A:\quad -T - F_A - F_{AB} + W_A\sin 25^\circ = m_A a_A$$

Substituting and using Eq. (1)

$$T = m_A g\ \sin 25^\circ - 0.15(m_A + m_B)g\ \cos 25^\circ$$
$$-0.15 m_B g\ \cos 25^\circ - m_A(-a_B)$$
$$= g[m_A\ \sin 25^\circ - 0.15(m_A + 2m_B)\cos 25^\circ] + m_A a_B$$
$$= 9.81[40\ \sin 25^\circ - 0.15(40 + 2\times 8)\cos 25^\circ] + 40a_B$$
$$= 91.15202 + 40a_B \qquad \text{(N)}$$

Equating the two expressions for T

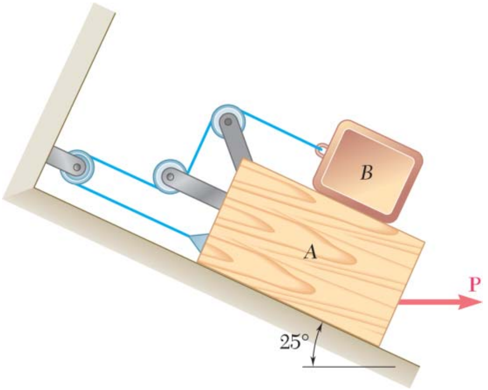
$$8(5.47952 - a_B) = 91.15202 + 40a_B$$

or $$a_B = -0.98575\ \text{m/s}^2$$

$\mathbf{a}_B = 0.986\ \text{m/s}^2 \searrow 25^\circ$ ◀

(*b*) We have $$T = 8[5.47952 - (-0.98575)]$$

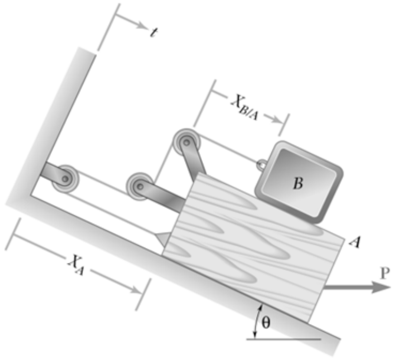
or $T = 51.7\ \text{N}$ ◀



PROBLEM 12.19

Block A has a mass of 40 kg, and block B has a mass of 8 kg. The coefficients of friction between all surfaces of contact are $\mu_s = 0.20$ and $\mu_k = 0.15$. If $P = 40\text{ N} \longrightarrow$, determine ($a$) the acceleration of block B, (b) the tension in the cord.

SOLUTION



From the constraint of the cord.

$$2x_A + x_{B/A} = \text{constant}$$

Then $$2v_A + v_{B/A} = 0$$

and $$2a_A + a_{B/A} = 0$$

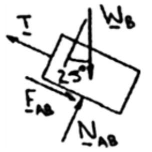
Now $$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

Then $$a_B = a_A + (-2a_A)$$

or $$a_B = -a_A \qquad (1)$$

First we determine if the blocks will move for the given value of P. Thus, we seek the value of P for which the blocks are in impending motion, with the impending motion of a down the incline.

B:

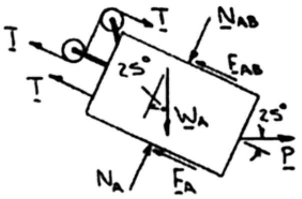


B: $$+\nearrow \Sigma F_y = 0: \quad N_{AB} - W_B \cos 25° = 0$$

or $$N_{AB} = m_B g \cos 25°$$

Now $$F_{AB} = \mu_s N_{AB}$$
$$= 0.2\, m_B g \cos 25°$$

A:



$$\overset{+}{\searrow} \Sigma F_x = 0: \quad -T + F_{AB} + W_B \sin 25° = 0$$

or $$T = 0.2\, m_B g \cos 25° + m_B g \sin 25°$$
$$= (8\text{ kg})(9.81\text{ m/s}^2)(0.2 \cos 25° + \sin 25°)$$
$$= 47.39249\text{ N}$$

A: $$+\nearrow \Sigma F_y = 0: \quad N_A - N_{AB} - W_A \cos 25° + P \sin 25° = 0$$

or $$N_A = (m_A + m_B) g \cos 25° - P \sin 25°$$

Now $$F_A = \mu_s N_A$$

or $$F_A = 0.2[(m_A + m_B)g\cos 25° - P\sin 25°]$$

$$\overset{+}{\searrow}\Sigma F_x = 0: \quad -T - F_A - F_{AB} + W_A \sin 25° + P\cos 25° = 0$$

or $$-T - 0.2[(m_A + m_B)g\cos 25° - P\sin 25°] - 0.2 m_B g\cos 25° + m_A g\sin 25° + P\cos 25° = 0$$

or $$P(0.2\sin 25° + \cos 25°) = T + 0.2[(m_A + 2m_B)g\cos 25°] - m_A g\sin 25°$$

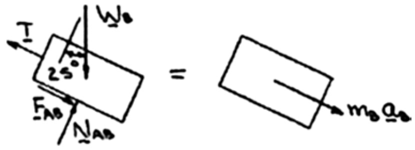
Then $$P(0.2\sin 25° + \cos 25°) = 47.39249\text{ N} + 9.81\text{ m/s}^2\{0.2[(40 + 2\times 8)\cos 25° - 40\sin 25°]\text{ kg}\}$$

or $$P = -19.04\text{ N} \text{ for impending motion.}$$

Since $P, < 40$ N, the blocks will move. Now consider the motion of the blocks.

(*a*) $$\overset{+}{\nearrow}\Sigma F_y = 0: \quad N_{AB} - W_B\cos 25° = 0$$

B:



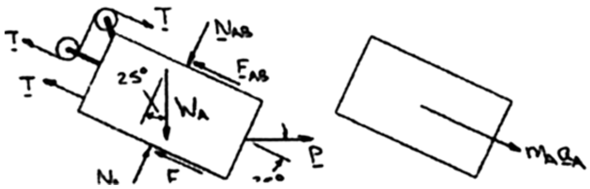
or $$N_{AB} = m_B g\cos 25°$$

Sliding: $$F_{AB} = \mu_k N_{AB}$$
$$= 0.15\, m_B g\cos 25°$$

$$\overset{+}{\searrow}\Sigma F_x = m_B a_B: \quad -T + F_{AB} + W_B\sin 25° = m_B a_B$$

or $$T = m_B[g(0.15\cos 25° + \sin 25°) - a_B]$$
$$= 8[9.81(0.15\cos 25° + \sin 25°) - a_B]$$
$$= 8(5.47952 - a_B) \qquad \text{(N)}$$

A:



$$\overset{+}{\nearrow}\Sigma F_y = 0: \quad N_A - N_{AB} - W_A\cos 25° + P\sin 25° = 0$$

or $$N_A = (m_A + m_B)g\cos 25° - P\sin 25°$$

Sliding: $$F_A = \mu_k N_A$$
$$= 0.15[(m_A + m_B)g\cos 25° - P\sin 25°]$$

$$\overset{+}{\searrow}\Sigma F_x = m_A a_A: \quad -T - F_A - F_{AB} + W_A\sin 25° + P\cos 25° = m_A a_A$$

PROBLEM 12.19 (Continued)

Substituting and using Eq. (1)

$$\begin{aligned}T &= m_A g \sin 25° - 0.15[(m_A + m_B)g \cos 25° - P \sin 25°]\\ &\quad - 0.15\, m_B g \cos 25° + P \cos 25° - m_A(-a_B)\\ &= g[m_A \sin 25° - 0.15(m_A + 2m_B)\cos 25°]\\ &\quad + P(0.15 \sin 25° + \cos 25°) + m_A a_B\\ &= 9.81[40 \sin 25° - 0.15(40 + 2 \times 8)\cos 25°]\\ &\quad + 40(0.15 \sin 25° + \cos 25°) + 40a_B\\ &= 129.94004 + 40a_B\end{aligned}$$

Equating the two expressions for T

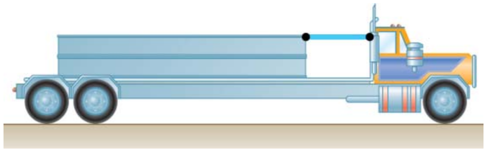
$$8(5.47952 - a_B) = 129.94004 + 40a_B$$

or $$a_B = -1.79383 \text{ m/s}^2$$

$$\mathbf{a}_B = 1.794 \text{ m/s}^2 \searrow 25° ◀$$

(*b*) We have $$T = 8[5.47952 - (-1.79383)]$$

or $$T = 58.2 \text{ N} ◀$$

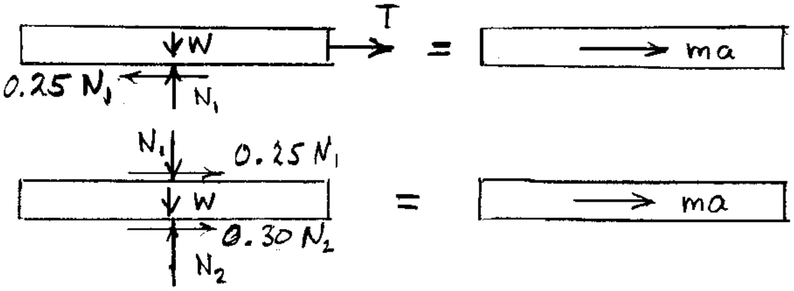


PROBLEM 12.20

The flat-bed trailer carries two 1500-kg beams with the upper beam secured by a cable. The coefficients of static friction between the two beams and between the lower beam and the bed of the trailer are 0.25 and 0.30, respectively. Knowing that the load does not shift, determine (*a*) the maximum acceleration of the trailer and the corresponding tension in the cable, (*b*) the maximum deceleration of the trailer.

SOLUTION

(*a*) Maximum acceleration. The cable secures the upper beam to the cab; the lower beam has impending slip.



For the upper beam, $+\uparrow \Sigma F_y = 0: \quad N_1 - W = 0$

$$N_1 = W = mg$$

For the lower beam, $+\uparrow \Sigma F_y = 0: \quad N_2 - N_1 - W = 0 \quad \text{or} \quad N_2 = 2W = 2mg$

$$\xrightarrow{+} \Sigma F_x = ma: \quad 0.25\,N_1 + 0.30\,N_2 = (0.25 + 0.60)mg = ma$$

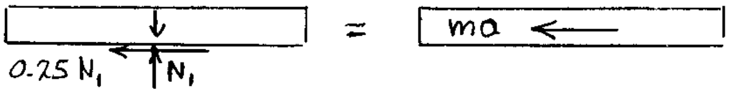
$a = 0.85g = (0.85)(9.81) \quad a = 8.34 \text{ m/s}^2 \longrightarrow$ ◀

For the upper beam, $\xrightarrow{+} \Sigma F_x = ma: \quad T - 0.25\,N_1 = ma$

$T = 0.25mg + ma = (0.25)(1500)(9.81) + (1500)(8.34) = 16.19 \times 10^3 \text{ N}$ $\qquad T = 16.19 \text{ kN}$ ◀

(*b*) Maximum deceleration of trailer.

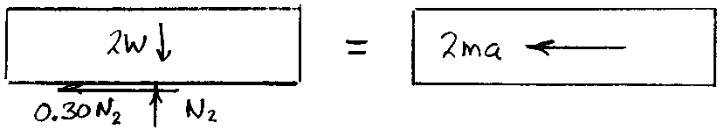
Case 1: Assume that only the top beam has impending motion. As in Part (*a*) $N_1 = mg$.



$$\xleftarrow{+} \Sigma F = ma: \quad 0.25mg = ma$$

$$a = 0.25g = 2.45 \text{ m/s}^2$$

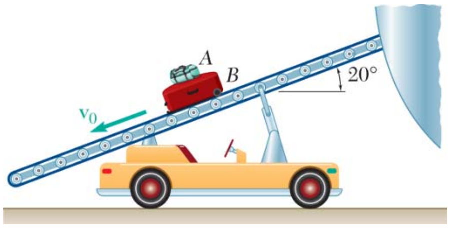
Case 2: Assume that both beams have impending slip. As before $N_2 = 2mg$.



$$\overset{+}{\longleftarrow} \Sigma F = (2m)a: \quad (0.30)(2mg) = (2m)a$$

$$a = 0.30g = 2.94 \text{ m/s}^2$$

The smaller of deceleration value of Case 1 governs. $a = 2.45 \text{ m/s}^2 \leftarrow$ ◀



PROBLEM 12.21

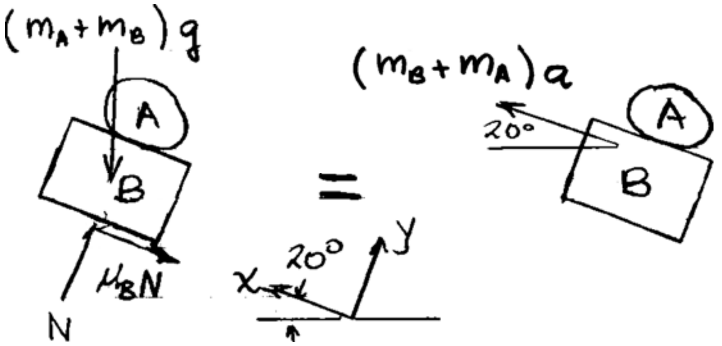
A baggage conveyor is used to unload luggage from an airplane. The 10-kg duffel bag *A* is sitting on top of the 20-kg suitcase *B*. The conveyor is moving the bags down at a constant speed of 0.5 m/s when the belt suddenly stops. Knowing that the coefficient of friction between the belt and *B* is 0.3 and that bag *A* does not slip on suitcase *B*, determine the smallest allowable coefficient of static friction between the bags.

SOLUTION

Since bag *A* does not slide on suitcase *B*, both have the same acceleration.

$$\mathbf{a} = a \nwarrow 20^\circ$$

Apply Newton's second law to the bag *A* – suitcase *B* combination treated as a single particle.



$$+\nearrow \Sigma F_y = ma_y: \quad -(m_B + m_A)g\cos 20^\circ + N = 0$$

$$N = (m_A + m_B)g\cos 30^\circ = (30)(9.81)\cos 20^\circ = 276.55 \text{ N}$$

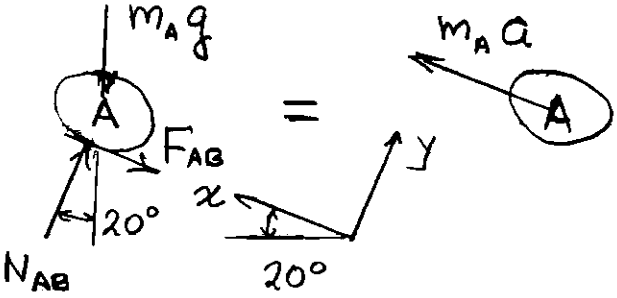
$$\mu_B N = (0.3)(276.55) = 82.965 \text{ N}$$

$$+\nwarrow \Sigma F_x = ma_x: \quad \mu_B N + (m_A + m_B)g\sin 20^\circ = (m_A + m_B)a$$

$$a = g\sin 20^\circ + \frac{\mu_B N}{m_A m_B} = 9.81\sin 20^\circ - \frac{82.965}{30}$$

$$a = 0.58972 \text{ m/s}^2 \qquad \mathbf{a} = 0.58972 \text{ m/s}^2 \swarrow 20^\circ$$

Apply Newton's second law to bag *A* alone.



PROBLEM 12.21 (Continued)

$$+\nearrow \ \Sigma F_y = ma_y: \quad N_{AB} - m_A g \cos 20° = 0$$

$$N_{AB} = m_a g \sin 20° = (10)(9.81)\cos 20° = 92.184 \text{ N}$$

$$+\nwarrow \ ZF_x = ma_x: \quad M_A g \sin 20° - F_{AB} = m_A a$$

$$F_{AB} = m_A(g \sin 20° - a) = (10)(9.81 \sin 20° - 0.58972)$$

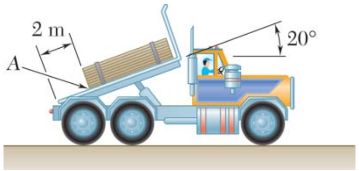
$$= 27.655 \text{ N}$$

Since bag *A* does not slide on suitcase *B*,

$$\mu_s > \frac{F_{AB}}{N_{AB}} = \frac{27.655}{92.184} = 0.300$$

$\mu_s > 0.300$ ◀

PROBLEM 12.22



To unload a bound stack of plywood from a truck, the driver first tilts the bed of the truck and then accelerates from rest. Knowing that the coefficients of friction between the bottom sheet of plywood and the bed are $\mu_s = 0.40$ and $\mu_k = 0.30$, determine (*a*) the smallest acceleration of the truck which will cause the stack of plywood to slide, (*b*) the acceleration of the truck which causes corner *A* of the stack to reach the end of the bed in 0.9 s.

SOLUTION

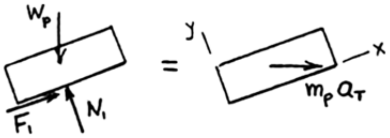
Let $\mathbf{a}_P$ be the acceleration of the plywood, $\mathbf{a}_T$ be the acceleration of the truck, and $\mathbf{a}_{P/T}$ be the acceleration of the plywood relative to the truck.

(*a*) Find the value of $\mathbf{a}_T$ so that the relative motion of the plywood with respect to the truck is impending.

$$a_P = a_T \text{ and } F_1 = \mu_s N_1 = 0.40 N_1$$

$$+\nwarrow \Sigma F_y = m_P a_y: \quad N_1 - W_P \cos 20^\circ = -m_P a_T \sin 20^\circ$$

$$N_1 = m_P (g \cos 20^\circ - a_T \sin 20^\circ)$$



$$+\nearrow \Sigma F_x = m a_x: \quad F_1 - W_P \sin 20^\circ = m_P a_T \cos 20^\circ$$

$$F_1 = m_P (g \sin 20^\circ + a_T \cos 20^\circ)$$

$$m_P (g \sin 20^\circ + a_T \cos 20^\circ) = 0.40\, m_P (g \cos 20^\circ - a_T \sin 20^\circ)$$

$$a_T = \frac{(0.40 \cos 20^\circ - \sin 20^\circ)}{\cos 20^\circ + 0.40 \sin 20^\circ} g$$

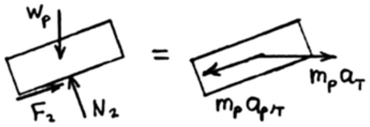
$$= (0.03145)(9.81)$$

$$= 0.309$$

$\mathbf{a}_T = 0.309 \text{ m/s}^2 \longrightarrow$ ◀

(*b*) $x_{P/T} = (x_{P/T})_o + (v_{P/T})t + \frac{1}{2} a_{P/T} t^2 = 0 + 0 + \frac{1}{2} a_{P/T} t^2$

$$a_{P/T} = \frac{2x_{P/T}}{t^2} = \frac{(2)(2)}{(0.9)^2} = 4.94 \text{ m/s}^2$$



$$\mathbf{a}_{P/T} = 4.94 \text{ m/s}^2 \swarrow 20^\circ$$

$$\mathbf{a}_P = \mathbf{a}_T + \mathbf{a}_{P/T} = (a_T \rightarrow) + (4.94 \text{ m/s}^2 \swarrow 20^\circ)$$

$$+\nwarrow F_y = m_P a_y: \quad N_2 - W_P \cos 20^\circ = -m_P a_T \sin 20^\circ$$

$$N_2 = m_P (g \cos 20^\circ - a_T \sin 20^\circ)$$

PROBLEM 12.22 (Continued)

$$+\nearrow \Sigma F_x = \Sigma m a_x: \quad F_2 - W_P \sin 20° = m_P a_T \cos 20° - m_P a_{P/T}$$

$$F_2 = m_P (g \sin 20° + a_T \cos 20° - a_{P/T})$$

For sliding with friction $F_2 = \mu_k N_2 = 0.30 N_2$

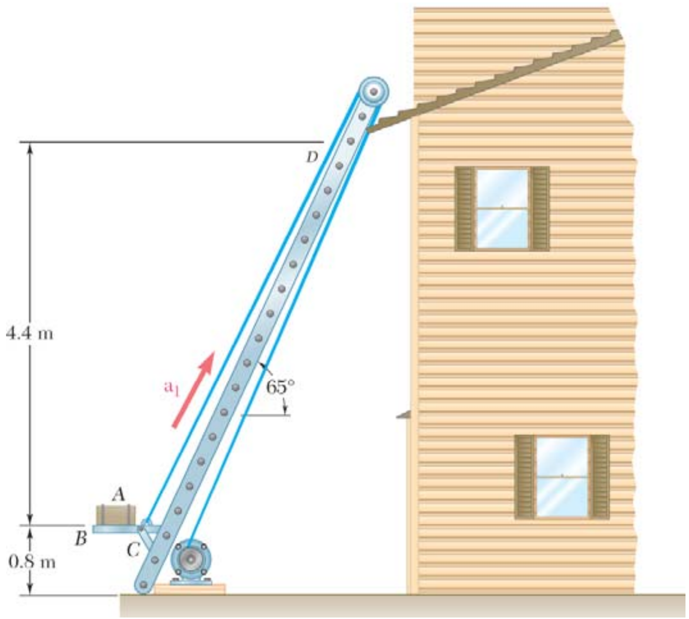
$$m_P (g \sin 20° + a_T \cos 20° - a_{P/T}) = 0.30 m_P (g \cos 20° + a_T \sin 20°)$$

$$a_T = \frac{(0.30 \cos 20° - \sin 20°) g + a_{P/T}}{\cos 20° + 0.30 \sin 20°}$$

$$= (-0.05767)(9.81) + (0.9594)(4.94)$$

$$= 4.17$$

$\mathbf{a}_T = 4.17 \text{ m/s}^2 \longrightarrow$ ◀

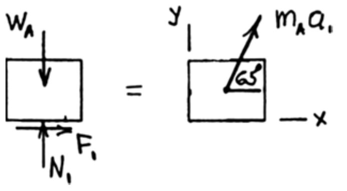


PROBLEM 12.23

To transport a series of bundles of shingles *A* to a roof, a contractor uses a motor-driven lift consisting of a horizontal platform *BC* which rides on rails attached to the sides of a ladder. The lift starts from rest and initially moves with a constant acceleration $\mathbf{a}_1$ as shown. The lift then decelerates at a constant rate $\mathbf{a}_2$ and comes to rest at *D*, near the top of the ladder. Knowing that the coefficient of static friction between a bundle of shingles and the horizontal platform is 0.30, determine the largest allowable acceleration $\mathbf{a}_1$ and the largest allowable deceleration $\mathbf{a}_2$ if the bundle is not to slide on the platform.

SOLUTION

Acceleration $\mathbf{a}_1$: Impending slip. $F_1 = \mu_s N_1 = 0.30 N_1$



$\Sigma F_y = m_A a_y$: $N_1 - W_A = m_A a_1 \sin 65°$

$$N_1 = W_A + m_A a_1 \sin 65°$$

$$= m_A (g + a_1 \sin 65°)$$

$\xrightarrow{+} \Sigma F_x = m_A a_x$: $F_1 = m_A a_1 \cos 65°$

$$F_1 = \mu_s N$$

or $m_A a_1 \cos 65° = 0.30 m_A (g + a_1 \sin 65°)$

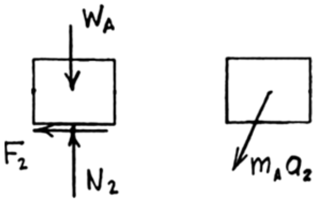
$$a_1 = \frac{0.30g}{\cos 65° - 0.30 \sin 65°}$$

$$= (1.990)(9.81)$$

$$= 19.53 \text{ m/s}^2$$

$\mathbf{a}_1 = 19.53 \text{ m/s}^2 \measuredangle 65°$ ◀

Deceleration $\mathbf{a}_2$: Impending slip. $F_2 = \mu_s N_2 = 0.30 N_2$



$\Sigma F_y = m a_y$: $N_1 - W_A = -m_A a_2 \sin 65°$

$$N_1 = W_A - m_A a_2 \sin 65°$$

$\xleftarrow{+} \Sigma F_x = m a_x$: $F_2 = m_A a_2 \cos 65°$

$$F_2 = \mu_s N_2$$

PROBLEM 12.23 (Continued)

or $$m_A a_2 \cos 65° = 0.30\, m_A (g - a_2 \cos 65°)$$

$$a_2 = \frac{0.30g}{\cos 65° + 0.30 \sin 65°}$$

$$= (0.432)(9.81)$$

$$= 4.24\ \text{m/s}^2$$

$\mathbf{a}_2 = 4.24\ \text{m/s}^2 \swarrow 65°$ ◀

PROBLEM 12.24

An airplane has a mass of 25 Mg and its engines develop a total thrust of 40 kN during take-off. If the drag **D** exerted on the plane has a magnitude $D = 2.25v^2$, where v is expressed in meters per second and D in newtons, and if the plane becomes airborne at a speed of 240 km/h, determine the length of runway required for the plane to take off.

SOLUTION

$$F = ma: \quad 40\times10^3 \text{ N} - 2.25v^2 = (25\times10^3 \text{ kg})a$$

Substituting

$$a = v\frac{dv}{dx}: \quad 40\times10^3 - 2.25\ v^2 = (25\times10^3)\ v\frac{dv}{dx}$$

$$\int_0^{x_1} dx = \int_0^{v_1} \frac{(25\times10^3)vdv}{40\times10^3 - 2.25v^2}$$

$$x_1 = -\frac{25\times10^3}{2(2.25)}[\ln(40\times10^3 - 2.25v^2)]_0^{v_1}$$

$$= \frac{25\times10^3}{4.5}\ln\frac{40\times10^3}{40\times10^3 - 2.25v_1^2}$$

For $v_1 = 240 \text{ km/h} = 66.67 \text{ m/s}$

$$x_1 = \frac{25\times10^3}{4.5}\ln\frac{40\times10^3}{40\times10^3 - 2.25(66.67)^2} = 5.556\ln 1.333$$

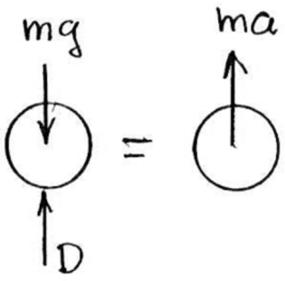
$$= 1.5982\times10^3 \text{ m}$$

$x_1 = 1.598 \text{ km}$ ◀

PROBLEM 12.25

A 4-kg projectile is fired vertically with an initial velocity of 90 m/s, reaches a maximum height and falls to the ground. The aerodynamic drag **D** has a magnitude $D = 0.0024\,v^2$ where D and v are expressed in newtons and m/s, respectively. Knowing that the direction of the drag is always opposite to the direction of the velocity, determine (*a*) the maximum height of the trajectory, (*b*) the speed of the projectile when it reaches the ground.

SOLUTION



Let y be the position coordinate of the projectile measured upward from the ground. The velocity and acceleration a taken to positive upward. $D = kv^2$.

(*a*) Upward motion. $+\!\uparrow \Sigma F_y = ma$:

$$-D - mg = ma$$

$$a = -\left(g + \frac{D}{m}\right) = -\left(g + \frac{kv^2}{m}\right)$$

$$v\frac{dv}{dy} = -\left(g + \frac{kv^2}{m}\right) = -\frac{k}{m}\left(v^2 + \frac{mg}{k}\right)$$

$$\frac{v\,dv}{v^2 + \frac{mg}{k}} = -\frac{k}{m}dy$$

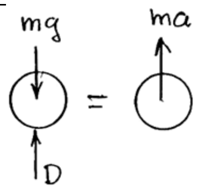
$$\int_{v_0}^{0} \frac{v\,dv}{v^2 + \frac{mg}{k}} = -\frac{k}{m}\int_0^h dy$$

$$\frac{1}{2}\ln\left(v^2 + \frac{mg}{k}\right)\Bigg|_{v_0}^{0} = -\frac{kh}{m}$$

$$\frac{1}{2}\ln\frac{\frac{mg}{k}}{v_0^2 + \frac{mg}{k}} = -\frac{1}{2}\ln\left(\frac{kv_0^2}{mg} + 1\right) = -\frac{kh}{m}$$

$$h = \frac{m}{2k}\ln\left(\frac{kv_0^2}{mg} + 1\right) = \frac{4}{(2)(0.0024)}\ln\left[\frac{(0.0024)(90)^2}{(4)(9.81)} + 1\right]$$

$= 335.36$ m $\qquad h = 335$ m ◀



PROBLEM 12.25 (Continued)

(*b*) Downward motion. $+\uparrow\ \Sigma F = ma$:

$$D - mg = ma$$

$$a = \frac{D}{m} - g = \frac{kv^2}{m} - g$$

$$v\frac{dv}{dy} = \frac{kv^2}{m} - g = -\frac{k}{m}\left(\frac{mg}{k} - v^2\right)$$

$$\frac{v\,dv}{\frac{mg}{k} - v^2} = -\frac{k}{m}dy$$

$$\int_0^{v_f} \frac{v\,dv}{mg} = -\frac{k}{m}\int_h^0 dy$$

$$\frac{1}{2}\ln\left(\frac{mg}{k} - v^2\right)\Bigg|_0^{v_f} = \frac{kh}{m}$$

$$-\frac{1}{2}\ln\left(\frac{\frac{mg}{k} - v_f^2}{\frac{mg}{k}}\right) = \frac{kh}{m}$$

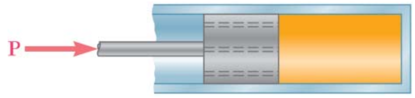
$$\ln\left(1 - \frac{kv_f^2}{mg}\right) = -\frac{2\,kh}{m}$$

$$1 - \frac{kv_f^2}{mg} = e^{-2\,kh/m}$$

$$v_f = \pm\sqrt{\frac{mg}{k}\left(1 - e^{-2\,kh/m}\right)}$$

$$v_f = \sqrt{\frac{(4)(9.81)}{0.0024}\left[1 - e^{-(2)(0.0024)(335.36)/4}\right]}$$

$$= 73.6 \text{ m/s} \qquad v_f = 73.6 \text{ m/s}\downarrow \blacktriangleleft$$

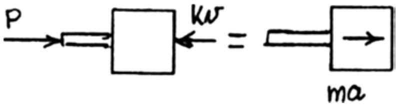


PROBLEM 12.26

A constant force **P** is applied to a piston and rod of total mass m to make them move in a cylinder filled with oil. As the piston moves, the oil is forced through orifices in the piston and exerts on the piston a force of magnitude kv in a direction opposite to the motion of the piston. Knowing that the piston starts from rest at $t = 0$ and $x = 0$, show that the equation relating x, v, and t, where x is the distance traveled by the piston and v is the speed of the piston, is linear in each of these variables.

SOLUTION

$$\xrightarrow{+} \Sigma F = ma: \quad P - kv = ma$$



$$\frac{dv}{dt} = a = \frac{P - kv}{m}$$

$$\int_0^t dt = \int_0^v \frac{m\,dv}{P - kv}$$

$$= -\frac{m}{k}\ln(P - kv)\Big|_0^v$$

$$= -\frac{m}{k}[\ln(P - kv) - \ln P]$$

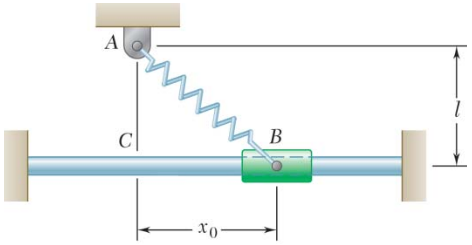
$$t = -\frac{m}{k}\ln\frac{P - kv}{P} \qquad \text{or} \qquad \ln\frac{P - kv}{m} = -\frac{kt}{m}$$

$$\frac{P - kv}{m} = e^{-kt/m} \qquad \text{or} \qquad v = \frac{P}{k}(1 - e^{-kt/m})$$

$$x = \int_0^t v\,dt = \frac{Pt}{k}\Bigg|_0^t - \frac{P}{k}\left(-\frac{k}{m}e^{-kt/m}\right)\Bigg|_0^t$$

$$= \frac{Pt}{k} + \frac{P}{m}(e^{-kt/m} - 1) = \frac{Pt}{k} - \frac{P}{m}(1 - e^{-kt/m})$$

$$x = \frac{Pt}{k} - \frac{kv}{m}, \quad \text{which is linear.} \qquad \blacktriangleleft$$



PROBLEM 12.27

A spring *AB* of constant *k* is attached to a support at *A* and to a collar of mass *m*. The unstretched length of the spring is ℓ. Knowing that the collar is released from rest at $x = x_0$ and neglecting friction between the collar and the horizontal rod, determine the magnitude of the velocity of the collar as it passes through Point *C*.

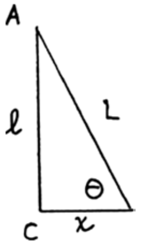
SOLUTION

Choose the origin at Point *C* and let *x* be positive to the right. Then *x* is a position coordinate of the slider *B* and x_0 is its initial value. Let *L* be the stretched length of the spring. Then, from the right triangle

$$L = \sqrt{\ell^2 + x^2}$$

The elongation of the spring is $e = L - \ell$, and the magnitude of the force exerted by the spring is

$$F_s = ke = k(\sqrt{\ell^2 + x^2} - \ell)$$

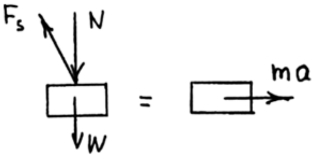


By geometry,

$$\cos\theta = \frac{x}{\sqrt{\ell^2 + x^2}}$$

$$\xrightarrow{+} \Sigma F_x = ma_x : \quad -F_s \cos\theta = ma$$

$$-k(\sqrt{\ell^2 + x^2} - \ell)\frac{x}{\sqrt{\ell^2 + x^2}} = ma$$



$$a = -\frac{k}{m}\left(x - \frac{\ell x}{\sqrt{\ell^2 + x^2}}\right)$$

$$\int_0^v v\,dv = \int_{x_0}^0 a\,dx$$

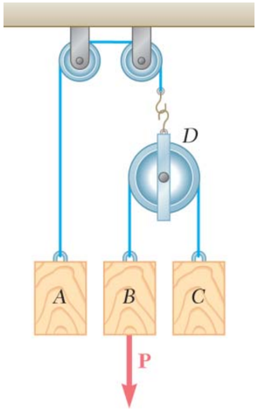
$$\frac{1}{2}v^2\bigg|_0^v = -\frac{k}{m}\int_{x_0}^0 \left(x - \frac{\ell x}{\sqrt{\ell^2 + x^2}}\right)dx = -\frac{k}{m}\left(\frac{1}{2}x^2 - \ell\sqrt{\ell^2 + x^2}\right)\bigg|_{x_0}^0$$

$$\frac{1}{2}v^2 = -\frac{k}{m}\left(0 - \ell^2 - \frac{1}{2}x_0^2 + \ell\sqrt{\ell^2 + x_0^2}\right)$$

$$v^2 = \frac{k}{m}\left(2\ell^2 + x_0^2 - 2\ell\sqrt{\ell^2 + x_0^2}\right)$$

$$= \frac{k}{m}\left[\left(\ell^2 + x_0^2\right) - 2\ell\sqrt{\ell^2 + x_0^2} + \ell^2\right]$$

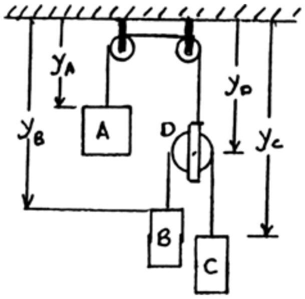
answer: $v = \sqrt{\frac{k}{m}}\left(\sqrt{\ell^2 + x_0^2} - \ell\right)$ ◀



PROBLEM 12.28

Block *A* has a mass of 10 kg, and blocks *B* and *C* have masses of 5 kg each. Knowing that the blocks are initially at rest and that *B* moves through 3 m in 2 s, determine (*a*) the magnitude of the force **P**, (*b*) the tension in the cord *AD*. Neglect the masses of the pulleys and axle friction.

SOLUTION



Let the position coordinate *y* be positive downward.

Constraint of cord *AD*: $y_A + y_D = \text{constant}$

$$v_A + v_D = 0, \quad a_A + a_D = 0$$

Constraint of cord *BC*: $(y_B - y_D) + (y_C - y_D) = \text{constant}$

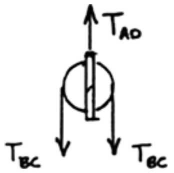
$$v_B + v_C - 2v_D = 0, \quad a_B + a_C - 2a_D = 0$$

Eliminate a_D. $$2a_A + a_B + a_C = 0 \tag{1}$$

We have uniformly accelerated motion because all of the forces are constant.

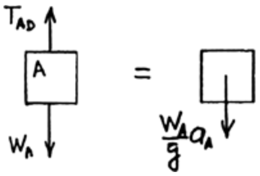
$$y_B = (y_B)_0 + (v_B)_0 t + \frac{1}{2}a_B t^2, \quad (v_B)_0 = 0$$

$$a_B = \frac{2[y_B - (y_B)_0]}{t^2} = \frac{(2)(3)}{(2)^2} = 1.5 \text{ m/s}^2$$



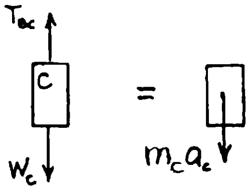
Pulley *D*: $+\!\downarrow \Sigma F_y = 0: \quad 2T_{BC} - T_{AD} = 0$

$$T_{AD} = 2T_{BC}$$

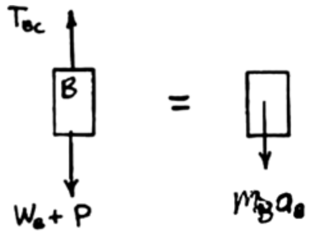


Block *A*: $+\!\downarrow \Sigma F_y = ma_y: \quad W_A - T_{AD} = m_A a_A$

or $$a_A = \frac{W_A - T_{AD}}{m_A} = \frac{W_A - 2T_{BC}}{m_A} \tag{2}$$



PROBLEM 12.28 (Continued)



Block *C*: $+\downarrow \Sigma F_y = ma_y: \quad W_C - T_{BC} = m_C a_C$

or
$$a_C = \frac{W_C - T_{BC}}{m_C} \qquad (3)$$

Substituting the value for a_B and Eqs. (2) and (3) into Eq. (1), and solving for T_{BC},

$$2\left(\frac{W_A - 2T_{BC}}{m_A}\right) + a_B + \left(\frac{W_C - T_{BC}}{m_C}\right) = 0$$

$$2\left(\frac{m_A g - 2T_{BC}}{m_A}\right) + a_B + \left(\frac{m_C g - T_{BC}}{m_C}\right) = 0$$

$$\left(\frac{4}{m_A} + \frac{1}{m_C}\right) T_{BC} = 3g + a_B$$

$$\left(\frac{4}{10} + \frac{1}{5}\right) T_{BC} = 3(9.81) + 1.5 \quad \text{or} \quad T_{BC} = 51.55 \text{ N}$$

Block *B*: $+\downarrow \Sigma F_y = ma_y: \quad P + W_B - T_{BC} = m_B a_B$

(*a*) Magnitude of *P*.

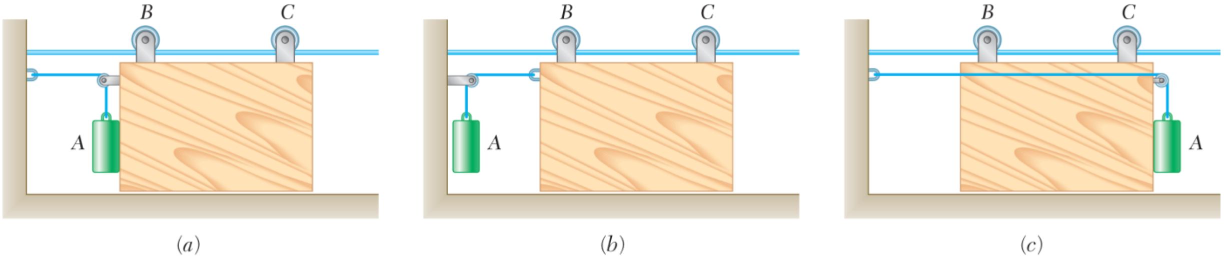
$$P = T_{BC} - W_B + m_B a_B$$
$$= 51.55 - 5(9.81) + 5(1.5) \qquad P = 10.00 \text{ N} \blacktriangleleft$$

(*b*) Tension in cord *AD*.

$$T_{AD} = 2T_{BC} = (2)(51.55) \qquad T_{AD} = 103.1 \text{ N} \blacktriangleleft$$

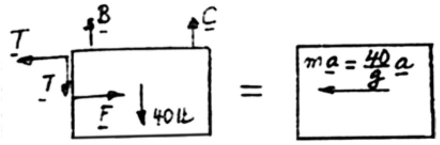
PROBLEM 12.29

A 40-lb sliding panel is supported by rollers at *B* and *C*. A 25-lb counterweight *A* is attached to a cable as shown and, in cases *a* and *c*, is initially in contact with a vertical edge of the panel. Neglecting friction, determine in each case shown the acceleration of the panel and the tension in the cord immediately after the system is released from rest.



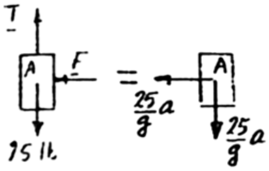
SOLUTION

(*a*) Panel: F = Force exerted by counterweight



$\xleftarrow{+}\ \Sigma F_x = ma$: $T - F = \dfrac{40}{g}a$ (1)

Counterweight *A*: Its acceleration has two components



$$\mathbf{a}_A = \mathbf{a}_P + \mathbf{a}_{A/P} = a \rightarrow + a \downarrow$$

$\xleftarrow{+}\ \Sigma F_x = ma_x$: $F = \dfrac{25}{g}a$ (2)

$+\!\downarrow \Sigma F_g = ma_g$: $25 - T = \dfrac{25}{g}a$ (3)

Adding (1), (2), and (3):

$$\cancel{T} - F + F + 25 - \cancel{T} = \frac{40+25+25}{g}a$$

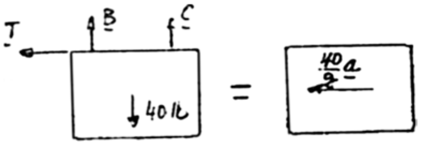
$$a = \frac{25}{90}g = \frac{25}{90}(32.2)$$ $\mathbf{a} = 8.94\ \text{ft/s}^2 \longleftarrow$ ◀

Substituting for *a* into (3):

$$25 - T = \frac{25}{g}\left(\frac{25}{90}g\right) \qquad T = 25 - \frac{625}{90}$$ $T = 18.06$ lb ◀

PROBLEM 12.29 (Continued)

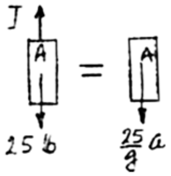
(*b*) Panel:



$\xleftarrow{+}\ \Sigma F_y = ma:$ $$T = \frac{40}{g}a \qquad (1)$$

Counterweight *A*:

$+\!\downarrow \Sigma F_y = ma:$ $$25 - T = \frac{25}{g}a \qquad (2)$$



Adding (1) and (2): $$\cancel{T} + 25 - \cancel{T} = \frac{40+25}{g}a$$

$$a = \frac{25}{65}g$$

$\mathbf{a} = 12.38\ \text{ft/s}^2 \longleftarrow$ ◀

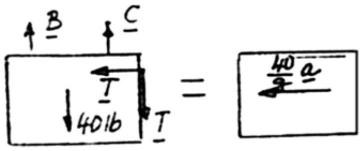
Substituting for *a* into (1):

$$T = \frac{40}{g}\left(\frac{25}{65}g\right) = \frac{1000}{65}$$

$T = 15.38$ lb ◀

(*c*) Since panel is accelerated to the left, there is no force exerted by panel on counterweight and vice versa.

Panel:



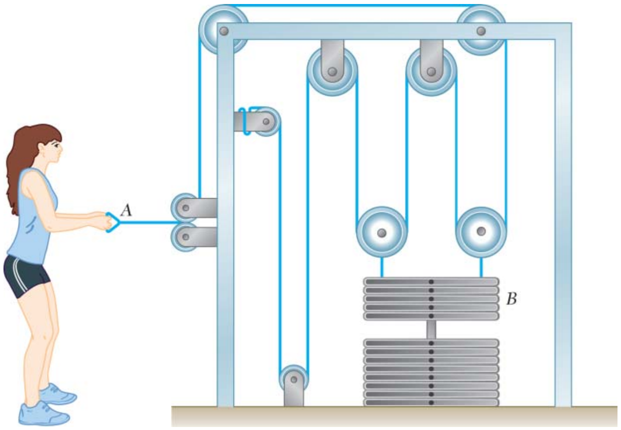
$\xleftarrow{+}\ \Sigma F_x = ma:$ $$T = \frac{40}{g}a \qquad (1)$$

Counterweight *A*: Same free body as in Part (*b*):

$+\!\downarrow \Sigma F_y = ma:$ $$25 - T = \frac{25}{g}a \qquad (2)$$

Since Eqs. (1) and (2) are the same as in (*b*), we get the same answers:

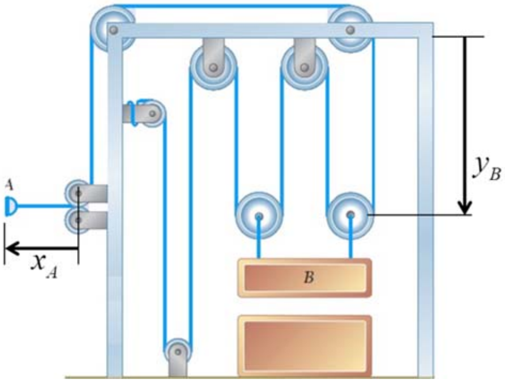
$\mathbf{a} = 12.38\ \text{ft/s}^2 \longleftarrow;\ T = 15.38$ lb ◀



PROBLEM 12.30

An athlete pulls handle *A* to the left with a constant force of *P* = 100 N. Knowing that after the handle *A* has been pulled 30 cm its velocity is 3 m/s, determine the mass of the weight stack *B*.

SOLUTION



Given: $P = 100\text{ N}$

$x_A = 0.30\text{ m}$

$\text{v}_A = 3\text{ m/s}$

Kinematics: $x_A + 4y_B = \text{constant}$

$v_A + 4v_B = 0$

$a_A + 4a_B = 0 \qquad (1)$

Uniform Acceleration of handle A:

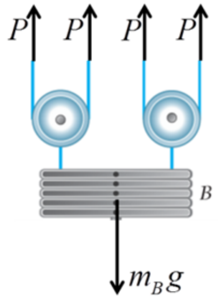
$$\text{v}_A{}^2 = (\text{v}_A)_o^2 + 2a_A\left(x_A - (x_A)_o\right)$$

$$3^2 = 0^2 + 2a_A(0.3 - 0)$$

$$a_A = 15\text{ m/s}^2$$

From (1): $a_B = 3.75\text{ m/s}^2 \uparrow$

Free Body Diagram:



From FBD:

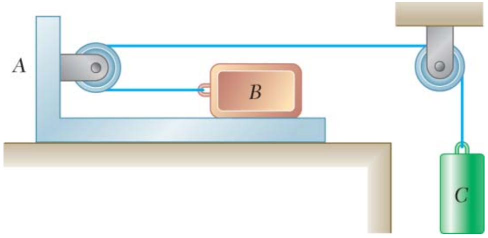
$$\sum F_y = m_B a_B$$

$$4P - m_B g = m_B a_B$$

$$m_B = \frac{4P}{g + a_B}$$

$$= \frac{4(100)}{9.81 + 3.75}\text{ kg}$$

$$m_B = 29.50\text{ kg} \blacktriangleleft$$

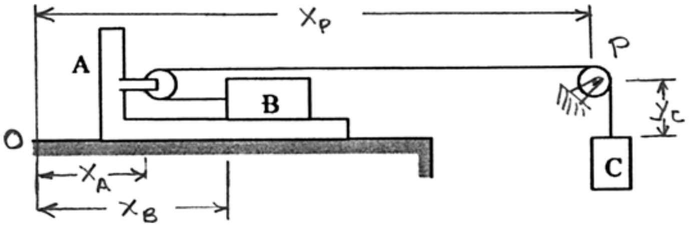


PROBLEM 12.31

A 10-lb block *B* rests as shown on a 20-lb bracket *A*. The coefficients of friction are $\mu_s = 0.30$ and $\mu_k = 0.25$ between block *B* and bracket *A*, and there is no friction in the pulley or between the bracket and the horizontal surface. (*a*) Determine the maximum weight of block *C* if block *B* is not to slide on bracket *A*. (*b*) If the weight of block *C* is 10% larger than the answer found in *a* determine the accelerations of *A*, *B* and *C*.

SOLUTION

Kinematics. Let x_A and x_B be horizontal coordinates of *A* and *B* measured from a fixed vertical line to the left of *A* and *B*. Let y_C be the distance that block *C* is below the pulley. Note that y_C increases when *C* moves downward. See figure.



The cable length *L* is fixed.

$$L = (x_B - x_A) + (x_P - x_A) + y_C + \text{constant}$$

Differentiating and noting that $\dot{x}_P = 0,$

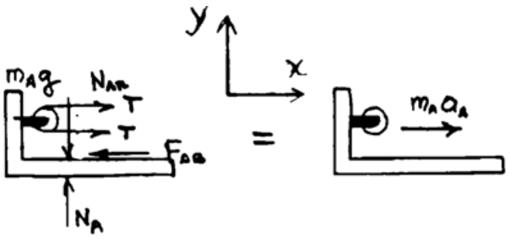
$$v_B - 2v_A + v_C = 0$$

$$-2a_A + a_B + a_C = 0 \qquad (1)$$

Here, a_A and a_B are positive to the right, and a_C is positive downward.

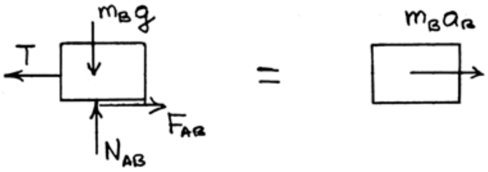
Kinetics. Let *T* be the tension in the cable and F_{AB} be the friction force between blocks *A* and *B*. The free body diagrams are:

Bracket *A*:

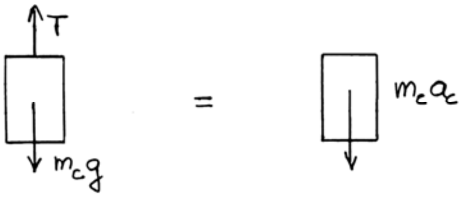


PROBLEM 12.31 (Continued)

Block *B*:



Block *C*:



Bracket *A*: $\xrightarrow{+} \Sigma F_x = ma_x: \quad 2T - F_{AB} = \dfrac{W_A}{g} a_A$ (2)

Block *B*: $\xrightarrow{+} \Sigma F_x = ma_x: \quad F_{AB} - T = \dfrac{W_B}{g} a_B$ (3)

$+\uparrow \Sigma F_y = ma_y: \quad N_{AB} - W_B = 0$

or $N_{AB} = W_B$

Block *C*: $+\downarrow \Sigma F_y = ma_y: \quad m_C - T = \dfrac{W_C}{g} a_C$ (4)

Adding Eqs. (2), (3), and (4), and transposing,

$$\frac{W_A}{g} a_A + \frac{W_B}{g} a_B + \frac{W_C}{g} a_C = W_C \tag{5}$$

Subtracting Eq. (4) from Eq. (3) and transposing,

$$\frac{W_B}{g} a_B - \frac{W_C}{g} a_C = F_{AB} - W_C \tag{6}$$

(*a*) No slip between *A* and *B*. $a_B = a_A$

From Eq. (1), $a_A = a_B = a_C = a$

From Eq. (5), $a = \dfrac{W_C g}{W_A + W_B + W_C}$

For impending slip, $F_{AB} = \mu_s N_{AB} = \mu_s W_B$

PROBLEM 12.31 (Continued)

Substituting into Eq. (6),

$$\frac{(W_B - W_C)(W_C g)}{W_A + W_B + W_C} = \mu_s W_B - W_C$$

Solving for W_C,

$$W_C = \frac{\mu_s W_B (W_A + W_B)}{W_A + 2W_B - \mu_s W_B}$$

$$= \frac{(0.30)(10)(20+10)}{20 + (2)(10) - (0.30)(10)}$$

$W_C = 2.43 \text{ lbs}$ ◀

(*b*) W_C increased by 10%. $W_C = 2.6757 \text{ lbs}$

Since slip is occurring, $F_{AB} = \mu_k N_{AB} = \mu_k W_B$

Eq. (6) becomes $\frac{W_B}{g} a_B - \frac{W_C}{g} a_C = \mu_k W_B - W_C$

or $10a_B - 2.6757a_C = [(0.25)(10) - 2.6757](32.2)$ (7)

With numerical data, Eq. (5) becomes

$$20a_A + 10a_B + 2.6757a_C = (2.6757)(32.2) \quad (8)$$

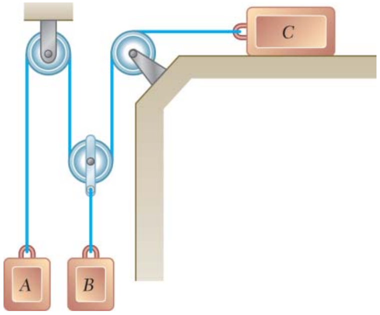
Solving Eqs. (1), (7), and (8) gives

$$a_A = 3.144 \text{ ft/s}^2, \quad a_B = 0.881 \text{ ft/s}^2, \quad a_C = 5.407 \text{ ft/s}^2$$

$\mathbf{a}_A = 3.14 \text{ ft/s}^2 \longrightarrow$ ◀

$\mathbf{a}_B = 0.881 \text{ ft/s}^2 \longrightarrow$ ◀

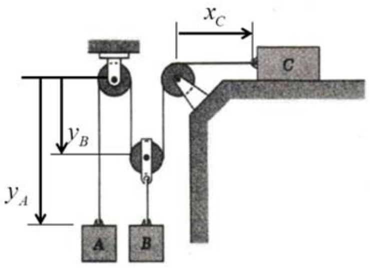
$\mathbf{a}_C = 5.41 \text{ ft/s}^2 \downarrow$ ◀



PROBLEM 12.32

Knowing that μ = 0.30, determine the acceleration of each block when $m_A = m_B = m_C$.

SOLUTION



Given: $\mu = 0.30$

$m_A = m_B = m_c = m$

Kinematics: $y_A + 2y_B + x_C = \text{constant}$

$v_A + 2v_B + v_C = 0$

$a_C + a_A + 2a_B = 0 \qquad (1)$

Free Body Diagram of Block A:

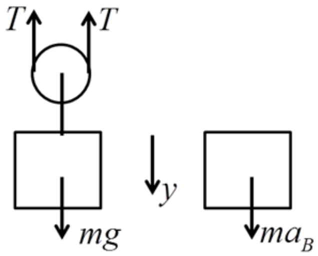
Equation of Motion:

$$\sum F_y = ma_A$$

$$mg - T = ma_A$$

$$a_A = g - \frac{T}{m} \qquad (2)$$

Free Body Diagram of Block B:



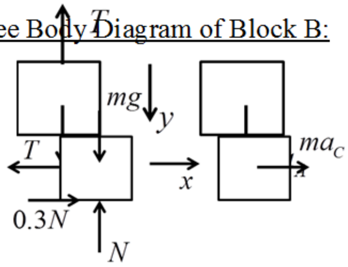
Equation of Motion:

$$\sum F_y = ma_B$$

$$mg - 2T = ma_B$$

$$a_B = g - \frac{2T}{m} \qquad (3)$$

Free Body Diagram of Block B:



Equations of Motion:

$$\sum F_y = 0$$

$$N = mg$$

$$\sum F_x = ma_C$$

$$0.3mg - T = ma_C$$

$$a_C = 0.3g - \frac{T}{m} \qquad (4)$$

Problem 12.32 (Continued)

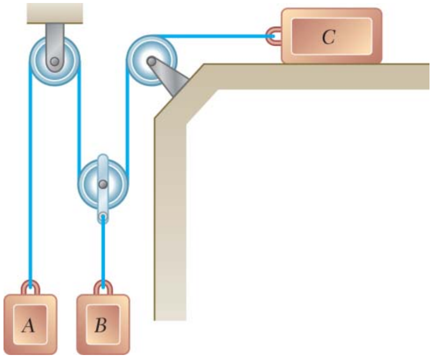
Substitute (2), (3) and (4) → (1)
$$3.3g - \frac{6T}{m} = 0$$

$$T = \frac{1.1}{2}mg \qquad (5)$$

Substitute (5) → (2): $a_A = g - \frac{1.1}{2}g$ $\qquad a_A = 0.45\text{g}\downarrow$ ◀

Substitute (5) → (3): $a_B = g - \frac{2.2}{2}g$ $\qquad a_B = 0.10\text{g}\uparrow$ ◀

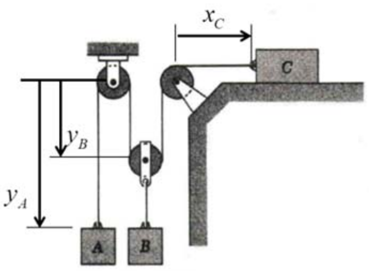
Substitute (5) → (4): $a_C = 0.3g - \frac{1.1}{2}g$ $\qquad a_C = 0.25\text{g}\leftarrow$ ◀



PROBLEM 12.33

Knowing that μ = 0.30, determine the acceleration of each block when $m_A = 5$ kg, $m_B = 30$ kg, and $m_C = 15$ kg.

SOLUTION



Given: $\mu = 0.30,\ g = 9.81\ \text{m/s}^2$

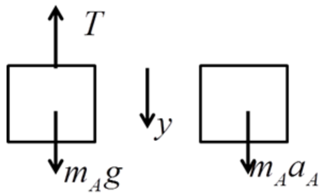
$m_A = 5\ \text{kg},\ m_B = 30\ \text{kg},\ m_C = 15\ \text{kg}$

Kinematics: $y_A + 2y_B + x_C = \text{constant}$

$$v_A + 2v_B + v_C = 0$$

$$a_A + 2a_B + a_C = 0 \qquad (1)$$

Free Body Diagram of Block A:



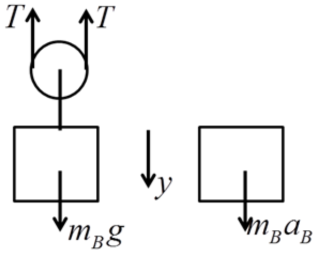
Equation of Motion:

$$\sum F_y = m_A a_A$$

$$m_A g - T = m_A a_A$$

$$a_A = g - \frac{T}{m_A} \qquad (2)$$

Free Body Diagram of Block B:



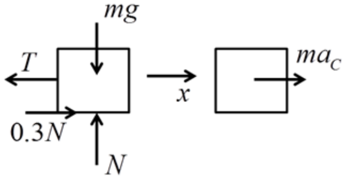
Equation of Motion:

$$\sum F_y = m_B a_B$$

$$m_B g - 2T = m_B a_B$$

$$a_B = g - \frac{2T}{m_B} \qquad (3)$$

Free Body Diagram of Block B:



Equations of Motion:

$$\sum F_y = 0 \qquad \sum F_x = m_C a_C$$

$$N = m_C g \qquad 0.3 m_C g - T = m_C a_C$$

Problem 12.33 (Continued)

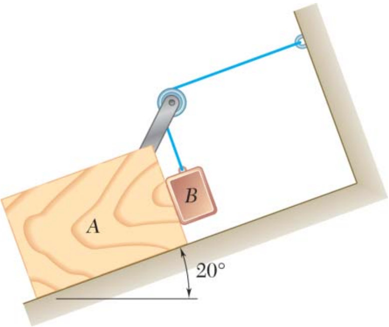
$$a_C = 0.3g - \frac{T}{m_C} \qquad (4)$$

Substitute (2), (3) and (4) → (1) $$3.3g - \frac{T}{m_A} - \frac{4T}{m_B} - \frac{T}{m_C} = 0$$

Substitute (5) → (2): $$a_A = g - \frac{8.25}{5}g \qquad a_A = 6.377 \text{ m/s}^2 \uparrow \blacktriangleleft$$

Substitute (5) → (3): $$a_B = g - \frac{16.5}{30}g \qquad a_B = 4.415 \text{ m/s}^2 \downarrow \blacktriangleleft$$

Substitute (5) → (4): $$a_C = 0.3g - \frac{8.25}{15}g \qquad a_C = -2.453 \text{ m/s}^2 \leftarrow \blacktriangleleft$$

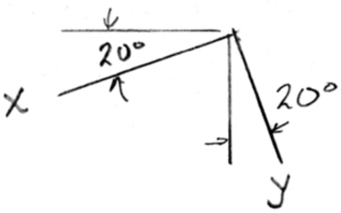


PROBLEM 12.34

A 25-kg block A rests on an inclined surface, and a 15-kg counterweight B is attached to a cable as shown. Neglecting friction, determine the acceleration of A and the tension in the cable immediately after the system is released from rest.

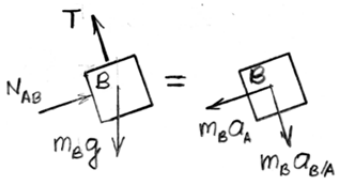
SOLUTION

Let the positive direction of x and y be those shown in the sketch, and let the origin lie at the cable anchor.



Constraint of cable: $x_A + y_{B/A} = \text{constant}$ or $a_A + a_{B/A} = 0$, where the positive directions of a_A and $a_{B/A}$ are respectively the x and the y directions. Then $a_{B/A} = -a_A$ (1)

First note that $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} = \left(a_A \nearrow 20°\right) + \left(a_{B/A} \searrow 20°\right)$



Block B: $\overset{+}{\swarrow}\ \Sigma F_x = m_B\left(a_B\right)_x: \quad m_Bg\sin 20° - N_{AB} = m_Ba_A$

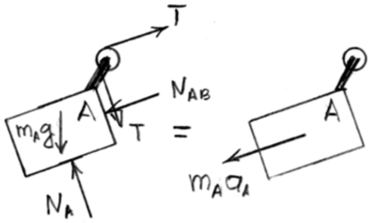
$$m_Ba_A + N_{AB} = m_Bg\sin 20°$$

$$15\,a_A + N_{AB} = 50.328 \tag{2}$$

$$\overset{+}{\searrow}\ \Sigma F_y = m_B\left(a_B\right)_y: \quad m_Bg\cos 20° - T = m_Ba_{B/A}$$

$$m_Ba_{B/A} + T = m_Bg\cos 20°$$

$$15\,a_{B/A} + T = 138.276 \tag{3}$$



Block A: $\overset{+}{\swarrow}\ \Sigma F_x = m_Aa_A: \quad m_Ag\sin 20° + N_{AB} + T = m_Aa_A$

$$m_Aa_A - N_{AB} + T = m_Ag\sin 20°$$

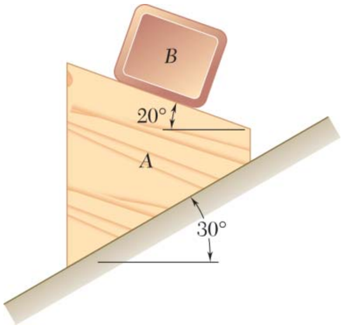
$$25\,a_{B/A} - N_{AB} + T = 83.880 \tag{4}$$

Eliminate $a_{B/A}$ using Eq. (1), then add Eq. (4) to Eq. (2) and subtract Eq. (3).

$55\,a_A = -4.068$ or $a_A = -0.0740\ \text{m/s}^2$, $\mathbf{a}_A = 0.0740\ \text{m/s}^2 \measuredangle 20°$ ◀

From Eq. (1), $a_{B/A} = 0.0740\ \text{m/s}^2$

From Eq. (3), $T = 137.2\ \text{N}$ $T = 137.2\ \text{N}$ ◀

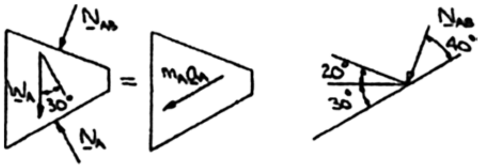


PROBLEM 12.35

Block B of mass 10-kg rests as shown on the upper surface of a 22-kg wedge A. Knowing that the system is released from rest and neglecting friction, determine (a) the acceleration of B, (b) the velocity of B relative to A at $t = 0.5$ s.

SOLUTION

A:

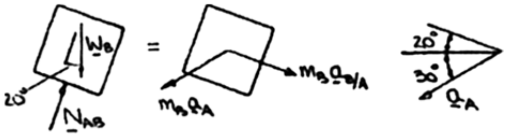

$\underline{N}_{AB}$
$\underline{W}_A$
30°
$\underline{N}_A$
=
$m_A\underline{a}_A$
20°
30°
$\underline{N}_{AB}$
40°

(a) $+\swarrow \Sigma F_x = m_A a_A$: $W_A \sin 30° + N_{AB} \cos 40° = m_A a_A$

or $$N_{AB} = \frac{22\left(a_A - \frac{1}{2} g\right)}{\cos 40°}$$

Now we note: $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$, where $\mathbf{a}_{B/A}$ is directed along the top surface of A.

B:


$\underline{W}_B$
20°
$\underline{N}_{AB}$
=
$m_B\underline{a}_{B/A}$
$m_B\underline{a}_A$
20°
30°
$\underline{a}_A$

$+\nearrow \Sigma F_{y'} = m_B a_{y'}$: $N_{AB} - W_B \cos 20° = -m_B a_A \sin 50°$

or $$N_{AB} = 10\,(g \cos 20° - a_A \sin 50°)$$

Equating the two expressions for N_{AB}

$$\frac{22\left(a_A - \frac{1}{2} g\right)}{\cos 40°} = 10(g \cos 20° - a_A \sin 50°)$$

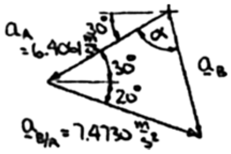
or $$a_A = \frac{(9.81)(1.1 + \cos 20° \cos 40°)}{2.2 + \cos 40° \sin 50°} = 6.4061 \text{ m/s}^2$$

$\overset{+}{\searrow} \Sigma F_{x'} = m_B a_{x'}$: $W_B \sin 20° = m_B a_{B/A} - m_B a_A \cos 50°$

or $$\begin{aligned} a_{B/A} &= g \sin 20° + a_A \cos 50° \\ &= (9.81 \sin 20° + 6.4061 \cos 50°) \text{ m/s}^2 \\ &= 7.4730 \text{ m/s}^2 \end{aligned}$$

PROBLEM 12.35 (Continued)

Finally $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$



$a_A = 6.4061 \frac{m}{s^2}$
30°
α
30°
20°
a_B
$a_{B/A} = 7.4730 \frac{m}{s^2}$

We have $a_B^2 = 6.4061^2 + 7.4730^2 - 2(6.4061 \times 7.4730)\cos 50°$

or $a_B = 5.9447 \text{ m/s}^2$

and $$\frac{7.4730}{\sin\alpha} = \frac{5.9447}{\sin 50°}$$

or $\alpha = 74.4°$

$\mathbf{a}_B = 5.94 \text{ m/s}^2$ ↘ 75.6° ◀

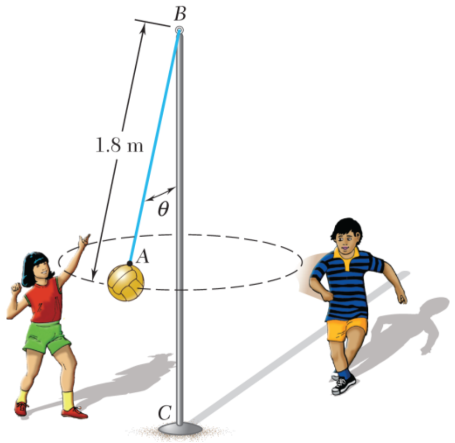
(*b*) *Note:* We have uniformly accelerated motion, so that

$$v = 0 + at$$

Now $\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A = \mathbf{a}_B t - \mathbf{a}_A t = \mathbf{a}_{B/A} t$

At $t = 0.5$ s: $v_{B/A} = 7.4730 \text{ m/s}^2 \times 0.5 \text{ s}$

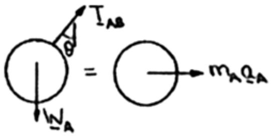
or $\mathbf{v}_{B/A} = 3.74 \text{ m/s}$ ↘ 20° ◀



PROBLEM 12.36

A 450-g tetherball A is moving along a horizontal circular path at a constant speed of 4 m/s. Determine (*a*) the angle θ that the cord forms with pole BC, (*b*) the tension in the cord.

SOLUTION



First we note $$a_A = a_n = \frac{v_A^2}{\rho}$$

where $$\rho = l_{AB}\sin\theta$$

(*a*) $+\!\uparrow \Sigma F_y = 0\text{:}\quad T_{AB}\cos\theta - W_A = 0$

or $$T_{AB} = \frac{m_A g}{\cos\theta}$$

$$\xrightarrow{+} \Sigma F_x = m_A a_A\text{:}\quad T_{AB}\sin\theta = m_A\frac{v_A^2}{\rho}$$

Substituting for T_{AB} and ρ

$$\frac{m_A g}{\cos\theta}\sin\theta = m_A\frac{v_A^2}{l_{AB}\sin\theta} \qquad \sin^2\theta = 1-\cos^2\theta$$

$$1-\cos^2\theta = \frac{(4\text{ m/s})^2}{1.8\text{ m}\times 9.81\text{ m/s}^2}\cos\theta$$

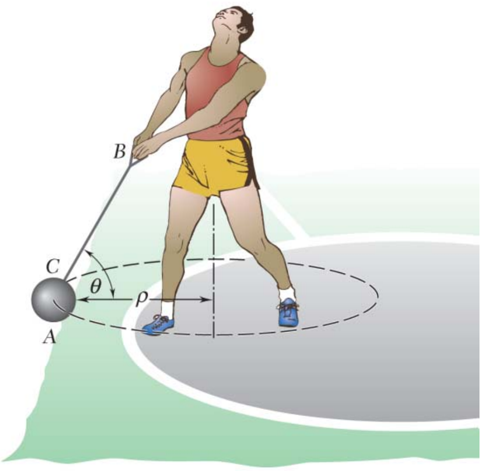
or $$\cos^2\theta + 0.906105\cos\theta - 1 = 0$$

Solving $$\cos\theta = 0.64479$$

or $$\theta = 49.9° \blacktriangleleft$$

(*b*) From above $$T_{AB} = \frac{m_A g}{\cos\theta} = \frac{0.450\text{ kg}\times 9.81\text{ m/s}^2}{0.64479}$$

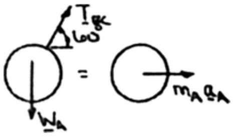
or $$T_{AB} = 6.85\text{ N} \blacktriangleleft$$



PROBLEM 12.37

During a hammer thrower's practice swings, the 7.1-kg head *A* of the hammer revolves at a constant speed *v* in a horizontal circle as shown. If $\rho = 0.93$ m and $\theta = 60°$, determine (*a*) the tension in wire *BC*, (*b*) the speed of the hammer's head.

SOLUTION



First we note $$a_A = a_n = \frac{v_A^2}{\rho}$$

(*a*) $+\uparrow \Sigma F_y = 0\text{:} \quad T_{BC} \sin 60° - W_A = 0$

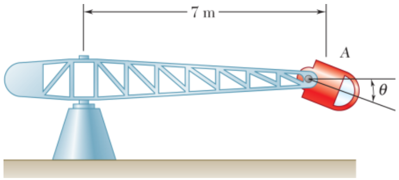
or $$T_{BC} = \frac{7.1 \text{ kg} \times 9.81 \text{ m/s}^2}{\sin 60°}$$
$$= 80.426 \text{ N}$$

$$T_{BC} = 80.4 \text{ N} \blacktriangleleft$$

(*b*) $\xrightarrow{+} \Sigma F_x = m_A a_A\text{:} \quad T_{BC} \cos 60° = m_A \frac{v_A^2}{\rho}$

or $$v_A^2 = \frac{(80.426 \text{ N}) \cos 60° \times 0.93 \text{ m}}{7.1 \text{ kg}}$$

or $$v_A = 2.30 \text{ m/s} \blacktriangleleft$$



PROBLEM 12.38

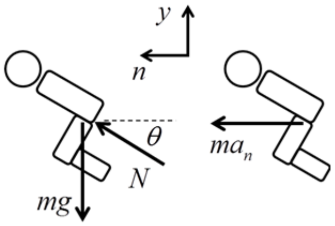
Human centrifuges are often used to simulate different acceleration levels for pilots. When aerospace physiologists say that a pilot is pulling 9*g's*, they mean that the resultant normal force on the pilot from the bottom of the seat is nine times their weight. Knowing that the centrifuge starts from rest and has a constant angular acceleration of 1.5 RPM per second until the pilot is pulling 9*g*'s and then continues with a constant angular velocity, determine (*a*) how long it will take for the pilot to reach 9*g's* (*b*) the angle θ of the normal force once the pilot reaches 9 g's. Assume that the force parallel to the seat is zero.

SOLUTION

Given:

$$\alpha = 1.5 \text{ RPM/s} = 0.157 \text{ rad/s}^2$$
$$\omega_0 = 0$$
$$\text{N} = 9\,\text{mg}$$
$$R = 7 \text{ m}$$

Free Body Diagram of Pilot:



Equations of Motion:

$$\sum F_y = ma_y$$
$$N\sin\theta - mg = m(0)$$
$$N\sin\theta = mg \qquad (1)$$

$$\sum F_n = ma_n$$
$$\text{N}\cos\theta = mR\omega^2$$
$$\omega = \sqrt{\frac{\text{N}\cos\theta}{mR}} \qquad (2)$$

Substitute N=9mg into (1): $9mg\sin\theta = mg$

$$\theta = \sin^{-1}\left(\frac{1}{9}\right)$$
$$\theta = 6.379°$$

Substitute N=9mg and θ into (2):

$$\omega = \sqrt{\frac{9*9.81\cos 6.379°}{7}}$$
$$\omega = 3.540 \text{ rad/s}$$

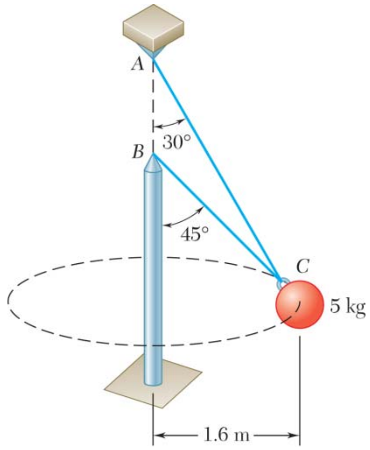
For constant angular acceleration:

$$\omega = \omega_0 + \alpha t$$
$$3.540 = 0 + 0.157 * t$$

(*a*) Solving for t: $t = 22.55 \text{ s}$ ◀

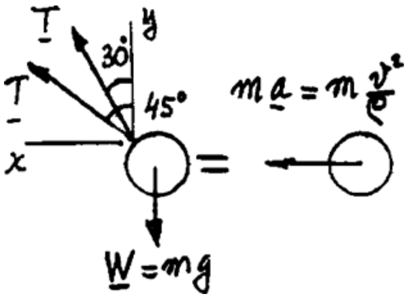
From earlier: $\theta = 6.379°$ ◀

PROBLEM 12.39



A single wire *ACB* passes through a ring at *C* attached to a sphere which revolves at a constant speed *v* in the horizontal circle shown. Knowing that the tension is the same in both portions of the wire, determine the speed *v*.

SOLUTION



$$\xleftarrow{+}\ \Sigma F_x = ma: \quad T(\sin 30° + \sin 45°) = \frac{mv^2}{\rho} \qquad (1)$$

$$+\uparrow\ \Sigma F_y = 0: \quad T\ (\cos 30° + \cos 45°) - mg = 0$$

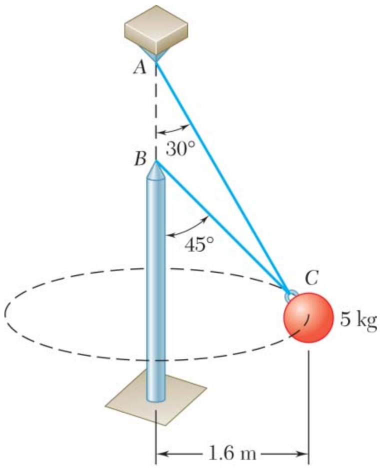
$$T\ (\cos 30° + \cos 45°) = mg \qquad (2)$$

Divide Eq. (1) by Eq. (2):

$$\frac{\sin 30° + \sin 45°}{\cos 30° + \cos 45°} = \frac{v^2}{\rho g}$$

$$v^2 = 0.76733\rho g = 0.76733\ (1.6\text{ m})(9.81\text{ m/s}^2) = 12.044\text{ m}^2/\text{s}^2$$

$v = 3.47$ m/s ◀



PROBLEM 12.40*

Two wires *AC* and *BC* are tied at *C* to a sphere which revolves at a constant speed *v* in the horizontal circle shown. Determine the range of the allowable values of *v* if both wires are to remain taut and if the tension in either of the wires is not to exceed 60 N.

SOLUTION

From the solution of Problem 12.39, we find that both wires remain taut for $3.01 \text{ m/s} \le v \le 3.96 \text{ m/s}$ ◁

To determine the values of *v* for which the tension in either wire will not exceed 60 N, we recall Eqs. (1) and (2) from Problem 12.39:

$$T_{AC}\sin 30° + T_{BC}\sin 45° = \frac{mv^2}{\rho} \qquad (1)$$

$$T_{AC}\cos 30° + T_{BC}\cos 45° = mg \qquad (2)$$

Subtract Eq. (1) from Eq. (2). Since sin 45° = cos 45°, we obtain

$$T_{AC}(\cos 30° - \sin 30°) = mg - \frac{mv^2}{\rho} \qquad (3)$$

Multiply Eq. (1) by cos 30°, Eq. (2) by sin 30°, and subtract:

$$T_{BC}(\sin 45°\cos 30° - \cos 45°\sin 30°) = \frac{mv^2}{\rho}\cos 30° - mg\sin 30°$$

$$T_{BC}\sin 15° = \frac{mv^2}{\rho}\cos 30° - mg\sin 30° \qquad (4)$$

Making $T_{AC} = 60$ N, $m = 5$ kg, $\rho = 1.6$ m, $g = 9.81 \text{ m/s}^2$ in Eq. (3), we find the value v_1 of v for which

$T_{AC} = 60$ N: $\quad 60(\cos 30° - \sin 30°) = 5(9.81) - \dfrac{5v_1^2}{1.6}$

$$21.962 = 49.05 - \frac{v_1^2}{0.32} \qquad v_1^2 = 8.668, \qquad v_1 = 2.94 \text{ m/s}$$

We have $T_{AC} \le 60$ N for $v \ge v_1$, that is, for $v \ge 2.94$ m/s ◁

PROBLEM 12.40* (Continued)

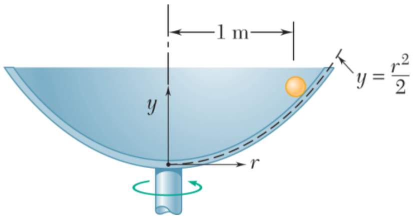
Making $T_{BC} = 60$ N, $m = 5$ kg, $\rho = 1.6$ m, $g = 9.81$ m/s^2 in Eq. (4), we find the value v_2 of v for which

$T_{BC} = 60$ N: $$60 \sin 15° = \frac{5v_2^2}{1.6} \cos 30° - 5(9.81) \sin 30°$$

$$15.529 = 2.7063v_2^2 - 24.523 \qquad v_2^2 = 14.80, \qquad v_2 = 3.85 \text{ m/s}$$

We have $T_{BC} \leq 60$ N for $v \leq v_2$, that is, for $v \leq 3.85$ m/s ◁

Combining the results obtained, we conclude that the range of allowable value is $3.01 \text{ m/s} \leq v \leq 3.85 \text{ m/s}$ ◀



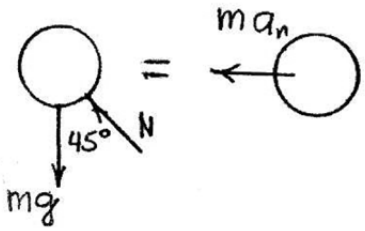
PROBLEM 12.41

A 1-kg sphere is at rest relative to a parabolic dish which rotates at a constant rate about a vertical axis. Neglecting friction and knowing that $r = 1$ m, determine (*a*) the speed v of the sphere, (*b*) the magnitude of the normal force exerted by the sphere on the inclined surface of the dish.

SOLUTION

Given: $$y = \frac{r^2}{2}, \quad \frac{dy}{dx} = r = 1 = \tan\theta \quad \text{or} \quad \theta = 45°$$

Free Body Diagram of Sphere:



Equations of Motion:

$$+\uparrow \Sigma F_y = 0: \quad N\cos\theta - mg = 0$$

$$N = \frac{mg}{\cos\theta}$$

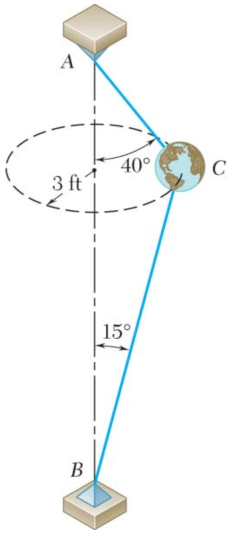
$$\overset{+}{\leftarrow} \Sigma F_n = ma_n: \quad N\sin\theta = ma_n$$

$$mg\tan\theta = m\frac{v^2}{r}$$

$$v^2 = gr\tan\theta$$

(*a*) $v^2 = (9.81)(1)(1.0000) = 9.81 \text{ m}^2/\text{s}^2$ $\quad v = 3.13$ m/s ◀

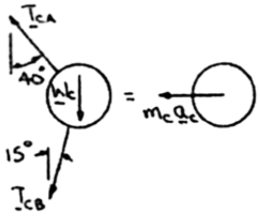
(*b*) $N = \dfrac{mg}{\cos 45°} = \dfrac{(1)(9.81)}{\cos 45°}$ $\quad N = 13.87$ N ◀



PROBLEM 12.42*

As part of an outdoor display, a 12-lb model *C* of the earth is attached to wires *AC* and *BC* and revolves at a constant speed *v* in the horizontal circle shown. Determine the range of the allowable values of *v* if both wires are to remain taut and if the tension in either of the wires is not to exceed 26 lb.

SOLUTION



First note $$a_C = a_n = \frac{v_C^2}{\rho}$$

where $$\rho = 3 \text{ ft}$$

$$\overset{+}{\leftarrow} \Sigma F_x = m_C a_C: \quad T_{CA}\sin 40° + T_{CB}\sin 15° = \frac{W_C}{g}\frac{v_C^2}{\rho} \qquad (1)$$

$$+\!\uparrow \Sigma F_y = 0: \quad T_{CA}\cos 40° - T_{CB}\cos 15° - W_C = 0 \qquad (2)$$

Note that Eq. (2) implies that

(*a*) when $T_{CB} = (T_{CB})_{\max}, \quad T_{CA} = (T_{CA})_{\max}$

(*b*) when $T_{CB} = (T_{CB})_{\min}, \quad T_{CA} = (T_{CA})_{\min}$

Case 1: T_{CA} is maximum.

Let $$T_{CA} = 26 \text{ lb}$$

Eq. (2) $$(26 \text{ lb})\cos 40° - T_{CB}\cos 15° - (12 \text{ lb}) = 0$$

or $$T_{CB} = 8.1964 \text{ lb}$$

$$(T_{CB})_{(T_{CA})_{\max}} < 26 \text{ lb} \qquad \text{OK}$$
$$[(T_{CB})_{\max} = 8.1964 \text{ lb}]$$

Eq. (1)

$$(v_C^2)_{(T_{CA})_{\max}} = \frac{(32.2\text{ ft/s}^2)(3\text{ ft})}{12\text{ lb}}(26\sin 40° + 8.1964\sin 15°)\text{ lb}$$

or $$(v_C)_{(T_{CA})_{\max}} = 12.31\text{ ft/s}$$

Now we form $$(\cos 15°)(\text{Eq. }1) + (\sin 15°)(\text{Eq. }2)$$

$$T_{CA}\sin 40°\cos 15° + T_{CA}\cos 40°\sin 15° = \frac{W_C}{g}\frac{v_C^2}{\rho}\cos 15° + W_C\sin 15°$$

or $$T_{CA}\sin 55° = \frac{W_C}{g}\frac{v_C^2}{\rho}\cos 15° + W_C\sin 15° \qquad (3)$$

$(v_c)_{\max}$ occurs when $T_{CA} = (T_{CA})_{\max}$

$(v_C)_{\max} = 12.31\text{ ft/s}$

Case 2: T_{CA} is minimum.

Because $(T_{CA})_{\min}$ occurs when $T_{CB} = (T_{CB})_{\min}$,

let $T_{CB} = 0$ (note that wire *BC* will not be taut).

Eq. (2) $$T_{CA}\cos 40° - (12\text{ lb}) = 0$$

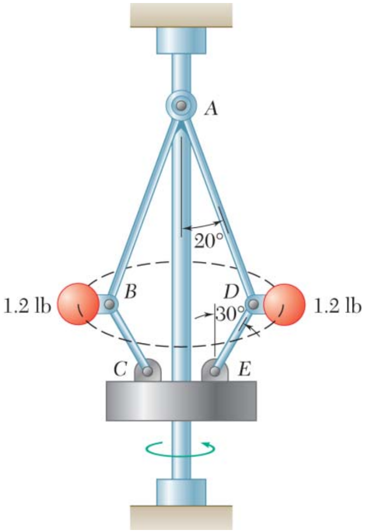
or $$T_{CA} = 15.6649\text{ lb}, \quad 26\text{ lb} \quad \text{OK}$$

Note: Eq. (3) implies that when $T_{CA} = (T_{CA})_{\min}$, $v_C = (v_C)_{\min}$. Then

Eq. (1) $$(v_C^2)_{\min} = \frac{(32.2\text{ ft/s}^2)(3\text{ ft})}{12\text{ lb}}(15.6649\text{ lb})\sin 40°$$

or $$(v_C)_{\min} = 9.00\text{ ft/s}$$

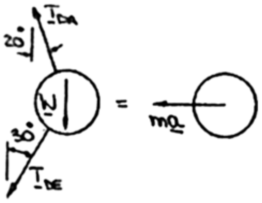
$0 < T_{CA} \le T_{CB} < 6\text{ lb}$ when $9.00\text{ ft/s} < v_C < 12.31\text{ ft/s}$ ◀



PROBLEM 12.43*

The 1.2-lb flyballs of a centrifugal governor revolve at a constant speed v in the horizontal circle of 6-in. radius shown. Neglecting the weights of links *AB*, *BC*, *AD*, and *DE* and requiring that the links support only tensile forces, determine the range of the allowable values of v so that the magnitudes of the forces in the links do not exceed 17 lb.

SOLUTION



First note $$a = a_n = \frac{v^2}{\rho}$$

where $$\rho = 0.5 \text{ ft}$$

$$\xleftarrow{+} \Sigma F_x = ma: \quad T_{DA} \sin 20° + T_{DE} \sin 30° = \frac{W}{g}\frac{v^2}{\rho} \qquad (1)$$

$$+\!\uparrow \Sigma F_y = 0: \quad T_{DA} \cos 20° - T_{DE} \cos 30° - W = 0 \qquad (2)$$

Note that Eq. (2) implies that

(*a*) when $T_{DE} = (T_{DE})_{\max}, \quad T_{DA} = (T_{DA})_{\max}$

(*b*) when $T_{DE} = (T_{DE})_{\min}, \quad T_{DA} = (T_{DA})_{\min}$

Case 1: T_{DA} is maximum.

Let $$T_{DA} = 17 \text{ lb}$$

Eq. (2) $$(17 \text{ lb}) \cos 20° - T_{DE} \cos 30° - (1.2 \text{ lb}) = 0$$

or $$T_{DE} = 17.06 \text{ lb} \quad \text{unacceptable } (> 17 \text{ lb})$$

Now let $$T_{DE} = 17 \text{ lb}$$

Eq. (2) $$T_{DA} \cos 20° - (17 \text{ lb}) \cos 30° - (1.2 \text{ lb}) = 0$$

or $$T_{DA} = 16.9443 \text{ lb} \quad \text{OK} \quad (\le 17 \text{ lb})$$

$$(T_{DA})_{\max} = 16.9443 \text{ lb}$$
$$(T_{DE})_{\max} = 17 \text{ lb}$$

Eq. (1) $$(v^2)_{(T_{DA})_{\max}} = \frac{(32.2 \text{ ft/s}^2)(0.5 \text{ ft})}{1.2 \text{ lb}}(16.9443 \sin 20° + 17 \sin 30°) \text{ lb}$$

or $$v_{(T_{DA})_{\max}} = 13.85 \text{ ft/s}$$

Now form $$(\cos 30°) \times [\text{Eq. (1)}] + (\sin 30°) \times [\text{Eq. (2)}]$$

$$T_{DA} \sin 20° \cos 30° + T_{DA} \cos 20° \sin 30° = \frac{W}{g}\frac{v^2}{\rho}\cos 30° + W \sin 30°$$

or $$T_{DA} \sin 50° = \frac{W}{g}\frac{v^2}{\rho}\cos 30° + W \sin 30° \qquad (3)$$

$v_{\max}$ occurs when $T_{DA} = (T_{DA})_{\max}$

$v_{\max} = 13.85$ ft/s

Case 2: T_{DA} is minimum.

Because $(T_{DA})_{\min}$ occurs when $T_{DE} = (T_{DE})_{\min}$,

let $T_{DE} = 0$.

Eq. (2) $$T_{DA} \cos 20° - (1.2 \text{ lb}) = 0$$

or $$T_{DA} = 1.27701 \text{ lb}, \ 17 \text{ lb} \qquad \text{OK}$$

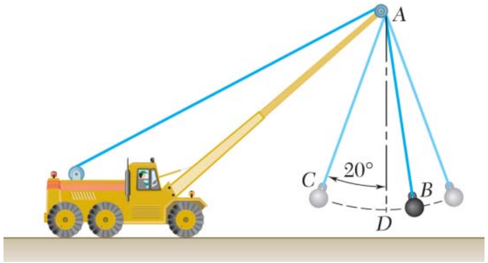
Note: Eq. (3) implies that when $T_{DA} = (T_{DA})_{\min}$, $v = v_{\min}$. Then

Eq. (1) $$(v^2)_{\min} = \frac{(32.2 \text{ ft/s}^2)(0.5 \text{ ft})}{1.2 \text{ lb}}(1.27701 \text{ lb}) \sin 20°$$

or $$v_{\min} = 2.42 \text{ ft/s}$$

$0 < T_{AB}, T_{BC}, T_{AD}, T_{DE} < 17$ lb

when $2.42 \text{ ft/s} < v < 13.85 \text{ ft/s}$ ◀

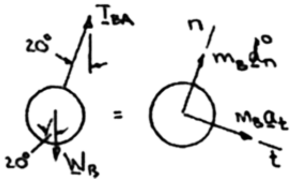


PROBLEM 12.44

A 130-lb wrecking ball B is attached to a 45-ft-long steel cable AB and swings in the vertical arc shown. Determine the tension in the cable (a) at the top C of the swing, (b) at the bottom D of the swing, where the speed of B is 13.2 ft/s.

SOLUTION

(a) At C, the top of the swing, $v_B = 0$; thus



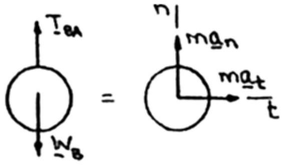
$$a_n = \frac{v_B^2}{L_{AB}} = 0$$

$$+\nearrow \ \Sigma F_n = 0: \quad T_{BA} - W_B \cos 20° = 0$$

or $$T_{BA} = (130 \text{ lb}) \times \cos 20°$$

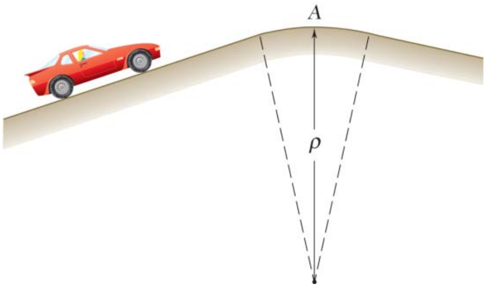
or $$T_{BA} = 122.2 \text{ lb} \blacktriangleleft$$

(b) $$+\uparrow \ \Sigma F_n = ma_n: \quad T_{BA} - W_B = m_B \frac{(v_B)_D^2}{L_{AB}}$$



or $$T_{BA} = (130 \text{ lb}) + \left[\left(\frac{130 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{(13.2 \text{ ft/s})^2}{45 \text{ ft}}\right)\right]$$

or $$T_{BA} = 145.6 \text{ lb} \blacktriangleleft$$



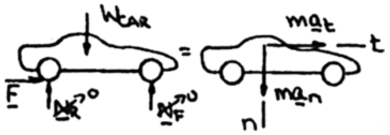
PROBLEM 12.45

During a high-speed chase, a 2400-lb sports car traveling at a speed of 100 mi/h just loses contact with the road as it reaches the crest *A* of a hill. (*a*) Determine the radius of curvature ρ of the vertical profile of the road at *A*. (*b*) Using the value of ρ found in part *a*, determine the force exerted on a 160-lb driver by the seat of his 3100-lb car as the car, traveling at a constant speed of 50 mi/h, passes through *A*.

SOLUTION

(*a*) *Note:* $100 \text{ mi/h} = 146.667 \text{ ft/s}$

$$+\!\downarrow \Sigma F_n = ma_n: \quad W_{\text{car}} = \frac{W_{\text{car}}}{g}\frac{v_A^2}{\rho}$$

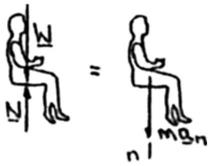


or $$\rho = \frac{(146.667 \text{ ft/s})^2}{32.2 \text{ ft/s}^2} = 668.05 \text{ ft}$$

or $$\rho = 668 \text{ ft} \blacktriangleleft$$

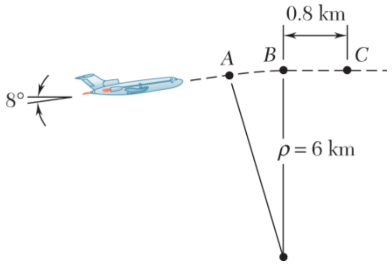
(*b*) *Note:* v is constant $\Rightarrow a_t = 0$; $50 \text{ mi/h} = 73.333 \text{ ft/s}$

$$+\!\downarrow \Sigma F_n = ma_n: \quad W - N = \frac{W}{g}\frac{v_A^2}{\rho}$$



or $$N = (160 \text{ lb})\left[1 - \frac{(73.333 \text{ ft/s})^2}{(32.2 \text{ ft/s}^2)(668.05 \text{ ft})}\right]$$

or $$\mathbf{N} = 120.0 \text{ lb}\uparrow \blacktriangleleft$$



PROBLEM 12.46

An airline pilot climbs to a new flight level along the path shown. Knowing that the speed of the airplane decreases at a constant rate from 180 m/s at point *A* to 160 m/s at point *C*, determine the magnitude of the abrupt change in the force exerted on a 90-kg passenger as the airplane passes point *B*.

SOLUTION

Angle change over arc *AB*. $\theta = \frac{8°}{180°}\pi = 0.13963 \text{ rad}$

Length of arc: $s_{AB} = \rho\theta = (6000)(0.13963) = 837.76 \text{ m}$

$$s_{BC} = 800 \text{ m}, \quad s_{AC} = 837.76 + 800 = 1637.76 \text{ m}$$

$$\int_{180}^{160} v\,dv = \int_0^{1637.76} a_t\,ds \qquad \text{or} \qquad \frac{160^2}{2} - \frac{180^2}{2} = a_t(1637.76)$$

$$a_t = -2.076 \text{ m/s}^2$$

$$\int_{180}^{v_B} v\,dv = \int_0^{837.76} a_t\,ds \qquad \text{or} \qquad \frac{v_B^2}{2} - \frac{180^2}{2} = (-2.076)(837.76)$$

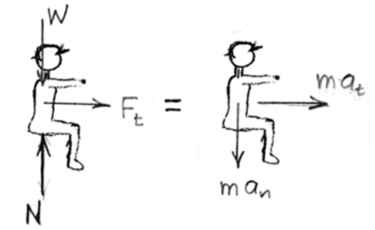
$$v_B^2 = 28922 \text{ m}^2/\text{s}^2 \qquad v_B = 170.06 \text{ m/s}$$

Weight of passenger: $mg = (90)(9.81) = 882.9 \text{ N}$

Just before point *B*. $v = 170.06 \text{ m/s}, \quad \rho = 6000 \text{ m}$

$$a_n = \frac{v^2}{\rho} = \frac{(170.06)^2}{6000} = 4.820 \text{ m/s}^2$$

Free Body Diagram of Pilot:



$$+\uparrow \Sigma F_n = N_1 - W = -m(a_n)_1: \quad N_1 = 882.9 - (90)(4.820) = 449.1 \text{ N}$$

$$\xrightarrow{+} \Sigma F_t = F_t = ma_t: \quad (F_t)_1 = (90)(-2.076) = -186.8 \text{ N}$$

Just after point *B*. $v = 170.06 \text{ m/s}, \quad \rho = \infty, \quad a_n = 0$

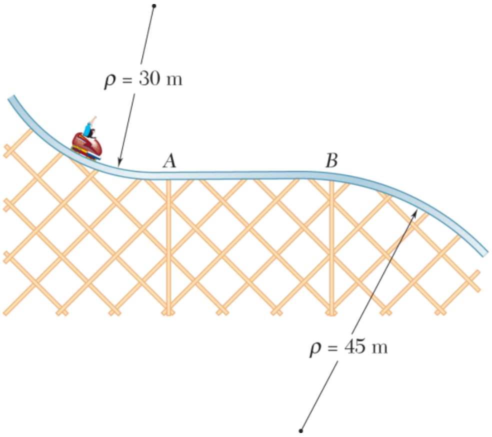
$$+\uparrow \Sigma F_y = 0: \quad N_2 - W = 0 \qquad N_2 = W = 882.9 \text{ N}$$

$$\xrightarrow{+} \Sigma F_x = ma_t: \quad F_t = ma_t = (90)(-2.076) = -186.8 \text{ N}$$

F_t does not change.

N increases by 433.8 N.

magnitude of change of force = 434 N ◀



$\rho = 30$ m
A
B
$\rho = 45$ m

PROBLEM 12.47

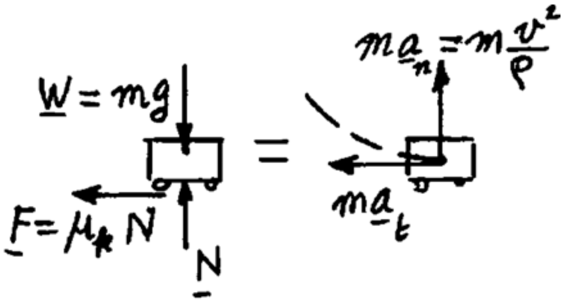
The roller-coaster track shown is contained in a vertical plane. The portion of track between A and B is straight and horizontal, while the portions to the left of A and to the right of B have radii of curvature as indicated. A car is traveling at a speed of 72 km/h when the brakes are suddenly applied, causing the wheels of the car to slide on the track ($\mu_k = 0.20$). Determine the initial deceleration of the car if the brakes are applied as the car (a) has almost reached A, (b) is traveling between A and B, (c) has just passed B.

SOLUTION

(a) $+\uparrow \Sigma F_n = ma_n$: $N - mg = m\dfrac{v^2}{\rho}$

$$N = m\left(g + \frac{v^2}{\rho}\right)$$

$$F = \mu_k N = \mu_k m\left(g + \frac{v^2}{\rho}\right)$$



$\underline{W} = mg$
$m\underline{a}_n = m\frac{v^2}{\rho}$
=
$\underline{F} = \mu_k N$
$\underline{N}$
$m\underline{a}_t$

$\overset{+}{\leftarrow} \Sigma F_t = ma_t$: $F = ma_t$

$$a_t = \frac{F}{m} = \mu_k\left(g + \frac{v^2}{\rho}\right)$$

Given data: $\mu_k = 0.20,\ v = 72 \text{ km/h} = 20 \text{ m/s}$

$$g = 9.81 \text{ m/s}^2, \quad \rho = 30 \text{ m}$$

$$a_t = 0.20\left[9.81 + \frac{(20)^2}{30}\right]$$

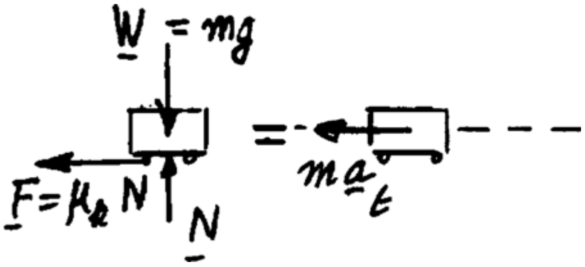
$a_t = 4.63 \text{ m/s}^2$ ◀

(b) $a_n = 0$ $+\uparrow \Sigma F_n = ma_n = 0$: $N - mg = 0$

$$N = mg$$

$$F = \mu_k N = \mu_k mg$$

$\overset{+}{\leftarrow} \Sigma F_t = ma_t$: $F = ma_t$



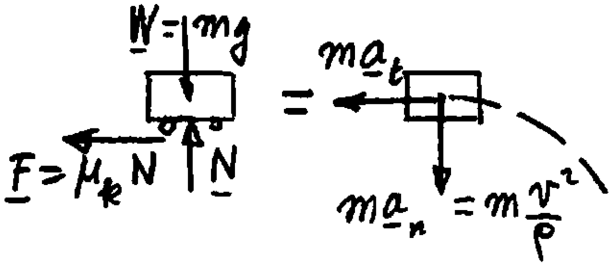
$\underline{W} = mg$
=
$\underline{F} = \mu_k N$
$\underline{N}$
$m\underline{a}_t$

$$a_t = \frac{F}{m} = \mu_k g = 0.20(9.81)$$

$a_t = 1.962 \text{ m/s}^2$ ◀

PROBLEM 12.47 (Continued)

(*c*) $\overset{+}{\downarrow} \Sigma F_n = ma_n: \quad mg - N = \dfrac{mv^2}{\rho}$



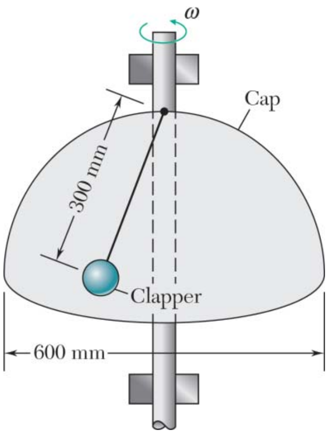
$$N = m\left(g - \frac{v^2}{\rho}\right)$$

$$F = \mu_k\, N = \mu_k m\left(g - \frac{v^2}{\rho}\right)$$

$\overset{+}{\leftarrow} \Sigma F_t = ma_t: \quad F = ma_t$

$$a_t = \frac{F}{m} = \mu_k\left(g - \frac{v^2}{\rho}\right) = 0.20\left[9.81 - \frac{(20)^2}{45}\right]$$

$a_t = 0.1842 \text{ m/s}^2$ ◀



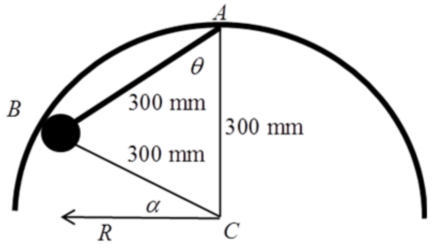
PROBLEM 12.48

A spherical-cap governor is fixed to a vertical shaft that rotates with angular velocity ω. When the string-supported clapper of mass m touches the cap, a cutoff switch is operated electrically to reduce the speed of the shaft. Knowing that the radius of the clapper is small relative to the cap, determine the minimum angular speed at which the cutoff switch operates.

SOLUTION

Given: $\omega_{\min}$ is the angular velocity when the clapper hits the cap

Geometry From Figure:



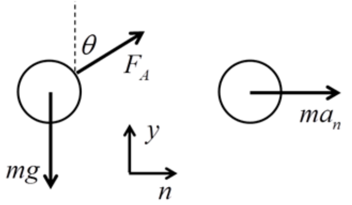
Distance AB is 300 mm, the length of the Clapper Arm

Distance BC and AC is 300 mm, the radius of the Spherical Cap.

Therefore $\triangle ABC$ is equilateral, so $\theta = 60°$ and $\alpha = 30°$

$$R = 0.3 * \cos\alpha$$

Free Body Diagram of Clapper:



Equations of Motion:

$$\sum F_y = ma_y \qquad \sum F_n = ma_n$$

$$\mathrm{F}_A \cos\theta - mg = m(0) \qquad F_A \sin\theta = mR\omega_{\min}^2$$

$$F_A = \frac{mg}{\cos\theta} \quad (1) \qquad \omega_{win} = \sqrt{\frac{F_A \sin\theta}{mR}} \quad (2)$$

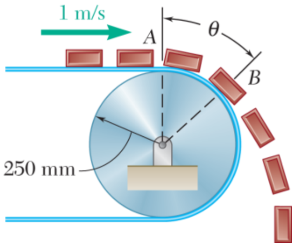
Substitute (1)into (2):

$$\omega_{win} = \sqrt{\frac{\cancel{m} g \tan\theta}{\cancel{m} R}}$$

$$= \sqrt{\frac{9.81 \tan 60°}{0.3 \cos 30°}}$$

$$= 8.087 \text{ rad/s}$$

ω_{win} =77.23 rpm ◀



PROBLEM 12.49

A series of small packages, each with a mass of 0.5 kg, are discharged from a conveyor belt as shown. Knowing that the coefficient of static friction between each package and the conveyor belt is 0.4, determine (*a*) the force exerted by the belt on a package just after it has passed Point *A*, (*b*) the angle θ defining the Point *B* where the packages first *slip* relative to the belt.

SOLUTION

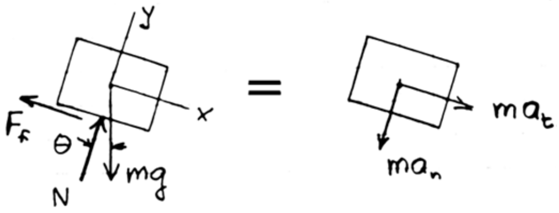
Assume package does not slip.

$$a_t = 0, \quad F_f \le \mu_s N$$

On the curved portion of the belt

$$a_n = \frac{v^2}{\rho} = \frac{(1\text{ m/s})^2}{0.250\text{ m}} = 4\text{ m/s}^2$$

For any angle θ



$$+\nearrow \Sigma F_y = ma_y: \quad N - mg\cos\theta = -ma_n = -\frac{mv^2}{\rho}$$

$$N = mg\cos\theta - \frac{mv^2}{\rho} \qquad (1)$$

$$+\searrow \Sigma F_x = ma_x: \quad -F_f + mg\sin\theta = ma_t = 0$$

$$F_f = mg\sin\theta \qquad (2)$$

(*a*) At Point *A*, $\theta = 0°$

$$N = (0.5)(9.81)(1.000) - (0.5)(4)$$

$N = 2.905\text{ N}$ ◀

(*b*) At Point *B*, $F_f = \mu_s N$

$$mg\sin\theta = \mu_s(mg\cos\theta - ma_n)$$

$$\sin\theta = \mu_s\left(\cos\theta - \frac{a_n}{g}\right) = 0.40\left[\cos\theta - \frac{4}{9.81}\right]$$

PROBLEM 12.49 (Continued)

Squaring and using trigonometic identities,

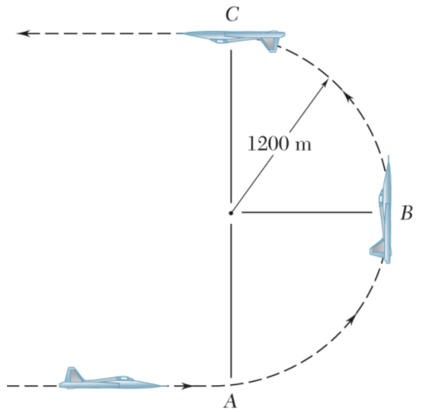
$$1-\cos^2\theta = 0.16\cos^2\theta - 0.130479\cos\theta + 0.026601$$

$$1.16\cos^2\theta - 0.130479\cos\theta - 0.97340 = 0$$

$$\cos\theta = 0.97402 \qquad \theta = 13.09° \blacktriangleleft$$

Check that package does not separate from the belt.

$$N = \frac{F_f}{\mu_s} = \frac{mg\sin\theta}{\mu_s} \qquad N > 0.$$



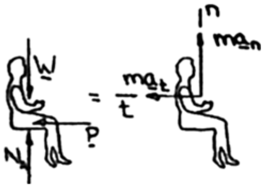
PROBLEM 12.50

A 54-kg pilot flies a jet trainer in a half vertical loop of 1200-m radius so that the speed of the trainer decreases at a constant rate. Knowing that the pilot's apparent weights at Points *A* and *C* are 1680 N and 350 N, respectively, determine the force exerted on her by the seat of the trainer when the trainer is at Point *B*.

SOLUTION

First we note that the pilot's apparent weight is equal to the vertical force that she exerts on the seat of the jet trainer.

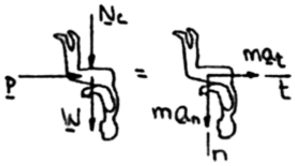
At *A*: $+\uparrow \Sigma F_n = ma_n\text{:}\quad N_A - W = m\dfrac{v_A^2}{\rho}$



or $$v_A^2 = (1200\text{ m})\left(\frac{1680\text{ N}}{54\text{ kg}} - 9.81\text{ m/s}^2\right)$$

$$= 25{,}561.3\text{ m}^2/\text{s}^2$$

At *C*: $+\downarrow \Sigma F_n = ma_n\text{:}\quad N_C + W = m\dfrac{v_C^2}{\rho}$



or $$v_C^2 = (1200\text{ m})\left(\frac{350\text{ N}}{54\text{ kg}} + 9.81\text{ m/s}^2\right)$$

$$= 19{,}549.8\text{ m}^2/\text{s}^2$$

Since $a_t = \text{constant,}$ we have from *A* to *C*

$$v_C^2 = v_A^2 + 2a_t\ \Delta s_{AC}$$

or $$19{,}549.8\text{ m}^2/\text{s}^2 = 25{,}561.3\text{ m}^2/\text{s}^2 + 2a_t(\pi \times 1200\text{ m})$$

or $$a_t = -0.79730\text{ m/s}^2$$

Then from *A* to *B*

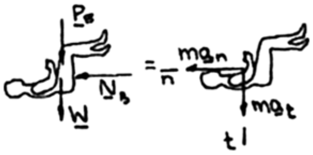
$$v_B^2 = v_A^2 + 2a_t\ \Delta s_{AB}$$

$$= 25{,}561.3\text{ m}^2/\text{s}^2 + 2(-0.79730\text{ m/s}^2)\left(\frac{\pi}{2} \times 1200\text{ m}\right)$$

$$= 22{,}555\text{ m}^2/\text{s}^2$$

PROBLEM 12.50 (Continued)

At B: $\overset{+}{\longleftarrow}\ \Sigma F_n = ma_n$: $N_B = m\dfrac{v_B^2}{\rho}$



or $$N_B = 54\ \text{kg}\,\frac{22{,}555\ \text{m}^2/\text{s}^2}{1200\ \text{m}}$$

or $$\mathbf{N}_B = 1014.98\ \text{N} \longleftarrow$$

$$+\big\downarrow \Sigma F_t = ma_t: \quad W + P_B = m\,|a_t|$$

or $$P_B = (54\ \text{kg})(0.79730 - 9.81)\ \text{m/s}^2$$

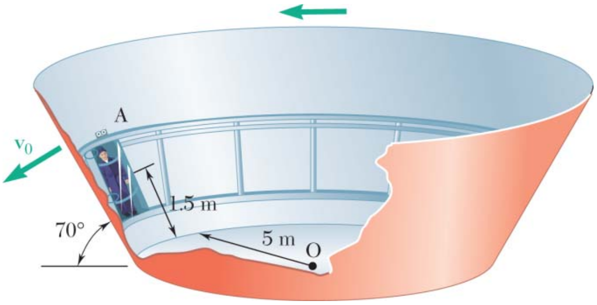
or $$\mathbf{P}_B = 486.69\ \text{N}\uparrow$$

Finally, $$(F_{\text{pilot}})_B = \sqrt{N_B^2 + P_B^2} = \sqrt{(1014.98)^2 + (486.69)^2}$$
$$= 1126\ \text{N}$$

or $$(\mathbf{F}_{\text{pilot}})_B = 1126\ \text{N} \searrow 25.6° \blacktriangleleft$$

PROBLEM 12.51

A carnival ride is designed to allow the general public to experience high acceleration motion. The ride rotates about Point O in a horizontal circle such that the rider has a speed v_0. The rider reclines on a platform A which rides on rollers such that friction is negligible. A mechanical stop prevents the platform from rolling down the incline. Determine (a) the speed v_0 at which the platform A begins to roll upwards, (b) the normal force experienced by an 80-kg rider at this speed.



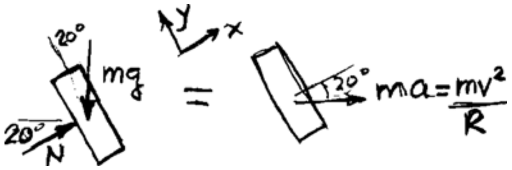
SOLUTION

Radius of circle: $R = 5 + 1.5\cos 70° = 5.513 \text{ m}$

$\Sigma\mathbf{F} = m\mathbf{a}$:

Components up the incline, ∡70°:

$$-m_A g\cos 20° = -\frac{mv_0^2}{R}\sin 20°$$

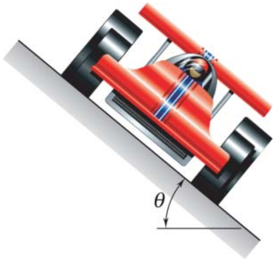


(a) Speed v_0: $v_0 = \left[\dfrac{gR}{\tan 20°}\right]^{\frac{1}{2}} = \left[\dfrac{(9.81 \text{ m/s})(5.513 \text{ m}}{\tan 20°}\right]^{\frac{1}{2}} = 12.1898 \text{ m/s}$ $v_0 = 12.19 \text{ m/s}$ ◀

Components normal to the incline, ∡ 20°.

$$N - mg\sin 20° = \frac{mv_0^2}{R}\cos 20°.$$

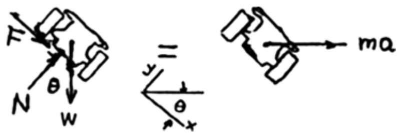
(b) Normal force: $N = (80)(9.81)\sin 20° + \dfrac{80(12.1898)^2}{5.513}\cos 20° = 2294 \text{ N}$ $N = 2290 \text{ N}$ ◀



PROBLEM 12.52

A curve in a speed track has a radius of 1000 ft and a rated speed of 120 mi/h. (See Sample Problem 12.7 for the definition of rated speed). Knowing that a racing car starts skidding on the curve when traveling at a speed of 180 mi/h, determine (*a*) the banking angle θ, (*b*) the coefficient of static friction between the tires and the track under the prevailing conditions, (*c*) the minimum speed at which the same car could negotiate that curve.

SOLUTION



Weight $W = mg$

Acceleration $a = \dfrac{v^2}{\rho}$

$$\Sigma F_x = ma_x: \quad F + W \sin\theta = ma \cos\theta$$

$$F = \frac{mv^2}{\rho}\cos\theta - mg\sin\theta \qquad (1)$$

$$\Sigma F_y = ma_y: \quad N - W\cos\theta = ma\sin\theta$$

$$N = \frac{mv^2}{\rho}\sin\theta + mg\cos\theta \qquad (2)$$

(*a*) Banking angle. Rated speed $v = 120$ mi/h $= 176$ ft/s. $F = 0$ at rated speed.

$$0 = \frac{mv^2}{\rho}\cos\theta - mg\sin\theta$$

$$\tan\theta = \frac{v^2}{\rho g} = \frac{(176)^2}{(1000)(32.2)} = 0.96199$$

$$\theta = 43.89^\circ$$

$\theta = 43.9^\circ$ ◀

(*b*) Slipping outward. $v = 180$ mi/h $= 264$ ft/s

$$F = \mu N \quad \mu = \frac{F}{N} = \frac{v^2\cos\theta - \rho g\sin\theta}{v^2\sin\theta + \rho g\cos\theta}$$

$$\mu = \frac{(264)^2\cos 43.89^\circ - (1000)(32.2)\sin 43.89^\circ}{(264)^2\sin 43.89^\circ + (1000)(32.2)\cos 43.89^\circ}$$

$$= 0.39009$$

$\mu = 0.390$ ◀

PROBLEM 12.52 (Continued)

(*c*) Minimum speed.

$$F = -\mu N$$

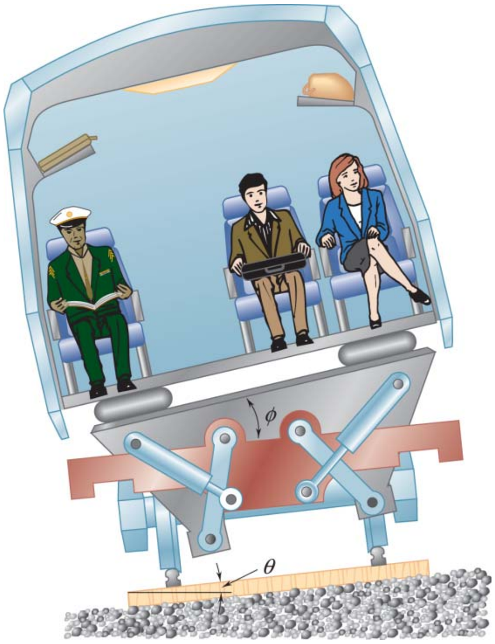
$$-\mu = \frac{v^2 \cos\theta - \rho g \sin\theta}{v^2 \sin\theta + \rho g \cos\theta}$$

$$v^2 = \frac{\rho g(\sin\theta - \mu\cos\theta)}{\cos\theta + \mu\sin\theta}$$

$$= \frac{(1000)(32.2)(\sin 43.89° - 0.39009 \cos 43.89°)}{\cos 43.89° + 0.39009 \sin 43.89°}$$

$$= 13.369 \text{ ft}^2/\text{s}^2$$

$$v = 115.62 \text{ ft/s} \qquad v = 78.8 \text{ mi/h} \blacktriangleleft$$



PROBLEM 12.53

Tilting trains, such as the *American Flyer* which will run from Washington to New York and Boston, are designed to travel safely at high speeds on curved sections of track which were built for slower, conventional trains. As it enters a curve, each car is tilted by hydraulic actuators mounted on its trucks. The tilting feature of the cars also increases passenger comfort by eliminating or greatly reducing the side force $\mathbf{F}_s$ (parallel to the floor of the car) to which passengers feel subjected. For a train traveling at 100 mi/h on a curved section of track banked through an angle $\theta = 6°$ and with a rated speed of 60 mi/h, determine (*a*) the magnitude of the side force felt by a passenger of weight W in a standard car with no tilt ($\phi = 0$), (*b*) the required angle of tilt ϕ if the passenger is to feel no side force. (See Sample Problem 12.7 for the definition of rated speed.)

SOLUTION

Rated speed: $v_R = 60 \text{ mi/h} = 88 \text{ ft/s}, \;\; 100 \text{ mi/h} = 146.67 \text{ ft/s}$

From Sample Problem 12.6,

$$v_R^2 = g\rho \tan\theta$$

or

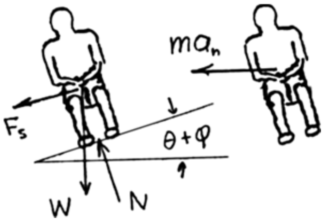
$$\rho = \frac{v_R^2}{g\tan\theta} = \frac{(88)^2}{32.2\tan 6°} = 2288 \text{ ft}$$

Let the *x*-axis be parallel to the floor of the car.

$$+\swarrow \Sigma F_x = ma_x: \quad F_s + W\sin(\theta+\phi) = ma_n\cos(\theta+\phi)$$

$$= \frac{mv^2}{\rho}\cos(\theta+\phi)$$

(*a*) $\phi = 0.$



$$F_s = W\left[\frac{v^2}{g\rho}\cos(\theta+\phi) - \sin(\theta+\phi)\right]$$

$$= W\left[\frac{(146.67)^2}{(32.2)(2288)}\cos 6° - \sin 6°\right]$$

$$= 0.1858W$$

$F_s = 0.1858W$ ◀

PROBLEM 12.53 (Continued)

(*b*) For $F_s = 0$,

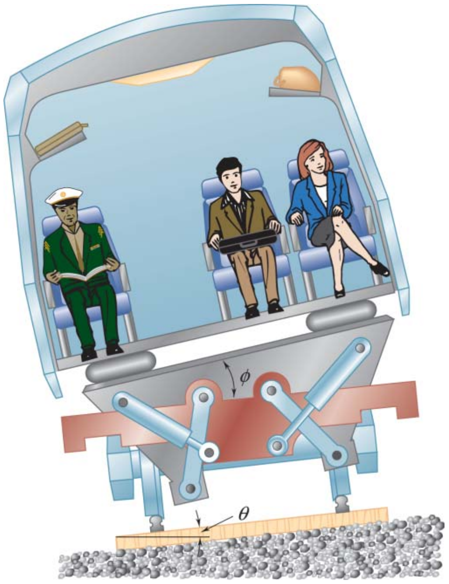
$$\frac{v^2}{g\rho}\cos(\theta+\phi) - \sin(\theta+\phi) = 0$$

$$\tan(\theta+\phi) = \frac{v^2}{g\rho} = \frac{(146.67)^2}{(32.2)(2288)} = 0.29199$$

$$\theta+\phi = 16.28°$$

$$\phi = 16.28° - 6°$$

$\phi = 10.28°$ ◀



PROBLEM 12.54

Tests carried out with the tilting trains described in Problem 12.53 revealed that passengers feel queasy when they see through the car windows that the train is rounding a curve at high speed, yet do not feel any side force. Designers, therefore, prefer to reduce, but not eliminate, that force. For the train of Problem 12.53, determine the required angle of tilt ϕ if passengers are to feel side forces equal to 10% of their weights.

SOLUTION

Rated speed: $v_R = 60 \text{ mi/h} = 88 \text{ ft/s}, \;\; 100 \text{ mi/h} = 146.67 \text{ ft/s}$

From Sample Problem 12.6,

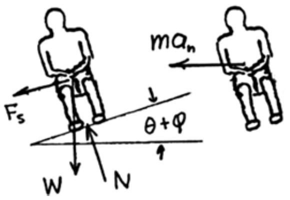
$$v_R^2 = g\rho \tan\theta$$

or $$\rho = \frac{v_R^2}{g\tan\theta} = \frac{(88)^2}{32.2\tan 6°} = 2288 \text{ ft}$$

Let the *x*-axis be parallel to the floor of the car.

$$+\swarrow \Sigma F_x = ma_x: \quad F_s + W\sin(\theta+\phi) = ma_n\cos(\theta+\phi)$$

$$= \frac{mv^2}{\rho}\cos(\theta+\phi)$$



Solving for F_s, $$F_s = W\left[\frac{v^2}{g\rho}\cos(\theta+\phi) - \sin(\theta+\phi)\right]$$

Now $$\frac{v^2}{g\rho} = \frac{(146.67)^2}{(32.2)(2288)} = 0.29199 \quad \text{and} \quad F_s = 0.10W$$

So that $$0.10W = W[0.29199\cos(\theta+\phi) - \sin(\theta+\phi)]$$

Let $$u = \sin(\theta+\phi)$$

Then $$\cos(\theta+\phi) = \sqrt{1-u^2}$$

$$0.10 = 0.29199\sqrt{1-u^2} - u \quad \text{or} \quad 0.29199\sqrt{1-u^2} = 0.10 + u$$

PROBLEM 12.54 (Continued)

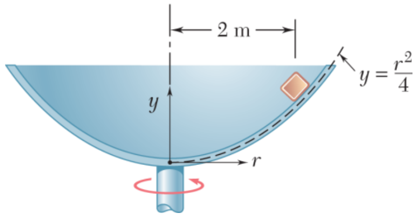
Squaring both sides, $0.08526(1-u^2) = 0.01 + 0.2u + u^2$

or $1.08526u^2 + 0.2u - 0.07526 = 0$

The positive root of the quadratic equation is $u = 0.18685$

Then, $\theta + \phi = \sin^{-1} u = 10.77°$

$\phi = 10.77° - 6°$ $\phi = 4.77°$ ◀



PROBLEM 12.55

A 3-kg block is at rest relative to a parabolic dish which rotates at a constant rate about a vertical axis. Knowing that the coefficient of static friction is 0.5 and that $r = 2$ m, determine the maximum allowable velocity v of the block.

SOLUTION

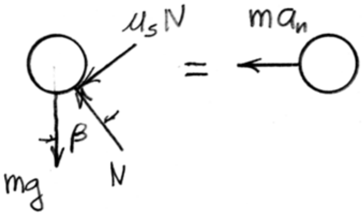
Let β be the slope angle of the dish. $\tan\beta = \frac{dy}{dr} = \frac{1}{2}r$

At $r = 2$ m, $\tan\beta = 1$ or $\beta = 45°$

Draw free body sketches of the sphere.

$$\Sigma F_y = 0: \quad N\cos\beta - \mu_S N\sin\beta - mg = 0$$

$$N = \frac{mg}{\cos\beta - \mu_S\sin\beta}$$

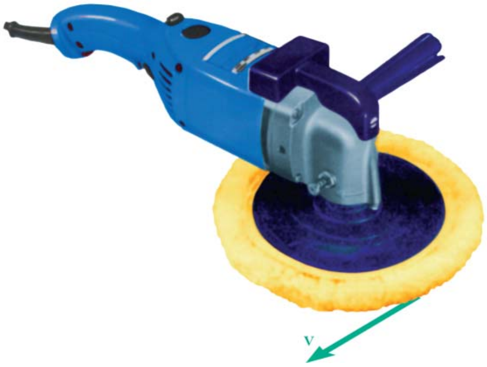


$$\overset{+}{\longleftarrow} \Sigma F_n = ma_n: \quad N\sin\beta + \mu_S N\cos\beta = \frac{mv^2}{\rho}$$

$$\frac{mg(\sin\beta + \mu_S N\cos\beta)}{\cos\beta - \mu_S\sin\beta} = \frac{mv^2}{\rho}$$

$$v^2 = \rho g\frac{\sin\beta + \mu_S\cos\beta}{\cos\beta - \mu_S\sin\beta} = (2)(9.81)\frac{\sin 45° + 0.5\cos 45°}{\cos 45° - 0.5\sin 45°} = 58.86 \text{ m}^2/\text{s}^2$$

$v = 7.67$ m/s ◀



PROBLEM 12.56

A polisher is started so that the fleece along the circumference undergoes a constant tangential acceleration of 4 m/s^2. Three seconds after a polisher is started from rest, small tufts of fleece from along the circumference of the 225-mm-diameter polishing pad are observed to fly free of the pad. At this instant, determine (*a*) the speed *v* of a tuft as it leaves the pad, (*b*) the magnitude of the force required to free a tuft if the average mass of a tuft is 1.6 mg.

SOLUTION

(*a*) $a_t = \text{constant} \Rightarrow$ uniformly acceleration motion

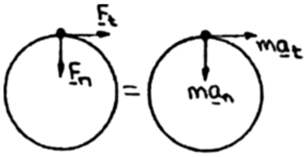
Then $\quad v = 0 + a_t t$

At $t = 3$ s: $\quad v = (4\text{ m/s}^2)(3\text{ s})$

or $\quad v = 12.00\text{ m/s}$ ◀

(*b*) $\Sigma F_t = ma_t: \quad F_t = ma_t$

or
$$F_t = (1.6\times10^{-6}\text{kg})(4\text{ m/s}^2) = 6.4\times10^{-6}\text{N}$$

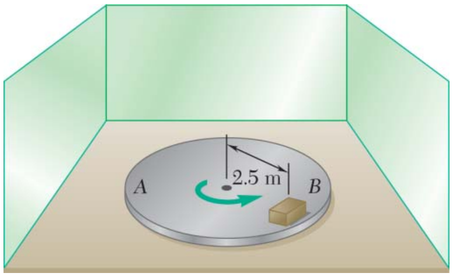


$$\Sigma F_n = ma_n: \quad F_n = m\frac{v^2}{\rho}$$

At $t = 3$ s:
$$F_n = (1.6\times10^{-6}\text{kg})\frac{(12\text{ m/s})^2}{\left(\frac{0.225}{2}\text{m}\right)} = 2.048\times10^{-3}\text{N}$$

Finally,
$$F_{\text{tuft}} = \sqrt{F_t^2 + F_n^2} = \sqrt{(6.4\times10^{-6}\text{N})^2 + (2.048\times10^{-3}\text{N})^2}$$

or $\quad F_{\text{tuft}} = 2.05\times10^{-3}\text{N}$ ◀



PROBLEM 12.57

A turntable A is built into a stage for use in a theatrical production. It is observed during a rehearsal that a trunk B starts to slide on the turntable 10 s after the turntable begins to rotate. Knowing that the trunk undergoes a constant tangential acceleration of 0.24 m/s^2, determine the coefficient of static friction between the trunk and the turntable.

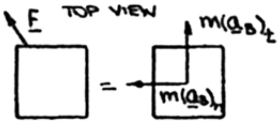
SOLUTION

First we note that $(a_B)_t =$ constant implies uniformly accelerated motion.

$$v_B = 0 + (a_B)_t t$$

At $t = 10$ s: $$v_B = (0.24\ \text{m/s}^2)(10\ \text{s}) = 2.4\ \text{m/s}$$

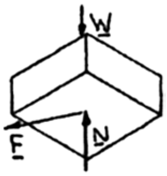
In the plane of the turntable



$$\Sigma\mathbf{F} = m_B\mathbf{a}_B: \quad \mathbf{F} = m_B(\mathbf{a}_B)_t + m_B(\mathbf{a}_B)_n$$

Then $$F = m_B\sqrt{(a_B)_t^2 + (a_B)_n^2}$$

$$= m_B\sqrt{(a_B)_t^2 + \left(\frac{v_B^2}{\rho}\right)^2}$$



$$+\uparrow \Sigma F_y = 0: \quad N - W = 0$$

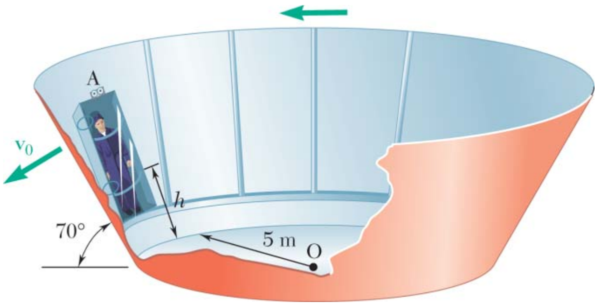
or $$N = m_B g$$

At $t = 10$ s: $$F = \mu_s N = \mu_s m_B g$$

Then $$\mu_s m_B g = m_B\sqrt{(a_B)_t^2 + \left(\frac{v_B^2}{\rho} -\right)^2}$$

or $$\mu_s = \frac{1}{9.81\ \text{m/s}^2}\left\{(0.24\ \text{m/s}^2)^2 + \left[\frac{(2.4\ \text{m/s})^2}{2.5\ \text{m}}\right]^2\right\}^{1/2}$$

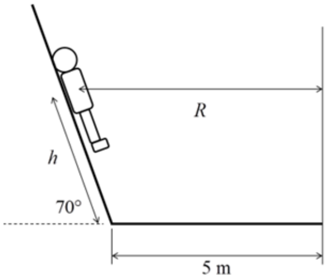
or $$\mu_s = 0.236 \blacktriangleleft$$



PROBLEM 12.58

The carnival ride from Prob 12.51 is modified so that the 80 kg riders can move up and down the inclined wall as the speed of the ride increases. Assuming that the friction between the wall and the carriage is negligible, determine the position h of the rider if the speed v_0 = 13 m/s.

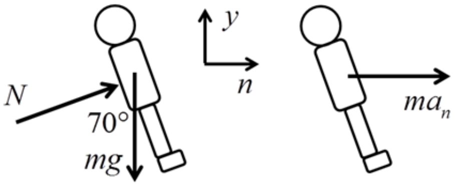
SOLUTION



Given: $m = 80\text{ kg},\ v_0 = 13\text{ m/s}$

Geometry from figure: $R = 5 + h\cos 70°$

Free Body Diagram of Rider:



Equations of Motion:

$$\sum F_y = ma_y$$

$$N\cos 70° - mg = m(0)$$

$$N = \frac{mg}{\cos 70°} \qquad (1)$$

$$\sum F_n = ma_n$$

$$N\sin 70° = \frac{mv^2}{R}$$

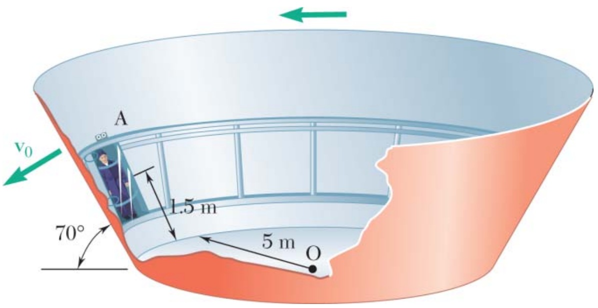
$$v = \sqrt{\frac{N\sin 70° R}{m}} \qquad (2)$$

Substitute (1)into (2):

$$v = \sqrt{\frac{\cancel{m}g\tan 70° R}{\cancel{m}}}$$

$$13 = \sqrt{9.81\tan 70°(5 + h\cos 70°)}$$

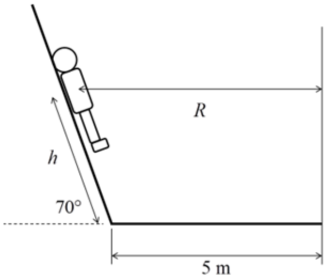
$$h = 3.714\text{ m} ◀$$



PROBLEM 12.59

The carnival ride from Prob 12.51 is modified so that the 80 kg riders can move up and down the inclined wall as the speed of the ride increases. Knowing that the coefficient of static friction between the wall and the platform is 0.2, determine the range of values of the constant speed v_0 for which the platform will remain in the position shown.

SOLUTION



Given: $m = 80 \text{ kg},\ \mu_s = 0.2,\ h = 1.5 \text{ m}$

Geometry from figure:

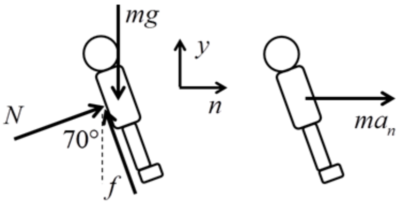
$$R = 5 + 1.5\cos 70^\circ$$
$$= 5.513 \text{ m}$$

Minimum and Maximum speeds occur if slipping is impending:

$$f = \mu_s N$$

Finding the minimum speed so that rider does not slide down the wall:

Free Body Diagram of Rider:



Equations of Motion:

$$\sum F_y = ma_y$$
$$N\cos 70^\circ + f\sin 70^\circ - mg = m(0)$$
$$N\cos 70^\circ + \mu_s \text{N}\sin 70^\circ - mg = 0$$
$$N = \frac{mg}{\cos 70^\circ + \mu_s \sin 70^\circ}$$
$$N = 1480.9 \text{ N}$$

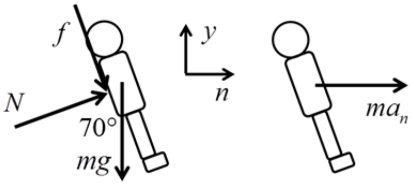
$$\sum F_n = ma_n$$
$$N\sin 70^\circ - f\cos 70^\circ = \frac{mv_{\min}^2}{R}$$
$$N\sin 70^\circ - \mu_s \text{N}\cos 70^\circ = \frac{mv_{\min}^2}{R}$$
$$v_{\min} = \sqrt{\frac{N(\sin 70^\circ - \mu_s\cos 70^\circ)R}{m}}$$

$$v_{\min} = 9.430 \text{ m/s} \blacktriangleleft$$

PROBLEM 12.59 (Continued)

Finding the maximum speed so that rider does not slide up the wall:

Free Body Diagram of Rider:



Equations of Motion:

$$\sum F_y = ma_y$$

$$N\cos 70° - f\sin 70° - mg = m(0)$$

$$N\cos 70° - \mu_s \mathrm{N}\sin 70° - mg = 0$$

$$N = \frac{mg}{\cos 70° - \mu_s \sin 70°}$$

$$N = 5093.4 \text{ N}$$

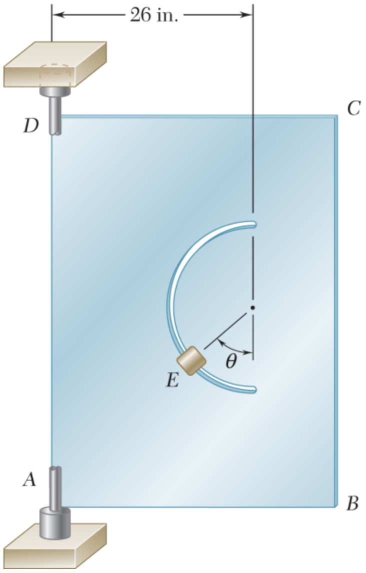
$$\sum F_n = ma_n$$

$$N\sin 70° + f\cos 70° = \frac{mv_{\min}^2}{R}$$

$$N\sin 70° + \mu_s \mathrm{N}\cos 70° = \frac{mv_{\min}^2}{R}$$

$$v_{\min} = \sqrt{\frac{N(\sin 70° + \mu_s \cos 70°)R}{m}}$$

$$v_{\min} = 18.81 \text{ m/s} ◀$$



PROBLEM 12.60

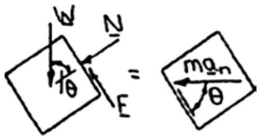
A semicircular slot of 10-in. radius is cut in a flat plate which rotates about the vertical *AD* at a constant rate of 14 rad/s. A small, 0.8-lb block *E* is designed to slide in the slot as the plate rotates. Knowing that the coefficients of friction are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine whether the block will slide in the slot if it is released in the position corresponding to (*a*) $\theta = 80°$, (*b*) $\theta = 40°$. Also determine the magnitude and the direction of the friction force exerted on the block immediately after it is released.

SOLUTION

First note
$$\rho = \frac{1}{12}(26 - 10\sin\theta)\ \text{ft}$$
$$v_E = \rho\dot{\phi}_{ABCD}$$

Then
$$a_n = \frac{v_E^2}{\rho} = \rho(\dot{\phi}_{ABCD})^2$$
$$= \left[\frac{1}{12}(26 - 10\sin\theta)\ \text{ft}\right](14\ \text{rad/s})^2$$
$$= \frac{98}{3}(13 - 5\sin\theta)\ \text{ft/s}^2$$

Assume that the block is at rest with respect to the plate.



$$+\swarrow\ \Sigma F_x = ma_x\text{:}\quad N + W\cos\theta = m\frac{v_E^2}{\rho}\sin\theta$$

or
$$N = W\left(-\cos\theta + \frac{v_E^2}{g\rho}\sin\theta\right)$$

$$+\searrow\ \Sigma F_y = ma_y\text{:}\quad -F + W\sin\theta = -m\frac{v_E^2}{\rho}\cos\theta$$

or
$$F = W\left(\sin\theta + \frac{v_E^2}{g\rho}\cos\theta\right)$$

PROBLEM 12.60 (Continued)

(*a*) We have $\theta = 80°$

Then

$$N = (0.8\text{ lb})\left[-\cos 80° + \frac{1}{32.2\text{ ft/s}^2} \times \frac{98}{3}(13 - 5\sin 80°)\text{ ft/s}^2 \times \sin 80°\right]$$
$$= 6.3159\text{ lb}$$

$$F = (0.8\text{ lb})\left[\sin 80° + \frac{1}{32.2\text{ ft/s}^2} \times \frac{98}{3}(13 - 5\sin 80°)\text{ ft/s}^2 \times \cos 80°\right]$$
$$= 1.92601\text{ lb}$$

Now $F_{\max} = \mu_s N = 0.35(6.3159\text{ lb}) = 2.2106\text{ lb}$

The block does not slide in the slot, and

$\mathbf{F} = 1.926\text{ lb} \searrow 80°$ ◀

(*b*) We have $\theta = 40°$

Then

$$N = (0.8\text{ lb})\left[-\cos 40° + \frac{1}{32.2\text{ ft/s}^2} \times \frac{98}{3}(13 - 5\sin 40°)\text{ ft/s}^2 \times \sin 40°\right]$$
$$= 4.4924\text{ lb}$$

$$F = (0.8\text{ lb})\left[\sin 40° + \frac{1}{32.2\text{ ft/s}^2} \times \frac{98}{3}(13 - 5\sin 40°)\text{ ft/s}^2 \times \cos 40°\right]$$
$$= 6.5984\text{ lb}$$

Now $F_{\max} = \mu_s N,$ from which it follows that

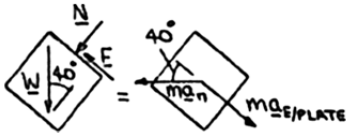
$$F > F_{\max}$$

Block *E* will slide in the slot

and
$$\mathbf{a}_E = \mathbf{a}_n + \mathbf{a}_{E/\text{plate}}$$
$$= \mathbf{a}_n + (\mathbf{a}_{E/\text{plate}})_t + (\mathbf{a}_{E/\text{plate}})_n$$

At $t = 0,$ the block is at rest relative to the plate, thus $(\mathbf{a}_{E/\text{plate}})_n = 0$ at $t = 0,$ so that $\mathbf{a}_{E/\text{plate}}$ must be directed tangentially to the slot.

$$+\nearrow \Sigma F_x = ma_x: \quad N + W\cos 40° = m\frac{v_E^2}{\rho}\sin 40°$$



or
$$N = W\left(-\cos 40° + \frac{v_E^2}{g\rho}\sin 40°\right) \quad \text{(as above)}$$
$$= 4.4924\text{ lb}$$

Sliding:

$$F = \mu_k N$$
$$= 0.25(4.4924 \text{ lb})$$
$$= 1.123 \text{ lb}$$

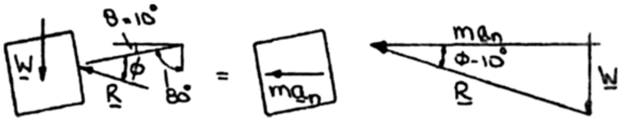
Noting that **F** and $\mathbf{a}_{E/\text{plane}}$ must be directed as shown (if their directions are reversed, then $\Sigma\mathbf{F}_x$ is $\searrow$ while $m\mathbf{a}_x$ is $\nwarrow$), we have the block slides downward in the slot and

$\mathbf{F} = 1.123 \text{ lb} \searrow 40°$ ◀

Alternative solutions.

(*a*) Assume that the block is at rest with respect to the plate.

$$\Sigma\mathbf{F} = m\mathbf{a}: \quad \mathbf{W} + \mathbf{R} = m\mathbf{a}_n$$



Then

$$\tan(\phi - 10°) = \frac{W}{ma_n} = \frac{W}{\frac{W}{g}\frac{v_E^2}{\rho}} = \frac{g}{\rho(\dot{\phi}_{ABCD})^2}$$

$$= \frac{32.2 \text{ ft/s}^2}{\frac{98}{3}(13 - 5\sin 80°) \text{ ft/s}^2} \qquad \text{(from above)}$$

or

$$\phi - 10° = 6.9588°$$

and

$$\phi = 16.9588°$$

Now

$$\tan\phi_s = \mu_s \quad \mu_s = 0.35$$

so that

$$\phi_s = 19.29°$$

$0 < \phi < \phi_s \Rightarrow$ Block does not slide and **R** is directed as shown.

Now

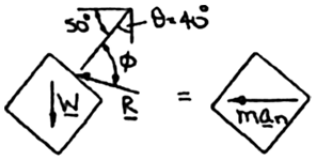
$$F = R\sin\phi \quad \text{and} \quad R = \frac{W}{\sin(\phi - 10°)}$$

Then

$$F = (0.8 \text{ lb})\frac{\sin 16.9588°}{\sin 6.9588°}$$
$$= 1.926 \text{ lb}$$

The block does not slide in the slot and

$\mathbf{F} = 1.926 \text{ lb} \searrow 80°$ ◀



(*b*) Assume that the block is at rest with respect to the plate.

$$\Sigma\mathbf{F} = m\mathbf{a}: \quad \mathbf{W} + \mathbf{R} = m\mathbf{a}_n$$

From Part *a* (above), it then follows that

$$\tan(\phi - 50°) = \frac{g}{\rho(\dot{\phi}_{ABCD})^2} = \frac{32.2 \text{ ft/s}^2}{\frac{98}{3}(13 - 5\sin 40°) \text{ ft/s}^2}$$

or $\phi - 50° = 5.752°$

and $\phi = 55.752°$

Now $\phi_s = 19.29°$

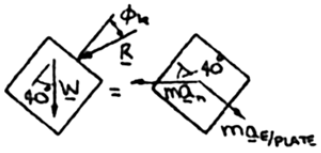
so that $\phi > \phi_s$

The block will slide in the slot and then

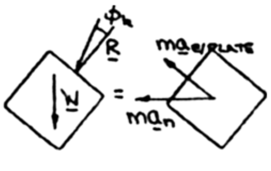
$$\phi = \phi_k, \quad \text{where} \quad \tan\phi_k = \mu_k \quad \mu_k = 0.25$$

or $\phi_k = 14.0362°$

To determine in which direction the block will slide, consider the free-body diagrams for the two possible cases.



SLIDING DOWNWARD



SLIDING UPWARD

Now $\Sigma\mathbf{F} = m\mathbf{a}: \quad \mathbf{W} + \mathbf{R} = m\mathbf{a}_n + m\mathbf{a}_{E/\text{plate}}$

From the diagrams it can be concluded that this equation can be satisfied only if the block is sliding downward. Then

$$+\swarrow \Sigma F_x = ma_x: \quad W\cos 40° + R\cos\phi_k = m\frac{v_E^2}{\rho}\sin 40°$$

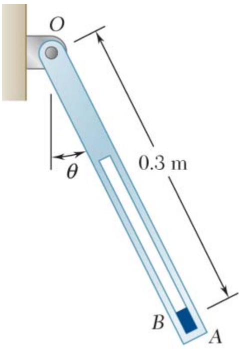
Now $F = R\sin\phi_k$

Then $$W\cos 40° + \frac{F}{\tan\phi_k} = \frac{W}{g}\frac{v_E^2}{\rho}\sin 40°$$

or $$F = \mu_k W\left(-\cos 40° + \frac{v_E^2}{g\rho}\sin 40°\right)$$

$$= 1.123 \text{ lb} \qquad \text{(see the first solution)}$$

The block slides downward in the slot and $\mathbf{F} = 1.123 \text{ lb} \measuredangle 40°$ ◀



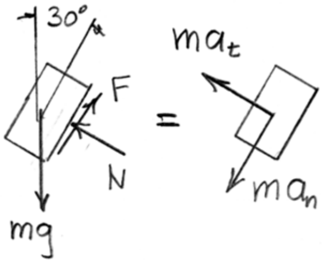
PROBLEM 12.61

A small block *B* fits inside a lot cut in arm *OA* which rotates in a vertical plane at a constant rate. The block remains in contact with the end of the slot closest to *A* and its speed is 1.4 m/s for $0 \le \theta \le 150°$. Knowing that the block begins to slide when $\theta = 150°$, determine the coefficient of static friction between the block and the slot.

SOLUTION

Draw the free body diagrams of the block *B* when the arm is at $\theta = 150°$.

$$\dot{v} = a_t = 0, \quad g = 9.81 \text{ m/s}^2$$



$$+\nwarrow \Sigma F_t = ma_t: \quad -mg\sin 30° + N = 0$$

$$N = mg\sin 30°$$

$$+\swarrow \Sigma F_n = ma_n: \quad mg\cos 30° - F = m\frac{v^2}{\rho}$$

$$F = mg\cos 30° - \frac{mv^2}{\rho}$$

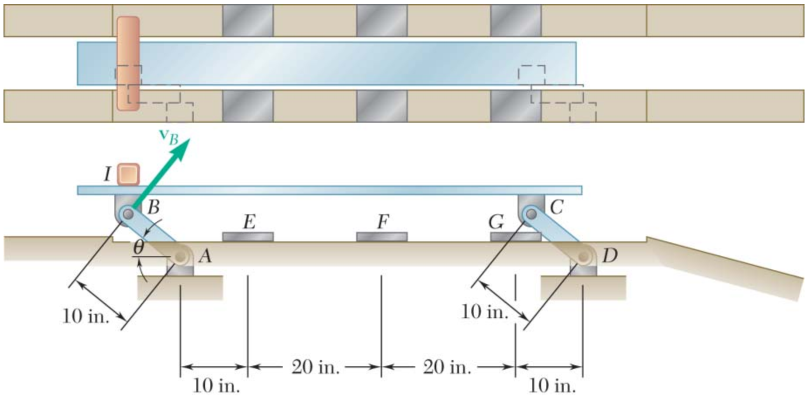
Form the ratio $\frac{F}{N}$, and set it equal to μ_s for impending slip.

$$\mu_s = \frac{F}{N} = \frac{g\cos 30° - v^2/\rho}{g\sin 30°} = \frac{9.81\cos 30° - (1.4)^2/0.3}{9.81\sin 30°}$$

$$\mu_s = 0.400 \blacktriangleleft$$

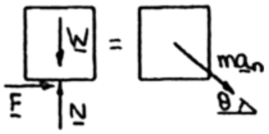
PROBLEM 12.62

The parallel-link mechanism *ABCD* is used to transport a component *I* between manufacturing processes at stations *E*, *F*, and *G* by picking it up at a station when $\theta = 0$ and depositing it at the next station when $\theta = 180°$. Knowing that member *BC* remains horizontal throughout its motion and that links *AB* and *CD* rotate at a constant rate in a vertical plane in such a way that $v_B = 2.2$ ft/s, determine (*a*) the minimum value of the coefficient of static friction between the component and *BC* if the component is not to slide on *BC* while being transferred, (*b*) the values of θ for which sliding is impending.



SOLUTION

$$\xrightarrow{+} \Sigma F_x = ma_x: \quad F = \frac{W}{g}\frac{v_B^2}{\rho}\cos\theta$$



$$+\!\uparrow \Sigma F_y = ma_y: \quad N - W = -\frac{W}{g}\frac{v_B^2}{\rho}\sin\theta$$

or

$$N = W\left(1 - \frac{v_B^2}{g\rho}\sin\theta\right)$$

Now

$$F_{\max} = \mu_s N = \mu_s W\left(1 - \frac{v_B^2}{g\rho}\sin\theta\right)$$

and for the component not to slide

$$F < F_{\max}$$

or

$$\frac{W}{g}\frac{v_B^2}{\rho}\cos\theta < \mu_s W\left(1 - \frac{v_B^2}{g\rho}\sin\theta\right)$$

or

$$\mu_s > \frac{\cos\theta}{\frac{g\rho}{v_B^2} - \sin\theta}$$

We must determine the values of θ which maximize the above expression. Thus

$$\frac{d}{d\theta}\left(\frac{\cos\theta}{\frac{g\rho}{v_B^2}-\sin\theta}\right)=\frac{-\sin\theta\left(\frac{g\rho}{v_B^2}-\sin\theta\right)-(\cos\theta)(-\cos\theta)}{\left(\frac{g\rho}{v_B^2}-\sin\theta\right)^2}=0$$

or $$\sin\theta=\frac{v_B^2}{g\rho}\qquad\text{for}\qquad \mu_s=(\mu_s)_{\min}$$

Now $$\sin\theta=\frac{(2.2\text{ ft/s})^2}{(32.2\text{ ft/s}^2)\left(\frac{10}{12}\text{ ft}\right)}=0.180373$$

or $$\theta=10.3915^\circ\quad\text{and}\quad\theta=169.609^\circ$$

(*a*) From above,

$$(\mu_s)_{\min}=\frac{\cos\theta}{\frac{g\rho}{v_B^2}-\sin\theta}\quad\text{where}\quad\sin\theta=\frac{v_B^2}{g\rho}$$

$$(\mu_s)_{\min}=\frac{\cos\theta}{\frac{1}{\sin\theta}-\sin\theta}=\frac{\cos\theta\sin\theta}{1-\sin^2\theta}=\tan\theta$$

$$=\tan 10.3915^\circ$$

or $$(\mu_s)_{\min}=0.1834 \blacktriangleleft$$

(*b*) We have impending motion

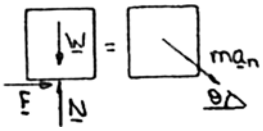
to the left for $\theta=10.39^\circ$ ◀

to the right for $\theta=169.6^\circ$ ◀

PROBLEM 12.63

Knowing that the coefficients of friction between the component *I* and member *BC* of the mechanism of Problem 12.62 are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine (*a*) the maximum allowable constant speed v_B if the component is not to slide on *BC* while being transferred, (*b*) the values of θ for which sliding is impending.

SOLUTION



$$\xrightarrow{+} \Sigma F_x = ma_x: \quad F = \frac{W}{g}\frac{v_B^2}{\rho}\cos\theta$$

$$+\uparrow \Sigma F_y = ma_y: \quad N - W = -\frac{W}{g}\frac{v_B^2}{\rho}\sin\theta$$

or

$$N = W\left(1 - \frac{v_B^2}{g\rho}\sin\theta\right)$$

Now

$$F_{\max} = \mu_s N$$

$$= \mu_s W\left(1 - \frac{v_B^2}{g\rho}\sin\theta\right)$$

and for the component not to slide

$$F < F_{\max}$$

or

$$\frac{W}{g}\frac{v_B^2}{\rho}\cos\theta < \mu_s W\left(1 - \frac{v_B^2}{g\rho}\sin\theta\right)$$

or

$$v_B^2 < \mu_s \frac{g\rho}{\cos\theta + \mu_s \sin\theta} \qquad (1)$$

To ensure that this inequality is satisfied, $\left(v_B^2\right)_{\max}$ must be less than or equal to the minimum value of $\mu_s g\rho/(\cos\theta + \mu_s\sin\theta)$, which occurs when $(\cos\theta + \mu_s\sin\theta)$ is maximum. Thus

$$\frac{d}{d\theta}(\cos\theta + \mu_s\sin\theta) = -\sin\theta + \mu_s\cos\theta = 0$$

or

$$\tan\theta = \mu_s$$

$$\mu_s = 0.35$$

or

$$\theta = 19.2900^\circ$$

PROBLEM 12.63 (Continued)

(*a*) The maximum allowed value of v_B is then

$$\left(v_B^2\right)_{\max} = \mu_s \frac{g\rho}{\cos\theta + \mu_s \sin\theta} \quad \text{where} \quad \tan\theta = \mu_s$$

$$= g\rho \frac{\tan\theta}{\cos\theta + (\tan\theta)\sin\theta} = g\rho\sin\theta$$

$$= (32.2 \text{ ft/s}^2)\left(\frac{10}{12}\text{ ft}\right)\sin 19.2900°$$

or $(v_B)_{\max} = 2.98 \text{ ft/s}$ ◀

(*b*) First note that for $90° < \theta < 180°$, Eq. (1) becomes

$$v_B^2 < \mu_s \frac{g\rho}{\cos\alpha + \mu_s \sin\alpha}$$

where $\alpha = 180° - \theta$. It then follows that the second value of θ for which motion is impending is

$$\theta = 180° - 19.2900°$$
$$= 160.7100°$$

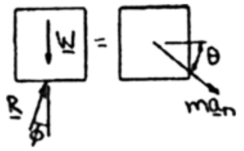
we have impending motion

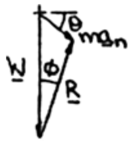
to the left for $\theta = 19.29°$ ◀

to the right for $\theta = 160.7°$ ◀

Alternative solution.

$$\Sigma\mathbf{F} = m\mathbf{a}: \quad \mathbf{W} + \mathbf{R} = m\mathbf{a}_n$$

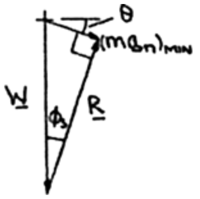




For impending motion, $\phi = \phi_s$. Also, as shown above, the values of θ for which motion is impending minimize the value of v_B, and thus the value of a_n is $\left(a_n = \frac{v_B^2}{\rho}\right)$. From the above diagram, it can be concluded that a_n is minimum when $m\mathbf{a}_n$ and $\mathbf{R}$ are perpendicular.

PROBLEM 12.63 (Continued)

Therefore, from the diagram


θ
$(m\underline{a}_n)_{MIN}$
$\underline{W}$
$\underline{R}$
ϕ_s

$$\theta = \phi_s = \tan^{-1}\mu_s \qquad \text{(as above)}$$

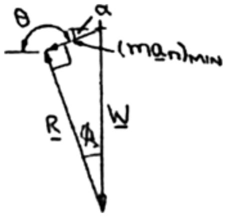
and $$ma_n = W\sin\phi_s$$

or $$m\frac{v_B^2}{\rho} = mg\sin\theta$$

or $$v_B^2 = g\rho\sin\theta \qquad \text{(as above)}$$

For $90° \le \theta \le 180°$, we have

from the diagram

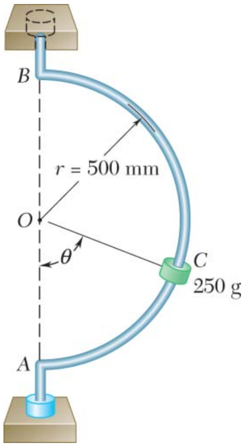

θ
α
$(m\underline{a}_n)_{MIN}$
$\underline{R}$
$\underline{W}$
ϕ_s

$$\alpha = 180° - \theta \qquad \text{(as above)}$$

$$\alpha = \phi_s$$

and $$ma_n = W\sin\phi_s$$

or $$v_B^2 = g\rho\sin\theta \qquad \text{(as above)}$$



PROBLEM 12.64

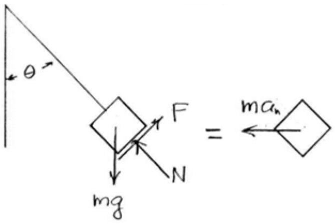
A small 250-g collar C can slide on a semicircular rod which is made to rotate about the vertical AB at a constant rate of 7.5 rad/s. Determine the three values of θ for which the collar will not slide on the rod, assuming no friction between the collar and the rod.

SOLUTION

If the collar is not sliding, it moves at constant speed on a circle of radius $\rho = r\sin\theta$. $v = \rho\omega$

Normal acceleration. $a_n = \dfrac{v^2}{\rho} = \dfrac{\rho^2\omega^2}{\rho} = (r\sin\theta)\omega^2$

$$+\uparrow \Sigma F_y = ma_y:$$



$$N\cos\theta - mg = 0$$

$$N = \frac{mg}{\cos\theta}$$

$$\overset{+}{\leftarrow} \Sigma F_x = ma_x:$$

$$N\sin\theta = ma$$

$$\frac{mg\sin\theta}{\cos\theta} = (m\,r\sin\theta)\omega^2 \quad (1)$$

To satisfy (1), either:

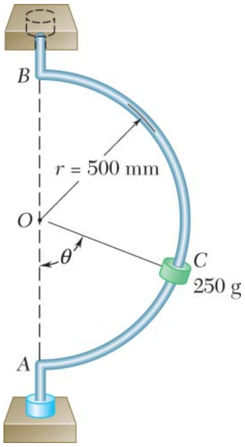
$$\sin\theta = 0$$

or $$\cos\theta = \frac{g}{r\omega^2}$$

$$\theta = 0° \text{ or } 180°$$

or $$\cos\theta = \frac{9.81}{(0.5)(7.5)^2} = 0.3488$$

$\theta = 0°, 180°, \text{ and } 69.6°$ ◀



PROBLEM 12.65

A small 250-g collar C can slide on a semicircular rod which is made to rotate about the vertical AB at a constant rate of 7.5 rad/s. Knowing that the coefficients of friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, indicate whether the collar will slide on the rod if it is released in the position corresponding to (*a*) $\theta = 75°$, (*b*) $\theta = 40°$. Also, determine the magnitude and direction of the friction force exerted on the collar immediately after release.

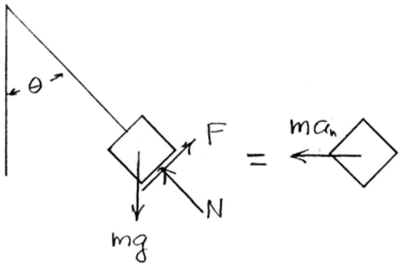
SOLUTION

If the collar is not sliding, it moves at constant speed on a circle of radius $\rho = r\sin\theta$. $v = \rho\omega$

Given: $r = 500\,\text{mm} = 0.500\,\text{m},\ \omega = 7.5\,\text{rad/s},$

$m = 250\,\text{g} = 0.250$ kg.

Normal acceleration: $a_n = \dfrac{v^2}{\rho} = \dfrac{\rho^2\omega^2}{\rho} = (r\sin\theta)\omega^2$



$+\nearrow\ \Sigma F = \nearrow\ ma$:

$$F - mg\sin\theta = -m(r\sin\theta)\,\omega^2\cos\theta$$

$$F = m(g - r\omega^2\cos\theta)\sin\theta$$

$+\nwarrow\ \Sigma F = \nwarrow\ ma$:

$$N - mg\cos\theta = m(r\sin\theta)\omega^2\sin\theta$$

$$N = m(g\cos\theta + r\,\omega^2\sin^2\theta)$$

(*a*) $\theta = 75°$.

$$F = (0.25)\left[9.81 - (0.500\cos 75°)(7.5)^2\right]\sin 75°$$

$$= 0.61112 \text{ N}$$

$$N = (0.25)\left[9.81\cos 75° + (0.500\sin^2 75°)(7.5)^2\right]$$

$$= 7.1950 \text{ N}$$

$$\mu_s N = (0.25)(7.1950) = 1.7987 \text{ N}$$

Since $|F| < \mu_s N$, the collar does not slide.

$F = 0.611\text{ N} \measuredangle 75°$ ◀

PROBLEM 12.65 (Continued)

(*b*) $\theta = 40°$.

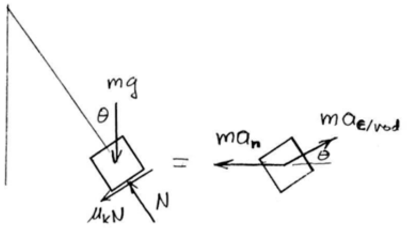
$$F = (0.25)\left[9.81 - (0.500\cos 40°)(7.5)^2\right]\sin 40°$$
$$= -1.8858 \text{ N}$$

$$N = (0.25)\left[9.81\cos 40° + (0.500\sin^2 40°)(7.5)^2\right]$$
$$= 4.7839 \text{ N}$$

$$\mu_s N = (0.25)(4.7839) = 1.1960 \text{ N}$$

Since $|F| > \mu_s N$, the collar slides.

Since the collar is sliding, $|F| = \mu_k N$.



$$+\nwarrow \sum F_n = +\nwarrow ma:$$

$$N - mg\cos\theta = ma_n \sin\theta$$

$$N = mg\cos\theta + m(r\sin\theta)\omega^2\sin\theta$$

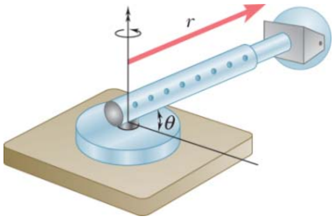
$$= m\left[g\cos\theta + (r\sin^2\theta)\omega^2\right]$$

$$= (0.25)\left[9.81\cos 40° + (0.500\sin^2 40°)(7.5)^2\right]$$

$$= 4.7839 \text{ N}$$

$$|F| = \mu_k N = (0.20)(4.7839) = 0.957 \text{ N}$$

$$F = 0.957 \text{ N} \swarrow 40° \blacktriangleleft$$



PROBLEM 12.66

An advanced spatial disorientation trainer allows the cab to rotate around multiple axes as well as extend inwards and outwards. It can be used to simulate driving, fixed wing aircraft flying, and helicopter maneuvering. In one training scenario, the trainer rotates and translates in the horizontal plane where the location of the pilot is defined by the relationships $r = 10 + 2\cos\left(\frac{\pi}{3}t\right)$ and $\theta = 0.1\left(2t^2 - t\right)$, where r, θ, and t are expressed in feet, radians, and seconds, respectively. Knowing that the pilot has a mass of 175 lbs, (a) find the magnitude of the resulting force acting on the pilot at t= 5 s (b) plot the magnitudes of the radial and transverse components of the force exerted on the pilot from 0 to 10 seconds.

SOLUTION

Given:

$$r = 10 + 2\cos\left(\frac{\pi}{3}t\right) \text{ ft}$$

$$\theta = 0.1\left(2t^2 - t\right) \text{ rad}$$

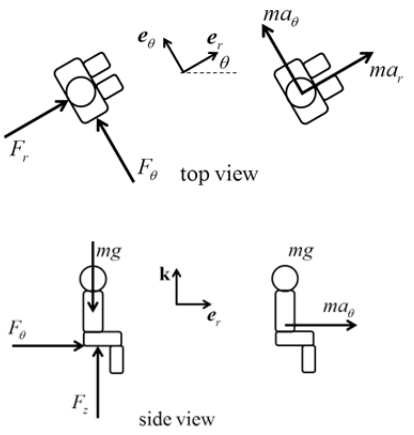
$$m = \frac{175 \text{ lbs}}{32.2 \text{ ft/s}^2} = 5.435 \text{ slugs}$$

Using Radial and Transverse Coordinates:

$$\dot{r} = -\frac{2\pi}{3}\sin\left(\frac{\pi}{3}t\right) \text{ ft/s} \qquad \dot{\theta} = 0.1\left(4t - 1\right) \text{ rad/s}$$

$$\ddot{r} = -\frac{2\pi^2}{9}\cos\left(\frac{\pi}{3}t\right) \text{ ft/s}^2 \qquad \ddot{\theta} = 0.4 \text{ rad/s}^2$$

Free Body Diagram of pilot (top and side view) showing net forces in the radial and transverse directions:



Equations of Motion:

$$\sum F_r = ma_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

$$F_r = m\left[-\frac{2\pi^2}{9}\cos\left(\frac{\pi}{3}t\right) - \left(10 + 2\cos\left(\frac{\pi}{3}t\right)\right)0.1^2\left(4t-1\right)^2\right]$$

$$\sum F_\theta = ma_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

$$F_\theta = m\left[\left(10 + 2\cos\left(\frac{\pi}{3}t\right)\right)(0.4) - \frac{0.4\pi}{3}\sin\left(\frac{\pi}{3}t\right)\left(4t-1\right)\right]$$

$$\sum F_z = 0 \Rightarrow F_z = mg$$

PROBLEM 12.66 (Continued)

(a) Evaluate forces at t=5 s:

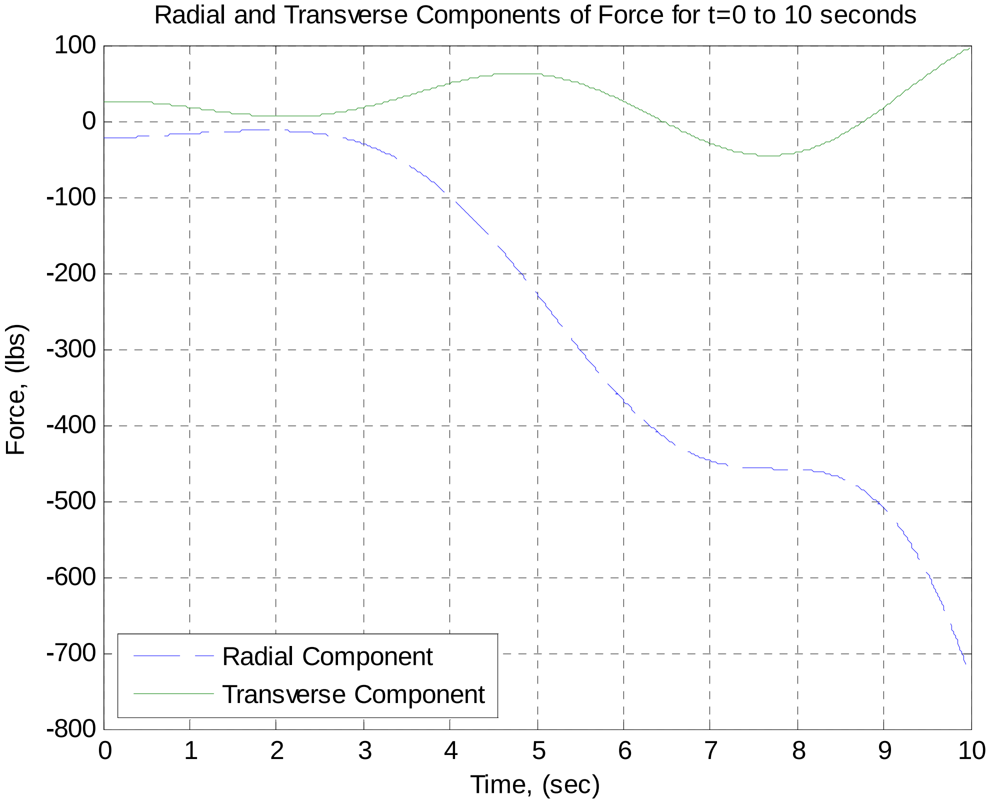
$$F_r = -221.78 \text{ lbs}$$

$$F_\theta = 61.37 \text{ lbs}$$

$$F_z = 175 \text{ lbs}$$

$$|\boldsymbol{F}| = \sqrt{F_r^2 + F_\theta^2 + F_z^2} \qquad |\boldsymbol{F}| = 289.1 \text{ lb} \blacktriangleleft$$

(b) Plot of F_r and F_θ from t=0 to t=10 s:



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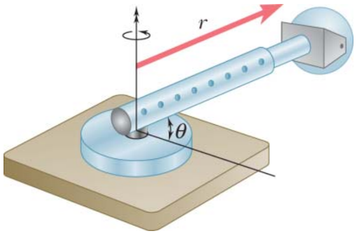


PROBLEM 12.67

An advanced spatial disorientation trainer is programmed to only rotate and translate in the horizontal plane. The pilot's location is defined by the relationships $r = 8\left(1-e^{-t}\right)$ and $\theta = 2/\pi\left(\sin\frac{\pi}{2}t\right)$, where r, θ, and t are expressed in feet, radians, and seconds, respectively. Determine the radial and transverse components of the force exerted on the 175 lb pilot at $t = 3$ s.

SOLUTION

Given:

$$r = 8\left(1-e^{-t}\right)\ \text{ft}$$

$$\theta = 2/\pi\left(\sin\frac{\pi}{2}t\right)\ \text{rad}$$

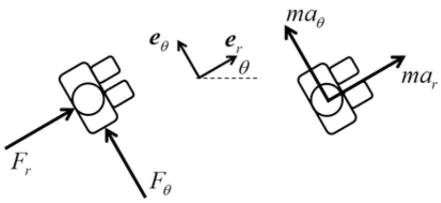
$$m = \frac{175\ \text{lbs}}{32.2\ \text{ft/s}^2} = 5.435\ \text{slugs}$$

Using Radial and Transverse Coordinates:

$$\dot{r} = 8e^{-t}\ \text{ft/s} \qquad \dot{\theta} = \cos\frac{\pi}{2}t\ \text{rad/s}$$

$$\ddot{r} = -8e^{-t}\ \text{ft/s}^2 \qquad \ddot{\theta} = -\frac{\pi}{2}\sin\frac{\pi}{2}\text{t rad/s}^2$$

Free Body Diagram of pilot (top view) showing net forces in the radial and transverse directions:



Equations of Motion:

$$\sum F_r = ma_r$$

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

$$F_r = m\left[-8e^{-t} - 8\left(1-e^{-t}\right)\cos^2\frac{\pi}{2}t\right]$$

$$\sum F_\theta = ma_\theta$$

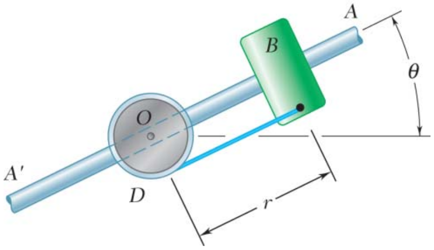
$$F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

$$F_\theta = m\left[8\left(1-e^{-t}\right)\left(-\frac{\pi}{2}\right)\sin\frac{\pi}{2}\text{t} - 16e^{-t}\cos\frac{\pi}{2}t\right]$$

Evaluate F_r and F_θ at t=3 s:

$$F_r = -2.165\ \text{lb} ◀$$

$$F_\theta = 64.90\ \text{lb} ◀$$



PROBLEM 12.68

The 3-kg collar B slides on the frictionless arm AA'. The arm is attached to drum D and rotates about O in a horizontal plane at the rate $\dot{\theta} = 0.75t$, where $\dot{\theta}$ and t are expressed in rad/s and seconds, respectively. As the arm-drum assembly rotates, a mechanism within the drum releases cord so that the collar moves outward from O with a constant speed of 0.5 m/s. Knowing that at $t = 0$, $r = 0$, determine the time at which the tension in the cord is equal to the magnitude of the horizontal force exerted on B by arm AA'.

SOLUTION

Kinematics

We have $$\frac{dr}{dt} = \dot{r} = 0.5 \text{ m/s}$$

At $t = 0, r = 0$: $$\int_0^r dr = \int_0^t 0.5\, dt$$

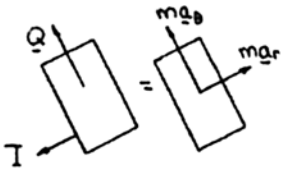
or $$r = (0.5t) \text{ m}$$

Also, $$\ddot{r} = 0 \qquad \dot{\theta} = (0.75t) \text{ rad/s}$$
$$\ddot{\theta} = 0.75 \text{ rad/s}^2$$

Now $$a_r = \ddot{r} - r\dot{\theta}^2 = 0 - [(0.5t) \text{ m}][(0.75t) \text{ rad/s}]^2 = -(0.28125t^3) \text{ m/s}^2$$

and $$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$= [(0.5t) \text{ m}][0.75 \text{ rad/s}^2] + 2(0.5 \text{ m/s})[(0.75t) \text{ rad/s}]$$
$$= (1.125t) \text{ m/s}^2$$

Kinetics



$$+\nearrow \Sigma F_r = ma_r: \quad -T = (3 \text{ kg})(-0.28125t^3) \text{ m/s}^2$$

or $$T = (0.84375t^3) \text{ N}$$

$$+\nwarrow \Sigma F_\theta = m_B a_\theta: \quad Q = (3 \text{ kg})(1.125t) \text{ m/s}^2$$

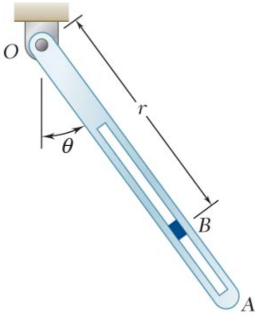
or $$Q = (3.375t) \text{ N}$$

Now require that $$T = Q$$

or $$(0.84375t^3) \text{ N} = (3.375t) \text{ N}$$

or $$t^2 = 4.000$$

or $$t = 2.00 \text{ s} \blacktriangleleft$$



PROBLEM 12.69

A 0.5-kg block B slides without friction inside a slot cut in arm OA which rotates in a vertical plane. The rod has a constant angular acceleration $\ddot{\theta} = 10\text{ rad/s}^2$. Knowing that when $\theta = 45°$ and $r = 0.8$ m the velocity of the block is zero, determine at this instant, (*a*) the force exerted on the block by the arm, (*b*) the relative acceleration of the block with respect to the arm.

SOLUTION

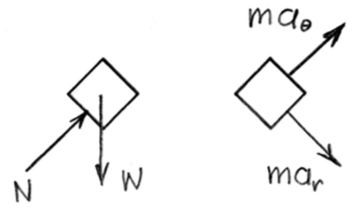
Given: $\theta = 45°, \quad r = 0.8\text{ m}, \quad \ddot{\theta} = 10\text{ rad/s}^2$
$m = 0.5\text{ kg}, \quad W = mg$

Using Radial and Transverse Components:

$$v_r = \dot{r} = 0, \quad v_\theta = r\dot{\theta} = 0$$

$$\dot{\theta} = 0$$

FBD of Block B



(*a*) $$\Sigma F_\theta = ma_\theta: \quad N - W\cos 45° = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

$$N = mg\cos 45° + m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

$$= (0.5)(9.81)\cos 45° + 0.5\left[(0.8)(10) + 0\right]$$

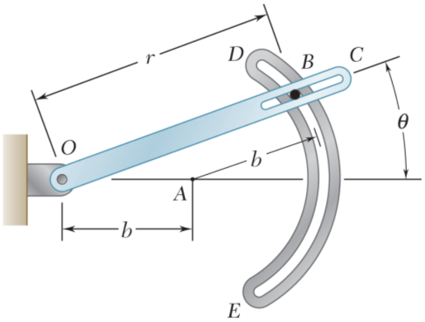
$$= 7.468$$

$N = 7.47\text{ N} \measuredangle 45°$ ◀

(*b*) $$\Sigma F_r = ma_r: \quad mg\sin 45° = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

$$\ddot{r} = \frac{mg}{m}\sin 45° + r\dot{\theta}^2$$
$$= g\sin 45° + r\dot{\theta}^2$$
$$= (9.81)\sin 45° + 0$$
$$= 6.937\text{ m/s}^2$$

$\mathbf{a}_{B/\text{rod}} = 6.94\text{ m/s}^2 \searrow 45°$ ◀



PROBLEM 12.70

Pin B weighs 4 oz and is free to slide in a horizontal plane along the rotating arm OC and along the circular slot DE of radius $b = 20$ in. Neglecting friction and assuming that $\dot{\theta} = 15$ rad/s and $\ddot{\theta} = 250$ rad/s^2 for the position $\theta = 20°$, determine for that position (*a*) the radial and transverse components of the resultant force exerted on pin B, (*b*) the forces **P** and **Q** exerted on pin B, respectively, by rod OC and the wall of slot DE.

SOLUTION

Kinematics.

From the geometry of the system, we have

Then $\quad r = 2b\cos\theta \qquad \dot{r} = -(2b\sin\theta)\dot{\theta} \qquad \ddot{r} = -2b(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$

and $\quad a_r = \ddot{r} - r\dot{\theta}^2 = -2b(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) - (2b\cos\theta)\dot{\theta}^2 = -2b(\ddot{\theta}\sin\theta + 2\dot{\theta}^2\cos\theta)$

Now $\quad = -2\left(\dfrac{20}{12}\text{ ft}\right)[(250\text{ rad/s}^2)\sin 20° + 2(15\text{ rad/s})^2\cos 20°] = -1694.56\text{ ft/s}^2$

and $\quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (2b\cos\theta)\ddot{\theta} + 2(-2b\dot{\theta}\sin\theta)\dot{\theta} = 2b(\ddot{\theta}\cos\theta - 2\dot{\theta}^2\sin\theta)$

$$= 2\left(\frac{20}{12}\text{ ft}\right)[(250\text{ rad/s}^2)\cos 20° - 2(15\text{ rad/s})^2\sin 20°] = 270.05\text{ ft/s}^2$$

Kinetics.

(*a*) We have $\quad F_r = ma_r = \dfrac{\frac{1}{4}\text{ lb}}{32.2\text{ ft/s}^2} \times (-1694.56\text{ ft/s}^2) = -13.1565\text{ lb} \qquad F_r = -13.16\text{ lb}$ ◀

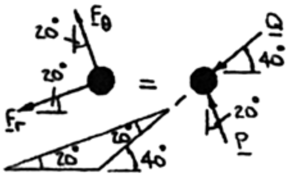
and $\quad F_\theta = ma_\theta = \dfrac{\frac{1}{4}\text{ lb}}{32.2\text{ ft/s}^2} \times (270.05\text{ ft/s}^2) = 2.0967\text{ lb} \qquad F_\theta = 2.10\text{ lb}$ ◀

(*b*) $\quad +\nearrow\ \Sigma F_r\text{:} \quad -F_r = -Q\cos 20°$

or $\quad Q = \dfrac{1}{\cos 20°}(13.1565\text{ lb}) = 14.0009\text{ lb}$

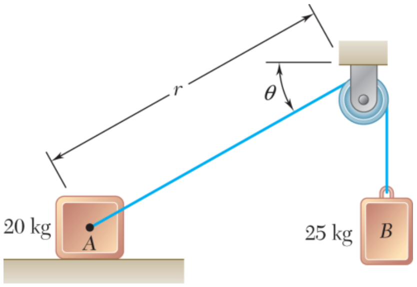
$+\nwarrow\ \Sigma F_\theta\text{:} \quad F_\theta = P - Q\sin 20°$

or $\quad P = (2.0967 + 14.0009\sin 20°)\text{ lb} = 6.89\text{ lb}$



$\mathbf{P} = 6.89\text{ lb}$ $\nwarrow$ 70° ◀

$\mathbf{Q} = 14.00\text{ lb}$ $\swarrow$ 40° ◀



PROBLEM 12.71

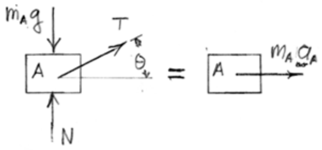
The two blocks are released from rest when $r = 0.8$ m and $\theta = 30°$. Neglecting the mass of the pulley and the effect of friction in the pulley and between block A and the horizontal surface, determine (*a*) the initial tension in the cable, (*b*) the initial acceleration of block A, (*c*) the initial acceleration of block B.

SOLUTION

Let r and θ be polar coordinates of block A as shown, and let y_B be the position coordinate (positive downward, origin at the pulley) for the rectilinear motion of block B.

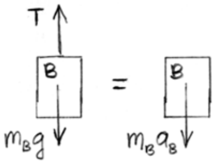
Constraint of cable: $r + y_B = \text{constant}$,

$$\dot{r} + v_B = 0, \quad \ddot{r} + a_B = 0 \quad \text{or} \quad \ddot{r} = -a_B \qquad (1)$$



For block A, $\xrightarrow{+} \Sigma F_x = m_A a_A$: $\quad T\cos\theta = m_A a_A$ or $T = m_A a_A \sec\theta \qquad (2)$

For block B, $+\!\downarrow \Sigma F_y = m_B a_B$: $\quad m_B g - T = m_B a_B \qquad (3)$

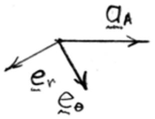


Adding Eq. (1) to Eq. (2) to eliminate T, $\quad m_B g = m_A a_A \sec\theta + m_B a_B \qquad (4)$

Radial and transverse components of $\mathbf{a}_A$.

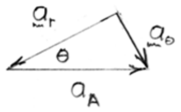
Use either the scalar product of vectors or the triangle construction shown, being careful to note the positive directions of the components.

$$\ddot{r} - r\dot{\theta}^2 = a_r = \mathbf{a}_A \cdot \mathbf{e}_r = -a_A \cos\theta \qquad (5)$$



Noting that initially $\dot{\theta} = 0$, using Eq. (1) to eliminate $\ddot{r}$, and changing signs gives

$$a_B = a_A \cos\theta \qquad (6)$$



Substituting Eq. (6) into Eq. (4) and solving for a_A,

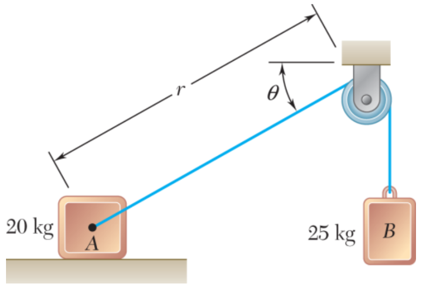
$$a_A = \frac{m_B g}{m_A \sec\theta + m_B \cos\theta} = \frac{(25)(9.81)}{20\sec 30° + 25\cos 30°} = 5.48 \text{ m/s}^2$$

From Eq. (6), $\quad a_B = 5.48\cos 30° = 4.75 \text{ m/s}^2$

(*a*) From Eq. (2), $T = (20)(5.48)\sec 30° = 126.6 \qquad T = 126.6 \text{ N}$ ◀

(*b*) Acceleration of block A. $\qquad \mathbf{a}_A = 5.48 \text{ m/s}^2 \longrightarrow$ ◀

(*c*) Acceleration of block B. $\qquad \mathbf{a}_B = 4.75 \text{ m/s}^2 \downarrow$ ◀



PROBLEM 12.72

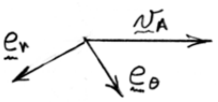
The velocity of block A is 2 m/s to the right at the instant when $r = 0.8$ m and $\theta = 30°$. Neglecting the mass of the pulley and the effect of friction in the pulley and between block A and the horizontal surface, determine, at this instant, (a) the tension in the cable, (b) the acceleration of block A, (c) the acceleration of block B.

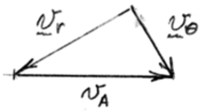
SOLUTION

Let r and θ be polar coordinates of block A as shown, and let y_B be the position coordinate (positive downward, origin at the pulley) for the rectilinear motion of block B.

Radial and transverse components of $\mathbf{v}_A$.

Use either the scalar product of vectors or the triangle construction shown, being careful to note the positive directions of the components.

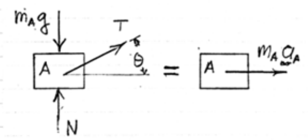


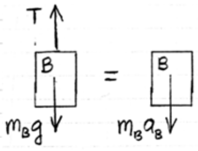


$$\dot{r} = v_r = \mathbf{v}_A \cdot \mathbf{e}_r = -v_A \cos 30°$$
$$= -2\cos 30° = -1.73205 \text{ m/s}$$

$$r\dot{\theta} = v_\theta = \mathbf{v}_A \cdot \mathbf{e}_\theta = -v_A \sin 30°$$
$$= 2\sin 30° = 1.000 \text{ m/s}^2$$

$$\dot{\theta} = \frac{v_\theta}{r} = \frac{1.000}{0.8} = 1.25 \text{ rad/s}$$





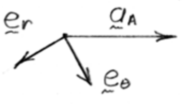
Constraint of cable: $r + y_B = \text{constant}$,

$$\dot{r} + v_B = 0, \quad \ddot{r} + a_B = 0 \quad \text{or} \quad \ddot{r} = -a_B \tag{1}$$

For block A, $\xrightarrow{+} \Sigma F_x = m_A a_A : T\cos\theta = m_A a_A$ or $T = m_A a_A \sec\theta$ (2)

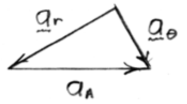
For block B, $+\!\downarrow \Sigma F_y = m_B a_B$: $m_B g - T = m_B a_B$ (3)

Adding Eq. (1) to Eq. (2) to eliminate T, $m_B g = m_A a_A \sec\theta + m_B a_B$ (4)



Radial and transverse components of $\mathbf{a}_A$.

Use a method similar to that used for the components of velocity.



$$\ddot{r} - r\dot{\theta}^2 = a_r = \mathbf{a}_A \cdot \mathbf{e}_r = -a_A \cos\theta \tag{5}$$

Using Eq. (1) to eliminate $\ddot{r}$ and changing signs gives

$$a_B = a_A \cos\theta - r\dot{\theta}^2 \tag{6}$$

Substituting Eq. (6) into Eq. (4) and solving for a_A,

$$a_A = \frac{m_B\left(g + r\dot{\theta}^2\right)}{m_A \sec\theta + m_B \cos\theta} = \frac{(25)[9.81 + (0.8)(1.25)^2]}{20\sec 30^\circ + 25\cos 30^\circ} = 6.18 \text{ m/s}^2$$

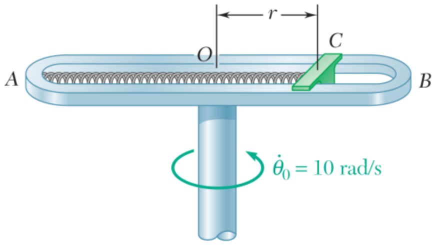
From Eq. (6), $a_B = 6.18\cos 30^\circ - (0.8)(1.25)^2 = 4.10 \text{ m/s}^2$

(*a*) From Eq. (2), $T = (20)(6.18)\sec 30^\circ = 142.7$ $T = 142.7 \text{ N}$ ◀

(*b*) Acceleration of block *A*. $\mathbf{a}_A = 6.18 \text{ m/s}^2 \longrightarrow$ ◀

(*c*) Acceleration of block *B*. $\mathbf{a}_B = 4.10 \text{ m/s}^2 \downarrow$ ◀

PROBLEM 12.73*



Slider *C* has a weight of 0.5 lb and may move in a slot cut in arm *AB*, which rotates at the constant rate $\dot{\theta}_0 = 10\text{ rad/s}$ in a horizontal plane. The slider is attached to a spring of constant *k* = 2.5 lb/ft, which is unstretched when *r* = 0. Knowing that the slider is released from rest with no radial velocity in the position *r* = 18 in. and neglecting friction, determine for the position *r* = 12 in. (*a*) the radial and transverse components of the velocity of the slider, (*b*) the radial and transverse components of its acceleration, (*c*) the horizontal force exerted on the slider by arm *AB*.

SOLUTION

Let l_0 be the radial coordinate when the spring is unstretched. Force exerted by the spring.

$$F_r = -k(r - l_0)$$

$$\Sigma F_r = ma_r: \quad -k(r - l_0) = m(\ddot{r} - r\dot{\theta}^2)$$

$$\ddot{r} = \left(\dot{\theta}^2 - \frac{k}{m}\right)r + \frac{kl_0}{m} \qquad (1)$$

But

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d\dot{r}}{dr}\frac{dr}{dt} = \dot{r}\frac{d\dot{r}}{dr}$$

$$\dot{r}d\dot{r} = \ddot{r}dr = \left[\left(\dot{\theta}^2 - \frac{k}{m}\right)r + \frac{kl_0}{m}\right]dr$$

Integrate using the condition $\dot{r} = \dot{r}_0$ when $r = r_0$.

$$\frac{1}{2}\dot{r}^2\Big|_{\dot{r}_0}^{\dot{r}} = \left[\frac{1}{2}\left(\dot{\theta}^2 - \frac{k}{m}\right)r^2 + \frac{kl_0}{m}r\right]\Bigg|_{r_0}^{r}$$

$$\frac{1}{2}\dot{r}^2 - \frac{1}{2}\dot{r}_0^2 = \frac{1}{2}\left(\dot{\theta}^2 - \frac{k}{m}\right)\left(r^2 - r_0^2\right) + \frac{kl_0}{m}(r - r_0)$$

$$\dot{r}^2 = \dot{r}_0^2 + \left(\dot{\theta}^2 - \frac{k}{m}\right)\left(r^2 - r_0^2\right) + \frac{2kl_0}{m}(r - r_0)$$

Data:

$$m = \frac{W}{g} = \frac{0.5\text{ lb}}{32.2\text{ ft/s}^2} = 0.01553\text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\dot{\theta} = 10\text{ rad/s}, \quad k = 2.5\text{ lb/ft}, \quad l_0 = 0$$

$$\dot{r}_0 = (v_r)_0 = 0, \quad r_0 = 18\text{ in.} = 1.5\text{ ft}, \quad r = 12\text{ in.} = 1.0\text{ ft}$$

PROBLEM 12.73* (Continued)

(*a*) Components of velocity when $r = 12$ in.

$$\dot{r}^2 = 0 + \left(10^2 - \frac{2.5}{0.01553}\right)(1.0^2 - 1.5^2) + 0$$

$$= 76.223\ \text{ft}^2/\text{s}^2$$

$$v_r = \dot{r} = \pm 8.7306\ \text{ft/s}$$

Since r is decreasing, v_r is negative

$$\dot{r} = -8.7306\ \text{ft/s} \qquad v_r = -8.73\ \text{ft/s} \blacktriangleleft$$

$$v_\theta = r\dot{\theta} = (1.0)(10) \qquad v_\theta = 10.00\ \text{ft/s} \blacktriangleleft$$

(*b*) Components of acceleration.

$$F_r = -kr + kl_0 = -(2.5)(1.0) + 0 = -2.5\ \text{lb}$$

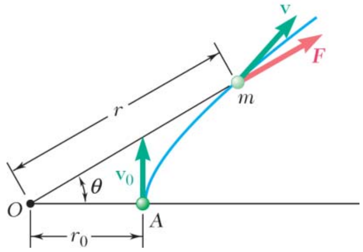
$$a_r = \frac{F_r}{m} = -\frac{2.5}{0.01553} \qquad a_r = 161.0\ \text{ft/s}^2 \blacktriangleleft$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 + (2)(-8.7306)(10)$$

$$a_\theta = -174.6\ \text{ft/s}^2 \blacktriangleleft$$

(*c*) Transverse component of force.

$$F_\theta = ma_\theta = (0.01553)(-174.6) \qquad F_\theta = -2.71\ \text{lb} \blacktriangleleft$$



PROBLEM 12.74

A particle of mass *m* is projected from Point *A* with an initial velocity $\mathbf{v}_0$ perpendicular to line *OA* and moves under a central force **F** directed away from the center of force *O*. Knowing that the particle follows a path defined by the equation $r = r_0/\sqrt{\cos 2\theta}$ and using Eq. (12.25), express the radial and transverse components of the velocity **v** of the particle as functions of θ.

SOLUTION

Since the particle moves under a central force, $h = \text{constant}$.

Using Eq. (12.27), $h = r^2\dot{\theta} = h_0 = r_0 v_0$

or
$$\dot{\theta} = \frac{r_0 v_0}{r^2} = \frac{r_0 v_0 \cos 2\theta}{r_0^{\,2}} = \frac{v_0}{r_0}\cos 2\theta$$

Radial component of velocity.

$$v_r = \dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{d}{d\theta}\left(\frac{r_0}{\sqrt{\cos 2\theta}}\right)\dot{\theta} = r_0 \frac{\sin 2\theta}{(\cos 2\theta)^{3/2}}\dot{\theta}$$

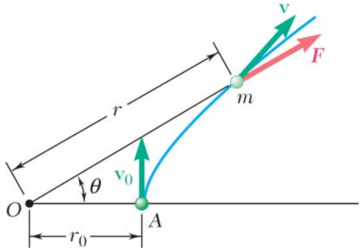
$$= r_0 \frac{\sin 2\theta}{(\cos 2\theta)^{3/2}} \frac{v_0}{r}\cos 2\theta$$

$$v_r = v_0 \frac{\sin 2\theta}{\sqrt{\cos 2\theta}} \blacktriangleleft$$

Transverse component of velocity.

$$v_\theta = \frac{h}{r} = \frac{r_0 v_0}{r_0}\sqrt{\cos 2\theta}$$

$$v_\theta = v_0\sqrt{\cos 2\theta} \blacktriangleleft$$



PROBLEM 12.75

For the particle of Problem 12.74, show (*a*) that the velocity of the particle and the central force **F** are proportional to the distance *r* from the particle to the center of force *O*, (*b*) that the radius of curvature of the path is proportional to r^3.

PROBLEM 12.74 A particle of mass *m* is projected from Point *A* with an initial velocity $\mathbf{v}_0$ perpendicular to line *OA* and moves under a central force **F** directed away from the center of force *O*. Knowing that the particle follows a path defined by the equation $r = r_0/\sqrt{\cos 2\theta}$ and using Eq. (12.25), express the radial and transverse components of the velocity **v** of the particle as functions of *θ*.

SOLUTION

Since the particle moves under a central force, $h =$ constant.

Using Eq. (12.27),

$$h = r^2\dot{\theta} = h_0 = r_0 v_0 \quad \text{or} \quad \dot{\theta} = \frac{r_0 v_0}{r^2} = \frac{r_0 v_0 \cos 2\theta}{r_0^{\,2}} = \frac{v_0}{r_0}\cos 2\theta$$

Differentiating the expression for *r* with respect to time,

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{d}{d\theta}\left(\frac{r_0}{\sqrt{\cos 2\theta}}\right)\dot{\theta} = r_0\frac{\sin 2\theta}{(\cos 2\theta)^{3/2}}\dot{\theta} = r_0\frac{\sin 2\theta}{(\cos 2\theta)^{3/2}}\frac{v_0}{r_0}\cos 2\theta = v_0\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$$

Differentiating again,

$$\ddot{r} = \frac{d\dot{r}}{d\theta}\dot{\theta} = \frac{d}{d\theta}\left(v_0\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}\right)\dot{\theta} = v_0\frac{2\cos^2 2\theta + \sin^2 2\theta}{(\cos 2\theta)^{3/2}}\dot{\theta} = \frac{v_0^{\,2}}{r_0}\frac{2\cos^2 2\theta + \sin^2 2\theta}{\sqrt{\cos 2\theta}}$$

(*a*) $\quad v_r = \dot{r} = v_0\dfrac{\sin 2\theta}{\sqrt{\cos 2\theta}} = \dfrac{v_0 r}{r_0}\sin 2\theta \qquad v_\theta = r\dot{\theta} = \dfrac{v_0 r}{r_0}\cos 2\theta$

$$v = \sqrt{(v_r)^2 + (v_\theta)^2} = \frac{v_0 r}{r_0}\sqrt{\sin^2 2\theta + \cos^2 2\theta}$$

$v = \dfrac{v_0 r}{r_0}$ ◀

$$a_r = \ddot{r} - r\dot{\theta}^2 = \frac{v_0^{\,2}}{r_0}\frac{2\cos^2 2\theta + \sin^2 2\theta}{\sqrt{\cos 2\theta}} - \frac{r_0}{\sqrt{\cos 2\theta}}\frac{v_0^{\,2}}{r_0^2}\cos^2 2\theta$$

$$= \frac{v_0^{\,2}}{r_0}\frac{\cos^2 2\theta + \sin^2 2\theta}{\sqrt{\cos 2\theta}} = \frac{v_0}{r_0\sqrt{\cos 2\theta}} = \frac{v_0^{\,2} r}{r_0^{\,2}}$$

$$F_r = ma_r = \frac{mv_0^2 r}{r_0^2}:$$

$F_r = \dfrac{mv_0^2 r}{r_0^2}$ ◀

PROBLEM 12.75 (Continued)

Since the particle moves under a central force, $a_\theta = 0.$

Magnitude of acceleration.

$$a = \sqrt{a_r{}^2 + a_\theta{}^2} = \frac{v_0{}^2 r}{r_0{}^2}$$

Tangential component of acceleration.

$$a_t = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{v_0 r}{r_0}\right) = \frac{v_0}{r_0}\dot{r} = \frac{v_0{}^2 r}{r_0{}^2}\sin 2\theta$$

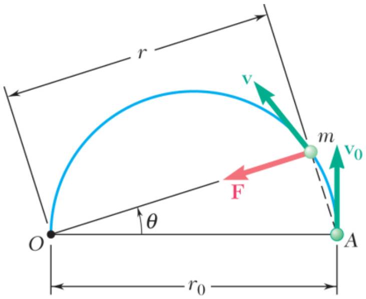
Normal component of acceleration.

$$a_t = \sqrt{a^2 - a_t{}^2} = \frac{v_0{}^2 r}{r_0{}^2}\sqrt{1-\sin^2 2\theta} = \frac{v_0{}^2 r \cos 2\theta}{r_0{}^2}$$

But $\qquad \cos 2\theta = \left(\frac{r_0}{r}\right)^2$

Hence, $\qquad a_n = \frac{v_0{}^2}{r}$

(*b*) But $a_n = \dfrac{v^2}{\rho}$ or $\rho = \dfrac{v^2}{a_n} = \dfrac{v_0{}^2 r^2}{r_0{}^2}\cdot\dfrac{r}{v_0{}^2}$ $\qquad \rho = \dfrac{r^3}{r_0^2}$ ◀



PROBLEM 12.76

A particle of mass *m* is projected from Point *A* with an initial velocity $\mathbf{v}_0$ perpendicular to line *OA* and moves under a central force **F** along a semicircular path of diameter *OA*. Observing that $r = r_0 \cos\theta$ and using Eq. (12.25), show that the speed of the particle is $v = v_0/\cos^2\theta$.

SOLUTION

Since the particle moves under a central force, $h =$ constant.

Using Eq. (12.27), $$h = r^2\dot{\theta} = h_0 = r_0 v_0$$

or $$\dot{\theta} = \frac{r_0 v_0}{r^2} = \frac{r_0 v_0}{r_0^2 \cos^2\theta} = \frac{v_0}{r_0 \cos^2\theta}$$

Radial component of velocity.

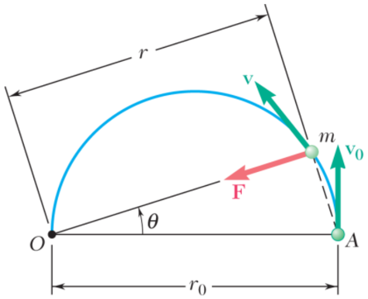
$$v_r = \dot{r} = \frac{d}{dt}(r_0 \cos\theta) = -(r_0 \sin\theta)\dot{\theta}$$

Transverse component of velocity.

$$v_\theta = r\dot{\theta} = (r_0 \cos\theta)\dot{\theta}$$

Speed. $$v = \sqrt{v_r^2 + v_\theta^2} = r_0\dot{\theta} = \frac{r_0 v_0}{r_0 \cos^2\theta}$$

$$v = \frac{v_0}{\cos^2\theta} \blacktriangleleft$$



PROBLEM 12.77

For the particle of Problem 12.76, determine the tangential component F_t of the central force **F** along the tangent to the path of the particle for (*a*) $\theta = 0$, (*b*) $\theta = 45°$.

PROBLEM 12.76 A particle of mass *m* is projected from Point *A* with an initial velocity $\mathbf{v}_0$ perpendicular to line *OA* and moves under a central force **F** along a semicircular path of diameter *OA*. Observing that $r = r_0 \cos\theta$ and using Eq. (12.25), show that the speed of the particle is $v = v_0/\cos^2\theta$.

SOLUTION

Since the particle moves under a central force, $h = \text{constant}$

Using Eq. (12.27),

$$h = r^2\dot{\theta} = h_0 = r_0 v_0$$

$$\dot{\theta} = \frac{r_0 v_0}{r^2} = \frac{r_0 v_0}{r_0^2 \cos^2\theta} = \frac{v_0}{r_0 \cos^2\theta}$$

Radial component of velocity.

$$v_r = \dot{r} = \frac{d}{dt}(r_0\cos\theta) = -(r_0\sin\theta)\dot{\theta}$$

Transverse component of velocity.

$$v_\theta = r\dot{\theta} = (r_0\cos\theta)\dot{\theta}$$

Speed.

$$v = \sqrt{v_r^2 + v_\theta^2} = r_0\dot{\theta} = \frac{r_0 v_0}{r_0\cos^2\theta} = \frac{v_0}{\cos^2\theta}$$

Tangential component of acceleration.

$$a_t = \frac{dv}{dt} = v_0\frac{(-2)(-\sin\theta)\dot{\theta}}{\cos^3\theta} = \frac{2v_0\sin\theta}{\cos^3\theta}\cdot\frac{v_0}{r_0\cos^2\theta}$$

$$= \frac{2v_0^2\sin\theta}{r_0\cos^5\theta}$$

Tangential component of force.

$$F_t = ma_t: \quad F_t = \frac{2mv_0^2\sin\theta}{r_0\cos^5\theta}$$

(*a*) $\theta = 0, \quad F_t = 0$ $\qquad F_t = 0$ ◀

(*b*) $\theta = 45°, \quad F_t = \frac{2mv_0\sin 45°}{\cos^5 45°}$ $\qquad F_t = \frac{8mv_0^2}{r_0}$ ◀

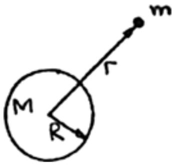
PROBLEM 12.78

Determine the mass of the earth knowing that the mean radius of the moon's orbit about the earth is 238,910 mi and that the moon requires 27.32 days to complete one full revolution about the earth.

SOLUTION

We have $F = G\dfrac{Mm}{r^2}$ [Eq. (12.28)]

and $F = F_n = ma_n = m\dfrac{v^2}{r}$



Then $G\dfrac{Mm}{r^2} = m\dfrac{v^2}{r}$

or $M = \dfrac{r}{G}v^2$

Now $v = \dfrac{2\pi r}{\tau}$

so that $M = \dfrac{r}{G}\left(\dfrac{2\pi r}{\tau}\right)^2 = \dfrac{1}{G}\left(\dfrac{2\pi}{\tau}\right)^2 r^3$

Noting that $\tau = 27.32\,\text{days} = 2.3604 \times 10^6\,\text{s}$

and $r = 238{,}910\text{ mi} = 1.26144 \times 10^9\,\text{ft}$

we have $M = \dfrac{1}{34.4\times 10^{-9}\,\text{ft}^4/\text{lb}\cdot\text{s}^4}\left(\dfrac{2\pi}{2.3604\times 10^6\,\text{s}}\right)^2 (1.26144 \times 10^9\,\text{ft})^3$

or $M = 413 \times 10^{21}\,\text{lb}\cdot\text{s}^2/\text{ft}$ ◀

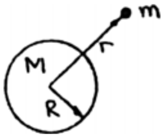
PROBLEM 12.79

Show that the radius r of the moon's orbit can be determined from the radius R of the earth, the acceleration of gravity g at the surface of the earth, and the time τ required for the moon to complete one full revolution about the earth. Compute r knowing that $\tau = 27.3$ days, giving the answer in both SI and U.S. customary units.

SOLUTION

We have $$F = G\frac{Mm}{r^2} \qquad \text{[Eq. (12.28)]}$$

and $$F = F_n = ma_n = m\frac{v^2}{r}$$



Then $$G\frac{Mm}{r^2} = m\frac{v^2}{r}$$

or $$v^2 = \frac{GM}{r}$$

Now $$GM = gR^2 \qquad \text{[Eq. (12.30)]}$$

so that $$v^2 = \frac{gR^2}{r} \quad \text{or} \quad v = R\sqrt{\frac{g}{r}}$$

For one orbit, $$\tau = \frac{2\pi r}{v} = \frac{2\pi r}{R\sqrt{\frac{g}{r}}}$$

or $$r = \left(\frac{g\tau^2R^2}{4\pi^2}\right)^{1/3} \qquad \text{Q.E.D.} \quad ◀$$

Now $$\tau = 27.3 \text{ days} = 2.35872\times10^6 \text{ s}$$
$$R = 3960 \text{ mi} = 20.9088\times10^6 \text{ ft}$$

SI: $$r = \left[\frac{9.81 \text{ m/s}^2\times(2.35872\times10^6 \text{ s})^2\times(6.37\times10^6 \text{ m})^2}{4\pi^2}\right]^{1/3} = 382.81\times10^6 \text{ m}$$

or $$r = 383\times10^3 \text{ km} \quad ◀$$

U.S. customary units:

$$r = \left[\frac{32.2 \text{ ft/s}^2\times(2.35872\times10^6 \text{ s})^2\times(20.9088\times10^6 \text{ ft})^2}{4\pi^2}\right]^{1/3} = 1256.52\times10^6 \text{ ft}$$

or $$r = 238\times10^3 \text{ mi} \quad ◀$$

PROBLEM 12.80

Communication satellites are placed in a geosynchronous orbit, i.e., in a circular orbit such that they complete one full revolution about the earth in one sidereal day (23.934 h), and thus appear stationary with respect to the ground. Determine (*a*) the altitude of these satellites above the surface of the earth, (*b*) the velocity with which they describe their orbit. Give the answers in both SI and U.S. customary units.

SOLUTION

For gravitational force and a circular orbit,

$$|F_r| = \frac{GMm}{r^2} = \frac{mv^2}{r} \qquad \text{or} \qquad v = \sqrt{\frac{GM}{r}}$$

Let τ be the period time to complete one orbit.

But $$v\tau = 2\pi r \qquad \text{or} \qquad v^2\tau^2 = \frac{GM\tau^2}{r} = 4\pi^2 r^2$$

Then $$r^3 = \frac{GM\tau^2}{4\pi^2} \qquad \text{or} \qquad r = \left(\frac{GM\tau^2}{4\pi^2}\right)^{1/3}$$

Data: $$\tau = 23.934\text{ h} = 86.1624\times 10^3\text{ s}$$

(*a*) In SI units: $$g = 9.81\text{ m/s}^2,\ R = 6.37\times 10^6\text{ m}$$

$$GM = gR^2 = (9.81)(6.37\times 10^6)^2 = 398.06\times 10^{12}\text{ m}^3/\text{s}^2$$

$$r = \left[\frac{(398.06\times 10^{12})(86.1624\times 10^3)^2}{4\pi^2}\right]^{1/3} = 42.145\times 10^6\text{ m}$$

altitude $h = r - R = 35.775\times 10^6$ m $\qquad h = 35{,}800$ km ◀

In U.S. units: $$g = 32.2\text{ ft/s}^2,\ R = 3960\text{ mi} = 20.909\times 10^6\text{ ft}$$

$$GM = gR^2 = (32.2)(20.909\times 10^6)^2 = 14.077\times 10^{15}\text{ ft}^3/\text{s}^2$$

$$r = \left[\frac{(14.077\times 10^{15})(86.1624\times 10^3)^2}{4\pi^2}\right]^{1/3} = 138.334\times 10^6\text{ ft}$$

altitude $h = r - R = 117.425\times 10^6$ ft $\qquad h = 22{,}200$ mi ◀

PROBLEM 12.80 (Continued)

(*b*) In SI units:

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{398.06 \times 10^{12}}{42.145 \times 10^{6}}} = 3.07 \times 10^{3} \text{ m/s} \qquad v = 3.07 \text{ km/s} \blacktriangleleft$$

In U.S. units:

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{14.077 \times 10^{15}}{138.334 \times 10^{6}}} = 10.09 \times 10^{3} \text{ ft/s} \qquad v = 10.09 \times 10^{3} \text{ ft/s} \blacktriangleleft$$

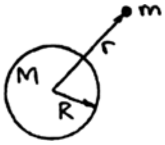
PROBLEM 12.81

Show that the radius r of the orbit of a moon of a given planet can be determined from the radius R of the planet, the acceleration of gravity at the surface of the planet, and the time τ required by the moon to complete one full revolution about the planet. Determine the acceleration of gravity at the surface of the planet Jupiter knowing that $R = 71{,}492$ km and that $\tau = 3.551$ days and $r = 670.9 \times 10^3$ km for its moon Europa.

SOLUTION

We have $$F = G\frac{Mm}{r^2} \qquad \text{[Eq. (12.28)]}$$

and $$F = F_n = ma_n = m\frac{v^2}{r}$$



Then $$G\frac{Mm}{r^2} = m\frac{v^2}{r}$$

or $$v^2 = \frac{GM}{r}$$

Now $$GM = gR^2 \qquad \text{[Eq. (12.30)]}$$

so that $$v^2 = \frac{gR^2}{r} \qquad \text{or} \qquad v = R\sqrt{\frac{g}{r}}$$

For one orbit, $$\tau = \frac{2\pi r}{v} = \frac{2\pi r}{R\sqrt{\frac{g}{r}}}$$

or $$r = \left(\frac{g\tau^2R^2}{4\pi^2}\right)^{1/3} \qquad \text{Q.E.D.} \quad ◀$$

Solving for g, $$g = 4\pi^2\frac{r^3}{\tau^2R^2}$$

and noting that $\tau = 3.551\,\text{days} = 306{,}806$ s, then

$$g_{\text{Jupiter}} = 4\pi^2\frac{r_{\text{Eur}}^3}{\tau_{\text{Eur}}^2 R_{\text{Jup}}} = 4\pi^2\frac{(670.9\times10^6\,\text{m})^3}{(306{,}806\,\text{s})^2(71.492\times10^6\,\text{m})^2}$$

or $$g_{\text{Jupiter}} = 24.8\ \text{m/s}^2 \quad ◀$$

Note: $g_{\text{Jupiter}} \approx 2.53 g_{\text{Earth}}$

PROBLEM 12.82

The orbit of the planet Venus is nearly circular with an orbital velocity of 126.5×10^3 km/h. Knowing that the mean distance from the center of the sun to the center of Venus is 108×10^6 km and that the radius of the sun is 695×10^3 km, determine (*a*) the mass of the sun, (*b*) the acceleration of gravity at the surface of the sun.

SOLUTION

Let M be the mass of the sun and m the mass of Venus.

For the circular orbit of Venus,

$$\frac{GMm}{r^2} = ma_n = \frac{mv^2}{r} \qquad GM = rv^2$$

where r is radius of the orbit.

Data: $$r = 108 \times 10^6 \text{ km} = 108 \times 10^9 \text{ m}$$

$$v = 126.5 \times 10^3 \text{ km/hr} = 35.139 \times 10^3 \text{ m/s}$$

$$GM = (108 \times 10^9)(35.139 \times 10^3)^2 = 1.3335 \times 10^{20} \text{ m}^3/\text{s}^2$$

(*a*) Mass of sun. $$M = \frac{GM}{G} = \frac{1.3335 \times 10^{20} \text{ m}^3/\text{s}^2}{66.73 \times 10^{-12}} \qquad M = 1.998 \times 10^{30} \text{ kg} \blacktriangleleft$$

(*b*) At the surface of the sun, $$R = 695.5 \times 10^3 \text{ km} = 695.5 \times 10^6 \text{ m}$$

$$\frac{GMm}{R^2} = mg$$

$$g = \frac{GM}{R^2} = \frac{1.3335 \times 10^{20}}{(695.5 \times 10^6)^2} \qquad g = 276 \text{ m/s}^2 \blacktriangleleft$$

PROBLEM 12.83

A satellite is placed into a circular orbit about the planet Saturn at an altitude of 2100 mi. The satellite describes its orbit with a velocity of 54.7×10^3 mi/h. Knowing that the radius of the orbit about Saturn and the periodic time of Atlas, one of Saturn's moons, are 85.54×10^3 mi and 0.6017 days, respectively, determine (*a*) the radius of Saturn, (*b*) the mass of Saturn. (The *periodic time* of a satellite is the time it requires to complete one full revolution about the planet.)

SOLUTION

Velocity of Atlas.

$$v_A = \frac{2\pi r_A}{\tau_A}$$

where

$$v_A = 85.54\times10^3 \text{ mi} = 451.651\times10^6 \text{ ft}$$

and

$$\tau_A = 0.6017 \text{ days} = 51{,}987 \text{ s}$$

$$v_A = \frac{(2\pi)(451.651\times10^6)}{51{,}987} = 54.587\times10^3 \text{ ft/s}$$

Gravitational force.

$$F = \frac{GMm}{r^2} = \frac{mv^2}{r}$$

from which

$$GM = rv^2 = \text{constant}$$

For the satellite,

$$r_s v_s^2 = r_A v_A^2$$

$$r_s = \frac{r_A v_A^2}{v_s{}^2}$$

where

$$v_s = 54.7\times10^3 \text{mi/h} = 80.227\times10^3 \text{ ft/s}$$

$$r_s = \frac{(451.651\times10^6)(54.587\times10^3)^2}{(80.227\times10^3)^2} = 209.09\times10^6 \text{ ft}$$

$$r_s = 39{,}600 \text{ mi}$$

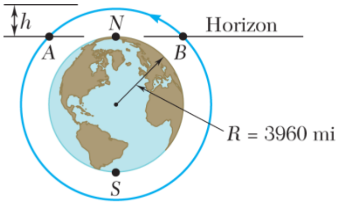
(*a*) Radius of Saturn.

$$R = r_s - (\text{altitude}) = 39{,}600 - 2100 \qquad R = 37{,}500 \text{ mi} \blacktriangleleft$$

(*b*) Mass of Saturn.

$$M = \frac{r_A v_A^2}{G} = \frac{(451.651\times10^6)(54.587\times10^3)^2}{34.4\times10^{-9}}$$

$$M = 39.1\times10^{24} \text{ lb}\cdot\text{s}^2/\text{ft} \blacktriangleleft$$



PROBLEM 12.84

The periodic time (see Prob. *12.83*) of an earth satellite in a circular polar orbit is 120 minutes. Determine (*a*) the altitude *h* of the satellite, (*b*) the time during which the satellite is above the horizon for an observer located at the north pole.

SOLUTION

For gravitational force and a circular orbit,

$$|F_r| = \frac{GMm}{r^2} = \frac{mv^2}{r} \qquad \text{or} \qquad v = \sqrt{\frac{GM}{r}}$$

Let τ be the periodic time to complete one orbit.

$$v\tau = 2\pi r \qquad \text{or} \qquad \tau\sqrt{\frac{GM}{r}} = 2\pi r$$

Solving for r, $\quad r = \left(\dfrac{GM\tau^2}{4\pi^2}\right)^{1/3}$

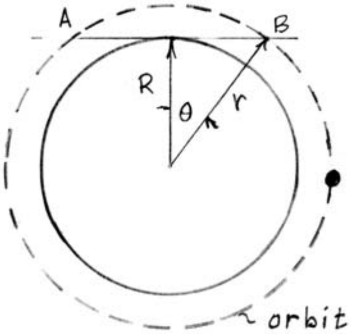
For earth, $R = 3960 \text{ mi} = 20.909 \times 10^6 \text{ ft}, \; g = 32.2 \text{ ft/s}^2$

$$GM = gR^2 = (32.2)\left(20.909 \times 10^6\right)^2 = 14.077 \times 10^{15} \text{ ft}^3/\text{s}^2$$

Data: $\tau = 120 \text{ min} = 7200 \text{ s}$

$$r = \left[\frac{\left(14.077 \times 10^{15}\right)(7200)^2}{4\pi^2}\right]^{1/3} = 26.441 \times 10^6 \text{ ft}$$

(*a*) altitude $h = r - R = 5.532 \times 10^6$ ft $\qquad h = 1048$ mi ◀



(*b*) $\cos\theta = \dfrac{R}{r} = \dfrac{20.909 \times 10^6}{26.441 \times 10^6} = 0.79078$

$$\theta = 37.74°$$

$$t_{AB} = \frac{2\theta}{360°}\tau = \frac{(75.48)(7200)}{360} = 1509.6 \text{ s}$$

$t_{AB} = 25.2$ min ◀

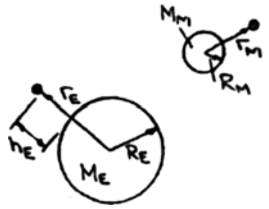
PROBLEM 12.85

A 500 kg spacecraft first is placed into a circular orbit about the earth at an altitude of 4500 km and then is transferred to a circular orbit about the moon. Knowing that the mass of the moon is 0.01230 times the mass of the earth and that the radius of the moon is 1737 km, determine (*a*) the gravitational force exerted on the spacecraft as it was orbiting the earth, (*b*) the required radius of the orbit of the spacecraft about the moon if the periodic times (see Problem 12.83) of the two orbits are to be equal, (*c*) the acceleration of gravity at the surface of the moon.

SOLUTION

First note that $R_E = 6.37\times 10^6 \text{ m}$

Then
$$r_E = R_E + h_E = (6.37\times 10^6 + 4.5\times 10^6)\text{ m}$$
$$= 10.87\times 10^6 \text{ m}$$



(*a*) We have $F = \dfrac{GMm}{r^2}$ [Eq. (12.28)]

and $GM = gR^2$ [Eq. (12.29)]

Then
$$F = gR^2\frac{m}{r^2} = W\left(\frac{R}{r}\right)^2$$

For the earth orbit,
$$F = (500\text{ kg})(9.81\text{ m/s}^2)\left(\frac{6.37\times 10^6\text{ m}}{10.87\times 10^6\text{ m}}\right)^2$$

or $F = 1684\text{ N}$ ◀

(*b*) From the solution to Problem 12.78, we have
$$M = \frac{1}{G}\left(\frac{2\pi}{\tau}\right)^2 r^3$$

Then
$$\tau = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

Now
$$\tau_E = \tau_M \Rightarrow \frac{2\pi r_E^{3/2}}{\sqrt{GM_E}} = \frac{2\pi r_M^{3/2}}{\sqrt{GM_M}} \qquad (1)$$

or
$$r_M = \left(\frac{M_M}{M_E}\right)^{1/3} r_E = (0.01230)^{1/3}(10.87\times 10^6\text{ m})$$

or $r_M = 2.509\times 10^6\text{ m}$ $r_M = 2510\text{ km}$ ◀

PROBLEM 12.85 (Continued)

(*c*) We have $GM = gR^2$ [Eq.(12.29)]

Substituting into Eq. (1)

$$\frac{2\pi r_E^{3/2}}{R_E\sqrt{g_E}} = \frac{2\pi r_M^{3/2}}{R_M\sqrt{g_M}}$$

or

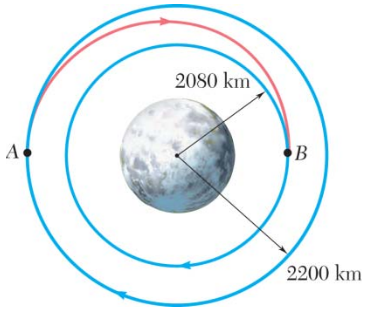
$$g_M = \left(\frac{R_E}{R_M}\right)^2 \left(\frac{r_M}{r_E}\right)^3 g_E = \left(\frac{R_E}{R_M}\right)^2 \left(\frac{M_M}{M_E}\right) g_E$$

using the results of Part (b). Then

$$g_M = \left(\frac{6370 \text{ km}}{1737 \text{ km}}\right)^2 (0.01230)(9.81 \text{ m/s}^2)$$

or $g_{\text{moon}} = 1.62 \text{ m/s}^2$ ◀

Note: $g_{\text{moon}} \approx \frac{1}{6} g_{\text{earth}}$



PROBLEM 12.86

A space vehicle is in a circular orbit of 2200-km radius around the moon. To transfer it to a smaller circular orbit of 2080-km radius, the vehicle is first placed on an elliptic path *AB* by reducing its speed by 26.3 m/s as it passes through *A*. Knowing that the mass of the moon is 73.49×10^{21} kg, determine (*a*) the speed of the vehicle as it approaches *B* on the elliptic path, (*b*) the amount by which its speed should be reduced as it approaches *B* to insert it into the smaller circular orbit.

SOLUTION

For a circular orbit, $\Sigma F_n = ma_n: \quad F = m\dfrac{v^2}{r}$

Eq. (12.28): $$F = G\frac{Mm}{r^2}$$

Then $$G\frac{Mm}{r^2} = m\frac{v^2}{r}$$

or $$v^2 = \frac{GM}{r}$$

Then $$(v_A)^2_{\text{circ}} = \frac{66.73\times 10^{-12}\ \text{m}^3/\text{kg}\cdot\text{s}^2 \times 73.49\times 10^{21}\ \text{kg}}{2200\times 10^3\ \text{m}}$$

or $$(v_A)_{\text{circ}} = 1493.0\ \text{m/s}$$

and $$(v_B)^2_{\text{circ}} = \frac{66.73\times 10^{-12}\ \text{m}^3/\text{kg}\cdot\text{s}^2 \times 73.49\times 10^{21}\ \text{kg}}{2080\times 10^3\ \text{m}}$$

or $$(v_B)_{\text{circ}} = 1535.5\ \text{m/s}$$

(*a*) We have $$(v_A)_{TR} = (v_A)_{\text{circ}} + \Delta v_A$$
$$= (1493.0 - 26.3)\ \text{m/s}$$
$$= 1466.7\ \text{m/s}$$

Conservation of angular momentum requires that

$$r_A m(v_A)_{TR} = r_B m(v_B)_{TR}$$

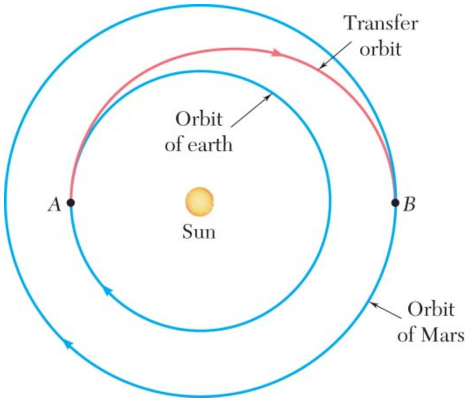
or $$(v_B)_{TR} = \frac{2200\ \text{km}}{2080\ \text{km}} \times 1466.7\ \text{m/s}$$
$$= 1551.3\ \text{m/s}$$

or $$(v_B)_{TR} = 1551\ \text{m/s} \blacktriangleleft$$

(*b*) Now $$(v_B)_{\text{circ}} = (v_B)_{TR} + \Delta v_B$$

or $$\Delta v_B = (1535.5 - 1551.3)\ \text{m/s}$$

or $$\Delta v_B = -15.8\ \text{m/s} \blacktriangleleft$$



PROBLEM 12.87

As a first approximation to the analysis of a space flight from the earth to Mars, assume the orbits of the earth and Mars are circular and coplanar. The mean distances from the sun to the earth and to Mars are 149.6×10^6 km and 227.8×10^6 km, respectively. To place the spacecraft into an elliptical transfer orbit at point *A*, its speed is increased over a short interval of time to v_A which is 2.94 km/s faster than the earth's orbital speed. When the spacecraft reaches point *B* on the elliptical transfer orbit, its speed v_B is increased to the orbital speed of Mars. Knowing that the mass of the sun is 332.8×10^3 times the mass of the earth, determine the increase in speed required at *B*.

SOLUTION

For earth, $(GM)_{\text{earth}} = gR^2 = (9.81)(6.37 \times 10^6)^2 = 398.06 \times 10^{12}\ \text{m}^3/\text{s}^2$

For sun, $(GM)_{\text{sun}} = (332.8 \times 10^3)(GM)_{\text{earth}} = 132.474 \times 10^{18}\text{m}^3/\text{s}^2$

For circular orbit of earth, $r_E = 149.6 \times 10^6\ \text{km} = 149.6 \times 10^9\ \text{m/s}$

$$v_E = \sqrt{\frac{(GM)_{\text{sun}}}{r_E}} = \sqrt{\frac{132.474 \times 10^{18}}{149.6 \times 10^9}} = 29.758 \times 10^3\ \text{m/s}$$

For transfer orbit *AB*, $r_A = r_E, \quad r_B = r_M = 227.8 \times 10^9\ \text{m}$

$$v_A = v_E + (\Delta v)_A = 29.758 \times 10^3 + 2.94 \times 10^3 = 32.698 \times 10^3\ \text{m/s}$$

$$mr_A v_A = mr_B v_B$$

$$v_B = \frac{r_A v_A}{r_B} = \frac{(149.6 \times 10^9)(32.698 \times 10^3)}{227.8 \times 10^9} = 21.473 \times 10^3\ \text{m/s}$$

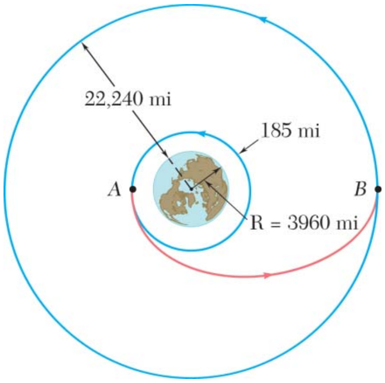
For circular orbit of Mars,

$$v_M = \sqrt{\frac{(GM)_{\text{sun}}}{r_M}} = \sqrt{\frac{132.474 \times 10^{18}}{227.8 \times 10^9}} = 24.115 \times 10^3\ \text{m/s}$$

Speed increase at *B*.

$$(\Delta v)_B = v_M - v_B = 24.115 \times 10^3 - 21.473 \times 10^3 = 2.643 \times 10^3\ \text{m/s}$$

$(\Delta v)_B = 2.64\ \text{km/s}$ ◀



PROBLEM 12.88

To place a communications satellite into a geosynchronous orbit (see Problem 12.80) at an altitude of 22,240 mi above the surface of the earth, the satellite first is released from a space shuttle, which is in a circular orbit at an altitude of 185 mi, and then is propelled by an upper-stage booster to its final altitude. As the satellite passes through *A*, the booster's motor is fired to insert the satellite into an elliptic transfer orbit. The booster is again fired at *B* to insert the satellite into a geosynchronous orbit. Knowing that the second firing increases the speed of the satellite by 4810 ft/s, determine (*a*) the speed of the satellite as it approaches *B* on the elliptic transfer orbit, (*b*) the increase in speed resulting from the first firing at *A*.

SOLUTION

For earth, $R = 3960 \text{ mi} = 20.909 \times 10^6 \text{ ft}$

$$GM = gR^2 = (32.2)(20.909 \times 10^6)^2 = 14.077 \times 10^{15} \text{ ft}^3/\text{s}^2$$

$$r_A = 3960 + 185 = 4145 \text{ mi} = 21.8856 \times 10^6 \text{ ft}$$

$$r_B = 3960 + 22,240 = 26,200 \text{ mi} = 138.336 \times 10^6 \text{ ft}$$

Speed on circular orbit through *A*.

$$(v_A)_{\text{circ}} = \sqrt{\frac{GM}{r_A}} = \sqrt{\frac{14.077 \times 10^{15}}{21.8856 \times 10^6}} = 25.362 \times 10^3 \text{ ft/s}$$

Speed on circular orbit through *B*.

$$(v_B)_{\text{circ}} = \sqrt{\frac{GM}{r_B}} = \sqrt{\frac{14.077 \times 10^{15}}{138.336 \times 10^6}} = 10.088 \times 10^3 \text{ ft/s}$$

PROBLEM 12.88 (Continued)

(*a*) Speed on transfer trajectory at *B*.

$$(v_B)_{\text{tr}} = 10.088 \times 10^3 - 4810$$

$$= 5.278 \times 10^3$$

5280 ft/s ◀

Conservation of angular momentum for transfer trajectory.

$$r_A (v_A)_{\text{tr}} = r_B (v_B)_{\text{tr}}$$

$$(v_A)_{\text{tr}} = \frac{r_B (v_B)_{\text{tr}}}{r_A}$$

$$= \frac{(138.336 \times 10^6)(5278)}{21.8856 \times 10^6}$$

$$= 33.362 \times 10^3 \text{ ft/s}$$

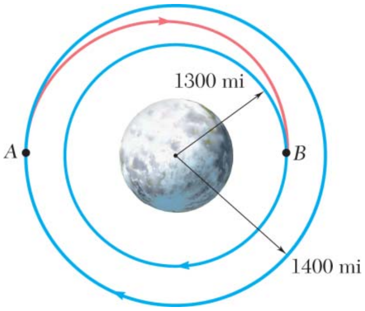
(*b*) Change in speed at *A*.

$$\Delta v_A = (v_A)_{\text{tr}} - (v_A)_{\text{circ}}$$

$$= 33.362 \times 10^3 - 25.362 \times 10^3$$

$$= 8.000 \times 10^3$$

$\Delta v_A = 8000$ ft/s ◀



PROBLEM 12.89

A space vehicle is in a circular orbit of 1400-mi radius around the moon. To transfer to a smaller orbit of 1300-mi radius, the vehicle is first placed in an elliptic path *AB* by reducing its speed by 86 ft/s as it passes through *A*. Knowing that the mass of the moon is 5.03×10^{21} lb·s^2/ft, determine (*a*) the speed of the vehicle as it approaches *B* on the elliptic path, (*b*) the amount by which its speed should be reduced as it approaches *B* to insert it into the smaller circular orbit.

SOLUTION

Circular orbits: $v = \sqrt{\dfrac{GM}{r}}$

$$r_A = 1400 \text{ mi} = 7.392 \times 10^6 \text{ ft}$$

$$(v_A)_1 = \sqrt{\frac{(34.4 \times 10^{-9})(5.03 \times 10^{21})}{7.392 \times 10^6}} = 4.8382 \times 10^3 \text{ ft/s}$$

$$r_B = 1300 \text{ mi} = 6.864 \times 10^6 \text{ ft}$$

$$(v_B)_2 = \sqrt{\frac{(34.4 \times 10^{-9})(5.03 \times 10^{21})}{6.864 \times 10^6}} = 5.0208 \times 10^3 \text{ ft/s}$$

(*a*) Transfer orbit *AB*.

$$(v_A)_2 = (v_A)_1 + (\Delta v)_A = 4.8382 \times 10^3 - 86 = 4.7522 \times 10^3 \text{ ft/s}$$

$$mr_A(v_A)_2 = mr_B(v_B)_1$$

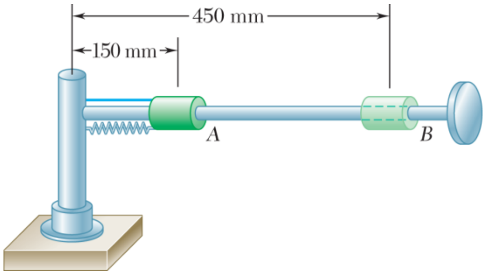
$$(v_B)_1 = \frac{r_A(v_A)_2}{r_B} = \frac{(7.392 \times 10^6)(4.7522 \times 10^3)}{6.864 \times 10^3} = 5.1178 \times 10^3 \text{ ft/s}$$

$$(v_B)_1 = 5.12 \times 10^3 \text{ ft/s} \blacktriangleleft$$

(*b*) Speed change at *B*.

$$(\Delta v_B) = (v_B)_2 - (v_B)_1 = 5.0208 \times 10^3 \text{ ft/s} - 5.1178 \times 10^3 \text{ ft/s} = -97.0 \text{ ft/s}$$

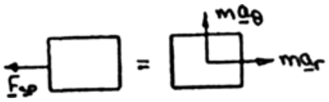
Speed reduction at *B*. $|\Delta v_B| = 97.0$ ft/s ◀



PROBLEM 12.90

A 1 kg collar can slide on a horizontal rod, which is free to rotate about a vertical shaft. The collar is initially held at *A* by a cord attached to the shaft. A spring of constant 30 N/m is attached to the collar and to the shaft and is undeformed when the collar is at *A*. As the rod rotates at the rate $\dot{\theta} = 16$ rad/s, the cord is cut and the collar moves out along the rod. Neglecting friction and the mass of the rod, determine (*a*) the radial and transverse components of the acceleration of the collar at *A*, (*b*) the acceleration of the collar relative to the rod at *A*, (*c*) the transverse component of the velocity of the collar at *B*.

SOLUTION



First note $\qquad F_{sp} = k(r - r_A)$

(*a*) $\quad F_\theta = 0$ and at *A*, $\qquad F_r = -F_{sp} = 0$

$$(a_A)_r = 0 \blacktriangleleft$$

$$(a_A)_\theta = 0 \blacktriangleleft$$

(*b*) $\quad \xrightarrow{+} \Sigma F_r = ma_r$: $\qquad -F_{sp} = m(\ddot{r} - r\dot{\theta}^2)$

Noting that $\qquad a_{\text{collar/rod}} = \ddot{r}$, we have at *A*

$$0 = m[a_{\text{collar/rod}} - (150 \text{ mm})(16 \text{ rad/s})^2]$$

$$a_{\text{collar/rod}} = 38400 \text{ mm/s}^2$$

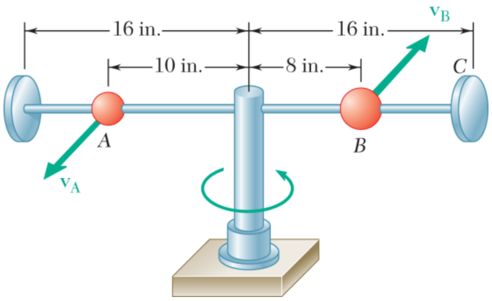
or $\qquad (a_{\text{collar/rod}})_A = 38.4 \text{ m/s}^2 \blacktriangleleft$

(*c*) After the cord is cut, the only horizontal force acting on the collar is due to the spring. Thus, angular momentum about the shaft is conserved.

$$r_A m(v_A)_\theta = r_B m(v_B)_\theta \quad \text{where} \quad (v_A)_\theta = r_A\dot{\theta}_0$$

Then $\qquad (v_B)_\theta = \dfrac{150 \text{ mm}}{450 \text{ mm}}[(150 \text{ mm})(16 \text{ rad/s})] = 800 \text{ mm/s}$

or $\qquad (v_B)_\theta = 0.800 \text{ m/s} \blacktriangleleft$



PROBLEM 12.91

A 1-lb ball A and a 2-lb ball B are mounted on a horizontal rod which rotates freely about a vertical shaft. The balls are held in the positions shown by pins. The pin holding B is suddenly removed and the ball moves to position C as the rod rotates. Neglecting friction and the mass of the rod and knowing that the initial speed of A is $v_A = 8$ ft/s, determine (*a*) the radial and transverse components of the acceleration of ball B immediately after the pin is removed, (*b*) the acceleration of ball B relative to the rod at that instant, (*c*) the speed of ball A after ball B has reached the stop at C.

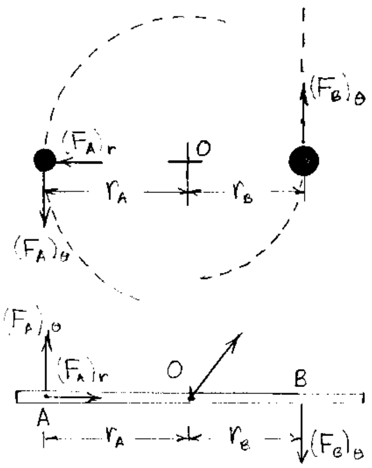
SOLUTION

Let r and θ be polar coordinates with the origin lying at the shaft.

Constraint of rod: $\theta_B = \theta_A + \pi$ radians; $\dot{\theta}_B = \dot{\theta}_A = \dot{\theta}$; $\ddot{\theta}_B = \ddot{\theta}_A = \ddot{\theta}$.

(*a*) Components of acceleration

Sketch the free body diagrams of the balls showing the radial and transverse components of the forces acting on them. Owing to frictionless sliding of B along the rod, $(F_B)_r = 0$.



Radial component of acceleration of B.

$$F_r = m_B(a_B)_r: \qquad (a_B)_r = 0 \blacktriangleleft$$

Transverse components of acceleration.

$$(a_A)_\theta = r_A\ddot{\theta} + 2\dot{r}_A\dot{\theta} = ra\ddot{\theta}$$

$$(a_B)_\theta = r_B\ddot{\theta} + 2\dot{r}_B\dot{\theta} \qquad (1)$$

Since the rod is massless, it must be in equilibrium. Draw its free body diagram, applying Newton's 3rd Law.

$$+\circlearrowleft \Sigma M_0 = 0: \quad r_A(F_A)_\theta + r_B(F_B)_\theta = r_A m_A(a_A)_\theta + r_B m_B(a_B)_\theta = 0$$

$$r_A m_A r_A\ddot{\theta} + r_B m_B(r_B\ddot{\theta} + 2\dot{r}_B\dot{\theta}) = 0$$

$$\ddot{\theta} = \frac{-2m_B\dot{r}_B\dot{\theta}}{m_A r_A^2 + m_B r_B^2}$$

At $t = 0$, $\dot{r}_B = 0$ so that $\ddot{\theta} = 0$.

From Eq. (1), $(a_B)_\theta = 0 \blacktriangleleft$

PROBLEM 12.91 (Continued)

(*b*) Acceleration of *B* relative to the rod.

At $t = 0,\ (v_A)_\theta = 8\text{ ft/s} = 96\text{ in./s},\ \dot{\theta} = \dfrac{(v_A)_\theta}{r_A} = \dfrac{96}{10} = 9.6\text{ rad/s}$

$$\ddot{r}_B - r_B\dot{\theta}^2 = (a_B)_r = 0$$

$$\ddot{r}_B = r_B\dot{\theta}^2 = (8)(9.6)^2 = 737.28\text{ in./s}^2$$

$$\ddot{r}_B = 61.4\text{ ft/s}^2 \blacktriangleleft$$

(*c*) Speed of *A*.

Substituting $\dfrac{d}{dt}(mr^2\dot{\theta})$ for rF_θ in each term of the moment equation gives

$$\frac{d}{dt}\left(m_A r_A{}^2\dot{\theta}\right) + \frac{d}{dt}\left(m_B r_B{}^2\dot{\theta}\right) = 0$$

Integrating with respect to time,

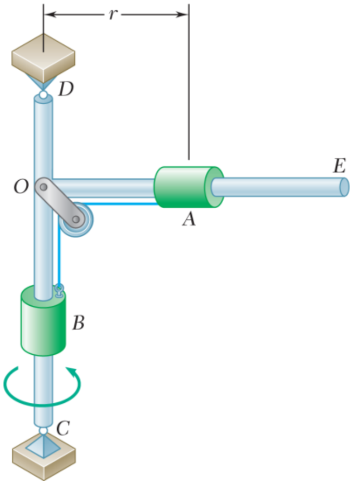
$$m_A r_A{}^2\dot{\theta} + m_B r_B{}^2\dot{\theta} = \left(m_A r_A{}^2\dot{\theta}\right)_0 + \left(m_B r_B{}^2\dot{\theta}\right)_0$$

Applying to the final state with ball *B* moved to the stop at *C*,

$$\left(\frac{W_A}{g}r_A^2 + \frac{W_B}{g}r_C^2\right)\dot{\theta}_f = \left[\frac{W_A}{g}r_A^2 + \frac{W_B}{g}(r_B)_0^2\right]\dot{\theta}_0$$

$$\dot{\theta}_f = \frac{W_A r_A{}^2 + W_B(r_B)_0^2}{W_A r_A{}^2 + W_B r_C{}^2}\dot{\theta}_0 = \frac{(1)(10)^2 + (2)(8)^2}{(1)(10)^2 + (2)(16)^2}(9.6) = 3.5765\text{ rad/s}$$

$$(v_A)_f = r_A\dot{\theta}_f = (10)(3.5765) = 35.765\text{ in./s} \qquad (v_A)_f = 2.98\text{ ft/s} \blacktriangleleft$$



PROBLEM 12.92

Two 2.6-lb collars *A* and *B* can slide without friction on a frame, consisting of the horizontal rod *OE* and the vertical rod *CD*, which is free to rotate about *CD*. The two collars are connected by a cord running over a pulley that is attached to the frame at *O* and a stop prevents collar *B* from moving. The frame is rotating at the rate $\dot{\theta} = 12$ rad/s and $r = 0.6$ ft when the stop is removed allowing collar *A* to move out along rod *OE*. Neglecting friction and the mass of the frame, determine, for the position $r = 1.2$ ft, (*a*) the transverse component of the velocity of collar *A*, (*b*) the tension in the cord and the acceleration of collar *A* relative to the rod *OE*.

SOLUTION

Masses: $m_A = m_B = \dfrac{2.6}{32.2} = 0.08075 \text{ lb}\cdot\text{s}^2/\text{ft}$

(*a*) Conservation of angular momentum of collar *A*: $(H_0)_2 = (H_0)_1$

$$m_A r_1 (v_\theta)_1 = m_A r_2 (v_\theta)_2$$

$$(v_\theta)_2 = \frac{r_1 (v_\theta)_1}{r_2} = \frac{r_1^2 \dot{\theta}_1}{r_2} = \frac{(0.6)^2(12)}{1.2} = 3.6$$

$(v_\theta)_2 = 3.60$ ft/s ◀

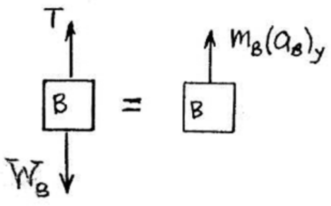
$$\dot{\theta}_2 = \frac{(v_\theta)_2}{r_A} = \frac{3.6}{1.2} = 3.00 \text{ rad/s}$$

(*b*) Let *y* be the position coordinate of *B*, positive upward with origin at *O*.

Constraint of the cord: $r - y = \text{constant}$ or $\ddot{y} = \ddot{r}$

Kinematics:

$$(a_B)_y = \ddot{y} = \ddot{r} \qquad \text{and} \qquad (a_A)_r = \ddot{r} - r\dot{\theta}^2$$

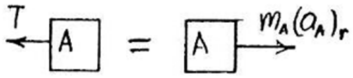


Collar *B*: $\Sigma F_y = m_B a_B: \quad T - W_B = m_B \ddot{y} = m_B \ddot{r}$ (1)

Collar *A*: $\xrightarrow{+} \Sigma F_r = m_A (a_A)_r: \quad -T = m_A(\ddot{r} - r\dot{\theta}^2)$ (2)

Adding (1) and (2) to eliminate *T*,

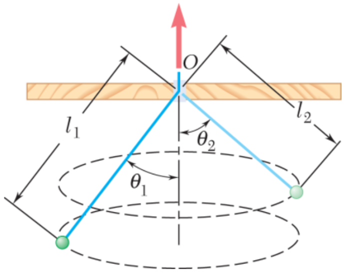
$$-W_B = (m_A + m_B)\ddot{r} + m_A r \dot{\theta}^2$$



$$a_{A/\text{rod}} = \ddot{r} = \frac{m_A r \dot{\theta}^2 - W_B}{m_A + m_B} = \frac{(0.08075)(1.2)(3.00)^2 - (2.6)}{0.08075 + 0.08075} = -10.70 \text{ ft/s}^2$$

$T = m_B(\ddot{r} + g) = (0.08075)(-10.70 + 32.2)$ $\qquad T = 1.736$ lb ◀

$a_{A/\text{rod}} = 10.70 \text{ ft/s}^2$ radially inward. ◀



PROBLEM 12.93

A small ball swings in a horizontal circle at the end of a cord of length l_1, which forms an angle θ_1 with the vertical. The cord is then slowly drawn through the support at O until the length of the free end is l_2. (*a*) Derive a relation among l_1, l_2, θ_1, and θ_2. (*b*) If the ball is set in motion so that initially $l_1 = 0.8$ m and $\theta_1 = 35°$, determine the angle θ_2 when $l_2 = 0.6$ m.

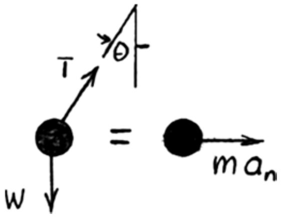
SOLUTION

(*a*) For state 1 or 2, neglecting the vertical component of acceleration,

$$+\!\uparrow \Sigma F_y = 0: \quad T\cos\theta - W = 0$$

$$T = W\cos\theta$$

$$\xrightarrow{+} \Sigma F_x = ma_n: \quad T\sin\theta = W\sin\theta\cos\theta = \frac{mv^2}{\rho}$$



But $\rho = \ell\sin\theta$ so that

$$v^2 = \frac{\rho W}{m}\sin^2\theta\cos\theta = \ell\, g\sin\theta\tan\theta$$

$$v_1 = \sqrt{\ell_1 g\sin\theta_1\tan\theta_1}$$

and $$v_2 = \sqrt{\ell_2\, g\sin\theta_2\tan\theta_2}$$

$$\Sigma M_y = 0: \quad H_y = \text{constant}$$

$$r_1 m v_1 = r_2 m v_2 \qquad \text{or} \qquad v_1\ell_1\sin\theta_1 = v_2\ell_2\sin\theta_2$$

$$\ell_1^{3/2} g\sin\theta_1\sqrt{\sin\theta_1\tan\theta_1} = \ell_2^{3/2}\sin\theta_2\sqrt{\sin\theta_2\tan\theta_2}$$

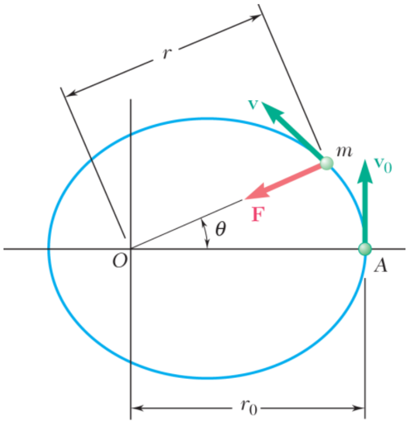
$$\ell_1^3\sin^3\theta_1\tan\theta_1 = \ell_2^3\sin^3\theta_2\tan\theta_2 \ ◀$$

(*b*) With $\theta_1 = 35°$, $\ell_1 = 0.8$ m, and $\ell_2 = 0.6$ m

$$(0.8)^3\sin^3 35°\tan 35° = (0.6)^3\sin^3\theta_2\tan\theta_2$$

$$\sin^3\theta_2\tan\theta_2 - 0.31320 = 0$$

$$\theta_2 = 43.6° \ ◀$$



PROBLEM 12.94

A particle of mass m is projected from Point A with an initial velocity $\mathbf{v}_0$ perpendicular to OA and moves under a central force $\mathbf{F}$ along an elliptic path defined by the equation $r = r_0/(2-\cos\theta)$. Using Eq. (12.35), show that $\mathbf{F}$ is inversely proportional to the square of the distance r from the particle to the center of force O.

SOLUTION

$$u = \frac{1}{r} = \frac{2-\cos\theta}{r_0}, \quad \frac{du}{d\theta} = \frac{\sin\theta}{r_0}, \quad \frac{d^2u}{d\theta^2} = \frac{\cos\theta}{r_0}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{2}{r_0} = \frac{F}{mh^2u^2} \quad \text{by Eq. (12.37).}$$

Solving for F,

$$F = \frac{2mh^2u^2}{r_0} = \frac{2mh^2}{r_0r^2}$$

Since m, h, and r_0 are constants, F is proportional to $\frac{1}{r^2}$, or inversely proportional to r^2.

PROBLEM 12.95

A particle of mass m describes the logarithmic spiral $r = r_0 e^{b\theta}$ under a central force **F** directed toward the center of force O. Using Eq. (12.35) show that **F** is inversely proportional to the cube of the distance r from the particle to O.

SOLUTION

$$u = \frac{1}{r} = \frac{1}{r_0} e^{-b\theta}$$

$$\frac{du}{d\theta} = -\frac{b}{r_0} e^{-b\theta}$$

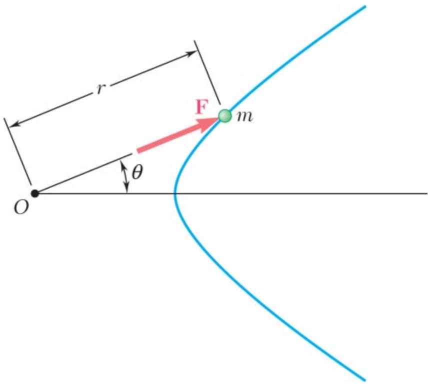
$$\frac{d^2u}{d\theta^2} = \frac{b^2}{r_0} e^{-b\theta}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{b^2 + 1}{r_0} e^{-b\theta} = \frac{F}{mh^2u^2}$$

$$F = \frac{(b^2 + 1)mh^2u^2}{r_0} e^{-b\theta}$$

$$= \frac{(b^2 + 1)mh^2u^2}{r} = \frac{(b^2 + 1)mh^2}{r^3}$$

Since b, m, and h are constants, **F** is proportional to $\frac{1}{r^3}$, or inversely proportional to r^3.



PROBLEM 12.96

A particle of mass m describes the path defined by the equation $r = r_0/(6\cos\theta - 5)$ under a central force **F** directed away from the center of force O. Using Eq. (12.35), show that **F** is inversely proportional to the square of the distance r from the particle to O.

SOLUTION

$$u = \frac{1}{r} = \frac{6\cos\theta - 5}{r_0}$$

$$\frac{du}{d\theta} = -\frac{6\sin\theta}{r_0}$$

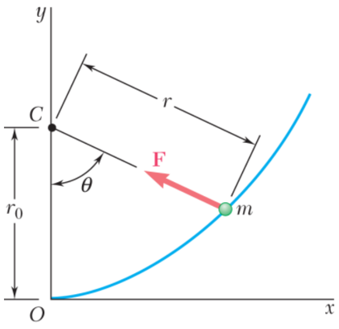
$$\frac{d^2u}{d\theta} = -\frac{6\cos\theta}{r_0}$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{5}{r_0} = \frac{F}{mh^2u^2} \quad \text{by Eq. (12.35).}$$

Solving for F, $$F = -\frac{5mh^2u^2}{r_0}$$

Substitute $u = \frac{1}{r}$ $$F = -\frac{5mh^2}{r_0 r^2} \blacktriangleleft$$

Since m, h, and r_0 are constants, F is proportional to $\frac{1}{r^2}$, or inversely proportional to r^2. The minus sign indicates that the force is repulsive, as shown in Fig. P12.96.



PROBLEM 12.97

A particle of mass m describes the parabola $y = x^2/4r_0$ under a central force $\mathbf{F}$ directed toward the center of force C. Using Eq. (12.35) and Eq. $(12.37')$ with $\varepsilon = 1$, show that $\mathbf{F}$ is inversely proportional to the square of the distance r from the particle to the center of force and that the angular momentum per unit mass $h = \sqrt{2GMr_0}$.

SOLUTION

From Fig. P12.97, $\quad x = r\sin\theta, \quad y = r_0 - r\cos\theta$

But $\quad y = \dfrac{x^2}{4r_0} \quad$ or $\quad r_0 - r\cos\theta = \dfrac{r^2\sin^2\theta}{4r_0} = \dfrac{r^2\left(1-\cos^2\theta\right)}{4r_0}$

$$\left(\frac{1-\cos^2\theta}{4r_0}\right)r^2 + (\cos\theta)r - r_0 = 0$$

Solving the quadratic equation for r,

$$r = \frac{2r_0}{1-\cos^2\theta}\left[-\cos\theta \pm \sqrt{\cos^2\theta - 4\left(\frac{1-\cos^2\theta}{4r_0}\right)(-r_0)}\right]$$

$$= \frac{2r_0}{1-\cos^2\theta}(-\cos\theta \pm 1) = \frac{2r_0(1-\cos\theta)}{1-\cos^2\theta}$$

since $r > 0$. Simplifying gives

$$r = \frac{2r_0}{1+\cos\theta} \quad \text{or} \quad \frac{1}{r} = \frac{1+\cos\theta}{2r_0} = u \qquad (1)$$

$$\frac{du}{d\theta} = -\frac{\sin\theta}{2r_0} \quad \text{and} \quad \frac{d^2u}{d\theta^2} = -\frac{\cos\theta}{2r_0}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{2r_0} = \frac{F}{mh^2u^2}$$

Solving for F, $\qquad F = \dfrac{mh^2u^2}{2r_0} \qquad\qquad F = \dfrac{mh^2}{2r_0r^2}$ ◀

Since m, h, and r_0 are constants, F is proportional to $\dfrac{1}{r^2}$, or inversely proportional to r^2.

By Eq. $(12.37')$, $\quad \dfrac{1}{r} = \dfrac{GM}{h^2}(1+\varepsilon\cos\theta) = u$

Comparing with (1) shows that $\varepsilon = 1$ and $\dfrac{GM}{h^2} = \dfrac{1}{2r_0} \qquad\qquad h = \sqrt{2GMr_0}$ ◀

PROBLEM 12.98

It was observed that during its second flyby of the earth, the Galileo spacecraft had a velocity of 14.1 km/s as it reached its minimum altitude of 303 km above the surface of the earth. Determine the eccentricity of the trajectory of the spacecraft during this portion of its flight.

SOLUTION

For earth, $R = 6.37 \times 10^6\ \text{m}$

$$r_0 = 6.37 \times 10^6 + 303. \times 10^3 = 6.673 \times 10^6\ \text{m}$$

$$h = r_0 v_0 = (6.673 \times 10^6)(14.1 \times 10^3) = 94.09 \times 10^9\ \text{m}^2/\text{s}$$

$$GM = gR^2 = (9.81)(6.37 \times 10^6)^2 = 398.06 \times 10^{12}\ \text{m}^3/\text{s}^2$$

$$\frac{1}{r_0} = \frac{GM}{h^2}(1 + \varepsilon)$$

$$1 + \varepsilon = \frac{h^2}{r_0 GM} = \frac{(94.09 \times 10^9)^2}{(6.673 \times 10^6)(398.06 \times 10^{12})} = 3.33$$

$$\varepsilon = 3.33 - 1 \qquad \varepsilon = 2.33 \blacktriangleleft$$

PROBLEM 12.99

It was observed that during the Galileo spacecraft's first flyby of the earth, its maximum altitude was 600 mi above the surface of the earth. Assuming that the trajectory of the spacecraft was parabolic, determine the maximum velocity of Galileo during its first flyby of the earth.

SOLUTION

For the earth: $R = 3960 \text{ mi} = 20.909 \times 10^6 \text{ ft}$

$$GM = gR^2 = (32.2)(20.909 \times 10^6)^2 = 14.077 \times 10^{15} \text{ ft}^3/\text{s}^2$$

For a parabolic trajectory, $\varepsilon = 1.$

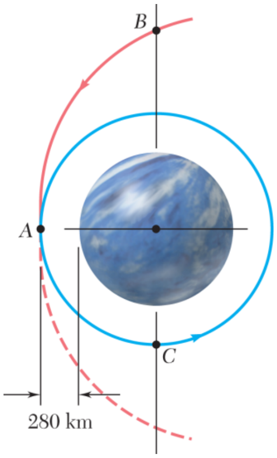
Eq. (12.39′): $\dfrac{1}{r} = \dfrac{GM}{h^2}(1 + \cos\theta)$

At $\theta = 0,$ $\dfrac{1}{r_0} = \dfrac{2GM}{h^2} = \dfrac{2GM}{r_0{}^2 v_0{}^2}$ or $v_0 = \sqrt{\dfrac{2GM}{r_0}}$

At $r_0 = 3960 + 600 = 4560 \text{ mi} = 24.077 \times 10^6 \text{ ft},$

$$v_0 = \sqrt{\frac{(2)(14.077 \times 10^{15})}{24.077 \times 10^6}} = 34.196 \times 10^3 \text{ ft/s}$$

$v_0 = 6.48 \text{ mi/s}$ ◀



PROBLEM 12.100

As a space probe approaching the planet Venus on a parabolic trajectory reaches Point A closest to the planet, its velocity is decreased to insert it into a circular orbit. Knowing that the mass and the radius of Venus are 4.87×10^{24} kg and 6052 km, respectively, determine (*a*) the velocity of the probe as it approaches A, (*b*) the decrease in velocity required to insert it into the circular orbit.

SOLUTION

First note $\quad r_A=(6052+280)\text{ km}=6332\text{ km}$

(*a*) From the textbook, the velocity at the point of closest approach on a parabolic trajectory is given by

$$v_0=\sqrt{\frac{2GM}{r_0}}$$

Thus, $$(v_A)_{\text{par}}=\left[\frac{2\times66.73\times10^{-12}\text{ m}^3/\text{kg}\cdot\text{s}^2\times4.87\times10^{24}\text{ kg}}{6332\times10^3\text{ m}}\right]^{1/2}$$

$$=10{,}131.4\text{ m/s}$$

or $\quad (v_A)_{\text{par}}=10.13\text{ km/s}$ ◀

(*b*) We have $\quad (v_A)_{\text{circ}}=(v_A)_{\text{par}}+\Delta v_A$

Now $$(v_A)_{\text{circ}}=\sqrt{\frac{GM}{r_0}}\qquad\text{Eq. (12.44)}$$

$$=\frac{1}{\sqrt{2}}(v_A)_{\text{par}}$$

Then $$\Delta v_A=\frac{1}{\sqrt{2}}(v_A)_{\text{par}}-(v_A)_{\text{par}}$$

$$=\left(\frac{1}{\sqrt{2}}-1\right)(10.1314\text{ km/s})$$

$$=-2.97\text{ km/s}$$

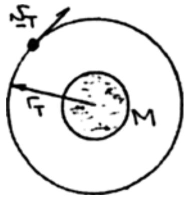
$|\Delta v_A|=2.97\text{ km/s}$ ◀

PROBLEM 12.101

It was observed that as the Voyager I spacecraft reached the point of its trajectory closest to the planet Saturn, it was at a distance of 185×10^3 km from the center of the planet and had a velocity of 21.0 km/s. Knowing that Tethys, one of Saturn's moons, describes a circular orbit of radius 295×10^3 km at a speed of 11.35 km/s, determine the eccentricity of the trajectory of Voyager I on its approach to Saturn.

SOLUTION

For a circular orbit,



Eq. (12.44) $$v = \sqrt{\frac{GM}{r}}$$

For the orbit of Tethys,

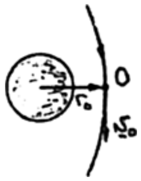
$$GM = r_T v_T^2$$

For Voyager's trajectory, we have

$$\frac{1}{r} = \frac{GM}{h^2}(1 + \varepsilon \cos\theta)$$

where $h = r_0 v_0$

At O, $$r = r_0, \ \theta = 0$$

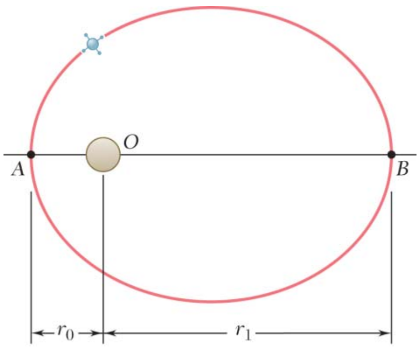


Then $$\frac{1}{r_0} = \frac{GM}{(r_0 v_0)^2}(1 + \varepsilon)$$

or $$\varepsilon = \frac{r_0 v_0^2}{GM} - 1 = \frac{r_0 v_0^2}{r_T v_T^2} - 1$$

$$= \frac{185 \times 10^3 \text{ km}}{295 \times 10^3 \text{ km}} \times \left(\frac{21.0 \text{ km/s}}{11.35 \text{ km/s}}\right)^2 - 1$$

or $$\varepsilon = 1.147 \blacktriangleleft$$



PROBLEM 12.102

A satellite describes an elliptic orbit about a planet of mass M. Denoting by r_0 and r_1, respectively, the minimum and maximum values of the distance r from the satellite to the center of the planet, derive the relation

$$\frac{1}{r_0}+\frac{1}{r_1}=\frac{2GM}{h^2}$$

where h is the angular momentum per unit mass of the satellite.

SOLUTION

Using Eq. (12.39),
$$\frac{1}{r_A}=\frac{GM}{h^2}+C\cos\theta_A$$

and
$$\frac{1}{r_B}=\frac{GM}{h^2}+C\cos\theta_B.$$

But
$$\theta_B=\theta_A+180°,$$

so that
$$\cos\theta_A=-\cos\theta_B.$$

Adding,
$$\frac{1}{r_A}+\frac{1}{r_B}=\frac{1}{r_0}+\frac{1}{r_1}=\frac{2GM}{h^2}$$

PROBLEM 12.103

A space probe is describing a circular orbit about a planet of radius R. The altitude of the probe above the surface of the planet is αR and its speed is v_0. To place the probe in an elliptic orbit which will bring it closer to the planet, its speed is reduced from v_0 to βv_0, where $\beta < 1$, by firing its engine for a short interval of time. Determine the smallest permissible value of β if the probe is not to crash on the surface of the planet.

SOLUTION

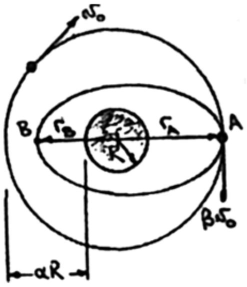
For the circular orbit, $v_0 = \sqrt{\dfrac{GM}{r_A}}$ Eq. (12.44),

where $r_A = R + \alpha R = R(1+\alpha)$

Then $GM = v_0^2 R(1+\alpha)$

From the solution to Problem 12.102, we have for the elliptic orbit,

$$\frac{1}{r_A} + \frac{1}{r_B} = \frac{2GM}{h^2}$$



Now
$$h = h_A = r_A (v_A)_{AB} = [R(1+\alpha)](\beta v_0)$$

Then
$$\frac{1}{R(1+\alpha)} + \frac{1}{r_B} = \frac{2v_0^2 R(1+\alpha)}{[R(1+\alpha)\beta v_0]^2} = \frac{2}{\beta^2 R(1+\alpha)}$$

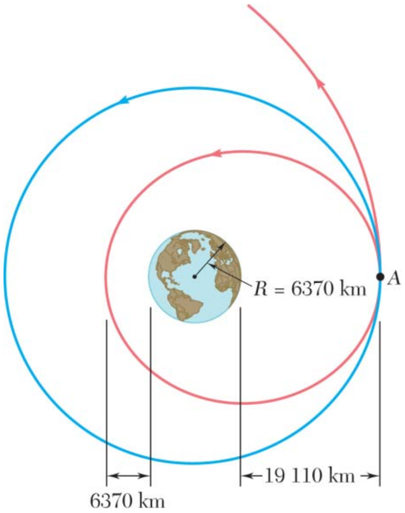
Now $\beta_{\min}$ corresponds to $r_B \to R.$

Then
$$\frac{1}{R(1+\alpha)} + \frac{1}{R} = \frac{2}{\beta_{\min}^2 R(1+\alpha)}$$

or
$$\beta_{\min} = \sqrt{\frac{2}{2+\alpha}} \blacktriangleleft$$

PROBLEM 12.104

A satellite describes a circular orbit at an altitude of 19 110 km above the surface of the earth. Determine (*a*) the increase in speed required at point *A* for the satellite to achieve the escape velocity and enter a parabolic orbit, (*b*) the decrease in speed required at point *A* for the satellite to enter an elliptic orbit of minimum altitude 6370 km, (*c*) the eccentricity ε of the elliptic orbit.



SOLUTION

For earth, $GM = gR^2 = (9.81)\left(6.37\times10^6\right) = 398.06\times10^{12}\ \text{m}^3/\text{s}^2$

$$r_A = 6370 + 19110 = 25480\ \text{km} = 25.48\times10^6\ \text{m}$$

$$v_{\text{circ}} = \sqrt{\frac{GM}{r_A}} = \sqrt{\frac{398.06\times10^{12}}{25.48\times10^6}} = 3.9525\times10^3\ \text{m/s}$$

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r_A}} = \sqrt{2}\,v_{\text{circ}} = 5.5897\times10^3\ \text{m/s}$$

(*a*) Increase in speed at *A*.

$$\Delta v = v_{\text{esc}} - v_{\text{circ}} = 1.637\times10^3\ \text{m/s} \qquad \Delta v = 1.637\times10^3\ \text{m/s} \blacktriangleleft$$

Elliptical orbit with $r_B = 6370 + 6370 = 12740\ \text{km} = 12.74\times10^6\ \text{m}.$

Using Eq. (12.39), $\dfrac{1}{r_A} = \dfrac{GM}{h^2} + C\cos\theta_A$ and $\dfrac{1}{r_B} = \dfrac{GM}{h^2} + C\cos\theta_B.$

But $\theta_B = \theta_A + 180°$, so that $\cos\theta_A = -\cos\theta_B$

Adding, $\dfrac{1}{r_A} + \dfrac{1}{r_B} = \dfrac{r_A + r_B}{r_A r_B} = \dfrac{2GM}{h^2}$

$$h = \sqrt{\frac{2GMr_Ar_B}{r_A + r_B}} = \sqrt{\frac{(2)\left(398.06\times10^{12}\right)\left(25.48\times10^6\right)\left(12.74\times10^6\right)}{38.22\times10^6}}$$

$$= 82.230\times10^9\ \text{m}^2/\text{s}$$

PROBLEM 12.104 (Continued)

$$v_A = \frac{h}{r_A} = \frac{82.230 \times 10^9}{25.48 \times 10^6} = 3.2272 \times 10^3 \text{ m/s}$$

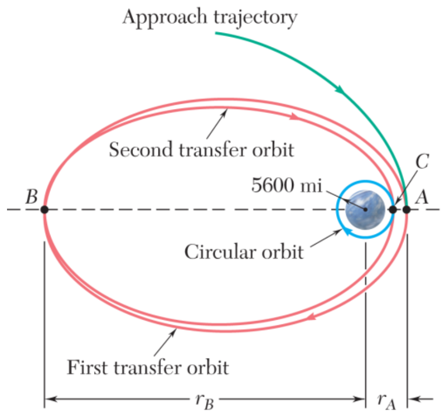
(*b*) Decrease in speed. $\Delta v = v_{\text{circ}} - v_A = 725 \text{ m/s}$ $\Delta v = 725 \text{ m/s}$ ◀

(*c*) $\dfrac{1}{r_B} - \dfrac{1}{r_A} = \dfrac{r_A - r_B}{r_A r_B} = C\cos\theta_B - C\cos_A = 2C$

$$C = \frac{r_A - r_B}{2r_A r_B} = \frac{12.74 \times 10^6}{(2)\left(25.48 \times 10^6\right)\left(12.74 \times 10^6\right)} = 19.623 \times 10^{-9} \text{ m}^{-1}$$

By Eq. (12.40), $\varepsilon = \dfrac{Ch^2}{GM} = \dfrac{\left(19.623 \times 10^{-9}\right)\left(82.230 \times 10^9\right)^2}{398.06 \times 10^{12}}$

$\varepsilon = 0.333$ ◀



PROBLEM 12.105

A space probe is to be placed in a circular orbit of 5600 mi radius about the planet Venus in a specified plane. As the probe reaches *A*, the point of its original trajectory closest to Venus, it is inserted in a first elliptic transfer orbit by reducing its speed by Δv_A. This orbit brings it to Point *B* with a much reduced velocity. There the probe is inserted in a second transfer orbit located in the specified plane by changing the direction of its velocity and further reducing its speed by Δv_B. Finally, as the probe reaches Point *C*, it is inserted in the desired circular orbit by reducing its speed by Δv_C. Knowing that the mass of Venus is 0.82 times the mass of the earth, that $r_A = 9.3 \times 10^3$ mi and $r_B = 190 \times 10^3$ mi, and that the probe approaches *A* on a parabolic trajectory, determine by how much the velocity of the probe should be reduced (*a*) at *A*, (*b*) at *B*, (*c*) at *C*.

SOLUTION

For Earth,

$$R = 3690 \text{ mi} = 20.9088 \times 10^6 \text{ ft}, \quad g = 32.2 \text{ ft/s}^2$$

$$GM_{\text{earth}} = gR^2 = (32.2)(20.9088 \times 10^6)^2 = 14.077 \times 10^{15} \text{ ft}^3/\text{s}^2$$

For Venus,

$$GM = 0.82 GM_{\text{earth}} = 11.543 \times 10^{15} \text{ ft}^3/\text{s}^2$$

For a parabolic trajectory with

$$r_A = 9.3 \times 10^3 \text{ mi} = 49.104 \times 10^6 \text{ ft}$$

$$(v_A)_1 = v_{\text{esc}} = \sqrt{\frac{2GM}{r_A}} = \sqrt{\frac{(2)(11.543 \times 10^{15})}{49.104 \times 10^6}} = 21.683 \times 10^3 \text{ ft/s}$$

First transfer orbit *AB*.

$$r_B = 190 \times 10^3 \text{ mi} = 1003.2 \times 10^6 \text{ ft}$$

At Point *A*, where $\theta = 180°$

$$\frac{1}{r_A} = \frac{GM}{h_{AB}^2} + C \cos 180° = \frac{GM}{h_{AB}^2} - C \qquad (1)$$

At Point *B*, where $\theta = 0°$

$$\frac{1}{r_B} = \frac{GM}{h_{AB}^2} + C \cos 0 = \frac{GM}{h_{AB}^2} + C \qquad (2)$$

Adding,

$$\frac{1}{r_A} + \frac{1}{r_B} = \frac{r_B + r_A}{r_A r_B} = \frac{2GM}{h_{AB}^2}$$

PROBLEM 12.105 (Continued)

Solving for h_{AB},

$$h_{AB} = \sqrt{\frac{2GMr_A r_B}{r_B + r_A}} = \sqrt{\frac{(2)(11.543\times10^{15})(49.104\times10^6)(1003.2\times10^6)}{1052.3\times10^6}} = 1.039575\times10^{12}\ \text{ft}^2/\text{s}$$

$$(v_A)_2 = \frac{h_{AB}}{r_A} = \frac{1.039575\times10^{12}}{49.104\times10^6} = 21.174\times10^3\ \text{ft/s}$$

$$(v_B)_1 = \frac{h_{AB}}{r_B} = \frac{1.039575\times10^{12}}{1003.2\times10^6} = 1.03626\times10^3\ \text{ft/s}$$

Second transfer orbit *BC*. $r_C = 5600\ \text{mi} = 29.568\times10^6\ \text{ft}$

At Point *B*, where $\theta = 0$

$$\frac{1}{r_B} = \frac{GM}{h_{BC}^2} + C\cos 0 = \frac{GM}{h_{BC}^2} + C$$

At Point *C*, where $\theta = 180°$

$$\frac{1}{r_C} = \frac{GM}{h_{BC}^2} + C\cos 180° = \frac{GM}{h_{BC}^2} - C$$

Adding, $$\frac{1}{r_B} + \frac{1}{r_C} = \frac{r_B + r_C}{r_B r_C} = \frac{2GM}{h_{BC}^2}$$

$$h_{BC} = \sqrt{\frac{2GMr_B r_C}{r_B + r_C}} = \sqrt{\frac{(2)(11.543\times10^{15})(1003.2\times10^6)(29.568\times10^6)}{1032.768\times10^6}} = 814.278\times10^9\ \text{ft}^2/\text{s}$$

$$(v_B)_2 = \frac{h_{BC}}{r_B} = \frac{814.278\times10^9}{1003.2\times10^6} = 811.69\ \text{ft/s}$$

$$(v_C)_1 = \frac{h_{BC}}{r_C} = \frac{814.278\times10^9}{29.568\times10^6} = 27.539\times10^3\ \text{ft/s}$$

Final circular orbit. $r_C = 29.568\times10^6\ \text{ft}$

$$(v_C)_2 = \sqrt{\frac{GM}{r_C}} = \sqrt{\frac{11.543\times10^{15}}{29.568\times10^6}} = 19.758\times10^3\ \text{ft/s}$$

Speed reductions.

(*a*) At *A*: $(v_A)_1 - (v_A)_2 = 21.683\times10^3 - 21.174\times10^3$ $\Delta v_A = 509\ \text{ft/s}$ ◀

(*b*) At *B*: $(v_B)_1 - (v_B)_2 = 1.036\times10^3 - 811.69$ $\Delta v_B = 224\ \text{ft/s}$ ◀

(*c*) At *C*: $(v_C)_1 - (v_C)_2 = 27.539\times10^3 - 19.758\times10^3$ $\Delta v_C = 7.78\times10^3\ \text{ft/s}$ ◀

PROBLEM 12.106

For the space probe of Problem 12.105, it is known that $r_A = 9.3 \times 10^3$ mi and that the velocity of the probe is reduced to $20{,}000$ ft/s as it passes through *A*. Determine (*a*) the distance from the center of Venus to Point *B*, (*b*) the amounts by which the velocity of the probe should be reduced at *B* and *C*, respectively.

SOLUTION

Data from Problem 12.105: $r_C = 29.568 \times 10^6$ ft, $M = 0.82\, M_{\text{earth}}$

For Earth, $R = 3960 \text{ mi} = 20.9088 \times 10^6 \text{ ft}, \quad g = 32.2 \text{ ft/s}^2$

$$GM_{\text{earth}} = gR^2 = (32.2)(20.9088 \times 10^6)^2 = 14.077 \times 10^{15} \text{ m}^3/\text{s}^2$$

For Venus, $GM = 0.82 GM_{\text{earth}} = 11.543 \times 10^{15} \text{ ft}^3/\text{s}^2$

Transfer orbit *AB*: $v_A = 20{,}000 \text{ ft/s}, \quad r_A = 9.3 \times 10^3 \text{ mi} = 49.104 \times 10^6 \text{ ft}$

$$h_{AB} = r_A v_A = (49.104 \times 10^6)(20{,}000) = 982.08 \times 10^9 \text{ ft}^2/\text{s}$$

At Point *A*, where $\theta = 180°$

$$\frac{1}{r_A} = \frac{GM}{h_{AB}^2} + C \cos 180° = \frac{GM}{h_{AB}^2} - C$$

At Point *B*, where $\theta = 0°$

$$\frac{1}{r_B} = \frac{GM}{h_{AB}^2} + C \cos 0 = \frac{GM}{h_{AB}^2} + C$$

Adding, $$\frac{1}{r_A} + \frac{1}{r_B} = \frac{2GM}{h_{AB}^2}$$

$$\frac{1}{r_B} = \frac{2GM}{h_{AB}^2} - \frac{1}{r_A}$$

$$= \frac{(2)(11.543 \times 10^{15})}{(982.08 \times 10^9)^2} - \frac{1}{49.104 \times 10^6}$$

$$= 3.57125 \times 10^{-9} \text{ ft}^{-1}$$

(*a*) Radial coordinate r_B. $r_B = 280.01 \times 10^6$ ft $\qquad r_B = 53.0 \times 10^3$ mi ◀

$$(v_B)_1 = \frac{h_{AB}}{r_B} = \frac{982.08 \times 10^9}{280.01 \times 10^6} = 3.5073 \times 10^3 \text{ ft/s}$$

PROBLEM 12.106 (Continued)

Second transfer orbit *BC*. $r_C = 5600 \text{ mi} = 29.568 \times 10^6 \text{ ft}$

At Point *B*, where $\theta = 0$

$$\frac{1}{r_B} = \frac{GM}{h_{BC}^2} + C\cos 0 = \frac{GM}{h_{BC}^2} + C$$

At Point *C*, where $\theta = 180°$

$$\frac{1}{r_C} = \frac{GM}{h_{BC}^2} + C\cos 180° = \frac{GM}{h_{BC}^2} - C$$

Adding, $$\frac{1}{r_B} + \frac{1}{r_C} = \frac{r_B + r_C}{r_B r_C} = \frac{2GM}{h_{BC}^2}$$

$$h_{BC} = \sqrt{\frac{2GMr_Br_C}{r_B + r_C}} = \sqrt{\frac{(2)(11.543\times10^{15})(280.01\times10^6)(29.568\times10^6)}{309.578\times10^6}}$$

$$= 785.755\times10^9 \text{ ft}^2/\text{s}$$

$$(v_B)_2 = \frac{h_{BC}}{r_B} = \frac{785.755\times10^9}{280.01\times10^6} = 2.8062\times10^3 \text{ ft/s}$$

$$(v_C)_1 = \frac{h_{BC}}{r_C} = \frac{785.755\times10^9}{29.568\times10^6} = 26.575\times10^3 \text{ ft/s}$$

Circular orbit with $r_C = 29.568\times10^6 \text{ ft}$

$$(v_C)_2 = \sqrt{\frac{GM}{r_C}} = \sqrt{\frac{11.543\times10^{15}}{29.568\times10^6}} = 19.758\times10^3 \text{ ft/s}$$

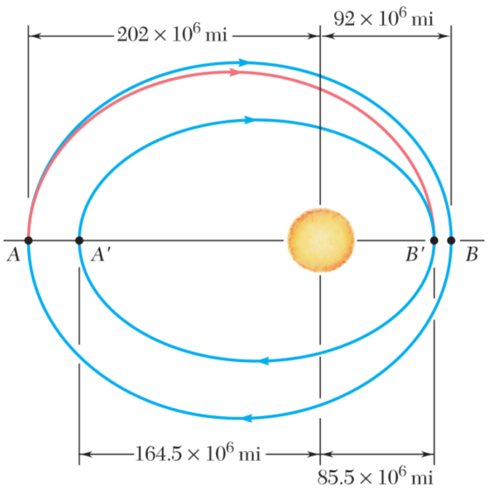
(*b*) Speed reductions at *B* and *C*.

At *B*: $(v_B)_1 - (v_B)_2 = 3.5073\times10^3 - 2.8062\times10^3$

$\Delta v_B = 701 \text{ ft/s}$ ◀

At *C*: $(v_C)_1 - (v_C)_2 = 26.575\times10^3 - 19.758\times10^3$

$\Delta v_C = 6.82\times10^3 \text{ ft/s}$ ◀



PROBLEM 12.107

As it describes an elliptic orbit about the sun, a spacecraft reaches a maximum distance of 202×10^6 mi from the center of the sun at Point *A* (called the aphelion) and a minimum distance of 92×10^6 mi at Point *B* (called the perihelion). To place the spacecraft in a smaller elliptic orbit with aphelion at A' and perihelion at B', where A' and B' are located 164.5×10^6 mi and 85.5×10^6 mi, respectively, from the center of the sun, the speed of the spacecraft is first reduced as it passes through *A* and then is further reduced as it passes through B'. Knowing that the mass of the sun is 332.8×10^3 times the mass of the earth, determine (*a*) the speed of the spacecraft at *A*, (*b*) the amounts by which the speed of the spacecraft should be reduced at *A* and B' to insert it into the desired elliptic orbit.

SOLUTION

First note

$$R_{\text{earth}} = 3960 \text{ mi} = 20.9088 \times 10^6 \text{ ft}$$

$$r_A = 202 \times 10^6 \text{ mi} = 1066.56 \times 10^9 \text{ ft}$$

$$r_B = 92 \times 10^6 \text{ mi} = 485.76 \times 10^9 \text{ ft}$$

From the solution to Problem 12.102, we have for any elliptic orbit about the sun

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2GM_{\text{sun}}}{h^2}$$

(*a*) For the elliptic orbit *AB*, we have

$$r_1 = r_A, \quad r_2 = r_B, \quad h = h_A = r_A v_A$$

Also,

$$GM_{\text{sun}} = G[(332.8 \times 10^3)M_{\text{earth}}]$$

$$= gR_{\text{earth}}^2(332.8 \times 10^3) \qquad \text{using Eq. (12.30).}$$

Then

$$\frac{1}{r_A} + \frac{1}{r_B} = \frac{2gR_{\text{earth}}^2(332.8 \times 10^3)}{(r_A v_A)^2}$$

or

$$v_A = \frac{R_{\text{earth}}}{r_A}\left(\frac{665.6g \times 10^3}{\frac{1}{r_A} + \frac{1}{r_B}}\right)^{1/2}$$

$$= \frac{3960 \text{ mi}}{202 \times 10^6 \text{ mi}}\left(\frac{665.6 \times 10^3 \times 32.2 \text{ ft/s}^2}{\frac{1}{1066.56 \times 10^9 \text{ ft}} + \frac{1}{485.76 \times 10^9 \text{ ft}}}\right)^{1/2}$$

$$= 52,431 \text{ ft/s}$$

or $v_A = 52.4 \times 10^3$ ft/s ◀

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PROBLEM 12.107 (Continued)

(*b*) From Part (*a*), we have

$$2GM_{\text{sun}} = (r_A v_A)^2\left(\frac{1}{r_A}+\frac{1}{r_B}\right)$$

Then, for any other elliptic orbit about the sun, we have

$$\frac{1}{r_1}+\frac{1}{r_2} = \frac{(r_A v_A)^2\left(\frac{1}{r_A}+\frac{1}{r_B}\right)}{h^2}$$

For the elliptic transfer orbit AB', we have

$$r_1 = r_A, \quad r_2 = r_{B'}, \quad h = h_{\text{tr}} = r_A (v_A)_{\text{tr}}$$

Then $$\frac{1}{r_A}+\frac{1}{r_{B'}} = \frac{(r_A v_A)^2\left(\frac{1}{r_A}+\frac{1}{r_B}\right)}{[r_A (v_A)_{\text{tr}}]^2}$$

or $$(v_A)_{\text{tr}} = v_A\left(\frac{\frac{1}{r_A}+\frac{1}{r_B}}{\frac{1}{r_A}+\frac{1}{r_{B'}}}\right)^{1/2} = v_A\left(\frac{1+\frac{r_A}{r_B}}{1+\frac{r_A}{r_{B'}}}\right)^{1/2}$$

$$= (52{,}431 \text{ ft/s})\left(\frac{1+\frac{202}{92}}{1+\frac{202}{85.5}}\right)^{1/2}$$

$$= 51{,}113 \text{ ft/s}$$

Now $$h_{\text{tr}} = (h_A)_{\text{tr}} = (h_{B'})_{\text{tr}}: \quad r_A (v_A)_{\text{tr}} = r_{B'} (v_{B'})_{\text{tr}}$$

Then $$(v_{B'})_{\text{tr}} = \frac{202\times 10^6 \text{ mi}}{85.5\times 10^6 \text{ mi}} \times 51{,}113 \text{ ft/s} = 120{,}758 \text{ ft/s}$$

For the elliptic orbit $A'B'$, we have

$$r_1 = r_{A'}, \quad r_2 = r_{B'}, \quad h = r_{B'} v_{B'}$$

Then $$\frac{1}{r_{A'}}+\frac{1}{r_{B'}} = \frac{(r_A v_A)^2\left(\frac{1}{r_A}+\frac{1}{r_B}\right)}{(r_{B'} v_{B'})^2}$$

or $$v_{B'} = v_A \frac{r_A}{r_{B'}}\left(\frac{\frac{1}{r_A}+\frac{1}{r_B}}{\frac{1}{r_{A'}}+\frac{1}{r_{B'}}}\right)^{1/2}$$

$$= (52{,}431 \text{ ft/s})\frac{202\times 10^6 \text{ mi}}{85.5\times 10^6 \text{ mi}}\left(\frac{\frac{1}{202\times 10^6}+\frac{1}{92\times 10^6}}{\frac{1}{164.5\times 10^6}+\frac{1}{85.5\times 10^6}}\right)^{1/2}$$

$$= 116{,}862 \text{ ft/s}$$

Finally, $(v_A)_{\text{tr}} = v_A + \Delta v_A$

or $\Delta v_A = (51{,}113 - 52{,}431) \text{ ft/s}$

or $|\Delta v_A| = 1318 \text{ ft/s}$ ◀

and $v_{B'} = (v_{B'})_{\text{tr}} + \Delta v_B$

or
$$\begin{aligned}\Delta v_{B'} &= (116{,}862 - 120{,}758) \text{ ft/s}\\ &= -3896 \text{ ft/s}\end{aligned}$$

or $|\Delta v_B| = 3900 \text{ ft/s}$ ◀

PROBLEM 12.108

Halley's comet travels in an elongated elliptic orbit for which the minimum distance from the sun is approximately $\frac{1}{2}r_E$, where $r_E = 150 \times 10^6$ km is the mean distance from the sun to the earth. Knowing that the periodic time of Halley's comet is about 76 years, determine the maximum distance from the sun reached by the comet.

SOLUTION

We apply Kepler's Third Law to the orbits and periodic times of earth and Halley's comet:

$$\left(\frac{\tau_H}{\tau_E}\right)^2 = \left(\frac{a_H}{a_E}\right)^3$$

Thus

$$\begin{aligned} a_H &= a_E\left(\frac{\tau_H}{\tau_E}\right)^{2/3} \\ &= r_E\left(\frac{76 \text{ years}}{1 \text{ year}}\right)^{2/3} \\ &= 17.94 r_E \end{aligned}$$

But

$$\begin{aligned} a_H &= \frac{1}{2}(r_{\min} + r_{\max}) \\ 17.94 r_E &= \frac{1}{2}\left(\frac{1}{2}r_E + r_{\max}\right) \\ r_{\max} &= 2(17.94 r_E) - \frac{1}{2}r_E \\ &= (35.88 - 0.5)r_E \\ &= 35.38\, r_E \\ r_{\max} &= (35.38)(150 \times 10^6 \text{ km}) \end{aligned}$$

$r_{\max} = 5.31 \times 10^9$ km ◀

PROBLEM 12.109

Based on observations made during the 1996 sighting of comet Hyakutake, it was concluded that the trajectory of the comet is a highly elongated ellipse for which the eccentricity is approximately $\varepsilon = 0.999887$. Knowing that for the 1996 sighting the minimum distance between the comet and the sun was $0.230R_E$, where R_E is the mean distance from the sun to the earth, determine the periodic time of the comet.

SOLUTION

For Earth's orbit about the sun,

$$v_0 = \sqrt{\frac{GM}{R_E}}, \quad \tau_0 = \frac{2\pi R_E}{v_0} = \frac{2\pi R_E^{3/2}}{\sqrt{GM}} \quad \text{or} \quad \sqrt{GM} = \frac{2\pi R_E^{3/2}}{\tau_0} \qquad (1)$$

For the comet Hyakutake,

$$\frac{1}{r_0} = \frac{GM}{h^2} = (1+\varepsilon), \quad \frac{1}{r_1} = \frac{GM}{h^2}(1+\varepsilon), \quad r_1 = \frac{1+\varepsilon}{1-\varepsilon} r_0$$

$$a = \frac{1}{2}(r_0 + r_1) = \frac{r_0}{1-\varepsilon}, \quad b = \sqrt{r_0 r_1} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\, r_0$$

$$h = \sqrt{GMr_0(1+\varepsilon)}$$

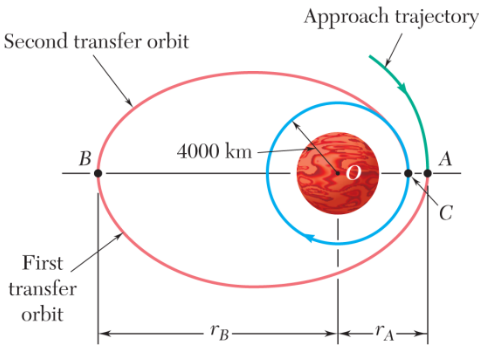
$$\tau = \frac{2\pi ab}{h} = \frac{2\pi r_0^2 (1+\varepsilon)^{1/2}}{(1-\varepsilon)^{3/2}\sqrt{GMr_0(1+\varepsilon)}}$$

$$= \frac{2\pi r_0^{3/2}}{\sqrt{GM}\,(1-\varepsilon)^{3/2}} = \frac{2\pi r_0^{3/2}\tau_0}{2\pi R_E^3 (1-\varepsilon)^{3/2}}$$

$$= \left(\frac{r_0}{R_E}\right)^{3/2} \frac{1}{(1-\varepsilon)^{3/2}}\,\tau_0$$

$$= (0.230)^{3/2} \frac{1}{(1-0.999887)^{3/2}}\,\tau_0 = 91.8 \times 10^3 \tau_0$$

Since $\tau_0 = 1$ yr, $\tau = (91.8 \times 10^3)(1.000)$ $\qquad \tau = 91.8 \times 10^3$ yr ◀



PROBLEM 12.110

A space probe is to be placed in a circular orbit of radius 4000 km about the planet Mars. As the probe reaches *A*, the point of its original trajectory closest to Mars, it is inserted into a first elliptic transfer orbit by reducing its speed. This orbit brings it to Point *B* with a much reduced velocity. There the probe is inserted into a second transfer orbit by further reducing its speed. Knowing that the mass of Mars is 0.1074 times the mass of the earth, that $r_A = 9000$ km and $r_B = 180{,}000$ km, and that the probe approaches *A* on a parabolic trajectory, determine the time needed for the space probe to travel from *A* to *B* on its first transfer orbit.

SOLUTION

For earth, $R = 6373 \text{ km} = 6.373 \times 10^6 \text{ m}$

$$GM = gR^2 = (9.81)(6.373 \times 10^6)^2 = 398.43 \times 10^{12} \text{ m}^3/\text{s}^2$$

For Mars, $GM = (0.1074)(398.43 \times 10^{12}) = 42.792 \times 10^{12} \text{ m}^3/\text{s}^2$

$$r_A = 9000 \text{ km} = 9.0 \times 10^6 \text{ m} \qquad r_B = 180000 \text{ km} = 180 \times 10^6 \text{ m}$$

For the parabolic approach trajectory at *A*,

$$(v_A)_1 = \sqrt{\frac{2GM}{r_A}} = \sqrt{\frac{(2)(42.792 \times 10^{12})}{9.0 \times 10^6}} = 3.0837 \times 10^3 \text{ m/s}$$

First elliptic transfer orbit *AB*.

Using Eq. (12.39), $\frac{1}{r_A} = \frac{GM}{h_{AB}^2} + C\cos\theta_A$ and $\frac{1}{r_B} = \frac{GM}{h_{AB}^2} + C\cos\theta_B$.

But $\theta_B = \theta_A + 180°$, so that $\cos\theta_A = -\cos\theta_B$.

Adding, $\frac{1}{r_A} + \frac{1}{r_B} = \frac{r_A + r_B}{r_A r_B} = \frac{2GM}{h_{AB}^2}$

$$h_{AB} = \sqrt{\frac{2GMr_A r_B}{r_A + r_B}} = \sqrt{\frac{(2)(42.792 \times 10^{12})(9.0 \times 10^6)(180 \times 10^6)}{189.0 \times 10^6}}$$

$$h_{AB} = 27.085 \times 10^9 \text{ m}^2/\text{s}$$

$$a = \frac{1}{2}(r_A + r_B) = \frac{1}{2}(9.0 \times 10^6 + 180 \times 10^6) = 94.5 \times 10^6 \text{ m}$$

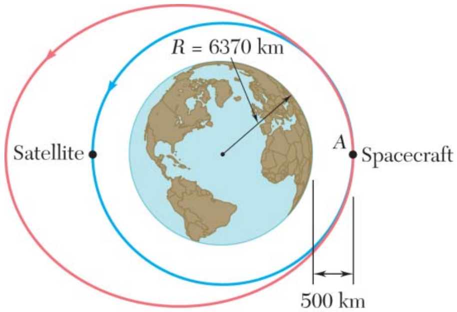
$$b = \sqrt{r_A r_B} = \sqrt{(9.0 \times 10^6)(180 \times 10^6)} = 40.249 \times 10^6 \text{ m}$$

PROBLEM 12.110 (Continued)

Periodic time for full ellipse: $\tau = \dfrac{2\pi ab}{h}$

For half ellipse *AB*, $\tau_{AB} = \dfrac{1}{2}\tau = \dfrac{\pi ab}{h}$

$$\tau_{AB} = \frac{\pi(94.5 \times 10^6)(40.249 \times 10^6)}{27.085 \times 10^9} = 444.81 \times 10^3 \text{ s} \qquad \tau_{AB} = 122.6 \text{ h} \blacktriangleleft$$



PROBLEM 12.111

A spacecraft and a satellite are at diametrically opposite positions in the same circular orbit of altitude 500 km above the earth. As it passes through point A, the spacecraft fires its engine for a short interval of time to increase its speed and enter an elliptic orbit. Knowing that the spacecraft returns to A at the same time the satellite reaches A after completing one and a half orbits, determine (a) the increase in speed required, (b) the periodic time for the elliptic orbit.

SOLUTION

For earth, $R = 6.37 \times 10^6$ m, $g = 9.81$ m/s^2

$$GM = gR^2 = (9.81)\left(6.37 \times 10^6\right)^2 = 398.06 \times 10^{12} \text{ m}^3/\text{s}^2$$

For circular orbit of satellite,

$$r_0 = 6370 + 500 = 6870 \text{ km} = 6.87 \times 10^6 \text{ m/s}$$

$$v_0 = \sqrt{\frac{GM}{r_0}} = \sqrt{\frac{398.06 \times 10^{12}}{6.87 \times 10^6}} = 7.6119 \times 10^3 \text{ m/s}$$

$$\tau_0 = \frac{2\pi r_0}{v_0} = \frac{(2\pi)\left(6.87 \times 10^6\right)}{7.6119 \times 10^3} = 5.6708 \times 10^3 \text{ s}$$

For elliptic orbit of spacecraft it is given that

$$\tau = \frac{3}{2}\tau_0 = 8.5062 \times 10^3 \text{ s}$$

$$a = \frac{1}{2}(r_A + r_B), \qquad b = \sqrt{r_A r_B}$$

Using Eq. (12.39), $\frac{1}{r_A} = \frac{GM}{h^2} + C\cos\theta_A$ and $\frac{1}{r_B} = \frac{GM}{h^2} + C\cos\theta_B$.

But $\theta_B = \theta_A + 180°$, so that $\cos\theta_A = -\cos\theta_B$.

Adding, $\frac{1}{r_A} + \frac{1}{r_B} = \frac{r_A + r_B}{r_A r_B} = \frac{2a}{b^2} = \frac{2GM}{h^2}$ or $h = \sqrt{\frac{GMb^2}{a}}$

Periodic time: $\tau = \frac{2\pi ab}{h} = \frac{2\pi ab\sqrt{a}}{\sqrt{GMb^2}} = \frac{2\pi a^{3/2}}{\sqrt{GM}}$

$$a^3 = \frac{GM\tau^2}{4\pi^2} = \frac{\left(398.06 \times 10^{12}\right)\left(8.5062 \times 10^3\right)^2}{4\pi^2} = 729.558 \times 10^{18} \text{ m}^3$$

$$a = 9.0023 \times 10^6 \text{ m}, \qquad r_A = r_0 = 6.87 \times 10^6 \text{ m}$$

$$r_B = 2a - r_A = 11.1346 \times 10^6 \text{ m}, \qquad b = \sqrt{r_A r_B} = 8.7461 \times 10^6 \text{ m}$$

$$h = \frac{2\pi ab}{\tau} = \frac{2\pi\left(9.0023 \times 10^6\right)\left(8.7461 \times 10^6\right)}{8.5062 \times 10^3} = 58.159 \times 10^9 \text{ m}^2\text{/s}$$

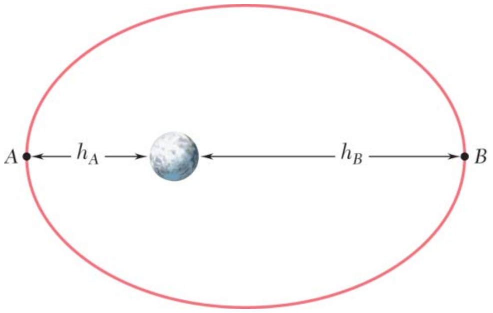
$$v_A = \frac{h}{r_A} = \frac{58.159 \times 10^9}{6.87 \times 10^6} = 8.4656 \times 10^3 \text{ m/s}$$

(*a*) Increase in speed at *A*.

$$\Delta v_A = v_A - v_0 = 8.4656 \times 10^3 - 7.6119 \times 10^3 \qquad \Delta v_A = 854 \text{ m/s} \blacktriangleleft$$

(*b*) Periodic time for elliptic orbit.

As calculated above $\tau = 8.5062$ s $\qquad \tau = 141.8$ min ◀



PROBLEM 12.112

The Clementine spacecraft described an elliptic orbit of minimum altitude $h_A = 400$ km and a maximum altitude of $h_B = 2940$ km above the surface of the moon. Knowing that the radius of the moon is 1737 km and that the mass of the moon is 0.01230 times the mass of the earth, determine the periodic time of the spacecraft.

SOLUTION

For earth, $R = 6370 \text{ km} = 6.370 \times 10^6 \text{ m}$

$$GM = gR^2 = (9.81)(6.370 \times 10^6)^2 = 398.06 \times 10^{12} \text{ m}^3/\text{s}^2$$

For moon, $GM = (0.01230)(398.06 \times 10^{12}) = 4.896 \times 10^{12} \text{ m}^3/\text{s}^2$

$$r_A = 1737 + 400 = 2137 \text{ km} = 2.137 \times 10^6 \text{ m}$$

$$r_B = 1737 + 2940 = 4677 \text{ km} = 4.677 \times 10^6 \text{ m}$$

Using Eq. (12.39), $\dfrac{1}{r_A} = \dfrac{GM}{h^2} + C\cos\theta_A \quad \text{and} \quad \dfrac{1}{r_B} = \dfrac{GM}{h^2} + C\cos\theta_B.$

But $\theta_B = \theta_A + 180°, \quad \text{so that} \quad \cos\theta_A = -\cos\theta_B.$

Adding, $\dfrac{1}{r_A} + \dfrac{1}{r_B} = \dfrac{r_A + r_B}{r_A r_B} = \dfrac{2GM}{h_{AB}^2}$

$$h_{AB} = \sqrt{\frac{2GMr_Ar_B}{r_A + r_B}} = \sqrt{\frac{(2)(4.896 \times 10^{12})(2.137 \times 10^6)(4.677 \times 10^6)}{6.814 \times 10^6}}$$

$$= 3.78983 \times 10^9 \text{ m}^2/\text{s}$$

$$a = \frac{1}{2}(r_A + r_B) = 3.402 \times 10^6 \text{ m}$$

$$b = \sqrt{r_A r_B} = 3.16145 \times 10^6 \text{ m}$$

Periodic time.

$$\tau = \frac{2\pi ab}{h_{AB}} = \frac{2\pi(3.402 \times 10^6)(3.16145 \times 10^6)}{3.78983 \times 10^9} = 17.831 \times 10^3 \text{ s}$$

$\tau = 4.95$ h ◀

PROBLEM 12.113

Determine the time needed for the space probe of Problem 12.100 to travel from *B* to *C*.

SOLUTION

From the solution to Problem 12.100, we have

$$(v_A)_{\text{par}} = 10{,}131.4 \text{ m/s}$$

and
$$(v_A)_{\text{circ}} = \frac{1}{\sqrt{2}}(v_A)_{\text{par}} = 7164.0 \text{ m/s}$$

Also,
$$r_A = (6052 + 280) \text{ km} = 6332 \text{ km}$$

For the parabolic trajectory *BA*, we have

$$\frac{1}{r} = \frac{GM_v}{h_{BA}^2}(1 + \varepsilon\cos\theta) \qquad \text{[Eq. (12.39′)]}$$

where $\varepsilon = 1$. Now

at A, $\theta = 0$:
$$\frac{1}{r_A} = \frac{GM_v}{h_{BA}^2}(1+1)$$

or
$$r_A = \frac{h_{BA}^2}{2GM_v}$$

at B, $\theta = -90°$:
$$\frac{1}{r_B} = \frac{GM_v}{h_{BA}^2}(1+0)$$

or
$$r_B = \frac{h_{BA}^2}{GM_v}$$

$$r_B = 2r_A$$

As the probe travels from *B* to *A*, the area swept out is the semiparabolic area defined by Vertex *A* and Point *B*. Thus,

$$(\text{Area swept out})_{BA} = A_{BA} = \frac{2}{3}r_A r_B = \frac{4}{3}r_A^2$$

Now
$$\frac{dA}{dt} = \frac{1}{2}h$$

where $h = \text{constant}$

Then $$A = \frac{1}{2}ht \quad \text{or} \quad t_{BA} = \frac{2A_{BA}}{h_{BA}} \qquad h_{BA} = r_A v_A$$

$$t_{BA} = \frac{2 \times \frac{4}{3} r_A^2}{r_A v_A} = \frac{8}{3}\frac{r_A}{v_A}$$

$$= \frac{8}{3}\frac{6332 \times 10^3 \text{ m}}{10{,}131.4 \text{ m/s}}$$

$$= 1666.63 \text{ s}$$

For the circular trajectory *AC*,

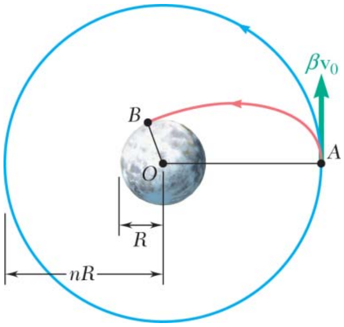
$$t_{AC} = \frac{\frac{\pi}{2} r_A}{(v_A)_{\text{circ}}} = \frac{\pi}{2}\frac{6332 \times 10^3 \text{ m}}{7164.0 \text{ m/s}} = 1388.37 \text{ s}$$

Finally, $$t_{BC} = t_{BA} + t_{AC}$$

$$= (1666.63 + 1388.37) \text{ s}$$

$$= 3055.0 \text{ s}$$

or $$t_{BC} = 50 \text{ min } 55 \text{ s} \blacktriangleleft$$



PROBLEM 12.114

A space probe is describing a circular orbit of radius nR with a velocity v_0 about a planet of radius R and center O. As the probe passes through Point A, its velocity is reduced from v_0 to βv_0, where $\beta < 1$, to place the probe on a crash trajectory. Express in terms of n and β the angle AOB, where B denotes the point of impact of the probe on the planet.

SOLUTION

For the circular orbit,

$$r_0 = r_A = nR$$

$$v_0 = \sqrt{\frac{GM}{r_0}} = \sqrt{\frac{GM}{nR}}$$

The crash trajectory is elliptic.

$$v_A = \beta v_0 = \sqrt{\frac{\beta^2 GM}{nR}}$$

$$h = r_A v_A = nRv_A = \sqrt{\beta^2 nGMR}$$

$$\frac{GM}{h^2} = \frac{1}{\beta^2 nR}$$

$$\frac{1}{r} = \frac{GM}{h^2}(1+\varepsilon\cos\theta) = \frac{1+\varepsilon\cos\theta}{\beta^2 nR}$$

At Point A, $\theta = 180°$

$$\frac{1}{r_A} = \frac{1}{nR} = \frac{1-\varepsilon}{\beta^2 nR} \quad \text{or} \quad \beta^2 = 1-\varepsilon \quad \text{or} \quad \varepsilon = 1-\beta^2$$

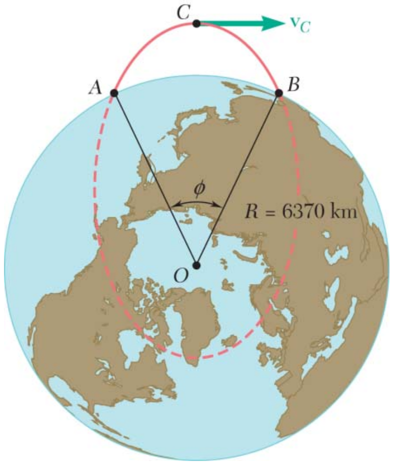
At impact Point B, $\theta = \pi - \phi$

$$\frac{1}{r_B} = \frac{1}{R}$$

$$\frac{1}{R} = \frac{1+\varepsilon\cos(\pi-\phi)}{\beta^2 nR} = \frac{1-\varepsilon\cos\phi}{\beta^2 nR}$$

$$\varepsilon\cos\phi = 1-n\beta^2 \quad \text{or} \quad \cos\phi = \frac{1-n\beta^2}{\varepsilon} = \frac{1-n\beta^2}{1-\beta^2}$$

$$\phi = \cos^{-1}[(1-n\beta^2)/(1-\beta^2)] \blacktriangleleft$$



PROBLEM 12.115

A long-range ballistic trajectory between Points *A* and *B* on the earth's surface consists of a portion of an ellipse with the apogee at Point *C*. Knowing that Point *C* is 1500 km above the surface of the earth and the range $R\phi$ of the trajectory is 6000 km, determine (*a*) the velocity of the projectile at *C*, (*b*) the eccentricity ε of the trajectory.

SOLUTION

For earth, $R = 6370 \text{ km} = 6.37 \times 10^6 \text{ m}$

$$GM = gR^2 = (9.81)(6.37 \times 10^6)^2 = 398.06 \times 10^{12} \text{ m}^3/\text{s}^2$$

For the trajectory, $r_C = 6370 + 1500 = 7870 \text{ km} = 7.87 \times 10^6 \text{ m}$

$$r_A = r_B = R = 6.37 \times 10^6 \text{ m}, \qquad \frac{r_C}{r_A} = \frac{7870}{6370} = 1.23548$$

Range *A* to *B*: $s_{AB} = 6000 \text{ km} = 6.00 \times 10^6 \text{ m}$

$$\varphi = \frac{s_{AB}}{R} = \frac{6.00 \times 10^6}{6.37 \times 10^6} = 0.94192 \text{ rad} = 53.968°$$

For an elliptic trajectory, $\dfrac{1}{r} = \dfrac{GM}{h^2}(1 + \varepsilon\cos\theta)$

At *A*, $\theta = 180° - \dfrac{\varphi}{2} = 153.016°, \qquad \dfrac{1}{r_A} = \dfrac{GM}{h^2}(1 + \varepsilon\cos 153.016°)$ (1)

At *C*, $\theta = 180°, \qquad \dfrac{1}{r_C} = \dfrac{GM}{h^2}(1 - \varepsilon)$ (2)

Dividing Eq. (1) by Eq. (2),

$$\frac{r_C}{r_A} = \frac{1 + \varepsilon\cos 153.016°}{1 - \varepsilon} = 1.23548$$

$$\varepsilon = \frac{1.23548 - 1}{1.23548 + \cos 153.016°} = 0.68384$$

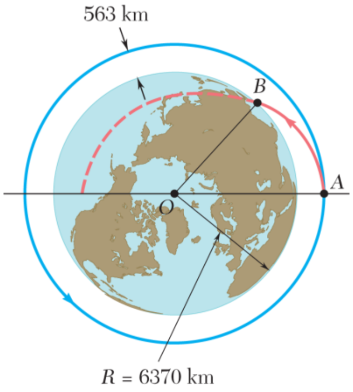
PROBLEM 12.115 (Continued)

From Eq. (2), $h = \sqrt{GM(1-\varepsilon)r_C}$

$$h = \sqrt{(398.06\times 10^{12})(0.31616)(7.87\times 10^{6})} = 31.471\times 10^{9}\ \text{m}^2/\text{s}$$

(*a*) Velocity at *C*. $v_C = \dfrac{h}{r_C} = \dfrac{31.471\times 10^{9}}{7.87\times 10^{6}} = 4.00\times 10^{3}\ \text{m/s}$ $v_C = 4\ \text{km/s}$ ◀

(*b*) Eccentricity of trajectory. $\varepsilon = 0.684$ ◀



PROBLEM 12.116

A space shuttle is describing a circular orbit at an altitude of 563 km above the surface of the earth. As it passes through Point *A*, it fires its engine for a short interval of time to reduce its speed by 152 m/s and begin its descent toward the earth. Determine the angle *AOB* so that the altitude of the shuttle at Point *B* is 121 km. (*Hint:* Point *A* is the apogee of the elliptic descent trajectory.)

SOLUTION

$$GM = gR^2 = (9.81)(6.37\times 10^6)^2 = 398.06\times 10^{12}\ \text{m}^3/\text{s}^2$$

$$r_A = 6370 + 563 = 6933\ \text{km} = 6.933\times 10^6\ \text{m}$$

$$r_B = 6370 + 121 = 6491\ \text{km} = 6.491\times 10^6\ \text{m}$$

For the circular orbit through Point *A*,

$$v_{\text{circ}} = \sqrt{\frac{GM}{r_A}} = \sqrt{\frac{398.06\times 10^{12}}{6.933\times 10^6}} = 7.5773\times 10^3\ \text{m/s}$$

For the descent trajectory,

$$v_A = v_{\text{circ}} + \Delta v = 7.5773\times 10^3 - 152 = 7.4253\times 10^3\ \text{m/s}$$

$$h = r_A v_A = (6.933\times 10^6)(7.4253\times 10^3) = 51.4795\times 10^9\ \text{m}^2/\text{s}$$

$$\frac{1}{r} = \frac{GM}{h^2}(1+\varepsilon\cos\theta)$$

At Point *A*, $\theta = 180°$, $r = r_A$

$$\frac{1}{r_A} = \frac{GM}{h^2}(1-\varepsilon)$$

$$1-\varepsilon = \frac{h^2}{GM\ r_A} = \frac{(51.4795\times 10^9)^2}{(398.06\times 10^{12})(6.933\times 10^6)} = 0.96028$$

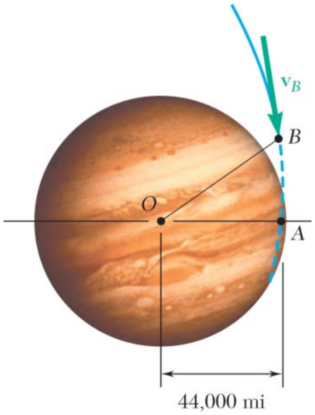
$$\varepsilon = 0.03972$$

$$\frac{1}{r_B} = \frac{GM}{h^2}(1+\varepsilon\cos\theta_B)$$

$$1+\varepsilon\cos\theta_B = \frac{h^2}{GM\ r_B} = \frac{(51.4795\times 10^9)^2}{(398.06\times 10^{12})(6.491\times 10^6)} = 1.02567$$

$$\cos\theta_B = \frac{1.02567-1}{\varepsilon} = 0.6463$$

$\theta_B = 49.7°$ $\qquad$ $\measuredangle AOB = 180° - \theta_B = 130.3°$ $\qquad$ $\measuredangle AOB = 130.3°$ ◀



PROBLEM 12.117

As a spacecraft approaches the planet Jupiter, it releases a probe which is to enter the planet's atmosphere at Point B at an altitude of 280 mi above the surface of the planet. The trajectory of the probe is a hyperbola of eccentricity $\varepsilon = 1.031$. Knowing that the radius and the mass of Jupiter are 44423 mi and 1.30×10^{26} slug, respectively, and that the velocity $\mathbf{v}_B$ of the probe at B forms an angle of 82.9° with the direction of OA, determine (a) the angle AOB, (b) the speed v_B of the probe at B.

SOLUTION

First we note $$r_B = (44.423 \times 10^3 + 280)\text{ mi} = 44.703 \times 10^3\text{ mi}$$

(a) We have $$\frac{1}{r} = \frac{GM_j}{h^2}(1 + \varepsilon\cos\theta) \qquad \text{[Eq. (12.39′)]}$$

At A, $\theta = 0$: $$\frac{1}{r_A} = \frac{GM_j}{h^2}(1 + \varepsilon)$$

or $$\frac{h^2}{GM_j} = r_A(1 + \varepsilon)$$

At B, $\theta = \theta_B = \measuredangle AOB$: $$\frac{1}{r_B} = \frac{GM_j}{h^2}(1 + \varepsilon\cos\theta_B)$$

or $$\frac{h^2}{GM_j} = r_B(1 + \varepsilon\cos\theta_B)$$

Then $$r_A(1 + \varepsilon) = r_B(1 + \varepsilon\cos\theta_B)$$

or $$\cos\theta_B = \frac{1}{\varepsilon}\left[\frac{r_A}{r_B}(1 + \varepsilon) - 1\right]$$
$$= \frac{1}{1.031}\left[\frac{44.0 \times 10^3\text{ mi}}{44.703 \times 10^3\text{ mi}}(1 + 1.031) - 1\right]$$
$$= 0.96902$$

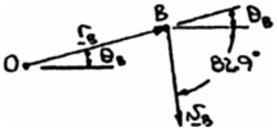
or $$\theta_B = 14.2988°$$ $$\measuredangle AOB = 14.30° \blacktriangleleft$$

PROBLEM 12.117 (Continued)

(*b*) From above $$h^2 = GM_j r_B(1+\varepsilon\cos\theta_B)$$

where $$h = \frac{1}{m}|\mathbf{r}_B \times m\mathbf{v}_B| = r_B v_B \sin\phi$$

$$\phi = (\theta_B + 82.9°) = 97.1988°$$



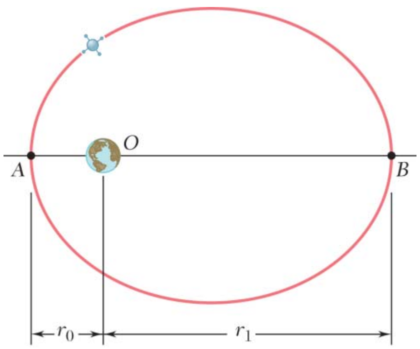
Then $$(r_B v_B \sin\phi)^2 = GM_j r_B(1+\varepsilon\cos\theta_B)$$

or

$$v_B = \frac{1}{\sin\phi}\left[\frac{GM_j}{r_B}(1+\varepsilon\cos\theta_B)\right]^{1/2}$$

$$= \frac{1}{\sin 97.1988°}\left\{\frac{34.4\times10^{-9}\ \text{ft}^4/\text{lb}\cdot\text{s}^4 \times (1.30\times10^{26}\ \text{slug})}{236.03\times10^6\ \text{ft}}\times[1+(1.031)(0.96902)]\right\}^{1/2}$$

or $$v_B = 196.2\ \text{ft/s} \blacktriangleleft$$



PROBLEM 12.118

A satellite describes an elliptic orbit about a planet. Denoting by r_0 and r_1 the distances corresponding, respectively, to the perigee and apogee of the orbit, show that the curvature of the orbit at each of these two points can be expressed as

$$\frac{1}{\rho}=\frac{1}{2}\left(\frac{1}{r_0}+\frac{1}{r_1}\right)$$

SOLUTION

Using Eq. (12.39),
$$\frac{1}{r_A}=\frac{GM}{h^2}+C\cos\theta_A$$

and
$$\frac{1}{r_B}=\frac{GM}{h^2}+C\cos\theta_B.$$

But
$$\theta_B=\theta_A+180°,$$

so that
$$\cos\theta_A=-\cos\theta_B$$

Adding,
$$\frac{1}{r_A}+\frac{1}{r_B}=\frac{2GM}{h^2}$$

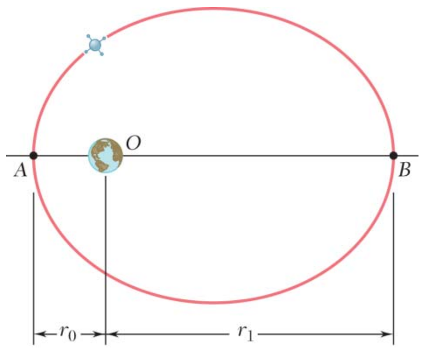
At Points *A* and *B*, the radial direction is normal to the path.

$$a_n=\frac{v^2}{\rho}=\frac{h^2}{r^2\rho}$$

But
$$F_n=\frac{GMm}{r^2}=ma_n=\frac{mh^2}{r^2\rho}$$

$$\frac{1}{\rho}=\frac{GM}{h^2}=\frac{1}{2}\left(\frac{1}{r_A}+\frac{1}{r_B}\right)$$

$$\frac{1}{\rho}=\frac{1}{2}\left(\frac{1}{r_0}+\frac{1}{r_1}\right) \blacktriangleleft$$



PROBLEM 12.119

(*a*) Express the eccentricity ε of the elliptic orbit described by a satellite about a planet in terms of the distances r_0 and r_1 corresponding, respectively, to the perigee and apogee of the orbit. (*b*) Use the result obtained in Part *a* and the data given in Problem 12.109, where $R_E = 149.6 \times 10^6$ km, to determine the approximate maximum distance from the sun reached by comet Hyakutake.

SOLUTION

(*a*) We have $$\frac{1}{r} = \frac{GM}{h^2}(1 + \varepsilon \cos \theta) \qquad \text{Eq. (12.39}'\text{)}$$

At A, $\theta = 0$: $$\frac{1}{r_0} = \frac{GM}{h^2}(1 + \varepsilon)$$

or $$\frac{h^2}{GM} = r_0(1 + \varepsilon)$$

At B, $\theta = 180°$: $$\frac{1}{r_1} = \frac{GM}{h^2}(1 - \varepsilon)$$

or $$\frac{h^2}{GM} = r_1(1 - \varepsilon)$$

Then $$r_0(1 + \varepsilon) = r_1(1 - \varepsilon)$$

or $$\varepsilon = \frac{r_1 - r_0}{r_1 + r_0} \blacktriangleleft$$

(*b*) From above, $$r_1 = \frac{1 + \varepsilon}{1 - \varepsilon} r_0$$

where $$r_0 = 0.230 R_E$$

Then $$r_1 = \frac{1 + 0.999887}{1 - 0.999887} \times 0.230(149.6 \times 10^9 \text{ m})$$

or $$r_1 = 609 \times 10^{12} \text{ m} \blacktriangleleft$$

Note: $r_1 = 4070 R_E$ or $r_1 = 0.064$ lightyears.

PROBLEM 12.120

Derive Kepler's third law of planetary motion from Eqs. (12.37) and (12.43).

SOLUTION

For an ellipse, $2a = r_A + r_B \quad \text{and} \quad b = \sqrt{r_A r_B}$

Using Eq. (12.37), $$\frac{1}{r_A} = \frac{GM}{h^2} + C\cos\theta_A$$

and $$\frac{1}{r_B} = \frac{GM}{h^2} + C\cos\theta_B.$$

But $\theta_B = \theta_A + 180°,$

so that $\cos\theta_A = -\cos\theta_B.$

Adding, $$\frac{1}{r_A} + \frac{1}{r_B} = \frac{r_A + r_B}{r_A r_B} = \frac{2a}{b^2} = \frac{2GM}{h^2}$$

$$h = b\sqrt{\frac{GM}{a}}$$

By Eq. (12.43),

$$\tau = \frac{2\pi ab}{h} = \frac{2\pi ab\sqrt{a}}{b\sqrt{GM}} = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

$$\tau^2 = \frac{4\pi^2 a^3}{GM}$$

For Orbits 1 and 2 about the same large mass,

$$\tau_1^2 = \frac{4\pi^2 a_1^3}{GM}$$

and $$\tau_2^2 = \frac{4\pi^2 a_2^3}{GM}$$

Forming the ratio, $$\left(\frac{\tau_1}{\tau_2}\right)^2 = \left(\frac{a_1}{a_2}\right)^3 \blacktriangleleft$$

PROBLEM 12.121

Show that the angular momentum per unit mass h of a satellite describing an elliptic orbit of semimajor axis a and eccentricity ε about a planet of mass M can be expressed as

$$h = \sqrt{GMa(1-\varepsilon^2)}$$

SOLUTION

By Eq. (12.39′),

$$\frac{1}{r} = \frac{GM}{h^2}(1+\varepsilon\cos\theta)$$

At A, $\theta = 0°$:

$$\frac{1}{r_A} = \frac{GM}{h^2} = (1+\varepsilon) \qquad \text{or} \qquad r_A = \frac{h^2}{GM(1+\varepsilon)}$$

At B, $\theta = 180°$:

$$\frac{1}{r_B} = \frac{GM}{h^2} = (1-\varepsilon) \qquad \text{or} \qquad r_B = \frac{h^2}{GM(1-\varepsilon)}$$

Adding,

$$r_A + r_B = \frac{h^2}{GM} = \left(\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}\right) = \frac{2h^2}{GM(1-\varepsilon^2)}$$

But for an ellipse,

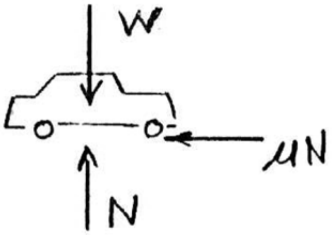
$$r_A + r_B = 2a$$

$$2a = \frac{2h^2}{GM(1-\varepsilon^2)} \qquad h = \sqrt{GMa(1-\varepsilon^2)} \blacktriangleleft$$

PROBLEM 12.122

In the braking test of a sports car its velocity is reduced from 70 mi/h to zero in a distance of 170 ft with slipping impending. Knowing that the coefficient of kinetic friction is 80 percent of the coefficient of static friction, determine (*a*) the coefficient of static friction, (*b*) the stopping distance for the same initial velocity if the car skids. Ignore air resistance and rolling resistance.

SOLUTION



(*a*) Coefficient of static friction.

$$\Sigma F_y = 0: \quad N - W = 0 \qquad N = W$$

$$v_0 = 70 \text{ mi/h} = 102.667 \text{ ft/s}$$

$$\frac{v^2}{2} - \frac{v_0^2}{2} = a_t(s - s_0)$$

$$a_t = \frac{v^2 - v_0^2}{2(s - s_0)} = \frac{0 - (102.667)^2}{(2)(170)} = -31.001 \text{ ft/s}^2$$

For braking without skidding $\mu = \mu_s$, so that $\mu_s N = m|a_t|$

$$\xrightarrow{+} \Sigma F_t = ma_t: \quad -\mu_s N = ma_t$$

$$\mu_s = -\frac{ma_t}{W} = -\frac{a_t}{g} = \frac{31.001}{32.2}$$

$\mu_s = 0.963$ ◀

(*b*) Stopping distance with skidding.

Use $\mu = \mu_k = (0.80)(0.963) = 0.770$

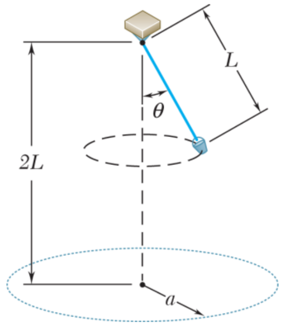
$$\xleftarrow{+} \Sigma F = ma_t: \quad \mu_k N = -ma_t$$

$$a_t = -\frac{\mu_k N}{m} = -\mu_k g = -24.801 \text{ ft/s}^2$$

Since acceleration is constant,

$$(s - s_0) = \frac{v^2 - v_0^2}{2a_t} = \frac{0 - (102.667)^2}{(2)(-24.801)}$$

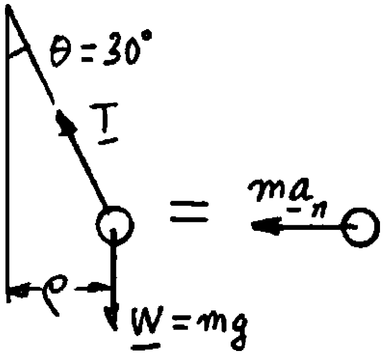
$s - s_0 = 212 \text{ ft}$ ◀

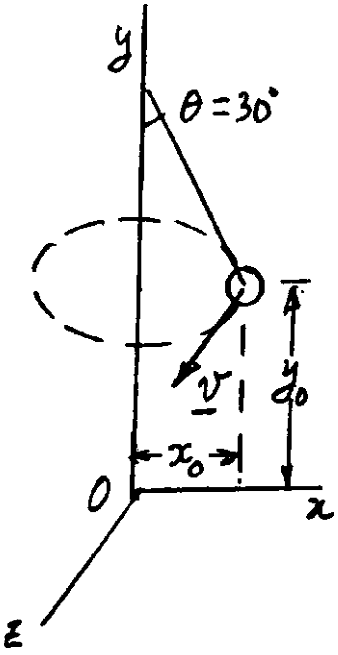


PROBLEM 12.123

A bucket is attached to a rope of length $L = 1.2$ m and is made to revolve in a horizontal circle. Drops of water leaking from the bucket fall and strike the floor along the perimeter of a circle of radius a. Determine the radius a when $\theta = 30°$.

SOLUTION





Initial velocity of drop = velocity of bucket

$$\Sigma F_y = 0: \qquad T\cos 30° = mg \qquad (1)$$

$$\xleftarrow{+} \Sigma F_x = ma_x: \qquad T\sin 30° = ma_n \qquad (2)$$

Divide (2) by (1): $\tan 30° = \dfrac{a_n}{g} = \dfrac{v^2}{\rho g}$

Thus $\quad v^2 = \rho g \tan 30°$

But $\quad \rho = L\sin 30° = (1.2 \text{ m}) \sin 30° = 0.6 \text{ m}$

Thus $\quad v^2 = 0.6(9.81)\tan 30° = 3.398 \text{ m}^2/\text{s}^2 \qquad v = 1.843 \text{ m/s}$

Assuming the bucket to rotate clockwise (when viewed from above), and using the axes shown, we find that the components of the initial velocity of the drop are

$$(v_0)_x = 0,\ (v_0)_y = 0,\ (v_0)_2 = 1.843 \text{ m/s}$$

Free fall of drop

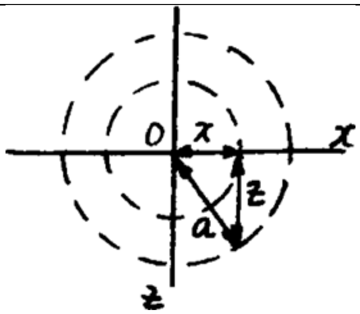
$$y = y_0 + (v_0)_y t - \frac{1}{2}gt^2 \qquad y = y_0 - \frac{1}{2}gt^2$$

When drop strikes floor:

$$y = 0 \qquad y_0 - \frac{1}{2}gt^2 = 0$$

But $\quad y_0 = 2L - L\cos 30° = 2(1.2) - 1.2\cos 30° = 1.361 \text{ m}$

Thus $\quad 1.361 - \dfrac{1}{2}(9.81)t^2 = 0 \qquad t = 0.5275$



PROBLEM 12.123 (Continued)

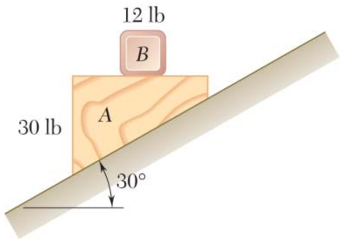
Projection on horizontal floor (uniform motion)

$$x = x_0 + (v_0)_z t = L \sin 30° + 0, \qquad x = 0.6 \text{ m}$$
$$z = z_0 + (v_0)_z t = 0 + 1.843(0.527) = 0.971 \text{ m}$$

Radius of circle: $a = \sqrt{x^2 + z^2}$

$$a = \sqrt{(0.6)^2 + (0.971)^2} \qquad a = 1.141 \text{ m} \blacktriangleleft$$

Note: The drop travels in a vertical plane parallel to the *yz* plane.



PROBLEM 12.124

A 12-lb block *B* rests as shown on the upper surface of a 30-lb wedge *A*. Neglecting friction, determine immediately after the system is released from rest (*a*) the acceleration of *A*, (*b*) the acceleration of *B* relative to *A*.

SOLUTION

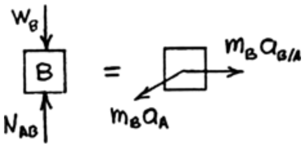
Acceleration vectors:

$$\mathbf{a}_A = a_A \swarrow 30^\circ, \quad \mathbf{a}_{B/A} = a_{B/A} \longrightarrow$$

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

Block *B*:

$$\xrightarrow{+} \Sigma F_x = ma_x: \quad m_B a_{B/A} - m_B a_A \cos 30^\circ = 0$$



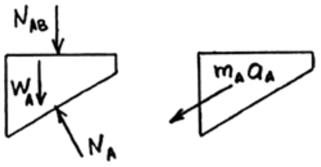
$$a_{B/A} = a_A \cos 30^\circ \tag{1}$$

$$+\!\downarrow \Sigma F_y = ma_y: \quad N_{AB} - W_B = -m_B a_A \sin 30^\circ$$

$$N_{AB} = W_B - (W_B \sin 30^\circ)\frac{a_A}{g} \tag{2}$$

Block *A*:

$$+\!\swarrow \Sigma F = ma: \quad W_A \sin 30^\circ + N_{AB} \sin 30^\circ = W_A \frac{a_A}{g}$$



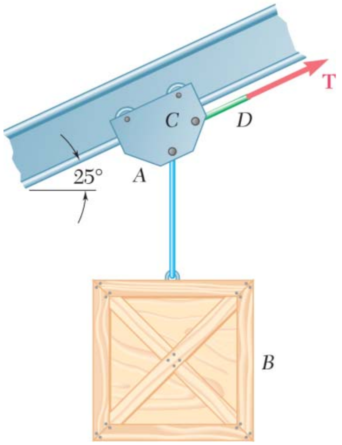
$$W_A \sin 30^\circ + W_B \sin 30^\circ - (W_B \sin^2 30^\circ)\frac{a_A}{g} = W_A \frac{a_A}{g}$$

$$a_A = \frac{(W_A + W_B)\sin 30^\circ}{W_A + W_B \sin^2 30^\circ} g = \frac{(30+12)\sin 30^\circ}{30 + 12\sin^2 30^\circ}(32.2) = 20.49 \text{ ft/s}^2$$

(*a*) $\mathbf{a}_A = 20.49 \text{ ft/s}^2 \swarrow 30^\circ$ ◀

$$a_{B/A} = (20.49)\cos 30^\circ = 17.75 \text{ ft/s}^2$$

(*b*) $\mathbf{a}_{B/A} = 17.75 \text{ ft/s}^2 \longrightarrow$ ◀



PROBLEM 12.125

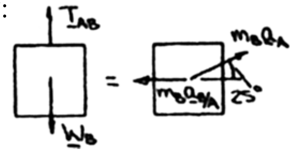
A 500-lb crate B is suspended from a cable attached to a 40-lb trolley A which rides on an inclined I-beam as shown. Knowing that at the instant shown the trolley has an acceleration of 1.2 ft/s^2 up and to the right, determine (a) the acceleration of B relative to A, (b) the tension in cable CD.

SOLUTION

(a) First we note: $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$, where $\mathbf{a}_{B/A}$ is directed perpendicular to cable AB.

$$\xrightarrow{+} \Sigma F_x = m_B a_x: \quad 0 = -m_B a_x + m_B a_A \cos 25°$$

B:



or
$$a_{B/A} = (1.2 \text{ ft/s}^2)\cos 25°$$

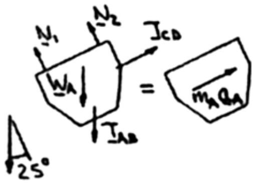
or
$$\mathbf{a}_{B/A} = 1.088 \text{ ft/s}^2 \longleftarrow \blacktriangleleft$$

(b) For crate B

$$+\!\uparrow \Sigma F_y = m_B a_y: \quad T_{AB} - W_B = \frac{W_B}{g} a_A \sin 25°$$

or
$$T_{AB} = (500 \text{ lb})\left[1 + \frac{(1.2 \text{ ft/s}^2)\sin 25°}{32.2 \text{ ft/s}^2}\right]$$
$$= 507.87 \text{ lb}$$

A:

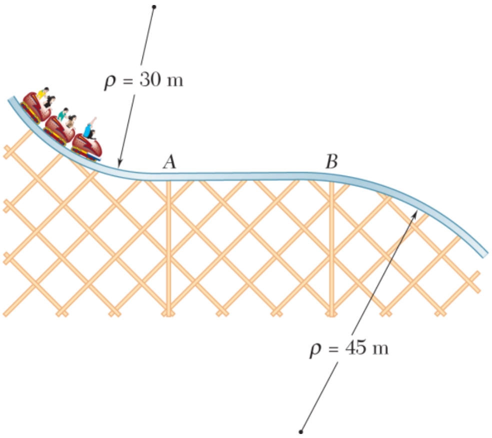


For trolley A

$$+\!\nearrow \Sigma F_x = m_A a_A: \quad T_{CD} - T_{AB}\sin 25° - W_A \sin 25° = \frac{W_A}{g} a_A$$

or
$$T_{CD} = (507.87 \text{ lb})\sin 25° + (40 \text{ lb})\left(\sin 25° + \frac{1.2 \text{ ft/s}^2}{32.2 \text{ ft/s}^2}\right)$$

or
$$T_{CD} = 233 \text{ lb} \blacktriangleleft$$

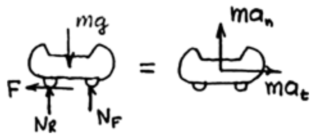


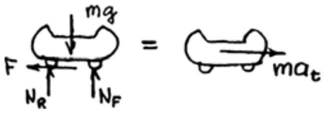
PROBLEM 12.126

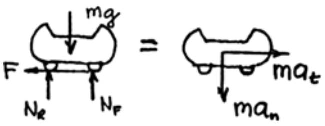
The roller-coaster track shown is contained in a vertical plane. The portion of track between *A* and *B* is straight and horizontal, while the portions to the left of *A* and to the right of *B* have radii of curvature as indicated. A car is traveling at a speed of 72 km/h when the brakes are suddenly applied, causing the wheels of the car to slide on the track ($\mu_k = 0.25$). Determine the initial deceleration of the car if the brakes are applied as the car (*a*) has almost reached *A*, (*b*) is traveling between *A* and *B*, (*c*) has just passed *B*.

SOLUTION

$$v = 72 \text{ km/h} = 20 \text{ m/s}$$







(*a*) Almost reached Point *A*. $\rho = 30 \text{ m}$

$$a_n = \frac{v^2}{\rho} = \frac{(20)^2}{30} = 13.333 \text{ m/s}^2 \uparrow$$

$$\Sigma F_y = ma_y: \quad N_R + N_F - mg = ma_n$$

$$N_R + N_F = m(g + a_n)$$

$$F = \mu_k(N_R + N_F) = \mu_k m(g + a_n)$$

$$\xrightarrow{+} \Sigma F_x = ma_x: \quad -F = ma_t$$

$$a_t = -\frac{F}{m} = -\mu_k(g + a_n)$$

$$|a_t| = \mu_k(g + a_n) = 0.25(9.81 + 13.33)$$

$|a_t| = 5.79 \text{ m/s}^2$ ◀

(*b*) Between *A* and *B*. $\rho = \infty$

$$a_n = 0$$

$$|a_t| = \mu_k g = (0.25)(9.81)$$

$|a_t| = 2.45 \text{ m/s}^2$ ◀

(*c*) Just passed Point *B*. $\rho = 45 \text{ m}$

$$a_n = \frac{v^2}{\rho} = \frac{(20)^2}{45} = 8.8889 \text{ m/s}^2 \downarrow$$

$$\Sigma F_y = ma_y: \quad N_R + N_F - mg = -ma_n$$

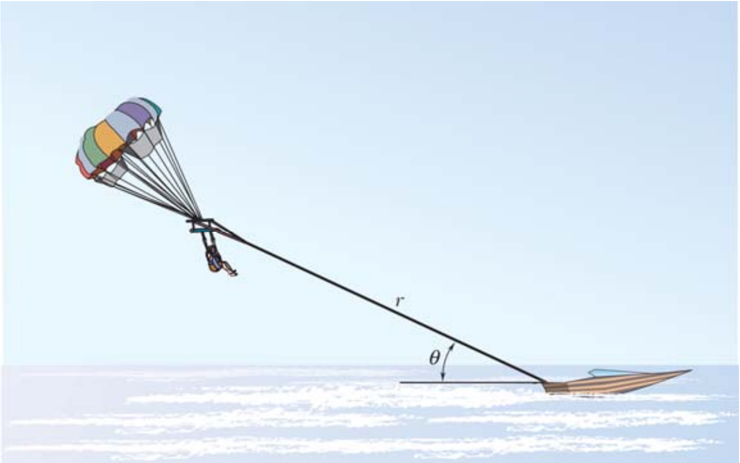
or
$$N_R + N_F = m(g - a_n)$$
$$F = \mu_k(N_R + N_F) = \mu_k m(g - a_n)$$

$$\xrightarrow{+} \Sigma F_x = ma_x: \quad -F = ma_t$$

$$a_t = -\frac{F}{m} = -\mu_k(g - a_n)$$

$$|a_t| = \mu_k(g - a_n) = (0.25)(9.81 - 8.8889)$$

$$|a_t| = 0.230 \text{ m/s}^2 \blacktriangleleft$$



PROBLEM 12.127

The parasailing system shown uses a winch to pull the rider in towards the boat, which is travelling with a constant velocity. During the interval when θ is between 20° and 40°, (where t = 0 at θ = 20°) the angle increases at the constant rate of 2 °/s. During this time, the length of the rope is defined by the relationship $r = 125 - \frac{1}{3}t^{3/2}$, where r and t are expressed in meters and seconds, respectively. At the instant when the rope makes a 30 degree angle with the water, the tension in the rope is 18 kN. At this instant, what is the magnitude and direction of the force of the parasail on the 75 kg parasailor?

SOLUTION

Given:

$$r = 125 - \frac{1}{3}t^{3/2} \text{ m}$$

$$\theta_0 = 20°,\ \theta = 30°,\ \dot{\theta} = 2°/s = 0.0349 \text{ rad/s},\ \ddot{\theta}=0$$

$$T = 18 \text{ kN},\ m = 75 \text{ kg}$$

Kinematics:

$$\theta = \theta_0 + \dot{\theta}t$$

$$30 = 20 + 2t \Rightarrow t = 5 \text{ s}$$

$$\mathbf{a}_P = \mathbf{a}_B + \mathbf{a}_{P/B}$$

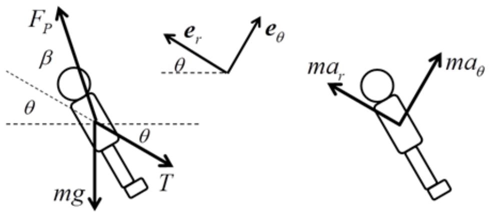
$$\mathbf{a}_P = 0 + a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta$$

Using Radial and Transverse Coordinates:

$$\dot{r} = -\frac{1}{2}t^{\frac{1}{2}} \text{ m/s}$$

$$\ddot{r} = -\frac{1}{4}t^{-\frac{1}{2}} \text{ m/s}^2$$

Free Body Diagram of parasailor (side view):



Equations of Motion:

$$\sum F_r = ma_r$$

$$F_P\cos\beta - T - mg\sin\theta = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

$$F_P\cos\beta = T + m\left[g\sin\theta + \ddot{r} - r\dot{\theta}^2\right] \qquad (1)$$

$$\sum F_\theta = ma_\theta$$

$$F_P\sin\beta - mg\cos\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

$$F_P\sin\beta = m\left[g\cos\theta + r\ddot{\theta} + 2\dot{r}\dot{\theta}\right] \qquad (2)$$

PROBLEM 12.127 (Continued)

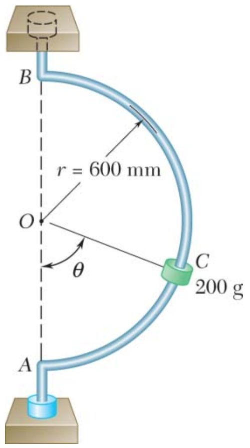
Evaluate at t=5 s: $r = 121.27\text{ m},\ \dot{r} = -1.118\text{ m/s},\ \ddot{r} = -0.1118\text{ m/s}^2$

(1) $F_P \cos\beta = 18000 + 75\left[9.81\sin 30° - 0.1118 - 121.27(0.0349)^2\right] \Rightarrow F_P \cos\beta = 18348.4\text{ N}$

(2) $F_P \sin\beta = 75\left[9.81\cos 30° + 0 + 2(-1.118)(0.0349)\right] \Rightarrow F_P \sin\beta = 631.33\text{ N}$

$$\frac{\cancel{F_P}\sin\beta}{\cancel{F_P}\cos\beta} = \frac{631.33}{18348.4} \Rightarrow \tan\beta = 0.03441 \Rightarrow \beta = 1.97°$$

$F_P \cos 1.97° = 18348.4\text{ N}$ $\qquad$ $F_P = 18.4\text{ kN} \measuredangle 31.97°$ ◀



PROBLEM 12.128

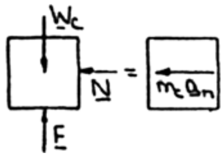
A small 200-g collar C can slide on a semicircular rod which is made to rotate about the vertical AB at the constant rate of 6 rad/s. Determine the minimum required value of the coefficient of static friction between the collar and the rod if the collar is not to slide when (a) $\theta = 90°$, (b) $\theta = 75°$, (c) $\theta = 45°$. Indicate in each case the direction of the impending motion.

SOLUTION

First note

$$\begin{aligned} v_C &= (r\sin\theta)\dot{\phi}_{AB} \\ &= (0.6\text{ m})(6\text{ rad/s})\sin\theta \\ &= (3.6\text{ m/s})\sin\theta \end{aligned}$$

(a) With $\theta = 90°$, $v_C = 3.6\text{ m/s}$



$$+\!\uparrow \Sigma F_y = 0: \quad F - W_C = 0$$

or $F = m_C g$

Now $F = \mu_s N$

or $N = \dfrac{1}{\mu_s} m_C g$

$$\xleftarrow{+} \Sigma F_n = m_C a_n: \quad N = m_C \frac{v_C^2}{r}$$

or $$\frac{1}{\mu_s} m_C g = m_C \frac{v_C^2}{r}$$

or $$\mu_s = \frac{gr}{v_C^2} = \frac{(9.81\text{ m/s}^2)(0.6\text{ m})}{(3.6\text{ m/s})^2}$$

or $(\mu_s)_{\min} = 0.454$ ◀

The direction of the impending motion is downward. ◀

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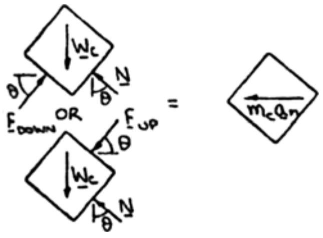
(*b*) and (*c*)

First observe that for an arbitrary value of θ, it is not known whether the impending motion will be upward or downward. To consider both possibilities for each value of θ, let F_{down} correspond to impending motion downward, F_{up} correspond to impending motion upward, then with the "top sign" corresponding to F_{down}, we have

$$+\uparrow \Sigma F_y = 0: \quad N\cos\theta \pm F\sin\theta - W_C = 0$$



Now $$F = \mu_s N$$

Then $$N\cos\theta \pm \mu_s N\sin\theta - m_C g = 0$$

or $$N = \frac{m_C g}{\cos\theta \pm \mu_s \sin\theta}$$

and $$F = \frac{\mu_s m_C g}{\cos\theta \pm \mu_s \sin\theta}$$

$$\xleftarrow{+} \Sigma F_n = m_C a_n: \quad N\sin\theta \mp F\cos\theta = m_C \frac{v_C^2}{\rho} \qquad \rho = r\sin\theta$$

Substituting for *N* and *F*

$$\frac{m_C g}{\cos\theta \pm \mu_s \sin\theta}\sin\theta \mp \frac{\mu_s m_C g}{\cos\theta \pm \mu_s \sin\theta}\cos\theta = m_C \frac{v_C^2}{r\sin\theta}$$

or $$\frac{\tan\theta}{1 \pm \mu_s \tan\theta} \mp \frac{\mu_s}{1 \pm \mu_s \tan\theta} = \frac{v_C^2}{gr\sin\theta}$$

or $$\mu_s = \pm \frac{\tan\theta - \frac{v_C^2}{gr\sin\theta}}{1 + \frac{v_C^2}{gr\sin\theta}\tan\theta}$$

Now $$\frac{v_C^2}{gr\sin\theta} = \frac{[(3.6\text{ m/s})\sin\theta]^2}{(9.81\text{ m/s}^2)(0.6\text{ m})\sin\theta} = 2.2018\sin\theta$$

Then $$\mu_s = \pm \frac{\tan\theta - 2.2018\sin\theta}{1 + 2.2018\sin\theta\tan\theta}$$

(*b*) $\theta = 75°$

$$\mu_s = \pm \frac{\tan 75° - 2.2018\sin 75°}{1 + 2.2018\sin 75°\tan 75°} = \pm 0.1796$$

Then

downward: $\mu_s = +0.1796$

upward: $\mu_s < 0$ not possible

$(\mu_s)_{\min} = 0.1796$ ◀

The direction of the impending motion is downward. ◀

(*c*) $\theta = 45°$

$$\mu_s = \pm \frac{\tan 45° - 2.2018 \sin 45°}{1 + 2.2018 \sin 45° \tan 45°} = \pm(-0.218)$$

Then

downward: $\mu_s < 0$ not possible

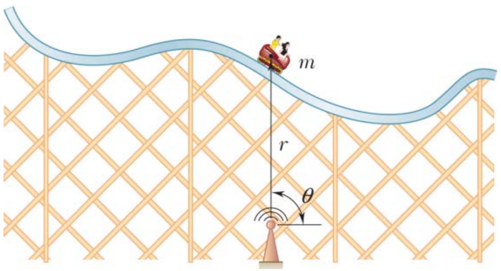
upward: $\mu_s = 0.218$ $(\mu_s)_{\min} = 0.218$ ◀

The direction of the impending motion is upward. ◀

Note: When $\tan\theta - 2.2018 \sin\theta = 0$

or $\theta = 62.988°$,

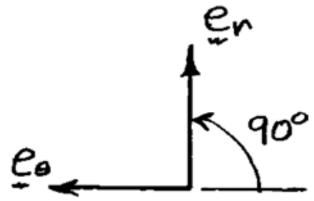
$\mu_s = 0$. Thus, for this value of θ, friction is not necessary to prevent the collar from sliding on the rod.

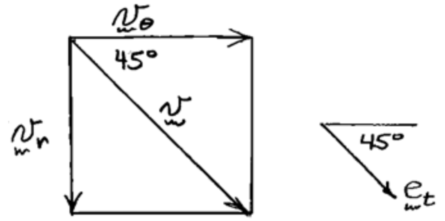


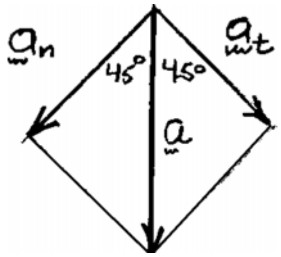
PROBLEM 12.129

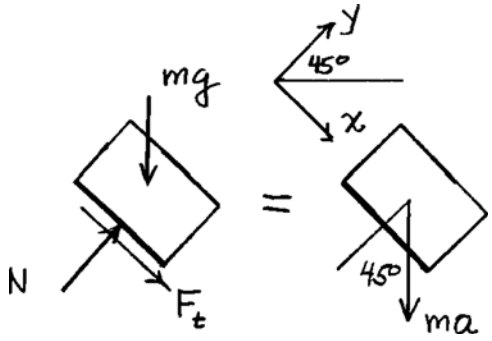
Telemetry technology is used to quantify kinematic values of a 200-kg roller coaster cart as it passes overhead. According to the system, $r = 25$ m, $\dot{r} = -10$ m/s, $\ddot{r} = -2$ m/s^2, $\theta = 90°$, $\dot{\theta} = -0.4$ rad/s, $\ddot{\theta} = -0.32$ rad/s^2. At this instant, determine (*a*) the normal force between the cart and the track, (*b*) the radius of curvature of the track.

SOLUTION


$\underset{\sim}{e}_r$
90°
$\underset{\sim}{e}_\theta$


$\underset{\sim}{v}_\theta$
45°
$\underset{\sim}{v}$
$\underset{\sim}{v}_r$
45°
$\underset{\sim}{e}_t$


$\underset{\sim}{a}_n$
$\underset{\sim}{a}_t$
45°
45°
$\underset{\sim}{a}$


y
45°
x
mg
=
N
F_t
45°
ma

Find the acceleration and velocity using polar coordinates.

$$v_r = \dot{r} = -10 \text{ m/s}$$

$$v_\theta = r\dot{\theta} = (25 \text{ m})(-0.4 \text{ rad/s}) = -10 \text{ m/s}$$

So the tangential direction is ↘ 45° and $v = 10\sqrt{2}$ m/s.

$$a_r = \ddot{r} - r\dot{\theta}^2 = -2 \text{ m/s} - (25 \text{ m})(-0.4 \text{ rad/s})^2$$
$$= -6 \text{ m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$= (25 \text{ m})(-0.32) \text{ rad/s}^2| + 2(-10 \text{ m/s})(-0.4 \text{ rad/s})$$
$$= 0$$

So the acceleration is vertical and downward.

(*a*) To find the normal force use Newton's second law.

y-direction

$$N - mg\sin 45° = -ma\cos 45°$$
$$N = m(g\sin 45° - a\cos 45°)$$
$$= (200 \text{ kg})(9.81) \text{ m/s}^2 - 6 \text{ m/s}^2)(0.70711)$$
$$= 538.815 \text{ N}$$

$N = 539$ N ◀

(*b*) Radius l curvature of the track.

$$a_n = \frac{v^2}{\rho}$$

$$\rho = \frac{v^2}{a_n} = \frac{(10\sqrt{2})^2}{6\cos 45°}$$

$\rho = 47.1$ m ◀

PROBLEM 12.130

The radius of the orbit of a moon of a given planet is equal to twice the radius of that planet. Denoting by ρ the mean density of the planet, show that the time required by the moon to complete one full revolution about the planet is $(24\ \pi/G\rho)^{1/2}$, where G is the constant of gravitation.

SOLUTION

For gravitational force and a circular orbit,

$$|F_r| = \frac{GMm}{r^2} = \frac{mv^2}{r} \qquad \text{or} \qquad v = \sqrt{\frac{GM}{r}}$$

Let τ be the periodic time to complete one orbit.

$$v\tau = 2\pi r \qquad \text{or} \qquad \tau\sqrt{\frac{GM}{r}} = 2\pi r$$

Solving for τ,

$$\tau = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

But

$$M = \frac{4}{3}\pi R^3 \rho, \quad \text{hence,} \quad \sqrt{GM} = 2\sqrt{\frac{\pi}{3}G\rho}\ R^{3/2}$$

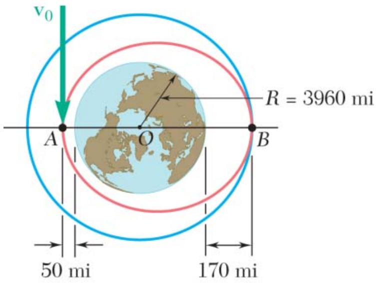
Then

$$\tau = \sqrt{\frac{3\pi}{G\rho}}\left(\frac{r}{R}\right)^{3/2}$$

Using $r = 2R$ as a given leads to

$$\tau = 2^{3/2}\sqrt{\frac{3\pi}{G\rho}} = \sqrt{\frac{24\pi}{G\rho}}$$

$\tau = (24\pi/G\rho)^{1/2}$ ◀



PROBLEM 12.131

At engine burnout on a mission, a shuttle had reached Point *A* at an altitude of 40 mi above the surface of the earth and had a horizontal velocity $\mathbf{v}_0$. Knowing that its first orbit was elliptic and that the shuttle was transferred to a circular orbit as it passed through Point *B* at an altitude of 170 mi, determine (*a*) the time needed for the shuttle to travel from *A* to *B* on its original elliptic orbit, (*b*) the periodic time of the shuttle on its final circular orbit.

SOLUTION

For Earth,
$$R = 3960 \text{ mi} = 20.909\times10^6 \text{ ft}, \quad g = 32.2 \text{ ft/s}^2$$

$$GM = gR^2 = (32.2)(20.909\times10^6)^2 = 14.077\times10^{15} \text{ ft}^3/\text{s}^2$$

(*a*) For the elliptic orbit,
$$r_A = 3960 + 40 = 4000 \text{ mi} = 21.12\times10^6 \text{ ft}$$
$$r_B = 3960 + 170 = 4130 \text{ mi} = 21.8064\times10^6 \text{ ft}$$

$$a = \frac{1}{2}(r_A + r_B) = 21.5032\times10^6 \text{ ft}$$
$$b = \sqrt{r_A r_B} = 21.4605\times10^6 \text{ ft}$$

Using Eq. 12.39,
$$\frac{1}{r_A} = \frac{GM}{h^2} + C\cos\theta_A$$

and
$$\frac{1}{r_B} = \frac{GM}{r_B^2} + C\cos\theta_B$$

But $\theta_B = \theta_A + 180°$, so that $\cos\theta_A = -\cos\theta_B$

Adding,
$$\frac{1}{r_A} + \frac{1}{r_B} = \frac{r_A + r_B}{r_A r_B} = \frac{2a}{b^2} = \frac{2GM}{h^2}$$

or
$$h = \sqrt{\frac{GMb^2}{a}}$$

Periodic time.
$$\tau = \frac{2\pi ab}{h} = \frac{2\pi ab\sqrt{a}}{\sqrt{GMb^2}} = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

$$\tau = \frac{2\pi(21.5032\times10^6)^{3/2}}{\sqrt{14.077\times10^{15}}} = 5280.6 \text{ s} = 1.4668 \text{ h}$$

The time to travel from *A* to *B* is one half the periodic time

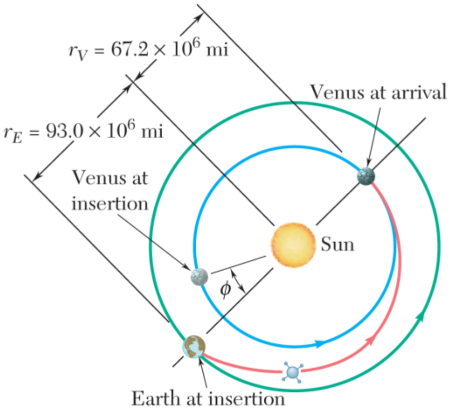
$$\tau_{AB} = 0.7334 \text{ h} \qquad \tau_{AB} = 44.0 \text{ min} \blacktriangleleft$$

PROBLEM 12.131 (Continued)

(*b*) For the circular orbit, $a = b = r_B = 21.8064 \times 10^6 \text{ ft}$

$$\tau_{\text{circ}} = \frac{2\pi a^{3/2}}{\sqrt{GM}} = \frac{2\pi(21.8064 \times 10^6)^{3/2}}{\sqrt{14.077 \times 10^{15}}} = 5393 \text{ s}$$

$\tau_{\text{circ}} = 1.498 \text{ h}$ $\tau_{\text{circ}} = 89.9 \text{ min}$ ◀



PROBLEM 12.132

A space probe in a low earth orbit is inserted into an elliptic transfer orbit to the planet Venus. Knowing that the mass of the sun is 332.8×10^3 times the mass of the earth and assuming that the probe is subjected only to the gravitational attraction of the sun, determine the value of ϕ, which defines the relative position of Venus with respect to the earth at the time the probe is inserted into the transfer orbit.

SOLUTION

First determine the time t_{probe} for the probe to travel from the earth to Venus:

$$t_{\text{probe}} = \frac{1}{2}\tau_{\text{tr}}$$

where τ_{tr} is the periodic time of the elliptic transfer orbit. Applying Kepler's Third Law to the orbits about the sun of the earth and the probe, we obtain

$$\frac{\tau_{\text{tr}}^2}{\tau_{\text{earth}}^2} = \frac{a_{\text{tr}}^3}{a_{\text{earth}}^3}$$

where
$$\begin{aligned} a_{\text{tr}} &= \frac{1}{2}(r_E + r_v) \\ &= \frac{1}{2}(93 \times 10^6 + 67.2 \times 10^6)\,\text{mi} \\ &= 80.1 \times 10^6\ \text{mi} \end{aligned}$$

and
$$a_{\text{earth}} \approx r_E \quad (\textit{Note:}\quad \varepsilon_{\text{earth}} = 0.0167)$$

Then
$$\begin{aligned} t_{\text{probe}} &= \frac{1}{2}\left(\frac{a_{\text{tr}}}{r_E}\right)^{3/2} \tau_{\text{earth}} \\ &= \frac{1}{2}\left(\frac{80.1 \times 10^6\ \text{mi}}{93.0 \times 10^6\ \text{mi}}\right)^{3/2} (365.25\ \text{days}) \\ &= 145.977\ \text{days} \\ &= 12.6124 \times 10^6\,\text{s} \end{aligned}$$

In time t_{probe}, Venus travels through the angle θ_v given by

$$\theta_v = \dot{\theta}_v t_{\text{probe}} = \frac{v_v}{r_v} t_{\text{probe}}$$

PROBLEM 12.132 (Continued)

Assuming that the orbit of Venus is circular (*note:* $\varepsilon_{\text{venus}} = 0.0068$), then, for a circular orbit

$$v_v = \sqrt{\frac{GM_{\text{sun}}}{r_v}} \qquad \text{[Eq. (12.44)]}$$

Now $$GM_{\text{sun}} = G(332.8\times 10^3 M_{\text{earth}})$$

$$= 332.8\times 10^3 \left(gR_{\text{earth}}^2\right) \qquad \text{using Eq. (12.30)}$$

Then $$\theta_v = \frac{t_{\text{probe}}}{r_v}\left[\frac{332.8\times 10^3\left(gR_{\text{earth}}^2\right)}{r_v}\right]^{1/2}$$

$$= t_{\text{probe}}\ R_{\text{earth}}\ \frac{(332.8g\times 10^3)^{1/2}}{r_v^{3/2}}$$

where $$R_{\text{earth}} = 3960 \text{ mi} = 20.9088\times 10^6 \text{ ft}$$

and $$r_v = 67.2\times 10^6 \text{ mi} = 354.816\times 10^9 \text{ ft}$$

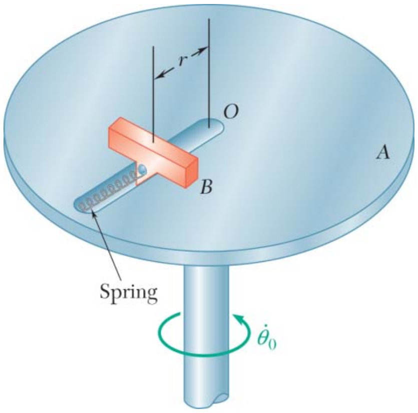
Then $$\theta_v = (12.6124\times 10^6 \text{ s})(20.9088\times 10^6 \text{ ft})\frac{(332.8\times 10^3\times 32.2 \text{ ft/s}^2)^{1/2}}{(354.816\times 10^9 \text{ ft})^{1/2}}$$

$$= 4.0845 \text{ rad}$$

$$= 234.02°$$

Finally, $$\phi = \theta_v - 180° = 234.02° - 180°$$

$$\varphi = 54.0° \blacktriangleleft$$



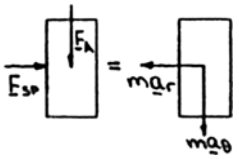
PROBLEM 12.133*

Disk *A* rotates in a horizontal plane about a vertical axis at the constant rate $\dot{\theta}_0 = 10$ rad/s. Slider *B* has mass 1 kg and moves in a frictionless slot cut in the disk. The slider is attached to a spring of constant *k*, which is undeformed when $r = 0$. Knowing that the slider is released with no radial velocity in the position $r = 500$ mm, determine the position of the slider and the horizontal force exerted on it by the disk at $t = 0.1$ s for (*a*) $k = 100$ N/m, (*b*) $k = 200$ N/m.

SOLUTION

First we note when $r = 0, \quad x_{sp} = 0 \Rightarrow F_{sp} = kr$

$$r_0 = 500 \text{ mm} = 0.5 \text{ m}$$



and $\dot{\theta} = \dot{\theta}_0 = 12$ rad/s

then $\ddot{\theta} = 0$

$$\overset{+}{\leftarrow} \Sigma F_r = m_B a_r: \quad -F_{sp} = m_B(\ddot{r} - r\dot{\theta}_0^2)$$

$$\ddot{r} + \left(\frac{k}{m_B} - \dot{\theta}_0^2\right) r = 0 \qquad (1)$$

$$+\!\downarrow \Sigma F_\theta = m_B a_\theta: \quad F_A = m_B(0 + 2\dot{r}\dot{\theta}_0) \qquad (2)$$

(*a*) $k = 100$ N/m

Substituting the given values into Eq. (1)

$$\ddot{r} + \left[\frac{100 \text{ N/m}}{1 \text{ kg}} - (10 \text{ rad/s})^2\right] r = 0$$

$$\ddot{r} = 0$$

Then $\dfrac{d\dot{r}}{dt} = \ddot{r} = 0$ and at $t = 0, \dot{r} = 0$:

$$\int_0^{\dot{r}} d\dot{r} = \int_0^{0.1} (0)\, dt$$

$$\dot{r} = 0$$

and $$\frac{dr}{dt} = \dot{r} = 0 \text{ and at } t = 0,\ r_0 = 0.5 \text{ m}$$

$$\int_{r_0}^{r} dr = \int_0^{0.1} (0)\, dt$$

$$r = r_0$$

$r = 0.5$ m ◀

Note: $\dot{r} = 0$ implies that the slider remains at its initial radial position.

With $\dot{r} = 0$, Eq. (2) implies

$F_H = 0$ ◀

(*b*) $k = 200$ N/m

Substituting the given values into Eq. (1)

$$\ddot{r} + \left[\frac{200}{1 \text{ kg}} - (10 \text{ rad/s})^2\right] r = 0$$

$$\ddot{r} + 100\, r = 0$$

Now $$\ddot{r} = \frac{d}{dt}(\dot{r}) \qquad \dot{r} = v_r \qquad \frac{d}{dt} = \frac{dr}{dt}\frac{d}{dr} = v_r \frac{d}{dr}$$

Then $$\ddot{r} = v_r \frac{dv_r}{dr}$$

so that $$v_r \frac{dv_r}{dr} + 100r = 0$$

At $t = 0,\ v_r = 0,\ r = r_0$: $$\int_0^{v_r} v_r\, d\, v_r = -100 \int_{r_0}^{r} r\, dr$$

$$v_r^2 = -100(r^2 - r_0^2)$$

$$v_r = 10\sqrt{r_0^2 - r^2}$$

Now $$v_r = \frac{dr}{dt} = 10\sqrt{r_0^2 - r^2}$$

At $t = 0,\ r = r_0$: $$\int_{r_0}^{r} \frac{dr}{\sqrt{r_0^2 - r^2}} = \int_0^t 10\, dt = 10t$$

PROBLEM 12.133* (Continued)

Let $$r = r_0 \sin\phi, \qquad dr = r_0 \cos\phi d\phi$$

Then $$\int_{\pi/2}^{\sin^{-1}(r/r_0)} \frac{r_0 \cos\phi d\phi}{\sqrt{r_0^2 - r_0^2 \sin^2\phi}} = 10t$$

$$\int_{\pi/2}^{\sin^{-1}(r/r_0)} d\phi = 10t$$

$$\sin^{-1}\left(\frac{r}{r_0}\right) - \frac{\pi}{2} = 10t$$

$$r = r_0 \sin\left(10t + \frac{\pi}{2}\right) = r_0 \cos 10t = (0.5\text{ ft})\cos 10t$$

Then $$\dot{r} = -(5\text{ m/s})\sin 10t$$

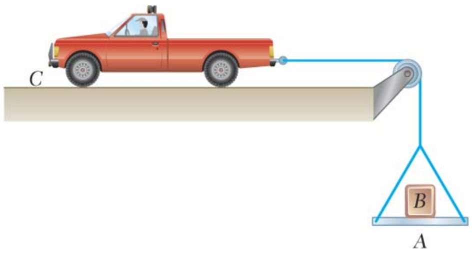
Finally, at $t = 0.1$ s:

$$r = (0.5\text{ ft})\cos(10 \times 0.1)$$

$$r = 0.270\text{ m} \blacktriangleleft$$

Eq. (2) $$F_H = 1\text{ kg} \times 2 \times [-(5\text{ ft/s})\sin(10 \times 0.1)]\ (10\text{ rad/s})$$

$$F_H = -84.1\text{ N} \blacktriangleleft$$



PROBLEM 12.CQ1

A 1000 lb boulder *B* is resting on a 200 lb platform *A* when truck *C* accelerates to the left with a constant acceleration. Which of the following statements are true (more than one may be true)?

(*a*) The tension in the cord connected to the truck is 200 lb

(*b*) The tension in the cord connected to the truck is 1200 lb

(*c*) The tension in the cord connected to the truck is greater than 1200 lb

(*d*) The normal force between *A* and *B* is 1000 lb

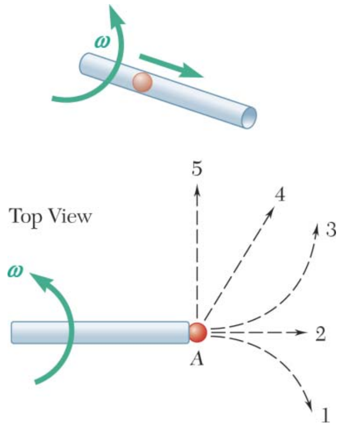
(*e*) The normal force between *A* and *B* is 1200 lb

(*f*) None of the above

SOLUTION

The boulder and platform are accelerating upwards; therefore it will require a force larger than their weight to achieve this acceleration:

Answers: (*c*) The tension will be greater than 1200 lb and (d) the normal force will be greater than 1000 lb.



PROBLEM 12.CQ2

Marble *A* is placed in a hollow tube, and the tube is swung in a horizontal plane causing the marble to be thrown out. As viewed from the top, which of the following choices best describes the path of the marble after leaving the tube?

(*a*) 1

(*b*) 2

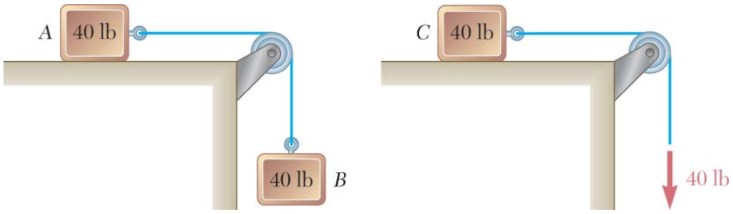
(*c*) 3

(*d*) 4

(*e*) 5

SOLUTION

Answer: (*d*) The particle will have velocity components along the tube and perpendicular to the tube when it leaves. After it leaves, it will not accelerate in the horizontal plane since there are no forces in that plane; therefore, it will travel in a straight line.



PROBLEM 12.CQ3

The two systems shown start from rest. On the left, two 40 lb weights are connected by an inextensible cord, and on the right, a constant 40 lb force pulls on the cord. Neglecting all frictional forces, which of the following statements is true?

(*a*) Blocks *A* and *C* will have the same acceleration

(*b*) Block *C* will have a larger acceleration than block *A*

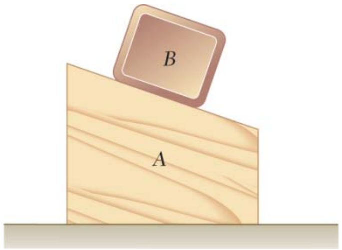
(*c*) Block *A* will have a larger acceleration than block *C*

(*d*) Block *A* will not move

(*e*) None of the above

SOLUTION

Answer: (*b*) If you draw a FBD of *B*, you will see that since it is accelerating downward, the tension in the cable will be less than 40 lb, so the acceleration of *A* will be less than the acceleration of *C*. Also, the system on the left has more inertia, so it will require more force to have the same acceleration as the system on the right.



PROBLEM 12.CQ4

The system shown is released from rest in the position shown. Neglecting friction, the normal force between block *A* and the ground is

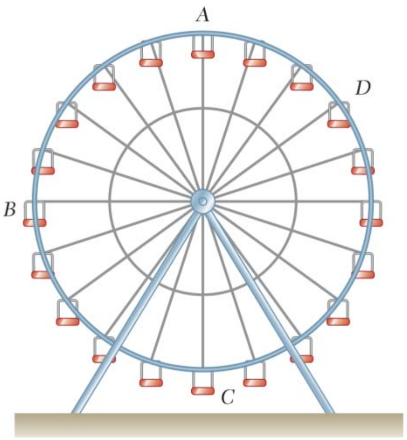
(*a*) less than the weight of *A* plus the weight of *B*

(*b*) equal to the weight of *A* plus the weight of *B*

(*c*) greater than the weight of *A* plus the weight of *B*

SOLUTION

Answer: (*a*) Since *B* has an acceleration component downward the normal force between *A* and the ground will be less than the sum of the weights. To see this, you can draw an FBD of both block A and B. From block A you will find that the normal force on the ground is the sum of the weight of block A plus the vertical component of the normal force of block B on A since block A does not accelerate in the vertical direction. From the FBD of block B and using F=ma, you will see that the vertical component of the normal force of B on A is less than the weight of block B since it is accelerating downwards.



PROBLEM 12.CQ5

People sit on a Ferris wheel at Points *A*, *B*, *C* and *D*. The Ferris wheel travels at a constant angular velocity. At the instant shown, which person experiences the largest force from his or her chair (back and seat)? Assume you can neglect the size of the chairs, that is, the people are located the same distance from the axis of rotation.

(*a*) *A*

(*b*) *B*

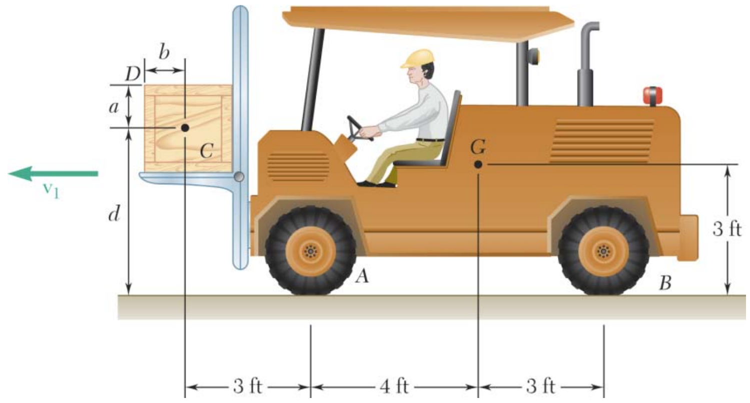
(*c*) *C*

(*d*) *D*

(*e*) The force is the same for all the passengers.

SOLUTION

Answer: (*c*) Draw a FBD and KD at each location and it will be clear that the maximum force will be experienced by the person at Point *C.* At this point the normal force isdirectly opposite the force of gravity and the acceleration is in the same direction as the normal force making the normal force the sum of *mg* and *ma*.



PROBLEM 12.CQ6

A uniform crate *C* with mass m_C is being transported to the left by a forklift with a constant speed v_1. What is the magnitude of the angular momentum of the crate about Point *D*, that is, the upper left corner of the crate?

(*a*) 0

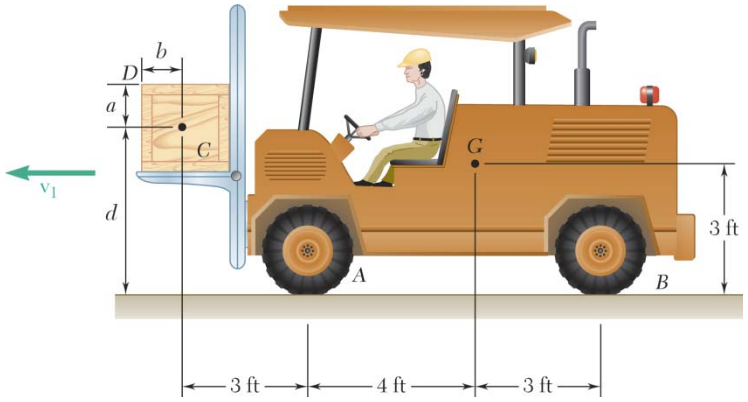
(*b*) mv_1a

(*c*) mv_1b

(*d*) $mv_1\sqrt{a^2+b^2}$

SOLUTION

Answers: (*b*) The angular momentum is the moment of the momentum, so simply take the linear momentum, mv_1, and multiply it by the perpendicular distance from the line of action of mv_1 and Point *D*.



PROBLEM 12.CQ7

A uniform crate C with mass m_C is being transported to the left by a forklift with a constant speed v_1. What is the magnitude of the angular momentum of the crate about Point A, that is, the point of contact between the front tire of the forklift and the ground?

(*a*) 0

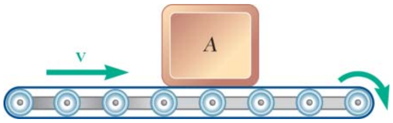
(*b*) mv_1d

(*c*) $3mv_1$

(*d*) $mv_1\sqrt{3^2 + d^2}$

SOLUTION

Answer: (*b*) The angular momentum is the moment of the momentum, so simply take the linear momentum, mv_1, and multiply it by the perpendicular distance from the line of action of mv_1 and Point A.

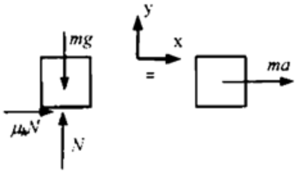


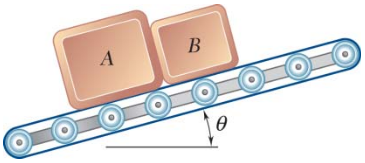
PROBLEM 12.F1

Crate *A* is gently placed with zero initial velocity onto a moving conveyor belt. The coefficient of kinetic friction between the crate and the belt is μ_k. Draw the FBD and KD for *A* immediately after it contacts the belt.

SOLUTION

Answer:





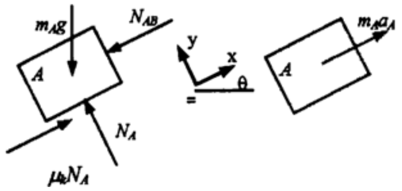
PROBLEM 12.F2

Two blocks weighing W_A and W_B are at rest on a conveyor that is initially at rest. The belt is suddenly started in an upward direction so that slipping occurs between the belt and the boxes. Assuming the coefficient of friction between the boxes and the belt is μ_k, draw the FBDs and KDs for blocks *A* and *B*. How would you determine if *A* and *B* remain in contact?

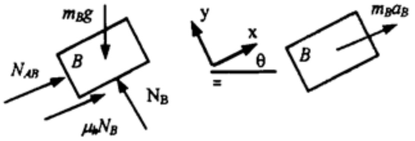
SOLUTION

Answer:

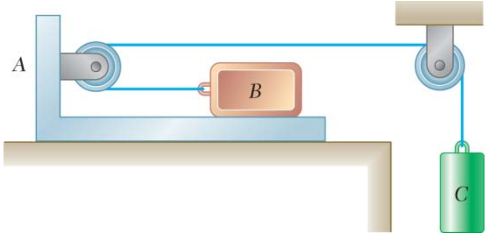
Block *A*


m_Ag
N_{AB}
A
y
x
θ
=
N_A
μ_kN_A
A
m_Aa_A

Block *B*


m_Bg
N_{AB}
B
N_B
μ_kN_B
y
x
θ
=
B
m_Ba_B

To see if they remain in contact assume $a_A = a_B$ and then check to see if N_{AB} is greater than zero.



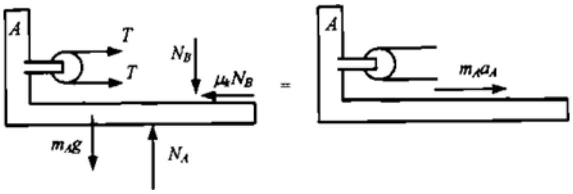
PROBLEM 12.F3

Objects *A*, *B*, and *C* have masses m_A, m_B, and m_C respectively. The coefficient of kinetic friction between *A* and *B* is μ_k, and the friction between *A* and the ground is negligible and the pulleys are massless and frictionless. Assuming *B* slides on *A* draw the FBD and KD for each of the three masses *A*, *B* and *C*.

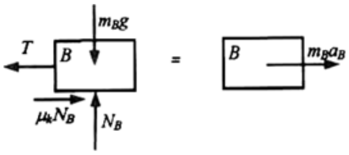
SOLUTION

Answer:

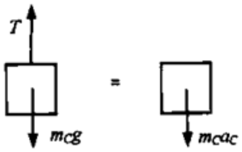
Block *A*

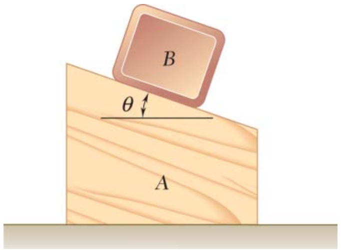

A
T
T
N_B
$\mu_k N_B$
$m_A g$
N_A
=
A
$m_A a_A$

Block *B*


$m_B g$
T
B
$\mu_k N_B$
N_B
=
B
$m_B a_B$

Block *C*



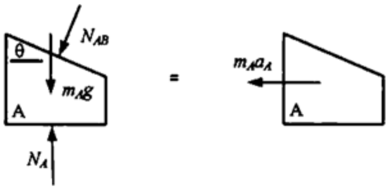


PROBLEM 12.F4

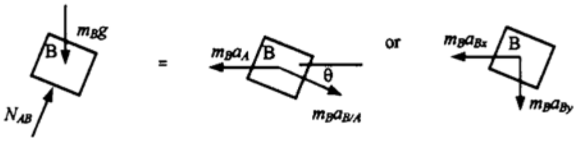
Blocks *A* and *B* have masses m_A and m_B respectively. Neglecting friction between all surfaces, draw the FBD and KD for each mass.

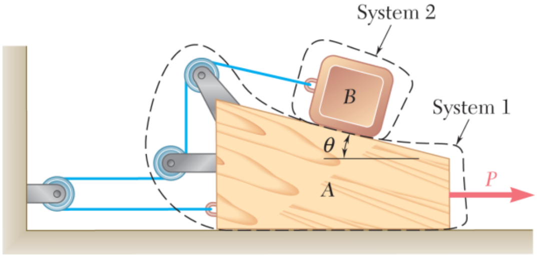
SOLUTION

Block *A*


N_{AB}
θ
$m_A g$
A
N_A
=
$m_A a_A$
A

Block *B*


$m_B g$
B
N_{AB}
=
$m_B a_A$
B
θ
$m_B a_{B/A}$
or
$m_B a_{Bx}$
B
$m_B a_{By}$

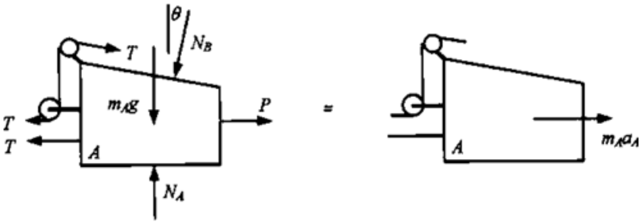

System 2
B
System 1
θ
A
P

PROBLEM 12.F5

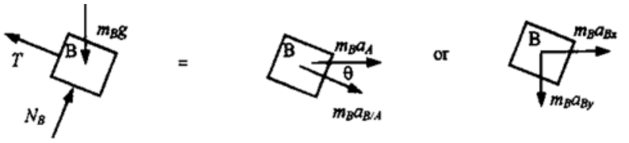
Blocks A and B have masses m_A and m_B respectively. Neglecting friction between all surfaces, draw the FBD and KD for the two systems shown.

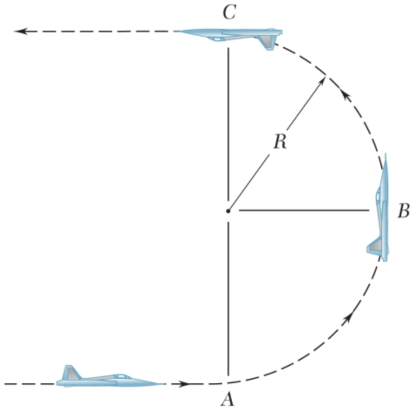
SOLUTION

System 1


T
θ
N_B
m_Ag
P
=
T
T
A
N_A
A
m_Aa_A

System 2

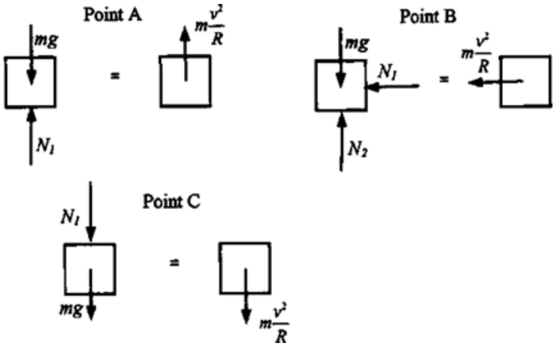

T
m_Bg
B
N_B
=
B
m_Ba_A
θ
$m_Ba_{B/A}$
or
B
m_Ba_{Bx}
m_Ba_{By}

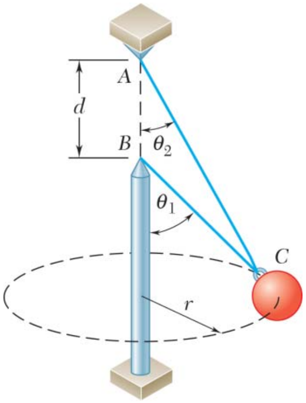


PROBLEM 12.F6

A pilot of mass *m* flies a jet in a half vertical loop of radius *R* so that the speed of the jet, *v*, remains constant. Draw a FBD and KD of the pilot at Points *A*, *B* and *C*.

SOLUTION

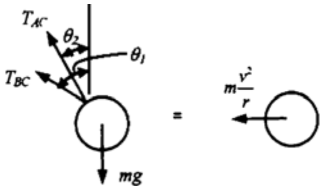

Point A
mg
=
$m\frac{v^2}{R}$
N_1
Point B
mg
N_1
=
$m\frac{v^2}{R}$
N_2
Point C
N_1
=
mg
$m\frac{v^2}{R}$

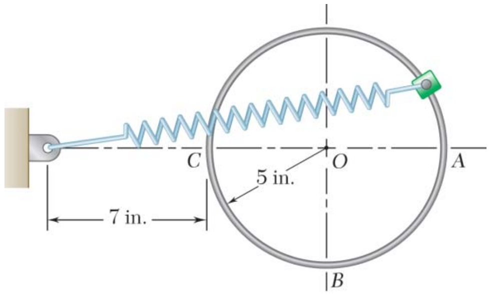

A
d
B
θ_2
θ_1
C
r

PROBLEM 12.F7

Wires *AC* and *BC* are attached to a sphere which revolves at a constant speed *v* in the horizontal circle of radius *r* as shown. Draw a FBD and KD of *C*.

SOLUTION

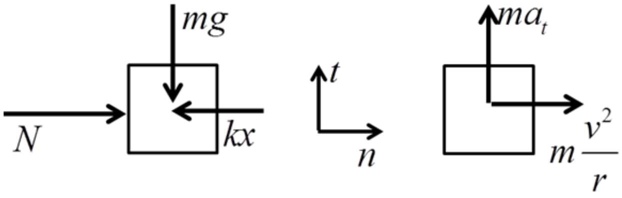

T_{AC}
θ_2
θ_1
T_{BC}
mg
=
$m\frac{v^2}{r}$


C
5 in.
O
A
7 in.
B

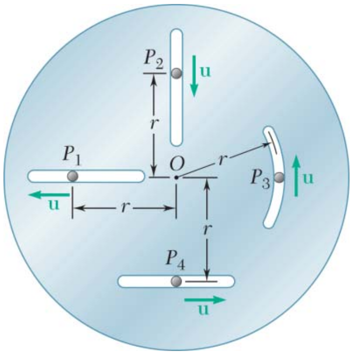
PROBLEM 12.F8

A collar of mass *m* is attached to a spring and slides without friction along a circular rod in a vertical plane. The spring has an undeformed length of 5 in. and a constant *k*. Knowing that the collar has a speed *v* at Point *C*, draw the FBD and KD of the collar at this point.

SOLUTION


mg
N
kx
t
n
ma_t
$m\frac{v^2}{r}$

where $x = 2/12$ ft and $r = 5/12$ ft.

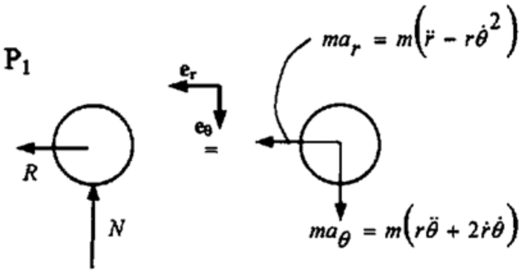

P_2
u
r
P_1
O
r
P_3
u
u
r
r
P_4
u

PROBLEM 12.F9

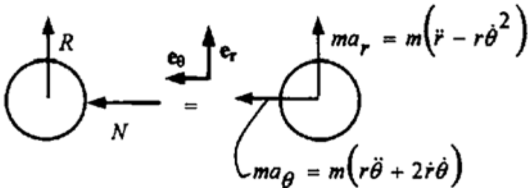
Four pins slide in four separate slots cut in a horizontal circular plate as shown. When the plate is at rest, each pin has a velocity directed as shown and of the same constant magnitude u. Each pin has a mass m and maintains the same velocity relative to the plate when the plate rotates about O with a constant counterclockwise angular velocity ω. Draw the FBDs and KDs to determine the forces on pins P_1 and P_2.

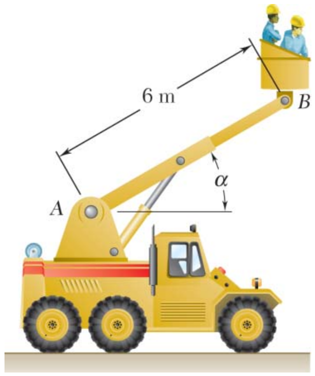
SOLUTION

Pin P_1


R
N
$\mathbf{e}_r$
$\mathbf{e}_\theta$
=
$ma_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$
$ma_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$

Pin P_2

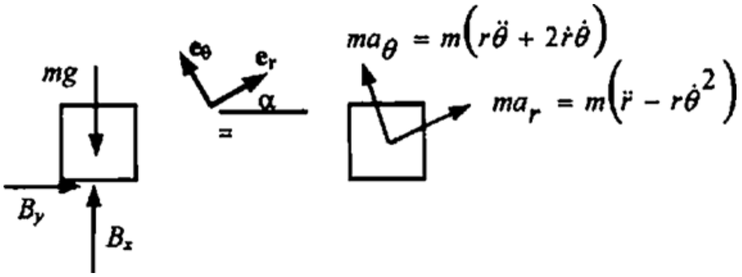

R
N
$\mathbf{e}_\theta$
$\mathbf{e}_r$
=
$ma_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$
$ma_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$



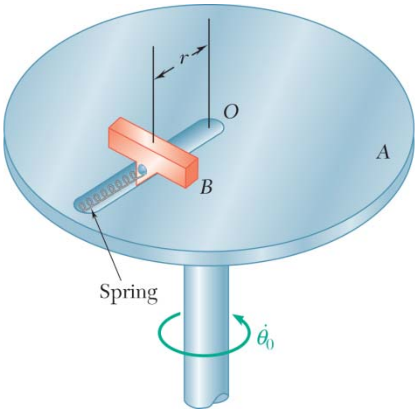
PROBLEM 12.F10

At the instant shown, the length of the boom *AB* is being *decreased* at the constant rate of 0.2 m/s, and the boom is being lowered at the constant rate of 0.08 rad/s. If the mass of the men and lift connected to the boom at Point *B* is *m*, draw the FBD and KD that could be used to determine the horizontal and vertical forces at *B*.

SOLUTION


$ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$
mg
$\mathbf{e}_\theta$
$\mathbf{e}_r$
α
$=$
$ma_r = m(\ddot{r} - r\dot{\theta}^2)$
B_y
B_x

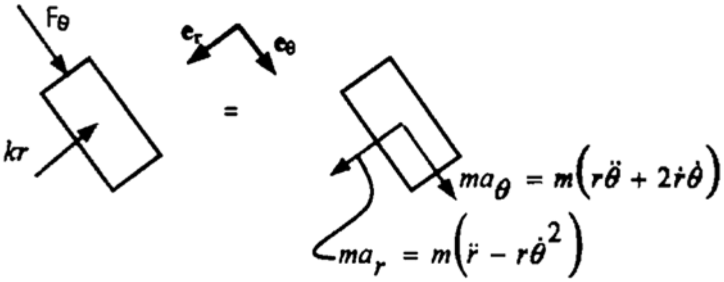
Where $r = 6$ m, $\dot{r} = -0.2$ m/s, $\ddot{r} = 0$, $\dot{\theta} = -0.08$ rad/s, $\ddot{\theta} = 0$

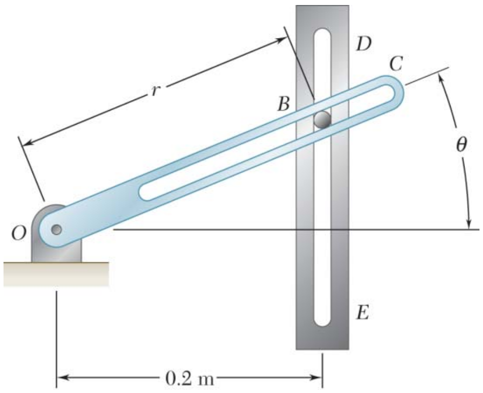

r
O
A
B
Spring
$\dot{\theta}_0$

PROBLEM 12.F11

Disk A rotates in a horizontal plane about a vertical axis at the constant rate $\dot{\theta}_0$. Slider B has a mass m and moves in a frictionless slot cut in the disk. The slider is attached to a spring of constant k, which is undeformed when $r = 0$. Knowing that the slider is released with no radial velocity in the position $r = r_0$, draw a FBD and KD at an arbitrary distance r from O.

SOLUTION

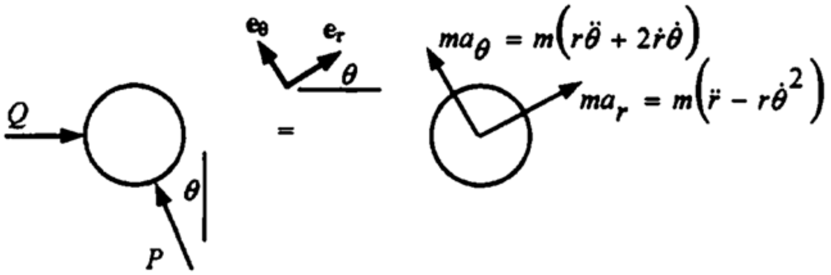

F_θ
kr
e_r
e_θ
=
$ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$
$ma_r = m(\ddot{r} - r\dot{\theta}^2)$


D
C
r
B
θ
O
E
0.2 m

PROBLEM 12.F12

Pin *B* has a mass *m* and slides along the slot in the rotating arm *OC* and along the slot *DE* which is cut in a fixed horizontal plate. Neglecting friction and knowing that rod *OC* rotates at the constant rate $\dot{\theta}_0$, draw a FBD and KD that can be used to determine the forces **P** and **Q** exerted on pin *B* by rod OC and the wall of slot *DE*, respectively.

SOLUTION


Q
P
θ
e_θ
e_r
θ
=
$ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$
$ma_r = m(\ddot{r} - r\dot{\theta}^2)$

CHAPTER 13

PROBLEM 13.1

A 400-kg satellite is placed in a circular orbit 6394 km above the surface of the earth. At this elevation the acceleration of gravity is $4.09\ \text{m/s}^2$. Knowing that its orbital speed is 20 000 km/h, determine the kinetic energy of the satellite.

SOLUTION

Given: Mass of satellite, $m = 400\ \text{kg}$

Speed of satellite, $v = 20.0 \times 10^3\ \text{km/h}$

Find: Kinetic energy, T

$$v = \left(20.0 \times 10^3\ \text{km/h}\right)\left(\frac{1\ \text{h}}{3600\ \text{s}}\right)\left(1000\ \text{m/km}\right)$$

$$= 5555\ \text{m/s}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}\left(400\ \text{kg}\right)\left(5.555 \times 10^3 \text{m/s}\right)^2$$

$$T = 6.17 \times 10^9\ \text{N}\cdot\text{m}$$

Note: Acceleration of gravity has no effect on the mass of the satellite.

$T = 6.17\ \text{GJ}$ ◀

PROBLEM 13.2

A 1-lb stone is dropped down the "bottomless pit" at Carlsbad Caverns and strikes the ground with a speed of 95 ft/s. Neglecting air resistance, determine (*a*) the kinetic energy of the stone as it strikes the ground and the height *h* from which it was dropped, (*b*) Solve Part *a* assuming that the same stone is dropped down a hole on the moon. (Acceleration of gravity on the moon = 5.31 ft/s^2.)

SOLUTION

Mass of stone: $m = \dfrac{W\text{ lb}}{g} = \dfrac{1\text{ lb}}{32.2\text{ ft/s}^2} = 0.031056\text{ lb}\cdot\text{s}^2/\text{ft}$

Initial kinetic energy: $T_1 = 0$ (rest)

(*a*) Kinetic energy at ground strike:

$$T_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(0.031056)(95)^2 = 140.14\text{ ft}\cdot\text{lb}$$

$T_2 = 140.1\text{ ft}\cdot\text{lb}$ ◀

Use work and energy: $T_1 + U_{1\to 2} = T_2$

where $U_{1\to 2} = wh = mgh$

$$0 + mgh = \frac{1}{2}mv_2^2$$

$$h = \frac{v_2^2}{2g} = \frac{(95)^2}{(2)(32.2)}$$

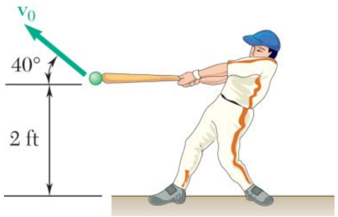
$h = 140.1\text{ ft}$ ◀

(*b*) On the moon: $g = 5.31\text{ ft/s}^2$

T_1 and T_2 will be the same, hence $T_2 = 140.1\text{ ft}\cdot\text{lb}$ ◀

$$h = \frac{v_2^2}{2g} = \frac{(95)^2}{(2)(5.31)}$$

$h = 850\text{ ft}$ ◀



PROBLEM 13.3

A baseball player hits a 5.1-oz baseball with an initial velocity of 140 ft/s at an angle of 40° with the horizontal as shown. Determine (*a*) the kinetic energy of the ball immediately after it is hit, (*b*) the kinetic energy of the ball when it reaches its maximum height, (*c*) the maximum height above the ground reached by the ball.

SOLUTION

Mass of baseball: $W = (5.1\text{ oz})\left(\dfrac{1\text{ lb}}{16\text{ oz}}\right) = 0.31875\text{ lb}$

$$m = \frac{W}{g} = \frac{0.31875\text{ lb}}{32.2\text{ ft/s}^2} = 0.009899\text{ lb}\cdot\text{s}^2/\text{ft}$$

(*a*) Kinetic energy immediately after hit.

$$v = v_0 = 140\text{ ft/s}$$

$$T_1 = \frac{1}{2}mv^2 = \frac{1}{2}(0.009899)(140)^2 \qquad T_1 = 97.0\text{ ft}\cdot\text{lb} \blacktriangleleft$$

(*b*) Kinetic energy at maximum height:

$$v = v_0\cos 40° = 140\cos 40° = 107.246\text{ ft/s}$$

$$T_2 = \frac{1}{2}mv^2 = \frac{1}{2}(0.009899)(107.246)^2 \qquad T_2 = 56.9\text{ ft}\cdot\text{lb} \blacktriangleleft$$

Principle of work and energy: $T_1 + U_{1\to 2} = T_2$

$$U_{1\to 2} = T_2 - T_1 = -40.082\text{ ft}\cdot\text{lb}$$

Work of weight: $U_{1\to 2} = -Wd$

Maximum height above impact point.

$$d = \frac{T_2 - T_1}{-W} = \frac{-40.082\text{ ft}\cdot\text{lb}}{-0.31875\text{ lb}} = 125.7\text{ ft} \qquad 125.7\text{ ft} \blacktriangleleft$$

(*c*) Maximum height above ground:

$$h = 125.7\text{ ft} + 2\text{ ft} \qquad h = 127.7\text{ ft} \blacktriangleleft$$

PROBLEM 13.4

A 500-kg communications satellite is in a circular geosynchronous orbit and completes one revolution about the earth in 23 h and 56 min at an altitude of 35800 km above the surface of the earth. Knowing that the radius of the earth is 6370 km, determine the kinetic energy of the satellite.

SOLUTION

Radius of earth: $R = 6370\text{ km}$

Radius of orbit: $r = R + h = 6370 + 35800 = 42170\text{ km} = 42.170 \times 10^6\text{ m}$

Time one revolution: $t = 23\text{ h} + 56\text{ min}$

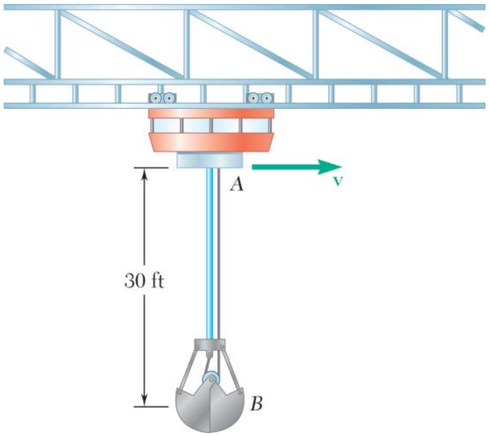
$$t = (23\text{ h})(3600\text{ s/h}) + (56\text{ min})(60\text{ s/min}) = 86.160 \times 10^3\text{ s}$$

Speed: $$v = \frac{2\pi r}{t} = \frac{2\pi(42.170 \times 10^6)}{86.160 \times 10^3} = 3075.2\text{ m/s}$$

Kinetic energy: $$T = \frac{1}{2}mv^2$$

$$T = \frac{1}{2}(500\text{ kg})(3075.2\text{ m/s})^2 = 2.3643 \times 10^9\text{ J}$$

$T = 2.36\text{ GJ}$ ◀



PROBLEM 13.5

In an ore-mixing operation, a bucket full of ore is suspended from a traveling crane which moves along a stationary bridge. The bucket is to swing no more than 10 ft horizontally when the crane is brought to a sudden stop. Determine the maximum allowable speed *v* of the crane.

SOLUTION

Let position ① be the position with bucket *B* directly below *A*, and position ② be that of maximum swing where $d = 10$ ft. Let *L* be the length *AB*.

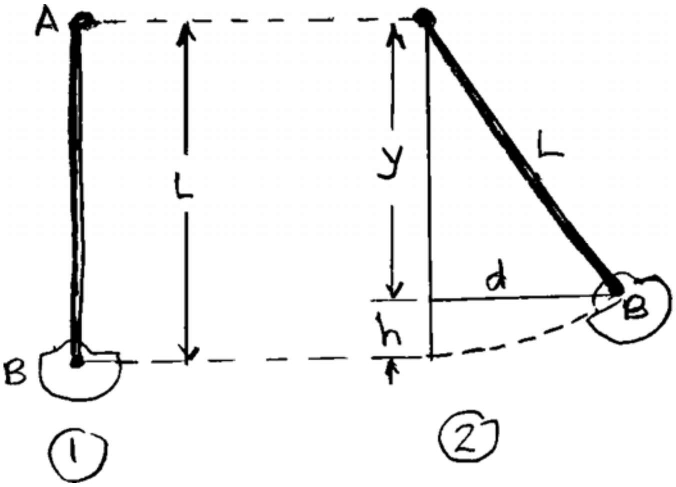
Kinetic energies: $T_1 = \frac{1}{2}mv^2,\ T_2 = 0$

Work of the weight: $U_{1\to 2} = -Wh = -mgh$

where *h* is the vertical projection of position ② above position ①

From geometry (see figure),

$$\begin{aligned} y &= \sqrt{L^2 - d^2} \\ h &= L - y \\ &= L - \sqrt{L^2 - d^2} \\ &= 30 - \sqrt{(30)^2 - (10)^2} \\ &= 1.7157 \text{ ft} \end{aligned}$$

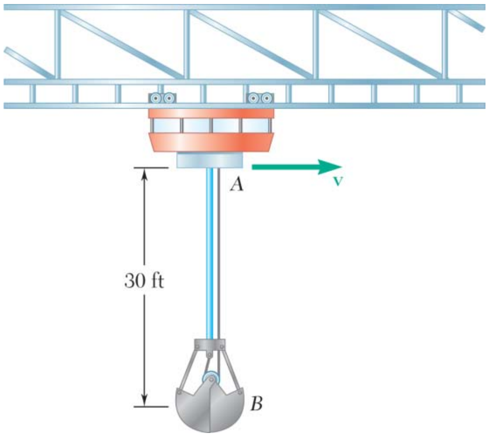


Principle of work and energy: $T_1 + U_{1\to 2} = T_2$

$$\frac{1}{2}mv^2 - mgh = 0$$

$$v^2 = 2gh = (2)(32.2 \text{ ft/s}^2)(1.7157 \text{ ft}) = 110.49 \text{ ft}^2/\text{s}^2$$

$v = 10.51$ ft/s ◀



PROBLEM 13.6

In an ore-mixing operation, a bucket full of ore is suspended from a traveling crane which moves along a stationary bridge. The crane is traveling at a speed of 10 ft/s when it is brought to a sudden stop. Determine the maximum horizontal distance through which the bucket will swing.

SOLUTION

Let position ① be the position with bucket *B* directly below *A*, and position ② be that of maximum swing where the horizontal distance is *d*. Let *L* be the length *AB*.

Kinetic energies: $$T_1 = \frac{1}{2}mv^2, \; T_2 = 0$$

Work of the weight: $$U_{1\to 2} = -Wh = -mgh$$

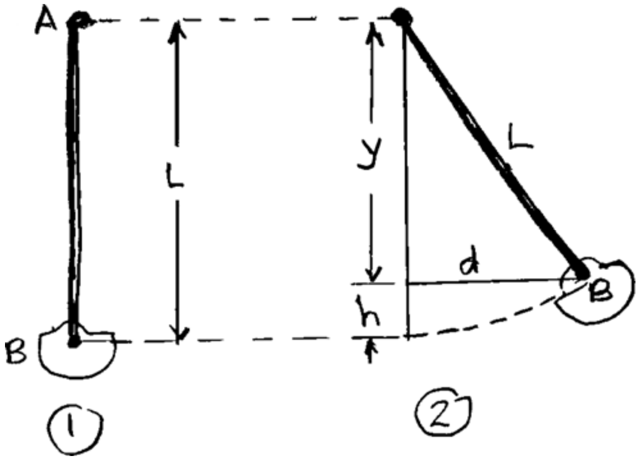
where *h* is the vertical projection of position ② above position ①.

Principle of work and energy: $$T_1 + U_{1\to 2} = T_2$$

$$\frac{1}{2}mv^2 - mgh = 0$$

$$h = \frac{v^2}{2g} = \frac{(10\text{ ft/s})^2}{(2)(32.2\text{ ft/s}^2)} = 1.5528\text{ ft}$$

From geometry (see figure),



$$\begin{aligned} d &= \sqrt{L^2 - y^2} \\ &= \sqrt{L^2 - (L-h)^2} \\ &= \sqrt{(30)^2 - (30-1.5528)^2} \\ &= 9.53\text{ ft} \end{aligned}$$

$d = 9.53$ ft ◀

PROBLEM 13.7

Determine the maximum theoretical speed that may be achieved over a distance of 110 m by a car starting from rest assuming there is no slipping. The coefficient of static friction between the tires and pavement is 0.75, and 60 percent of the weight of the car is distributed over its front wheels and 40 percent over its rear wheels. Assume (*a*) front-wheel drive, (*b*) rear-wheel drive.

SOLUTION

Let W be the weight and m the mass. $W = mg$

(*a*) *Front wheel drive*: $N = 0.60W = 0.60mg$

$\mu_s = 0.75$

Maximum friction force without slipping:

$$F = \mu_s N = (0.75)(0.60W) = 0.45mg$$

$$U_{1\to 2} = Fd = 0.45mgd$$

$$T_1 = 0, \qquad T_2 = \frac{1}{2}mv_2^2$$

Principle of work and energy: $T_1 + U_{1\to 2} = T_2$

$$0 + 0.45mgd = \frac{1}{2}mv_2^2$$

$$v_2^2 = (2)(0.45gd) = (2)(0.45)(9.81\text{ m/s}^2)(110\text{ m}) = 971.19\text{ m}^2/\text{s}^2$$

$v_2 = 31.164$ m/s $\qquad v_2 = 112.2$ km/h ◀

(*b*) *Rear wheel drive*: $N = 0.40W = 0.40mg$

$\mu_s = 0.75$

Maximum friction force without slipping:

$$F = \mu_s N = (0.75)(0.40W) = 0.30mg$$

$$U_{1\to 2} = Fd = 0.30mgd$$

$$T_1 = 0, \qquad T_2 = \frac{1}{2}mv_2^2$$

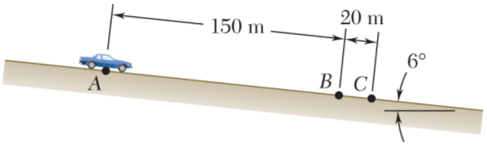
Principle of work and energy: $T_1 + U_{1\to 2} = T_2$

$$0 + 0.30mgd = \frac{1}{2}mv_2^2$$

$$v_2^2 = (2)(0.30)gd = (2)(0.30)(9.81\text{ m/s}^2)(110\text{ m}) = 647.46\text{ m}^2/\text{s}^2$$

$v_2 = 25.445$ m/s $\qquad v_2 = 91.6$ km/h ◀

Note: The car is treated as a particle in this problem. The weight distribution is assumed to be the same for static and dynamic conditions. Compare with sample Problem 16.1 where the vehicle is treated as a rigid body.



PROBLEM 13.8

A 2000-kg automobile starts from rest at point *A* on a 6° incline and coasts through a distance of 150 m to point *B*. The brakes are then applied, causing the automobile to come to a stop at point *C*, 20 m from *B*. Knowing that slipping is impending during the braking period and neglecting air resistance and rolling resistance, determine (*a*) the speed of the automobile at point *B*, (*b*) the coefficient of static friction between the tires and the road.

SOLUTION

Given: Automobile Weight $W = mg = (2000\text{ kg})\,(9.81)$

$$W = 19{,}620\text{ N}$$

Initial Velocity *A*, $v_A = 0\,\text{m/s}$

Incline Angle, $\alpha = 6°$

Vehicle brakes at impending slip for 20 m from *B* to *C*

$v_C = 0$

Find; speed of automobile at point *B*, v_B

Coefficient of static friction, μ

(*a*)

$$U_{A\to B} = Wh_{A\to B} - (19620\text{ N})\,(150\text{ m})\sin 6°$$

$$= 307.63\times 10^3\ \text{N}\cdot\text{m}$$

$$U_{A\to B} = T_B - T_A = \frac{1}{2}mv^2 - 0$$

$$307.63\times 10^3\ \text{N}\cdot\text{m} = \frac{1}{2}\,(2000\,\text{kg})\,v_B^2 - 0$$

$v_B = 17.54$ m/s ◀

(*b*)

$$U_{A\to C} = Wh_{A\to C} - Fd_{B\to C} = T_C - T_A = 0$$

$$d_{B\to C} = 20\text{ m} \qquad F = \mu\,\text{N}$$

Where μ = coefficient of static friction

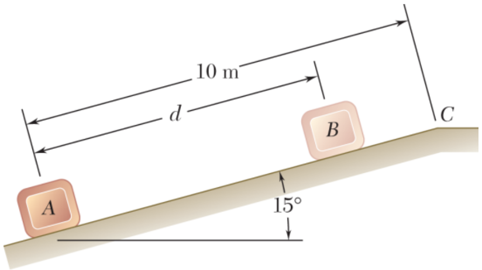
$$U_{A\to C} = (19620\,\text{N})(\sin 6°)(170\,\text{m}) - F(20\,\text{m})$$

$$F = \mu(19620\,\text{N})\,\cos 6°$$

$$(19620\,\text{N})(\sin 6°)(170\,\text{m}) - \mu(19620\,\text{N})(\cos 6°)\,(20\,\text{m}) = 0$$

$$\mu = \frac{170}{20}\tan 6° = 0.893$$

$\mu = 0.893$ ◀



PROBLEM 13.9

A package is projected up a 15° incline at *A* with an initial velocity of 8 m/s. Knowing that the coefficient of kinetic friction between the package and the incline is 0.12, determine (*a*) the maximum distance *d* that the package will move up the incline, (*b*) the velocity of the package as it returns to its original position.

SOLUTION

(*a*) Up the plane from *A* to *B*:

$$T_A = \frac{1}{2}mv_A^2 = \frac{1}{2}\frac{W}{g}(8\text{ m/s})^2 = 32\frac{W}{g} \qquad T_B = 0$$

$$U_{A-B} = (-W\ \sin 15° - F)d \qquad F = \mu_k N = 0.12\,\text{N}$$

$$\nwarrow \Sigma F = 0 \quad N - W\cos 15° = 0 \quad N = W\cos 15°$$

$$U_{A-B} = -W(\sin 15° + 0.12\cos 15°)d = -Wd(0.3747)$$

$$T_A + U_{A-B} = T_B\text{:} \quad 32\frac{W}{g} - Wd(0.3743) = 0$$

$$d = \frac{32}{(9.81)(0.3747)}$$

$d = 8.70$ m ◀

(*b*) Down the plane from *B* to *A*: (*F* reverses direction)

$$T_A = \frac{1}{2}\frac{W}{g}v_A^2 \qquad T_B = 0 \qquad d = 8.71\text{ m/s}$$

$$U_{B-A} = (W\ \sin 15° - F)d$$

$$= W(\sin 15° - 0.12\cos 15°)(8.70\text{ m/s})$$

$$U_{B-A} = 1.245W$$

$$T_B + U_{B-A} = T_A \quad 0 + 1.245W = \frac{1}{2}\frac{W}{g}v_A^2$$

$$v_A^2 = (2)(9.81)(1.245)$$

$$= 24.43$$

$$v_A = 4.94\text{ m/s}$$

$\mathbf{v}_A = 4.94$ m/s $\swarrow$ 15° ◀

PROBLEM 13.10

A 1.4 kg model rocket is launched vertically from rest with a constant thrust of 25 N until the rocket reaches an altitude of 15 m and the thrust ends. Neglecting air resistance, determine (*a*) the speed of the rocket when the thrust ends, (*b*) the maximum height reached by the rocket, (*c*) the speed of the rocket when it returns to the ground.

SOLUTION

Weight: $W = mg = (1.4)(9.81) = 13.734 \text{ N}$

(*a*) First stage: $T_1 = 0$

$$U_{1\to 2} = (25 - 13.734)(15) = 169.0 \text{ N}\cdot\text{m}$$

$$T_1 + U_{1\to 2} = T_2$$

$$T_2 = \frac{1}{2}mv^2 = U_{1\to 2} = 169.0 \text{ N}\cdot\text{m}$$

$$v_2 = \sqrt{\frac{2U_{1\to 2}}{m}} = \sqrt{\frac{(2)(169.0)}{1.4}}$$

$v_2 = 15.54 \text{ m/s}$ ◀

(*b*) Unpowered flight to maximum height h:

$$T_2 = 169.0 \text{ N}\cdot\text{m} \qquad T_3 = 0$$

$$U_{2\to 3} = -W(h - 15)$$

$$T_2 + U_{2\to 3} = T_3$$

$$W(h-15) = T_2$$

$$h - 15 = \frac{T_2}{W} = \frac{169.0}{13.734}$$

$h = 27.3 \text{ m}$ ◀

(*c*) Falling from maximum height:

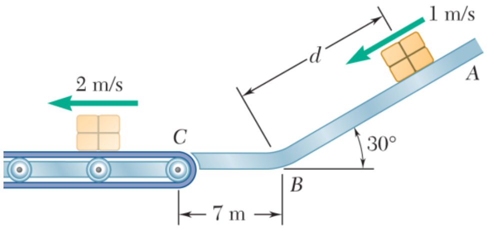
$$T_3 = 0 \qquad T_4 = \frac{1}{2}mv_4^2$$

$$U_{3\to 4} = Wh = mgh$$

$$T_3 = U_{3\to 4} = T_4: \qquad 0 + mgh = \frac{1}{2}mv_4^2$$

$$v_4^2 = 2\,gh = (2)(9.81 \text{ m/s}^2)(27.3 \text{ m}) = 535.6 \text{ m}^2/\text{s}^2$$

$v_4 = 23.1 \text{ m/s}$ ◀



PROBLEM 13.11

Packages are thrown down an incline at *A* with a velocity of 1 m/s. The packages slide along the surface *ABC* to a conveyor belt which moves with a velocity of 2 m/s. Knowing that $\mu_k = 0.25$ between the packages and the surface *ABC*, determine the distance *d* if the packages are to arrive at *C* with a velocity of 2 m/s.

SOLUTION

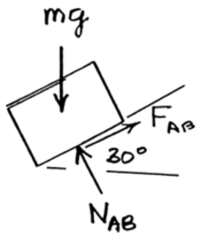
On incline *AB*:

$$N_{AB} = mg\cos 30°$$

$$F_{AB} = \mu_k N_{AB} = 0.25\,mg\,\cos 30°$$

$$U_{A\to B} = mgd\sin 30° - F_{AB}\,d$$

$$= mgd(\sin 30° - \mu_k\cos 30°)$$

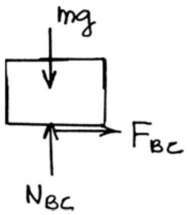


On level surface *BC*:

$$N_{BC} = mg \qquad x_{BC} = 7\text{ m}$$

$$F_{BC} = \mu_k mg$$

$$U_{B\to C} = -\mu_k mg\,x_{BC}$$



At *A*,

$$T_A = \frac{1}{2}mv_A^2 \quad\text{and}\quad v_A = 1\text{ m/s}$$

At *C*,

$$T_C = \frac{1}{2}mv_C^2 \quad\text{and}\quad v_C = 2\text{ m/s}$$

Assume that no energy is lost at the corner *B*.

Work and energy.

$$T_A + U_{A\to B} + U_{B\to C} = T_C$$

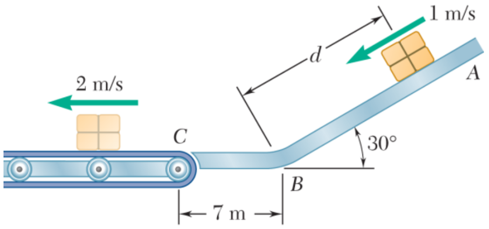
$$\frac{1}{2}mv_A^2 + mgd(\sin 30° - \mu_k\cos 30°) - \mu_k mg\,x_{BC} = \frac{1}{2}mv_0^2$$

Dividing by *m* and solving for *d*,

$$d = \frac{\left[v_C^2/2g + \mu_k x_{BC} - v_A^2/2g\right]}{(\sin 30° - \mu_k\cos 30°)}$$

$$= \frac{(2)^2/(2)(9.81) + (0.25)(7) - (1)^2/(2)(9.81)}{\sin 30° - 0.25\cos 30°}$$

$d = 6.71$ m ◀



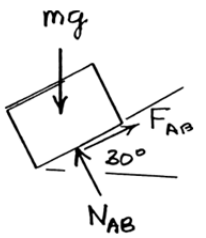
PROBLEM 13.12

Packages are thrown down an incline at *A* with a velocity of 1 m/s. The packages slide along the surface *ABC* to a conveyor belt which moves with a velocity of 2 m/s. Knowing that $d = 7.5 \text{ m}$ and $\mu_k = 0.25$ between the packages and all surfaces, determine (*a*) the speed of the package at *C*, (*b*) the distance a package will slide on the conveyor belt before it comes to rest relative to the belt.

SOLUTION

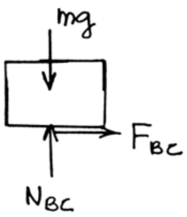
(*a*) On incline *AB*:

$$\begin{aligned} N_{AB} &= mg\cos 30^\circ \\ F_{AB} &= \mu_k N_{AB} = 0.25\,mg\cos 30^\circ \\ U_{A\to B} &= mgd\sin 30^\circ - F_{AB}\,d \\ &= mgd(\sin 30^\circ - \mu_k \cos 30^\circ) \end{aligned}$$



On level surface *BC*:

$$\begin{aligned} N_{BC} &= mg \qquad x_{BC} = 7 \text{ m} \\ F_{BC} &= \mu_k mg \\ U_{B\to C} &= -\mu_k mg\,x_{BC} \end{aligned}$$



At *A*, $T_A = \frac{1}{2}mv_A^2$ and $v_A = 1 \text{ m/s}$

At *C*, $T_C = \frac{1}{2}mv_C^2$ and $v_C = 2 \text{ m/s}$

Assume that no energy is lost at the corner *B*.

Work and energy. $T_A + U_{A\to B} + U_{B\to C} = T_C$

$$\frac{1}{2}mv_A^2 + mgd(\sin 30^\circ - \mu_k\cos 30^\circ) - \mu_k mg\,x_{BC} = \frac{1}{2}mv_0^2$$

Solving for v_C^2,

$$\begin{aligned} v_C^2 &= v_A^2 + 2gd(\sin 30^\circ - \mu_k\cos 30^\circ) - 2\mu_k g\,x_{BC} \\ &= (1)^2 + (2)(9.81)(7.5)(\sin 30^\circ - 0.25\cos 30^\circ) - (2)(0.25)(9.81)(7) \\ &= 8.3811 \text{ m}^2/\text{s}^2 \end{aligned}$$

$v_C = 2.90 \text{ m/s}$ ◀

(*b*) Box on belt: Let x_{belt} be the distance moves by a package as it slides on the belt.

$$+\uparrow \Sigma F_y = ma_y \qquad N - mg = 0 \qquad N = mg$$

$$F_x = \mu_k N = \mu_k mg$$

At the end of sliding, $v = v_{\text{belt}} = 2 \text{ m/s}$

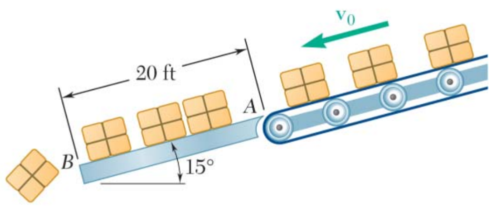
PROBLEM 13.12 (Continued)

Principle of work and energy:

$$\frac{1}{2}mv_C^2 - \mu_k mg\,x_{\text{belt}} = \frac{1}{2}mv_{\text{belt}}^2$$

$$x_{\text{belt}} = \frac{v_C^2 - v_{\text{belt}}^2}{2\,\mu_k g}$$

$$= \frac{8.3811 - (2)^2}{(2)(0.25)(9.81)} \qquad x_{\text{belt}} = 0.893\text{ m} ◀$$

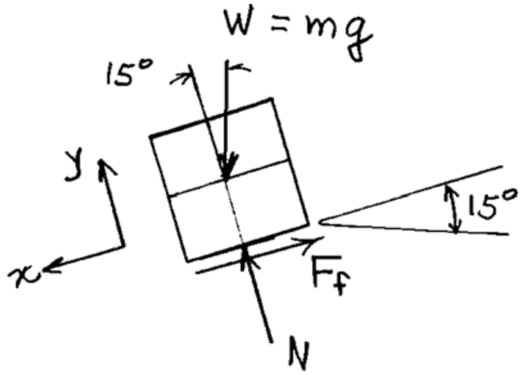


PROBLEM 13.13

Boxes are transported by a conveyor belt with a velocity $\mathbf{v}_0$ to a fixed incline at A where they slide and eventually fall off at B. Knowing that $\mu_k = 0.40$, determine the velocity of the conveyor belt if the boxes leave the incline at B with a velocity of 8 ft/s.

SOLUTION

Forces when box is on AB.



$$\Sigma F_y = 0: \quad N - W\cos 15° = 0$$

$$N = W\cos 15°$$

Box is sliding on AB.

$$F_f = \mu_k N = \mu_k W\cos 15°$$

Distance $\quad AB = d = 20$ ft

Work of gravity force: $\quad (U_{A-B})_g = Wd\sin 15°$

Work of friction force: $\quad -F_f d = -\mu_k Wd\cos 15°$

Total work $\quad U_{A\to B} = Wd(\sin 15° - \mu_k \sin 15°)$

Kinetic energy:

$$T_A = \frac{1}{2}\frac{W}{g}v_0^2$$

$$T_B = \frac{1}{2}\frac{W}{g}v_B^2$$

Principle of work and energy: $\quad T_A + U_{A\to B} = T_B$

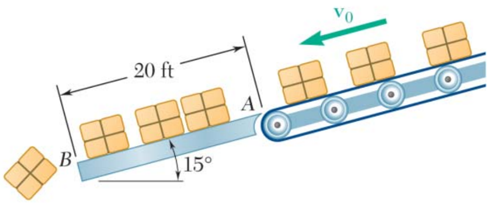
$$\frac{1}{2}\frac{W}{g}v_0^2 + Wd(\sin 15° - \mu_k \cos 15°) = \frac{1}{2}\frac{W}{g}v_B^2$$

$$v_0^2 = v_B^2 - 2gd(\sin 15° - \mu_k \cos 15°)$$

$$= (8)^2 - (2)(32.2)(20)[\sin 15° - (0.40)(\cos 15°)]$$

$$= 228.29 \text{ ft}^2/\text{s}^2$$

$\mathbf{v}_0 = 15.11$ ft/s ↙ 15° ◀

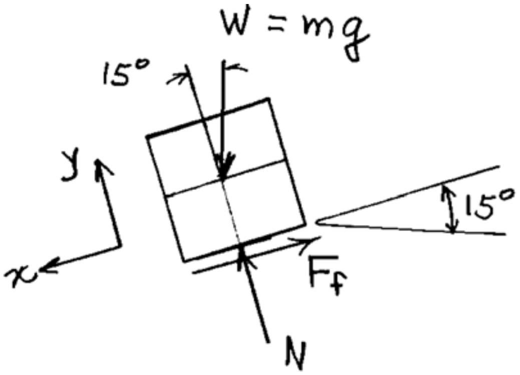


PROBLEM 13.14

Boxes are transported by a conveyor belt with a velocity $\mathbf{v}_0$ to a fixed incline at *A* where they slide and eventually fall off at *B*. Knowing that $\mu_k = 0.40$, determine the velocity of the conveyor belt if the boxes are to have zero velocity at *B*.

SOLUTION

Forces when box is on *AB*.



$$\Sigma F_y = 0: \quad N - W\cos 15° = 0$$

$$N = W\cos 15°$$

Box is sliding on AB.

$$F_f = \mu_k N = \mu_k W\cos 15°$$

Distance $\quad AB = d = 20 \text{ ft}$

Work of gravity force: $\quad (U_{A-B})_g = Wd\sin 15°$

Work of friction force: $\quad -F_f d = -\mu_k Wd\cos 15°$

Total work $\quad U_{A-B} = Wd(\sin 15° - \mu_k\cos 15°)$

Kinetic energy: $\quad T_A = \frac{1}{2}\frac{W}{g}v_0^2$

$$T_B = \frac{1}{2}\frac{W}{g}v_B^2$$

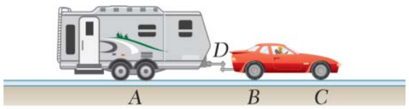
Principle of work and energy: $\quad T_A + U_{A-B} = T_B$

$$\frac{1}{2}\frac{W}{g}v_0^2 + Wd(\sin 15° - \mu_k\cos 15°) = \frac{1}{2}\frac{W}{g}v_B^2$$

$$\begin{aligned} v_0^2 &= v_B^2 - 2gd(\sin 15° - \mu_k\cos 15°) \\ &= 0 - (2)(32.2)(20)[\sin 15° - (0.40)(\cos 15°)] \\ &= 164.29 \text{ ft}^2/\text{s}^2 \end{aligned}$$

$\mathbf{v}_0 = 12.81 \text{ ft/s} \nearrow 15°$ ◀

PROBLEM 13.15



A 1200-kg trailer is hitched to a 1400-kg car. The car and trailer are traveling at 72 km/h when the driver applies the brakes on both the car and the trailer. Knowing that the braking forces exerted on the car and the trailer are 5000 N and 4000 N, respectively, determine (*a*) the distance traveled by the car and trailer before they come to a stop, (*b*) the horizontal component of the force exerted by the trailer hitch on the car.

SOLUTION

Let position 1 be the initial state at velocity $v_1 = 72\text{ km/h} = 20\text{ m/s}$ and position 2 be at the end of braking $(v_2 = 0)$. The braking forces and $F_C = 5000\text{ N}$ for the car and 4000 N for the trailer.

(*a*) Car and trailer system. (d = braking distance)

$$T_1 = \frac{1}{2}(m_C + m_T)v_1^2 \qquad T_2 = 0$$

$$U_{1\to 2} = -(F_C + F_T)d$$

$$T_1 + U_{1\to 2} = T_2$$

$$\frac{1}{2}(m_C + m_T)v_1^2 - (F_C + F_T)d = 0$$

$$d = \frac{(m_C + m_T)v_1^2}{2(F_C + F_T)} = \frac{(2600)(20)^2}{(2)(9000)} = 57.778 \qquad d = 57.8\text{ m} ◀$$

(*b*) Car considered separately.

Let H be the horizontal pushing force that the trailer exerts on the car through the hitch.

$$T_1 = \frac{1}{2}m_C v_1^2 \qquad T_2 = 0$$

$$U_{1\to 2} = (H - F_C)d$$

$$T_1 + U_{1\to 2} = T_2$$

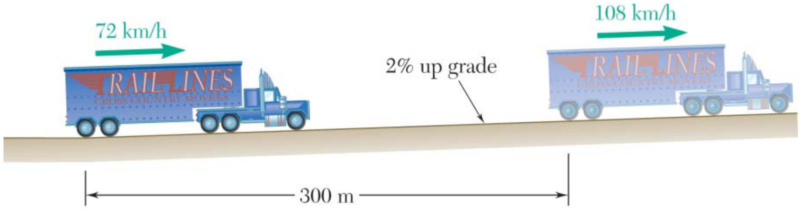
$$\frac{1}{2}m_C v_1^2 + (H - F_C)d = 0$$

$$H = F_C - \frac{m_C v_1^2}{2d} = 5000 - \frac{(1400)(20)^2}{(2)(57.778)}$$

Trailer hitch force on car: $\mathbf{H} = 154\text{ N} \longrightarrow$ ◀

PROBLEM 13.16

A trailer truck enters a 2 percent uphill grade traveling at 72 km/h and reaches a speed of 108 km/h in 300 m. The cab has a mass of 1800 kg and the trailer 5400 kg. Determine (*a*) the average force at the wheels of the cab, (*b*) the average force in the coupling between the cab and the trailer.



SOLUTION

Initial speed: $v_1 = 72 \text{ km/h} = 20 \text{ m/s}$

Final speed: $v_2 = 108 \text{ km/h} = 30 \text{ m/s}$

Vertical rise: $h = (0.02)(300) = 6.00 \text{ m}$

Distance traveled: $d = 300 \text{ m}$

(*a*) Traction force. Use cab and trailer as a free body.

$m = 1800 + 5400 = 7200 \text{ kg}$ $\qquad W = mg = (7200)(9.81) = 70.632 \times 10^3 \text{ N}$

Work and energy: $T_1 + U_{1\to 2} = T_2 \qquad \frac{1}{2}mv_1^2 - Wh + F_t d = \frac{1}{2}mv_2^2$

$$F_t = \frac{1}{d}\left[\frac{1}{2}mv_1^2 + Wh - mv_1^2\right] = \frac{1}{300}\left[\frac{1}{2}(7200)(30)^2 + (70.632\times 10^3)(6.00) - \frac{1}{2}(7200)(20)^2\right]$$

$= 7.4126 \times 10^3 \text{ N}$ $\qquad F_t = 7.41 \text{ kN}$ ◀

(*b*) Coupling force F_c. Use the trailer alone as a free body.

$m = 5400 \text{ kg}$ $\qquad W = mg = (5400)(9.81) = 52.974 \times 10^3 \text{ N}$

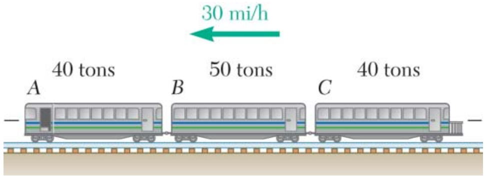
Assume that the tangential force at the trailer wheels is zero.

Work and energy: $T_1 + U_{1\to 2} = T_2 \qquad \frac{1}{2}mv_1^2 - Wh + F_c d = \frac{1}{2}mv_2^2$

The plus sign before F_c means that we have assumed that the coupling is in tension.

$$F_c = \frac{1}{d}\left[\frac{1}{2}mv_2^2 + Wh - \frac{1}{2}mv_1^2\right] = \frac{1}{300}\left[\frac{1}{2}(5400)(30)^2 + (52.974\times 10^3)(6.00) - \frac{1}{2}(5400)(20)^2\right]$$

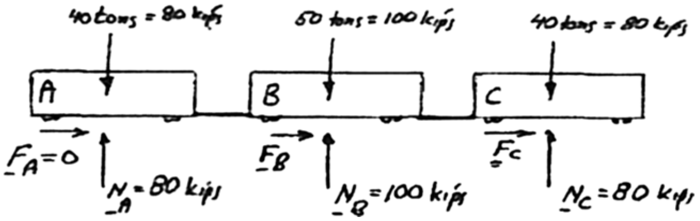
$= 5.5595 \times 10^3 \text{ N}$ $\qquad F_c = 5.56 \text{ kN (tension)}$ ◀


30 mi/h
40 tons
50 tons
40 tons
A
B
C

PROBLEM 13.17

The subway train shown is traveling at a speed of 30 mi/h when the brakes are fully applied on the wheels of cars *B* and *C*, causing them to slide on the track, but are not applied on the wheels of car *A*. Knowing that the coefficient of kinetic friction is 0.35 between the wheels and the track, determine (*a*) the distance required to bring the train to a stop, (*b*) the force in each coupling.

SOLUTION


40 tons = 80 kips
50 tons = 100 kips
40 tons = 80 kips
A
B
C
$F_A = 0$
$N_A = 80$ kips
F_B
$N_B = 100$ kips
F_C
$N_C = 80$ kips

$$\mu_k = 0.35 \quad F_B = (0.35)(100 \text{ kips}) = 35 \text{ kips}$$

$$F_C = (0.35)(80 \text{ kips}) = 28 \text{ kips}$$

$$v_1 = 30 \text{ mi/h} = 44 \text{ ft/s} \longleftarrow \qquad v_2 = 0 \quad T_2 = 0$$

(*a*) Entire train:

$$T_1 + U_{1-2} = T_2$$

$$\frac{1}{2}\frac{(80 \text{ kips} + 100 \text{ kips } + 80 \text{ kips})}{32.2 \text{ ft/s}^2}(44 \text{ ft/s})^2 - (28 \text{ kips} + 35 \text{ kips})\, x = 0$$

$$x = 124.07 \text{ ft} \qquad x = 124.1 \text{ ft} \blacktriangleleft$$

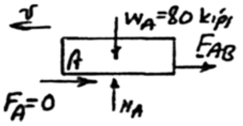
(*b*) Force in each coupling: Recall that $x = 124.07$ ft

Car *A*: Assume F_{AB} to be in tension

$$T_1 + V_{1-2} = T_2$$

$$\frac{1}{2}\frac{80 \text{ kips}}{32.2}(44)^2 - F_{AB}(124.07 \text{ ft}) = 0$$

$$F_{AB} = +19.38 \text{ kips}$$

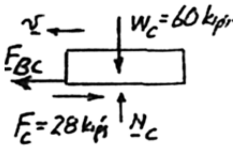

v
$W_A = 80$ kips
A
F_{AB}
$F_A = 0$
N_A

$$F_{AB} = 19.38 \text{ kips (tension)} \blacktriangleleft$$

Car *C*:

$$T_1 + U_{1-2} = T_2$$

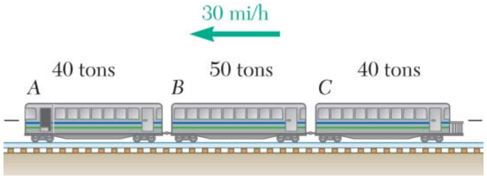
$$\frac{1}{2}\frac{80 \text{ kips}}{32.2}(44)^2 + (F_{BC} - 28 \text{ kips})(124.07 \text{ ft}) = 0$$


v
$W_C = 60$ kips
F_{BC}
$F_C = 28$ kips
N_C

$$F_{BC} - 28 \text{ kips} = -19.38 \text{ kips}$$

$$F_{BC} = +8.62 \text{ kips} \qquad F_{BC} = 8.62 \text{ kips (tension)} \blacktriangleleft$$

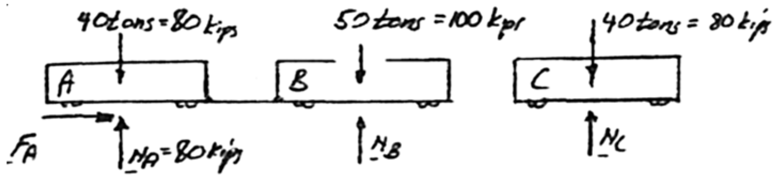
PROBLEM 13.18


30 mi/h
40 tons
50 tons
40 tons
A
B
C

The subway train shown is traveling at a speed of 30 mi/h when the brakes are fully applied on the wheels of cars *A*, causing it to slide on the track, but are not applied on the wheels of cars *A* or *B*. Knowing that the coefficient of kinetic friction is 0.35 between the wheels and the track, determine (*a*) the distance required to bring the train to a stop, (*b*) the force in each coupling.

SOLUTION

(*a*) Entire train:


40 tons = 80 kips
50 tons = 100 kps
40 tons = 80 kips
A
B
C
F_A
N_A = 80 kips
N_B
N_C

$$F_A = \mu N_A = (0.35)(80\text{ kips}) = 28\text{ kips}$$

$$v_1 = 30\text{ mi/h} = 44\text{ ft/s} \leftarrow \qquad v_2 = 0 \quad T_2 = 0$$

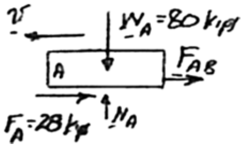
$$T_1 + V_{1-2} = T_2$$

$$\frac{1}{2}\frac{(80\text{ kips} + 100\text{ kips} + 80\text{ kips})}{32.2\text{ ft/s}^2}(44\text{ ft/s})^2 - (28\text{ kips})\,x = 0$$

$$x = 279.1\text{ ft} \qquad x = 279\text{ ft} \blacktriangleleft$$

(*b*) Force in each coupling:

Car *A*: Assume F_{AB} to be in tension


v
W_A = 80 kips
A
F_{AB}
F_A = 28 kips
N_A

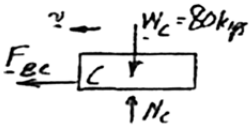
$$T_1 + V_{1-2} = T_2$$

$$\frac{1}{2}\frac{80\text{ kips}}{32.2\text{ ft/s}^2}(44\text{ ft/s})^2 - (28\text{ kips} + F_{AB})(279.1\text{ ft}) = 0$$

$$28\text{ kips} + F_{AB} = +8.62\text{ kips}$$

$$F_{AB} = -19.38\text{ kips} \qquad F_{AB} = 19.38\text{ kips (compression)} \blacktriangleleft$$

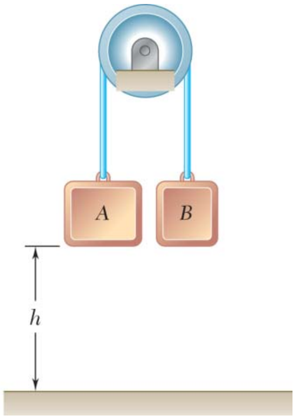
Car *C*:


v
W_C = 80 kips
F_{BC}
C
N_C

$$T_1 + V_{1-2} = T_2$$

$$\frac{1}{2}\frac{80\text{ kips}}{32.2\text{ ft/s}^2}(44\text{ ft/s})^2 + F_{BC}(279.1\text{ ft}) = 0$$

$$F_{BC} = -8.617\text{ kips} \qquad F_{BC} = 8.62\text{ kips (compression)} \blacktriangleleft$$



PROBLEM 13.19

Blocks A and B weigh 25 lbs and 10 lbs, respectively, and they are both at a height 6 ft above the ground when the system is released from rest. Just before hitting the ground block A is moving at a speed of 9 ft/s. Determine (*a*) the amount of energy dissipated in friction by the pulley, (*b*) the tension in each portion of the cord during the motion.

SOLUTION

By constraint of the cable block B moves up a distance h when block A moves down a distance h. $(h = 6 \text{ ft})$ Their speeds are equal.

Let F_A and F_B be the tensions on the A and B sides, respectively, of the pulley.

Masses:

$$M_A = \frac{W_A}{g} = \frac{25}{32.2} = 0.7764 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$M_B = \frac{W_B}{g} = \frac{10}{32.2} = 0.31056 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Let position 1 be the initial position with both blocks a distance h above the ground and position 2 be just before block A hits the ground.

Kinetic energies:

$$(T_1)_A = 0, \qquad (T_1)_B = 0$$

$$(T_2)_A = \frac{1}{2}m_A v^2 = \frac{1}{2}(0.7764)(9)^2 = 31.444 \text{ ft} \cdot \text{lb}$$

$$(T_2)_B = \frac{1}{2}m_B v^2 = \frac{1}{2}(0.31056)(9)^2 = 12.578 \text{ ft} \cdot \text{lb}$$

Principle of work and energy: $T_1 + U_{1\to 2} = T_2$

Block A:

$$U_{1\to 2} = (W_A - F_A)h$$

$$0 + (25 - F_A)(6) = 31.444 \qquad F_A = 19.759 \text{ lb}$$

Block B:

$$U_{1\to 2} = (F_B - W_B)h$$

$$0 + (F_B - 10)(6) = 12.578 \qquad F_B = 12.096 \text{ lb}$$

At the pulley F_A moves a distance h down, and F_B moves a distance h up. The work done is

$$U_{1\to 2} = (F_A - F_B)h = (19.759 - 12.096)(6) = 46.0 \text{ ft} \cdot \text{lb}$$

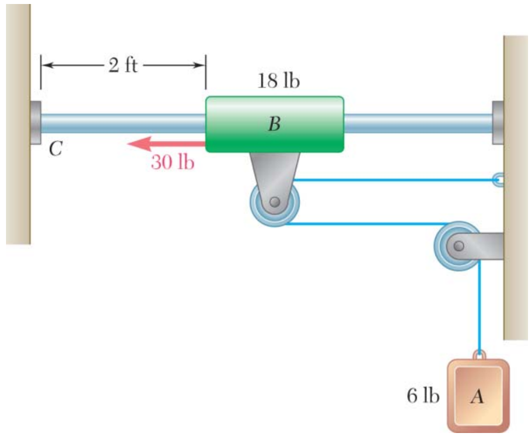
PROBLEM 13.19 (Continued)

Since the pulley is assumed to be massless, it cannot acquire kinetic energy; hence,

(*a*) Energy dissipated by the pulley: $E_p = 46.0 \text{ ft} \cdot \text{lb}$ ◀

(b) Tension in each portion of the cord: $A: 19.76 \text{ lb}$ ◀

$B: 12.10 \text{ lb}$ ◀



PROBLEM 13.20

The system shown is at rest when a constant 30 lb force is applied to collar *B*. (*a*) If the force acts through the entire motion, determine the speed of collar *B* as it strikes the support at *C*. (*b*) After what distance *d* should the 30 lb force be removed if the collar is to reach support *C* with zero velocity?

SOLUTION

Let *F* be the cable tension and v_B be the velocity of collar *B* when it strikes the support. Consider the collar *B*. Its movement is horizontal so only horizontal forces acting on *B* do work. Let *d* be the distance through which the 30 lb applied force moves.

$$(T_1)_B + (U_{1\to 2})_B = (T_2)_B$$

$$0 + 30d - (2F)(2) = \frac{1}{2}\frac{18}{32.2}v_B^2$$

$$30d - 4F = 0.27950 v_B^2 \qquad (1)$$

Now consider the weight *A*. When the collar moves 2 ft to the left, the weight moves 4 ft up, since the cable length is constant. Also, $v_A = 2v_B$.

$$(T_1)_A + (U_{1-2})_A = (T_2)_B$$

$$0 + (F - W_A)(4) = \frac{1}{2}\frac{W_A}{g}v_A^2$$

$$4F - (6)(4) = \frac{1}{2}\frac{6}{32.2}(2v_B)^2$$

$$4F - 24 = 0.37267\, v_B^2 \qquad (2)$$

Add Eqs. (1) and (2) to eliminate *F*.

$$30d - 24 = 0.65217 v_B^2 \qquad (3)$$

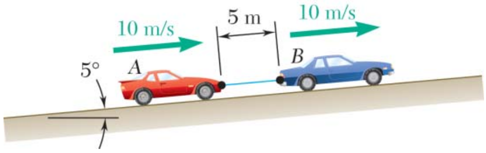
(*a*) Case *a*: $d = 2$ ft, $v_B = ?$

$$(30)(2) - (24) = 0.65217 v_B^2$$

$$v_B^2 = 55.2 \text{ ft}^2/\text{s}^2 \qquad v_B = 7.43 \text{ ft/s} \blacktriangleleft$$

(*b*) Case *b*: $d = ?$, $v_B = 0.$

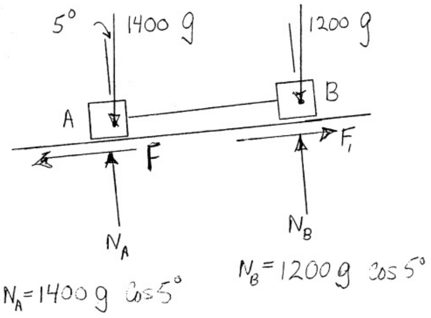
$$30d - 24 = 0 \qquad d = 0.800 \text{ ft} \blacktriangleleft$$



PROBLEM 13.21

Car *B* is towing car *A* at a constant speed of 10 m/s on an uphill grade when the brakes of car *A* are fully applied causing all four wheels to skid. The driver of car *B* does not change the throttle setting or change gears. The masses of the cars *A* and *B* are 1400 kg and 1200 kg, respectively, and the coefficient of kinetic friction is 0.8. Neglecting air resistance and rolling resistance, determine (*a*) the distance traveled by the cars before they come to a stop, (*b*) the tension in the cable.

SOLUTION



$F = 0.8N_A$

Given: Car *B* tows car *A* at 10 m/s uphill.

Car *A* brakes so 4 wheels skid. $\mu_k = 0.8$

Car *B* continues in same gear and throttle setting.

Find: (*a*) Distance *d*, traveled to stop

(*b*) Tension in cable

(*a*) F_1 = traction force (from equilibrium)

$$F_1 = (1400g)\sin 5° + (1200g)\sin 5°$$

$$= 2600(9.81)\sin 5°$$

For system: $A + B$

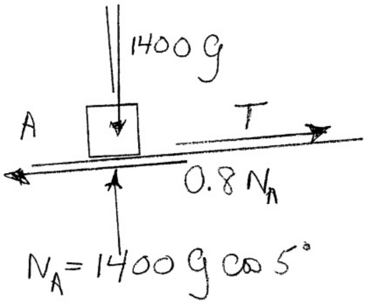
$$U_{1-2} = [(F_1 - 1400g\sin 5° - 1200g\sin 5°) - F]d$$

$$= T_2 - T_1 = 0 - \frac{1}{2}m_{A+B}v^2 = -\frac{1}{2}(2600)(10)^2$$

Since $(F_1 - 1400g\sin 5° - 1200g\sin 5°) = 0$

$$-Fd = -0.8[1400(9.81)\cos 5°]d = -130{,}000\ \text{N}\cdot\text{m}$$

$d = 11.88$ m ◀



(*b*) Cable tension, *T*

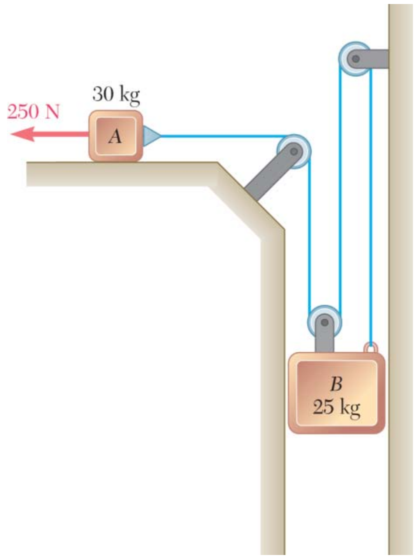
$$U_{1-2} = [T - 0.8N_A](11.88) = T_2 - T_1$$

$$(T - 0.8(1400)(9.81)\cos 5°)11.88 = -\frac{1400}{2}(10)^2$$

$$(T - 10945) = -5892$$

$$= 5.053\ \text{kN}$$

$T = 5.05$ kN ◀

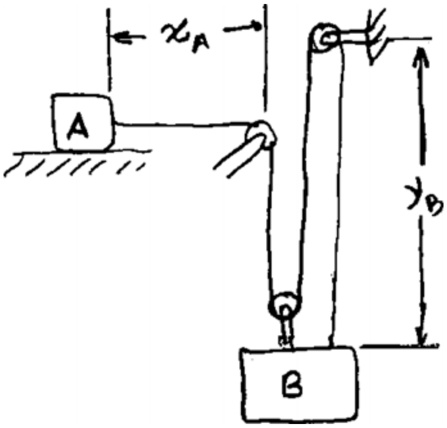


PROBLEM 13.22

The system shown is at rest when a constant 250-N force is applied to block *A*. Neglecting the masses of the pulleys and the effect of friction in the pulleys and between block *A* and the horizontal surface, determine (*a*) the velocity of block *B* after block *A* has moved 2 m, (*b*) the tension in the cable.

SOLUTION

Constraint of cable:



$$x_A + 3y_B = \text{constant}$$

$$\Delta x_A + 3\Delta y_B = 0$$

$$v_A + 3v_B = 0$$

Let *F* be the tension in the cable.

Block *A*: $m_A = 30$ kg, $P = 250$ N, $(T_1)_A = 0$

$$(T_1)_A + (U_{1\to 2})_A = (T_2)_A$$

$$0 + (P - F)(\Delta x_A) = \frac{1}{2}m_A v_A^2$$

$$0 + (250 - F)(2) = \frac{1}{2}(30)(3v_B)^2$$

$$500 - 2F = 135v_B^2 \quad (1)$$

Block *B*: $m_B = 25$ kg, $W_B = m_B g = 245.25$ N

$$(T_1)_B + (U_{1\to 2})_B = (T_2)_B$$

$$0 + (3F - W_B)(-\Delta y_B) = \frac{1}{2}m_B v_B^2$$

$$(3F) - 245.25)\left(\frac{2}{3}\right) = \frac{1}{2}(25)\,v_B^2$$

$$2F - 163.5 = 12.5\ v_B^2 \quad (2)$$

PROBLEM 13.22 (Continued)

Add Eqs. (1) and (2) to eliminate F.

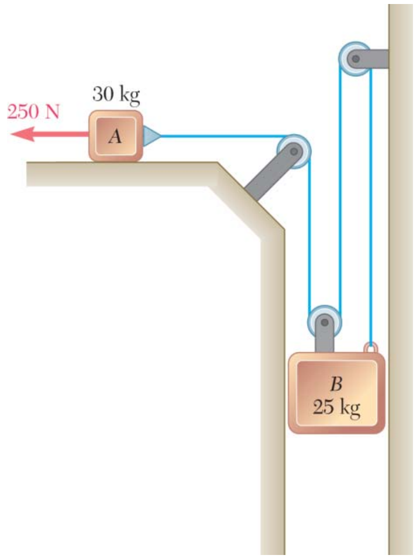
$$500 - 163.5 = 147.5 v_B^2$$

$$v_B^2 = 2.2814 \text{ m}^2/\text{s}^2$$

(a) Velocity of B. $\mathbf{v}_B = 1.510 \text{ m/s} \longleftarrow$ ◀

(b) Tension in the cable.

From Eq. (2), $2F - 163.5 = (12.5)(2.2814)$ $F = 96.0 \text{ N}$ ◀



PROBLEM 13.23

The system shown is at rest when a constant 250-N force is applied to block *A*. Neglecting the masses of the pulleys and the effect of friction in the pulleys and assuming that the coefficients of friction between block *A* and the horizontal surface are $\mu_s = 0.25$ and $\mu_k = 0.20$, determine (*a*) the velocity of block *B* after block *A* has moved 2 m, (*b*) the tension in the cable.

SOLUTION

Check the equilibrium position to see if the blocks move. Let *F* be the tension in the cable.

Block *B*: $3F - m_B g = 0$

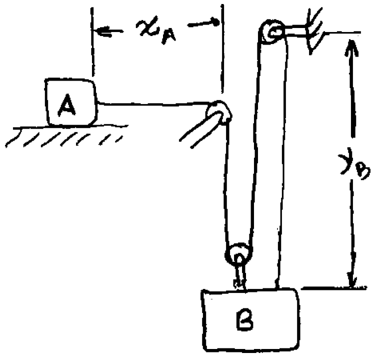
$$F = \frac{m_B g}{3} = \frac{(25)(9.81)}{3} = 81.75 \text{ N}$$

Block *A*: $+\uparrow \Sigma F_y = 0$: $N_A - m_A g = 0$

$$N_A = m_A g = (30)(9.81) = 294.3 \text{ N}$$

$\overset{+}{\longleftarrow} \Sigma F_x = 0$: $250 - F_A - F = 0$

$$F_A = 250 - 81.75 = 168.25 \text{ N}$$



Available static friction force: $\mu_s N_A = (0.25)(294.3) = 73.57 \text{ N}$

Since $F_A > \mu_s N_A$, the blocks move.

The friction force, F_A, during sliding is

$$F_A = \mu_k N_A = (0.20)(294.3) = 58.86 \text{ N}$$

Constraint of cable:

$$x_A + 3y_B = \text{constant}$$
$$\Delta x_A + 3\Delta y_B = 0$$
$$v_A + 3v_B = 0$$

PROBLEM 13.23 (Continued)

Block *A*: $m_A = 30\text{ kg},\quad P = 250\text{ N},\quad (T_1)_A = 0.$

$$(T_1)_A + (U_{1\to 2})_A = (T_2)_A$$

$$0 + (P - F_A - F)(\Delta x_A) = \frac{1}{2} m_A v_A^2$$

$$0 + (250 - 58.86 - F)(2) = \frac{1}{2}(30)(3v_B)^2$$

$$382.28 - 2F = 135 v_B^2 \qquad (1)$$

Block *B*: $M_B = 25\text{ kg},\quad W_B = m_B g = 245.25\text{ N}$

$$(T_1)_B + (U_{1\to 2})_B = (T_2)_B$$

$$0 + (3F - W_B)(-\Delta y_B) = \frac{1}{2} m_B v_B^2$$

$$(3F - 245.25)\left(\frac{2}{3}\right) = \frac{1}{2}(25) v_B^2$$

$$2F - 163.5 = 12.5 v_B^2 \qquad (2)$$

Add Eqs. (1) and (2) to eliminate *F*.

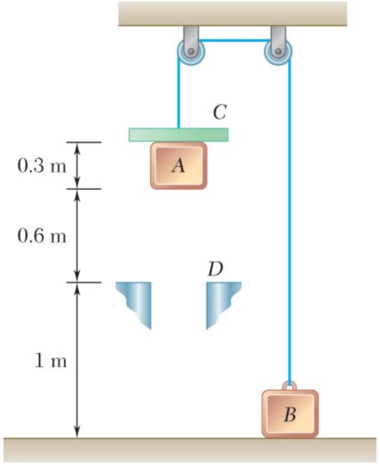
$$382.28 - 163.5 = 147.5 v_B^2$$

$$v_B^2 = 1.48325\text{ m}^2/\text{s}^2$$

(*a*) Velocity of *B*: $\mathbf{v}_B = 1.218\text{ m/s} \longleftarrow$ ◀

(*b*) Tension in the cable:

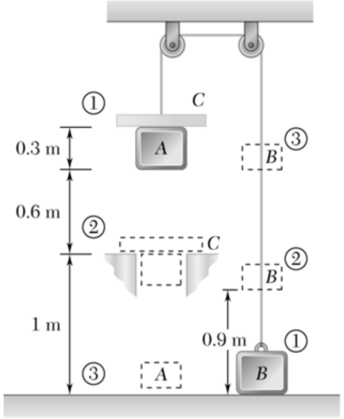
From Eq. (2), $2F - 163.5 = (12.5)(1.48325)$ $F = 91.0\text{ N}$ ◀



PROBLEM 13.24

Two blocks *A* and *B*, of mass 4 kg and 5 kg, respectively, are connected by a cord which passes over pulleys as shown. A 3 kg collar *C* is placed on block *A* and the system is released from rest. After the blocks have moved 0.9 m, collar *C* is removed and blocks *A* and *B* continue to move. Determine the speed of block *A* just before it strikes the ground.

SOLUTION



Position ① to Position ②. $v_1 = 0 \quad T_1 = 0$

At ② before *C* is removed from the system

$$T_2 = \frac{1}{2}(m_A + m_B + m_C)v_2^2 = \frac{1}{2}(12\text{ kg})v_2^2 = 6v_2^2$$

$$U_{1-2} = (m_A + m_C - m_B)g(0.9\text{ m})$$

$$U_{1-2} = (4 + 3 - 5)(g)(0.9\text{ m}) = (2\text{ kg})(9.81\text{ m/s}^2)(0.9\text{ m})$$

$$U_{1-2} = 17.658\text{ J}$$

$$T_1 + U_{1-2} = T_2:$$

$$0 + 17.658 = 6v_2^2 \qquad v_2^2 = 2.943$$

At Position ②, collar *C* is removed from the system.

Position ② to Position ③. $T_2' = \frac{1}{2}(m_A + m_B)v_2^2 = \left(\frac{9}{2}\text{ kg}\right)(2.943) = 13.244\text{ J}$

$$T_3 = \frac{1}{2}(m_A + m_B)(v_3)^2 = \frac{9}{2}v_3^2$$

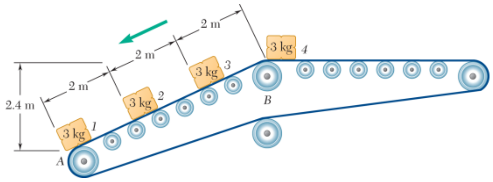
$$U_{2'-3} = (m_A - m_B)(g)(0.7\text{ m}) = (-1\text{ kg})(9.81\text{ m/s}^2)(0.7\text{ m}) = -6.867\text{ J}$$

$$T_2' + U_{2-3} = T_3$$

$$13.244 - 6.867 = 4.5v_3^2 \qquad v_3^2 = 1.417$$

$$v_A = v_3 = 1.190\text{ m/s}$$

$v_A = 1.190\text{ m/s}$ ◀


2 m
2 m
2 m
3 kg 4
3 kg 3
3 kg 2
3 kg 1
2.4 m
A
B

PROBLEM 13.25

Four 3-kg packages are held in place by friction on a conveyor which is disengaged from its drive motor. When the system is released from rest, package 1 leaves the belt at *A* just as package 4 comes onto the inclined portion of the belt at *B*. Determine (*a*) the velocity of package 2 as it leaves the belt at *A*, (*b*) the velocity of package 3 as it leaves the belt at *A*. Neglect the mass of the belt and rollers.

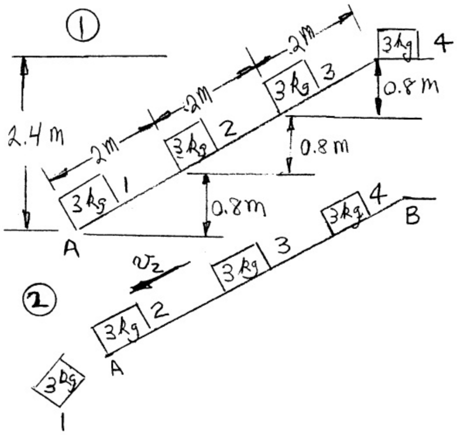
SOLUTION

Given: Conveyor is disengaged, packages held by friction and system is released from rest. Neglect mass of belt and rollers. Package 1 leaves the belt as package 4 comes onto the belt.

Find: (*a*) Velocity of package 2 as it leaves the belt at *A*.

(*b*) Velocity of package 3 as it leaves the belt at *A*.

(*a*) Package 1 falls off the belt, and 2, 3, 4 move down.


①
2.4 m
2m
2m
2m
3kg 4
3kg 3
3kg 2
3kg 1
0.8 m
0.8 m
0.8m
A
v_2
②
3kg 4
3kg 3
3kg 2
B
A
3kg
1

$$\frac{2.4}{3} = 0.8 \text{ m}$$

$$T_2 = 3\left[\frac{1}{2}mv_2^2\right]$$

$$T_2 = \frac{3}{2}(3 \text{ kg})v_2^2$$

$$T_2 = 4.5v_2^2$$

$$U_{1-2} = (3)(W)(0.8) = (3)(3 \text{ kg}) \times 9.81 \text{ m/s}^2 (0.8)$$

$$U_{1-2} = 70.632 \text{ J}$$

$$T_1 + U_{1-2} = T_2 \qquad 0 + 70.632 = 4.5v_2^2$$

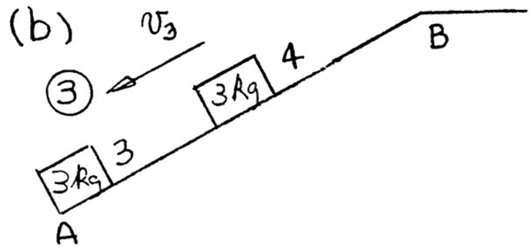
$$v_2^2 = 15.696$$

$$v_2 = 3.9618$$

$v_2 = 3.96$ m/s ◀

PROBLEM 13.25 (Continued)

(*b*) Package 2 falls off the belt and its energy is lost to the system, and 3 and 4 move down 2 ft.



$$T_2' = (2)\left[\frac{1}{2}mv_2^2\right]$$

$$T_2' = (3\text{ kg})(15.696)$$

$$T_2' = 47.088\text{ J}$$

$$T_3 = (2)\left[\frac{1}{2}mv_3^2\right] = (3\text{ kg})\left(v_3^2\right)$$

$$T_3 = 3v_3^2$$

$$U_{2-3} = 2(W)(0.8) = (2)\left(3\text{ kg} \times 9.81\text{ m/s}^2\right)(0.8\text{ m})$$

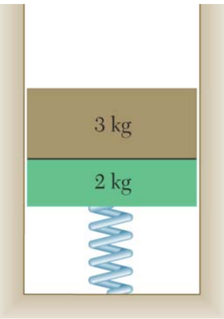
$$U_{2-3} = 47.088\text{ J}$$

$$T_2' + U_{2-3} = T_3 = 47.088 + 47.088 = 3v_3^2$$

$$v_3^2 = 31.392$$

$$v_3 = 5.6029$$

$v_3 = 5.60$ m/s ◀



PROBLEM 13.26

A 3-kg block rests on top of a 2-kg block supported by, but not attached to, a spring of constant 40 N/m. The upper block is suddenly removed. Determine (*a*) the maximum speed reached by the 2-kg block, (*b*) the maximum height reached by the 2-kg block.

SOLUTION

Call blocks A and B. $m_A = 2\text{ kg}, \quad m_B = 3\text{ kg}$

(*a*) Position 1: Block B has just been removed.

Spring force: $F_S = -(m_A + m_B)g = -kx \uparrow$

Spring stretch:

$$x_1 = -\frac{(m_A + m_B)g}{k} = -\frac{(5\text{ kg})(9.81\text{ m/s}^2)}{40\text{ N/m}} = -1.22625\text{ m}$$

Let position 2 be a later position while the spring still contacts block A.

Work of the force exerted by the spring:

$$(U_{1\to 2})_e = -\int_{x_1}^{x_2} kx\,dx$$

$$= -\frac{1}{2}kx^2\Big|_{x_1}^{x_2} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$$

$$= \frac{1}{2}(40)(-1.22625)^2 - \frac{1}{2}(40)x_2^2 = 30.074 - 20x_2^2$$

Work of the gravitational force:

$$(U_{1\to 2})_g = -m_A g(x_2 - x_1)$$

$$= -(2)(9.81)(x_2 + 1.22625) = -19.62x_2 - 24.059$$

Total work: $U_{1\to 2} = -20x_2^2 + 19.62x_2 + 6.015$

Kinetic energies: $T_1 = 0$

$$T_2 = \frac{1}{2}m_A v_2^2 = \frac{1}{2}(2)v_2^2 = v_2^2$$

Principle of work and energy: $T_1 + U_{1\to 2} = T_2$

$$0 + 20x_2^2 - 19.62x_2 + 6.015 = v_2^2$$

Speed squared:

$$v_2^2 = -20x_2^2 - 19.62x_2 + 6.015 \qquad (1)$$

At maximum speed,

$$\frac{dv_2}{dx_2} = 0$$

PROBLEM 13.26 (Continued)

Differentiating Eq. (1), and setting equal to zero,

$$2v_2 \frac{dv_2}{dx} = -40x_2 = -19.62 = 0$$

$$x_2 = -\frac{19.62}{40} = -0.4905 \text{ m}$$

Substituting into Eq. (1), $v_2^2 = -(20)(-0.4905)^2 - (19.62)(-0.4905) + 6.015 = 10.827 \text{ m}^2/\text{s}^2$

Maximum speed: $v^2 = 3.29 \text{ m/s}$ ◀

(*b*) Position 3: Block *A* reaches maximum height. Assume that the block has separated from the spring. Spring force is zero at separation.

Work of the force exerted by the spring:

$$(U_{1\to 3})_e = -\int_{x_1}^{0} kxdx = \frac{1}{2}kx_1^2 = \frac{1}{2}(40)(1.22625)^2 = 30.074 \text{ J}$$

Work of the gravitational force:

$$(U_{1\to 3})_g = -m_A gh = -(2)(9.81)h = -19.62\ h$$

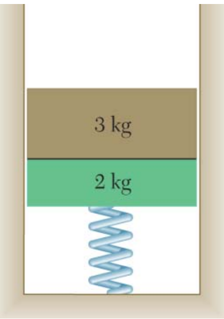
Total work: $U_{1\to 3} = 30.074 - 19.62\ h$

At maximum height, $v_3 = 0, \quad T_3 = 0$

Principle of work and energy: $T_1 + U_{1\to 3} = T_3$

$$0 + 30.074 - 19.62\ h = 0$$

Maximum height: $h = 1.533 \text{ m}$ ◀



PROBLEM 13.27

Solve Problem 13.26, assuming that the 2-kg block is attached to the spring.

PROBLEM 13.26 A 3-kg block rests on top of a 2-kg block supported by, but not attached to, a spring of constant 40 N/m. The upper block is suddenly removed. Determine (*a*) the maximum speed reached by the 2-kg block, (*b*) the maximum height reached by the 2-kg block.

SOLUTION

Call blocks *A* and *B*. $m_A = 2\text{ kg}, \quad m_B = 3\text{ kg}$

(*a*) Position 1: Block *B* has just been removed.

Spring force: $F_S = -(m_A + m_B)g = -kx_1$

Spring stretch: $x_1 = -\dfrac{(m_A + m_B)g}{k} = -\dfrac{(5\text{ kg})(9.81\text{ m/s}^2)}{40\text{ N/m}} = -1.22625\text{ m}$

Let position 2 be a later position. Note that the spring remains attached to block *A*.

Work of the force exerted by the spring: $(U_{1\to 2})_e = -\int_{x_1}^{x_2} kx dx$

$$= -\frac{1}{2}kx^2\Big|_{x_1}^{x_2} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$$

$$= \frac{1}{2}(40)(-1.22625)^2 - \frac{1}{2}(40)x_2^2 = 30.074 - 20x_2^2$$

Work of the gravitational force: $(U_{1\to 2})_g = -m_A g(x_2 - x_1)$

$$= -(2)(9.81)(x_2 + 1.22625) = -19.62x_2 - 24.059$$

Total work: $U_{1\to 2} = -20x_2^2 - 19.62x_2 + 6.015$

Kinetic energies: $T_1 = 0$

$$T_2 = \frac{1}{2}m_A v_2^2 = \frac{1}{2}(2)v_2^2 = v_2^2$$

Principle of work and energy: $T_1 + U_{1\to 2} = T_2$

$$0 + 20x_2^2 - 19.62x_2 + 6.015 = v_2^2$$

Speed squared: $v_2^2 = -20x_2^2 - 19.62x_2 + 6.015$ (1)

At maximum speed, $\dfrac{dv_2}{dx_2} = 0$

PROBLEM 13.27 (Continued)

Differentiating Eq. (1) and setting equal to zero,

$$2v_2 \frac{dv_2}{dx_2} = -40x_2 = -19.62 = 0$$

$$x_2 = -\frac{19.62}{40} = -0.4905 \text{ m}$$

Substituting into Eq. (1), $v_2^2 = -(20)(-0.4905)^2 - (19.62)(-0.4905) + 6.015 = 10.827 \text{ m}^1/\text{s}^2$

Maximum speed: $v_2 = 3.29 \text{ m/s}$ ◀

(*b*) Maximum height occurs when $v_2 = 0$.

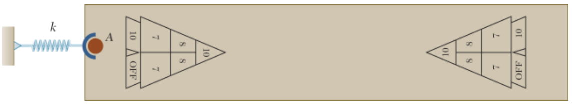
Substituting into Eq. (1), $0 = -20x_2^2 - 19.62x_2 + 6.015$

Solving the quadratic equation

$$x_2 = -1.22625 \text{ m} \quad \text{and} \quad 0.24525 \text{ m}$$

Using the larger value, $x_2 = 0.24525 \text{ m}$

Maximum height: $h = x_2 - x_1 = 0.24525 + 1.22625$ $h = 1.472 \text{ m}$ ◀



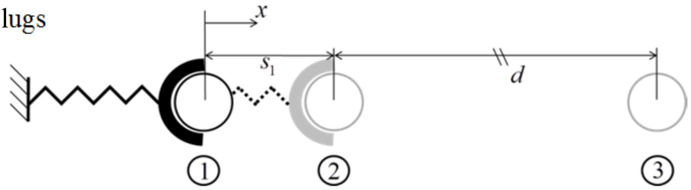
PROBLEM 13.28

People with mobility impairments can gain great health and social benefits from participating in different recreational activities. You are tasked with designing an adaptive spring-powered shuffleboard attachment that can be utilized by people who use wheelchairs. Knowing that the coefficient of kinetic friction between the 15 ounce puck *A* and the wooden surface is 0.3, the maximum spring displacement you desire is 6 inches, and that you want the puck to travel at least 30 ft/s, determine (*a*) the spring constant *k*, (*b*) how far the athlete should pull back the spring to make the puck come to rest after 34 ft.

SOLUTION

Given: $\mu_k = 0.3$

Sketch:



$$m = \frac{15\text{ oz}}{16\ \frac{\text{oz}}{\text{lbs}}} \div 32.2\text{ft/s}^2 = 0.02911\text{ slugs}$$

$$s_{1,\max} = \frac{6\text{ in}}{12\ \frac{\text{in}}{\text{ft}}} = 0.5\text{ ft}$$

$$v_{2,\max} = 30\text{ ft/s}$$

$$d_3 = 34\text{ ft}$$

(a) Find spring constant k, for maximum conditions:

$T_1 + U_{1\to 2} = T_2$ where: $T_1 = 0$ and $T_2 = \frac{1}{2}mv_{2,\max}^2$

$$U_{1\to 2} = \int_0^{s_{1,\max}} kxdx - \int_0^{s_{1,\max}} \mu_k mgdx$$

$$= \frac{1}{2}ks_{1,\max}^2 - \mu_k mgs_{1,\max}$$

$$0 + \frac{1}{2}k(0.5)^2 - 0.3(0.02911)(32.2)(0.5) = \frac{1}{2}(0.02911)(30)^2$$

$k = 105.9$ lb/ft $\qquad$ $k = 8.828$ lb/in ◀

(b) Find pull back distance, s_1 so that puck stops after d=34 ft.

$T_1 + U_{1\to 3} = T_3$ where: $T_1 = 0$ and $T_3 = 0$

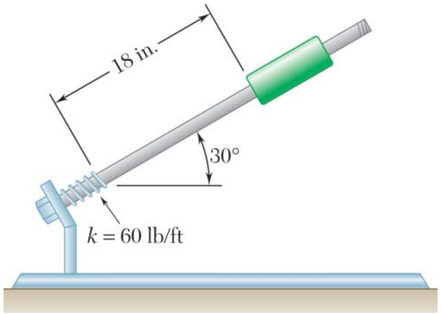
$$U_{1\to 3} = \int_0^{s_1} kxdx - \int_0^{s_1+d} \mu_k mgdx$$

$$= \frac{1}{2}ks_{1,\max}^2 - \mu_k mg(s_1 + d)$$

$$0 + \frac{1}{2}(105.9)s_1^2 - 0.3(0.02911)(32.2)(s_1 + 34) = 0$$

$$52.95s_1^2 - 0.28125s_1 - 9.5625 = 0$$

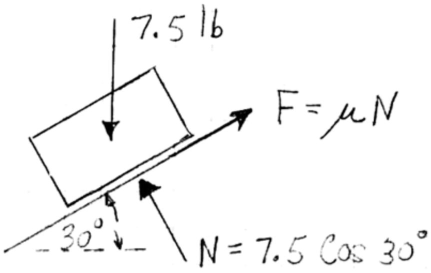
$s_1 = 0.4276$ ft or ~~$s_1 = -0.4221$ ft~~ $\qquad$ s_1=5.13 in ◀



PROBLEM 13.29

A 7.5-lb collar is released from rest in the position shown, slides down the inclined rod, and compresses the spring. The direction of motion is reversed and the collar slides up the rod. Knowing that the maximum deflection of the spring is 5 in., determine (*a*) the coefficient of kinetic friction between the collar and the rod, (*b*) the maximum speed of the collar.

SOLUTION



Position 1, initial condition

Position 2, spring deflected 5 inches

Position 3, initial contact of spring with collar

$$U_{1\to 2} = -F\underbrace{\left(\frac{18+5}{12}\right)}_{\text{(Friction)}} - \underbrace{\frac{1}{2}(60)\left(\frac{5}{12}\right)^2}_{\text{(Spring)}} + \underbrace{7.5\left(\frac{18+5}{12}\right)\sin 30°}_{\text{(Gravity)}}$$

$$T_1 = T_2 = 0, \therefore U_{1-2} = 0$$

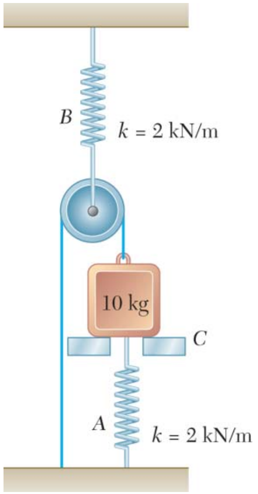
$$0 = -\mu(7.5)(0.866)\left(\frac{23}{12}\right) - \frac{1}{2}(60)\left(\frac{5}{12}\right)^2 + 7.5\left(\frac{23}{12}\right)(0.5)$$

(*a*) $\mu = 0.1590$ ◀

(*b*) Max speed occurs just before contact with the spring

$$U_{1\to 3} = -\mu(7.5)(0.866)\left(\frac{18}{12}\right) + 7.5\left(\frac{18}{12}\right)(0.5) = T_3 = \frac{1}{2}\left(\frac{7.5}{32.2}\right)v_{\max}^2$$

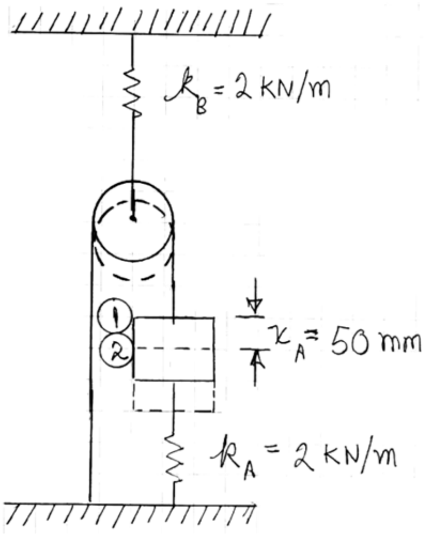
$v_{\max} = 5.92$ ft/s ◀



PROBLEM 13.30

A 10-kg block is attached to spring *A* and connected to spring *B* by a cord and pulley. The block is held in the position shown with both springs unstretched when the support is removed and the block is released with no initial velocity. Knowing that the constant of each spring is 2 kN/m, determine (*a*) the velocity of the block after it has moved down 50 mm, (*b*) the maximum velocity achieved by the block.

SOLUTION



(*a*) W = weight of the block $= 10\,(9.81) = 98.1$ N

$$x_B = \frac{1}{2}x_A$$

$$U_{1-2} = W(x_A) - \frac{1}{2}k_A(x_A)^2 - \frac{1}{2}k_B(x_B)^2$$

(Gravity) (Spring *A*) (Spring *B*)

$$U_{1-2} = (98.1\text{ N})(0.05\text{ m}) - \frac{1}{2}(2000\text{ N/m})(0.05\text{ m})^2$$

$$-\frac{1}{2}(2000\text{ N/m})(0.025\text{ m})^2$$

$$U_{1-2} = \frac{1}{2}(m)v^2 = \frac{1}{2}(10\text{ kg})v^2$$

$$4.905 - 2.5 - 0.625 = \frac{1}{2}(10)v^2$$

$v = 0.597$ m/s ◀

(*b*) Let x = distance moved down by the 10 kg block

$$U_{1-2} = W(x) - \frac{1}{2}k_A(x)^2 - \frac{1}{2}k_B\left(\frac{x}{2}\right)^2 = \frac{1}{2}(m)v^2$$

$$\frac{d}{dx}\left[\frac{1}{2}(m)v^2\right] = 0 = W - k_A(x) - \frac{k_B}{8}(2x)$$

PROBLEM 13.30 (Continued)

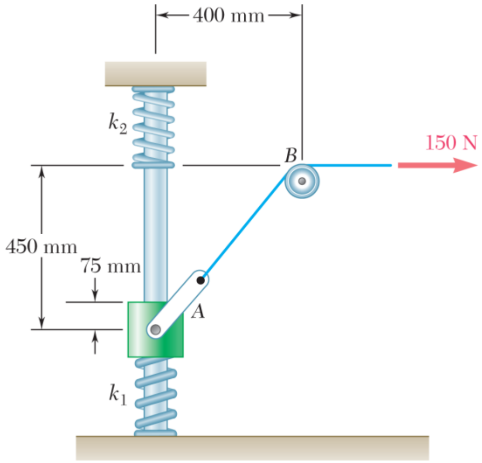
$$0 = 98.1 - 2000\,(x) - \frac{2000}{8}\,(2x) = 98.1 - (2000 + 250)x$$

$x = 0.0436$ m (43.6 mm)

For $x = 0.0436,\ U = 4.2772 - 1.9010 - 0.4752 = \frac{1}{2}(10)v^2$

$v_{\max} = 0.6166$ m/s

$v_{\max} = 0.617$ m/s ◀



PROBLEM 13.31

A 5-kg collar *A* is at rest on top of, but not attached to, a spring with stiffness k_1 = 400 N/m; when a constant 150-N force is applied to the cable. Knowing *A* has a speed of 1 m/s when the upper spring is compressed 75 mm, determine the spring stiffness k_2. Ignore friction and the mass of the pulley.

SOLUTION

Use the method of work and energy applied to the collar *A*.

$$T_1 + U_{1\to 2} = T_2$$

Since collar is initially at rest, $T_1 = 0.$

In position 2, where the upper spring is compressed 75 mm and $v_2 = 1.00$ m/s, the kinetic energy is

$$T_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(5 \text{ kg})(1.00 \text{ m/s})^2 = 2.5 \text{ J}$$

As the collar is raised from level *A* to level *B*, the work of the weight force is

$$(U_{1\to 2})_g = -mgh$$

where $m = 5$ kg, $g = 9.81$ m/s^2 and $h = 450$ mm $= 0.450$ m

Thus, $(U_{1\to 2})_g = -(5)(9.81)(0.450) = -22.0725$ J

In position 1, the force exerted by the lower spring is equal to the weight of collar *A*.

$$F_1 = mg = -(5 \text{ kg})(9.81 \text{ m/s}) = -49.05 \text{ N}$$

As the collar moves up a distance x_1, the spring force is

$$F = F_1 - k_1 x_2$$

until the collar separates from the spring at

$$x_f = \frac{F_1}{k_1} = \frac{49.05 \text{ N}}{400 \text{ N/m}} = 0.122625 \text{ m} = 122.625 \text{ mm}$$

Work of the force exerted by the lower spring:

$$(U_{1\to 2})_1 = \int_0^{x_f} (F_1 - k_1 x)dx$$

$$= F_1 x_f - \frac{1}{2}kx_f^2 = k_1 x_f^2 - \frac{1}{2}k_1 x_f^2 = \frac{1}{2}k_1 x_f^2$$

$$= \frac{1}{2}(400 \text{ N/m})(0.122625)^2 = 3.0074 \text{ J}$$

In position 2, the upper spring is compressed by $y = 75$ mm $= 0.075$ m. The work of the force exerted by this spring is

$$(U_{1\to 2})_2 = -\frac{1}{2}k_2 y^2 = -\frac{1}{2}k_2(0.075)^2 = -0.0028125 \text{ k}_2$$

Finally, we must calculate the work of the 150 N force applied to the cable. In position 1, the length *AB* is

$$(l_{AB})_1 = \sqrt{(450)^2 + (400)^2} = 602.08 \text{ mm}$$

In position 2, the length *AB* is $(l_{AB})_2 = 400$ mm.

The displacement *d* of the 150 N force is

$$d = (l_{AB})_1 - (l_{AB})_2 = 202.08 \text{ mm} = 0.20208 \text{ m}$$

The work of the 150 N force *P* is

$$(U_{1\to 2})_P = Pd = (150 \text{ N})(0.20208 \text{ m}) = 30.312 \text{ J}$$

Total work:

$$U_{1\to 2} = -22.0725 + 3.0074 - 0.0028125k_2 + 30.312$$

$$= 11.247 - 0.0028125k_2$$

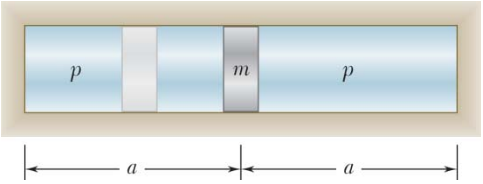
Principle of work and energy:

$$T_1 + U_{1\to 2} = T_2$$

$$0 + 11.247 - 0.0028125k_2 = 2.5$$

$$k_2 = 3110 \text{ N/m}$$

$k_2 = 3110$ N/m ◀

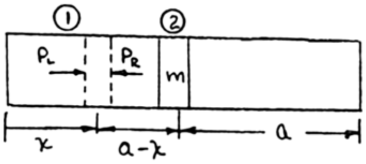


PROBLEM 13.32

A piston of mass m and cross-sectional area A is equilibrium under the pressure p at the center of a cylinder closed at both ends. Assuming that the piston is moved to the left a distance $a/2$ and released, and knowing that the pressure on each side of the piston varies inversely with the volume, determine the velocity of the piston as it again reaches the center of the cylinder. Neglect friction between the piston and the cylinder and express your answer in terms of m, a, p, and A.

SOLUTION

Pressures vary inversely as the volume



$$\frac{p_L}{P} = \frac{Aa}{Ax} \qquad p_L = \frac{pa}{x}$$

$$\frac{p_R}{P} = \frac{Aa}{A(2a-x)} \quad p_R = \frac{pa}{(2a-x)}$$

Initially at ①, $\quad v = 0 \quad x = \dfrac{a}{2}$

$$T_1 = 0$$

At ②, $\quad x = a, \quad T_2 = \dfrac{1}{2}mv^2$

$$U_{1-2} = \int_{a/2}^{a} (p_L - p_R)A\,dx = \int_{a/2}^{a} paA\left[\frac{1}{x} - \frac{1}{2a-x}\right]dx$$

$$U_{1-2} = paA[\ln x + \ln (2a-x)]_{a/2}^{a}$$

$$U_{1-2} = paA\left[\ln a + \ln a - \ln\left(\frac{a}{2}\right) - \ln\left(\frac{3a}{2}\right)\right]$$

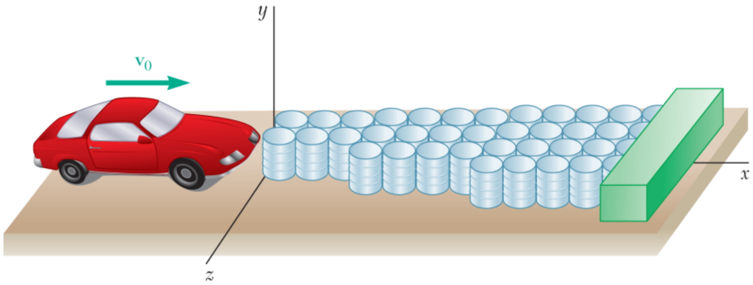
$$U_{1-2} = paA\left[\ln a^2 - \ln\frac{3a^2}{4}\right] = paA\,\ln\left(\frac{4}{3}\right)$$

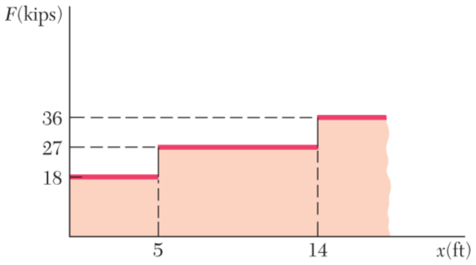
$$T_1 + U_{1-2} = T_2 \quad 0 + paA\,\ln\left(\frac{4}{3}\right) = \frac{1}{2}mv^2$$

$$v^2 = \frac{2paA\,\ln\left(\frac{4}{3}\right)}{m} = 0.5754\frac{paA}{m} \qquad v = 0.759\sqrt{\frac{paA}{m}} \blacktriangleleft$$

PROBLEM 13.33

An uncontrolled automobile traveling at 65 mph strikes squarely a highway crash cushion of the type shown in which the automobile is brought to rest by successively crushing steel barrels. The magnitude *F* of the force required to crush the barrels is shown as a function of the distance *x* the automobile has moved into the cushion. Knowing that the weight of the automobile is 2250 lb and neglecting the effect of friction, determine (*a*) the distance the automobile will move into the cushion before it comes to rest, (*b*) the maximum deceleration of the automobile.

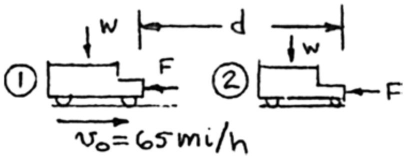




SOLUTION

(*a*) 65 mi/h = 95.3 ft/s

Assume auto stops in $5 \le d \le 14$ ft.



$$v_1 = 95.33 \text{ ft/s}$$

$$T_1 = \frac{1}{2}mv_1^2\,\frac{1}{2}\left(\frac{2250 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(95.3 \text{ ft/s})^2$$

$$T_1 = 317{,}530 \text{ lb}\cdot\text{ft}$$

$$= 317.63 \text{ k}\cdot\text{ft}$$

$$v_2 = 0$$

$$T_2 = 0$$

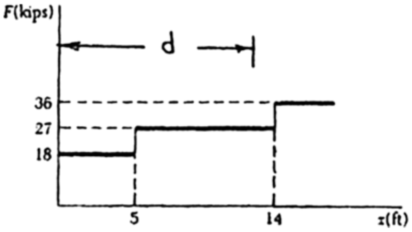
$$U_{1-2} = (18 \text{ k})(5 \text{ ft}) + (27 \text{ k})(d-5)$$

$$= 90 + 27d - 135$$

$$= 27d - 45 \text{ k}\cdot\text{ft}$$

$$T_1 + U_{1-2} = T_2$$

$$317.53 = 27d - 45$$



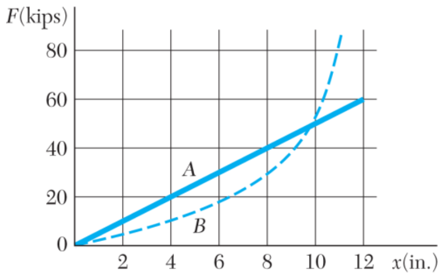
$d = 13.43$ ft ◀

Assumption that $d \le 14$ ft is ok.

(*b*) Maximum deceleration occurs when *F* is largest. For $d = 13.43$ ft, $F = 27$ k. Thus, $F = ma_D$

$$(27{,}000 \text{ lb}) = \left(\frac{2250 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(a_D)$$

$a_D = 386 \text{ ft/s}^2$ ◀



PROBLEM 13.34

Two types of energy-absorbing fenders designed to be used on a pier are statically loaded. The force-deflection curve for each type of fender is given in the graph. Determine the maximum deflection of each fender when a 90-ton ship moving at 1 mi/h strikes the fender and is brought to rest.

SOLUTION

Weight: $W_1 = (90 \text{ ton})(2000 \text{ lb/ton}) = 180 \times 10^3 \text{ lb}$

Mass: $m = \dfrac{W}{g} = \dfrac{180 \times 10^3}{32.2} = 5590 \text{ lb} \cdot \text{s}^2/\text{ft}$

Speed: $v_1 = 1 \text{ mi/h} = \dfrac{5280 \text{ ft}}{3600 \text{ s}} = 1.4667 \text{ ft/s}$

Kinetic energy:

$$T_1 = \frac{1}{2} m v_1^2 = \frac{1}{2}(5590)(1.4667)^2$$

$$= 6012 \text{ ft} \cdot \text{lb}$$

$$T_2 = 0 \qquad \text{(rest)}$$

Principle of work and energy: $T_1 + U_{1\to2} = T_2$

$$6012 + U_{1\to2} = 0$$

$$U_{1\to2} = -6012 \text{ ft} \cdot \text{lb} = -72.15 \text{ kip} \cdot \text{in.}$$

The area under the force-deflection curve up to the maximum deflection is equal to $72.15 \text{ kip} \cdot \text{in.}$

Fender *A*: From the force-deflection curve $F = kx$ $\quad k = \dfrac{F_{\max}}{x_{\max}} = \dfrac{60}{12} = 5 \text{ kip/in.}$

$$\text{Area} = \int_0^x f dx = \int_0^x kx\, dx = \frac{1}{2} kx^2$$

$$\frac{1}{2}(5)x^2 = 72.51$$

$$x^2 = 28.86 \text{ in.}^2$$

$x = 5.37 \text{ in.}$ ◀

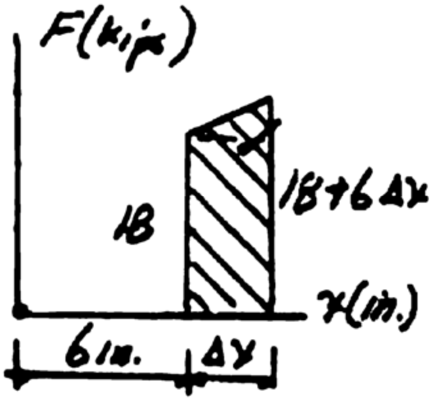
Fender *B*: We divide area under curve *B* into trapezoids

		Partial area	Total Area
From $x = 0$	to $x = 2$ in.:	$\frac{1}{2}(2 \text{ in.})(4 \text{ kips}) = 4 \text{ kip} \cdot \text{in.}$	$4 \text{ kip} \cdot \text{in.}$
From $x = 2$ in.	to $x = 4$ in.:	$\frac{1}{2}(2 \text{ in.})(4 + 10) = 14 \text{ kip} \cdot \text{in.}$	$18 \text{ kip} \cdot \text{in.}$
From $x = 4$ in.	to $x = 6$ in.:	$\frac{1}{2}(2 \text{ in.})(10 + 18) = 28 \text{ kip} \cdot \text{in.}$	$46 \text{ kip} \cdot \text{in.}$

PROBLEM 13.34 (Continued)

We still need $\Delta U = 72.15 - 46 = 26.15 \text{ kip}\cdot\text{in.}$

Equation of straight line approximating curve B from $x = 6$ in. to $x = 8$ in. is



$$\frac{\Delta x}{2} = \frac{F-18}{30-18} \quad F = 18 + 6\Delta x$$

$$\Delta U = 18\Delta x + \frac{1}{2}(6\Delta x)\Delta x = 26.15 \text{ kip}\cdot\text{in.}$$

$$(\Delta x)^2 + 6\Delta x - 8.716 = 0$$

$$\Delta x = 1.209 \text{ in.}$$

Thus: $x = 6 \text{ in.} + 1.209 \text{ in.} = 7.209 \text{ in.}$ $x = 7.21 \text{ in.}$ ◀

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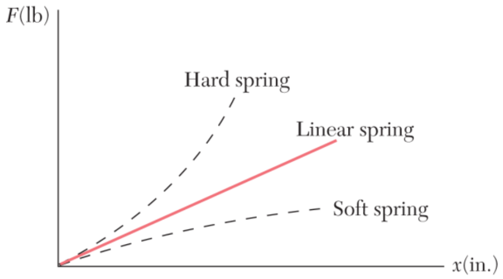
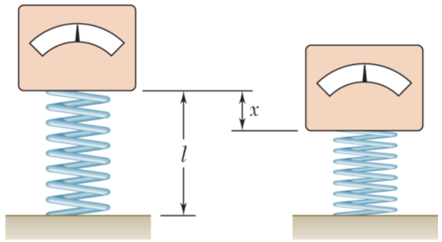
PROBLEM 13.35

Nonlinear springs are classified as hard or soft, depending upon the curvature of their force-deflection curve (see figure). If a delicate instrument having a mass of 5 kg is placed on a spring of length l so that its base is just touching the undeformed spring and then inadvertently released from that position, determine the maximum deflection x_m of the spring and the maximum force F_m exerted by the spring, assuming (*a*) a linear spring of constant $k = 3$ kN/m, (*b*) a hard, nonlinear spring, for which $F = (3 \text{ kN/m})(x + 160x^2)$.

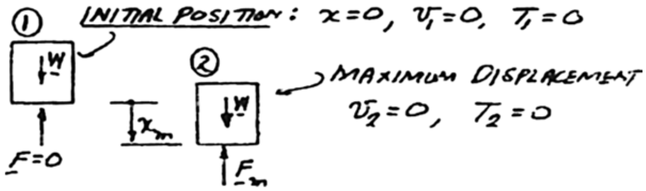




SOLUTION

$W = mg = (5 \text{ kg})g$

$W = 49.05 \text{ N}$



Since $T_1 = T_2 = 0, \quad T_1 + U_{1-2} = T_2$ yields $U_{1-2} = 0$

$$U_{1-2} = Wx_m - \int_0^{x_m} F dx = 49.05x_m - \int_0^{x_m} F dx = 0 \qquad (1)$$

(*a*) For $F = kx = (300 \text{ N/m})x$

Eq. (1): $49.05x_m - \int_0^{x_m} 3000x\, dx = 0$

$$49.05x_m - 1500x_m^2 = 0$$

$x_m = 32.7 \times 10^{-3} \text{ m} = 32.7 \text{ mm}$ ◀

$F_m = 3000x_m = 3000(32.7 \times 10^{-3})$

$\mathbf{F}_m = 98.1 \text{ N} \uparrow$ ◀

(*b*) For $F = (3000 \text{ N/m})x(1 + 160x^2)$

Eq. (1) $49.05x_m - \int_0^{x_m} 3000(x + 160x^3)dx = 0$

$$49.05x_m - 3000\left(\frac{1}{2}x_m^2 + 40x_m^4\right) = 0 \qquad (2)$$

Solve by trial: $x_m = 30.44 \times 10^{-3} \text{ m}$

$x_m = 30.4 \text{ mm}$ ◀

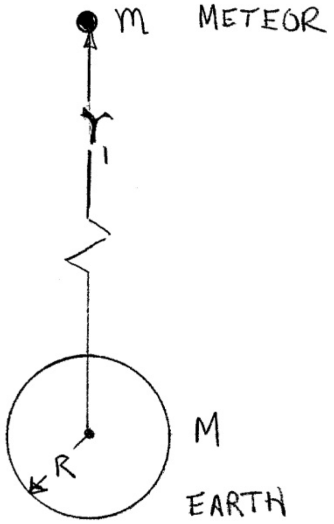
$F_m = (3000)(30.44 \times 10^{-3})[1 + 160(30.44 \times 10^{-3})^2]$

$\mathbf{F}_m = 104.9 \text{ N} \uparrow$ ◀

PROBLEM 13.36

A meteor starts from rest at a very great distance from the earth. Knowing that the radius of the earth is 6370 km and neglecting all forces except the gravitational attraction of the earth, determine the speed of the meteor (*a*) when it enters the ionosphere at an altitude of 1000 km, (*b*) when it enters the stratosphere at an altitude of 50 km, (*c*) when it strikes the earth's surface.

SOLUTION



Given: $R = 6370$ km

$$U_{1-2} = T_2 - T_1$$
$$= \frac{GMm}{r_2} - \frac{GMm}{r_1}$$

Since r_1 is very large,

$$\frac{GMm}{r_1} \approx 0 \qquad \text{thus} \qquad T_1 \approx 0$$

$$\frac{1}{2}\cancel{m}v^2 = \frac{GM\cancel{m}}{r_2}$$

$$\frac{v^2}{2} = g\frac{R^2}{r_2}$$

$$v^2 = \frac{2gR^2}{r_2}$$
$$= \frac{2\left(9.81 \text{ m/s}^2\right)\left(6,370,000 \text{ m}\right)^2}{r_2}$$

(*a*) For $r_2 = 6,370,000 \text{ m} + 1,000,000 \text{ m} = 7,370,000 \text{ m}$

$v = 10.3933$ km/s

$v = 10.39$ km/s ◀

(*b*) For $r_2 = 6,370,000 \text{ m} + 50,000 \text{ m} = 6,420,000 \text{ m}$

$v = 11.1358$ km/s

$v = 11.14$ km/s ◀

(*c*) For $r_2 = 6,370,000 \text{ m}$

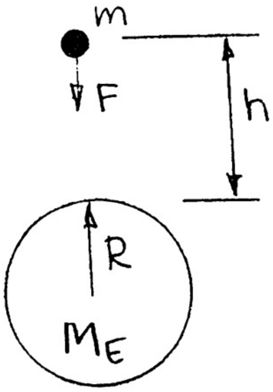
$v = 11.1794$ km/s

$v = 11.18$ km/s ◀

PROBLEM 13.37

Express the acceleration of gravity g_h at an altitude h above the surface of the earth in terms of the acceleration of gravity g_0 at the surface of the earth, the altitude h, and the radius R of the earth. Determine the percent error if the weight that an object has on the surface of the earth is used as its weight at an altitude of (*a*) 0.625 mi, (*b*) 625 mi.

SOLUTION



$$F = \frac{GM_E m}{(h+R)^2} = \frac{GM_E m/R^2}{\left(\frac{h}{R}+1\right)^2} = mg_h$$

At earths' surface $(h = 0)$ $$\frac{GM_E\, m}{R^2} = mg_0$$

$$\frac{GM_E}{R^2} = g_0 \qquad g_h = \frac{\frac{GM_E}{R^2}}{\left(\frac{h}{R}+1\right)^2}$$

Thus $$g_h = \frac{g_0}{\left(\frac{h}{R}+1\right)^2}$$

$$R = 3960 \text{ mi}$$

At altitude *h*, "true" weight $F = mg_h = W_T$

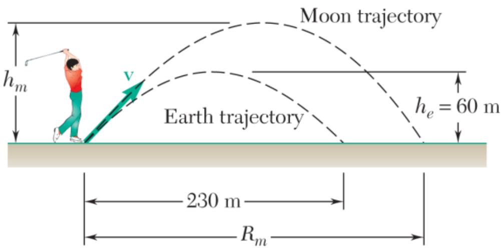
Assume weight $W_0 = mg_0$

$$\text{Error} = E = \frac{W_0 - W_T}{W_0} = \frac{mg_0 - mg_h}{mg_0} = \frac{g_0 - g_h}{g_0}$$

$$g_h = \frac{g_0}{\left(\frac{h}{R}+1\right)^2} \qquad E = \frac{g_0 - \frac{g_0}{\left(1+\frac{h}{R}\right)^2}}{g_0} = \left[1 - \frac{1}{\left(1+\frac{h}{R}\right)^2}\right]$$

(*a*) $h = 0.625 \text{ mi}: \quad P = 100E = 100\left[1 - \frac{1}{\left(1+\frac{0.625}{3960}\right)^2}\right]$ $P = 0.0316\%$ ◀

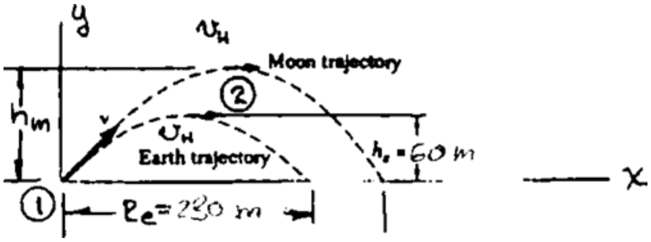
(*b*) $h = 625 \text{ mi}: \quad P = 100E = 100\left[1 - \frac{1}{\left(1+\frac{625}{3960}\right)^2}\right]$ $P = 25.4\%$ ◀


Moon trajectory
h_m
v
Earth trajectory
$h_e = 60$ m
230 m
R_m

PROBLEM 13.38

A golf ball struck on earth rises to a maximum height of 60 m and hits the ground 230 m away. How high will the same golf ball travel on the moon if the magnitude and direction of its velocity are the same as they were on earth immediately after the ball was hit? Assume that the ball is hit and lands at the same elevation in both cases and that the effect of the atmosphere on the earth is neglected, so that the trajectory in both cases is a parabola. The acceleration of gravity on the moon is 0.165 times that on earth.

SOLUTION


y
v_H
Moon trajectory
②
h_m
v
v_H
Earth trajectory
$h_e = 60$ m
x
①
$R_e = 230$ m

Solve for h_m.

At maximum height, the total velocity is the horizontal component of the velocity, which is constant and the same in both cases.

$$T_1 = \frac{1}{2}mv^2 \qquad T_2 = \frac{1}{2}mv_H^2$$

$$U_{1-2} = -mg_e h_e \quad \text{Earth}$$

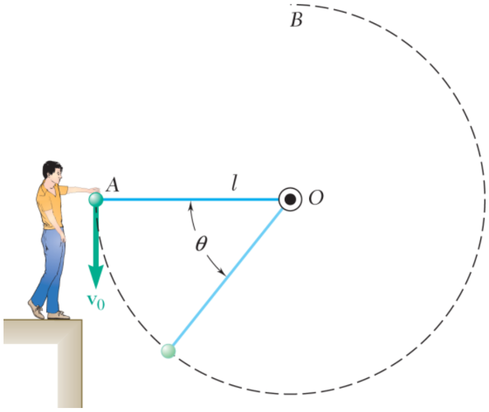
$$U_{1-2} = -mg_m h_m \quad \text{Moon}$$

Earth $$\frac{1}{2}mv^2 - mg_e h_e = \frac{1}{2}mv_H^2$$

Moon $$\frac{1}{2}mv^2 - mg_m h_m = \frac{1}{2}mv_H^2$$

$$-g_e h_e + g_m h_m = 0 \qquad \frac{h_m}{h_e} = \frac{g_e}{g_m}$$

Subtracting $$h_m = (60\text{ m})\left(\frac{g_e}{0.165 g_e}\right)$$ $h_m = 364$ m ◀



PROBLEM 13.39

The sphere at A is given a downward velocity $\mathbf{v}_0$ of magnitude 5 m/s and swings in a vertical plane at the end of a rope of length $l = 2$ m attached to a support at O. Determine the angle θ at which the rope will break, knowing that it can withstand a maximum tension equal to twice the weight of the sphere.

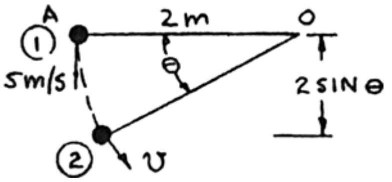
SOLUTION

$$T_1 = \frac{1}{2}mv_0^2 = \frac{1}{2}m\,(5)^2$$

$$T_1 = 12.5\text{ m}$$

$$T_2 = \frac{1}{2}mv^2$$

$$U_{1-2} = mg(l)\sin\theta$$

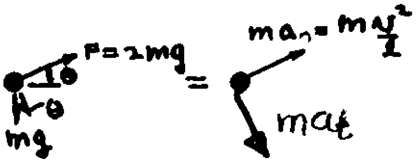


$$T_1 + U_{1-2} = T_2 \qquad 12.5m + 2mg\sin\theta = \frac{1}{2}mv^2$$

$$25 + 4g\sin\theta = v^2 \qquad (1)$$

Newton's law at ②.

$$+\nearrow\ 2mg - mg\sin\theta = m\frac{v^2}{\ell} = m\frac{v^2}{2} \qquad (2)$$



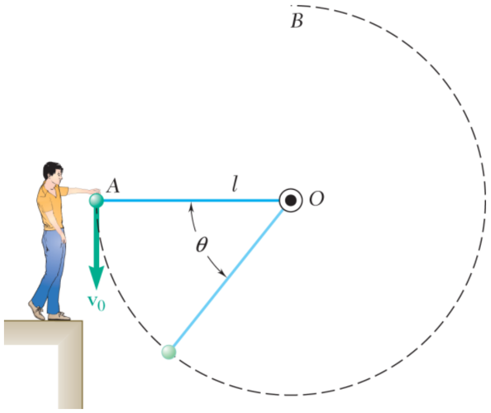
$$v^2 = 4g - 2g\sin\theta$$

Substitute for v^2 from Eq. (2) into Eq. (1)

$$25 + 4g\sin\theta = 4g - 2g\sin\theta$$

$$\sin\theta = \frac{(4)(9.81) - 25}{(6)(9.81)} = 0.2419$$

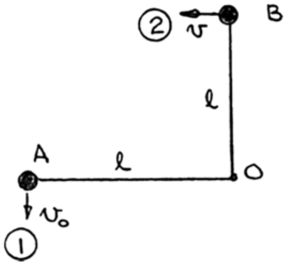
$\theta = 14.00°$ ◀



PROBLEM 13.40

The sphere at A is given a downward velocity $\mathbf{v}_0$ and swings in a vertical circle of radius l and center O. Determine the smallest velocity $\mathbf{v}_0$ for which the sphere will reach Point B as it swings about Point O (a) if AO is a rope, (b) if AO is a slender rod of negligible mass.

SOLUTION



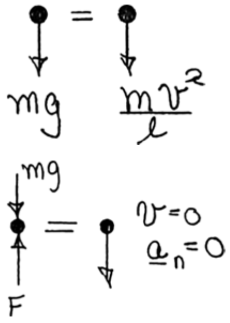
$$T_1 = \frac{1}{2}mv_0^2$$

$$T_2 = \frac{1}{2}mv^2$$

$$U_{1-2} = -mgl$$

$$T_1 + U_{1-2} = T_2 \qquad \frac{1}{2}mv_0^2 - mgl = \frac{1}{2}mv^2$$

$$v_0^2 = v^2 + 2gl$$



Newton's law at ②

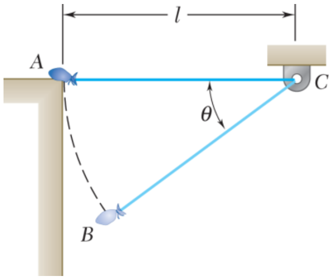
(a) For minimum v, tension in the cord must be zero.

Thus, $v^2 = gl$

$$v_0^2 = v^2 + 2gl = 3gl \qquad v_0 = \sqrt{3gl} \blacktriangleleft$$

(b) Force in the rod can support the weight so that v can be zero.

Thus, $v_0^2 = 0 + 2gl \qquad v_0 = \sqrt{2gl} \blacktriangleleft$



PROBLEM 13.41

A bag is gently pushed off the top of a wall at *A* and swings in a vertical plane at the end of a rope of length *l* Determine the angle θ for which the rope will break, knowing that it can withstand a maximum tension equal to twice the weight of the bag.

SOLUTION

Use work - energy: position 1 is at *A*, position 2 is at *B*.

$$T_1 + U_{1\to 2} = T_2 \qquad (1)$$

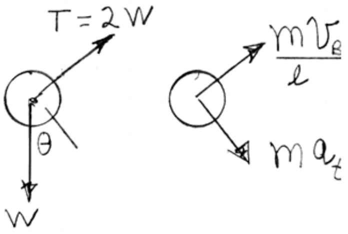
Where $T_1 = 0;\quad U_{1\to 2} = mgl\sin\theta;\quad T_2 = \frac{1}{2}mv_B^2$

Substitute

$$0 + mgl\sin\theta = \frac{1}{2}mv_B^2$$

$$v_B^2 = 2gl\sin\theta \qquad (2)$$

For $T = 2\,W$ use Newton's 2nd law.



$$\nearrow\ \Sigma F_n = ma_n \Rightarrow 2W - W\sin\theta = \frac{mv_B^2}{l} \qquad (3)$$

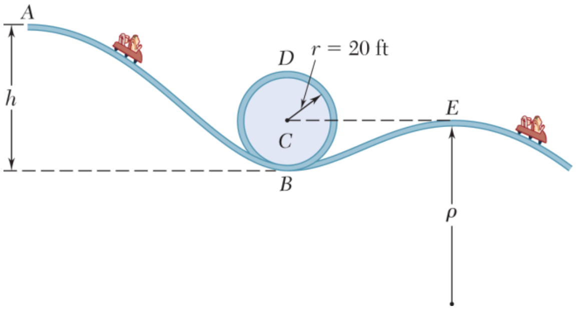
Substitute (2) into (3)

$$2\cancel{m}g - \cancel{m}g\sin\theta = 2\cancel{m}g\frac{\cancel{l}\sin\theta}{\cancel{l}}$$

$$2 = 3\sin\theta$$

$$\text{or } \sin\theta = \frac{2}{3} \Rightarrow \theta = 41.81°$$

$\theta = 41.8°$ ◀



PROBLEM 13.42

A roller coaster starts from rest at A, rolls down the track to B, describes a circular loop of 40-ft diameter, and moves up and down past Point E. Knowing that $h = 60$ ft and assuming no energy loss due to friction, determine (a) the force exerted by his seat on a 160-lb rider at B and D, (b) the minimum value of the radius of curvature at E if the roller coaster is not to leave the track at that point.

SOLUTION

Let y_p be the vertical distance from Point A to any Point P on the track. Let position 1 be at A and position 2 be at P. Apply the principle of work and energy.

$$T_1 = 0 \qquad T_2 = \frac{1}{2}mv_P^2$$

$$U_{1\to 2} = mgy_P$$

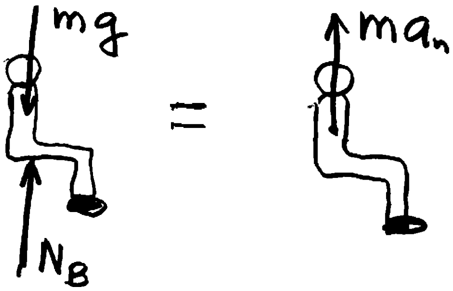
$$T_1 + U_{1\to 2} = T_2: \quad 0 + mgy_P = \frac{1}{2}mv_P^2$$

$$v_P^2 = 2gy_P$$

Magnitude of normal acceleration at P:

$$(a_P)_n = \frac{v_P^2}{\rho_P} = \frac{2gy_P}{\rho_P}$$

(a) Rider at Point B.



$$y_B = h = 60 \text{ ft}$$

$$\rho_B = r = 20 \text{ ft}$$

$$a_n = \frac{(2g)(60)}{20} = 6g$$

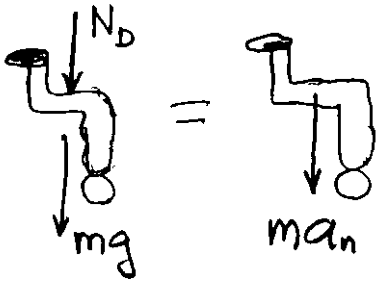
$$+\uparrow \Sigma F = ma:$$

$$N_B - mg = m(6g)$$

$$N_B = 7mg = 7W = (7)(160 \text{ lb})$$

$$\mathbf{N}_B = 1120 \text{ lb} \uparrow \blacktriangleleft$$

Rider at Point *D*.



$$y_D = h - 2r = 20 \text{ ft}$$

$$\rho_D = 20 \text{ ft}$$

$$a_n = \frac{(2g)(20)}{20} = 2g$$

$+\downarrow \Sigma F = ma$:

$$N_D + mg = m(2g)$$

$$N_D = mg = W = 160 \text{ lb}$$

$\mathbf{N}_D = 160 \text{ lb} \downarrow$ ◀

(*b*) Car at Point *E*.

$$y_E = h - r = 40 \text{ ft}$$

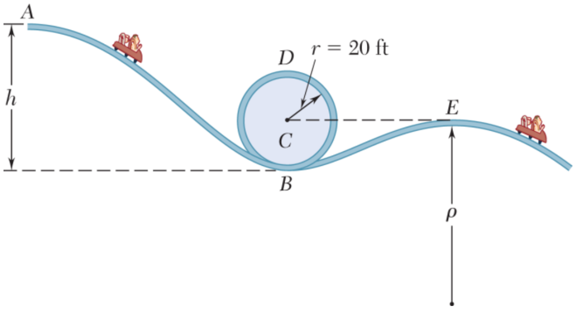
$$N_E = 0$$

$+\uparrow \Sigma F = ma_n$:

$$mg = m \cdot \frac{2gy_E}{\rho_E}$$

$$\rho_E = 2y_E$$

$\rho = 80.0 \text{ ft}$ ◀



PROBLEM 13.43

In Problem 13.42, determine the range of values of h for which the roller coaster will not leave the track at D or E, knowing that the radius of curvature at E is $\rho = 75$ ft. Assume no energy loss due to friction.

PROBLEM 13.42 A roller coaster starts from rest at A, rolls down the track to B, describes a circular loop of 40-ft diameter, and moves up and down past Point E. Knowing that h = 60 ft and assuming no energy loss due to friction, determine (a) the force exerted by his seat on a 160-lb rider at B and D, (b) the minimum value of the radius of curvature at E if the roller coaster is not to leave the track at that point.

SOLUTION

Let y_p be the vertical distance from Point A to any Point P on the track. Let position 1 be at A and position 2 be at P. Apply the principle of work and energy.

$$T_1 = 0 \qquad T_2 = \frac{1}{2} m v_P^2$$

$$U_{1\to 2} = mg\,y_P$$

$$T_1 + U_{1\to 2} = T_2: \quad 0 + mgy_p = \frac{1}{2} m v_P^2$$

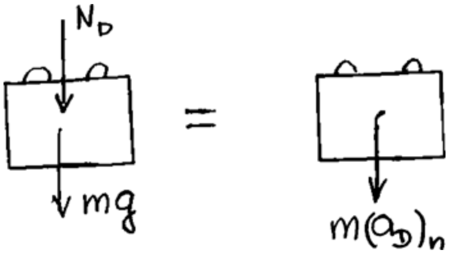
$$v_P^2 = 2gy_P$$

Magnitude of normal acceleration of P:

$$(a_P)_n = \frac{v_P^2}{\rho_P} = \frac{2gy_P}{\rho_P}$$

The condition of loss of contact with the track at P is that the curvature of the path is equal to ρ_p and the normal contact force $N_P = 0$.

Car at Point D.



$$\rho_D = r = 20 \text{ ft}$$

$$y_D = h - 2r$$

$$(a_D)_n = \frac{2\,g\,(h-2r)}{r}$$

$$+\!\downarrow \Sigma F = ma$$

$$N_D + mg = m\frac{2g(h-2r)}{r}$$

$$N_D = mg\frac{2h-5r}{r}$$

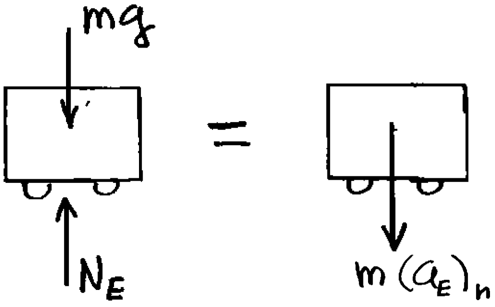
PROBLEM 13.43 (Continued)

For $N_D > 0$

$$2h - 5r > 0$$

$$h > \frac{5}{2}r = 50 \text{ ft}$$

Car at Point *E*.



$$\rho_E = \rho = 75 \text{ ft}$$

$$y_E = h - r = h - 20 \text{ ft}$$

$$(a_E)_n = \frac{2g(h-20)}{75}$$

$$+\uparrow \Sigma F = ma$$

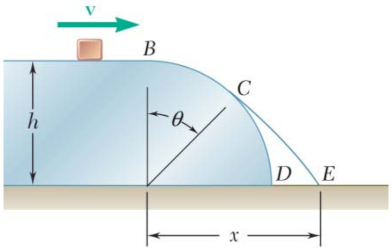
$$N_E - mg = -\frac{2mg(h-20)}{75}$$

$$N_E = mg\frac{115-2h}{75}$$

For $N_E > 0, \quad 115 - 2h > 0$

$$h < 57.5 \text{ ft}$$

Range of values for *h*: $50.0 \text{ ft} \le h \le 57.5 \text{ ft}$ ◀

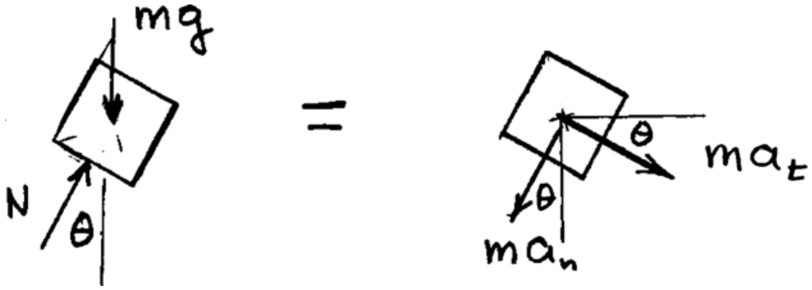


PROBLEM 13.44

A small block slides at a speed v on a horizontal surface. Knowing that $h = 0.9$ m, determine the required speed of the block if it is to leave the cylindrical surface BCD when $\theta = 30°$.

SOLUTION

At Point C where the block leaves the surface BCD the contact force is reduced to zero. Apply Newton's second law at Point C.



n-direction: $$N - mg\cos\theta = -ma_n = -\frac{mv_C^2}{h}$$

With $N = 0$, we get $$v_c^2 = gh\cos\theta$$

Apply the work-energy principle to the block sliding over the path BC. Let position 1 correspond to Point B and position 2 to C.

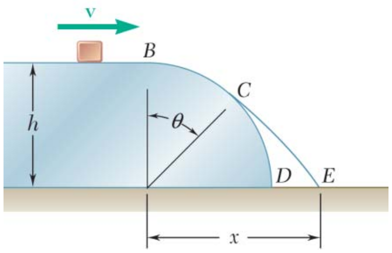
$$T_1 = \frac{1}{2}mv_B^2 \qquad T_2 = \frac{1}{2}\,mv_C^2 = \frac{1}{2}mgh\cos\theta$$

$$U_{1\to2} = \text{ weight} \times \text{change in vertical distance}$$
$$= mgh(1-\cos\theta)$$

$$T_1 + U_{1\to2} = T_2 : \frac{1}{2}mv_B^2 + mg(1-\cos\theta) = \frac{1}{2}mgh\cos\theta$$
$$v_B^2 = gh\cos\theta - 2gh(1-\cos\theta) = gh(3\cos\theta - 2)$$

Data: $$g = 9.81 \text{ m/s}^2,\ h = 0.9 \text{ m},\ \theta = 30°.$$
$$v_B^2 = (9.81)(0.9)(3\cos30° - 2) = 5.2804 \text{ m}^2/\text{s}^2$$

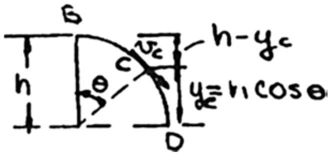
$$v_B = 2.30 \text{ m/s} \blacktriangleleft$$

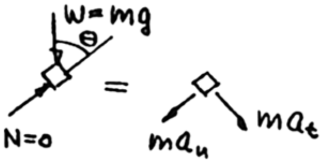


PROBLEM 13.45

A small block slides at a speed $v = 8$ ft/s on a horizontal surface at a height $h = 3$ ft above the ground. Determine (*a*) the angle θ at which it will leave the cylindrical surface *BCD*, (*b*) the distance *x* at which it will hit the ground. Neglect friction and air resistance.

SOLUTION





Block leaves surface at *C* when the normal force $N = 0$.

$$+\swarrow \; mg\cos\theta = ma_n$$

$$g\cos\theta = \frac{v_C^2}{h}$$

$$v_C^2 = gh\cos\theta = gy \qquad (1)$$

Work-energy principle.

(*a*)
$$T_B = \frac{1}{2}mv^2 = \frac{1}{2}m(8)^2 = 32m$$

$$T_C = \frac{1}{2}mv_C^2 \qquad U_{B-C} = W(h-g) = mg(h-y_C)$$

$$T_B + U_{B-C} = T_C$$

$$32m + mg(h-y) = \frac{1}{2}mv_C^2$$

Use Eq. (1)
$$32 + g(h-y_C) = \frac{1}{2}gy_C \qquad (2)$$

$$32 + gh = \frac{3}{2}gy_C$$

$$y_C = \frac{(32+gh)}{\left(\frac{3}{2}g\right)}$$

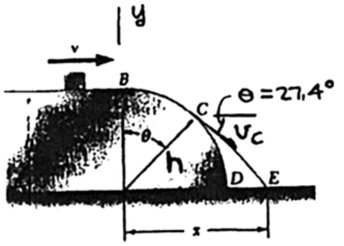
$$y_C = \frac{(32+(32.2)(3))}{\frac{3}{2}(32.2)}$$

$$y_C = 2.6625 \text{ ft} \qquad (3)$$

$$y_C = h\cos\theta \qquad \cos\theta = \frac{y_C}{h} = \frac{2.6625}{3} = 0.8875$$

$\theta = 27.4°$ ◀

PROBLEM 13.45 (Continued)



(*b*) From (1) and (3)

$$v_C = \sqrt{gy}$$
$$v_C = \sqrt{(32.2)(2.6625)}$$
$$v_C = 9.259 \text{ ft/s}$$

At *C*: $(v_C)_x = v_C \cos\theta = (9.259)(\cos 27.4°) = 8.220 \text{ ft/s}$

$$(v_C)_y = -v_C \sin\theta = -(9.259)(\sin 27.4°) = 4.261 \text{ ft/s}$$

$$y = y_C + (v_C)_y\, t - \frac{1}{2} g t^2 = 2.6625 - 4.261t - 16.1t^2$$

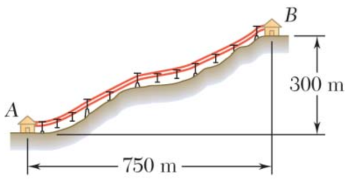
At *E*: $y_E = 0$: $t^2 + 0.2647t - 0.1654 = 0$

$$t = 0.2953 \text{ s}$$

At *E*: $x = h(\sin\theta) + (v_C)_x\, t = (3)(\sin 27.4°) + (8.220)(0.2953)$

$$x = 1.381 + 2.427 = 3.808 \text{ ft}$$

$x = 3.81$ ft ◀



PROBLEM 13.46

A chair-lift is designed to transport 1000 skiers per hour from the base *A* to the summit *B*. The average mass of a skier is 70 kg and the average speed of the lift is 75 m/min. Determine (*a*) the average power required, (*b*) the required capacity of the motor if the mechanical efficiency is 85 percent and if a 300 percent overload is to be allowed.

SOLUTION

Note: Solution is independent of speed.

(*a*)

$$\text{Average power } = \frac{\Delta U}{\Delta t} = \frac{(1000)(70\text{ kg})(9.81\text{ m/s}^2)(300\text{ m})}{3600\text{ s}} = 57{,}225\ \frac{\text{N}\cdot\text{m}}{\text{s}}$$

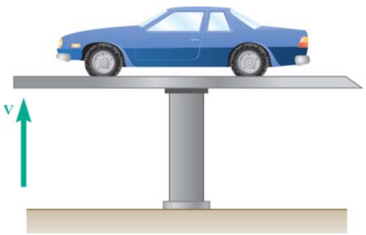
$$\text{Average power } = 57.2\text{ kW} \blacktriangleleft$$

(*b*) Maximum power required with 300% over load

$$= \frac{100+300}{100}(57.225\text{ kW}) = 229\text{ kW}$$

Required motor capacity (85% efficient)

$$\text{Motor capacity } = \frac{229\text{ kW}}{0.85} = 269\text{ kW} \blacktriangleleft$$



PROBLEM 13.47

It takes 15 s to raise a 1200-kg car and the supporting 300-kg hydraulic car-lift platform to a height of 2.8 m. Determine (*a*) the average output power delivered by the hydraulic pump to lift the system, (*b*) the average power electric required, knowing that the overall conversion efficiency from electric to mechanical power for the system is 82 percent.

SOLUTION

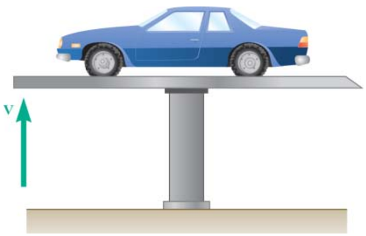
(*a*) $(P_P)_A = (F)(v_A) = (m_C + m_L)(g)(v_A)$

$v_A = \text{s/t} = (2.8\text{ m})/(15\text{ s}) = 0.18667\text{ m/s}$

$(P_P)_A = [(1200\text{ kg}) + (300\text{ kg})](9.81\text{ m/s}^2)(0.18667\text{ m/s})^3$

$(P_P)_A = 2.747\text{ kJ/s}$ $\qquad (P_P)_A = 2.75\text{ kW}$ ◀

(*b*) $(P_E)_A = (P_P)/\eta = (2.75\text{ kW})/(0.82)$ $\qquad (P_E)_A = 3.35\text{ kW}$ ◀



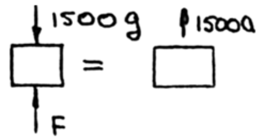
PROBLEM 13.48

The velocity of the lift of Problem 13.47 increases uniformly from zero to its maximum value at mid-height 7.5 s and then decreases uniformly to zero in 7.5 s. Knowing that the peak power output of the hydraulic pump is 6 kW when the velocity is maximum, determine the maximum life force provided by the pump.

PROBLEM 13.47 It takes 15 s to raise a 1200-kg car and the supporting 300-kg hydraulic car-lift platform to a height of 2.8 m. Determine (*a*) the average output power delivered by the hydraulic pump to lift the system, (*b*) the average power electric required, knowing that the overall conversion efficiency from electric to mechanical power for the system is 82 percent.

SOLUTION

Newton's law



$$Mg = (M_C + M_L)g = (1200 + 300)g$$

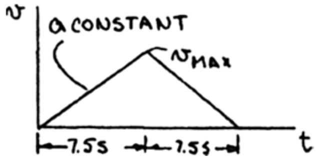
$$Mg = 1500g$$

$$+\uparrow \quad \Sigma F = F - 1500g = 1500a \qquad (1)$$

Since motion is uniformly accelerated, a = constant

Thus, from (1), F is constant and peak power occurs when the velocity is a maximum at 7.5 s.

$$a = \frac{v_{\max}}{7.5\text{ s}}$$



$$P = (6000\text{ W}) = (F)(v_{\max})$$

$$v_{\max} = (6000)/F$$

Thus,

$$a = (6000)/(7.5)(F) \qquad (2)$$

Substitute (2) into (1)

$$F - 1500g = (1500)(6000)/(7.5)(F)$$

$$F^2 - (1500\text{ kg})(9.81\text{ m/s}^2)F - \frac{(1500\text{ kg})(6000\text{ N}\cdot\text{m/s})}{(7.5\text{ s})} = 0$$

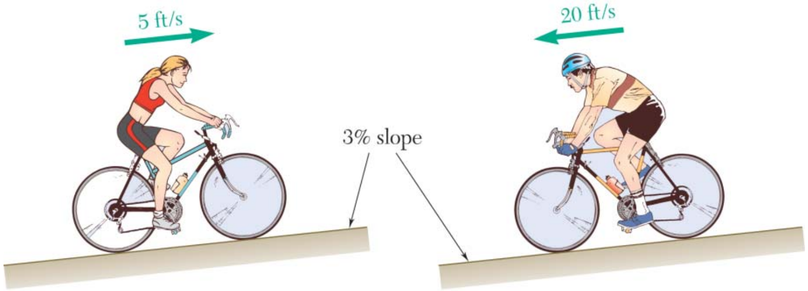
$$F^2 - 14{,}715F - 1.2\times10^6 = 0$$

$$F = 14{,}800\text{ N}$$

$$F = 14.8\text{ kN} \blacktriangleleft$$

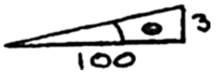
PROBLEM 13.49

(a) A 120-lb woman rides a 15-lb bicycle up a 3-percent slope at a constant speed of 5 ft/s. How much power must be developed by the woman? (*b*) A 180-lb man on an 18-lb bicycle starts down the same slope and maintains a constant speed of 20 ft/s by braking. How much power is dissipated by the brakes? Ignore air resistance and rolling resistance.

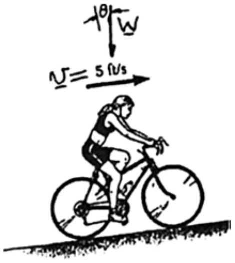


(*a*) (*b*)

SOLUTION



$$\tan\theta = \frac{3}{100} \qquad \theta = 1.718°$$



(*a*)

(*a*)

$$W = W_B + W_W = 15 + 120$$

$$W = 135 \text{ lb}$$

$$P_W = \mathbf{W}\cdot\mathbf{v} = (W\sin\theta)(v)$$

$$P_W = (135)(\sin 1.718°)(5)$$

$$P_W = 20.24 \text{ ft}\cdot\text{lb/s}$$

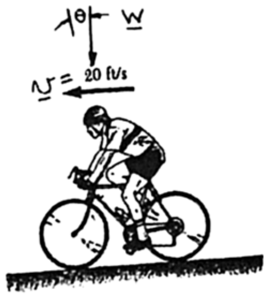
$P_W = 20.2 \text{ ft}\cdot\text{lb/s}$ ◀

(*b*)

$$W = W_B + W_m = 18 + 180$$

$$W = 198 \text{ lb}$$

Brakes must dissipate the power generated by the bike and the man going down the slope at 20 ft/s.

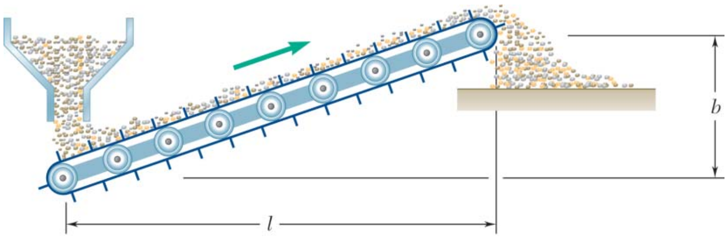


(*b*)

$$P_B = \mathbf{W}\cdot\mathbf{v} = (W\sin\theta)(v)$$

$$P_B = (198)(\sin 1.718)(20)$$

$P_B = 118.7 \text{ ft}\cdot\text{lb/s}$ ◀



PROBLEM 13.50

A power specification formula is to be derived for electric motors which drive conveyor belts moving solid material at different rates to different heights and distances. Denoting the efficiency of the motors by η and neglecting the power needed to drive the belt itself, derive a formula (*a*) in the SI system of units for the power P in kW, in terms of the mass flow rate m in kg/h, the height b and horizontal distance l in meters, and (*b*) in U.S. customary units, for the power in hp, in terms of the material flow rate w in tons/h, and the height b and horizontal distance l in feet.

SOLUTION

(*a*) Material is lifted to a height b at a rate, $(m \text{ kg/h})(g \text{ m/s}^2) = [mg(\text{N/h})]$

Thus,

$$\frac{\Delta U}{\Delta t} = \frac{[mg(\text{N/h})][b(m)]}{(3600 \text{ s/h})} = \left(\frac{mgb}{3600}\right) \text{N} \cdot \text{m/s}$$

$$1000 \text{ N} \cdot \text{m/s} = 1 \text{ kW}$$

Thus, including motor efficiency, η

$$P(\text{kw}) = \frac{mgb(\text{N} \cdot \text{m/s})}{(3600)\left(\frac{1000 \text{ N} \cdot \text{m/s}}{\text{kW}}\right)(\eta)}$$

$$P(\text{kW}) = 0.278 \times 10^{-6} \frac{mgb}{\eta} \blacktriangleleft$$

(*b*)

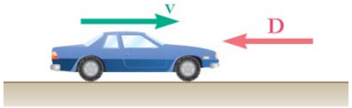
$$\frac{\Delta U}{\Delta t} = \frac{[W(\text{tons/h})(2000 \text{ lb/ton})][b(\text{ft})]}{3600 \text{ s/h}}$$

$$= \frac{Wb}{1.8} \text{ ft} \cdot \text{lb/s}; \quad 1 \text{ hp} = 550 \text{ ft} \cdot \text{lb/s}$$

With η,

$$hp = \left[\frac{Wb}{1.8}(\text{ft} \cdot \text{lb/s})\right]\left[\frac{1 \text{ hp}}{550 \text{ ft} \cdot \text{lb/s}}\right]\left[\frac{1}{\eta}\right]$$

$$hp = \frac{1.010 \times 10^{-3} Wb}{\eta} \blacktriangleleft$$



PROBLEM 13.51

A 1400-kg automobile starts from rest and travels 400 m during a performance test. The motion of the automobile is defined by the relation $x = 4000\ln(\cosh 0.03t)$, where x and t are expressed in meters and seconds, respectively. The magnitude of the aerodynamic drag is $D = 0.35v^2$, where D and v are expressed in newtons and m/s, respectively. Determine the power dissipated by the aerodynamic drag when (*a*) $t = 10$ s, (*b*) $t = 15$ s.

SOLUTION

Motion is determined as a function of time as

$$x = 4000\ln(\cosh 0.03t)$$

Velocity
$$v = \frac{dx}{dt} = 4000\left(\frac{1}{\cosh 0.03t}\right)(\sinh 0.03t)(0.03)$$

$$v = \frac{120\sinh 0.03t}{\cosh 0.03t}$$

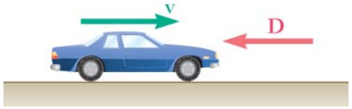
Power dissipated
$$P = Dv = (0.35v^2)v = 0.35v^3$$

$$P = 0.35(120)^3\left[\frac{\sinh 0.03t}{\cosh 0.03t}\right]^3$$

$$= 604.8(10)^3\left[\frac{e^{0.03t} - e^{-0.03t}}{e^{0.03t} + e^{-0.03t}}\right]^3$$

(*a*) $t = 10$ s, $P = 14.95(10)^3 \text{ W} = 14.95 \text{ kW}$ ◀

(*b*) $t = 15$ s, $P = 45.4(10)^3 \text{ W} = 45.4 \text{ kW}$ ◀



PROBLEM 13.52

A 1400-kg automobile starts from rest and travels 400 m during a performance test. The motion of the automobile is defined by the relation $a = 3.6e^{-0.0005x}$, where a and x are expressed in $\mathrm{m/s^2}$ and meters, respectively. The magnitude of the aerodynamic drag is $D = 0.35v^2$, where D and v are expressed in newtons and m/s, respectively. Determine the power dissipated by the aerodynamic drag when (*a*) $x = 200$ m, (*b*) $x = 400$ m.

SOLUTION

Motion is defined by the following function:

$$a = 3.6e^{-0.0005x}$$

$$= v\frac{dv}{dx}$$

$$\int_0^v v\,dv = 3.6\int_0^x e^{-0.0005x}dx$$

$$= \frac{-3.6}{0.0005}\int_0^{-0.0005x} e^u du$$

$$\frac{v^2}{2} = -7200\left(e^{-0.0005x} - 1\right)$$

$$v^2 = 14400\left(1 - e^{-0.0005x}\right)$$

$$v = 120\left(1 - e^{-0.0005x}\right)^{\frac{1}{2}}$$

$$\text{Power dissipated} = P = Dv = 0.35v^3$$

$$P = 604.8(10)^3\left[1 - e^{-0.0005x}\right]^{\frac{3}{2}}$$

(a) $x = 200$ m, $P = 17.75(10)^3\,W = 17.75$ kW ◀

(*b*) $x = 400$ m, $P = 46.7(10)^3\,W = 46.7$ kW ◀

PROBLEM 13.53

The fluid transmission of a 15-Mg truck allows the engine to deliver an essentially constant power of 50 kW to the driving wheels. Determine the time required and the distance traveled as the speed of the truck is increased (*a*) from 36 km/h to 54 km/h, (*b*) from 54 km/h to 72 km/h.

SOLUTION

For constant power, P:

$$\begin{aligned} P &= Fv \\ &= mav \\ &= m\frac{dv}{dt}v \end{aligned}$$

Separate variables

$$\int_0^t dt = \int_0^{v_1} \frac{m}{P}\frac{dv}{v} \Rightarrow t = \frac{m}{P}\left(\frac{v_1^2}{2} - \frac{v_0^2}{2}\right) \tag{1}$$

Distance

$$P = mv\frac{dv}{dx}\frac{dx}{dt} = mv^2\frac{dv}{dx}$$

Separate variables

$$\int_0^x dx = \frac{m}{P}\int_{v_0}^{v_1} v^2 dv \Rightarrow x = \frac{m}{P}\left(\frac{v_1^3}{3} - \frac{v_0^3}{3}\right) \tag{2}$$

with numbers

(*a*) $v_0 = 36 \text{ km/h} = 10 \text{ m/s};\ \ v_1 = 54 \text{ km/h} = 15 \text{ m/s},$ so,

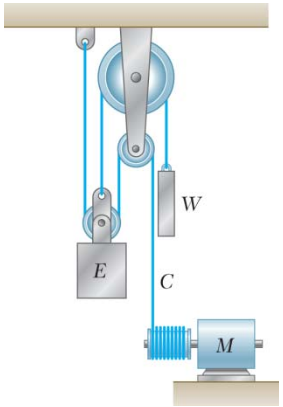
$$t = \frac{15 \times 10^3 \text{ kg}}{2\left(50 \times 10^3 \text{ W}\right)}\left[(15 \text{ m/s})^2 - (10 \text{ m/s})^2\right] \qquad \Rightarrow t = 18.75 \text{ s} \blacktriangleleft$$

$$x = \frac{15 \times 10^3}{(3)\left(50 \times 10^3\right)}\left[15^3 - 10^3\right] = 237.5 \text{ m} \qquad \Rightarrow x = 238 \text{ m} \blacktriangleleft$$

(*b*) $v_0 = 54 \text{ km/h} = 15 \text{ m/s};\ \ v_1 = 72 \text{ km/h} = 20 \text{ m/s}$

$$t = \frac{15 \times 10^3}{(2)\left(50 \times 10^3\right)}\left[20^2 - 15^2\right] = 26.25 \text{ s} \qquad \Rightarrow t = 26.2 \text{ s} \blacktriangleleft$$

$$x = \frac{15 \times 10^3}{(3)\left(50 \times 10^3\right)}\left[20^3 - 15^3\right] = 462.5 \text{ m} \qquad \Rightarrow x = 462 \text{ m} \blacktriangleleft$$



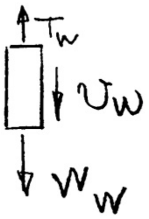
PROBLEM 13.54

The elevator E has a weight of 6600 lbs when fully loaded and is connected as shown to a counterweight W of weight of 2200 lb. Determine the power in hp delivered by the motor (a) when the elevator is moving down at a constant speed of 1 ft/s, (b) when it has an upward velocity of 1 ft/s and a deceleration of 0.18 ft/s^2.

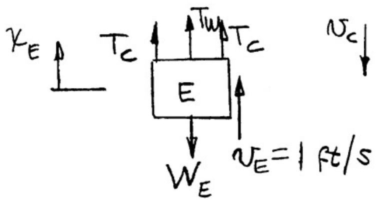
SOLUTION

(a) Acceleration = 0

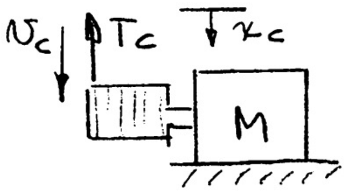
Counterweight



Elevator


x_E
T_C
T_W
T_C
E
W_E
$v_E = 1$ ft/s
v_C

Motor



$$+\uparrow \Sigma F_y = 0: \quad T_W - W_W = 0 \qquad\qquad +\uparrow \Sigma F = 0: \quad 2T_C + T_W - 6600 = 0$$

$$T_W = 2200 \text{ lb} \qquad\qquad T_C = 2200 \text{ lb}$$

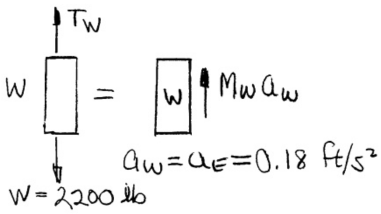
Kinematics: $2x_E = x_C, \; 2\dot{x}_E = \dot{x}_C, \; v_C = 2v_E = 2$ ft/s

$$P = T_C \cdot v_C = (2200 \text{ lb})(2 \text{ ft/s}) = 4400 \text{ lb}\cdot\text{ft/s} = 8.00 \text{ hp}$$

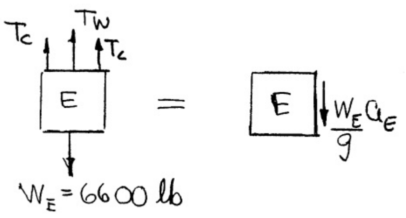
$P = 8.00$ hp ◀

(b) $a_E = 0.18 \text{ ft/s}^2 \uparrow, \; v_E = 1 \text{ ft/s} \downarrow$

Counterweight


T_W
W
=
W
$M_W a_W$
$a_W = a_E = 0.18$ ft/s^2
W = 2200 lb

Elevator


T_C
T_W
T_C
E
=
E
$\frac{W_E}{g} a_E$
$W_E = 6600$ lb

Counterweight: $+\uparrow \Sigma F = Ma: \quad T_W - W = \dfrac{W}{g}(a_W)$

PROBLEM 13.54 (Continued)

$$T_W = (2200 \text{ lb}) + \frac{(2200 \text{ lb})(0.18 \text{ ft/s}^2)}{(32.2 \text{ ft/s}^2)}$$

$$T_W = 2212 \text{ lb}$$

Elevator

$$+\uparrow \Sigma F = ma \qquad 2T_C + T_W - W_E = \frac{-W_E}{g}(a_E)$$

$$2T_C = (-2212 \text{ lb}) + (6600 \text{ lb}) - \frac{(6600 \text{ lb})(0.18 \text{ ft/s}^2)}{(32.2 \text{ ft/s}^2)}$$

$$2T_C = 4351 \text{ lb}$$

$$T_C = 2175.6 \text{ lb}$$

$$v_C = 2 \text{ ft/s (see part}(a))$$

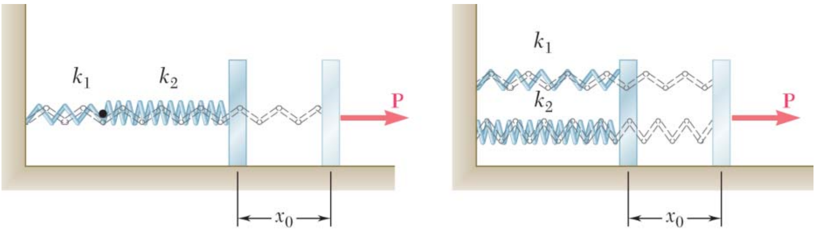
$$P = T_C \cdot v_C = (2175.6 \text{ lb})(2 \text{ ft/s}) = 4351.2 \text{ lb}\cdot\text{ft/s}$$

$$= 7.911 \text{ hp}$$

$P = 7.91$ hp ◀

PROBLEM 13.55

A force **P** is slowly applied to a plate that is attached to two springs and causes a deflection x_0. In each of the two cases shown, derive an expression for the constant k_e, in terms of k_1 and k_2, of the single spring equivalent to the given system, that is, of the single spring which will undergo the same deflection x_0 when subjected to the same force **P**.

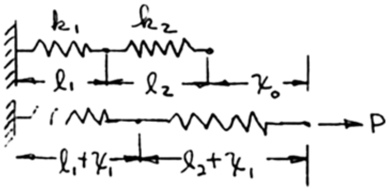


(*a*) (*b*)

SOLUTION

System is in equilibrium in deflected x_0 position.

Case (*a*) Force in both springs is the same $= P$



$$x_0 = x_1 + x_2$$

$$x_0 = \frac{P}{k_e}$$

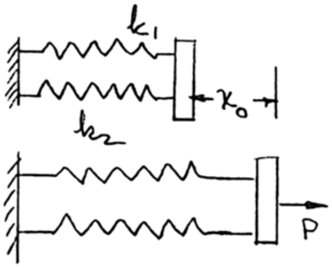
$$x_1 = \frac{P}{k_1} \qquad x_2 = \frac{P}{k_2}$$

Thus,
$$\frac{P}{k_e} = \frac{P}{k_1} + \frac{P}{k_2}$$

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$k_e = \frac{k_1 k_2}{k_1 + k_2} \blacktriangleleft$$

Case (*b*) Deflection in both springs is the same $= x_0$

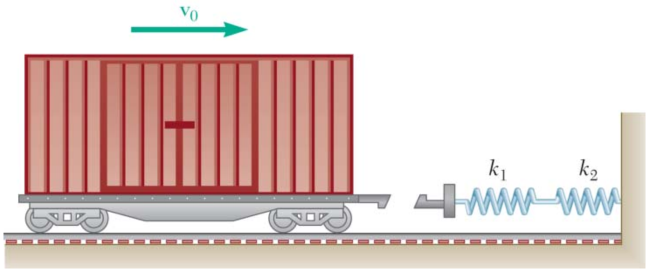


$$P = k_1 x_0 + k_2 x_0$$
$$P = (k_1 + k_2) x_0$$
$$P = k_e x_0$$

Equating the two expressions for

$$P = (k_1 + k_2) x_0 = k_e x_0$$

$$k_e = k_1 + k_2 \blacktriangleleft$$



PROBLEM 13.56

A loaded railroad car of mass m is rolling at a constant velocity $\mathbf{v_0}$ when it couples with a massless bumper system. Determine the maximum deflection of the bumper assuming the two springs are (*a*) in series (as shown), (*b*) in parallel.

SOLUTION

Let position A be at the beginning of contact and position B be at maximum deflection.

$$T_A = \frac{1}{2}mv_0^2$$

$$V_A = 0 \qquad \text{(zero force in springs)}$$

$$T_B = 0 \qquad (v = 0 \text{ at maximum deflection})$$

$$V_B = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2$$

where x_1 is deflection of spring k_1 and x_2 is that of spring k_2.

Conservation of energy: $T_A + V_A = T_B + V_B$

$$\frac{1}{2}mv_0^2 + 0 = 0 + \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2$$

$$k_1x_1^2 + k_2x_2^2 = mv_0^2 \qquad (1)$$

(*a*) *Springs are in series.*

Let F be the force carried by the two springs.

Then, $x_1 = \dfrac{F}{k_1}$ and $x_2 = \dfrac{F}{k_2}$

Eq. (1) becomes $F^2\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) = mv_0^2$

so that $F = v_0\sqrt{m/\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right)}$

The maximum deflection is

$$\delta = x_1 + x_2 = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)F$$

$$= \left(\frac{1}{k_1} + \frac{1}{k_2}\right)v_0\sqrt{m/\left(\frac{1}{k_1} + \frac{1}{k_2}\right)}$$

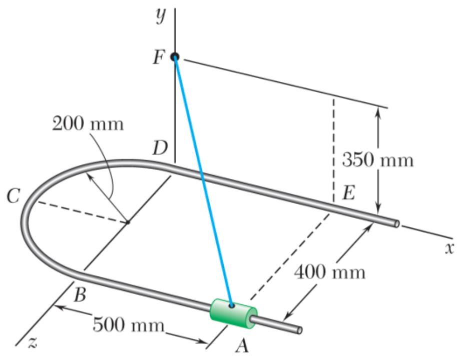
$$= v_0\sqrt{m\left(\frac{1}{k_1} + \frac{1}{k_2}\right)} \qquad \delta = v_0\sqrt{m(k_1 + k_2)/k_1k_2} \blacktriangleleft$$

PROBLEM 13.56 (Continued)

(*b*) *Springs are in parallel.*

$$x_1 = x_2 = \delta$$

Eq. (1) becomes $(k_1 + k_2)\delta^2 = mv_0^2$ $\delta = v_0\sqrt{\dfrac{m}{k_1 + k_2}}$ ◀



PROBLEM 13.57

A 750-g collar can slide along the horizontal rod shown. It is attached to an elastic cord with an undeformed length of 300 mm and a spring constant of 150 N/m. Knowing that the collar is released from rest at A and neglecting friction, determine the speed of the collar (*a*) at B, (*b*) at E.

SOLUTION

Use conservation of energy $$T_1 + V_1 = T_2 + V_2 \qquad (1)$$

(*a*) Position 1 is at A and position 2 is at B

$$T_1 = 0; \qquad V_1 = \frac{1}{2}kx_1^2 \qquad \text{where} \quad x_1 = \ell - \ell_0$$

$$\ell = \left[500^2 + 400^2 + 350^2\right]^{\frac{1}{2}} = 729.726 \text{ mm}$$

So $\ell - \ell_0 = 429.726$ mm

$$V_1 = \frac{1}{2}(1500 \text{ N/m})(0.429726 \text{ m})^2 = 13.8498 \text{ J}$$

At B $$\ell = \left[350^2 + 400^2\right]^{\frac{1}{2}} = 531.507 \text{ mm} \Rightarrow \ell - \ell_0 = 231.507 \text{ mm}$$

$$T_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(0.75)v_2^2 = 0.375\ v_2^2$$

$$V_2 = \frac{1}{2}(150)(0.231507)^2 = 4.01966 \text{ J}$$

Substituting into (1) $$0 + 13.8498 = 0.375\ v_2^2 + 4.01966$$

$$v_2 = v_B = 5.12 \text{ m/s} \blacktriangleleft$$

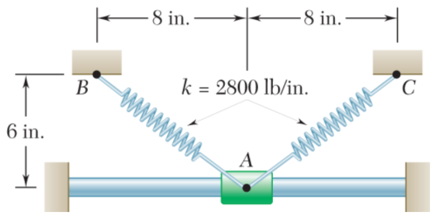
(*b*) At E $$T_1 = 0; \quad V_1 = 13.8498 \text{ (same as before)}$$

At E $$\ell = \left[350^2 + 500^2\right]^{\frac{1}{2}} = 610.328 \text{ mm} \Rightarrow \ell - \ell_0 = 310.328 \text{ mm}$$

$$T_3 = \frac{1}{2}mv_E^2 = 0.375\ v_E^2$$

$$v_E = \frac{1}{2}kx^2 = \frac{1}{2}(150)(0.310328)^2 = 7.223 \text{ J}$$

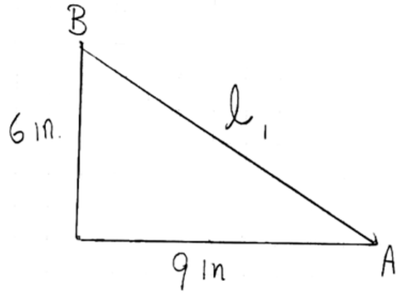
Substituting into (1) $$0 + 13.8498 = 0.375\ v_E^2 + 7.223 \Rightarrow v_E = 4.20 \text{ m/s} \blacktriangleleft$$



PROBLEM 13.58

A 4-lb collar can slide without friction along a horizontal rod and is in equilibrium at *A* when it is pushed 1 in. to the right and released from rest. The springs are undeformed when the collar is at *A* and the constant of each spring is 2800 lb/in. Determine the maximum velocity of the collar.

SOLUTION

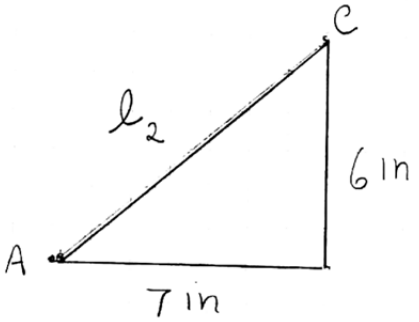


$$\ell_1 = \sqrt{6^2 + 9^2} = 10.817 \text{ in.}$$

$$\ell_0 = \sqrt{(6)^2 + (8)^2} = 10 \text{ in.} = 0.8333 \text{ ft}$$

$$\text{Stretch} = 10.817 - 10 = 0.817 \text{ in.}$$

$$S_1 = 0.06805 \text{ ft}$$



$$\ell_2 = \sqrt{(7)^2 + (6)^2} = 9.215 \text{ in.}$$

$$\text{Stretch} = 9.2195 - 10 = -0.7805 \text{ in.}$$

$$S_2 = 0.06504 \text{ ft}$$

$$T_1 = 0, \quad V_2 = 0$$

$$T_2 = \frac{1}{2}mv_2^2$$

$$= \frac{1}{2}\left(\frac{4}{32.2}\right)v_2^2$$

$$V_1 = \frac{1}{2}(33{,}600 \text{ lb/ft})\left(S_1^2 + S_2^2\right)$$

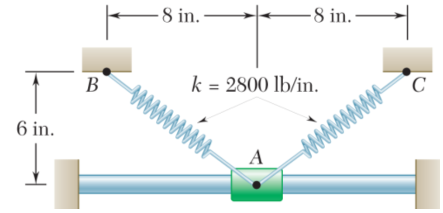
$$V_1 = (16{,}800)(0.008861) = 148.86 \text{ ft}\cdot\text{lb}$$

$$\cancel{T_1} + V_1 = T_2 + \cancel{V_2}$$

$$148.86 = \frac{1}{2}\left(\frac{4}{32.2}\right)v_2^2$$

$$v_2^2 = 2396.7$$

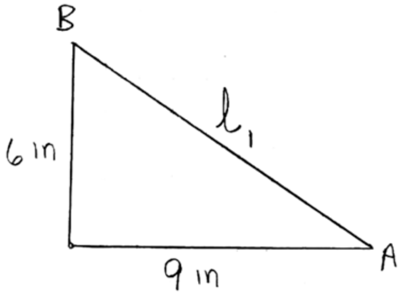
$v_2 = 49.0$ ft/s ◀

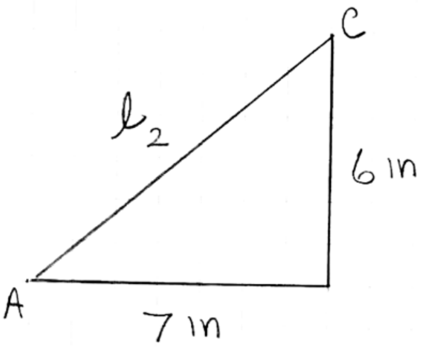


PROBLEM 13.59

A 4-lb collar can slide without friction along a horizontal rod and is released from rest at *A*. The undeformed lengths of springs *BA* and *CA* are 10 in. and 9 in., respectively, and the constant of each spring is 2800 lb/in. Determine the velocity of the collar when it has moved 1 in. to the right.

SOLUTION





$$k = 2800 \text{ lb/in.} = 33{,}600 \text{ lb/ft}$$

$$T_1 = 0$$

$$V_1 = \frac{1}{2}k\left(\Delta l_1\right)^2 = \frac{1}{2}(33{,}600)\left(\frac{1}{12}\right)^2$$

$$= 116.667 \text{ ft} \cdot \text{lb}$$

$$\ell_1 = \sqrt{6^2 + 9^2} = 10.817 \text{ in.} = 0.9014 \text{ ft}$$

$$S_1 = \text{Stretch} = 10.817 - 10 = 0.817 \text{ in.} = 0.06808 \text{ ft}$$

$$\ell_2 = \sqrt{6^2 + 7^2} = 9.2195 \text{ in.}$$

$$S_2 = \text{Stretch} = 9.2195 - 9 = 0.2195 \text{ in.} = 0.018295 \text{ ft}$$

$$T_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}\left(\frac{4}{32.2}\right)v_2^2 = 0.0621\, v_2^2$$

$$V_2 = \frac{1}{2}(33{,}600)\left(S_1^2 + S_2^2\right)$$

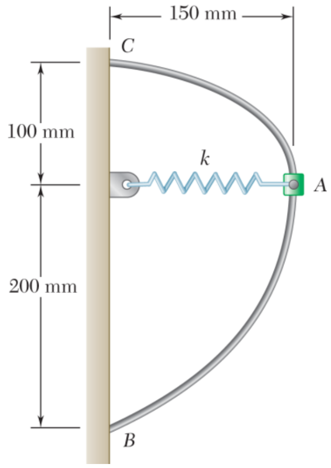
$$= 16800\left[(0.06808)^2 + (0.018295)^2\right]$$

$$= 83.489 \text{ ft}\cdot\text{lb}$$

$$\cancel{T_1} + V_1 = T_2 + V_2$$

$$116.667 = 0.06211\, v_2^2 + 83.489$$

$v_2 = 23.1$ ft/s ◀

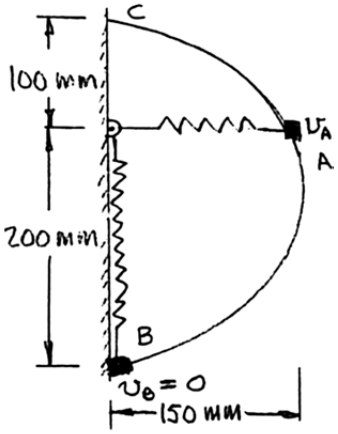


PROBLEM 13.60

A 500-g collar can slide without friction on the curved rod BC in a *horizontal* plane. Knowing that the undeformed length of the spring is 80 mm and that $k = 400$ kN/m, determine (*a*) the velocity that the collar should be given at A to reach B with zero velocity, (*b*) the velocity of the collar when it eventually reaches C.

SOLUTION

(*a*) Velocity at A:



$$T_A = \frac{1}{2}mv_A^2 = \left(\frac{0.5}{2}\text{ kg}\right)v_A^2$$

$$T_A = (0.25)v_A^2$$

$$\Delta L_A = 0.150\text{ m} - 0.080\text{ m}$$

$$\Delta L_A = 0.070\text{ m}$$

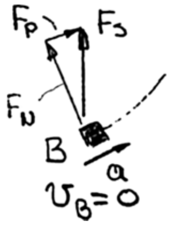
$$V_A = \frac{1}{2}k(\Delta L_A)^2$$

$$V_A = \frac{1}{2}(400\times 10^3\text{ N/m})(0.070\text{ m})^2$$

$$V_A = 980\text{ J}$$

$$v_B = 0 \qquad T_B = 0$$

$$\Delta L_B = 0.200\text{ m} - 0.080\text{ m} = 0.120\text{ m}$$



$$V_B = \frac{1}{2}k(\Delta L_B)^2 = \frac{1}{2}(400\times 10^3\text{ N/m})(0.120\text{ m})^2$$

$$V_B = 2880\text{ J}$$

Substitute into conservation of energy.

$$T_A + V_A = T_B + V_B \qquad 0.25v_A^2 + 980 = 0 + 2880$$

$$v_A^2 = \frac{(2880 - 980)}{(0.25)}$$

$$v_A^2 = 7600\text{ m}^2/\text{s}^2$$

$v_A = 87.2$ m/s ◀

(*b*) Velocity at C:

Since slope at B is positive, the component of the spring force F_P, parallel to the rod, causes the block to move back toward A.

$$T_B = 0, \quad V_B = 2880 \text{ J} \quad [\text{from part } (a)]$$

$$T_C = \frac{1}{2}mv_C^2 = \frac{(0.5 \text{ kg})}{2}v_C^2 = 0.25v_C^2$$

$$\Delta L_C = 0.100 \text{ m} - 0.080 \text{ m} = 0.020 \text{ m}$$

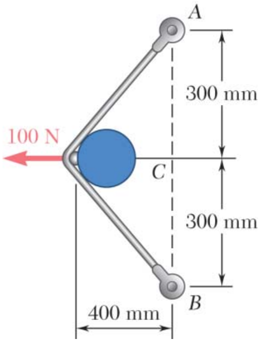
$$V_C = \frac{1}{2}k(\Delta L_C)^2 = \frac{1}{2}(400 \times 10^3 \text{ N/m})(0.020 \text{ m})^2 = 80.0 \text{ J}$$

Substitute into conservation of energy.

$$T_B + V_B = T_C + V_C \quad 0 + 2880 = 0.25v_C^2 + 80.0$$

$$v_C^2 = 11{,}200 \text{ m}^2/\text{s}^2$$

$$v_C = 105.8 \text{ m/s} \blacktriangleleft$$



PROBLEM 13.61

For the adapted shuffleboard device in Prob 13.28, you decide to utilize an elastic cord instead of a compression spring to propel the puck forward. When the cord is stretched directly between points *A* and *B*, the tension is 20 N. The 425 gram puck is placed in the center and pulled back through a distance of 400 mm; a force of 100 N is required to hold it at this location. Knowing that the coefficient of friction is 0.3, determine how far the puck will travel.

SOLUTION

Given:

$\mu_k = 0.3$

$m = 0.425\text{ kg}$

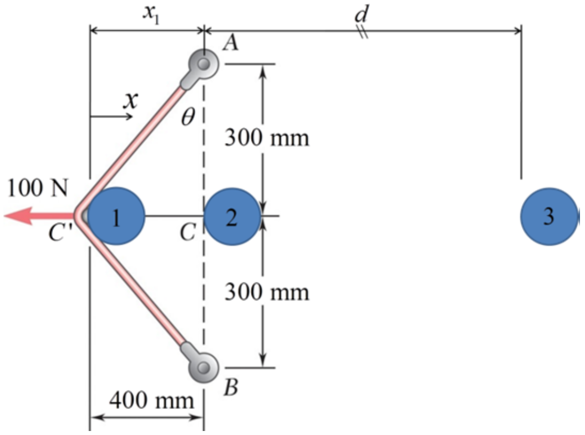
$F_{C,2} = 20\text{ N}$, Force in cord at positon 2

l_i = length of cord at positon i

s_i = stretch of cord in position i

l_O = unstretched length of cord

Sketch:



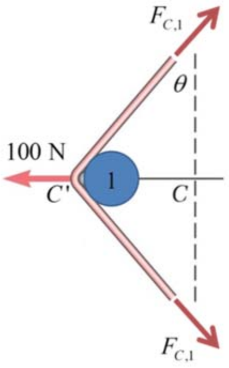
At Position 1:

$$l_1 = 2\left(0.3^2 + 0.4^2\right) \Rightarrow l_1 = 1\text{ m}$$

$$\theta = \tan^{-1}\left(\frac{40}{30}\right) = 53.13°$$

Find Force in cord:

FBD:



For Equilibrium:

$$\sum F_x = 0$$

$$2F_{C,1}\sin\theta - 100 = 0$$

$$F_{C,1} = 62.5\text{ N}$$

At Position 2:

$$l_2 = 0.6\text{ m}$$

PROBLEM 13.61 (Continued)

Find Stiffness of cord, *k*:

$$F_{C,1} = ks_1, \quad F_{C,2} = ks_2$$

$$F_{C,1} - F_{C,2} = k(s_1 - s_2)$$
$$62.5 - 20 = k[(1.0 - l_0) - (0.6 - l_0)]$$
$$k = 106.25 \text{ N/m}$$

Find stretch in cord at positions 1, 2 and 3:

$$s_1 = \frac{F_{C,1}}{k} = \frac{62.5 \text{ N}}{106.25 \text{ N/m}} = 0.5882 \text{ m} \qquad s_2 = s_3 = \frac{F_{C,2}}{k} = \frac{20 \text{ N}}{106.25 \text{ N/m}} = 0.1882 \text{ m}$$

Work Energy from position 1 to 3:

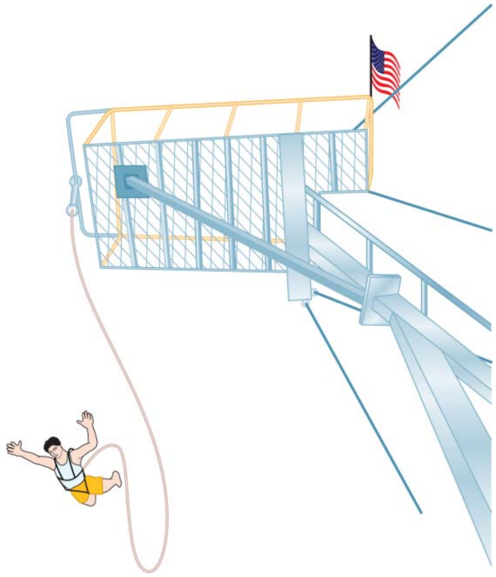
$$\cancel{T_1} + \cancel{V_{g_1}} + V_{e_1} + U_{1\to3}^{NC} = \cancel{T_3} + \cancel{V_{g_3}} + V_{e_3} \quad \text{where:} \quad V_{e_1} = \frac{1}{2}ks_1^2$$
$$U_{1\to3}^{NC} = \int_0^{x_1+d} -\mu_k mg\,dx$$
$$V_{e_3} = \frac{1}{2}ks_3^2$$

Therefore: $$\frac{1}{2}ks_1^2 - \mu_k mg(x_1 + d) = \frac{1}{2}ks_3^2$$

Substitute known values:

$$\frac{1}{2}(106.25)(0.5882)^2 - 0.3(0.425)(9.81)(0.4 + d) = \frac{1}{2}(106.25)(0.1882)^2$$

Solve for *d*: $d = 12.79 \text{ m}$ ◀



PROBLEM 13.62

An elastic cable is to be designed for bungee jumping from a tower 130 ft high. The specifications call for the cable to be 85 ft long when unstretched, and to stretch to a total length of 100 ft when a 600-lb weight is attached to it and dropped from the tower. Determine (*a*) the required spring constant *k* of the cable, (*b*) how close to the ground a 186-lb man will come if he uses this cable to jump from the tower.

SOLUTION

(*a*) Conservation of energy:

$$V_1 = 0 \quad T_1 = 0 \quad V_1 = 100\,W$$

Datum at ②:

$$V_1 = (100 \text{ ft})(600 \text{ lb})$$

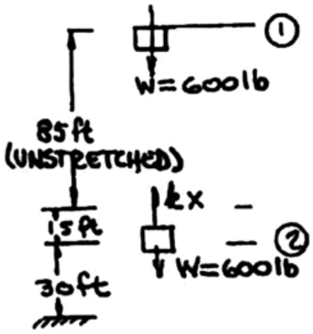
$$= 6 \times 10^4 \text{ ft} \cdot \text{lb}$$

$$V_2 = 0 \quad T_2 = 0$$

$$V_2 = V_g + V_e = 0 + \frac{1}{2}k(15 \text{ ft})^2$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 6 \times 10^4 = 0 + (112.5)k$$


①
W=600lb
85ft
(UNSTRETCHED)
x
15ft
②
W=600lb
30ft

$k = 533$ lb/ft ◀

(*b*) From (*a*), $k = 533$ lb/ft

$$T_1 = 0$$

$$W = 186 \text{ lb}$$

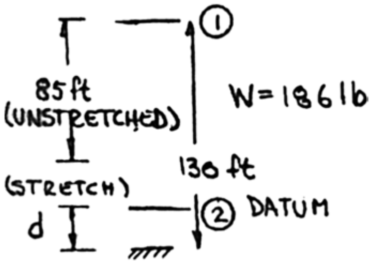
$$V_1 = (186)(130 - d)$$

$$T_2 = 0$$

Datum:

$$V_2 = V_g + V_e = 0 + \frac{1}{2}(533)(130 - 85 - d)^2$$

$$V_2 = (266.67)(45 - d)^2$$


①
85ft
(UNSTRETCHED)
W=186lb
130 ft
(STRETCH)
d
② DATUM

PROBLEM 13.62 (Continued)

d = distance from the ground $\qquad T_1 + V_1 = T_2 + V_2$

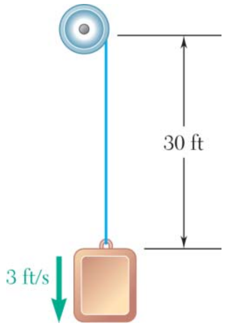
$$0 + (186)(130 - d) = 0 + (266.67)(45 - d)^2$$

$$266.7d^2 - 23815d + 515827 = 0$$

$$d = \frac{23815 \mp \sqrt{(23815)^2 - 4(266.7)(515827)}}{(2)(266.7)} = \begin{matrix} 36.99 \text{ ft} \\ 52.3 \text{ ft} \end{matrix}$$

Discard 52.3 ft (since the cord acts in compression when rebound occurs).

$d = 37.0$ ft ◀



PROBLEM 13.63

It is shown in mechanics of materials that the stiffness of an elastic cable is $k = AE/L$ where A is the cross sectional area of the cable, E is the modulus of elasticity and L is the length of the cable. A winch is lowering a 4000-lb piece of machinery using at a constant speed of 3ft/s when the winch suddenly stops. Knowing that the steel cable has a diameter of 0.4 in., $E = 29 \times 10^6$ lb/in^2, and when the winch stops $L = 30$ ft, determine the maximum downward displacement of the piece of machinery from the point it was when the winch stopped.

SOLUTION

Mass of machinery:
$$m = \frac{W}{g} = \frac{4000}{32.2} = 124.22\ \text{lb}\cdot\text{s}^2/\text{ft}$$

Let position 1 be the state just before the winch stops and the gravitational potential V_g be equal to zero at this state.

For the cable,
$$A = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}(0.4\ \text{in.})^2 = 0.12566\ \text{in}^2$$

$$AE = (0.12556\ \text{in.}^2)(29\times 10^6\ \text{lb/in}^2) = 3.6442\times 10^6\ \text{lb}$$

For
$$L = 30\ \text{ft},\quad k = \frac{AE}{L} = \frac{3.6442\times 10^6\ \text{lb}}{30\ \text{ft}} = 121.47\times 10^3\ \text{lb/ft}$$

Initial force in cable (equilibrium):
$$F_1 = W = 4000\ \text{lb.}$$

Elongation in position 1:
$$x_1 = \frac{F_1}{k} = \frac{4000}{121.47\times 10^3} = 0.03293\ \text{ft}$$

Potential energy:
$$V_1 = \frac{1}{2}kx_1^2 = \frac{F_1^2}{2k}$$

$$V_1 = \frac{(4000\ \text{lb})^2}{(2)(121.47\times 10^3\ \text{lb/ft})} = 65.860\ \text{ft}\cdot\text{lb}$$

Kinetic energy:
$$T_1 = \frac{1}{2}mv_1^2$$

$$T_1 = \frac{1}{2}(124.22\ \text{lb}\cdot\text{s}^2/\text{ft})(3\ \text{ft/s}^2) = 558.99\ \text{ft}\cdot\text{lb}$$

Let position 2 be the position of maximum downward displacement. Let x_2 be the elongation in this position.

Potential energy:
$$V_2 = \frac{1}{2}kx_2^2 - W(x_2 - x_1)$$

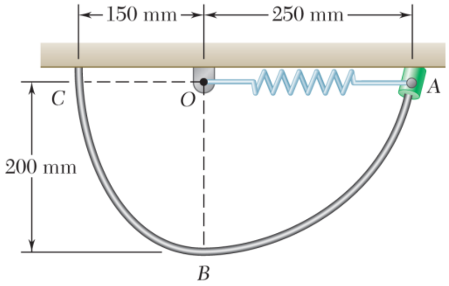
$$V_2 = \frac{1}{2}(121.47\times10^3)x_2^2 - (4000)(x_2 - 0.03293)$$
$$= 60.735\times10^3 x^2 - 4000x_2 + 131.72$$

Kinetic energy: $T_2 = 0$ (since $v_2 = 0$)

Principle of work and energy: $T_1 + V_1 = T_2 + V_2$

$$558.99 + 65.860 = 60.735\times10^3 x_2^2 - 4000x_2 + 131.72$$
$$60.735\times10^3 x_2^2 - 4000x_2 - 493.13 = 0$$
$$x_2 = 0.12887\text{ ft}$$

Maximum displacement: $\delta = x_2 - x_1 = 0.09594\text{ ft}$ $\delta = 1.151\text{ in.}$



PROBLEM 13.64

A 2-kg collar is attached to a spring and slides without friction in a vertical plane along the curved rod *ABC*. The spring is undeformed when the collar is at *C* and its constant is 600 N/m. If the collar is released at *A* with no initial velocity, determine its velocity (*a*) as it passes through *B*, (*b*) as it reaches *C*.

SOLUTION

Spring elongations:

At *A*, $x_A = 250 \text{ mm} - 150 \text{ mm} = 100 \text{ mm} = 0.100 \text{ m}$

At *B*, $x_B = 200 \text{ mm} - 150 \text{ mm} = 50 \text{ mm} = 0.050 \text{ m}$

At *C*, $x_C = 0$

Potential energies for springs.

$$(V_A)_e = \frac{1}{2}kx_A^2 = \frac{1}{2}(600)(0.100)^2 = 3.00 \text{ J}$$

$$(V_B)_e = \frac{1}{2}kx_B^2 = \frac{1}{2}(600)(0.050)^2 = 0.75 \text{ J}$$

$$(V_C)_e = 0$$

Gravitational potential energies: Choose the datum at level *AOC*.

$$(V_A)_g = (V_C)_g = 0$$

$$(V_B)_g = -mg\,y = -(2)(9.81)(0.200) = -3.924 \text{ J}$$

Kinetic energies: $T_A = 0$

$$T_B = \frac{1}{2}mv_B^2 = 1.00\,v_B^2$$

$$T_C = \frac{1}{2}mv_C^2 = 1.00\,v_C^2$$

(*a*) *Velocity as the collar passes through B.*

Conservation of energy: $T_A + V_A = T_B + V_B$

$$0 + 3.00 + 0 = 1.00\,v_B^2 + 0.75 - 3.924$$

$$v_B^2 = 6.174 \text{ m}^2/\text{s}^2$$

$\mathbf{v}_B = 2.48 \text{ m/s} \longleftarrow$ ◀

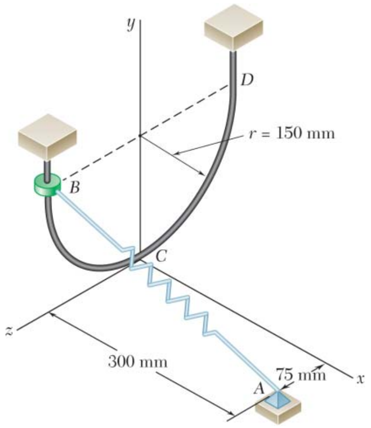
PROBLEM 13.64 (Continued)

(*b*) *Velocity as the collar reaches C.*

Conservation of energy: $T_A + V_A = T_C + V_C$

$$0 + 3.00 + 0 = 1.00 v_C^2 + 0 + 0$$

$$v_C^2 = 3.00\ \text{m}^2/\text{s}^2 \qquad \mathbf{v}_C = 1.732\ \text{m/s} \uparrow \blacktriangleleft$$



PROBLEM 13.65

A 500-g collar can slide without friction along the semicircular rod *BCD*. The spring is of constant 320 N/m and its undeformed length is 200 mm. Knowing that the collar is released form rest at *B*, determine (*a*) the speed of the collar as it passes through *C*, (*b*) the force exerted by the rod on the collar at *C*.

SOLUTION

Given:

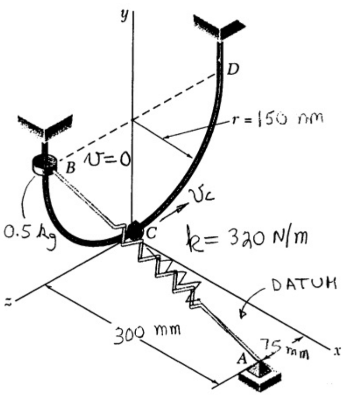
$m = 500$ g

$k = 320$ N/m

Undeformed length of spring $= 200$ mm

$v_B = 0$

Sketch:



Finding Speed at C:

$$L_{AB} = \sqrt{(300)^2 + (150)^2 + (75)^2} = 343.69318 \text{ mn}$$

$$k = 320 \text{ N/m}$$

At B

$$T_B = 0$$

$$V_B = (V_B)_e + (V_B)_g$$

$$\Delta L_{AB} = 343.69318 \text{ mm} - 200 \text{ mm}$$

$$\Delta L_{AB} = 143.69318 \text{ mm} = 0.14369318 \text{ m}$$

$$(V_B)_e = \frac{1}{2}k(\Delta L_{AB})^2 = \frac{1}{2}(320 \text{ N/m})(0.1436932 \text{ m})^2$$

$$(V_B)_e = 3.303637 \text{ J}$$

$$(V_B)_g = Wr = (0.5 \text{ kg})(9.81 \text{ m/s}^2)(0.15 \text{ m}) = 0.73575 \text{ J}$$

$$V_B = (V_B)_e + (V_B)_g = 3.303637 \text{ J} + 0.73575 \text{ J} = 4.03939 \text{ J}$$

At C

$$T_C = \frac{1}{2}mv_C^2 = \frac{1}{2}(0.5 \text{ kg})(v_C^2)$$

$$T_C = 0.25v_C^2$$

$$(V_C)_e = \frac{1}{2}k(\Delta L_{AC})^2$$

PROBLEM 13.65 (Continued)

$$\Delta L_{AC} = 309.23\text{ mm} - 200\text{ mm} = 109.23\text{ mm} = 0.10923\text{ m}$$

$$(V_C)_e = \frac{1}{2}(320\text{ N/m})(0.10923\text{ m})^2 = 1.90909\text{ J}$$

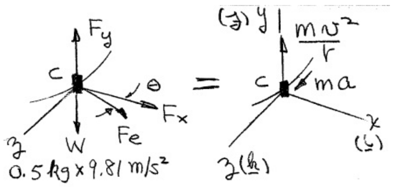
$$T_B + V_B = T_C + V_C$$

$$0 + 4.0394 = 0.25v_C^2 + 1.90909$$

(a) $$v_C^2 = \frac{4.0394 - 1.90909}{0.25} = 8.5212\text{ m}^2/\text{s}^2$$ $v_C = 2.92\text{ m/s}$ ◀

Find force of rod on collar *AC*

Free Body Diagram



$$F_z = 0 \quad \text{(no friction)}$$

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$$

$$\theta = \tan^{-1}\frac{75}{300} = 14.04°$$

$$\mathbf{F_e} = (k\Delta L_{AC})(\cos\theta\mathbf{i} + \sin\theta\mathbf{k})$$

$$\mathbf{F_e} = (320)(0.10923)(\cos 14.04°\mathbf{i} + \sin 14.04°\mathbf{k})$$

$$\mathbf{F_e} = 33.909\mathbf{i} + 8.4797\mathbf{k}\text{ (N)}$$

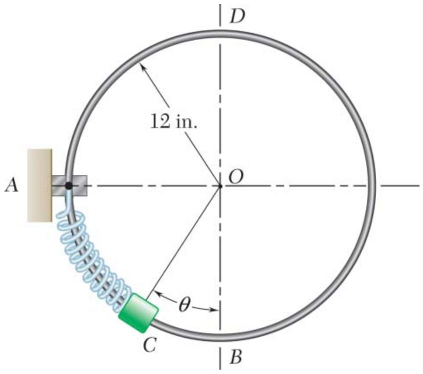
$$\Sigma\mathbf{F} = (F_x + 33.909)\mathbf{i} + (F_y - 4.905)\mathbf{j} + 8.4797\mathbf{k} = \frac{mv^2}{r}\mathbf{j} + mg\mathbf{k}$$

$$F_x + 33.909\text{ N} = 0 \qquad F_y = 4.905\text{ N} + (0.5)\frac{(8.5212\text{ m}^2/\text{s}^2)}{0.15\text{ m}}$$

$$F_x = -33.909\text{ N}$$

$$F_y = 33.309\text{ N}$$

(b) $\mathbf{F} = -33.9\text{ N}\,\mathbf{i} + 33.3\text{ N}\,\mathbf{j}$ ◀

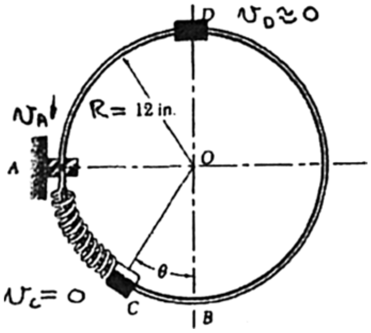


PROBLEM 13.66

A thin circular rod is supported in a *vertical plane* by a bracket at *A*. Attached to the bracket and loosely wound around the rod is a spring of constant $k = 3$ lb/ft and undeformed length equal to the arc of circle *AB*. An 8-oz collar *C*, not attached to the spring, can slide without friction along the rod. Knowing that the collar is released from rest at an angle θ with the vertical, determine (*a*) the smallest value of θ for which the collar will pass through *D* and reach Point *A*, (*b*) the velocity of the collar as it reaches Point *A*.

SOLUTION

(*a*) Smallest angle θ occurs when the velocity at *D* is close to zero.



$R = 12$ in. $= 1$ ft

$$v_C = 0 \qquad v_D = 0$$
$$T_C = 0 \qquad T_D = 0$$
$$V = V_e + V_g$$

Point *C*:

$$\Delta L_{BC} = (1\text{ ft})(\theta) = \theta\text{ ft}$$
$$(V_C)_e = \frac{1}{2}k(\Delta L_{BC})^2$$
$$(V_C)_e = \frac{3}{2}\theta^2$$
$$(V_C)_g = WR(1-\cos\theta)$$
$$(V_C)_g = \left(\frac{8\text{ oz}}{16\text{ oz/lb}}\right)(1\text{ ft})(1-\cos\theta)$$
$$(V_C)_g = \frac{1}{2}(1-\cos\theta)$$
$$V_C = (V_C)_e + (V_C)_g = \frac{3}{2}\theta^2 + \frac{1}{2}(1-\cos\theta)$$

Point *D*:

$$(V_D)_e = 0 \quad \text{(spring is unattached)}$$
$$(V_D)_g = W(2R) = (2)(0.5\text{ lb})(1\text{ ft}) = 1\text{ lb}\cdot\text{ft}$$
$$T_C + V_C = T_D + V_D \qquad 0 + \frac{3}{2}\theta^2 + \frac{1}{2}(1-\cos\theta) = 1$$
$$(1.5)\theta^2 - (0.5)\cos\theta = 0.5$$

By trial, $\theta = 0.7592$ rad $\qquad \theta = 43.5°$ ◀

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PROBLEM 13.66 (Continued)

(*b*) Velocity at *A*:

Point *D*:

$$V_D = 0 \quad T_D = 0 \quad V_D = 1\ \text{lb}\cdot\text{ft}[\text{see Part }(a)]$$

Point *A*:

$$T_A = \frac{1}{2}mv_A^2 = \frac{1}{2}\frac{(0.5\ \text{lb})}{(32.2\ \text{ft/s}^2)}v_A^2$$

$$T_A = 0.0077640v_A^2$$

$$V_A = (V_A)_g = W(R) = (0.5\ \text{lb})(1\ \text{ft}) = 0.5\ \text{lb}\cdot\text{ft}$$

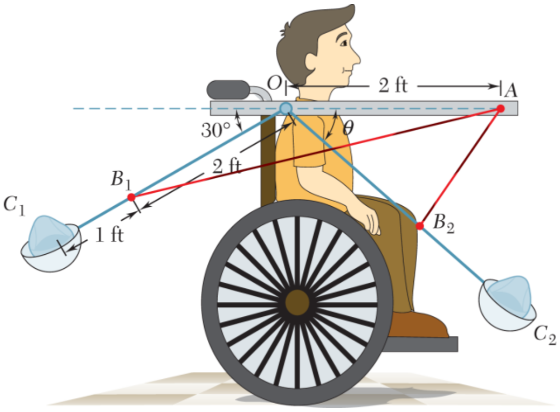
$$T_A + V_A = T_D + V_D$$

$$0.0077640v_A^2 + 0.5 = 0 + 1$$

$$v_A^2 = 64.4\ \text{ft}^2/\text{s}^2$$

$\mathbf{v}_A = 8.02$ ft/s ↓ ◀

PROBLEM 13.67



Cornhole is a game that requires you to toss beanbags through a hole in a wooden board. People with limited arm mobility often have difficulty enjoying this favorite tailgating activity. An adapted launching device attaches to a wheelchair so that points O and A are fixed. The device mimics an underhand throw by utilizing an elastic band to power the arm OC, which rotates about pin O. The elastic cord has an unstretched length of 1 ft and is attached to fixed point A and to point B on the arm. The combined weight of the beanbag and holder at C is 4 lbs, and you can neglect the weight of the rod OB. Knowing that the starting position is 30 degrees from the horizontal as shown in the figure, determine the spring constant if the velocity of the bean bag is 31 ft/s when the bag is released at an angle of θ = 45 degrees.

SOLUTION

Given:

$l_O = 1$ ft

$m = \dfrac{4 \text{ lbs}}{32.2 \text{ ft/s}^2} = 0.1242$ slugs

$v_2 = 31$ ft/s

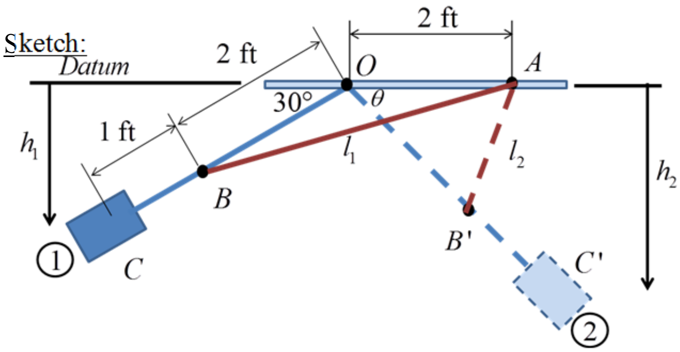
$\theta_2 = 45°$

l_i = length of cord at positon i

s_i = stretch of cord in position i

h_i = height of beanbag at position i

Sketch:



At Position 1:

From law of cosines:

$$l_1^2 = 2^2 + 2^2 + 2(2)(2)\cos 30°$$
$$l_1 = 3.864 \text{ ft}$$

Stretch in cord:

$$s_1 = l_1 - l_0$$
$$s_1 = 2.864 \text{ ft}$$

Height of beanbag:

$$h_1 = -3\sin 30°$$
$$h_1 = -1.5 \text{ ft}$$

At Position 2:

From law of cosines:

$$l_2^2 = 2^2 + 2^2 - 2(2)(2)\cos 45°$$
$$l_2 = 1.531 \text{ ft}$$

PROBLEM 13.67 (Continued)

Stretch in cord: $s_2 = l_2 - l_0$

$s_2 = 0.5307 \text{ ft}$

Height of beanbag: $h_2 = -3\sin 45°$

$h_2 = -2.1213 \text{ ft}$

Work Energy from position 1 to 2:

$$\cancel{T_1} + V_{g_1} + V_{e_1} + \cancel{U_{1\to 2}^{NC}} = T_2 + V_{g_2} + V_{e_2} \qquad \text{where:}$$

$$V_{g_1} = mgh_1,\ V_{e_1} = \frac{1}{2}ks_1^2$$

$$T_2 = \frac{1}{2}mv_2^2,\ V_{g_2} = mgh_2$$

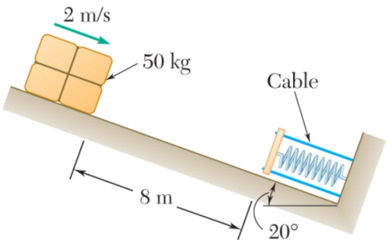
$$V_{e_2} = \frac{1}{2}ks_2^2$$

Therefore: $mgh_1 + \frac{1}{2}ks_1^2 = \frac{1}{2}mv_2^2 + mgh_2 + \frac{1}{2}ks_2^2$

Substitute known values:

$$(0.1242)(32.2)(-1.5) + \frac{1}{2}k(2.864)^2 = \frac{1}{2}(0.1242)31^2 + (0.1242)(32.2)(-2.1213) + \frac{1}{2}k(0.5307)^2$$

Solve for *k*: $k = 14.447 \text{ lb/ft}$ $\qquad k = 1.204 \text{ lb/in}$ ◀



PROBLEM 13.68

A spring is used to stop a 50-kg package which is moving down a 20º incline. The spring has a constant $k = 30$ kN/m and is held by cables so that it is initially compressed 50 mm. Knowing that the velocity of the package is 2 m/s when it is 8 m from the spring and neglecting friction, determine the maximum additional deformation of the spring in bringing the package to rest.

SOLUTION

Let position 1 be the starting position 8 m from the end of the spring when it is compressed 50 mm by the cable. Let position 2 be the position of maximum compression. Let x be the additional compression of the spring. Use the principle of conservation of energy. $T_1 + V_1 = T_2 + V_2$.

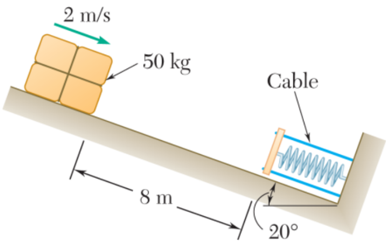
Position 1:
$$T_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(50)(2)^2 = 100 \text{ J}$$
$$V_{1g} = mgh_1 = (50)(9.81)(8 \sin 20°) = 1342.09 \text{ J}$$
$$V_{1e} = \frac{1}{2}ke_1^2 = \frac{1}{2}(30 \times 10^3)(0.050)^2 = 37.5 \text{ J}$$

Position 2:
$$T_2 = \frac{1}{2}mv_2^2 = 0 \quad \text{since} \quad v_2 = 0.$$
$$V_{2g} = mgh_2 = (50)(9.81)(-x \sin 20°) = -167.76x$$
$$V_{2e} = \frac{1}{2}ke_2^2 = \frac{1}{2}(30 \times 10^3)(0.05 + x)^2 = 37.5 + 1500x + 15000x^2$$

Principle of conservation of energy:
$$100 + 1342.09 + 37.5 = -167.61x + 37.5 + 1500x + 15000x^2$$
$$15{,}000x^2 + 1332.24x - 1442.09 = 0$$

Solving for x,
$$x = 0.26882 \quad \text{and} \quad -0.357\,64$$

$x = 0.269$ m ◀



PROBLEM 13.69

Solve Problem 13.68 assuming the kinetic coefficient of friction between the package and the incline is 0.2.

PROBLEM 13.68 A spring is used to stop a 50-kg package which is moving down a 20° incline. The spring has a constant $k = 30$ kN/m and is held by cables so that it is initially compressed 50 mm. Knowing that the velocity of the package is 2 m/s when it is 8 m from the spring and neglecting friction, determine the maximum additional deformation of the spring in bringing the package to rest.

SOLUTION

Let position 1 be the starting position 8 m from the end of the spring when it is compressed 50 mm by the cable. Let position 2 be the position of maximum compression. Let x be the additional compression of the spring. Use the principle of work and energy. $T_1 + V_1 + U_{1\to 2} = T_2 + V_2$

Position 1.

$$T_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(50)(2)^2 = 100 \text{ J}$$

$$V_{1g} = mgh_1 = (50)(9.81)(8\sin 20°) = 1342.09 \text{ J}$$

$$V_{1e} = \frac{1}{2}ke_1^2 = \frac{1}{2}(30\times 10^3)(0.05)^2 = 37.5 \text{ J}$$

Position 2.

$$T_2 = \frac{1}{2}mv_2^2 = 0 \quad \text{since} \quad v_2 = 0.$$

$$V_{2g} = mgh_2 = (50)(9.81)(-x\sin 20°) = -167.76x$$

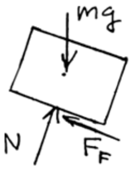
$$V_{2e} = \frac{1}{2}ke_2^2 = \frac{1}{2}(30\times 10^3)(0.05 + x)^2 = 37.5 + 1500x + 15{,}000x^2$$

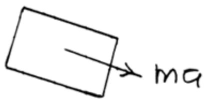
Work of the friction force.

$$+\nearrow \; \Sigma F_n = 0$$

$$N - mg\cos 20° = 0$$

$$\begin{aligned} N &= mg\cos 20° \\ &= (50)(9.81)\cos 20° \\ &= 460.92 \text{ N} \\ F_f &= \mu_k N \\ &= (0.2)(460.92) \\ &= 92.184 \\ U_{1\to 2} &= -F_f d \\ &= -92.184(8 + x) \\ &= -737.47 - 92.184x \end{aligned}$$





PROBLEM 13.69 (Continued)

Principle of work and energy: $T_1 + V_1 + U_{1-2} = T_2 + V_2$

$$100 + 1342.09 + 37.5 - 737.47 - 92.184x$$

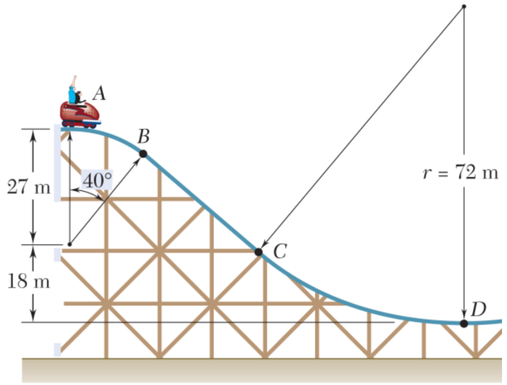
$$= -167.76x + 37.5 + 1500x + 15,000x^2$$

$$15,000\,x^2 + 1424.42x - 704.62 = 0$$

Solving for *x*,

$$x = 0.17440 \quad \text{and} \quad -0.26936$$

$x = 0.1744$ m ◀



PROBLEM 13.70

A section of track for a roller coaster consists of two circular arcs *AB* and *CD* joined by a straight portion *BC*. The radius of *AB* is 27 m and the radius of *CD* is 72 m. The car and its occupants, of total mass 250 kg, reach Point *A* with practically no velocity and then drop freely along the track. Determine the normal force exerted by the track on the car as the car reaches point *B*. Ignore air resistance and rolling resistance.

SOLUTION

Calculate the speed of the car as it reaches Point *B* using the principle of conservation of energy as the car travels from position *A* to position *B*.

Position *A*: $v_A = 0, \quad T_A = \frac{1}{2}mv_A^2 = 0, \quad V_A = 0$ (datum)

Position *B*: $V_B = -mgh$

where *h* is the decrease in elevation between *A* and *B*.

$$T_B = \frac{1}{2}mv_B^2$$

Conservation of energy: $T_A + V_A = T_B + V_B$:

$$0 + 0 = \frac{1}{2}mv_B^2 - mgh$$

$$v_B^2 = 2gh$$

$$= (2)(9.81\ \text{m/s}^2)(27\ \text{m})(1-\cos 40^\circ)$$

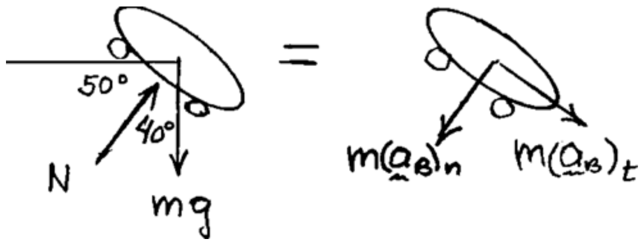
$$= 123.94\ \text{m}^2/\text{s}^2$$

Normal acceleration at *B*:

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{123.94\ \text{m}^2/\text{s}^2}{27\ \text{m}} = 4.59\ \text{m/s}^2$$

$$(\mathbf{a}_B)_n = 4.59\ \text{m/s}^2 \swarrow 50^\circ$$

Apply Newton's second law to the car at *B*.



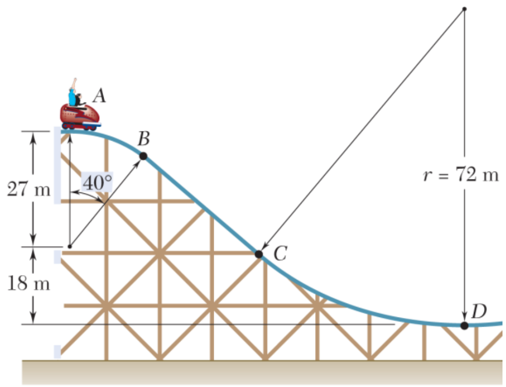
PROBLEM 13.70 (Continued)

$$\measuredangle 50^\circ \Sigma F_n = ma_n: \quad N - mg\cos 40^\circ = -ma_n$$

$$N = mg\cos 40^\circ - ma_n = m(g\cos 40^\circ - a_n)$$

$$= (250\text{ kg})[(9.81\text{ m/s}^2)\cos 40^\circ - 4.59\text{ m/s}^2]$$

$$= 1878.7 - 1147.5 \qquad N = 731\text{ N} \blacktriangleleft$$



PROBLEM 13.71

A section of track for a roller coaster consists of two circular arcs *AB* and *CD* joined by a straight portion *BC*. The radius of *AB* is 27 m and the radius of *CD* is 72 m. The car and its occupants, of total mass 250 kg, reach Point *A* with practically no velocity and then drop freely along the track. Determine the maximum and minimum values of the normal force exerted by the track on the car as the car travels from *A* to *D*. Ignore air resistance and rolling resistance.

SOLUTION

Calculate the speed of the car as it reaches Point *P*, any point on the roller coaster track. Apply the principle of conservation of energy.

Position *A*: $v_A = 0, \quad T_A = \frac{1}{2}mv_A^2 = 0, \quad V_A = 0$ (datum)

Position *P*: $V_P = -mgh$

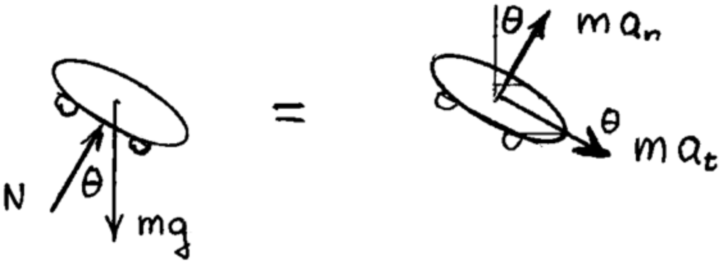
where *h* is the decrease in elevation along the track.

$$T_P = \frac{1}{2}mv^2$$

Conservation of energy: $T_A + V_A = T_P + V_P$

$$0 + 0 = \frac{1}{2}mv^2 - mgh \qquad v^2 = 2gh \qquad (1)$$

Calculate the normal force using Newton's second law. Let θ be the slope angle of the track.



$$\Sigma F_n = ma_n: \quad N - mg\cos\theta = ma_n$$

$$N = mg\cos\theta + ma_n \qquad (2)$$

Over portion *AB* of the track, $h = \rho(1 - \cos\theta)$

and $a_n = -\frac{mv^2}{\rho}$

where ρ is the radius of curvature. ($\rho = 27$ m)

$$N = mg\cos\theta - \frac{2mg\rho(1-\cos\theta)}{\rho} = mg(3\cos\theta - 2)$$

At Point A ($\theta = 0$) $\quad N_A = mg = (250)(9.81) = 2452.5 \text{ N}$

At Point B ($\theta = 40°$) $\quad N_B = (2452.5)(3\cos 40° - 2)$

$$N_B = 731 \text{ N}$$

Over portion BC, $\quad \theta = 40°, \quad a_n = 0 \quad$ (straight track)

$$N_{BC} = mg\cos 40° = 2452.5\cos 40°$$
$$N_{BC} = 1879 \text{ N}$$

Over portion CD, $\quad h = h_{\max} - r(1-\cos\theta)$

and $\quad a_n = \dfrac{mv^2}{r}$

where r is the radius of curvature. ($r = 72$ m)

$$\begin{aligned} N &= mg\cos\theta + \frac{2mgh}{r} \\ &= mg\cos\theta + 2mg\left(\frac{h_{\max}}{r} - 1 - \cos\theta\right) \\ &= mg\left(3\cos\theta - 2 + \frac{2h_{\max}}{r}\right) \end{aligned}$$

which is maximum at Point D, where

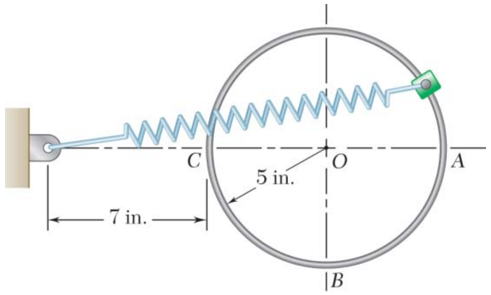
$$N_D = mg\left(1 + \frac{2h_{\max}}{r}\right)$$

Data: $\quad h_{\max} = 27 + 18 = 45 \text{ m}, \quad r = 72 \text{ m}$

$$N_D = (2452.5)\left[1 + \frac{(2)(45)}{72}\right] = 5520 \text{ N}$$

Summary: minimum (just above B): 731 N ◀

maximum (at D): 5520 N ◀



PROBLEM 13.72

A 1-lb collar is attached to a spring and slides without friction along a circular rod in a *vertical* plane. The spring has an undeformed length of 5 in. and a constant $k = 10$ lb/ft. Knowing that the collar is released from being held at A determine the speed of the collar and the normal force between the collar and the rod as the collar passes through B.

SOLUTION

For the collar, $$m = \frac{W}{g} = \frac{1}{32.2} = 0.031056 \text{ lb}\cdot\text{s}^2/\text{ft}$$

For the spring, $$k = 10 \text{ lb/ft} \quad l_0 = 5 \text{ in.}$$

At A: $$\ell_A = 7 + 5 + 5 = 17 \text{ in.}$$
$$\ell_\Delta - \ell_0 = 12 \text{ in.} = 1 \text{ ft}$$

At B: $$\ell_B = \sqrt{(7+5)^2 + 5^2} = 13 \text{ in.}$$
$$\ell_B - \ell_0 = 1.8 \text{ in.} = \frac{2}{3}\text{ft}$$

Velocity of the collar at B.

Use the principle of conservation of energy.

$$T_A + V_A = T_B + V_B$$

Where $$T_A = \frac{1}{2}mv_A^2 = 0$$

$$V_A = \frac{1}{2}k(\ell_A - \ell_0)^2 + W(0)$$
$$= \frac{1}{2}(10)(1)^2 + 0 = 5 \text{ ft}\cdot\text{lb}$$
$$T_B = \frac{1}{2}mv_B^2 = \frac{1}{2}(0.031056)v_B^2 = 0.015528v_B^2$$
$$V_B = \frac{1}{2}k(\ell_B - \ell_0)^2 + Wh$$
$$= \frac{1}{2}(10)\left(\frac{2}{3}\right)^2 + (1)\left(-\frac{5}{12}\right)$$
$$= 1.80556 \text{ ft}\cdot\text{lb}$$
$$0 + 5 = 0.015528v_B^2 = 1.80556$$
$$v_B^2 = 205.72 \text{ ft}^2/\text{s}^2$$

$v_B = 14.34$ ft/s ◀

PROBLEM 13.72 (Continued)

Forces at *B*.

$$F_s = k(\ell_B - \ell_0) = (10)\left(\frac{2}{3}\right) = 6.6667 \text{ lb.}$$

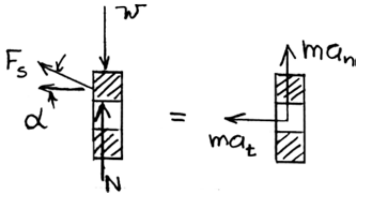
$$\sin\alpha = \frac{5}{13}$$

$$\rho = 5 \text{ in.} = \frac{5}{12}\text{ft}$$

$$ma_n = \frac{mv_B^2}{\rho}$$

$$= \frac{(0.031056)(205.72)}{5/12}$$

$$= 15.3332 \text{ lb}$$



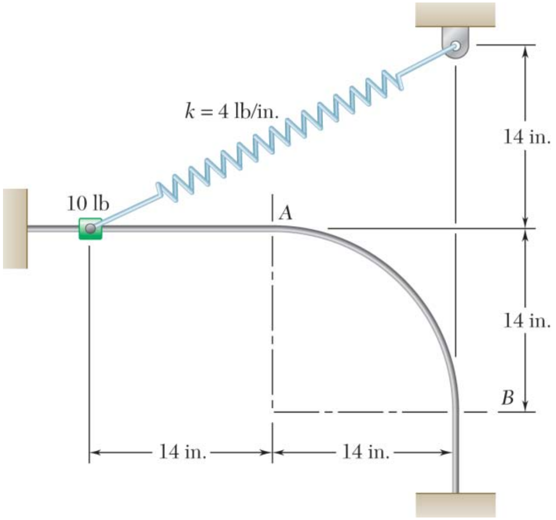
$$+\uparrow \Sigma F_y = ma_y: \quad F_s \sin\alpha - W + N = ma_n$$

$$N = ma_n + W - F_s \sin\alpha$$

$$= 15.3332 + 1 - (6.6667)\left(\frac{5}{13}\right)$$

$$N = 13.769 \text{ lb}$$

$\mathbf{N} = 13.77 \text{ lb} \uparrow$ ◀



PROBLEM 13.73

A 10-lb collar is attached to a spring and slides without friction along a fixed rod in a vertical plane. The spring has an undeformed length of 14 in. and a constant $k = 4$ lb/in. Knowing that the collar is released from rest in the position shown, determine the force exerted by the rod on the collar at (*a*) Point *A*, (*b*) Point *B*. Both these points are on the curved portion of the rod.

SOLUTION

Mass of collar: $m = \dfrac{W}{g} = \dfrac{10}{32.2} = 0.31056 \text{ lb}\cdot\text{s}^2/\text{ft}$

Let position 1 be the initial position shown, and calculate the potential energies of the spring for positions 1, *A*, and *B*. $l_0 = 14$ in.

$$l_1 = \sqrt{(14+14)^2 + (14)^2} = 31.305 \text{ in.}$$

$$x_1 = l_1 - l_0 = 31.305 - 14 = 17.305 \text{ in.}$$

$$(V_1)_e = \frac{1}{2}kx_1^2 = \frac{1}{2}(4)(17.305)^2 = 598.92 \text{ in}\cdot\text{lb} = 49.910 \text{ ft}\cdot\text{lb}$$

$$l_A = \sqrt{(14)^2 + (14)^2} = 19.799 \text{ in.}$$

$$x_A = l_A - l_0 = 19.799 - 14 = 5.799 \text{ in.}$$

$$(V_A)_e = \frac{1}{2}kx_A^2 = \frac{1}{2}(4)(5.799)^2 = 67.257 \text{ in}\cdot\text{lb} = 5.605 \text{ ft}\cdot\text{lb}$$

$$l_B = 14 + 14 = 28 \text{ in.}$$

$$x_B = l_B - l_0 = 28 - 14 = 14 \text{ in.}$$

$$(V_B)_e = \frac{1}{2}kx_B^2 = \frac{1}{2}(4)(14)^2 = 392 \text{ in}\cdot\text{lb} = 32.667 \text{ ft}\cdot\text{lb}$$

Gravitational potential energies: Datum at level *A*.

$$(V_1)_g = 0 \quad (V_A)_g = 0$$

$$(V_B)g = Wy = (10 \text{ lb})(-14 \text{ in.}) = -140 \text{ in}\cdot\text{lb} = -11.667 \text{ ft}\cdot\text{lb}$$

PROBLEM 13.73 (Continued)

Total potential energies: $V = V_e + V_g$

$$V_1 = 49.910\text{ ft}\cdot\text{lb},\ \ V_A = 5.605\text{ ft}\cdot\text{lb},\ \ V_B = 21.0\text{ ft}\cdot\text{lb}$$

Kinetic energies: $T_1 = 0$

$$T_A = \frac{1}{2}mv_A^2 = \frac{1}{2}(0.31056)v_A^2 = 0.15528v_A^2$$

$$T_B = \frac{1}{2}mv_B^2 = \frac{1}{2}(0.31056)v_B^2 = 0.15528v_B^2$$

Conservation of energy: $T_1 + V_1 = T_A + V_A$:

$$0 + 49.910 = 0.15528v_A^2 + 5.605 \qquad v_A^2 = 285.32\text{ ft}^2/\text{s}^2$$

Conservation of energy: $T_1 + V_1 = T_B + V_B$

$$0 + 49.910 = 0.15528v_B^2 + 21.0 \qquad v_B^2 = 186.18\text{ ft}^2/\text{s}^2$$

Normal accelerations at *A* and *B*. $a_n = v^2/\rho$

$$\rho = 14\text{ in.} = 1.16667\text{ ft}$$

$$(a_A)_n = \frac{285.32\text{ ft}^2/\text{s}^2}{1.16667\text{ ft}} \qquad (\mathbf{a}_A)_n = 244.56\text{ ft/s}^2 \downarrow$$

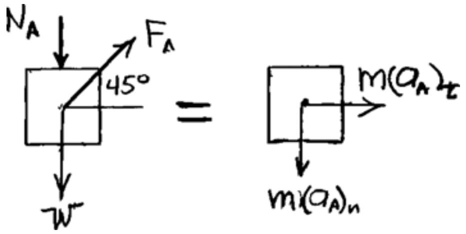
$$(a_B)_n = \frac{189.10\text{ ft}^2/\text{s}^2}{1.16667\text{ ft}} \qquad (\mathbf{a}_B)_n = 159.58\text{ ft/s}^2 \longleftarrow$$

Spring forces at *A* and *B*: $F = kx$

$$F_A = (4\text{ lb/in.})(5.799\text{ in.}) \qquad \mathbf{F}_A = 23.196\text{ lb} \measuredangle 45^\circ$$

$$F_B = (4\text{ lb/in.})(14\text{ in.}) \qquad \mathbf{F}_B = 56.0\text{ lb} \uparrow$$

To determine the forces $(\mathbf{N}_A$ and $\mathbf{N}_B)$ exerted by the rod on the collar, apply Newton's second law.



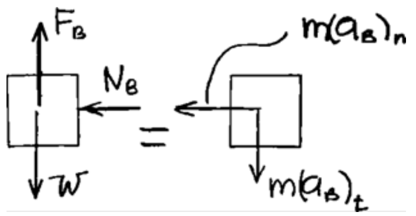
(*a*) At Point *A*:

$$\downarrow \Sigma F = m(a_A)_n:$$

$$W + N_A - F_A \sin 45^\circ = m(a_A)_n$$

$$10 + N_A - 23.196 \sin 45^\circ = (0.31056)(244.56)$$

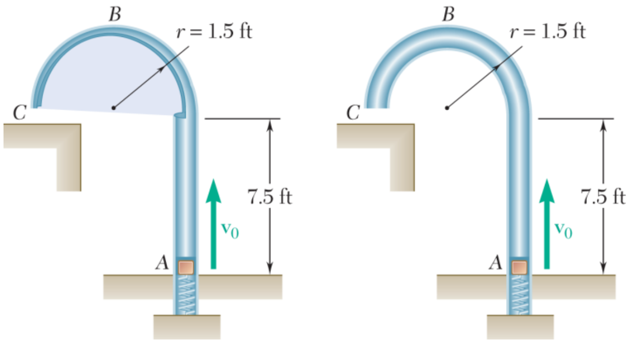
$$\mathbf{N}_A = 82.4\text{ lb} \downarrow \blacktriangleleft$$



(*b*) At Point *B*:

$$\overset{+}{\longleftarrow} \Sigma F = m(a_B)_n:$$

$$N_B = (0.31056)(159.58) \qquad \mathbf{N}_B = 49.6\text{ lb} \longleftarrow \blacktriangleleft$$

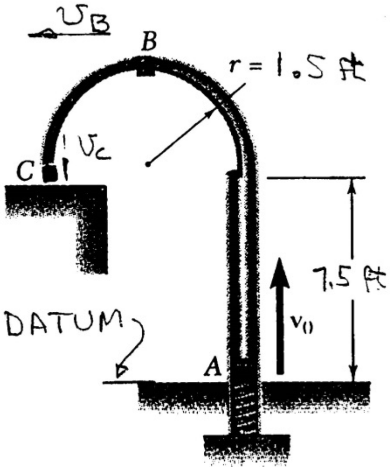


PROBLEM 13.74

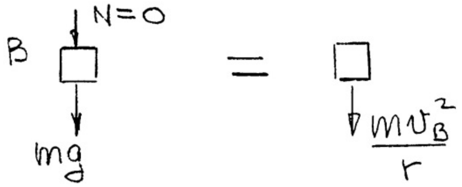
An 8-oz package is projected upward with a velocity $\mathbf{v}_0$ by a spring at *A*; it moves around a frictionless loop and is deposited at *C*. For each of the two loops shown, determine (*a*) the smallest velocity $\mathbf{v}_0$ for which the package will reach *C*, (*b*) the corresponding force exerted by the package on the loop just before the package leaves the loop at *C*.

SOLUTION

Loop 1



(*a*) The smallest velocity at *B* will occur when the force exerted by the tube on the package is zero.



$$+\downarrow\ \Sigma F = 0 + mg = \frac{mv_B^2}{r}$$

$$v_B^2 = rg = 1.5\text{ ft}(32.2\text{ ft/s}^2)$$

$$v_B^2 = 48.30$$

At *A* $\quad T_A = \frac{1}{2}mv_0^2$

$$V_A = 0 \quad \left(8\text{ oz} = 0.5\text{ lb} \Rightarrow = \frac{0.5}{32.2} = 0.01553\right)$$

At *B* $\quad T_B = \frac{1}{2}mv_B^2 = \frac{1}{2}m(48.30) = 24.15\text{ m}$

$$V_B = mg(7.5 + 1.5) = 9mg = 9(0.5) = 4.5\text{ lb}\cdot\text{ft}$$

$$T_A + V_A = T_B + V_B:\quad \frac{1}{2}(0.01553)v_0^2 = 24.15(0.01553) + 4.5$$

$$v_0^2 = 627.82 \qquad v_0 = 25.056 \qquad\qquad v_0 = 25.1\text{ ft/s} \blacktriangleleft$$

At *C*

$$T_C = \frac{1}{2}mv_C^2 = 0.007765v_C^2 \qquad V_C = 7.5mg = 7.5(0.5) = 3.75$$

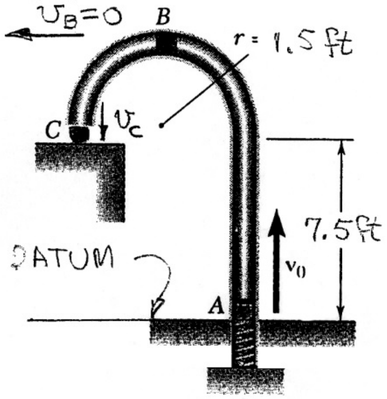
$$T_A + V_A = T_C + V_C:\quad 0.007765v_0^2 = 0.007765v_C^2 + 3.75$$

$$0.007765(25.056)^2 - 3.75 = 0.007765v_C^2$$

$$v_C^2 = 144.87$$

PROBLEM 13.74 (Continued)

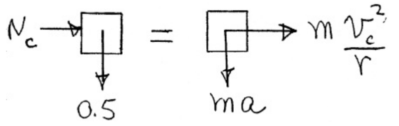
Loop 2



$v_B = 0$
B
r = 1.5 ft
C
v_c
7.5 ft
DATUM
A
v_0

(2)

(*b*)



N_c
0.5
=
$m\frac{v_c^2}{r}$
ma

$$\xrightarrow{+} \Sigma F = ma_n: \quad N = 0.01553\frac{(144.87)}{1.5}$$

$$N = 1.49989$$

{Package in tube} $N_C = 1.500 \text{ lb} \longleftarrow$ ◀

(*a*) At *B*, tube supports the package so,

$$v_B \approx 0$$

$$v_B = 0, \; T_B = 0 \qquad V_B = mg(7.5 + 1.5)$$

$$= 4.5 \text{ lb}\cdot\text{ft}$$

$$T_A + V_A = T_B + V_B$$

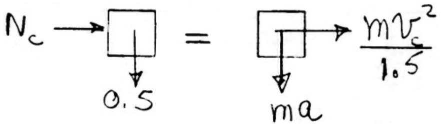
$$\frac{1}{2}(0.01553)v_A^2 = 4.5 \Rightarrow v_A = 24.073$$

$v_A = 24.1$ ft/s ◀

(*b*) At *C* $\qquad T_C = 0.007765v_C^2, \; V_C = 7.5mg = 3.75$

$$T_A + V_A = T_C + V_C: \quad 0.007765(24.073)^2 = 0.007765v_C^2 + 3.75$$

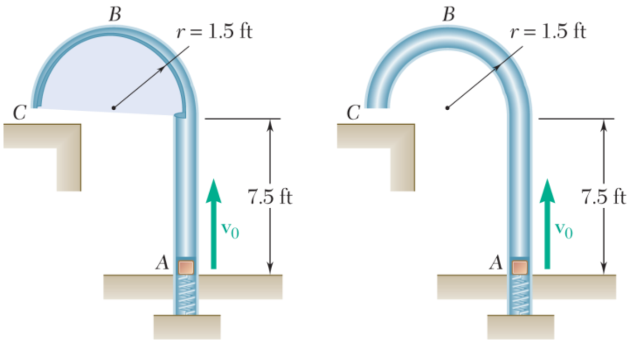
$$v_C^2 = 96.573$$



N_c
0.5
=
$\frac{mv_c^2}{1.5}$
ma

$$N_C = 0.01553\left(\frac{96.573}{1.5}\right) = 0.99985$$

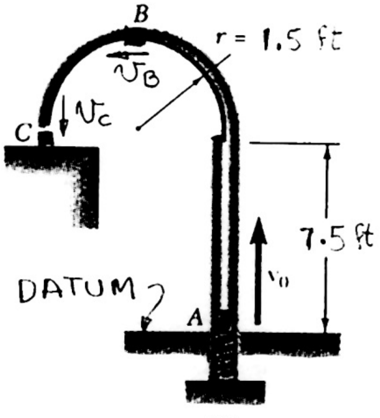
{Package on tube} $N_C = 1.000$ lb ◀


B
$r = 1.5$ ft
C
7.5 ft
$\mathbf{v}_0$
A
B
$r = 1.5$ ft
C
7.5 ft
$\mathbf{v}_0$
A

PROBLEM 13.75

If the package of Problem 13.74 is not to hit the horizontal surface at C with a speed greater than 10 ft/s, (*a*) show that this requirement can be satisfied only by the second loop, (*b*) determine the largest allowable initial velocity $\mathbf{v}_0$ when the second loop is used.

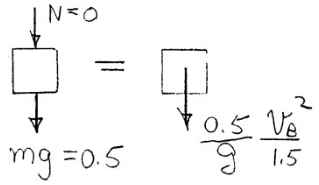
SOLUTION


B
r = 1.5 ft
v_B
C
v_C
7.5 ft
DATUM
A
v_0

(1)

(*a*) Loop 1

From Problem 13.74, at B


N=0
=
mg = 0.5
$\frac{0.5}{g}\frac{v_B^2}{1.5}$

$$v_B^2 = gr = 48.3 \text{ ft}^2/\text{s}^2 \Rightarrow v_B = 6.9498 \text{ ft/s}$$

$$T_B = \frac{1}{2}mv_B^2 = \frac{1}{2}(0.01553)(48.3) = 0.37505$$

$$V_B = mg(7.5+1.5) = (0.5)(9) = 4.5 \text{ lb}\cdot\text{ft}$$

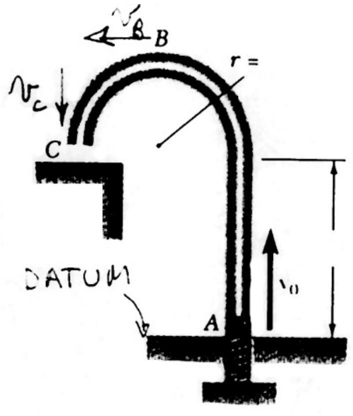
$$T_C = \frac{1}{2}mv_C^2 = \frac{1}{2}(0.01553)v_C^2 = 0.007765v_C^2$$

$$V_C = 7.5(0.5) = 3.75 \text{ lb}\cdot\text{ft}$$

$$T_B + V_B = T_C + V_C: \quad 0.37505 + 4.5 = 0.007765v_C^2 + 3.75$$

$$v_C^2 = 144.887 \Rightarrow v_C = 12.039 \text{ ft/s}$$

12.04 ft/s > 10 ft/s ⇒ Loop (1) does not work ◀


v_B
B
r =
v_C
C
DATUM
A
v_0

(2)

(*b*) Loop 2 at A $\quad T_A = \frac{1}{2}mv_0^2 = 0.007765v_0^2$

$$V_A = 0$$

At C assume $\quad v_C = 10$ ft/s

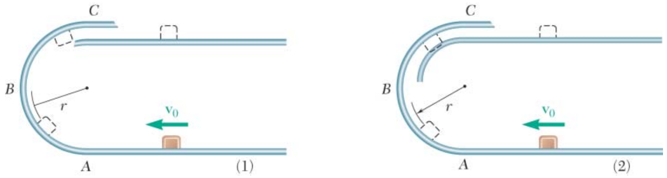
$$T_C = \frac{1}{2}mv_C^2 = 0.007765(10)^2 = 0.7765$$

$$v_C = 7.5(0.5) = 3.75$$

$$T_A + V_A = T_C + V_C: \quad 0.007765v_0^2 = 0.7765 + 3.75$$

$v_0 = 24.144$ $\qquad$ $v_0 = 24.1$ ft/s ◀

PROBLEM 13.76

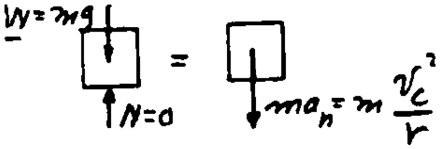


A small package of weight W is projected into a vertical return loop at A with a velocity $\mathbf{v}_0$. The package travels without friction along a circle of radius r and is deposited on a horizontal surface at C. For each of the two loops shown, determine (a) the smallest velocity $\mathbf{v}_0$ for which the package will reach the horizontal surface at C, (b) the corresponding force exerted by the loop on the package as it passes Point B.

SOLUTION

Loop 1:

(a) Newton's second law at position C:



$$+\downarrow \Sigma F = ma:$$

$$mg = m\frac{v_c^2}{r} \qquad v_c^2 = gr$$

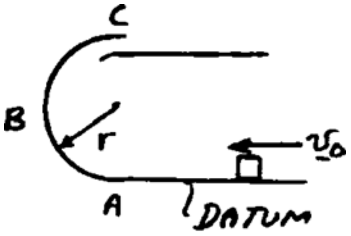
Conservation of energy between position A and B.

$$T_A = \frac{1}{2}mv_0^2$$

$$V_A = 0$$

$$T_C = \frac{1}{2}mv_C^2 = \frac{1}{2}mgr$$

$$V_C = mg(2r) = 2mgr$$



$$T_A + V_A = T_C + V_C: \quad \frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mgr + 2mgr$$

$$v_0^2 = 5gr$$

Smallest velocity $\mathbf{v}_0$: $v_0 = \sqrt{5gr} \longleftarrow$ ◀

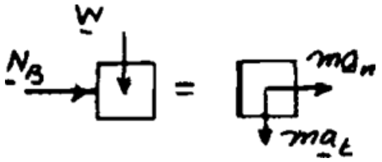
(b) Conservation of energy between positions A and B.

$$(b)\ T_B = \frac{1}{2}mv_B^2; \quad V_B = mg(r)$$

$$T_A + V_A = T_B + V_B: \quad \frac{1}{2}mv_A^2 + 0 = \frac{1}{2}mv_B^2 + mgr$$

$$\frac{1}{2}m(5gr) + 0 = \frac{1}{2}mv_B^2 + mgr \qquad v_B^2 = 3gr$$

PROBLEM 13.76 (Continued)



Newton's second law at position *B*.

$$ma_n = m\frac{v_B^2}{r} = m\frac{3gr}{r} = 3\ mg$$

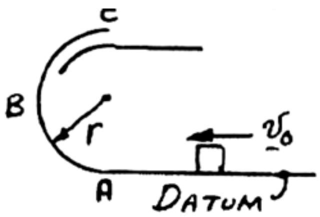
$$\xrightarrow{+} \Sigma F = \Sigma F_{eff}: \qquad N_B = 3\ mg$$

Force exerted by the loop: $N_B = 3W \longrightarrow$ ◀

Loop 2:

(*a*) At point *C*, $v_c = 0$

Conservation of energy between positions *A* and *C*.



$$T_C = \frac{1}{2}mv_C^2 = 0$$

$$V_C = mg(2r) = 2mgr$$

$$T_A + V_A = T_C + V_C:$$

$$\frac{1}{2}mv_0^2 + 0 = 0 + 2mgr$$

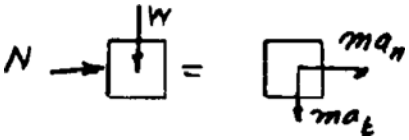
$$v_0^2 = 4gr$$

Smallest velocity $\mathbf{v}_0$: $v_0 = \sqrt{4gr} \longleftarrow$ ◀

(*b*) Conservation of energy between positions *A* and *B*.

$$T_A + V_A - T_B + V_B: \ \frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv_B^2 + mgr$$

$$\frac{1}{2}m(4gr) = \frac{1}{2}mv_B^2 + mgr \qquad v_B^2 = 2gr$$

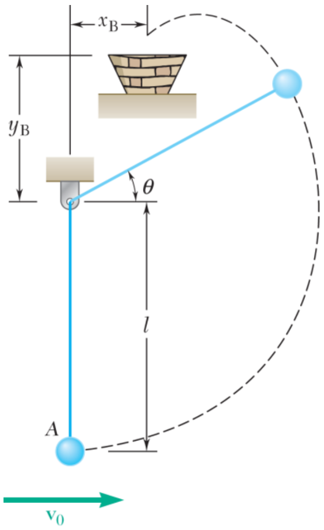


Newton's second law at position *B*.

$$ma_n = m\frac{v_B^2}{r} = m\frac{2gr}{r} = 2\ mg$$

$$\xrightarrow{+} \Sigma F = \Sigma_{\text{eff}}: \ N = 2\ mg$$

Force exerted by loop: $\mathbf{N} = 2W \longrightarrow$ ◀



PROBLEM 13.77

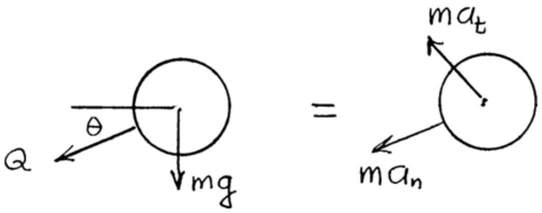
The 1 kg ball at A is suspended by an inextensible cord and given an initial horizontal velocity of 5 m/s. If $l = 0.6$ m and $x_B = 0$, determine y_B so that the ball will enter the basket.

SOLUTION

Let position 1 be at A. $v_1 = v_0$

Let position 2 be the point described by the angle where the path of the ball changes from circular to parabolic. At position 2, the tension Q in the cord is zero.

Relationship between v_2 and θ based on $Q = 0$. Draw the free body diagram.



$$+\swarrow \Sigma F = 0: \quad Q + mg\sin\theta = ma_n = \frac{mv_2^2}{l}$$

With $Q = 0, \quad v_2^2 = gl\sin\theta \quad$ or $\quad v_2 = \sqrt{gl\sin\theta}$ (1)

Relationship among v_0, v_2 and θ based on conservation of energy.

$$T_1 + V_1 = T_2 + V_2$$

$$\frac{1}{2}mv_0^2 - mgl = \frac{1}{2}mv_2^2 + mgl\sin\theta$$

$$v_0^2 - v_2^2 = 2gl(1 + \sin\theta) \qquad (2)$$

Eliminating v_2 from Eqs. (1) and (2),

$$v_0^2 - gl\sin\theta = 2gl(1+\sin\theta)$$

$$\sin\theta = \frac{1}{3}\left[\frac{v_0^2}{gl} - 2\right] = \frac{1}{3}\left[\frac{(5)^2}{(9.81)(0.6)} - 2\right] = 0.74912$$

$$\theta = 48.514°$$

From Eq. (1),

$$v_2^2 = (9.81)(0.6)\sin 48.514° = 4.4093\ \text{m}^2/\text{s}^2$$

$$v_2 = 2.0998\ \text{m/s}$$

x and y coordinates at position 2.

$$x_2 = l\cos\theta = 0.6\cos 48.514° = 0.39746\ \text{m}$$

$$y_2 = l\sin\theta = 0.6\sin 48.514° = 0.44947\ \text{m}$$

Let t_2 be the time when the ball is a position 2.

Motion on the parabolic path. The horizontal motion is

$$\dot{x} = -v_2\sin\theta = -2.0998\sin 48.514°$$

$$= -1.5730\ \text{m/s}$$

$$x = x_2 - 1.5730(t - t_2)$$

At Point B,

$$x_B = 0$$

$$0 = 0.39746 - 1.5730(t_B - t_2) \quad t_B - t_2 = 0.25267\ \text{s}$$

The vertical motion is

$$y = y_2 + v_2\cos\theta(t - t_2) - \frac{1}{2}g(t - t_2)^2$$

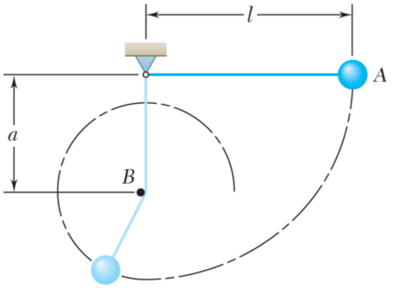
At Point B,

$$y_B = y_2 + v_2\cos\theta(t_B - t_2) - \frac{1}{2}g(t_B - t_2)^2$$

$$y_B = 0.44947 + (2.0998\cos 48.514°)(0.25267) - \frac{1}{2}(9.81)(0.25267)^2$$

$$= 0.48779\ \text{m}$$

$y_B = 0.448$ m ◀



PROBLEM 13.78

The pendulum shown is released from rest at *A* and swings through 90° before the cord touches the fixed peg *B*. Determine the smallest value of *a* for which the pendulum bob will describe a circle about the peg.

SOLUTION

Use conservation of energy from the point of release (*A*) and the top of the circle.

$$T_1 + V_1 = T_2 + V_2 \qquad \text{(1) (datum at lowest point)}$$

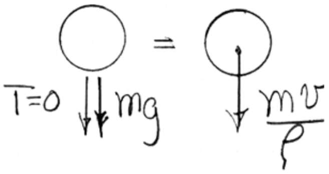
where

$$T_1 = 0; \quad V_1 = mg\ell$$

At 2 $$T_2 = \frac{1}{2}mv^2; \; V_2 = mgz = mg(2)(\ell - a)$$

Substituting into (1) $$0 + mg\ell = \frac{1}{2}mv^2 + 2mg(\ell - a) \qquad (2)$$

We need another equation – use Newton's 2nd law at the top. $\left(\text{Tension, } T_0 = 0 \text{ at top}\right)$



$$+\!\downarrow \sum F_n = ma_n \Rightarrow \cancel{m}g = \frac{\cancel{m}v^2}{\rho}$$

$$v^2 = g\rho = g(\ell - a)$$

Substituting into (2)

$$mg\ell = \frac{1}{2}mg(\ell - a) + 2mg(\ell - a)$$

$$2\ell = \ell - a + 4\ell - 4a$$

$$5a = 3\ell$$

$$a = \frac{3}{5}\ell \blacktriangleleft$$

PROBLEM 13.79*

Prove that a force $F(x, y, z)$ is conservative if, and only if, the following relations are satisfied:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \quad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y} \quad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}$$

SOLUTION

For a conservative force, Equation (13.22) must be satisfied.

$$F_x = -\frac{\partial V}{\partial x} \quad F_y = -\frac{\partial V}{\partial y} \quad F_z = -\frac{\partial V}{\partial z}$$

We now write $$\frac{\partial F_x}{\partial y} = -\frac{\partial^2 V}{\partial x \partial y} \quad \frac{\partial F_y}{\partial x} = -\frac{\partial^2 V}{\partial y \partial x}$$

Since $\dfrac{\partial^2 V}{\partial x \partial y} = \dfrac{\partial^2 V}{\partial y \partial x}$: $$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \blacktriangleleft$$

We obtain in a similar way

$$\frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y} \quad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z} \blacktriangleleft$$

PROBLEM 13.80

The force $\mathbf{F} = (yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k})/xyz$ acts on the particle $P(x, y, z)$ which moves in space. (*a*) Using the relation derived in Problem 13.79, show that this force is a conservative force. (*b*) Determine the potential function associated with **F**.

SOLUTION

(*a*)

$$F_x = \frac{yz}{xyz} \quad F_y = \frac{zx}{xyz} \quad F_z = \frac{xy}{xyz}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial\left(\frac{1}{x}\right)}{\partial y} = 0 \quad \frac{\partial F_y}{\partial x} = \frac{\partial\left(\frac{1}{y}\right)}{\partial x} = 0$$

Thus,

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

The other two equations derived in Problem 13.79 are checked in a similar way.

(*b*) Recall that

$$F_x = -\frac{\partial V}{\partial x}, \qquad F_y = -\frac{\partial V}{\partial y}, \qquad F_z = -\frac{\partial V}{\partial z}$$

$$F_x = \frac{1}{x} = -\frac{\partial V}{\partial x} \qquad V = -\ln x + f(y, z) \tag{1}$$

$$F_y = \frac{1}{y} = -\frac{\partial V}{\partial y} \qquad V = -\ln y + g(z, x) \tag{2}$$

$$F_z = \frac{1}{z} = -\frac{\partial V}{\partial z} \qquad V = -\ln z + h(x, y) \tag{3}$$

Equating (1) and (2)

$$-\ln x + f(y, z) = -\ln y + g(z, x)$$

Thus,

$$f(y, z) = -\ln y + k(z) \tag{4}$$

$$g(z, x) = -\ln x + k(z) \tag{5}$$

Equating (2) and (3)

$$-\ln z + h(x, y) = -\ln y + g(z, x)$$

$$g(z, x) = -\ln z + l(x)$$

From (5),

$$g(z, x) = -\ln x + k(z)$$

PROBLEM 13.80 (Continued)

Thus,

$$k(z) = -\ln z$$

$$l(x) = -\ln x$$

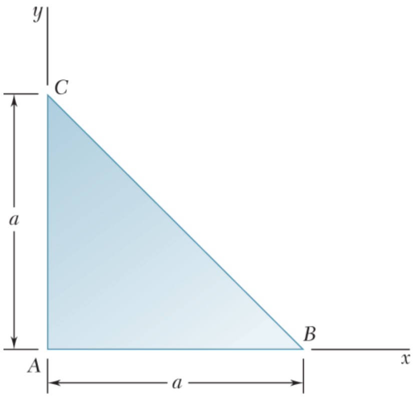
From (4),

$$f(y,z) = -\ln y - \ln z$$

Substitute for $f(y,z)$ in (1)

$$V = -\ln x - \ln y - \ln z$$

$V = -\ln xyz$ ◀

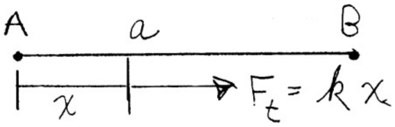


PROBLEM 13.81*

A force **F** acts on a particle $P(x, y)$ which moves in the xy plane. Determine whether **F** is a conservative force and compute the work of **F** when P describes the path $ABCA$ knowing that (*a*) $\mathbf{F} = (kx + y)\mathbf{i} + (kx + y)\mathbf{j}$, (*b*) $\mathbf{F} = (kx + y)\mathbf{i} + (x + ky)\mathbf{j}$.

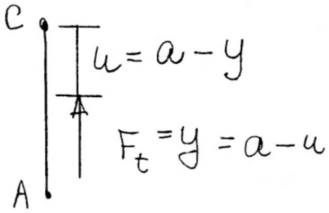
SOLUTION

(*a*)



$$U_{AB} = \int_0^a kx\,dx = k\frac{a^2}{2}$$

$$F_x = F_y, \quad \therefore \quad F \text{ is normal } to\ BC, \quad U_{BC} = 0$$



$$U_{CA} = \int_0^a -(a-u)\,du = \frac{-a^2}{2}$$

$$U_{ABCA} = (k-1)\frac{a^2}{2}, \text{ not conservative} ◀$$

(*b*) From Problem 13.79,

$$\frac{\partial F_x}{\partial y} = 1 = \frac{\partial F_y}{\partial x}$$

Conservative, $U_{ABCA} = 0$ ◀

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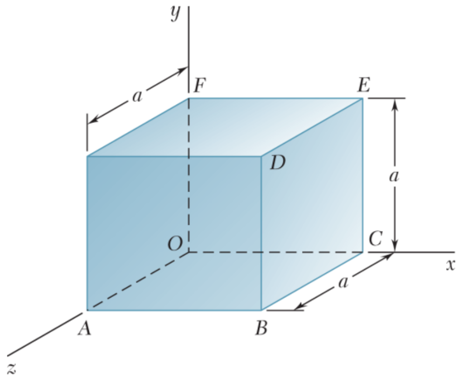


PROBLEM 13.82*

The potential function associated with a force **P** in space is known to be $V(x, y, z) = -(x^2 + y^2 + z^2)^{1/2}$. (*a*) Determine the *x*, *y*, and *z* components of **P**. (*b*) Calculate the work done by **P** from *O* to *D* by integrating along the path *OABD*, and show that it is equal to the negative of the change in potential from *O* to *D*.

SOLUTION

(*a*) $$P_x = -\frac{\partial V}{\partial x} = -\frac{\partial[-(x^2 + y^2 + z^2)^{1/2}]}{\partial x} = x(x^2 + y^2 + z^2)^{-1/2}$$ ◀

$$P_y = -\frac{\partial V}{\partial y} = -\frac{\partial[-(x^2 + y^2 + z^2)^{1/2}]}{\partial y} = y(x^2 + y^2 + z^2)^{-1/2}$$ ◀

$$P_z = -\frac{\partial V}{\partial z} = -\frac{\partial[-(x^2 + y^2 + z^2)^{1/2}]}{\partial z} = z(x^2 + y^2 + z^2)^{-1/2}$$ ◀

(*b*) $U_{OABD} = U_{OA} + U_{AB} + U_{BD}$

O–A: P_y and P_x are perpendicular to *O–A* and do no work.

Also, on *O–A* $\quad x = y = 0$ and $P_z = 1$

Thus, $$U_{O-A} = \int_0^a P_z dz = \int_0^a dz = a$$

A–B: P_z and P_y are perpendicular to *A–B* and do no work.

Also, on *A–B* $\quad y = 0, \quad z = a$ and $P_x = \dfrac{x}{(x^2 + a^2)^{1/2}}$

Thus, $$U_{A-B} = \int_0^a \frac{xdx}{(x^2 + a^2)^{1/2}}$$
$$= a(\sqrt{2} - 1)$$

B–D: P_x and P_z are perpendicular to *B–D* and do no work.

On *B–D*, $\quad k = a$

$z = a$

$$P_y = \frac{y}{(y^2 + 2a^2)^{1/2}}$$

PROBLEM 13.82* (Continued)

Thus, $$U_{BD} = \int_0^a \frac{y}{(y^2 + 2a^2)^{1/2}}\,dy = (y^2 + 2a^2)^{1/2}\Big|_0^a$$

$$U_{BD} = (a^2 + 2a^2)^{1/2} - (2a^2)^{1/2} = a(\sqrt{3} - \sqrt{2})$$

$$U_{OABD} = U_{O-A} + U_{A-B} + U_{B-D}$$

$$= a + a(\sqrt{2} - 1) + a(\sqrt{3} - \sqrt{2})$$

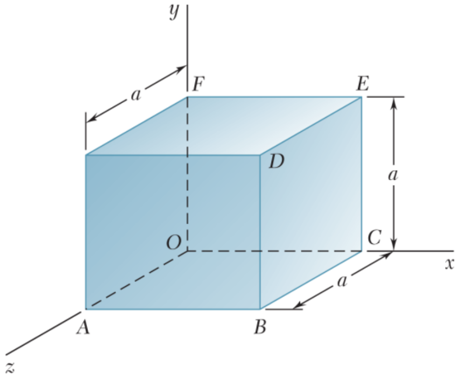
$U_{OABD} = a\sqrt{3}$ ◀

$$\Delta V_{OD} = V(a,a,a) - V(0,0,0)$$

$$= -(a^2 + a^2 + a^2)^{1/2} - 0$$

$\Delta V_{OD} = -a\sqrt{3}$ ◀

Thus, $$U_{OABD} = -\Delta V_{OD}$$

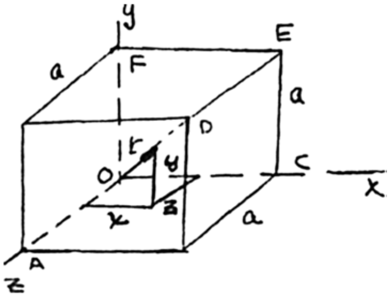


PROBLEM 13.83*

(*a*) Calculate the work done from *D* to *O* by the force **P** of Problem 13.82 by integrating along the diagonal of the cube. (*b*) Using the result obtained and the answer to part *b* of Problem 13.82, verify that the work done by a conservative force around the closed path *OABDO* is zero.

PROBLEM 13.82 The potential function associated with a force **P** in space is known to be $V(x, y, z) = -(x^2 + y^2 + z^2)^{1/2}$. (*a*) Determine the *x*, *y*, and *z* components of **P**. (*b*) Calculate the work done by **P** from *O* to *D* by integrating along the path *OABD*, and show that it is equal to the negative of the change in potential from *O* to *D*.

SOLUTION



From solution to (*a*) of Problem 13.82

$$\mathbf{P} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{1/2}}$$

(*a*) $U_{OD} = \int_O^D \mathbf{P} \cdot d\mathbf{r}$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

$$\mathbf{P} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{1/2}}$$

Along the diagonal. $x = y = z$

Thus, $\mathbf{P} \cdot dr = \frac{3x}{(3x^2)^{1/2}} = \sqrt{3}$

$$U_{O-D} = \int_0^a \sqrt{3}\, dx = \sqrt{3}a$$

$U_{OD} = \sqrt{3}a$ ◀

(*b*) $U_{OABDO} = U_{OABD} + U_{DO}$

From Problem 13.82

$U_{OABD} = \sqrt{3}a$ at left

The work done from *D* to *O* along the diagonal is the negative of the work done from *O* to *D*.

$U_{DO} = -U_{OD} = -\sqrt{3}a$ [see part (*a*)]

Thus,

$U_{OABDO} = \sqrt{3}a - \sqrt{3}a = 0$ ◀

PROBLEM 13.84*

The force $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2 + y^2 + z^2)^{3/2}$ acts on the particle $P(x, y, z)$ which moves in space. (*a*) Using the relations derived in Problem 13.79, prove that **F** is a conservative force. (*b*) Determine the potential function $V(x, y, z)$ associated with **F**.

SOLUTION

(*a*)

$$F_x = \frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$F_y = \frac{y}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\frac{\partial F_x}{\partial y} = \frac{x\left(-\frac{3}{2}\right)(2y)}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial F_y}{\partial x} = \frac{y\left(-\frac{3}{2}\right)2y}{(x^2 + y^2 + z^2)^{5/2}}$$

Thus,

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

The other two equations derived in Problem 13.79 are checked in a similar fashion.

(*b*) Recalling that

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}$$

$$F_x = -\frac{\partial V}{\partial x} \quad V = -\int \frac{x}{(x^2 + y^2 + z^2)^{3/2}} dx$$

$$V = (x^2 + y^2 + z^2)^{-1/2} + f(y, z)$$

Similarly integrating $\partial V/\partial y$ and $\partial V/\partial z$ shows that the unknown function $f(x, y)$ is a constant.

$$V = \frac{1}{(x^2 + y^2 + z^2)^{1/2}} \blacktriangleleft$$

PROBLEM 13.85

(*a*) Determine the kinetic energy per unit mass which a missile must have after being fired from the surface of the earth if it is to reach an infinite distance from the earth. (*b*) What is the initial velocity of the missile (called the *escape velocity*)? Give your answers in SI units and show that the answer to part *b* is independent of the firing angle.

SOLUTION

At the surface of the earth, $g = 9.81 \text{ m/s}^2$

$$r_1 = R = 6370 \text{ km} = 6.37 \times 10^6 \text{ m}$$

Centric force at the surface of the earth,

$$F = mg = \frac{GMm}{R^2}$$

$$GM = gR^2 = (9.81)(6.37 \times 10^6)^2 = 398.06 \times 10^{12} \text{ m}^3/\text{s}^2$$

Let position 1 be on the surface of the earth $(r_1 = R)$ and position 2 be at $r_2 = OD$. Apply the conservation of energy principle.

$$T_1 + V_1 = T_2 + V_2$$

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 + \frac{GMm}{r_2}$$

$$T_1 = T_2 + \frac{GMm}{R} - \frac{GMm}{\infty}$$

$$\frac{T_1}{m} = \frac{T_2}{m} + \frac{GM}{R} = \frac{T_2}{m} + gR$$

For the escape condition set $\frac{T_2}{m} = 0$

$$\frac{T_1}{m} = gR = (9.81 \text{ m/s}^2)(6.37 \times 10^6 \text{ m}) = 62.49 \times 10^6 \text{ m}^2/\text{s}^2$$

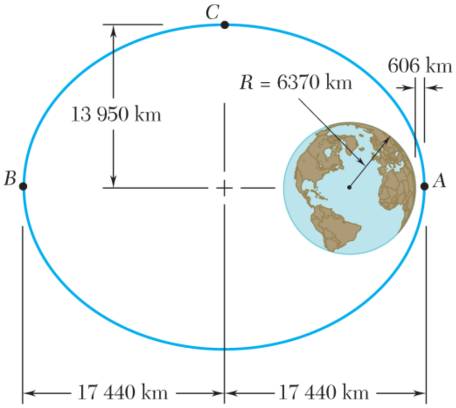
(*a*) $\frac{T_1}{m} = 62.5 \text{ MJ/kg}$ ◀

$$\frac{1}{2}mv_{\text{esc}}^2 = mgr$$

$$v_{\text{esc}} = \sqrt{2gR}$$

(*b*) $v_{\text{esc}} = \sqrt{(2)(9.81)(6.37 \times 10^6)} = 11.18 \times 10^3 \text{ m/s}$ $\quad v_{\text{esc}} = 11.18 \text{ km/s}$ ◀

Note that the escape condition depends only on the speed in position 1 and is independent of the direction of the velocity (firing angle).



PROBLEM 13.86

A satellite describes an elliptic orbit of minimum altitude 606 km above the surface of the earth. The semimajor and semiminor axes are 17,440 km and 13,950 km, respectively. Knowing that the speed of the satellite at Point C is 4.78 km/s, determine (*a*) the speed at Point A, the perigee, (*b*) the speed at Point B, the apogee.

SOLUTION

$$r_A = 6370 + 606 = 6976 \text{ km} = 6.976\times 10^6 \text{ m}$$

$$r_C = \sqrt{(17440-6976)^2 + (13950)^2} = 17438.4 \text{ km} = 17.4384\times 10^6 \text{ m}$$

$$r_B = (2)(17440) - 6976 = 27904 \text{ km} = 27.904\times 10^6 \text{ m}$$

For earth, $R = 6370 \text{ km} = 6.37\times 10^6 \text{ m}$

$$GM = gR^2 = (9.81 \text{ m/s}^2)(6.370\times 10^6)^2 = 398.06\times 10^{12} \text{ m}^3/\text{s}^2$$

$$v_C = 4.78 \text{ km/s} = 4780 \text{ m/s}$$

(*a*) *Speed at Point A*: Use conservation of energy.

$$T_A + V_A = T_C + V_C$$

$$\frac{1}{2}mv_A^2 - \frac{GMm}{r_A} = \frac{1}{2}mv_C^2 - \frac{GMm}{r_C}$$

$$v_A^2 = v_C^2 + 2GM\left(\frac{1}{r_A} - \frac{1}{r_C}\right)$$

$$= (4780)^2 + (2)(398.06\times 10^{12})\left[\frac{1}{6.976\times 10^6} - \frac{1}{17.4384\times 10^6}\right]$$

$$= 91.318\times 10^6 \text{ m}^2/\text{s}^2$$

$$V_A = 9.556\times 10^3 \text{ m/s}$$

$v_A = 9.56$ km/s ◀

(*b*) *Speed at Point B*: Use conservation of energy.

$$T_B + V_B = T_C + V_C$$

$$\frac{1}{2}mv_B^2 - \frac{GMm}{r_B} = \frac{1}{2}mv_C^2 - \frac{GMm}{r_C}$$

$$v_B^2 = v_C^2 + 2GM\left(\frac{1}{r_B} - \frac{1}{r_C}\right)$$

$$= (4780)^2 + (2)(398.06\times 10^{12})\left[\frac{1}{27.904\times 10^6} - \frac{1}{17.4384\times 10^6}\right]$$

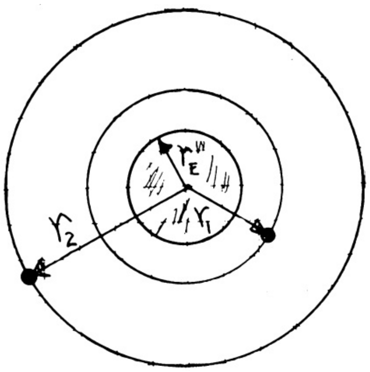
$$= 5.7258\times 10^6 \text{ m}^2/\text{s}^2$$

$v_B = 2.39\times 10^3$ m/s $\qquad$ $v_B = 2.39$ km/s ◀

PROBLEM 13.87

While describing a circular orbit 200 mi above the earth a space vehicle launches a 6000-lb communications satellite. Determine (*a*) the additional energy required to place the satellite in a geosynchronous orbit at an altitude of 22,000 mi above the surface of the earth, (*b*) the energy required to place the satellite in the same orbit by launching it from the surface of the earth, excluding the energy needed to overcome air resistance. (A *geosynchronous orbit* is a circular orbit in which the satellite appears stationary with respect to the ground).

SOLUTION



Geosynchronous orbit

$$r_1 = 3960 + 200 = 4160\text{ mi} = 21.965 \times 10^6\text{ ft}$$

$$r_2 = 3960 + 22{,}000 = 25{,}960\text{ mi} = 137.07 \times 10^6\text{ ft}$$

Total energy $\quad E = T + V = \frac{1}{2}mv^2 - \frac{GMm}{r}$

M = mass of earth

m = mass of satellite

Newton's second law $\quad F = ma_n\!: \quad \frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{GM}{r}$

$$T = \frac{1}{2}mv^2 = m\frac{GM}{2r} \qquad V = -\frac{GMm}{r}$$

$$E = T + V = \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r} = -\frac{1}{2}\frac{GMm}{r}$$

$$GM = gR_E^2 \qquad E = -\frac{1}{2}\frac{gR_E^2 m}{r} = -\frac{1}{2}\frac{R_E^2 W}{r} \text{ where } (W = mg)$$

$$E = -\frac{1}{2}\frac{(6000)(20.9088 \times 10^6\text{ ft})^2}{r} = -\frac{1.3115 \times 10^{18}}{r}\text{ ft}\cdot\text{lb}$$

Geosynchronous orbit at $r_2 = 137.07 \times 10^6$ ft

$$E_{Gs} = \frac{-1.3115 \times 10^{18}}{137.07 \times 10^6} = -9.5681 \times 10^9\text{ ft}\cdot\text{lb}$$

(*a*) At 200 mi, $\quad r_1 = 21.965 \times 10^6$ ft

$$E_{200} = -\frac{1.3115 \times 10^{18}}{21.965 \times 10^6} = -5.9709 \times 10^{10}$$

$$\Delta E_{300} = E_{Gs} - E_{200} = 5.0141 \times 10^{10}$$

$$\Delta E_{300} = 50.1 \times 10^9\text{ ft}\cdot\text{lb} \blacktriangleleft$$

PROBLEM 13.87 (Continued)

(*b*) Launch from earth

At launch pad $E_E = -\dfrac{GMm}{R_E} = \dfrac{-gR_E^2 m}{R_E} = -WR_E$

$$E_E = -6000(3960 \times 5280) = -1.25453 \times 10^{11}$$

$$\Delta E_E = E_{Gs} - E_E = -9.5681 \times 10^9 + 125.453 \times 10^9$$

$$\Delta E_E = 115.9 \times 10^9 \text{ ft} \cdot \text{lb} \blacktriangleleft$$

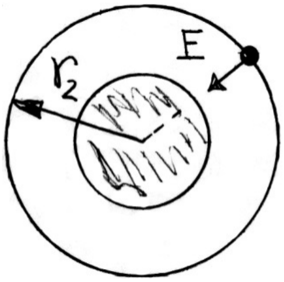
PROBLEM 13.88

How much energy per pound should be imparted to a satellite in order to place it in a circular orbit at an altitude of (*a*) 400 mi, (*b*) 4000 mi?

SOLUTION

Before launching: $r_1 = R = 3960\text{ mi}: \quad v_1 = 0, \ T = 0$

$$E_1 = -\frac{GMm}{R} = -\frac{gR^2m}{R} = -mgR$$



In circular orbit of radius $r_2 \left[cf.\text{ Equation 12.30}\right]$

Newton's second law

$$F = ma_n \qquad \frac{GMm}{r_2^2} = \frac{mv_2^2}{r_2} \qquad v_2^2 = \frac{GM}{r_2} = \frac{gR^2}{r_2}$$

$$E_2 = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2} = \frac{1}{2}mg\frac{R^2}{r_2} - \frac{gR^2m}{r_2} = -\frac{1}{2}\frac{gRm}{r_2}$$

Energy imparted is $\Delta E = E_2 - E_1$

$$\Delta E = -\frac{1}{2}\frac{gmR^2}{r_2} - \left(-mgR\right) = mgR\left(1 - \frac{R}{2r_2}\right)$$

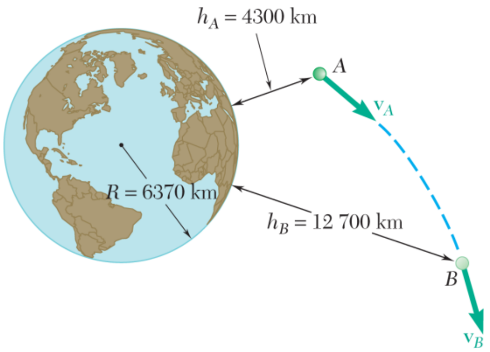
Unit weight $W = mg \qquad \dfrac{\Delta E}{W} = R\left(1 - \dfrac{R}{2r_2}\right)$

(*a*) $r_2 = 3960 + 400 = 4360\text{ mi}$

$$\frac{\Delta E}{W} = 3960\left(1 - \frac{3960}{4360}\right) \times 5280\text{ ft/mi} \qquad \frac{\Delta E}{W} = 1.918 \times 10^6\text{ ft}\cdot\text{lb/lb} \blacktriangleleft$$

(*b*) $r_2 = 3960 + 4000 = 7960\text{ mi}$

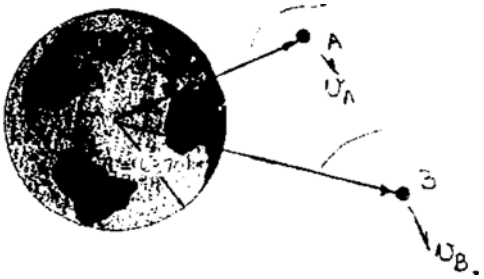
$$\frac{\Delta E}{W} = 3960\left(1 - \frac{3960}{7960}\right) \times 5280\text{ ft/mi} \qquad \frac{\Delta E}{W} = 10.51 \times 10^6\text{ ft}\cdot\text{lb/lb} \blacktriangleleft$$



PROBLEM 13.89

Knowing that the velocity of an experimental space probe fired from the earth has a magnitude $v_A = 32.5$ Mm/h at Point A, determine the speed of the probe as it passes through Point B.

SOLUTION



$$r_A = R + h_A = 6370 + 4300 = 10740 \text{ km} = 10.670 \times 10^6 \text{ m}$$

$$r_B = 6370 + 12700 = 19070 \text{ km} = 19.070 \times 10^6 \text{ m}$$

$$GM = gR^2 = (9.81)(6.370 \times 10^6)^2 = 398.06 \times 10^{12} \text{ m}^3/\text{s}^2$$

$$v_A = 32.5 \text{ Mm/h} = 9.0278 \times 10^3 \text{ m/s}$$

Use conservation of energy.

$$T_B + V_B = T_A + V_A$$

$$\frac{1}{2}mv_B^2 - \frac{GMm}{r_B} = \frac{1}{2}mv_A^2 - \frac{GMm}{r_A}$$

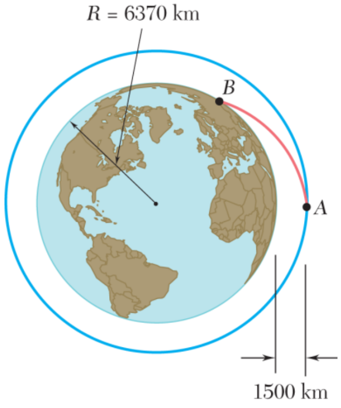
$$v_B^2 = v_A^2 + 2GM\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

$$= (9.0278 \times 10^3)^2 + (2)(398.06 \times 10^{12})\left[\frac{1}{19.070 \times 10^6} - \frac{1}{10.670 \times 10^6}\right]$$

$$= 48.635 \times 10^6 \text{ m}^2/\text{s}^2$$

$$v_B = 6.97 \times 10^3 \text{ m/s}$$

$v_B = 25.1$ Mm/h ◀



PROBLEM 13.90

A spacecraft is describing a circular orbit at an altitude of 1500 km above the surface of the earth. As it passes through Point *A*, its speed is reduced by 40 percent and it enters an elliptic crash trajectory with the apogee at Point *A*. Neglecting air resistance, determine the speed of the spacecraft when it reaches the earth's surface at Point *B*.

SOLUTION

Circular orbit velocity

$$\frac{v_C^2}{r} = \frac{GM}{r^2}, \quad GM = gR^2$$

$$v_C^2 = \frac{GM}{r} = \frac{gR^2}{r} = \frac{(9.81\text{ m/s}^2)(6.370\times 10^6\text{ m})^2}{(6.370\times 10^6\text{ m} + 1.500\times 10^6\text{ m})}$$

$$v_C^2 = 50.579\times 10^6\text{ m}^2/\text{s}^2$$

$$v_C = 7112\text{ m/s}$$

Velocity reduced to 60% of v_C gives $v_A = 4267$ m/s.

Conservation of energy:

$$T_A + V_A = T_B + V_B$$

$$\frac{1}{2}\cancel{m}v_A^2 - \frac{GM\cancel{m}}{r_A} = \frac{1}{2}\cancel{m}v_B^2 - \frac{GM\cancel{m}}{r_B}$$

$$\frac{1}{2}(4.267\times 10^3)^2 - \frac{9.81(6.370\times 10^6)^2}{(7.870\times 10^6)} = \frac{v_B^2}{2} - \frac{9.81(6.370\times 10^6)^2}{(6.370\times 10^6)}$$

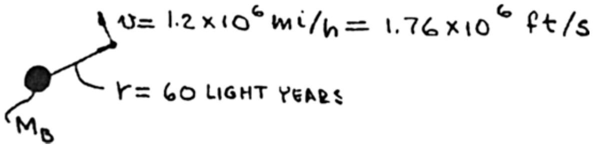
$$v_B = 6.48\times 10^3\text{ m/s}$$

$v_B = 6.48$ km/s ◀

PROBLEM 13.91

Observations show that a celestial body traveling at 1.2×10^6 mi/h appears to be describing about Point *B* a circle of radius equal to 60 light years. Point *B* is suspected of being a very dense concentration of mass called a black hole. Determine the ratio M_B/M_S of the mass at *B* to the mass of the sun. (The mass of the sun is 330,000 times the mass of the earth, and a light year is the distance traveled by light in one year at a velocity of 186,300 mi/s.)

SOLUTION



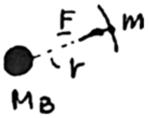
One light year is the distance traveled by light in one year.

Speed of light = 186,300 mi/s

$$r = (60\text{ yr})(186{,}300\text{ mi/s})(5280\text{ ft/mi})(365\text{ days/yr})(24\text{ h/day})(3600\text{ s/h})$$

$$r = 1.8612\times10^{18}\text{ ft}$$

Newton's second law



$$F = \frac{GM_Bm}{r^2} = m\frac{v^2}{r}$$

$$M_B = \frac{rv^2}{G}$$

$$GM_{\text{earth}} = gR_{\text{earth}}^2$$

$$= (32.2\text{ ft/s}^2)(3960\text{ mi}\times5280\text{ ft/mi})^2$$

$$= 14.077\times10^{15}(\text{ft}^3/\text{s}^2)$$

$$M_{\text{sun}} = 330{,}000M_E:\quad GM_{\text{sun}} = 330{,}000\,GM_{\text{earth}}$$

$$GM_{\text{sun}} = (330{,}000)(14.077\times10^{15})$$

$$= 4.645\times10^{21}\text{ ft}^3/\text{s}^2$$

$$G = \frac{4.645\times10^{21}}{M_{\text{sun}}}$$

$$M_B = \frac{rv^2}{G} = \frac{rv^2M_{\text{sun}}}{4.645\times10^{21}}$$

$$\frac{M_B}{M_{\text{sun}}} = \frac{(1.8612\times10^{18})(1.76\times10^6)^2}{4.645\times10^{21}}$$

$$\frac{M_B}{M_{\text{sun}}} = 1.241\times10^9 \blacktriangleleft$$

PROBLEM 13.92

(*a*) Show that, by setting $r = R + y$ in the right-hand member of Eq. (13.17′) and expanding that member in a power series in y/R, the expression in Eq. (13.16) for the potential energy V_g due to gravity is a first-order approximation for the expression given in Eq. (13.17′). (*b*) Using the same expansion, derive a second-order approximation for V_g.

SOLUTION

$$V_g = -\frac{WR^2}{r} \quad \text{setting } r = R + y\text{:} \quad V_g = -\frac{WR^2}{R+y} = -\frac{WR}{1+\frac{y}{R}}$$

$$V_g = -WR\left(1+\frac{y}{R}\right)^{-1} = -WR\left[1+\frac{(-1)}{1}\frac{y}{R}+\frac{(-1)(-2)}{1\cdot 2}\left(\frac{y}{R}\right)^2+\cdots\right]$$

We add the constant *WR*, which is equivalent to changing the datum from $r = \infty$ to $r = R$:

$$V_g = WR\left[\frac{y}{R}-\left(\frac{y}{R}\right)^2+\cdots\right]$$

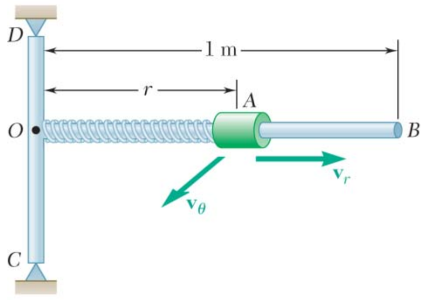
(*a*) First order approximation:

$$V_g = WR\left(\frac{y}{R}\right) = Wy \quad ◀$$

[Equation 13.16]

(*b*) Second order approximation: $V_g = WR\left[\frac{y}{R}-\left(\frac{y}{R}\right)^2\right]$

$$V_g = Wy - \frac{Wy^2}{R} \quad ◀$$



PROBLEM 13.93

Collar *A* has a mass of 3 kg and is attached to a spring of constant 1200 N/m and of undeformed length equal to 0.5 m. The system is set in motion with $r = 0.3$ m, $v_\theta = 2$ m/s, and $v_r = 0$. Neglecting the mass of the rod and the effect of friction, determine the radial and transverse components of the velocity of the collar when $r = 0.6$ m.

SOLUTION

Let position 1 be the initial position.

$$r_1 = 0.3 \text{ m}$$

$$(v_r)_1 = 0, \qquad (v_\theta)_1 = 2 \text{ m/s}, \qquad v_1 = 2 \text{ m/s}$$

$$x_1 = r_1 - l_0 = (0.3 - 0.5) = -0.2 \text{ m}$$

Let position 2 be when $r = 0.6$ m.

$$r_2 = 0.6 \text{ m}$$

$$(v_r)_2 = ?, \qquad (v_\theta)_2 = ?, \qquad v_2 = ?$$

$$x_2 = r_2 - l_0 = (0.6 - 0.5) = 0.1 \text{ m}$$

Conservation of angular momentum: $r_1 m(v_\theta)_1 = v_2 m(v_\theta)_2$

$$(v_\theta)_2 = \frac{r_1(v_\theta)_1}{r_2} = \frac{(0.3)(2)}{0.6} = 1.000 \text{ m/s}$$

Conservation of energy: $T_1 + V_1 = T_2 + V_2$

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2$$

$$v_2^2 = v_1^2 + \frac{k}{m}\left(x_1^2 - x_2^2\right)$$

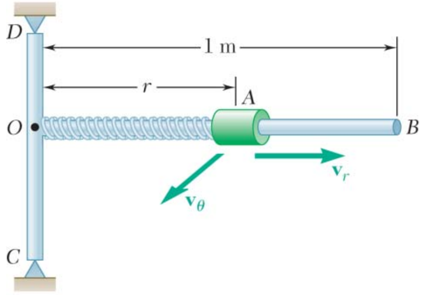
$$= (2)^2 + \frac{1200}{3}\left[(0.2)^2 - (0.1)^2\right] = 16 \text{ m}^2/\text{s}^2$$

$$(v_r)_2^2 = v_2^2 - (v_\theta)^2 = 16 - 1 = 15 \text{ m}^2/\text{s}^2$$

$$v_r = \pm 3.87 \text{ m/s}$$

$v_r = \pm 3.87$ m/s ◀

$v_\theta = 1.000$ m/s ◀



PROBLEM 13.94

Collar *A* has a mass of 3 kg and is attached to a spring of constant 1200 N/m and of undeformed length equal to 0.5 m. The system is set in motion with $r = 0.3$ m, $v_\theta = 2$ m/s, and $v_r = 0$. Neglecting the mass of the rod and the effect of friction, determine (*a*) the maximum distance between the origin and the collar, (*b*) the corresponding speed. (*Hint:* Solve the equation obtained for *r* by trial and error.)

SOLUTION

Let position 1 be the initial position.

$$r_1 = 0.3 \text{ m}$$

$$(v_r)_1 = 0, \qquad (v_\theta)_1 = 2 \text{ m/s}, \qquad v_1 = 2 \text{ m/s}$$

$$x_1 = r_1 - l_0 = 0.3 - 0.5 = -0.2 \text{ m}$$

$$T_1 = \frac{1}{2}mv^2 = \frac{1}{2}(3)(2)^2 = 6 \text{ J}$$

$$V_1 = \frac{1}{2}kx_1^2 = \frac{1}{2}(1200)(-0.2)^2 = 24 \text{ J}$$

Let position 2 be when *r* is maximum. $(v_r)_2 = 0$

$$r_2 = r_m$$

$$x_2 = (r_m - 0.5)$$

$$T_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(3)(v_\theta)_2^2 = 1.5(v_\theta)_2^2$$

$$V_2 = \frac{1}{2}kx_2^2 = \frac{1}{2}(1200)(r_m - 0.5)^2$$

$$= 600(r_m - 0.5)^2$$

Conservation of angular momentum: $r_1 m(v_\theta)_1 = v_2 m(v_\theta)_2$

$$(v_\theta)_2 = \frac{r_1}{r_2}(v_\theta),= \frac{(0.3)}{r_m}(2) = \frac{0.6}{r_m}$$

Conservation of energy: $T_1 + V_1 = T_2 + V_2$

$$6 + 24 = 1.5(v_\theta)_2^2 + 600(r_m - 0.5)^2$$

$$30 = (1.5)\left(\frac{0.6}{r_m}\right)^2 + 600(r_m - 0.5)^2$$

$$f(r_m) = \frac{0.54}{r_m^2} + 600(r_m - 0.5)^2 - 30 = 0$$

PROBLEM 13.94 (Continued)

Solve for r_m by trial and error.

r_m (m)	0.5	1.0	0.8	0.7	0.72	0.71
$f(r_m)$	–27.8	120.5	24.8	–4.9	0.080	–2.469

$$r_m = 0.72 - \frac{(0.01)(0.08)}{2.467 + 0.08} = 0.7197 \text{ m}$$

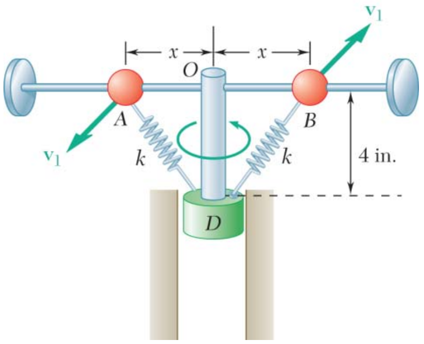
(*a*) *Maximum distance.* $r_m = 0.720 \text{ m}$ ◀

(*b*) *Corresponding speed.*

$$(v_\theta)_2 = \frac{0.6}{0.7197} = 0.8337 \text{ m/s}$$

$$(v_r)_2 = 0$$

$v_2 = 0.834 \text{ m/s}$ ◀



PROBLEM 13.95

A governor is designed so that the valve of negligible mass at *D* will open once a vertical force greater than 20 lbs is exerted on it. In initial testing of the device, the two 1 lb masses are at *x* = 1 in., and are prevented from sliding along the rod by stops. Each mass is connected to the valve by a 10 lb/in spring, which are both unstretched at *x* = 1 in. The governor is rotating so that $v_1 = 30$ ft/s when the stops are removed. When the valve opens, determine the position and velocity of the masses.

SOLUTION

Given:

$$x_1 = 1\text{ in} = \frac{1}{12}\text{ ft}$$

$$m = \frac{1\text{ lbs}}{32.2\text{ ft/s}^2} = 0.03106\text{ slugs}$$

$$v_{\theta,1} = 30\text{ ft/s},\ v_{r,1} = 0$$

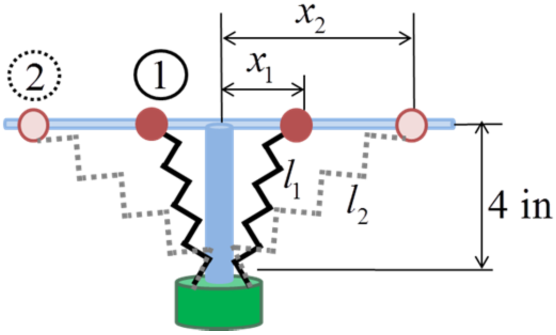
l_i = length of spring at positon i

s_i = stretch of spring in position i

$$F_{y,2} = 20\text{ lbs}$$

$$k = 10\text{ lb/in} = 120\text{ lb/ft}$$

Sketch: Pos. 1 with stops, Pos. 2 when valve opens.

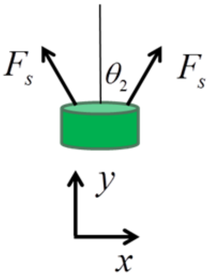


At Position 1: $s_1 = 0$

$$l_1 = \frac{\sqrt{4^2 + x_1^2}}{12}$$

$$l_1 = \frac{\sqrt{17}}{12}\text{ ft}$$

At Position 2: FBD of valve where: $F_s = ks_2$



Equilibrium in y-direction:

$$\sum F_y = 20\text{ lbs}$$

$$2F_s\cos\theta_2 = 20\text{ lbs}$$

$$2ks_2\left(\frac{4/12}{l_2}\right) = 20$$

$$s_2 = 30\frac{l_2}{k}$$

Note $s_2 = l_2 - l_1$ so:

$$l_2 - l_1 = 30\frac{l_2}{k}$$

$$l_2 = \frac{l_1 k}{k - 30} \Rightarrow l_2 = 0.4581\text{ ft}$$

$$s_2 = 0.1145\text{ ft}$$

PROBLEM 13.95 (Continued)

Location of mass, $x_2 = \sqrt{l_2^2 - \left(\frac{4}{12}\right)^2} \quad \Rightarrow \quad x_2 = 0.3143 \text{ ft}$ $\qquad x_2 = 3.771 \text{ in}$ ◀

Conservation of angular momentum for each mass:

$$mx_1 v_{\theta,1} = mx_2 v_{\theta,2}$$

$$v_{\theta,2} = \frac{x_1}{x_2} v_{\theta,1}$$

$$v_{\theta,2} = 7.955 \text{ ft/s}$$

Work Energy from position 1 to 2:

$$T_1 + \cancel{V_{g_1}} + \cancel{V_{e_1}} + \cancel{U_{1\to 2}^{NC}} = T_2 + \cancel{V_{g_2}} + V_{e_2}$$

where:

$$T_2 = 2\left(\frac{1}{2}mv_1^2\right)$$

$$T_2 = 2\left(\frac{1}{2}mv_2^2\right)$$

$$V_{e_2} = 2\left(\frac{1}{2}ks_2^2\right)$$

Therefore:

$$2\left(\frac{1}{2}mv_1^2\right) = 2\left(\frac{1}{2}mv_2^2\right) + 2\left(\frac{1}{2}ks_2^2\right)$$

$$v_2 = \sqrt{v_1^2 - \frac{k}{m}s_2^2}$$

Substitute known values:

$$v_2 = \sqrt{30^2 - \frac{120}{0.03106}(0.1145)^2}$$

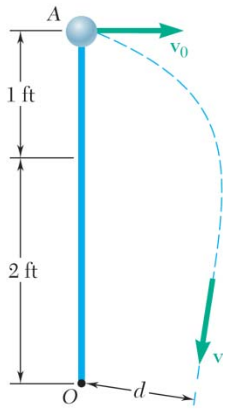
$$v_2 = 29.14 \text{ ft/s}$$

Solve for $v_{r,2}$:

$$v_{r,2} = \sqrt{v_2^2 - v_{\theta,2}^2}$$

$$v_{r,2} = 28.04 \text{ ft/s}$$

$\mathbf{v}_2 = (28.04 \text{ ft/s})\ \mathbf{e}_r + (7.96 \text{ ft/s})\ \mathbf{e}_r$ ◀



PROBLEM 13.96

A 1.5-lb ball that can slide on a *horizontal* frictionless surface is attached to a fixed Point *O* by means of an elastic cord of constant $k = 1$ lb/in. and undeformed length 2 ft. The ball is placed at Point *A*, 3 ft from *O*, and given an initial velocity $\mathbf{v}_0$ perpendicular to *OA*. Determine (*a*) the smallest allowable value of the initial speed v_0 if the cord is not to become slack, (*b*) the closest distance *d* that the ball will come to Point *O* if it is given half the initial speed found in part *a*.

SOLUTION

Let L_1 be the initial stretched length of the cord and L_2 the length of the closest approach to Point *O* if the cord does not become slack. Let position 1 be the initial state and position 2 be that of closest approach to Point *O*. The only horizontal force acting on the ball is the conservative central force due to the elastic cord. At the point of closest approach the velocity of the ball is perpendicular to the cord.

Conservation of angular momentum:

$$r_1 m v_1 = r_2 m v_2$$

$$L_1 m v_0 = L_2 m v_2 \quad \text{or} \quad v_2 = \frac{L_1 v_0}{L_2}$$

Conservation of energy:

$$T_1 + V_1 = T_2 + V_2$$

$$\frac{1}{2}mv_1^2 + \frac{1}{2}k(L_1 - L_0)^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}k(L_2 - L_0)^2$$

$$v_1^2 - v_2^2 = -\frac{k}{m}[(L_1 - L_0)^2 + (L_2 - L_0)^2]$$

$$v_0^2 - \frac{L_1^2}{L_2^2}v_0^2 = -\frac{k}{m}[(L_1 - L_0)^2 + (L_2 - L_0)^2]$$

Data:

$$L_0 = 2 \text{ ft}, \quad L_1 = 3 \text{ ft}$$

$L_2 = L_0 = 2$ ft for zero tension in the cord at the point of closest approach.

$$k = 1 \text{ lb/in.} = 12 \text{ lb/ft}$$

$$m = W/g = 1.5/32.2 = 0.04658 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$v_0^2 - \frac{(3)^2}{(2)^2}v_0 = -\frac{12}{0.04658}[(3-2)^2 + (2-2)^2]$$

$$-1.25\, v_0^2 = -257.6$$

(*a*) $$v_0^2 = 206.1 \text{ ft}^2/\text{s}^2$$ $v_0 = 14.36$ ft/s ◀

PROBLEM 13.96 (Continued)

(*b*) Let $v_0 = \frac{1}{2}(14.36 \text{ ft/s}) = 7.18 \text{ ft/s}^2$ so that the cord is slack in the position of closest approach to Point *O*.

Let position 1 be the initial position and position 2 be position of closest approach with the cord being slack.

Conservation of energy: $T_1 + V_1 = T_2 + V_2$

$$\frac{1}{2}mv_0^2 + \frac{1}{2}k(L_1 - L_0)^2 = \frac{1}{2}mv_2^2$$

$$v_2^2 = v_0^2 + \frac{k}{m}(L - L_0)^2$$

$$= (7.18)^2 + \frac{12}{0.04658}(3 - 2)^2 = 309.17 \text{ ft}^2/\text{s}^2$$

$$v_2 = 17.583 \text{ ft/s}$$

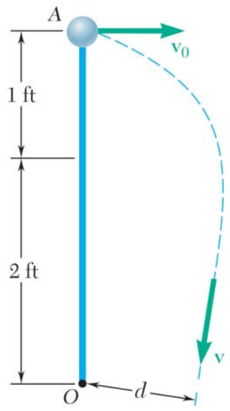
Conservation of angular momentum:

$$r_1 m v_1 = r_2 m v_2 \sin\phi$$

$$r_2 \sin\phi = d = \frac{r_1 v_1}{v_2} = \frac{L_1 v_0}{v_2}$$

$$d = \frac{(3)(7.18)}{17.583}$$

$d = 1.225$ ft ◀



PROBLEM 13.97

A 1.5-lb ball that can slide on a *horizontal* frictionless surface is attached to a fixed Point O by means of an elastic cord of constant $k = 1$ lb/in. and undeformed length 2 ft. The ball is placed at Point A, 3 ft from O, and given an initial velocity $\mathbf{v}_0$ perpendicular to OA, allowing the ball to come within a distance $d = 9$ in. of Point O after the cord has become slack. Determine (*a*) the initial speed v_0 of the ball, (*b*) its maximum speed.

SOLUTION

Let L_1 be the initial stretched length of the cord. Let position 1 be the initial position. Let position 2 be the position of closest approach to point after the cord has become slack. While the cord is slack there are no horizontal forces acting on the ball, so the velocity remains constant. While the cord is stretched, the only horizontal force acting on the ball is the conservative central force due to the elastic cord. At the point of closest approach the velocity of the ball is perpendicular to the radius vector.

Conservation of angular momentum: $r_1 m v_1 = r_2 m v_2$

$$L_1 v_0 = d\, v_2 \qquad \text{or} \qquad v_2 = \frac{L_1}{d} v_0$$

Conservation of energy: $T_1 + V_1 = T_2 + V_2$

$$\frac{1}{2} m v_0^2 + \frac{1}{2} k (L_1 - L_0)^2 = \frac{1}{2} m v_2^2 + 0$$

$$v_0^2 - v_2^2 = -\frac{k}{m}(L_1 - L_0)^2$$

$$v_0^2 - \left(\frac{L_1}{d^2} v_0\right)^2 = -\frac{k}{m}(L_1 - L_0)$$

Data: $L_0 = 2$ ft, $L_1 = 3$ ft, $d = 9$ in. $= 0.75$ ft

$k = 1$ lb/in. $= 12$ lb/ft

$m = W/g = 1.5/32.2 = 0.04658$ lb $\cdot$ s^2/ft

$$v_0^2 - \left(\frac{3\, v_0}{0.75}\right)^2 = -\frac{12}{0.04658}(3-2)^2$$

$$-15 v_0^2 = -257.6$$

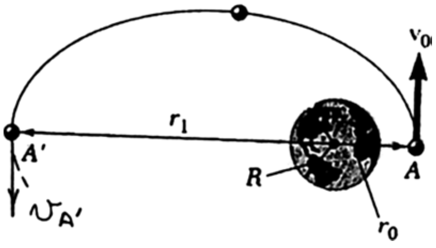
(*a*) $v_0^2 = 17.17$ ft^2/s^2 $\qquad v_0 = 4.14$ ft/s ◀

(*b*) Maximum speed. $v_m = v_2 = \dfrac{3 v_0}{0.75}$ $\qquad v_m = 16.58$ ft/s ◀

PROBLEM 13.98

Using the principles of conservation of energy and conservation of angular momentum, solve part *a* of Sample Problem 12.14.

SOLUTION



$$R = 6370 \text{ km}$$
$$r_0 = 500 \text{ km} + 6370 \text{ km}$$
$$r_0 = 6870 \text{ km} = 6.87 \times 10^6 \text{ m}$$
$$v_0 = 36{,}900 \text{ km/h}$$
$$= \frac{36.9 \times 10^6 \text{ m}}{3.6 \times 10^3 \text{ s}}$$
$$= 10.25 \times 10^3 \text{ m/s}$$

Conservation of angular momentum:

$$r_0 m v_0 = r_1 m v_A, \qquad r_0 = r_{\min},\ r_1 = r_{\max}$$
$$v_{A'} = \left(\frac{r_0}{r_1}\right) v_0 = \left(\frac{6.870 \times 10^6}{r_1}\right)(10.25 \times 10^3)$$
$$v_{A'} = \frac{70.418 \times 10^9}{r_1} \qquad (1)$$

Conservation of energy:

Point A:

$$v_0 = 10.25 \times 10^3 \text{ m/s}$$
$$T_A = \frac{1}{2} m v_0^2 = \frac{1}{2} m (10.25 \times 10^3)^2$$
$$T_A = (m)(52.53 \times 10^6)(\text{J})$$
$$V_A = -\frac{GMm}{r_0}$$
$$GM = gR^2 = (9.81 \text{ m/s}^2)(6.37 \times 10^6 \text{ m})^2$$
$$GM = 398 \times 10^{12} \text{ m}^3/\text{s}^2$$
$$r_0 = 6.87 \times 10^6 \text{ m}$$
$$V_A = -\frac{(398 \times 10^{12} \text{ m}^3/\text{s}^2) m}{(6.87 \times 10^6 \text{ m})}$$
$$= -57.93 \times 10^6\, m \text{ (J)}$$

PROBLEM 13.98 (Continued)

Point A':

$$T_{A'} = \frac{1}{2} m v_{A'}^2$$

$$V_{A'} = -\frac{GMm}{r_1}$$

$$= -\frac{398 \times 10^{12}\, m}{r_1}\,(\mathrm{J})$$

$$T_A + V_A = T_{A'} + V_{A'}$$

$$52.53 \times 10^6\, \cancel{m} - 57.93 \times 10^6\, \cancel{m} = \frac{1}{2}\cancel{m} v_{A'}^2 - \frac{398 \times 10^{12}\, \cancel{m}}{r_1}$$

Substituting for $v_{A'}$ from (1)

$$-5.402 \times 10^6 = \frac{(70.418 \times 10^9)^2}{(2)(r_1)^2} - \frac{398 \times 10^{12}}{r_1}$$

$$-5.402 \times 10^6 = \frac{(2.4793 \times 10^{21})}{r_1^2} - \frac{398 \times 10^{12}}{r_1}$$

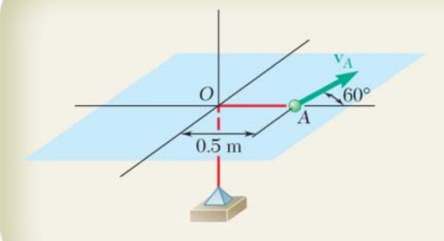
$$(5.402 \times 10^6) r_1^2 - (398 \times 10^{12}) r_1 + 2.4793 \times 10^{21} = 0$$

$$r_1 = 66.7 \times 10^6 \text{ m},\ 6.87 \times 10^6 \text{ m}$$

$r_{\max} = 66{,}700$ km ◀

PROBLEM 13.99

Solve sample Problem 13.11, assuming that the elastic cord is replaced by a central force **F** of magnitude $(80/r^2)$ N directed toward *O*.



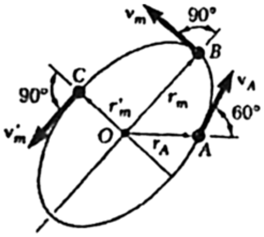
SAMPLE PROBLEM 13.11 A sphere of mass $m = 6$ kg is attached to an elastic cord of constant $k = 100$ N/m, which is undeformed when the sphere is located at the origin *O*. Knowing that the sphere may slide without friction on the horizontal surface and that in the position shown its velocity $\mathbf{v}_A$ has a magnitude of 20 m/s, determine *(a)* the maximum and minimum distances from the sphere to the origin *O*, *(b)* the corresponding values of its speed.

SOLUTION

(*a*) The force exerted on the sphere passes through *O*. Angular momentum about *O* is conserved.

Minimum velocity is at *B*, where the distance from *O* is maximum.

Maximum velocity is at *C*, where distance from *O* is minimum.



$$r_A m v_A \sin 60° = r_m m v_m$$

$$(0.5 \text{ m})(0.6 \text{ kg})(20 \text{ m/s})\sin 60° = r_m (0.6 \text{ kg}) v_m$$

$$v_m = \frac{8.66}{r_m} \tag{1}$$

Conservation of energy:

At Point *A*, $$T_A = \frac{1}{2} m v_A^2 = \frac{1}{2}(0.6 \text{ kg})(20 \text{ m/s})^2 = 120 \text{ J}$$

$$V = \int F dr = \int \frac{80}{r^2} dr = \frac{-80}{r},$$

$$V_A = \frac{-80}{0.5} = -160 \text{ J}$$

At Point *B*, $$T_B = \frac{1}{2} m v_m^2 = \frac{1}{2}(0.6 \text{ kg}) v_m^2 = 0.3 v_m^2$$

and Point *C*: $$V_B = \frac{-80}{r_m}$$

$$T_A + V_A = T_B + V_B$$

$$120 - 160 = 0.3 v_m^2 - \frac{80}{r_m} \tag{2}$$

PROBLEM 13.99 (Continued)

Substitute (1) into (2)

$$-40 = (0.3)\left(\frac{8.66}{r_m}\right)^2 - \frac{80}{r_m}$$

$$r_m^2 - 2r_m + 0.5625 = 0$$

$$r_m' = 0.339 \text{ m} \quad \text{and} \quad r_m = 1.661 \text{ m}$$

$r_{\text{max}} = 1.661 \text{ m}$ ◀

$r_{\text{min}} = 0.339 \text{ m}$ ◀

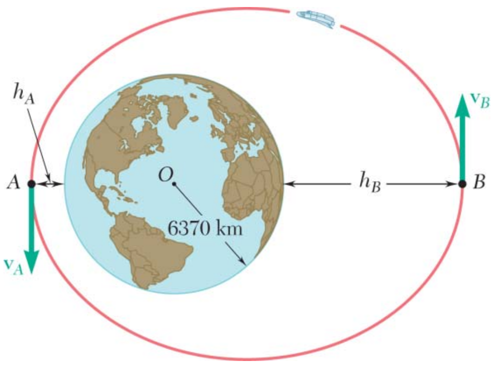
(*b*) Substitute r_m' and r_m from results of part (*a*) into (1) to get corresponding maximum and minimum values of the speed.

$$v_m' = \frac{8.66}{0.339} = 25.6 \text{ m/s}$$

$v_{\text{max}} = 25.6 \text{ m/s}$ ◀

$$v_m = \frac{8.66}{1.661} = 5.21 \text{ m/s}$$

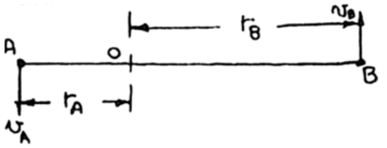
$v_{\text{min}} = 5.21 \text{ m/s}$ ◀



PROBLEM 13.100

A spacecraft is describing an elliptic orbit of minimum altitude $h_A = 2400$ km and maximum altitude $h_B = 9600$ km above the surface of the earth. Determine the speed of the spacecraft at A.

SOLUTION



$$r_A = 6370 \text{ km} + 2400 \text{ km}$$
$$r_A = 8770 \text{ km}$$
$$r_B = 6370 \text{ km} + 9600 \text{ km}$$
$$= 15{,}970 \text{ km}$$

Conservation of momentum: $r_A m v_A = r_B m v_B$

$$v_B = \frac{r_A}{r_B} v_A = \frac{8770}{15{,}970} v_A = 0.5492 v_A \qquad (1)$$

Conservation of energy: $T_A = \frac{1}{2} m v_A^2 \quad V_A = \frac{-GMm}{r_A} \quad T_B = \frac{1}{2} m v_B^2 \quad V_B = \frac{-GMm}{r_B}$

$$GM = gR^2 = (9.81 \text{ m/s}^2)(6370 \times 10^3 \text{ m})^2 = 398.1 \times 10^{12} \text{ m}^3/\text{s}^2$$

$$V_A = \frac{-(398.1 \times 10^{12})m}{8770 \times 10^3} = -45.39 \times 10^6 \, m$$

$$V_B = \frac{-(398.1 \times 10^{12})m}{(15{,}970 \times 10^3)} = -24.93 \, m$$

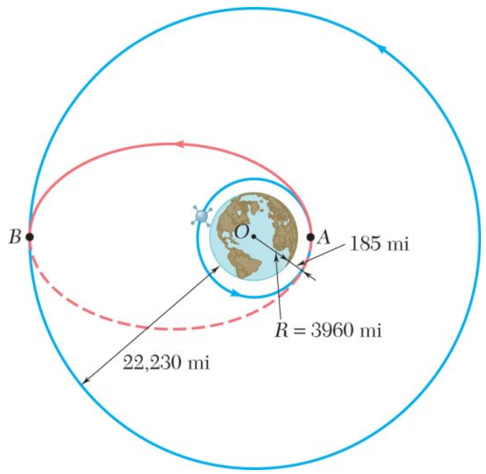
$T_A + V_A = T_B + V_B$:

$$\frac{1}{2} \cancel{m} v_A^2 - 45.39 \times 10^6 \cancel{m} = \frac{1}{2} \cancel{m} v_B^2 - 24.93 \times 10^6 \, m \qquad (2)$$

Substituting for v_B in (2) from (1)

$$v_A^2 [1 - (0.5492)^2] = 40.92 \times 10^6$$
$$v_A^2 = 58.59 \times 10^6 \text{ m}^2/\text{s}^2$$
$$v_A = 7.65 \times 10^3 \text{ m/s}$$

$v_A = 27.6 \times 10^3$ km/h ◀



PROBLEM 13.101

While describing a circular orbit, 185 mi above the surface of the earth, a space shuttle ejects at Point *A* an inertial upper stage (IUS) carrying a communication satellite to be placed in a geosynchronous orbit (see Problem 13.87) at an altitude of 22,230 mi above the surface of the earth. Determine (*a*) the velocity of the IUS relative to the shuttle after its engine has been fired at *A*, (*b*) the increase in velocity required at *B* to place the satellite in its final orbit.

SOLUTION

For earth,

$$R = 3960 \text{ mi} = 20.909\times10^6 \text{ ft}$$

$$g = 32.2 \text{ ft}^2$$

$$GM = gR^2 = (32.2)(20.909\times10^6)^2 = 14.077\times10^{15} \text{ ft}^3/\text{s}^2$$

Speed on a circular orbits of radius r, r_A, and r_B.

$$F = ma_n$$

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$v^2 = \frac{GM}{r} \qquad v = \sqrt{\frac{GM}{r}}$$

$$r_A = 3960 + 185 = 4145 \text{ mi} = 21.886\times10^6 \text{ ft}$$

$$(v_A)_{\text{circ}} = \sqrt{\frac{14.077\times10^{15}}{21.886\times10^6}} = 25.362\times10^3 \text{ ft/s}$$

$$r_B = 3960 + 22230 = 26190 \text{ mi} = 138.283\times10^6 \text{ ft}$$

$$(v_B)_{\text{circ}} = \sqrt{\frac{14.077\times10^{15}}{138.283\times10^6}} = 10.089\times10^3 \text{ ft/s}$$

Calculate speeds at *A* and *B* for path *AB*.

Conservation of angular momentum: $mr_Av_A \sin\phi_A = mr_Bv_B \sin\phi_A$

$$v_B = \frac{r_Av_A \sin 90^\circ}{r_B \sin 90^\circ} = \frac{21.886\times10^6 v_A}{138.283\times10^6} = 0.15816 v_A$$

PROBLEM 13.101 (Continued)

Conservation of energy: $$T_A + V_A = T_B + V_B$$

$$\frac{1}{2}mv_A^2 + \frac{GMm}{r_A} = \frac{1}{2}mv_B^2 - \frac{GMm}{r_B}$$

$$v_A^2 - v_B^2 = 2GM\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = \frac{2GM(r_B - r_A)}{r_A r_B}$$

$$v_A^2 - (0.15816v_A)^2 = \frac{(2)(14.077\times10^{15})(116.397\times10^6)}{(21.886\times10^6)(138.283\times10^6)}$$

$$0.97499v_A^2 = 1.082796\times10^9$$

$$v_A = 33.325\times10^3 \text{ ft/s}$$

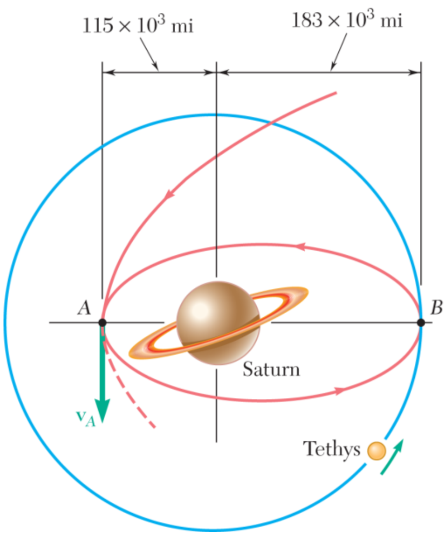
$$v_B = (0.15816)(33.325\times10^6) = 5.271\times10^3 \text{ ft/s}$$

(*a*) Increase in speed at *A*:

$$\Delta v_A = 33.325\times10^3 - 25.362\times10^3 = 7.963\times10^3 \text{ ft/s} \qquad \Delta v_A = 7960 \text{ ft/s} ◀$$

(*b*) Increase in speed at *B*:

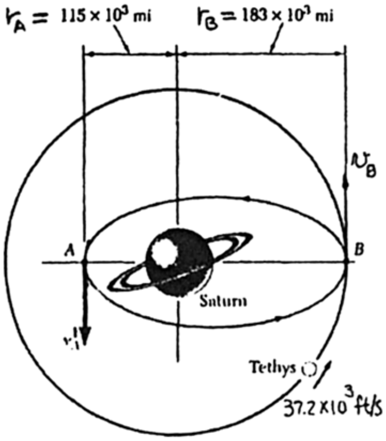
$$\Delta v_B = 10.089\times10^3 - 5.271\times10^3 = 4.818\times10^3 \text{ ft/s} \qquad \Delta v_B = 4820 \text{ ft/s} ◀$$



PROBLEM 13.102

A spacecraft approaching the planet Saturn reaches Point A with a velocity $\mathbf{v}_A$ of magnitude 68.8×10^3 ft/s. It is to be placed in an elliptic orbit about Saturn so that it will be able to periodically examine Tethys, one of Saturn's moons. Tethys is in a circular orbit of radius 183×10^3 mi about the center of Saturn, traveling at a speed of 37.2×10^3 ft/s. Determine (a) the decrease in speed required by the spacecraft at A to achieve the desired orbit, (b) the speed of the spacecraft when it reaches the orbit of Tethys at B.

SOLUTION



(a)

$$r_A = 607.2\times10^6 \text{ ft}$$
$$r_B = 966.2\times10^6 \text{ ft}$$

$v_A' =$ speed of spacecraft in the elliptical orbit after its speed has been decreased.

Elliptical orbit between A and B.

Conservation of energy

Point A: $$T_A = \frac{1}{2}mv_A'^2$$
$$V_A = \frac{-GM_{\text{sat}}m}{r_A}$$

$M_{\text{sa}} =$ Mass of Saturn, determine GM_{sa} from the speed of Tethys in its circular orbit.

(Eq. 12.44) $$v_{\text{circ}} = \sqrt{\frac{GM_{\text{sat}}}{r}} \quad GM_{\text{sat}} = r_B v_{\text{circ}}^2$$

$$GM_{\text{sat}} = (966.2\times10^6 \text{ ft}^2)(37.2\times10^3 \text{ ft/s})^2$$
$$= 1.337\times10^{18} \text{ ft}^3/\text{s}^2$$
$$V_A = -\frac{(1.337\times10^{18} \text{ ft}^3/\text{s}^2)\, m}{(607.2\times10^6 \text{ ft})}$$
$$= -2.202\times10^9\, m$$

PROBLEM 13.102 (Continued)

Point B:
$$T_B = \frac{1}{2}mv_B^2 \qquad V_B = \frac{-GM_{\text{sat}}m}{r_B} = -\frac{(1.337\times10^{18}\ \text{ft}^3/\text{s}^2)\,m}{(966.2\times10^6\ \text{ft})}$$
$$V_B = 1.384\times10^9$$

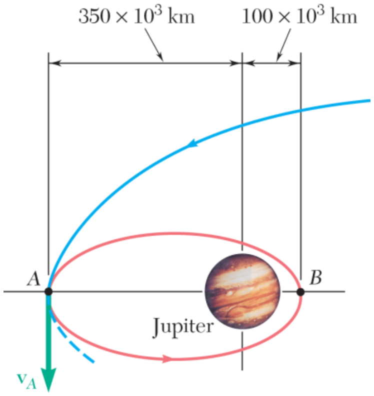
$$T_A + V_A = T_B + V_B;$$
$$\frac{1}{2}\cancel{m}v_A'^2 - 2.202\times10^9\,\cancel{m} = \frac{1}{2}\cancel{m}v_B^2 - 1.384\times10^9\,\cancel{m}$$
$$v_A'^2 - v_B^2 = 1.636\times10^9$$

Conservation of angular momentum:

$$r_Amv_A' = r_Bmv_B \quad v_B = \frac{r_A}{r_B}v_A' = \frac{607.2\times10^6}{966.2\times10^6}v_A' = 0.6284v_A'$$
$$v_A'^2[1-(0.6284)^2] = 1.636\times10^9$$
$$v_A' = 52{,}005\ \text{ft/s}$$

(*a*) $\Delta v_A = v_A - v_A' = 68{,}800 - 52{,}005$ $\qquad \Delta v_A = 16{,}795\ \text{ft/s}$ ◀

(*b*) $v_B = \dfrac{r_A}{r_B}v_A' = (0.6284)(52{,}005)$ $\qquad v_B = 32{,}700\ \text{ft/s}$ ◀



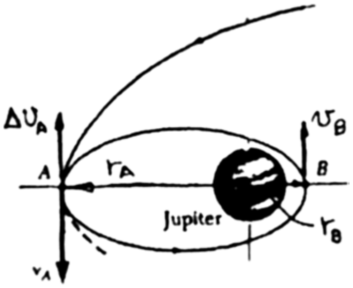
PROBLEM 13.103

A spacecraft traveling along a parabolic path toward the planet Jupiter is expected to reach Point A with a velocity $\mathbf{v}_A$ of magnitude 26.9 km/s. Its engines will then be fired to slow it down, placing it into an elliptic orbit which will bring it to within 100×10^3 km of Jupiter. Determine the decrease in speed Δv at Point A which will place the spacecraft into the required orbit. The mass of Jupiter is 319 times the mass of the earth.

SOLUTION

Conservation of energy.

Point A:



$$T_A = \frac{1}{2}m(v_A - \Delta v_A)^2$$

$$V_A = \frac{-GM_J m}{r_A}$$

$$GM_J = 319 GM_E = 319 g R_E^2$$

$$R_E = 6.37 \times 10^6 \text{ m}$$

$$GM_J = (319)(9.81 \text{ m/s}^2)(6.37 \times 10^6 \text{ m})^2$$

$$GM_J = 126.98 \times 10^{15} \text{ m}^3/\text{s}^2$$

$$r_A = 350 \times 10^6 \text{ m}$$

$$V_A = \frac{-(126.98 \times 10^{15} \text{ m}^3/\text{s}^2)m}{(350 \times 10^6 \text{ m})}$$

$$V_A = -(362.8 \times 10^6)m$$

Point B:

$$T_B = \frac{1}{2}mv_B^2$$

$$V_B = \frac{-GM_J m}{r_B} = \frac{-(126.98 \times 10^{15} \text{ m}^3/\text{s}^2)m}{(100 \times 10^6 m)}$$

$$V_B = -(1269.8 \times 10^6)m$$

$$T_A + V_A = T_B + V_B$$

$$\frac{1}{2}m(v_A - \Delta v_A)^2 - 362.8 \times 10^6 m = \frac{1}{2}mv_B^2 - 1269.8 \times 10^6 \text{ m}$$

$$(v_A - \Delta v_A)^2 - v_B^2 = -1814 \times 10^6 \qquad (1)$$

PROBLEM 13.103 (Continued)

Conservation of angular momentum.

$$r_A = 350 \times 10^6 \text{ m}$$
$$r_B = 100 \times 10^6 \text{ m}$$
$$r_A m(v_A - \Delta v_A) = r_B m v_B$$
$$v_B = \left(\frac{r_A}{r_B}\right)(v_A - \Delta v_A)$$
$$= \left(\frac{350}{100}\right)(v_A - \Delta v_A) \qquad (2)$$

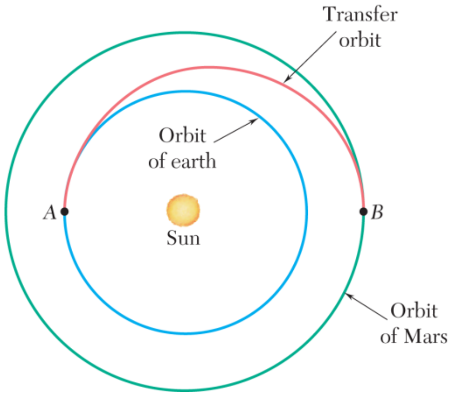
Substitute v_B in (2) into (1)

$$(v_A - \Delta v_A)^2[1-(3.5)^2] = -1814 \times 10^6$$
$$(v_A - \Delta v_A)^2 = 161.24 \times 10^6$$
$$(v_A - \Delta v_A) = \mp 12.698 \times 10^3 \text{ m/s}$$

(Take positive root; negative root reverses flight direction.)

$$v_A = 26.9 \times 10^3 \text{ m/s} \qquad \text{(given)}$$
$$\Delta v_A = (26.9 \times 10^3 \text{ m/s} - 12.698 \times 10^3 \text{ m/s})$$

$\Delta v_A = 14.20$ km/s ◀



PROBLEM 13.104

As a first approximation to the analysis of a space flight from the earth to Mars, it is assumed that the orbits of the earth and Mars are circular and coplanar. The mean distances from the sun to the earth and to Mars are 149.6×10^6 km and 227.8×10^6 km, respectively. To place the spacecraft into an elliptical transfer orbit at Point *A*, its speed is increased over a short interval of time to v_A which is faster than the earth's orbital speed. When the spacecraft reaches Point *B* on the elliptical transfer orbit, its speed v_B is increased to the orbital speed of Mars. Knowing that the mass of the sun is 332.8×10^3 times the mass of the earth, determine the increase in velocity required (*a*) at *A*, (*b*) at *B*.

SOLUTION

M = mass of the sun

$$GM = 332.8(10)^3(9.81\text{ m/s}^2)(6.37 \times 10^6\text{ m})^2 = 1.3247(10)^{20}\text{ m}^3/\text{s}^2$$

Circular orbits

$$\text{Earth } v_E = \sqrt{\frac{GM}{149.6(10)^9}} = 29.758\text{ m/s}$$

$$\text{Mars } v_M = \sqrt{\frac{GM}{227.8(10)^9}} = 24.115\text{ m/s}$$

Conservation of angular momentum

Elliptical orbit $\quad v_A(149.6) = v_B(227.8)$

Conservation of energy

$$\frac{1}{2}v_A^2 - \frac{GM}{149.6(10)^9} = \frac{1}{2}v_B^2 - \frac{GM}{227.8(10)^9}$$

$$v_A = v_B\frac{(227.8)}{(149.6)} = 1.52273v_B$$

$$\frac{1}{2}(1.52273)^2v_B^2 - \frac{1.3247(10)^{20}}{149.6(10)^9} = \frac{1}{2}v_B^2 - \frac{1.3247(10)^{20}}{227.8(10)^9}$$

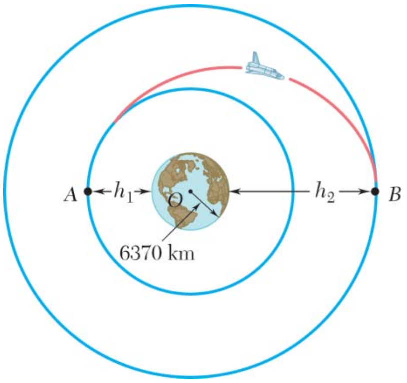
$$0.65935v_B^2 = 3.0398(10)^8$$

$$v_B^2 = 4.6102(10)^8$$

$$v_B = 21{,}471\text{ m/s},\; v_A = 32{,}695\text{ m/s}$$

(*a*) Increase at *A*, $\quad v_A - v_E = 32.695 - 29.758 = 2.94\text{ km/s}$ ◀

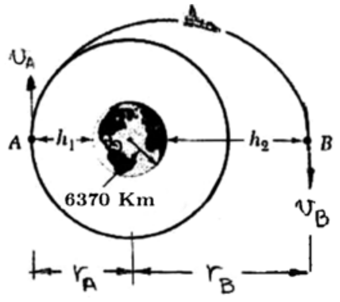
(*b*) Increase at *B*, $\quad v_B - v_M = 24.115 - 21.471 = 2.64\text{ km/s}$ ◀



PROBLEM 13.105

The optimal way of transferring a space vehicle from an inner circular orbit to an outer coplanar circular orbit is to fire its engines as it passes through *A* to increase its speed and place it in an elliptic transfer orbit. Another increase in speed as it passes through *B* will place it in the desired circular orbit. For a vehicle in a circular orbit about the earth at an altitude $h_1 = 200$ mi, which is to be transferred to a circular orbit at an altitude $h_2 = 500$ mi, determine (*a*) the required increases in speed at *A* and at *B*, (*b*) the total energy per unit mass required to execute the transfer.

SOLUTION



Elliptical orbit between *A* and *B*

Conservation of angular momentum

$$mr_A v_A = mr_B v_B$$

$$v_A = \frac{r_B}{r_A} v_B = \frac{7.170}{6.690} v_B$$

$$r_A = 6370\text{ km} + 320\text{ km} = 6690\text{ km}, \qquad r_A = 6.690 \times 10^6\text{ m}$$

$$v_A = 1.0718 v_B \qquad (1)$$

$$r_B = 6370\text{ km} + 800\text{ km} = 7170\text{ km}, \qquad r_B = 7.170 \times 10^6\text{ m}$$

$$R = (6370\text{ km}) = 6.37 \times 10^6\text{ m}$$

Conservation of energy

$$GM = gR^2 = (9.81\text{ m/s}^2)(6.37 \times 10^6\text{ m})^2 = 398.060 \times 10^{12}\text{ m}^3/\text{s}^2$$

Point A:

$$T_A = \frac{1}{2} m v_A^2 \qquad V_A = -\frac{GMm}{r_A} = -\frac{(398.060 \times 10^{12})m}{(6.690 \times 10^6)}$$

$$V_A = 59.501 \times 10^6\text{ m}$$

Point B:

$$T_B = \frac{1}{2} m v_B^2 \qquad V_B = -\frac{GMm}{r_B} = -\frac{(398.060 \times 10^{12})m}{(7.170 \times 10^6)}$$

$$V_B = 55.5 \times 10^6\text{ m}$$

PROBLEM 13.105 (Continued)

$$T_A + V_A = T_B + V_B$$

$$\frac{1}{2}mv_A^2 - 59.501 \times 10^6 \text{ m} = \frac{1}{2}mv_B^2 - 55.5 \times 10^6 \text{ m}$$

$$v_A^2 - v_B^2 = 8.002 \times 10^6$$

From (1) $\quad v_A = 1.0718v_B \qquad v_B^2[(1.0718)^2 - 1] = 8.002 \times 10^6$

$$v_B^2 = 53.79 \times 10^6 \text{ m}^2/\text{s}^2, \qquad v_B = 7334 \text{ m/s}$$

$$v_A = (1.0718)(7334 \text{ m/s}) = 7861 \text{ m/s}$$

Circular orbit at *A* and *B*

(Equation 12.44)

$$(v_A)_C = \sqrt{\frac{GM}{r_A}} = \sqrt{\frac{398.060 \times 10^{12}}{6.690 \times 10^6}} = 7714 \text{ m/s}$$

$$(v_B)_C = \sqrt{\frac{GM}{r_B}} = \sqrt{\frac{398.060 \times 10^{12}}{7.170 \times 10^6}} = 7451 \text{ m/s}$$

(*a*) Increases in speed at *A* and *B*

$$\Delta v_A = v_A - (v_A)_C = 7861 - 7714 = 147 \text{ m/s} \blacktriangleleft$$

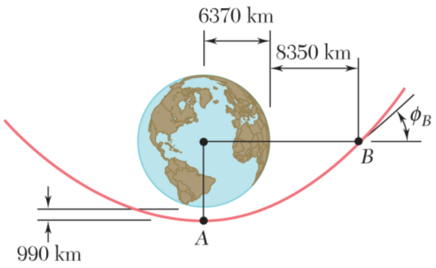
$$\Delta v_B = (v_B)_C - v_B = 7451 - 7334 = 117 \text{ m/s} \blacktriangleleft$$

(*b*) Total energy per unit mass

$$E/m = \frac{1}{2}[(v_A)^2 - (v_A)_C^2 + (v_B)_C^2 - (v_B)^2]$$

$$E/m = \frac{1}{2}[(7861)^2 - (7714)^2 + (7451)^2 - (7334)^2]$$

$$E/m = 2.01 \times 10^6 \text{ J/kg} \blacktriangleleft$$



PROBLEM 13.106

During a flyby of the earth, the velocity of a spacecraft is 10.4 km/s as it reaches its minimum altitude of 990 km above the surface at Point A. At Point B the spacecraft is observed to have an altitude of 8350 km. Determine (a) the magnitude of the velocity at Point B, (b) the angle ϕ_B.

SOLUTION

At A:

$$h_A = vr = [1.04(10)^4 \text{ m/s}][6.37(10)^6 \text{ m} + 0.990(10)^6 \text{ m}]$$

$$h_A = 76.544(10)^9 \text{ m}^2/\text{s}$$

$$\frac{1}{m}(T_A + V_A) = \frac{1}{2}v^2 - \frac{GM}{r}$$

$$= \frac{1}{2}[1.04(10)^4]^2 - \frac{(9.81)[6.37(10)^6]^2}{[6.37(10)^6 + 0.990(10)^6]} \cong 0$$

(Parabolic orbit)

At B:

$$\frac{1}{m}(T_B + V_B) = \frac{1}{2}v_B^2 - \frac{GM}{r_B} = 0$$

$$\frac{1}{2}v_B^2 = \frac{(9.81)[6.37(10^6)]^2}{[6.37(10)^6 + 8.35(10)^6]}$$

$$v_B^2 = 54.084(10)^6$$

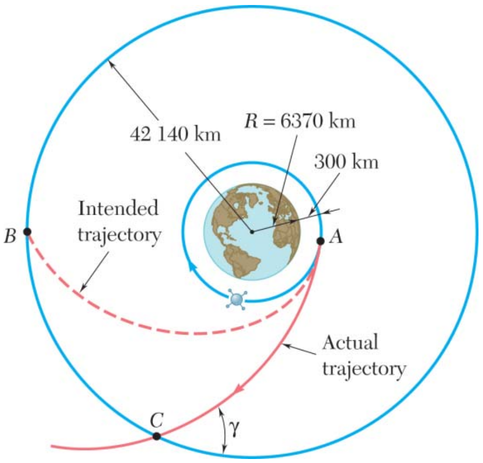
(a) $v_B = 7.35$ km/s ◀

$$h_B = v_B \sin\phi_B r_B = 76.544(10)^9$$

$$\sin\phi_B = \frac{76.544(10)^9}{7.35(10^6)[6.37(10)^6 + 8.35(10)^6]}$$

$$= 0.707483$$

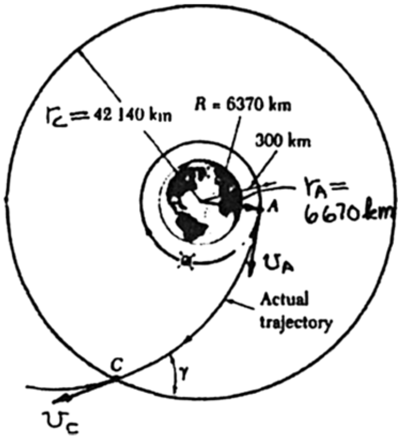
(b) $\phi_B = 45.0°$ ◀


42 140 km
R = 6370 km
300 km
Intended trajectory
B
A
Actual trajectory
C
γ

PROBLEM 13.107

A space platform is in a circular orbit about the earth at an altitude of 300 km. As the platform passes through *A*, a rocket carrying a communications satellite is launched from the platform with a relative velocity of magnitude 3.44 km/s in a direction tangent to the orbit of the platform. This was intended to place the rocket in an elliptic transfer orbit bringing it to Point *B*, where the rocket would again be fired to place the satellite in a geosynchronous orbit of radius 42,140 km. After launching, it was discovered that the relative velocity imparted to the rocket was too large. Determine the angle γ at which the rocket will cross the intended orbit at Point *C*.

SOLUTION


r_C = 42 140 km
R = 6370 km
300 km
r_A = 6670 km
A
υ_A
Actual trajectory
C
γ
υ_C

$$R = 6370 \text{ km}$$

$$r_A = 6370 \text{ km} + 300 \text{ km}$$

$$r_A = 6.67 \times 10^6 \text{ m}$$

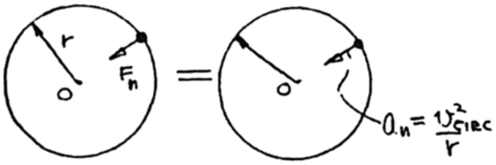
$$r_C = 42.14 \times 10^6 \text{ m}$$

$$GM = gR^2$$

$$GM = (9.81 \text{ m/s})(6.37 \times 10^6 \text{ m})^2$$

$$GM = 398.1 \times 10^{12} \text{ m}^3/\text{s}^2$$

For any circular orbit:


r
O
F_n
=
O
$a_n = \frac{v^2_{\text{circ}}}{r}$

$$F_n = ma_n = \frac{mv^2_{\text{circ}}}{r}$$

$$F_n = \frac{GMm}{r^2} = m\frac{mv^2_{\text{circ}}}{r}$$

$$v_{\text{circ}} = \sqrt{\frac{GM}{r}}$$

Velocity at *A*:

$$(v_A)_{\text{circ}} = \sqrt{\frac{GM}{r_A}} = \sqrt{\frac{(398.1 \times 10^{12} \text{ m}^3/\text{s}^3)}{(6.67 \times 10^6 \text{ m})}} = 7.726 \times 10^3 \text{ m/s}$$

$$v_A = (v_A)_{\text{circ}} + (v_A)_R = 7.726 \times 10^3 + 3.44 \times 10^3 = 11.165 \times 10^3 \text{ m/s}$$

PROBLEM 13.107 (Continued)

Velocity at *C*:

Conservation of energy: $T_A + V_A = T_C + V_C$

$$\frac{1}{2}\cancel{m}v_A^2 - \frac{GM\cancel{m}}{r_A} = \frac{1}{2}\cancel{m}v_C^2 - \frac{GM\cancel{m}}{r_C}$$

$$v_C^2 = v_A^2 + 2GM\left(\frac{1}{r_C} - \frac{1}{r_A}\right)$$

$$= (11.165\times10^3)^2 + 2(398.1\times10^{12})\left(\frac{1}{42.14\times10^6} - \frac{1}{6.67\times10^6}\right)$$

$$v_C^2 = 124.67\times10^6 - 100.48\times10^6$$

$$= 24.19\times10^6 \text{ m}^2/\text{s}^2$$

$$v_C = 4.919\times10^3 \text{ m/s}$$

Conservation of angular momentum:

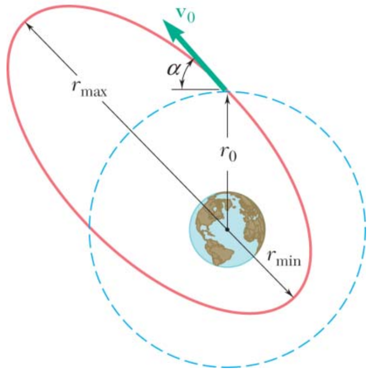
$$r_A m v_A = r_C m v_C \cos\gamma$$

$$\cos\gamma = \frac{r_A v_A}{r_C v_C}$$

$$= \frac{(6.67\times10^6)(11.165\times10^3)}{(42.14\times10^6)(4.919\times10^3)}$$

$$\cos\gamma = 0.35926$$

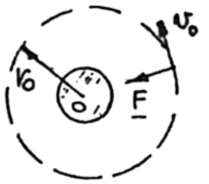
$\gamma = 68.9°$ ◀



PROBLEM 13.108

A satellite is projected into space with a velocity $\mathbf{v}_0$ at a distance r_0 from the center of the earth by the last stage of its launching rocket. The velocity $\mathbf{v}_0$ was designed to send the satellite into a circular orbit of radius r_0. However, owing to a malfunction of control, the satellite is not projected horizontally but at an angle α with the horizontal and, as a result, is propelled into an elliptic orbit. Determine the maximum and minimum values of the distance from the center of the earth to the satellite.

SOLUTION



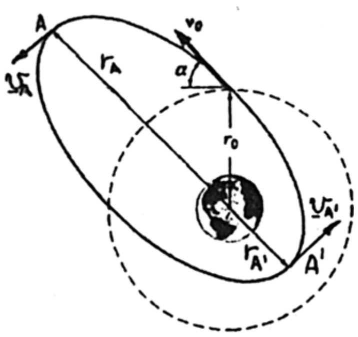
For circular orbit of radius r_0

$$F = ma_n \qquad \frac{GMm}{r_0^2} = m\frac{v_0^2}{r_0}$$

$$v_0^2 = \frac{GM}{r_0}$$

But v_0 forms an angle α with the intended circular path.

For elliptic orbit.



Conservation of angular momentum:

$$r_0 m v_0 \cos\alpha = r_A m v_A$$

$$v_A = \left(\frac{r_0}{r_A}\cos\alpha\right)v_0 \tag{1}$$

Conservation of energy:

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \frac{1}{2}mv_A^2 - \frac{GMm}{r_A}$$

$$v_0^2 - v_A^2 = \frac{2GM}{r_0}\left(1 - \frac{r_0}{r_A}\right)$$

Substitute for v_A from (1)

$$v_0^2\left[1 - \left(\frac{r_0}{r_A}\right)^2 \cos^2\alpha\right] = \frac{2GM}{r_0}\left(1 - \frac{r_0}{r_A}\right)$$

But $\quad v_0^2 = \dfrac{GM}{r_0}, \qquad$ thus $\quad 1 - \left(\dfrac{r_0}{r_A}\right)^2 \cos^2\alpha = 2\left(1 - \dfrac{r_0}{r_A}\right)$

$$\cos^2\alpha\left(\frac{r_0}{r_A}\right)^2 - 2\left(\frac{r_0}{r_A}\right) + 1 = 0$$

PROBLEM 13.108 (Continued)

Solving for $\frac{r_0}{r_A}$

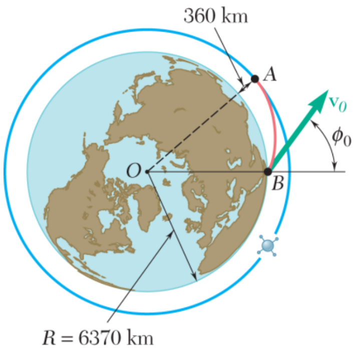
$$\frac{r_0}{r_A} = \frac{+2 \pm \sqrt{4 - 4\cos^2\alpha}}{2\cos^2\alpha} = \frac{1 \pm \sin\alpha}{1 - \sin^2\alpha}$$

$$r_A = \frac{(1+\sin\alpha)(1-\sin\alpha)}{1 \pm \sin\alpha} r_0 = (1 \mp \sin\alpha) r_0$$

also valid for Point A'

Thus,

$$r_{\max} = (1+\sin\alpha) r_0 \qquad r_{\min} = (1-\sin\alpha) r_0$$ ◀



PROBLEM 13.109

A space vehicle is to rendezvous with an orbiting laboratory which circles the earth at a constant altitude of 360 km. The shuttle has reached an altitude of 60 km when its engine is shut off, and its velocity $\mathbf{v}_0$ forms an angle $\phi_0 = 50°$ with the vertical OB at that time. What magnitude should $\mathbf{v}_0$ have if the shuttle's trajectory is to be tangent at A to the orbit of the laboratory?

SOLUTION

Conservation of energy

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r_B} = \frac{1}{2}mv_A^2 - \frac{GMm}{r_A}$$

So

$$v_A^2 = v_0^2 - \frac{2GM}{r_B}\left(1 - \frac{r_B}{r_A}\right)$$

(1)

Given

$$R = 6370 \text{ km} = 6.37 \times 10^6 \text{ m}$$

$$GM = gR^2 = (9.81)\left(6.37 \times 10^6\right)^2 = 398 \times 10^{12}$$

$$r_A = 6370 + 360 = 6730 \text{ km} = 6.73 \times 10^6 \text{ m}$$

$$r_B = 6370 + 60 = 6430 \text{ km} = 6.43 \times 10^6 \text{ m}$$

Substitute into (1)

$$v_A^2 = v_0^2 - \frac{2\left(398 \times 10^{12}\right)}{6.43 \times 10^6}\left(1 - \frac{6.43 \times 10^6}{6.73 \times 10^6}\right)$$

$$v_A^2 = v_0^2 - 5.518 \times 10^6 \qquad (2)$$

We need another equation → conservation of angular momentum

$$r_B mv \sin\phi = r_A m v_A$$

$$v_A = \frac{r_B v_0 \sin\phi}{r_A} = \left(\frac{6.43 \times 10^6}{6.73 \times 10^6}\right) v_0 \sin 50°$$

$$v_A = 0.7319 v_0$$

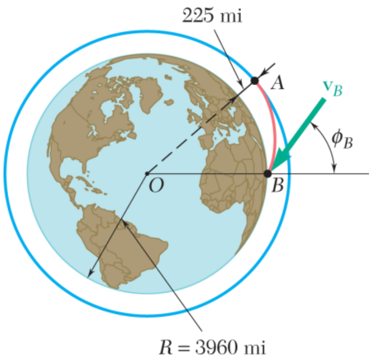
Substitute into (2)

$$\left(0.7319 v_0\right)^2 = v_0^2 - 5.518 \times 10^6$$

$$0.46433 v_0^2 = 5.518 \times 10^6$$

$$v_0 = 3477 \text{ m/s}$$

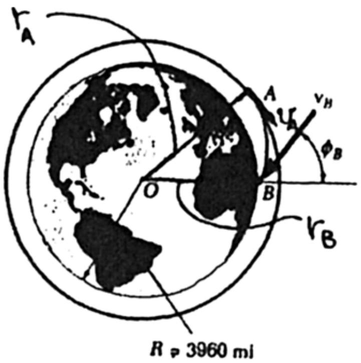
$v_0 = 3450$ m/s ◀



PROBLEM 13.110

A space vehicle is in a circular orbit at an altitude of 225 mi above the earth. To return to earth, it decreases its speed as it passes through *A* by firing its engine for a short interval of time in a direction opposite to the direction of its motion. Knowing that the velocity of the space vehicle should form an angle $\phi_B = 60°$ with the vertical as it reaches Point *B* at an altitude of 40 mi, determine (*a*) the required speed of the vehicle as it leaves its circular orbit at *A*, (*b*) its speed at Point *B*.

SOLUTION



(*a*)

$$r_A = 3960 \text{ mi} + 225 \text{ mi} = 4185 \text{ mi}$$

$$r_A = 4185 \text{ mi} \times 5280 \text{ ft/mi} = 22{,}097 \times 10^3 \text{ ft}$$

$$r_B = 3960 \text{ mi} + 40 \text{ mi} = 4000 \text{ mi}$$

$$r_B = 4000 \times 5280 = 21{,}120 \times 10^3 \text{ ft}$$

$$R = 3960 \text{ mi} = 20{,}909 \times 10^3 \text{ ft}$$

$$GM = gR^2 = (32.2 \text{ ft/s}^2)(20{,}909 \times 10^3 \text{ ft})^2$$

$$GM = 14.077 \times 10^{15} \text{ ft}^3/\text{s}^2$$

Conservation of energy:

$$T_A = \frac{1}{2}mv_A^2$$

$$V_A = \frac{-GMm}{r_A}$$

$$= \frac{-14.077 \times 10^{15} m}{22{,}097 \times 10^3}$$

$$= -637.1 \times 10^6 m$$

$$T_B = \frac{1}{2}mv_B^2$$

$$V_B = \frac{-GMm}{r_B}$$

$$= \frac{-14.077 \times 10^{15} m}{21{,}120 \times 10^3}$$

$$= -666.5 \times 10^6 m$$

$$T_A + V_A = T_B + V_B$$

$$\frac{1}{2}\cancel{m}v_A^2 - 637.1 \times 10^6 \cancel{m} = \frac{1}{2}\cancel{m}v_B^2 - 666.5 \times 10^6 \cancel{m}$$

$$v_A^2 = v_B^2 - 58.94 \times 10^6 \qquad (1)$$

PROBLEM 13.110 (Continued)

Conservation of angular momentum:

$$r_A m v_A = r_B m v_B \sin\phi_B$$

$$v_B = \frac{(r_A)v_A}{(r_B)(\sin\phi_B)} = \frac{4185}{4000}\left(\frac{1}{\sin 60°}\right)v_A$$

$$v_B = 1.208\, v_A \qquad (2)$$

Substitute v_B from (2) in (1)

$$v_A^2 = (1.208 v_A)^2 - 58.94\times 10^6$$

$$v_A^2[(1.208)^2 - 1] = 58.94\times 10^6$$

$$v_A^2 = 128.27\times 10^6 \text{ ft}^2/\text{s}^2$$

(*a*) $v_A = 11.32\times 10^3$ ft/s ◀

(*b*) From (2)

$$v_B = 1.208 v_A$$

$$= 1.208(11.32\times 10^6)$$

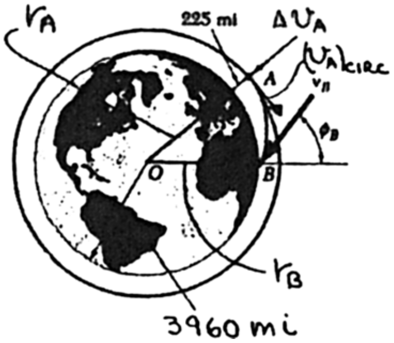
$$= 13.68\times 10^3 \text{ ft/s}$$

$v_B = 13.68\times 10^3$ ft/s ◀

PROBLEM 13.111*

In Problem 13.110, the speed of the space vehicle was decreased as it passed through *A* by firing its engine in a direction opposite to the direction of motion. An alternative strategy for taking the space vehicle out of its circular orbit would be to turn it around so that its engine would point away from the earth and then give it an incremental velocity $\Delta\mathbf{v}_A$ toward the center *O* of the earth. This would likely require a smaller expenditure of energy when firing the engine at *A*, but might result in too fast a descent at *B*. Assuming this strategy is used with only 50 percent of the energy expenditure used in Problem 13.109, determine the resulting values of ϕ_B and v_B.

SOLUTION



$$r_A = 3960 \text{ mi} + 225 \text{ mi}$$

$$r_A = 4185 \text{ mi} = 22.097 \times 10^6 \text{ ft}$$

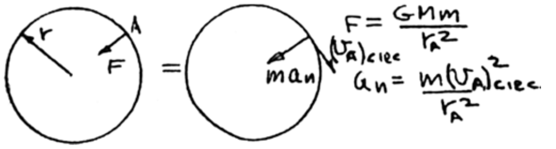
$$r_B = 3960 \text{ mi} + 40 \text{ mi} = 4000 \text{ mi}$$

$$r_B = 21.120 \times 10^6 \text{ ft}$$

$$GM = gR^2 = (32.2 \text{ ft/s}^2)[(3960)(5280) \text{ ft}^2]$$

$$GM = 14.077 \times 10^{15} \text{ ft}^3/\text{s}^2$$

Velocity in circular orbit at 225 m altitude:



Newton's second law

$$F = ma_n: \quad \frac{GMm}{r_A^2} = \frac{m(v_A)_{\text{circ}}^2}{r_A}$$

$$(v_A)_{\text{circ}} = \sqrt{\frac{GM}{r_A}} = \sqrt{\frac{14.077 \times 10^{15}}{22.097 \times 10^6}}$$

$$= 25.24 \times 10^3 \text{ ft/s}$$

Energy expenditure:

From Problem 13.110, $\quad v_A = 11.32 \times 10^3 \text{ ft/s}$

Energy, $\quad \Delta E_{109} = \frac{1}{2}m(v_A)_{\text{circ}}^2 - \frac{1}{2}mv_A^2$

$$\Delta E_{109} = \frac{1}{2}m(25.24 \times 10^3)^2 - \frac{1}{2}m(11.32 \times 10^3)^2$$

$$\Delta E_{109} = 254.46 \times 10^6\, m \text{ ft} \cdot \text{lb}$$

$$\Delta E_{110} = (0.50)\Delta E_{109} = \frac{(254.46 \times 10^6\, m)}{2} \text{ft} \cdot \text{lb}$$

PROBLEM 13.111* (Continued)

Thus, additional kinetic energy at A is

$$\frac{1}{2}m(\Delta v_A)^2 = \Delta E_{110} = \frac{(254.46\times 10^6 m)}{2}\text{ ft}\cdot\text{lb} \qquad (1)$$

Conservation of energy between A and B:

$$T_A = \frac{1}{2}m[(v_A)^2_{\text{circ}} + (\Delta v_A)^2] \qquad V_A = \frac{-GMm}{r_A}$$

$$T_B = \frac{1}{2}mv_B^2 \qquad V_B = \frac{-GMm}{r_A}$$

$$T_A + V_A = T_B + V_B$$

$$\frac{1}{2}m(25.24\times 10^3)^2 + \frac{254.46\times 10^6 m}{2} - \frac{14.077\times 10^{15} m}{22.097\times 10^6} = \frac{1}{2}mv_B^2 - \frac{14.077\times 10^{15} m}{21.120\times 10^6}$$

$$v_B^2 = 637.06\times 10^6 + 254.46\times 10^6 - 1274.1\times 10^6 + 1333\times 10^6$$
$$v_B^2 = 950.4\times 10^3$$

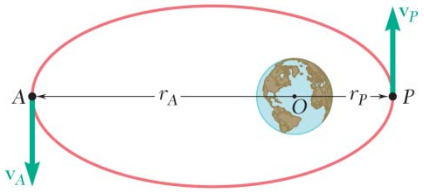
$$v_B = 30.88\times 10^3\text{ ft/s} \blacktriangleleft$$

Conservation of angular momentum between A and B:

$$r_A m(v_A)_{\text{circ}} = r_B m v_B \sin\phi_B$$

$$\sin\phi_B = \left(\frac{r_A}{r_B}\right)\frac{(v_A)_{\text{circ}}}{(v_B)} = \frac{(4185)}{(4000)}\frac{(25.24\times 10^3)}{(30.88\times 10^3)} = 0.8565$$

$$\phi_B = 58.9° \blacktriangleleft$$



PROBLEM 13.112

Show that the values v_A and v_P of the speed of an earth satellite at the apogee A and the perigee P of an elliptic orbit are defined by the relations

$$v_A^2 = \frac{2GM}{r_A + r_P}\frac{r_P}{r_A} \qquad v_P^2 = \frac{2GM}{r_A + r_P}\frac{r_A}{r_P}$$

where M is the mass of the earth, and r_A and r_P represent, respectively, the maximum and minimum distances of the orbit to the center of the earth.

SOLUTION

Conservation of angular momentum:

$$r_A m v_A = r_P m v_P$$

$$v_A = \frac{r_P}{r_A} v_P \qquad (1)$$

Conservation of energy:

$$\frac{1}{2} m v_P^2 - \frac{GMm}{r_P} = \frac{1}{2} m v_A^2 - \frac{GMm}{r_A} \qquad (2)$$

Substituting for v_A from (1) into (2)

$$v_P^2 - \frac{2GM}{r_P} = \left(\frac{r_P}{r_A}\right)^2 v_P^2 - \frac{2GM}{r_A}$$

$$\left(1 - \left(\frac{r_P}{r_A}\right)^2\right) v_P^2 = 2GM\left(\frac{1}{r_P} - \frac{1}{r_A}\right)$$

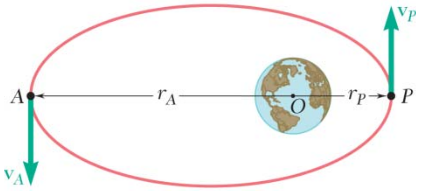
$$\frac{r_A^2 - r_P^2}{r_A^2} v_P^2 = 2GM \frac{r_A - r_P}{r_A r_P}$$

with $r_A^2 - r_P^2 = (r_A - r_P)(r_A + r_P)$

$$v_P^2 = \frac{2GM}{r_A + r_P}\left(\frac{r_A}{r_P}\right) \quad (3)□ \quad ◀$$

Exchanging subscripts P and A

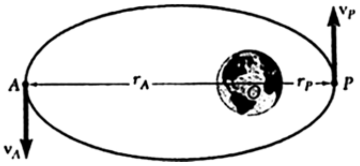
$$v_A^2 = \frac{2GM}{r_A + r_P}\left(\frac{r_P}{r_A}\right) \qquad \text{Q.E.D.} \quad ◀$$



PROBLEM 13.113

Show that the total energy E of an earth satellite of mass m describing an elliptic orbit is $E = -GMm/(r_A + r_P)$, where M is the mass of the earth, and r_A and r_P represent, respectively, the maximum and minimum distances of the orbit to the center of the earth. (Recall that the gravitational potential energy of a satellite was defined as being zero at an infinite distance from the earth.)

SOLUTION



See solution to Problem 13.112 (above) for derivation of Equation (3).

$$v_P^2 = \frac{2GM}{(r_A + r_P)} \frac{r_A}{r_P}$$

Total energy at Point P is

$$E = T_P + V_P = \frac{1}{2} m v_P^2 - \frac{GMm}{r_P}$$

$$= \frac{1}{2}\left[\frac{2GMm}{(r_A + r_0)} \frac{r_A}{r_P}\right] - \frac{GMm}{r_P}$$

$$= GMm\left[\frac{r_A}{r_P(r_A + r_P)} - \frac{1}{r_P}\right]$$

$$= GMm \frac{(r_A - r_A - r_P)}{r_P(r_A + r_P)}$$

$$E = -\frac{GMm}{r_A + r_P} \blacktriangleleft$$

Note: Recall that gravitational potential of a satellite is defined as being zero at an infinite distance from the earth.

PROBLEM 13.114*

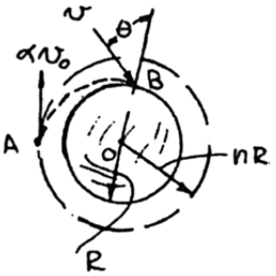
A space probe describes a circular orbit of radius nR with a velocity $\mathbf{v}_0$ about a planet of radius R and center O. Show that (a) in order for the probe to leave its orbit and hit the planet at an angle θ with the vertical, its velocity must be reduced to $\alpha\mathbf{v}_0$, where

$$\alpha = \sin\theta\sqrt{\frac{2(n-1)}{n^2-\sin^2\theta}}$$

(b) the probe will not hit the planet if α is larger than $\sqrt{2/(1+n)}$.

SOLUTION

(a) Conservation of energy:



At A:

$$T_A = \frac{1}{2}m(\alpha v_0)^2$$

$$V_A = -\frac{GMm}{nR}$$

At B:

$$T_B = \frac{1}{2}mv^2$$

$$V_B = -\frac{GMm}{R}$$

M = mass of planet

m = mass of probe

$$T_A + V_A = T_B + V_B$$

$$\frac{1}{2}\cancel{m}(\alpha v_0)^2 - \frac{GM\cancel{m}}{nR} = \frac{1}{2}\cancel{m}v^2 - \frac{GM\cancel{m}}{R} \qquad (1)$$

Conservation of angular momentum:

$$nRm\alpha v_0 = Rmv\sin\theta$$

$$v = \frac{n\alpha v_0}{\sin\theta} \qquad (2)$$

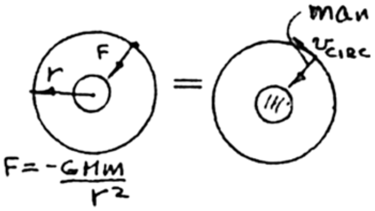
Replacing v in (1) by (2)

$$(\alpha v_0)^2 - \frac{2GM}{nR} = \left(\frac{n\alpha v_0}{\sin\theta}\right)^2 - \frac{2GM}{R} \qquad (3)$$

PROBLEM 13.114* (Continued)

For any circular orbit.

$$a_n = \frac{v^2}{r}$$



Newton's second law

$$\frac{-GMm}{r^2} = \frac{m(v)^2_{\text{circ}}}{r}$$

$$v_{\text{circ}} = \sqrt{\frac{GM}{r}}$$

For $r = nR$, $$v_0 = v_{\text{circ}} = \sqrt{\frac{GM}{nR}}$$

Substitute for v_0 in (3)

$$\alpha^2 \frac{G\cancel{M}}{n\cancel{R}} - \frac{2G\cancel{M}}{n\cancel{R}} = \frac{n^2\alpha^2}{\sin^2\theta}\left(\frac{G\cancel{M}}{n\cancel{R}}\right) - \frac{2G\cancel{M}}{\cancel{R}}$$

$$\alpha^2\left[1 - \frac{n^2}{\sin^2\theta}\right] = 2(1-n)$$

$$\alpha^2 = \frac{2(1-n)(\sin^2\theta)}{(\sin^2\theta - n^2)} = \frac{2(n-1)\sin^2\theta}{(n^2 - \sin^2\theta)}$$

$$\alpha = \sin\theta\sqrt{\frac{2(n-1)}{n^2 - \sin^2\theta}} \qquad \text{Q.E.D.}$$ ◀

(*b*) Probe will just miss the planet if $\theta > 90°$,

$$\alpha = \sin 90°\sqrt{\frac{2(n-1)}{n^2 - \sin^2 90°}} = \sqrt{\frac{2}{n+1}}$$ ◀

Note: $$n^2 - 1 = (n-1)(n+1)$$

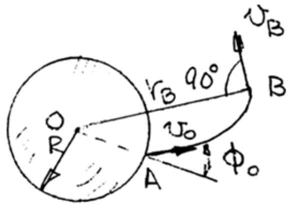
PROBLEM 13.115

A missile is fired from the ground with an initial velocity $\mathbf{v}_0$ forming an angle ϕ_0 with the vertical. If the missile is to reach a maximum altitude equal to αR, where R is the radius of the earth, (*a*) show that the required angle ϕ_0 is defined by the relation

$$\sin\phi_0 = (1+\alpha)\sqrt{1-\frac{\alpha}{1+\alpha}\left(\frac{v_{\text{esc}}}{v_0}\right)^2}$$

where v_{esc} is the escape velocity, (*b*) determine the range of allowable values of v_0.

SOLUTION



(*a*) $$r_A = R$$

Conservation of angular momentum:

$$Rmv_0\sin\phi_0 = r_B m v_B$$

$$r_B = R+\alpha R = (1+\alpha)R$$

$$v_B = \frac{Rv_0\sin\phi_0}{(1+\alpha)R} = \frac{v_0\sin\phi_0}{(1+\alpha)} \qquad (1)$$

Conservation of energy:

$$T_A + V_A = T_B + V_B \qquad \frac{1}{2}mv_0^2 - \frac{GMm}{R} = \frac{1}{2}mv_B^2 - \frac{GMm}{(1+\alpha)R}$$

$$v_0^2 - v_B^2 = \frac{2GMm}{R}\left(1-\frac{1}{1+\alpha}\right) = \frac{2GMm}{R}\left(\frac{\alpha}{1+\alpha}\right)$$

Substitute for v_B from (1)

$$v_0^2\left(1-\frac{\sin^2\phi_0}{(1+\alpha)^2}\right) = \frac{2GMm}{R}\left(\frac{\alpha}{1+\alpha}\right)$$

From Equation (12.43): $$v_{\text{esc}}^2 = \frac{2GM}{R}$$

$$v_0^2\left(1-\frac{\sin^2\phi_0}{(1+\alpha)^2}\right) = v_{\text{esc}}^2\left(\frac{\alpha}{1+\alpha}\right)$$

$$\frac{\sin^2\phi_0}{(1+\alpha)^2} = 1-\left(\frac{v_{\text{esc}}}{v_0}\right)^2\frac{\alpha}{1+\alpha} \qquad (2)$$

$$\sin\phi_0 = (1+\alpha)\sqrt{1-\frac{\alpha}{1+\alpha}\left(\frac{v_{\text{esc}}}{v_0}\right)^2} \qquad \text{Q.E.D.}$$

(*b*) Allowable values of v_0 (for which maximum altitude = αR)

$$0 < \sin^2 \phi_0 < 1$$

For $\sin \phi_0 = 0$, from (2)

$$0 = 1 - \left(\frac{v_{\text{esc}}}{v_0}\right)^2 \frac{\alpha}{1+\alpha}$$

$$v_0 = v_{\text{esc}} \sqrt{\frac{\alpha}{1+\alpha}}$$

For $\sin \phi_0 = 1$, from (2)

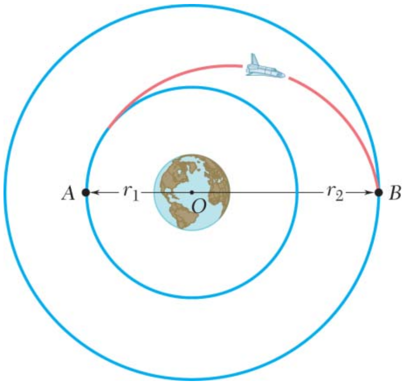
$$\frac{1}{(1+\alpha)^2} = 1 - \left(\frac{v_{\text{esc}}}{v_0}\right)^2 \frac{\alpha}{1+\alpha}$$

$$\left(\frac{v_{\text{esc}}}{v_0}\right)^2 = \frac{1}{\alpha}\left(1+\alpha - \frac{1}{1+\alpha}\right) = \frac{1+2\alpha+\alpha^2-1}{\alpha(1+\alpha)} = \frac{2+\alpha}{1+\alpha}$$

$$v_0 = v_{\text{esc}} \sqrt{\frac{1+\alpha}{2+\alpha}}$$

$$v_{\text{esc}} \sqrt{\frac{\alpha}{1+\alpha}} < v_0 < v_{\text{esc}} \sqrt{\frac{1+\alpha}{2+\alpha}} \blacktriangleleft$$

PROBLEM 13.116



A spacecraft of mass m describes a circular orbit of radius r_1 around the earth. (*a*) Show that the additional energy ΔE which must be imparted to the spacecraft to transfer it to a circular orbit of larger radius r_2 is

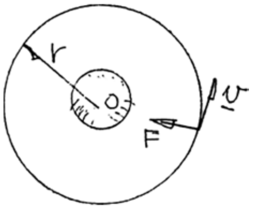
$$\Delta E = \frac{GMm(r_2 - r_1)}{2r_1r_2}$$

where M is the mass of the earth. (*b*) Further show that if the transfer from one circular orbit to the other is executed by placing the spacecraft on a transitional semielliptic path AB, the amounts of energy ΔE_A and ΔE_B which must be imparted at A and B are, respectively, proportional to r_2 and r_1:

$$\Delta E_A = \frac{r_2}{r_1 + r_2}\Delta E \qquad \Delta E_B = \frac{r_1}{r_1 + r_2}\Delta E$$

SOLUTION

(*a*) For a *circular* orbit of radius r



$$F = ma_n: \quad \frac{GMm}{r^2} = m\frac{v^2}{r}$$

$$v^2 = \frac{GM}{r}$$

$$E = T + V = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{1}{2}\frac{GMm}{r} \qquad (1)$$

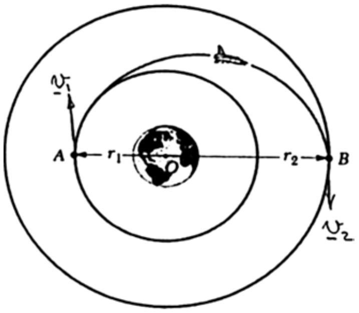
Thus ΔE required to pass from circular orbit of radius r_1 to circular orbit of radius r_2 is

$$\Delta E = E_1 - E_2 = -\frac{1}{2}\frac{GMm}{r_1} + \frac{1}{2}\frac{GMm}{r_2}$$

$$\Delta E = \frac{GMm(r_2 - r_1)}{2r_1r_2} \qquad \text{Q.E.D.} \qquad (2)$$

(*b*) For an elliptic orbit, we recall Equation (3) derived in Problem 13.113 (with $v_P = v_1$)

$$v_1^2 = \frac{2Gm}{(r_1 + r_2)}\frac{r_2}{r_1}$$



PROBLEM 13.116 (Continued)

At Point *A*: Initially spacecraft is in a circular orbit of radius r_1.

$$v_{\text{circ}}^2 = \frac{GM}{r_1}$$

$$T_{\text{circ}} = \frac{1}{2}mv_{\text{circ}}^2 = \frac{1}{2}m\frac{GM}{r_1}$$

After the spacecraft engines are fired and it is placed on a semi-elliptic path *AB*, we recall

$$v_1^2 = \frac{2GM}{(r_1 + r_2)} \cdot \frac{r_2}{r_1}$$

and

$$T_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}m\frac{2GMr_2}{r_1(r_1 + r_2)}$$

At Point *A*, the increase in energy is

$$\Delta E_A = T_1 - T_{\text{circ}} = \frac{1}{2}m\frac{2GMr_2}{r_1(r_1 + r_2)} - \frac{1}{2}m\frac{GM}{r_1}$$

$$\Delta E_A = \frac{GMm(2r_2 - r_1 - r_2)}{2r_1(r_1 + r_2)} = \frac{GMm(r_2 - r_1)}{2r_1(r_1 + r_2)}$$

$$\Delta E_A = \frac{r_2}{r_1 + r_2}\left[\frac{GMm(r_2 - r_1)}{2r_1r_2}\right]$$

Recall Equation (2):

$$\Delta E_A = \frac{r_2}{(r_1 + r_2)}\Delta E \qquad \text{Q.E.D.}$$

A similar derivation at Point *B* yields,

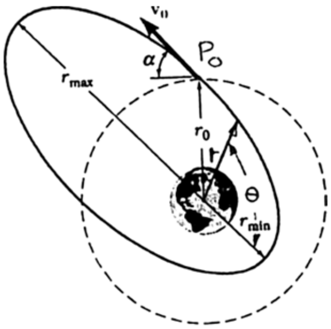
$$\Delta E_B = \frac{r_1}{(r_1 + r_2)}\Delta E \qquad \text{Q.E.D.}$$

PROBLEM 13.117*

Using the answers obtained in Problem 13.108, show that the intended circular orbit and the resulting elliptic orbit intersect at the ends of the minor axis of the elliptic orbit.

PROBLEM 13.108 A satellite is projected into space with a velocity $\mathbf{v}_0$ at a distance r_0 from the center of the earth by the last stage of its launching rocket. The velocity $\mathbf{v}_0$ was designed to send the satellite into a circular orbit of radius r_0. However, owing to a malfunction of control, the satellite is not projected horizontally but at an angle α with the horizontal and, as a result, is propelled into an elliptic orbit. Determine the maximum and minimum values of the distance from the center of the earth to the satellite.

SOLUTION



If the point of intersection P_0 of the circular and elliptic orbits is at an end of the minor axis, then v_0 is parallel to the major axis. This will be the case only if $\alpha + 90° = \theta_0$, that is if $\cos\theta_0 = -\sin\alpha$. We must therefore prove that

$$\cos\theta_0 = -\sin\alpha \tag{1}$$

We recall from Equation (12.39):

$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta \tag{2}$$

When $\theta = 0$, $r = r_{\text{min}}$ and $r_{\text{min}} = r_0(1-\sin\alpha)$

$$\frac{1}{r_0(1-\sin\alpha)} = \frac{GM}{h^2} + C \tag{3}$$

For $\theta = 180°$, $r = r_{\text{max}} = r_0(1+\sin\alpha)$

$$\frac{1}{r_0(1+\sin\alpha)} = \frac{GM}{h^2} - C \tag{4}$$

Adding (3) and (4) and dividing by 2:

$$\frac{GM}{h^2} = \frac{1}{2r_0}\left(\frac{1}{1-\sin\alpha} + \frac{1}{1+\sin\alpha}\right)$$
$$= \frac{1}{r_0\cos^2\alpha}$$

Subtracting (4) from (3) and dividing by 2:

$$C = \frac{1}{2r_0}\left(\frac{1}{1-\sin\alpha} - \frac{1}{1+\sin\alpha}\right) = \left(\frac{1}{2r_0}\right)\frac{2\sin\alpha}{1-\sin^2\alpha}$$
$$C = \frac{\sin\alpha}{r_0\cos^2\alpha}$$

PROBLEM 13.117* (Continued)

Substituting for $\frac{GM}{h^2}$ and C into Equation (2)

$$\frac{1}{r}=\frac{1}{r_0\cos^2\alpha}(1+\sin\alpha\cos\theta) \tag{5}$$

Letting $r=r_0$ and $\theta=\theta_0$ in Equation (5), we have

$$\cos^2\alpha=1+\sin\alpha\cos\theta_0$$

$$\cos\theta_0=\frac{\cos^2\alpha-1}{\sin\alpha}$$

$$=-\frac{\sin^2\alpha}{\sin\alpha}$$

$$=-\sin\alpha$$

This proves the validity of Equation (1) and thus P_0 is an end of the minor axis of the elliptic orbit.

PROBLEM 13.118*

(*a*) Express in terms of $r_{\min}$ and $v_{\max}$ the angular momentum per unit mass, *h*, and the total energy per unit mass, *E*/*m*, of a space vehicle moving under the gravitational attraction of a planet of mass *M* (Figure 13.15).
(*b*) Eliminating $v_{\max}$ between the equations obtained, derive the formula

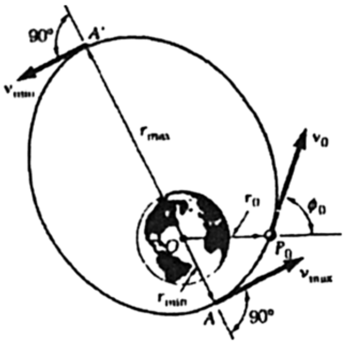
$$\frac{1}{r_{\min}} = \frac{GM}{h^2}\left[1+\sqrt{1+\frac{2E}{m}\left(\frac{h}{GM}\right)^2}\right]$$

(*c*) Show that the eccentricity ε of the trajectory of the vehicle can be expressed as

$$\varepsilon = \sqrt{1+\frac{2E}{m}\left(\frac{h}{GM}\right)^2}$$

(*d*) Further show that the trajectory of the vehicle is a hyperbola, an ellipse, or a parabola, depending on whether *E* is positive, negative, or zero.

SOLUTION



(*a*) Point *A*:

Angular momentum per unit mass.

$$h = \frac{H_0}{m}$$

$$= \frac{r_{\min} m v_{\max}}{m}$$

$$h = r_{\min} v_{\max} \qquad (1) \blacktriangleleft$$

Energy per unit mass

$$\frac{E}{m} = \frac{1}{m}(T+V)$$

$$\frac{E}{m} = \frac{1}{m}\left(\frac{1}{2}mv_{\max}^2 - \frac{GMm}{r_{\min}}\right) = \frac{1}{2}v_{\max}^2 - \frac{GM}{r_{\min}} \qquad (2) \blacktriangleleft$$

(*b*) From Eq. (1): $v_{\max} = h/r_{\min}$ substituting into (2)

$$\frac{E}{m} = \frac{1}{2}\frac{h^2}{r_{\min}^2} - \frac{GM}{r_{\min}}$$

$$\left(\frac{1}{r_{\min}}\right)^2 - \frac{2GM}{h^2}\cdot\frac{1}{r_{\min}} - \frac{2\left(\frac{E}{m}\right)}{h^2} = 0$$

Solving the quadratic: $$\frac{1}{r_{\min}} = \frac{GM}{h^2} + \sqrt{\left(\frac{GM}{h^2}\right)^2 + \frac{2\left(\frac{E}{m}\right)}{h^2}}$$

Rearranging

$$\frac{1}{r_{\min}} = \frac{GM}{h^2}\left[1 + \sqrt{1 + \frac{2E}{m}\left(\frac{h}{GM}\right)^2}\right] \tag{3}$$ ◀

(*c*) Eccentricity of the trajectory:

Eq. (12.39′) $$\frac{1}{r} = \frac{GM}{h^2}(1 + \varepsilon\cos\theta)$$

When $\theta = 0$, $\cos\theta = 1$ and $r = r_{\min}$

Thus,

$$\frac{1}{r_{\min}} = \frac{GM}{h^2}(1 + \varepsilon) \tag{4}$$

Comparing (3) and (4), $$\varepsilon = \sqrt{1 + \frac{2E}{m}\left(\frac{h}{GM}\right)^2} \tag{5}$$

(*d*) Recalling discussion in section 12.12 and in view of Eq. (5)

1. Hyperbola if $\varepsilon > 1$, that is, if $E > 0$ ◀
2. Parabola if $\varepsilon = 1$, that is, if $E = 0$ ◀
3. Ellipse if $\varepsilon < 1$, that is, if $E < 0$ ◀

Note: For circular orbit $\varepsilon = 0$ and

$$1 + \frac{2E}{m}\left(\frac{h}{GM}\right)^2 = 0 \qquad \text{or} \qquad E = -\left(\frac{GM}{h}\right)^2 \frac{m}{2},$$

but for circular orbit $$v^2 = \frac{GM}{r} \qquad \text{and} \qquad h^2 = v^2 r^2 = GMr,$$

thus $$E = -\frac{1}{2}m\frac{(GM)^2}{GMr} = -\frac{1}{2}\frac{GMm}{r}$$

PROBLEM 13.119

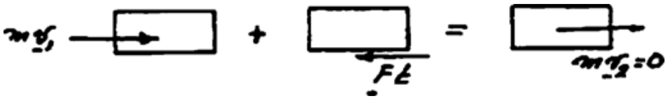
A 35,000 Mg ocean liner has an initial velocity of 4 km/h. Neglecting the frictional resistance of the water, determine the time required to bring the liner to rest by using a single tugboat which exerts a constant force of 150 kN.

SOLUTION

$$m = 35{,}000\ \ Mg = 35 \times 10^6 \text{ kg}$$

$$F = 150 \times 10^3 \text{ N}$$

$$v_1 = 4 \text{ km/hr} = 1.1111 \text{ m/s}$$



$$mv_1 - Ft = 0$$

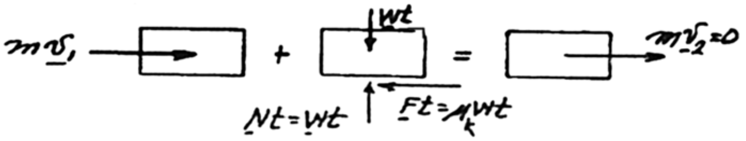
$$(35 \times 10^6 \text{ kg})(1.1111 \text{ m/s}) - (150 \times 10^3 \text{ N})t = 0$$

$$t = 259.26 \text{ s} \qquad t = 4 \text{ min} 19 \text{ s} ◀$$

PROBLEM 13.120

A 2500-lb automobile is moving at a speed of 60 mi/h when the brakes are fully applied, causing all four wheels to skid. Determine the time required to stop the automobile (*a*) on dry pavement ($\mu_k = 0.75$), (*b*) on an icy road ($\mu_k = 0.10$).

SOLUTION



$m\underline{v}_1$
+
Wt
=
$m\underline{v}_2=0$
$\underline{N}t=\underline{W}t$
$\underline{F}t=\mu_k Wt$

$$v_1 = 60 \text{ mph} = 88 \text{ ft/s}$$

$$mv_1 - \mu_k Wt = 0$$

$$t = \frac{mv_1}{\mu_k W} = \frac{mv_1}{\mu_k mg} = \frac{v_1}{\mu_k g}$$

(*a*) For $\mu_k = 0.75$

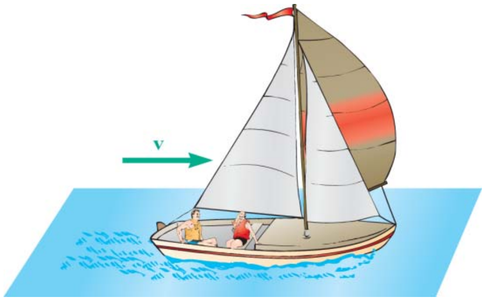
$$t = \frac{88 \text{ ft/s}}{(0.75)(32.2 \text{ ft/s}^2)}$$

$t = 3.64$ s ◀

(*b*) For $\mu_k = 0.10$

$$t = \frac{88 \text{ ft/s}}{(0.10)(32.2 \text{ ft/s}^2)}$$

$t = 27.3$ s ◀



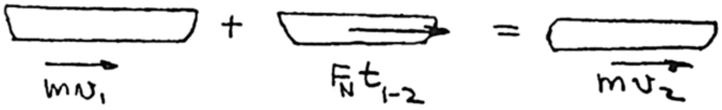
PROBLEM 13.121

A sailboat weighing 980 lb with its occupants is running down wind at 8 mi/h when its spinnaker is raised to increase its speed. Determine the net force provided by the spinnaker over the 10-s interval that it takes for the boat to reach a speed of 12 mi/h.

SOLUTION

$$v_1 = 8 \text{ mi/h} = 11.73 \text{ ft/s} \quad t_{1-2} = 10 \text{ sec}$$

$$v_2 = 12 \text{ mi/h} = 17.60 \text{ ft/s}$$

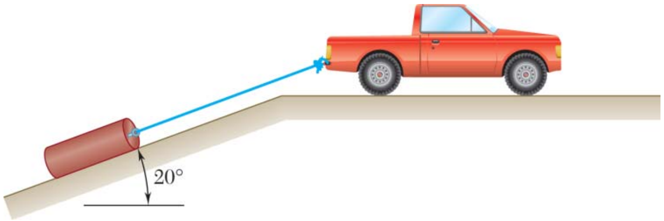


$$m \cdot v_1 + \text{imp}_{1-2} = mv_2$$

$$m(11.73 \text{ ft/s}) + F_n(10 \text{ s}) = m(17.60 \text{ ft/s})$$

$$F_n = \frac{(980 \text{ lb})(17.60 \text{ ft/s} - 11.73 \text{ ft/s})}{(32.2 \text{ ft/s}^2)(10 \text{ s})} \qquad F_n = 17.86 \text{ lb} \blacktriangleleft$$

Note: F_n is the *net* force *provided* by the sails. The force on the sails is actually greater and includes the force needed to overcome the water resistance on the hull.

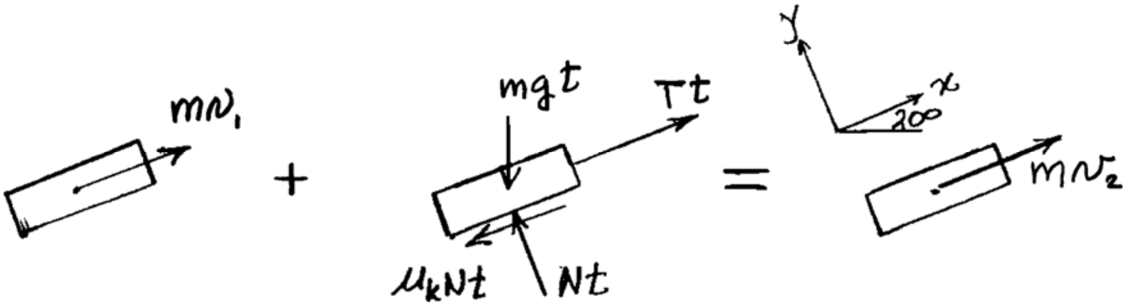


PROBLEM 13.122

A truck is hauling a 300-kg log out of a ditch using a winch attached to the back of the truck. Knowing the winch applies a constant force of 2500 N and the coefficient of kinetic friction between the ground and the log is 0.45, determine the time for the log to reach a speed of 0.5m/s.

SOLUTION

Apply the principle of impulse and momentum to the log.



$$m\mathbf{v}_1 + \Sigma\mathbf{Imp}_{1\to 2} = m\mathbf{v}_2$$

Components in y-direction:

$$0 + Nt - mgt\cos 20° = 0$$

$$N = mg\cos 20°$$

Components in x-direction:

$$0 + Tt - mgt\sin 20° - \mu_k Nt = mv_2$$

$$(T - mg\sin 20° - \mu_k mg\cos 20°)t = mv_2$$

$$[T - mg(\sin 20° + \mu_k \cos 20°)]t = mv_2$$

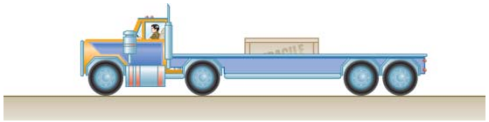
Data: $T = 2500\text{ N}, \quad m = 300\text{ kg}, \quad \mu_k = 0.45,$

$g = 9.81\text{ m/s}^2, \quad v_2 = 0.5\text{ m/s}$

$$[2500 - (300)(9.81)(\sin 20° + 0.45\cos 20°)]t = (300)(0.5)$$

$$248.95\, t = 150$$

$t = 0.603$ s ◀



PROBLEM 13.123

The coefficients of friction between the load and the flatbed trailer shown are $\mu_s = 0.40$ and $\mu_k = 0.35$. Knowing that the speed of the rig is 55 mi/h, determine the shortest time in which the rig can be brought to a stop if the load is not to shift.

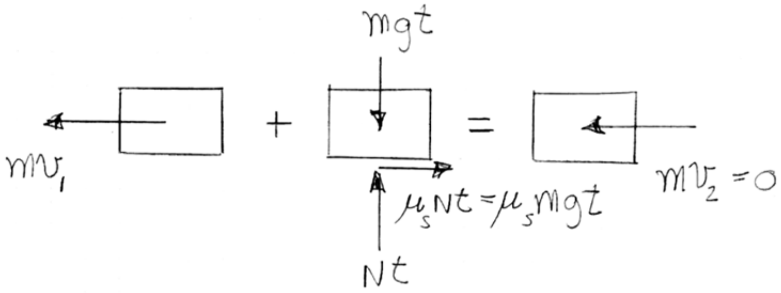
SOLUTION

Apply the principle impulse-momentum to the crate, knowing that, if the crate does not shift, the velocity of the crate matches that of the truck. For impending slip the friction and normal components of the contact force between the crate and the flatbed trailer satisfy the following equation:

$$F_f = \mu_s N$$

Given:
$$v_1 = 55\,\text{mi/h} = 80.667\ \text{ft/s}$$
$$\mu_s = 0.40,\ \mu_k = 0.35$$

Impulse Momentum Diagram:



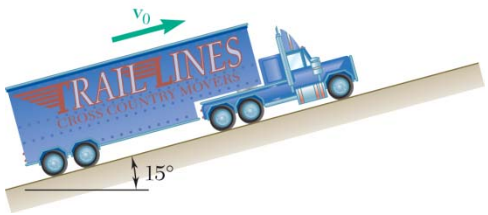
Using impulse momentum in the x-Direction ⟵

$$\cancel{m} v_1 - \mu_s \cancel{m} gt = 0$$

$$t = \frac{v_1}{\mu_s g} = \frac{80.667\ \text{ft/s}}{(0.4)\left(32.2\ \text{ft/s}^2\right)}$$

$t = 6.26$ s ◀

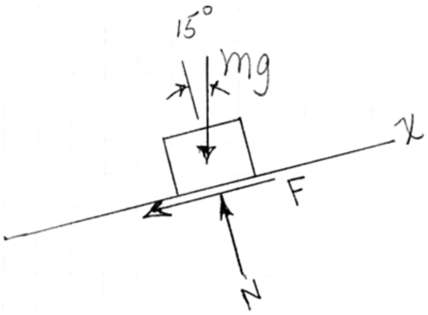
Since this is the shortest time the load can be brought to rest and the load does not slide it is also the shortest time the rig can be brought to rest.



PROBLEM 13.124

Steep safety ramps are built beside mountain highways to enable vehicles with defective brakes to stop. A 10-ton truck enters a 15° ramp at a high speed $v_0 = 108$ ft/s and travels for 6 s before its speed is reduced to 36 ft/s. Assuming constant deceleration, determine (*a*) the magnitude of the braking force, (*b*) the additional time required for the truck to stop. Neglect air resistance and rolling resistance.

SOLUTION



$$W = 20{,}000 \text{ lb}$$

$$m = \frac{20{,}000}{32.2} = 621.118 \text{ lb}\cdot\text{s}^2/\text{ft}$$

Momentum in the *x* direction

$$x\text{: } mv_0 - (F + mg\sin 15°)t = mv_1$$

$$621.118(108) - (F + mg\sin 15°)6 = (621.118)(36)$$

$$F + mg\sin 15° = 7453.4$$

(*a*) $F = 7453.4 - 20{,}000 \sin 15° = 2277$ lb

$F = 2280$ lb ◀

(*b*) $mv_0 - (F + mg\sin 15°)t = 0$ $t =$ total time

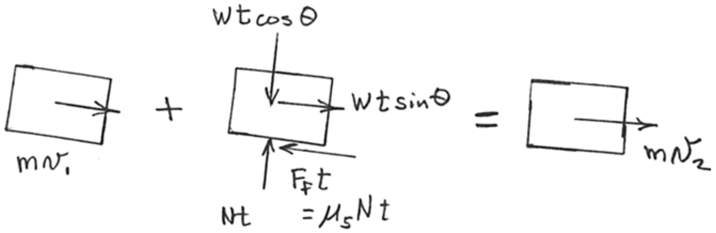
$621.118(108) - 7453.4t = 0;$ $t = 9.00$ s

Additional time = 9 – 6 $t = 3.00$ s ◀

PROBLEM 13.125

Baggage on the floor of the baggage car of a high-speed train is not prevented from moving other than by friction. The train is travelling down a 5 percent grade when it decreases its speed at a constant rate from 120 mi/h to 60 mi/h in a time interval of 12 s. Determine the smallest allowable value of the coefficient of static friction between a trunk and the floor of the baggage car if the trunk is not to slide.

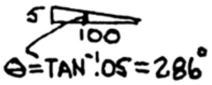
SOLUTION



$$v_1 = 120 \text{ mi/h} = 176 \text{ ft/s}$$

$$v_2 = 60 \text{ mi/h} = 88 \text{ ft/s}$$

$$t_{1-2} = 12 \text{ s}$$



$$Nt_{1-2} = Wt_{1-2}\cos\theta$$

$$\overset{+}{\searrow}\ \cancel{m}v_1 - \mu_s \cancel{m} g t_{1-2}\cos\theta + \cancel{m} g t_{1-2}\sin\theta = \cancel{m}v_2$$

$$(176 \text{ ft/s}) - \mu_s(32.2 \text{ ft/s}^2)(12 \text{ s})(\cos 2.86°) + (32.2 \text{ ft/s}^2)(12 \text{ s})(\sin 2.86°) = 88 \text{ ft/s}$$

$$\mu_s = \frac{176 - 88 + (32.2)(12)(\sin 2.86°)}{(32.2)(12)(\cos 2.86°)}$$

$\mu_s = 0.278$ ◀

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PROBLEM 13.126

The 18000-kg F-35B uses thrust vectoring to allow it to take off vertically. In one maneuver, the pilot reaches the top of her static hover at 200 m. The combined thrust and lift force on the airplane applied at the end of the static hover can be expressed as F= $(44t + 2500t^2)$**i** + $(250t^2 + t + 176580)$**j,** where *F* and *t* are expressed in newtons and seconds, respectively. Determine (a) how long it will take the airplane to reach a cruising speed of 1000 km/hr (cruising speed is defined to be in the x-direction only), (b) the altitude of the plane at this time.

SOLUTION

Given:

$y_1 = 200$ m

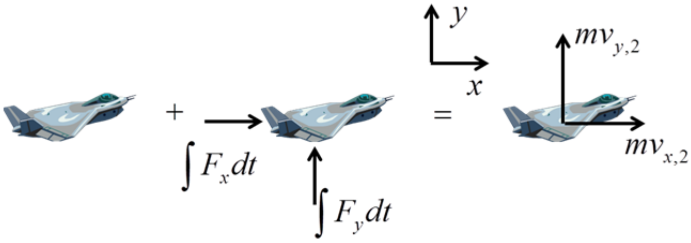
$m = 18000$ kg

$F_x = 44t + 2500t^2$

$F_y = 250t^2 + t + 176580$

$v_{x,2} = 1000$ km/hr $= 277.78$ m/s

Impulse-Momentum Diagram:



Impulse-Momentum in the x-direction:

$$0 + \int F_x dt = mv_{x,2}$$

$$\int_0^t \left(44t + 2500t^2\right) dt = 18000 * 277.78$$

$$\frac{2500}{3}t^3 + 22t^2 = 5{,}000{,}000$$

$$t^3 + \frac{33}{1250}t^2 - 6000 = 0$$

(a) Solve for *t*: $t = 18.162$ s ◀

Impulse-Momentum in the y-direction:

$$0 + \int F_y dt = mv_{\mathrm{y},2}$$

$$\int_0^t \left(250t^2 + t + 176580\right) dt = mv_{\mathrm{y},2}$$

$$\frac{250}{3}t^3 + \frac{1}{2}t^2 + 176580t = mv_{\mathrm{y},2}$$

Solve for velocity as function of time:

$$v_{\mathrm{y},2} = \frac{1}{m}\left(\frac{250}{3}t^3 + \frac{1}{2}t^2 + 176580t\right)$$

From kinematics:

$$v_{\mathrm{y},2} = \frac{dy}{dt} \Rightarrow v_{\mathrm{y},2}dt = dy$$

PROBLEM 13.126 (Continued)

Integrating:

$$\int_0^t v_{y,2}\,dt = \int_{200}^{y_2} dy$$

$$\frac{1}{m}\int_0^{18.162}\left(\frac{250}{3}t^3 + \frac{1}{2}t^2 + 176580t\right)dt = \int_{200}^{y_2} dy$$

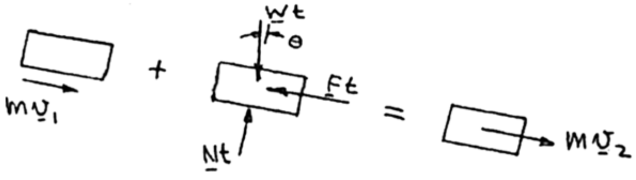
$$\frac{1}{18000}\left(\frac{125}{6}t^4 + \frac{1}{6}t^3 + 88290t^2\right) = y_2 - 200$$

(b) Find y_2 for t=18.162 s: $y_2 = 1944$ m $\qquad y_2 = 1.944$ km ◀

PROBLEM 13.127

A truck is traveling down a road with a 4-percent grade at a speed of 60 mi/h when its brakes are applied to slow it down to 20 mi/h. An antiskid braking system limits the braking force to a value at which the wheels of the truck are just about to slide. Knowing that the coefficient of static friction between the road and the wheels is 0.60, determine the shortest time needed for the truck to slow down.

SOLUTION



$$\theta = \tan^{-1}\frac{4}{100} = 2.29^\circ$$

$$m\mathbf{v}_1 + \Sigma\,\mathbf{imp}_{1-2} = m\mathbf{v}_2$$

$$+\searrow \quad mv_1 + Wt\sin\theta - Ft = mv_2$$

$$v_1 = 60 \text{ mi/h} = 88 \text{ ft/s} \qquad N = W\cos\theta \quad W = mg$$

$$v_2 = 20 \text{ mi/h} = 29.33 \text{ ft/s} \qquad F = \mu_s N = \mu_s W\cos\theta$$

$$(\cancel{m})(88 \text{ ft/s}) + (\cancel{m})(32.2 \text{ ft/s}^2)(t)(\sin 2.29^\circ) - (0.60)(\cancel{m})(32.2 \text{ ft/s}^2)(\cos 2.29^\circ)(t) = (\cancel{m})(29.33 \text{ ft/s})$$

$$t = \frac{88 - 29.33}{32.2[(0.60)\cos 2.29^\circ - \sin 2.29^\circ]} \qquad t = 3.26 \text{ s} \blacktriangleleft$$

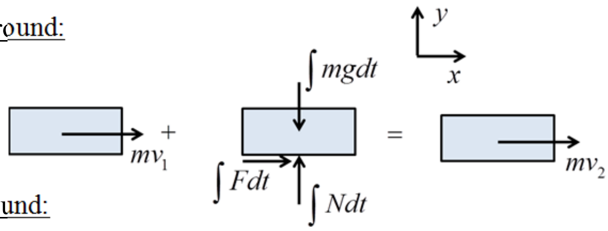
PROBLEM 13.128

In anticipation of a long 6° upgrade, a bus driver accelerates at a constant rate from 80 km/h to 100 km/h in 8 s while still on a level section of the highway. Knowing that the speed of the bus is 100 km/h as it begins to climb the grade at time $t = 0$ and that the driver does not change the setting of the throttle or shift gears, determine (*a*) the speed of the bus when $t = 10$ s, (*b*) the time when the speed is 60 km/h.

SOLUTION

Given: $v_2 = 100\ \text{km/h} = 27.778\ \text{m/s}$, $v_1 = 80\ \text{km/h} = 22.222\ \text{m/s}$, $t = 8\ \text{s}$

Impulse Momentum Diagram for acceleration on level ground:

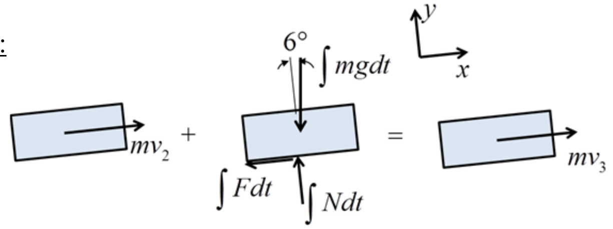


Impulse-Momentum in the x-direction while on level ground:

$$mv_1 + \int_0^8 Fdt = mv_2$$

$$F(8) = m(v_2 - v_1) \Rightarrow F = 0.6944m$$

Impulse Momentum Diagram for acceleration on incline:



Impulse-Momentum in the x-direction while on incline:

$$mv_2 + \int_0^{10} (F - mg\sin 6°)dt = mv_3$$

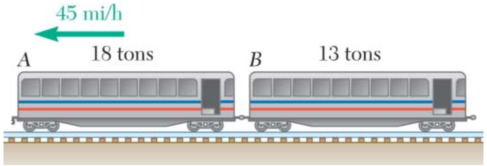
$$\cancel{m}(0.6944 - (9.81)\sin 6°)(10) = \cancel{m}(v_3 - 27.778)$$

(*a*) Solve for v_3: $v_3 = 24.47$ m/s $v_3 = 88.08$ km/h ◀

$$mv_2 + \int_0^{t} (F - mg\sin 6°)dt = mv_3$$

$$\cancel{m}(0.6944 - (9.81)\sin 6°)(t) = \cancel{m}(16.667 - 27.778)$$

(b) Solve for t: $t = 33.57$ s ◀

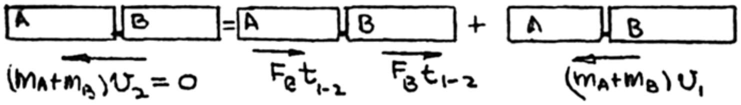

45 mi/h
A
18 tons
B
13 tons

PROBLEM 13.129

A light train made of two cars travels at 45 mi/h. Car *A* weighs 18 tons, and car *B* weighs 13 tons. When the brakes are applied, a constant braking force of 4300 lb is applied to each car. Determine (*a*) the time required for the train to stop after the brakes are applied, (*b*) the force in the coupling between the cars while the train is slowing down.

SOLUTION

(*a*) Entire train: $v_1 = 45 \text{ mi/h} = 66 \text{ ft/s}$


A
B
=
A
B
+
A
B
$(m_A+m_B)v_2 = 0$
$F_B t_{1-2}$
$F_B t_{1-2}$
$(m_A+m_B)v_1$

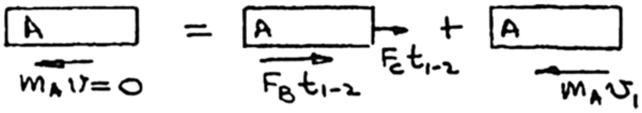
$$W_A + W_B = 18 + 13 = 31 \text{ tons} = 62{,}000 \text{ lb}$$

$$\overset{+}{\leftarrow}\ 0 = -(4300+4300)t_{1-2} + \frac{62{,}000}{g}(66)$$

$$t_{1-2} = \frac{(62{,}000 \text{ lb})(66 \text{ ft/s})}{(32.2 \text{ ft/s}^2)(8600 \text{ lb})}$$

$t_{1-2} = 14.78 \text{ s}$ ◀

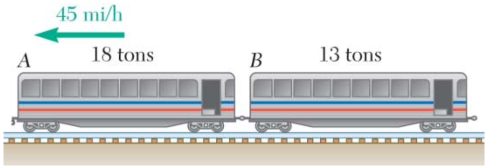
(*b*) Car *A*: $W_A = 18 \text{ tons} = 36{,}000 \text{ lb}, \quad t_{1-2} = 14.78 \text{ s}$


A
=
A
+
A
$m_A v = 0$
$F_B t_{1-2}$
$F_C t_{1-2}$
$m_A v_1$

$$\overset{+}{\leftarrow}\ 0 - [(4300 \text{ lb}) + F_C][14.78 \text{ s}] = \frac{(36{,}000 \text{ lb})}{(32.2 \text{ ft/s}^2)}(66 \text{ ft/s})$$

$$F_C = 692.5 \text{ lb}$$

$F_C = 693 \text{ lb (tension)}$ ◀



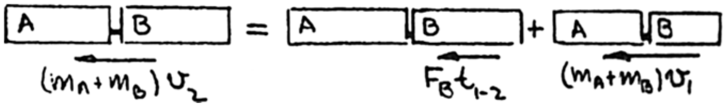
PROBLEM 13.130

Solve Problem 13.129, assuming that a constant braking force of 4300 lb is applied to car *B*, but the brakes on car *A* are not applied.

PROBLEM 13.129 A light train made of two cars travels at 45 mi/h. Car *A* weighs 18 tons, and car *B* weighs 13 tons. When the brakes are applied, a constant braking force of 4300 lb is applied to each car. Determine (*a*) the time required for the train to stop after the brakes are applied, (*b*) the force in the coupling between the cars while the train is slowing down.

SOLUTION

(*a*) Entire train: $v_1 = 45 \text{ mi/h} = 66 \text{ ft/s}$

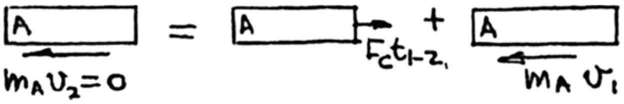


$$W_A + W_B = 18 + 13 = 31 \text{ tons} = 62{,}000 \text{ lb}$$

$$\overset{+}{\longleftarrow} \; 0 = -(4300 \text{ lb})t_{1-2} + \frac{(62{,}000 \text{ lb})}{(32.2 \text{ ft/s}^2)}(66 \text{ ft/s})$$

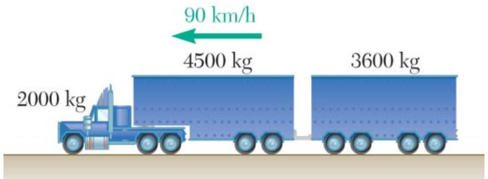
$$t_{1-2} = 29.55 \text{ s} \qquad\qquad t_{1-2} = 29.6 \text{ s} \blacktriangleleft$$

(*b*) Car *A*:



$$\overset{+}{\longleftarrow} \; 0 = -F_C(t_{1-2}) + m_A v_1 \quad t_{1-2} = 29.55 \text{ s}$$

$$F_C = \frac{(36{,}000 \text{ lb})(66 \text{ ft/s})}{(32.2 \text{ ft/s}^2)(29.55 \text{ s})} = 2497 \text{ lb} \qquad F_C = 2500 \text{ lb (tension)} \blacktriangleleft$$



PROBLEM 13.131

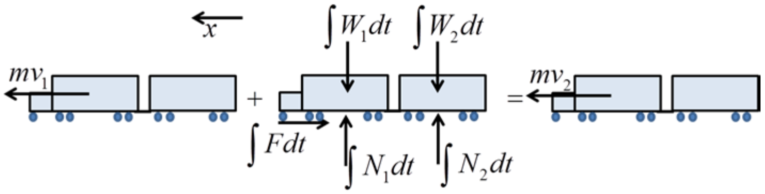
A tractor-trailer rig with a 2000-kg tractor, a 4500-kg trailer, and a 3600-kg trailer is traveling on a level road at 90 km/h. The brakes on the rear trailer fail and the antiskid system of the tractor and front trailer provide the largest possible force which will not cause the wheels to slide. Knowing that the coefficient of static friction is 0.75, determine (*a*) the shortest time for the rig to a come to a stop, (*b*) the force in the coupling between the two trailers during that time. Assume that the force exerted by the coupling on each of the two trailers is horizontal.

SOLUTION

Given: $v_1 = 90\,\text{km/h} = 25\ \text{m/s}\,,\ v_2 = 0$

$$W_1 = (6500)(9.81) = 63765\ \text{N};\quad W_2 = (3600)(9.81) = 35316\ \text{N}$$

Impulse Momentum diagram for combined cab and trailer:



From Equilibrium: $N_1 = W_1;\quad N_2 = W_2 \qquad F = 0.75 N_1$

Impulse-Momentum in the x-direction:

$$mv_1 - \int_0^t 0.75 N_1 dt = mv_2$$

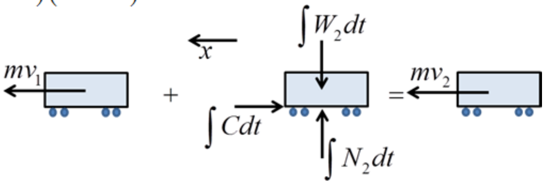
$$0.75 N_1 t = m(v_2 - v_1)$$

$$t = \frac{m(v_1 - \cancel{v_2})}{0.75 N_1}$$

(*a*) $$t = \frac{10{,}100(25)}{(0.75)(63765)} = 5.2798\ \text{s}$$ $t = 5.28\ \text{s}$ ◀

Impulse Momentum diagram for the trailer alone:

Impulse-Momentum in the x-direction:

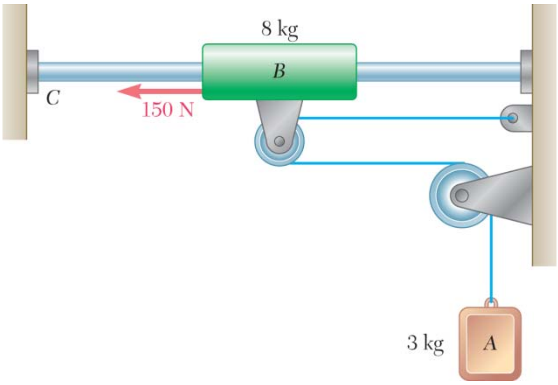


$$mv_1 - \int_0^{5.2798} C dt = mv_2$$

$$5.2798 C = m(v_1 - 0)$$

$$C = \frac{3600(25 - \cancel{v_2})}{5.2798} \Rightarrow C = 17046\ \text{N}$$

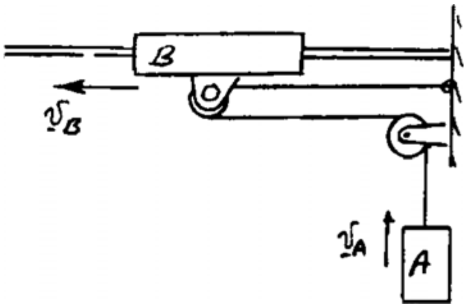
(*b*) $C = 17.05\ \text{kN}$ Compression ◀



PROBLEM 13.132

The system shown is at rest when a constant 150-N force is applied to collar *B*. Neglecting the effect of friction, determine (*a*) the time at which the velocity of collar *B* will be 2.5 m/s to the left, (*b*) the corresponding tension in the cable.

SOLUTION



Constraint of cord. When the collar *B* moves 1 unit to the left, the weight *A* moves up 2 units. Thus

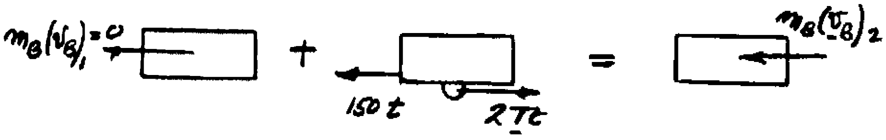
$$v_A = 2v_B \qquad v_B = \frac{1}{2}v_A$$

Masses and weights: $m_A = 3\text{ kg}$ $\quad W_A = 29.43\text{ N}$

$m_B = 8\text{ kg}$

Let *T* be the tension in the cable.

Principle of impulse and momentum applied to collar *B*.



$$\overset{+}{\leftarrow}:\ 0+150t-2Tt=m_B(v_B)_2$$

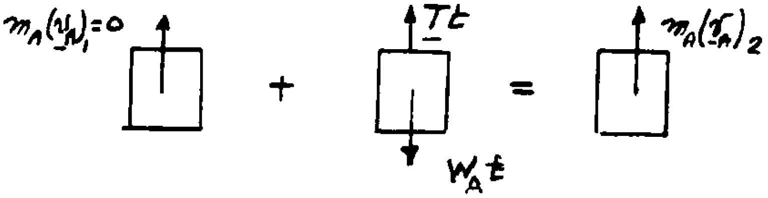
For $(v_B)_2 = 2.5\text{ m/s}$

$$150t-2Tt=(8\text{ kg})(2.5\text{ m/s})$$

$$150t-2Tt=20 \qquad (1)$$

PROBLEM 13.132 (Continued)

Principle of impulse and momentum applied to weight *A*.



$$+\!\uparrow:\quad 0 + Tt - W_A t = m_A(v_A)_2$$

$$Tt + W_A t = m_A(2V_{B2})$$

$$Tt - 29.43t = (3\text{ kg})(2)(2.5\text{ m/s})$$

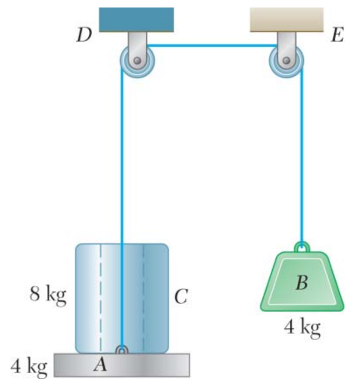
$$Tt - 29.43t = 15 \qquad (2)$$

To eliminate *T* multiply Eq. (2) by 2 and add to Eq. (1).

(*a*) Time: $91.14t = 50$ $\quad t = 0.549\text{ s}$ ◀

From Eq. (2), $T = \dfrac{15}{t} + 29.43$

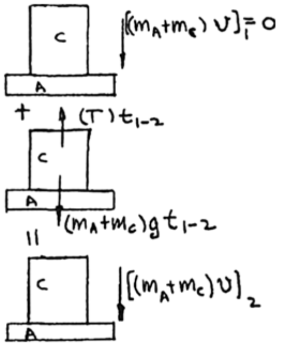
(*b*) Tension in the cable. $\quad T = 56.8\text{ N}$ ◀


D
E
8 kg
C
B
4 kg
4 kg
A

PROBLEM 13.133

An 8-kg cylinder *C* rests on a 4-kg platform *A* supported by a cord which passes over the pulleys *D* and *E* and is attached to a 4-kg block *B*. Knowing that the system is released from rest, determine (*a*) the velocity of block *B* after 0.8 s, (*b*) the force exerted by the cylinder on the platform.

SOLUTION


[(mA+mc) v]1 = 0
c
A
+
(T) t1-2
c
A
(mA+mc) g t1-2
=
c
A
[(mA+mc) v]2

(*a*) Blocks *A* and *C*:

$$\downarrow +$$

$$[(m_A + m_C)v]_1 - T(t_{1-2}) + (m_A + m_C)gt_{1-2} = [(m_A + m_C)v]_2$$

$$0 + (12g - T)(0.8) = 12v \qquad (1)$$

Block *B*:

$$\uparrow +$$

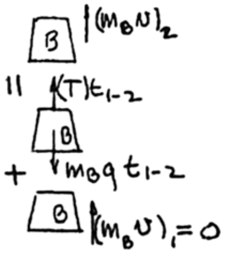
$$[m_B v]_1 + (T)t_{1-2} - m_B g t_{1-2} = (m_B v)_2$$

$$0 + (T - 4g)(0.8) = 4v \qquad (2)$$

Adding (1) and (2), (eliminating *T*)

$$(12g - 4g)(0.8) = (12 + 4)$$

$$v = \frac{(8\text{ kg})(9.81\text{ m/s}^2)(0.8\text{ s})}{16\text{ kg}} \qquad v = 3.92\text{ m/s} \blacktriangleleft$$


B
(mB v)2
=
(T) t1-2
B
+
mB g t1-2
B
(mB v)1 = 0

(*b*) Collar *A*:

$$\downarrow +$$

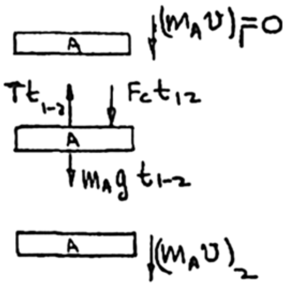
$$(m_A v)_1 = 0 \quad 0 + (F_C + m_A g) \qquad (3)$$

From Eq. (2) with $v = 3.92$ m/s

$$T = \frac{4v}{0.8} + 4g$$

$$T = \frac{(4\text{ kg})(3.92\text{ m/s})}{(0.8\text{ s})} + (4\text{ kg})(9.81\text{ m/s}^2)$$

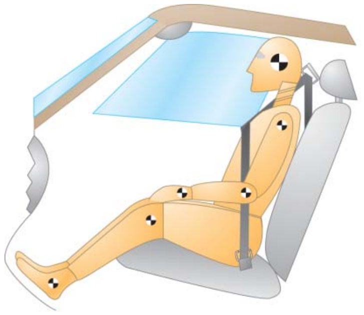
$$T = 58.84\text{ N}$$

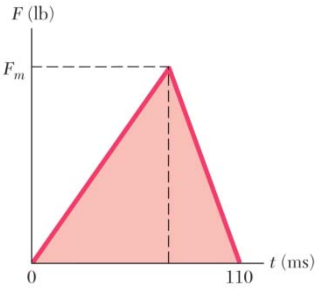

A
(mA v)1 = 0
T t1-2
Fc t12
A
mA g t1-2
A
(mA v)2

Solving for F_C in (3)

$$F_C = \frac{(4\text{ kg})(3.92\text{ m/s})}{(0.8\text{ s})} - (4\text{ kg})(9.81\text{ m/s}^2) + 58.84\text{ N}$$

$$F_C = 39.2\text{ N} \blacktriangleleft$$



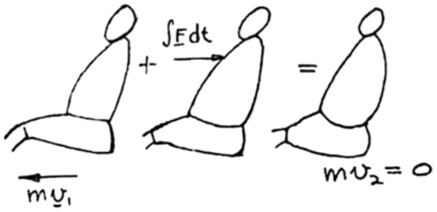


PROBLEM 13.134

An estimate of the expected load on over-the-shoulder seat belts is to be made before designing prototype belts that will be evaluated in automobile crash tests. Assuming that an automobile traveling at 45 mi/h is brought to a stop in 110 ms, determine (*a*) the average impulsive force exerted by a 200-lb man on the belt, (*b*) the maximum force F_m exerted on the belt if the force-time diagram has the shape shown.

SOLUTION

(*a*) Force on the belt is opposite to the direction shown.



$$v_1 = 45 \text{ mi/h} = 66 \text{ ft/s},$$

$$W = 200 \text{ lb}$$

$$m\mathbf{v}_1 - \int \mathbf{F} dt = m\mathbf{v}_2$$

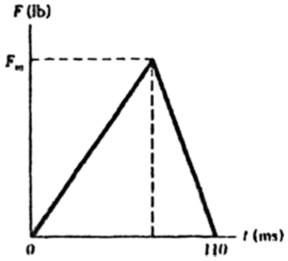
$$\int F dt = F_{\text{ave}} \Delta t$$

$$\Delta t = 0.110 \text{ s}$$

$$\frac{(200 \text{ lb})(66 \text{ ft/s})}{(32.2 \text{ ft/s}^2)} - F_{\text{ave}}(0.110 \text{ s}) = 0$$

$$F_{\text{ave}} = \frac{(200)(66)}{(32.2)(0.110)} = 3727 \text{ lb}$$

$F_{\text{ave}} = 3730 \text{ lb}$ ◀

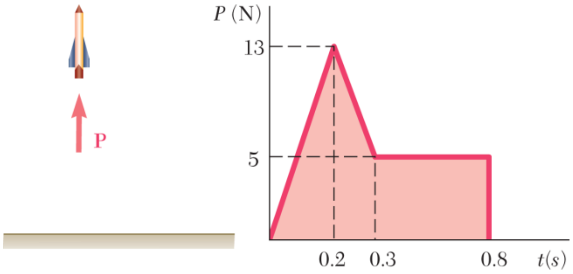


(*b*) Impulse = area under $F-t$ diagram $= \frac{1}{2} F_m (0.110 \text{ s})$

From (*a*), impulse $= F_{\text{ave}} \Delta t = (3727 \text{ lb})(0.110 \text{ s})$

$$\frac{1}{2} F_m (0.110) = (3727)(0.110)$$

$F_m = 7450 \text{ lb}$ ◀



PROBLEM 13.135

A 60-g model rocket is fired vertically. The engine applies a thrust **P** which varies in magnitude as shown. Neglecting air resistance and the change in mass of the rocket, determine (*a*) the maximum speed of the rocket as it goes up, (*b*) the time for the rocket to reach its maximum elevation.

SOLUTION

Mass: $m = 0.060\text{ kg}$

Weight: $mg = (0.060)(9.81) = 0.5886\text{ N}$

Forces acting on the model rocket:

Thrust: $P(t)(\text{given function of } t)\uparrow$

Weight: $W\ (\text{constant})\downarrow$

Support: $S\ (\text{acts until } P > W)\uparrow$

Over $0 < t < 0.2\text{ s}$:

$$P = \frac{13}{0.2}t = 65t$$

$$W = 0.5886\text{ N}$$

Before the rocket lifts off, $S = W - P = 0.5886 - 65t$

S become zero when $t = t_1$.

$$0 = 0.5886 - 65t_1 \qquad t_1 = 0.009055\text{ s.}$$

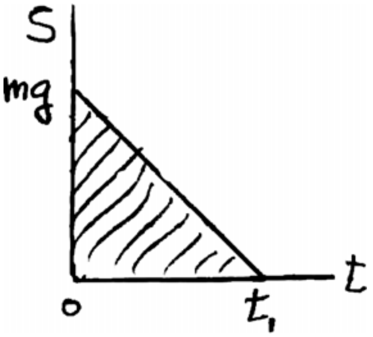
Impulse due to S: $(t > t_1)$

$$\int_0^t S dt = \int_0^{t_1} S dt$$

$$= \frac{1}{2}mgt_1$$

$$= (0.5)(0.5886)(0.009055)$$

$$= 0.00266\text{ N}\cdot\text{s}$$



The maximum speed occurs when $\frac{dv}{dt} = a = 0.$

At this time, $W - P = 0,$ which occurs at $t_2 = 0.8\text{ s.}$

PROBLEM 13.135 (Continued)

(*a*) Maximum speed (upward motion):

Apply the principle of impulse-momentum to the rocket over $0 \le t \le t_2$.

$$\int_0^{0.8} P dt = \text{area under the given thrust-time plot.}$$

$$\begin{aligned} &= \frac{1}{2}(0.2)(13) + \frac{1}{2}(0.1)(13+5) + (0.8-0.3)(5) \\ &= 4.7 \text{ N} \cdot \text{s} \end{aligned}$$

$$\int_0^{0.8} W dt = (0.5886)(0.8) = 0.47088 \text{ N} \cdot \text{S}$$

$$m_1 v_1 + \int_0^{0.8} P dt + \int_0^{0.8} S dt - \int_0^{0.8} W dt = m v_2$$

$$0 + 4.7 + 0.00266 - 0.47088 = 0.060 \, v_2$$

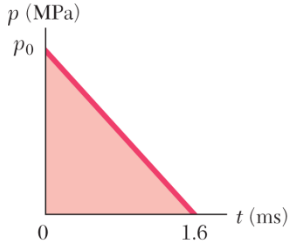
$v_2 = 70.5$ m/s ◀

(*b*) Time t_3 to reach maximum height: $(v_3 = 0)$

$$m v_1 + \int_0^{t_3} P dt + \int_0^{t_3} S dt - W t_3 = m v_3$$

$$0 + 4.7 + 0.00266 - 0.5886 t_3 = 0$$

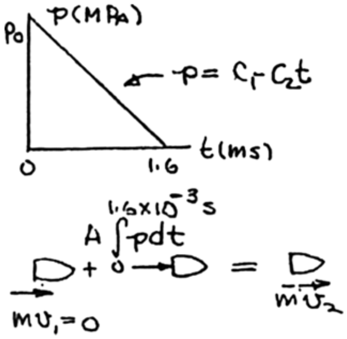
$t_3 = 7.99$ s ◀



PROBLEM 13.136

A simplified model consisting of a single straight line is to be obtained for the variation of pressure inside the 10-mm-diameter barrel of a rifle as a 20-g bullet is fired. Knowing that it takes 1.6 ms for the bullet to travel the length of the barrel and that the velocity of the bullet upon exit is 700 m/s, determine the value of p_0.

SOLUTION



At $t = 0$,

$$p = p_0 = c_1 - c_2 t$$
$$c_1 = p_0$$

At $t = 1.6 \times 10^{-3}$ s,

$$p = 0$$
$$0 = c_1 - c_2 (1.6 \times 10^{-3} \text{ s})$$
$$c_2 = \frac{p_0}{1.6 \times 10^{-3} \text{ s}}$$
$$m = 20 \times 10^{-3} \text{ kg}$$
$$0 + A\int_0^{1.6\times 10^{-3}\text{ s}} p\,dt = m v_2$$
$$A = \frac{\pi (10 \times 10^{-3})^2}{4}$$
$$A = 78.54 \times 10^{-6} \text{ m}^2$$
$$0 + A\int_0^{1.6\times 10^{-3}\text{ s}} (c_1 - c_2 t)\,dt = \frac{20 \times 10^{-3}}{g}$$
$$(78.54 \times 10^{-6} \text{ m}^2)\left[(c_1)(1.6 \times 10^{-3} \text{ s}) - \frac{(c_2)(1.6 \times 10^{-3} \text{ s})^2}{2}\right] = (20 \times 10^{-3} \text{ kg})(700 \text{ m/s})$$
$$1.6 \times 10^{-3} c_1 - 1.280 \times 10^{-6} c_2 = 178.25 \times 10^3$$
$$(1.6 \times 10^{-3} \text{ m}^2 \cdot \text{s}) p_0 - \frac{(1.280 \times 10^{-6} \text{ m}^2 \cdot \text{s}^2)}{(1.6 \times 10^{-3} \text{ s})} p_0 = 178.25 \times 10^3 \text{ kg} \cdot \text{m/s}$$

$p_0 = 222.8 \times 10^6 \text{ N/m}^2$ $\qquad p_0 = 223$ MPa ◀

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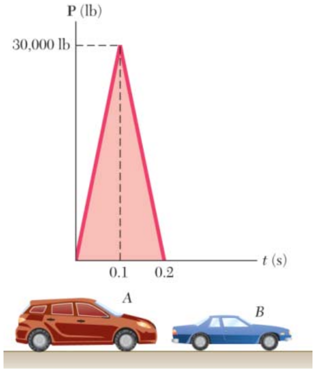


PROBLEM 13.137

A crash test is performed between an SUV *A* and a 2500 lb compact car *B*. The compact car is stationary before the impact and has its brakes applied. A transducer measures the force during the impact, and the force **P** varies as shown. Knowing that the coefficients of friction between the tires and road are $\mu_s = 0.9$ and $\mu_k = 0.7$, determine (a) the time at which the compact car will start moving, (b) the maximum speed of the car, (c) the time at which the car will come to a stop.

SOLUTION

Given: $m_B = \dfrac{2500\text{ lb}}{32.2\text{ ft/s}^2} = 77.64\text{ slugs}$

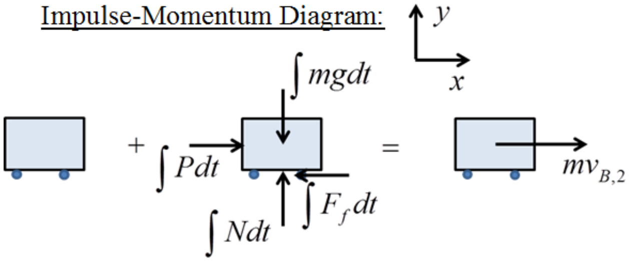
$\mu_s = 0.9,\ \mu_k = 0.7$

$0 \le t \le 0.1 \quad \Rightarrow \quad P = 300000t\text{ lb}$

$0.1 < t \le 0.2 \quad \Rightarrow \quad P = 300000(0.2 - t)\text{ lb}$

$v_{B,1} = 0$

Impulse-Momentum Diagram:



Impulse-Momentum in the x-direction: $\quad 0 + \int (P - F_f)\,dt = m_B v_{\text{B},2} \qquad (1)$

From Equilibrium: $\quad N = m_B g$

(a) Compact car starts moving when: $\quad F_f = \mu_s N = \mu_s m_B g$

Substitute into (1): $\quad 0 + \int (P - \mu_s m_B g)\,dt = 0 \text{ or } P - \mu_s m_B g = 0$

Solving for P: $\quad P = 2250$ lbs when car starts to move

For $0 \le t \le 0.1,\ P = 300000t$ $\quad 2250 = 300000t \qquad t_s = 0.0075\text{ s}$ ◀

(b) Maximum Velocity will occur near the end of the impulse when: $\quad P = F_f = \mu_k N = \mu_k m_B g$

For $\quad 0.1 < t \le 0.2,\ P = 300000(0.2 - t)$

$0.7(2500) = 300000(0.2 - t)$

$t_m = 0.19417\text{ s}$

Use (1) to find maximum velocity:

$$0 + \int_{t_s}^{0.1} 300000t\,dt + \int_{0.1}^{t_m} 300000(0.2 - t)\,dt - \int_{t_s}^{t_m} \mu_k m_B g\,dt = m_B v_{\text{B,max}}$$

$$v_{B,\max} = \frac{1}{m_B}\left[150000t^2\Big|_{t_s}^{0.1} + \left(60000t - 150000t^2\right)\Big|_{0.1}^{t_m}\right] - \mu_k g\Big|_{t_s}^{t_m} \qquad v_{B,\max} = 34.26\text{ ft/s} ◀$$

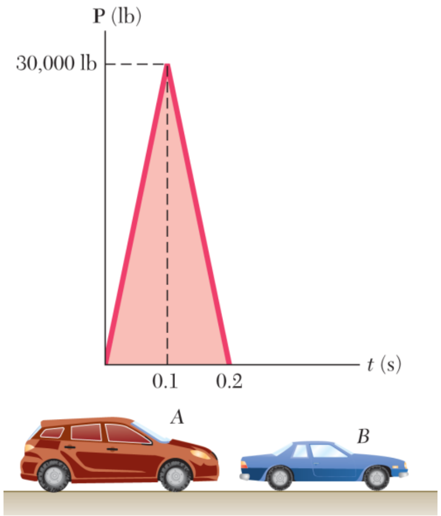
PROBLEM 13.137 (Continued)

(c) Use impulse momentum from t_m until car stops at t_{stp}:

$$m_B v_{B,\max} + \int_{t_m}^{t_{stp}} \left(P - F_f\right) dt = 0$$

$$m_B v_{\mathrm{B,max}} + \int_{t_m}^{0.2} 300000\left(0.2 - t\right) dt - \int_{t_m}^{t_{stp}} \mu_k m_B g dt = 0$$

$$m_B v_{\mathrm{B,max}} + \left(60000t - 150000t^2\right)\Big|_{t_m}^{0.2} - \mu_k m_B g\Big|_{t_m}^{t_{stp}} = 0 \qquad t_{stp} = 1.717 \text{ s} \blacktriangleleft$$



PROBLEM 13.138

A crash test is performed between a 4500 lb SUV *A* and a compact car *B*. A transducer measures the force during the impact, and the force **P** varies as shown. Knowing that the SUV is travelling 30 mph when it hits the car, determine the speed of the SUV immediately after the impact.

SOLUTION

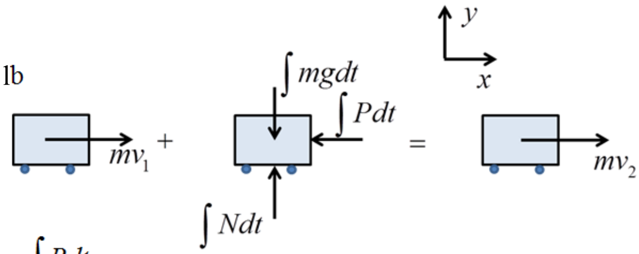
Given: $m = \dfrac{4500 \text{ lb}}{32.2 \text{ ft/s}^2} = 139.75 \text{ slugs}$

$0 \le t \le 0.1 \quad \Rightarrow \quad P = 300000t \text{ lb}$

$0.1 < t \le 0.2 \quad \Rightarrow \quad P = 300000(0.2 - t) \text{ lb}$

$v_1 = 30 \text{ mph} = 44.0 \text{ ft/s}$

Impulse-Momentum Diagram of SUV:



Impulse-Momentum in the x-direction: $mv_1 - \int Pdt = mv_2$

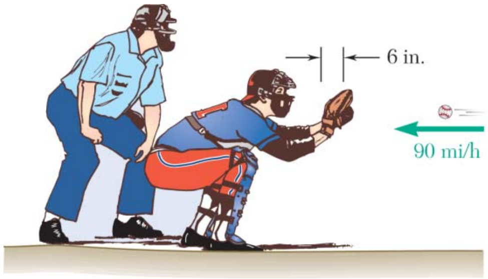
Put in load function P(t): $mv_1 - \int_0^{0.1} 300000t - \int_{0.1}^{0.2} 300000(0.2 - t) = mv_2$

Integrate: $mv_1 - 150000t^2\Big|_0^{0.1} - \left(60000t - 150000t^2\right)\Big|_{0.1}^{0.2} = mv_2$

Solve for v_2: $v_2 = v_1 - \dfrac{1}{m}[3000]$

$v_2 = 22.533 \text{ ft/s}$

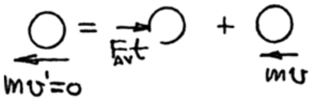
$v_2 = 15.36 \text{ mi/h}$ ◀



PROBLEM 13.139

A baseball player catching a ball can soften the impact by pulling his hand back. Assuming that a 5-oz ball reaches his glove at 90 mi/h and that the player pulls his hand back during the impact at an average speed of 30 ft/s over a distance of 6 in., bringing the ball to a stop, determine the average impulsive force exerted on the player's hand.

SOLUTION



$$v = 90 \text{ mi/h} = 132 \text{ ft/s}$$

$$m = \frac{5}{16}/\text{g}$$

$$t = \frac{d}{v_{av}} = \frac{\left(\frac{6}{12}\right)}{30} = \left(\frac{1}{60}\right) \text{ s}$$

$$\xrightarrow{+} \quad 0 = F_{av}t + mv \qquad F_{av} = \frac{Wv}{gt}$$

$$F_{av} = \frac{mv}{t}$$

$$= \frac{\left(\frac{5}{16}\text{ lb}\right)(132 \text{ ft/s})}{(32.2 \text{ ft/s}^2)\left(\frac{1}{60}\text{ s}\right)}$$

$F_{av} = 76.9$ lb ◀

PROBLEM 13.140

A 1.62 ounce golf ball is hit with a golf club and leaves it with a velocity of 100 mi/h. We assume that for $0 \le t \le t_0$, where t_0 is the duration of the impact, the magnitude F of the force exerted on the ball can be expressed as $F = F_m \sin(\pi t/t_0)$. Knowing that $t_0 = 0.5$ ms, determine the maximum value F_m of the force exerted on the ball.

SOLUTION

$$W = 1.62 \text{ ounces} = 0.10125 \text{ lb} \quad m = 3.1444 \times 10^{-3} \text{ slug}$$

$$t = 0.5 \text{ ms} = 0.5 \times 10^{-3} \text{ s}$$

$$v = 100 \text{ mi/h} = 146.67 \text{ ft/s}$$

The impulse applied to the ball is

$$\int_0^{t_0} F dt = \int_0^{t_0} F_m \sin\frac{\pi t}{t_0} dt = -\frac{F_m t_0}{\pi} \cos\frac{\pi t}{t_0}\bigg|_0^{t_0}$$

$$= -\frac{F_m t_0}{\pi}(\cos\pi - \cos 0) = \frac{2F_m t_0}{\pi}$$

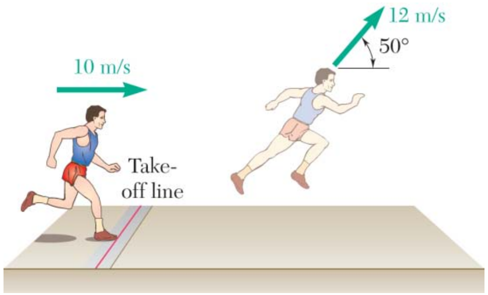
Principle of impulse and momentum.

$$m\mathbf{v}_1 + \int_0^{t_0} \mathbf{F}\, dt = m\mathbf{v}_2$$

with $\mathbf{v}_1 = 0$,

$$0 + \frac{2F_m t_0}{\pi} = mv_2$$

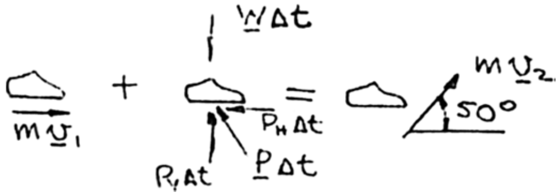
Solving for F_m, $$F_m = \frac{\pi m v_2}{2t_0} = \frac{\pi(3.1444 \times 10^{-3})(146.67)}{(2)(0.5 \times 10^{-3})} = 1.4488 \times 10^3 \text{ lb} \qquad F_m = 1.45 \text{ kip} ◀$$


12 m/s
50°
10 m/s
Take-off line

PROBLEM 13.141

The triple jump is a track-and-field event in which an athlete gets a running start and tries to leap as far as he can with a hop, step, and jump. Shown in the figure is the initial hop of the athlete. Assuming that he approaches the takeoff line from the left with a horizontal velocity of 10 m/s, remains in contact with the ground for 0.18 s, and takes off at a 50° angle with a velocity of 12 m/s, determine the vertical component of the average impulsive force exerted by the ground on his foot. Give your answer in terms of the weight W of the athlete.

SOLUTION


$m\underline{v}_1$
+
$\underline{W}\Delta t$
$P_H\Delta t$
$P_v\Delta t$
$\underline{P}\Delta t$
=
$m\underline{v}_2$
50°

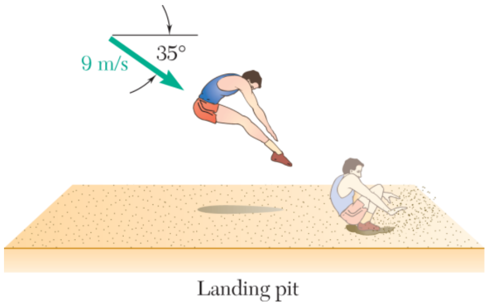
$$m\mathbf{v}_1 + (\mathbf{P} - \mathbf{W})\Delta t = m\mathbf{v}_2 \qquad \Delta t = 0.18 \text{ s}$$

Vertical components

$$0 + (P_v - W)(0.18) = \frac{W}{g}(12)(\sin 50°)$$

$$P_v = W + \frac{(12)(\sin 50°)}{(9.81)(0.18)}W$$

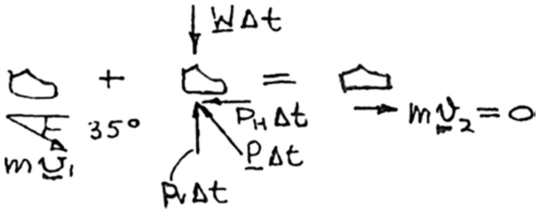
$$P_v = 6.21W \quad ◀$$



PROBLEM 13.142

The last segment of the triple jump track-and-field event is the jump, in which the athlete makes a final leap, landing in a sand-filled pit. Assuming that the velocity of a 80-kg athlete just before landing is 9 m/s at an angle of 35° with the horizontal and that the athlete comes to a complete stop in 0.22 s after landing, determine the horizontal component of the average impulsive force exerted on his feet during landing.

SOLUTION



$$m = 80 \text{ kg}$$

$$\Delta t = 0.22 \text{ s}$$

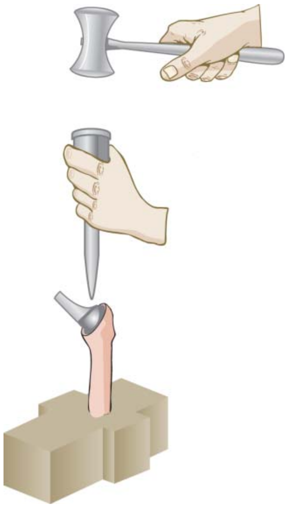
$$m\mathbf{v}_1 + (\mathbf{P} - \mathbf{W})\Delta t = m\mathbf{v}_2$$

Horizontal components

$$m(9)(\cos 35°) - P_H(0.22) = 0$$

$$P_H = \frac{(80 \text{ kg})(9 \text{ m/s})(\cos 35°)}{(0.22 \text{ s})} = 2.6809 \text{ kN}$$

$P_H = 2.68$ kN ◀



PROBLEM 13.143

The design for a new cementless hip implant is to be studied using an instrumented implant and a fixed simulated femur. Assuming the punch applies an average force of 2 kN over a time of 2 ms to the 200 g implant determine (*a*) the velocity of the implant immediately after impact, (*b*) the average resistance of the implant to penetration if the implant moves 1 mm before coming to rest.

SOLUTION

$$m = 200 \text{ g} = 0.200 \text{ kg}$$
$$F_{\text{ave}} = 2 \text{ kN} = 2000 \text{ N}$$
$$\Delta t = 2 \text{ ms} = 0.002 \text{ s}$$

(*a*) Velocity immediately after impact:

Use principle of impulse and momentum:

$$v_1 = 0 \qquad v_2 = ? \qquad \text{Imp}_{1\to 2} = F_{\text{ave}}(\Delta t)$$
$$m\mathbf{v}_1 + \mathbf{Imp}_{1\to 2} = m\mathbf{v}_2$$
$$0 + F_{\text{ave}}(\Delta t) = mv_2$$
$$v_2 = \frac{F_{\text{ave}}(\Delta t)}{m} = \frac{(2000)(0.002)}{0.200}$$

$v_2 = 20.0$ m/s ◀

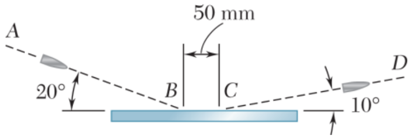
(*b*) Average resistance to penetration:

$$\Delta x = 1 \text{ mm} = 0.001 \text{ m}$$
$$v_2 = 20.0 \text{ ft/s}$$
$$v_3 = 0$$

Use principle of work and energy.

$$T_2 + U_{2\to 3} = T_3 \qquad \text{or} \qquad \frac{1}{2}mv_2^2 - R_{\text{ave}}(\Delta x) = 0$$
$$R_{\text{ave}} = \frac{mv_2^2}{2(\Delta x)} = \frac{(0.200)(20.0)^2}{(2)(0.001)} = 40\times 10^3 \text{ N}$$

$R_{\text{ave}} = 40.0$ kN ◀



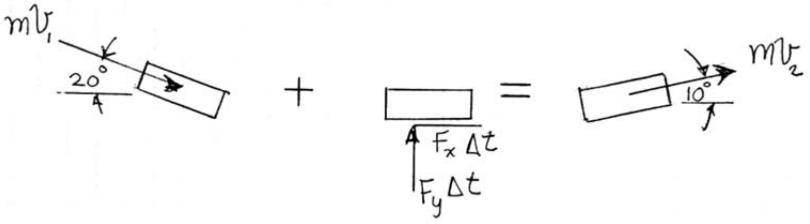
PROBLEM 13.144

A 28-g steel-jacketed bullet is fired with a velocity of 650 m/s toward a steel plate and ricochets along path *CD* with a velocity 500 m/s. Knowing that the bullet leaves a 50-mm scratch on the surface of the plate and assuming that it has an average speed of 600 m/s while in contact with the plate, determine the magnitude and direction of the impulsive force exerted by the plate on the bullet.

SOLUTION

Given: $m = 0.028$ kg, $v_1 = 650$ m/s, $v_2 = 500$ m/s

Impulse-momentum diagram of the bullet:



Impulse momentum in the x-dir ⟶

$$mv_1 \cos 20° - F_x \Delta t = mv_2 \cos 10°$$

So,
$$F_x \Delta t = mv_1 \cos 20° - mv_2 \cos 10°$$

$$= 0.028\,(650)\,\cos 20° - 0.028\,(500)\,\cos 10° = 3.3151 \text{ N}\cdot\text{s}$$

y-dir ↑

$$-mv_1 \sin 20° + F_y \Delta t = mv_2 \sin 10°$$

So,
$$F_y \Delta t = mv_2 \sin 10° + mv_1 \sin 20°$$

$$= 0.028\,(500)\,\sin 10° + 0.028\,(650)\,\sin 20° = 8.6558 \text{ N}\cdot\text{s}$$

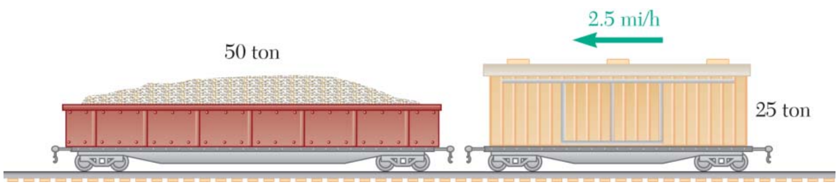
We need Δt. The average velocity is 600 m/s

$$\Delta x = v_{\text{ave}}\,\Delta t;\ \Delta t = \frac{\Delta x}{v_{\text{ave}}} = \frac{0.05 \text{ m}}{600 \text{ m/s}} = 83.33 \times 10^{-6} \text{ s}$$

So

$$\left.\begin{aligned} F_x &= \frac{3.3151}{83.33 \times 10^{-6}} = 39.78 \text{ kN} \\ F_y &= \frac{8.6558}{83.33 \times 10^{-6}} = 103.87 \text{ kN} \end{aligned}\right\}$$

$\overline{F} = 111.2 \text{ kN} \measuredangle 69.0°$ ◀



PROBLEM 13.145

A 25-ton railroad car moving at 2.5 mi/h is to be coupled to a 50 ton car which is at rest with locked wheels ($\mu_k = 0.30$). Determine (*a*) the velocity of both cars after the coupling is completed, (*b*) the time it takes for both cars to come to rest.

SOLUTION

Weight and mass: (Label cars *A* and *B*.)

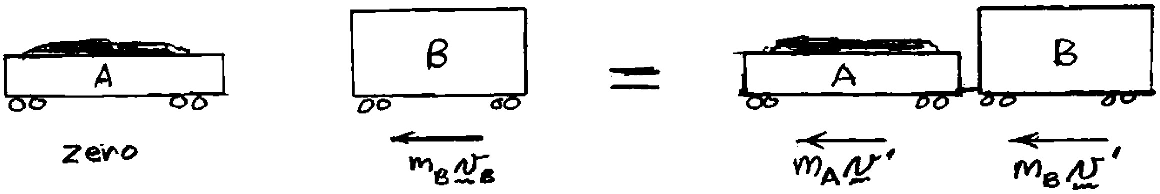
$$\text{Car } A: \quad W_A = 50 \text{ tons} = 100{,}000 \text{ lb}, \quad m_A = 3106 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\text{Car } B: \quad W_B = 25 \text{ tons} = 50{,}000 \text{ lb}, \quad m_B = 1553 \text{ lb}\cdot\text{s}^2/\text{ft}$$

Initital velocities: $\mathbf{v}_A = 0$

$$v_B = 2.5 \text{ mi/h} = 3.6667 \text{ ft/s} \qquad \mathbf{v}_B = 3.6667 \text{ ft/s} \leftarrow$$

(*a*) The momentum of the system consisting of the two cars is conserved immediately before and after coupling.



Let $\mathbf{v}'$ be the common velocity of that cars immediately after coupling. Apply conservation of momentum.

$$+\leftarrow: \; m_B v_B = m_A v' + m_B v'$$

$$v' = \frac{m_B v_B}{m_A + m_B} = \frac{(3106)(3.6667)}{4569} = 2.444 \text{ ft/s} \qquad \mathbf{v}' = 1.667 \text{ mi/h} \leftarrow \blacktriangleleft$$

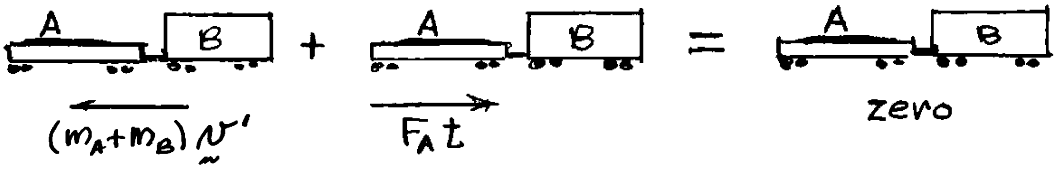
(*b*) After coupling: The friction force acts only on car *A*.

$$+\uparrow \Sigma F = 0_A: \quad N_A - W_A = 0 \qquad N_A = W_A$$

$$F_A = \mu_k N_A = \mu_k W_A \quad \text{(sliding)}$$

$$F_B = 0 \quad \text{(Car } B \text{ is rolling.)}$$

Apply impuslse-momentum to the coupled cars.



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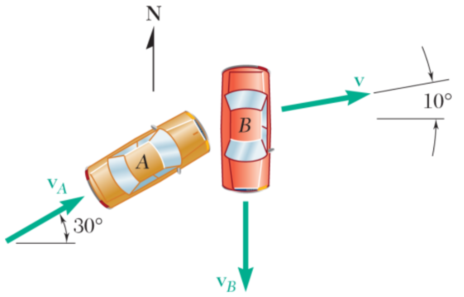
$$\xrightarrow{+}: -(m_A + m_B)v' + F_A t = 0$$

$$t = \frac{(m_A + m_B)v_1^1}{F_A} = \frac{m_B v_B}{\mu_k W_A}$$

$$t = \frac{(1553)(3.6667)}{(0.30)(100,000)} = 0.1898$$

$t = 0.190$ s ◀

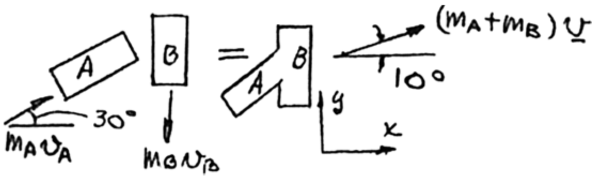


PROBLEM 13.146

At an intersection car *B* was traveling south and car *A* was traveling 30° north of east when they slammed into each other. Upon investigation it was found that after the crash the two cars got stuck and skidded off at an angle of 10° north of east. Each driver claimed that he was going at the speed limit of 50 km/h and that he tried to slow down but couldn't avoid the crash because the other driver was going a lot faster. Knowing that the masses of cars *A* and *B* were 1500 kg and 1200 kg, respectively, determine (*a*) which car was going faster, (*b*) the speed of the faster of the two cars if the slower car was traveling at the speed limit.

SOLUTION

(*a*) Total momentum of the two cars is conserved.



$$\Sigma mv,\ x: \qquad m_A v_A \cos 30° = (m_A + m_B) v \cos 10° \qquad (1)$$

$$\Sigma mv,\ y: \qquad m_A v_A \sin 30° - m_B v_B = (m_A + m_B) v \sin 10° \qquad (2)$$

Dividing (1) into (2),

$$\frac{\sin 30°}{\cos 30°} - \frac{m_B v_B}{m_A v_A \cos 30°} = \frac{\sin 10°}{\cos 10°}$$

$$\frac{v_B}{v_A} = \frac{(\tan 30° - \tan 10°)(m_A \cos 30°)}{m_B}$$

$$\frac{v_B}{v_A} = (0.4010)\frac{(1500)}{(1200)}\cos 30°$$

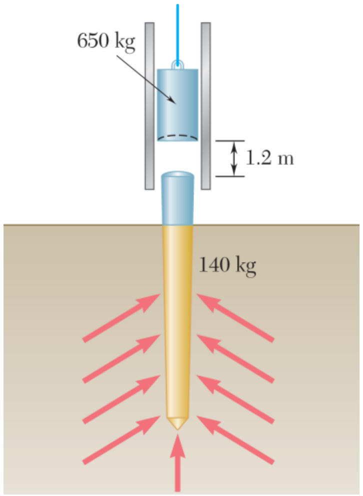
$$\frac{v_B}{v_A} = 0.434 \qquad v_A = 2.30\, v_B$$

Thus, *A* was going faster. ◀

(*b*) Since v_B was the slower car.

$$v_B = 50 \text{ km/h}$$

$$v_A = (2.30)(50) \qquad v_A = 115.2 \text{ km/h} \ ◀$$



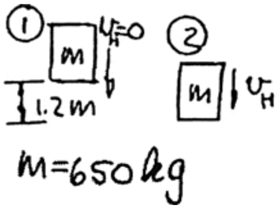
PROBLEM 13.147

The 650-kg hammer of a drop-hammer pile driver falls from a height of 1.2 m onto the top of a 140-kg pile, driving it 110 mm into the ground. Assuming perfectly plastic impact $(e = 0)$, determine the average resistance of the ground to penetration.

SOLUTION

Velocity of the hammer at impact:

Conservation of energy.



$$T_1 = 0$$

$$V_H = mg\ (1.2\text{ m})$$

$$V_H = (650\text{ kg})(9.81\text{ m/s}^2)(1.2\text{ m})$$

$$V_1 = 7652\text{ J}$$

$$T_2 = \frac{1}{2}m$$

$$V_H^2 = \frac{650}{2}v^2 = 325\,v_H^2$$

$$V_2 = 0$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 7652 = 325\,v^2$$

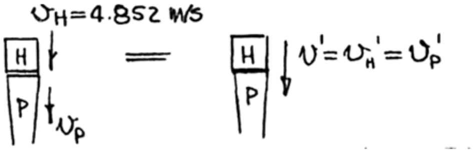
$$v^2 = 23.54\text{ m}^2/\text{s}^2$$

$$v = 4.852\text{ m/s}$$

Velocity of pile after impact:

Since the impact is plastic $(e = 0)$, the velocity of the pile and hammer are the same after impact.

Conservation of momentum:



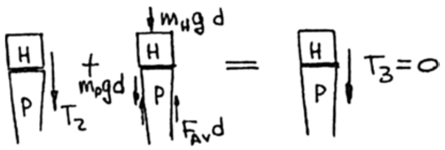
The ground reaction and the weights are non-impulsive.

PROBLEM 13.147 (Continued)

Thus, $m_H v_H = (m_H + m_p)v'$

$$v' = \frac{m_H v_H}{(m_H + m_p)} = \frac{(650)}{(650+140)}(4.852 \text{ m/s}) = 3.992 \text{ m/s}$$

Work and energy: $d = 0.110 \text{ m}$



$$T_2 + U_{2-3} = T_3$$

$$T_2 = \frac{1}{2}(m_H + m_H)(v')^2$$

$$T_3 = 0$$

$$T_2 = \frac{1}{2}(650+140)(3.992)^2$$

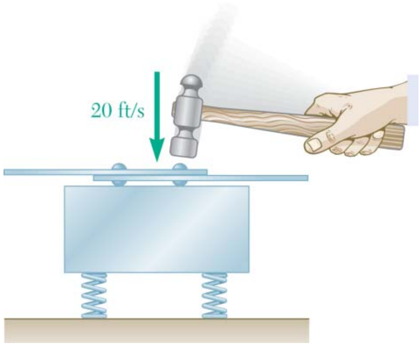
$$T_2 = 6.295 \times 10^3 \text{ J}$$

$$U_{2-3} = (m_H + m_p)gd - F_{AV}d$$
$$= (650+140)(9.81)(0.110) - F_{AV}(0.110)$$
$$U_{2-3} = 852.49 - (0.110)F_{AV}$$

$$T_2 + U_{2-3} = T_3$$

$$6.295 \times 10^3 + 852.49 - (0.110)F_{AV} = 0$$

$$F_{AV} = (7147.5)/(0.110) = 64.98 \times 10^3 \text{ N}$$ $F_{AV} = 65.0 \text{ kN}$ ◀



PROBLEM 13.148

A small rivet connecting two pieces of sheet metal is being clinched by hammering. Determine the impulse exerted on the rivet and the energy absorbed by the rivet under each blow, knowing that the head of the hammer has a weight of 1.5 lbs and that it strikes the rivet with a velocity of 20 ft/s. Assume that the hammer does not rebound and that the anvil is supported by springs and (*a*) has an infinite mass (rigid support), (*b*) has a weight of 9 lb.

SOLUTION

Weight and mass:

Hammer: $W_H = 1.5\text{ lb}$ $\quad m_H = 0.04658\text{ lb}\cdot\text{s}^2/\text{ft}$

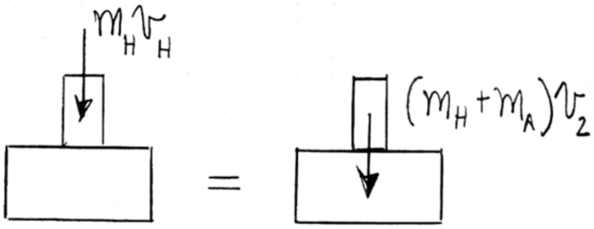
Anvil: Part *a*: $W_A = \infty$ $\quad m_A = \infty$

Part *b*: $W_A = 9\text{ lb}$ $\quad m_A = 0.2795\text{ lb}\cdot\text{s}^2/\text{ft}$

Kinetic energy before impact:

$$T_1 = \frac{1}{2}m_H v_H^2 = \frac{1}{2}(0.04658)(20)^2 = 9.316\text{ ft}\cdot\text{lb}$$

Let v_2 be the velocity common to the hammer and anvil immediately after impact. Apply the principle of conservation of momentum to the hammer and anvil over the duration of the impact.



$$+\downarrow: \Sigma m v_1 = \Sigma m v_2$$

$$m_H v_H = (m_H + m_A)v_2$$

$$v_2 = \frac{m_H v_H}{m_H + m_A} \qquad (1)$$

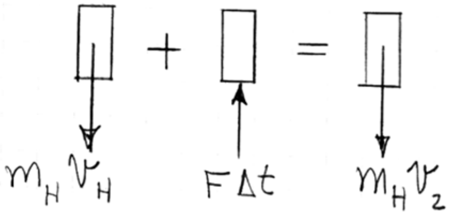
Kinetic energy after impact:

$$T_A = \frac{1}{2}(m_H + m_A)v_2^2 = \frac{1}{2}\frac{m_H^2 v_H^2}{m_H + m_A}$$

$$T_2 = \frac{m_H}{m_H + m_A}T_1 \qquad (2)$$

PROBLEM 13.148 (Continued)

Impulse exerted on the hammer:



$$+\downarrow: m_H v_H - F(\Delta t) = m_H v_2$$

$$F\Delta t = m_H(v_H - v_2) \quad (3)$$

(*a*) $W_A = \infty$:

By Eq. (1), $v_2 = 0$

By Eq. (2), $T_2 = 0$

Energy absorbed: $T_1 - T_2 = 9.32\ \text{ft}\cdot\text{lb}$ ◀

By Eq. (3), $F(\Delta t) = (0.04658)(20-0) = 0.932\ \text{lb}\cdot\text{s}$

The impulse exerted on the rivet the same magnitude but opposite to direction.

$F\Delta t = 0.932\ \text{lb}\cdot\text{s}$ ◀

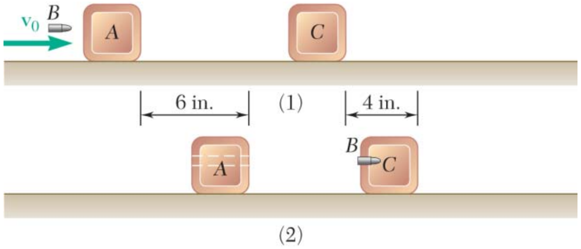
(*b*) $W_A = 9$ lb:

By Eq. (1), $v_2 = \dfrac{(0.04658)(20)}{0.32608} = 2.857$ ft/s

By Eq. (2), $T_2 = \dfrac{(0.04658)(9.316)}{0.32608} = 1.331\ \text{ft}\cdot\text{lb}$

Energy absorbed: $T_1 - T_2 = 7.99\ \text{ft}\cdot\text{lb}$ ◀

By Eq. (3), $F(\Delta t) = (0.04658)(20 - 2.857)$ $F(\Delta t) = 0.799\ \text{lb}\cdot\text{s}$ ◀



PROBLEM 13.149

Bullet B weighs 0.5 oz and blocks A and C both weigh 3 lb. The coefficient of friction between the blocks and the plane is $\mu_k = 0.25$. Initially the bullet is moving at v_0 and blocks A and C are at rest (Figure 1). After the bullet passes through A it becomes embedded in block C and all three objects come to stop in the positions shown (Figure 2). Determine the initial speed of the bullet v_0.

SOLUTION

Masses:

Bullet: $m_B = \dfrac{0.5}{(16)(32.2)} = 970.5 \times 10^{-6}\,\text{lb}\cdot\text{s}^2/\text{ft}$

Blocks A and C: $m_A = m_C = \dfrac{3}{32.2} = 93.168 \times 10^{-3}\,\text{lb}\cdot\text{s}^2/\text{ft}$

Block C + bullet: $m_C + m_B = 94.138 \times 10^{-3}\,\text{lb}\cdot\text{s}^2/\text{ft}$

Normal forces for sliding blocks from $N - mg = 0$

Block A: $N_A = m_A g = 3.00$ lb.

Block C + bullet: $N_C = (m_C + m_B)g = 3.03125$ lb.

Let v_0 be the initial speed of the bullet;

v_1 be the speed of the bullet after it passes through block A;

v_A be the speed of block A immediately after the bullet passes through it;

v_C be the speed block C immediately after the bullet becomes embedded in it.

Four separate processes and their governing equations are described below.

1. The bullet hits block A and passes through it. Use the principle of conservation of momentum.

$$(v_A)_0 = 0$$

$$m_B v_0 + m_A (v_A)_0 = m_B v_1 + m_A v_A$$

$$v_0 = v_1 + \frac{m_A v_A}{m_B} \tag{1}$$

2. The bullet hits block C and becomes embedded in it. Use the principle of conservation of momentum.

$$(v_C)_0 = 0$$

$$m_B v_1 + m_C (v_C)_0 = (m_B + m_C) v_C$$

$$v_1 = \frac{(m_B + m_C) v_C}{m_B} \tag{2}$$

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3. Block *A* slides on the plane. Use principle of work and energy.

$$T_1 + U_{1\to 2} = T_2$$

$$\frac{1}{2}m_A v_A^2 - \mu_k N_A d_A = 0 \quad \text{or} \quad v_A = \sqrt{\frac{2\mu_k N_A d_A}{m_A}} \tag{3}$$

4. Block *C* with embedded bullet slides on the plane. Use principle of work and energy.

$$d_C = 4\text{ in.} = 0.33333\text{ ft}$$

$$T_1 + U_{1\to 2} = T_2$$

$$\frac{1}{2}(m_C + m_B)v_C^2 - \mu_k N_C d_C = 0 \quad \text{or} \quad v_C = \sqrt{\frac{2\mu_k N_C d_C}{m_C + m_B}} \tag{4}$$

Applying the numerical data:

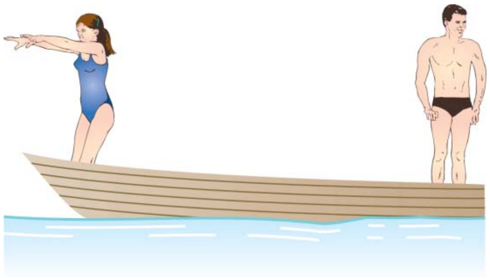
From Eq. (4), $$v_C = \sqrt{\frac{(2)(0.25)(3.03125)(0.33333)}{94.138\times 10^{-3}}} = 2.3166\text{ ft/s}$$

From Eq. (3), $$v_A = \sqrt{\frac{(2)(0.25)(3.00)(0.5)}{93.168\times 10^{-3}}} = 2.8372\text{ ft/s}$$

From Eq. (2), $$v_1 = \frac{(94.138\times 10^{-3})(2.3166)}{970.5\times 10^{-6}} = 224.71\text{ ft/s}$$

From Eq. (1), $$v_0 = 224.71 + \frac{(93.138\times 10^{-3})(2.8372)}{970.5\times 10^{-6}}$$

$v_0 = 497$ ft/s ◀



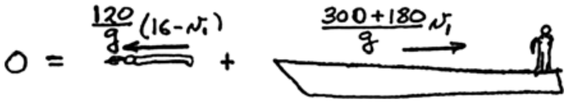
PROBLEM 13.150

A 180-lb man and a 120-lb woman stand at opposite ends of a 300-lb boat, ready to dive, each with a 16-ft/s velocity relative to the boat. Determine the velocity of the boat after they have both dived, if (*a*) the woman dives first, (*b*) the man dives first.

SOLUTION

(*a*) Woman dives first:

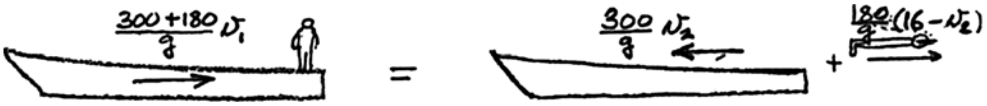
Conservation of momentum:



$$-\frac{120}{g}(16-v_1)+\frac{300+180}{g}v_1=0$$

$$v_1=\frac{(120)(16)}{600}=3.20\text{ ft/s}\longrightarrow$$

Man dives next. Conservation of momentum:



$$\frac{300+180}{g}v_1=-\frac{300}{g}v_2+\frac{180}{g}(16-v_2)$$

$$v_2=\frac{480v_1-(180)(16)}{480}=2.80\text{ ft/s}$$

$$\mathbf{v}_2=2.80\text{ ft/s}\longleftarrow \blacktriangleleft$$

(*b*) Man dives first:

Conservation of momentum:

$$\frac{180}{g}(16-v_1')-\frac{300+120}{g}v_1'=0$$

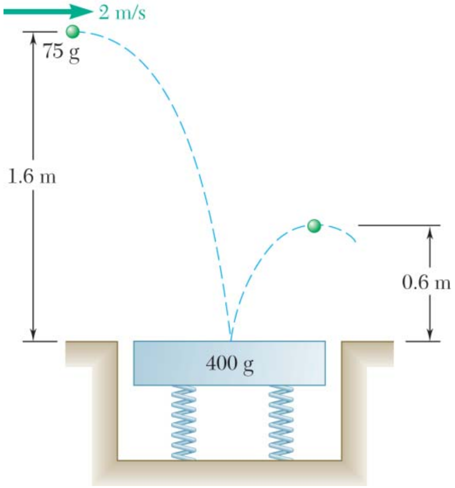
$$v_1'=\frac{(180)(16)}{600}=4.80\text{ ft/s}\longleftarrow$$

Woman dives next. Conservation of momentum:

$$-\frac{300+120}{g}v_1'=\frac{300}{g}v_2'+\frac{120}{g}(16-v_2')$$

$$v_2'=\frac{-420v_1'+(120)(16)}{420}=-0.229\text{ ft/s}$$

$$\mathbf{v}_2'=0.229\text{ ft/s}\longleftarrow \blacktriangleleft$$

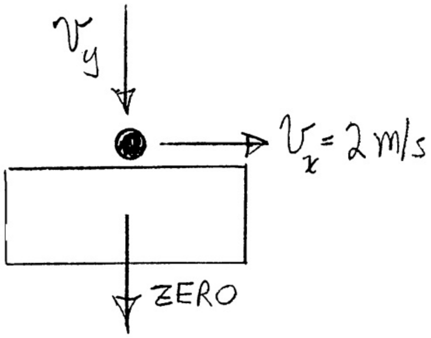


PROBLEM 13.151

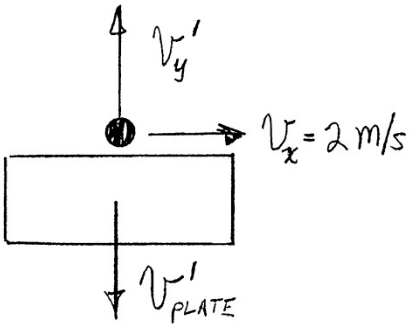
A 75-g ball is projected from a height of 1.6 m with a horizontal velocity of 2 m/s and bounces from a 400-g smooth plate supported by springs. Knowing that the height of the rebound is 0.6 m, determine (*a*) the velocity of the plate immediately after the impact, (*b*) the energy lost due to the impact.

SOLUTION

Just before impact



Just after impact



$$v_y = \sqrt{2g(1.6)} = 5.603 \text{ m/s}$$

$$v_y = \sqrt{2g(0.6)} = 3.431 \text{ m/s}$$

(*a*) Conservation of momentum: $(+y\downarrow)$

$$m_{\text{ball}} v_y + 0 = -m_{\text{ball}} v_y' + m_{\text{plate}} v'_{\text{plate}}$$

$$(0.075)(5.603) + 0 = -0.075(3.431) + 0.4 v'_{\text{plate}}$$

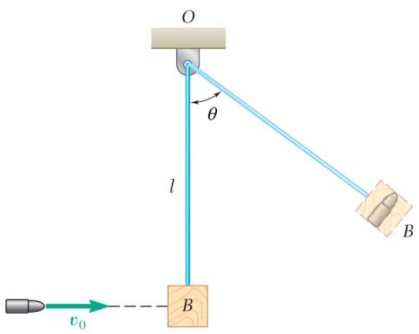
$$v'_{\text{plate}} = 1.694 \text{ m/s} \downarrow \blacktriangleleft$$

(*b*) Energy loss

Initial energy $$(T+V)_1 = \frac{1}{2}(0.075)(2)^2 + 0.075g(1.6)$$

Final energy $$(T+V)_2 = \frac{1}{2}(0.075)(2)^2 + 0.075g(0.6) + \frac{1}{2}(0.4)(1.694)^2$$

$$\text{Energy lost} = (1.3272 - 1.1653)\text{ J} = 0.1619 \text{ J} \blacktriangleleft$$



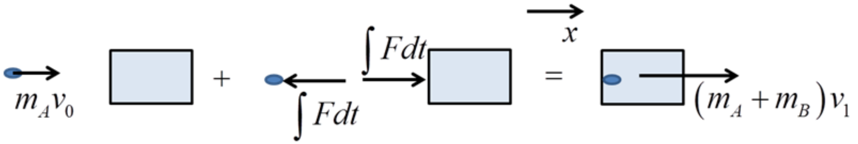
PROBLEM 13.152

A ballistic pendulum is used to measure the speed of high-speed projectiles. A 6-g bullet *A* is fired into a 1-kg wood block *B* suspended by a cord of length *l*= 2.2 m. The block then swings through a maximum angle of θ= 60°. Determine (*a*) the initial speed of the bullet v_o, (b) the impulse imparted by the bullet on the block, (c) the force on the cord immediately after the impact

SOLUTION

Given:
$m_A = 6 \text{ g} = 0.006 \text{ kg}$
$m_B = 1 \text{ kg}$
$l = 2.2 \text{ m}$
$\theta = 60°$

Impulse-Momentum Diagram in x-direction:



Impulse-Momentum in the x-direction during impact:

$$m_A v_0 = (m_A + m_B)v_1 \qquad (1)$$

Apply Work-Energy between the impact location and the maximum swing angle (Datum at the pivot, O):

$$T_1 + V_{g_1} + \cancel{V_{e_1}} + \cancel{U^{NC}_{1\to3}} = \cancel{T_2} + V_{g_2} + \cancel{V_{e_2}}$$

where:
$T_1 = \frac{1}{2}(m_A + m_B)v_1^2$
$V_{g_1} = -(m_A + m_B)gl$
$V_{g_1} = -(m_A + m_B)gl\cos\theta$

Hence:

$$\frac{1}{2}\cancel{(m_A + m_B)}v_1^2 - \cancel{(m_A + m_B)}g = -\cancel{(m_A + m_B)}gl\cos\theta$$

Solving for v_1:

$$v_1 = \sqrt{2gl(1-\cos\theta)}$$
$$= 4.646 \text{ m/s}$$

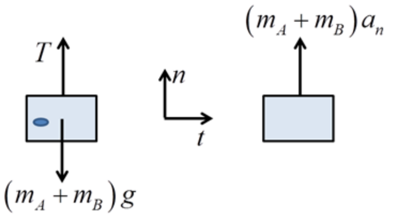
(*a*) Substitute into (1) and solve for v_0:

$$v_0 = \frac{(m_A + m_B)}{m_A}v_1 \qquad v_0 = 778.92 \text{ m/s} ◀$$

(*b*) Write the Impulse Moment Equation for the block during impact:

$$0 + \int Fdt = m_B v_1 \qquad \int Fdt = 4.646J ◀$$

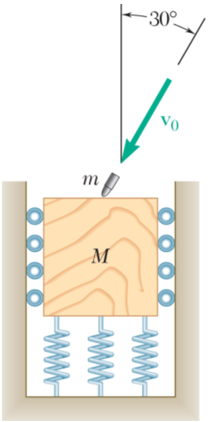
(*c*) FBD of Block just after impact:



$$\sum F_n = ma_n$$

$$T - (m_A + m_B)g = (m_A + m_B)\frac{v_1^2}{l} \qquad T = 19.74 \text{ N} ◀$$

$$T = (m_A + m_B)\left(\frac{v_1^2}{l} + g\right)$$



PROBLEM 13.153

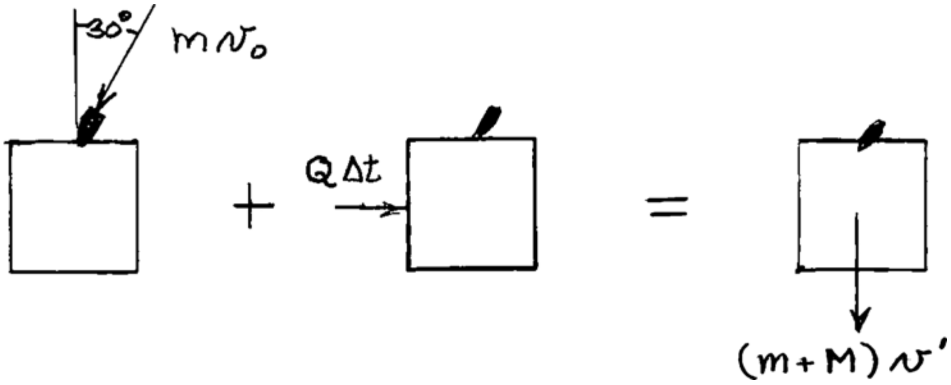
A 1-oz bullet is traveling with a velocity of 1400 ft/s when it impacts and becomes embedded in a 5-lb wooden block. The block can move vertically without friction. Determine (*a*) the velocity of the bullet and block immediately after the impact, (*b*) the horizontal and vertical components of the impulse exerted by the block on the bullet.

SOLUTION

Weight and mass. Bullet: $w = 1 \text{ oz} = \dfrac{1}{16} \text{ lb} \qquad m = 0.001941 \text{ lb} \cdot \text{s}^2/\text{ft.}$

Block: $W = 5 \text{ lb} \qquad M = 0.15528 \text{ lb} \cdot \text{s}^2/\text{ft.}$

(*a*) Use the principle of impulse and momentum applied to the bullet and the block together.



$$\Sigma m\mathbf{v}_1 + \Sigma \mathbf{Imp}_{1\to 2} = m\mathbf{v}_2$$

Components $\downarrow$: $\quad mv_0 \cos 30° + 0 = (m + M)v'$

$$v' = \frac{mv_0 \cos 30°}{m + M} = \frac{(0.001941)(1400)\cos 30°}{0.157221}$$

$$v' = 14.968 \text{ ft/s}$$

$\mathbf{v}' = 14.97 \text{ ft/s} \downarrow$ ◀

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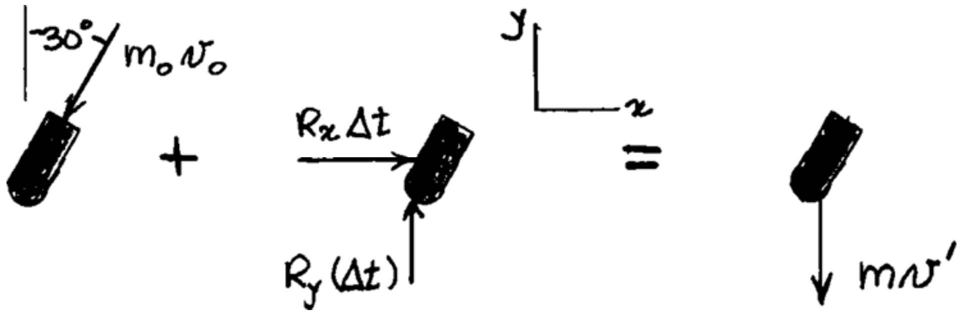
PROBLEM 13.153 (Continued)

(*b*) Use the principle of impulse and momentum applied to the bullet alone.



x-components: $-mv_0\sin 30° + R_x\Delta t = 0$

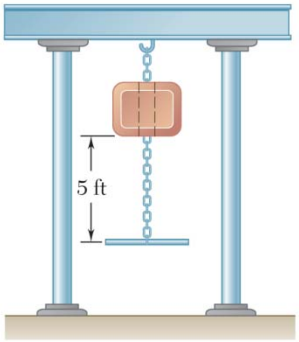
$$R_x\Delta t = mv_0\sin 30° = (0.001941)(1400)\sin 30°$$

$$= 1.3587 \text{ lb}\cdot\text{s} \qquad R_x\Delta t = 1.359 \text{ lb}\cdot\text{s} \blacktriangleleft$$

y-components: $-mv_0\cos 30° + R_y\Delta t = -mv'$

$$R_y\Delta t = m(v_0\cos 30° - v')$$

$$= (0.001941)(1400\cos 30° - 14.968) \qquad R_y\Delta t = 2.32 \text{ lb}\cdot\text{s} \blacktriangleleft$$



PROBLEM 13.154

In order to test the resistance of a chain to impact, the chain is suspended from a 240-lb rigid beam supported by two columns. A rod attached to the last link is then hit by a 60-lb block dropped from a 5-ft height. Determine the initial impulse exerted on the chain and the energy absorbed by the chain, assuming that the block does not rebound from the rod and that the columns supporting the beam are (*a*) perfectly rigid, (*b*) equivalent to two perfectly elastic springs.

SOLUTION

Velocity of the block just before impact:

$$T_1 = 0 \qquad V_1 = Wh = (60\text{ lb})(5\text{ ft}) = 300\text{ lb}\cdot\text{ft}$$

$$T_2 = \frac{1}{2}mv^2 \quad V_2 = 0$$

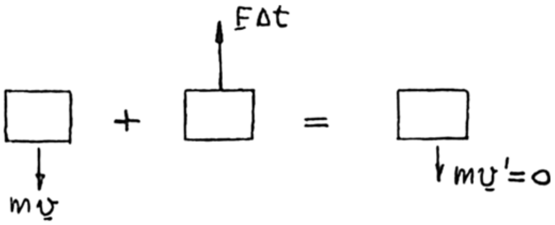
$$T_1 + V_1 = T_2 + V_2$$

$$0 + 300 = \frac{1}{2}\left(\frac{60}{g}\right)v^2$$

$$v = \sqrt{\frac{(600)(32.2)}{60}}$$

$$= 17.94\text{ ft/s}$$

(*a*) Rigid columns:



$$+\!\uparrow\; -mv + F\Delta t = 0 \qquad \left(\frac{60}{g}\right)(17.94) = F\Delta t$$

$$F\Delta t = 33.43\text{ lb}\cdot\text{s}\uparrow \text{ on the block.}$$

$F\Delta t = 33.4$ lb·s ◀

All of the kinetic energy of the block is absorbed by the chain.

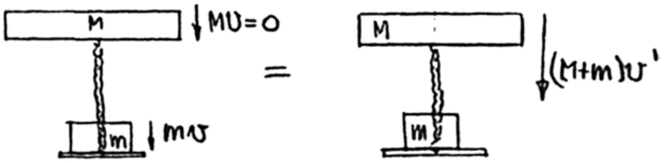
$$T = \frac{1}{2}\left(\frac{60}{g}\right)(17.94)^2$$

$$= 300\text{ ft}\cdot\text{lb}$$

$E = 300$ ft · lb ◀

(*b*) Elastic columns:

Momentum of system of block and beam is conserved.



$$mv = (M + m)v'$$

$$v' = \frac{m}{(m+M)}v = \frac{60}{300}(17.94 \text{ ft/s}) \qquad v' = 3.59 \text{ ft/s}$$

Referring to figure in part (*a*),

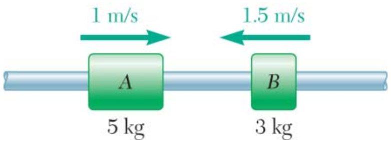
$$-mv + F\Delta t = -mv'$$

$$F\Delta t = m(v - v')$$

$$= \left(\frac{60}{g}\right)(17.94 - 3.59) \qquad F\Delta t = 26.7 \text{ lb}\cdot\text{s} \blacktriangleleft$$

$$E = \frac{1}{2}mv^2 - \frac{1}{2}mv'^2 - \frac{1}{2}Mv'^2$$

$$= \frac{60}{2g}[(17.94)^2 - (3.59)^2] - \frac{240}{2g}(3.59)^2 \qquad E = 240 \text{ ft}\cdot\text{lb} \blacktriangleleft$$



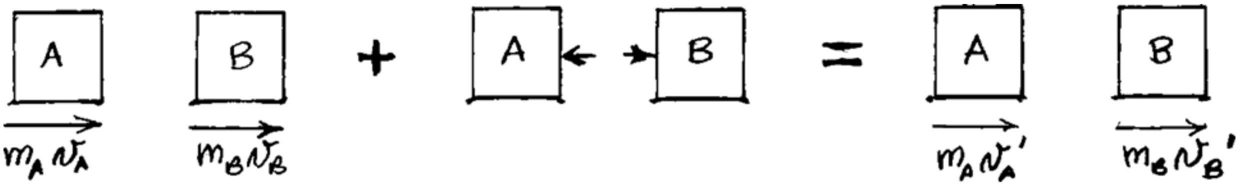
PROBLEM 13.155

The coefficient of restitution between the two collars is known to be 0.70. Determine (*a*) their velocities after impact, (*b*) the energy loss during impact.

SOLUTION

Impulse-momentum principle (collars *A* and *B*):

$$\Sigma m\mathbf{v}_1 + \Sigma \mathbf{Imp}_{1\to 2} = \Sigma m\mathbf{v}_2$$



Horizontal components $\xrightarrow{+}$: $m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$

Using data, $(5)(1) + (3)(-1.5) = 5v_A' + 3v_B'$

or $5v_A' + 3v_B' = 0.5$ (1)

Apply coefficient of restitution.

$$v_B' - v_A' = e(v_A - v_B)$$
$$v_B' - v_A' = 0.70[1 - (-0.5)]$$
$$v_B' - v_A' = 1.75 \quad (2)$$

(*a*) Solving Eqs. (1) and (2) simultaneously for the velocities,

$v_A' = -0.59375$ m/s $\qquad \mathbf{v}_A = 0.594$ m/s ⟵ ◀

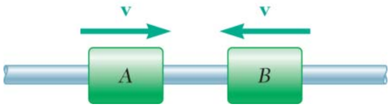
$v_B' = 1.15625$ m/s $\qquad \mathbf{v}_B = 1.156$ m/s ⟶ ◀

Kinetic energies: $T_1 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 = \frac{1}{2}(5)(1)^2 + \frac{1}{2}(3)(-1.5)^2 = 5.875 \text{ J}$

$$T_2 = \frac{1}{2}m_A(v_A')^2 + \frac{1}{2}m_B(v_B')^2 = \frac{1}{2}(5)(-0.59375)^2 + \frac{1}{2}(3)(1.15625)^2 = 2.8867 \text{ J}$$

(*b*) Energy loss: $T_1 - T_2 = 2.99 \text{ J}$ ◀

PROBLEM 13.156

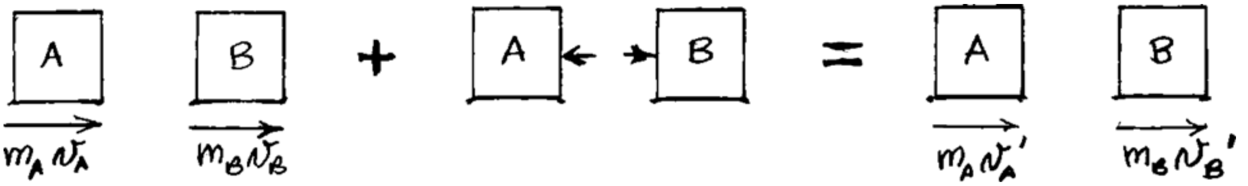


Collars *A* and *B*, of the same mass *m*, are moving toward each other with identical speeds as shown. Knowing that the coefficient of restitution between the collars is *e*, determine the energy lost in the impact as a function of *m*, *e* and *v*.

SOLUTION

Impulse-momentum principle (collars *A* and *B*):

$$\Sigma m\mathbf{v}_1 + \Sigma \mathbf{Imp}_{1\to 2} = \Sigma m\mathbf{v}_2$$



Horizontal components $\xrightarrow{+}$: $m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$

Using data, $mv + m(-v) = mv_A' + mv_B'$

or $v_A' + v_B' = 0$ (1)

Apply coefficient of restitution.

$$v_B' - v_A' = e(v_A - v_B)$$
$$v_B' - v_A' = e[v - (-v)]$$
$$v_B' - v_A' = 2ev \qquad (2)$$

Subtracting Eq. (1) from Eq. (2), $-2v_A = 2ev$

$v_A = -ev$ $\qquad \mathbf{v}_A = ev \longleftarrow$

Adding Eqs. (1) and (2), $2v_B = 2ev$

$v_B = ev$ $\qquad \mathbf{v}_B = ev \longrightarrow$

Kinetic energies: $T_1 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 = \frac{1}{2}mv^2 + \frac{1}{2}m(-v)^2 = mv^2$

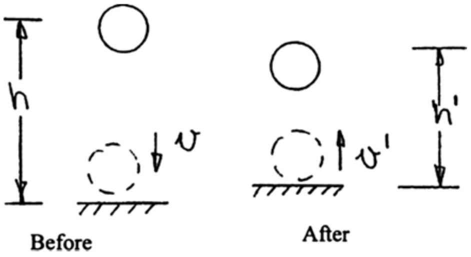
$$T_2 = \frac{1}{2}m_A(v_A')^2 + \frac{1}{2}m_B(v_B')^2 = \frac{1}{2}m(ev)^2 + \frac{1}{2}m(ev)^2 = e^2 mv^2$$

Energy loss: $T_1 - T_2 = (1 - e^2)\, mv^2$ ◀

PROBLEM 13.157

One of the requirements for tennis balls to be used in official competition is that, when dropped onto a rigid surface from a height of 100 in., the height of the first bounce of the ball must be in the range 53 in. $\leq h \leq 58$ in. Determine the range of the coefficient of restitution of the tennis balls satisfying this requirement.

SOLUTION



Uniform accelerated motion:

$$v = \sqrt{2gh}$$

$$v' = \sqrt{2gh'}$$

Coefficient of restitution: $$e = \frac{v'}{v}$$

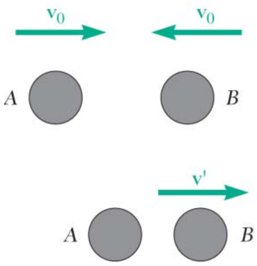
$$e = \sqrt{\frac{h'}{h}}$$

Height of drop $$h = 100 \text{ in.}$$

Height of bounce $$53 \text{ in.} \leq h' \leq 58 \text{ in.}$$

Thus, $$\sqrt{\frac{53}{100}} \leq e \leq \sqrt{\frac{58}{100}}$$

$0.728 \leq e \leq 0.762$ ◀

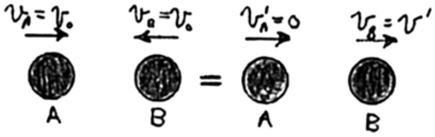


PROBLEM 13.158

Two disks sliding on a frictionless horizontal plane with opposite velocities of the same magnitude v_0 hit each other squarely. Disk *A* is known to have a weight of 6-lb and is observed to have zero velocity after impact. Determine (*a*) the weight of disk *B*, knowing that the coefficient of restitution between the two disks is 0.5, (*b*) the range of possible values of the weight of disk *B* if the coefficient of restitution between the two disks is unknown.

SOLUTION

Total momentum conserved:



$$\xrightarrow{+} m_A v_A + m_B v_B = m_A v'_A + m_B v'$$

$$(m_A)v_0 + m_B(-v_0) = 0 + m_B v'$$

$$v' = \left(\frac{m_A}{m_B} - 1\right)v_0 \qquad (1)$$

Relative velocities:

$$v'_B - v'_A = e(v_A - v_B)$$

$$v' = 2ev_0 \qquad (2)$$

Subtracting Eq. (2) from Eq. (1) and dividing by v_0,

$$\frac{m_A}{m_B} - 1 - 2e = 0 \qquad \frac{m_A}{m_B} = 1 + 2e \qquad m_B = \frac{m_A}{1+2e}$$

Since weight is proportional to mass, $$W_B = \frac{W_A}{1+2e} \qquad (3)$$

(*a*) With $W_A = 6$ lb and $e = 0.5$,

$$W_B = \frac{6}{1+(2)(0.5)}$$

$W_B = 3.00$ lb ◀

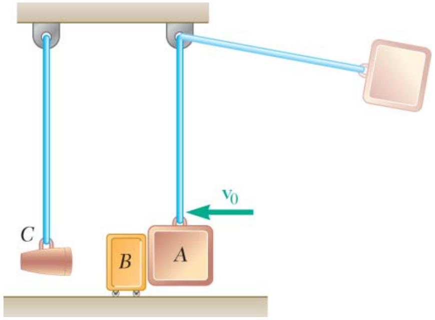
(*b*) With $W_A = 6$ lb and $e = 1$,

$$W_B = \frac{6}{1+(2)(1)} = 2 \text{ lb}$$

With $W_A = 6$ lb and $e = 0$,

$$W_B = \frac{6}{1+(2)(0)} = 6 \text{ lb}$$

Range: $2.00 \text{ lb} \le W_B \le 6.00 \text{ lb}$ ◀



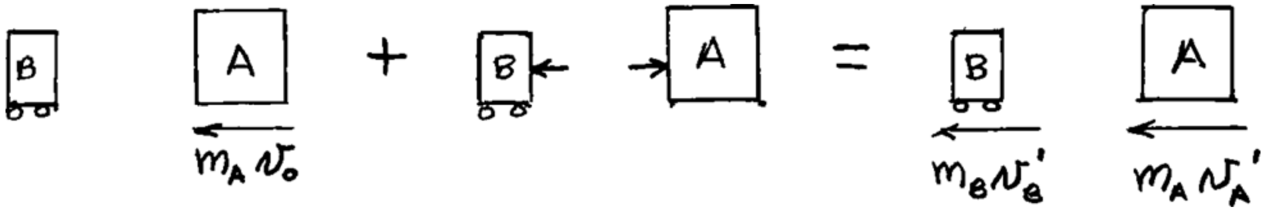
PROBLEM 13.159

To apply shock loading to an artillery shell, a 20-kg pendulum *A* is released from a known height and strikes impactor *B* at a known velocity $\mathbf{v_0}$. Impactor *B* then strikes the 1-kg artillery shell *C*. Knowing the coefficient of restitution between all objects is *e*, determine the mass of *B* to maximize the impulse applied to the artillery shell *C*.

SOLUTION

First impact: *A* impacts *B*. $m_A = 20 \text{ kg}, \ m_B = ?$

Impulse-momentum: $\Sigma m\mathbf{v} + \Sigma \mathbf{Imp}_{1\to 2} = \Sigma m\mathbf{v}_2$



Components directed left:

$$m_A v_0 = m_A v_A' + m_B v_B'$$

$$20v_0 = 20v_A' + m_B v_B' \qquad (1)$$

Coefficient of restitution:

$$v_B' - v_A' = e(v_A - v_B)$$

$$v_B' - v_A' = ev_0$$

$$v_A' = v_B' - ev_0 \qquad (2)$$

Substituting Eq. (2) into Eq. (1) yields

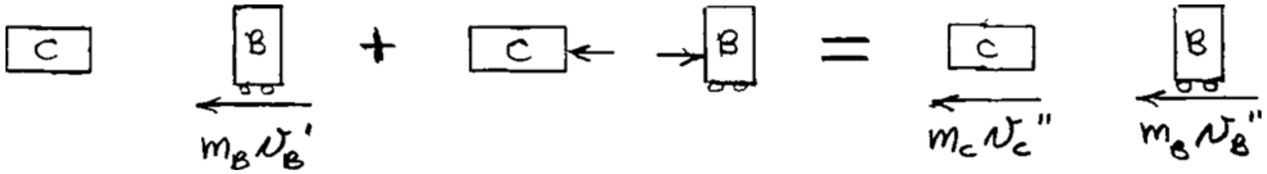
$$20v_0 = 20(v_B' - ev_0) + m_B v_B'$$

$$20v_0(1+e) = (+m_B)v_B'$$

$$v_B' = \frac{20v_0(1+e)}{20 + m_B} \qquad (3)$$

Second impact: *B* impacts *C*. $m_B = ?, \ m_C = 1 \text{ kg}$

Impulse-momentum: $\Sigma m\mathbf{v}_2 + \Sigma \mathbf{Imp}_{2\to 3} = \Sigma m\mathbf{v}_3$



Components directed left:

$$m_B v_B' = m_B v_B'' + m_C v_C''$$

$$m_B v_B' = m_B v_B'' + v_C'' \qquad (4)$$

Coefficient of restitution: $v_C'' - v_B'' = e(v_B' - v_C')$

$$v_C'' - v_B'' = e v_B'$$

$$v_B'' - v_C'' = e v_C' \qquad (5)$$

Substituting Eq. (4) into Eq. (5) yields

$$m_B v_B' = m_B (v_C'' - e v_B') + m_C v_C''$$

$$m_B v_B' (1+e) = (1+m_B) v_C''$$

$$v_C'' = \frac{m_B v_B' (1+e)}{1+m_B} \qquad (6)$$

Substituting Eq. (3) for v_B' in Eq. (6) yields

$$v_C'' = \frac{20 m_B v_0 (1+e)^2}{(20+m_B)(1+m_B)}$$

The impulse applied to the shell C is

$$m_C v_C'' = \frac{(1)(20)\, m_B v_0 (1+e)^2}{(20+m_B)(1+m_B)}$$

To maximize this impulse choose m_B such that

$$Z = \frac{m_B}{(20+m_B)(1+m_B)}$$

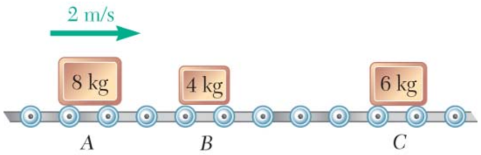
is maximum. Set dZ/dm_B equal to zero.

$$\frac{dZ}{dm_B} = \frac{(20+m_B)(1+m_B) - m_B[(20+m_B)+(1+m_B)]}{(20+m_B)^2(1+m_B)^2} = 0$$

$$20 + 21 m_B + m_B^2 - m_B(21 + 2m_B) = 0$$

$$20 - m_B^2 = 0$$

$m_B = 4.47$ kg ◀

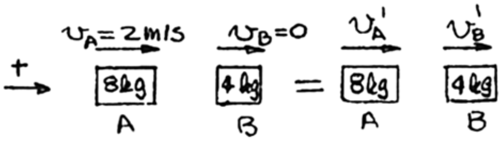

2 m/s
8 kg
4 kg
6 kg
A
B
C

PROBLEM 13.160

Packages in an automobile parts supply house are transported to the loading dock by pushing them along on a roller track with very little friction. At the instant shown, packages *B* and *C* are at rest and package *A* has a velocity of 2 m/s. Knowing that the coefficient of restitution between the packages is 0.3, determine (*a*) the velocity of package *C* after *A* hits *B* and *B* hits *C*, (*b*) the velocity of *A* after it hits *B* for the second time.

SOLUTION

(*a*) Packages *A* and *B*.


+
$v_A = 2$ m/s
$v_B = 0$
v_A'
v_B'
8 kg
4 kg
=
8 kg
4 kg
A
B
A
B

Total momentum conserved.

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

$$(8 \text{ kg})(2 \text{ m/s}) + 0 = (8 \text{ kg})v_A' + (4 \text{ kg})v_B'$$

$$4 = 2v_A' + v_B' \qquad (1)$$

Relative velocities.

$$(v_A - v_B)e = (v_B' - v_A')$$

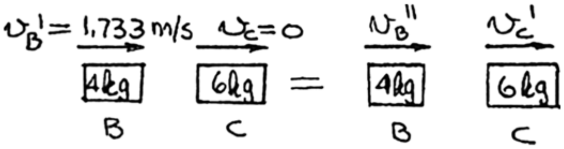
$$(2)(0.3) = v_B' - v_A' \qquad (2)$$

Solving Equations (1) and (2) simultaneously,

$$v_A' = 1.133 \text{ m/s} \rightarrow$$

$$v_B' = 1.733 \text{ m/s} \rightarrow$$

Packages *B* and *C*.


$v_B' = 1.733$ m/s
$v_C = 0$
v_B''
v_C'
4 kg
6 kg
=
4 kg
6 kg
B
C
B
C

$$\xrightarrow{+}\ m_B v_B' + m_C v_C = m_B v_B'' + m_C v_C'$$

$$(4 \text{ kg})(1.733 \text{ m/s}) + 0 = 4v_B'' + 6v_c'$$

$$6.932 = 4v_B'' + 6v_C' \qquad (3)$$

Relative velocities.

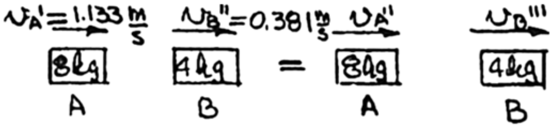
$$(v_B' - v_C)e = v_C' - v_B''$$

$$(1.733)(0.3) = 0.5199 = v_C' - v_B'' \qquad (4)$$

Solving equations (3) and (4) simultaneously,

$\mathbf{v}_C' = 0.901$ m/s → ◀

(*b*) Packages *A* and *B* (second time),


vA' = 1.133 m/s
vB'' = 0.381 m/s
vA'
vB'''
8kg
4kg
=
8kg
4kg
A
B
A
B

Total momentum conserved.

$$(8)(1.133) + (4)(0.381) = 8v_A'' + 4v_B'''$$

$$10.588 = 8v_A'' + 4v_B'' \qquad (5)$$

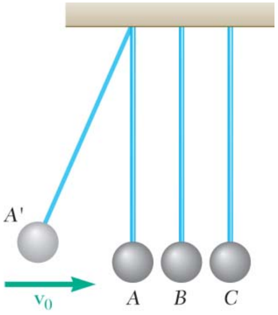
Relative velocities.

$$(v_A' - v_B'')e = v_B''' - v_A''$$

$$(1.133 - 0.381)(0.3) = 0.2256 = v_B''' - v_A'' \qquad (6)$$

Solving (5) and (6) simultaneously,

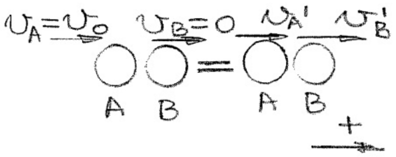
$v_A'' = 0.807$ m/s $\qquad$ $\mathbf{v}_A'' = 0.807$ m/s → ◀



PROBLEM 13.161

Three steel spheres of equal mass are suspended from the ceiling by cords of equal length which are spaced at a distance slightly greater than the diameter of the spheres. After being pulled back and released, sphere *A* hits sphere *B*, which then hits sphere *C*. Denoting by *e* the coefficient of restitution between the spheres and by $\mathbf{v}_0$ the velocity of *A* just before it hits *B*, determine (*a*) the velocities of *A* and *B* immediately after the first collision, (*b*) the velocities of *B* and *C* immediately after the second collision. (*c*) Assuming now that *n* spheres are suspended from the ceiling and that the first sphere is pulled back and released as described above, determine the velocity of the last sphere after it is hit for the first time. (*d*) Use the result of part *c* to obtain the velocity of the last sphere when $n = 8$ and $e = 0.9$.

SOLUTION



(*a*) First collision (between *A* and *B*)

The total momentum is conserved

$$mv_A + mv_B = mv_A' + mv_B'$$

$$v_0 = v_A' + v_B' \qquad (1)$$

Relative velocities

$$(v_A - v_B)e = (v_B' - v_A')$$

$$v_0 e = v_B' - v_A' \qquad (2)$$

Solving equations (1) and (2) simultaneously

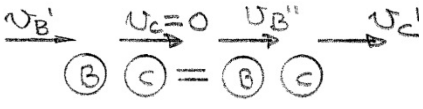
$$v_A' = \frac{v_0(1-e)}{2} \blacktriangleleft$$

$$v_B' = \frac{v_0(1+e)}{2} \blacktriangleleft$$

(*b*) Second collision (Between *B* and *C*)

The total momentum is conserved.

$$mv_B' + mv_C = mv_B'' + mv_C'$$



Using the result from (*a*) for v_B'

$$\frac{v_0(1+e)}{2} + 0 = v_B'' + v_C' \qquad (3)$$

Relative velocities

$$(v_B' - 0)e = v_C' - v_B''$$

Substituting again for v_B' from (*a*)

$$v_0\frac{(1+e)}{2}(e) = v_C' - v_B'' \qquad (4)$$

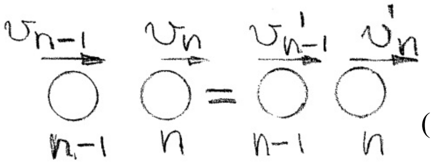
PROBLEM 13.161 (Continued)

Solving equations (3) and (4) simultaneously

$$v'_C = \frac{1}{2}\left[\frac{v_0(1+e)}{2} + v_0(1+e)\frac{(e)}{2}\right]$$

$$v'_C = \frac{v_0(1+e)^2}{4} \blacktriangleleft$$

$$v''_B = \frac{v_0(1-e)^2}{4} \blacktriangleleft$$



(*c*) For *n* spheres

n Balls

$n-1th$ collision

We note from the answer to part (*b*), with $n = 3$

$$v'_n = v'_3 = v'_C = \frac{v_0(1+e)^2}{4}$$

or

$$v'_3 = \frac{v_0(1+e)^{(3-1)}}{2^{(3-1)}}$$

Thus for *n* balls

$$v'_n = \frac{v_0(1+e)^{(n-1)}}{2^{(n-1)}} \blacktriangleleft$$

(*d*) For $n = 8,\ e = 0.90$

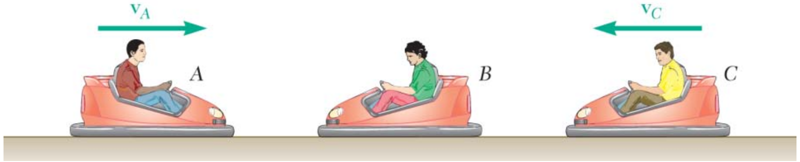
From the answer to part (*c*) with $n = 8$

$$v'_B = \frac{v_0(1+0.9)^{(8-1)}}{2^{(8-1)}} = \frac{v_0(1.9)^7}{(2)^7}$$

$$v'_8 = 0.698v_0 \blacktriangleleft$$

PROBLEM 13.162

At an amusement park there are 200-kg bumper cars *A*, *B*, and *C* that have riders with masses of 40 kg, 60 kg, and 35 kg respectively. Car *A* is moving to the right with a velocity $\mathbf{v}_A = 2$ m/s and car *C* has a velocity $\mathbf{v}_B = 1.5$ m/s to the left, but car *B* is initially at rest. The coefficient of restitution between each car is 0.8. Determine the final velocity of each car, after all impacts, assuming (*a*) cars *A* and *C* hit car *B* at the same time, (*b*) car *A* hits car *B* before car *C* does.



SOLUTION

Assume that each car with its rider may be treated as a particle. The masses are:

$$m_A = 200 + 40 = 240 \text{ kg},$$
$$m_B = 200 + 60 = 260 \text{ kg},$$
$$m_C = 200 + 35 = 235 \text{ kg}.$$

Assume velocities are positive to the right. The initial velocities are:

$$v_A = 2 \text{ m/s} \quad v_B = 0 \quad v_C = -1.5 \text{ m/s}$$

Let v'_A, v'_B, and v'_C be the final velocities.

(*a*) Cars *A* and *C* hit *B* at the same time. Conservation of momentum for all three cars.

$$m_A v_A + m_B v_B + m_C v_C = m_A v'_A + m_B v'_B + m_C v'_C$$
$$(240)(2) + 0 + (235)(-1.5) = 240v'_A + 260v'_B + 235v'_C \qquad (1)$$

Coefficient of restition for cars *A* and *B*.

$$v'_B - v'_A = e(v_A - v_B) = (0.8)(2 - 0) = 1.6 \qquad (2)$$

Coefficient of restitution for cars *B* and *C*.

$$v'_C - v'_B = e(v_B - v_C) = (0.8)[0 - (-1.5)] = 1.2 \qquad (3)$$

Solving Eqs. (1), (2), and (3) simultaneously,

$$v'_A = -1.288 \text{ m/s} \quad v'_B = 0.312 \text{ m/s} \quad v'_C = 1.512 \text{ m/s}$$

$\mathbf{v}'_A = 1.288$ m/s ⟵ ◀

$\mathbf{v}'_B = 0.312$ m/s ⟶ ◀

$\mathbf{v}'_C = 1.512$ m/s ⟶ ◀

(*b*) Car *A* hits car *B* before *C* does.

First impact. Car *A* hits car *B*. Let v'_A and v'_B be the velocities after this impact. Conservation of momentum for cars *A* and *B*.

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B$$
$$(240)(2) + 0 = 240 v'_A + 260 v'_B \qquad (4)$$

Coefficient of restitution for cars *A* and *B*.

$$v'_B - v'_A = e(v_A - v_B) = (0.8)(2-0) = 1.6 \qquad (5)$$

Solving Eqs. (4) and (5) simultaneously,

$$v'_A = 0.128 \text{ m/s}, \quad v'_B = 1.728 \text{ m/s}$$

$$\mathbf{v}'_A = 0.128 \text{ m/s} \longrightarrow$$

$$\mathbf{v}'_B = 1.728 \text{ m/s} \longrightarrow$$

Second impact. Cars *B* and *C* hit. Let v''_B and v''_C be the velocities after this impact. Conservation of momentum for cars *B* and *C*.

$$m_B v'_B + m_C v_C = m_B v''_B + m_C v''_C$$

$$(260)(1.728) + (235)(-1.5) = 260 v''_B + 235 v''_C \qquad (6)$$

Coefficient of restitution for cars *B* and *C*.

$$v''_C - v''_B = e(v'_B - v_C) = (0.8)[1.728 - (-1.5)] = 2.5824 \qquad (7)$$

Solving Eqs. (6) and (7) simultaneously,

$$v''_B = -1.03047 \text{ m/s} \quad v''_C = 1.55193 \text{ m/s}$$

$$\mathbf{v}''_B = 1.03047 \text{ m/s} \longleftarrow$$

$$\mathbf{v}''_C = 1.55193 \text{ m/s} \longrightarrow$$

Third impact. Cars *A* and *B* hit again. Let v'''_A and v'''_B be the velocities after this impact. Conservation of momentum for cars *A* and *B*.

$$m_A v'_A + m_B v''_B = m_A v'''_A + m_B v'''_B$$
$$(240)(0.128) + (260)(-1.03047) = 240 v'''_A + 260 v'''_B \qquad (8)$$

Coefficient of restitution for cars *A* and *B*.

$$v'''_B - v'''_A = e(v'_A - v''_B) = (0.8)[0.128 - (-1.03047)] = 0.926776 \qquad (9)$$

Solving Eqs. (8) and (9) simultaneously,

$$v'''_A = -0.95633 \text{ m/s}$$
$$v'''_B = -0.02955 \text{ m/s}$$

$$\mathbf{v}'''_A = 0.95633 \text{ m/s} \longleftarrow$$

$$\mathbf{v}'''_B = 0.02955 \text{ m/s} \longleftarrow$$

PROBLEM 13.162 (Continued)

There are no more impacts. The final velocities are:

$\mathbf{v}_A''' = 0.956 \text{ m/s} \longleftarrow$ ◀

$\mathbf{v}_B''' = 0.0296 \text{ m/s} \longleftarrow$ ◀

$\mathbf{v}_C'' = 1.552 \text{ m/s} \longrightarrow$ ◀

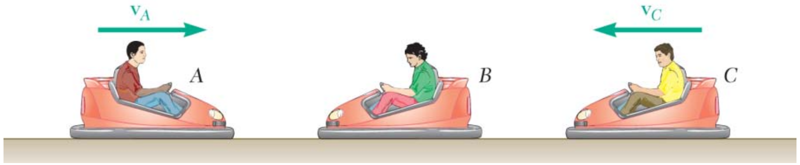
We may check our results by considering conservation of momentum of all three cars over all three impacts.

$$\begin{aligned} m_A v_A + m_B v_B + m_C v_C &= (240)(2) + 0 + (235)(-1.5) \\ &= 127.5 \text{ kg} \cdot \text{m/s} \end{aligned}$$

$$\begin{aligned} m_A v_A''' + m_B v_B''' + m_C v_C'' &= (240)(-0.95633) + (260)(-0.02955) + (235)(1.55193) \\ &= 127.50 \text{ kg} \cdot \text{m/s.} \end{aligned}$$

PROBLEM 13.163

At an amusement park there are 200-kg bumper cars *A*, *B*, and *C* that have riders with masses of 40 kg, 60 kg, and 35 kg respectively. Car *A* is moving to the right with a velocity $\mathbf{v}_A = 2$ m/s when it hits stationary car *B*. The coefficient of restitution between each car is 0.8. Determine the velocity of car *C* so that after car *B* collides with car *C* the velocity of car *B* is zero.



SOLUTION

Assume that each car with its rider may be treated as a particle. The masses are:

$$m_A = 200 + 40 = 240 \text{ kg}$$
$$m_B = 200 + 60 = 260 \text{ kg}$$
$$m_C = 200 + 35 = 235 \text{ kg}$$

Assume velocities are positive to the right. The initial velocities are:

$$v_A = 2 \text{ m/s}, \quad v_B = 0, \quad v_C = ?$$

First impact. Car *A* hits car *B*. Let v_A' and v_B' be the velocities after this impact. Conservation of momentum for cars *A* and *B*.

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$
$$(240)(2) + 0 = 240 v_A' + 260 v_B' \qquad (1)$$

Coefficient of restitution for cars *A* and *B*.

$$v_B' - v_A' = e(v_A - v_B) = (0.8)(2 - 0) = 1.6 \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously,

$$v_A' = 0.128 \text{ m/s}$$
$$v_B' = 1.728 \text{ m/s}$$

$$\mathbf{v}_A' = 0.128 \text{ m/s} \longrightarrow$$

$$\mathbf{v}_B' = 1.728 \text{ m/s} \longrightarrow$$

Second impact. Cars *B* and *C* hit. Let v_B'' and v_C'' be the velocities after this impact. $v_B'' = 0$. Coefficient of restitution for cars *B* and *C*.

$$v_C'' - v_B'' = e(v_B' - v_C) = (0.8)(1.728 - v_C)$$
$$v_C'' = 1.3824 - 0.8 v_C$$

PROBLEM 13.163 (Continued)

Conservation of momentum for cars *B* and *C*.

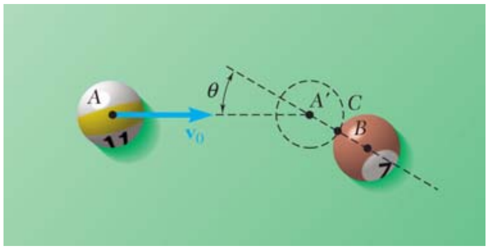
$$m_B v'_B + m_C v_C = m_B v''_B + m_C v''_C$$

$$(260)(1.728) + 235 v_C = (260)(0) + (235)(1.3824 - 0.8 v_C)$$

$$(235)(1.8) v_C = (235)(1.3824) - (260)(1.728)$$

$$v_C = -0.294 \text{ m/s} \qquad \mathbf{v}_C = 0.294 \text{ m/s} \longleftarrow \blacktriangleleft$$

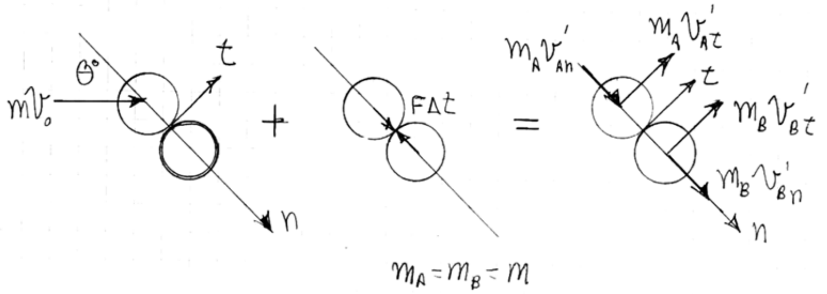
Note: There will be another impact between cars *A* and *B*.



PROBLEM 13.164

Two identical billiard balls can move freely on a horizontal table. Ball *A* has a velocity $\mathbf{v}_0$ as shown and hits ball *B*, which is at rest, at a Point *C* defined by $\theta = 45°$. Knowing that the coefficient of restitution between the two balls is $e = 0.8$ and assuming no friction, determine the velocity of each ball after impact.

SOLUTION



Ball *A*: *t-dir*

$$\cancel{m}v_0 \sin\theta = \cancel{m}v'_{At} \Rightarrow v'_{At} = v_0 \sin\theta$$

Ball *B*: *t-dir*

$$0 = m_B v'_{Bt} \Rightarrow v'_{Bt} = 0$$

Balls *A* + *B*: *n-dir*

$$\cancel{m}v_0 \cos\theta + 0 = \cancel{m}\, v'_{An} + \cancel{m}\, v'_{Bn} \tag{1}$$

Coefficient of restitution

$$v'_{Bn} - v'_{An} = e(v_{An} - v_{Bn})$$

$$v'_{Bn} - v'_{An} = e(v_0 \cos\theta - 0) \tag{2}$$

Solve (1) and (2)

$$v'_{An} = v_0\left(\frac{1-e}{2}\cos\theta\right); \quad v'_{Bn} = v_0\left(\frac{1+e}{2}\right)\cos\theta$$

With numbers

$$e = 0.8; \quad \theta = 45°$$

$$v'_{At} = v_0 \sin 45° = 0.707\, v_0$$

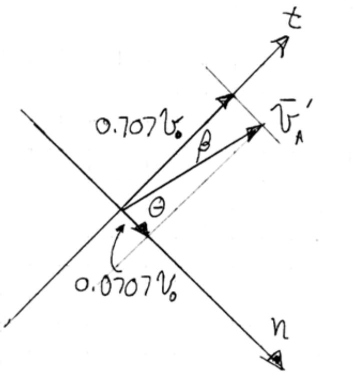
$$v'_{An} = v_0\left(\frac{1-0.8}{2}\cos 45°\right) = 0.0707\ v_0$$

$$v'_{Bt} = 0$$

$$v'_{Bn} = v_0\left(\frac{1+0.8}{2}\right)\cos 45° = 0.6364\, v_0$$

PROBLEM 13.164 (Continued)

(*A*)

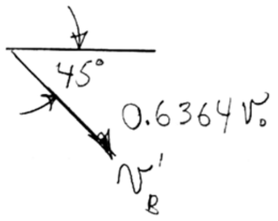

t
0.707v₀
v'_A
β
θ
0.0707v₀
n

$$\left|v''_A\right| = [(0.707\,v_0)^2 + (0.0707 v_0)^2]^{\frac{1}{2}} = 0.711 v_0$$

$$\beta = \tan^{-1}\left(\frac{0.0707}{0.707}\right) = 5.7106°$$

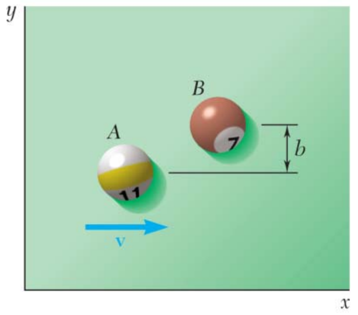
So $\theta = 45 - 5.7106 = 39.3°$

(*B*)


45°
0.6364v₀
v'_B

$\vec{v}'_A = 0.711 v_0$ ∠39.3° ◀

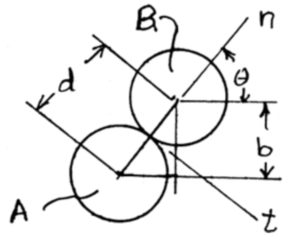
$\vec{v}'_B = 0.636 v_0$ ∠45° ◀



PROBLEM 13.165

Two identical pool balls of 2.37-in. diameter may move freely on a pool table. Ball *B* is at rest and ball *A* has an initial velocity $\mathbf{v} = v_0\mathbf{i}$. (*a*) Knowing that $b = 2$ in. and $e = 0.7$, determine the velocity of each ball after impact. (*b*) Show that if $e = 1$, the final velocities of the balls form a right angle for all values of *b*.

SOLUTION



Geometry at instant of impact.

$$\sin\theta = \frac{b}{d} = \frac{2}{2.37}$$

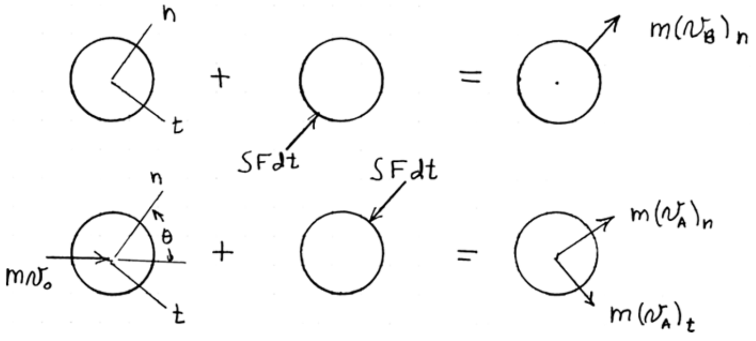
$$\theta = 57.552°$$

Directions *n* and *t* are shown in the figure.

Principle of impulse and momentum.

Ball *B*

Ball *A*



Ball *A*, *t*-direction: $\quad mv_0\sin\theta + 0 = m(v_A)_t \quad (v_A)_t = v_0\sin\theta \quad (1)$

Ball *B*, *t*-direction: $\quad 0 + 0 = m(v_B)_t \quad (v_B)_t = 0 \quad (2)$

Balls *A* and *B*, *n*-direction: $\quad mv_0\cos\theta + 0 = m(v_A)_n + m(v_B)_n$

$$(v_A)_n + (v_B)_n = v_0\cos\theta \quad (3)$$

Coefficient of restitution. $\quad (v_B)_n - (v_A)_n = e[v_0\cos\theta] \quad (4)$

(*a*) $\quad e = 0.7.$ From Eqs. (1) and (2),

$$(v_A)_t = 0.84388v_0 \quad (1)'$$

$$(v_B)_t = 0 \quad (2)'$$

PROBLEM 13.165 (Continued)

From Eqs. (3) and (4),

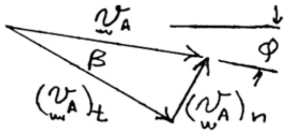
$$(v_A)_n + (v_B)_n = 0.53653v_0 \qquad (3)'$$

$$(v_B)_n - (v_A)_n = (0.7)(0.53653v_0) \qquad (4)'$$

Solving Eqs. (5) and (6) simultaneously,

$$(v_A)_n = 0.08048v_0 \quad (v_B)_n = 0.45605v_0$$

$$\begin{aligned} v_A &= \sqrt{(v_A)_n^2 + (v_A)_t^2} \\ &= \sqrt{(0.08048v_0)^2 + (0.84388v_0^2} \\ &= 0.84771v_0 \end{aligned}$$



$$\tan\beta = \frac{(v_A)_n}{(v_A)_t} = \frac{0.08048v_0}{0.84388v_0} = 0.09537$$

$$\begin{aligned} \beta &= 5.448° \\ \varphi &= 90° - \theta - \beta \\ &= 90° - 57.552° - 5.448° \\ &= 27.000° \end{aligned}$$

$\mathbf{v}_A = 0.848v_0 \searrow 27.0°$ ◀

$\mathbf{v}_B = 0.456v_0 \angle 57.6°$ ◀

(*b*) $e = 1.$ Eqs. (3) and (4) become

$$(v_A)_n + (v_B)_n = v_0 \cos\theta \qquad (3)''$$

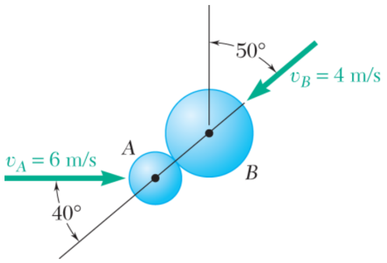
$$(v_B)_n - (v_A)_n = v_0 \cos\theta \qquad (4)''$$

Solving Eqs. $(3)''$ and $(4)''$ simultaneously,

$$(v_A)_n = 0, \quad (v_B)_t = v_0 \cos\theta$$

But $\qquad (v_A)_t = v_0 \sin\theta, \quad \text{and} \quad (v_B)_t = 0$

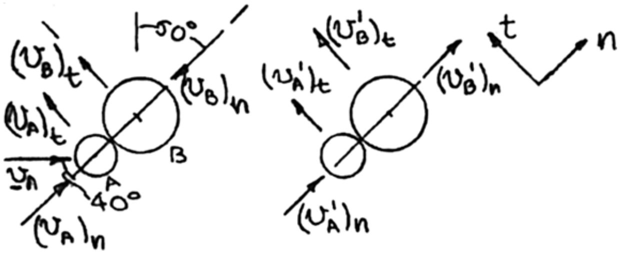
$\mathbf{v}_A$ is in the *t*-direction and $\mathbf{v}_B$ is in the *n*-direction; therefore, the velocity vectors form a right angle.



PROBLEM 13.166

A 600-g ball *A* is moving with a velocity of magnitude 6 m/s when it is hit as shown by a 1-kg ball *B* which has a velocity of magnitude 4 m/s. Knowing that the coefficient of restitution is 0.8 and assuming no friction, determine the velocity of each ball after impact.

SOLUTION



Before After

$$v_A = 6 \text{ m/s}$$
$$(v_A)_n = (6)(\cos 40°) = 4.596 \text{ m/s}$$
$$(v_A)_t = -6(\sin 40°) = -3.857 \text{ m/s}$$
$$v_B = (v_B)_n = -4 \text{ m/s}$$
$$(v_B)_t = 0$$

t-direction:

Total momentum conserved:

$$m_A(v_A)_t + m_B(v_B)_t = m_A(v'_B)_t + m_B(v'_B)_t$$
$$(0.6 \text{ kg})(-3.857 \text{ m/s}) + 0 = (0.6 \text{ kg})(v'_A)_t + (1 \text{ kg})(v'_B)t$$
$$-2.314 \text{ m/s} = 0.6\,(v'_A)_t + (v'_B)_t \qquad (1)$$

Ball *A* alone:

Momentum conserved:

$$m_A(v_A)_t = m_A(v'_A)_t \qquad -3.857 = (v'_A)_t$$
$$(v'_A)_t = -3.857 \text{ m/s} \qquad (2)$$

Replacing $(v'_A)_t$ in (2) in Eq. (1)

$$-2.314 = (0.6)(-3.857) + (v'_B)_t$$
$$-2.314 = -2.314 + (v'_B)_t$$
$$(v'_B)_t = 0$$

PROBLEM 13.166 (Continued)

n-direction:

Relative velocities:

$$[(v_A)_n - (v_B)_n]e = (v'_B)_n - (v'_A)_n$$
$$[(4.596) - (-4)](0.8) = (v'_B)_n - (v'_A)_n$$
$$6.877 = (v'_B)_n - (v'_A)_n \qquad (3)$$

Total momentum conserved:

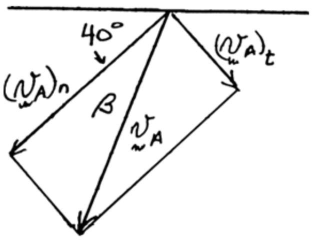
$$m_A(v_A)_n + m_B(v_B)_n = m_A(v'_A)_n + m_B(v'_B)_n$$
$$(0.6\text{ kg})(4.596\text{ m/s}) + (1\text{ kg})(-4\text{ m/s}) = (1\text{ kg})(v'_B)_n + (0.6\text{ kg})(v'_A)_n$$
$$-1.2424 = (v'_B)_n + 0.6(v'_A)_n \qquad (4)$$

Solving Eqs. (4) and (3) simultaneously,

$$(v'_A)_n = 5.075\text{ m/s}$$
$$(v'_B)_n = 1.802\text{ m/s}$$

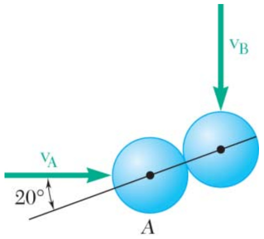
Velocity of A:

$$\tan\beta = \frac{|(v_A)_t|}{|(v_A)_n|} = \frac{3.857}{5.075}$$
$$\beta = 37.2° \qquad \beta + 40° = 77.2°$$
$$v'_A = \sqrt{(3.857)^2 + (5.075)^2} = 6.37\text{ m/s}$$



$\mathbf{v}'_A = 6.37\text{ m/s} \measuredangle 77.2°$ ◀

Velocity of B: $\mathbf{v}'_B = 1.802\text{ m/s} \measuredangle 40°$ ◀

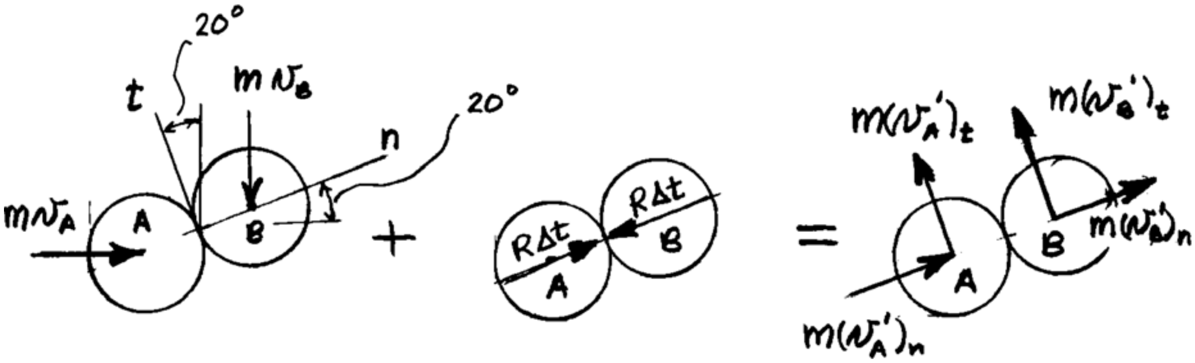


PROBLEM 13.167

Two identical hockey pucks are moving on a hockey rink at the same speed of 3 m/s and in perpendicular directions when they strike each other as shown. Assuming a coefficient of restitution $e = 0.9$, determine the magnitude and direction of the velocity of each puck after impact.

SOLUTION

Use principle of impulse-momentum: $\Sigma m\mathbf{v}_1 + \Sigma \mathbf{Imp}_{1\to 2} = \Sigma m\mathbf{v}_2$



t-direction for puck *A*:

$$-mv_A \sin 20° + 0 = m(v'_A)_t$$
$$(v'_A)_t = v_A \sin 20° = 3\sin 20° = 1.0261 \text{ m/s}$$

t-direction for puck *B*:

$$-mv_B \cos 20° + 0 = m(v'_B)_t$$
$$(v'_B)_t = v_B \cos 20° = -3\cos 20° = -2.8191 \text{ m/s}$$

n-direction for both pucks:

$$mv_A \cos 20° - mv_B \sin 20° = m(v'_A)_n + m(v'_B)_n$$
$$(v'_A)_n + (v'_B)_n = v_A \cos 20° - v_B \sin 20°$$
$$= 3\cos 20° - 3\sin 20° \qquad (1)$$

Coefficient of restitution: $e = 0.9$

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n]$$
$$= 0.9[3\cos 20° - (-3)\sin 20°] \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously,

$$(v'_A)_n = -0.8338 \text{ m/s} \qquad (v'_B)_n = 2.6268 \text{ m/s}$$

PROBLEM 13.167 (Continued)

Summary:

$(\mathbf{v}'_A)_n = 0.8338 \text{ m/s} \swarrow 20°$

$(\mathbf{v}'_A)_t = 1.0261 \text{ m/s} \searrow 70°$

$(\mathbf{v}'_B)_n = 2.6268 \text{ m/s} \nearrow 20°$

$(\mathbf{v}'_B)_t = 2.8191 \text{ m/s} \searrow 70°$

$$v_A = \sqrt{(0.8338)^2 + (1.0261)^2} = 1.322 \text{ m/s}$$

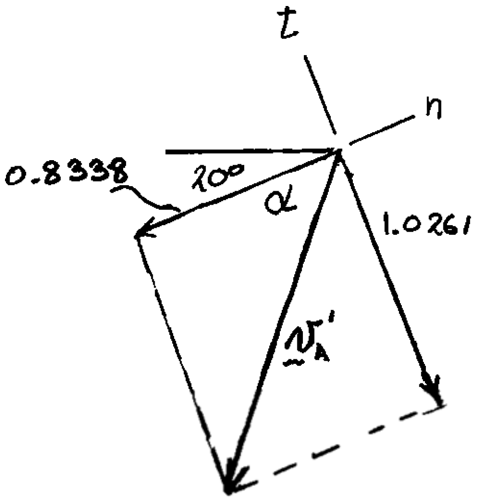
$$\tan\alpha = \frac{1.0261}{0.8338} \qquad \alpha = 50.9° \qquad \alpha + 20° = 70.9°$$

$\mathbf{v}'_A = 1.322 \text{ m/s} \swarrow 70.9°$ ◀

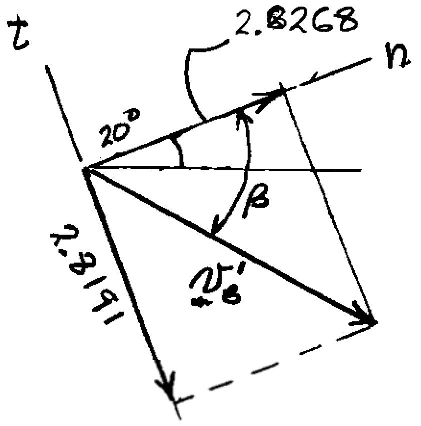
$$v'_B = \sqrt{(2.6268)^2 + (2.8191)^2} = 3.85 \text{ m/s}$$

$$\tan\beta = \frac{2.8191}{2.6268} \qquad \beta = 47.0° \qquad \beta - 20° = 27.0°$$

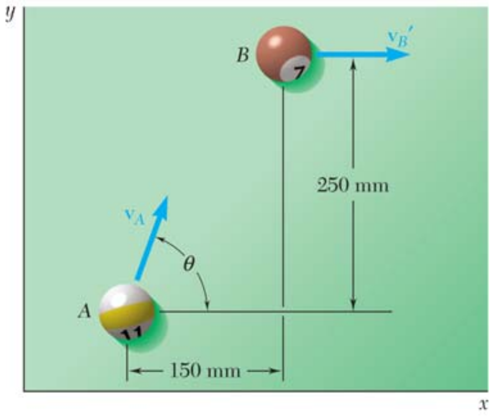
$\mathbf{v}'_B = 3.85 \text{ m/s} \searrow 27.0°$ ◀



Velocity of Puck A



Velocity of Puck B

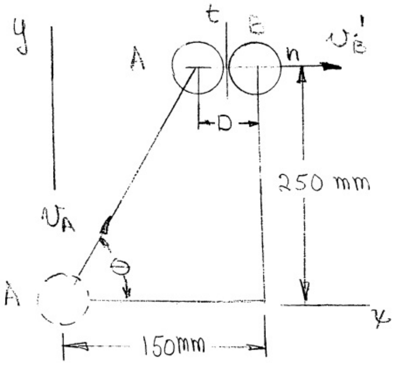


PROBLEM 13.168

The coefficient of restitution is 0.9 between the two 60-mm-diameter billiard balls *A* and *B*. Ball *A* is moving in the direction shown with a velocity of 1 m/s when it strikes ball *B*, which is at rest. Knowing that after impact *B* is moving in the *x* direction, determine (*a*) the angle θ, (*b*) the velocity of *B* after impact.

SOLUTION

(*a*) Since v_B' is in the *x*-direction and (assuming no friction), the common tangent between *A* and *B* at impact must be parallel to the *y*-axis



Thus $$\tan\theta = \frac{250}{150 - D}$$

$$\theta = \tan^{-1}\frac{250}{150 - 60} = 70.20°$$

$$\theta = 70.2° \blacktriangleleft$$

(*b*) Conservation of momentum in *x*(*n*) direction

$$mv_A\cos\theta + m(v_B)_n = m(v_A')_n + mv_B'$$

$$(1)\cos(70.20) + 0 = (v_A')_n + v_B'$$

$$0.3387 = (v_A')_n + (v_B') \qquad (1)$$

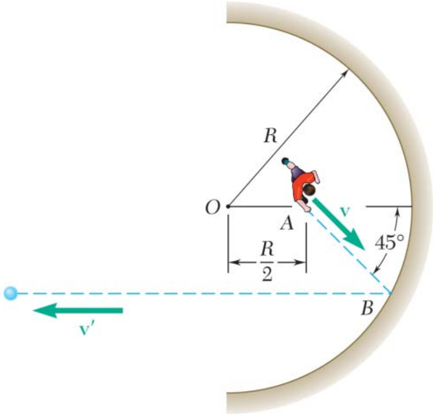
Relative velocities in the *n* direction

$$e = 0.9 \qquad (v_A\cos\theta - (v_B)_n)e = v_B' - (v_A')_n$$

$$(0.3387 - 0)(0.9) = v_B' - (v_A')_n \qquad (2)$$

(1) + (2)

$$2v_B' = 0.3387(1.9) \qquad v_B' = 0.322 \text{ m/s} \blacktriangleleft$$

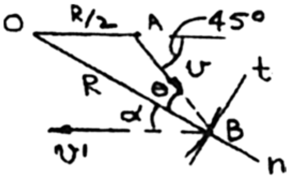


PROBLEM 13.169

A boy located at Point *A* halfway between the center *O* of a semicircular wall and the wall itself throws a ball at the wall in a direction forming an angle of 45° with *OA*. Knowing that after hitting the wall the ball rebounds in a direction parallel to *OA*, determine the coefficient of restitution between the ball and the wall.

SOLUTION

Law of sines:



$$\frac{\sin\theta}{\frac{R}{2}} = \frac{\sin 135^\circ}{R}$$

$$\theta = 20.705^\circ$$

$$\alpha = 45^\circ - 20.705^\circ$$

$$= 24.295^\circ$$

Conservation of momentum for ball in *t*-direction:

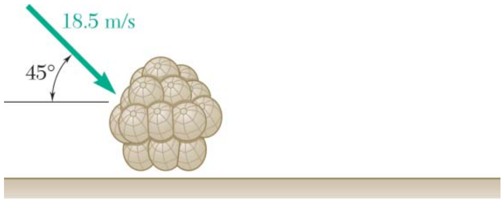
$$-v\sin\theta = -v'\sin\alpha$$

Coefficient of restitution in *n*: $v(\cos\theta)e = v'\cos\alpha$

Dividing, $$\frac{\tan\theta}{e} = \tan\alpha$$

$$e = \frac{\tan 20.705^\circ}{\tan 24.295^\circ}$$

$e = 0.837$ ◀



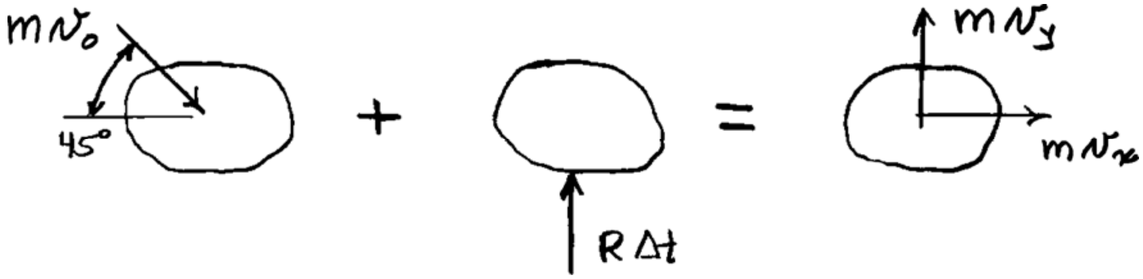
PROBLEM 13.170

The Mars Pathfinder spacecraft used large airbags to cushion its impact with the planet's surface when landing. Assuming the spacecraft had an impact velocity of 18 m/s at an angle of 45° with respect to the horizontal, the coefficient of restitution is 0.85 and neglecting friction, determine (*a*) the height of the first bounce, (*b*) the length of the first bounce. (Acceleration of gravity on the Mars = 3.73 m/s^2.)

SOLUTION

Use impulse-momentum principle.

$$\Sigma m\mathbf{v}_1 + \Sigma \mathbf{Imp}_{1\to 2} = \Sigma m\mathbf{v}_2$$



The horizontal direction (*x* to the right) is the tangential direction and the vertical direction (*y* upward) is the normal direction.

Horizontal components: $mv_0 \sin 45° = 0 = mv_x$

$$v_x = v_0 \sin 45°. \qquad \mathbf{v}_x = 12.728 \text{ m/s} \longrightarrow$$

Vertical components, using coefficient of restitution $e = 0.85$

$$v_y - 0 = e[0 - (-v_0 \cos 45°)]$$

$$v_y = (0.85)(18 \cos 45°) \qquad \mathbf{v}_y = 10.819 \text{ m/s} \uparrow$$

The motion during the first bounce is projectile motion.

Vertical motion: $y = (v_y)_0 t - \frac{1}{2} g t^2$

$$v_y = (v_y)_0 - gt$$

Horizontal motion: $x = v_x t$

PROBLEM 13.170 (Continued)

(*a*) Height of first bounce:

$$v_y = 0: \qquad 0 = (v_y)_0 = gt$$

$$t = \frac{(v_y)_0}{g} = \frac{10.819 \text{ m/s}}{3.73 \text{ m/s}^2} = 2.901 \text{ s}$$

$$y = (10.819)(2.901) - \frac{1}{2}(3.73)(2.901)^2$$

$y = 15.69$ m ◀

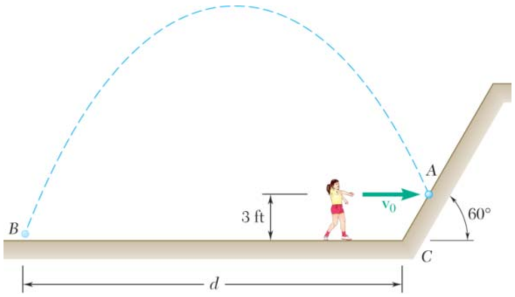
(*b*) Length of first bounce:

$$y = 0: \qquad 10.819t - \frac{1}{2}(3.73)t^2 = 0$$

$$t = 5.801 \text{ s}$$

$$x = (12.728)(5.801)$$

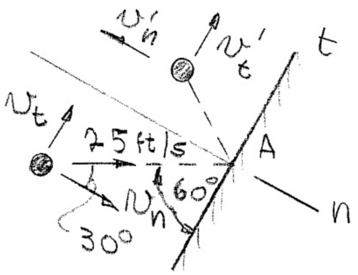
$x = 73.8$ m ◀



PROBLEM 13.171

A girl throws a ball at an inclined wall from a height of 3 ft, hitting the wall at *A* with a horizontal velocity $\mathbf{v}_0$ of magnitude 25 ft/s. Knowing that the coefficient of restitution between the ball and the wall is 0.9 and neglecting friction, determine the distance *d* from the foot of the wall to the Point *B* where the ball will hit the ground after bouncing off the wall.

SOLUTION



Momentum in *t* direction is conserved

$$mv\sin 30° = mv_t'$$

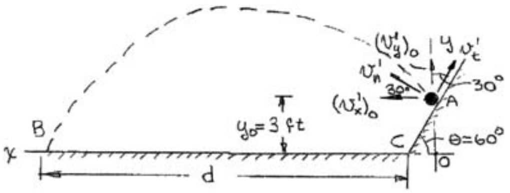
$$(25)(\sin 30°) = v_t'$$

$$v_t' = 12.5 \text{ ft/s}$$

Coefficient of restitution in *n*-direction

$$(v\cos 30°)e = v_n'$$

$$(25)(\cos 30°)(0.9) = v_n' \qquad v_n' = 19.49 \text{ ft/s}$$



Write v' in terms of *x* and *y* components

$$(v_x')_0 = v_n'(\cos 30°) - v_t'(\sin 30°) = 19.49(\cos 30°) - 12.5(\sin 30°)$$
$$= 10.63 \text{ ft/s}$$

$$(v_y')_0 = v_n'(\sin 30°) + v_t'(\cos 30°) = 19.49(\sin 30°) + 12.5(\cos 30°)$$
$$= 20.57 \text{ ft/s}$$

Projectile motion

$$y = y_0 + (v_y')_0 t - \frac{1}{2}gt^2 = 3 \text{ ft} + (20.57 \text{ ft/s})t - (32.2 \text{ ft/s}^2)\frac{t^2}{2}$$

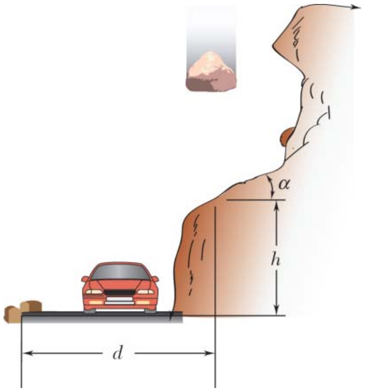
PROBLEM 13.171 (Continued)

At B, $$y = 0 = 3 + 20.57t_B - 16.1t_B^2; \qquad t_B = 1.4098 \text{ s}$$

$$x_B = x_0 + (v'_x)_0 t_B = 0 + 10.63(1.4098); \qquad x_B = 14.986 \text{ ft}$$

$$d = x_B - 3\cot 60^\circ = (14.986 \text{ ft}) - (3 \text{ ft})\cot 60^\circ = 13.254 \text{ ft}$$

$d = 13.25$ ft ◀



PROBLEM 13.172

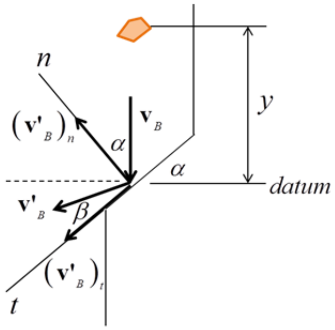
Rockfalls can cause major damage to roads and infrastructure. To design mitigation bridges and barriers, engineers use the coefficient of restitution to model the behavior of the rocks. The rock A falls a distance of 20 m before striking an incline with slope $\alpha = 40^\circ$. Knowing that the coefficient of restitution between A and the incline is 0.2, determine the velocity of the rock after the impact.

SOLUTION

Given:

$y = 20 \text{ m}$
$e = 0.2$
$\alpha = 40^\circ$

Sketch:



Apply Work-Energy to determine speed at impact:

$$\cancel{T_1} + V_{g_1} + \cancel{V_{e_1}} + \cancel{U^{NC}_{1\to3}} = T_2 + \cancel{V_{g_2}} + \cancel{V_{e_2}} \qquad \text{where: } V_{g_1} = mgy \text{ and } T_2 = \frac{1}{2}mv_B^2$$

$$v_B = \sqrt{40g} \text{ m/s}$$

Components in n and t directions:

$$(\mathbf{v}_B)_n = -v_B \cos\alpha$$
$$(\mathbf{v}_B)_t = v_B \sin\alpha$$

Conservation of momentum in the t-direction:

$$(\mathbf{v'}_B)_t = (\mathbf{v}_B)_t = -v_B \sin\alpha$$

n-components using coefficient of restitution:

$$e\big((\mathbf{v}_B)_n - 0\big) = 0 - (\mathbf{v'}_B)_n$$
$$(\mathbf{v'}_B)_n = -e(\mathbf{v}_B)_n$$

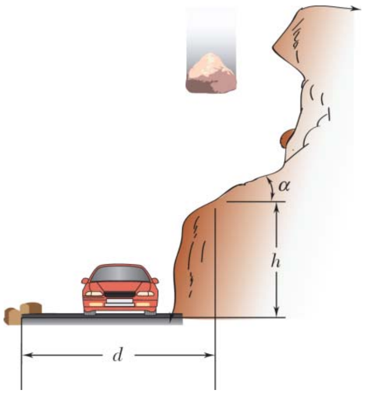
Speed of rock after impact:

$$|\mathbf{v'}_B| = \sqrt{(\mathbf{v'}_B)_t^2 + (\mathbf{v'}_B)_n^2}$$
$$= \sqrt{e^2(\mathbf{v}_B)_n^2 + (\mathbf{v}_B)_t^2}$$
$$= v_B\sqrt{e^2\cos^2\alpha + \sin^2\alpha} = 13.09 \text{ m/s}$$

Direction after impact:

$$\tan\beta = \frac{(\mathbf{v'}_B)_n}{(\mathbf{v'}_B)_t} == \frac{e\cancel{v_B}\cos\alpha}{\cancel{v_B}\sin\alpha} \Rightarrow \beta = 13.4^\circ$$

$$\mathbf{v'}_B = 13.09 \text{ m/s} \swarrow 26.6^\circ \blacktriangleleft$$



PROBLEM 13.173

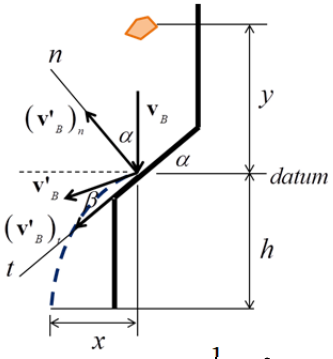
From experimental tests, smaller boulders tend to have a greater coefficient of restitution than larger boulders. The rock *A* falls a distance of 20 meters before striking an incline with slope α = 45°. Knowing that *h*= 30 m and *d*= 20 m, determine if a boulder will land on the road or beyond the road for a coefficient of restitution of (a) e= 0.2, (b) e= 0.1.

SOLUTION

Given:

$y = 20$ m
$h = 30$ m
$d = 20$ m
$\alpha = 45°$

Sketch:



Apply Work-Energy to determine speed of the rock at impact:

$$\cancel{T_1} + V_{g_1} + \cancel{V_{e_1}} + \cancel{U_{1\to3}^{NC}} = T_2 + \cancel{V_{g_2}} + \cancel{V_{e_2}}$$

where: $V_{g_1} = mgy$ and $T_2 = \frac{1}{2}mv_B^2$

$$v_B = \sqrt{40g} = 19.809 \text{ m/s}$$

Components in n and t directions:

$$(\mathbf{v}_B)_n = -v_B \cos\alpha$$
$$(\mathbf{v}_B)_t = v_B \sin\alpha$$

Conservation of momentum in the t-direction:

$$(\mathbf{v}'_B)_t = (\mathbf{v}_B)_t = -v_B \sin\alpha$$

n-components using coefficient of restitution:

$$e\left((\mathbf{v}_B)_n - 0\right) = 0 - (\mathbf{v}'_B)_n$$
$$(\mathbf{v}'_B)_n = -e(\mathbf{v}_B)_n$$

Speed of rock after impact:

$$|\mathbf{v}'_B| = \sqrt{(\mathbf{v}'_B)_t^2 + (\mathbf{v}'_B)_n^2}$$
$$= \sqrt{e^2(\mathbf{v}_B)_n^2 + (\mathbf{v}_B)_t^2}$$
$$= v_B\sqrt{e^2\cos^2\alpha + \sin^2\alpha}$$

Direction after impact:

$$\tan\beta = \frac{(\mathbf{v}'_B)_n}{(\mathbf{v}'_B)_t} == \frac{e\cancel{v_B}\cos\alpha}{\cancel{v_B}\sin\alpha} \Rightarrow \beta = \tan^{-1} e$$

x and *y* components of velocity after impact:

$$(\mathbf{v}'_B)_x = v'_B\cos(\alpha - \beta)$$
$$(\mathbf{v}'_B)_y = -v'_B\sin(\alpha - \beta)$$

PROBLEM 13.173 (Continued)

Projectile Motion in y-dir: $y = y_0 + (v'_B)_y t - \frac{1}{2}gt^2$

$$0 = 30 + (v'_B)_y t - \frac{1}{2}gt^2 \qquad (1)$$

Projectile Motion in x-dir: $x = \cancel{x_0} + (v'_B)_x t \qquad (2)$

(a) For e=0.2:

$|\mathbf{v'}_B| = 14.285$ m/s

$\beta = 11.31°$

$(\mathbf{v'}_B)_x = 11.885$ m/s

$(\mathbf{v'}_B)_y = -7.924$ m/s

Solve (1) for t: $t = 1.7939$ s or ~~$t = -3.4094$ s~~

Solve (2) for x: $x = 21.32$ m $> d$ Rock will land beyond the road◀

(b) For e=0.1:

$|\mathbf{v'}_B| = 14.077$ m/s

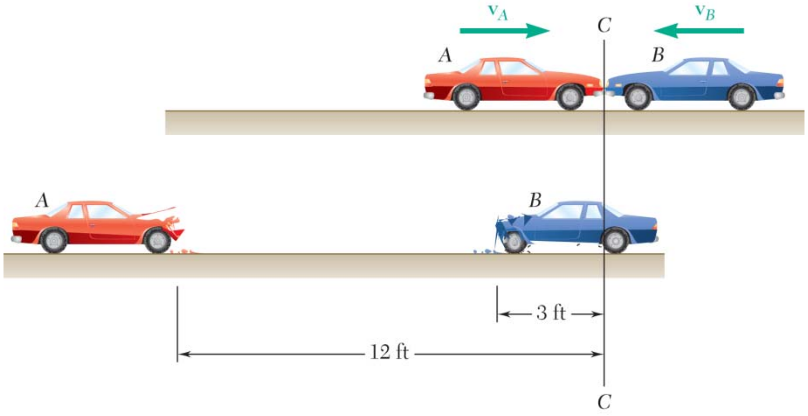
$\beta = 5.711°$

$(\mathbf{v'}_B)_x = 10.894$ m/s

$(\mathbf{v'}_B)_y = -8.914$ m/s

Solve (1) for t: $t = 1.7261$ s or ~~$t = -3.5434$ s~~

Solve (2) for x: $x = 18.81$ m $< d$ Rock will land on the road◀

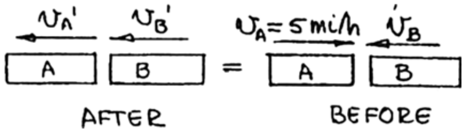


PROBLEM 13.174

Two cars of the same mass run head-on into each other at *C*. After the collision, the cars skid with their brakes locked and come to a stop in the positions shown in the lower part of the figure. Knowing that the speed of car *A* just before impact was 5 mi/h and that the coefficient of kinetic friction between the pavement and the tires of both cars is 0.30, determine (*a*) the speed of car *B* just before impact, (*b*) the effective coefficient of restitution between the two cars.

SOLUTION

(*a*) At *C*:



Conservation of total momentum:

$$m_A = m_B = m$$

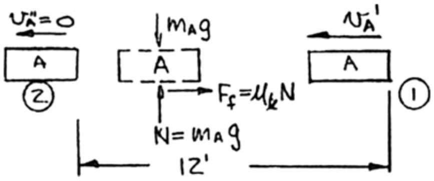
$$5 \text{ mi/h} = 7.333 \text{ ft/s}$$

$$\overset{+}{\longleftarrow} \quad m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

$$-7.333 + v_B = v_A' + v_B' \tag{1}$$

Work and energy.

Care *A* (after impact):



$$T_1 = \frac{1}{2} m_A (v_A')^2$$

$$T_2 = 0$$

$$U_{1-2} = F_f (12)$$

$$U_{1-2} = \mu_k m_A g (12 \text{ ft})$$

$$T_1 + U_{1-2} = T_2$$

$$\frac{1}{2} m_A (v_A')^2 - m_k m_A g (12) = 0$$

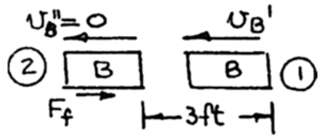
$$(v_A')^2 = (2)(12 \text{ ft})(0.3)(32.2 \text{ ft/s}^2)$$

$$= 231.84 \text{ ft/s}^2$$

$$v_A' = 15.226 \text{ ft/s}$$

PROBLEM 13.174 (Continued)

Car *B* (after impact):



$$T_1 = \frac{1}{2} m_B (v_B')^2$$

$$T_2 = 0$$

$$U_{1-2} = \mu_k\, m_B\, g(3)$$

$$T_1 + U_{1-2} = T_2$$

$$\frac{1}{2} m_B (v_B')^2 - \mu_k m_B g(3)$$

$$v_B'^2 = (2)(3 \text{ ft})(0.3)(32.2 \text{ ft/s}^2)$$

$$(v_B')^2 = 57.96 \text{ ft/s}^2$$

$$v_B' = 7.613 \text{ ft/s}$$

From (1)

$$v_B = 7.333 + v_A' + v_B'$$

$$= 7.333 + 15.226 + 7.613$$

$$v_B = 30.2 \text{ ft/s} = 20.6 \text{ mi/h} \blacktriangleleft$$

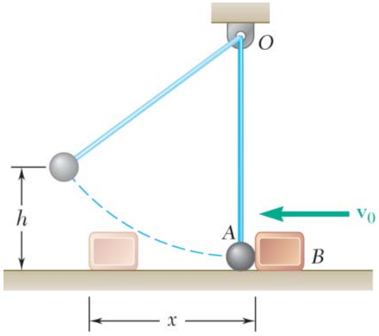
(*b*) Relative velocities:

$$\overset{+}{\longleftarrow} \quad (-v_A - v_B)e = v_B' - v_A'$$

$$(-7.333 - 30.2)e = 7.613 - 15.226$$

$$e = \frac{-(7.613)}{-(37.53)} = 0.2028$$

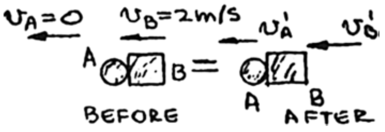
$$e = 0.203 \blacktriangleleft$$



PROBLEM 13.175

A 1-kg block B is moving with a velocity $\mathbf{v}_0$ of magnitude $v_0 = 2$ m/s as it hits the 0.5-kg sphere A, which is at rest and hanging from a cord attached at O. Knowing that $\mu_k = 0.6$ between the block and the horizontal surface and $e = 0.8$ between the block and the sphere, determine after impact (*a*) the maximum height h reached by the sphere, (*b*) the distance x traveled by the block.

SOLUTION



Velocities just after impact

Total momentum in the horizontal direction is conserved:

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

$$0 + (1\text{ kg})(2\text{ m/s}) = (0.5\text{ kg})(v_A') + (1\text{ kg})(v_B')$$

$$4 = v_A' + 2v_B' \qquad (1)$$

Relative velocities:

$$(v_A - v_B)e = (v_B' - v_A')$$

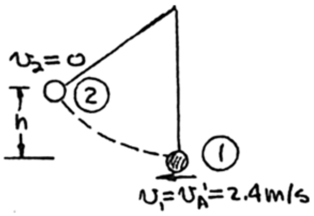
$$(0-2)(0.8) = v_B' - v_A'$$

$$-1.6 = v_B' - v_A' \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously:

$$v_B' = 0.8\text{ m/s}$$

$$v_A' = 2.4\text{ m/s}$$



(*a*) Conservation of energy:

$$T_1 = \frac{1}{2}m_A v_1^2 \qquad V_1 = 0$$

$$T_1 = \frac{1}{2}m_A(2.4\text{ m/s})^2 = 2.88\,m_A$$

$$T_2 = 0$$

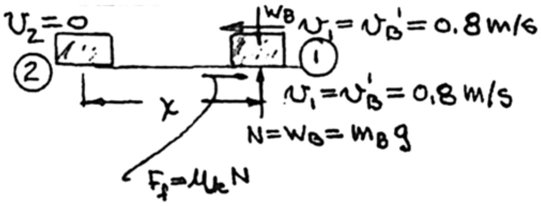
$$V_2 = m_A gh$$

$$T_1 + V_1 = T_2 + V_2 \qquad 2.88\cancel{m}_A + 0 = 0 + \cancel{m}_A(9.81)h$$

$h = 0.294$ m ◀

PROBLEM 13.175 (Continued)

(*b*) Work and energy:


$v_2 = 0$
②
W_B
$v_1 = v_B' = 0.8$ m/s
①
x
$v_1 = v_B' = 0.8$ m/s
$N = W_B = m_B g$
$F_f = \mu_k N$

$$T_1 = \frac{1}{2} m_B v_1^2 = \frac{1}{2} m_B (0.8 \text{ m/s})^2 = 0.32 m_B \qquad T_2 = 0$$

$$U_{1-2} = -F_f x = -\mu_k N x = -\mu_x m_B g x = -(0.6)(m_B)(9.81)x$$

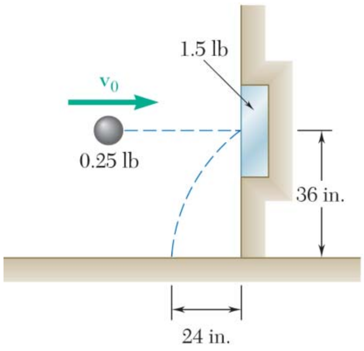
$$U_{1-2} = -5.886 m_B x$$

$$T_1 + U_{1-2} = T_2: \qquad 0.32 m_B - 5.886 m_B x = 0$$

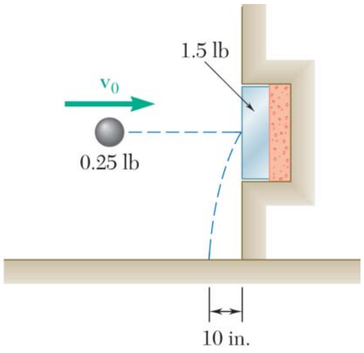
$$x = 0.0544 \text{ m} \qquad\qquad x = 54.4 \text{ mm} \blacktriangleleft$$

PROBLEM 13.176

A 0.25-lb ball thrown with a horizontal velocity $\mathbf{v}_0$ strikes a 1.5-lb plate attached to a vertical wall at a height of 36 in. above the ground. It is observed that after rebounding, the ball hits the ground at a distance of 24 in. from the wall when the plate is rigidly attached to the wall (Figure 1) and at a distance of 10 in. when a foam-rubber mat is placed between the plate and the wall (Figure 2). Determine (*a*) the coefficient of restitution *e* between the ball and the plate, (*b*) the initial velocity $\mathbf{v}_0$ of the ball.

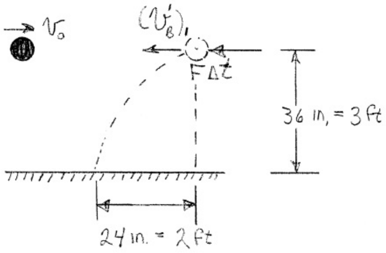

1.5 lb
$\mathbf{v}_0$
0.25 lb
36 in.
24 in.

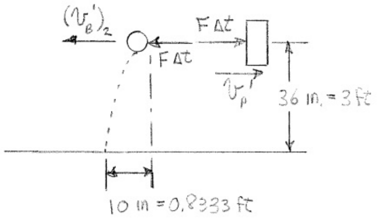
(1)


1.5 lb
$\mathbf{v}_0$
0.25 lb
10 in.

(2)

SOLUTION


v_0
$(v_B')_1$
$F\Delta t$
36 in. = 3 ft
24 in. = 2 ft


$(v_B')_2$
$F\Delta t$
$F\Delta t$
v_P'
36 in. = 3 ft
10 in = 0.8333 ft

(*a*) Figure (1), ball alone relative velocities

$$v_0 e = (v_B')_1$$

Projectile motion

$$t = \text{time for the ball to hit ground}$$

$$2\text{ ft} = v_0 e t \qquad (1)$$

Figure (2), ball and plate relative velocities

$$(v_B - v_A)e = v_P' + (v_B')_2$$

$$v_B = v_0, \qquad v_P = 0$$

$$v_0 e = v_P' + (v_B')_2 \qquad (2)$$

Conservation of momentum

$$\xrightarrow{+} m_B v_B + m_P v_P = m_B v_B' + m_P v_P'$$

$$\frac{0.25}{g} v_0 + 0 = \frac{0.25}{g}(-v_B')_2 + \frac{1.5}{g} v_P'$$

$$0.25v_0 = -0.25(v'_B)_2 + 1.5v'_p \Rightarrow v_0 = -(v'_B)_2 - 6v'_p \qquad (3)$$

Solving (2) and (3) for $(v'_B)_2$, $\qquad (v'_B)_2 = \frac{(6e-1)}{7}v_0$

Projectile motion

$$0.8333 = \frac{(6e-1)}{7}v_0 t \qquad (4)$$

Dividing Equation (4) by Equation (1)

$$\frac{0.8333}{2} = \frac{6e-1}{7e};\ 2.91655e = 6e - 1$$

$e = 0.324$ ◀

(*b*) From Figure (1)

Projectile motion, $\qquad h = \frac{1}{2}gt^2;\ 3 = \frac{1}{2}(32.2)t^2$

$$6 = 32.2t^2 \qquad (5)$$

From Equation (1),

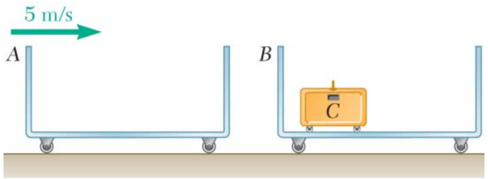
$$2 = v_0 e t \Rightarrow t = \frac{2}{0.324v_0} = \frac{6.1728}{v_0}$$

Using Equation (5)

$$6 = 32.2\left(\frac{6.1728}{v_0}\right)^2 \Rightarrow 6v_0^2 = 1226.947$$

$$v_0^2 = 204.49$$

$v_0 = 14.30$ ft/s ◀



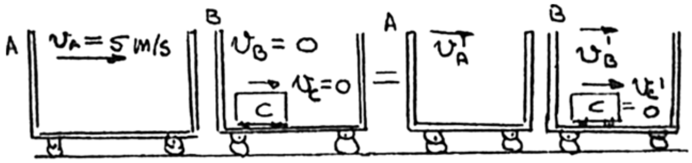
PROBLEM 13.177

After having been pushed by an airline employee, an empty 40-kg luggage carrier *A* hits with a velocity of 5 m/s an identical carrier *B* containing a 15-kg suitcase equipped with rollers. The impact causes the suitcase to roll into the left wall of carrier *B*. Knowing that the coefficient of restitution between the two carriers is 0.80 and that the coefficient of restitution between the suitcase and the wall of carrier *B* is 0.30, determine (*a*) the velocity of carrier *B* after the suitcase hits its wall for the first time, (*b*) the total energy lost in that impact.

SOLUTION

(*a*) Impact between *A* and *B*:

Total momentum conserved:



$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B' \qquad m_A = m_B = 40 \text{ kg}$$

$$\xrightarrow{+} 5 \text{ m/s} + 0 = v_A' + v_B' \tag{1}$$

Relative velocities:

$$(v_A - v_B)e_{AB} = v_B' - v_A'$$

$$(5 - 0)(0.80) = v_B' - v_A' \tag{2}$$

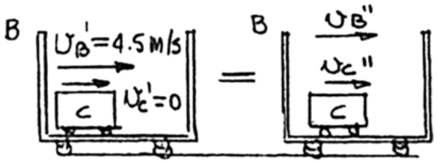
Adding Eqs. (1) and (2)

$$(5 \text{ m/s})(1 + 0.80) = 2v_B'$$

$$v_B' = 4.5 \text{ m/s} \longrightarrow$$

Impact between *B* and *C* (after *A* hits *B*)

Total momentum conserved:



$$\xrightarrow{+} m_B v_B' + m_C v_C' = m_B v_B'' + m_C v_C''$$

$$(40 \text{ kg})(4.5 \text{ m/s}) + 0 = (40 \text{ kg})\, v_B'' + (15 \text{ kg})\, v_C''$$

$$4.5 = v_B'' + 0.375 v_C'' \tag{3}$$

Relative velocities:

$$(v_B' - v_C')\, e_{BC} = v_C'' - v_B''$$

$$(4.5 - 0)(0.30) = v_C'' - v_B'' \tag{4}$$

Adding Eqs. (4) and (3)

$$(4.5)(1+0.3)=(1.375)v_C''$$

$$v_C''=4.2545\text{ m/s}$$

$$v_B''=4.5-0.375(4.2545)\,v_B''=2.90\text{ m/s} \qquad v_B'=2.90\text{ m/s} \blacktriangleleft$$

(*b*)

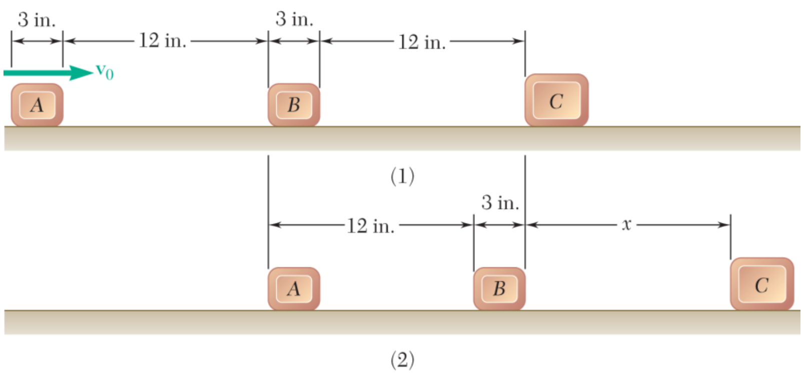
$$\Delta T_L=(T_B'+T_C')-(T_B''+T_C'')$$

$$T_B'=\frac{1}{2}m_B(v_B')^2=\left(\frac{40}{2}\text{ kg}\right)(4.5\text{ m/s})^2=405\text{ J}$$

$$T_C'=0 \qquad T_B''=\frac{1}{2}m_B(v_B'')^2=\left(\frac{40}{2}\text{ kg}\right)(2.90)^2=168.72\text{ J}$$

$$T_C''=\frac{1}{2}m_C(v_C'')^2=\left(\frac{15}{2}\text{ kg}\right)(4.2545\text{ m/s})^2=135.76\text{ J}$$

$$\Delta T_L=(405+0)-(168.72+135.76)=100.5\text{ J} \qquad \Delta T_L=100.5\text{ J} \blacktriangleleft$$



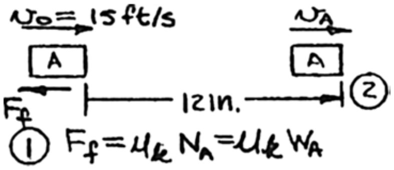
PROBLEM 13.178

Blocks *A* and *B* each weigh 0.8 lb and block *C* weighs 2.4 lb. The coefficient of friction between the blocks and the plane is $\mu_k = 0.30$. Initially block *A* is moving at a speed $v_0 = 15$ ft/s and blocks *B* and *C* are at rest (Fig. 1). After *A* strikes *B* and *B* strikes *C*, all three blocks come to a stop in the positions shown (Fig. 2). Determine (*a*) the coefficients of restitution between *A* and *B* and between *B* and *C*, (*b*) the displacement *x* of block *C*.

SOLUTION

(*a*) Work and energy

Velocity of A just before impact with *B*:



$$T_1 = \frac{1}{2}\frac{W_A}{g}v_0^2 \qquad T_2 = \frac{1}{2}\frac{W_A}{g}\left(v_A^2\right)_2$$

$$U_{1-2} = -\mu_k W_A(1\text{ ft})$$

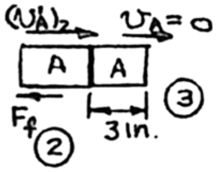
$$T_1 + U_{1-2} = T_2$$

$$\frac{1}{2}\frac{W_A}{g}v_0^2 - \mu_k W_A(1) = \frac{1}{2}\frac{W_A}{g}\left(v_A^2\right)_2$$

$$(v_A^2)_2 = v_0^2 - 2\mu_k g = (15\text{ ft/s})^2 - 2(0.3)(32.2\text{ ft/s}^2)(1\text{ ft})$$

$$(v_A^2)_2 = 205.68\text{ ft/s}^2, \qquad (v_A)_2 = 14.342\text{ ft/s}$$

Velocity of *A* after impact with *B*: $(v_A')_2$



$$T_2' = \frac{1}{2}\frac{W_A}{g}(v_A')_2^2 \qquad T_3 = 0$$

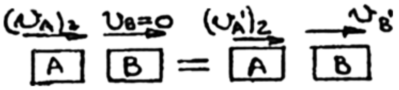
$$U_{2-3} = -\mu_k W_A(3/12)$$

$$T_2' + U_{2-3} = T_3, \quad \frac{1}{2}\frac{W_A}{g}(v_A')_2^2 - (\mu_k)(W_A/4) = 0$$

$$(v_A')_2^2 = 2(0.3)(32.2\text{ ft/s}^2)\left(\frac{1}{4}\text{ft}\right) = 4.83\text{ ft}^2/\text{s}^2$$

$$(v_A')_2 = 2.198\text{ ft/s}$$

Conservation of momentum as *A* hits *B*:



$$(v_A)_2 = 14.342 \text{ ft/s}$$
$$(v_A')_2 = 2.198 \text{ ft/s}$$

$$\xrightarrow{+}\; m_A(v_A)_2 + m_B v_B = m_B(v_A')_2 + m_B v_B' \quad m_A = m_B$$
$$14.342 + 0 = 2.198 + v_B' \qquad v_B' = 12.144 \text{ ft/s}$$

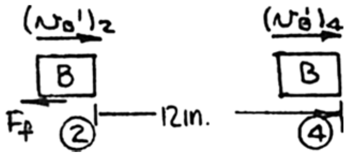
Relative velocities *A* and *B*:

$$\xrightarrow{+}\; [(v_A)_2 - v_B]e_{AB} = v_B' - (v_A')_2$$
$$(14.342 - 0)e_{AB} = 12.144 - 2.198$$

$e_{AB} = 0.694$ ◀

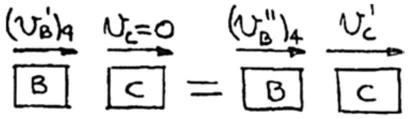
Work and energy.

Velocity of *B* just before impact with *C*:



$$T_2 = \frac{1}{2}\frac{W_B}{g}(v_B')_2^2 = \frac{W_B}{2g}(12.144)^2$$
$$T_4 = \frac{1}{2}\frac{W_B}{g}(v_B')_4^2 = \frac{W_B}{2g}(v_B')_4^2$$
$$(v_B') = 12.144 \text{ ft/s} \qquad U_{2-4} = -\mu_k W_B(1 \text{ ft}) = (0.3)W_B$$
$$F_f = \mu_k W_B \qquad T_2 + U_{2-4} = T_4, \quad \frac{(12.144)^2}{2g} - 0.3 = \frac{(v_B')_4^2}{2g}$$
$$(v_B')_4 = 11.321 \text{ ft/s}$$

Conservation of momentum as *B* hits *C:*



$$m_B = \frac{0.8}{g}$$
$$m_C = \frac{2.4}{g}$$
$$(v_B')_4 = 11.321 \text{ ft/s}$$

$$\xrightarrow{+}\; m_B(v_B')_4 + m_C v_C = m_B(v_B'')_4 + m_C v_C'$$
$$\frac{0.8}{g}(11.321) + 0 = \frac{0.8}{g}(v_B'')_4 + \frac{(2.4)}{g}(v_C')$$
$$11.321 = (v_B'')_4 + 3v_C'$$

PROBLEM 13.178 (Continued)

Velocity of B after B hits $C, (v''_B)_4 = 0.$

(Compare Figures (1) and (2).)

$v'_C = 3.774$ ft/s

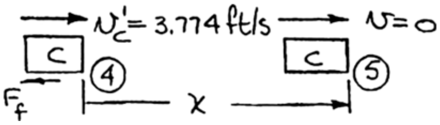
Relative velocities B and C:

$$((v'_B)_4 - v_C)e_{BC} = v'_C - (v''_B)_4$$
$$(11.321 - 0)e_{BC} = 3.774 - 0$$

$e_{BC} = 0.333$ ◀

(*b*) Work and energy, Block C:

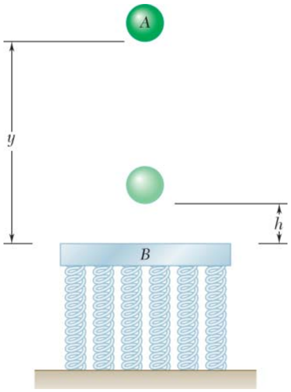
$$T_4 = \frac{1}{2}\frac{W_C}{g}(v_C)^2 \qquad T_5 = 0 \qquad U_{4-5} = -\mu_k W_C(x)$$



$$T_4 + U_{4-5} = T_5 \qquad \frac{1}{2}\frac{W_C}{g}(3.774)^2 - (0.3)W_C(x) = 0$$

$$x = \frac{(3.774)^2}{2(32.2)(0.3)} = 0.737 \text{ ft}$$

$x = 8.84$ in. ◀



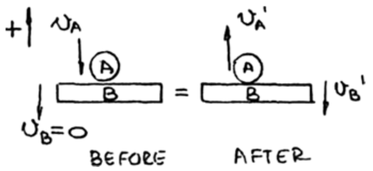
PROBLEM 13.179

A 5 kg sphere is dropped from a height *y*=2 m to test newly designed spring floors used in gymnastics. The mass of the floor section is 10 kg, and the effective stiffness of the floor is *k*= 120 kN/m. Knowing that the coefficient of restitution between the ball and the platform is 0.6, determine *(a)* the height *h* reached by the sphere after rebound, *(b)* the maximum force in the springs.

SOLUTION

Given:
$m_A = 5$ kg
$m_B = 10$ kg
$y = 2$ m
$e = 0.6$
$k = 120$ kN/m

Sketch before/after impact:



Apply Work-Energy to determine speed of the sphere just before impact:

$$\cancel{T_1} + V_{g_1} + \cancel{V_{e_1}} + \cancel{U_{1\to2}^{NC}} = T_2 + \cancel{V_{g_2}} + \cancel{V_{e_2}}$$

where: $V_{g_1} = m_A g y$ and $T_2 = \frac{1}{2} m_A v_A^2$

$$v_A = \sqrt{2gy} = 6.264 \text{ m/s} \downarrow$$

Conservation of momentum in the y-direction: $m_A v_A + m_B \cancel{v_B} = m_A v'_A + m_B v'_B$

Dividing by m_A with ($m_B/m_A = 2$): $-6.264 = v'_A + 2v'_B$ (1)

Coefficient of restitution: $v'_B - v'_A = e\left(v_A - \cancel{v_B}\right)$ (2)

$$= -6.264e$$

Solving Eqs. (1) and (2) simultaneously yields:
$v'_A = 0.4176$ m/s
$v'_B = -3.341$ m/s

(a) Apply Work-Energy to sphere A to determine height it rises:

$$T_1 + \cancel{V_{g_1}} + \cancel{V_{e_1}} + \cancel{U_{1\to2}^{NC}} = \cancel{T_2} + V_{g_2} + \cancel{V_{e_2}}$$

where: $T_1 = \frac{1}{2} m_A v'^2_A$ and $V_{g_2} = m_A g h$

$$h = \frac{v'^2_A}{2g} \text{ m/s}$$

$h = 8.89$ mm ◀

(b) Let δ_0 be the initial compression of the spring and h_B be the additional compression of the spring after impact. In the initial equilibrium state: $+\uparrow \ \Sigma F_y = 0: \ k\delta_0 - W_B = 0$ or $k\delta_0 = m_B g$

Apply Work-Energy to determine maximum deflection of the springs (when the plate reaches its lowest point and the speed is zero):

PROBLEM 13.179 (Continued)

$T_1 + V_{g_1} + V_{e_1} + \cancel{U^{NC}_{1\to 2}} = \cancel{T_2} + V_{g_2} + V_{e_2}$ where: $T_1 = \frac{1}{2} m_B v'^2_B,\ V_{g_1} = -m_B g \delta_0,\ V_{e_1} = \frac{1}{2} k \delta_0^2$

$V_{g_2} = -m_B g \left(\delta_0 + h_B\right),\ V_{e_2} = \frac{1}{2} k \left(\delta_0 + h_B\right)^2$

$$\frac{1}{2} m_B v'^2_B - m_B g \delta_0 + \frac{1}{2} k \delta_0^2 = -m_B g \left(\delta_0 + h_B\right) + \frac{1}{2} k \delta_0^2 + k \delta_0 h_B + \frac{1}{2} k h_B^2$$

Simplify: $\frac{1}{2} m_B v'^2_B = -m_B g h_B + k \delta_0 h_B + \frac{1}{2} k h_B^2$

Substitute $k\delta_0 = m_B g$: $\cancel{\frac{1}{2}} m_B v'^2_B = \cancel{-m_B g h_B} + \cancel{m_B g h_B} + \cancel{\frac{1}{2}} k h_B^2$

Put in known values:

$$120000 h_B^2 - 111.616 = 0$$

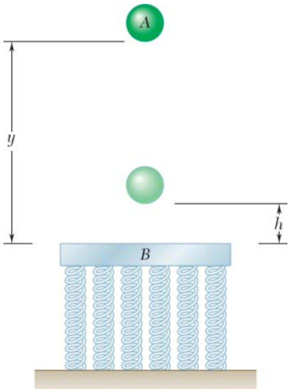
Solving for h_B: $h_B = 0.0305 \text{ m}$

Maximum force in spring $F = k\left(\delta_0 + h_B\right)$

$= k\delta_0 + k h_B$

$= m_B g + k h_B$

$F_{\max} = 3758 \text{ N}$ ◀



PROBLEM 13.180

A 5 kg sphere is dropped from a height y=3 m to test a new spring floor used in gymnastics. The mass of the floor section B is 12 kg, and the sphere bounces back upwards a distance of 44 mm. Knowing that the maximum deflection of the floor section is 33 mm from its equilibrium position, determine *(a)* the coefficient of restitution between the sphere and the floor, *(b)* the effective spring constant k of the floor section.

SOLUTION

Given:

$m_A = 5$ kg

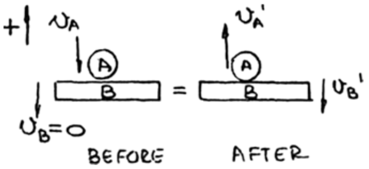
$m_B = 12$ kg

$y = 3$ m

$h = 44$ mm

$s_{\max} = 33$ mm

Sketch before/after impact:



Apply Work-Energy to determine speed of the sphere just before impact:

$$\cancel{T_1} + V_{g_1} + \cancel{V_{e_1}} + \cancel{U_{1\to 2}^{NC}} = T_2 + \cancel{V_{g_2}} + \cancel{V_{e_2}} \qquad \text{where: } V_{g_1} = m_A g y \text{ and } T_2 = \frac{1}{2} m_A v_A^2$$

$$v_A = \sqrt{2gy} = 7.672 \text{ m/s} \downarrow \text{ or } v_A - 7.672 \text{ m/s}$$

Apply Work Energy to determine the speed of the sphere just after impact:

$$T_2 + \cancel{V_{g_1}} + \cancel{V_{e_1}} + \cancel{U_{2\to 3}^{NC}} = \cancel{T_3} + V_{g_3} + \cancel{V_{e_3}} \qquad \text{where: } T_2 = \frac{1}{2} m_A v'^2_A \text{ and } V_{g_3} = m_A g h$$

$$v'_A = \sqrt{2gh} = 0.9291 \text{ m/s} \uparrow$$

(a) Conversation of momentum in the y-direction: $m_A v_A + m_B \cancel{v_B} = m_A v'_A + m_B v'_B$

Put in known values and solve:

$$v'_B = \frac{m_A}{m_B}\left(v_A - v'_A\right)$$
$$= -3.584 \text{ m/s}$$

Coefficient of restitution:

$$v'_B - v'_A = e\left(v_A - \cancel{v_B}\right)$$
$$e = \frac{v'_B - v'_A}{v_A}$$

Put in known values and solve: $e = 0.588$ ◀

(b) Let δ_0 be the initial compression of the spring and $s_{\max}$ be the additional compression of the spring after impact. In the initial equilibrium state: $+\uparrow\ \Sigma F_y = 0$: $k\delta_0 - W_B = 0$ or $\delta_0 = \frac{m_B g}{k}$

PROBLEM 13.180 (Continued)

Apply Work-Energy to determine the stiffness of the springs (when the plate reaches its lowest point and the speed is zero). Take the equilibrium position as the datum.

$$T_1 + \cancel{V_{g_1}} + V_{e_1} + \cancel{U_{1\to 2}^{NC}} = \cancel{T_2} + V_{g_2} + V_{e_2} \qquad \text{where: } T_1 = \frac{1}{2}m_B v'^2_B,\ V_{e_1} = \frac{1}{2}k\delta_0^2$$

$$V_{g_2} = -m_B g\left(s_{\max}\right),\ V_{e_2} = \frac{1}{2}k\left(s_{\max}\right)^2$$

Substitute $\delta_0 = \dfrac{m_B g}{k}$:

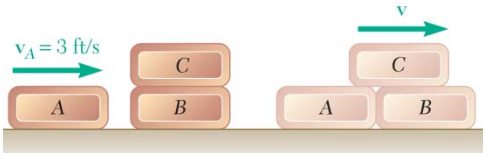
$$\frac{1}{2}m_B v'^2_B + \frac{1}{2}k\left(\frac{m_B g}{k}\right)^2 = -m_B g s_{\max} + \frac{1}{2}k\left(s_{\max}\right)^2$$

$$m_B v'^2_B k + \left(m_B g\right)^2 = -2m_B g\left(s_{\max}\right)k + k^2\left(s_{\max}\right)^2$$

$$k^2\left(s_{\max}\right)^2 - \left(m_B v'^2_B + 2m_B g s_{\max}\right)k - \left(m_B g\right)^2 = 0$$

Put in known values and solve for k:

$k = 148.7$ kN/m or ~~$k = -85.5$ N/m~~ $\qquad k = 148.7\,\text{kN/m}$ ◀



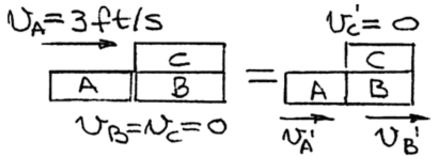
PROBLEM 13.181

The three blocks shown are identical. Blocks *B* and *C* are at rest when block *B* is hit by block *A*, which is moving with a velocity $\mathbf{v}_A$ of 3 ft/s. After the impact, which is assumed to be perfectly plastic $(e = 0)$, the velocity of blocks *A* and *B* decreases due to friction, while block *C* picks up speed, until all three blocks are moving with the same velocity **v**. Knowing that the coefficient of kinetic friction between all surfaces is $\mu_k = 0.20$, determine (*a*) the time required for the three blocks to reach the same velocity, (*b*) the total distance traveled by each block during that time.

SOLUTION

(*a*) Impact between *A* and *B*, conservation of momentum

$$mv_A + mv_B + mv_C = mv'_A + mv'_B + mv'_C$$



$$3 + 0 = v'_A + v'_B + 0$$

Relative velocities $(e = 0)$

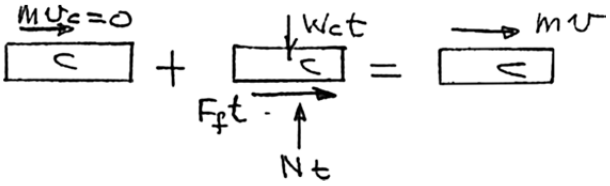
$$(v_A - v_B)e = v'_B - v'_A \qquad 3 = 2v'_B$$

$$0 = v'_B - v'_A \qquad v'_B = 1.5 \text{ ft/s}$$

$$v'_A = v'_B$$

$v =$ Final (common) velocity

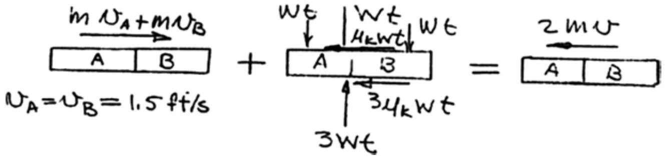
Block *C*: Impulse and momentum



$$\xrightarrow{+} W_C v_C + F_f t = \frac{W_C}{g} v \qquad F_f = \mu_k W_C$$

$$0 + (0.2)t = \frac{v}{g} \qquad v = (0.2)gt \qquad (1)$$

Blocks *A* and *B*: Impulse and momentum


$m v_A + m v_B$
A
B
$v_A = v_B = 1.5$ ft/s
+
Wt
Wt
$\mu_k Wt$
Wt
A
B
$3\mu_k Wt$
3Wt
=
2mv
A
B

$$2\frac{W}{g}(1.5) - 4(0.2)Wt = 2\frac{W}{g}v$$

$$1.5 - 0.4gt = v \qquad (2)$$

Substitute *v* from Eq. (1) into Eq. (2)

$$1.5 - 0.4gt = 0.2gt$$

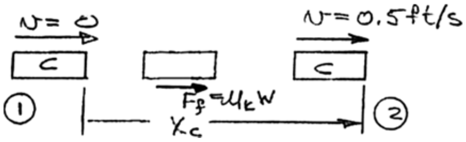
$$t = \frac{(1.5\text{ ft/s})}{0.6(32.2\text{ ft/s}^2)} \qquad t = 0.0776\text{ s} \blacktriangleleft$$

(*b*) Work and energy:

From Eq. (1) $v = (0.2)(32.2)(0.0776) = 0.5$ ft/s

Block *C*:

$$T_1 = 0 \qquad T_2 = \frac{1}{2}\frac{W}{g}(v)^2 = \frac{W}{2g}(0.5)^2$$


v = 0
C
v = 0.5 ft/s
C
$F_f = \mu_k W$
①
x_C
②

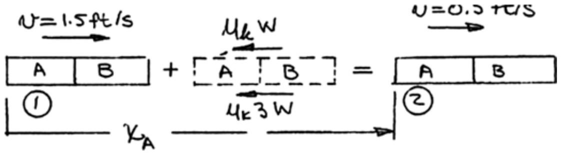
$$U_{1-2} = F_f x_C = \mu_k W x_C = 0.2W x_C$$

$$T_1 + U_{1-2} = T_2 \qquad 0 + (0.2)(W)x_C = \frac{1}{2}\frac{W}{g}v^2$$

$$x_C = \frac{(0.5\text{ ft/s})^2}{0.2(2)(32.2\text{ ft/s}^2)} = 0.01941\text{ ft}$$

$$x_C = 0.01941\text{ ft} \blacktriangleleft$$

Blocks *A* and *B*:


v = 1.5 ft/s
A
B
①
+
$\mu_k W$
A
B
$\mu_k 3W$
=
v = 0.5 ft/s
A
B
②
x_A

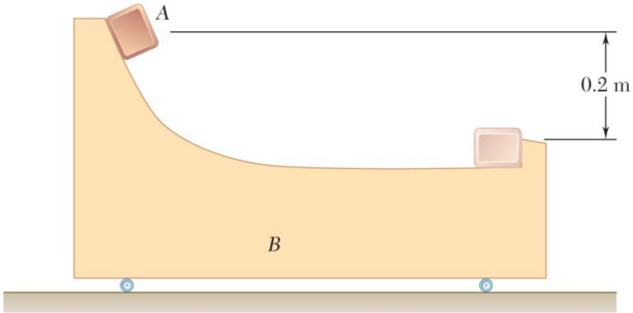
$$T_1 = \frac{1}{2}\left(2\frac{W}{g}\right)(1.5)^2 = 2.25W \qquad T_2 = \frac{1}{2}\left(2\frac{W}{g}\right)(0.5)^2 = 0.25W$$

$$U_{1-2} = -4\mu_k W g x_A = -0.8 W g x_A$$

$$T_1 + U_{1-2} = T_2$$

$$2.25W - 4(0.2)W(32.2)x_A = 0.25W$$

$$x_A = 0.07764\text{ ft} \qquad x_A = 0.0776\text{ ft} \blacktriangleleft$$



PROBLEM 13.182

Block A is released from rest and slides down the frictionless surface of B until it hits a bumper on the right end of B. Block A has a mass of 10 kg and object B has a mass of 30 kg and B can roll freely on the ground. Determine the velocities of A and B immediately after impact when (a) $e = 0$, (b) $e = 0.7$.

SOLUTION

Let the x-direction be positive to the right and the y-direction vertically upward.

Let $(v_A)_x$, $(v_A)_y$, $(v_A)_x$ and $(v_B)_y$ be velocity components just before impact and $(v'_A)_x, (v'_A)'_y, (v'_B)_x$, and $(v'_B)_y$ those just after impact. By inspection,

$$(v_A)_y = (v_B)_y = (v'_A)_y = (v'_B)_y = 0$$

Conservation of momentum for x-direction:

While block is sliding down: $$0 + 0 = m_A(v_A)_x + m_B(v_B)_x \qquad (v_B)_x = -\beta(v_A)_x \tag{1}$$

Impact: $$0 + 0 = m_A(v'_A)_x + m_B(v'_B)_x \qquad (v'_B)_x = -\beta(v'_A)_x \tag{2}$$

where $$\beta = m_A/m_B$$

Conservation of energy during frictionless sliding:

Initial potential energies: $m_A gh$ for A, 0 for B.

Potential energy just before impact: $V_1 = 0$

Initial kinetic energy: $T_0 = 0$ (rest)

Kinetic energy just before impact: $$T_1 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$$

$$T_0 + V_0 = T_1 + V_1$$

$$m_A gh = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 = \frac{1}{2}(m_A + m_B\beta^2)v_A^2$$

$$= \frac{1}{2}m_A(1+\beta)v_A^2$$

$$v_A^2 = (v_A)_x^2 = \frac{2gh}{1+\beta} \qquad v_A = \sqrt{\frac{2gh}{1+\beta}} \tag{3}$$

PROBLEM 13.182 (Continued)

Velocities just before impact: $\mathbf{v}_A = \sqrt{\dfrac{2gh}{1+\beta}} \longrightarrow$

$$\mathbf{v}_B = \beta\sqrt{\frac{2gh}{1+\beta}} \longleftarrow$$

Analysis of impact. Use Eq. (2) together with coefficient of restitution.

$$(v'_B)_x - (v'_A)_x = e[(v_A)_x - (v_B)_x]$$
$$-\beta(v'_A)_x - (v'_A)_x = e[(v_A)_x + \beta(v_A)_x]$$
$$(v'_A)_x = -e(v_A)_x \qquad (4)$$

Data: $m_A = 10 \text{ kg}$

$m_B = 30 \text{ kg}$

$h = 0.2 \text{ m}$

$g = 9.81 \text{ m/s}^2$

$$\beta = \frac{10\,\text{kg}}{30\,\text{kg}} = 0.33333$$

From Eq. (3),
$$v_A = \sqrt{\frac{(2)(9.81)(0.2)}{1.33333}} = 1.71552 \text{ m/s}$$

(*a*) $e = 0$: $(v'_A)_x = 0 \qquad (v'_B)_x = 0$

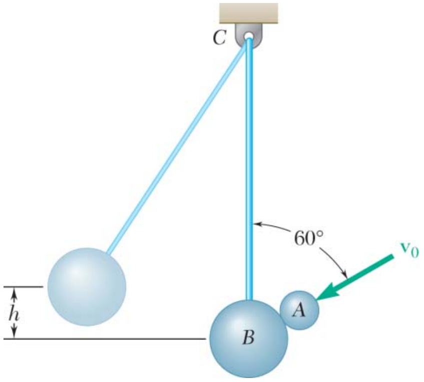
$\mathbf{v}'_A = 0$ ◀

$\mathbf{v}'_B = 0$ ◀

(*b*) $e = 0.7$:
$$(v'_A)_x = -(0.7)(1.71552) = -1.20086 \text{ m/s}$$
$$(v'_B)_x = -(0.33333)(1.20086) = 0.40029 \text{ m/s}$$

$\mathbf{v}'_A = 1.201 \text{ m/s} \longleftarrow$ ◀

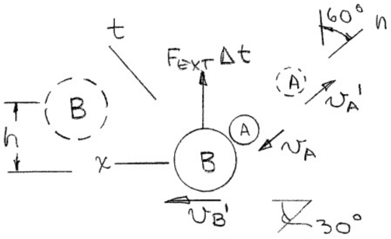
$\mathbf{v}'_B = 0.400 \text{ m/s} \longrightarrow$ ◀



PROBLEM 13.183

A 340-g ball *B* is hanging from an inextensible cord attached to a support *C*. A 170-g ball *A* strikes *B* with a velocity $\mathbf{v}_0$ of magnitude 1.5 m/s at an angle of 60° with the vertical. Assuming perfectly elastic impact ($e = 1$) and no friction, determine the height *h* reached by ball *B*.

SOLUTION

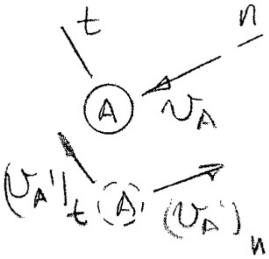


Ball *A* alone

Momentum in *t*-direction conserved

$$m_A(v_A)_t = m_A(v'_A)_t$$

$$(v_A)_t = 0 = (v'_A)_t$$



Thus $\quad (v'_A)_n = v'_A \nearrow 60^\circ$

Total momentum in the *x*-direction is conserved.

$$m_A v_A \sin 60^\circ + m_B(v_B)_x = m_A(-v'_A)\sin 60 + m_B v'_B$$

$$v_A = v_0 = 1.5 \text{ m/s} \qquad (v_B)_x = 0$$

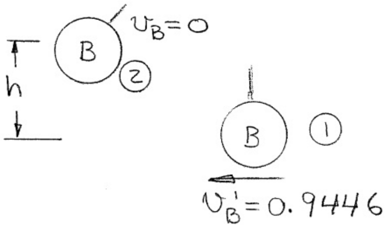
$$0.17(1.5)(\sin 60^\circ) + 0 = -(0.17)(v'_A)(\sin 60^\circ) + (0.34)v'_B$$

$$0.2208 = -0.1472v'_A + 0.34v'_B \qquad (1)$$

Relative velocity in the *n*-direction

$$[-v_A - (v_B)_n]e = -v'_B \cos 30^\circ - v'_A;$$

$$(-1.5 - 0)(1) = -0.866v'_B - v'_A \qquad (2)$$



Solving Equations (1) *and* (2) simultaneously

$$v'_B = 0.9446 \text{ m/s}, \; v'_A = 0.6820 \text{ m/s}$$

Conservation of energy ball *B*

$$T_1 = \frac{1}{2}m_B(v'_B)^2$$

$$T_1 = \frac{1}{2}\frac{W_B}{g}(3.0232)^2 \qquad T_2 = 0$$

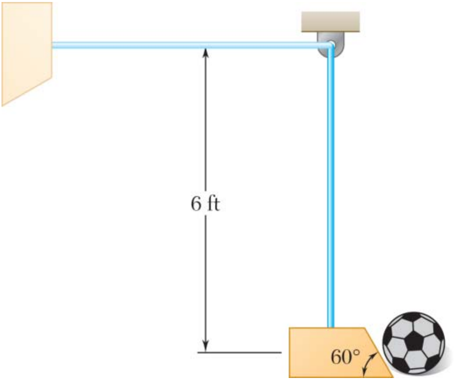
PROBLEM 13.183 (Continued)

$$V_1 = 0 \qquad V_2 = W_B h$$

$$T_1 + V_1 = T_2 + V_2;\ \frac{1}{2}\frac{\cancel{W}_B}{g}(0.9446)^2 = 0 + \cancel{W}_B h;$$

$$h = \frac{(0.9446)^2}{(2)(9.81)} = 0.0455 \text{ m}$$

$h = 45.5$ mm ◀



PROBLEM 13.184

A test machine that kicks soccer balls has a 5-lb simulated foot attached to the end of a 6-ft long pendulum arm of negligible mass. Knowing that the arm is released from the horizontal position and that the coefficient of restitution between the foot and the 1 lb ball is 0.8, determine the exit velocity of the ball (a) if the ball is stationary, (b) if the ball is struck when it is rolling towards the foot with a velocity of 10 ft/s.

SOLUTION

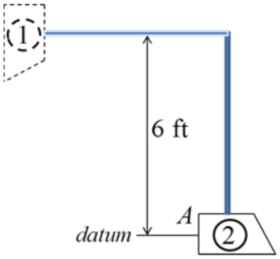
Given: $m_A = 5/32.2$ slugs

$m_B = 1/32.2$ slugs

$l = 6$ ft

$e = 0.8$

Pendulum in two positions:



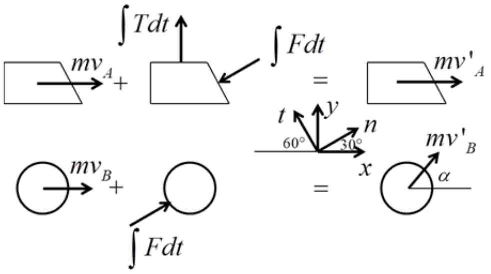
Apply Work-Energy to determine speed of foot before impact:

$$\cancel{T_1} + V_{g_1} + \cancel{V_{e_1}} + \cancel{U_{1\to 2}^{NC}} = T_2 + \cancel{V_{g_2}} + \cancel{V_{e_2}}$$

where: $V_{g_1} = m_A g l$ and $T_2 = \frac{1}{2} m_A v_A^2$

$$v_A = \sqrt{2gl} = 19.66 \text{ ft/s} \rightarrow \text{m/s}$$

Impulse momentum diagrams for ball and foot:



Conservation of momentum of the ball in the t-direction:

$$(v'_B)_t = (v_B)_t$$
$$= -v_B \cos 60° \quad (1)$$

Conservation of momentum of the systems in the x-direction:

$$m_A (v_A)_x + m_B (v_B)_x = m_A (v'_A)_x + m_B (v'_B)_x$$

where:

$$(v_A)_x = v_A$$
$$(v_B)_x = v_B$$
$$(v'_A)_x = v'_A$$
$$(v'_B)_x = (v'_B)_n \sin 60° - (v'_B)_t \cos 60°$$

therefore:

$$m_A v_A + m_B v_B = m_A v'_A + m_B \left[(v'_B)_n \sin 60° + v_B \cos^2 60° \right] \quad (2)$$

PROBLEM 13.184 (Continued)

Coefficient of Restitution: $\left(v'_B\right)_n - \left(v'_A\right)_n = e\left[\left(v_A\right)_n - \left(v_B\right)_n\right]$

where:

$$\left(v'_A\right)_n = v'_A \sin 60°$$
$$\left(v_A\right)_n = v_A \sin 60°$$
$$\left(v_B\right)_n = v_B \sin 60°$$

therefore: $\left(v'_B\right)_n - v'_A \sin 60° = e\left(v_A \sin 60° - v_B \sin 60°\right)$ (3)

Solving equations (2) and (3)

$$\left(v'_B\right)_n = \frac{\left[m_A v_A\left(1+e\right) - e m_A v_B + m_B v_B\left(1-\cos^2 60°\right)\right]\sin 60°}{m_A + m_B \sin^2 60°} \qquad (4)$$

$$v'_B = \sqrt{\left(v'_B\right)_t^2 + \left(v'_B\right)_n^2} \qquad (5)$$

$$\alpha = 30° + \tan^{-1}\left(\frac{\left(v'_B\right)_t}{\left(v'_B\right)_n}\right) \qquad (6)$$

(a) Evaluating equations (1), (4) for $v_B = 0$

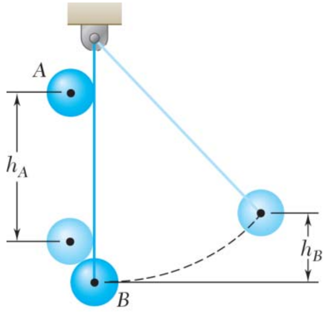
$$\left(v'_B\right)_t = 0$$
$$\left(v'_B\right)_n = 26.65 \text{ ft/s}$$

Using equations (5) and (6) yields: 26.65 ft/s ∠ 30° ◀

(b) Evaluating equations (1), (4) for $v_B = -10$ ft/s

$$\left(v'_B\right)_t = 5 \text{ ft/s}$$
$$\left(v'_B\right)_n = 31.54 \text{ ft/s}$$

Using equations (5) and (6) yields: 31.93 ft/s ∠ 39.0° ◀



PROBLEM 13.185

Ball *B* is hanging from an inextensible cord. An identical ball *A* is released from rest when it is just touching the cord and drops through the vertical distance $h_A = 8$ in. before striking ball *B*. Assuming $e = 0.9$ and no friction, determine the resulting maximum vertical displacement h_B of ball *B*.

SOLUTION

Ball *A* falls

$$T_1 = 0 \quad V_2 = 0$$

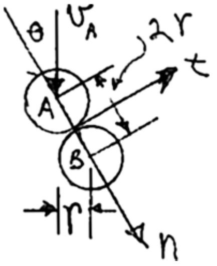
$$T_1 + V_1 = T_2 + V_2 \qquad \text{(Put datum at 2)}$$

$$h = 8 \text{ in.} = 0.66667 \text{ ft}$$

$$mgh = \frac{1}{2} m v_A^2$$

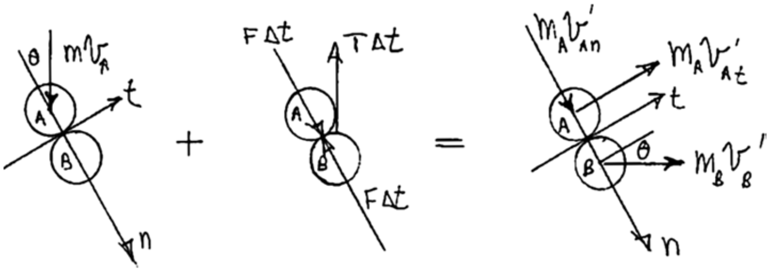
$$v_A = \sqrt{2gh} = \sqrt{(2)(32.2)(0.66667)} = 6.5524 \text{ ft/s}$$

Impact



$$\theta = \sin^{-1} \frac{r}{2r} = 30°$$

Impulse-Momentum



Unknowns: v'_B, v'_{At}, v'_{An}

x-dir ⟶

$$0 + 0 = m_B v'_B + m_A v'_{An} \sin 30° + m_A v'_{At} \cos 30° \qquad (1)$$

Noting that $m_A = m_B$ and dividing by m_A

$$v'_B + v'_{An} \sin 30° + v'_{At} \cos 30° = 0 \qquad (1)$$

Ball *A* alone:

Momentum in *t*-direction:

$$-m_A v_A \sin 30° + 0 = m_A v_{At}$$

$$v'_{At} = -v_A \sin 30° = -6.5524 \sin 30° = -3.2762 \text{ ft/s} \qquad (2)$$

Coefficient of restitution:

$$v'_{Bn} - v'_{An} = e(v_{An} - v_{en})$$

$$v'_B \sin 30° - v'_{An} = 0.9(v_A \cos 30° - 0) \qquad (3)$$

With known value for v_{At}, Eqs. (1) and (3) become

$$v'_B + v'_{An} \sin 30° = 3.2762 \cos 30°$$
$$v'_B \sin 30° - v'_{An} = (0.9)(6.5524) \cos 30°$$

Solving the two equations simultaneously,

$$v'_B = 4.31265 \text{ ft/s}$$
$$v'_{An} = -2.9508 \text{ ft/s}$$

After the impact, ball *B* swings upward. Using *B* as a free body

$$T' + V' = T_B + V_B$$

where $T' = \frac{1}{2} m_B (v'_B)^2,$

$V' = 0,$

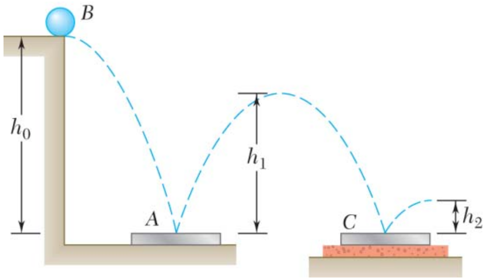
$T_B = 0$

and $V_B = m_B g h_B$

$$\frac{1}{2} m_B (v'_B)^2 = m_B g h_B$$

$$h_B = \frac{1}{2} \frac{(v'_B)^2}{g}$$
$$= \frac{1}{2} \frac{(4.31265)^2}{32.2}$$
$$= 0.2888 \text{ ft}$$

$h_B = 3.47$ in. ◀

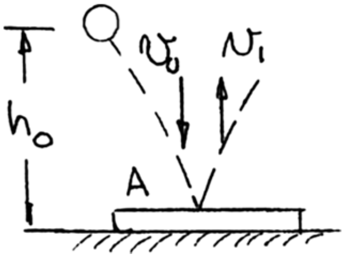


PROBLEM 13.186

A 70 g ball B dropped from a height $h_0 = 1.5$ m reaches a height $h_2 = 0.25$ m after bouncing twice from identical 210-g plates. Plate A rests directly on hard ground, while plate C rests on a foam-rubber mat. Determine (*a*) the coefficient of restitution between the ball and the plates, (*b*) the height h_1 of the ball's first bounce.

SOLUTION

(*a*) Plate on hard ground (first rebound):



Conservation of energy:

$$\cancel{\frac{1}{2} m_B v_y^2} + \frac{1}{2} m_B v_0^2 = m_B g h_0 + \cancel{\frac{1}{2} m_B v_x^2}$$

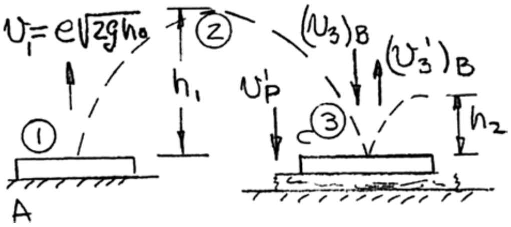
$$v_0 = \sqrt{2 g h_0}$$

Relative velocities., *n*-direction:

$$v_0 e = v_1 \qquad v_1 = e\sqrt{2 g h_0}$$

t-direction $\qquad v'_{Bx} = v_{Bx}$

Plate on foam rubber support at *C*.



Conservation of energy:

Points ① and ③: $\qquad V_1 = V_3 = 0$

$$\cancel{\frac{1}{2} m_B (v'_{Bx})^2} \times \frac{1}{2} m_B v_1^2 = \frac{1}{2} m_B (v_3)_B^2 + \cancel{\frac{1}{2} m_B (v'_{Bx})^2}$$

$$(v_3)_B = e\sqrt{2 g h_0}$$

Conservation of momentum:

$\uparrow +$ At ③: $\qquad m_B(-v_3)_B + m_P v_P = m_B (v'_3)_B - m_P v'_P$

$$\frac{m_P}{m_B} = \frac{210}{70} = 3 \qquad -e\sqrt{2 g h_0} = (v'_3)_B - 3 v'_P \tag{1}$$

PROBLEM 13.186 (Continued)

Relative velocities:

$$[(-v_3)_B - (v_P)]e = -v'_P - (v'_3)_B$$

$$e^2\sqrt{2gh_0} + 0 = v'_P + (v'_3)_B \qquad (2)$$

Multiplying (2) by 3 and adding to (1)

$$4(v'_3)_B = \sqrt{2gh_0}\,(3e^2 - e)$$

Conservation of energy at ③, $(v'_3)_B = \sqrt{2gh_2}$

Thus, $4\sqrt{2gh_2} = \sqrt{2gh_0}\,(3e^2 - e)$

$$3e^2 - e = 4\sqrt{\frac{h_2}{h_0}} = 4\sqrt{\frac{0.25}{1.5}} = 1.63299$$

$$3e^2 - e - 1.633 = 0 \qquad e = 0.923 \blacktriangleleft$$

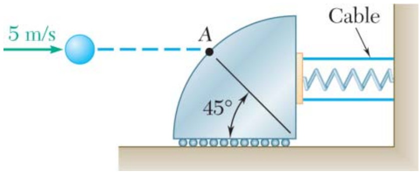
(*b*) Points ① and ②:

Conservation of energy.

$$\cancel{\frac{1}{2}m_B(v'_{Bx})^2} + \frac{1}{2}m_B v_1^2 = \cancel{\frac{1}{2}m_B(v'_{Bx})^2}; \quad \frac{1}{2}e^2(2gh_0) = gh_1$$

$$h_1 = e^2 h_0 = (0.923)^2(1.5)$$

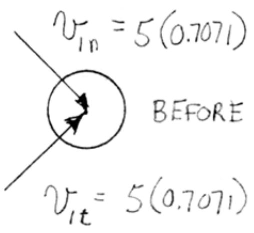
$$h_1 = 1.278 \text{ m} \blacktriangleleft$$

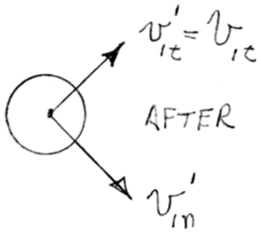

5 m/s
A
Cable
45°

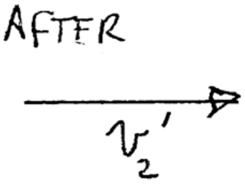
PROBLEM 13.187

A 2-kg sphere moving to the right with a velocity of 5 m/s strikes at *A* the surface of a 9-kg quarter cylinder which is initially at rest and in contact with a spring of constant 20 kN/m. The spring is held by cables so that it is initially compressed 50 mm. Neglecting friction and knowing that the coefficient of restitution is 0.6, determine (*a*) the velocity of the sphere immediately after impact, (*b*) the maximum compressive force in the spring.

SOLUTION


$v_{1n} = 5(0.7071)$
BEFORE
$v_{1t} = 5(0.7071)$


$v'_{1t} = v_{1t}$
AFTER
v'_{1n}



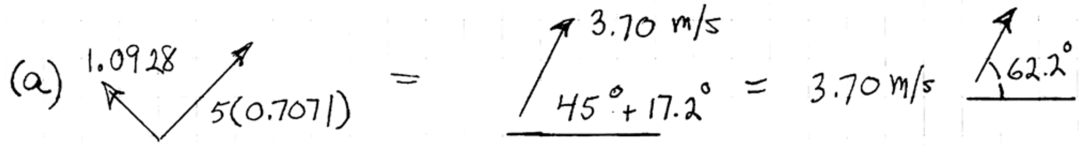
Momentum: $+\searrow \; mv_{1n} + 0 = mv'_{1n} + Mv'_2(0.7071)$

$$(2\text{ kg})(5\text{ m/s})(0.7071) = 2\text{ kg}\; v'_{1n} + 9\text{ kg}\; v'_2(0.7071)$$

Restitution: $v'_2(0.7071) - v'_{1n} = 0.6\,v_{1n} = 0.6(5)(0.7071)$

Solve for $v'_2 = \left(\frac{16}{11}\right)$ m/s, $\quad v'_{1n} = -\left(\frac{17}{11}\right)(0.7071)$

$$v'_{1n} = -1.092801\text{ m/s}$$


(a)
1.0928
5(0.7071)
=
3.70 m/s
45° + 17.2°
=
3.70 m/s
62.2°

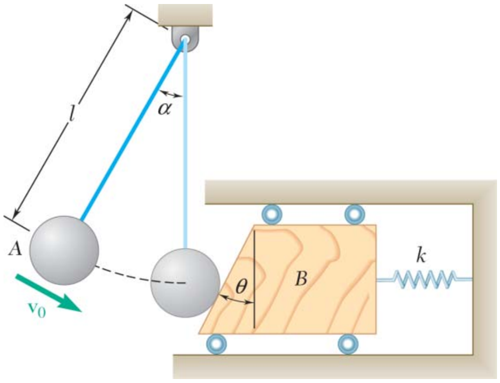
◀

(*b*) Conservation of energy – cylinder + spring:

$$\frac{1}{2}kx_0^2 + \frac{1}{2}M(v'_2)^2 = \frac{1}{2}kx_2^2$$

$$\frac{20{,}000}{2}(0.05)^2 + \frac{1}{2}(9)\left(\frac{16}{11}\right)^2 = \frac{20{,}000}{2}x_2^2 = 34.52$$

$x_2 = 0.05875$ m, $\quad F = kx_2 = 20{,}000\,\frac{\text{N}}{\text{m}}(0.0587\text{ m}) = 1175\text{ N}$ ◀



PROBLEM 13.188

When the rope is at an angle of $\alpha = 30°$ the 1-lb sphere A has a speed $v_0 = 4$ ft/s. The coefficient of restitution between A and the 2-lb wedge B is 0.7 and the length of rope $l = 2.6$ ft. The spring constant has a value of 2 lb/in. and $\theta = 20°$. Determine (*a*) the velocities of A and B immediately after the impact, (*b*) the maximum deflection of the spring assuming A does not strike B again before this point.

SOLUTION

Masses: $m_A = (1/32.2)\ \text{lb}\cdot\text{s}^2/\text{ft} \qquad m_B = (2/32.2)\ \text{lb}\cdot\text{s}^2/\text{ft}$

Analysis of sphere A as it swings down:

Initial state: $\alpha = 30°, \quad h_0 = l(1-\cos\alpha) = (2.6)(1-\cos 30°) = 0.34833\ \text{ft}$

$$V_0 = m_A g h_0 = (1)(0.34833) = 0.34833\ \text{lb}\cdot\text{ft}$$

$$T_0 = \frac{1}{2}m_A v_0^2 = \frac{1}{2}\left(\frac{1.0}{32.2}\right)(4)^2 = 0.24845\ \text{lb}\cdot\text{ft}$$

Just before impact: $\alpha = 0, \quad h_1 = 0, \quad V_1 = 0$

$$T_1 = \frac{1}{2}m_A v_A^2 = \frac{1}{2}\left(\frac{1.0}{32.2}\right)v_A^2 = \left(\frac{1.0}{64.4}\right)v_A^2$$

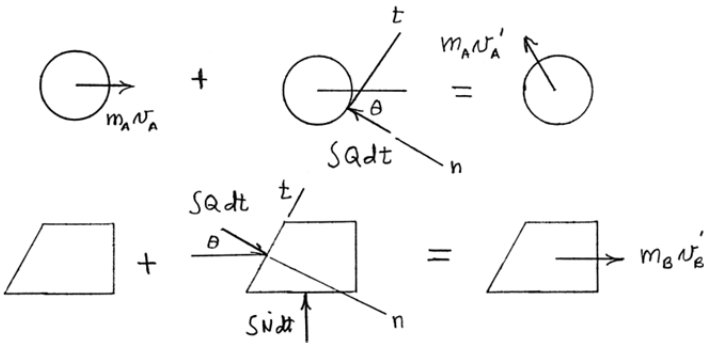
Conservation of energy: $T_0 + V_0 = T_1 + V_1$

$$0.24845 + 0.34833 = \frac{1}{64.4}v_A^2 + 0$$

$$v_A^2 = 38.433\ \text{ft}^2/\text{s}^2$$

$$\mathbf{v}_A = 6.1994\ \text{ft/s} \longrightarrow$$

Analysis of the impact. Use conservation of momentum together with the coefficient of restitution. $e = 0.7$.



Note that the rope does not apply an impulse since it becomes slack.

PROBLEM 13.188 (Continued)

Sphere A: Momentum in t-direction:

$$m_A v_A \sin\theta + 0 = m_A (v'_A)_t$$
$$(v'_A)_t = v_A \sin\theta = 6.1994 \sin 20^\circ = 2.1203 \text{ m/s}$$
$$(\mathbf{v}_A)_t = 2.1203 \text{ m/s} \ \angle 70^\circ$$

Both A and B: Momentum in x-direction:

$$m_A v_A + 0 = m_A (v'_A)_n \cos\theta + m_A (v'_A)_t \sin\theta + m_B v'_B$$
$$(1/32.2)(6.1994) = (1/32.2)(v_A)_n \cos 20^\circ + (1/32.2)(2.120323)\sin 20^\circ + (2/32.2)v'_B$$
$$(1/32.2)(v'_A)_n \cos 20^\circ + (2/32.2)v'_B = 0.17001 \qquad (1)$$

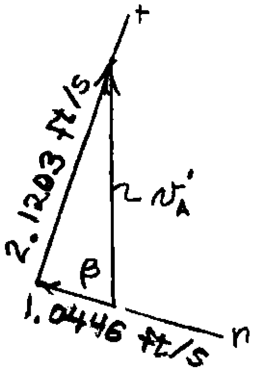
Coefficient of restitution:

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n]$$
$$v'_B \cos\theta - (v'_A)_n = e[v_A \cos\theta - 0]$$
$$v'_B \cos 20^\circ - (v'_A)_n = (0.7)(6.1994)\cos 20^\circ \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously for $(v'_A)_n$ and v'_B,

$$(v'_A)_n = -1.0446 \text{ ft/s}$$
$$v'_B = 3.2279 \text{ ft/s}$$

Resolve $\mathbf{v}_A$ into horizontal and vertical components.



$$\tan\beta = \frac{(v'_A)_t}{-(v'_A)_n} = \frac{2.1203}{1.0446}$$
$$\beta = 63.77^\circ \qquad \beta + 20^\circ = 83.8^\circ$$
$$v'_A = \sqrt{(2.1203)^2 + (1.0446)^2} = 2.3637 \text{ ft/s}$$

(*a*) *Velocities immediately after impact.* $\mathbf{v}'_A = 2.36 \text{ ft/s} \searrow 83.8^\circ$ ◀

$\mathbf{v}'_B = 3.23 \text{ ft/s} \longrightarrow$ ◀

(*b*) *Maximum deflection of wedge B.*

Use conservation of energy: $T_{B1} + V_{B1} = T_{B2} + V_{B2}$

$$T_{B1} = \frac{1}{2} m_B v_B^2$$
$$V_{B1} = 0$$
$$T_{B2} = 0$$
$$V_{B2} = \frac{1}{2} k (\Delta x)^2$$

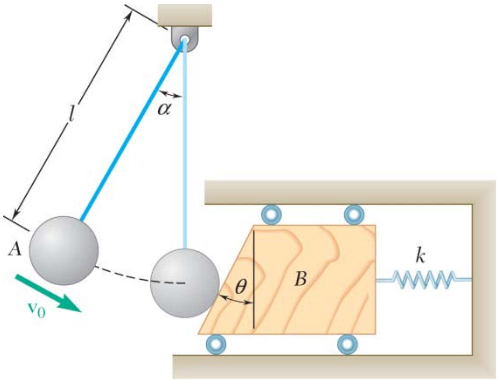
PROBLEM 13.188 (Continued)

The maximum deflection will occur when the block comes to rest (ie, no kinetic energy)

$$\frac{1}{2}m_B v_B^2 = \frac{1}{2}k(\Delta x)^2$$

$$(\Delta x)^2 = \frac{m_B v_B^2}{k} = \frac{\left(\frac{2\text{ lb}}{32.2\text{ ft/s}^2}\right)(3.2279\text{ ft/s})^2}{2\text{ lb/in }(12\text{ in/ft})}$$

$$(\Delta x) = 0.1642118\text{ ft} \qquad (\Delta x) = 1.971\text{ in.} \blacktriangleleft$$



PROBLEM 13.189

When the rope is at an angle of $\alpha = 30°$ the 1-kg sphere A has a speed $v_0 = 0.6$ m/s. The coefficient of restitution between A and the 2-kg wedge B is 0.8 and the length of rope $l = 0.9$ m The spring constant has a value of 1500 N/m and $\theta = 20°$. Determine, (*a*) the velocities of A and B immediately after the impact (*b*) the maximum deflection of the spring assuming A does not strike B again before this point.

SOLUTION

Masses: $m_A = 1$ kg

$m_B = 2$ kg

Analysis of sphere *A* as it swings down:

Initial state: $\alpha = 30°, \quad h_0 = l(1-\cos\alpha) = (0.9)(1-\cos 30°) = 0.12058$ m

$$V_0 = m_A g h_0 = (1)(9.81)(0.12058) = 1.1829 \text{ N}\cdot\text{m}$$

$$T_0 = \frac{1}{2}mv_0^2 = \frac{1}{2}(1)(0.6)^2 = 0.180 \text{ N}\cdot\text{m}$$

Just before impact: $\alpha = 0, \quad h_1 = 0, \quad V_1 = 0$

$$T_1 = \frac{1}{2}m_A v_A^2 = \frac{1}{2}(1)v_A^2 = 0.5v_A^2$$

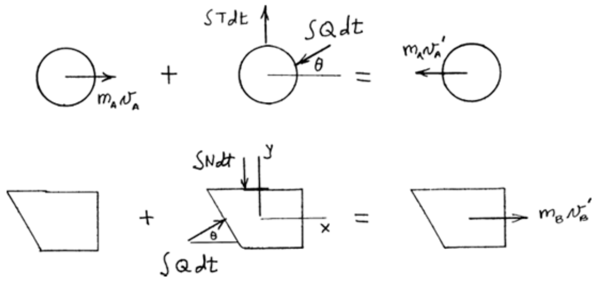
Conservation of energy: $T_0 + V_0 = T_1 + V_1$

$$0.180 + 1.1829 = 0.5v_A^2 + 0$$

$$v_A^2 = 2.7257 \text{ m}^2/\text{s}^2$$

$$\mathbf{v}_A = 1.6510 \text{ m/s} \longrightarrow$$

Analysis of the impact: Use conservation of momentum together with the coefficient of restitution. $e = 0.8$.



Note that the ball rebounds horizontally and that an impulse $\int T dt$ is applied by the rope. Also, an impulse $\int N dt$ is applied to B through its supports.

Both A and B:

Momentum in x-direction:

$$m_A(v_A)_x + 0 = m_A(v'_A)_x + m_B(v'_B)_x$$

$$(1)(1.6510) = (1)(v'_A)_x + (2)(v'_B)_x \qquad (1)$$

Coefficient of restitution: $(v_A)_n = (v_A)_x \cos\theta$

$$(v_B)_n = 0, \quad (v'_A)_n = (v'_A)_x \cos\theta, \quad (v'_B)_x \cos 30°$$

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n]$$

$$(v'_B)_x \cos\theta - (v'_A)_x \cos\theta = e[(v_A)_x \cos\theta]$$

Dividing by $\cos\theta$ and applying $e = 0.8$ gives

$$(v'_B)_x - (v'_A)_x = (0.8)(1.6510) \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously,

$$(v'_A)_x = -0.33020 \text{ m/s}$$

$$(v'_B)_x = 0.99059 \text{ m/s}$$

$\mathbf{v}'_A = 0.330$ m/s ← ◀

(*a*) *Velocities immediately after impact.* $\mathbf{v}'_B = 0.991$ m/s → ◀

(*b*) Maximum deflection of wedge B.

Use conservation of energy: $T_{B1} + V_{B1} = T_{B2} + V_{B2}$

$$T_{B1} = \frac{1}{2} m_B v_B^2$$

$$V_{B1} = 0$$

$$T_{B2} = 0$$

$$V_{B2} = \frac{1}{2} k(\Delta x)^2$$

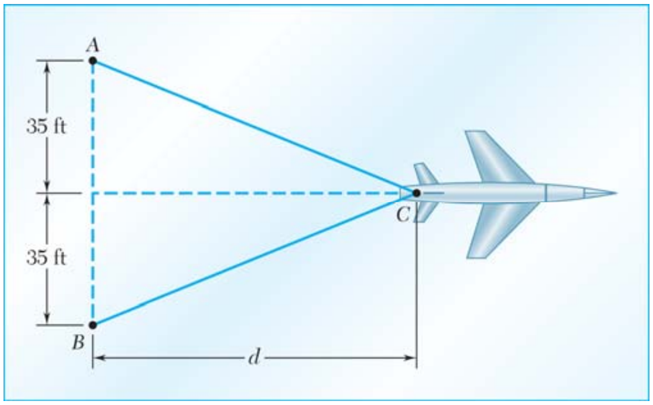
The maximum deflection will occur when the block comes to rest (ie, no kinetic energy)

$$\frac{1}{2} m_B v_B^2 = \frac{1}{2} k(\Delta x)^2$$

$$(\Delta x)^2 = \frac{m_B v_B^2}{k} = \frac{(2)(0.99059 \text{ m/s})^2}{1500 \text{ N/m})}$$

$$(\Delta x) = 0.0362 \text{ m}$$

$\Delta x = 36.2$ mm ◀

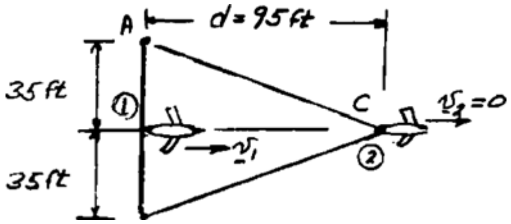


PROBLEM 13.190

A 32,000-lb airplane lands on an aircraft carrier and is caught by an arresting cable. The cable is inextensible and is paid out at A and B from mechanisms located below dock and consisting of pistons moving in long oil-filled cylinders. Knowing that the piston-cylinder system maintains a constant tension of 85 kips in the cable during the entire landing, determine the landing speed of the airplane if it travels a distance $d = 95$ ft after being caught by the cable.

SOLUTION

Mass of airplaine: $m = \frac{W}{g} = \frac{32000 \text{ lb}}{32.2 \text{ ft/s}^2} = 993.79 \text{ lb}\cdot\text{s}^2/\text{ft}$



Work of arresting cable force.

$$Q = 85 \text{ kips} = 85000 \text{ lb.}$$

As the cable is pulled out, the cable tension acts paralled to the cable at the airplane hook. For a small displacement

$$\Delta U = -Q(\Delta l_{AC}) - Q(\Delta l_{BC})$$

Since Q is constant,

$$U_{1\to2} = -Q\left[\overline{AC} + \overline{BC} - \overline{AB}\right]$$

For $d = 95$ ft, $\overline{AC} = \overline{BC} = \sqrt{(35)^2 + (95)^2} = 101.24$ ft

$$U_{1\to2} = -(85000)(101.24 + 101.24 - 70) = -11.261 \text{ ft}\cdot\text{lb}$$

Principle of work and energy: $T_1 + U_{1\to2} = T_2$

$$\frac{1}{2}mv_1^2 + U_{1\to2} = \frac{1}{2}mv_2^2$$

Since $v_2 = 0$, we get

$$v_1^2 = -\frac{2U_{1\to2}}{m} = -\frac{(2)(-11.261)}{993.79} = 22.663\times10^3 \text{ ft}^2/\text{s}^2$$

Initial speed: $v_1 = 150.54$ ft/s $\quad v_1 = 102.6$ mi/h ◀

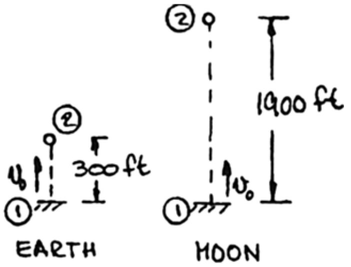
PROBLEM 13.191

A 2-oz pellet shot vertically from a spring-loaded pistol on the surface of the earth rises to a height of 300 ft. The same pellet shot from the same pistol on the surface of the moon rises to a height of 1900 ft. Determine the energy dissipated by aerodynamic drag when the pellet is shot on the surface of the earth. (The acceleration of gravity on the surface of the moon is 0.165 times that on the surface of the earth.)

SOLUTION

Since the pellet is shot from the same pistol the initial velocity v_0 is the same on the moon and on the earth.

Work and energy.



Earth: $$T_1 = \frac{1}{2}mv_0^2$$

$$U_{1-2} = -mg_E(300\text{ ft}) - E_L$$

$(E_L =$ Loss of energy due to drag$)$

Moon: $$T_1 = \frac{1}{2}mv_0^2 \qquad T_2 = 0$$

$$U_{1-2} = -mg_M(1900) \qquad T_1 - 300mg_E - E_L = 0 \tag{1}$$

$$T_2 = 0$$

$$T_1 - 1900mg_M = 0 \tag{2}$$

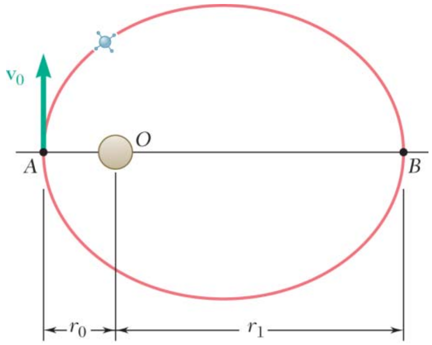
Subtracting (1) from (2) $$-1900mg_M + 300mg_E + E_L = 0$$

$$g_M = 0.165g_E$$

$$m = \frac{(2/16)}{g_E}$$

$$E_L = (1900)\frac{(2/16)}{g_E}(0.165g_E) - 300\frac{(2/16)}{g_E}g_E$$

$E_L = 1.688\text{ ft}\cdot\text{lb}$ ◀



PROBLEM 13.192

A satellite describes an elliptic orbit about a planet of mass *M*. The minimum and maximum values of the distance *r* from the satellite to the center of the planet are, respectively, r_0 and r_1. Use the principles of conservation of energy and conservation of angular momentum to derive the relation

$$\frac{1}{r_0}+\frac{1}{r_1}=\frac{2GM}{h^2}$$

where *h* is the angular momentum per unit mass of the satellite and *G* is the constant of gravitation.

SOLUTION

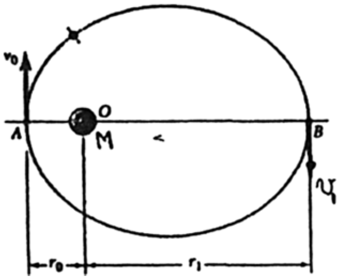
Angular momentum:

$$h = r_0, \quad v_0 = r_1 v_1$$

$$b = r_0 v_0 = r_1 v_1$$

$$v_0 = \frac{h}{r_0} \quad v_1 = \frac{h}{r_1} \tag{1}$$

Conservation of energy:



$$T_A = \frac{1}{2}mv_0^2$$

$$V_A = -\frac{GMm}{r_0}$$

$$T_B = \frac{1}{2}mv_1^2$$

$$V_B = -\frac{GMm}{r_1}$$

$$T_A + V_A = T_B + V_B$$

$$\frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \frac{1}{2}mv_1^2 - \frac{GMm}{r_1}$$

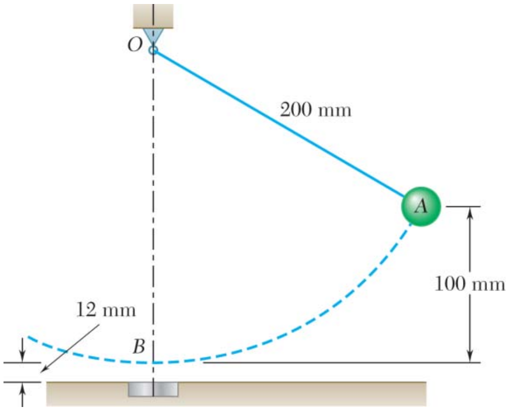
$$v_0^2 - v_1^2 = 2GM\left[\frac{1}{r_0}-\frac{1}{r_1}\right] = 2GM\left[\frac{r_1-r_0}{r_1r_0}\right]$$

Substituting for v_0 and v_1 from Eq. (1)

$$h^2\left[\frac{1}{r_0^2}-\frac{1}{r_1^2}\right] = 2GM\left[\frac{r_1-r_0}{r_1r_0}\right]$$

$$h^2\left[\frac{r_1^2-r_0^2}{r_1^2r_0^2}\right] = \frac{h^2}{r_1^2r_0^2}(r_1-r_0)(r_1+r_0) = 2GM\left[\frac{r_1-r_0}{r_1r_0}\right]$$

$$h^2\left(\frac{1}{r_0}+\frac{1}{r_1}\right) = 2GM \qquad \left(\frac{1}{r_0}+\frac{1}{r_1}\right) = \frac{2GM}{h^2} \quad \text{Q.E.D.}$$



PROBLEM 13.193

A 60-g steel sphere attached to a 200-mm cord can swing about Point *O* in a vertical plane. It is subjected to its own weight and to a force **F** exerted by a small magnet embedded in the ground. The magnitude of that force expressed in newtons is $F = 3000/r^2$ where r is the distance from the magnet to the sphere expressed in millimeters. Knowing that the sphere is released from rest at *A*, determine its speed as it passes through Point *B*.

SOLUTION

Mass and weight:

$$m = 0.060 \text{ kg}$$

$$W = mg = (0.060)(9.81) = 0.5886 \text{ N}$$

Gravitational potential energy:

$$V_g = Wh$$

where *h* is the elevation above level at *B*.

Potential energy of magnetic force:

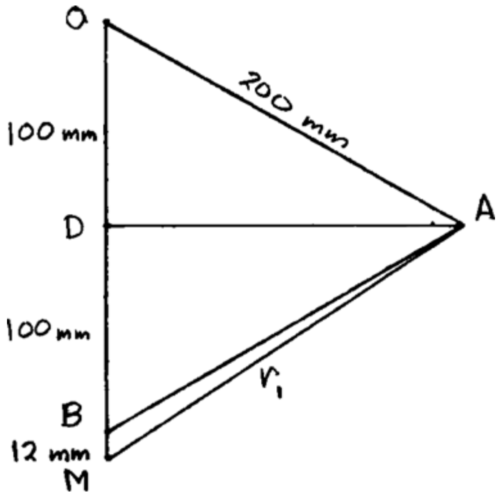
$$F = \frac{3000}{r^2} = -\frac{dV}{dr} \quad (F, \text{ in newtons, } r \text{ in mm})$$

$$V_m = -\int_{\infty}^{r} \frac{3000}{r^2} = \frac{3000}{r} \text{ N} \cdot \text{mm}$$

Use conservation of energy:

$$T_1 + V_1 = T_2 + V_2$$

Position 1: (Rest at *A*.)



$$v_1 = 0 \qquad T_1 = 0$$

$$h_1 = 100 \text{ mm}$$

$$(V_g)_1 = (0.5886 \text{ N})(100 \text{ mm}) = 58.86 \text{ N} \cdot \text{mm}$$

From the figure, $\overline{AD}^2 = 200^2 - 100^2 \text{ (mm}^2)$

$$\overline{MD} = 100 + 12 = 112 \text{ mm}$$

$$r_1^2 = \overline{AD}^2 + \overline{MD}^2$$

$$= 200^2 - 100^2 + 112^2$$

$$= 42544 \text{ mm}^2$$

$$r_1 = 206.26 \text{ mm}$$

$$(V_r)_1 = -\frac{3000}{r_1} = -14.545 \text{ N} \cdot \text{mm}$$

$$V_1 = 58.86 - 14.545 = 44.3015 \text{ N} \cdot \text{mm} = 44.315 \times 10^{-3} \text{ N} \cdot \text{m}$$

PROBLEM 13.193 (Continued)

Position 2. (Sphere at Point *B*.)

$$T_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(0.060)v_2^2 = 0.030\ v_2^2$$

$$(V_g)_2 = 0 \qquad \text{(since } h_2 = 0\text{)}$$

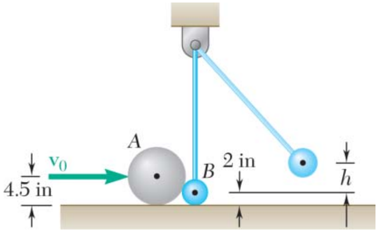
$$r_2 = \overline{MB} = 12\ \text{mm} \qquad \text{(See figure.)}$$

$$(V_m)_2 = -\frac{3000}{12} = -250\ \text{N}\cdot\text{mm} = -250\times10^{-3}\ \text{N}\cdot\text{mm}$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 44.315\times10^{-3} = 0.030v_2^2 - 250\times10^{-3}$$

$$v_2^2 = 9.8105\ \text{m}^2/\text{s}^2 \qquad v_2 = 3.13\ \text{m/s}\ ◀$$

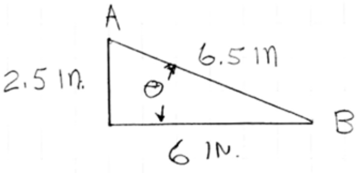

A
v_0
4.5 in
B
2 in
h

PROBLEM 13.194

A 50-lb sphere *A* of radius 4.5 in. moving with a velocity of magnitude $v_0 = 6$ ft/s strikes a 4.6-lb sphere *B* of radius 2 in. which is hanging from an inextensible cord and is initially at rest. Knowing that sphere *B* swings to a maximum height $h = 0.75$ ft, determine the coefficient of restitution between the two spheres.

SOLUTION

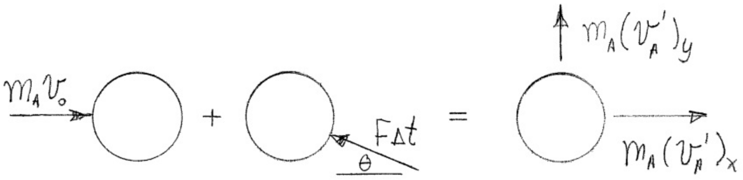
Angle of impulse force from geometry of *A* and *B*


A
6.5 in
2.5 in.
θ
B
6 in.

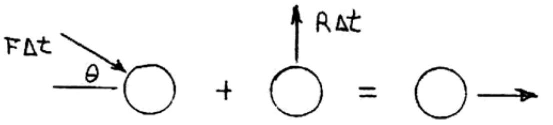
$$\theta = \cos^{-1}\left(\frac{6}{6.5}\right) = 22.62°$$

Total momentum conserved

Ball *A:*

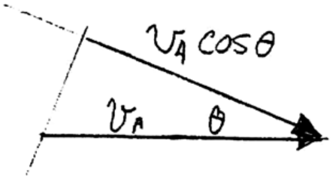

$m_A v_0$
+
$F\Delta t$
θ
=
$m_A(v_A')_y$
$m_A(v_A')_x$

Ball *B:*

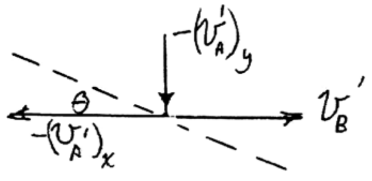

$F\Delta t$
θ
+
$R\Delta t$
=

$$m_B v_B' = m_B\sqrt{2gh} \qquad (1)$$

Restitution


$v_A \cos\theta$
v_A
θ

Approach



Separation

$$e = \frac{v_B'\cos\theta - (v_A')_x\cos\theta + (v_A')_y\sin\theta}{v_A\cos\theta} \qquad v_A = v_0 = 6 \text{ ft/s}$$

$$e = \frac{v_B' - (v_A')_x + (v_A')_y\tan\theta}{v_A} = \frac{v_B' - (v_A')_x + (v_A')_y\left(\frac{25}{6}\right)}{2}$$

PROBLEM 13.194 (Continued)

$$A: \quad \cancel{m}_A v_A \sin\theta = \cancel{m}_A (v'_A)_x (\sin\theta) + \cancel{m}_A (v'_A)_y (\cos\theta)$$

$$v_A \tan\theta = (v'_A)_x \tan\theta + (v'_A)_y; \quad 6\left(\frac{2.5}{6}\right) = (v'_A)_x \left(\frac{2.5}{6}\right) + (v'_A)_y$$

$$15 = 2.5(v'_A)_x + 6(v'_A)_y \tag{2}$$

$$A + B: \quad m_A v_A = m_A (v'_A)_x + m_B v'_B; \quad \left(\frac{50}{g}\right)(6 \text{ ft/s}) = \left(\frac{50}{g}\right)(v'_A)_x + \left(\frac{4.6}{g}\right) v'_B \tag{3}$$

g's cancel

From equation (1) $\quad v'_B = \sqrt{2(32.2 \text{ ft/s}^2)(0.75 \text{ ft})} = 6.9498 \text{ ft/s}$

From equation (3) $\quad (50)(6) = 50(v'_A)_x + 4.6(6.9498)$

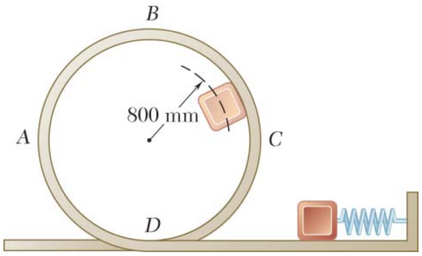
$$(v'_A)_x = 5.3606 \text{ ft/s}$$

From equation (2) $\quad 15 = 2.5(5.3606) + 6(v'_A)_y$

$$(v'_A)_y = 0.2664 \text{ ft/s}$$

$$e = \frac{6.9498 - 5.3606 + 0.2664\left(\frac{2.5}{6}\right)}{6} = 0.2834$$

$e = 0.283$ ◀

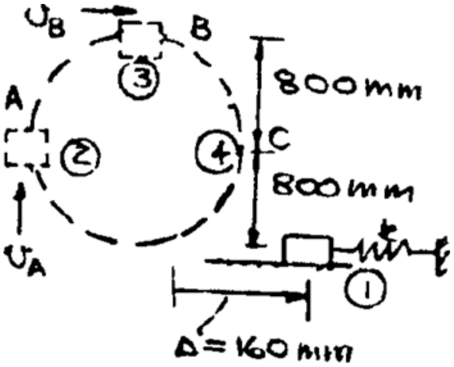


PROBLEM 13.195

A 300-g block is released from rest after a spring of constant $k = 600$ N/m has been compressed 160 mm. Determine the force exerted by the loop *ABCD* on the block as the block passes through (*a*) Point *A*, (*b*) Point *B*, (*c*) Point *C*. Assume no friction.

SOLUTION

Conservation of energy to determine speeds at locations *A*, *B*, and *C*.



Mass: $m = 0.300$ kg

Initial compression in spring: $x_1 = 0.160$ m

Place datum for gravitational potential energy at position 1.

Position 1: $v_1 = 0 \qquad T_1 = \frac{1}{2}mv_1^2 = 0$

$$V_1 = \frac{1}{2}kx_1^2 = \frac{1}{2}(600 \text{ N/m})(0.160 \text{ m})^2 = 7.68 \text{ J}$$

Position 2: $T_2 = \frac{1}{2}mv_A^2 = \frac{1}{2}(0.3)v_A^2 = 0.15v_A^2$

$$V_2 = mgh_2 = (0.3 \text{ kg})(9.81 \text{ m/s}^2)(0.800 \text{ m}) = 2.3544 \text{ J}$$

$$T_1 + V_1 = T_2 + V_2: \quad 0 + 7.68 = 0.15v_A^2 + 2.3544$$

$$v_A^2 = 35.504 \text{ m}^2/\text{s}^2$$

Position 3: $T_3 = \frac{1}{2}mv_B^2 = \frac{1}{2}(0.3)v_B^2 = 0.15v_B^2$

$$V_3 = mgh_3 = (0.3 \text{ kg})(9.81 \text{ m/s}^2)(1.600 \text{ m}) = 4.7088 \text{ J}$$

$$T_1 + V_1 = T_3 + V_3: \quad 0 + 7.68 = 0.15v_B^2 + 4.7088$$

$$v_B^2 = 19.808 \text{ m}^2/\text{s}^2$$

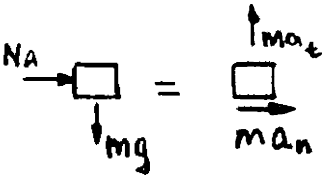
Position 4: $T_2 = \frac{1}{2}mv_C^2 = \frac{1}{2}(0.3)v_C^2 = 0.15v_C^2$

$$V_4 = mgh_4 = (0.3 \text{ kg})(9.81 \text{ m/s})(0.800 \text{ m}) = 2.3544 \text{ J}$$

$$T_1 + V_1 = T_4 + V_4: \quad 0 + 7.68 = 0.15v_C^2 + 2.3544$$

$$v_C^2 = 35.504 \text{ m}^2/\text{s}^2$$

(*a*) Newton's second law at *A*:



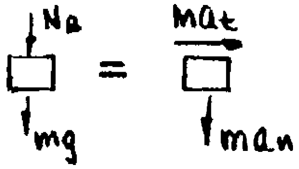
$$a_n = \frac{v_A^2}{\rho} = \frac{35.504\ \text{m}^2/\text{s}^2}{0.800\ \text{m}} = 44.38\ \text{m/s}^2$$

$$\mathbf{a}_n = 44.38\ \text{m/s}^2 \longrightarrow$$

$$\xrightarrow{+} \Sigma F = ma_n: \quad N_A = ma_n$$

$$N_A = (0.3\ \text{kg})(44.38\ \text{m/s}^2) \qquad \mathbf{N}_A = 13.31\ \text{N} \longrightarrow \blacktriangleleft$$

(*b*) Newton's second law at *B*:



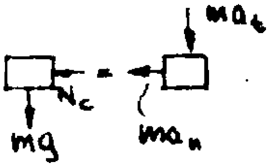
$$a_n = \frac{v_B^2}{\rho} = \frac{19.808\ \text{m}^2/\text{s}^2}{0.800\ \text{m}} = 24.76\ \text{m/s}^2$$

$$\mathbf{a}_n = 24.76\ \text{m/s}^2 \downarrow$$

$$+\!\downarrow \Sigma F = ma_n: \quad N_B = mg = ma_n$$

$$N_B = m(a_n - g) = (0.3\ \text{kg})(24.76\ \text{m/s}^2 - 9.81\ \text{m/s}^2) \qquad \mathbf{N}_B = 4.49\ \text{N} \downarrow \blacktriangleleft$$

(*c*) Newton's second law at *C*:

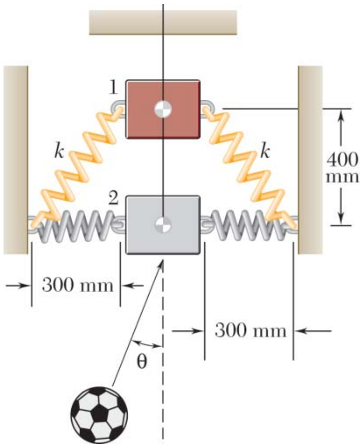


$$a_n = \frac{v_C^2}{\rho} = \frac{35.504\ \text{m}^2/\text{s}^2}{0.800\ \text{m}} = 44.38\ \text{m/s}^2$$

$$\mathbf{a}_n = 44.38\ \text{m/s}^2 \longleftarrow$$

$$\xleftarrow{+} \Sigma F = ma_n: \quad N_C = ma_n$$

$$N_C = (0.3\ \text{kg})(44.38\ \text{m/s}^2) \qquad \mathbf{N}_C = 13.31\ \text{N} \longleftarrow \blacktriangleleft$$



PROBLEM 13.196

A soccer kicking machine has a 5 lb “simulated foot” attached to A “kicking” attachment goes on the front of a wheelchair, allowing athletes with mobility impairments to play soccer. The athletes load up the spring shown through a ratchet mechanism that pulls the 2 kg “foot” back to the position 1. They then release the “foot” to impact the 0.45 kg soccer ball, which is rolling towards the “foot” with a speed of 2 m/s at an angle $\theta = 30^{\circ}$ as shown. The impact occurs with a coefficient of restitution $e = 0.75$ when the foot is at position 2, where the spring is unstretched. Knowing that the effective friction coefficient during rolling is $\mu_k = 0.1$, determine *(a)* the necessary spring coefficient to make the ball roll 30 meters, *(b)* the direction the ball will travel after it is kicked.

SOLUTION

Given: $m_A = 2 \text{ kg}, m_B = 0.45 \text{ kg}, v_B = 2 \text{ m/s}, e = 0.75, \mu_k = 0.1, s_2 = 0$

Conservation of momentum of the ball in the t-direction:

$$(v'_B)_t = (v_B)_t$$
$$= -v_B \sin 30^{\circ}$$
$$= 1 \text{ m/s}$$

Apply Work-Energy to the ball after impact to when it comes to rest:

$$T_1 + \cancel{V_{g_1}} + \cancel{V_{e_1}} + U_{1\to 2}^{NC} = \cancel{T_2} + \cancel{V_{g_2}} + \cancel{V_{e_2}} \qquad \text{where:} \qquad T_2 = \frac{1}{2} m_B v'^2_B$$

$$U_{1\to 2}^{NC} = \mu_k m_B g(30)$$

Therefore:

$$\frac{1}{2}\cancel{m_B} v'^2_B - \mu_k \cancel{m_B} g(30) = 0$$

$$v'_B = 7.672 \text{ m/s}$$

Find $(v'_B)_n$ using:

$$v'_B = \sqrt{(v'_B)_n^2 + (v'_B)_t^2}$$

$$(v'_B)_n = \sqrt{v'^2_B - (v'_B)_t^2}$$

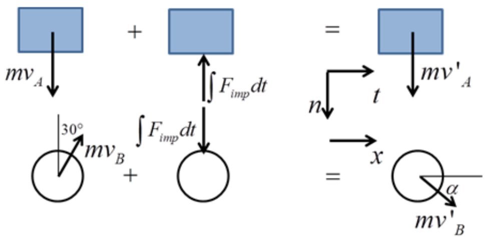
$$= 7.607 \text{ m/s} \downarrow$$

Conservation of momentum of the systems in the n-direction foot:

$$m_A(v_A)_n + m_B(v_B)_n = m_A(v'_A)_n + m_B(v'_B)_n$$

$$(v'_A)_n = \frac{1}{m_A}\left[m_A(v_A)_n + m_B(v_B)_n - m_B(v'_B)_n\right] \qquad (1)$$

Impulse momentum diagrams for ball and



Coefficient of Restitution:

$$(v'_B)_n - (v'_A)_n = e\left[(v_A)_n - (v_B)_n\right] \qquad (2)$$

PROBLEM 13.196 (Continued)

Sub (1) into (2) and solve for $(v_A)_n$:

$$(v'_B)_n - \frac{1}{m_A}\left[m_A(v_A)_n + m_B(v_B)_n - m_B(v'_B)_n\right] = e\left[(v_A)_n - (v_B)_n\right]$$

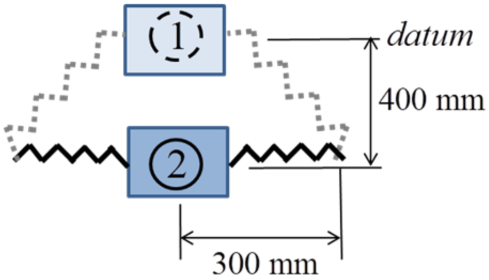
$$(v_A)_n = \frac{(m_A+m_B)(v'_B)_n - m_B(v_B)_n + em_A(v_B)_n}{m_A(e+1)}$$

Sub in known quantities: $$(v_A)_n = \frac{(2+0.45)(7.607) - 0.45(-1.732) + 0.75(2)(-1.732)}{2(0.75+1)}$$

$$= 4.805 \text{ m/s}$$

(a) Apply Work-Energy to determine speed of foot before impact:

“Foot in two positions:



$$\cancel{T_1} + \cancel{V_{g_1}} + V_{e_1} + \cancel{U^{NC}_{1\to 2}} = T_2 + \cancel{V_{g_2}} + \cancel{V_{e_2}}$$

where: $V_{e_1} = 2\left(\frac{1}{2}ks_1^2\right)$, and $T_2 = \frac{1}{2}m_A(v_A)_n^2$

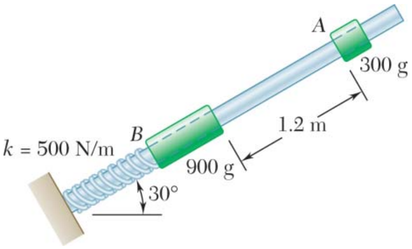
$$s_1 = \sqrt{0.4^2 + 0.3^2} - 0.3$$
$$= 0.2 \text{ m}$$

Therefore: $$k = \frac{m_A(v_A)_n^2}{2s_1^2}$$
$$= \frac{2(4.805)^2}{2(0.2)^2}$$

$k = 577.3$ N/m ◀

(b) Direction of the ball: $$\tan\alpha = \frac{(v'_B)_n}{(v'_B)_t}$$
$$= \frac{7.607}{1}$$

$\alpha = 82.5°$ ◀

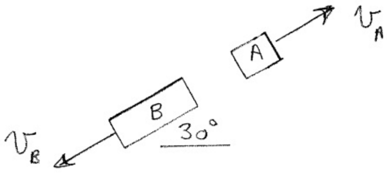


PROBLEM 13.197

A 300-g collar *A* is released from rest, slides down a frictionless rod, and strikes a 900-g collar *B* which is at rest and supported by a spring of constant 500 N/m. Knowing that the coefficient of restitution between the two collars is 0.9, determine (*a*) the maximum distance collar *A* moves up the rod after impact, (*b*) the maximum distance collar *B* moves down the rod after impact.

SOLUTION

After impact



Velocity of *A* just before impact, v_0

$$v_0 = \sqrt{2gh} = \sqrt{2(9.81\text{ m/s}^2)(1.2\text{ m})\sin 30^\circ}$$

$$= \sqrt{2(9.81)(1.2)(0.5)} = 3.431\text{ m/s}$$

Conservation of momentum

$$\overset{+}{\swarrow}_{30^\circ}\ m_A v_0 = m_B v_B - m_A v_A: \quad 0.3v_0 = 0.9v_B - 0.3v_A \quad (1)$$

Restitution

$$(v_A + v_B) = e(v_0 + 0) = 0.9v_0\,(2)$$

Substituting for v_B from (2) in (1)

$$0.3v_0 = 0.9(0.9v_0 - v_A) - 0.3v_A \qquad 1.2v_A = 0.51v_0$$

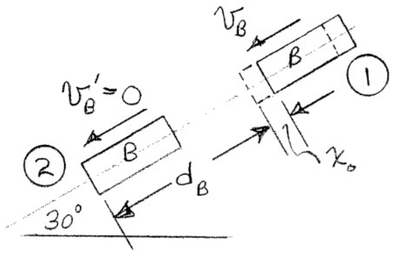
$$v_A = 1.4582\text{ m/s}, \ v_B = 1.6297\text{ m/s}$$

(*a*) *A* moves up the distance *d* where:

$$\frac{1}{2}\cancel{m_A}v_A^2 = \cancel{m_A}gd\sin 30^\circ; \quad \frac{1}{2}(1.4582\text{ m/s})^2 = (9.81\text{ m/s}^2)d(0.5)$$

$$\nearrow_{30^\circ}\ d_A = 0.21675\text{ m} = 217\text{ mm} \blacktriangleleft$$

(*b*) Static deflection = x_0, *B* moves down $\overset{d_B}{\swarrow}_{30^\circ}$



Conservation of energy (1) to (2)

Position (1) – spring deflected, x_0

$$kx_0 = m_B g\sin 30^\circ$$

$$T_1 + V_1 = T_2 + V_2: \quad T_1 = \frac{1}{2}m_B v_B^2, \ T_2 = 0$$

PROBLEM 13.197 (Continued)

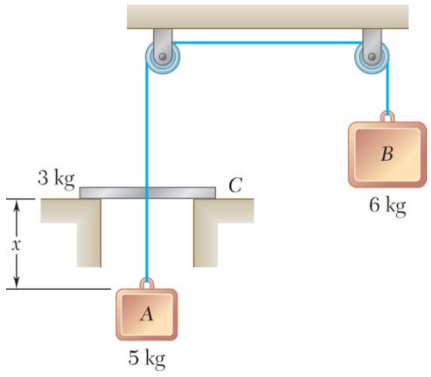
$$V_1 = V_e + V_g = \frac{1}{2}kx_0^2 + m_B g d_B \sin 30°$$

$$V_2 = V_e' + V_g' = \int_0^{x_0 + d_B} kx dx = \frac{1}{2}k\left(d_B^2 + 2d_B x_0 + x_0^2\right)$$

$$\frac{1}{2}kx_0^2 + mgd_B \sin 30° + \frac{1}{2}m_B v_B^2 = \frac{1}{2}k\left(d_B^2 + 2d_B x_0 + x_0^2\right) + 0 + 0$$

$$\therefore \quad kd_B^2 = m_B v_B^2; \qquad 500d_B^2 = 0.9(1.6297)^2 \qquad d_B = 0.0691 \text{ m}$$

$$d_B = 69.1 \text{ mm} \blacktriangleleft$$



PROBLEM 13.198

Blocks *A* and *B* are connected by a cord which passes over pulleys and through a collar *C*. The system is released from rest when $x = 1.7$ m. As block *A* rises, it strikes collar *C* with perfectly plastic impact ($e = 0$). After impact, the two blocks and the collar keep moving until they come to a stop and reverse their motion. As *A* and *C* move down, *C* hits the ledge and blocks *A* and *B* keep moving until they come to another stop. Determine (*a*) the velocity of the blocks and collar immediately after *A* hits *C*, (*b*) the distance the blocks and collar move after the impact before coming to a stop, (*c*) the value of *x* at the end of one compete cycle.

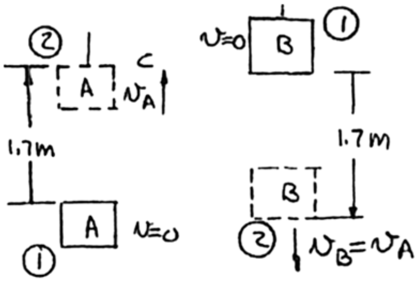
SOLUTION

(*a*) Velocity of *A* just before it hits *C*:

Conservations of energy:

Datum at ①:

Position ①:



$$(v_A)_1 = (v_B)_1 = 0$$
$$T_1 = 0$$
$$V_1 = 0$$

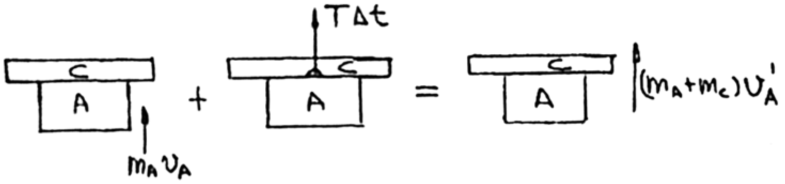
Position ②:

$$T_2 = \frac{1}{2}m_A(v_A)^2 + \frac{1}{2}m_B v_B^2$$
$$v_A = v_B \quad \text{(kinematics)}$$
$$T_2 = \frac{1}{2}(5+6)v_A^2 = \frac{11}{2}v_A^2$$
$$V_2 = m_A g(1.7) - m_B g(1.7)$$
$$= (5-6)(g)(1.7)$$
$$V_2 = -1.7g$$

$$T_1 + V_1 = T_2 + V_2$$
$$0 + 0 = \frac{11}{2}v_A^2 - 1.7g$$
$$v_A^2 = \left(\frac{3.4}{11}\right)(9.81)$$
$$= 3.032 \text{ m}^2/\text{s}^2$$
$$v_A = 1.741 \text{ m/s}$$

PROBLEM 13.198 (Continued)

Velocity of *A* and *C* after *A* hits *C*:


C
A
$m_A v_A$
+
$T\Delta t$
C
A
=
C
A
$(m_A+m_C)v_A'$

$v_A' = v_C'$ (plastic impact)

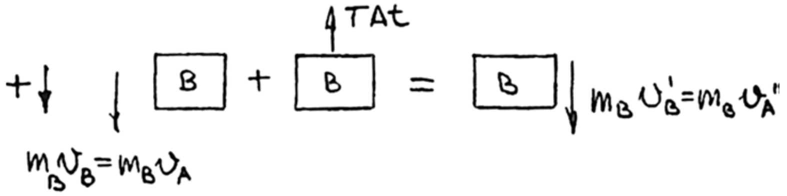
Impulse-momentum *A* and *C*:

$$+\uparrow \quad m_A v_A + T\Delta t = (m_A + m_C)v_A'$$

$$(5)(1.741) + T\Delta t = 8v_A' \tag{1}$$

$$v_B = v_A; \; v_B' = v_A' \text{ (cord remains taut)}$$

B alone:


$+\downarrow$
B
$m_B v_B = m_B v_A$
+
$T\Delta t$
B
=
B
$m_B v_B' = m_B v_A'$

$$m_B v_A - T\Delta t = m_B v_A'$$

$$(6)(1.741) - T\Delta t = 6v_A' \tag{2}$$

Adding Equations (1) and (2), $\quad 11(1.741) = 14v_A'$

$$v_A' = 1.3679 \text{ m/s}$$

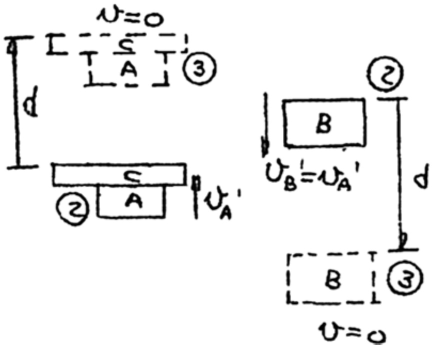
$$v_A' = v_B' = v_C' = 1.368 \text{ m/s} \blacktriangleleft$$

(*b*) Distance *A* and *C* move before stopping:

Conservations of energy:

Datum at ②:

Position ②:


v=0
C
A
③
d
C
②
A
v_A'
②
B
$v_B' = v_A'$
d
B
③
v=0

$$T_2 = \frac{1}{2}(m_A + m_B + m_C)(v_A')$$

$$T_2 = \left(\frac{14}{2}\right)(1.3681)^2$$

$$T_2 = 13.103 \text{ J}$$

$$V_2 = 0$$

PROBLEM 13.198 (Continued)

Position ③:

$T_3 = 0$

$$V_3 = (m_A + m_C)gd - m_B gd$$
$$V_3 = (8-6)gd = 2gd$$

$T_2 + V_2 = T_3 + V_3$

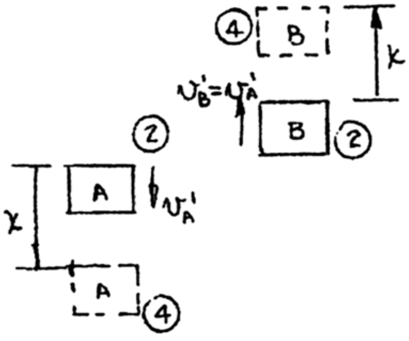
$$13.103 + 0 = 0 + 2gd$$

$$d = (13.103)/(2)(9.81) = 0.6679 \text{ m} \qquad d = 0.668 \text{ m} \blacktriangleleft$$

(*c*) As the system returns to position ② after stopping in position ③, energy is conserved, and the velocities of *A*, *B*, and *C* before the collar at *C* is removed are the same as they were in Part (*a*) above with the directions reversed. Thus, $v'_A = v'_C = v'_B = 1.3679$ m/s. After the collar *C* is removed, the velocities of *A* and *B* remain the same since there is no impulsive force acting on either.

Conversation of energy:

Datum at ②:



$$T_2 = \frac{1}{2}(m_A + m_B)(v'_A)^2$$
$$T_2 = \frac{1}{2}(5+6)(1.3679)^2$$
$$T_2 = 10.291 \text{ J}$$

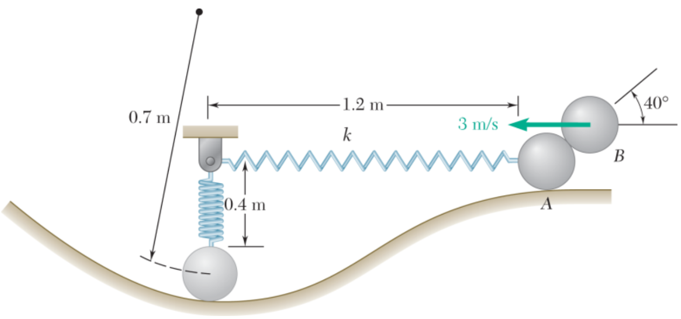
$V_2 = 0$

$$T_4 = 0 \quad V_4 = m_B gx - m_A gx$$
$$V_4 = (6-5)gx$$

$T_2 + V_2 = T_4 + V_4$

$$10.291 + 0 = (1)(9.81)x$$

$$x = 1.049 \text{ m} \blacktriangleleft$$



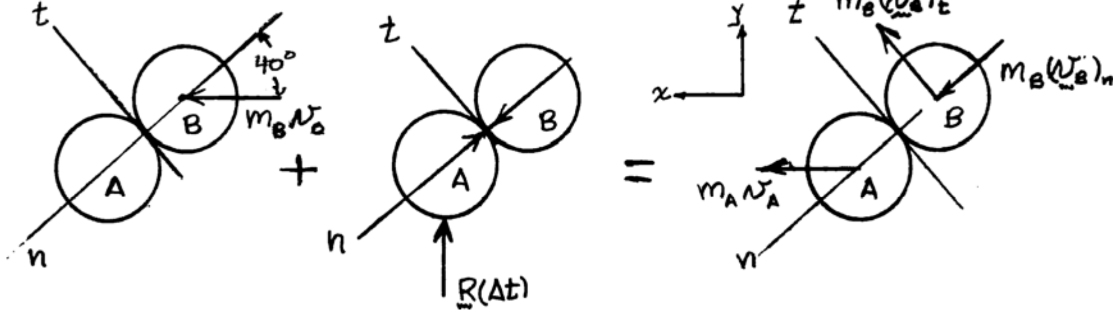
PROBLEM 13.199

A 2-kg ball *B* is traveling horizontally at 10 m/s when it strikes 2-kg ball *A*. Ball *A* is initially at rest and is attached to a spring with constant 100 N/m and an unstretched length of 1.2 m. Knowing the coefficient of restitution between *A* and *B* is 0.8 and friction between all surfaces is negligible, determine the normal force between *A* and the ground when it is at the bottom of the hill.

SOLUTION

Ball *B* impacts on ball *A*. Use the principle of impulse and momentum.

$$\Sigma m\mathbf{v}_1 + \Sigma \mathbf{Imp}_{1\to 2} = \Sigma m\mathbf{v}_2$$



Velocity components: $v_0 = 10$ m/s

$$(v_0)_x = v_0 \quad (v_0)_n = v_0 \cos 40° \quad (v_0)_t = v_0 \sin 40°$$

$$(v_A)_x = v_A \quad (v_A)_n = v_A \cos 40°$$

$$(v_B)_x = (v_B)_n \cos 40° + (v_B)_t \sin 40°$$

Impulse-momentum for ball *B* alone.

t-direction:

$$m_B(v_0)_t = m_B(v_B)_t$$

$$(v_B)_t = (v_0)_t = 10\sin 40° = 6.4279 \text{ m/s} \qquad (1)$$

Impulse-momentum for balls *A* and *B*.

x-direction ←

$$m_B v_0 + 0 = m_A v_A + m_B(v_B)_x + m_B(v_B)_t$$

$$(2)(10) + 0 = 2v_A + 2[(v_B)_n \cos 40° + 6.4279\sin 40°]$$

$$2v_A + 2(v_B)_n \cos 40° = 11.7365 \qquad (1)$$

Coefficient of restitution. $(e = 0.8)$

$$(v_B)_n = (v_A)_n = e[0 - (v_0)_n]$$

$$(v_B)_n - v_A \cos 40° = -(0.8)(10)\cos 40° \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously,

$$v_A = 6.6566 \text{ m/s} \qquad (v_B)_n = -1.0291 \text{ m/s}$$

As ball *A* moves from the impact location to the lowest point on the path, the spring compresses and the elevation decreases. Since friction is negligible, energy is conserved.

$$T_1 + V_1 = T_2 + V_2$$

$$\frac{1}{2}m_A v_A^2 + (V_e)_1 + (V_g)_1 = \frac{1}{2}m_A v_2^2 + (V_e)_2 + (V_g)_2$$

Position 1: (Just after impact.)

$$T_1 = \frac{1}{2}m_A v_A^2 = \frac{1}{2}(2)(6.6566)^2 = 44.3101 \text{ J}$$

$$(V_e)_1 = 0 \quad \text{(The spring is unstretched.)}$$

$$(V_g)_1 = 0 \quad \text{(Datum)}$$

Position 2: (Lowest point on path.)

$$T_2 = \frac{1}{2}m_A v_2^2 = \frac{1}{2}(2)v_2^2 = v_2^2$$

For the spring, $x_2 = l_2 - l_0 = 0.4 \text{ m} - 1.2 \text{ m} = 0.8 \text{ m}$

$$F_e = kx_2 = (100)(0.8) = 80 \text{ N}$$

$$(V_2)_e = \frac{1}{2}kx_2^2 = \frac{1}{2}(100)(0.8)^2 = 32 \text{ J}$$

Elevation above datum: $h_2 = -0.4 \text{ m}$

$$(V_2)_g = m_A g\, h_2 = (2)(9.81)(-0.4) = -7.848$$

Conservation of energy:

$$44.310 + 0 + 0 = v_2^2 + 32 - 7.848$$

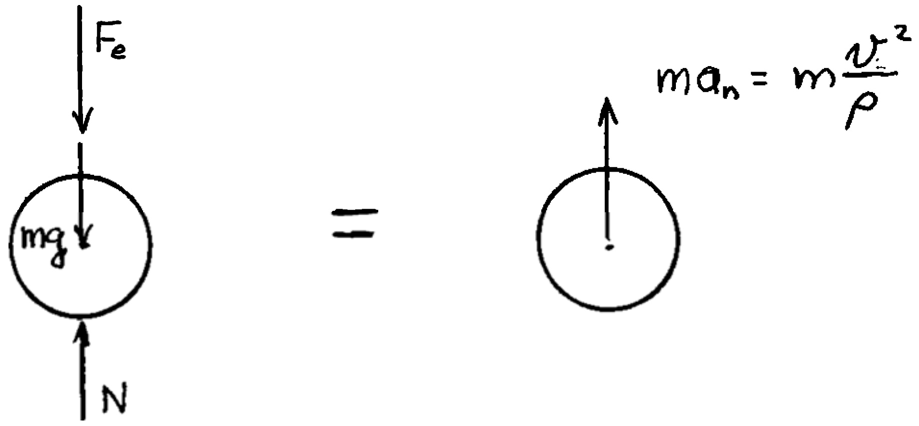
$$v_2^2 = 20.158 \text{ m}^2/\text{s}^2 \qquad v_2 = 4.489 \text{ m/s}$$

Normal acceleration at lowest point on path:

$$a_n = \frac{v_2^2}{\rho} = \frac{20.158}{0.7} = 28.798 \text{ m/s}^2 \qquad \mathbf{a}_n = 28.8 \text{ m/s}^2 \uparrow$$

PROBLEM 13.199 (Continued)

Apply Newton's second law to the ball.

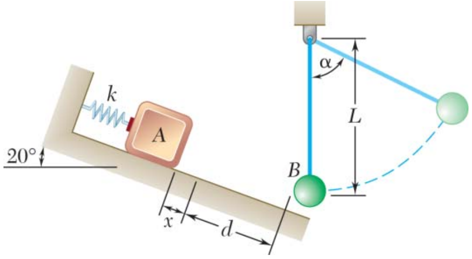

F_e
mg
N
=
$ma_n = m\frac{v^2}{\rho}$

$$+\uparrow \Sigma F = ma_n: N - mg - F_e = ma_n$$

$$N = mg + F_e + ma_n$$

$$= (2)(9.81) + 80 + (2)(28.798)$$

$N = 157.2$ N ◀



PROBLEM 13.200

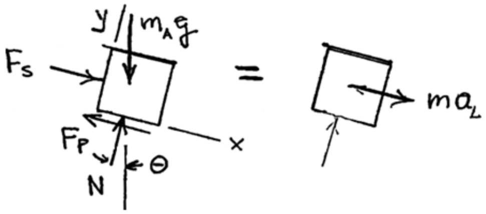
A 2-kg block A is pushed up against a spring compressing it a distance x. The block is then released from rest and slides down the 20° incline until it strikes a 1-kg sphere B which is suspended from a 1 m inextensible rope. The spring constant $k = 800$ N/m, the coefficient of friction between A and the ground is 0.2, the distance A slides from the unstretched length of the spring $d = 1.5$ m and the coefficient of restitution between A and B is 0.8. Knowing the tension in the rope is 20 N when $\alpha = 30°$, determine initial compression x of the spring.

SOLUTION

Data: $m_A = 2\text{ kg}, \quad m_B = 1\text{ kg}, \quad k = 800\text{ N/m}, \; d = 1.5\text{ m}, \; T = 20\text{ N}$

$\mu_k = 0.2, \quad e = 0.8, \quad \theta = 20°, \quad \alpha = 30°, \quad l = 1.0\text{ m}$

Block slides down the incline:



$$+\nearrow \; \Sigma F_y = 0$$

$$N - m_A g\cos\theta = 0$$

$$\begin{aligned} N &= m_A g\cos\theta \\ &= (2)(9.81)\cos 20° \\ &= 18.4368\text{ N} \\ F_f &= \mu_k N = (0.2)(18.4368) \\ &= 3.6874\text{ N} \end{aligned}$$

Use work and energy. Datum for V_g is the impact point near B.

$$T_1 = 0, \quad (V_1)_e = \frac{1}{2}kx^2 = \frac{1}{2}(800)x^2 = 400x^2\text{ J}$$

$$(V_1)_g = m_A g h_1 = m_A g(x+d)\sin\theta = (2)(9.81)(x+1.5)\sin 20° = 6.7104(x+1.5)\text{ J}$$

$$U_{1\to 2} = -F_f(x+d) = -(3.6874)(x+1.5)\text{ J}$$

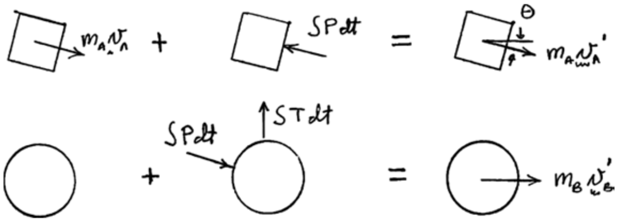
$$T_2 = \frac{1}{2}m_A v_A^2 = \frac{1}{2}(1)(v_A^2) = 1.000\,v_A^2 \qquad V_2 = 0$$

$$T_1 + V_1 + U_{1\to 2} = T_2 + V_2: \quad 0 + 400x^2 + 6.7104(x+1.5) - (3.6874)(x+1.5) = 1.000\,v_A^2 + 0$$

$$400x^2 + 3.0231x + 4.5346 = v_A^2 \qquad (1)$$

PROBLEM 13.200 (Continued)

Impact: Conservation of momentum.



Both A and B, horizontal components $\xrightarrow{+}$:

$$m_A v_A \cos\theta + 0 = m_A v_A' \cos\theta + m_B v_B'$$

$$2v_A \cos 20° = 2v_A' \cos 20° + v_B' \tag{2}$$

Relative velocities:

$$(v_B')_n - (v_A')_n = e[(v_A)_n - (v_B)_n]$$

$$v_B' \cos\theta - v_A' = e[v_A - 0]$$

$$v_B' \cos 20° - v_A' = (0.8)v_A \tag{3}$$

Solving Eqs. (2) and (3) simultaneously,

$$v_B' = \frac{3.6\cos 20°}{2\cos^2 20° + 1} v_A \tag{4}$$

$$= 1.2230 v_A$$

Sphere *B* rises: Use conservation of energy.

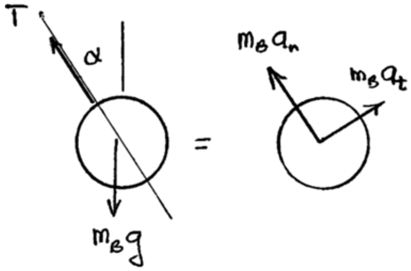
$$T_1 = \frac{1}{2} m_B (v_B')^2 \quad V_1 = 0$$

$$T_2 = \frac{1}{2} m_B v_2^2 \qquad V_2 = m_B g h_2 = m_B g l (1 - \cos\alpha)$$

$$T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2} m_B (v_B')^2 + 0 = \frac{1}{2} m_B v_2^2 + m_B g(1 - \cos)$$

$$v_2^2 = (1.2230 v_A)^2 - 2gl(1 - \cos\alpha)$$

Tension in the rope: $\rho = 1.00\text{ m}$



$$a_n = \frac{v_2^2}{\rho} = (1.2230 v_A)^2 - 2gl(1 - \cos\alpha)$$

$$+\nwarrow \Sigma F_n = m_B a_n:$$

$$T - m_B g \cos\alpha = m_B a_n$$

$$T = m_B (a_n + g\cos\alpha)$$

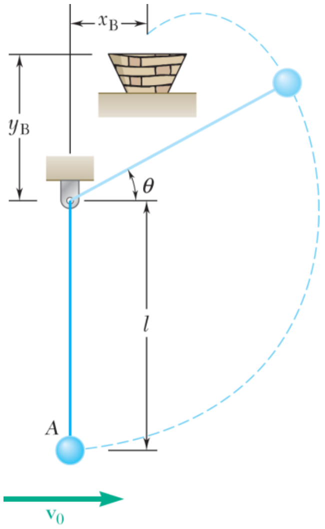
$$T = m_B ((1.2230 v_A)^2 - 2gl(1 - \cos\alpha) + g\cos\alpha)$$

$$20 = (1.0)((1.2230 v_A)^2 - 2(9.81)(1)(1 - \cos 30°) + 9.81\cos 30°) \tag{5}$$

Solving (5) $\qquad v_A = 3.0739\text{ m/s}$ $\qquad$ (6)

Sub (6) into (1) $\qquad 400x^2 + 3.0231x - 4.9141 = 0$

Solve for x: $\qquad x = 0.107\text{ m}$ ◀



PROBLEM 13.201*

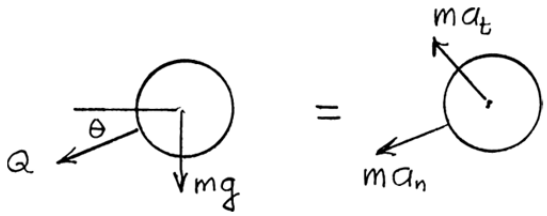
The 2-lb ball at A is suspended by an inextensible cord and given an initial horizontal velocity of $\mathbf{v}_0$. If $l = 2$ ft, $x_B = 0.3$ ft and $y_B = 0.4$ ft determine the initial velocity $\mathbf{v}$ so that the ball will enter in the basket. *Hint*: use a computer to solve the resulting set of equations.

SOLUTION

Let position 1 be at A. $\quad v_1 = v_0$

Let position 2 be the point described by the angle θ where the path of the ball changes from circular to parabolic. At position 2 the tension Q in the cord is zero.

Relationship between v_2 and θ based on $Q = 0$. Draw the free body diagram.



$$+\swarrow \Sigma F = 0: \quad Q + mg\sin\theta = ma_n = \frac{mv_2^2}{\ell}$$

With $Q = 0$,
$$v_2^2 = g\ell\sin\theta \quad \text{or} \quad v_2 = \sqrt{g\ell\sin\theta} \tag{1}$$

Relationship among $v_0, v_2,$ and θ based on conservation of energy.

$$T_1 + V_1 = T_2 + V_2$$

$$\frac{1}{2}mv_0^2 - mg\ell = \frac{1}{2}mv_2^2 + mg\ell\sin\theta$$

$$v_0^2 = v_2^2 + 2g\ell(1 + \sin\theta) \tag{2}$$

PROBLEM 13.201* (Continued)

x and *y* coordinates at position 2:

$$x_2 = \ell \cos\theta \tag{3}$$

$$y_2 = \ell \sin\theta \tag{4}$$

Let t_2 be the time when the ball is in position 2.

Motion on the parabolic path. The horizontal motion is

$$\dot{x} = -v_2 \sin\theta$$

$$x = x_2 - (v_2 \sin\theta)(t - t_2) \tag{5}$$

At Point *B*, $x = x_B$ and $t = t_B$. From Eq. (5),

$$(t_B - t_2) = \frac{\ell \cos\theta - x_B}{v_\theta \sin\theta} \tag{6}$$

Vertical motion: $\dot{y} = v_2 \cos\theta - g(t - t_2)$

$$y = y_2 + (v_2 \cos\theta)(t - t_2) - \frac{1}{2} g(t - t_2)^2$$

At Point *B*,

$$y_B = \ell \sin\theta + (v_2 \cos\theta)(t_B - t_2) - \frac{1}{2} g(t_B - t_2)^2 \tag{7}$$

Data: $\ell = 2 \text{ ft}, \quad x_B = 0.3 \text{ ft}, \quad y_B = 0.4 \text{ ft}, \quad g = 32.2 \text{ ft/s}^2$

With the numerical data,

Eq. (1) becomes $$v_2 = \sqrt{64.4 \sin\theta} \tag{1$'$}$$

Eq. (6) becomes $$t_B - t_2 = \frac{2\cos\theta - 0.3}{v_2 \sin\theta} \tag{6$'$}$$

Eq. (7) becomes $$y_B = 2\sin\theta + (v_2 \cos\theta)(t_B - t_2) - 16.1(t_B - t_2)^2 \tag{7$'$}$$

Method of solution. From a trial value of θ, calculate v_2 from Eq. (1)′, $t_B - t_2$ from Eq. (6)′, and y_B from Eq. (7)′. Repeat until $y_B = 0.4$ ft as required.

Try $\theta = 30°$. $$v_2 = \sqrt{64.4 \sin 30°} = 5.6745 \text{ ft/s}$$

$$t_B - t_2 = \frac{2\cos 30° - 0.3}{5.6745 \sin 30°} = 0.50473 \text{s}$$

$$y_B = 2\sin 30° + (5.6745 \cos 30°)(0.50473) - (16.1)(0.50473)^2$$
$$= -0.62116 \text{ ft}$$

PROBLEM 13.201* (Continued)

Try $\theta = 45°$.

$$v_2 = \sqrt{64.4 \sin 45°} = 6.7482$$

$$t_B - t_2 = \frac{2\cos 45° - 0.3}{6.7482 \sin 45°} = 0.23351 \text{ s}$$

$$y_B = 2\sin 45° + (6.7482 \cos 45°)(0.23351) - (16.1)(0.23351)^2$$
$$= 1.65060 \text{ ft}$$

Try $\theta = 37.5°$.

$$v_2 = \sqrt{64.4 \sin 37.5°} = 6.2613 \text{ ft/s}$$

$$t_B - t_2 = \frac{2\cos 37.5° - 0.3}{6.2613 \sin 37.5°} = 0.33757 \text{ s}$$

$$y_B = 2\sin 37.5° + (6.2613 \cos 37.5°)(0.33757) - (16.1)(0.33757)^2$$
$$= 1.05972 \text{ ft}$$

Let $u = \theta - 30°$. The following sets of data points have be determined:

$$(u, y_B) = (0°, -0.62114 \text{ ft}), (7.5°, 1.05972 \text{ ft}), (15°, 1.65060 \text{ ft})$$

The quadratic curve fit of this data gives

$$y_B = -0.62114 + 0.29678\,u - 0.009688711\,u^2$$

Setting $y_B = 0.4$ ft gives the quadratic equation

$$-0.009688711\,u^2 + 0.29678\,u - 1.02114 = 0$$

Solving for u, $u = 3.95°$ and $26.68°$

Rejecting the second value gives $\theta = 30° + u = 33.95°$.

Try $\theta = 33.95°$.

$$v_2 = \sqrt{64.4 \sin 33.95°} = 5.997 \text{ ft/s}$$

$$t_B - t_2 = \frac{2\cos 33.95° - 0.3}{5.9971 \sin 33.95°} = 0.40578 \text{ s}$$

$$y_B = 2\sin 33.95° + (5.997 \cos 33.95°)(0.40578) - (16.1)(0.40578)^2$$
$$= 0.48462 \text{ ft}$$

The new quadratic curve-fit is based on the data points

$$(u, y_B) = (0°, -0.62114 \text{ ft}), (3.95°, 0.48462 \text{ ft}), (7.5°, 1.05972 \text{ ft}).$$

The quadratic curve fit of this data is

$$y_B = -0.62114 + 0.342053907\,u - 0.015725232\,u^2$$

Setting $y_B = 0.4$ ft gives

$$-0.015725232\,u^2 + 0.342053907\,u - 1.02114 = 0$$

PROBLEM 13.201* (Continued)

Solving for u,

$$u = 3.572° \qquad \theta = 30° + 3.572° = 33.572°$$

Try $\theta = 33.572°$. $\quad v_2 = \sqrt{64.4 \sin 33.572°} = 5.9676 \text{ ft/s}$

$$t_B - t_2 = \frac{2 \cos 33.572° - 0.3}{5.9676 \sin 33.572°} = 0.41406 \text{ s}$$

$$y_B = 2 \sin 33.572° + (5.9676 \cos 33.572°)(0.41406) - (16.1)(0.41406)^2$$
$$= 0.40445 \text{ ft}$$

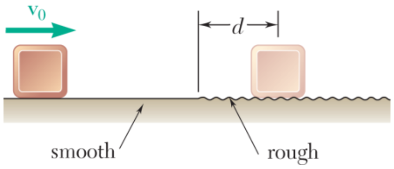
which is close enough to 0.4 ft.

Substituting $\theta = 33.572°$ and $v_2 = 5.9676$ ft/s into Eq. (2) along with other data gives

$$v_0^2 = (5.9676)^2 + (2)(32.2)(2)(1 + \sin 33.572°) = 235.64 \text{ ft}^2/\text{s}^2$$

$$\mathbf{v}_0 = 15.35 \text{ ft/s} \longrightarrow \blacktriangleleft$$

PROBLEM 13.CQ1



Block A is traveling with a speed v_0 on a smooth surface when the surface suddenly becomes rough with a coefficient of friction of μ causing the block to stop after a distance d. If block A were traveling twice as fast, that is, at a speed $2v_0$, how far will it travel on the rough surface before stopping?

(*a*) $d/2$

(*b*) d

(*c*) $\sqrt{2}d$

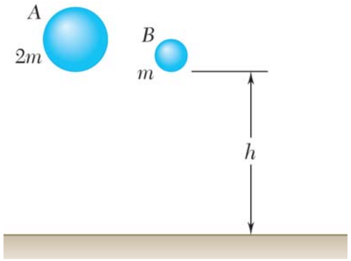
(*d*) $2d$

(*e*) $4d$

SOLUTION

If the block were traveling twice as fast, its kinetic energy would be 4x as much ($T = \frac{1}{2}mv^2$). The work done by friction to stop the block is $U = Fd$. This work must equal the amount of kinetic energy before the block hits the rough patch. Since the kinetic energy is 4x as much it will take 4x the distance to stop.

Answer: (*e*)



PROBLEM 13.CQ2

Two small balls *A* and *B* with masses 2*m* and *m* respectively are released from rest at a height *h* above the ground. Neglecting air resistance, which of the following statements are true when the two balls hit the ground?

(*a*) The kinetic energy of *A* is the same as the kinetic energy of *B*.

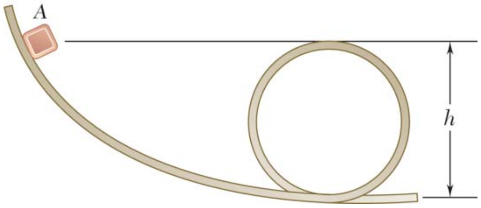
(*b*) The kinetic energy of *A* is half the kinetic energy of *B*.

(*c*) The kinetic energy of *A* is twice the kinetic energy of *B*.

(*d*) The kinetic energy of *A* is four times the kinetic energy of *B*.

SOLUTION

Since each ball accelerates due to gravity at the same rate, they have the same speed when they hit the ground. Since kinetic energy, $T = \frac{1}{2}mv^2$, and ball *A* has twice the mass as ball *B*, then A will have twice the kinetic energy. Answer: (*c*)



PROBLEM 13.CQ3

Block *A* is released from rest and slides down the frictionless ramp to the loop. The maximum height *h* of the loop is the same as the initial height of the block. Will *A* make it completely around the loop without losing contact with the track?

(*a*) Yes

(*b*) No

(*c*) need more information

SOLUTION

Answer: (*b*) In order for *A* to not maintain contact with the track, the normal force must remain greater than zero, which requires a non-zero speed at the top of the loop.

PROBLEM 13.CQ4

A large insect impacts the front windshield of a sports car traveling down a road. Which of the following statements is true during the collision?

(*a*) The car exerts a greater force on the insect than the insect exerts on the car.

(*b*) The insect exerts a greater force on the car than the car exerts on the insect.

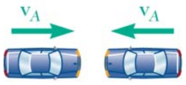
(*c*) The car exerts a force on the insect, but the insect does not exert a force on the car.

(*d*) The car exerts the same force on the insect as the insect exerts on the car.

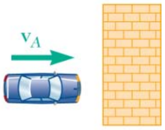
(*e*) Neither exerts a force on the other; the insect gets smashed simply because it gets in the way of the car.

SOLUTION

Newton’s third law states that when a body exerts a force on a second body, the second body exerts a force equal in magnitude and opposite in direction on the first body. Therefore the answer is: (*d*)



Case 1



Case 2

PROBLEM 13.CQ5

The expected damages associated with two types of perfectly plastic collisions are to be compared. In the first case, two identical cars traveling at the same speed impact each other head on. In the second case, the car impacts a massive concrete wall. In which case would you expect the car to be more damaged?

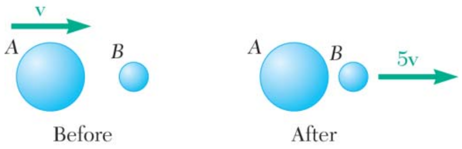
(*a*) Case 1

(*b*) Case 2

(*c*) The same damage in each case

SOLUTION

In both cases the car will come to a complete stop, so the applied impulse will be the same. With the same impulse, the same damage should occur. Answer: (*c*)



PROBLEM 13.CQ6

A 5 kg ball *A* strikes a 1 kg ball *B* that is initially at rest. Is it possible that after the impact *A* is not moving and *B* has a speed of 5*v*?

(*a*) Yes

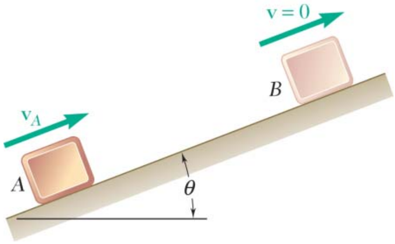
(*b*) No

Explain your answer.

SOLUTION

Answer: (*b*) No.

Conservation of momentum is satisfied, but the coefficient of restitution equation is not. The coefficient of restitution must be less than 1.

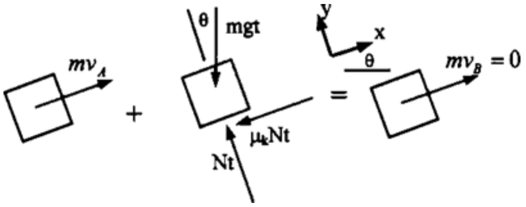


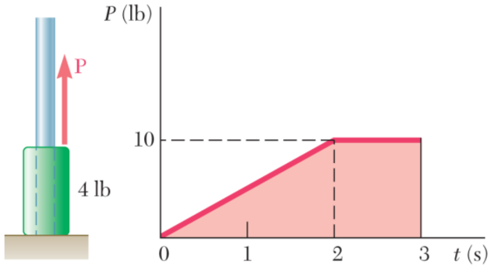
PROBLEM 13.F1

The initial velocity of the block in position *A* is 30 ft/s. The coefficient of kinetic friction between the block and the plane is $\mu_k = 0.30$. Draw impulse-momentum diagrams that could be used to determine the time it takes for the block to reach *B* with zero velocity, if $\theta = 20°$.

SOLUTION

Answer:



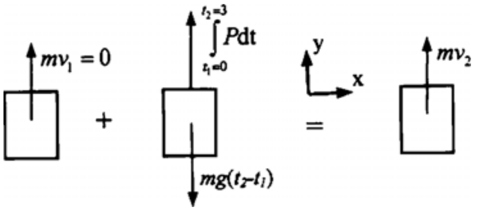

P (lb)
P
10
4 lb
0
1
2
3
t (s)

PROBLEM 13.F2

A 4-lb collar which can slide on a frictionless vertical rod is acted upon by a force **P** which varies in magnitude as shown. Knowing that the collar is initially at rest, draw impulse-momentum diagrams that could be used to determine its velocity at $t = 3$ s.

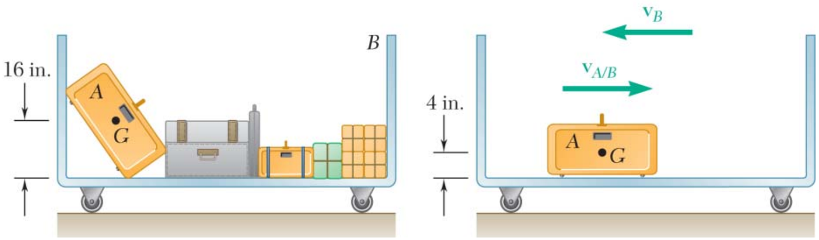
SOLUTION

Answer:


$mv_1 = 0$
+
$\int_{t_1=0}^{t_2=3} P\,dt$
$mg(t_2-t_1)$
y
x
=
mv_2

Where $\int_{t_1=0}^{t_2=3} P\,dt$ is the area under the curve.

PROBLEM 13.F3

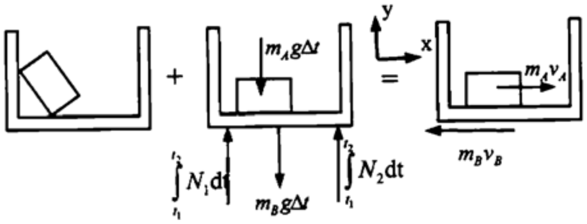

16 in.
A
G
B
4 in.
v_B
$v_{A/B}$
A
G

The 15-kg suitcase *A* has been propped up against one end of a 40-kg luggage carrier *B* and is prevented from sliding down by other luggage. When the luggage is unloaded and the last heavy trunk is removed from the carrier, the suitcase is free to slide down, causing the 40-kg carrier to move to the left with a velocity v_B of magnitude 0.8 m/s. Neglecting friction, draw impulse-momentum diagrams that could be used to determine (*a*) the velocity of *A* as it rolls on the carrier and (*b*) the velocity of the carrier after the suitcase hits the right side of the carrier without bouncing back.

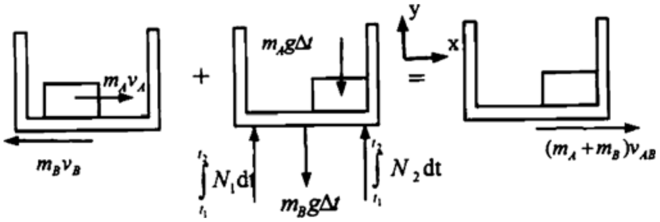
SOLUTION

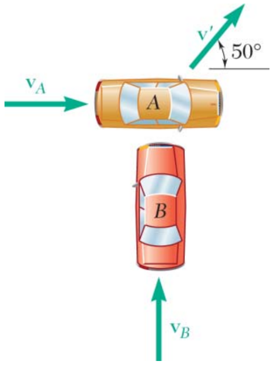
Answer:

(*a*)


+
$m_A g\Delta t$
y
x
=
$m_A v_A$
$m_B v_B$
$\int_{t_1}^{t_2} N_1 dt$
$m_B g\Delta t$
$\int_{t_1}^{t_2} N_2 dt$

(*b*)


$m_A v_A$
$m_B v_B$
+
$m_A g\Delta t$
y
x
=
$\int_{t_1}^{t_2} N_1 dt$
$m_B g\Delta t$
$\int_{t_1}^{t_2} N_2 dt$
$(m_A + m_B)v_{AB}$

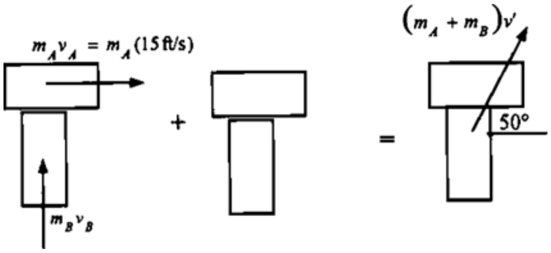


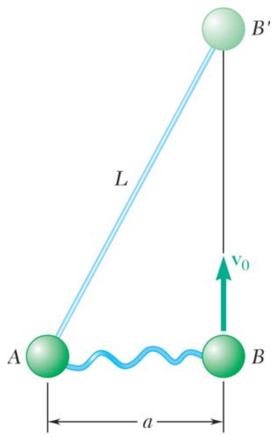
PROBLEM 13.F4

Car *A* was traveling west at a speed of 15 m/s and car *B* was traveling north at an unknown speed when they slammed into each other at an intersection. Upon investigation it was found that after the crash the two cars got stuck and skidded off at an angle of 50° north of east. Knowing the masses of *A* and *B* are m_A and m_B respectively, draw impulse-momentum diagrams that could be used to determine the velocity of *B* before impact.

SOLUTION

Answer:



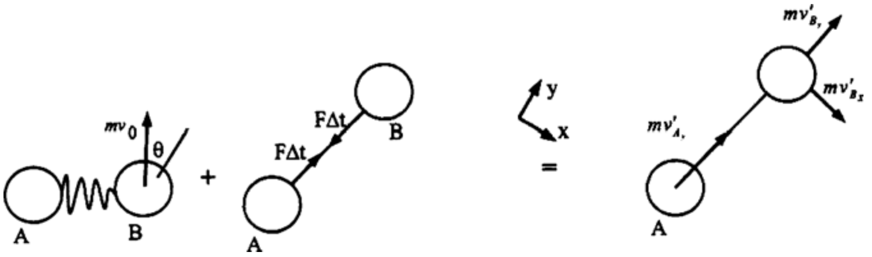


PROBLEM 13.F5

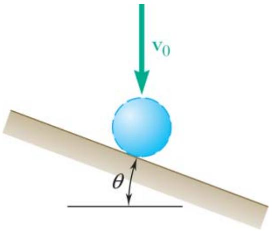
Two identical spheres *A* and *B*, each of mass *m*, are attached to an inextensible inelastic cord of length *L* and are resting at a distance *a* from each other on a frictionless horizontal surface. Sphere *B* is given a velocity $\mathbf{v}_0$ in a direction perpendicular to line *AB* and moves it without friction until it reaches *B′* where the cord becomes taut. Draw impulse-momentum diagrams that could be used to determine the magnitude of the velocity of each sphere immediately after the cord has become taut.

SOLUTION

Answer:



Where $v'_{A_y} = v'_{B_y}$ since the cord is inextensible.

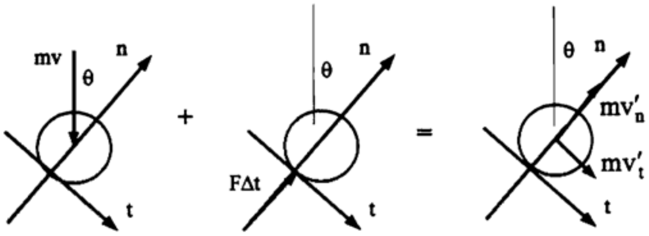


PROBLEM 13.F6

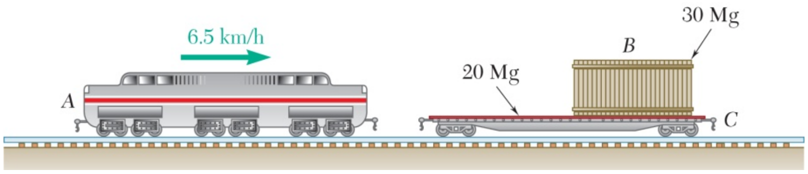
A sphere with a speed v_0 rebounds after striking a frictionless inclined plane as shown. Draw impulse-momentum diagrams that could be used to find the velocity of the sphere after the impact.

SOLUTION

Answer:



mv
θ
n
t
+
θ
n
FΔt
t
=
θ
n
mv'_n
mv'_t
t


6.5 km/h
A
B
30 Mg
20 Mg
C

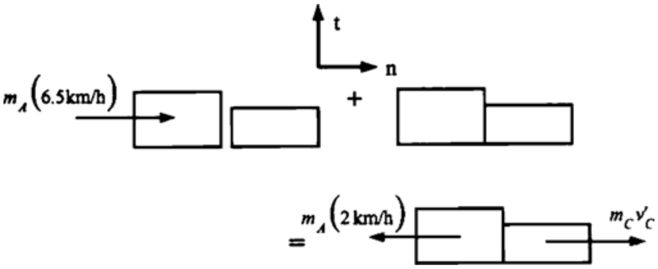
PROBLEM 13.F7

An 80-Mg railroad engine *A* coasting at 6.5 km/h strikes a 20-Mg flatcar *C* carrying a 30-Mg load *B* which can slide along the floor of the car ($\mu_k = 0.25$). The flatcar was at rest with its brakes released. Instead of *A* and *C* coupling as expected, it is observed that *A* rebounds with a speed of 2 km/h after the impact. Draw impulse-momentum diagrams that could be used to determine (*a*) the coefficient of restitution and the speed of the flatcar immediately after impact, and (*b*) the time it takes the load to slide to a stop relative to the car.

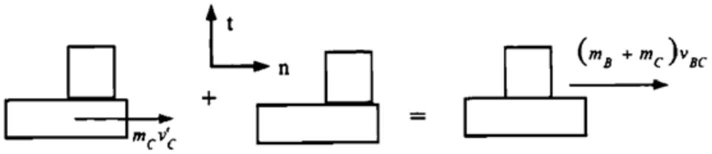
SOLUTION

Answer:

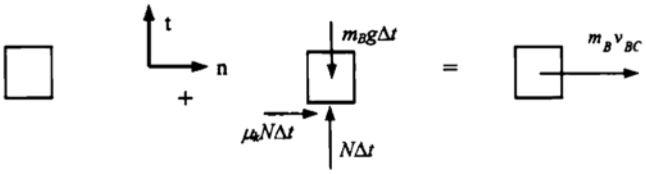
(*a*) Look at *A* and *C* (the friction force between *B* and *C* is not impulsive) to find the velocity after impact.

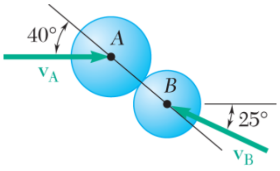

t
n
$m_A(6.5\text{ km/h})$
+
=
$m_A(2\text{ km/h})$
$m_C v'_C$

(*b*) Consider just *B* and *C* to find their final velocity.


t
n
$m_C v'_C$
+
=
$(m_B + m_C)v_{BC}$

Consider just *B* to find the time.


t
n
+
$m_B g\Delta t$
$\mu_k N\Delta t$
$N\Delta t$
=
$m_B v_{BC}$

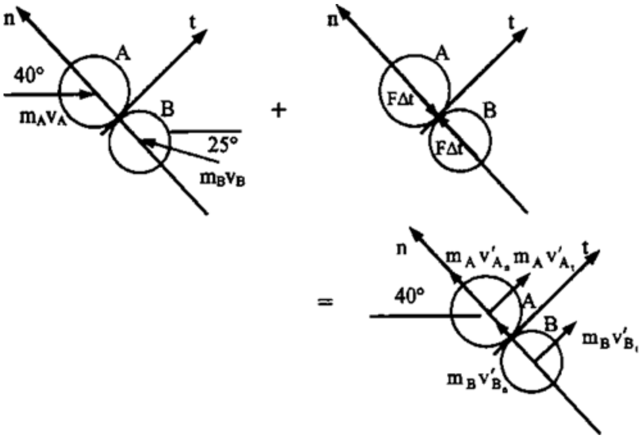

40°
A
v_A
B
25°
v_B

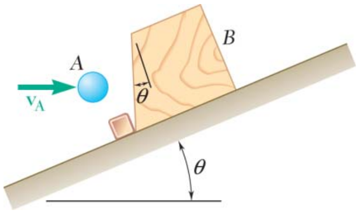
PROBLEM 13.F8

Two frictionless balls strike each other as shown. The coefficient of restitution between the balls is *e*. Draw the impulse-momentum diagrams that could be used to find the velocities of *A* and *B* after the impact.

SOLUTION

Answer:


n
t
A
40°
$m_A v_A$
B
25°
$m_B v_B$
+
n
t
A
$F\Delta t$
B
$F\Delta t$
=
n
$m_A v'_{A_n}$
$m_A v'_{A_t}$
t
40°
A
B
$m_B v'_{B_t}$
$m_B v'_{B_n}$

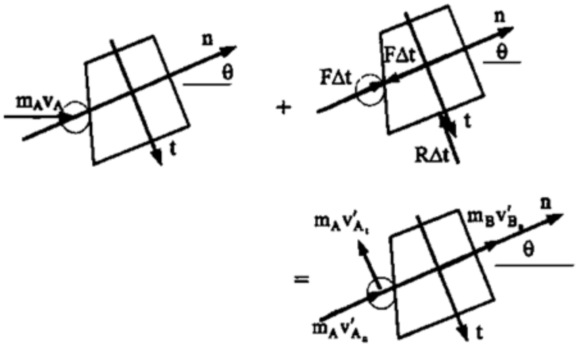


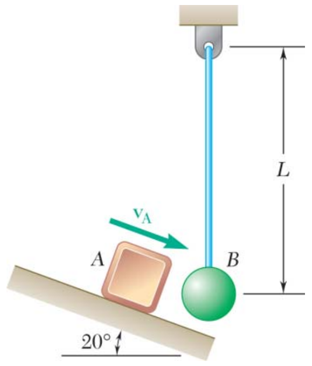
PROBLEM 13.F9

A 10-kg ball *A* moving horizontally at 12 m/s strikes a 10-kg block *B*. The coefficient of restitution of the impact is 0.4 and the coefficient of kinetic friction between the block and the inclined surface is 0.5. Draw impulse-momentum diagrams that could be used to determine the speeds of *A* and *B* after the impact.

SOLUTION

Answer:


$m_A v_A$
n
θ
t
+
FΔt
FΔt
n
θ
t
RΔt
=
$m_A v'_{A_t}$
$m_B v'_{B_n}$
n
θ
$m_A v'_{A_n}$
t



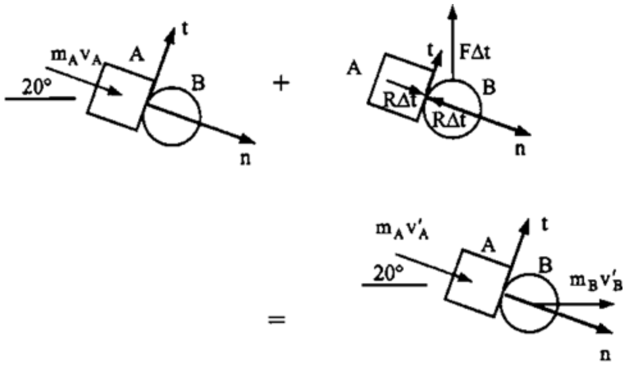
L
$\mathbf{v}_A$
A
B
20°

PROBLEM 13.F10

Block A of mass m_A strikes ball B of mass m_B with a speed of v_A as shown. Draw impulse-momentum diagrams that could be used to determine the speeds of A and B after the impact and the impulse during the impact.

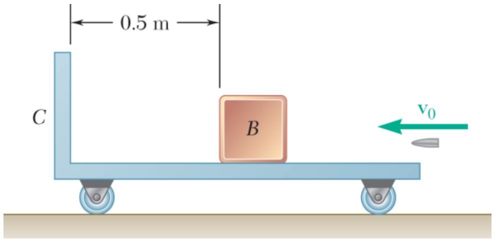
SOLUTION

Answer:



$m_A v_A$
20°
A
B
t
n
+
A
t
FΔt
B
RΔt
RΔt
n
=
$m_A v'_A$
20°
A
t
B
$m_B v'_B$
n

CHAPTER 14



PROBLEM 14.1

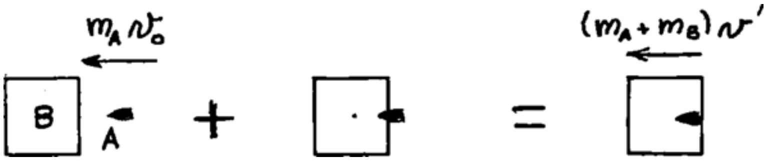
A 30-g bullet is fired with a horizontal velocity of 450 m/s and becomes embedded in block *B* which has a mass of 3 kg. After the impact, block *B* slides on 30-kg carrier *C* until it impacts the end of the carrier. Knowing the impact between *B* and *C* is perfectly plastic and the coefficient of kinetic friction between *B* and *C* is 0.2, determine (*a*) the velocity of the bullet and *B* after the first impact, (*b*) the final velocity of the carrier.

SOLUTION

For convenience, label the bullet as particle *A* of the system of three particles *A*, *B*, and *C*.

(*a*) *Impact between A and B*: Use conservation of linear momentum of *A* and *B*. Assume that the time period is so short that any impulse due to the friction force between *B* and *C* may be neglected.

$$\Sigma m\bar{\mathbf{v}}_1 + \Sigma \mathbf{Imp}_{1\rightarrow 2} = \Sigma m\bar{\mathbf{v}}_2$$



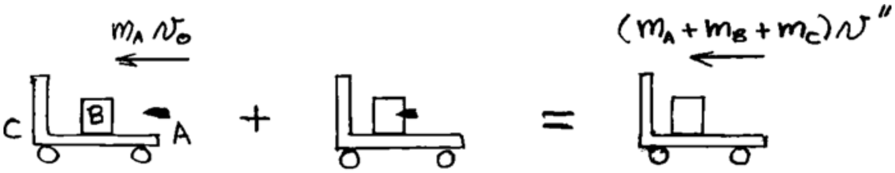
Components $\longleftarrow$: $\quad m_A v_0 + 0 = (m_A + m_B)v'$

$$v' = \frac{m_A v_0}{m_A + m_B} = \frac{(30\times 10^{-3}\text{ kg})(450\text{ m/s})}{(30\times 10^{-3}\text{ kg} + 3\text{ kg}} = 4.4554\text{ m/s}$$

$\mathbf{v}' = 4.46\text{ m/s} \longleftarrow$ ◀

(*b*) *Final velocity of the carrier*: Particles *A*, *B*, and *C* have the same velocity v'' to the left. Use conservation of linear momentum of all three particles. The friction forces between *B* and *C* are internal forces. Neglect friction at the wheels of the carrier.

$$\Sigma m\bar{\mathbf{v}}_2 + \Sigma \mathbf{Imp}_{2\rightarrow 3} = \Sigma m\bar{\mathbf{v}}_3$$

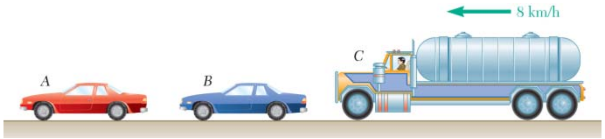


Components $\longleftarrow$: $\quad (m_A + m_B)v' + 0 = (m_A + m_B + m_C)v$

$$v'' = \frac{(m_A + m_B)v'}{m_A + m_B + m_C} = \frac{m_A v_0}{m_A + m_B + m_C}$$

$$= \frac{(30\times 10^{-3}\text{ kg})(450\text{ m/s})}{30\times 10^{-3}\text{ kg} + 3\text{ kg} + 30\text{ kg}} = 0.4087\text{ m/s}$$

$\mathbf{v}'' = 0.409\text{ m/s} \longleftarrow$ ◀



PROBLEM 14.2

Two identical 1350-kg automobiles *A* and *B* are at rest with their brakes released when *B* is struck by a 5400-kg truck *C* which is moving to the left at 8 km/h. A second collision then occurs when *B* strikes *A*. Assuming the first collision is perfectly plastic and the second collision is perfectly elastic, determine the velocities of the three vehicles just after the second collision.

SOLUTION

Given: $m_A = m_B = 1350$ kg and $m_C = 5400$ kg.

Let v_A, v_B, and v_C be the sought after final velocities, positive to the left.

Initial velocities: $(v_A)_0 = (v_B)_0 = 0, \; (v_C)_0 = 8 \text{ km/h} = 2.2222 \text{ m/s}$

First collision. Truck C strikes car B. Plastic impact: $e = 0$

Let $(v_{BC})_0$ be the common velocity of B and C after impact.

Conservation of momentum for B and C:

$$(m_B + m_C)v_{BC} = m_B(v_B)_0 + m_C(v_C)_0$$

$$6750\, v_{BC} = 0 + (5400)(2.2222) \qquad v_{BC} = 1.77778 \text{ m/s}$$

Second collision. Car-truck BC strikes car A.

Elastic impact. $e = 1$

$$v_A - v_{BC} = -e\left[(v_A)_0 - (v_{BC})_0\right] = -1.77778 \text{ m/s} \qquad (1)$$

Conservation of momentum for A, B, and C.

$$m_A v_A + (m_B + m_C)(v_{BC}) = m_A(v_A)_0 + (m_B + m_C)(v_{BC})_0$$

$$1350\, v_A + 6750\, v_{BC} = 0 + (6750)(1.77778) \qquad (2)$$

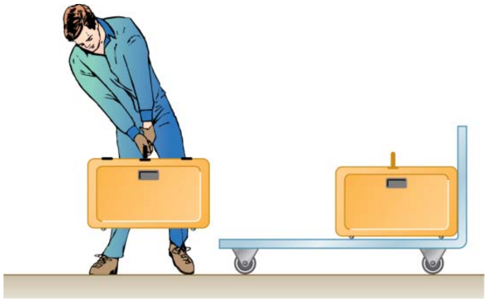
Solving (1) and (2) simultaneously for v_A and v_{BC},

$$v_A = 2.9630 \text{ m/s}, \quad v_{BC} = 1.18519 \text{ m/s} = v_B = v_C$$

$\mathbf{v}_A = 10.67$ km/h ← ◀

$\mathbf{v}_B = 4.27$ km/h ← ◀

$\mathbf{v}_C = 4.27$ km/h ← ◀



PROBLEM 14.3

An airline employee tosses two suitcases, of weight 30 lb and 40 lb, respectively, onto a 50-lb baggage carrier in rapid succession. Knowing that the carrier is initially at rest and that the employee imparts a 9-ft/s horizontal velocity to the 30-lb suitcase and a 6-ft/s horizontal velocity to the 40-lb suitcase, determine the final velocity of the baggage carrier if the first suitcase tossed onto the carrier is (*a*) the 30-lb suitcase, (*b*) the 40-lb suitcase.

SOLUTION

The weights are $W_A = 30$ lb, $W_B = 40$ lb, and $W_C = 50$ lb.

Initial velocities: $(v_A)_0 = 9 \text{ ft/s} \longrightarrow,\quad (v_B)_0 = 6 \text{ ft/s} \longrightarrow,\quad$ and $\quad (v_C)_0 = 0.$

There are no horizontal external forces acting during the impacts, and the baggage carrier is free to coast between the impacts.

(*a*) *Suitcase A is thrown first.*

Let v_1 be the common velocity of suitcase *A* and the carrier after the first impact and v_2 be the common velocity of the two suitcases and the carrier after the second impact.

Initial momenta: $\dfrac{W_A}{g}(v_A)_0,\quad \dfrac{W_B}{g}(v_B)_0,\quad$ and $\quad 0.$

Suitcase *A* impacts carrier. Conservation of momentum:

$$\frac{W_A}{g}(v_A)_0 + 0 = \frac{W_A + W_C}{g}v_1 \qquad v_1 = \frac{W_A(v_A)_0}{W_A + W_C} = \frac{(30)(9)}{80} = 3.375 \text{ ft/s}$$

Suitcase *B* impacts on suitcase *A* and carrier. Conservation of momentum:

$$\frac{W_B}{g}(v_B)_0 + \frac{W_A + W_C}{g}v_1 = \frac{W_A + W_B + W_C}{g}v_2$$

$$v_2 = \frac{W_B(v_B)_0 + (W_A + W_C)v_1}{W_A + W_B + W_C} = \frac{(40)(6) + (80)(3.375)}{120} = 4.25 \text{ ft/s} \qquad \mathbf{v}_2 = 4.25 \text{ ft/s} \longrightarrow \blacktriangleleft$$

(*b*) *Suitcase B is thrown first.*

Let v_3 be the common velocity of suitcase *B* and the carrier after the first impact and v_4 be the common velocity of all after the second impact.

Suitcase *B* impacts the carrier. Conservation of momentum:

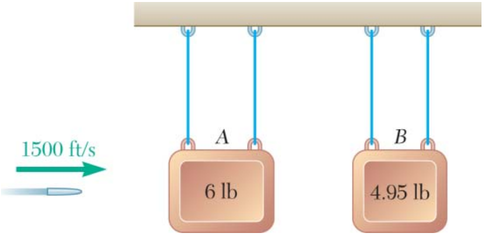
$$\frac{W_B}{g}(v_B)_0 + 0 = \frac{W_B + W_C}{g}v_3 \qquad v_3 = \frac{W_B(v_B)_0}{W_B + W_C} = \frac{(40)(6)}{90} = 2.6667 \text{ ft/s}$$

PROBLEM 14.3 (Continued)

Suitcase *A* impacts on suitcase *B* and carrier. Conservation of momentum:

$$\frac{W_A}{g}(v_A)_0 + \frac{W_B + W_C}{g} v_3 = \frac{W_A + W_B + W_C}{g} v_4$$

$$v_4 = \frac{W_A(v_A)_0 + (W_B + W_C)v_3}{W_A + W_B + W_C} = \frac{(30)(9) + (90)(2.6667)}{120} = 4.25 \text{ ft/s} \qquad \mathbf{v}_4 = 4.25 \text{ ft/s} \longrightarrow \blacktriangleleft$$



PROBLEM 14.4

A bullet is fired with a horizontal velocity of 1500 ft/s through a 6-lb block *A* and becomes embedded in a 4.95-lb block *B*. Knowing that blocks *A* and *B* start moving with velocities of 5 ft/s and 9 ft/s, respectively, determine (*a*) the weight of the bullet, (*b*) its velocity as it travels from block *A* to block *B*.

SOLUTION

The masses are m for the bullet and m_A and m_B for the blocks.

(*a*) The bullet passes through block *A* and embeds in block *B*. Momentum is conserved.

Initial momentum: $mv_0 + m_A(0) + m_B(0) = mv_0$

Final momentum: $mv_B + m_A v_A + m_B v_B$

Equating, $mv_0 = mv_B + m_A v_A + m_B v_B$

$$m = \frac{m_A v_A + m_B v_B}{v_0 - v_B} = \frac{(6)(5) + (4.95)(9)}{1500 - 9} = 0.0500 \text{ lb}$$

$m = 0.800$ oz ◀

(*b*) The bullet passes through block *A*. Momentum is conserved.

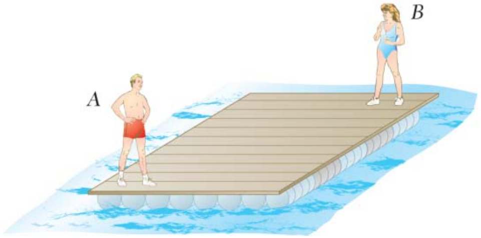
Initial momentum: $mv_0 + m_A(0) = mv_0$

Final momentum: $mv_1 + m_A v_A$

Equating, $mv_0 = mv_1 + m_A v_A$

$$v_1 = \frac{mv_0 - m_A v_A}{m} = \frac{(0.0500)(1500) - (6)(5)}{0.0500} = 900 \text{ ft/s}$$

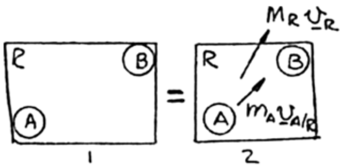
$\mathbf{v}_1 = 900 \text{ ft/s} \longrightarrow$ ◀



PROBLEM 14.5

Two swimmers *A* and *B*, of weight 190 lb and 125 lb, respectively, are at diagonally opposite corners of a floating raft when they realize that the raft has broken away from its anchor. Swimmer *A* immediately starts walking toward *B* at a speed of 2 ft/s relative to the raft. Knowing that the raft weighs 300 lb, determine (*a*) the speed of the raft if *B* does not move, (*b*) the speed with which *B* must walk toward *A* if the raft is not to move.

SOLUTION



(*a*) The system consists of *A* and *B* and the raft *R*.

Momentum is conserved.

$$(\Sigma m\mathbf{v})_1 = (\Sigma m\mathbf{v})_2$$

$$0 = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_R\mathbf{v}_R \qquad (1)$$

$$\mathbf{v}_A = \mathbf{v}_{A/R} + \mathbf{v}_R \qquad \mathbf{v}_B - \mathbf{v}_{B/R} + \mathbf{v}_R \qquad v_{B/R} = 0$$

$$\mathbf{v}_A = 2\text{ ft/s}\overset{B}{\underset{A}{\nearrow}} + \mathbf{v}_R \qquad \mathbf{v}_B = \mathbf{v}_R$$

$$0 = m_A[2\overset{B}{\underset{A}{\nearrow}} + \mathbf{v}_R] + m_B\mathbf{v}_R + m_R\mathbf{v}_e$$

$$\mathbf{v}_R = \frac{-2m_A}{(m_A + m_B + m_R)} = \frac{-(2\text{ ft/s})(190\text{ lb})}{(190\text{ lb} + 125\text{ lb} + 300\text{ lb})}$$

$v_R = 0.618$ ft/s ◀

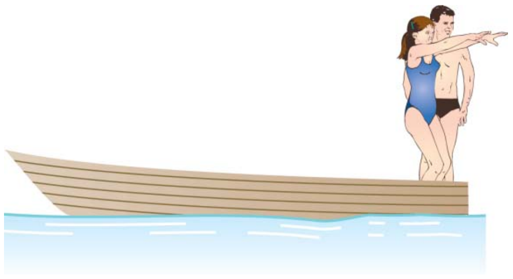
(*b*) From Eq. (1),

$$0 = m_A v_A + m_B v_B + 0 \qquad (v_R = 0)$$

$$v_B = -\frac{m_A v_A}{m_B} \qquad v_A = v_{A/R} + \cancel{v}_R^{\,0} = 2\text{ ft/s}$$

$$v_B = -\frac{(2\text{ ft/s})(190\text{ lb})}{(125\text{ lb})} = 3.04\text{ ft/s}$$

$v_B = 3.04$ ft/s ◀



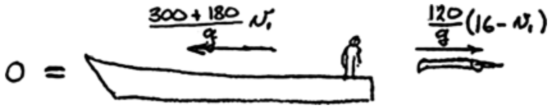
PROBLEM 14.6

A 180-lb man and a 120-lb woman stand side by side at the same end of a 300-lb boat, ready to dive, each with a 16-ft/s velocity relative to the boat. Determine the velocity of the boat after they have both dived, if (*a*) the woman dives first, (*b*) the man dives first.

SOLUTION

(*a*) Woman dives first.

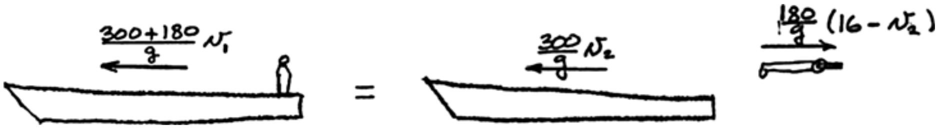
Conservation of momentum:



$$\frac{120}{g}(16 - v_1) - \frac{300+180}{g} v_1 = 0$$

$$v_1 = \frac{(120)(16)}{600} = 3.20 \text{ ft/s} \longleftarrow$$

Man dives next. Conservation of momentum:



$$-\frac{300+180}{g} v_1 = -\frac{300}{g} v_2 + \frac{180}{g}(16 - v_2)$$

$$v_2 = \frac{480 v_1 + (180)(16)}{480} = 9.20 \text{ ft/s}$$

$\mathbf{v}_2 = 9.20$ ft/s ⟵ ◀

(*b*) Man dives first.

Conservation of momentum:

$$\frac{180}{g}(16 - v_1') - \frac{300+120}{g} v_1' = 0$$

$$v_1' = \frac{(180)(16)}{600} = 4.80 \text{ ft/s}$$

Woman dives next. Conservation of momentum:

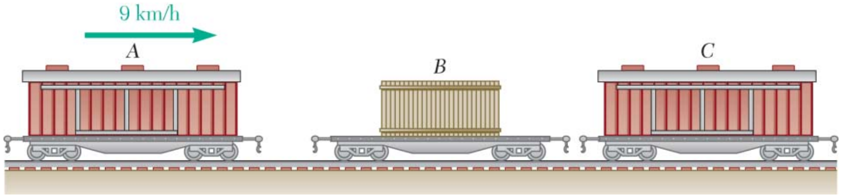
$$-\frac{300+120}{g} v_1' = -\frac{300}{g} v_2' + \frac{120}{g}(16 - v_2')$$

$$v_2' = \frac{420 v_1' + (120)(16)}{420} = 9.37 \text{ ft/s}$$

$\mathbf{v}_2' = 9.37$ ft/s ⟵ ◀

PROBLEM 14.7

A 40-Mg boxcar *A* is moving in a railroad switchyard with a velocity of 9 km/h toward cars *B* and *C*, which are both at rest with their brakes off at a short distance from each other. Car *B* is a 25-Mg flatcar supporting a 30-Mg container, and car *C* is a 35-Mg boxcar. As the cars hit each other they get automatically and tightly coupled. Determine the velocity of car *A* immediately after each of the two couplings, assuming that the container (*a*) does not slide on the flatcar, (*b*) slides after the first coupling but hits a stop before the second coupling occurs, (*c*) slides and hits the stop only after the second coupling has occurred.



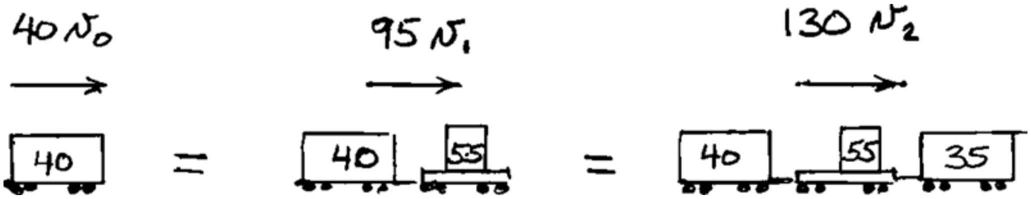
SOLUTION

Each term of the conservation of momentum equation is mass times velocity. As long as the same units are used in all terms, any unit may be used for mass and for velocity. We use Mg for mass and km/h for velocity and apply conservation of momentum.

Note: Only moving masses are shown in the diagrams.

Initial momentum: $m_A v_0 = (40)(9) = 360$

(*a*) Container does not slide

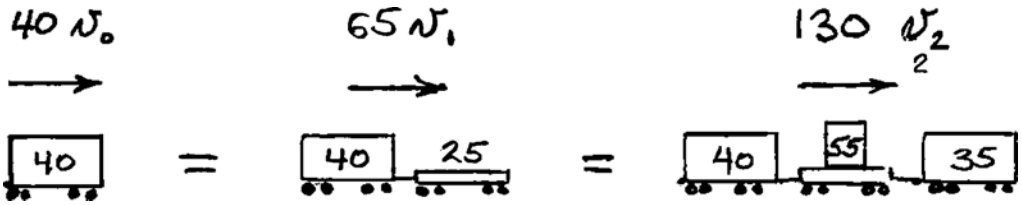


$360 = 95v_1 = 130v_2$

$\mathbf{v}_1 = 3.79$ km/h → ◀

$\mathbf{v}_2 = 2.77$ km/h → ◀

(*b*) Container slides after 1st coupling, stops before 2nd



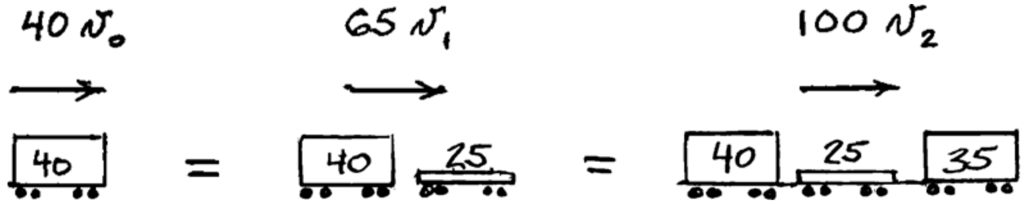
$360 = 65v_1 = 130v_2$

$\mathbf{v}_1 = 5.54$ km/h → ◀

$\mathbf{v}_2 = 2.77$ km/h → ◀

PROBLEM 14.7 (Continued)

(*c*) Container slides and stops only after 2nd coupling


40 v_0
65 v_1
100 v_2
40
=
40
25
=
40
25
35

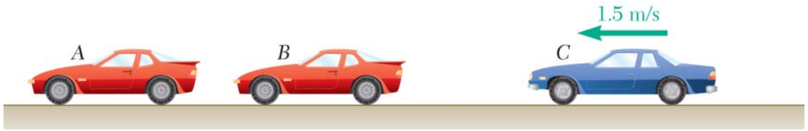
$360 = 65v_1 = 100v_2$

$\mathbf{v}_1 = 5.54$ km/h ⟶ ◀

$\mathbf{v}_2 = 3.60$ km/h ⟶ ◀

PROBLEM 14.8

Two identical cars *A* and *B* are at rest on a loading dock with brakes released. Car *C*, of a slightly different style but of the same weight, has been pushed by dockworkers and hits car *B* with a velocity of 1.5 m/s. Knowing that the coefficient of restitution is 0.8 between *B* and *C* and 0.5 between *A* and *B*, determine the velocity of each car after all collisions have taken place.

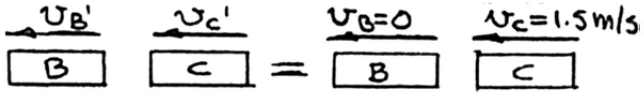


SOLUTION

$$m_A = m_B = m_C = m$$

Collision between *B* and *C*:

The total momentum is conserved:



$$\overset{+}{\leftarrow} \quad mv_B' + mv_C' = mv_B + mv_C$$

$$v_B' + v_C' = 0 + 1.5 \qquad (1)$$

Relative velocities:

$$(v_B - v_C)(e_{BC}) = (v_C' - v_B')$$
$$(-1.5)(0.8) = (v_C' - v_B')$$
$$-1.2 = v_C' - v_B' \qquad (2)$$

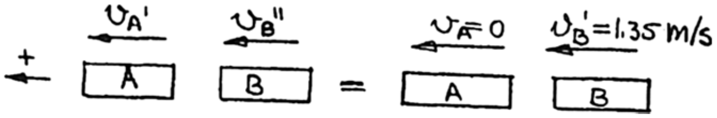
Solving (1) and (2) simultaneously,

$$v_B' = 1.35 \text{ m/s}$$
$$v_C' = 0.15 \text{ m/s}$$

$\mathbf{v}_C' = 0.150 \text{ m/s} \leftarrow$ ◀

Since $v_B' > v_C'$, car *B* collides with car *A*.

Collision between *A* and *B*:



$$mv_A' + mv_B'' = mv_A + mv_B'$$

$$v_A' + v_B'' = 0 + 1.35 \qquad (3)$$

PROBLEM 14.8 (Continued)

Relative velocities:

$$(v_A - v'_B)e_{AB} = (v''_B - v'_A)$$
$$(0 - 1.35)(0.5) = v''_B - v'_A$$
$$v'_A - v''_B = 0.675 \qquad (4)$$

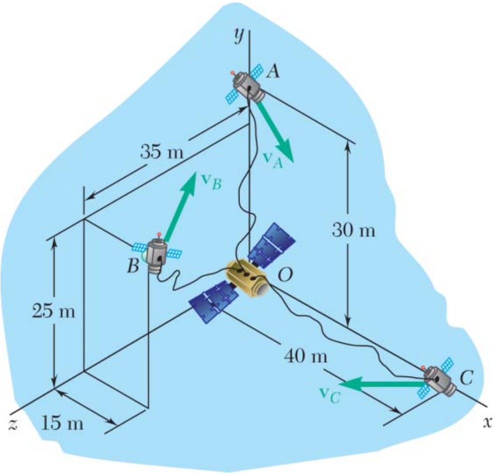
Solving (3) and (4) simultaneously,

$$2v'_A = 1.35 + 0.675$$

$$\mathbf{v}'_A = 1.013 \text{ m/s} \longleftarrow \blacktriangleleft$$

$$\mathbf{v}''_B = 0.338 \text{ m/s} \longleftarrow \blacktriangleleft$$

Since $v'_C < v''_B < v'_A$, there are no further collisions.



PROBLEM 14.9

A 20-kg base satellite deploys three sub-satellites, each which has its own thrust capabilities, to perform research on tether propulsion. The weights of sub-satellite *A*, *B*, and *C* are 4 kg, 6 kg, and 8 kg, respectively, and their velocities expressed in m/s are given by $\mathbf{v_A} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v_B} = \mathbf{i} + 4\mathbf{j}$, $\mathbf{v_C} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. At the instant shown, what is the angular momentum $\mathbf{H_O}$ of the system about the base satellite?

SOLUTION

Given: $m_A = 4\text{ kg},\ m_B = 6\text{ kg},\ m_C = 8\text{ kg}$

$\mathbf{v}_\text{A} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\text{ m/s},\ \mathbf{v}_\text{B} = \mathbf{i} + 4\mathbf{j}\text{ m/s},\ \mathbf{v}_\text{C} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}\text{ m/s}$

Linear momentum of each particle expressed in kg·m/s :

$$m_A\mathbf{v}_A = 16.0\mathbf{i} - 8.0\mathbf{j} + 8.0\mathbf{k}$$
$$m_B\mathbf{v}_B = 6.0\mathbf{i} + 24.0\mathbf{j} + 0\mathbf{k}$$
$$m_C\mathbf{v}_C = 16.0\mathbf{i} + 16.0\mathbf{j} + 32.0\mathbf{k}$$

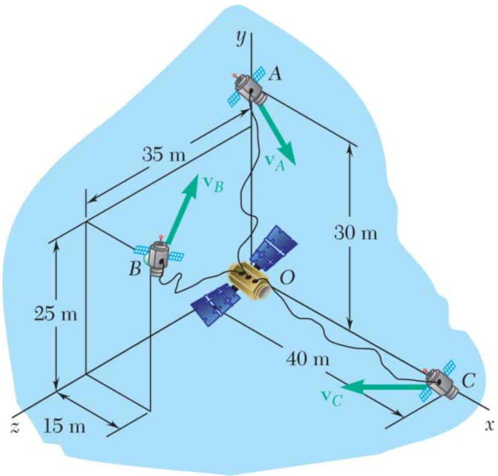
Position Vectors from point O to each satellite in meters:

$$\mathbf{r}_A = 30\mathbf{j}$$
$$\mathbf{r}_B = 15\mathbf{i} + 25\mathbf{j} + 35\mathbf{k}$$
$$\mathbf{r}_C = 40\mathbf{i}$$

Angular Momentum about O, $\text{kg}\cdot\text{m}^2/\text{s}$:

$$\mathbf{H}_O = \mathbf{r}_A \times (m_A\mathbf{v}_A) + \mathbf{r}_B \times (m_B\mathbf{v}_B) + \mathbf{r}_C \times (m_C\mathbf{v}_C)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 30 & 0 \\ 16.0 & -8.0 & 8.0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & 25 & 35 \\ 6.0 & 24.0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 40 & 0 & 0 \\ 16.0 & 16.0 & 32.0 \end{vmatrix}$$
$$= (240.0\mathbf{i} - 480.0\mathbf{k}) + (-840.0\mathbf{i} + 210.0\mathbf{j} + 210.0\mathbf{k}) + (-1280.0\mathbf{j} + 640.0\mathbf{k})$$
$$= -600.0\mathbf{i} - 1070.0\mathbf{j} + 370.0\mathbf{k}$$

$$\mathbf{H}_O = -(600\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (1070.0\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (370.0\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}\ \blacktriangleleft$$



PROBLEM 14.10

For the satellite system of Prob. 14.9, assuming that the velocity of the base satellite is zero, determine (*a*) the position vector **r** of the mass center *G* of the system, (*b*) the linear momentum $m\bar{\mathbf{v}}$ of the system, (*c*) the angular momentum $\mathbf{H}_G$ of the system about *G*. Also, verify that the answers to this problem and to Prob. 14.9 satisfy the equation given in Prob. 14.27.

PROBLEM 14.9 A 20-kg base satellite deploys three sub-satellites, each which has its own thrust capabilities, to perform research on tether propulsion. The weights of sub-satellite *A*, *B*, and *C* are 4 kg, 6 kg, and 8 kg, respectively, and their velocities expressed in m/s are given by $\mathbf{v_A} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v_B} = \mathbf{i} + 4\mathbf{j}$, $\mathbf{v_C} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. At the instant shown, what is the angular momentum $\mathbf{H_O}$ of the system about the base satellite?

SOLUTION

Given: $m_A = 4\text{ kg}$, $m_B = 6\text{ kg}$, $m_C = 8\text{ kg}$

$\mathbf{v}_\text{A} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\text{ m/s}$, $\mathbf{v}_\text{B} = 1\mathbf{i} + 4\mathbf{j}\text{ m/s}$, $\mathbf{v}_\text{C} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}\text{ m/s}$

$\mathbf{r}_A = 30\mathbf{j}\text{ m}$, $\mathbf{r}_B = 15\mathbf{i} + 25\mathbf{j} + 35\mathbf{k}\text{ m}$, $\mathbf{r}_C = 40\mathbf{i}\text{ m}$

(*a*) The mass center must satisfy:

$$\bar{\mathbf{r}} = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n} m_i}$$

$$\bar{\mathbf{r}} = \frac{m_A\mathbf{r}_A + m_B\mathbf{r}_B + m_C\mathbf{r}_C}{(m_A + m_B + m_C)}$$

$$\bar{\mathbf{r}} = \frac{(4)(30\mathbf{j}) + (6)(15\mathbf{i} + 25\mathbf{j} + 35\mathbf{k}) + (8)(40\mathbf{i})}{18}$$

$$\bar{\mathbf{r}} = 22.778\mathbf{i} + 15.000\mathbf{j} + 11.667\mathbf{k}\text{ m} \blacktriangleleft$$

(*b*) Linear momentum of the system:

$$m\bar{\mathbf{v}} = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C$$

$$= 16.0\mathbf{i} - 8.0\mathbf{j} + 8.0\mathbf{k} + 6.0\mathbf{i} + 24.0\mathbf{j} + 0\mathbf{k} + 16.0\mathbf{i} + 16.0\mathbf{j} + 32.0\mathbf{k}$$

$$m\bar{\mathbf{v}} = (38.0\text{ kg}\cdot\text{m/s})\mathbf{i} + (32.0\text{ kg}\cdot\text{m/s})\mathbf{j} + (40.0\text{ kg}\cdot\text{m/s})\mathbf{k} \blacktriangleleft$$

(*c*) Angular Momentum about the mass center, G:

$$\mathbf{H}_\text{G} = \sum_{i=1}^{n} \mathbf{r}_i' \times m_i \mathbf{v}_i$$

PROBLEM 14.10 (Continued)

where: $$\mathbf{r}'_A = \mathbf{r}_A - \overline{\mathbf{r}} = -22.778\mathbf{i} + 15.000\mathbf{j} - 11.667\mathbf{k} \text{ ft}$$
$$\mathbf{r}'_B = \mathbf{r}_B - \overline{\mathbf{r}} = -7.778\mathbf{i} + 10.000\mathbf{j} + 23.333\mathbf{k} \text{ ft}$$
$$\mathbf{r}'_C = \mathbf{r}_C - \overline{\mathbf{r}} = 17.222\mathbf{i} - 15.000\mathbf{j} - 11.667\mathbf{k} \text{ ft}$$

$$\mathbf{H}_G = \mathbf{r}'_A \times m_A\mathbf{v}_A + \mathbf{r}'_B \times m_B\mathbf{v}_B + \mathbf{r}'_C \times m_C\mathbf{v}_C$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -22.778 & 15.000 & -11.667 \\ 16.0 & -8.0 & 8.0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7.778 & 10.000 & 23.333 \\ 6.0 & 24.0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 17.222 & -15.000 & -11.667 \\ 16.0 & 16.0 & 32.0 \end{vmatrix}$$
$$= (26.667\mathbf{i} - 4.444\mathbf{j} - 57.778\mathbf{k}) + (-560.0\mathbf{i} + 140.0\mathbf{j} - 246.667\mathbf{k}) + (-293.333\mathbf{i} - 737.778\mathbf{j} + 515.556\mathbf{k})$$
$$= -826.667\mathbf{i} - 602.222\mathbf{j} + 211.111\mathbf{k}$$

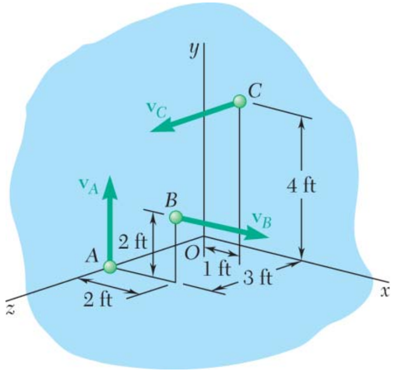
$$\mathbf{H}_G = -826.667\mathbf{i} - 602.222\mathbf{j} + 211.111\mathbf{k} \text{ kg} \cdot \text{m}^2/\text{s} \blacktriangleleft$$

From Prob. 14.27 show that: $\mathbf{H}_O = \mathbf{H}_G + \overline{\mathbf{r}} \times m\overline{\mathbf{v}}$

$$\overline{\mathbf{r}} \times m\overline{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 22.778 & 15.000 & 11.667 \\ 38.0 & 32.0 & 40.0 \end{vmatrix}$$
$$= 226.667\mathbf{i} - 467.778\mathbf{j} + 158.889\mathbf{k} \quad \text{kg}\cdot\text{m}^2/\text{s}$$

$$\mathbf{H}_G + \overline{\mathbf{r}} \times m\overline{\mathbf{v}} = -600.0\mathbf{i} - 1070.0\mathbf{j} + 370.0\mathbf{k} \quad \text{kg}\cdot\text{m}^2/\text{s}$$

This is the identical answer to Problem 14.9.



PROBLEM 14.11

A system consists of three identical 19.32-lb particles *A, B,* and *C.* The velocities of the particles are, respectively, $\mathbf{v}_A = v_A\mathbf{j}$, $\mathbf{v}_B = v_B\mathbf{i}$, and $\mathbf{v}_C = v_C\mathbf{k}$. Knowing that the angular momentum of the system about *O*, expressed in $\text{ft}\cdot\text{lb}\cdot\text{s}$ is $\mathbf{H}_O = -1.2\mathbf{k}$, determine (*a*) the velocities of the particles, (*b*) the angular momentum of the system about its mass center *G*.

SOLUTION

The masses are $m_A = m_B = m_C = \dfrac{19.32}{32.2} = 0.6\ \text{lb}\cdot\text{s}^2/\text{ft}$.

Position vectors (ft): $\mathbf{r}_A = 3\mathbf{k}, \qquad \mathbf{r}_B = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \qquad \mathbf{r}_C = \mathbf{i} + 4\mathbf{j}$

In units of $\text{ft}\cdot\text{lb}\cdot\text{s}$, $\quad \mathbf{H}_O = \mathbf{r}_A \times (m_A\mathbf{v}_A) + \mathbf{r}_B \times (m_B\mathbf{v}_B) + \mathbf{r}_C \times (m_C\mathbf{v}_C)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 3 \\ 0 & (0.6v_A) & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 3 \\ (0.6v_B) & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 0 & 0 & (0.6v_C) \end{vmatrix}$$

$$= (-1.8v_A\mathbf{i}) + (1.8v_B\mathbf{j} - 1.2v_B\mathbf{k}) + (2.4v_C\mathbf{i} - 0.6v_C\mathbf{j})$$

$$= (-1.8v_A + 2.4v_C)\mathbf{i} + (1.8v_B - 0.6v_C)\mathbf{j} + (-1.2v_B)\mathbf{k}$$

But, $\mathbf{H}_O$ is given as $-1.2\mathbf{k}\ \text{ft}\cdot\text{lb}\cdot\text{s}$.

Equating the two expressions for $\mathbf{H}_O$ and resolving into components,

$$\mathbf{i}: \quad -1.8v_A + 2.4v_C = 0 \qquad (1)$$

$$\mathbf{j}: \quad 1.8v_B - 0.6v_C = 0 \qquad (2)$$

$$\mathbf{k}: \quad -1.2v_B = -1.2 \qquad (3)$$

(*a*) Solving, $\quad v_A = 4\ \text{ft/s} \qquad \mathbf{v}_A = (4.00\ \text{ft/s})\mathbf{j}$ ◀

$v_B = 1\ \text{ft/s} \qquad \mathbf{v}_B = (1.000\ \text{ft/s})\mathbf{i}$ ◀

$v_C = 3\ \text{ft/s} \qquad \mathbf{v}_C = (3.00\ \text{ft/s})\mathbf{k}$ ◀

Coordinates of mass center *G* in ft.

$$\bar{\mathbf{r}} = \frac{m_A\mathbf{r}_A + m_B\mathbf{r}_B + m_C\mathbf{r}_C}{m_A + m_B + m_C}$$

$$= \frac{(0.6)(3\mathbf{k}) + (0.6)(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + (0.6)(\mathbf{i} + 4\mathbf{j})}{1.8}$$

$$= \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

PROBLEM 14.11 (Continued)

Position vectors relative to the mass center in ft.

$$\mathbf{r}'_A = \mathbf{r}_A - \overline{\mathbf{r}} = (3\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}'_B = \mathbf{r}_B - \overline{\mathbf{r}} = (2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \mathbf{i} + \mathbf{k}$$

$$\mathbf{r}'_C = \mathbf{r}_C - \overline{\mathbf{r}} = (\mathbf{i} + 4\mathbf{j}) - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 2\mathbf{j} - 2\mathbf{k}$$

$$m_A\mathbf{v}_A = (0.6)(4\mathbf{j}) = (2.4\text{ lb}\cdot\text{s})\mathbf{j}$$

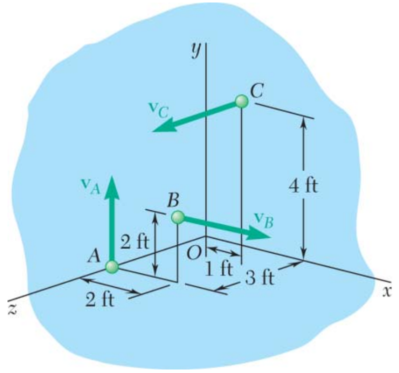
$$m_B\mathbf{v}_B = (0.6)(\mathbf{i}) = (0.6\text{ lb}\cdot\text{s})\mathbf{i}$$

$$m_C\mathbf{v}_C = (0.6)(3\mathbf{k}) = (1.8\text{ lb}\cdot\text{s})\mathbf{k}$$

(*b*)

$$\begin{aligned}\mathbf{H}_G &= \mathbf{r}'_A \times (m_A\mathbf{v}_A) + \mathbf{r}'_B \times (m_B\mathbf{v}_B) + \mathbf{r}'_C \times (m_C\mathbf{v}_C)\\ &= (-\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (2.4\mathbf{j}) + (\mathbf{i} + \mathbf{k}) \times (0.6\mathbf{i}) + (2\mathbf{j} - 2\mathbf{k}) \times (1.8\mathbf{k})\\ &= (-2.4\mathbf{i} - 2.4\mathbf{k}) + (0.6\mathbf{j}) + (3.6\mathbf{i}) = 1.2\mathbf{i} + 0.6\mathbf{j} - 2.4\mathbf{k}\end{aligned}$$

$$\mathbf{H}_G = (1.20\text{ ft}\cdot\text{lb}\cdot\text{s})\mathbf{i} + (0.60\text{ ft}\cdot\text{lb}\cdot\text{s})\mathbf{j} - (2.40\text{ ft}\cdot\text{lb}\cdot\text{s})\mathbf{k} \blacktriangleleft$$



PROBLEM 14.12

A system consists of three identical 19.32-lb particles *A, B,* and *C.* The velocities of the particles are, respectively, $\mathbf{v}_A = v_A\mathbf{j}$, $\mathbf{v}_B = v_B\mathbf{i}$, and $\mathbf{v}_C = v_C\mathbf{k}$, and the magnitude of the linear momentum **L** of the system is 9 lb·s. Knowing that $\mathbf{H}_G = \mathbf{H}_O$, where $\mathbf{H}_G$ is the angular momentum of the system about its mass center *G* and $\mathbf{H}_O$ is the angular momentum of the system about *O*, determine (*a*) the velocities of the particles, (*b*) the angular momentum of the system about *O.*

SOLUTION

The masses are $m_A = m_B = m_C = \dfrac{19.32}{32.2} = 0.6\text{ lb}\cdot\text{s}^2/\text{ft}.$

Position vectors (ft): $\mathbf{r}_A = 3\mathbf{k}, \qquad \mathbf{r}_B = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \qquad \mathbf{r}_C = \mathbf{i} + 4\mathbf{j}$

Coordinates of mass center *G* expressed in ft.

$$\begin{aligned}\overline{\mathbf{r}} &= \frac{m_A\mathbf{r}_A + m_B\mathbf{r}_B + m_C\mathbf{r}_C}{m_A + m_B + m_C}\\ &= \frac{(0.6)(3\mathbf{k}) + (0.6)(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + (0.6)(\mathbf{i} + 4\mathbf{j})}{1.8}\\ &= \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

Position vectors relative to the mass center expressed in ft.

$$\begin{aligned}\mathbf{r}'_A &= \mathbf{r}_A - \overline{\mathbf{r}} = (3\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}\\ \mathbf{r}'_B &= \mathbf{r}_B - \overline{\mathbf{r}} = (2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \mathbf{i} + \mathbf{k}\\ \mathbf{r}'_C &= \mathbf{r}_C - \overline{\mathbf{r}} = (\mathbf{i} + 4\mathbf{j}) - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 2\mathbf{j} - 2\mathbf{k}\end{aligned}$$

Angular momenta.

$$\begin{aligned}\mathbf{H}_O &= \mathbf{r}_A \times (m_A\mathbf{v}_A) + \mathbf{r}_B \times (m_B\mathbf{v}_B) + \mathbf{r}_C \times (m_C\mathbf{v}_C)\\ \mathbf{H}_G &= \mathbf{r}'_A \times (m_A\mathbf{v}_A) + \mathbf{r}'_B \times (m_B\mathbf{v}_B) + \mathbf{r}'_C \times (m_C\mathbf{v}_C)\end{aligned}$$

Subtracting,

$$\mathbf{H}_O - \mathbf{H}_G = (\mathbf{r}_A - \mathbf{r}'_A) \times (m_A\mathbf{v}_A) + (\mathbf{r}_B - \mathbf{r}'_B) \times m_B\mathbf{v}_B + (\mathbf{r}_C - \mathbf{r}'_C) \times m_C\mathbf{v}_C$$

$$\begin{aligned}0 &= \overline{\mathbf{r}} \times (m_A\mathbf{v}_A) + \overline{\mathbf{r}} \times (m_B\mathbf{v}_B) + \overline{\mathbf{r}} \times (m_C\mathbf{v}_C)\\ &= \overline{\mathbf{r}} \times (m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C) = \overline{\mathbf{r}} \times \mathbf{L}\end{aligned}$$

L is parallel to $\overline{\mathbf{r}}$. $\qquad \mathbf{L} = \lambda\overline{\mathbf{r}} \qquad \mathbf{L}\cdot\mathbf{L} = \lambda^2\overline{\mathbf{r}}\cdot\overline{\mathbf{r}}$

$$\lambda^2 = \frac{\mathbf{L}\cdot\mathbf{L}}{\overline{\mathbf{r}}\cdot\overline{\mathbf{r}}} = \frac{(9)^2}{(3)^2} = 3^2, \qquad \lambda = \pm 3\text{ lb}\cdot\text{s/ft}$$

PROBLEM 14.12 (Continued)

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C = \lambda\overline{\mathbf{r}}$$

$$(0.6)(v_A\mathbf{j}) + (0.6)(v_B\mathbf{i}) + (0.6)(v_C\mathbf{k}) = \pm 3(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

(*a*) Resolve into components and solve for v_A, v_B, and v_C.

$$v_A = 10 \text{ ft/s} \qquad \mathbf{v}_A = (10.00 \text{ ft/s})\mathbf{j} \blacktriangleleft$$

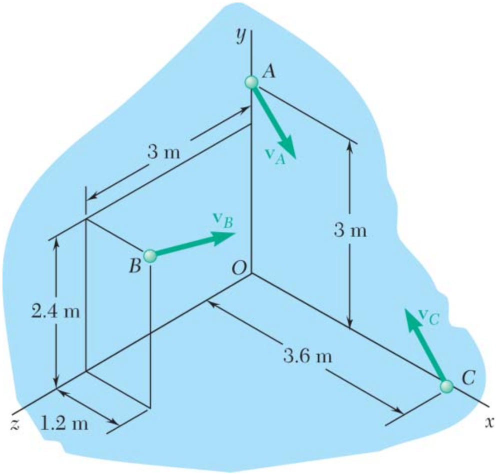
$$v_B = 5 \text{ ft/s} \qquad \mathbf{v}_B = (5.00 \text{ ft/s})\mathbf{i} \blacktriangleleft$$

$$v_C = 10 \text{ ft/s} \qquad \mathbf{v}_C = (10.00 \text{ ft/s})\mathbf{k} \blacktriangleleft$$

(*b*) Angular momentum about *O* expressed in ft·lb·s.

$$\begin{aligned}
\mathbf{H}_O &= \mathbf{r}_A \times (m_A\mathbf{v}_A) + \mathbf{r}_B \times (m_B\mathbf{v}_B) + \mathbf{r}_C \times (m_C\mathbf{v}_C) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 3 \\ 0 & (0.6v_A) & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 3 \\ (0.6v_B) & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 0 & 0 & (0.6v_C) \end{vmatrix} \\
&= (-1.8v_A\mathbf{i}) + (1.8v_B\mathbf{j} - 1.2v_B\mathbf{k}) + (2.4v_C\mathbf{i} - 0.6v_C\mathbf{j}) \\
&= (-1.8v_A + 2.4v_C)\mathbf{i} + (1.8v_B - 0.6v_C)\mathbf{j} + (-1.2v_B)\mathbf{k} \\
&= 6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}
\end{aligned}$$

$$\mathbf{H}_O = (6.00 \text{ ft}\cdot\text{lb}\cdot\text{s})\mathbf{i} + (3.00 \text{ ft}\cdot\text{lb}\cdot\text{s})\mathbf{j} - (6.00 \text{ ft}\cdot\text{lb}\cdot\text{s})\mathbf{k} \blacktriangleleft$$



PROBLEM 14.13

A system consists of three particles A, B, and C. We know that $m_A = 3$ kg, $m_B = 2$ kg, and $m_C = 4$ kg and that the velocities of the particles expressed in m/s are, respectively, $\mathbf{v}_A = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v}_B = 4\mathbf{i} + 3\mathbf{j}$, and $\mathbf{v}_C = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. Determine the angular momentum $\mathbf{H}_O$ of the system about O.

SOLUTION

Linear momentum of each particle expressed in $\text{kg}\cdot\text{m/s}$.

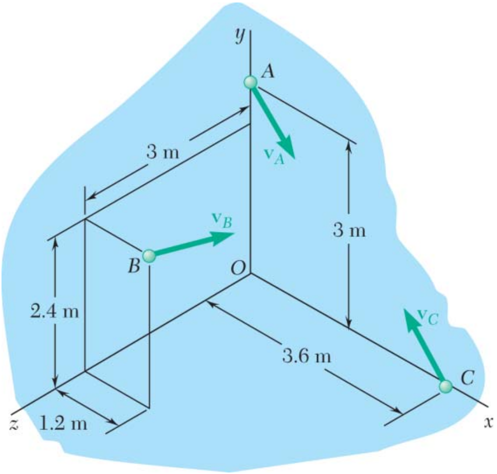
$$m_A\mathbf{v}_A = 12\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$$
$$m_B\mathbf{v}_B = 8\mathbf{i} + 6\mathbf{j}$$
$$m_C\mathbf{v}_C = -8\mathbf{i} + 16\mathbf{j} + 8\mathbf{k}$$

Position vectors, (meters): $\mathbf{r}_A = 3\mathbf{j}$, $\mathbf{r}_B = 1.2\mathbf{i} + 2.4\mathbf{j} + 3\mathbf{k}$, $\mathbf{r}_C = 3.6\mathbf{i}$

Angular momentum about O, $(\text{kg}\cdot\text{m}^2/\text{s})$.

$$\begin{aligned}
\mathbf{H}_O &= \mathbf{r}_A \times (m_A\mathbf{v}_A) + \mathbf{r}_B \times (m_B\mathbf{v}_B) + \mathbf{r}_C \times (m_C\mathbf{v}_C) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 0 \\ 12 & 6 & 6 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.2 & 2.4 & 3 \\ 8 & 6 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3.6 & 0 & 0 \\ -8 & 16 & 8 \end{vmatrix} \\
&= (18\mathbf{i} - 36\mathbf{k}) + (-18\mathbf{i} + 24\mathbf{j} - 12\mathbf{k}) + (-28.8\mathbf{j} + 57.6\mathbf{k}) \\
&= 0\mathbf{i} - 4.8\mathbf{j} + 9.6\mathbf{k}
\end{aligned}$$

$$\mathbf{H}_O = -(4.80\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (9.60\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{k} \blacktriangleleft$$



PROBLEM 14.14

For the system of particles of Prob. 14.13, determine (*a*) the position vector $\overline{\mathbf{r}}$ of the mass center *G* of the system, (*b*) the linear momentum $m\overline{\mathbf{v}}$ of the system, (*c*) the angular momentum $\mathbf{H}_G$ of the system about *G*. Also verify that he answers to this problem and to problem 14.13 satisfy the equation given in Prob. 14.27.

SOLUTION

Position vectors, (meters): $\mathbf{r}_A = 3\mathbf{j}, \quad \mathbf{r}_B = 1.2\mathbf{i} + 2.4\mathbf{j} + 3\mathbf{k}, \quad \mathbf{r}_C = 3.6\mathbf{i}$

(*a*) Mass center: $(m_A + m_B + m_C)\overline{\mathbf{r}} = m_A\mathbf{r}_A + m_B\mathbf{r}_B + m_C\mathbf{r}_C$

$$9\overline{\mathbf{r}} = (3)(3\mathbf{j}) + (2)(1.2\mathbf{i} + 2.4\mathbf{j} + 3\mathbf{k}) + (4)(3.6\mathbf{i})$$

$$\overline{\mathbf{r}} = 1.86667\mathbf{i} + 1.53333\mathbf{j} + 0.66667\mathbf{k}$$

$$\overline{\mathbf{r}} = (1.867\text{ m})\mathbf{i} + (1.533\text{ m})\mathbf{j} + (0.667\text{ m})\mathbf{k} \blacktriangleleft$$

Linear momentum of each particle, $(\text{kg}\cdot\text{m}^2/\text{s})$.

$$m_A\mathbf{v}_A = 12\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$$

$$m_B\mathbf{v}_B = 8\mathbf{i} + 6\mathbf{j}$$

$$m_C\mathbf{v}_C = -8\mathbf{i} + 16\mathbf{j} + 8\mathbf{k}$$

(*b*) Linear momentum of the system, $(\text{kg}\cdot\text{m/s.})$

$$m\overline{\mathbf{v}} = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C = 12\mathbf{i} + 28\mathbf{j} + 14\mathbf{k}$$

$$m\overline{\mathbf{v}} = (12.00\text{ kg}\cdot\text{m/s})\mathbf{i} + (28.0\text{ kg}\cdot\text{m/s})\mathbf{j} + (14.00\text{ kg}\cdot\text{m/s})\mathbf{k} \blacktriangleleft$$

Position vectors relative to the mass center, (meters).

$$\mathbf{r}'_A = \mathbf{r}_A - \overline{\mathbf{r}} = -1.86667\mathbf{i} + 1.46667\mathbf{j} - 0.66667\mathbf{k}$$

$$\mathbf{r}'_B = \mathbf{r}_B - \overline{\mathbf{r}} = -0.66667\mathbf{i} + 0.86667\mathbf{j} + 2.33333\mathbf{k}$$

$$\mathbf{r}'_C = \mathbf{r}_C - \overline{\mathbf{r}} = 1.73333\mathbf{i} - 1.53333\mathbf{j} - 0.66667\mathbf{k}$$

(*c*) Angular momentum about *G*, $\left(\text{kg}\cdot\text{m}^2/\text{s}\right)$.

$$\begin{aligned}
\mathbf{H}_G &= \mathbf{r}'_A \times m_A\mathbf{v}_A + \mathbf{r}'_B \times m_B\mathbf{v}_B + \mathbf{r}'_C \times m_C\mathbf{v}_C \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1.86667 & 1.46667 & -0.66667 \\ 12 & 6 & 6 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.66667 & 0.86667 & 2.33333 \\ 8 & 6 & 0 \end{vmatrix} \\
&\quad + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.73333 & -1.53333 & -0.66667 \\ -8 & 16 & 8 \end{vmatrix} \\
&= \left(12.8\mathbf{i} + 3.2\mathbf{j} - 28.8\mathbf{k}\right) + \left(-14\mathbf{i} + 18.6667\mathbf{j} - 10.9333\mathbf{k}\right) \\
&\quad + \left(-1.6\mathbf{i} - 8.5333\mathbf{j} + 15.4667\mathbf{k}\right) \\
&= -2.8\mathbf{i} + 13.3333\mathbf{j} - 24.2667\mathbf{k}
\end{aligned}$$

$$\mathbf{H}_G = -\left(2.80\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{i} + \left(13.33\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{j} - \left(24.3\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{k} \blacktriangleleft$$

$$\begin{aligned}
\bar{\mathbf{r}} \times m\bar{\mathbf{v}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.86667 & 1.53333 & 0.66667 \\ 12 & 28 & 14 \end{vmatrix} \\
&= \left(2.8\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{i} - \left(18.1333\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{j} + \left(33.8667\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{k}
\end{aligned}$$

$$\mathbf{H}_G + \bar{\mathbf{r}} \times m\bar{\mathbf{v}} = -\left(4.8\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{j} + \left(9.6\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{k}$$

Angular momentum about *O*.

$$\begin{aligned}
\mathbf{H}_O &= \mathbf{r}_A \times \left(m_A\mathbf{v}_A\right) + \mathbf{r}_B \times \left(m_B\mathbf{v}_B\right) + \mathbf{r}_C \times \left(m_C\mathbf{v}_C\right) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 0 \\ 12 & 6 & 6 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.2 & 2.4 & 3 \\ 8 & 6 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3.6 & 0 & 0 \\ -8 & 16 & 8 \end{vmatrix} \\
&= \left(18\mathbf{i} - 36\mathbf{k}\right) + \left(-18\mathbf{i} + 24\mathbf{j} - 12\mathbf{k}\right) + \left(-28.8\mathbf{j} + 57.6\mathbf{k}\right) \\
&= -\left(4.8\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{j} + \left(9.6\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{k}
\end{aligned}$$

Note that

$$\mathbf{H}_O = \mathbf{H}_G + \bar{\mathbf{r}} \times m\bar{\mathbf{v}}$$

PROBLEM 14.15

A 13-kg projectile is passing through the origin *O* with a velocity $\mathbf{v}_0 = (35\text{ m/s})\mathbf{i}$ when it explodes into two fragments *A* and *B*, of mass 5 kg and 8 kg, respectively. Knowing that 3 s later the position of fragment *A* is (90 m, 7 m, –14 m), determine the position of fragment *B* at the same instant. Assume $a_y = -g = -9.81\,\text{m/s}^2$ and neglect air resistance.

SOLUTION

Motion of mass center:

It moves as if projectile had not exploded.

$$\begin{aligned}\bar{\mathbf{r}} &= v_0 t\mathbf{i} - \frac{1}{2}gt^2\mathbf{j}\\ &= (35\text{ m/s})(3\text{ s})\mathbf{i} - \frac{1}{2}(9.81\text{ m/s}^2)(3\text{ s})^2\,\mathbf{j}\\ &= (105\text{ m})\mathbf{i} - (44.145\text{ m})\,\mathbf{j}\end{aligned}$$

Equation (14.12):

$$m\bar{\mathbf{r}} = \Sigma m_i \mathbf{r}_i:$$

$$m\bar{\mathbf{r}} = m_A\mathbf{r}_A + m_B\mathbf{r}_B$$

$$13(105\mathbf{i} - 44.145\mathbf{j}) = 5(90\mathbf{i} + 7\,\mathbf{j} - 14\mathbf{k}) + 8\mathbf{r}_B$$

$$\begin{aligned}8\mathbf{r}_B &= (13\times 105 - 5\times 90)\mathbf{i}\\ &\quad + (-13\times 44.145 - 5\times 7)\,\mathbf{j} + (5\times 14)\mathbf{k}\\ &= 915\mathbf{i} - 608.89\,\mathbf{j} + 70\mathbf{k}\end{aligned}$$

$$\mathbf{r}_B = (114.4\text{ m})\mathbf{i} - (76.1\text{ m})\,\mathbf{j} + (8.75\text{ m})\mathbf{k} \ \blacktriangleleft$$

PROBLEM 14.16

A 300-kg space vehicle traveling with a velocity $\mathbf{v}_0 = (360 \text{ m/s})\mathbf{i}$ passes through the origin *O* at $t = 0$. Explosive charges then separate the vehicle into three parts *A*, *B*, and *C*, with mass, respectively, 150 kg, 100 kg, and 50 kg. Knowing that at $t = 4$ s, the positions of parts *A* and *B* are observed to be *A* (1170 m, –290 m, –585 m) and *B* (1975 m, 365 m, 800 m), determine the corresponding position of part *C*. Neglect the effect of gravity.

SOLUTION

Motion of mass center:

Since there is no external force,

$$\overline{\mathbf{r}} = \mathbf{v}_0 t = (360 \text{ m/s})\mathbf{i}(4 \text{ s}) = (1440 \text{ m})\mathbf{i}$$

Equation (14.12):

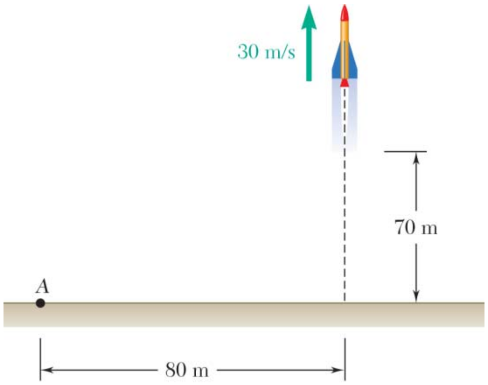
$$m\overline{\mathbf{r}} = \Sigma m_i \mathbf{r}_i:$$

$$\begin{aligned}(300)(1440\mathbf{i}) &= (150)(1170\mathbf{i} - 290\mathbf{j} - 585\mathbf{k})\\ &\quad + (100)(1975\mathbf{i} + 365\mathbf{j} + 800\mathbf{k})\\ &\quad + (50)\mathbf{r}_C\end{aligned}$$

$$\begin{aligned}50\mathbf{r}_C &= (300\times 1440 - 150\times 1170 - 100\times 1975)\mathbf{i}\\ &\quad + (150\times 290 - 100\times 365)\mathbf{j} + (150\times 585 - 100\times 800)\mathbf{k}\end{aligned}$$

$$= 59,000\mathbf{i} + 7,000\mathbf{j} + 7,750\mathbf{k}$$

$$\mathbf{r}_C = (1180 \text{ m})\mathbf{i} + (140 \text{ m})\mathbf{j} + (155 \text{ m})\mathbf{k} \blacktriangleleft$$



PROBLEM 14.17

A 2-kg model rocket is launched vertically and reaches an altitude of 70 m with a speed of 30 m/s at the end of powered flight, time $t = 0$. As the rocket approaches its maximum altitude it explodes into two parts of masses $m_A = 0.7\text{ kg}$ and $m_B = 1.3\text{ kg}$. Part A is observed to strike the ground 80 m west of the launch point at $t = 6$ s. Determine the position of part B at that time.

SOLUTION

Choose a planar coordinate system having coordinates x and y with the origin at the launch point on the ground and the x-axis pointing east and the y-axis vertically upward.

Let subscript E refer to the point where the explosion occurs, and A and B refer to the fragments A and B.
Let t be the time elapsed after the explosion.

Motion of the mass center:

$$\bar{x} = x_E + (\bar{v}_x)t = 0$$

$$\bar{y} = y_E + (\bar{v}_y)_0^t - \frac{1}{2}gt^2$$

where $\quad y_E = 70\text{ m} \quad \text{and} \quad (\bar{v}_y)_0 = 30\text{ m/s}$

At $\quad t = 6\text{ s}, \quad \bar{x} = 0$

$$\bar{y} = 70 + (30)(6) - \frac{1}{2}(9.81)(6)^2 = 73.42\text{ m}$$

Definition of mass center:

$$m_A\bar{x} = m_Ax_A + m_Bx_B$$

$$0 = (0.7\text{ kg})(-80\text{ m}) + (1.3\text{ kg})x_B$$

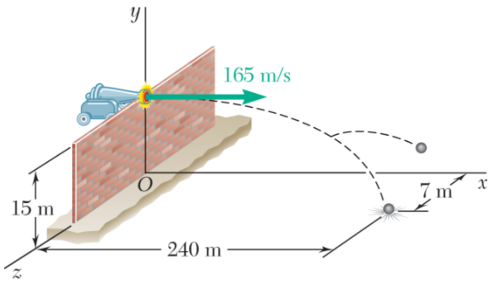
$$x_B = 43.1\text{ m}$$

$$m\bar{y} = m_Ay_A + m_By_B$$

$$(2\text{ kg})(73.42\text{ m}) = (0.7\text{ kg})(0) + (1.3\text{ kg})y_B$$

$$y_B = 113.0\text{ m}$$

Position of part B: $\quad 43.1\text{ m (east)}, \quad 113.0\text{ m (up)}$ ◀



PROBLEM 14.18

An 18-kg cannonball and a 12-kg cannonball are chained together and fired horizontally with a velocity of 165 m/s from the top of a 15-m wall. The chain breaks during the flight of the cannonballs and the 12-kg cannonball strikes the ground at $t = 1.5$ s, at a distance of 240 m from the foot of the wall, and 7 m to the right of the line of fire. Determine the position of the other cannonball at that instant. Neglect the resistance of the air.

SOLUTION

Let subscript A refer to the 12-kg cannonball and B to the 18-kg cannonball.

The motion of the mass center of A and B is uniform in the x-direction, uniformly accelerated with acceleration $-g = -9.81\ \text{m/s}^2$ in the y-direction, and zero in the z-direction.

$$\bar{x} = (v_0)_x t = (165\ \text{m/s})(1.5\ \text{s}) = 247.5\ \text{m}$$

$$\bar{y} = \bar{y}_0 + (\bar{v}_0)_y t - \frac{1}{2}gt^2$$

$$= 15\ \text{m} + 0 - \frac{1}{2}(9.81\ \text{m/s}^2)(1.55)^2 = 3.964\ \text{m}$$

$$\bar{z} = 0$$

$$\bar{\mathbf{r}} = (247.5\ \text{m})\mathbf{i} + (3.964\ \text{m})\mathbf{j}$$

Definition of mass center:

$$m\bar{\mathbf{r}} = m_A\mathbf{r}_A + m_B\mathbf{r}_B$$

Data:

$$m_A = 12\ \text{kg}, \quad m_B = 18\ \text{kg}, \quad m = m_A + m_B = 30\ \text{kg}$$

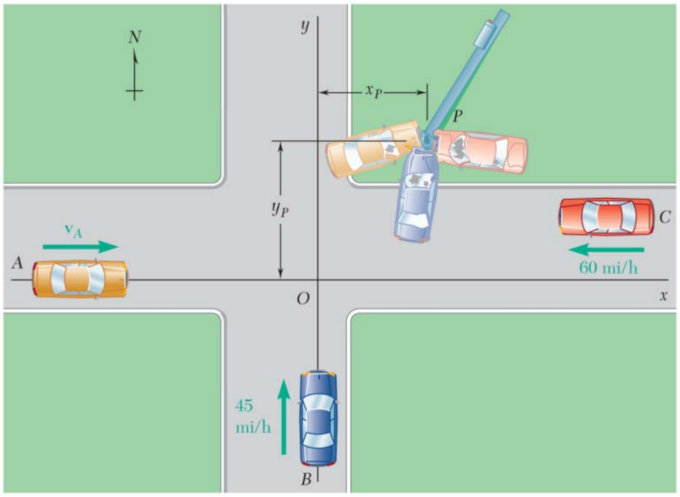
$$t = 1.5\ \text{s}, \quad x_A = 240\ \text{m}, \quad y_A = 0, \quad z_A = 7\ \text{m}$$

$$(30)(247.5\mathbf{i} + 3.964\ \mathbf{j}) = (12)(240\mathbf{i} + 7\mathbf{k}) + (18)(x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k})$$

$$\mathbf{i}: \quad (30)(247.5) = (12)(240) + 18\,x_B \qquad x_B = 253\ \text{m} \ \blacktriangleleft$$

$$\mathbf{j}: \quad (30)(3.964) = (12)(0) + 18\,y_B \qquad y_B = 6.61\ \text{m} \ \blacktriangleleft$$

$$\mathbf{k}: \quad (30)(0) = (12)(7) + 18\,z_B \qquad z_B = -4.67\ \text{m} \ \blacktriangleleft$$



PROBLEM 14.19

Car A was traveling east at high speed when it collided at Point O with car B, which was traveling north at 45 mi/h. Car C, which was traveling west at 60 mi/h, was 32 ft east and 10 ft north of Point O at the time of the collision. Because the pavement was wet, the driver of car C could not prevent his car from sliding into the other two cars, and the three cars, stuck together, kept sliding until they hit the utility pole P. Knowing that the weights of cars A, B, and C are, respectively, 3000 lb, 2600 lb, and 2400 lb, and neglecting the forces exerted on the cars by the wet pavement, solve the problems indicated.

Knowing that the speed of car A was 75 mi/h and that the time elapsed from the first collision to the stop at P was 2.4 s, determine the coordinates of the utility pole P.

SOLUTION

Let t be the time elapsed since the first collision. No external forces in the xy plane act on the system consisting of cars A, B, and C during the impacts with one another. The mass center of the system moves at the velocity it had before the collision.

Setting the origin at O, we can find the initial mass center $\overline{\mathbf{r}}_0$: at the moment of the first collision:

$$(m_A + m_B + m_C)(\overline{x}_0\mathbf{i} + \overline{y}_0\mathbf{j}) = m_A(0) + m_B(0) + m_C(x_C\mathbf{i} + y_C\mathbf{j})$$

$$\overline{x}_0 = 0.3x_C = (0.3)(32) = 9.6 \text{ ft}, \quad \overline{y}_0 = 0.3y_C = (0.3)(10) = 3 \text{ ft}$$

Given velocities:

$$\mathbf{v}_A = (75 \text{ mi/h})\mathbf{i} = (110 \text{ ft/s})\mathbf{i}, \quad \mathbf{v}_B = (45 \text{ mi/h})\mathbf{i} = (66 \text{ ft/s})\mathbf{j}, \quad \mathbf{v}_C = (60 \text{ mi/h})\mathbf{i} = (88 \text{ ft/s})\mathbf{i}$$

Velocity of mass center:

$$(m_A + m_B + m_C)\overline{\mathbf{v}} = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C$$

$$\overline{\mathbf{v}} = 0.375\mathbf{v}_A + 0.325\mathbf{v}_B + 0.3\mathbf{v}_C$$

Since the collided cars hit the pole at

$$\mathbf{r}_P = x_P\mathbf{i} + y_P\mathbf{j}$$

$$\mathbf{x}_P\mathbf{i} + y_P\mathbf{j} = \overline{x}_0\mathbf{i} + \overline{y}_0\mathbf{j} + \overline{\mathbf{v}}t \qquad \text{Resolve into components.}$$

$$x: \quad x_P = \overline{x}_0 + 0.375v_At_P - 0.3v_Ct_P \tag{1}$$

$$y: \quad y_P = \overline{y}_0 + 0.325v_Bt_P \tag{2}$$

PROBLEM 14.19 (Continued)

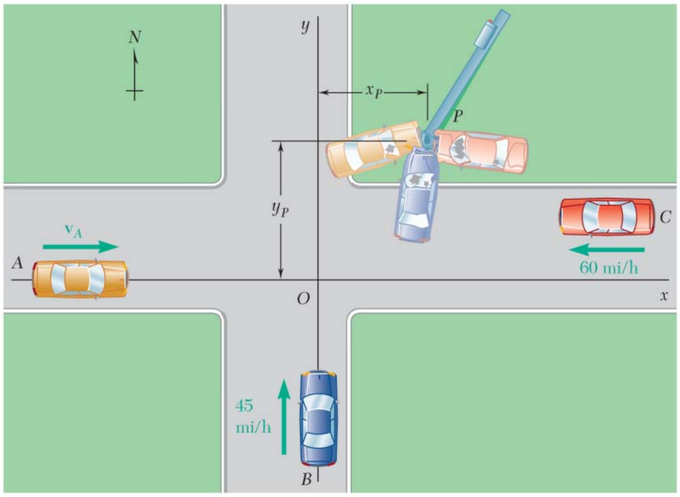
Data: $t_P = 2.4\text{ s}$

From (1), $x_P = 9.6 + (0.375)(110)(2.4) - (0.3)(88)(2.4) = 45.240$

From (2), $y_P = 3.0 + (0.325)(66)(2.4) = 54.480\text{ ft}$

$x_P = 45.2\text{ ft}$ ◀

$y_P = 54.5\text{ ft}$ ◀



PROBLEM 14.20

Car *A* was traveling east at high speed when it collided at Point *O* with car *B*, which was traveling north at 45 mi/h. Car *C*, which was traveling west at 60 mi/h, was 32 ft east and 10 ft north of Point *O* at the time of the collision. Because the pavement was wet, the driver of car *C* could not prevent his car from sliding into the other two cars, and the three cars, stuck together, kept sliding until they hit the utility pole *P*. Knowing that the weights of cars *A*, *B*, and *C* are, respectively, 3000 lb, 2600 lb, and 2400 lb, and neglecting the forces exerted on the cars by the wet pavement, solve the problems indicated. Knowing that the coordinates of the utility pole are $x_P = 46$ ft and $y_P = 59$ ft, determine (*a*) the time elapsed from the first collision to the stop at *P*, (*b*) the speed of car *A*.

SOLUTION

Let *t* be the time elapsed since the first collision. No external forces in the *xy* plane act on the system consisting of cars *A*, *B*, and *C* during the impacts with one another. The mass center of the system moves at the velocity it had before the collision.

Setting the origin at *O*, we can find the initial mass center $\bar{\mathbf{r}}_0$: at the moment of the first collision:

$$(m_A + m_B + m_C)(\bar{x}_0\mathbf{i} + \bar{y}_0\mathbf{j}) = m_A(0) + m_B(0) + m_C(x_C\mathbf{i} + y_C\mathbf{j})$$

$$\bar{x}_0 = 0.3x_C = (0.3)(32) = 9.6\text{ ft},\quad \bar{y}_0 = 0.3y_C = (0.3)(10) = 3\text{ ft}$$

Given velocities:

$$\mathbf{v}_A = v_A\mathbf{i},\quad \mathbf{v}_B = (45\text{ mi/h})\mathbf{i} = (66\text{ ft/s})\mathbf{j},\quad \mathbf{v}_C = (60\text{ mi/h})\mathbf{i} = (88\text{ ft/s})\mathbf{i}$$

Velocity of mass center:

$$(m_A + m_B + m_C)\mathbf{v} = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C$$

$$\bar{\mathbf{v}} = 0.375\mathbf{v}_A + 0.325\mathbf{v}_B + 0.3\mathbf{v}_C$$

Since the collided cars hit the pole at

$$\mathbf{r}_P = x_P\mathbf{i} + y_P\mathbf{j}$$

$$x_P\mathbf{i} + y_P\mathbf{j} = \bar{x}_0\mathbf{i} + \bar{y}_0\mathbf{j} + \bar{\mathbf{v}}t$$ Resolve into components.

$$x:\quad x_P = \bar{x}_0 + 0.375v_At_P - 0.3v_Ct_P \tag{1}$$

$$y:\quad y_P = \bar{y}_0 + 0.325v_Bt_P \tag{2}$$

PROBLEM 14.20 (Continued)

Data: $x_P = 46\text{ ft},\ y_P = 59\text{ ft}$

(a) From (2), $59 = 3 + 0.325(66)t_P$

$t_P = 2.611\text{ s}$ $\qquad t_P = 2.61\text{ s}$ ◀

(b) From (1), $46 = 9.6 + (0.375)v_A(2.611) - (0.3)(88)(2.611)$

$v_A = 107.58\text{ ft/s}$ $\qquad v_A = 73.3\text{ mi/h}$ ◀

PROBLEM 14.21

An expert archer demonstrates his ability by hitting tennis balls thrown by an assistant. A 2-oz tennis ball has a velocity of (32 ft/s)**i** – (7 ft/s)**j** and is 33 ft above the ground when it is hit by a 1.2-oz arrow traveling with a velocity of (165 ft/s)**j** + (230 ft/s)**k** where **j** is directed upwards. Determine the position *P* where the ball and arrow will hit the ground, relative to Point *O* located directly under the point of impact.

SOLUTION

Assume that the ball and arrow move together after the hit.

Conservation of momentum of ball and arrow during the hit.

$$m_A = \frac{1.2/16}{32.2} = 2.3292\times 10^{-3}\ \text{slug} \qquad m_B = \frac{2/16}{32.2} = 3.8820\times 10^{-3}\ \text{slug}$$

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = (m_A + m_B)\overline{\mathbf{v}}$$

$$(2.3292\times 10^{-3})(165\mathbf{j} + 230\mathbf{k}) + (3.8820\times 10^{-3})(32\mathbf{i} - 7\mathbf{j}) = (2.3292\times 10^{-3} + 3.8820\times 10^{-3})\overline{\mathbf{v}}$$

$$\overline{\mathbf{v}} = (20.0\ \text{ft/s})\mathbf{i} + (57.5\ \text{ft/s})\mathbf{j} + (86.25\ \text{ft/s})\mathbf{k}$$

After the hit, the ball and arrow move as a projectile.

Vertical motion:

$$y = y_0 + (v_y)_0 t - \frac{1}{2}gt^2$$

$$y = 33 + 57.5t - \frac{1}{2}(32.2)t^2$$

$y = 0$ at ground.

$$-16.1t^2 + 57.5t + 33 = 0 \qquad \text{Solve for } t.$$

After rejecting the negative root, $t = 4.0745\ \text{s}$

Horizontal motion:

$$x = x_0 + (v_x)_0 t$$

$$z = z_0 + (v_2)_0 t$$

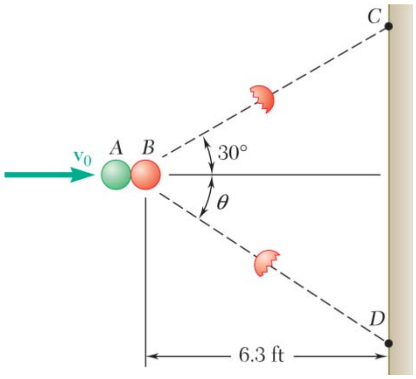
$$x = 0 + (20)(4.07448)$$

$$= 81.490\ \text{ft}$$

$$z = 0 + (86.25)(4.07448)$$

$$= 351.42\ \text{ft}$$

$$\mathbf{r}_P = (81.5\ \text{ft})\mathbf{i} + (351\ \text{ft})\mathbf{k}\ ◀$$



PROBLEM 14.22

Two spheres, each of mass m, can slide freely on a frictionless, horizontal surface. Sphere A is moving at a speed $v_0 = 16$ ft/s when it strikes sphere B which is at rest and the impact causes sphere B to break into two pieces, each of mass $m/2$. Knowing that 0.7 s after the collision one piece reaches Point C and 0.9 s after the collision the other piece reaches Point D, determine (a) the velocity of sphere A after the collision, (b) the angle θ and the speeds of the two pieces after the collision.

SOLUTION

Velocities of pieces C and D after impact and fracture.

$$(v'_C)_x = \frac{x_C}{t_C} = \frac{6.3}{0.7} = 9 \text{ ft/s}, \qquad (v'_C)_y = 9\tan 30° \text{ ft/s}$$

$$(v'_D)_x = \frac{x_D}{t_D} = \frac{6.3}{0.9} = 7 \text{ ft/s}, \qquad (v'_D)_y = -7\tan\theta \text{ ft/s}$$

Assume that during the impact the impulse between spheres A and B is directed along the x-axis. Then, the y component of momentum of sphere A is conserved.

$$0 = m(v'_A)_y$$

Conservation of momentum of system:

$$\xrightarrow{+}: \quad m_A v_0 + m_B(0) = m_A v'_A + m_C(v'_C)_x + m_D(v'_D)_x$$

$$m(16) + 0 = mv'_A + \frac{m}{2}(9) + \frac{m}{2}(7)$$

(a) $\mathbf{v}'_A = 8.00 \text{ ft/s} \longrightarrow$ ◀

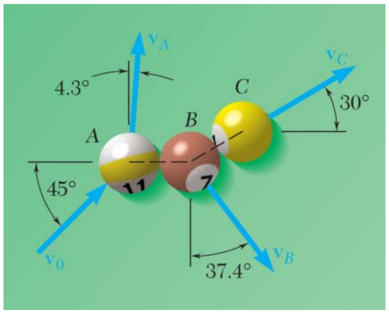
$$+\uparrow: \quad m_A(0) + m_B(0) = m_A(v'_A)_y + m_C(v'_C)_y + m_D(v'_D)_y$$

$$0 + 0 = 0 + \frac{m}{2}(9\tan 30°) - \frac{m}{2}(7\tan\theta)$$

(b) $$\tan\theta = \frac{9}{7}\tan 30° = 0.7423$$ $\theta = 36.6°$ ◀

$$v_C = \sqrt{(v_C)_x^2 + (v_C)_y^2} = \sqrt{(9)^2 + (9\tan 30°)^2}$$ $v_C = 10.39$ ft/s ◀

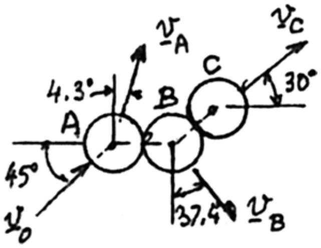
$$v_D = \sqrt{(v_D)_x^2 + (v_D)_y^2} = \sqrt{(7)^2 + (7\tan 36.6°)^2}$$ $v_D = 8.72$ ft/s ◀



PROBLEM 14.23

In a game of pool, ball *A* is moving with a velocity $\mathbf{v}_0$ when it strikes balls *B* and *C* which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated, and that $v_0 = 12$ ft/s and $v_C = 6.29$ ft/s, determine the magnitude of the velocity of (*a*) ball *A*, (*b*) ball *B*.

SOLUTION



Conservation of linear momentum. In *x* direction:

$$m(12 \text{ ft/s})\cos 45° = mv_A \sin 4.3° + mv_B \sin 37.4°$$
$$+m(6.29)\cos 30°$$
$$0.07498v_A + 0.60738v_B = 3.0380 \qquad (1)$$

In *y* direction:

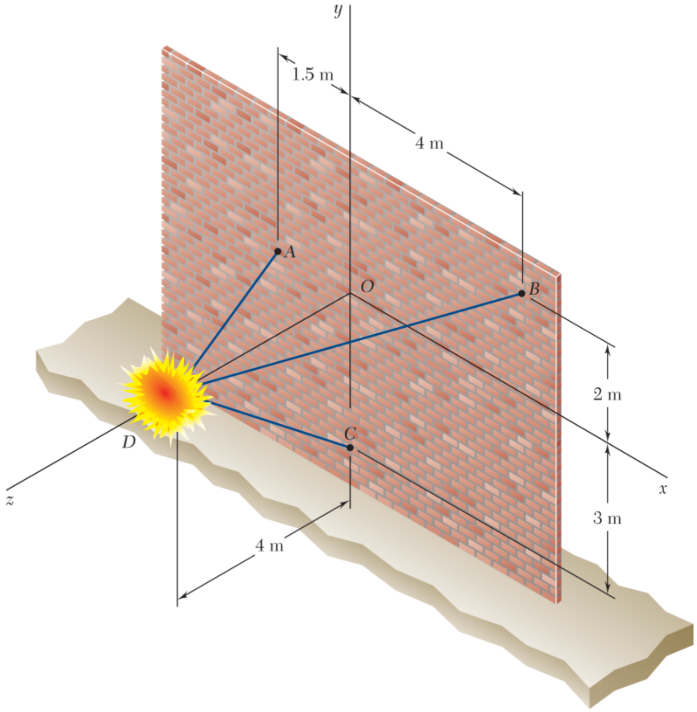
$$m(12 \text{ ft/s})\sin 45° = mv_A \cos 4.3° - mv_B \cos 37.4°$$
$$+m(6.29)\sin 30°$$
$$0.99719v_A - 0.79441v_B = 5.3403 \qquad (2)$$

(*a*) Multiply (1) by 0.79441, (2) by 0.60738, and add:

$$0.66524v_A = 5.6570 \qquad v_A = 8.50 \text{ ft/s} \blacktriangleleft$$

(*b*) Multiply (1) by 0.99719, (2) by –0.07498, and add:

$$0.66524v_B = 2.6290 \qquad v_B = 3.95 \text{ ft/s} \blacktriangleleft$$



PROBLEM 14.24

A 6-kg shell moving with a velocity $\mathbf{v}_0 = (12\text{ m/s})\mathbf{i} - (9\text{ m/s})\mathbf{j} - (360\text{ m/s})\mathbf{k}$ explodes at Point *D* into three fragments *A*, *B*, and *C* of mass, respectively, 3 kg, 2 kg, and 1 kg. Knowing that the fragments hit the vertical wall at the points indicated, determine the speed of each fragment immediately after the explosion. Assume that elevation changes due to gravity may be neglected.

SOLUTION

Position vectors (m): $\mathbf{r}_D = 4\mathbf{k}$

$$\mathbf{r}_A = -1.5\mathbf{i} \qquad \mathbf{r}_{A/D} = -1.5\mathbf{i} - 4\mathbf{k} \qquad r_{A/D} = 4.272$$

$$\mathbf{r}_B = 4\mathbf{i} + 2\mathbf{j} \qquad \mathbf{r}_{B/D} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} \qquad r_{B/D} = 6$$

$$\mathbf{r}_C = -3\mathbf{j} \qquad \mathbf{r}_{C/D} = -3\mathbf{j} - 4\mathbf{k} \qquad r_{C/D} = 5$$

Unit vectors: Along $\mathbf{r}_{A/D}$, $\boldsymbol{\lambda}_A = \dfrac{1}{4.272}(-1.5\mathbf{i} - 4\mathbf{k})$

Along $\mathbf{r}_{B/D}$, $\boldsymbol{\lambda}_B = \dfrac{1}{6}(4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$

Along $\mathbf{r}_{C/D}$, $\boldsymbol{\lambda}_C = \dfrac{1}{5}(-3\mathbf{j} - 4\mathbf{k})$

Assume that elevation changes due to gravity may be neglected. Then, the velocity vectors after the explosion have the directions of the unit vectors.

$$\mathbf{v}_A = v_A \boldsymbol{\lambda}_A$$

$$\mathbf{v}_B = v_B \boldsymbol{\lambda}_B$$

$$\mathbf{v}_C = v_C \boldsymbol{\lambda}_C$$

Conservation of momentum: $m\mathbf{v}_0 = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C$

$$6(12\mathbf{i} - 9\mathbf{j} - 360\mathbf{k}) = 3\left(\frac{v_A}{4.272}\right)(-1.5\mathbf{i} - 4\mathbf{k}) + 2\left(\frac{v_B}{6}\right)(4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$$

$$+1\left(\frac{v_C}{5}\right)(-3\mathbf{j} - 4\mathbf{k})$$

PROBLEM 14.24 (Continued)

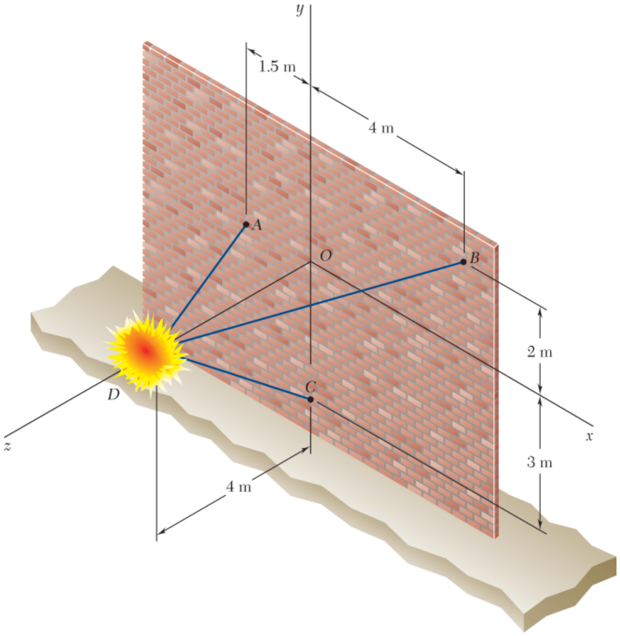
Resolve into components.

$$72 = -1.0534v_A + 1.3333v_B$$
$$-54 = 0.66667v_B - 0.60000v_C$$
$$-2160 = -2.8090v_A - 1.3333v_B - 0.80000v_C$$

Solving, $v_A = 431\text{ m/s}$ ◀

$v_B = 395\text{ m/s}$ ◀

$v_C = 528\text{ m/s}$ ◀



PROBLEM 14.25

A 6-kg shell moving with a velocity $\mathbf{v}_0 = (12\text{ m/s})\mathbf{i} - (9\text{ m/s})\mathbf{j} - (360\text{ m/s})\mathbf{k}$ explodes at Point D into three fragments A, B, and C of mass, respectively, 2 kg, 1 kg, and 3 kg. Knowing that the fragments hit the vertical wall at the points indicated, determine the speed of each fragment immediately after the explosion. Assume that elevation changes due to gravity may be neglected.

SOLUTION

Position vectors (m):

$$\mathbf{r}_D = 4\mathbf{k}$$

$$\mathbf{r}_A = -1.5\mathbf{i} \qquad \mathbf{r}_{A/D} = -1.5\mathbf{i} - 4\mathbf{k} \qquad r_{A/D} = 4.272$$

$$\mathbf{r}_B = 4\mathbf{i} + 2\mathbf{j} \qquad \mathbf{r}_{B/D} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} \qquad r_{B/D} = 6$$

$$\mathbf{r}_C = -3\mathbf{j} \qquad \mathbf{r}_{C/D} = -3\mathbf{j} - 4\mathbf{k} \qquad r_{C/D} = 5$$

Unit vectors:

Along $\mathbf{r}_{A/D}$, $$\boldsymbol{\lambda}_A = \frac{1}{4.272}(-1.5\mathbf{i} - 4\mathbf{k})$$

Along $\mathbf{r}_{B/D}$, $$\boldsymbol{\lambda}_B = \frac{1}{6}(4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$$

Along $\mathbf{r}_{C/D}$, $$\boldsymbol{\lambda}_C = \frac{1}{5}(-3\mathbf{j} - 4\mathbf{k})$$

Assume that elevation changes due to gravity may be neglected. Then the velocity vectors after the explosion have the directions of the unit vectors.

$$\mathbf{v}_A = v_A\boldsymbol{\lambda}_A$$
$$\mathbf{v}_B = v_B\boldsymbol{\lambda}_B$$
$$\mathbf{v}_C = v_C\boldsymbol{\lambda}_C$$

Conservation of momentum: $$m\mathbf{v}_0 = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C$$

Resolve into components.

$$6(12\mathbf{i} - 9\mathbf{j} - 360\mathbf{k}) = 2\left(\frac{v_A}{4.272}\right)(-1.5\mathbf{i} - 4\mathbf{k}) + 1\left(\frac{v_B}{6}\right)(4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$$
$$+3\left(\frac{v_C}{5}\right)(3\mathbf{j} - 4\mathbf{k})$$

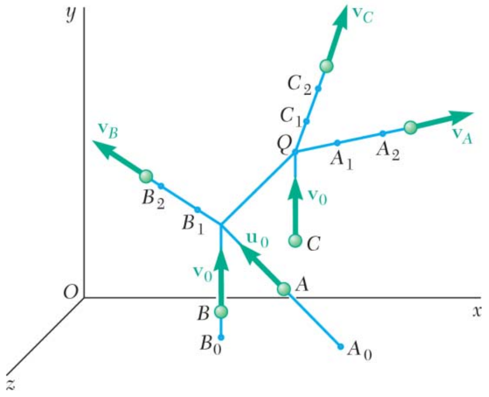
PROBLEM 14.25 (Continued)

$$72 = -0.70225 v_A + 0.66667 v_B$$
$$-54 = 0.33333 v_B - 1.8000 v_C$$
$$-2160 = -1.8727 v_A - 0.66667 v_B - 2.40000 v_C$$

Solving, $v_A = 646\text{ m/s}$ ◀

$v_B = 789\text{ m/s}$ ◀

$v_C = 176\text{ m/s}$ ◀



PROBLEM 14.26

In a scattering experiment, an alpha particle A is projected with the velocity $\mathbf{u}_0 = -(600\text{ m/s})\mathbf{i} + (750\text{ m/s})\mathbf{j} - (800\text{ m/s})\mathbf{k}$ into a stream of oxygen nuclei moving with a common velocity $\mathbf{v}_0 = (600\text{ m/s})\mathbf{j}$. After colliding successively with nuclei B and C, particle A is observed to move along the path defined by the Points $A_1(280, 240, 120)$and $A_2(360, 320, 160)$, while nuclei B and C are observed to move along paths defined, respectively, by $B_1(147, 220, 130)$, $B_2(114, 290, 120)$,and by $C_1(240, 232, 90)$ and $C_2(240, 280, 75)$. All paths are along straight lines and all coordinates are expressed in millimeters. Knowing that the mass of an oxygen nucleus is four times that of an alpha particle, determine the speed of each of the three particles after the collisions.

SOLUTION

Position vectors (mm): $\overrightarrow{A_1A_2} = 80\mathbf{i} + 80\mathbf{j} + 40\mathbf{k}$ $\quad (A_1A_2) = 120$

$\overrightarrow{B_1B_2} = -33\mathbf{i} + 70\mathbf{j} - 10\mathbf{k}$ $\quad (B_1B_2) = 78.032$

$\overrightarrow{C_1C_2} = 48\mathbf{j} - 15\mathbf{k}$ $\quad (C_1C_2) = 50.289$

Unit vectors: Along A_1A_2, $\quad \lambda_A = 0.66667\mathbf{i} + 0.66667\mathbf{j} + 0.33333\mathbf{k}$

Along B_1B_2, $\quad \lambda_B = -0.42290\mathbf{i} + 0.89707\mathbf{j} - 0.12815\mathbf{k}$

Along C_1C_2, $\quad \lambda_C = 0.95448\mathbf{j} - 0.29828\mathbf{k}$

Velocity vectors after the collisions:

$$\mathbf{v}_A = v_A\lambda_A$$
$$\mathbf{v}_B = v_B\lambda_B$$
$$\mathbf{v}_C = v_C\lambda_C$$

Conservation of momentum:

$$m\mathbf{u}_0 + 4m\mathbf{v}_0 + 4m\mathbf{v}_0 = m\mathbf{v}_A + 4m\mathbf{v}_B + 4m\mathbf{v}_C$$

Divide by m and substitute data.

$$(-600\mathbf{i} + 750\mathbf{j} - 800\mathbf{k}) + 2400\mathbf{j} + 2400\mathbf{j} = v_A\lambda_A + 4v_B\lambda_B + 4v_C\lambda_C$$

Resolving into components,

$$\mathbf{i}: \quad -600 = 0.66667v_A - 1.69160v_B$$
$$\mathbf{j}: \quad 5550 = 0.66667v_A + 3.58828v_B + 3.81792v_C$$
$$\mathbf{k}: \quad -800 = 0.33333v_A - 0.51260v_B - 1.19312v_C$$

PROBLEM 14.26 (Continued)

Solving the three equations simultaneously,

$$v_A = 919.26 \text{ m/s}$$
$$v_B = 716.98 \text{ m/s}$$
$$v_C = 619.30 \text{ m/s}$$

$v_A = 919$ m/s ◀

$v_B = 717$ m/s ◀

$v_C = 619$ m/s ◀

PROBLEM 14.27

Derive the relation

$$\mathbf{H}_O = \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + \mathbf{H}_G$$

between the angular momenta $\mathbf{H}_O$ and $\mathbf{H}_G$ defined in Eqs. (14.7) and (14.24), respectively. The vectors $\bar{\mathbf{r}}$ and $\bar{\mathbf{v}}$ define, respectively, the position and velocity of the mass center G of the system of particles relative to the newtonian frame of reference *Oxyz*, and m represents the total mass of the system.

SOLUTION

From Eq. (14.7),

$$\begin{aligned} \mathbf{H}_O &= \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{v}_i) \\ &= \sum_{i=1}^{n} \left[\left(\bar{\mathbf{r}} + \mathbf{r}_i' \right) \times m_i \mathbf{v}_i \right] \\ &= \bar{\mathbf{r}} \times \sum_{i=1}^{n} (m_i v_i) + \sum_{i=1}^{n} \left(r_i' \times m_i \mathbf{v}_i \right) \\ &= \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + \mathbf{H}_G \end{aligned}$$

PROBLEM 14.28

Show that Eq. (14.23) may be derived directly from Eq. (14.11) by substituting for $\mathbf{H}_O$ the expression given in Problem 14.27.

SOLUTION

From Eq. (14.7),

$$\begin{aligned}\mathbf{H}_O &= \sum_{i=1}^{n}(\mathbf{r}_i \times m_i\mathbf{v}_i)\\ &= \sum_{i=1}^{n}\left[\left(\overline{\mathbf{r}} + \mathbf{r}_i'\right)\times m_i\mathbf{v}_i\right]\\ &= \overline{\mathbf{r}} \times \sum_{i=1}^{n}(m_i v_i) + \sum_{i=1}^{n}\left(r_i' \times m_i\mathbf{v}_i\right)\\ &= \overline{\mathbf{r}} \times m\overline{\mathbf{v}} + \mathbf{H}_G\end{aligned}$$

Differentiating,

$$\dot{\mathbf{H}}_O = \dot{\overline{\mathbf{r}}} \times m\overline{\mathbf{v}} + \overline{\mathbf{r}} \times m\dot{\overline{\mathbf{v}}} + \dot{\mathbf{H}}_G$$

Using Eq. (14.11),

$$\begin{aligned}\Sigma\mathbf{M}_O &= \dot{\overline{\mathbf{r}}} \times m\overline{\mathbf{v}} + \overline{\mathbf{r}} \times m\dot{\overline{\mathbf{v}}} + \dot{\mathbf{H}}_G\\ &= \overline{\mathbf{v}} \times m\overline{\mathbf{v}} + \overline{\mathbf{r}} \times m\overline{\mathbf{a}} + \dot{\mathbf{H}}_G\\ &= 0 + \overline{\mathbf{r}} \times \left(\sum_{i=1}^{n}\mathbf{F}_i\right) + \dot{\mathbf{H}}_G \qquad (1)\end{aligned}$$

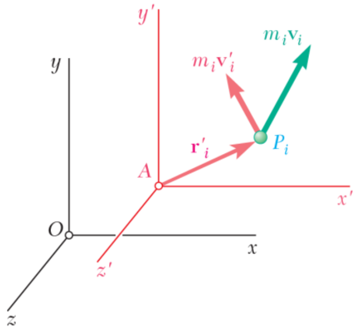
But

$$\sum_{i=1}^{n}\mathbf{M}_O = \sum_{i=1}^{n}\mathbf{M}_G + \sum_{i=1}^{n}\overline{\mathbf{r}} \times \left(\sum_{i=1}^{n}F_i\right)$$

Subtracting $\overline{\mathbf{r}} \times \left(\Sigma_{i=1}^{n}F_i\right)$ from each side of Eq. (1) gives

$$\Sigma\mathbf{M}_G = \dot{\mathbf{H}}_G$$

PROBLEM 14.29



Consider the frame of reference $Ax'y'z'$ in translation with respect to the newtonian frame of reference $Oxyz$. We define the angular momentum $\mathbf{H}'_A$ of a system of n particles about A as the sum

$$\mathbf{H}'_A = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{v}'_i \qquad (1)$$

of the moments about A of the momenta $m_i\mathbf{v}'_i$ of the particles in their motion relative to the frame $Ax'y'z'$. Denoting by $\mathbf{H}_A$ the sum

$$\mathbf{H}_A = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{v}_i \qquad (2)$$

of the moments about A of the momenta $m_i\mathbf{v}_i$ of the particles in their motion relative to the newtonian frame $Oxyz$, show that $\mathbf{H}_A = \mathbf{H}'_A$ at a given instant if, and only if, one of the following conditions is satisfied at that instant: (*a*) A has zero velocity with respect to the frame $Oxyz$, (*b*) A coincides with the mass center G of the system, (*c*) the velocity $\mathbf{v}_A$ relative to $Oxyz$ is directed along the line AG.

SOLUTION

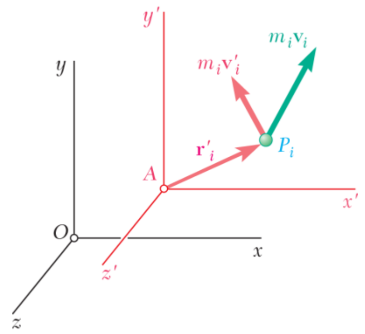
$$\mathbf{v}_i = \mathbf{v}_A + \mathbf{v}'_i$$

$$\begin{aligned}
\mathbf{H}_A &= \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{v}_i \\
&= \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \left(\mathbf{v}_A + \mathbf{v}'_i\right) \\
&= \sum_{i=1}^{n} \left(\mathbf{r}'_i \times m_i \mathbf{v}_A\right) + \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{v}_i \\
&= \sum_{i=1}^{n} \left(m_i \mathbf{r}'_i\right) \times \mathbf{v}_A + \mathbf{H}'_A \\
&= \sum_{i=1}^{n} m_i (\mathbf{r}_i - \mathbf{r}_A) \times \mathbf{v}_A + \mathbf{H}'_A \\
&= m(\overline{\mathbf{r}} - \mathbf{r}_A) \times \mathbf{v}_A + \mathbf{H}'_A
\end{aligned}$$

$$\mathbf{H}_A = \mathbf{H}'_A \quad \text{if, and only if,} \qquad m(\mathbf{r} - \mathbf{r}_A) \times \mathbf{v}_A = 0$$

This condition is satisfied if

	(*a*)	$\mathbf{v}_A = 0$	Point A has zero velocity.
or	(*b*)	$\overline{\mathbf{r}} = \mathbf{r}_A$	Point A coincides with the mass center.
or	(*c*)	$\mathbf{v}_A$ is parallel to $\overline{\mathbf{r}} - \mathbf{r}_A$.	Velocity $\mathbf{v}_A$ is directed along line AG.



PROBLEM 14.30

Show that the relation $\Sigma\mathbf{M}_A = \dot{\mathbf{H}}'_A$, where $\mathbf{H}'_A$ is defined by Eq. (1) of Problem 14.29 and where $\Sigma\mathbf{M}_A$ represents the sum of the moments about A of the external forces acting on the system of particles, is valid if, and only if, one of the following conditions is satisfied: (*a*) the frame $Ax'y'z'$ is itself a newtonian frame of reference, (*b*) A coincides with the mass center G, (*c*) the acceleration $\mathbf{a}_A$ of A relative to $Oxyz$ is directed along the line AG.

SOLUTION

From equation (1),

$$\mathbf{H}'_A = \sum_{i=1}^{n}\left(\mathbf{r}'_i \times m_i\mathbf{v}'_i\right)$$

$$\mathbf{H}'_A = \sum_{i=1}^{n}[(\mathbf{r}_i - \mathbf{r}_A)\times m_i(\mathbf{v}_i - \mathbf{v}_A)]$$

Differentiate with respect to time.

$$\dot{\mathbf{H}}'_A = \sum_{i=1}^{n}[(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_A)\times m_i(\mathbf{v}_i - \mathbf{v}_A)] + \sum_{i=1}^{n}[(\mathbf{r}_i - \mathbf{r}_A)\times m_i(\dot{\mathbf{v}}_i - \dot{\mathbf{v}}_A)]$$

But $\dot{\mathbf{r}}_i = \mathbf{v}_i$

$\dot{\mathbf{v}}_i = \mathbf{a}_i$

$\dot{\mathbf{r}}_A = \mathbf{v}_A$

and $\dot{\mathbf{v}}_A = \mathbf{a}_A$

Hence,

$$\dot{\mathbf{H}}'_A = 0 + \sum_{i=1}^{n}[(\mathbf{r}_i - \mathbf{r}_A)\times m_i(\mathbf{a}_i - \mathbf{a}_A)]$$

$$= \sum_{i=1}^{n}[(\mathbf{r}_i - \mathbf{r}_A)\times(\mathbf{F}_i - m_i\mathbf{a}_A)]$$

$$= \sum_{i=1}^{n}[(\mathbf{r}_i - \mathbf{r}_A)\times\mathbf{F}_i] - \sum_{i=1}^{n}[m_i(\mathbf{r}_i - \mathbf{r}_A)]\times\mathbf{a}_A$$

$$= \mathbf{M}_A - m(\bar{\mathbf{r}} - r_A)\times\mathbf{a}_A$$

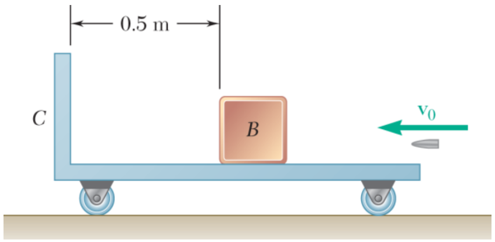
$$\dot{\mathbf{H}}'_A = M_A \qquad \text{if, and only if,} \qquad m(\bar{\mathbf{r}} - \mathbf{r}_A)\times\mathbf{a}_A = 0$$

This condition is satisfied if

(*a*) $\mathbf{a}_A = 0$ The frame is newtonian.

or (*b*) $\bar{\mathbf{r}} = \mathbf{r}_A$ Point A coincides with the mass center.

or (*c*) $\mathbf{a}_A$ is parallel to $\bar{\mathbf{r}} - \mathbf{r}_A$. Acceleration $\mathbf{a}_A$ is directed along line AG.



PROBLEM 14.31

Determine the energy lost due to friction and the impacts for Problem 14.1.

PROBLEM 14.1 A 30-g bullet is fired with a horizontal velocity of 450 m/s and becomes embedded in block *B* which has a mass of 3 kg. After the impact, block *B* slides on 30-kg carrier *C* until it impacts the end of the carrier. Knowing the impact between *B* and *C* is perfectly plastic and the coefficient of kinetic friction between *B* and *C* is 0.2, determine (*a*) the velocity of the bullet and *B* after the first impact, (*b*) the final velocity of the carrier.

SOLUTION

From the solution to Problem 4.1 the velocity of *A* and *B* after the first impact is $v' = 4.4554$ m/s and the velocity common to *A*, *B*, and *C* after the sliding of block *B* and bullet *A* relative to the carrier *C* has ceased in $v'' = 0.4087$ m/s.

Friction loss due to sliding:

Normal force:
$$N = W_A + W_B = (m_A + m_B)g$$
$$= (0.030 \text{ kg} + 3 \text{ kg})(9.81 \text{ m/s}) = 29.724 \text{ N}$$

Friction force:
$$F_f = \mu_k N = (0.2)(29.724) = 5.945 \text{ N}$$

Relative sliding distance: Assume $d = 0.5$ m.

Energy loss due to friction: $F_f d = (5.945)(0.5)$ $\qquad F_f d = 2.97 \text{ J}$ ◀

Kinetic energy of block with embedded bullet immediately after first impact:

$$T'_{AB} = \frac{1}{2}(m_A + m_B)(v')^2 = \frac{1}{2}(3.03 \text{ kg})(4.4554 \text{ m/s})^2 = 30.07 \text{ J}$$

Final kinetic energy of *A*, *B*, and *C* together

$$T''_{ABC} = \frac{1}{2}(m_A + m_B + m_C)(v'')^2 = \frac{1}{2}(33.03 \text{ kg})(0.4087 \text{ m/s})^2 = 2.76 \text{ J}$$

Loss due to friction and stopping impact: $T'_{AB} - T''_{ABC} = 30.07 - 2.76 = 27.31 \text{ J}$

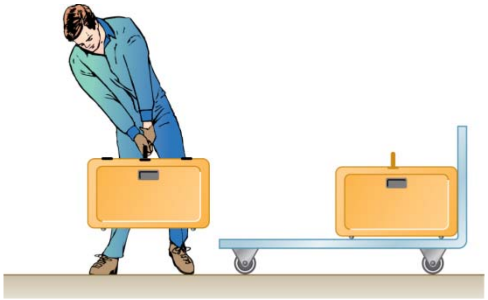
Since $27.31 \text{ J} \geq 2.97 \text{ J}$, the block slides 0.5 m relative to the carrier as assumed above.

Impact loss due to *AB* impacting the carrier: $27.31 - 2.97 = 24.34$ $\qquad \text{Loss} = 24.3 \text{ J}$ ◀

Initial kinetic energy of system *ABC*.

$$T_0 = \frac{1}{2}m_A v_0^2 = \frac{1}{2}(0.030 \text{ kg})(450 \text{ m/s})^2 = 3037.5 \text{ J}$$

Impact loss at first impact: $T_0 - T'_{AB} = 3037.5 - 30.07$ $\qquad \text{Loss} = 3007 \text{ J}$ ◀



PROBLEM 14.32

Assuming that the airline employee of Prob. 14.3 first tosses the 30-lb suitcase on the baggage carrier, determine the energy lost (*a*) as the first suitcase hits the carrier, (*b*) as the second suitcase hits the carrier.

PROBLEM 14.3 An airline employee tosses two suitcases, of weight 30 lb and 40 lb, respectively, onto a 50-lb baggage carrier in rapid succession. Knowing that the carrier is initially at rest and that the employee imparts a 9-ft/s horizontal velocity to the 30-lb suitcase and a 6-ft/s horizontal velocity to the 40-lb suitcase, determine the final velocity of the baggage carrier if the first suitcase tossed onto the carrier is (a) the 30-lb suitcase,(b) the 40-lb suitcase.

SOLUTION

The weights are $W_A = 30\text{ lb}, \quad W_B = 40\text{ lb}, \quad \text{and} \quad W_C = 50\text{ lb}.$

Initial velocities: $(v_A)_0 = 9\text{ ft/s} \longrightarrow, \quad (v_B)_0 = 6\text{ ft/s} \longrightarrow, \quad \text{and} \quad (v_C)_0 = 0.$

There are no horizontal external forces acting during the impacts, and the baggage carrier is free to coast between the impacts.

Let v_1 be the common velocity of suitcase *A* and the carrier after the first impact and v_2 be the common velocity of the two suitcases and the carrier after the second impact.

Initial momenta: $\frac{W_A}{g}(v_A)_0, \quad \frac{W_B}{g}(v_B)_0, \quad \text{and} \quad 0.$

Suitcase A impacts carrier. Conservation of momentum.

$$\frac{W_A}{g}(v_A)_0 + 0 = \frac{W_A + W_C}{g}v_1 \qquad v_1 = \frac{W_A(v_A)_0}{W_A + W_C} = \frac{(30)(9)}{80} = 3.375\text{ ft/s}$$

Kinetic energies:

Before: $$T_0 = \frac{1}{2}\frac{W_A}{g}(v_A)_0^2 = \frac{1}{2}\frac{30}{32.2}(9)^2 = 37.733\text{ ft}\cdot\text{lb}$$

After: $$T_1 = \frac{1}{2}\frac{W_A + W_C}{g}v_1^2 = \frac{1}{2}\frac{30+50}{32.2}(3.375)^2 = 14.150\text{ ft}\cdot\text{lb}$$

(*a*) Lost: $T_0 - T_1 = 23.583\text{ ft}\cdot\text{lb}$ energy lost $= 23.6\text{ ft}\cdot\text{lb}$ ◀

Suitcase B impacts on suitcase A and carrier. Conservation of momentum.

$$\frac{W_B}{g}(v_B)_0 + \frac{W_A + W_C}{g}v_1 = \frac{W_A + W_B + W_C}{g}v_2$$

$$v_2 = \frac{W_B(v_B)_0 + (W_A + W_C)v_1}{W_A + W_B + W_C} = \frac{(40)(6) + (80)(3.375)}{120} = 4.25\text{ ft/s}$$

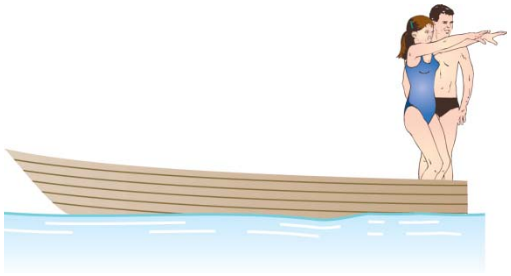
PROBLEM 14.32 (Continued)

Kinetic energies:

Before: $T_1' = \frac{1}{2}\frac{W_A + W_C}{g}v_1^2 + \frac{1}{2}\frac{W_B}{g}(v_B)_0^2 = 14.150 + \frac{1}{2}\frac{40}{32.2}(6)^2 = 36.510 \text{ ft}\cdot\text{lb}$

After: $T_2 = \frac{1}{2}\frac{W_A + W_B + W_C}{g}v_2^2 = \frac{1}{2}\frac{120}{32.2}(4.25)^2 = 33.657 \text{ ft}\cdot\text{lb}$

(*b*) Lost: $T_1' - T_2 = 2.853 \text{ ft}\cdot\text{lb}$ energy lost $= 2.85 \text{ ft}\cdot\text{lb}$ ◀



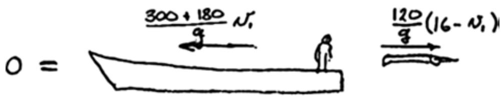
PROBLEM 14.33

In Problem 14.6, determine the work done by the woman and by the man as each dives from the boat, assuming that the woman dives first.

PROBLEM 14.6 A 180-lb man and a 120-lb woman stand side by side at the same end of a 300-lb boat, ready to dive, each with a 16-ft/s velocity relative to the boat. Determine the velocity of the boat after they have both dived, if (*a*) the woman dives first, (*b*) the man dives first.

SOLUTION

Woman dives first.



Conservation of momentum: $\frac{120}{g}(16-v_1)-\frac{300+180}{g}v_1=0$

$$v_1=\frac{(120)(16)}{600}=3.20\text{ ft/s}\leftarrow \quad\text{and}\quad 16-v_1=12.80\text{ ft/s}\rightarrow$$

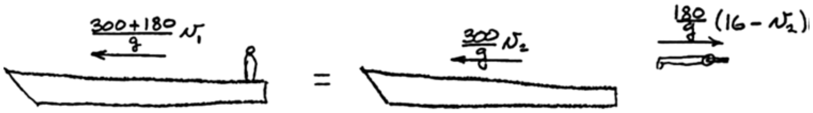
Kinetic energy before dive: $T_0=0$

Kinetic energy after dive: $T_1=\frac{1}{2}\frac{300+180}{32.2}(3.20)^2+\frac{1}{2}\frac{120}{32.2}(12.80)^2$

$$=381.61\text{ ft}\cdot\text{lb}$$

Work of woman: $T_1-T_0=381.61\text{ ft}\cdot\text{lb}$ $\qquad T_1-T_0=382\text{ ft}\cdot\text{lb}$ ◀

Man dives next. Conservation of momentum:



$$-\frac{300+180}{g}v_1=-\frac{300}{g}v_2+\frac{180}{g}(16-v_2)$$

$$v_2=\frac{480v_1+(180)(16)}{480}=9.20\text{ ft/s}\longleftarrow$$

$$16-9.20=6.80\text{ ft/s}\longrightarrow$$

Kinetic energy before dive: $T_1'=\frac{1}{2}\frac{300+180}{32.2}(3.20)^2$

$$=76.323\text{ ft}\cdot\text{lb}$$

Kinetic energy after dive: $T_2'=\frac{1}{2}\frac{300}{32.2}(9.20)^2+\frac{1}{2}\frac{180}{32.2}(6.80)^2$

$$=523.53\text{ ft}\cdot\text{lb}$$

Work of man: $T_2'-T_1'=447.2\text{ ft}\cdot\text{lb}$ $\qquad T_2'-T_1'=447\text{ ft}\cdot\text{lb}$ ◀

PROBLEM 14.34

Determine the energy lost as a result of the series of collisions described in Problem 14.8.

PROBLEM 14.8 Two identical cars *A* and *B* are at rest on a loading dock with brakes released. Car *C*, of a slightly different style but of the same weight, has been pushed by dockworkers and hits car *B* with a velocity of 1.5 m/s. Knowing that the coefficient of restitution is 0.8 between *B* and *C* and 0.5 between *A* and *B*, determine the velocity of each car after all collisions have taken place.

SOLUTION

From the solution to Problem 14.8

$$v_A = 2 \text{ m/s}, \quad v_B = v_C = 0,$$

$$v'_B = 1.35 \text{ m/s}, \quad v'_C = 0.15 \text{ m/s}$$

$$v'_A = 1.013 \text{ m/s} \quad v''_B = 0.338 \text{ m/s},$$

$$m_A = m_B = m_C = m$$

C hits *B*:

$$T_1 = \frac{1}{2} m_C v_C^2 = \frac{1}{2} m(1.5 \text{ m/s})^2 = 1.125m \text{ J}$$

$$T_2 = \frac{1}{2} m_B (v'_B)^2 + \frac{1}{2} m_C (v'_C)^2$$

$$T = \frac{1}{2} m(1.35 \text{ m/s})^2 + \frac{1}{2} \text{m}(0.15 \text{ m/s})^2 = 0.9m \text{ J}$$

$\text{Loss} = T_1 - T_2$: $\text{Loss} = 0.225m \text{ J}$ ◀

B hits *A*:

$$T_3 = \frac{1}{2} m_B (v'_B)^2 = \frac{1}{2} m(1.35)^2 = 0.9113m \text{ J}$$

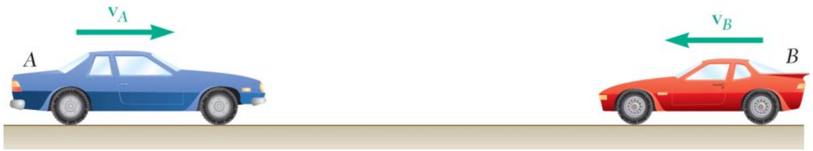
$$T_4 = \frac{1}{2} m_B (v''_B)^2 + \frac{1}{2} m_A (v'_A)^2$$

$$= \frac{1}{2} \text{m}(0.338 \text{ m/s})^2 + \frac{1}{2} m(1.013 \text{ m/s})^2 = 0.5702 \text{m J}$$

$\text{Loss} = T_3 - T_4$: $\text{Loss} = 0.341m \text{ J}$ ◀

PROBLEM 14.35

Two automobiles *A* and *B*, of mass m_A and m_B, respectively, are traveling in opposite directions when they collide head on. The impact is assumed perfectly plastic, and it is further assumed that the energy absorbed by each automobile is equal to its loss of kinetic energy with respect to a moving frame of reference attached to the mass center of the two-vehicle system. Denoting by E_A and E_B, respectively, the energy absorbed by automobile *A* and by automobile *B*, (*a*) show that $E_A/E_B = m_B/m_A$, that is, the amount of energy absorbed by each vehicle is inversely proportional to its mass, (*b*) compute E_A and E_B, knowing that $m_A = 1600$ kg and $m_B = 900$ kg and that the speeds of *A* and *B* are, respectively, 90 km/h and 60 km/h.



SOLUTION

Velocity of mass center: $(m_A + m_B)\overline{\mathbf{v}} = m_A\mathbf{v}_A + m_B\mathbf{v}_B$

$$\overline{\mathbf{v}} = \frac{m_A\mathbf{v}_A + m_B\mathbf{v}_B}{m_A + m_B}$$

Velocities relative to the mass center:

$$\mathbf{v}'_A = \mathbf{v}_A - \overline{\mathbf{v}} = \mathbf{v}_A - \frac{m_A\mathbf{v}_A + m_B\mathbf{v}_B}{m_A + m_B} = \frac{m_B(\mathbf{v}_A + \mathbf{v}_B)}{m_A + m_B}$$

$$\mathbf{v}'_B = \mathbf{v}_B - \overline{\mathbf{v}} = \mathbf{v}_B - \frac{m_A\mathbf{v}_A + m_B\mathbf{v}_B}{m_A + m_B} = \frac{m_A(\mathbf{v}_A + \mathbf{v}_B)}{m_A + m_B}$$

Energies:

$$E_A = \frac{1}{2}m_A\mathbf{v}'_A \cdot \mathbf{v}'_A = \frac{m_A m_B^2(\mathbf{v}_A + \mathbf{v}_B)\cdot(\mathbf{v}_A + \mathbf{v}_B)}{2(m_A + m_B)^2}$$

$$E_B = \frac{1}{2}m_B\mathbf{v}'_B \cdot \mathbf{v}'_B = \frac{m_A^2 m_B(\mathbf{v}_A + \mathbf{v}_B)\cdot(\mathbf{v}_A + \mathbf{v}_B)}{2(m_A + m_B)^2}$$

(*a*) Ratio: $$\frac{E_A}{E_B} = \frac{m_B}{m_A} \blacktriangleleft$$

(*b*)

$$\mathbf{v}_A = 90 \text{ km/h} = 25 \text{ m/s} \longrightarrow$$

$$\mathbf{v}_B = 60 \text{ km/h} = 16.667 \text{ m/s} \longleftarrow$$

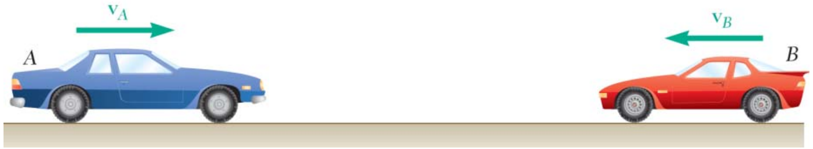
$$\mathbf{v}_A + \mathbf{v}_B = 41.667 \text{ m/s} \longrightarrow$$

$$E_A = \frac{(1600)(900)^2(41.667)^2}{(2)(2500)^2} = 180.0\times10^3 \text{ J} \qquad E_A = 180.0 \text{ kJ} \blacktriangleleft$$

$$E_B = \frac{(1600)^2(900)(41.667)^2}{(2)(2500)^2} = 320\times10^3 \text{ J} \qquad E_B = 320 \text{ kJ} \blacktriangleleft$$

PROBLEM 14.36

It is assumed that each of the two automobiles involved in the collision described in Problem 14.35 had been designed to safely withstand a test in which it crashed into a solid, immovable wall at the speed v_0. The severity of the collision of Problem 14.35 may then be measured for each vehicle by the ratio of the energy it absorbed in the collision to the energy it absorbed in the test. On that basis, show that the collision described in Problem 14.35 is $(m_A/m_B)^2$ times more severe for automobile B than for automobile A.



SOLUTION

Velocity of mass center:

$$(m_A + m_B)\overline{\mathbf{v}} = m_A\mathbf{v}_A + m_B\mathbf{v}_B$$

$$\overline{\mathbf{v}} = \frac{m_A\mathbf{v}_A + m_B\mathbf{v}_B}{m_A + m_B}$$

Velocities relative to the mass center:

$$\mathbf{v}'_A = \mathbf{v}_A - \overline{\mathbf{v}} = \mathbf{v}_A - \frac{m_A\mathbf{v}_A + m_B\mathbf{v}_B}{m_A + m_B} = \frac{m_B(\mathbf{v}_A + \mathbf{v}_B)}{m_A + m_B}$$

$$\mathbf{v}'_B = \mathbf{v}_B - \overline{\mathbf{v}} = \mathbf{v}_B - \frac{m_A\mathbf{v}_A + m_B\mathbf{v}_B}{m_A + m_B} = \frac{m_A(\mathbf{v}_A + \mathbf{v}_B)}{m_A + m_B}$$

Energies:

$$E_A = \frac{1}{2}m_A\mathbf{v}'_A \cdot \mathbf{v}'_A = \frac{m_A m_B^2(\mathbf{v}_A + \mathbf{v}_B)\cdot(\mathbf{v}_A + \mathbf{v}_B)}{2(m_A + m_B)^2}$$

$$E_B = \frac{1}{2}m_B\mathbf{v}'_B \cdot \mathbf{v}'_B = \frac{m_A^2 m_B(\mathbf{v}_A + \mathbf{v}_B)\cdot(\mathbf{v}_A + \mathbf{v}_B)}{2(m_A + m_B)^2}$$

Energies from tests:

$$(E_A)_0 = \frac{1}{2}m_A v_0^2, \quad (E_B)_0 = \frac{1}{2}m_B v_0^2$$

Severities:

$$S_A = \frac{E_A}{(E_A)_0} = \frac{m_B^2(\mathbf{v}_A + \mathbf{v}_B)\cdot(\mathbf{v}_A + \mathbf{v}_B)}{(m_A + m_B)^2 v_0^2}$$

$$S_B = \frac{E_B}{(E_B)_0} = \frac{m_A^2(\mathbf{v}_A + \mathbf{v}_B)\cdot(\mathbf{v}_A + \mathbf{v}_B)}{(m_A + m_B)^2 v_0^2}$$

Ratio:

$$\frac{S_A}{S_B} = \frac{m_B^2}{m_A^2} \blacktriangleleft$$

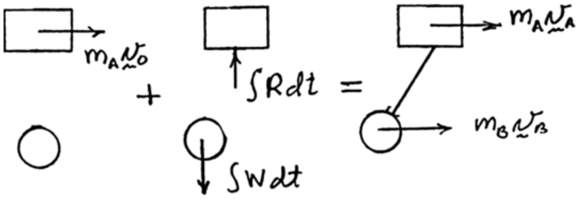
PROBLEM 14.37

Solve Sample Problem 14.5, assuming that cart *A* is given an initial horizontal velocity $\mathbf{v}_0$ while ball *B* is at rest.

SOLUTION

(*a*) Velocity of *B* at maximum elevation: At maximum elevation, ball *B* is at rest relative to cart *A*. $\mathbf{v}_B = \mathbf{v}_A$

Use impulse-momentum principle.



$m_A \underset{\sim}{v}_0$
+
$\int R\,dt$
=
$\int W\,dt$
$m_A \underset{\sim}{v}_A$
$m_B \underset{\sim}{v}_B$

x components:

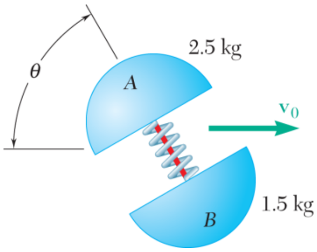
$$m_A v_0 + 0 = m_A v_A + m_B v_B$$
$$= (m_A + m_B) v_B$$

$$v_B = \frac{m_A v_0}{m_A + m_B} \longrightarrow \blacktriangleleft$$

(*b*) Conservation of energy:

$$T_1 = \frac{1}{2} m_A v_0^2, \quad V_1 = 0$$
$$T_2 = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$$
$$= \frac{1}{2}(m_A + m_B) v_B^2$$
$$= \frac{m_A^2 v_0^2}{2(m_A + m_B)}$$
$$V_2 = m_B g h$$
$$T_2 + V_2 = T_1 + V_1$$
$$\frac{m_A^2 v_0^2}{2(m_A + m_B)} + m_B g h = \frac{1}{2} m_A v_0^2$$

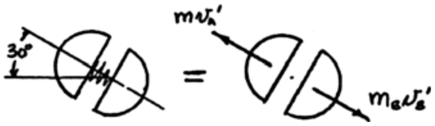
$$h = \frac{1}{2 m_B g}\left[m_A v_0^2 - \frac{m_A^2 v_0^2}{m_A + m_B} \right] \qquad h = \frac{m_A}{m_A + m_B} \frac{v_0^2}{2g} \blacktriangleleft$$



PROBLEM 14.38

Two hemispheres are held together by a cord which maintains a spring under compression (the spring is not attached to the hemispheres). The potential energy of the compressed spring is 120 J and the assembly has an initial velocity $\mathbf{v}_0$ of magnitude $v_0 = 8$ m/s. Knowing that the cord is severed when $\theta = 30°$, causing the hemispheres to fly apart, determine the resulting velocity of each hemisphere.

SOLUTION



Use a frame of reference moving with the mass center.

Conservation of momentum:

$$0 = -m_A v_A' + m_B v_B'$$

$$v_A' = \frac{m_B}{m_A} v_B'$$

Conservation of energy:

$$V = \frac{1}{2} m_A (v_A')^2 + \frac{1}{2} m_B (v_B')^2$$

$$= \frac{1}{2} m_A \left(\frac{m_B}{m_A} v_B' \right)^2 + \frac{1}{2} m_B (v_B')^2$$

$$= \frac{m_B (m_A + m_B)}{2m_A} (v_B')^2$$

$$v_B' = \sqrt{\frac{2m_A V}{m_B (m_A + m_B)}}$$

Data: $m_A = 2.5$ kg $\quad m_B = 1.5$ kg

$$V = 120 \text{ J}$$

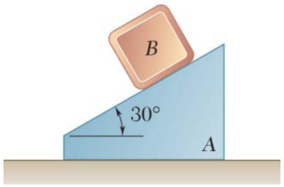
$$v_B' = \sqrt{\frac{(2)(2.5)(120)}{(1.5)(4.0)}} = 10 \qquad v_B' = 10 \text{ m/s} \searrow 30°$$

$$v_A' = \frac{1.5}{2.5}(10) = 6 \qquad v_A' = 6 \text{ m/s} \nwarrow 30°$$

Velocities of *A* and *B*.

$$\mathbf{v}_A = [8 \text{ m/s} \rightarrow] + [6 \text{ m/s} \nwarrow 30°] \qquad \mathbf{v}_A = 4.11 \text{ m/s} \nearrow 46.9° \blacktriangleleft$$

$$\mathbf{v}_B = [8 \text{ m/s} \rightarrow] + [10 \text{ m/s} \searrow 30°] \qquad \mathbf{v}_B = 17.39 \text{ m/s} \searrow 16.7° \blacktriangleleft$$



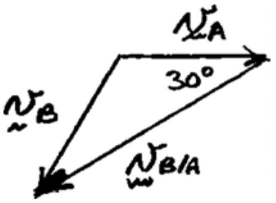
PROBLEM 14.39

A 15-lb block *B* starts from rest and slides on the 25-lb wedge *A*, which is supported by a horizontal surface. Neglecting friction, determine (*a*) the velocity of *B* relative to *A* after it has slid 3 ft down the inclined surface of the wedge, (*b*) the corresponding velocity of *A*.

SOLUTION

Kinematics:

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

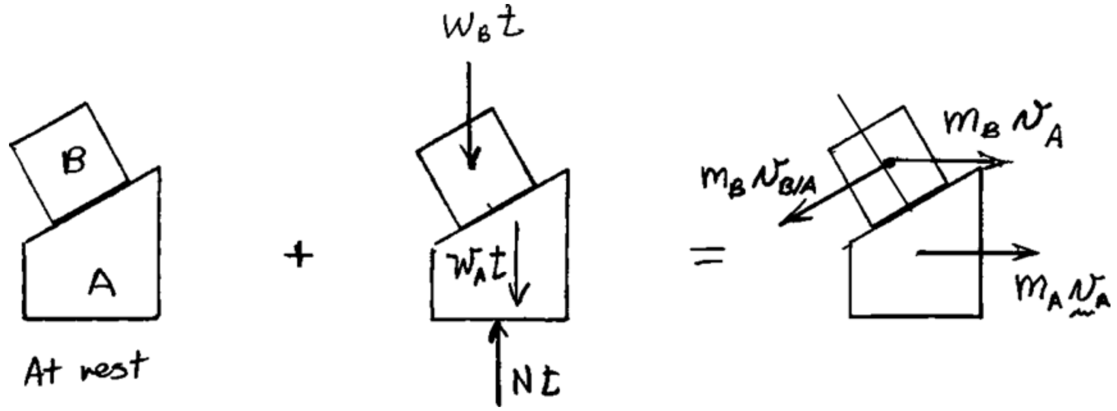


Law of cosines:

$$v_B^2 = v_A^2 + v_{B/A}^2 - 2v_A v_{B/A}\cos 30° \tag{1}$$

Principle of impulse and momentum:

$$\Sigma m\mathbf{v}_0 + \Sigma \mathbf{F}t = \Sigma m\mathbf{v}$$



Components →:

$$0 + 0 = m_A v_A + m_B(v_A - v_{B/A}\cos 30°)$$

$$v_A = \frac{m_B v_{B/A}\cos 30°}{m_A + m_B} = \frac{15\cos 30°}{25+15}v_{B/A}$$
$$= 0.32476\, v_{B/A}$$

From Eq. (1)

$$v_B^2 = (0.32476)^2\, v_{B/A}^2 + v_{B/A}^2 - (2)(0.32476)\cos 30° v_{B/A}^2$$
$$= 0.54297\ v_{B/A}^2$$

Principle of conservation of energy:

$$T_0 + V_0 = T_1 + V_1$$
$$0 + 0 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 - W_B d\sin 30°$$
$$\frac{1}{2}\frac{W_A}{g}(0.32476 v_{B/A})^2 + \frac{1}{2}\frac{W_B}{g}(0.54297)\, v_{B/A}^2 = W_B d\sin 30°$$

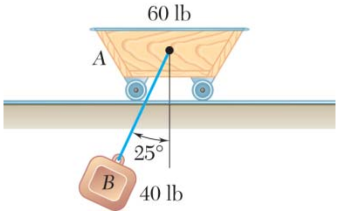
PROBLEM 14.39 (Continued)

$$\left[\frac{1}{2}\frac{25}{32.2}(0.32476)^2+\frac{1}{2}\frac{15}{32.2}(0.54297)\right]v_{B/A}^2=(15)(3)\sin 30°$$

$$0.16741v_{B/A}^2=22.5$$

(*a*) $\mathbf{v}_{B/A}=11.59\text{ ft/s}$ ↙ 30° ◀

(b) $v_A=(0.32476)(11.59)$ $\mathbf{v}_A=3.76\text{ ft/s}$ → ◀


60 lb
A
25°
B
40 lb

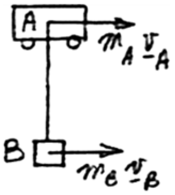
PROBLEM 14.40

A 40-lb block B is suspended from a 6-ft cord attached to a 60-lb cart A, which may roll freely on a frictionless, horizontal track. If the system is released from rest in the position shown, determine the velocities of A and B as B passes directly under A.

SOLUTION

Conservation of linear momentum:

Since block and cart are initially at rest,


A
$m_A \underline{v}_A$
B
$m_B \underline{v}_B$

$$\mathbf{L}_0 = 0$$

Thus, as B passes under A,

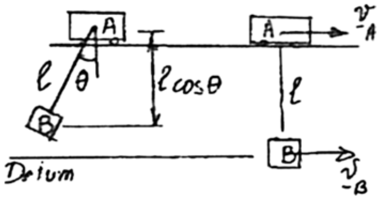
$$\mathbf{L} = m_A \mathbf{v}_A + m_B \mathbf{v}_B = 0$$

$$\xrightarrow{+} m_A v_A + m_B v_B = 0$$

$$v_A = -\frac{m_B}{m_A} v_B \qquad (1)$$

Conservation of energy:

Initially,


A
ℓ
θ
ℓ cos θ
Datum
A
$\underline{v}_A$
ℓ
B
$\underline{v}_B$
B

$$T_0 = 0$$

$$V_0 = m_B gl(1 - \cos\theta)$$

As B passes under A,

$$T = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$$

$$V = 0$$

Thus,

$$T_0 + V_0 = T + V: \quad m_B gl(1 - \cos\theta) = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$$

Substituting for v_A from (1) and multiplying by 2:

$$2m_B gl(1 - \cos\theta) = m_A \left(\frac{m_B^2}{m_A^2} v_B^2 \right) + m_B v_B^2$$

$$= \left(\frac{m_B^2}{m_A} + m_B \right) v_B^2 = m_B \frac{m_B + m_A}{m_A} v_B^2$$

$$v_B = \sqrt{\frac{2m_A}{m_A + m_B} gl(1 - \cos\theta)} \qquad (2)$$

PROBLEM 14.40 (Continued)

Given data:

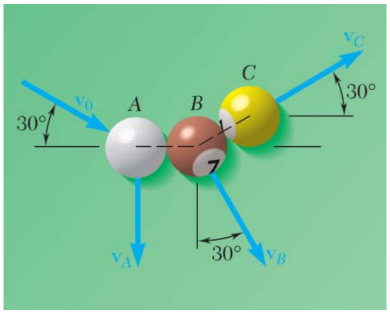
$$w_A = 60 \text{ lb} \quad w_B = 40 \text{ lb}, \quad l = 6 \text{ ft}$$

$$g = 32.2 \text{ ft/s}^2 \quad \theta = 25°$$

$$\frac{2m_A}{m_A + m_B} = \frac{2w_A}{w_A + w_B} = \frac{(2)(60)}{60+40} = 1.2$$

From Eq. (2), $v_B = \sqrt{(1.2)(32.2)(6)(1-\cos 25°)}$ $\mathbf{v}_B = 4.66 \text{ ft/s} \longrightarrow$ ◀

From Eq. (1), $v_A = -\dfrac{w_B}{w_A} v_B = -\dfrac{40}{60}(4.66)$ $\mathbf{v}_A = 3.11 \text{ ft/s} \longleftarrow$ ◀



PROBLEM 14.41

In a game of pool, ball A is moving with a velocity $\mathbf{v}_0$ of magnitude $v_0 = 15$ ft/s when it strikes balls B and C, which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated and assuming frictionless surfaces and perfectly elastic impact (that is, conservation of energy), determine the magnitudes of the velocities $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$.

SOLUTION

Velocity vectors:

$$\mathbf{v}_0 = v_0(\cos 30°\mathbf{i} - \sin 30°\mathbf{j}) \qquad v_0 = 15 \text{ ft/s}$$

$$\mathbf{v}_A = -v_A\mathbf{j}$$

$$\mathbf{v}_B = v_B(\sin 30°\mathbf{i} - \cos 30°\mathbf{j})$$

$$\mathbf{v}_C = v_C(\cos 30°\mathbf{i} + \sin 30°\mathbf{j})$$

Conservation of momentum:

$$m\mathbf{v}_0 = m\mathbf{v}_A + m\mathbf{v}_B + m\mathbf{v}_C$$

Divide by m and resolve into components.

$$\mathbf{i}: \quad v_0 \cos 30° = v_B \sin 30° + v_C \cos 30°$$

$$\mathbf{j}: \quad -v_0 \sin 30° = -v_A - v_B \cos 30° + v_C \sin 30°$$

Solving for v_B and v_C,

$$v_B = \frac{\sqrt{3}}{2}(v_0 - v_A) \qquad v_C = \frac{1}{2}(v_0 + v_A)$$

Conservation of energy: $$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}mv_C^2$$

Divide by $\frac{1}{2}m$ and substitute for v_B and v_C.

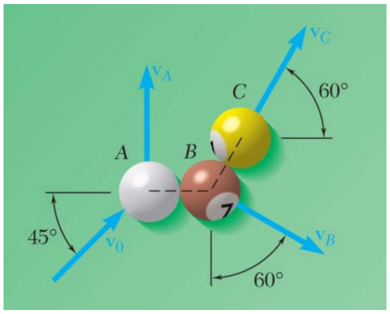
$$v_0^2 = v_A^2 + \frac{3}{4}(v_0 - v_A)^2 + \frac{1}{4}(v_0 + v_A)^2$$

$$= 2v_A^2 + v_0^2 - v_0 v_A$$

$$v_A = \frac{1}{2}v_0 = 7.5 \text{ ft/s} \qquad v_A = 7.50 \text{ ft/s} \blacktriangleleft$$

$$v_B = \frac{\sqrt{3}}{2}(15 - 7.5) = 6.4952 \text{ ft/s} \qquad v_B = 6.50 \text{ ft/s} \blacktriangleleft$$

$$v_C = \frac{1}{2}(15 + 7.5) = 11.25 \text{ ft/s} \qquad v_C = 11.25 \text{ ft/s} \blacktriangleleft$$



PROBLEM 14.42

In a game of pool, ball *A* is moving with a velocity $\mathbf{v}_0$ of magnitude $v_0 = 15$ ft/s when it strikes balls *B* and *C*, which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated and assuming frictionless surfaces and perfectly elastic impact (that is, conservation of energy), determine the magnitudes of the velocities $\mathbf{v}_A$, $\mathbf{v}_B$ and $\mathbf{v}_C$.

SOLUTION

Velocity vectors:

$$\mathbf{v}_0 = v_0(\cos 45°\,\mathbf{i} + \sin 45°\,\mathbf{j}) \qquad v_0 = 15 \text{ ft/s}$$

$$\mathbf{v}_A = v_A\mathbf{j}$$

$$\mathbf{v}_B = v_B(\sin 60°\,\mathbf{i} - \cos 60°\,\mathbf{j})$$

$$\mathbf{v}_C = v_C(\cos 60°\,\mathbf{i} + \sin 60°\,\mathbf{j})$$

Conservation of momentum:

$$m\mathbf{v}_0 = m\mathbf{v}_A + m\mathbf{v}_B + m\mathbf{v}_C$$

Divide by *m* and resolve into components.

$$\mathbf{i}\text{:} \quad v_0 \cos 45° = v_B \sin 60° + v_C \cos 60°$$

$$\mathbf{j}\text{:} \quad v_0 \sin 45° = v_A - v_B \cos 60° + v_C \sin 60°$$

Solving for v_B and v_C,

$$v_B = 0.25882 v_0 - 0.5 v_A \qquad v_C = 0.96593 v_0 - 0.86603 v_A$$

Conservation of energy:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}mv_C^2$$

Divide by m and substitute for v_B and v_C.

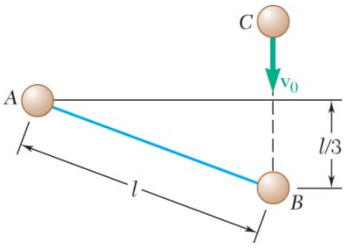
$$v_0^2 = v_A^2 + (0.25882 v_0 + 0.5 v_A)^2 + (0.96593 v_0 - 0.86603 v_A)^2$$

$$= v_0^2 + 1.4142 v_0 v_A + 2v_A^2$$

$$v_A = 0.70711 v_0 = 10.607 \text{ ft/s} \qquad v_A = 10.61 \text{ ft/s} ◀$$

$$v_B = 0.61237 v_0 = 9.1856 \text{ ft/s} \qquad v_B = 9.19 \text{ ft/s} ◀$$

$$v_C = 0.35356 v_0 = 5.303 \text{ ft/s} \qquad v_C = 5.30 \text{ ft/s} ◀$$



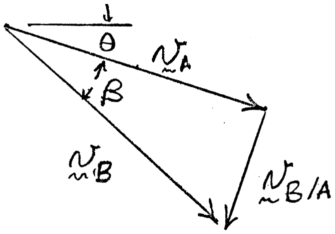
PROBLEM 14.43

Three spheres, each of mass *m*, can slide freely on a frictionless, horizontal surface. Spheres *A* and *B* are attached to an inextensible, inelastic cord of length *l* and are at rest in the position shown when sphere *B* is struck squarely by sphere *C* which is moving with a velocity $\mathbf{v}_0$. Knowing that the cord is taut when sphere *B* is struck by sphere *C* and assuming perfectly elastic impact between *B* and *C*, and thus conservation of energy for the entire system, determine the velocity of each sphere immediately after impact.

SOLUTION

$$\sin\theta = \frac{1}{3}, \qquad \cos\theta = \frac{\sqrt{8}}{3}, \qquad \theta = 19.471°$$

Velocity vectors



$$\mathbf{v}_0 = -v_0\mathbf{j}$$

$$\mathbf{v}_A = v_A(\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j})$$

$$\mathbf{v}_{B/A} = u_B(-\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$\mathbf{v}_C = v_C\mathbf{j}$$

Conservation of momentum: $m\mathbf{v}_0 = m\mathbf{v}_A + m\mathbf{v}_B + m\mathbf{v}_C = 2m\mathbf{v}_A + m\mathbf{v}_{B/A} + m\mathbf{v}_C$

Divide by *m* and resolve into components.

$$\mathbf{i}: \quad 0 = 2v_A\cos\theta - u_B\sin\theta$$

$$-\mathbf{j}: \quad v_0 = +2v_A\sin\theta + u_B\cos\theta - v_C$$

Solving for v_A and u_B, $\quad v_A = \frac{1}{6}(v_0 + v_C) \qquad u_B = 0.94281(v_0 + v_C)$

Conservation of energy: $\frac{1}{2}mv_0^2 = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}mv_C^2$

$$= \frac{1}{2}mv_A^2 + \frac{1}{2}m\left(v_A^2 + u_B^2\right) + \frac{1}{2}mv_C^2$$

Divide by $\frac{1}{2}m$ and substitute for v_A and u_B.

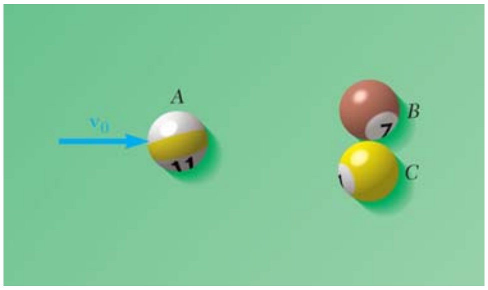
$$v_0^2 = 2\left(\frac{1}{6}\right)^2(v_0 + v_C)^2 + (0.94281)^2(v_0 + v_C)^2 + v_C^2$$

$$v_0^2 - v_C^2 = 0.94445(v_0 + v_C)^2 \qquad v_C = 0.02857v_0 \qquad \mathbf{v}_C = 0.0286v_0 \uparrow ◀$$

$$v_A = 0.17143v_0 \qquad \mathbf{v}_A = [0.17143v_0 \searrow 19.471°], \qquad \mathbf{v}_A = 0.1714v_0 \searrow 19.5° ◀$$

$$u_B = 0.96975v_0 \qquad \mathbf{v}_{B/A}[0.96975v_0 \swarrow 19.471°]$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \qquad \mathbf{v}_B = 0.985 \searrow 80.1° ◀$$



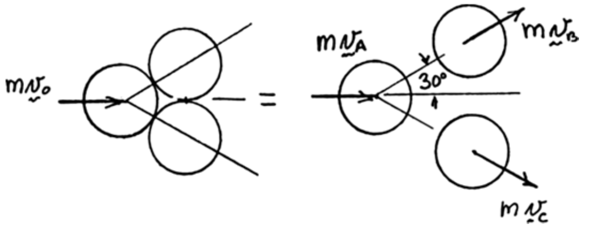
PROBLEM 14.44

In a game of pool, ball *A* is moving with the velocity $\mathbf{v}_0 = v_0\mathbf{i}$ when it strikes balls *B* and *C*, which are at rest side by side. Assuming frictionless surfaces and perfectly elastic impact (i.e., conservation of energy), determine the final velocity of each ball, assuming that the path of *A* is (*a*) perfectly centered and that *A* strikes *B* and *C* simultaneously, (*b*) not perfectly centered and that *A* strikes *B* slightly before it strikes *C*.

SOLUTION

(*a*) *A* strikes *B* and *C* simultaneously:

During the impact, the contact impulses make 30° angles with the velocity $\mathbf{v}_0$.



Thus, $\mathbf{v}_B = v_B(\cos 30°\mathbf{i} + \sin 30°\mathbf{j})$

$\mathbf{v}_C = v_C(\cos 30°\mathbf{i} - \sin 30°\mathbf{j})$

By symmetry, $\mathbf{v}_A = v_A\mathbf{i}$

Conservation of momentum: $m\mathbf{v}_0 = m\mathbf{v}_A + m\mathbf{v}_B + m\mathbf{v}_C$

y component: $0 = 0 + mv_B \sin 30° - mv_C \sin 30° \qquad v_C = v_B$

x component: $mv_0 = mv_A + mv_B \cos 30° + mv_C \cos 30°$

$$v_B + v_C = \frac{v_0 - v_A}{\cos 30°} = \frac{2}{\sqrt{3}}(v_0 - v_A)$$

$$v_B = v_C = \frac{v_0 - v_A}{\sqrt{3}}$$

Conservation of energy: $\frac{1}{2}mv_0^2 = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}mv_C^2$

$$v_0^2 = v_A^2 + \frac{2}{3}(v_0 - v_A)^2$$

$$v_0^2 - v_A^2 = (v_0 - v_A)(v_0 + v_A) = \frac{2}{3}(v_0 - v_A)^2$$

$$v_0 + v_A = \frac{2}{3}(v_0 - v_A) \qquad \frac{1}{3}v_0 = -\frac{5}{3}v_A \qquad v_A = -\frac{1}{5}v_0$$

$$v_B = v_C = \frac{6}{5\sqrt{3}}v_0 = \frac{2\sqrt{3}}{5}v_0$$

$\mathbf{v}_A = 0.200 v_0 \longleftarrow$ ◀

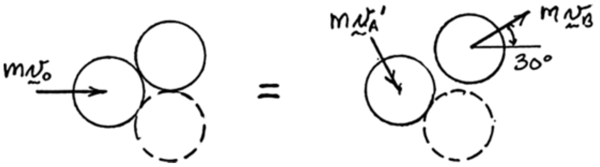
$\mathbf{v}_B = 0.693 v_0 \measuredangle 30^\circ$ ◀

$\mathbf{v}_C = 0.693 v_0 \searrow 30^\circ$ ◀

(*b*) *A* strikes *B* before it strikes *C*:

First impact: *A* strikes *B*.

During the impact, the contact impulse makes a 30° angle with the velocity $\mathbf{v}_0$.



Thus, $\mathbf{v}_B = v_B(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$

Conservation of momentum: $m\mathbf{v}_0 = m\mathbf{v}_A + m\mathbf{v}_B$

y component: $0 = m(v_A')_y + mv_B \sin 30^\circ \qquad (v_A')_y = -v_B \sin 30^\circ$

x component: $v_0 = m(v_A')_x + mv_B \cos 30^\circ \qquad (v_A')_x = v_0 - v_B \cos 30^\circ$

Conservation of energy:

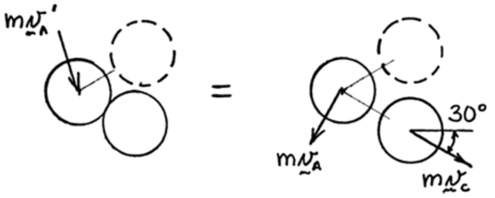
$$\begin{aligned}\frac{1}{2}mv_0^2 &= \frac{1}{2}m(v_A')_x^2 + \frac{1}{2}m(v_A')_y^2 + \frac{1}{2}mv_B^2\\ &= \frac{1}{2}m(v_0 - v_B\cos 30^\circ)^2 + \frac{1}{2}(v_B \sin 30^\circ)^2 + \frac{1}{2}v_B^2\\ &= \frac{1}{2}m\left(v_0^2 - 2v_0v_B + v_B^2\cos^2 30^\circ + v_B^2 \sin^2 30^\circ + v_B^2\right)\end{aligned}$$

$$v_B = v_0\cos 30^\circ = \frac{\sqrt{3}}{2}v_0, \quad (v_A')_x = v_0 \sin^2 30^\circ = \frac{1}{4}v_0,$$

$$(v_A')_y = -v_0 \cos 30^\circ \sin 30^\circ = -\frac{\sqrt{3}}{4}v_0$$

PROBLEM 14.44 (Continued)

Second impact: *A* strikes *C*.
During the impact, the contact impulse makes a 30° angle with the velocity $\mathbf{v}_0$.



Thus, $\mathbf{v}_C = v_C(\cos 30°\mathbf{i} - \sin 30°\mathbf{j})$

Conservation of momentum: $m\mathbf{v}_A' = m\mathbf{v}_A + m\mathbf{v}_C$

x component: $m(v_A')_x = m(v_A)_x + mv_C\cos 30°,$

$$(v_A)_x = (v_A')_x - v_C\cos 30° = \frac{1}{4}v_0 - v_C\cos 30°$$

y component: $m(v_A')_y = m(v_A)_y - mv_C\sin 30°$

$$(v_A)_y = (v_A')_y + v_C\sin 30° = -\frac{\sqrt{3}}{4}v_0 + v_C\sin 30°$$

Conservation of energy:

$$\frac{1}{2}m(v_A')_x^2 + \frac{1}{2}m(v_A')_y^2 = \frac{1}{2}m(v_A)_x^2 + \frac{1}{2}m(v_A)_y^2 + \frac{1}{2}mv_C^2$$

$$\frac{1}{2}m\left[\frac{1}{16}v_0^2 + \frac{3}{16}v_0^2\right] = \frac{1}{2}m\left[\left(\frac{1}{4}v_0 - v_C\cos 30°\right)^2 + \left(-\frac{\sqrt{3}}{4}v_0 + v_C\sin 30°\right)^2 + v_C^2\right]$$

$$= \frac{1}{2}m\left[\frac{1}{16}v_0^2 - \frac{1}{2}v_0v_C\cos 30° + v_C^2\cos^2 30° + \frac{3}{16}v_0^2 - \frac{\sqrt{3}}{2}v_0v_C\sin 30° + v_C^2\sin^2 30° + v_C^2\right]$$

$$0 = -v_0v_C\left(\frac{1}{2}\cos 30° + \frac{\sqrt{3}}{2}\sin 30°\right) + 2v_C^2$$

$$v_C = v_0\left(\frac{1}{4}\cos 30° + \frac{\sqrt{3}}{4}\sin 30°\right) = \frac{\sqrt{3}}{4}v_0$$

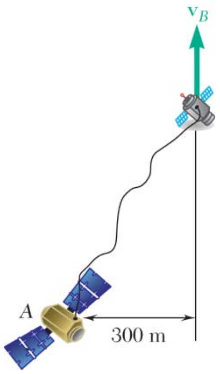
$$(v_A)_x = \frac{1}{4}v_0 - \frac{\sqrt{3}}{4}v_0\cos 30° = -\frac{1}{8}v_0$$

$$(v_A)_y = -\frac{\sqrt{3}}{4}v_0 + \frac{\sqrt{3}}{4}v_0\sin 30° = -\frac{\sqrt{3}}{8}v_0$$

$\mathbf{v}_A = 0.250v_0$ ↙ 60° ◀

$\mathbf{v}_B = 0.866v_0$ ↗ 30° ◀

$\mathbf{v}_C = 0.433v_0$ ↘ 30° ◀



PROBLEM 14.45

The 2-kg sub-satellite B has an initial velocity $\mathbf{v}_B = 3$ m/s $\mathbf{j}$. It is connected to the 20-kg base-satellite A by a 500-m space tether. Determine the velocity of the base satellite and sub-satellite immediately after the tether becomes taut (assuming no rebound).

SOLUTION

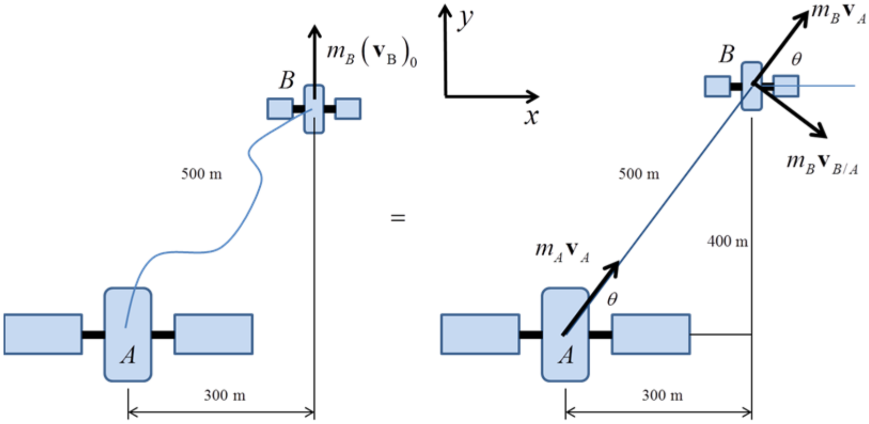
Given:

$$m_A = 20 \text{ kg},\ m_B = 2 \text{ kg}$$

$$(\mathbf{v}_A)_0 = 0,\ (\mathbf{v}_B)_0 = 3\mathbf{j} \text{ m/s}$$

Because the cord is inextensible, after it becomes taut, the component of $\mathbf{v}_B$ along AB must be equal to $\mathbf{v}_A$.

Impulse-Momentum Diagram:



Linear momentum is conserved:

x-components:

$$0 = m_A v_A \cos\theta + m_B v_A \cos\theta + m_B v_{B/A} \sin\theta$$

$$0 = 13.2 v_A + 1.6 v_{B/A} \qquad (1)$$

y-components:

$$2 \text{ kg}(3 \text{ m/s}) = m_A v_A \sin\theta + m_B v_A \sin\theta - m_B v_{B/A} \cos\theta$$

$$6 = 17.6 v_A - 1.2 v_{B/A} \qquad (2)$$

Solving (1) and (2) simultaneously: $v_A = 0.2182$ m/s, $v_{B/A} = -1.8$ m/s

$$\mathbf{v}_A = 0.218 \text{ m/s} \measuredangle 53.1° \blacktriangleleft$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$= (v_A \cos\theta + v_{B/A} \sin\theta)\mathbf{i} + (v_A \sin\theta - v_{B/A} \cos\theta)\mathbf{j}$$

$$= -1.3091\mathbf{i} + 1.2545\mathbf{j} \text{ m/s}$$

$$\mathbf{v}_B = 1.813 \text{ m/s} \measuredangle 43.8° \blacktriangleleft$$

PROBLEM 14.46

A 900-lb space vehicle traveling with a velocity $\mathbf{v}_0 = (1500\text{ ft/s})\mathbf{k}$ passes through the origin *O*. Explosive charges then separate the vehicle into three parts *A*, *B*, and *C*, with masses of 150 lb, 300 lb, and 450 lb, respectively. Knowing that shortly thereafter the positions of the three parts are, respectively, *A*(250, 250, 2250), *B* (600, 1300, 3200), and *C* (–475, –950, 1900), where the coordinates are expressed in ft, that the velocity of *B* is $\mathbf{v}_B = (500\text{ ft/s})\mathbf{i} + (1100\text{ ft/s})\mathbf{j} + (2100\text{ ft/s})\mathbf{k}$, and that the *x* component of the velocity of *C* is -400 ft/s, determine the velocity of part *A*.

SOLUTION

Position vectors (ft):
$$\mathbf{r}_A = 250\mathbf{i} + 250\mathbf{j} + 2250\mathbf{k}$$
$$\mathbf{r}_B = 600\mathbf{i} + 1300\mathbf{j} + 3200\mathbf{k}$$
$$\mathbf{r}_C = -475\mathbf{i} - 950\mathbf{j} + 1900\mathbf{k}$$

Since there are no external forces, linear momentum is conserved.

$$(m_A + m_B + m_C)\mathbf{v}_0 = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C$$

$$\mathbf{v}_A = \frac{m_A + m_B + m_C}{m_A}\mathbf{v}_0 - \frac{m_B}{m_A}\mathbf{v}_B - \frac{m_C}{m_A}\mathbf{v}_C = 6\mathbf{v}_0 - 2\mathbf{v}_B - 3\mathbf{v}_C \qquad (1)$$

$$= (6)(1500\mathbf{k}) - (2)(500\mathbf{i} + 1100\mathbf{j} + 2100\mathbf{k}) - (3)[-400\mathbf{i} + (v_C)_y\,\mathbf{j} + (v_C)_z\,\mathbf{k}]$$

$$= -3(v_C)_y\,\mathbf{j} - 3(v_C)_z\,\mathbf{k} + 200\mathbf{i} - 2200\mathbf{j} + 4800\mathbf{k}$$

$$(v_A)_x = 200, \quad (v_C)_y = -3(v_C)_y - 2200, \quad (v_A)_z = -3(v_C)_z + 4800$$

Conservation of angular momentum about *O*:

$$(\mathbf{H}_O)_2 = (\mathbf{H}_O)_1$$

Since the vehicle passes through the origin, $(\mathbf{H}_O)_1 = 0.$

$$(\mathbf{H}_O)_2 = \mathbf{r}_A \times (m_A\mathbf{v}_A) + \mathbf{r}_B \times (m_B\mathbf{v}_B) + \mathbf{r}_C \times (m_C\mathbf{v}_C) = 0$$

Divide by m_A.

$$\begin{aligned}
\mathbf{r}_A \times \mathbf{v}_A + \frac{m_B}{m_A}\mathbf{r}_B \times \mathbf{v}_B + \frac{m_C}{m_A}\mathbf{r}_C \times \mathbf{v}_C &= \mathbf{r}_A \times \mathbf{v}_A + 2\mathbf{r}_B \times \mathbf{v}_B + 3\mathbf{r}_C \times \mathbf{v}_C \\
&= \mathbf{r}_A \times (6\mathbf{v}_0 - 2\mathbf{v}_B - 3\mathbf{v}_C) + 2\mathbf{r}_B \times \mathbf{v}_B + 3\mathbf{r}_C \times \mathbf{v}_C \\
&= 3(\mathbf{r}_C - \mathbf{r}_A) \times \mathbf{v}_C + 6\mathbf{r}_A \times \mathbf{v}_0 + 2(\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{v}_B \\
&= (-2175\mathbf{i} - 3600\mathbf{j} - 1050\mathbf{k}) \times \mathbf{v}_C + (1500\mathbf{i} + 1500\mathbf{j} + 13500\mathbf{k}) \times \mathbf{v}_0 \\
&\quad + (700\mathbf{i} + 2100\mathbf{j} + 1900\mathbf{k}) \times \mathbf{v}_B = 0
\end{aligned}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2175 & -3600 & -1050 \\ (v_C)_x & (v_C)_y & (v_C)_z \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1500 & 1500 & 13500 \\ 0 & 0 & 1500 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 700 & 2100 & 1900 \\ 500 & 1100 & 2100 \end{vmatrix} = 0$$

Resolve into components.

i: $1050(v_C)_y - 3600(v_C)_z + 2,250,000 + 2,320,000 = 0$ (2)

j: $2175(v_C)_z - 1050(v_C)_x - 2,250,000 - 520,000 = 0$ (3)

k: $3600(v_C)_x - 2175(v_C)_y + 0 - 280,000 = 0$ (4)

Set $(v_C)_x = -400$ ft/s

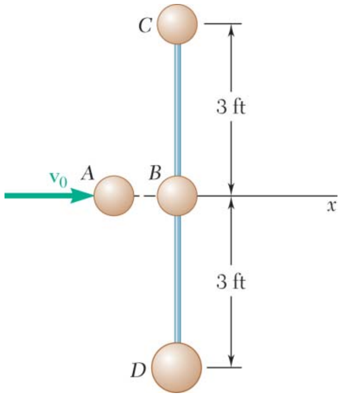
From Eq. (4), $(v_C)_y = -790.80$ ft/s

From Eq. (3), $(v_C)_z = 1080.5$ ft/s

From Eq. (1),

$$\begin{aligned} \mathbf{v}_A &= (6)(1500\mathbf{k}) - (2)(500\mathbf{i} + 1100\mathbf{j} + 2100\mathbf{k}) - (3)[-400\mathbf{i} + (790.80)\mathbf{j} + (1080.5)\mathbf{k}] \\ &= 200\mathbf{i} + 172.4\mathbf{j} + 1558.6\mathbf{k} \end{aligned}$$

$\mathbf{v}_A = (200 \text{ ft/s})\mathbf{i} + (172 \text{ ft/s})\mathbf{j} + (1560 \text{ ft/s})\mathbf{k}$ ◀

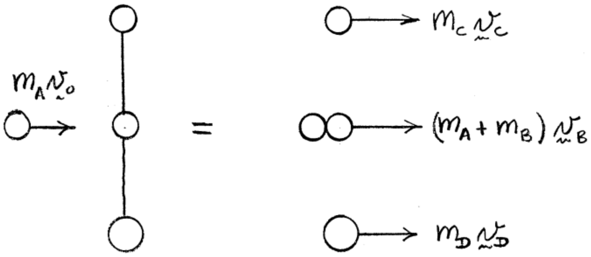


PROBLEM 14.47

Four small disks A, B, C, and D can slide freely on a frictionless horizontal surface. Disks B, C, and D are connected by light rods and are at rest in the position shown when disk B is struck squarely by disk A, which is moving to the right with a velocity $\mathbf{v}_0 = (38.5\ \text{ft/s})\mathbf{i}$. The weights of the disks are $W_A = W_B = W_C = 15$ lb, and $W_D = 30$ lb. Knowing that the velocities of the disks immediately after the impact are $\mathbf{v}_A = \mathbf{v}_B = (8.25\ \text{ft/s})\mathbf{i}$, $\mathbf{v}_C = v_C\,\mathbf{i}$, and $\mathbf{v}_D = v_D\,\mathbf{i}$, determine (*a*) the speeds v_C and v_D, (*b*) the fraction of the initial kinetic energy of the system which is dissipated during the collision.

SOLUTION

There are no external forces. Momentum is conserved.



$m_A \underset{\sim}{v}_0$
=
$m_C \underset{\sim}{v}_C$
$(m_A + m_B)\underset{\sim}{v}_B$
$m_D \underset{\sim}{v}_D$

(*a*) Moments about D $+\circlearrowright$: $\quad 3m_A v_0 = 6m_C v_C + 3(m_A + m_B)v_B$

$$v_C = \frac{3m_A}{6m_C}v_0 - \frac{3(m_A + m_B)}{6m_C}v_B = (0.5)(38.5) - (8.25) = 11$$

$v_C = 11.00$ ft/s ◀

Moments about C $+\circlearrowleft$: $\quad 3m_A v_0 = 3(m_A + m_B)v_B + 6m_D v_D$

$$v_D = \frac{3m_A v_0}{6m_D} - \frac{3(m_A + m_B)}{6m_D}v_B = (0.25)(38.5) - (0.5)(8.25) = 5.5\ \text{ft/s}$$

$v_D = 5.50$ ft/s ◀

(*b*) Initial kinetic energy:

$$T_1 = \frac{1}{2}\frac{W_A}{g}v_0^2 = \frac{1}{2}\frac{15}{32.2}(38.5)^2 = 345.24\ \ \text{ft}\cdot\text{lb}$$

Final kinetic energy:

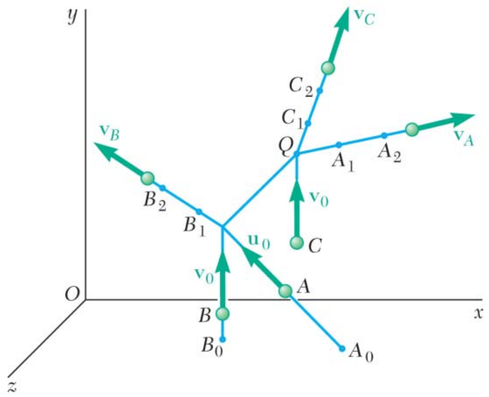
$$T_2 = \frac{1}{2}\frac{W_A + W_B}{g}v_B^2 + \frac{1}{2}\frac{W_C}{g}v_C^2 + \frac{1}{2}\frac{W_D}{g}v_D^2$$

$$= \frac{1}{2}\frac{30}{32.2}(8.25)^2 + \frac{1}{2}\frac{15}{32.2}(11.00)^2 + \frac{1}{2}\frac{30}{32.2}(5.50)^2 = 73.98\ \text{ft}\cdot\text{lb}$$

Energy lost: $\quad 345.24 - 73.98 = 271.26\ \text{ft}\cdot\text{lb}$

Fraction of energy lost $= \dfrac{271.26}{345.24} = 0.786$

$$\frac{(T_1 - T_2)}{T_1} = 0.786 \blacktriangleleft$$



PROBLEM 14.48

In the scattering experiment of Problem 14.26, it is known that the alpha particle is projected from $A_0(300, 0, 300)$ and that it collides with the oxygen nucleus C at $Q(240, 200, 100)$, where all coordinates are expressed in millimeters. Determine the coordinates of Point B_0 where the original path of nucleus B intersects the zx plane. (*Hint:* Express that the angular momentum of the three particles about Q is conserved.)

PROBLEM 14.26 In a scattering experiment, an alpha particle A is projected with the velocity $\mathbf{u}_0 = -(600 \text{ m/s})\mathbf{i} + (750 \text{ m/s})\mathbf{j} - (800 \text{ m/s})\mathbf{k}$ into a stream of oxygen nuclei moving with a common velocity $\mathbf{v}_0 = (600 \text{ m/s})\mathbf{j}$. After colliding successively with nuclei B and C, particle A is observed to move along the path defined by the Points $A_1(280, 240, 120)$ and $A_2(360, 320, 160)$, while nuclei B and C are observed to move along paths defined, respectively, by $B_1(147, 220, 130)$, $B_2(114, 290, 120)$, and by $C_1(240, 232, 90)$ and $C_2(240, 280, 75)$. All paths are along straight lines and all coordinates are expressed in millimeters. Knowing that the mass of an oxygen nucleus is four times that of an alpha particle, determine the speed of each of the three particles after the collisions.

SOLUTION

Conservation of angular momentum about Q:

$$\overrightarrow{QA}_0 \times (m\mathbf{u}_0) + \overrightarrow{QB}_0 \times (4m\mathbf{v}_0) + \overrightarrow{QC}_0 \times (4m\mathbf{v}_0) = \overrightarrow{QA}_1 \times (m\mathbf{v}_A) + \overrightarrow{QB}_1 \times (4m\mathbf{v}_B) + \overrightarrow{QC}_1 \times (4m\mathbf{v}_C)$$

$$\overrightarrow{QA}_0 \times (m\mathbf{u}_0) + \overrightarrow{QB}_0 \times (4m\mathbf{v}_0) + 0 = 0 + \overrightarrow{QB}_1 \times (4m\mathbf{v}_B) + 0 \qquad (1)$$

where

$$\overrightarrow{QA}_0 = \mathbf{r}_{A_0} - \mathbf{r}_Q = (300\mathbf{i} + 300\mathbf{k}) - (240\mathbf{i} + 200\mathbf{j} + 100\mathbf{k})$$

$$= (60 \text{ mm})\mathbf{i} - (200 \text{ mm})\mathbf{j} + (200 \text{ mm})\mathbf{k}$$

$$\overrightarrow{QB}_0 = (\Delta x)\mathbf{i} + (\Delta y)\mathbf{j} + (\Delta z)\mathbf{k}$$

$$\overrightarrow{QB}_1 = \mathbf{r}_{B_1} - \mathbf{r}_Q = (147\mathbf{i} + 220\mathbf{j} + 130\mathbf{k}) - (240\mathbf{i} + 200\mathbf{j} + 100\mathbf{k})$$

$$-(93 \text{ mm})\mathbf{i} + (20 \text{ mm})\mathbf{j} + (30 \text{ mm})\mathbf{k}$$

$$\mathbf{u}_0 = -(600 \text{ m/s})\mathbf{i} + (750 \text{ m/s})\mathbf{j} - (800 \text{ m/s})\mathbf{k} \qquad \mathbf{v}_0 = (600 \text{ m/s})\mathbf{j}$$

and from the solution to Problem 14.26,

$$\mathbf{v}_B = v_B \boldsymbol{\lambda}_B = (716.98)(-0.42290\mathbf{i} + 0.89707\mathbf{j} - 0.12815\mathbf{k})$$

$$= -(303.21 \text{ m/s})\mathbf{i} + (643.18 \text{ m/s})\mathbf{j} - (91.88 \text{ m/s})\mathbf{k}$$

PROBLEM 14.48 (Continued)

Calculating each term and dividing by *m*,

$$\overrightarrow{QA}_0 \times \mathbf{u}_0 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 60 & -200 & 200 \\ -600 & 750 & -800 \end{vmatrix} = 10,000\,\mathbf{i} - 72,000\,\mathbf{j} - 75,000\,\mathbf{k}$$

$$\begin{aligned} \overrightarrow{QB}_0 \times (4\mathbf{v}_0) &= [(\Delta x)\mathbf{i} + (\Delta y)\mathbf{j} + (\Delta z)\mathbf{k}] \times (2400\mathbf{j}) \\ &= -2400(\Delta z)\mathbf{i} + 2400(\Delta x)\mathbf{k} \end{aligned}$$

$$\overrightarrow{QB}_1 \times (4\mathbf{v}_B) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -93 & 20 & 30 \\ -1212.84 & 2572.72 & -367.52 \end{vmatrix} = -84,532\,\mathbf{i} - 70,565\,\mathbf{j} - 215,006\,\mathbf{k}$$

Collect terms and resolve into components.

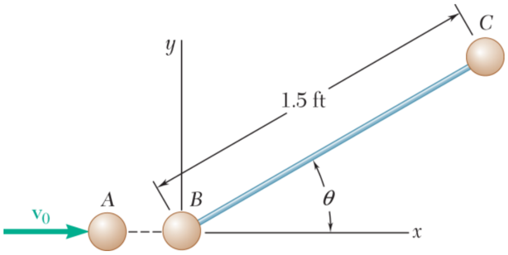
$\mathbf{i}$: $\quad 10,000 - 2400(\Delta z) = -84,532 \qquad \Delta z = 39.388 \text{ mm}$

$\mathbf{k}$: $\quad -75,000 + 2400(\Delta x) = -215,006 \qquad \Delta x = -58.336 \text{ mm}$

Coordinates: $\quad x_{B_0} = x_Q + \Delta x = 240 - 58.336 \qquad x_{B_0} = 181.7 \text{ mm}$ ◀

$y_{B_0} = 0$ ◀

$z_{B_0} = z_Q + \Delta z = 100 + 39.388 \qquad z_{B_0} = 139.4 \text{ mm}$ ◀



PROBLEM 14.49

Three identical small spheres, each of weight 2 lb, can slide freely on a horizontal frictionless surface. Spheres *B* and *C* are connected by a light rod and are at rest in the position shown when sphere *B* is struck squarely by sphere *A* which is moving to the right with a velocity $\mathbf{v}_0 = (8\text{ ft/s})\mathbf{i}$. Knowing that $\theta = 45°$ and that the velocities of spheres *A* and *B* immediately after the impact are $\mathbf{v}_A = 0$ and $\mathbf{v}_B = (6\text{ ft/s})\mathbf{i} + (\mathbf{v}_B)_y\,\mathbf{j}$, determine $(v_B)_y$ and the velocity of *C* immediately after impact.

SOLUTION

Let *m* be the mass of one ball.

Conservation of linear momentum: $(\Sigma m\mathbf{v}) = (\Sigma m\mathbf{v})_0$

$$m\mathbf{v}_A + m\mathbf{v}_B + m\mathbf{v}_C = m(\mathbf{v}_A)_0 + (m\mathbf{v}_B)_0 + (m\mathbf{v}_C)_0$$

Dividing by *m* and applying numerical data,

$$0 + [(6\text{ ft/s})\mathbf{i} + (v_B)_y\mathbf{j}] + [(v_C)_x\mathbf{i} + (v_C)_y\mathbf{j}] = (8\text{ ft/s})\mathbf{i} + 0 + 0$$

Components:

$$x:\quad 6 + (v_C)_x = 8 \qquad\qquad (v_C)_x = 2\text{ ft/s}$$

$$y:\quad (v_B)_y + (v_C)_y = 0 \qquad (1)$$

Conservation of angular momentum about *O*:

$$\Sigma[\mathbf{r} \times (m\mathbf{v})] = \Sigma[\mathbf{r} \times (m\mathbf{v}_0)]$$

where $\mathbf{r}_A = 0, \quad \mathbf{r}_B = 0, \quad \mathbf{r}_C = (1.5\text{ ft})(\cos 45°\mathbf{i} + \sin 45°\mathbf{j})$

$$(1.5)(\cos 45°\mathbf{i} + \sin 45°\mathbf{j}) \times [m(v_C)_x\mathbf{i} + m(v_C)_y\mathbf{j}] = 0$$

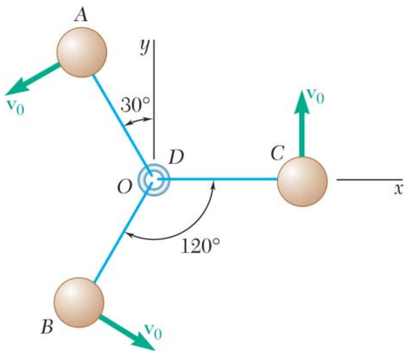
Since their cross product is zero, the two vectors are parallel.

$$(v_C)_y = (v_C)_x \tan 45° = 2\tan 45° = 2\text{ ft/s}$$

From (1), $(v_B)_y = -2$ ft/s

$$(v_B)_y = -2.00\text{ ft/s} \blacktriangleleft$$

$$\mathbf{v}_C = (2.00\text{ ft/s})\mathbf{i} + (2.00\text{ ft/s})\mathbf{j} \blacktriangleleft$$



PROBLEM 14.50

Three small spheres *A, B,* and *C*, each of mass *m*, are connected to a small ring *D* of negligible mass by means of three inextensible, inelastic cords of length *l*. The spheres can slide freely on a frictionless horizontal surface and are rotating initially at a speed v_0 about ring *D* which is at rest. Suddenly the cord *CD* breaks. After the other two cords have again become taut, determine (*a*) the speed of ring *D*, (*b*) the relative speed at which spheres *A* and *B* rotate about *D*, (*c*) the fraction of the original energy of spheres *A* and *B* that is dissipated when cords *AD* and *BD* again became taut.

SOLUTION

Let the system consist of spheres *A* and *B*.

State 1: Instant cord *DC* breaks.

$$m(\mathbf{v}_A)_1 = mv_0\left(-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)$$

$$m(\mathbf{v}_B)_1 = mv_0\left(\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)$$

$$\mathbf{L}_1 = m(\mathbf{v}_A)_1 + m(\mathbf{v}_B)_1 = -mv_0\mathbf{j}$$

$$\overline{\mathbf{v}} = \frac{\mathbf{L}_1}{2m} = -\frac{1}{2}v_0\mathbf{j}$$

Mass center lies at Point *G* midway between balls *A* and *B*.

$$(\mathbf{H}_G)_1 = \frac{\sqrt{3}}{2}l\mathbf{j} \times (m\mathbf{v}_A)_1 + -\frac{\sqrt{3}}{2}l\mathbf{j} \times (m\mathbf{v}_B)_1$$

$$= \frac{3}{2}lmv_0\mathbf{k}$$

$$T_1 = \frac{1}{2}mv_0^2 + \frac{1}{2}mv_0^2 = mv_0^2$$

State 2: The cord is taut. Conservation of linear momentum:

(*a*) $$\mathbf{v}_D = \overline{\mathbf{v}} = -\frac{1}{2}v_0\mathbf{j}$$ $v_D = 0.500v_0$ ◀

Let $(\mathbf{v}_A)_2 = \overline{\mathbf{v}} + \mathbf{u}_A$ and $\mathbf{v}_B = \overline{\mathbf{v}} + \mathbf{u}_B$

$$\mathbf{L}_2 = 2m\overline{\mathbf{v}} + m\mathbf{u}_A + m\mathbf{u}_B = \mathbf{L}_1$$

$$\mathbf{u}_B = -\mathbf{u}_A \qquad u_B = u_A$$

$$(\mathbf{H}_G)_2 = lmu_A\mathbf{k} + lmu_B\mathbf{k} = 2lmu_A\mathbf{k}$$

PROBLEM 14.50 (Continued)

(*b*) Conservation of angular momentum:

$$(\mathbf{H}_G)_2 = (\mathbf{H}_G)_1$$

$$2lmu_A\mathbf{k} = \frac{3}{2}lmv_0\mathbf{k} \qquad u_A = u_B = \frac{3}{4}v_0$$

$$u = 0.750v_0 \blacktriangleleft$$

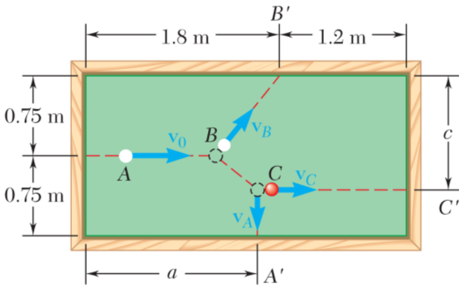
$$T_2 = \frac{1}{2}(2m)\bar{v}^2 + \frac{1}{2}mu_A^2 + \frac{1}{2}mu_B^2$$

$$= \frac{1}{2}mv_0^2\left(\frac{1}{2} + \frac{9}{16} + \frac{9}{16}\right) = \frac{13}{16}mv_0^2$$

(*c*) Fraction of energy lost:

$$\frac{T_1 - T_2}{T_1} = \frac{1 - \frac{13}{16}}{1} = \frac{3}{16}$$

$$\frac{T_1 - T_2}{T_1} = 0.1875 \blacktriangleleft$$



PROBLEM 14.51

In a game of billiards, ball A is given an initial velocity $\mathbf{v}_0$ along the longitudinal axis of the table. It hits ball B and then ball C, which are both at rest. Balls A and C are observed to hit the sides of the table squarely at A' and C', respectively, and ball B is observed to hit the side obliquely at B'. Knowing that $v_0 = 4$ m/s, $v_A = 1.92$ m/s, and $a = 1.65$ m, determine (*a*) the velocities $\mathbf{v}_B$ and $\mathbf{v}_C$ of balls B and C, (*b*) the Point C' where ball C hits the side of the table. Assume frictionless surfaces and perfectly elastic impacts (that is, conservation of energy).

SOLUTION

Velocities in m/s. Lengths in meters. Assume masses are 1.0 for each ball.

Before impacts: $(\mathbf{v}_A)_0 = v_0\mathbf{i} = 4\mathbf{i}, \qquad (\mathbf{v}_B)_0 = (\mathbf{v}_C)_0 = 0$

After impacts: $\mathbf{v}_A = -1.92\mathbf{j}, \qquad \mathbf{v}_B = (v_B)_x\mathbf{i} + (v_B)_y\mathbf{j}, \qquad \mathbf{v}_C = v_C\mathbf{i}$

Conservation of linear momentum: $\mathbf{v}_0 = \mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C$

$$\mathbf{i}: \quad 4 = 0 + (v_B)_x + v_C \qquad (v_B)_x = 4 - v_C$$

$$\mathbf{j}: \quad 0 = -1.92 + (v_B)_y + 0 \qquad (v_B)_y = 1.92$$

Conservation of energy: $\frac{1}{2}v_0^2 = \frac{1}{2}v_A^2 + \frac{1}{2}v_B^2 + \frac{1}{2}v_C^2$

$$\frac{1}{2}(4)^2 = \frac{1}{2}(1.92)^2 + \frac{1}{2}(1.92)^2 + \frac{1}{2}(4 - v_C)^2 + \frac{1}{2}v_C^2$$

$$v_C^2 - 4v_C + 3.6864 = 0$$

$$v_C = \frac{4 \pm \sqrt{(4)^2 - (4)(3.6864)}}{2} = 2 \pm 0.56 = 2.56 \qquad \text{or} \qquad 1.44$$

Conservation of angular momentum about B':

$$0.75v_0 = (1.8 - a)v_A + cv_C$$

$$cv_C = (0.75)(4) - (1.8 - 1.65)(1.92) = 2.712$$

$$c = \frac{2.712}{v_C}$$

If $v_C = 1.44$, $\qquad c = 1.8833$ off the table. Reject.

If $v_C = 2.56$, $\qquad c = 1.059$

PROBLEM 14.51 (Continued)

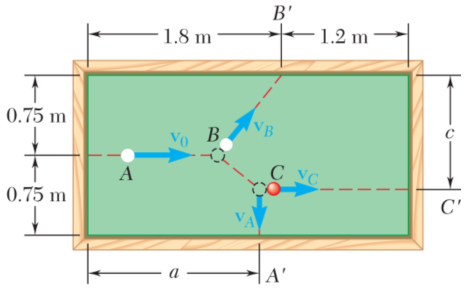
Then, $(v_B)_x = 4 - 2.56 = 1.44, \qquad \mathbf{v}_B = 1.44\mathbf{i} + 1.92\mathbf{j}$

Summary.

(*a*) $\mathbf{v}_B = 2.40\text{ m/s} \measuredangle 53.1°$ ◀

$\mathbf{v}_C = 2.56\text{ m/s} \longrightarrow$ ◀

(*b*) $c = 1.059\text{ m}$ ◀



PROBLEM 14.52

For the game of billiards of Problem 14.51, it is now assumed that $v_0 = 5$ m/s, $v_C = 3.2$ m/s, and $c = 1.22$ m. Determine (*a*) the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of balls *A* and *B*, (*b*) the Point *A*′ where ball *A* hits the side of the table.

PROBLEM 14.51 In a game of billiards, ball *A* is given an initial velocity $\mathbf{v}_0$ along the longitudinal axis of the table. It hits ball *B* and then ball *C*, which are both at rest. Balls *A* and *C* are observed to hit the sides of the table squarely at *A*′ and *C*′, respectively, and ball *B* is observed to hit the side obliquely at *B*′. Knowing that $v_0 = 4$ m/s, $v_A = 1.92$ m/s, and $a = 1.65$ m, determine (*a*) the velocities $\mathbf{v}_B$ and $\mathbf{v}_C$ of balls *B* and *C*, (*b*) the Point *C*′ where ball *C* hits the side of the table. Assume frictionless surfaces and perfectly elastic impacts (that is, conservation of energy).

SOLUTION

Velocities in m/s. Lengths in meters. Assume masses are 1.0 for each ball.

Before impacts: $(\mathbf{v}_A)_0 = v_0\mathbf{i} = 5\mathbf{i}, \qquad (\mathbf{v}_B)_0 = (\mathbf{v}_C)_0 = 0$

After impacts: $\mathbf{v}_A = -v_A\mathbf{j}, \qquad \mathbf{v}_B = (v_B)_x\mathbf{i} + (v_B)_y\mathbf{j}, \qquad \mathbf{v}_C = 3.2\mathbf{i}$

Conservation of linear momentum: $\mathbf{v}_0 = \mathbf{v}_A + \mathbf{v}_B + \mathbf{v}_C$

$$\mathbf{i}: \quad 5 = 0 + (v_B)_x + 3.2 \qquad (v_B)_x = 1.8$$

$$\mathbf{j}: \quad 0 = -v_A + (v_B)_y + 0 \qquad (v_B)_y = v_A$$

Conservation of energy: $\frac{1}{2}v_0^2 = \frac{1}{2}v_A^2 + \frac{1}{2}v_B^2 + \frac{1}{2}v_C^2$

$$\frac{1}{2}(5)^2 = \frac{1}{2}(v_A)^2 + \frac{1}{2}(1.8)^2 + \frac{1}{2}(v_A)^2 + \frac{1}{2}(3.2)^2$$

(*a*) $v_A^2 = 11.52 \qquad v_A = 2.4$ $\qquad \mathbf{v}_A = 2.40 \text{ m/s}\downarrow$ ◀

$(v_B)_y = 2.4 \qquad \mathbf{v}_B = 1.8\mathbf{i} + 2.4\mathbf{j}$ $\qquad \mathbf{v}_B = 3.00 \text{ m/s} \measuredangle 53.1°$ ◀

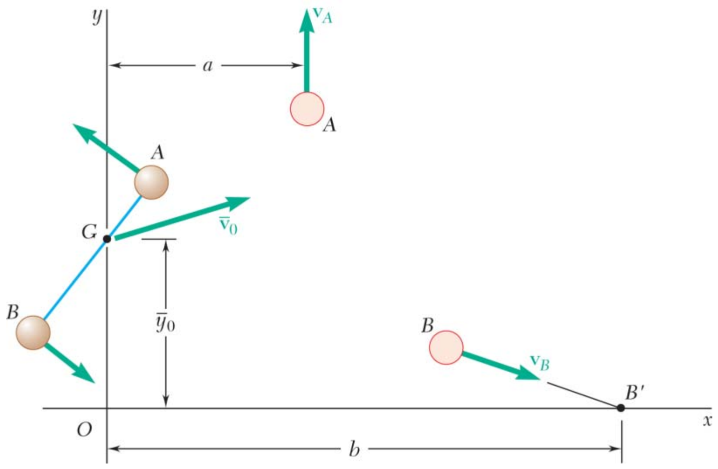
Conservation of angular momentum about B':

$$0.75v_0 = (1.8 - a)v_A + cv_C$$

$$av_A = 1.8v_A + cv_C - 0.75v_0$$

$$= (1.8)(2.4) + (1.22)(3.2) - (0.75)(5) = 4.474$$

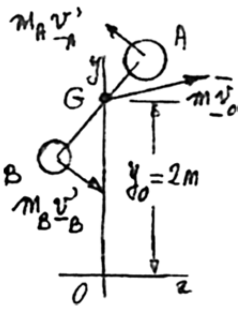
(*b*) $a = \dfrac{4.474}{v_A} = \dfrac{4.474}{2.4}$ $\qquad a = 1.864$ m ◀



PROBLEM 14.53

Two small disks A and B, of mass 3 kg and 1.5 kg, respectively, may slide on a horizontal, frictionless surface. They are connected by a cord, 600 mm, long, and spin counterclockwise about their mass center G at the rate of 10 rad/s. At $t = 0$, the coordinates of G are $\overline{x}_0 = 0$, $\overline{y}_0 = 2$ m, and its velocity is $\overline{\mathbf{v}}_0 = (1.2 \text{ m/s})\mathbf{i} + (0.96 \text{ m/s})\mathbf{j}$. Shortly thereafter, the cord breaks; disk A is then observed to move along a path parallel to the y axis and disk B along a path which intersects the x axis at a distance $b = 7.5$ m from O. Determine (a) the velocities of A and B after the cord breaks, (b) the distance a from the y-axis to the path of A.

SOLUTION



Initial conditions.

Location of G:

$$\frac{AG}{m_B} = \frac{BG}{m_A} = \frac{AG + GB}{m_B + m_A} = \frac{AB}{m} = \frac{0.6 \text{ m}}{4.5 \text{ kg}}$$

$$AG = 1.5\left(\frac{0.6}{4.5}\right) = 0.2 \text{ m}$$

$$BG = 0.4 \text{ m}$$

Linear momentum:

$$\mathbf{L}_0 = m\overline{\mathbf{v}}_0 = (4.5 \text{ kg})(1.2\,\mathbf{i} + 0.96\,\mathbf{j})$$
$$= 5.4\mathbf{i} + 4.32\mathbf{j}$$

Angular momentum:

About G:

$$(\mathbf{H}_G)_0 = \overrightarrow{GA} \times m_A \mathbf{v}'_A + \overrightarrow{GB} \times m_B \mathbf{v}'_B$$
$$= (0.2 \text{ m})(3 \text{ kg})(0.2 \text{ m} \times 10 \text{ rad/s})\mathbf{k}$$
$$+ (0.4 \text{ m})(1.5 \text{ kg})(0.4 \text{ m} \times 10 \text{ rad/s})\mathbf{k}$$
$$(\mathbf{H}_G)_0 = (3.6 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{k}$$

About O: Using formula derived in Problem 14.27,

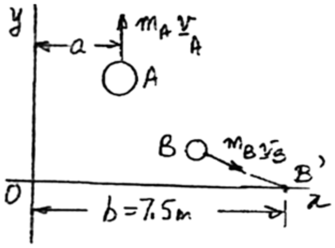
$$(\mathbf{H}_O)_0 = \overline{\mathbf{r}} \times m\overline{\mathbf{v}}_0 + (\mathbf{H}_G)_0$$
$$= 2\mathbf{j} \times (5.4\mathbf{i} + 4.32\mathbf{j}) + 3.6\mathbf{k}$$
$$= -10.8\mathbf{k} + 3.6\mathbf{k} = -(7.2 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{k}$$

Kinetic energy: Using Eq. (14.29),

$$T_0 = \frac{1}{2}m\bar{v}_0^2 + \frac{1}{2}\sum_i m: \quad v_i'^2 = \frac{1}{2}m\bar{v}_0^2 + \frac{1}{2}m_A v_A'^2 + \frac{1}{2}m_B v_B'^2$$

$$= \frac{1}{2}(4.5)[(1.2)^2 + (0.96)^2] + \frac{1}{2}(3)(0.2\times 10)^2 + \frac{1}{2}(1.5)(0.4\times 10)^2$$

$$= (5.3136 + 6 + 12) = 23.314 \text{ J}$$



(*a*) Conservation of linear momentum:

$$\mathbf{L}_0 = \mathbf{L}$$

$$5.4\mathbf{i} + 4.32\mathbf{j} = m_A\mathbf{v}_A + m_B\mathbf{v}_B$$

$$= 3(v_A\mathbf{j}) + 1.5[(v_B)_z\mathbf{i} + (v_B)_y\mathbf{j}]$$

Equating coefficients of **i**: $5.4 = 1.5(v_B)_x$

$$(v_B)_x = 3.6 \text{ m/s} \qquad (1)$$

Equating coefficients of **j**: $4.32 = 3v_A + 1.5(v_B)_y$

$$(v_B)_y = 2.88 - 2v_A \qquad (2)$$

Conservation of energy:

$$T_0 = T: \quad T_0 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$$

$$23.314 \text{ J} = \frac{1}{2}(3)\left(v_A^2\right) + \frac{1}{2}(1.5)[(v_B)_x^2 + (v_B)_y^2]$$

Substituting from Eqs. (1) and (2):

$$23.314 = 1.5v_A^2 + 0.75(3.6)^2 + 0.75(2.88 - 2v_A)^2$$

$$4.5v_A^2 - 8.64v_A - 7.373 = 0$$

$$v_A^2 - 1.92v_A - 1.6389 = 0$$

$$v_A = 0.96 + 1.60 = 2.56 \text{ m/s}$$

$\mathbf{v}_A = 2.56$ m/s ↑ ◀

and $v_A = 0.96 - 1.60 = -0.64$ m/s (rejected, since $\mathbf{v}_A$ is shown directed up)

From Eqs. (1) and (2):

$$(v_B)_x = 3.6 \text{ m/s}$$

$$(v_B)_y = 2.88 - 2(2.56) = -2.24 \text{ m/s}$$

$$\mathbf{v}_B = 3.6\mathbf{i} - 2.24\mathbf{j}$$

$\mathbf{v}_B = 4.24$ m/s ↘ 31.9° ◀

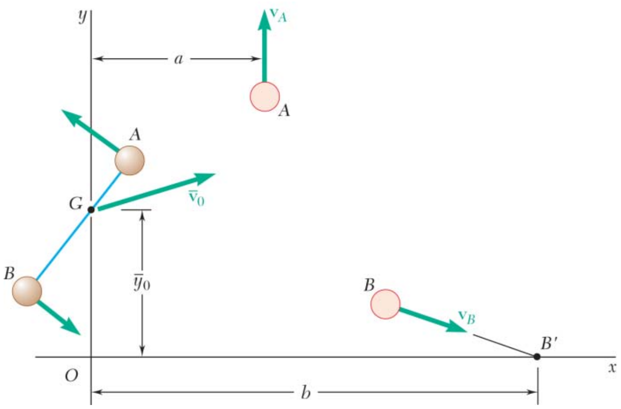
(*b*) Conservation of angular momentum about *O*:

$$(\mathbf{H}_O)_0 = \mathbf{H}_O: \quad -7.2\mathbf{k} = a\mathbf{i}\times m_A\mathbf{v}_A + b\mathbf{i}\times m_B\mathbf{v}_B$$

$$-7.2\mathbf{k} = a\mathbf{i}\times 3(2.56\mathbf{j}) + 7.5\mathbf{i}\times 1.5(3.6\mathbf{i} - 2.24\mathbf{j})$$

$$-7.2\mathbf{k} = 7.68a\mathbf{k} - 25.2\mathbf{k}$$

$a = 2.34$ m ◀



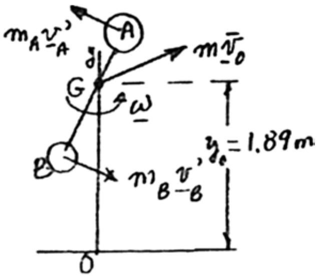
PROBLEM 14.54

Two small disks A and B, of mass 2 kg and 1 kg, respectively, may slide on a horizontal and frictionless surface. They are connected by a cord of negligible mass and spin about their mass center G. At $t = 0$, G is moving with the velocity $\overline{\mathbf{v}}_0$ and its coordinates are $\overline{x}_0 = 0$, $\overline{y}_0 = 1.89$ m. Shortly thereafter, the cord breaks and disk A is observed to move with a velocity $\mathbf{v}_A = (5 \text{ m/s})\mathbf{j}$ in a straight line and at a distance $a = 2.56$ m from the y-axis, while B moves with a velocity $\mathbf{v}_B = (7.2 \text{ m/s})\mathbf{i} - (4.6 \text{ m/s})\mathbf{j}$ along a path intersecting the x-axis at a distance $b = 7.48$ m from the origin O. Determine (a) the initial velocity $\overline{\mathbf{v}}_0$ of the mass center G of the two disks, (b) the length of the cord initially connecting the two disks, (c) the rate in rad/s at which the disks were spinning about G.

SOLUTION

Initial conditions.

Location of G:



$$\frac{AG}{m_B} = \frac{BG}{m_A} = \frac{AG + GB}{m_B + m_A} = \frac{l}{m}$$

$$AG = \frac{m_B}{m} l = \frac{1}{3} l$$

$$BG = \frac{m_A}{m} l = \frac{2}{3} l$$

Linear momentum:

$$\mathbf{L}_0 = m\overline{\mathbf{v}}_0 = 3\overline{\mathbf{v}}_0$$

Angular momentum about G:

$$(\mathbf{H}_G)_0 = \overrightarrow{GA} \times m_A \mathbf{v}'_A + \overrightarrow{GB} \times m_B \mathbf{v}'_B$$

$$= \left(\frac{1}{3} l\right)(2 \text{ kg})\left(\frac{1}{3} l\omega\right)\mathbf{k} + \left(\frac{2}{3} l\right)(1 \text{ kg})\left(\frac{2}{3} l\omega\right)\mathbf{k}$$

$$= \frac{2}{3} l^2 \omega \mathbf{k}^2$$

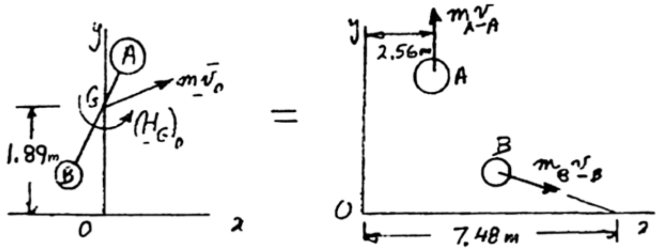
Kinetic energy: Using Eq. (14.29),

$$T_0 = \frac{1}{2} m\overline{v}_0^2 + \frac{1}{2} \sum_i m_i v_i^2 = \frac{1}{2} m\overline{v}_0^2 + \frac{1}{2} m_A v_A'^2 + \frac{1}{2} m_B v_B'^2$$

$$= \frac{1}{2}(3)\overline{v}_0^2 + \frac{1}{2}(2)\left(\frac{1}{3} l\omega\right)^2 + \frac{1}{2}(1)\left(\frac{2}{3} l\omega\right)^2$$

$$T_0 = \frac{3}{2}\overline{v}_0^2 + \frac{1}{3} l^2 \omega^2$$

PROBLEM 14.54 (Continued)



Conservation of linear momentum:

$$m\bar{\mathbf{v}}_0 = m_A\mathbf{v}_A + m_B\mathbf{v}_B$$

$$3\bar{\mathbf{v}}_0 = (2)(5\mathbf{j}) + (1)(7.2\mathbf{i} - 4.6\mathbf{j}) = 7.2\mathbf{i} + 5.4\mathbf{j}$$

(*a*) $\bar{\mathbf{v}}_0 = (2.4 \text{ m/s})\mathbf{i} + (1.8 \text{ m/s})\mathbf{j}$ ◀

Conservation of angular momentum about *O*:

$$+\circlearrowleft: \quad (1.89\mathbf{j}) \times m\bar{\mathbf{v}}_0 + (\mathbf{H}_G)_0 = (2.56\mathbf{i}) \times m_A\mathbf{v}_A + (7.48\mathbf{i}) \times m_B\mathbf{v}_A$$

Substituting for $\bar{\mathbf{v}}_0$, $(\mathbf{H}_G)_0, \mathbf{v}_A, \mathbf{v}_B$ and masses:

$$(1.89\mathbf{j}) \times 3(2.4\mathbf{i} + 1.8\mathbf{j}) + \frac{2}{3}l^2\omega\mathbf{k} = (2.56\mathbf{i}) \times 2(5\mathbf{j}) + (7.48\mathbf{i}) \times (7.2\mathbf{i} - 4.6\mathbf{j})$$

$$-13.608\mathbf{k} + \frac{2}{3}l^2\omega\mathbf{k} = 25.6\mathbf{k} - 34.408\mathbf{k}$$

$$\frac{2}{3}l^2\omega = 4.80 \qquad l^2\omega = 7.20 \tag{1}$$

Conservation of energy:

$$T_0 = T: \quad \frac{3}{2}\bar{v}_0^2 + \frac{1}{3}l^2\omega^2 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$$

$$\frac{3}{2}[(2.4)^2 + (1.8)^2] + \frac{1}{3}l^2\omega^2 = \frac{1}{2}(2)(5)^2 + \frac{1}{2}(1)[(7.2)^2 + (4.6)^2]$$

$$13.5 + \frac{1}{3}l^2\omega^2 = 25 + 36.5 \qquad l^2\omega^2 = 144.0 \tag{2}$$

Dividing Eq. (2) by Eq. (1), member by member:

$$\omega = \frac{144.0}{7.20} = 20.0 \text{ rad/s}$$

(*c*) Original rate of spin = 20.0 rad/s ◀

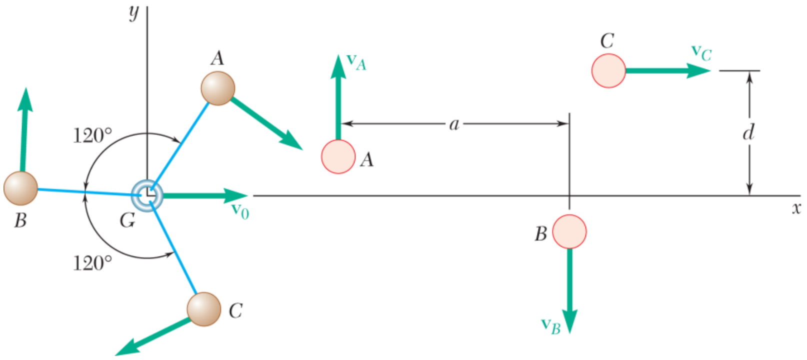
Substituting for ω into Eq. (1):

$$l^2(20.0) = 7.20 \qquad l^2 = 0.360 \qquad l = 0.600 \text{ m}$$

(*b*) Length of cord = 600 mm ◀

PROBLEM 14.55

Three small identical spheres *A*, *B*, and *C*, which can slide on a horizontal, frictionless surface, are attached to three 9-in-long strings, which are tied to a ring *G*. Initially, the spheres rotate clockwise about the ring with a relative velocity of 2.6 ft/s and the ring moves along the *x*-axis with a velocity $\mathbf{v}_0 = (1.3 \text{ ft/s})\mathbf{i}$. Suddenly, the ring breaks and the three spheres move freely in the *xy* plane with *A* and *B*, following paths parallel to the *y*-axis at a distance $a = 1.0$ ft from each other and *C* following a path parallel to the *x*-axis. Determine (*a*) the velocity of each sphere, (*b*) the distance *d*.



SOLUTION

Conservation of linear momentum:

Before break: $\mathbf{L}_0 = (3m)\overline{\mathbf{v}} = 3m(1.3\mathbf{i}) = m(3.9 \text{ ft/s})\mathbf{i}$

After break: $\mathbf{L} = mv_A\mathbf{j} - mv_B\mathbf{j} + mv_C\mathbf{i}$

$\mathbf{L} = \mathbf{L}_0$: $mv_C\mathbf{i} + m(v_A - v_B)\mathbf{j} = m(3.9 \text{ ft/s})\mathbf{i}$

Therefore, $v_A = v_B$ (1)

$v_C = 3.9000 \text{ ft/s} \quad \mathbf{v}_C = 3.90 \text{ ft/s} \longrightarrow$ (2)

Conservation of angular momentum:

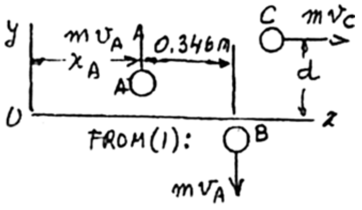
Before break: $+\circlearrowright (H_O)_0 = 3mlv' = 3m(0.75 \text{ ft})(2.6 \text{ ft/s})$

$= 5.85m$

After break: $H_O = -mv_Ax_A$

$+ mv_A(x_A + 0.346)$

$+ mv_Cd$



$+\circlearrowright H_O = -m_Ax_A + mv_A(x_A + 1.0) + mv_Cd$

$H_O = (H_O)_0$: $1.0mv_A + mv_Cd = 5.85m$

Recalling Eq. (2):

$v_A + 3.9d = 5.85$

$d = 1.5 - 0.25641v_A$ (3)

PROBLEM 14.55 (Continued)

Conservation of energy.

Before break:

$$T_0 = \frac{1}{2}(3m)\overline{v}^2 + 3\left(\frac{1}{2}mv'^2\right)$$

$$= \frac{3}{2}m\left(v_0^2 + v'^2\right) = \frac{3}{2}[(1.3)^2 + (2.6)^2]m = 12.675m$$

After break:

$$T = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}mv_C^2$$

$T = T_0$: Substituting for v_B from Eq. (1) and v_C from Eq. (2),

$$\frac{1}{2}\left[v_A^2 + v_A^2 + (3.900)^2\right] = 12.675$$

$$v_A^2 = 5.0700$$

$$v_A = v_B = 2.2517 \text{ ft/s}$$

(*a*) *Velocities*:

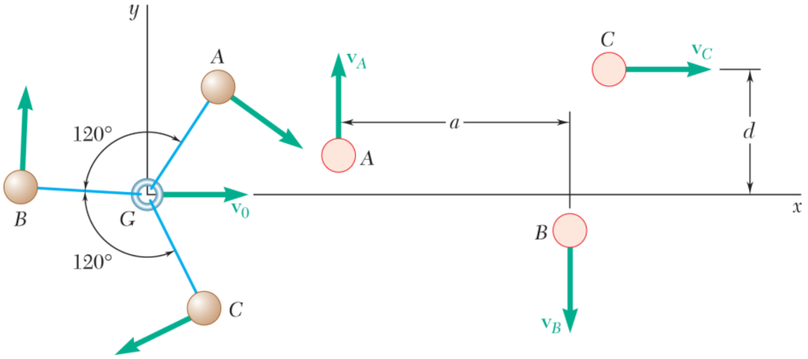
$\mathbf{v}_A = 2.25$ ft/s ↑; $\mathbf{v}_B = 2.25$ ft/s ↓; $\mathbf{v}_C = 3.9$ ft/s ⟶ ◀

(*b*) *Distance d*:

From Eq. (3): $d = 1.5 - 0.25641(2.2517) = 0.92265$ ft $d = 11.1$ in. ◀

PROBLEM 14.56

Three small identical spheres *A*, *B*, and *C*, which can slide on a horizontal, frictionless surface, are attached to three strings of length *l* which are tied to a ring *G*. Initially, the spheres rotate clockwise about the ring which moves along the *x* axis with a velocity $\mathbf{v}_0$. Suddenly the ring breaks and the three spheres move freely in the *xy* plane. Knowing that $\mathbf{v}_A = (3.5\text{ ft/s})\mathbf{j}$, $\mathbf{v}_C = (6.0\text{ ft/s})\mathbf{i}$, $a = 16$ in. and $d = 9$ in., determine (*a*) the initial velocity of the ring, (*b*) the length *l* of the strings, (*c*) the rate in rad/s at which the spheres were rotating about *G*.



SOLUTION

Conservation of linear momentum:

$$(3m)\overline{\mathbf{v}} = m\mathbf{v}_A + m\mathbf{v}_B + m\mathbf{v}_C$$

$$3mv_0\mathbf{i} = m(3.5\text{ ft/s})\mathbf{j} - mv_B\mathbf{j} + m(6.0\text{ ft/s})\mathbf{i}$$

Equating coefficients of unit vectors:

$$3v_0 = 6.00\text{ ft/s}$$

$$0 = 3.5\text{ ft/s} - \mathbf{v}_B \qquad v_B = 3.5\text{ ft/s} \qquad (1)$$

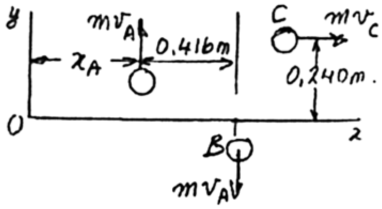
$$v_0 = 2.00\text{ ft/s} \longrightarrow ◀$$

(*a*) Conservation of angular momentum:

Before break: $\circlearrowright\!+\;(H_O)_0 = 3ml^2\dot{\theta}$

After break:

$$\begin{aligned} H_O &= -mv_Ax_A + mv_A(x_A + 16/12) + mv_C(9/12) \\ &= m(3.5)(16/12) + m(6.0)(9/12) \\ &= m(9.1667) \end{aligned}$$



$(H_O)_0 = H_0$: $\qquad 3ml^2\dot{\theta} = m(9.1667) \qquad l^2\dot{\theta} = 3.0556 \qquad (2)$

PROBLEM 14.56 (Continued)

Conservation of energy:

Before break:

$$T_0 = \frac{1}{2}(3m)\overline{v}^2 + 3\left(\frac{1}{2}mv'^2\right) = \frac{3}{2}mv_0^2 + \frac{3}{2}m(l\dot{\theta})^2$$

$$= \frac{3}{2}m(2.0)^2 + \frac{3}{2}ml^2\dot{\theta}^2$$

After break:

$$T = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}mv_C^2$$

$$= \frac{1}{2}m[(3.5)^2 + (6.0)^2] = \frac{1}{2}m(60.5)$$

$T = T_0$: $\quad \frac{1}{2}m(60.5) = \frac{3}{2}m(2.0)^2 + \frac{3}{2}ml^2\dot{\theta}^2$

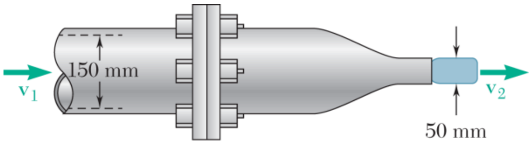
$$l^2\dot{\theta}^2 = 16.167 \qquad (3)$$

Dividing Eq. (3) by Eq. (2): $\quad \dot{\theta} = \frac{16.167}{3.0556} = 5.2909$

(*b*) From Eq. (2): $\quad l^2 = \frac{3.0556}{5.2909} \qquad l = 0.75994 \text{ ft}$

$l = 0.76$ ft ◀

(*c*) Rate of rotation: $\dot{\theta} = 5.29$ rad/s ↻ ◀



PROBLEM 14.57

A stream of water with density ρ = 1000 kg/m^3 is discharged from a nozzle at the rate of 0.06 m^3/s. Using Bernoulli's equation the gage pressure P in the pipe just upstream from the nozzle is $P = \frac{\rho}{2}\left(v_2^2 - v_1^2\right)$. Knowing the nozzle is held to the pipe by six flange bolts, determine the tension in each bolt, neglecting the initial tension caused by the tightening of the nuts.

SOLUTION

Given:

$\rho = 1000 \text{ kg/m}^3$

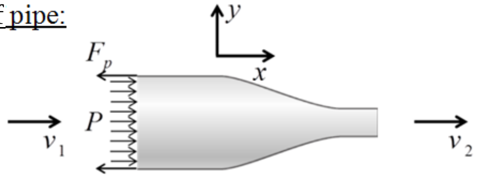
$Q = 0.06 \text{ m}^3\text{/s}$

$d_1 = 0.150 \text{ m}$

$d_2 = 0.05 \text{ m}$

$F_p =$ Force in the pipe wall

FBD of pipe:



Mass flow rates.

$$\frac{dm_1}{dt} = \frac{dm_2}{dt} = \rho Q = 60 \text{ kg/s}$$

Velocity of fluid upstream and downstream from the nozzle:

$$Q = A_1 v_1 \qquad Q = A_2 v_2$$

$$v_1 = \frac{4Q}{\pi d_1^2} \qquad v_2 = \frac{4Q}{\pi d_2^2}$$

$$v_1 = 3.395 \text{ m/s} \qquad v_2 = 30.56 \text{ m/s}$$

The gauge pressure in the pipe upstream of the nozzle is:

$$P = \frac{\rho}{2}\left(v_2^2 - v_1^2\right) = 461.12 \text{ kN/m}^2$$

Apply principle of Impulse-momentum in x-direction

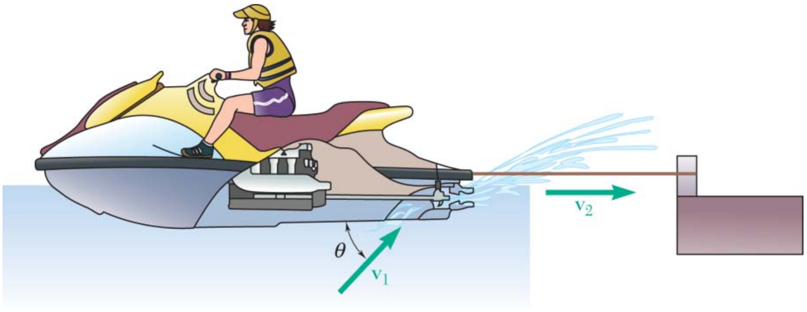
$$\sum F = \frac{dm}{dt}\left(v_2 - v_1\right) \quad \text{where: } \sum F = PA_1 - F_p$$

$$F_p = 461.12 \bullet \frac{\pi}{4}\left(0.15\right)^2 - 60\left(30.56 - 3.395\right)$$

$$= 6519.0 \text{ N}$$

Force per bolt: $F_b = \frac{F_p}{6}$

$F_b = 1086.5 \text{ N}$ ◀



PROBLEM 14.58

A jet ski is placed in a channel and is tethered so that it is stationary. Water enters the jet ski with velocity $\mathbf{v}_1$ and exits with velocity $\mathbf{v}_2$. Knowing the inlet area is A_1 and the exit area is A_2, determine the tension in the tether.

SOLUTION

Mass flow rates. Consider a cylindrical portion of the fluid lying in a section of pipe of cross sectional area A and length Δl.

The volume and mass are

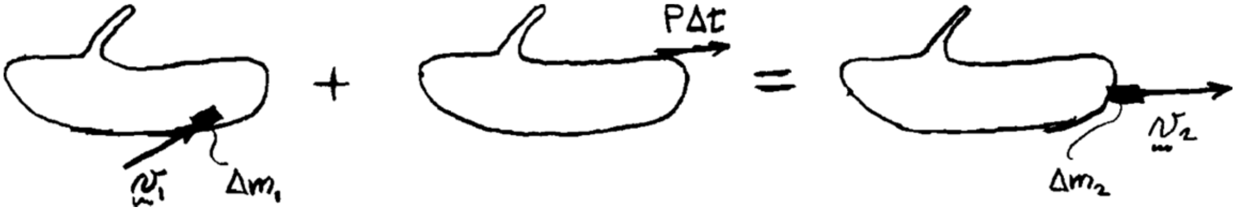
$$\Delta m = \rho A(\Delta l)$$

Then
$$\frac{\Delta m}{\Delta t} = \rho A \frac{\Delta l}{\Delta t} = \rho A v$$

At the pipe inlet and outlet, we get

$$\frac{\Delta m_1}{\Delta t} = \rho A_1 v_1, \qquad \frac{\Delta m_2}{\Delta t} = \rho A_2 v_2$$

Impulse and momentum principle:



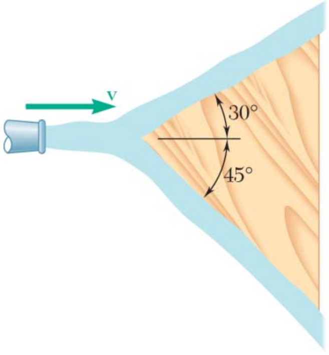
$$\Sigma m\mathbf{v}_1 + \text{Imp}_{1\to 2} = \Sigma m\mathbf{v}_2$$

Using horizontal components $(+\rightarrow)$,

$$(\Delta m_1)v_1 \cos\theta + P(\Delta t) = (\Delta m_2)v_2$$

$$P = \frac{\Delta m_1}{\Delta t} v_2 - \frac{\Delta m_1}{\Delta t} v_1 \cos\theta$$
$$= \rho A_2 v_2^2 - \rho A_1 v_1^2 \cos\theta$$

$$P = \rho A_2 v_2^2 - \rho A_1 v_1^2 \cos\theta \blacktriangleleft$$



PROBLEM 14.59

The nozzle shown discharges a stream of water at a flow rate Q = 475 gal/min with a velocity **v** of magnitude 60 ft/s. The stream is split into two streams with equal flow rates by a wedge that is kept in a fixed position. Determine the components (drag and lift) of the force exerted by the stream on the wedge. (1 $\text{ft}^3 = 7.48$ gal.)

SOLUTION

Let **F** be the force that the wedge exerts on the stream. Assume that the fluid speed is constant. $v = 60$ ft/s.

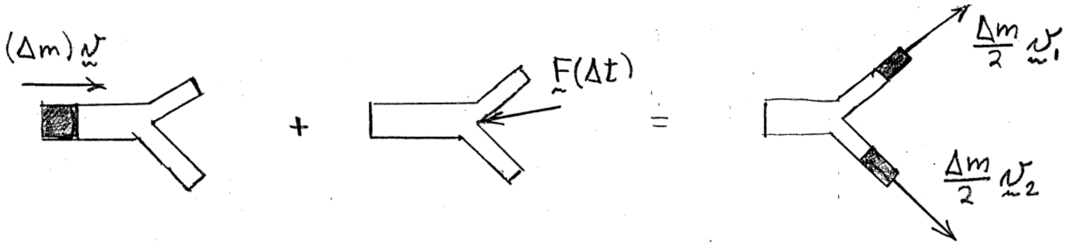
Volumetric flow rate: $Q = 475 \text{ gal/min} = 1.0583 \text{ ft}^3\text{/s}$

Mass flow rate: $$\frac{dm}{dt} = \rho Q = \left(\frac{62.4}{32.2} \text{ slug/ft}^3\right)\left(1.0584 \text{ ft}^3\text{/s}\right) = 2.051 \text{ slug/s}$$

Velocity vectors: $$\mathbf{v} = v\mathbf{i}, \quad \mathbf{v}_1 = v\left(\cos 30°\,\mathbf{i} + \sin 30°\,\mathbf{j}\right)$$

$$\mathbf{v}_2 = v\left(\cos 45°\,\mathbf{i} - \sin 45°\,\mathbf{j}\right)$$

Impulse – momentum principle:



$$(\Delta m)\mathbf{v} + \mathbf{F}(\Delta t) = \frac{\Delta m}{2}\mathbf{v}_1 + \frac{\Delta m}{2}\mathbf{v}_2$$

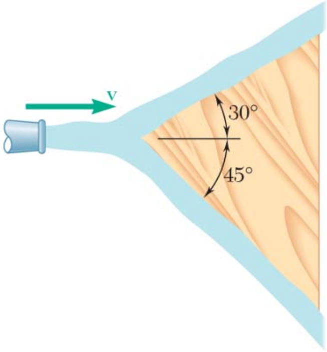
$$\begin{aligned}
\mathbf{F} &= \frac{\Delta m}{\Delta t}\left(\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 - \mathbf{v}\right)\\
&= \frac{dm}{dt}v\left[\frac{1}{2}\left(\cos 30°\,\mathbf{i} + \sin 30°\,\mathbf{j}\right) + \frac{1}{2}\left(\cos 45°\,\mathbf{i} - \sin 45°\,\mathbf{j}\right) - \mathbf{i}\right]\\
&= (2.051)(60 \text{ ft/s})(-0.21343\mathbf{i} - 0.10355\mathbf{j})\\
&= -(26.26 \text{ lb})\mathbf{i} - (12.74 \text{ lb})\mathbf{j}
\end{aligned}$$

Force that the stream exerts on the wedge:

$$-\mathbf{F} = (26.26 \text{ lb})\mathbf{i} + (12.74 \text{ lb})\mathbf{j}$$

drag = 26.3 lb → ◀

lift = 12.74 lb ↑ ◀



PROBLEM 14.60

The nozzle shown discharges a stream of water at a flow rate $Q = 500$ gal/min with a velocity **v** of magnitude 48 ft/s. The stream is split into two streams of equal flow rates by a wedge that is moving to the left at a constant speed of 12 ft/s. Determine the components (drag and lift) of the force exerted by the stream on the wedge. ($1\text{ft}^3 = 7.48$ gal.)

SOLUTION

For a fixed observer, the upstream velocity is $\mathbf{v} = (48 \text{ ft/s})\mathbf{i}$.

Volumetric flow rate: $Q = 500 \text{ gal/min} = 1.1141 \text{ ft}^3/\text{s}$

Mass flow rate:
$$\frac{dm}{dt} = \rho Q = \left(\frac{62.4}{32.2} \text{ slug/ft}^3\right)\left(1.1141 \text{ ft}^3/\text{s}\right) = 2.1590 \text{ slug/s}$$

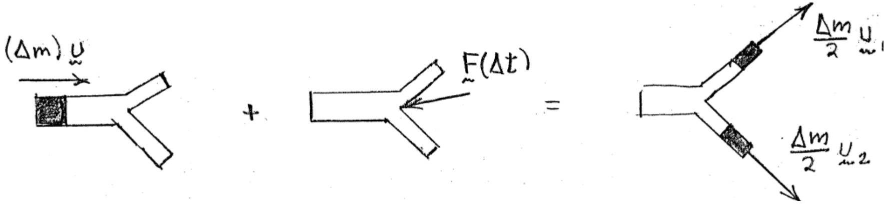
Use a frame of reference that is moving with the wedge to the left at 12 ft/s. In this frame of reference the upstream velocity vector is

$$\mathbf{u} = 48\mathbf{i} - (-12\mathbf{i}) = (60 \text{ ft/s})\mathbf{i}.$$

For the moving frame of reference the mass flow rate is
$$\frac{dm'}{dt} = \frac{u}{v}\frac{dm}{dt} = \frac{60}{48}(2.1590) = 2.6987 \text{ slug/s}.$$

Velocity vectors:
$$\mathbf{u} = u\mathbf{i}, \quad \mathbf{u}_1 = u(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$$
$$\mathbf{u}_2 = u(\cos 45^\circ \mathbf{i} - \sin 45^\circ \mathbf{j})$$

Let **F** be the force that the wedge exerts on the stream.
Impulse-momentum principle:


$(\Delta m)\underset{\sim}{u}$
+
$\underset{\sim}{F}(\Delta t)$
=
$\frac{\Delta m}{2}\underset{\sim}{u}_1$
$\frac{\Delta m}{2}\underset{\sim}{u}_2$

$$(\Delta m)\mathbf{u} + \mathbf{F}(\Delta t) = \frac{\Delta m}{2}\mathbf{u}_1 + \frac{\Delta m}{2}\mathbf{u}_2$$

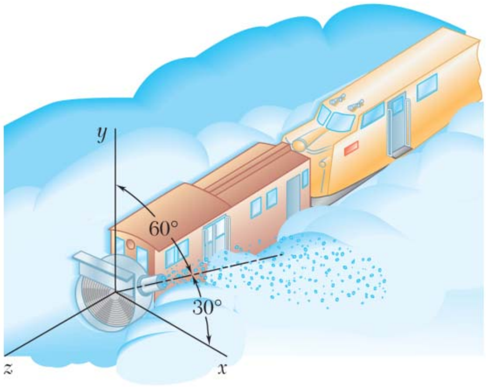
$$\mathbf{F} = \frac{\Delta m}{\Delta t}\left(\frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 - \mathbf{u}\right)$$
$$= \frac{dm'}{dt}u\left[\frac{1}{2}(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) + \frac{1}{2}(\cos 45^\circ \mathbf{i} - \sin 45^\circ \mathbf{j}) - \mathbf{i}\right]$$
$$= (2.6987)(60)(-0.21343\mathbf{i} - 0.10355\mathbf{j})$$
$$= -(34.6 \text{ lb})\mathbf{i} - (16.76 \text{ lb})\mathbf{j}$$

Force that the stream exerts on the wedge

$$-\mathbf{F} = (34.6 \text{ lb})\mathbf{i} + (16.76 \text{ lb})\mathbf{j}$$

drag = 34.6 lb ⟶ ◀

lift = 16.76 lb ↑ ◀



PROBLEM 14.61

A rotary power plow is used to remove snow from a level section of railroad track. The plow car is placed ahead of an engine that propels it at a constant speed of 20 km/h. The plow car clears 160 Mg of snow per minute, projecting it in the direction shown with a velocity of 12 m/s relative to the plow car. Neglecting friction, determine (*a*) the force exerted by the engine on the plow car, (*b*) the lateral force exerted by the track on the plow.

SOLUTION

Velocity of the plow: $v_P = 20 \text{ km/h} = 5.5556 \text{ m/s}$

Velocity of thrown snow:

$$\mathbf{v}_s = (12 \text{ m/s})(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) + (5.5556 \text{ m/s})\mathbf{k}$$

Mass flow rate:

$$\frac{dm}{dt} = \frac{(160000 \text{ kg/min})}{(60 \text{ s/min})} = 2666.7 \text{ kg/s}$$

Let F be the force exerted on the plow and the snow.

Apply impulse-momentum, noting that the snow is initially at rest and that the velocity of the plow is constant. Neglect gravity.

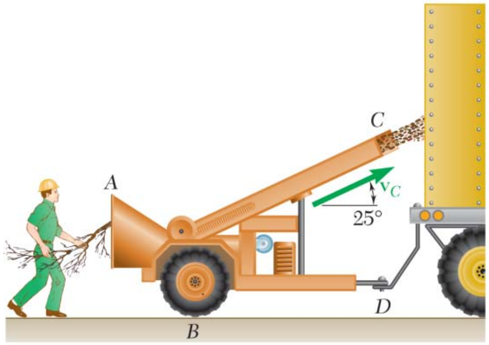
$$\mathbf{F}(\Delta t) = (\Delta m)\mathbf{v}_s$$

$$\mathbf{F} = \left(\frac{dm}{dt}\right)v_s = (2.666.7)(12\cos 30°\mathbf{i} + 12\sin 30°\mathbf{j} + 5.5556\mathbf{k})$$

$$= (27713 \text{ N})\mathbf{i} + (16000 \text{ N})\mathbf{j} + (14815 \text{ N})\mathbf{k}$$

(*a*) Force exerted by engine. $F_z = 14.8 \text{ kN}$ ◀

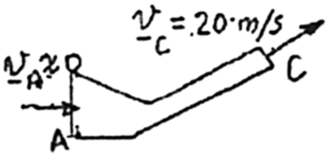
(*b*) Lateral force exerted by track. $F_x = 27.7 \text{ kN}$ ◀



PROBLEM 14.62

Tree limbs and branches are being fed at *A* at the rate of 5 kg/s into a shredder which spews the resulting wood chips at *C* with a velocity of 20 m/s. Determine the horizontal component of the force exerted by the shredder on the truck hitch at *D*.

SOLUTION

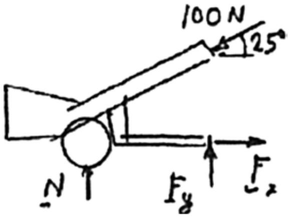


Eq. (14.38):

$$(\Delta m)\mathbf{v}_A + \Sigma\mathbf{F}\Delta t = (\Delta m)\mathbf{v}_C$$

$$\Sigma\mathbf{F} = \frac{\Delta m}{\Delta t}\mathbf{v}_C = (5\ \text{kg/s})(20\ \text{m/s} \measuredangle 25°)$$

Force exerted on chips $= \Sigma\mathbf{F} = 100\ \text{N} \measuredangle 25°$

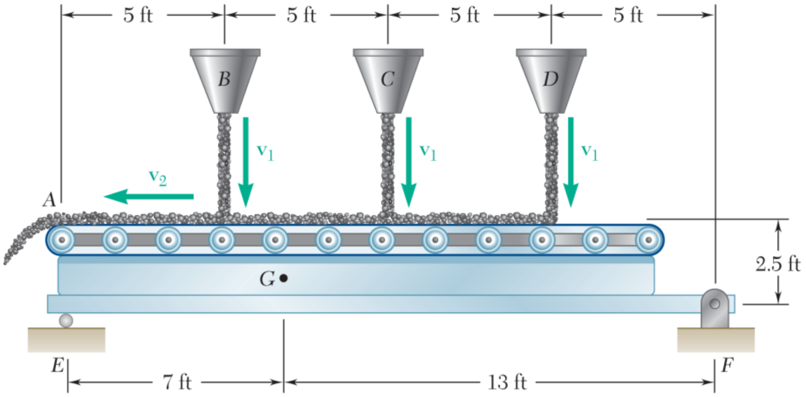


Free body: shredder:

$$\xrightarrow{+} \Sigma F_x = 0: \quad F_x - (100\ \text{N})\cos 25° = 0$$

$$\mathbf{F}_x = 90.6\ \text{N} \longrightarrow$$

On hitch: $\mathbf{F}_x = 90.6\ \text{N} \longleftarrow$ ◀

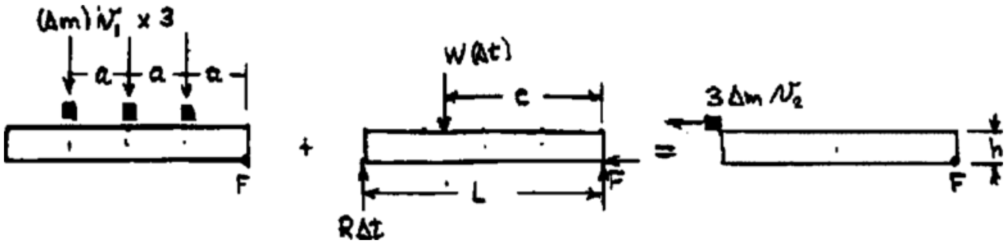

5 ft
5 ft
5 ft
5 ft
B
C
D
v_1
v_1
v_1
v_2
A
G
2.5 ft
E
F
7 ft
13 ft

PROBLEM 14.63

Sand falls from three hoppers onto a conveyor belt at a rate of 90 lb/s for each hopper. The sand hits the belt with a vertical velocity $v_1 = 10$ ft/s and is discharged at A with a horizontal velocity $v_2 = 13$ ft/s. Knowing that the combined mass of the beam, belt system, and the sand it supports is 1300 lb with a mass center at G, determine the reaction at E.

SOLUTION

Principle of impulse and momentum:


$(\Delta m)v_1 \times 3$
a
a
a
F
+
$W(\Delta t)$
c
L
F
$R\Delta t$
=
$3\Delta m\, v_2$
F
h

$+\circlearrowright$ Moments about F:

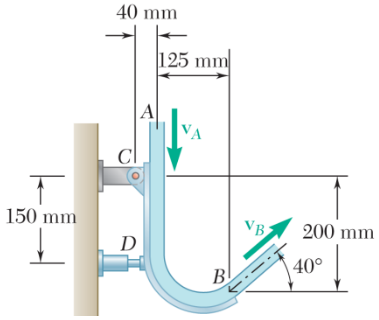
$$(\Delta m)v_1(3a) + \Delta m\, v_1(2a) + (\Delta m)v_1 a + (W\Delta t)c - (R\Delta t)L = 3(\Delta m)v_2 h$$

$$R = \frac{1}{L}\left[cW + 6av_1\frac{\Delta m}{\Delta t} - 3hv_2\frac{\Delta m}{\Delta t}\right]$$

Data: $L = 20$ ft, $c = 13$ ft, $a = 5$ ft, $h = 2.5$ ft, $\dfrac{\Delta m}{\Delta t} = \dfrac{\Delta W/g}{dt} = \dfrac{1}{g}\dfrac{dW}{dt} = 2.7950$ slug/s

$$R = \frac{1}{20}[13(1300) + (6)(5)(10)(2.7950) - (3)(2.5)(13)(2.7950)]$$

$R = 873$ lb $\uparrow$ ◀



PROBLEM 14.64

The stream of water shown flows at a rate of 550 liters/min and moves with a velocity of magnitude 18 m/s at both A and B. The vane is supported by a pin and bracket at C and by a load cell at D that can exert only a horizontal force. Neglecting the weight of the vane, determine the components of the reactions at C and D.

SOLUTION

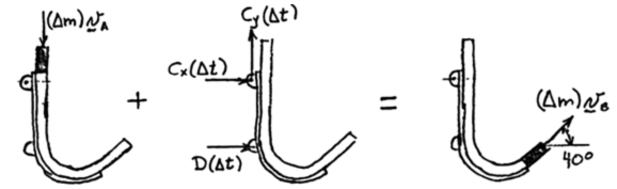
Mass flow rate:

$$\frac{dm}{dt} = \rho Q = \frac{(1000\ \text{kg/m}^3)(550\ \text{liters/min})(1\ \text{min})}{(1000\ \text{liters/m}^3)(60\ \text{sec})}$$

$$\frac{dm}{dt} = 9.1667\ \text{kg/s}$$

Velocity vectors: $\mathbf{v}_A = 18\ \text{m/s}\downarrow \qquad \mathbf{v}_B = 18\ \text{m/s} \measuredangle\ 40^\circ$

Apply the impulse-momentum principle.



+↺ Moments about C: $-0.040(\Delta m)v_A + 0.150D(\Delta t) = 0.200(\Delta m)v_B\cos 40^\circ + 0.165(\Delta m)v_B\sin 40^\circ$

$$D = \frac{1}{0.150}\left(\frac{\Delta m}{\Delta t}\right)[0.200v_B\cos 40^\circ + 0.165v_B\sin 40^\circ + 0.040v_A]$$

$$= \frac{1}{0.150}(9.1667)[(0.200)(18)\cos 40^\circ + 0.165(18)\sin 40^\circ + 0.040(18)]$$

$$= 329.20\ \text{N}$$

$D_x = 329$ N ◀

$D_y = 0$ ◀

$\xrightarrow{+}$ x components: $C_x(\Delta t) + D(\Delta t) = (\Delta m)v_B\cos 40^\circ$

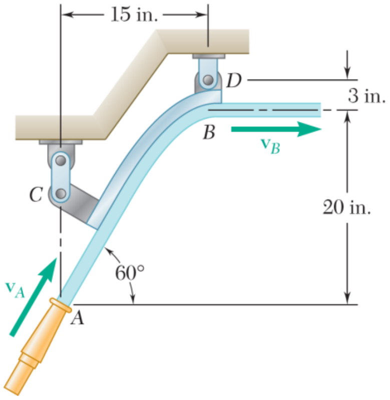
$$C_x = \left(\frac{\Delta m}{\Delta t}\right)v_B\cos 40^\circ - D = (9.1667)(18\cos 40^\circ) - 329.20 = -202.79\ \text{N}$$

$C_x = -203$ N ◀

+↑ y components: $-(\Delta m)v_A + C_y(\Delta t) = (\Delta m)v_B\sin 40^\circ$

$$C_y = \left(\frac{\Delta m}{\Delta t}\right)v_A + \frac{\Delta m}{\Delta t}v_B\sin 40^\circ = (9.1667)(18 + 18\sin 40^\circ) = 271.06\ \text{N}$$

$C_y = 271$ N ◀



PROBLEM 14.65

The nozzle shown discharges water at the rate of 40 ft^3/min. Knowing that at both A and B the stream of water moves with a velocity of magnitude 75 ft/s and neglecting the weight of the vane, determine the components of the reactions at C and D.

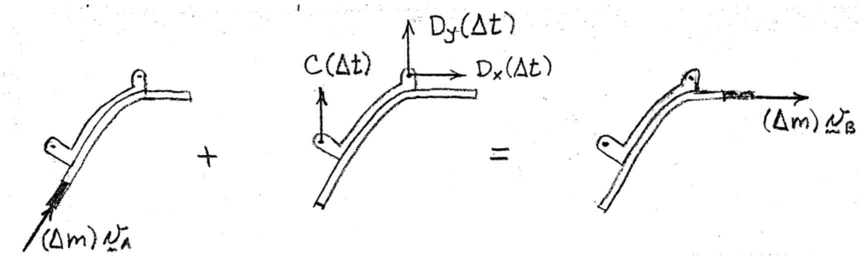
SOLUTION

Mass flow rate:

$$\frac{dm}{dt} = \frac{\gamma}{g}Q = \frac{62.4\ \text{lb/ft}^3}{32.2\ \text{ft/s}^2}\,\frac{40\ \text{ft}^3/\text{min}}{60\ \text{s/min}} = 1.29193\ \text{lb}\cdot\text{s/ft}$$

$$v_A = v_B = 75\ \text{ft/s}$$

Use impulse - momentum principle.



+) moments about D:

$$-(\Delta m)(v_A \sin 60°)\left(\frac{15}{12}\right) + (\Delta m)(v_A \cos 60°)\left(\frac{23}{12}\right) - C(\Delta t)\left(\frac{15}{12}\right)$$

$$= (\Delta m)v_B\left(\frac{3}{12}\right)$$

$$\frac{15}{12}C = \left(\frac{\Delta m}{\Delta t}\right)v_A\left[-\frac{15}{12}\sin 60° + \frac{23}{12}\cos 60° - \frac{3}{12}\right] = (1.29193)(75)(-0.37420)$$

$$C = -29.006\ \text{lb} \qquad C_x = 0, \quad C_y = -29.0\ \text{lb} ◀$$

$\xrightarrow{+}$ x component:

$$(\Delta m)v_A \cos 60° + D_x(\Delta t) = (\Delta m)v_B$$

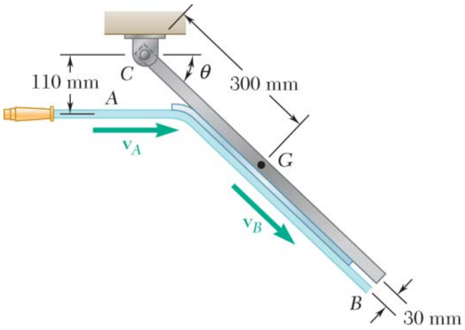
$$D_x = \left(\frac{\Delta m}{\Delta t}\right)(v_B - v_A \cos 60°) = (1.29193)(75 - 75\cos 60°) \qquad D_x = 48.4\ \text{lb} ◀$$

+↑ y component:

$$(\Delta m)v_A \sin 60° + C(\Delta t) + D_y(\Delta t) = 0$$

$$D_y = -C - \frac{\Delta m}{\Delta t}v_A \sin 60° = +29.006 - (1.29193)(75)\sin 60°$$

$$D_y = -54.9\ \text{lb} ◀$$



PROBLEM 14.66

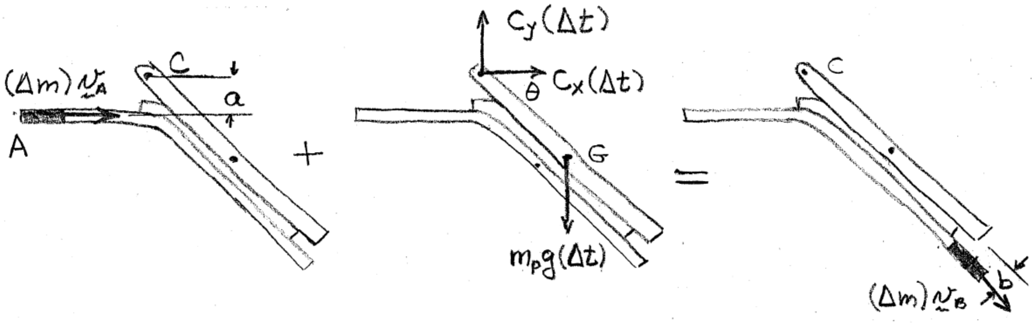
A stream of water flowing at a rate of 1.2 m^3/min and moving with a velocity of magnitude 30 m/s at both *A* and *B* is deflected by a vane welded to a hinged plate. Knowing that the combined mass of the vane and plate is 20 kg with the mass center at point *G*, determine (*a*) the angle θ, (*b*) the reaction at *C*.

SOLUTION

Mass flow rate:
$$\frac{dm}{dt} = \rho Q = \left(1000\frac{\text{kg}}{\text{m}^3}\right)\left(1.2\frac{\text{m}^3}{\text{min}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 20\text{ kg/s}$$

$$v_A = v_B = 30\text{ m/s}$$

Use the impulse-momentum principle.



+↻ Moments about *C*:
$$(\Delta m)v_A\, a - m_\rho g(\Delta t)\; l\cos\theta = (\Delta m)v_B b$$

(*a*)
$$\cos\theta = \frac{\Delta m}{\Delta t}\frac{v_A a - v_B b}{m_p g l} = (20)\frac{(30)(0.110)-(30)(0.030)}{(20)(9.81)(0.300)} = 0.81549$$

$$\theta = 35.364°$$

$\theta = 35.4°$ ◀

$\xrightarrow{+}$ *x* components:
$$(\Delta m)v_A + C_x(\Delta t) = (\Delta m)v_B\cos\theta$$

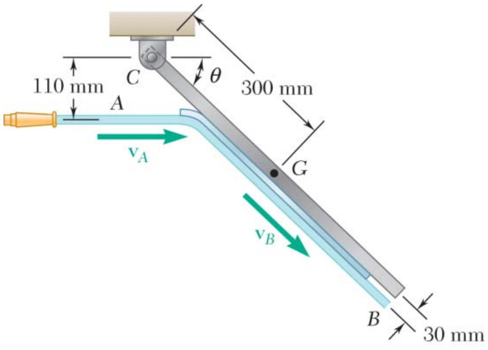
$$C_x = \frac{\Delta m}{\Delta t}(v_B\cos\theta - v_A) = (20)(30\cos\theta - 30) = -110.70\text{ N}$$

+↑ *y* components:
$$0 + C_y(\Delta t) - m_p g(\Delta t) = -(\Delta m)v_B\sin\theta$$

$$C_y = m_p g - \frac{\Delta m}{\Delta t}v_B\sin\theta = (20)(9.81) - (20)(30)\sin\theta = -151.06\text{ N}$$

(*b*)
$$\mathbf{C} = [110.70\text{ N} \leftarrow] + [151.06\text{ N} \downarrow]$$

$\mathbf{C} = 187.3$ N ↙ 53.8° ◀



PROBLEM 14.67

A stream of water flowing at a rate of 1.2 m^3/min and moving with a velocity of magnitude v at both A and B is deflected by a vane welded to a hinged plate. The combined mass of the vane and plate is 20 kg with the mass center at point G. Knowing that $\theta = 45°$, determine (*a*) the speed v of the flow, (*b*) the reaction at C.

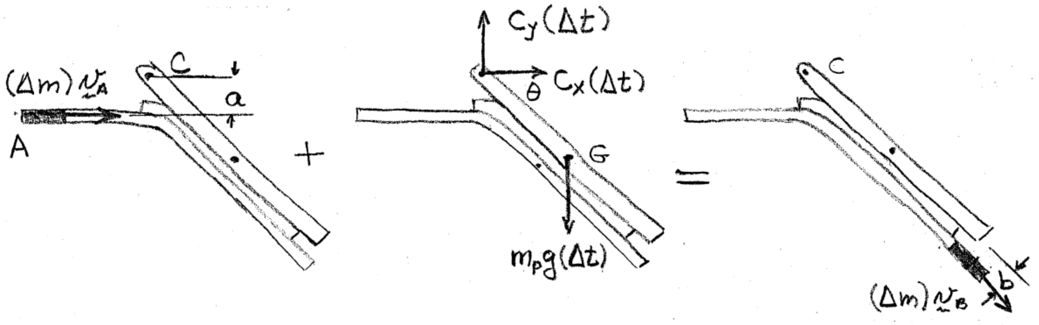
SOLUTION

Mass flow rate:

$$\frac{dm}{dt} = \rho Q = \left(1000\frac{\text{kg}}{\text{m}^3}\right)\left(1.2\frac{\text{m}^3}{\text{min}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 20\text{ kg/s}$$

$$v_A = v_A = v, \qquad \theta = 45°$$

Use the impulse-momentum principle.


$(\Delta m)\underset{\sim}{v}_A$
C
a
A
+
$C_y(\Delta t)$
$C_x(\Delta t)$
θ
G
$m_p g(\Delta t)$
=
C
$(\Delta m)\underset{\sim}{v}_B$
b

+↻ moments about C:

$$(\Delta m)v_A a - m_p g(\Delta t)\, l\cos\theta = (\Delta m)v_B b$$

(*a*)

$$v_A = v_B = \frac{m_p g\, l\cos\theta}{(a-b)(\Delta m)/(\Delta t)} = \frac{(20)(9.81)(0.300)\cos 45°}{(0.110-0.030)(20)} = 26.013\text{ m/s}$$

$v = 26.0$ m/s ◀

$\xrightarrow{+}$ x components:

$$(\Delta m)v_A + C_x(\Delta t) = (\Delta m)v_B\cos\theta$$

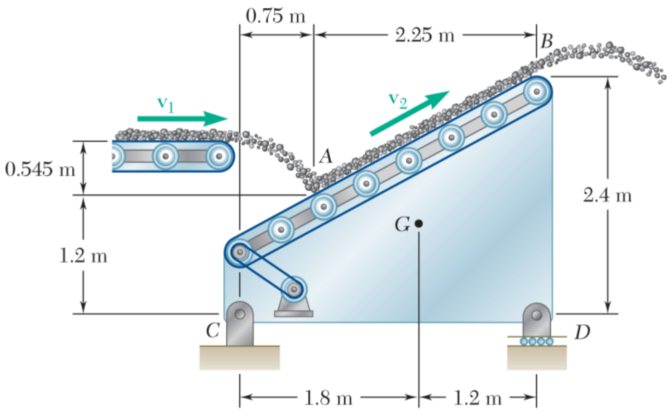
$$C_x = \frac{\Delta m}{\Delta t}(v_B\cos\theta - v_A) = (20)[26.013\cos 45° - 26.013] = -152.38\text{ N}$$

+↑ y components:

$$0 + C_y(\Delta t) - m_p g(\Delta t) = -(\Delta m)v_B\sin\theta$$

$$C_y = m_p g - \frac{\Delta m}{\Delta t}v_B\sin\theta = (20)(9.81) - (20)(26.013)\sin 45° = -171.68\text{ N}$$

(*b*) $\mathbf{C} = [-152.38\text{ N} \longleftarrow] + [-171.68\text{ N} \downarrow]$ $\qquad$ $\mathbf{C} = 230$ N ↙ 48.4° ◀



PROBLEM 14.68

Coal is being discharged from a first conveyor belt at the rate of 120 kg/s. It is received at *A* by a second belt that discharges it again at *B*. Knowing that $v_1 = 3$ m/s and $v_2 = 4.25$ m/s and that the second belt assembly and the coal it supports have a total mass of 472 kg, determine the components of the reactions at *C* and *D*.

SOLUTION

Velocity before impact at *A*:

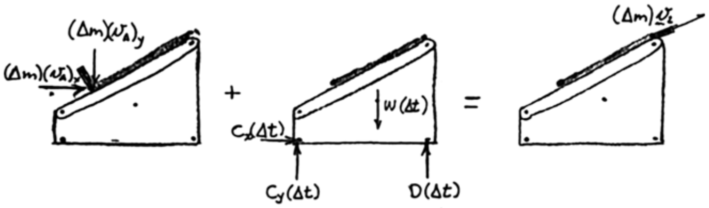
$$(v_A)_x = v_1 = 3 \text{ m/s} \longrightarrow$$

$$(v_A)_y^2 = 2g(\Delta y) = (2)(9.81)(0.545) = 10.693 \text{ m}^2/\text{s}^2 \qquad (v_A)_y = 3.270 \text{ m/s} \downarrow$$

Slope of belt: $\tan\theta = \dfrac{2.4 - 1.2}{2.25}, \qquad \theta = 28.07°$

Velocity of coal leaving at *B*: $\mathbf{v}_2 = 4.25\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$

Apply the impulse-momentum principle.



$\xrightarrow{+}$ x components: $(\Delta m)(v_A)_x + C_x(\Delta t) = (\Delta m)v_2\cos\theta$

$$C_x = \frac{\Delta m}{\Delta t}[v_2\cos\theta - (v_A)_x] = (120)(4.25\cos 28.07° - 3)$$

$$C_x = 90.0 \text{ N} \longrightarrow$$

$\circlearrowleft +$ moments about C: $(\Delta m)[-1.2(v_A)_x - 0.75(v_A)_y] + 3.00D(\Delta t) - 1.8W(\Delta t) = (\Delta m)[-2.4\,v_2\cos\theta + 3v_2\sin\theta]$

$$\frac{\Delta m}{\Delta t}[-(1.2)(3) - (0.75)(3.270)] + 3D - (1.8)(472)(9.81)$$

$$= \frac{\Delta m}{\Delta t}[-(2.4)(4.25\cos\theta) + (3)(4.25\sin\theta)]$$

$$D = 2775 + 1.0168\frac{dm}{dt} = 2775 + (1.0168)(120) = 2897 \text{ N} \uparrow$$

PROBLEM 14.68 (Continued)

$+\uparrow$ y components: $$(\Delta m)(-v_A)_y + (C_y + D - W)(\Delta t) = (\Delta m)v_2 \sin\theta$$

$$C_y + D - W = \frac{\Delta m}{\Delta t}(3.270 + 4.25\sin\theta)$$

$$= (120)(5.268) = 632.2 \text{ N}$$

$$C_y = 4625.6 - 2897 + 632.2 \qquad C_y = 2361 \text{ N}\uparrow$$

$$C_x = 90.0 \text{ N}, \quad C_y = 2360 \text{ N} \blacktriangleleft$$

$$D_x = 0, \qquad D_y = 2900 \text{ N} \blacktriangleleft$$

PROBLEM 14.69

The total drag due to air friction on a jet airplane traveling at 900 km/h is 35 kN. Knowing that the exhaust velocity is 600 m/s relative to the airplane, determine the mass of air that must pass through the engine per second to maintain the speed of 900 km/h in level flight.

SOLUTION

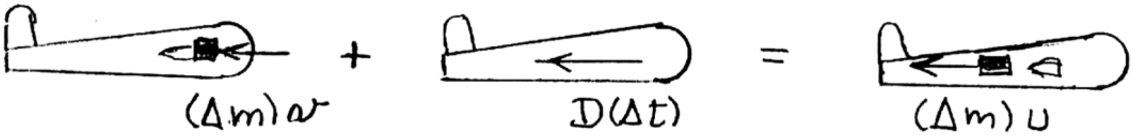
Symbols: $\frac{dm}{dt}$ = mass flow rate

u = exhaust relative to the airplane

v = speed of airplane

D = drag force

Principle of impulse and momentum:



$$\xleftarrow{+} \quad (\Delta m)v + D(\Delta t) = (\Delta m)u$$

$$\frac{\Delta m}{\Delta t} = \frac{dm}{dt} = \frac{D}{u - v}$$

Data: $v = 900 \text{ km/h} = 250 \text{ m/s}$

$u = 600 \text{ m/s}$

$D = 35 \text{ kN} = 35000 \text{ N}$

$$\frac{dm}{dt} = \frac{35000}{600 - 250}$$

$$\frac{dm}{dt} = 100 \text{ kg/s} \blacktriangleleft$$

PROBLEM 14.70

While cruising in level flight at a speed of 600 mi/h, a jet plane scoops in air at the rate of 200 lb/s and discharges it with a velocity of 2100 ft/s relative to the airplane. Determine the total drag due to air friction on the airplane.

SOLUTION

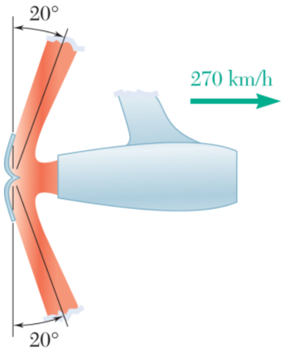
Flight speed: $v = 600 \text{ mi/h} = 880 \text{ ft/s}$

Mass flow rate: $\frac{dm}{dt} = \frac{200 \text{ lb/s}}{32.2 \text{ ft/s}^2} = 6.2112 \text{ slug/s}$

$$\Sigma \mathbf{F} = \mathbf{D} = \frac{dm}{dt}(\mathbf{u} - \mathbf{v}) \qquad \text{or} \qquad D = \frac{dm}{dt}(v - u)$$

where, for a frame of reference moving with the plane, v is the free stream velocity (equal to the air speed) and u is the relative exhaust velocity.

$$D = (6.2112)(2100 - 880) = 7577.6 \text{ lb} \qquad D = 7580 \text{ lb} \blacktriangleleft$$


20°
270 km/h
20°

PROBLEM 14.71

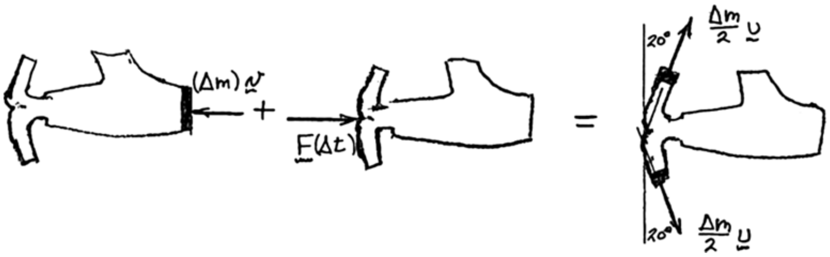
In order to shorten the distance required for landing, a jet airplane is equipped with movable vanes, that partially reverse the direction of the air discharged by each of its engines. Each engine scoops in the air at a rate of 120 kg/s and discharges it with a velocity of 600 m/s relative to the engine. At an instant when the speed of the airplane is 270 km/h, determine the reverse thrust provided by each of the engines.

SOLUTION

Apply the impulse-momentum principle to the moving air. Use a frame of reference that is moving with the airplane. Let **F** be the force on the air.

$$v = 270 \text{ km/h} = 75 \text{ m/s}$$

$$u = 600 \text{ m/s}$$


$(\Delta m)\underset{\sim}{v}$
+
$\underset{\sim}{F}(\Delta t)$
=
20°
$\frac{\Delta m}{2}\underset{\sim}{u}$
20°
$\frac{\Delta m}{2}\underset{\sim}{u}$

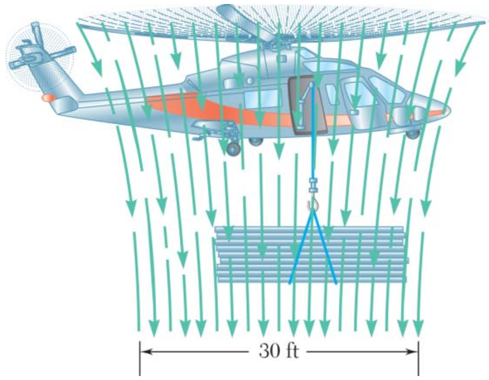
$$-(\Delta m)v + F(\Delta t) = 2\frac{(\Delta m)}{2}u\sin 20°$$

$$F = \frac{\Delta m}{\Delta t}(v + u\sin 20°) = \frac{dm}{dt}(v + u\sin 20°)$$

$$F = (120)(75 + 600\sin 20°) = 33.6 \times 10^3 \text{ N}$$

Force on airplane is $-\mathbf{F}$.

$\mathbf{F} = 33.6$ kN ← ◀



PROBLEM 14.72

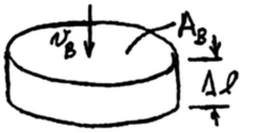
The helicopter shown can produce a maximum downward air speed of 80 ft/s in a 30-ft-diameter slipstream. Knowing that the weight of the helicopter and its crew is 3500 lb and assuming $\gamma = 0.076\ \text{lb/ft}^3$ for air, determine the maximum load that the helicopter can lift while hovering in midair.

SOLUTION

The thrust is
$$F = \frac{dm}{dt}(v_B - v_A)$$

Calculation of $\frac{dm}{dt}$.

$$\text{mass} = \text{density} \times \text{volume} = \text{density} \times \text{area} \times \text{length}$$



$$\Delta m = \rho A_B(\Delta l) = \rho A_B v_B(\Delta t)$$
$$\frac{\Delta m}{\Delta t} = \rho A_B v_B = \frac{\gamma}{g} A_B v_B = \frac{dm}{dt}$$

where A_B is the area of the slipstream well below the helicopter and v_B is the corresponding velocity in the slipstream. Well above the blade, $v_A \approx 0$.

Hence,
$$F = \frac{\gamma}{g} A_B v_B^2$$
$$= \left(\frac{0.076\ \text{lb/ft}^3}{32.2\ \text{ft/s}^2}\right)\left(\frac{\pi}{4}\right)(30\ \text{ft})^2(80\ \text{ft/s})^2$$
$$= 10{,}678\ \text{lb}$$

$$F = 10{,}678\ \text{lb}\downarrow$$

The force on the helicopter is 10,678 lb ↑.

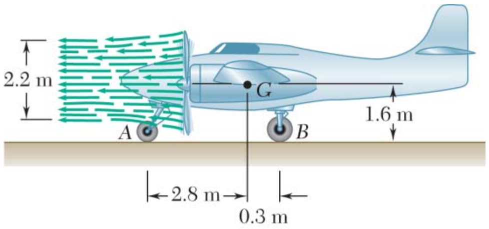
Weight of helicopter: $\mathbf{W}_H = 3500\ \text{lb}\downarrow$

Weight of payload: $\mathbf{W}_P = W_P\downarrow$

Statics: $+\uparrow\ \Sigma F_y = F - W_H - W_P = 0$

$$W_P = F - W_H = 10{,}678 - 3500 = 7178\ \text{lb}$$

$W = 7180\ \text{lb}$ ◀



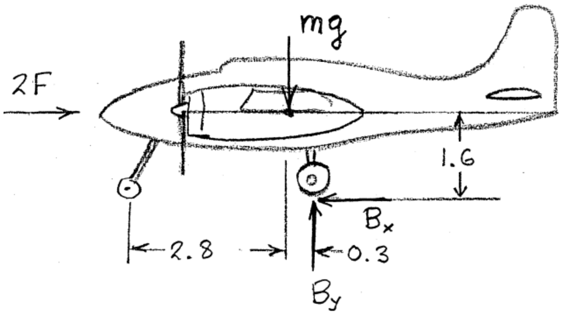
PROBLEM 14.73

Prior to take-off the pilot of a 3000-kg twin-engine airplane tests the reversible-pitch propellers by increasing the reverse thrust with the brakes at point *B* locked. Knowing that point *G* is the center of gravity of the airplane, determine the velocity of the air in the two 2.2-meter-diameter slipstreams when the nose wheel *A* begins to lift off the ground. Assume $\rho = 1.21 \text{ kg/m}^3$ and neglect the approach velocity of the air.

SOLUTION

Let **F** be the force exerted on the slipstream of one engine. Then, the force exerted on the airplane is –2**F** as shown.

Statics.



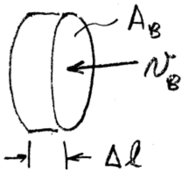
$$+\circlearrowleft \Sigma M_B = 0$$

$$(0.3)\text{mg} - (1.6)(2F) = 0$$

$$F = \frac{(0.3)(3000)(9.81)}{(2)(1.6)}$$

$$= 2.5791 \times 10^3 \text{ N}$$

Calculation of $\frac{dm}{dt}$.



$$\text{mass} = \text{density} \times \text{volume} = \text{density} \times \text{area} \times \text{length}$$

$$\Delta m = \rho A_B(\Delta l) = \rho A_B v_B(\Delta t)$$

$$\frac{\Delta m}{\Delta t} = \frac{dm}{dt} = \rho A_B v_B$$

Force exerted on the slipstream:

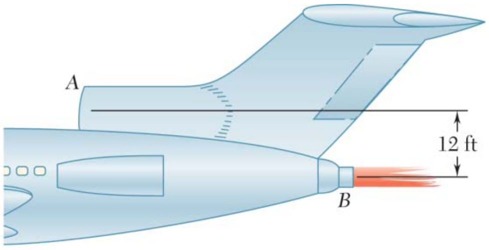
$$F = \frac{dm}{dt}(v_B - v_A)$$

Assume that v_A, the speed far upstream, is negligible.

$$F = \rho A_B v_B(v_B - 0) = \rho\left(\frac{\pi}{4}D^2\right)v_B^2$$

$$v_B^2 = \frac{4F}{\pi D^2 \rho} - \frac{(4)(2.5791 \times 10^3)}{\pi(2.2)^2(1.21)} = 599.8 \text{ m}^2/\text{s}^2$$

$\mathbf{v}_B = 24.5 \text{ m/s} \longleftarrow$ ◀

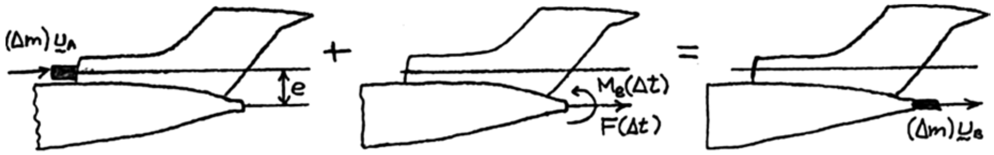


PROBLEM 14.74

The jet engine shown scoops in air at A at a rate of 200 lb/s and discharges it at B with a velocity of 2000 ft/s relative to the airplane. Determine the magnitude and line of action of the propulsive thrust developed by the engine when the speed of the airplane is (*a*) 300 mi/h, (*b*) 600 mi/h.

SOLUTION

Use a frame of reference moving with the plane. Apply the impulse-momentum principle. Let F be the force that the plane exerts on the air.



$\xrightarrow{+}$ x components: $(\Delta m)u_A + F(\Delta t) = (\Delta m)u_B$

$$F = \frac{\Delta m}{\Delta t}(u_B - u_A) = \frac{dm}{dt}(u_B - u_A) \qquad (1)$$

$+\circlearrowleft$ moments about B: $-e(\Delta m)u_A + M_B(\Delta t) = 0$

$$M_B = e\frac{dm}{dt}u_A \qquad (2)$$

Let d be the distance that the line of action is below B.

$$Fd = M_B \qquad d = \frac{M_B}{F} = \frac{eu_A}{u_B - u_A} \qquad (3)$$

Data: $\frac{dm}{dt} = 200 \text{ lb/s} = \frac{200}{32.2} = 6.2112 \text{ slugs/s}, \quad u_B = 2000 \text{ ft/s}, \quad e = 12 \text{ ft}$

(*a*) $u_A = 300 \text{ mi/h} = 440 \text{ ft/s}$

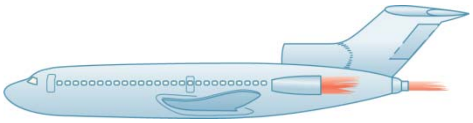
From Eq. (1), $F = (6.2112)(2000 - 440)$ $\qquad F = 9690 \text{ lb}$ ◀

From Eq. (3), $d = \frac{(12)(440)}{2000 - 440}$ $\qquad d = 3.38 \text{ ft}$ ◀

(*b*) $u_A = 600 \text{ mi/h} = 880 \text{ ft/s}$

From Eq. (1), $F = (6.2112)(2000 - 880)$ $\qquad F = 6960 \text{ lb}$ ◀

From Eq. (3), $d = \frac{(12)(880)}{2000 - 880}$ $\qquad d = 9.43 \text{ ft}$ ◀



PROBLEM 14.75

A jet airliner is cruising at a speed of 900 km/h with each of its three engines discharging air with a velocity of 800 m/s relative to the plane. Determine the speed of the airliner after it has lost the use of (*a*) one of its engines, (*b*) two of its engines. Assume that the drag due to air friction is proportional to the square of the speed and that the remaining engines keep operating at the same rate.

SOLUTION

Let v be the airliner speed and u be the discharge relative velocity.

$$u = 800 \text{ m/s.}$$

Thrust formula for one engine: $$F = \frac{dm}{dt}(u - v)$$

Drag formula: $$D = kv^2$$

Three engines working. Cruising speed $= v_0 = 900$ km/h $= 250$ m/s

$$3F - D = 3\frac{dm}{dt}(u - v_0) - kv_0^2 = 0$$

$$\frac{dm}{dt} = \frac{kv_0^2}{3(u - v_0)} = \frac{k(250)^2}{3(800 - 250)} = 37.879k$$

(*a*) *One engine fails*. Two engines working. Cruising speed $= v_1$

$$2F - D = 2\frac{dm}{dt}(u - v_1) - kv_1^2 = 0$$

$$(2)(37.879k)(800 - v_1) - kv_1^2 = 0$$

$$v_1^2 + 75.758v_1 - 60.606 \times 10^3 = 0$$

$$v_1 = 211.20 \text{ m/s} \qquad v_1 = 760 \text{ km/h} \blacktriangleleft$$

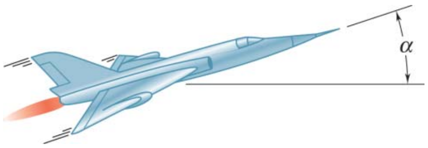
(*b*) *Two engines fail*. One engine working. Cruising speed $= v_2$

$$F - D = \frac{dm}{dt}(u - v_2) - kv_2^2 = 0$$

$$(37.879k)(800 - v_2) - kv_2^2 = 0$$

$$v_2^2 + 37.879v_2 - 30.303 \times 10^3 = 0$$

$$v_2 = 156.17 \text{ m/s} \qquad v_2 = 562 \text{ km/h} \blacktriangleleft$$



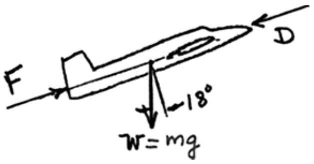
PROBLEM 14.76

A 16-Mg jet airplane maintains a constant speed of 774 km/h while climbing at an angle $\alpha = 18°$. The airplane scoops in air at a rate of 300 kg/s and discharges it with a velocity of 665 m/s relative to the airplane. If the pilot changes to a horizontal flight while maintaining the same engine setting, determine (*a*) the initial acceleration of the plane, (*b*) the maximum horizontal speed that will be attained. Assume that the drag due to air friction is proportional to the square of the speed.

SOLUTION

Calculate the propulsive force using velocities relative to the airplane.

$$F = \frac{dm}{dt}(v_B - v_A)$$



Data:

$$\frac{dm}{dt} = 300 \text{ kg/s}$$

$$v_A = 774 \text{ km/h} = 215 \text{ m/s}$$

$$v_B = 665 \text{ m/s}$$

$$F = (300)(665 - 215) = 135{,}000 \text{ N}$$

Since there is no acceleration while the airplane is climbing, the forces are in equilibrium.

$$+\nearrow 18°\ \Sigma F = 0: \quad F - D - mg\sin\alpha = 0$$

$$F - D = mg\sin\alpha = (16{,}000)(9.8)\sin 18° = 48{,}454 \text{ N}$$

(*a*) *Initial acceleration of airplane in horizontal flight*:

$$ma = F - D: \quad 16{,}000a = 48.454 \times 10^3$$

$\mathbf{a} = 3.03 \text{ m/s}^2 \nearrow 18°$ ◀

Corresponding drag force:

$$D = 135{,}000 - 48{,}454 = 86{,}546 \text{ N}$$

Drag force factor:

$$D = kv_A^2$$

or

$$k = \frac{D}{v_A^2} = \frac{86{,}546}{(215)^2} = 1.87228 \text{ N}\cdot\text{s}^2/\text{m}^2$$

PROBLEM 14.76 (Continued)

(*b*) *Maximum speed in horizontal flight*:

Since the acceleration is zero, the forces are in equilibrium.

$$F - D = 0$$

$$\frac{dm}{dt}(v_B - v_A) - kv_A^2 = 0 \quad kv_A^2 + \frac{dm}{dt}v_A - \frac{dm}{dt}v_B = 0$$

$$1.87228v_A^2 + 300v_A - (300)(665) = 0$$

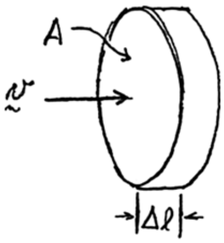
$$v_A = 256.0 \text{ m/s} \qquad v_A = 922 \text{ km/h} \blacktriangleleft$$

PROBLEM 14.77

The propeller of a small airplane has a 2-m-diameter slipstream and produces a thrust of 3600 N when the airplane is at rest on the ground. Assuming $\rho = 1.225\ \text{kg/m}^3$ for air, determine (*a*) the speed of the air in the slipstream, (*b*) the volume of air passing through the propeller per second, (*c*) the kinetic energy imparted per second to the air in the slipstream.

SOLUTION

Calculation of $\frac{dm}{dt}$ at a section in the airstream:



$$\begin{aligned}\text{mass} &= \text{density} \times \text{volume}\\ &= \text{density} \times \text{area} \times \text{length}\\ \Delta m &= \rho A(\Delta l) = \rho A v\,\Delta t\\ \frac{\Delta m}{\Delta t} &= \frac{dm}{dt} = \rho A v\end{aligned}$$

(*a*) Thrust $= \frac{dm}{dt}(\mathbf{v}_B - \mathbf{v}_A)$ where $\mathbf{v}_B$ is the velocity just downstream of propeller and $\mathbf{v}_A$ is the velocity far upstream. Assume $\mathbf{v}_A$ is negligible.

$$\text{Thrust} = (\rho A v)v = \rho\left(\frac{\pi}{4}D^2\right)v^2$$

$$3600 = 1.225\left(\frac{\pi}{4}\right)(2)^2 v^2$$

$$v^2 = 935.44$$

$$v = 30.585\ \text{m/s}$$

$v = 30.6\ \text{m/s}$ ◀

(*b*)

$$Q = \frac{1}{\rho}\frac{dm}{dt} = Av = \left(\frac{\pi}{4}D^2\right)v = \frac{\pi}{4}(2)^2(30.585) = 96.086$$

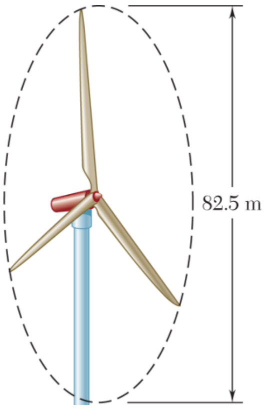
$Q = 96.1\ \text{m}^3/\text{s}$ ◀

(*c*) Kinetic energy of mass Δm:

$$\Delta T = \frac{1}{2}(\Delta m)v^2 = \frac{1}{2}\rho A(\Delta l)v^2 = \frac{1}{2}\rho A v(\Delta t)v^3$$

$$\begin{aligned}\frac{\Delta T}{\Delta t} &= \frac{dT}{dt}\\ &= \frac{1}{2}\rho A v^3\\ &= \frac{1}{2}\rho\left(\frac{\pi}{4}D^2\right)v^3\\ &= \frac{1}{2}(1.225)\left(\frac{\pi}{4}\right)(2)^2(30.585)^3\\ &= 55{,}053\ \text{N}\cdot\text{m/s}\end{aligned}$$

$\frac{dT}{dt} = 55{,}100\ \text{N}\cdot\text{m/s}$ ◀



PROBLEM 14.78

The wind turbine-generator shown has an output-power rating of 1.5 MW for a wind speed of 36 km/h. For the given wind speed, determine (*a*) the kinetic energy of the air particles entering the 82.5-m-diameter circle per second, (*b*) the efficiency of this energy conversion system. Assume $\rho = 1.21\ \text{kg/m}^3$ for air.

SOLUTION

(*a*) Rate of kinetic energy in the slipstream.

Let Δm be the mass moving through the slipstream of area A in the time Δt. Then,

$$\Delta m = \rho A(\Delta l) = \rho A v(\Delta t)$$

The kinetic energy carried by this mass is

$$\Delta t = \frac{1}{2}(\Delta m)v^2 = \frac{1}{2}\rho A v^3(\Delta t)$$

$$\frac{dT}{dt} = \frac{\Delta T}{\Delta t} = \frac{1}{2}\rho A v^3$$

Data:

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}(82.5\ \text{m})^2 = 5345.6\ \text{m}^2$$

$$v = 36\ \text{km/h} = 10\ \text{m/s}$$

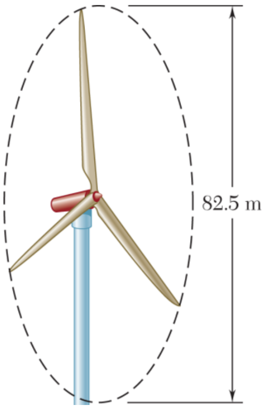
$$\frac{dT}{dt} = \frac{1}{2}(1.21\ \text{kg/m}^3)(5345.6\ \text{m}^2)(10\ \text{m/s})^3$$

$$= 3.234 \times 10^6\ \text{kg} \cdot \text{m}^2/\text{s}^3$$

$$\frac{dT}{dt} = 3.234\ \text{MW} \blacktriangleleft$$

(*b*) Efficiency *n*:

$$\eta = \frac{\text{output power}}{\text{available input power}} = \frac{1.5\ \text{MW}}{3.234\ \text{MW}} \qquad \eta = 0.464 \blacktriangleleft$$



PROBLEM 14.79

A wind turbine-generator system having a diameter of 82.5 m produces 1.5 MW at a wind speed of 12 m/s. Determine the diameter of blade necessary to produce 10 MW of power assuming the efficiency is the same for both designs and $\rho = 1.21\ \text{kg/m}^3$ for air.

SOLUTION

Rate of kinetic energy in the slipstream.

Let Δm be the mass moving through the slipstream of area A in time Δt. Then

$$\Delta m = \rho A(\Delta l) = \rho A v(\Delta t)$$

The kinetic energy carried by this mass is

$$\Delta T = \frac{1}{2}(\Delta m)v^2 = \frac{1}{2}\rho A v^3(\Delta t)$$

$$\frac{dT}{dt} = \frac{\Delta T}{\Delta t} = \frac{1}{2}\rho A v^3$$

This is the available input power for the wind turbine. For a wind turbine of efficiency η, the output power P is

$$P = \eta\frac{dT}{dt} = \frac{\eta}{2}\rho A v^3$$

We want to compare two turbines having $P_1 = 1.5\ \text{MW}$ and $P_2 = 10\ \text{MW}$, respectively. Then

$$\frac{P_2}{P_1} = \frac{\eta_2\rho_2 A_2 v_2^3}{\eta_1\rho_1 A_1 v_1^3}$$

Since $\eta_2 = \eta_1$, $\rho_2 = \rho_1$, and $v_2 = v_1$, we get

$$\frac{P_2}{P_1} = \frac{A_2}{A_1} = \frac{d_2^2}{d_1^2} = \frac{10}{1.5} = 6.6667$$

$$d_2^2 = 6.6667 d_1^2 = (6.6667)(82.5\ \text{in})^2$$

$d_2 = 213\ \text{m}$ ◀

PROBLEM 14.80

While cruising in level flight at a speed of 570 mi/h, a jet airplane scoops in air at a rate of 240 lb/s and discharges it with a velocity of 2200 ft/s relative to the airplane. Determine (*a*) the power actually used to propel the airplane, (*b*) the total power developed by the engine, (*c*) the mechanical efficiency of the airplane.

SOLUTION

Data:

$$\frac{dm}{dt} = \frac{240}{32.2} = 7.4534 \text{ slugs/s}$$

$$u = 2200 \text{ ft/s}$$

$$v = 570 \text{ mi/h} = 836 \text{ ft/s}$$

$$F = \frac{dm}{dt}(u - v)$$

$$= (7.4534)(2200 - 836)$$

$$= 10{,}166 \text{ lb}$$

(*a*) Power used to propel airplane:

$$P_1 = Fv$$

$$= (10{,}166)(836)$$

$$= 8.499 \times 10^6 \text{ ft} \cdot \text{lb/s}$$

Propulsion power $= 15{,}450$ hp ◀

Power of kinetic energy of exhaust:

$$P_2(\Delta t) = \frac{1}{2}(\Delta m)(u - v)^2$$

$$P_2 = \frac{1}{2}\frac{dm}{dt}(u - v)^2$$

$$= \frac{1}{2}(7.4534)(2200 - 836)^2$$

$$= 6.934 \times 10^6 \text{ ft} \cdot \text{lb/s}$$

(*b*) Total power:

$$P = P_1 + P_2$$

$$= 15.433 \times 10^6 \text{ ft} \cdot \text{lb/s}$$

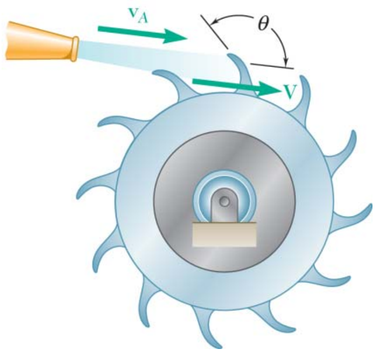
Total power $= 28{,}060$ hp ◀

(*c*) Mechanical efficiency:

$$\frac{P_1}{P} = \frac{8.499 \times 10^6}{15.433 \times 10^6}$$

$$= 0.551$$

Mechanical efficiency $= 0.551$ ◀



PROBLEM 14.81

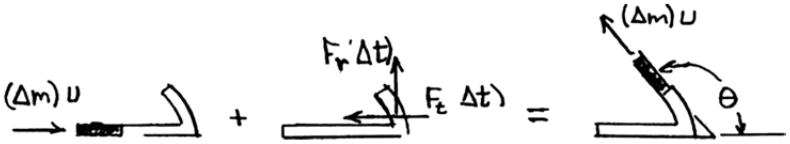
In a Pelton-wheel turbine, a stream of water is deflected by a series of blades so that the rate at which water is deflected by the blades is equal to the rate at which water issues from the nozzle $(\Delta m/\Delta t = A\rho v_A)$. Using the same notation as in Sample Problem 14.8, (*a*) determine the velocity **V** of the blades for which maximum power is developed, (*b*) derive an expression for the maximum power, (*c*) derive an expression for the mechanical efficiency.

SOLUTION

Let **u** be the velocity of the stream relative to the velocity of the blade.

$$u = (v - V)$$

Mass flow rate:
$$\frac{\Delta m}{\Delta t} = \rho A v_A$$



Principle of impulse and momentum:

$$\xrightarrow{+} (\Delta m)u - F_t(\Delta t) = (\Delta m)u \cos\theta$$

$$F_t = \frac{\Delta m}{\Delta t} u(1 - \cos\theta)$$
$$= \rho A v_A (v_A - V)(1 - \cos\theta)$$

where F_t is the tangential force on the fluid.

The force F_t on the fluid is directed to the left as shown. By Newton's law of action and reaction, the tangential force on the blade is F_t to the right.

Output power:
$$P_{\text{out}} = F_t V$$
$$= \rho A v_A (v_A - V)V(1 - \cos\theta)$$

(*a*) V for maximum power output:

$$\frac{dP_{\text{out}}}{dV} = \rho A(v_A - 2V)(1 - \cos\theta) = 0$$

$$v_A = \frac{1}{2}V \quad ◀$$

(*b*) Maximum power:

$$(P_{\text{out}})_{\max} = \rho A v_A \left(v_A - \frac{1}{2}v_A\right)\left(\frac{1}{2}v_A\right)(1 - \cos\theta)$$

$$(P_{\text{out}})_{\max} = \frac{1}{4}\rho A v_A^3 (1 - \cos\theta) \quad ◀$$

PROBLEM 14.81 (Continued)

Input power = rate of supply of kinetic energy of the stream

$$P_{\text{in}} = \frac{1}{\Delta t}\left[\frac{1}{2}(\Delta m)v_A^2\right]$$

$$= \frac{1}{2}\frac{\Delta m}{\Delta t}v_A^2$$

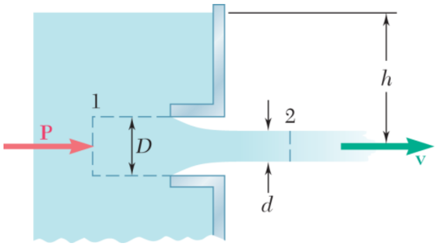
$$= \frac{1}{2}\rho A v_A^3$$

(*c*) Efficiency:

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}}$$

$$\eta = \frac{\rho A v_A (v_A - V)V(1-\cos\theta)}{\frac{1}{2}\rho A v_A^3}$$

$$\eta = 2\left(1-\frac{V}{v_A}\right)\frac{V}{v_A}(1-\cos\theta) \quad ◀$$



PROBLEM 14.82

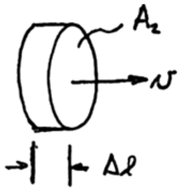
A circular reentrant orifice (also called Borda's mouthpiece) of diameter D is placed at a depth h below the surface of a tank. Knowing that the speed of the issuing stream is $v=\sqrt{2gh}$ and assuming that the speed of approach v_1 is zero, show that the diameter of the stream is $d=D/\sqrt{2}$. (*Hint*: Consider the section of water indicated, and note that P is equal to the pressure at a depth h multiplied by the area of the orifice).

SOLUTION

From hydrostatics, the pressure at section 1 is $p_1=\gamma h=\rho gh.$

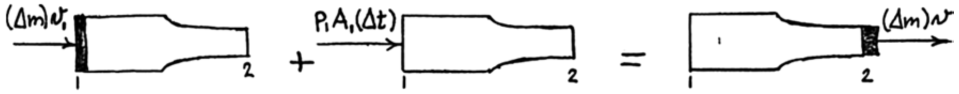
The pressure at section 2 is $p_2=0.$

Calculate the mass flow rate using section 2.



$$\begin{aligned}\text{mass} &= \text{density}\times\text{volume}\\ &= \text{density}\times\text{area}\times\text{length}\\ \Delta m &= \rho A_2(\Delta l)=\rho A_2 v(\Delta t)\\ \frac{dm}{dt} &= \frac{\Delta m}{\Delta t}=\rho A_2 v\end{aligned}$$

Apply the impulse-momentum principle to fluid between sections 1 and 2.

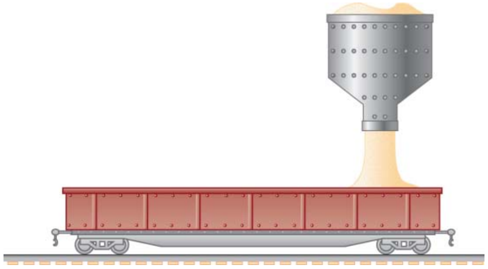


$$\begin{aligned}(\Delta m)v_1+p_1A_1(\Delta t) &= (\Delta m)v\\ \frac{dm}{dt}v_1+p_1A_1 &= \frac{dm}{dt}v\\ p_1A_1 &= \frac{dm}{dt}(v-v_1)\\ &= \rho A_2 v(v-v_1)\end{aligned}$$

But v_1 is negligible, $\qquad p_1=\rho gh \quad \text{and} \quad v=\sqrt{2gh}$

$$\rho ghA_1=\rho A_2(2gh) \quad \text{or} \quad A_1=2A_2$$

$$\frac{\pi}{4}D^2=2\left(\frac{\pi}{4}d^2\right) \qquad\qquad d=\frac{D}{\sqrt{2}} \blacktriangleleft$$



PROBLEM 14.83

A railroad car of length L and mass m_0 when empty is moving freely on a horizontal track while being loaded with sand from a stationary chute at a rate $dm/dt = q$. Knowing that the car was approaching the chute at a speed v_0, determine (*a*) the mass of the car and its load after the car has cleared the chute, (*b*) the speed of the car at that time.

SOLUTION

Consider the conservation of the horizontal component of momentum of the railroad car of mass m_0 and the sand mass qt.

$m_0 v_0$
☐ qt
=
$(m_0+qt)v$

$$m_0 v_0 = (m_0 + qt)v \qquad v = \frac{m_0 v_0}{m_0 + qt} \tag{1}$$

$$\frac{dx}{dt} = v = \frac{m_0 v_0}{m_0 + qt}$$

Integrating, using $x_0 = 0$ and $x = L$ when $t = t_L$,

$$L = \int_0^{t_L} v\,dt = \int_0^{t_L} \frac{m_0 v_0}{m_0 + qt}dt = \frac{m_0 v_0}{q}\left[\ln\left(m_0 + qt_L\right) - \ln m_0\right]$$

$$= \frac{m_0 v_0}{q} \ln \frac{m_0 + qt_L}{m_0}$$

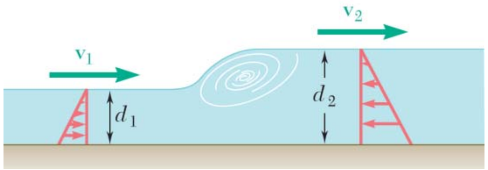
$$\ln \frac{m_0 + qt_L}{m_0} = \frac{qL}{m_0 v_0} \qquad \frac{m_0 + qt_L}{m_0} = e^{qL/m_0 v_0}$$

(*a*) Final mass of railroad car and sand $\quad m_0 + qt_L = m_0 e^{qL/m_0 v_0}$ ◀

(*b*) Using (1),

$$v_L = \frac{m_0 v_0}{m_0 + qt_L} = \frac{m_0 v_0}{m_0} e^{-qL/m_0 v_0}$$

$v_L = v_0 e^{-qL/m_0 v_0}$ ◀



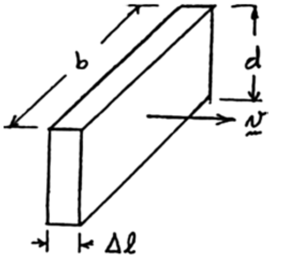
PROBLEM 14.84*

The depth of water flowing in a rectangular channel of width b at a speed v_1 and a depth d_1 increases to a depth d_2 at a *hydraulic jump*. Express the rate of flow Q in terms of b, d_1, and d_2.

SOLUTION

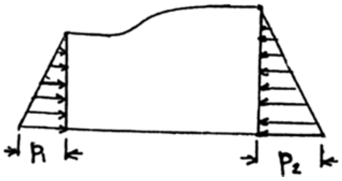
Mass flow rate:

$$\text{mass} = \text{density} \times \text{volume}$$
$$= \text{density} \times \text{area} \times \text{length}$$
$$\Delta m = \rho b d(\Delta l) = \rho b d v(\Delta t)$$
$$\frac{dm}{dt} = \frac{\Delta m}{\Delta t} = \rho b d v$$
$$Q = \frac{1}{\rho}\frac{dm}{dt} = b d v$$



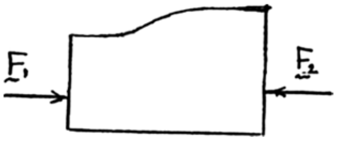
Continuity of flow:

$$Q_1 = Q_2 = Q$$
$$v_1 = \frac{Q}{b d_1} \qquad v_2 \frac{Q}{b d_2}$$

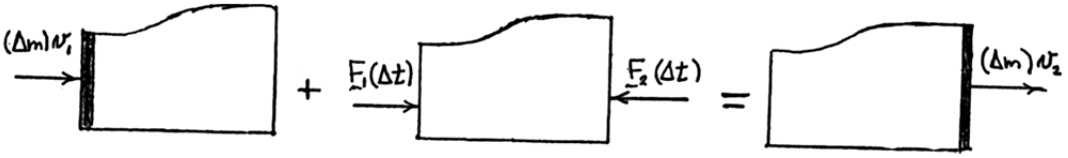


Resultant pressure forces:

$$p_1 = \gamma d_1 \qquad p_2 = \gamma d_2$$
$$F_1 = \frac{1}{2} p_1 b d_1 = \frac{1}{2}\gamma b d_1^2$$
$$F_2 = \frac{1}{2} p_2 b d_2 = \frac{1}{2}\gamma b d_2^2$$



Apply impulse-momentum principle to water between sections 1 and 2.



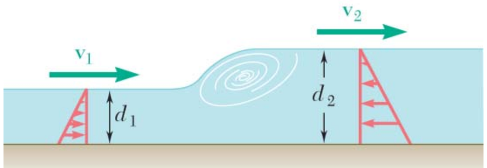
$$(\Delta m)v_1 + F_1(\Delta t) - F_2(\Delta t) = (\Delta m)v_2$$

$$\frac{\Delta m}{\Delta t}(v_1 - v_2) = F_2 - F_1 \qquad \rho Q \cdot \left(\frac{Q}{b d_1} - \frac{Q}{b d_2}\right) = \frac{1}{2}\gamma b\left(d_2^2 - d_1^2\right)$$

$$\frac{\rho Q^2 (d_2 - d_1)}{b d_1 d_2} = \frac{1}{2}\gamma b (d_1 + d_2)(d_2 - d_1)$$

Noting that $\gamma = \rho g$,

$$Q = b\sqrt{\frac{1}{2} g d_1 d_2 (d_1 + d_2)} \blacktriangleleft$$

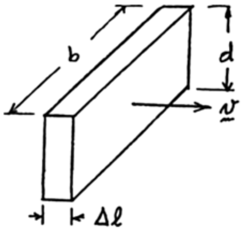


PROBLEM 14.85*

Determine the rate of flow in the channel of Problem 14.84, knowing that $b = 12$ ft, $d_1 = 4$ ft, and $d_2 = 5$ ft.

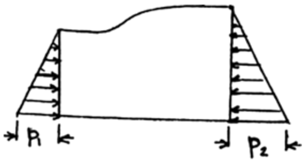
PROBLEM 14.84 The depth of water flowing in a rectangular channel of width b at a speed v_1 and a depth d_1 increases to a depth d_2 at a *hydraulic jump*. Express the rate of flow Q in terms of b, d_1, and d_2.

SOLUTION



Mass flow rate:

$$\text{mass} = \text{density} \times \text{volume}$$
$$= \text{density} \times \text{area} \times \text{length}$$
$$\Delta m = \rho b d(\Delta l) = \rho b d v(\Delta t)$$
$$\frac{dm}{dt} = \frac{\Delta m}{\Delta t} = \rho b d v$$
$$Q = \frac{1}{\rho}\frac{dm}{dt} = bdv$$

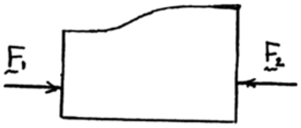


Continuity of flow:

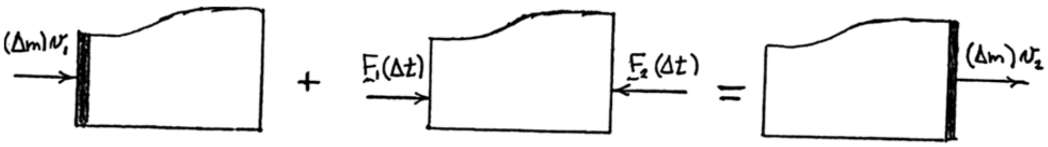
$$Q_1 = Q_2 = Q$$
$$v_1 = \frac{Q}{bd_1} \qquad v_2 \frac{Q}{bd_2}$$

Resultant pressure forces:

$$p_1 = \gamma d_1 \qquad p_2 = \gamma d_2$$
$$F_1 = \frac{1}{2} p_1 b d_1 = \frac{1}{2}\gamma b d_1^2$$
$$F_2 = \frac{1}{2} p_2 b d_2 = \frac{1}{2}\gamma b d_2^2$$



Apply impulse-momentum principle to water between sections 1 and 2.



$$(\Delta m)v_1 + F_1(\Delta t) - F_2(\Delta t) = (\Delta m)v_2$$
$$\frac{\Delta m}{\Delta t}(v_1 - v_2) = F_2 - F_1 \qquad \rho Q \cdot \left(\frac{Q}{bd_1} - \frac{Q}{bd_2}\right) = \frac{1}{2}\gamma b\left(d_2^2 - d_1^2\right)$$
$$\frac{\rho Q^2 (d_2 - d_1)}{bd_1 d_2} = \frac{1}{2}\gamma b(d_1 + d_2)(d_2 - d_1)$$

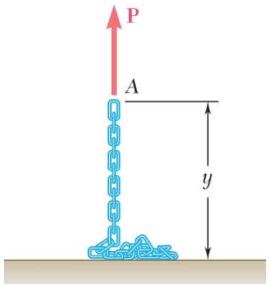
PROBLEM 14.85* (Continued)

Noting that $\gamma = \rho g$, $$Q = b\sqrt{\frac{1}{2} g d_1 d_2 (d_1 + d_2)}$$

Data: $g = 32.2\ \text{ft/s,}^2 \quad b = 12\ \text{ft}, \quad d_1 = 4\ \text{ft}, \quad d_2 = 5\ \text{ft}$

$$Q = 12\sqrt{\frac{1}{2}(32.2)(4)(5)(9)}$$

$Q = 646\ \text{ft}^3/\text{s}$ ◀



PROBLEM 14.86

A chain of length l and mass m lies in a pile on the floor. If its end A is raised vertically at a constant speed v, express in terms of the length y of chain that is off the floor at any given instant (a) the magnitude of the force **P** applied at A, (b) the reaction of the floor.

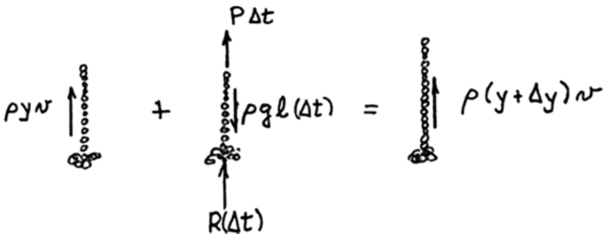
SOLUTION

Let ρ be the mass per unit length of chain. Apply the impulse-momentum to the entire chain. Assume that the reaction from the floor is equal to the weight of chain still in contact with the floor.

Calculate the floor reaction.

$$R = \rho g(l - y)$$

$$R = mg\left(1 - \frac{y}{l}\right)$$



Apply the impulse-momentum principle.

$$\rho y v + P(\Delta t) + R(\Delta t) - \rho g L(\Delta t) = \rho(y + \Delta y)v$$

$$P\Delta t = \rho(\Delta y)v + \rho g L(\Delta t) - R(\Delta t)$$

(a)

$$P = \rho\frac{\Delta y}{\Delta t}v + \rho g L - \rho(L - y)g$$

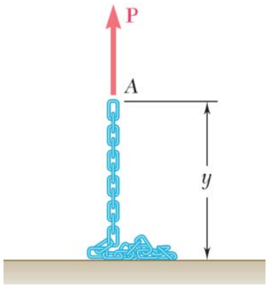
$$= \rho v^2 + \rho g y \qquad P = \frac{m}{l}(v^2 + gy) ◀$$

Let

$$\frac{\Delta y}{\Delta t} = \frac{dy}{dt} = v$$

(b) From above,

$$\mathbf{R} = mg\left(1 - \frac{y}{l}\right) \uparrow ◀$$



PROBLEM 14.87

Solve Problem 14.86, assuming that the chain is being *lowered* to the floor at a constant speed v.

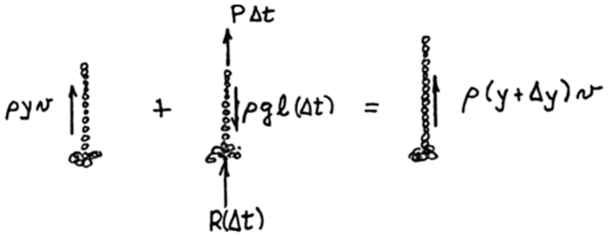
PROBLEM 14.86 A chain of length l and mass m lies in a pile on the floor. If its end A is raised vertically at a constant speed v, express in terms of the length y of chain that is off the floor at any given instant (a) the magnitude of the force **P** applied at A, (b) the reaction of the floor.

SOLUTION

(a) Let ρ be the mass per unit length of chain. The force P supports the weight of chain still off the floor.

$$P = \rho g y \qquad\qquad P = \frac{mgy}{l} \blacktriangleleft$$

(b) Apply the impulse-momentum principle to the entire chain.



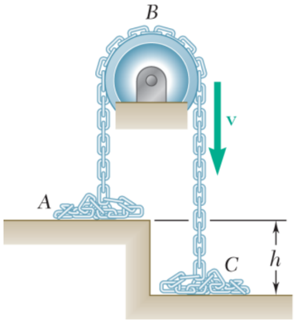
$$-\rho yv + P(\Delta t) + R(\Delta t) - \rho gL(\Delta t) = -\rho g(y+\Delta y)v$$

$$R(\Delta t) = \rho gL(\Delta t) - P(\Delta t) - \rho g(\Delta y)v$$

$$R = \rho gL - \rho gy - \rho\frac{\Delta y}{\Delta t}v$$

Let $\Delta t \to 0$. Then $\quad \dfrac{\Delta y}{\Delta t} = \dfrac{dy}{dt} = -v$

$$R = \rho g(L-y) + \rho v^2 \qquad\qquad \mathbf{R} = \frac{m}{l}[g(L-y)+v^2]\uparrow \blacktriangleleft$$



PROBLEM 14.88

The ends of a chain lie in piles at A and C. When released from rest at time $t = 0,$ the chain moves over the pulley at B, which has a negligible mass. Denoting by L the length of chain connecting the two piles and neglecting friction, determine the speed v of the chain at time t.

SOLUTION

Let m be the mass of the portion of the chain between the two piles. This is the portion of the chain that is moving with speed v. The remainder of the chain lies in either of the two piles. Consider the time period between t and $t + \Delta t$ and apply the principle impulse and momentum. Let Δm be the amount of chain that is picket up at A and deposited at C during the time period Δt.

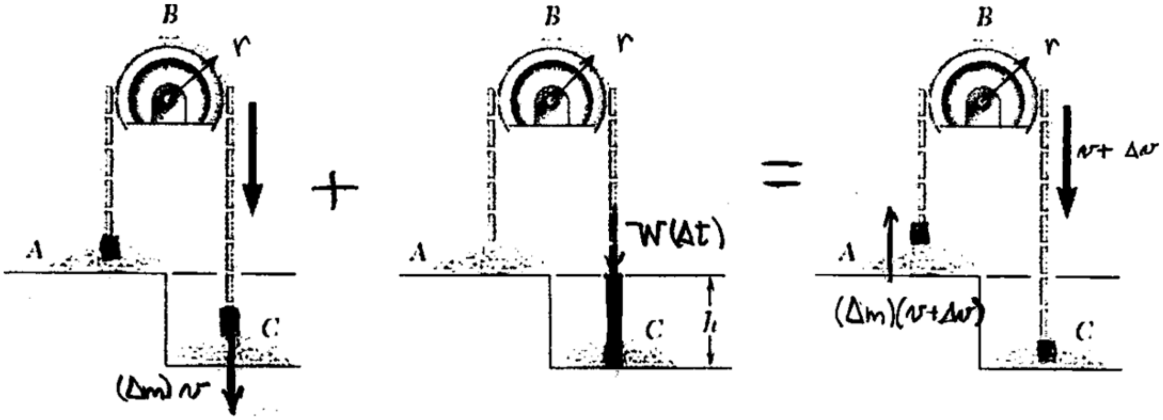
At time t, Δm is still in pile A while Δm has a downward at speed v just above pile C. The remaining mass $(m - \Delta m)$ is moving with speed v.

At time $t + \Delta t$, Δm is moving with speed $v + \Delta v$ just above pile A and Δm is at rest in pile C.

Over the time period an unbalanced weight of chain acts on the system. The weight is

$$W = \frac{mgh}{L}$$

Apply the impulse-momentum principle to the system.



Consider moments about the pulley axle.

$$r[(\Delta m)v + (m - \Delta m)v] + rW(\Delta t)$$
$$= r[(\Delta m)(v + \Delta v) + (m - \Delta m)(v + \Delta v)]$$

Dividing by r and canceling the terms $(\Delta m)v$ and $(m-\Delta m)v$

$$0+\frac{mgh}{L}(\Delta t)=(\Delta m)(v+\Delta v)+(m-\Delta m)(\Delta v)$$
$$=v(\Delta m)+m(\Delta v)$$

But $$\Delta m=\frac{m}{L}v(\Delta t)$$

Hence, $$\frac{mgh}{L}(\Delta t)=\frac{mv^2}{L}(\Delta t)+m(\Delta v)$$

Solving for Δt, $$\Delta t=\frac{L(\Delta v)}{gh-v^2}$$

Letting $c^2=gh,$ and considering the limit as Δt and Δv become infinitesimal, gives

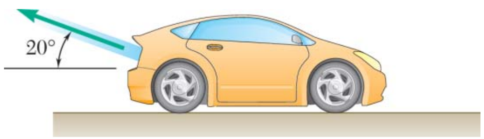
$$dt=\frac{L\,dv}{c^2-v^2}$$

Integrate, noting that $v=0$ when $t=0$

$$t=L\int_0^v\frac{dv}{c^2-v^2}=\frac{L}{c}\tanh^{-1}\frac{v}{c}\bigg|_0^v$$

$$\tanh\frac{ct}{L}=\frac{v}{c}$$

$$v=\sqrt{gh}\tanh\left(\frac{\sqrt{gh}}{L}t\right) \blacktriangleleft$$

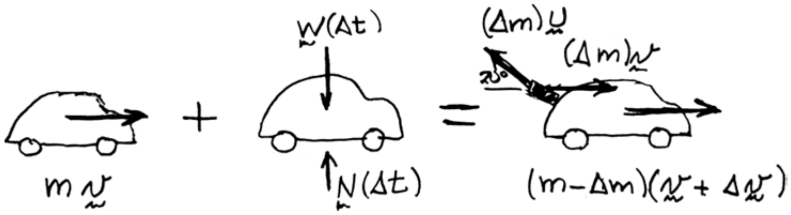


PROBLEM 14.89

A toy car is propelled by water that squirts from an internal tank at a constant 6 ft/s relative to the car. The weight of the empty car is 0.4 lb and it holds 2 lb of water. Neglecting other tangential forces, determine the top speed of the car.

SOLUTION

Consider a time interval Δt. Let m be the mass of the car plus the water in the tank at the beginning of the interval and $(m - \Delta m)$ the corresponding mass at the end of the interval. m_0 is the initial value of m. Let **v** be the velocity of the car. Apply the impulse and momentum principle over the time interval.



Horizontal components $\xrightarrow{+}$:

$$mv + 0 = (\Delta m)(\cancel{v} - u\cos 20°) + (\cancel{m} - \Delta m)(\cancel{v} + \Delta v)$$

$$\Delta v = u\cos 20° \frac{\Delta m}{m - \Delta m}$$

Let Δv be replaced by differential dv and Δm be replaced by the small differential $-dm$, the minus sign meaning that dm is the infinitesimal increase in m.

$$dv = -u\cos 20° \frac{dm}{m}$$

Integrating, $$v = v_0 - u\cos 20° \ln\frac{m}{m_0}$$

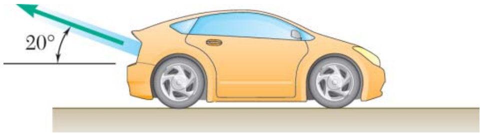
Since $v_0 = 0$, $$v = u\cos 20° \ln\frac{m_0}{m}$$

The velocity is maximum when $m = m_f$, the value of m when all of the water is expelled.

$$v_{\max} = u\cos 20° \ln\frac{m_0}{m_f}$$

$$v_{\max} = (6\text{ ft/s})\cos 20° \ln\frac{0.4 + 2}{0.4}$$

$v_{\max} = 10.10$ ft/s ◀

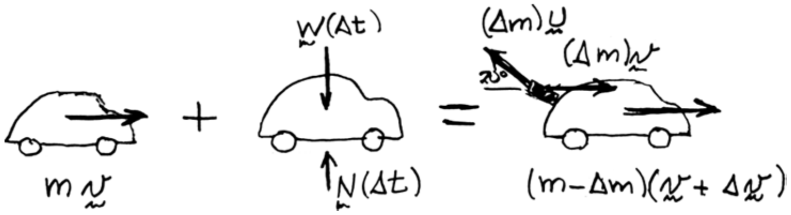


PROBLEM 14.90

A toy car is propelled by water that squirts from an internal tank. The weight of the empty car is 0.4 lb and it holds 2 lb of water. Knowing the top speed of the car is 8 ft/s, determine the relative velocity of the water that is being ejected.

SOLUTION

Consider a time interval Δt. Let m be the mass of the car plus the water in the tank at the beginning of the interval and $(m - \Delta m)$ the corresponding mass at the end of the interval. m_0 is the initial value of m. Let **v** be the velocity of the car. Apply the impulse and momentum principle over the time interval.



Horizontal components $\xrightarrow{+}$:

$$mv + 0 = (\Delta m)(\cancel{v} - u\cos 20°) + (\cancel{m} - \Delta m)(\cancel{v} + \Delta v)$$

$$\Delta v = u\cos 20° \frac{\Delta m}{m - \Delta m}$$

Let Δv be replaced by differential dv and Δm be replaced by the small differential $-dm$, the minus sign meaning that dm is the infinitesimal increase in m.

$$dv = -u\cos 20° \frac{dm}{m}$$

Integrating, $$v = v_0 - u\cos 20° \ln\frac{m}{m_0}$$

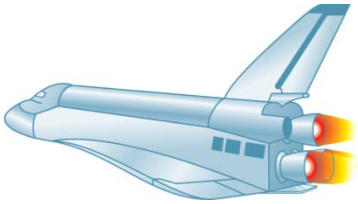
Since $v_0 = 0$, $$v = u\cos 20° \ln\frac{m_0}{m}$$

The velocity is maximum when $m = m_f$, the value of m when all of the water is expelled.

$$v_{\max} = u\cos 20° \ln\frac{m_0}{m_f}$$

$$8\text{ ft/s} = u\cos 20° \ln\frac{0.4 + 2}{0.4}$$

$u = 4.75$ ft/s ◀



PROBLEM 14.91

The main propulsion system of a space shuttle consists of three identical rocket engines which provide a total thrust of 6 MN. Determine the rate at that the hydrogen-oxygen propellant is burned by each of the three engines, knowing that it is ejected with a relative velocity of 3750 m/s.

SOLUTION

Thrust of each engine:

$$P = \frac{1}{3}(6\text{ MN}) = 2\times 10^6\text{ N}$$

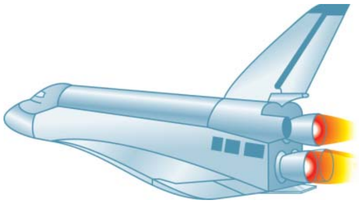
Eq. (14.44):

$$P = \frac{dm}{dt}u$$

$$2\times 10^6\text{ N} = \frac{dm}{dt}(3750\text{ m/s})$$

$$\frac{dm}{dt} = \frac{2\times 10^6\text{ N}}{3750\text{ m/s}}$$

$$\frac{dm}{dt} = 533\text{ kg/s} \blacktriangleleft$$



PROBLEM 14.92

The main propulsion system of a space shuttle consists of three identical rocket engines, each of which burns the hydrogen-oxygen propellant at the rate of 750 lb/s and ejects it with a relative velocity of 12000 ft/s. Determine the total thrust provided by the three engines.

SOLUTION

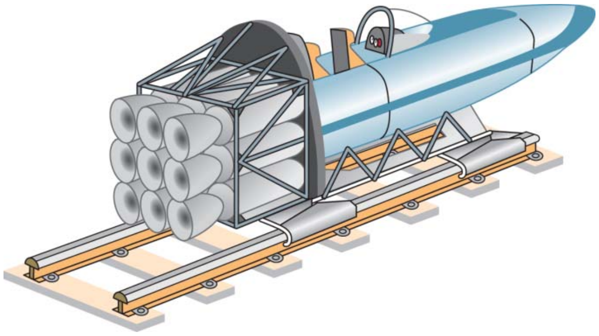
From Eq. (14.44) for each engine:

$$\begin{aligned} P &= \frac{dm}{dt}u \\ &= \frac{(750\text{ lb/s})}{32.2\text{ ft/s}^2}(12000\text{ ft/s}) \\ &= 279.50\times 10^3\text{ lb} \end{aligned}$$

For the 3 engines:

$$\text{Total thrust} = 3(279.50\times 10^3\text{ lb})$$

$$\text{Total thrust} = 839{,}000\text{ lb} \blacktriangleleft$$



PROBLEM 14.93

A rocket sled burns fuel at the constant rate of 120 lb/s. The initial weight of the sled is 1800 lb, including 360 lb of fuel. Assume that the track is lubricated and the sled is aerodynamically designed so that air resistance and friction are negligible. *(a)* Derive a formula for the acceleration a of the sled as a function of time t and the exhaust velocity v_{ex} of the burned fuel relative to the sled. Plot the ratio a/v_{ex} versus time t for the range $0 < t < 4$ s, and check the slope of the graph at $t = 0$ and $t = 4$ s using the formula for a. (*b*) Determine the ratio of the velocity v_b of the sled at burnout to the exhaust velocity v_{ex}.

SOLUTION

Given: $\frac{dm}{dt} = 120/32.2 = 3.727 \text{ slug/s}$

$m_i = 1800/32.2 = 55.90 \text{ slugs}, \; m_f = (1800-360)/32.2 = 44.72 \text{ slugs}$

Mass of rocket sled over time: $m = m_i - \frac{dm}{dt}t = 55.90 - 3.727t$

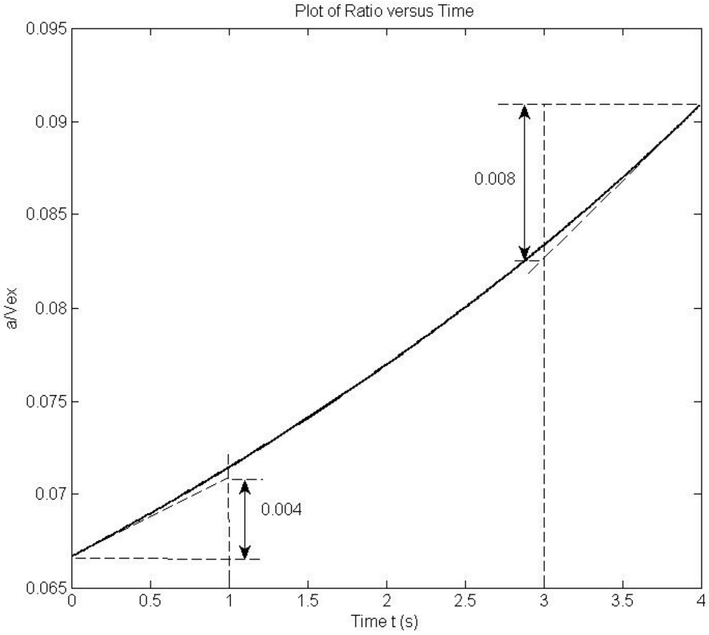
From Equation 14.42 applied to the direction of motion:

$$\sum F = ma - \frac{dm}{dt}u \qquad \text{where: } \sum F = 0, \; u = v_{ex}$$

$$\text{or} \quad a = \frac{1}{m}\frac{dm}{dt}v_{ex}$$

(a) Putting in known values: $a = \frac{3.727v_{ex}}{(55.90 - 3.727t)}$ $\qquad a/v_{ex} = \frac{1}{15-t}$ ◀

Plot of a/v_{ex} vs. time:



Slope of graph from figure:

at t=0 s : slope ≈ 0.004◀

at t=4 s: slope ≈ 0.008◀

Find slope analytically: $$\frac{d}{dt}\left(\frac{a}{v_{ex}}\right)=\frac{1}{(15-t)^2}$$

Evaluating the function:

at t=0 s : slope $= 0.00444$◀

at t=4 s: slope $= 0.00826$◀

(b) Burnout of fuel occurs when $m{=}m_f$:

$$m_f = m_i - \frac{dm}{dt}t_f$$

$$44.72 = 55.90 - 3.727 t_f$$

$$t_f = 3.0 \text{ s}$$

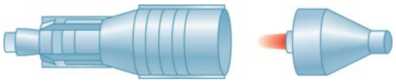
Velocity of sled: $$a=\frac{dv}{dt} \Rightarrow adt = dv$$

$$\int_0^{t_f} adt = \int_0^{v_b} dv$$

$$\int_0^{t_f} \frac{v_{ex}}{15-t}dt = \int_0^{v_b} dv$$

$$\frac{v_b}{v_{ex}} = \ln\left(\frac{15}{15-t_f}\right)$$

Evaluate with $t_f = 3.0$ s $$\frac{v_b}{v_{ex}} = 0.223$$◀



PROBLEM 14.94

A space vehicle describing a circular orbit at a speed of 24×10^3 km/h releases its front end, a capsule that has a gross mass of 600 kg, including 400 kg of fuel. If the fuel is consumed at the rate of 18 kg/s and ejected with a relative velocity of 3000 m/s, determine (*a*) the tangential acceleration of the capsule as the engine is fired, (*b*) the maximum speed attained by the capsule.

SOLUTION

Thrust:

$$P = \left|\frac{dm}{dt}\right|u$$

$$= (18\text{ kg/s})(3000\text{ m/s})$$

$$= 54\times10^3\text{ N}$$

(*a*)

$$(a_t)_0 = \frac{P}{m_0} = \frac{54\times10^3}{600} = 90\text{ m/s}^2$$

$$(a_t)_0 = 90.0\text{ m/s}^2 \blacktriangleleft$$

(*b*) Maximum speed is attained when all the fuel is used up:

$$v_1 = v_0 + \int_0^{t_1} a_t\,dt = v_0 + \int_0^{t_1}\frac{P}{m}dt$$

$$= v_0 + \int_0^{t_1}\frac{u\left(\frac{dm}{dt}\right)}{m}dt = v_0 + u\int_{m_0}^{m_1}\left(-\frac{dm}{m}\right)$$

$$v_1 = v_0 + u\left(-\ln\frac{m_1}{m_0}\right) = v_0 + u\ln\frac{m_0}{m_1}$$

Data:

$$v_0 = 24\times10^3\text{ km/h}$$

$$= 6.6667\times10^3\text{ m/s}$$

$$u = 3000\text{ m/s}$$

$$m_0 = 600\text{ kg}$$

$$m_1 = 600 - 400 = 200\text{ kg}$$

$$v_1 = 6.6667\times10^3 + 3000\ln\frac{600}{200}$$

$$= 9.9625\times10^3\text{ m/s}$$

$$v_1 = 35.9\times10^3\text{ km/h} \blacktriangleleft$$

PROBLEM 14.95

A 540-kg spacecraft is mounted on top of a rocket with a mass of 19 Mg, including 17.8 Mg of fuel. Knowing that the fuel is consumed at a rate of 225 kg/s and ejected with a relative velocity of 3600 m/s, determine the maximum speed imparted to the spacecraft if the rocket is fired vertically from the ground.

SOLUTION

See sample Problem 14.8 for derivation of

$$v = u \ln \frac{m_0}{m_0 - qt} - gt \qquad (1)$$

Data:

$$u = 3600 \text{ m/s} \quad q = 225 \text{ kg/s}, \quad m_{\text{fuel}} = 17{,}800 \text{ kg}$$

$$m_0 = 19{,}000 \text{ kg} + 540 \text{ kg} = 19{,}540 \text{ kg}$$

We have

$$m_{\text{fuel}} = qt, \quad 17{,}800 \text{ kg} = (225 \text{ kg/s})t$$

$$t = \frac{17{,}800 \text{ kg}}{225 \text{ kg/s}} = 79.111 \text{ s}$$

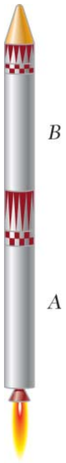
Maximum velocity is reached when all fuel has been consumed, that is, when $qt = m_{\text{fuel}}$. Eq. (1) yields

$$v_m = u \ln \frac{m_0}{m_0 - m_{\text{fuel}}} - gt$$

$$= (3600 \text{ m/s}) \ln \frac{19{,}540}{19{,}540 - 17{,}800} - (9.81 \text{ m/s}^2)(79.111 \text{ s})$$

$$= (3600 \text{ m/s}) \ln 11.230 - 776.1 \text{ m/s}$$

$$= 7930.8 \text{ m/s} \qquad v_m = 7930 \text{ m/s} \blacktriangleleft$$



PROBLEM 14.96

The rocket used to launch the 540-kg spacecraft of Problem 14.95 is redesigned to include two stages *A* and *B*, each of mass 9.5 Mg, including 8.9 Mg of fuel. The fuel is again consumed at a rate of 225 kg/s and ejected with a relative velocity of 3600 m/s. Knowing that when stage *A* expels its last particle of fuel its casing is released and jettisoned, determine (*a*) the speed of the rocket at that instant, (*b*) the maximum speed imparted to the spacecraft.

SOLUTION

Thrust force: $P = u\dfrac{dm}{dt} = uq$

Mass of rocket + unspent fuel: $m = m_0 - qt$

Corresponding weight force: $W = mg$

Acceleration: $a = \dfrac{F}{m} = \dfrac{P-W}{m} = \dfrac{P}{m} - g = \dfrac{uq}{m_0 - qt} - g$

Integrating with respect to time to obtain the velocity,

$$v = v_0 + \int_0^t a\,dt = v_0 + u\int_0^t \frac{q\,dt}{m_0 - qt} - gt$$

$$= v_0 - u\ln\frac{m_0 - qt}{m_0} - gt \qquad (1)$$

For each stage, $m_{\text{fuel}} = 8900\text{ kg} \qquad u = 3600\text{ m/s}$

$$q = 225\text{ kg/s} \qquad t = \frac{m_{\text{fuel}}}{q} = \frac{8900}{225} = 39.556\text{ s}$$

For the first stage, $v_0 = 0 \qquad m_0 = 540 + (2)(9500) = 19{,}540\text{ kg}$

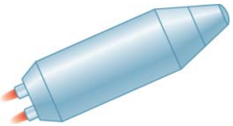
(*a*) $$v_1 = 0 - 3600\ \ln\frac{19{,}540 - 8900}{19{,}540} - (9.81)(39.556) = 1800.1\text{ m/s}$$

$v_1 = 1800\text{ m/s}$ ◀

For the second stage, $v_0 = 1800.1\text{ m/s}, \qquad m_0 = 540 + 9500 = 10{,}040\text{ kg}$

(*b*) $$v_2 = 1800.1 - 3600\ \ln\frac{10{,}040 - 8900}{10{,}040} - (9.81)(39.556) = 9244\text{ m/s}$$

$v_2 = 9240\text{ m/s}$ ◀

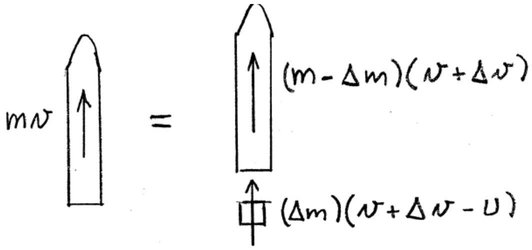


PROBLEM 14.97

The weight of a spacecraft, including fuel, is 11,600 lb when the rocket engines are fired to increase its velocity by 360 ft/s. Knowing that 1000 lb of fuel is consumed, determine the relative velocity of the fuel ejected.

SOLUTION

Apply conservation of momentum to the rocket plus the fuel.



$$mv = (m-\Delta m)(v+\Delta v) + (\Delta m)(v+\Delta v - v)$$
$$= mv + m(\Delta v) - (\Delta m)v - (\Delta m)(\Delta v) + (\Delta m)v + (\Delta m)(\Delta v) - (\Delta m)v$$

$$m(\Delta v) - u(\Delta m) = 0$$

$$\Delta m = -\frac{dm}{dt}(\Delta t)$$

$$\frac{\Delta v}{\Delta t} = \frac{dv}{dt} = -\frac{u}{m}\frac{dm}{dt}$$

$$\int_{v_0}^{v_1} dv = -\int_0^{t_1} \frac{u}{m}\frac{dm}{dt}dt = -\int_{m_0}^{m_1} u\frac{dm}{m}$$

$$v_1 - v_0 = -u\ln\frac{m_1}{m_0} = u\ln\frac{m_0}{m_1} = u\ \ln\frac{W_0}{W_1}$$

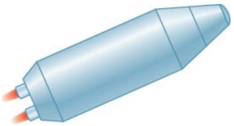
Data:

$$v_1 - v_0 = 360 \text{ ft/s}$$

$$W_0 = 11{,}600 \text{ lb}, \quad W_1 = (11{,}600 - 1000) = 10{,}600 \text{ lb}$$

$$360 = u\ln\frac{11{,}600}{10{,}600}$$

$u = 3990$ ft/s ◀

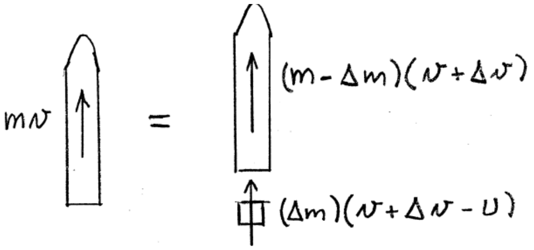


PROBLEM 14.98

The rocket engines of a spacecraft are fired to increase its velocity by 450 ft/s. Knowing that 1200 lb of fuel is ejected at a relative velocity of 5400 ft/s, determine the weight of the spacecraft after the firing.

SOLUTION

Apply conservation of momentum to the rocket plus the fuel.



$$mv = (m - \Delta m)(v + \Delta v) + (\Delta m)(v + \Delta v - v)$$

$$= mv + m(\Delta v) - (\Delta m)v - (\Delta m)(\Delta v) + (\Delta m)v + (\Delta m)(\Delta v) - (\Delta m)v$$

$$m(\Delta v) - u(\Delta m) = 0$$

$$\Delta m = -\frac{dm}{dt}(\Delta t)$$

$$\frac{\Delta v}{\Delta t} = \frac{dv}{dt} = -\frac{u}{m}\frac{dm}{dt}$$

$$\int_{v_0}^{v_1} dv = -\int_0^{t_1} \frac{u}{m}\frac{dm}{dt}\,dt = -\int_{m_0}^{m_1} u\frac{dm}{m}$$

$$v_1 - v_0 = -u\ln\frac{m_1}{m_0} = u\ln\frac{m_0}{m_1}$$

$$\frac{m_0}{m_1} = \exp\left(\frac{v_1 - v_0}{u}\right) = \frac{W_0}{W_1}$$

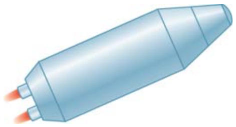
Data:
$$v_1 - v_0 = 450 \text{ ft/s}, \qquad u = 5400 \text{ ft/s}$$

$$W_0 = W_1 + W_{\text{fuel}} = W_1 + 1200 \text{ lb}$$

$$\frac{W_1 + 1200}{W_1} = \exp\frac{450}{5400} = 1.08690$$

$$\frac{1200}{W_1} = 0.08690$$

$W_1 = 13810$ lb ◀



PROBLEM 14.99

Determine the distance traveled by the spacecraft of Prob. 14.97 during the rocket engine firing, knowing that its initial speed was 7500 ft/s and the duration of the firing was 60 s.

SOLUTION

Thrust force $$P = u\frac{dm}{dt} = uq$$

Mass of rocket plus unspent fuel $$m = m_0 - qt$$

Acceleration: $$a = \frac{P}{m} = \frac{uq}{m_0 - qt}$$

Integrating with respect to time to obtain the velocity,

$$v = v_0 + \int_0^t a dt = v_0 + u\int_0^t \frac{q}{m_0 - qt} dt$$

$$= v_0 - u\left[\ln\left(m_0 - qt\right) - \ln m_0\right] = v_0 - u\ln\frac{m_0 - qt}{m_0} \qquad (1)$$

Integrating again to obtain the displacement,

$$s = s_0 + v_0 t - u\int_0^t \ln\frac{m_0 - qt}{m_0} dt$$

Let $$z = \frac{m_0 - qt}{m_0} \qquad dz = -\frac{q}{m_0}dt \qquad \text{or} \qquad dt = -\frac{m_0}{q}dz$$

$$s = s_0 + v_0 t + \frac{m_0 u}{q}\int_{z_0}^{z} \ln z dz = \frac{m_0 u}{q}\left(z\ln z - z\right)\Big]_{z_0}^{z}$$

$$= s_0 + v_0 t + \frac{m_0 u}{q}\left[\frac{m_0 - qt}{m_0}\left(\ln\frac{m_0 - qt}{m_0} - 1\right) - \frac{m_0}{m_0}\left(\ln\frac{m_0}{m_0} - 1\right)\right]$$

$$= s_0 + v_0 t + \frac{m_0 u}{q}\left[\left(1 - \frac{qt}{m_o}\right)\left(\ln\frac{m_0 - qt}{m_0} - 1\right) + 1\right]$$

$$= s_0 + v_0 t + \frac{m_0 u}{q}\left[\ln\frac{m_0 - qt}{m_0} - 1 + 1\right] - ut\left[\ln\frac{m_0 - qt}{m_0} - 1\right]$$

$$= s_0 + v_0 t + ut + \left(\frac{m_0 u}{q} - ut\right)\ln\frac{m_0 - qt}{m_0}$$

$$s = s_0 + v_0 t + u\left[t - \left(\frac{m_0}{q} - t\right)\ln\frac{m_0}{m_0 - qt}\right] \qquad (2)$$

PROBLEM 14.99 (Continued)

Data: $v_0 = 7500 \text{ ft/s} \qquad v = 7500 + 360 = 7860 \text{ ft/s}$

$$t = 60 \text{ s} \qquad q = \frac{m_{\text{fuel}}}{t} = \frac{W_{\text{fuel}}}{gt} = \frac{1000}{(32.2)(60)} = 0.5176 \text{ slug/s}$$

$$m_0 = \frac{W_0}{g} = \frac{11600}{32.2} = 360.25 \text{ slugs}$$

$$m_0 - qt = \frac{11600}{32.2} - \frac{1000}{32.2} = 329.19 \text{ slugs}$$

$$s_0 = 0$$

From (1),

$$7860 = 7500 - u\ln\frac{329.19}{360.25}$$

$$360 = u\ln\frac{360.25}{329.19} \qquad u = 3993 \text{ ft/s}$$

From (2),

$$s = 0 + (7500)(60) + 3993\left[60 - \left(\frac{360.25}{0.5176} - 60\right)\ln\frac{360.25}{329.19}\right]$$

$$= 460.6 \times 10^6 \text{ ft} \qquad s = 87.2 \text{ mi} \blacktriangleleft$$

PROBLEM 14.100

A rocket weighs 2600 lb, including 2200 lb of fuel, which is consumed at the rate of 25 lb/s and ejected with a relative velocity of 13000 ft/s. Knowing that the rocket is fired vertically from the ground, determine *(a)* its acceleration as it is fired, *(b)* its acceleration as the last particle of fuel is being consumed, *(c)* the altitude at which all the fuel has been consumed, *(d)* the velocity of the rocket at that time.

SOLUTION

Given initial conditions: $W = W_0 = 2600 \text{ lb}$

$$g = 32.2 \text{ ft/s}^2 \qquad m = \frac{2600}{32.2} = 80.745 \text{ slug}$$

Thrust Force:

$$P = \frac{dm}{dt}u$$

$$= \frac{(25 \text{ lb/s})}{32.2 \text{ ft/s}^2}(13000 \text{ ft/s})$$

$$= 10.093 \times 10^3 \text{ lb}$$

$$\Sigma F = ma$$

$$P - mg = ma$$

$$a = \frac{P}{m} - g \qquad (1)$$

(a) Initial Acceleration:

$$a = \frac{10.093 \times 10^3 \text{ lb}}{80.745 \text{ slug}} - 32.2 = 92.80 \text{ ft/s}^2$$

$\mathbf{a} = 92.8 \text{ ft/s}^2 \uparrow$ ◀

(b) As the last particle of fuel is consumed:

$$W = 2600 - 2200 = 400 \text{ lb}$$

$$m = \frac{400}{32.2} = 12.422 \text{ slug}$$

From (1):

$$a = \frac{10.093 \times 10^3 \text{ lb}}{12.422} - 32.2 = 780.30 \text{ ft/s}^2$$

$\mathbf{a} = 780 \text{ ft/s}^2 \uparrow$ ◀

See Sample Problem 14.8 for derivation of

$$v = u \ln \frac{m_0}{m_0 - qt} - gt = -u \ln \frac{m_0 - qt}{m_0} - gt \qquad (2)$$

Set $\frac{dy}{dt} = v$ in Eq. (2) and integrate with respect to time:

$$h = \int_0^h dy = \int_0^t v dt = \int_0^t \left(u \ln \frac{m_0}{m_0 - qt} - gt \right) dt$$

$$= -u \int_0^t \ln \frac{m_0 - qt}{m_0} dt - \frac{1}{2} g t^2$$

Making the following substitutions:

$$z = \frac{m_0 - qt}{m_0} \qquad dz = -\frac{q}{m_0} dt \qquad \text{or} \qquad dt = -\frac{m_0}{q} dz$$

PROBLEM 14.100 (Continued)

$$h = \frac{m_0 u}{q}\int_{z_0}^{z} \ln z\, dz - \frac{1}{2}gt^2 = \frac{m_0 u}{q}[(z\ln z + z)]_{z_0}^{z} - \frac{1}{2}gt^2$$

$$= \frac{m_0 u}{q}\left[\frac{m_0 - qt}{m_0}\left(\ln\frac{m_0 - qt}{m_0} - 1\right) - \frac{m_0}{m_0}\left(\ln\frac{m_0}{m_0} - 1\right)\right] - \frac{1}{2}gt^2$$

$$= \frac{m_0 u}{q}\left[\left(1 - \frac{qt}{m_0}\right)\left(\ln\frac{m_0 - qt}{m_0} - 1\right) + 1\right] - \frac{1}{2}gt^2$$

$$= \frac{m_0 u}{q}\left[\ln\frac{m_0 - qt}{m_0} - 1 + 1\right] - ut\left[\ln\frac{m_0 - qt}{m_0} - 1\right] - \frac{1}{2}gt^2$$

$$= ut + \left(\frac{m_0 u}{q} - ut\right)\ln\frac{m_0 - qt}{m_0} - \frac{1}{2}gt^2$$

$$h = u\left[t - \left(\frac{m_0}{q} - t\right)\ln\frac{m_0}{m_0 - qt}\right] - \frac{1}{2}gt^2 \qquad (3)$$

(c) From Eq. (3),

$$h = (13000)\left[88 - \left(\frac{2600}{25} - 88\right)\ln\frac{2600}{2600 - 2200}\right] - \frac{1}{2}(32.2)(88)^2$$

$$= (13000)(88 - 16\ln 6.5) - 124680$$

$$= 630{,}000 \text{ ft}$$

$h = 119.3$ mi ◀

(d) From Eq. (2),

$$v = -13000\ln\frac{2600 - 2200}{2600} - (32.2)(88)$$

$$= 13000\ln 6.5 - 2834$$

$$= 21500 \text{ ft/s}$$

$v = 14{,}660$ mi/h ◀

PROBLEM 14.101

Determine the altitude reached by the spacecraft of Problem 14.95 when all the fuel of its launching rocket has been consumed.

SOLUTION

See Sample Problem 14.8 for derivation of

$$v = u\ln\frac{m_0}{m_0 - qt} - gt = -u\ln\frac{m_0 - qt}{m_0} - gt \qquad (1)$$

Note that g is assumed to be constant.

Set $\frac{dy}{dt} = v$ in Eq. (1) and integrate with respect to time.

$$h = \int_0^h dy = \int_0^t v\,dt = \int_0^t \left(u\ln\frac{m_0}{m_0 - qt} - gt\right)dt$$

$$= -u\int_0^t \ln\frac{m_0 - qt}{m_0}\,dt - \frac{1}{2}gt^2$$

Let $$z = \frac{m_0 - qt}{m_0} \qquad dz = -\frac{q}{m_0}dt \qquad \text{or} \qquad dt = -\frac{m_0}{q}dz$$

$$h = \frac{m_0 u}{q}\int_{z_0}^{z} \ln z\,dz - \frac{1}{2}gt^2 = \frac{m_0 u}{q}\left[(z\ln z + z)\right]_{z_0}^{z} - \frac{1}{2}gt^2$$

$$= \frac{m_0 u}{q}\left[\frac{m_0 - qt}{m_0}\left(\ln\frac{m_0 - qt}{m_0} - 1\right) - \frac{m_0}{m_0}\left(\ln\frac{m_0}{m_0} - 1\right)\right] - \frac{1}{2}gt^2$$

$$= \frac{m_0 u}{q}\left[\left(1 - \frac{qt}{m_0}\right)\left(\ln\frac{m_0 - qt}{m_0} - 1\right) + 1\right] - \frac{1}{2}gt^2$$

$$= \frac{m_0 u}{q}\left[\ln\frac{m_0 - qt}{m_0} - 1 + 1\right] - ut\left[\ln\frac{m_0 - qt}{m_0} - 1\right] - \frac{1}{2}gt^2$$

$$= ut + \left(\frac{m_0 u}{q} - ut\right)\ln\frac{m_0 - qt}{m_0} - \frac{1}{2}gt^2$$

$$h = u\left[t - \left(\frac{m_0}{q} - t\right)\ln\frac{m_0}{m_0 - qt}\right] - \frac{1}{2}gt^2 \qquad (2)$$

PROBLEM 14.101 (Continued)

Data:

$$u = 3600 \text{ m/s} \qquad m_0 = 19{,}000 + 540 = 19{,}540 \text{ kg}$$

$$q = 225 \text{ kg/s} \qquad m_{\text{fuel}} = 17{,}800 \text{ kg}$$

$$t = \frac{m_{\text{fuel}}}{q} = \frac{17{,}800}{225} = 79.111 \text{ s} \qquad g = 9.81 \text{ m/s}^2$$

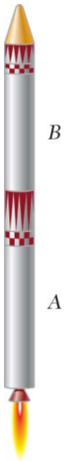
$$m_0 - qt = 1740 \text{ kg}$$

From Eq. (2),

$$h = (3600)\left[79.111 - \left(\frac{19{,}540}{225} - 79.111\right)\ln\frac{19{,}540}{1740}\right] - \frac{1}{2}(9.81)(79.111)^2$$

$$= 186{,}766 \text{ m}$$

$h = 186.8$ km ◀



PROBLEM 14.102

For the spacecraft and the two-stage launching rocket of Problem 14.96, determine the altitude at which (*a*) stage *A* of the rocket is released, (*b*) the fuel of both stages has been consumed.

SOLUTION

Thrust force: $$P = u\frac{dm}{dt} = uq$$

Mass of rocket + unspent fuel: $$m = m_0 - qt$$

Corresponding weight force: $$W = mg$$

Acceleration: $$a = \frac{F}{m} = \frac{P - W}{m} = \frac{P}{m} - g = \frac{uq}{m_0 - qt} - g$$

Integrating with respect to time to obtain velocity,

$$v = v_0 + \int_0^t a\,dt = v_0 + u\int_0^t \frac{q\,dt}{m_0 - qt} - gt$$

$$= v_0 - u\ln\frac{m_0 - qt}{m_0} - gt \qquad (1)$$

Integrating again to obtain the displacement,

$$s = s_0 + v_0 t - u\int_0^t \ln\frac{m_0 - qt}{m_0}\,dt - \frac{1}{2}gt^2$$

Let $$z = \frac{m_0 - qt}{m_0} \qquad dz = -\frac{q}{m_0}\,dt \qquad dt = -\frac{m_0}{q}\,dz$$

Then $$s = s_0 + v_0 t + \frac{m_0 u}{q}\int_{z_0}^{z} \ln z\,dz - \frac{1}{2}gt^2$$

$$= s_0 + v_0 t + \frac{m_0 u}{q}(z\ln z + z)\Big|_{z_0}^{z} - \frac{1}{2}gt^2$$

$$= s_0 + v_0 t + \frac{m_0 u}{q}\left[\frac{m_0 - qt}{m_0}\ln\frac{m_0 - qt}{m_0} + \frac{m_0 - qt}{m_0} - \frac{m_0}{m_0}\ln\frac{m_0}{m_0} + \frac{m_0}{m_0}\right] - \frac{1}{2}gt^2$$

$$= s_0 + v_0 t + u\left[t + \left(\frac{m_0}{q} - t\right)\ln\frac{m_0 - qt}{m_0}\right] - \frac{1}{2}gt^2 \qquad (2)$$

PROBLEM 14.102 (Continued)

For each stage, $m_{\text{fuel}} = 8900 \text{ kg} \qquad u = 3600 \text{ m/s}$

$$q = 225 \text{ kg/s} \qquad t = \frac{m_{\text{fuel}}}{q} = \frac{8900}{225} = 39.556 \text{ s}$$

For the first stage, $v_0 = 0 \qquad s_0 = 0$

$$m_0 = 540 + (2)(9500) = 19,540 \text{ kg}$$

From Eq. (1),

$$v_1 = 0 - 3600 \ln \frac{19,540 - 8900}{19,540} - (9.81)(39.556)$$
$$= 1800.1 \text{ m/s}$$

From Eq. (2),

(*a*)
$$s_1 = 0 + 0 + 3600\left[39.556 + \left(\frac{19,540}{225} - 39.556\right)\ln\frac{19,540 - 8900}{19,540}\right] - \frac{1}{2}(9.81)(39.556)^2$$

$$= 31,249 \text{ m}$$

$h_1 = 31.2 \text{ km}$ ◀

For the second stage, $v_0 = 1800.1 \text{ m/s} \quad s_0 = 31,249 \text{ m}$

$$m_0 = 540 + 9500 = 10,040 \text{ kg}$$

From Eq. (2),

(*b*)
$$s_2 = 31,249 + (1800.1)(39.556) + 3600\left[39.556 + \left(\frac{10,040}{225} - 39.556\right)\ln\frac{10,040 - 8900}{10,040}\right]$$
$$- \frac{1}{2}(9.81)(39.556)^2$$
$$= 197,502 \text{ m}$$

$h_2 = 197.5 \text{ km}$ ◀

PROBLEM 14.103

In a jet airplane, the kinetic energy imparted to the exhaust gases is wasted as far as propelling the airplane is concerned. The useful power is equal to the product of the force available to propel the airplane and the speed of the airplane. If v is the speed of the airplane and u is the relative speed of the expelled gases, show that the mechanical efficiency of the airplane is $\eta = 2v/(u+v)$. Explain why $\eta = 1$ when $u = v$.

SOLUTION

Let F be the thrust force, and $\frac{dm}{dt}$ be the mass flow rate.

Absolute velocity of exhaust: $v_e = u - v$

Thrust force: $F = \dfrac{dm}{dt}(u-v)$

Power of thrust force: $P_1 = Fv = \dfrac{dm}{dt}(u-v)v$

Power associated with exhaust: $P_2(\Delta t) = \dfrac{1}{2}(\Delta m)v_e^2 = \dfrac{1}{2}(\Delta m)(u-v)^2$

$$P_2 = \frac{1}{2}\frac{dm}{dt}(u-v)^2$$

Total power supplied by engine: $P = P_1 + P_2$

$$P = \frac{dm}{dt}\left[(u-v)v - \frac{1}{2}(u-v)^2\right]$$

$$= \frac{1}{2}\frac{dm}{dt}(u^2 - v^2)$$

Mechanical efficiency: $\eta = \dfrac{\text{useful power}}{\text{total power}} = \dfrac{P_1}{P}$

$$\eta = \frac{2(u-v)v}{u^2 - v^2}$$

$$\eta = \frac{2v}{(u+v)} \blacktriangleleft$$

$\eta = 1$ when $u = v$. The exhaust, having zero velocity, carries no power away.

PROBLEM 14.104

In a rocket, the kinetic energy imparted to the consumed and ejected fuel is wasted as far as propelling the rocket is concerned. The useful power is equal to the product of the force available to propel the rocket and the speed of the rocket. If v is the speed of the rocket and u is the relative speed of the expelled fuel, show that the mechanical efficiency of the rocket is $\eta = 2uv/(u^2 + v^2)$. Explain why $\eta = 1$ when $u = v$.

SOLUTION

Let F be the thrust force and $\frac{dm}{dt}$ be the mass flow rate.

Absolute velocity of exhaust: $v_e = u - v$

Thrust force: $F = \dfrac{dm}{dt}u$

Power of thrust force: $P_1 = Fv = \dfrac{dm}{dt}uv$

Power associated with exhaust:

$$P_2(\Delta t) = \frac{1}{2}(\Delta m)v_e^2 = \frac{1}{2}(\Delta m)(u-v)^2$$

$$P_2 = \frac{1}{2}\frac{dm}{dt}(u-v)^2$$

Total power supplied by engine:

$$P = P_1 + P_2$$

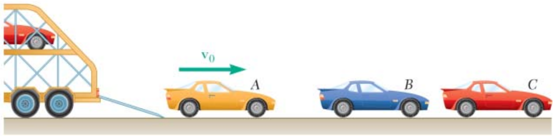
$$P = \frac{dm}{dt}\left[uv - \frac{1}{2}(u-v)^2\right] = \frac{1}{2}\frac{dm}{dt}(u^2 - v^2)$$

Mechanical efficiency:

$$\eta = \frac{\text{useful power}}{\text{total power}} = \frac{P_1}{P}$$

$$\eta = \frac{2uv}{(u^2 + v^2)} \blacktriangleleft$$

$\eta = 1$ when $u = v$. The exhaust, having zero velocity, carries no power away.



PROBLEM 14.105

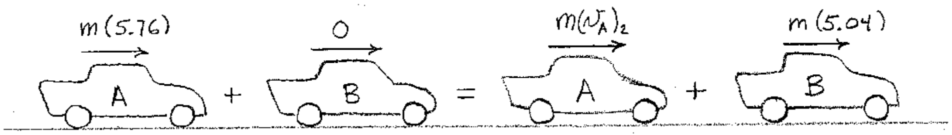
Three identical cars are being unloaded from an automobile carrier. Cars *B* and *C* have just been unloaded and are at rest with their brakes off when car *A* leaves the unloading ramp with a velocity of 5.76 ft/s and hits car *B*, which hits car *C*. Car *A* then again hits car *B*. Knowing that the velocity of car *B* is 5.04 ft/s after the first collision, 0.630 ft/s after the second collision, and 0.709 ft/s after the third collision, determine (*a*) the final velocities of cars *A* and *C*, (*b*) the coefficient of restitution for each of the collisions.

SOLUTION

There are no horizontal forces acting. Horizontal momentum is conserved.

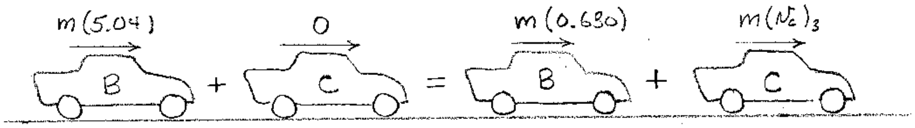
(a) *Velocities*:

Event 1 → 2: Car *A* hits car *B*.



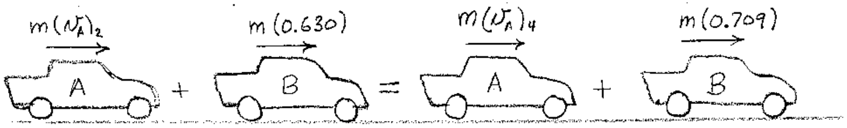
$$m(5.76) + 0 = m(v_A)_2 + m(5.04) \qquad (\mathbf{v}_A)_2 = 0.720 \text{ ft/s} \longrightarrow$$

Event 2 → 3: Car *B* hits car *C*.



$$m(5.04) + 0 = m(0.630) + m(v_C)_3 \qquad (\mathbf{v}_C)_3 = 4.41 \text{ ft/s} \longrightarrow ◀$$

Event 3 → 4: Car *A* hits car *B* again.



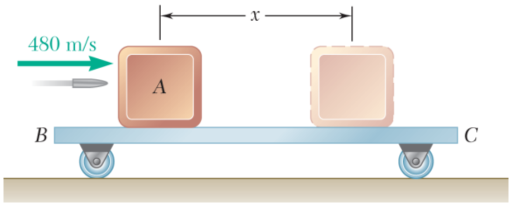
$$m(0.720) + m(0.630) = m(v_A)_4 + m(0.709) \qquad (\mathbf{v}_A)_4 = 0.641 \text{ ft/s} \longrightarrow ◀$$

(*b*) *Coefficients of restitution*:

Event 1 → 2: $$e_{1\to 2} = \left|\frac{(v_A)_2 - (v_B)_2}{(v_A)_1 - (v_B)_1}\right| = \frac{5.04 - 0.720}{5.76 - 0} \qquad e_{1\to 2} = 0.750 ◀$$

Event 2 → 3: $$e_{2\to 3} = \left|\frac{(v_B)_3 - (v_C)_3}{(v_B)_2 - (v_C)_2}\right| = \frac{4.41 - 0.630}{5.04 - 0} \qquad e_{2\to 3} = 0.750 ◀$$

Event 3 → 4: $$e_{3\to 4} = \left|\frac{(v_A)_4 - (v_B)_4}{(v_A)_3 - (v_B)_3}\right| = \frac{0.709 - 0.641}{0.720 - 0.630} \qquad e_{3\to 4} = 0.756 ◀$$

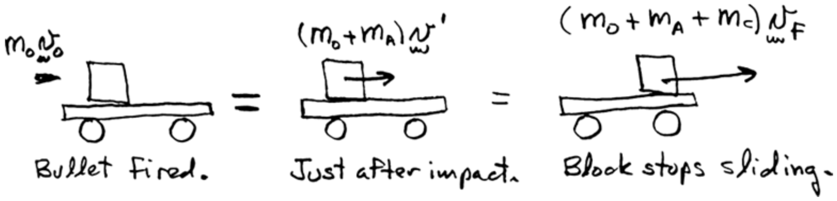


PROBLEM 14.106

A 30-g bullet is fired with a velocity of 480 m/s into block *A*, which has a mass of 5 kg. The coefficient of kinetic friction between block *A* and cart *BC* is 0.50. Knowing that the cart has a mass of 4 kg and can roll freely, determine (*a*) the final velocity of the cart and block, (*b*) the final position of the block on the cart.

SOLUTION

(*a*) Conversation of linear momentum:



$$m_0 v_0 = (m_0 + m_A)v' = (m_0 + m_A + m_C)v_f$$

$$(0.030\text{ kg})(480\text{ m/s}) = (5.030\text{ kg})v' = (9.030\text{ kg})v_f$$

$$v' = \frac{0.030}{5.030}(480\text{ m/s}) = 2.863\text{ m/s}$$

$$v_f = \frac{0.030}{9.030}(480\text{ m/s}) = 1.5947\text{ m/s}$$

$v_f = 1.595$ m/s ◀

(*b*) Work-energy principle:

Just after impact:
$$T' = \frac{1}{2}(m_0 + m_A)v'^2$$
$$= \frac{1}{2}(5.030\text{ kg})(2.863\text{ m/s})^2$$
$$= 20.615\text{ J}$$

Final kinetic energy:
$$T_f = \frac{1}{2}(m_0 + m_A + m_C)\frac{1}{2}v_f^2$$
$$= \frac{1}{2}(9.030\text{ kg})(1.5947\text{ m/s})^2$$
$$= 11.482\text{ J}$$

Work of friction force:
$$F = \mu_k N$$
$$= \mu_k (m_0 + m_A)g$$
$$= 0.50(5.030)(9.81)$$
$$= 24.672\text{ N}$$

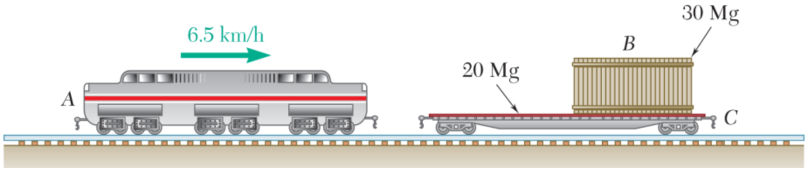
Work $= U = -Fx = -24.672x$

$$T' + U = T_f: \quad 20.615 - 24.672x = 11.482$$

$x = 0.370$ m ◀

PROBLEM 14.107

An 80-Mg railroad engine A coasting at 6.5 km/h strikes a 20-Mg flatcar C carrying a 30-Mg load B that can slide along the floor of the car $(\mu_k = 0.25)$. Knowing that the car was at rest with its brakes released and that it automatically coupled with the engine upon impact, determine the velocity of the car (*a*) immediately after impact, (*b*) after the load has slid to a stop relative to the car.



SOLUTION

The masses are the engine $(m_A = 80\times 10^3 \text{ kg})$, the load $(m_B = 30\times 10^3 \text{ kg})$, and the flat car $(m_C = 20\times 10^3 \text{ kg})$.

Initial velocities:
$$(v_A)_0 = 6.5 \text{ km/h}$$
$$= 1.80556 \text{ m/s}$$
$$(v_B)_0 = (v_C)_0 = 0.$$

No horizontal external forces act on the system during the impact and while the load is sliding relative to the flat car. Momentum is conserved.

Initial momentum:
$$m_A(v_A)_0 + m_B(0) + m_C(0) = m_A(v_A)_0 \qquad (1)$$

(*a*) Let v' be the common velocity of the engine and flat car immediately after impact. Assume that the impact takes place before the load has time to acquire velocity.

Momentum immediately after impact:
$$m_A v' + m_B(0) + m_C v' = (m_A + m_C)v' \qquad (2)$$

Equating (1) and (2) and solving for v',
$$v' = \frac{m_A(v_A)_0}{m_A + m_C}$$
$$= \frac{(80\times 10^3)(1.80556)}{(100\times 10^3)}$$
$$= 1.44444 \text{ m/s}$$

$$\mathbf{v}' = 5.20 \text{ km/h} \longrightarrow \blacktriangleleft$$

(*b*) Let v_f be the common velocity of all three masses after the load has slid to a stop relative to the car.

Corresponding momentum:
$$m_A v_f + m_B v_f + m_C v_f = (m_A + m_B + m_C)v_f \qquad (3)$$

PROBLEM 14.107 (Continued)

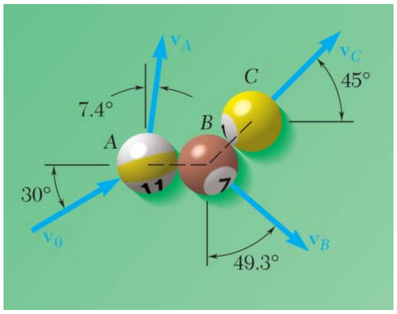
Equating (1) and (3) and solving for v_f,

$$v_f = \frac{m_A (v_A)_0}{m_A + m_B + m_C}$$

$$= \frac{(80 \times 10^3)(1.80556)}{(130 \times 10^3)}$$

$$= 1.11111\ \text{m/s}$$

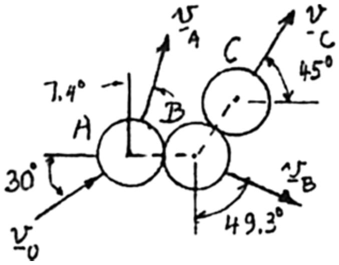
$\mathbf{v}_f = 4.00\ \text{km/h} \longrightarrow$ ◀


$\mathbf{v}_A$
7.4°
C
$\mathbf{v}_C$
45°
B
A
30°
$\mathbf{v}_0$
$\mathbf{v}_B$
49.3°

PROBLEM 14.108

In a game of pool, ball *A* is moving with a velocity $\mathbf{v}_0$ when it strikes balls *B* and *C* which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated and that $v_0 = 12$ ft/s and $v_C = 6.29$ ft/s, determine the magnitude of the velocity of (*a*) ball *A*, (*b*) ball *B*.

SOLUTION


$\underline{v}_A$
$\underline{v}_C$
C
45°
7.4°
B
A
30°
$\underline{v}_B$
49.3°
$\underline{v}_0$

Conservation of linear momentum. In *x* direction:

$$m(12\text{ ft/s})\cos 30° = mv_A \sin 7.4° + mv_B \sin 49.3° + m(6.29)\cos 45°$$

$$0.12880v_A + 0.75813v_B = 5.9446 \qquad (1)$$

In *y* direction:

$$m(12\text{ ft/s})\sin 30° = mv_A \cos 7.4° - mv_B \cos 49.3° + m(6.29)\sin 45°$$

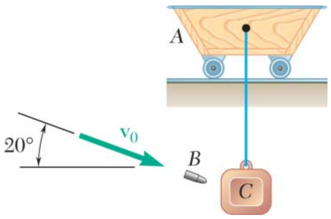
$$0.99167v_A - 0.65210v_B = 1.5523 \qquad (2)$$

(*a*) Multiply (1) by 0.65210, (2) by 0.75813, and add:

$$0.83581\, v_A = 5.0533 \qquad v_A = 6.05\text{ ft/s} \blacktriangleleft$$

(*b*) Multiply (1) by 0.99167, (2) by –0.12880, and add:

$$0.83581\, v_B = 5.6951 \qquad v_B = 6.81\text{ ft/s} \blacktriangleleft$$

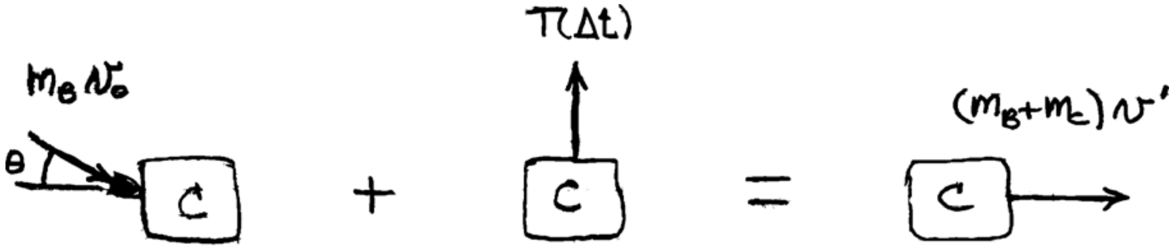


PROBLEM 14.109

Mass *C*, which has a mass of 4 kg, is suspended from a cord attached to cart *A*, which has a mass of 5 kg and can roll freely on a frictionless horizontal track. A 60-g bullet is fired with a speed $v_0 = 500$ m/s and gets lodged in block *C*. Determine (*a*) the velocity of *C* as it reaches its maximum elevation, (*b*) the maximum vertical distance *h* through which *C* will rise.

SOLUTION

Consider the impact as bullet *B* hits mass *C*. Apply the principle of impulse-momentum to the two particle system.



$$\Sigma m\mathbf{v}_1 + \Sigma\mathbf{Imp}_{1\to 2} = \Sigma m\mathbf{v}_2$$

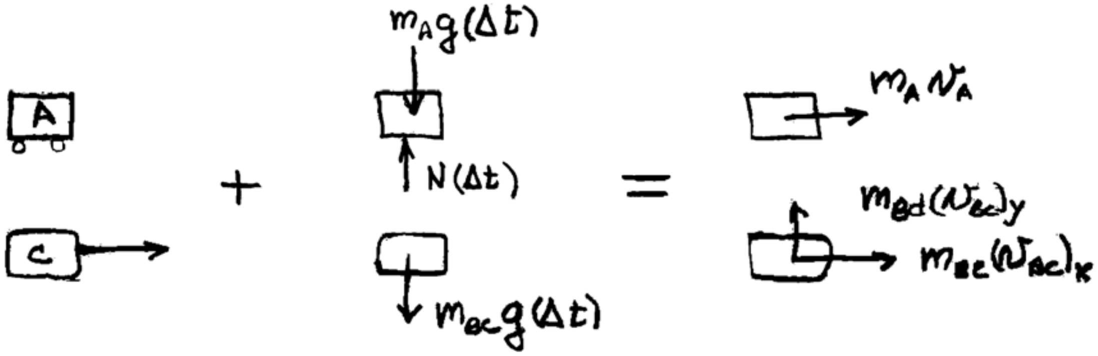
Using both *B* and *C* and taking horizontal components gives

$$m_B v_0 \cos\theta + O = (m_B + m_C)v' = m_{BC}v'$$

$$v' = \frac{m_B v_0 \cos\theta}{m_{BC}}$$

$$= \frac{(0.060\text{ kg})(500\text{ m/s})\cos 20°}{(4.06\text{ kg})} = 6.9435\text{ m/s}$$

Now consider the system of m_A and m_{BC} after the impact, and apply to impulse momentum principle.



PROBLEM 14.109 (Continued)

$$\Sigma m\mathbf{v}_2 + \Sigma \mathbf{Imp}_{2\to 3} = \Sigma m\mathbf{v}_3$$

Horizontal components: $+\longrightarrow$

$$m_{BC}v' + 0 = m_A v_A + m_{BC} v_{cx}$$

$$v_A = \frac{m_{BC}}{m_A}(v' - v_{cx})$$

$$= \frac{4.06}{5}(6.9435 - v_{cx})$$

$$v_A = 5.6381 - 0.812 v_{cx} \quad \text{in m/s} \qquad (1)$$

(*a*) At maximum elevation.

Both particles have the same velocity, thus

$$v_{cx} = v_A$$
$$v_A = 5.6381 - 0.812\, v_A$$

$$v_A = 3.1115 \text{ m/s} \qquad v_A = 3.11 \text{ m/s} \blacktriangleleft$$

(*b*) Conservation of energy: $T_2 + V_2 = T_3 + V_3$

$$T_2 = \frac{1}{2}m_A(0) + \frac{1}{2}m_{BC}(v')^2$$

$$= \frac{1}{2}(4.06)(6.9435)^2 = 97.871 \text{ J}$$

$$V_2 = 0 \qquad \text{(datum)}$$

$$T_3 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_{BC}(v_{Bx}^2 + v_{By}^2)^2$$

$$= \frac{1}{2}(5)(3.1115)^2 + \frac{1}{2}(4.06)[(3.1115)^2 + 0] = 43.857 \text{ J}$$

$$V_3 = m_{BC}gh = (4.06)(9.81)h = 39.829 \text{ h}$$

$$97.871 + 0 = 43.857 + 39.829 \text{ h}$$

$$h = 1.356 \text{ m} \blacktriangleleft$$

Another method: We observe that no external horizontal forces are exerted on the system consisting of *A*, *B*, and *C*. Thus the horizontal component of the velocity of the mass center remains constant.

$$m = m_A + m_B + m_C = 5 + 0.06 + 4 = 9.06 \text{ kg}$$

$$\overline{v}_x = \frac{m_B v_0 \cos\theta}{m_A + m_B + m_C} = \frac{(0.060 \text{ kg})(500 \text{ m/s})\cos 20°}{9.06 \text{ kg}} = 3.1115 \text{ m/s}$$

(*a*) At maximum elevation, v_A and v_{BC} are equal.

$$v_A = 3.1115 \text{ m/s} \qquad \mathbf{v}_A = 3.11 \text{ m/s} \longrightarrow \blacktriangleleft$$

PROBLEM 14.109 (Continued)

Immediately after the impact of B on C, the velocity v_A is zero.

$$(m_B + m_C)v' = (m_A + m_B + m_C)\overline{v}_x$$

$$v' = \frac{m_A + m_B + m_C}{m_B + m_C}\overline{v}_x = \frac{9.06}{4.06}(3.1115 \text{ m/s}) = 6.9435 \text{ m/s}$$

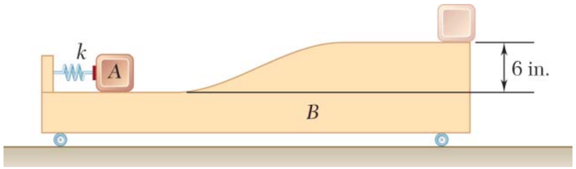
(*b*) Principle of work and energy: $T_2 + V_2 = T_3 + V_3$

$T_2, V_2,$ and V_3 are calculated as before.

For T_3 we note that the velocities $\mathbf{v}'_A$ and $\mathbf{v}'_{BC}$ relative to the mass center are zero. Thus, T_3 is given by

$$T_3 = \frac{1}{2}m\overline{\mathbf{v}}^2 = \frac{1}{2}(9.06)(3.1115)^2 = 43.857 \text{ J}$$

As before, h is found to be $h = 1.356 \text{ m}$ ◀



PROBLEM 14.110

A 15-lb block *B* is at rest and a spring of constant $k = 72$ lb/in. is held compressed 3 in. by a cord. After 5-lb block *A* is placed against the end of the spring, the cord is cut causing *A* and *B* to move. Neglecting friction, determine the velocities of blocks *A* and *B* immediately after *A* leaves *B*.

SOLUTION

$$m_A = \frac{5}{32.2} = 0.15528 \text{ lb}\cdot\text{s}^2/\text{ft}$$

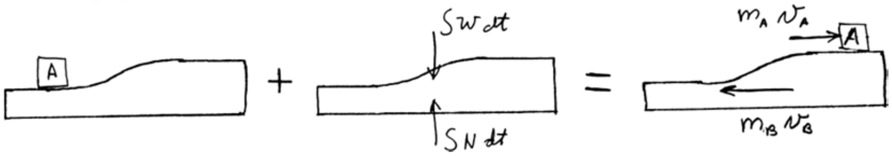
$$m_B = \frac{15}{32.2} = 0.46584 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$k = 72 \text{ lb/in} = 864 \text{ lb/ft}$$

$$e = 3 \text{ in.} = 0.25 \text{ ft}$$

$$h = 6 \text{ in.} = 0.5 \text{ ft}$$

Conservation of linear momentum:

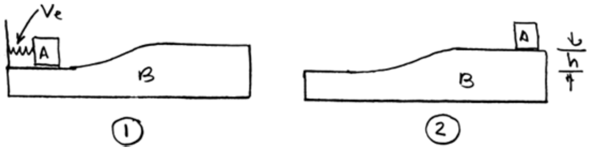


Horizontal components $\xrightarrow{+}$:

$$0 + 0 = m_A v_A - m_B v_B$$

$$v_B = \frac{m_A}{m_B} v_A = \frac{1}{3} v_A$$

Conservation of energy:



State 1:

$$V_{1e} = \frac{1}{2}ke^2 = \frac{1}{2}(864)(0.25)^2 = 27 \text{ ft}\cdot\text{lb}$$

$$V_{1g} = 0$$

$$T_1 = 0$$

State 2:

$$V_{2e} = 0$$

$$V_{2g} = W_A h = (5)(0.5) = 2.5 \text{ ft}\cdot\text{lb}$$

$$T_2 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$$

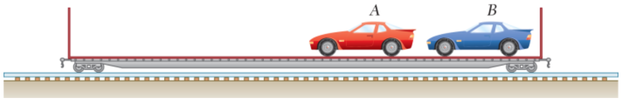
$$= \frac{1}{2}(0.15528)v_A^2 + \frac{1}{2}(0.46584)\left(\frac{v_A}{3}\right)^2 = 0.10352 v_A^2$$

PROBLEM 14.110 (Continued)

$T_1 + V_1 = T_2 + V_2$: $\quad 0 + 27 = 0.10352 v_A^2 + 2.5$

$$v_A^2 = 236.67 \text{ ft}^2 \qquad \mathbf{v}_A = 15.38 \text{ ft/s} \longrightarrow \blacktriangleleft$$

$$\mathbf{v}_B = 5.13 \text{ ft/s} \longleftarrow \blacktriangleleft$$



PROBLEM 14.111

Car *A* of mass 1800 kg and car *B* of mass 1700 kg are at rest on a 20-Mg flatcar which is also at rest. Cars *A* and *B* then accelerate and quickly reach constant speeds relative to the flatcar of 2.35 m/s and 1.175 m/s, respectively, before decelerating to a stop at the opposite end of the flatcar. Neglecting friction and rolling resistance, determine the velocity of the flatcar when the cars are moving at constant speeds.

SOLUTION

The masses are $m_A = 1800$ kg, $m_B = 1700$ kg, and $m_F = 20 \times 10^3$ kg.

Let v_A, v_B, and v_F be the sought after velocities in m/s, positive to the right.

Initial values: $(v_A)_0 = (v_B)_0 = (v_F)_0 = 0.$

Initial momentum of system: $m_A(v_A)_0 + m_B(v_B)_0 + m_F(v_F)_0 = 0.$

There are no horizontal external forces acting during the time period under consideration. Momentum is conserved.

$$0 = m_A v_A + m_B v_B + m_F v_F$$

$$1800 v_A + 1700 v_B + 20 \times 10^3 v_F = 0 \qquad (1)$$

The relative velocities are given as

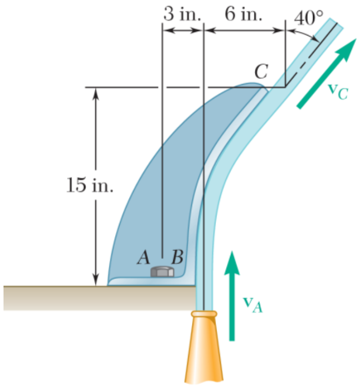
$$v_{A/F} = v_A - v_F = -2.35 \text{ m/s} \qquad (2)$$

$$v_{B/F} = v_B - v_F = -1.117 \text{ m/s} \qquad (3)$$

Solving (1), (2), and (3) simultaneously,

$$v_A = -2.085 \text{ m/s}, \quad v_B - 0.910 \text{ m/s}, \quad v_F = 0.265 \text{ m/s}$$

$$\mathbf{v}_F = 0.265 \text{ m/s} \longrightarrow \blacktriangleleft$$



PROBLEM 14.112

The nozzle shown discharges water at the rate of 200 gal/min. Knowing that at both *B* and *C* the stream of water moves with a velocity of magnitude 100 ft/s, and neglecting the weight of the vane, determine the force-couple system that must be applied at *A* to hold the vane in place $(1\text{ ft}^3 = 7.48\text{ gal})$.

SOLUTION

$$Q = \frac{200\text{ gal/min}}{(7.48\text{ gal/ft}^3)(60\text{ s/min})}$$

$$= 0.44563\text{ ft}^3\text{/s}$$

$$\frac{dm}{dt} = \frac{\gamma Q}{g}$$

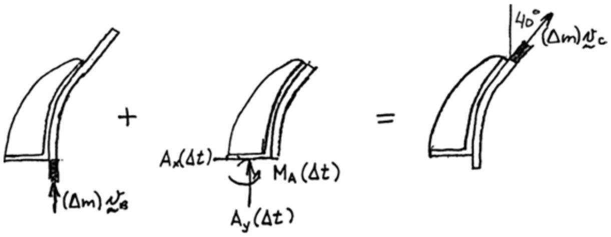
$$= \frac{(62.4\text{ lb/ft}^3)(0.44563\text{ ft}^3\text{/s})}{32.2\text{ ft/s}^2}$$

$$= 0.8636\text{ lb}\cdot\text{s/ft}$$

$$\mathbf{v}_B = (100\text{ ft/s})\mathbf{j}$$

$$\mathbf{v}_C = (100\text{ ft/s})(\sin 40°\,\mathbf{i} + \cos 40°\,\mathbf{j})$$

Apply the impulse-momentum principle.



$\xrightarrow{+}$ x components:

$$0 + A_x(\Delta t) = (\Delta m)(100\sin 40°)$$

$$A_x = \frac{\Delta m}{\Delta t}(100\sin 40°)$$

$$= (0.8636)(100\sin 40°) \qquad A_x = 55.5\text{ lb} \longrightarrow$$

$+\!\uparrow$ y components:

$$(\Delta m)(100) + A_y(\Delta t) = (\Delta m)(100\cos 40°)$$

$$A_y = \frac{\Delta m}{\Delta t}(100)(\cos 40° - 1)$$

$$= (0.8636)(100)(\cos 40° - 1)$$

$$= -20.2\text{ lb} \qquad A_y = 20.2\text{ lb}\downarrow$$

PROBLEM 14.112 (Continued)

+↺ Moments about A:

$$\left(\frac{3}{12}\right)(\Delta m)(100) + M_A(\Delta t) = \left(\frac{9}{12}\right)(\Delta m)(100\cos 40^\circ)$$

$$-\left(\frac{15}{12}\right)(\Delta m)(100\sin 40^\circ)$$

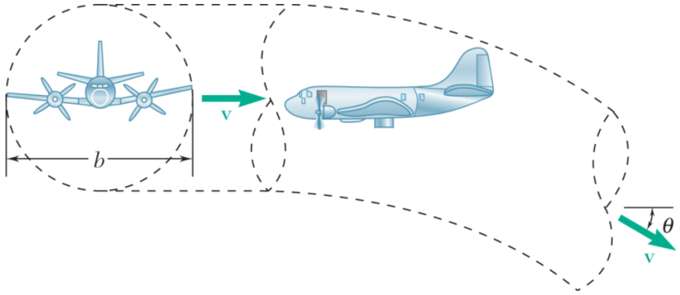
$$M_A = \left(\frac{\Delta m}{\Delta t}\right)(75\cos 40^\circ - 125\sin 40^\circ - 25)$$

$$= (0.8636)(-47.895)$$

$$= -41.36\ \text{lb}\cdot\text{ft}$$

$\mathbf{M}_A = 41.4\ \text{lb}\cdot\text{ft}$ ↻ ◀

$\mathbf{A} = 59.1\ \text{lb}$ ↘ 20.0° ◀



PROBLEM 14.113

An airplane with weight W and total wing span b flies horizontally at a constant speed v. Use the airplane as a reference frame; that is, consider the airplane to be motionless and the air to flow past it with speed v. Suppose that a cylinder of air with diameter b is deflected downward by the wing (the cross section of the cylinder is the dashed circle in in the figure). Show that the angle through which the cylinder stream is deflected (called the *downwash angle*) is determined by the formula $sin\theta = 4W/(\pi b^2 \rho v^2)$ where ρ is the mass density of the air.

SOLUTION

Given: W = Weight, b = wing span, v = speed, ρ = mass density of air

At constant speed, mass flow is steady:

$$\frac{dm}{dt} = \rho v A \qquad \text{where: } A = \frac{\pi b^2}{4}$$

$$\frac{dm}{dt} = \frac{\rho v \pi b^2}{4}$$

Velocity of the airstream after the plane in the vertical direction:

$$v_{2,y} = -v\sin\theta$$

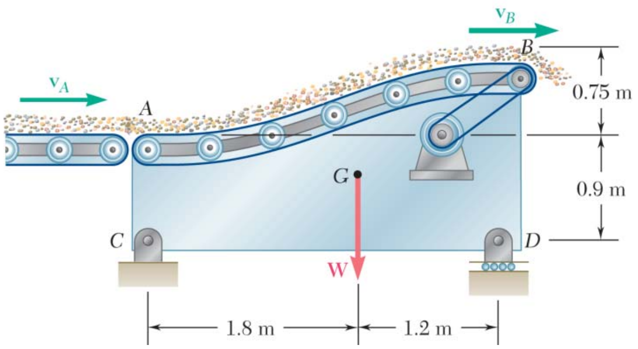
For equilibrium of the plane in vertical direction the lift provided by the wings equals the weight of the plane. This is also the downward force on the airstream

$$\sum F_y = -W$$

Apply Impulse-Momentum in the vertical direction:

$$\sum F_y = \frac{dm}{dt}\left(v_{2,y} - \cancel{v_{1,y}}\right)$$

$$-W = \frac{-\rho v \pi b^2 v \sin\theta}{4} \qquad \text{or} \qquad \sin\theta = \frac{4W}{\pi b^2 \rho v^2} \blacktriangleleft$$

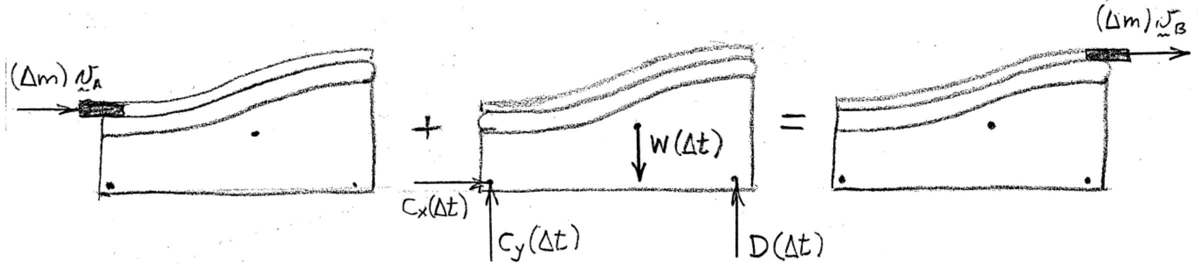

$\mathbf{v}_B$
B
0.75 m
$\mathbf{v}_A$
A
G
0.9 m
C
D
$\mathbf{W}$
1.8 m
1.2 m

PROBLEM 14.114

The final component of a conveyor system receives sand at a rate of 100 kg/s at A and discharges it at B. The sand is moving horizontally at A and B with a velocity of magnitude $v_A = v_B = 4.5$ m/s. Knowing that the combined weight of the component and of the sand it supports is $W = 4$ kN, determine the reactions at C and D.

SOLUTION

Apply the impulse-momentum principle.


$(\Delta m)\underset{\sim}{v}_A$
+
$W(\Delta t)$
=
$(\Delta m)\underset{\sim}{v}_B$
$C_x(\Delta t)$
$C_y(\Delta t)$
$D(\Delta t)$

$+\circlearrowright$ Moments about C:

$$-(0.9)(\Delta m)v_A + 3D(\Delta t) - 1.8W(\Delta t) = -(1.65)(\Delta m)v_B$$

$$D = \frac{1.8}{3}W + \frac{1}{3}\frac{\Delta m}{\Delta t}(0.9v_A - 1.65v_B)$$

$$= \frac{(1.8)(4000)}{3} + \frac{1}{3}(100)\left[(0.9)(4.5) - (1.65)(4.5)\right] = 2287.5\text{ N}$$

$\mathbf{D} = 2.29\text{ kN}\uparrow$ ◀

$\xrightarrow{+}$ x components:

$$(\Delta m)v_A + C_x(\Delta t) = (\Delta m)v_B$$

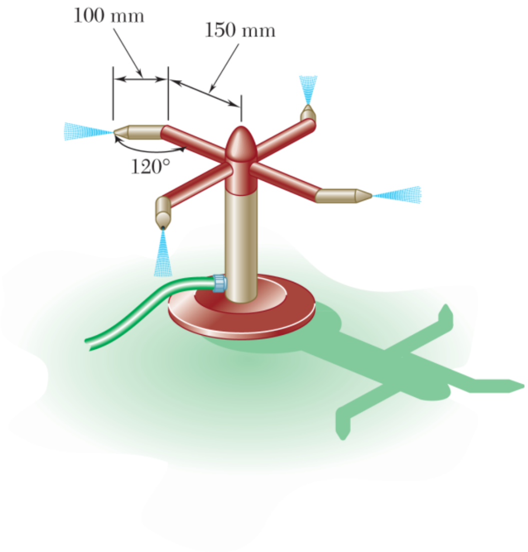
$$C_x = \left(\frac{\Delta m}{\Delta t}\right)(v_B - v_A) = (100)(4.5 - 4.5) = 0$$

$+\uparrow$ y components:

$$0 + C_y(\Delta t) + D(\Delta t) - W(\Delta t) = 0$$

$$C_y = W - D = 4000 - 2287.5 = 1712.5\text{ N}$$

$\mathbf{C} = 1.712\text{ kN}\uparrow$ ◀



PROBLEM 14.115

A garden sprinkler has four rotating arms, each of which consists of two horizontal straight sections of pipe forming an angle of 120° with each other. Each arm discharges water at a rate of 20 L/min with a velocity of 18 m/s relative to the arm. Knowing that the friction between the moving and stationary parts of the sprinkler is equivalent to a couple of magnitude $M = 0.375\ \text{N}\cdot\text{m}$, determine the constant rate at which the sprinkler rotates.

SOLUTION

The flow through each arm is 20 L/min.

$$Q = \frac{20\ \text{L/min}}{1000\ \text{L/m}^3} \times \frac{1\text{min}}{60\ \text{s}} = 333.33 \times 10^{-6}\ \text{m}^3\text{/s}$$

$$\frac{dm}{dt} = \rho Q = (1000\ \text{kg/m}^3)(333.33 \times 10^{-6})$$

$$= 0.33333\ \text{kg/s}$$

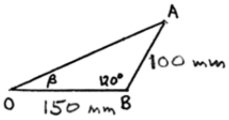
Consider the moment about O exerted on the fluid stream of one arm. Apply the impulse-momentum principle. Compute moments about O. First, consider the geometry of triangle OAB. Using first the law of cosines,

$$(OA)^2 = 150^2 + 100^2 - (2)(150)(100)\cos 120°$$

$$OA = 217.95\ \text{mm} = 0.21795\ \text{m}$$

Law of sines: $$\frac{\sin\beta}{100} = \frac{\sin 120°}{217.95}$$

$$\beta = 23.413°, \quad \alpha = 60° - \beta = 36.587°$$



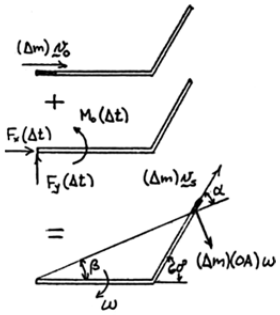
$+\circlearrowleft$ Moments about O:

$$(\Delta m)(v_O)(0) + M_O(\Delta t) = (OA)(\Delta m)v_s \sin\alpha - (OA)(\Delta m)(OA)\omega$$

$$M_O = \frac{\Delta m}{\Delta t}[(OA)v_s \sin\alpha - (OA)^2\omega]$$

$$= (0.33333)[(0.21795)(18)\sin 36.587° - (0.21795)^2\omega]$$

$$= 0.77945 - 0.015834\omega$$



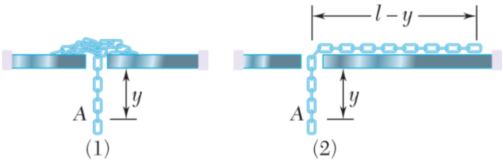
Moment that the stream exerts on the arm is $-M_O$.

Balance of the friction couple and the four streams

$$M_F - 4M_O = 0$$

$$0.375 - 4(0.77945 - 0.015834\omega) = 0$$

$$\omega = 43.305\ \text{rad/s} \qquad \omega = 414\ \text{rpm} \blacktriangleleft$$



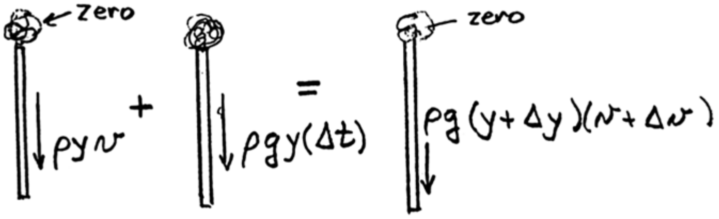
PROBLEM 14.116

A chain of length l and mass m falls through a small hole in a plate. Initially, when y is very small, the chain is at rest. In each case shown, determine (a) the acceleration of the first link A as a function of y, (b) the velocity of the chain as the last link passes through the hole. In case 1, assume that the individual links are at rest until they fall through the hole; in case 2, assume that at any instant all links have the same speed. Ignore the effect of friction.

SOLUTION

Let ρ be the mass per unit length of chain. Assume that the weight of any chain above the hole is supported by the floor. It and the corresponding upward reaction of the floor are not shown in the diagrams.

Case 1: Apply the impulse-momentum principle to the entire chain.



$$\rho yv + \rho gy\Delta t = \rho(y+\Delta y)(v+\Delta v)$$
$$= \rho yv + \rho(\Delta y)v + \rho y(\Delta v) + \rho(\Delta y)(\Delta v)$$
$$\rho gy = \rho\frac{\Delta y}{\Delta t}v + \rho y\frac{\Delta v}{\Delta t} + \rho\frac{(\Delta y)(\Delta v)}{\Delta t}$$

Let $\Delta t \to 0$.
$$\rho gy = \rho\frac{dy}{dt}v + \rho y\frac{dv}{dt}$$
$$= \rho\frac{d}{dt}(yv)$$

Multiply both sides by yv.
$$\rho gy^2 v = \rho yv\frac{d}{dt}(yv)$$

Let $v = \frac{dy}{dt}$ on left hand side.
$$\rho gy^2\frac{dy}{dt} = \rho yv\frac{d}{dt}(yv)$$

Integrate with respect to time.
$$\rho g\int y^2 dy = \rho\int(yv)\,d(yv)$$

$$\frac{1}{3}\rho gy^3 = \frac{1}{2}\rho(yv)^2 \quad \text{or} \quad v^2 = \frac{2}{3}gy \qquad (1)$$

Differentiate with respect to time.
$$2v\frac{dv}{dt} = \frac{2}{3}g\frac{dy}{dt} = \frac{2}{3}gv$$

PROBLEM 14.116 (Continued)

(*a*) $$a = \frac{dv}{dt} = \frac{1}{3}g \qquad \mathbf{a} = 0.333g \downarrow ◀$$

(*b*) Set $y = l$ in Eq. (1). $$v^2 = \frac{2}{3}gl \qquad \mathbf{v} = 0.817\sqrt{gl} \downarrow ◀$$

Case 2: Apply conservation of energy using the floor as the level from which the potential energy is measured.

$$T_1 = 0 \qquad V_1 = 0$$

$$T_2 = \frac{1}{2}mv^2 \quad V_2 = -\rho g y \frac{y}{2}$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 = \frac{1}{2}mv^2 - \frac{1}{2}\rho g y^2 \qquad v^2 = \frac{\rho g y^2}{m} = \frac{g y^2}{l} \tag{2}$$

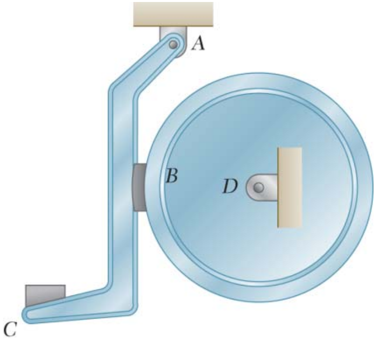
Differentiating with respect to *y*, $$2v\frac{dv}{dy} = \frac{2gy}{l}$$

(*a*) Acceleration: $$a = v\frac{dv}{dy} = \frac{gy}{l} \qquad \mathbf{a} = \frac{gy}{l} \downarrow ◀$$

(*b*) Setting $y = l$ in Eq. (2), $$v^2 = gl \qquad \mathbf{v} = \sqrt{gl} \downarrow ◀$$

Note: The impulse-momentum principle may be used to obtain the force that the edge of the hole exerts on the chain.

CHAPTER 15



PROBLEM 15.1

The brake drum is attached to a larger flywheel that is not shown. The motion of the brake drum is defined by the relation $\theta = 36t - 1.6t^2$, where θ is expressed in radians and t in seconds. Determine (*a*) the angular velocity at $t = 2$ s, (*b*) the number of revolutions executed by the brake drum before coming to rest.

SOLUTION

Given: $\theta = 36t - 1.6t^2$ radians

Differentiate to obtain the angular velocity.

$$\omega = \frac{d\theta}{dt} = 36 - 3.2t \quad \text{rad/s}$$

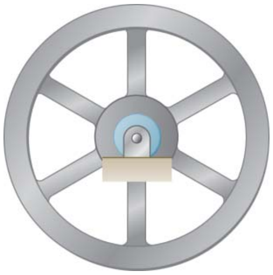
(*a*) At $t = 2$ s, $\omega = 36 - (3.2)(2)$ $\omega = 29.6$ rad/s ◀

(*b*) When the rotor stops, $\omega = 0.$

$0 = 36 - 3.2t$ $t = 11.25$ s

$$\theta = (36)(11.25) - (1.6)(11.25)^2 = 202.5 \text{ radians}$$

In revolutions, $\theta = \dfrac{202.5}{2\pi}$ $\theta = 32.2$ rev ◀



PROBLEM 15.2

The motion of an oscillating flywheel is defined by the relation $\theta = \theta_0 e^{-3\pi t} \cos 4\pi t$, where θ is expressed in radians and t in seconds. Knowing that $\theta_0 = 0.5$ rad, determine the angular coordinate, the angular velocity, and the angular acceleration of the flywheel when (*a*) $t = 0$, (*b*) $t = 0.125$ s.

SOLUTION

$$\theta = 0.5\, e^{-3\pi t} \cos 4\pi t$$

$$\omega = \frac{d\theta}{dt} = 0.5\left(-3\pi e^{-3\pi t} \cos 4\pi t - 4\pi e^{-3\pi t} \sin 4\pi t\right)$$

$$\alpha = \frac{d\omega}{dt} = 0.5\left(9\pi^2 e^{-3\pi t} \cos 4\pi t + 12\pi^2 e^{-3\pi t} \sin 4\pi t + 12\pi^2 e^{-3\pi t} \sin 4\pi t - 16\pi^2 e^{-3\pi t} \cos 4\pi t\right)$$

$$= 0.5\left(24\pi^2 e^{-3\pi t} \sin 4\pi t - 7\pi^2 e^{-3\pi t} \cos 4\pi t\right)$$

(*a*) $t = 0,$ $\quad \theta = (0.5)$ $\qquad \theta = 0.500 \text{ rad}$ ◀

$\omega = (0.5)(-3\pi) = -4.71$ $\qquad \omega = -4.71 \text{ rad/s}$ ◀

$\alpha = (0.5)(-7\pi^2) = -34.5$ $\qquad \alpha = -34.5 \text{ rad/s}^2$ ◀

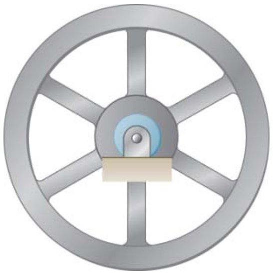
(*b*) $t = 0.125 \text{ s}, \qquad \cos 4\pi t = \cos\frac{\pi}{2} = 0, \qquad \sin 4\pi t = \sin\frac{\pi}{2} = 1$

$$e^{-3\pi t} = 0.30786$$

$\theta = (0.5)(0.30786)(0) = 0$ $\qquad \theta = 0$ ◀

$\omega = (0.5)(0.30786)(-4\pi) = -1.93437$ $\qquad \omega = -1.934 \text{ rad/s}$ ◀

$\alpha = (0.5)(0.30786)(24\pi^2) = 36.461$ $\qquad \alpha = 36.5 \text{ rad/s}^2$ ◀



PROBLEM 15.3

The motion of an oscillating flywheel is defined by the relation $\theta = \theta_0 e^{-7\pi t/6} \sin 4\pi t$, where θ is expressed in radians and t in seconds. Knowing that $\theta_0 = 0.4$ rad, determine the angular coordinate, the angular velocity, and the angular acceleration of the flywheel when (*a*) $t = 0.125$ s, (*b*) $t = \infty$.

SOLUTION

$$\theta = \theta_0 e^{-7\pi t/6} \sin 4\pi t$$

$$\omega = \frac{d\theta}{dt} = \theta_0\left(-\frac{7\pi}{6}e^{-7\pi t/6}\sin 4\pi t + 4\pi e^{-7\pi t/6}\cos 4\pi t\right)$$

$$\alpha = \frac{d\omega}{dt} = \theta_0\left(\frac{49\pi^2}{36}e^{-7\pi t/6}\sin 4\pi t - \frac{28\pi^2}{6}e^{-7\pi t/6}\cos 4\pi t - \frac{28\pi^2}{6}e^{-7\pi t/6}\cos 4\pi t - 16\pi^2 e^{-7\pi t/6}\sin 4\pi t\right)$$

$$= -\theta_0 e^{-7\pi t/6}\left[\left(16 - \frac{49}{36}\right)\pi^2 \sin 4\pi t + \frac{28}{3}\pi^2 \cos 4\pi t\right]$$

(*a*) $\theta_0 = 0.4$ rad, $t = 0.125$ s

$$e^{-7\pi(0.125)/6} = 0.63245, \quad 4\pi t = \frac{\pi}{2}, \quad \sin\frac{\pi}{2} = 1, \quad \cos\frac{\pi}{2} = 0$$

$\theta = (0.4)(0.63245)(1) = 0.25298$ radians $\qquad \theta = 0.253$ rad ◀

$\omega = (0.4)(0.63245)\left(-\frac{7\pi}{6}\right)(1) = -0.92722$ rad/s $\qquad \omega = -0.927$ rad/s ◀

$\alpha = -(0.4)(0.63245)\left(16 - \frac{49}{36}\right)\pi^2(1) = -36.551$ rad/s^2 $\qquad \alpha = -36.6$ rad/s^2 ◀

(*b*) $t = \infty$, $\quad e^{-7\pi t/6} = 0$ $\qquad \theta = 0$ ◀

$\omega = 0$ ◀

$\alpha = 0$ ◀

PROBLEM 15.4

The rotor of a gas turbine is rotating at a speed of 6900 rpm when the turbine is shut down. It is observed that 4 min is required for the rotor to coast to rest. Assuming uniformly accelerated motion, determine (*a*) the angular acceleration, (*b*) the number of revolutions that the rotor executes before coming to rest.

SOLUTION

$$\omega_0 = 6900 \text{ rpm}$$

$$= 722.57 \text{ rad/s}$$

$$t = 4 \text{ min} = 240 \text{ s}$$

(*a*) $$\omega = \omega_0 + \alpha t; \quad 0 = 722.57 + \alpha(240)$$

$$\alpha = -3.0107 \text{ rad/s}$$

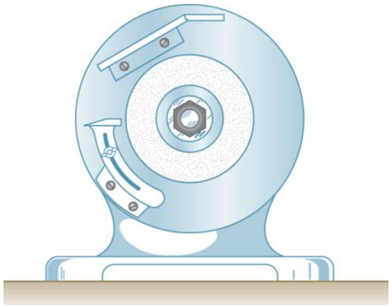
$\alpha = -3.01 \text{ rad/s}^2$ ◀

(*b*) $$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = (722.57)(240) + \frac{1}{2}(-3.0107)(240)^2$$

$$\theta = 173{,}416 - 86{,}708 = 86{,}708 \text{ rad}$$

$$\theta = 86{,}708 \text{ rad}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)$$

$\theta = 13{,}800 \text{ rev}$ ◀



PROBLEM 15.5

A small grinding wheel is attached to the shaft of an electric motor which has a rated speed of 3600 rpm. When the power is turned on, the unit reaches its rated speed in 5 s, and when the power is turned off, the unit coasts to rest in 70 s. Assuming uniformly accelerated motion, determine the number of revolutions that the motor executes (*a*) in reaching its rated speed, (*b*) in coasting to rest.

SOLUTION

For uniformly accelerated motion,

$$\omega = \omega_0 + \alpha t \qquad (1)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \qquad (2)$$

(*a*) Data for start up: $\theta_0 = 0,\ \omega_0 = 0,$

At $t = 5$ s, $\omega = 3600 \text{ rpm} = \dfrac{2\pi(3600)}{60} = 120\pi$ rad/s

From Eq. (1), $120\pi = \alpha(5) \qquad \alpha = 24\pi \text{ rad/s}^2$

From Eq. (2), $\theta = 0 + 0 + \dfrac{1}{2}(24\pi)(5)^2 = 300\pi$ radians

In revolutions, $\theta = \dfrac{300\pi}{2\pi}$ $\qquad \theta = 150$ rev ◀

(*b*) Data for coasting to rest:

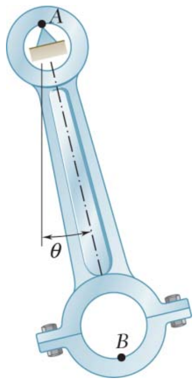
$\theta_0 = 0, \quad \omega_0 = 120\pi$ rad/s

At $t = 70$ s, $\omega = 0$

From Eq. (1), $0 = 120\pi - \alpha(70) \qquad \alpha = \dfrac{120\pi}{70}$ rad/s

From Eq. (2), $\theta = 0 + (120\pi)(70) - \dfrac{(120\pi)(70)^2}{2(70)} = 4200\pi$ radians

In revolutions, $\theta = \dfrac{4200\pi}{2\pi}$ $\qquad \theta = 2100$ rev ◀



PROBLEM 15.6

A connecting rod is supported by a knife-edge at Point A. For small oscillations the angular acceleration of the connecting rod is governed by the relation $\alpha = -6\theta$ where α is expressed in rad/s^2 and θ in radians. Knowing that the connecting rod is released from rest when $\theta = 20°$, determine (*a*) the maximum angular velocity, (*b*) the angular position when $t = 2$ s.

SOLUTION

Angular motion relations:

$$\alpha = \frac{d\omega}{dt} = \frac{\omega d\omega}{d\theta} = -6\theta \qquad (1)$$

Separation of variables ω and θ gives

$$\omega d\omega = -6\theta d\theta$$

Integrating, using $\omega = 0$ when $\theta = \theta_0$,

$$\int_0^\omega \omega d\omega = -6\int_{\theta_0}^\theta \theta d\theta$$

$$\frac{1}{2}\omega^2 = -3(\theta^2 - \theta_0^2) = 3(\theta_0^2 - \theta^2)$$

$$\omega^2 = 6(\theta_0^2 - \theta^2) \qquad \omega = \sqrt{6(\theta_0^2 - \theta^2)}$$

(*a*) ω is maximum when $\theta = 0$.

Data: $\theta_0 = 20° = 0.34907$ radians

$$\omega_{\text{max}}^2 = 6(0.34907^2 - 0) = 0.73108 \text{ rad}^2/\text{s}$$

$\omega_{\text{max}} = 0.855$ rad/s ◀

(*b*) From $\omega = \dfrac{d\theta}{dt}$ we get $dt = \dfrac{d\theta}{\omega} = \dfrac{1}{\sqrt{6}}\dfrac{d\theta}{\sqrt{\theta_0^2 - \theta^2}}$

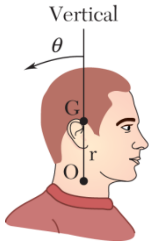
Integrating, using $t = 0$ when $\theta = \theta_0$,

$$\int_0^t dt = \frac{1}{\sqrt{6}}\int_{\theta_0}^\theta \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}}$$

$$t = -\frac{1}{\sqrt{6}}\cos^{2-1}\frac{\theta}{\theta_0}\Bigg|_{\theta_0}^{\theta} = -\frac{1}{\sqrt{6}}\left[0 - \cos^{-1}\frac{\theta}{\theta_0}\right] = -\frac{1}{\sqrt{6}}\cos^{-1}\frac{\theta}{\theta_0}$$

$$\theta = \theta_0\cos(\sqrt{6}t) = 0.34907\cos[(\sqrt{6})2] = (0.34907)(0.18551) = 0.064756 \text{ radians}$$

$\theta = 3.71°$ ◀



PROBLEM 15.7

When studying whiplash resulting from rear end collisions, the rotation of the head is of primary interest. An impact test was performed, and it was found that the angular acceleration of the head is defined by the relation $\alpha = 700\cos\theta + 70\sin\theta$ where α is expressed in rad/s^2 and θ in radians. Knowing that the head is initially at rest, determine the angular velocity of the head when $\theta = 30°$.

SOLUTION

Angular motion relations:

$$\alpha = \frac{d\omega}{dt} = \frac{\omega d\omega}{d\theta} = 700\cos\theta + 70\sin\theta$$

Separating variables ω and θ gives

$$\omega d\omega = (700\cos\theta + 70\sin\theta)d\theta$$

Integrating, using $\omega = 0$ when $\theta = 0$,

$$\int_0^\omega \omega d\omega = \int_0^\theta (700\cos\theta + 70\sin\theta)d\theta$$

$$\frac{1}{2}\omega^2 = (700\sin\theta - 70\cos\theta)\Big|_0^\theta$$

$$= 700\sin\theta + 70(1 - \cos\theta)$$

$$\omega = \sqrt{1400\sin\theta + 140(1 - \cos\theta)}$$

Data: $\theta = 30° = \frac{\pi}{6}$ rad

With calculator set to "degrees" for trigonometric functions,

$$\omega = \sqrt{1400\sin 30° + 140(1 - \cos 30°)} = 26.8 \text{ rad/s}$$

$\omega = 26.8$ rad/s ◀

With calculator set to "radians" for trigonometric functions,

$$\omega = \sqrt{1400\sin(\pi/6) + 140(1 - \cos(\pi/6))} = 26.8 \text{ rad/s}$$

PROBLEM 15.8

The angular acceleration of an oscillating disk is defined by the relation $\alpha = -k\theta$. Determine (*a*) the value of k for which $\omega = 12$ rad/s when $\theta = 0$ and $\theta = 6$ rad when $\omega = 0$, (*b*) the angular velocity of the disk when $\theta = 3$ rad.

SOLUTION

Given: $$\alpha = -k\theta \text{ and } \alpha = \omega \frac{d\omega}{d\theta}$$

$$\omega\, d\omega = \alpha d\theta \quad \Rightarrow \quad \omega\, d\omega = -k\theta d\theta$$

Integrating, $$\int_{12}^{0} \omega\, d\omega = -k \int_{0}^{6} \theta d\theta$$

$$0 - \frac{12^2}{2} = -k\left(\frac{6^2}{2} - 0\right)$$

(*a*) $$k = \frac{12^2}{6^2} = 9 \text{ s}^{-2}$$ $k = 9.00 \text{ s}^{-2}$ ◀

$$\int_{12}^{\omega} \omega\, d\omega = -k \int_{0}^{3} \theta d\theta$$

$$\frac{\omega^2}{2} - \frac{12^2}{2} = -9\left(\frac{3^2}{2} - 0\right)$$

(*b*) $$\omega^2 = 12^2 - (9)(3)^2 = 63 \text{ rad}^2/\text{s}^2$$ $\omega = 7.94$ rad/s ◀

PROBLEM 15.9

The angular acceleration of a shaft is defined by the relation $\alpha = -0.5\omega$, where α is expressed in rad/s^2 and ω in rad/s. Knowing that at $t = 0$ the angular velocity of the shaft is 30 rad/s, determine (*a*) the number of revolutions the shaft will execute before coming to rest, (*b*) the time required for the shaft to come to rest, (*c*) the time required for the angular velocity of the shaft to reduce to 2 percent of its initial value.

SOLUTION

(*a*) Given: $\alpha = -0.5\omega, \qquad \omega\dfrac{d\omega}{d\theta} = \alpha$

$$\omega\frac{d\omega}{d\theta} = -0.5\omega \quad \Rightarrow \quad d\omega = -0.5d\theta$$

Integrating, $\int_{30}^{0} d\omega = -0.5\int_{0}^{\theta} d\theta \qquad -30 = -0.5\theta$

$$\theta = 60 \text{ radians} = \frac{60}{2\pi} = 9.55 \text{ rev} \qquad\qquad \theta = 9.55 \text{ rev} \blacktriangleleft$$

(*b*) $\dfrac{d\omega}{dt} = -0.5\omega \qquad dt = -2\dfrac{d\omega}{\omega}$

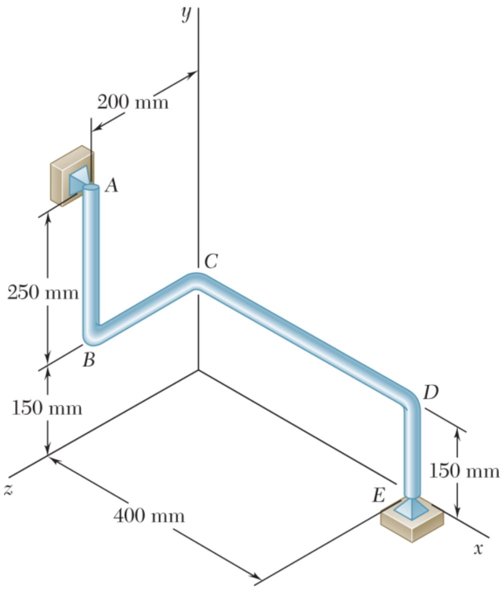
Integrating, $\int_{0}^{t} dt = -2\int_{30}^{0} \dfrac{d\omega}{\omega}$

$$t = -2 \ln\frac{0}{30} = \infty \qquad\qquad t = \infty \blacktriangleleft$$

(*c*) $\omega = (0.02)(30) = 0.6 \text{ rad/s}$

$$\int_{0}^{t} dt = -2\int_{30}^{0.6} \frac{d\omega}{\omega}$$

$$t = -2 \ln\frac{0.6}{30} = 2 \ln 50 \qquad\qquad t = 7.82 \text{ s} \blacktriangleleft$$



PROBLEM 15.10

The bent rod *ABCDE* rotates about a line joining Points *A* and *E* with a constant angular velocity of 9 rad/s. Knowing that the rotation is clockwise as viewed from *E*, determine the velocity and acceleration of corner *C*.

SOLUTION

$$EA^2 = 0.4^2 + 0.4^2 + 0.2^2$$

$$EA = 0.6 \text{ m}$$

$$\mathbf{r}_{C/E} = -(0.4 \text{ m})\mathbf{i} + (0.15 \text{ m})\mathbf{j}$$

$$\overrightarrow{EA} = -(0.4 \text{ m})\mathbf{i} + (0.4 \text{ m})\mathbf{j} + (0.2 \text{ m})\mathbf{k}$$

$$\boldsymbol{\lambda}_{EA} = \frac{\overrightarrow{EA}}{EA} = \frac{1}{0.6}(-0.4\mathbf{i} + 0.4\mathbf{j} + 0.2\mathbf{k}) = \frac{1}{3}(-2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\boldsymbol{\omega} = \omega_{AE}\boldsymbol{\lambda}_{EA} = (9 \text{ rad/s})\frac{1}{3}(-2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\boldsymbol{\omega} = -(6 \text{ rad/s})\mathbf{i} + (6 \text{ rad/s})\mathbf{j} + (3 \text{ rad/s})\mathbf{k}$$

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_{C/E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 6 & 3 \\ -0.4 & 0.15 & 0 \end{vmatrix} = -0.45\mathbf{i} - 1.2\mathbf{j} + (-0.9 + 2.4)\mathbf{k}$$

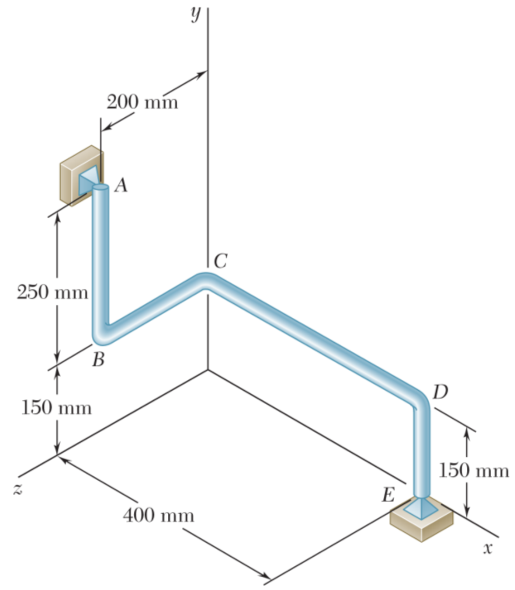
$$\mathbf{v}_C = -(0.45 \text{ m/s})\mathbf{i} - (1.2 \text{ m/s})\mathbf{j} + (1.5 \text{ m/s})\mathbf{k} \blacktriangleleft$$

$$\mathbf{a}_C = \boldsymbol{\alpha}_{AC} \times \mathbf{r}_{C/E} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{C/E}) = \boldsymbol{\alpha}_{AC} \times \mathbf{r}_{C/E} + \boldsymbol{\omega} \times \mathbf{v}_C$$

$$\mathbf{a}_C = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 6 & 3 \\ -0.45 & -1.2 & 1.5 \end{vmatrix}$$

$$= (9 + 3.6)\mathbf{i} + (-1.35 + 9)\mathbf{j} + (7.2 + 2.7)\mathbf{k}$$

$$\mathbf{a}_C = (12.60 \text{ m/s}^2)\mathbf{i} + (7.65 \text{ m/s}^2)\mathbf{j} + (9.90 \text{ m/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.11

In Problem 15.10, determine the velocity and acceleration of corner B, assuming that the angular velocity is 9 rad/s and increases at the rate of 45 rad/s^2.

PROBLEM 15.10 The bent rod *ABCDE* rotates about a line joining Points A and E with a constant angular velocity of 9 rad/s. Knowing that the rotation is clockwise as viewed from E, determine the velocity and acceleration of corner C.

SOLUTION

$$EA^2 = 0.4^2 + 0.4^2 + 0.2^2$$

$$EA = 0.6 \text{ m}$$

$$\mathbf{r}_{B/A} = -(0.25 \text{ m})\mathbf{j}$$

$$\overrightarrow{EA} = -(0.4 \text{ m})\mathbf{i} + (0.4 \text{ m})\mathbf{j} + (0.2 \text{ m})\mathbf{k}$$

$$\boldsymbol{\lambda}_{EA} = \frac{\overrightarrow{EA}}{EA} = \left(\frac{-0.4\mathbf{i} + 0.4\mathbf{j} + 0.2\mathbf{k}}{0.6}\right) = \frac{1}{3}(-2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\boldsymbol{\omega} = \omega_{AE}\boldsymbol{\lambda}_{EA} = (9 \text{ rad/s})\frac{1}{3}(-2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\boldsymbol{\omega} = -(6 \text{ rad/s})\mathbf{i} + (6 \text{ rad/s})\mathbf{j} + (3 \text{ rad/s})\mathbf{k}$$

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = (-6\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}) \times (-0.25)\mathbf{j} = 1.5\mathbf{k} + 0.75\mathbf{i}$$

$$\mathbf{v}_B = (0.75 \text{ m/s})\mathbf{i} + (1.5 \text{ m/s})\mathbf{k} \blacktriangleleft$$

$$\boldsymbol{\alpha} = \alpha_{AE}\boldsymbol{\lambda}_{EA} = (45 \text{ rad/s}^2)\frac{1}{3}(-2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

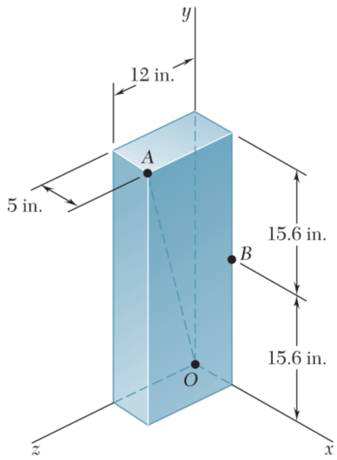
$$\boldsymbol{\alpha} = -(30 \text{ rad/s}^2)\mathbf{i} + (30 \text{ rad/s}^2)\mathbf{j} + (15 \text{ rad/s}^2)\mathbf{k}$$

$$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times \mathbf{v}_B$$

$$\mathbf{a}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -30 & 30 & 15 \\ 0 & -0.25 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 6 & 3 \\ 0.75 & 0 & 1.5 \end{vmatrix}$$

$$= 3.75\mathbf{i} + 7.5\mathbf{k} + 9\mathbf{i} + (2.25 + 9)\mathbf{j} - 4.5\mathbf{k}$$

$$\mathbf{a}_B = (12.75 \text{ m/s}^2)\mathbf{i} + (11.25 \text{ m/s}^2)\mathbf{j} + (3.00 \text{ m/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.12

The rectangular block shown rotates about the diagonal *OA* with a constant angular velocity of 6.76 rad/s. Knowing that the rotation is counterclockwise as viewed from *A*, determine the velocity and acceleration of point *B* at the instant shown.

SOLUTION

Given:

$$\omega = 6.76 \text{ rad/s}$$

$$\mathbf{r}_{A/O} = (5 \text{ in.})\mathbf{i} + (31.2 \text{ in.})\mathbf{j} + (12 \text{ in.})\mathbf{k}$$

$$\mathbf{r}_{B/O} = (5 \text{ in.})\mathbf{i} + (15.6 \text{ in.})\mathbf{j}$$

$$l_{OA} = \sqrt{(5)^2 + (31.2)^2 + (12)^2} = 33.8 \text{ in.}$$

Angular velocity:

$$\boldsymbol{\omega} = \frac{\omega}{l_{OA}}\mathbf{r}_{A/O} = \frac{6.76}{33.8}(5\mathbf{i} + 31.2\mathbf{j} + 12\mathbf{k})$$

$$\boldsymbol{\omega} = (1.0 \text{ rad/s})\mathbf{i} + (6.24 \text{ rad/s})\mathbf{j} + (2.4 \text{ rad/s})\mathbf{k}$$

Velocity of point B:

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/O}$$

$$\mathbf{v}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.0 & 6.24 & 2.4 \\ 5 & 15.6 & 0 \end{vmatrix} = -37.44\mathbf{i} + 12\mathbf{j} - 15.6\mathbf{k}$$

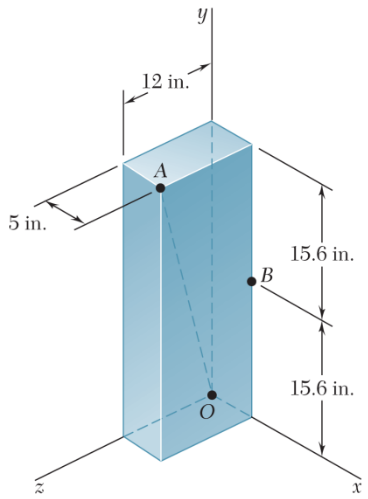
$$\mathbf{v}_B = -(37.4 \text{ in./s})\mathbf{i} + (12.00 \text{ in./s})\mathbf{j} - (15.60 \text{ in./s})\mathbf{k} \blacktriangleleft$$

Acceleration of point B:

$$\mathbf{a}_B = \boldsymbol{\omega} \times \mathbf{v}_B$$

$$\mathbf{a}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.0 & 6.24 & 2.4 \\ -37.4 & 12 & -15.60 \end{vmatrix} = -126.1\mathbf{i} - 74.26\mathbf{j} + 245.6\mathbf{k}$$

$$\mathbf{a}_B = -(126.1 \text{ in./s}^2)\mathbf{i} - (74.3 \text{ in./s}^2)\mathbf{j} + (246 \text{ in./s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.13

The rectangular block shown rotates about the diagonal *OA* with an angular velocity of 3.38 rad/s that is decreasing at the rate of 5.07 rad/s^2. Knowing that the rotation is counterclockwise as viewed from *A*, determine the velocity and acceleration of point *B* at the instant shown.

SOLUTION

Given:

$$\omega = 3.38\ \text{rad/s},\ \alpha = -5.07\ \text{rad/s}^2$$

$$\mathbf{r}_{A/O} = (5\ \text{in.})\mathbf{i} + (31.2\ \text{in.})\mathbf{j} + (12\ \text{in.})\mathbf{k}$$

$$\mathbf{r}_{B/O} = (5\ \text{in.})\mathbf{i} + (15.6\ \text{in.})\mathbf{j}$$

$$l_{OA} = \sqrt{(5)^2 + (31.2)^2 + (12)^2} = 33.8\ \text{in.}$$

Angular velocity:

$$\boldsymbol{\omega} = \frac{\omega}{l_{OA}}\mathbf{r}_{A/O} = \frac{3.38}{33.8}(5\mathbf{i} + 31.2\mathbf{j} + 12\mathbf{k})$$

$$\boldsymbol{\omega} = (0.5\ \text{rad/s})\mathbf{i} + (3.12\ \text{rad/s})\mathbf{j} + (1.2\ \text{rad/s})\mathbf{k}$$

Velocity of point B:

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/O}$$

$$\mathbf{v}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.5 & 3.12 & 1.2 \\ 5 & 15.6 & 0 \end{vmatrix} = -18.72\mathbf{i} + 6\mathbf{j} - 7.80\mathbf{k}$$

$$\mathbf{v}_B = -(18.72\ \text{in./s})\mathbf{i} + (6.00\ \text{in./s})\mathbf{j} - (7.80\ \text{in./s})\mathbf{k} \blacktriangleleft$$

Angular Acceleration:

$$\boldsymbol{\alpha} = \frac{\alpha}{l_{OA}}\mathbf{r}_{A/O} = \frac{-5.07}{33.8}(5\mathbf{i} + 31.2\mathbf{j} + 12\mathbf{k})$$

$$\boldsymbol{\alpha} = -(0.75\ \text{rad/s}^2)\mathbf{i} - (4.68\ \text{rad/s}^2)\mathbf{j} - (1.8\ \text{rad/s}^2)\mathbf{k}$$

PROBLEM 15.13 (Continued)

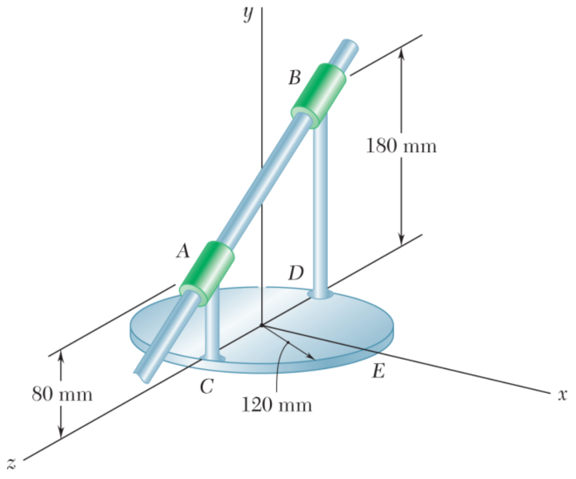
Acceleration of point B:

$$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_{B/O} + \boldsymbol{\omega} \times \mathbf{v}_B$$

$$\mathbf{a}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.75 & -4.68 & -1.8 \\ 5 & 15.6 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.5 & 3.12 & 1.2 \\ -18.72 & 6 & -7.8 \end{vmatrix}$$

$$= 28.08\mathbf{i} - 9\mathbf{j} + 11.7\mathbf{k} - 31.536\mathbf{i} - 18.564\mathbf{j} + 61.406\mathbf{k}$$

$$\mathbf{a}_B = -\left(3.46 \text{ in./s}^2\right)\mathbf{i} - \left(27.6 \text{ in./s}^2\right)\mathbf{j} + \left(73.1 \text{ in./s}^2\right)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.14

A circular plate of 120 mm radius is supported by two bearings A and B as shown. The plate rotates about the rod joining A and B with a constant angular velocity of 26 rad/s. Knowing that, at the instant considered, the velocity of Point C is directed to the right, determine the velocity and acceleration of Point E.

SOLUTION

$$\overrightarrow{BA} = -(100\text{ mm})\mathbf{j} + (240\text{ mm})\mathbf{k} \qquad BA = 260\text{ mm}$$

$$\lambda_{BA} = \frac{\overrightarrow{BA}}{BA} = \frac{-(100)\mathbf{j} + (240)\mathbf{k}}{260}, \qquad \alpha = 0$$

$$\boldsymbol{\omega} = \omega\lambda_{BA} = 26\left(\frac{1}{260}\right)(-(100)\mathbf{j} + (240)\mathbf{k})$$

Point E:

$$\mathbf{r}_{E/A} = (120\text{ mm})\mathbf{i} - (80\text{ mm})\mathbf{j} - (120\text{ mm})\mathbf{k}$$

$$\mathbf{v}_E = \boldsymbol{\omega} \times \mathbf{r}_{E/A}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -10 & 24 \\ 120 & -80 & -120 \end{vmatrix}$$

$$= (3120\text{ mm/s})\mathbf{i} + (2880\text{ mm/s})\mathbf{j} + (1200\text{ mm/s})\mathbf{k}$$

$$\mathbf{v}_E = (3.12\text{ m/s})\mathbf{i} + (2.88\text{ m/s})\mathbf{j} + (1.200\text{ m/s})\mathbf{k} \blacktriangleleft$$

$$\mathbf{a}_E = \boldsymbol{\alpha} \times \mathbf{r}_{E/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{E/A}) = 0 + \boldsymbol{\omega} \times \mathbf{v}_B$$

$$\mathbf{a}_E = \boldsymbol{\omega} \times \mathbf{v}_B$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -10 & 24 \\ 3120 & 2880 & 1200 \end{vmatrix}$$

$$= -(81120\text{ mm/s}^2)\mathbf{i} + (74880\text{ mm/s}^2)\mathbf{j} + (31200\text{ mm/s}^2)$$

$$\mathbf{a}_E = -(81.1\text{ m/s}^2)\mathbf{i} + (74.9\text{ m/s}^2)\mathbf{j} + (31.2\text{ m/s}^2)\mathbf{k} \blacktriangleleft$$

PROBLEM 15.15

In Problem 15.14, determine the velocity and acceleration of Point E, assuming that the angular velocity is 26 rad/s and increases at the rate of 65 rad/s^2.

SOLUTION

See Problem 15.14 for $\boldsymbol{\lambda}_{BA}$ and $\boldsymbol{\omega}$

$$\boldsymbol{\lambda}_{BA} = \frac{-(100)\mathbf{j} + (240)\mathbf{k}}{260}$$

$$\omega = (10 \text{ rad/s})\mathbf{j} = (24 \text{ rad/s})\mathbf{k}$$

$$\boldsymbol{\alpha} = +65 \text{ rad/s}^2; \qquad \boldsymbol{\alpha} = \boldsymbol{\alpha}\lambda_{BA} = (65 \text{ rad/s}^2)\left(\frac{1}{260}\right) - (100)\mathbf{j} + (240)\mathbf{k}$$

$$\boldsymbol{\alpha} = -(25 \text{ rad/s}^2)\mathbf{j} + (60 \text{ rad/s}^2)\mathbf{k}$$

Point E:

$$\mathbf{r}_{E/A} = (120 \text{ mm})\mathbf{i} - (80)\mathbf{j} - (120)\mathbf{k}$$

$$\mathbf{v}_E = \boldsymbol{\omega} \times \mathbf{r}_{E/A}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -10 & 24 \\ 120 & -80 & -120 \end{vmatrix}$$

$$= (3120 \text{ mm/s})\mathbf{i} + (2880 \text{ mm/s})\mathbf{j} + (1200 \text{ mm/s})\mathbf{k}$$

$$\mathbf{v}_E = (3.12 \text{ m/s})\mathbf{i} + (2.88 \text{ m/s})\mathbf{j} + (1.200 \text{ m/s})\mathbf{k} \blacktriangleleft$$

$$\mathbf{a}_D = \boldsymbol{\alpha} \times \mathbf{r}_{E/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{E/A}) = \boldsymbol{\alpha} \times \mathbf{r}_{E/A} + \boldsymbol{\omega} \times \mathbf{v}_E$$

$$\mathbf{a}_D = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -25 & 60 \\ 120 & -80 & -120 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -10 & 24 \\ 3120 & 2880 & 1200 \end{vmatrix}$$

$$= -(7800 \text{ mm/s}^2)\mathbf{i} + (7200 \text{ mm/s}^2)\mathbf{j} + (3000 \text{ mm/s}^2)\mathbf{k}$$
$$- (81120 \text{ mm/s}^2)\mathbf{i} + (74880 \text{ mm/s}^2)\mathbf{j} + (31200 \text{ mm/s}^2)\mathbf{k}$$

$$\mathbf{a}_B = -(73320 \text{ mm/s}^2)\mathbf{i} + (82080 \text{ mm/s}^2)\mathbf{j} + (34200 \text{ mm/s}^2)\mathbf{k}$$
$$= -(73.3 \text{ m/s}^2)\mathbf{i} + (82.1 \text{ m/s}^2)\mathbf{j} + (34.2 \text{ m/s}^2)\mathbf{k} \quad \blacktriangleleft$$

PROBLEM 15.16

The earth makes one complete revolution around the sun in 365.24 days. Assuming that the orbit of the earth is circular and has a radius of 93,000,000 mi, determine the velocity and acceleration of the earth.

SOLUTION

$$\omega = \frac{2\pi \text{ rad}}{(365.24 \text{ days})\left(\frac{24 \text{ h}}{\text{day}}\right)\left(\frac{3600 \text{ s}}{\text{h}}\right)}$$

$$= 199.11 \times 10^{-9} \text{ rad/s}$$

$$v = r\omega$$

$$= (93 \times 10^{6} \text{ mi})\left(\frac{5280 \text{ ft}}{\text{mi}}\right)(199.11 \times 10^{-9} \text{ rad/s})$$

$$v = 97{,}770 \text{ ft/s} \qquad v = 66{,}700 \text{ mi/h} \blacktriangleleft$$

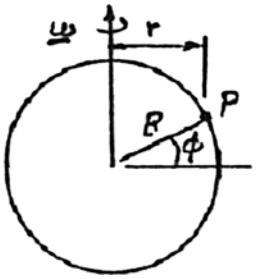
$$a = r\omega^{2}$$

$$= (93 \times 10^{6})(5280)(199.11 \times 10^{-9} \text{ rad/s})^{2} \qquad a = 19.47 \times 10^{-3} \text{ ft/s}^{2} \blacktriangleleft$$

PROBLEM 15.17

The earth makes one complete revolution on its axis in 23 h 56 min. Knowing that the mean radius of the earth is 3960 mi, determine the linear velocity and acceleration of a point on the surface of the earth (*a*) at the equator, (*b*) at Philadelphia, latitude 40° north, (*c*) at the North Pole.

SOLUTION



$$23\text{ h }56\text{ m} = 23.933\text{ h}$$

$$\boldsymbol{\omega} = \frac{2\pi\text{ rad}}{(23.933\text{ h})\left(\frac{3600\text{ s}}{\text{h}}\right)}$$

$$= 72.925\times 10^{-6}\text{ rad/s}$$

$$R = (3960\text{ mi})\left(\frac{5280\text{ ft}}{\text{mi}}\right)$$

$$= 20.91\times 10^{6}\text{ ft}$$

$$r = \text{radius of path}$$

$$= R\cos\phi$$

(*a*) Equator: Latitude $=\phi = 0$

$$v = r\omega$$

$$= R(\cos 0)\omega$$

$$= (20.91\times 10^{6}\text{ ft})(1)(72.925\times 10^{-6}\text{ rad/s}) \qquad v = 1525\text{ ft/s} \blacktriangleleft$$

$$a = r\omega^2$$

$$= R(\cos 0)\omega^2$$

$$= (20.91\times 10^{6}\text{ ft})(1)(72.925\times 10^{-6}\text{ rad/s})^2 \qquad a = 0.1112\text{ ft/s}^2 \blacktriangleleft$$

(*b*) Philadelphia: Latitude $=\phi = 40°$

$$v = r\omega$$

$$= R(\cos 40°)\omega$$

$$= (20.91\times 10^{6}\text{ ft})(\cos 40°)(72.925\times 10^{-6}\text{ rad/s}) \qquad v = 1168\text{ ft/s} \blacktriangleleft$$

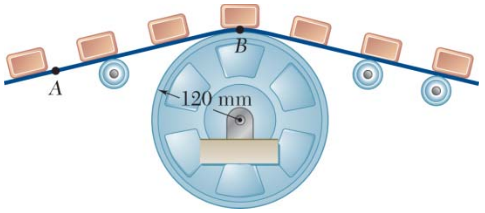
$$a = r\omega^2$$

$$= R(\cos 40°)\omega^2$$

$$= (20.91\times 10^{6}\text{ ft})(\cos 40°)(72.925\times 10^{-6}\text{ rad/s})^2 \qquad a = 0.0852\text{ ft/s}^2 \blacktriangleleft$$

(*c*) North Pole: Latitude $=\phi = 0$

$$r = R\cos 0 = 0 \qquad v = a = 0 \blacktriangleleft$$



PROBLEM 15.18

A series of small machine components being moved by a conveyor belt pass over a 120 mm radius idler pulley. At the instant shown, the velocity of Point *A* is 300 mm/s to the left and its acceleration is 180 mm/s^2 to the right. Determine (*a*) the angular velocity and angular acceleration of the idler pulley, (*b*) the total acceleration of the machine component at *B*.

SOLUTION

$$v_B = v_A = 300\text{ mm/s} \longleftarrow \qquad r_B = 120\text{ mm}$$

$$(a_B)_t = a_A = 180\text{ mm/s} \longrightarrow$$

(*a*) $v_B = \omega r_B,$ $\qquad \omega = \dfrac{v_B}{r_B} = \dfrac{300}{120} = 2.5\text{ rad/s} \circlearrowleft$ $\qquad \boldsymbol{\omega} = 2.50\text{ rad/s} \circlearrowleft$ ◀

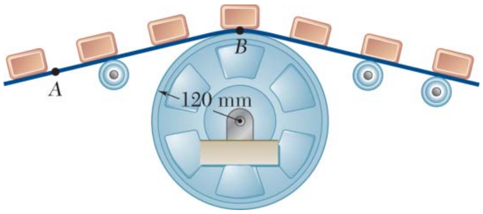
$(a_B)_t = \alpha r_B,$ $\qquad \alpha = \dfrac{(a_B)_t}{r_B} = \dfrac{180}{120} = 1.5\text{ rad/s} \circlearrowright$ $\qquad \boldsymbol{\alpha} = 1.500\text{ rad/s}^2 \circlearrowright$ ◀

(*b*) $(a_B)_n = r_B\omega^2 = (120)(2.5)^2 = 750\text{ mm/s}^2 \downarrow$

$$a_B = \sqrt{(a_B)_t^2 + (a_B)_n^2} = \sqrt{(180)^2 + (750)^2} = 771\text{ mm/s}^2$$

$$\tan\beta = \frac{750}{180}, \qquad \beta = 76.5^\circ$$

$\mathbf{a}_B = 771\text{ mm/s}^2 \searrow 76.5^\circ$ ◀



PROBLEM 15.19

A series of small machine components being moved by a conveyor belt pass over a 120-mm-radius idler pulley. At the instant shown, the angular velocity of the idler pulley is 4 rad/s clockwise. Determine the angular acceleration of the pulley for which the magnitude of the total acceleration of the machine component at B is $2400\ \text{mm/s}^2$.

SOLUTION

$$\omega_B = 4\ \text{rad/s}\ \circlearrowright, \quad r_B = 120\ \text{mm}$$

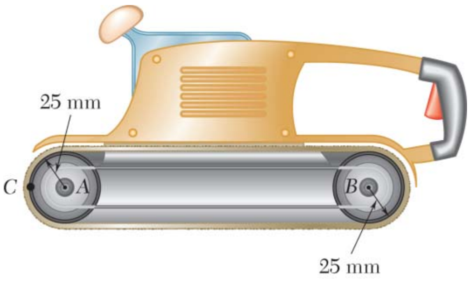
$$(a_B)_n = r_B\omega_B^2 = (120)(4)^2 = 1920\ \text{mm/s}^2$$

$$a_B = 2400\ \text{mm/s}^2$$

$$(a_B)_t = \sqrt{a_B^2 - (a_B)_n^2} = \sqrt{2400^2 - 1920^2} = \pm 1440\ \text{mm/s}^2$$

$$(a_B)_t = r_B\alpha, \qquad \alpha = \frac{(a_B)_t}{r_B} = \frac{\pm 1440}{120} = \pm 12\ \text{rad/s}^2$$

$12.00\ \text{rad/s}^2\ \circlearrowleft$ or $\circlearrowright$ ◀



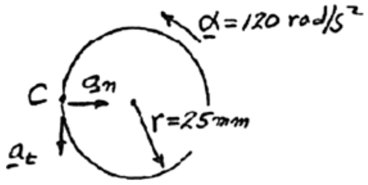
PROBLEM 15.20

The belt sander shown is initially at rest. If the driving drum *B* has a constant angular acceleration of 120 rad/s^2 counterclockwise, determine the magnitude of the acceleration of the belt at Point *C* when (*a*) $t = 0.5$ s, (*b*) $t = 2$ s.

SOLUTION

$$a_t = r\alpha = (0.025 \text{ m})(120 \text{ rad/s}^2)$$

$$\mathbf{a}_t = 3 \text{ m/s}^2 \downarrow$$



(*a*) $t = 0.5$ s:

$$\omega = \alpha t = (120 \text{ rad/s}^2)(0.5 \text{ s}) = 60 \text{ rad/s}$$

$$a_n = r\omega^2 = (0.025 \text{ m})(60 \text{ rad/s})^2$$

$$\mathbf{a}_n = 90 \text{ m/s}^2 \longrightarrow$$

$$a_B^2 = a_t^2 + a_n^2 = 3^2 + 90^2$$

$a_B = 90.05 \text{ m/s}^2$ ◀

(*b*) $t = 2$ s:

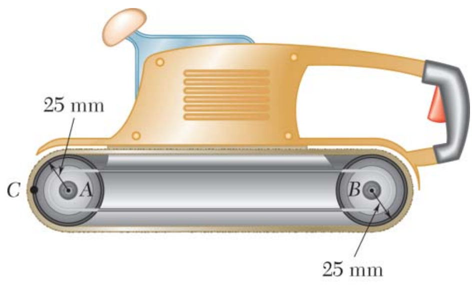
$$\omega = \alpha t = (120 \text{ rad/s}^2)(2 \text{ s}) = 240 \text{ rad/s}$$

$$a_n = r\omega^2 = (0.025 \text{ m})(240 \text{ rad/s})^2$$

$$a_n = 1440 \text{ m/s}^2$$

$$a_B^2 = a_t^2 + a_n^2 = 3^2 + 1440^2$$

$a_B = 1440 \text{ m/s}^2$ ◀



PROBLEM 15.21

The rated speed of drum *B* of the belt sander shown is 2400 rpm. When the power is turned off, it is observed that the sander coasts from its rated speed to rest in 10 s. Assuming uniformly decelerated motion, determine the velocity and acceleration of Point *C* of the belt, (*a*) immediately before the power is turned off, (*b*) 9 s later.

SOLUTION

$$\omega_0 = 2400 \text{ rpm}$$

$$= 251.3 \text{ rad/s}$$

$$r = 0.025 \text{ m}$$

(*a*)

$$v_C = r\omega = (0.025 \text{ m})(251.3 \text{ rad/s})$$

$v_C = 6.28 \text{ m/s}$ ◀

$$a_C = r\omega^2 = (0.025 \text{ m})(251.3 \text{ rad/s})^2$$

$a_C = 1579 \text{ m/s}^2$ ◀

(*b*) When $t = 10$ s:

$$\omega = 0.$$

$$\omega = \omega_0 + \alpha t$$

$$0 = 251.3 \text{ rad/s} + \alpha(10 \text{ s})$$

$$\alpha = -25.13 \text{ rad/s}^2$$

When $t = 9$ s:

$$\omega = \omega_0 + \alpha t$$

$$\omega_9 = 251.3 \text{ rad/s} - (25.13 \text{ rad/s}^2)(9 \text{ s})$$

$$= 25.13 \text{ rad/s}$$

$$v_C = r\omega_9$$

$$= (0.025 \text{ m})(25.13 \text{ rad/s})$$

$v_9 = 0.628 \text{ m/s}$ ◀

$$(a_C)_t = r\alpha_n$$

$$= (0.025 \text{ m})(-25.13 \text{ rad/s}^2)$$

$$(a_C)_t = 0.628 \text{ m/s}^2$$

$$(a_C)_n = r\omega_9^2$$

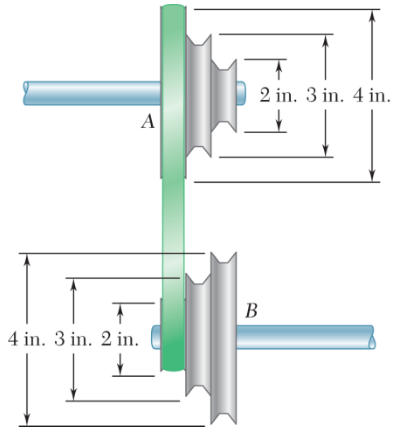
$$= (0.025 \text{ m})(25.13 \text{ rad/s})^2$$

$$(a_C)_n = 15.79 \text{ m/s}^2$$

$$a_C^2 = (a_C)_t^2 + (a_C)_n^2$$

$$= (0.628 \text{ m/s}^2)^2 + (15.79 \text{ m/s}^2)^2$$

$a_C = 15.80 \text{ m/s}^2$ ◀



PROBLEM 15.22

The two pulleys shown may be operated with the V belt in any of three positions. If the angular acceleration of shaft A is 6 rad/s^2 and if the system is initially at rest, determine the time required for shaft B to reach a speed of 400 rpm with the belt in each of the three positions.

SOLUTION

Angular velocity of shaft A: $\omega_A = \alpha_A t$

Belt speed: $v = r_A\omega_A = r_B\omega_B$

Angular speed of shaft B: $\omega_B = \dfrac{v}{r_B} = \dfrac{r_A\alpha_A t}{r_B}$

Solving for t, $t = \dfrac{r_B\omega_B}{r_A\alpha_A}$

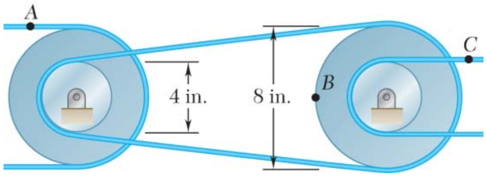
Data: $\alpha_A = 6$ rad/s, $\omega_B = 400 \text{ rpm} = 41.889 \text{ rad/s}$

$$t = \frac{r_B}{r_A}\cdot\frac{41.889}{6} = 6.9813\frac{r_B}{r_A} = 6.9813\frac{d_B}{d_A}$$

Belt at left: $\dfrac{d_B}{d_A} = \dfrac{2 \text{ in.}}{4 \text{ in.}}$ $t = 3.49$ s ◀

Belt in middle: $\dfrac{d_B}{d_A} = \dfrac{3 \text{ in.}}{3 \text{ in.}}$ $t = 6.98$ s ◀

Belt at right: $\dfrac{d_B}{d_A} = \dfrac{4 \text{ in.}}{2 \text{ in.}}$ $t = 13.96$ s ◀

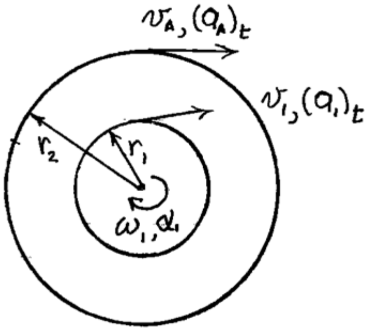


PROBLEM 15.23

Three belts move over two pulleys without slipping in the speed reduction system shown. At the instant shown the velocity of Point *A* on the input belt is 2 ft/s to the right, decreasing at the rate of 6 ft/s^2. Determine, at this instant, (*a*) the velocity and acceleration of Point *C* on the output belt, (*b*) the acceleration of Point *B* on the output pulley.

SOLUTION

Left pulley.



Inner radius $r_1 = 2$ in.

Outer radius $r_2 = 4$ in.

$$v_A = 2 \text{ ft/s} \longrightarrow$$

$$(a_A)_t = -6 \text{ ft/s}^2 = 6 \text{ ft/s}^2 \longleftarrow$$

$$\omega_1 = \frac{v_A}{r_2} = \frac{2}{\frac{4}{12}} = 6 \text{ rad/s} \circlearrowright$$

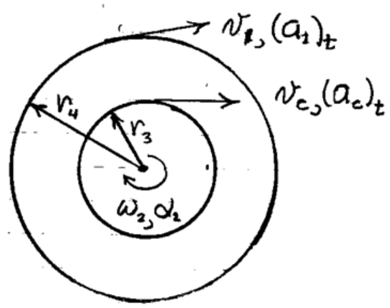
$$\alpha_1 = \frac{(a_A)_t}{r_2} = \frac{6}{\frac{4}{12}} = 18 \text{ rad/s}^2 \circlearrowleft$$

Intermediate belt.

$$v_1 = r_1\omega_1 = \left(\frac{2}{12}\right)(6) = 1 \text{ ft/s}$$

$$(a_1)_t = r_1\alpha_1 = \left(\frac{2}{12}\right)(18) = 3 \text{ ft/s}^2$$

Right pulley.



Inner radius $r_3 = 2$ in.

Outer radius $r_4 = 4$ in.

$$\omega_2 = \frac{v_1}{r_4} = \frac{1}{\left(\frac{4}{12}\right)} = 3 \text{ rad/s} \circlearrowright$$

$$\alpha_2 = \frac{(a_1)_t}{r_4} = \frac{3}{\left(\frac{4}{12}\right)} = 9 \text{ rad/s}^2 \circlearrowleft$$

PROBLEM 15.23 (Continued)

(*a*) *Velocity and acceleration of Point C.*

$$v_C = r_3\omega_2 = \left(\frac{2}{12}\right)(3) = 0.5 \text{ ft/s}$$

$\mathbf{v}_C = 0.5$ ft/s → ◀

$$(a_C)_t = r_3\alpha_2 = \left(\frac{2}{12}\right)(9) = 1.5 \text{ ft/s}^2$$

$\mathbf{a}_C = 1.5 \text{ ft/s}^2$ ← ◀

(*b*) *Acceleration of Point B.*

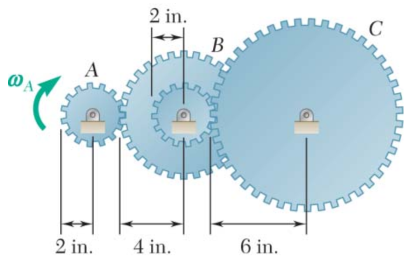
$$(a_B)_n = r_4\omega_2^2 = \left(\frac{4}{12}\right)(3)^2 = 3 \text{ ft/s}^2$$

$(\mathbf{a}_B)_n = 3 \text{ ft/s}^2$ →

$$(a_B)_t = r_4\alpha_2 = \left(\frac{4}{12}\right)(9) = 3 \text{ ft/s}^2$$

$(\mathbf{a}_B)_t = 3 \text{ ft/s}^2$ ↓

$\mathbf{a}_B = 4.24 \text{ ft/s}^2$ ↘ 45° ◀



PROBLEM 15.24

A gear reduction system consists of three gears A, B, and C. Knowing that gear A rotates clockwise with a constant angular velocity $\omega_A = 600$ rpm, determine (*a*) the angular velocities of gears B and C, (*b*) the accelerations of the points on gears B and C which are in contact.

SOLUTION

(*a*)

$$\omega_A = 600 \text{ rpm} = \frac{(600)(2\pi)}{60} = 20\pi \text{ rad/s.}$$

Let Points A, B, and C lie at the axles of gears A, B, and C, respectively.

Let D be the contact point between gears A and B.

$$v_D = r_{D/A}\omega_A = (2)(20\pi) = 40\pi \text{ in./s} \downarrow$$

$$\omega_B = \frac{v_D}{r_{D/B}} = \frac{40\pi}{4} = 10\pi \text{ rad/s} = 10\pi \cdot \frac{60}{2\pi} = 300 \text{ rpm} \circlearrowleft$$

$\boldsymbol{\omega}_B = 300 \text{ rpm} \circlearrowleft$ ◀

Let E be the contact point between gears B and C.

$$v_E = r_{E/B}\omega_B = (2)(10\pi) = 20\pi \text{ in./s} \uparrow$$

$$\omega_C = \frac{v_E}{r_{E/C}} = \frac{20\pi}{6} = 3.333\pi \text{ rad/s} = (3.333\pi)\frac{60}{2\pi} = 100 \text{ rpm} \circlearrowright$$

$\boldsymbol{\omega}_C = 100 \text{ rpm} \circlearrowright$ ◀

(*b*) *Accelerations at Point E.*

On gear B:

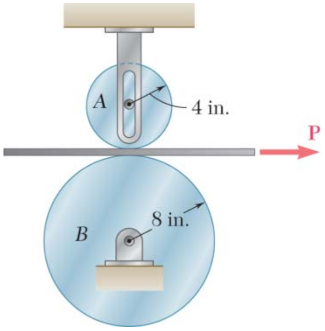
$$a_B = \frac{v_E^2}{r_{E/B}} = \frac{(20\pi)^2}{2} = 1973.9 \text{ in./s}^2 \longleftarrow$$

$\mathbf{a}_B = 1974 \text{ in./s}^2 \longleftarrow$ ◀

On gear C:

$$a_C = \frac{v_E^2}{r_{E/C}} = \frac{(20\pi)^2}{6} = 658 \text{ in./s}^2 \longrightarrow$$

$\mathbf{a}_C = 658 \text{ in./s}^2 \longrightarrow$ ◀



PROBLEM 15.25

A belt is pulled to the right between cylinders *A* and *B*. Knowing that the speed of the belt is a constant 5 ft/s and no slippage occurs, determine (*a*) the angular velocities of *A* and *B*, (*b*) the accelerations of the points which are in contact with the belt.

SOLUTION

(*a*) Angular velocities.

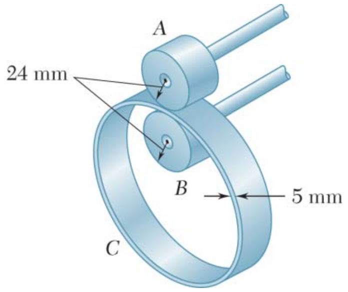
Disk A: $\omega_A = \dfrac{v_P}{r_A} = \dfrac{5\text{ ft/s}}{(4/12)\text{ ft}}$ $\boldsymbol{\omega}_A = 15.00\text{ rad/s} \circlearrowleft$ ◀

Disk B: $\omega_B = \dfrac{v_P}{r_B} = \dfrac{5\text{ ft/s}}{(8/12)\text{ ft}}$ $\boldsymbol{\omega}_B = 7.50\text{ rad/s} \circlearrowleft$ ◀

(*b*) Accelerations of contact points.

Disk A: $a_A = \omega_A^2 r_A = (15.00\text{ rad/s})^2((4/12)\text{ ft})$ $\mathbf{a}_A = 75.00\text{ rad/s}^2 \uparrow$ ◀

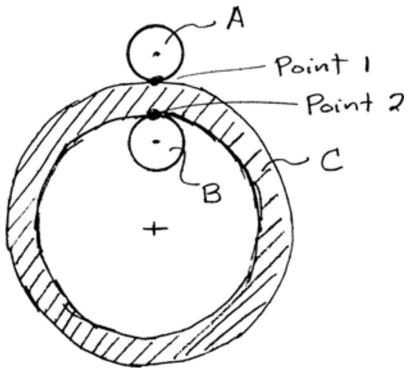
Disk B: $a_B = \omega_B^2 r_B = (7.50\text{ rad/s})^2((8/12)\text{ ft})$ $\mathbf{a}_B = 37.5\text{ rad/s}^2 \downarrow$ ◀



PROBLEM 15.26

Ring *C* has an inside radius of 55 mm and an outside radius of 60 mm and is positioned between two wheels *A* and *B*, each of 24-mm outside radius. Knowing that wheel *A* rotates with a constant angular velocity of 300 rpm and that no slipping occurs, determine (*a*) the angular velocity of the ring *C* and of wheel *B*, (*b*) the acceleration of the Points on *A* and *B* that are in contact with *C*.

SOLUTION



$$\omega_A = 300 \text{ rpm}\left(\frac{2\pi}{60}\right)$$

$$= 31.416 \text{ rad/s}$$

$$r_A = 24 \text{ mm}$$

$$r_B = 24 \text{ mm}$$

$$r_1 = 60 \text{ mm}$$

$$r_2 = 55 \text{ mm}$$

[We assume senses of rotation shown for our computations.]

(*a*) Velocities:

Point 1 (Point of contact of *A* and *C*)

$$v_1 = r_A\omega_A = r_1\omega_C$$

$$\omega_C = \frac{r_A}{r_1}\omega_A$$

$$= \frac{24 \text{ mm}}{60 \text{ mm}}(300 \text{ rpm})$$

$$= 120 \text{ rpm}$$

$\omega_C = 120$ rpm ◀

Point 2 (Point of contact of *B* and *C*)

$$v_2 = r_B\omega_B = r_2\omega_C$$

$$\omega_B = \frac{r_2}{r_B}\omega_C$$

$$= \frac{r_2}{r_B}\left(\frac{r_A}{r_1}\right)\omega_A$$

$$= \frac{55 \text{ mm}}{24 \text{ mm}}\left(\frac{24 \text{ mm}}{60 \text{ mm}}\right)300 \text{ rpm}$$

$$\omega_B = 275 \text{ mm}$$

$\omega_B = 275$ rpm ◀

PROBLEM 15.26 (Continued)

(*b*) Accelerations:

Point on rim of *A*:

$$r_A = 24\text{ mm} = 0.024\text{ m}$$

$$a_A = r_A\omega_A^2$$

$$= (0.024\text{ m})(31.416\text{ rad/s})^2$$

$$= 23.687\text{ m/s}^2 \qquad \mathbf{a}_A = 23.7\text{ m/s}^2 \uparrow \blacktriangleleft$$

Point on rim of *B*:

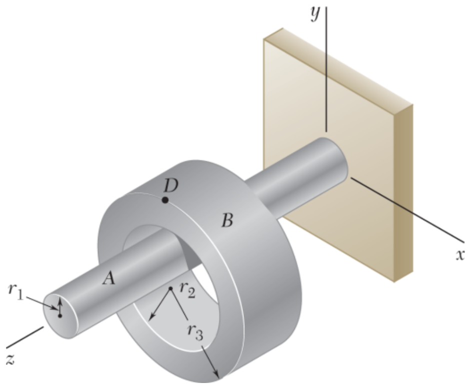
$$\omega_B = 275\text{ rpm}\left(\frac{2\pi}{60}\right)$$

$$= 28.798\text{ rad/s}$$

$$a_B = r_B\omega_B^2$$

$$= (0.024\text{ m})(28.798\text{ rad/s})^2$$

$$= 19.904\text{ m/s}^2 \qquad \mathbf{a}_B = 19.90\text{ m/s}^2 \downarrow \blacktriangleleft$$



PROBLEM 15.27

Ring B has an inside radius r_2 and hangs from the horizontal shaft A as shown. Shaft A rotates with a constant angular velocity of 25 rad/s and no slipping occurs. Knowing that $r_1 = 12$ mm, $r_2 = 30$ mm, and $r_3 = 40$ mm, determine (a) the angular velocity of ring B, (b) the accelerations of the points of shaft A and ring B which are in contact, (c) the magnitude of the acceleration of point D.

SOLUTION

Let Point C be the point of contact between the shaft and the ring.

$$v_C = r_1\omega_A$$

$$\omega_B = \frac{v_C}{r_2}$$

$$= \frac{r_1\omega_A}{r_2} \qquad \omega_B = \frac{r_1\omega_A}{r_2} \downarrow$$

On shaft A: $$a_A = r_1\omega_A^2 \qquad \mathbf{a}_A = r_1\omega_A^2 \downarrow$$

On ring B: $$a_B = r_2\omega_B^2 = r_2\left(\frac{r_1\omega_A}{r_2}\right)^2 \qquad \mathbf{a}_B = \frac{r_1^2\omega_A^2}{r_2} \downarrow$$

Acceleration of Point *D* on outside of ring.

$$a_D = r_3\omega_B^2 = r_3\left(\frac{r_1}{r_2}\omega_A\right)^2$$

$$a_D = r_3\left(\frac{r_1}{r_2}\right)^2\omega_A^2 \downarrow$$

Data:

$$\omega_A = 25 \text{ rad/s}$$
$$r_1 = 12 \text{ mm}$$
$$r_2 = 30 \text{ mm}$$
$$r_3 = 40 \text{ mm}$$

(a)

$$\omega_B = \frac{r_1}{r_2}\omega_A$$

$$= \frac{12 \text{ mm}}{30 \text{ mm}}(25 \text{ rad/s}) \qquad \boldsymbol{\omega}_B = 10 \text{ rad/s} \circlearrowright \blacktriangleleft$$

PROBLEM 15.27 (Continued)

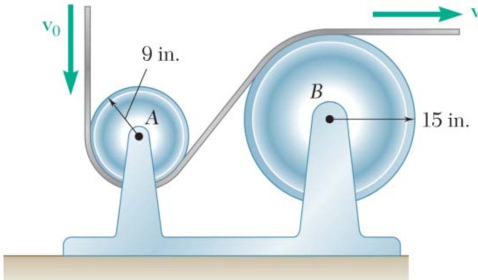
(*b*)

$$a_A = r_1\omega_A^2$$
$$= (12\text{ mm})(25\text{ rad/s})^2$$
$$= 7.5\times 10^3\text{ mm/s}^2 \qquad \mathbf{a}_A = 7.50\text{ m/s}^2 \downarrow \blacktriangleleft$$

$$a_B = \frac{r_1^2}{r_2}\omega_A^2$$
$$= \frac{(12\text{ mm})^2}{(30\text{ mm})}(25\text{ rad/s})^2$$
$$= 3\times 10^3\text{ mm/s}^2 \qquad \mathbf{a}_B = 3.00\text{ m/s}^2 \downarrow \blacktriangleleft$$

(*c*)

$$a_D = r_3\left(\frac{r_1}{r_2}\right)^2\omega_A^2$$
$$= (40\text{ mm})\left(\frac{12\text{ mm}}{30\text{ mm}}\right)^2(25\text{ rad/s})^2$$
$$a_D = 4\times 10^3\text{ mm/s}^2 \qquad \mathbf{a}_D = 4.00\text{ m/s}^2 \downarrow \blacktriangleleft$$



PROBLEM 15.28

A plastic film moves over two drums. During a 4-s interval the speed of the tape is increased uniformly from v_0 = 2 ft/s to v_1 = 4 ft/s. Knowing that the tape does not slip on the drums, determine (*a*) the angular acceleration of drum *B*, (*b*) the number of revolutions executed by drum *B* during the 4-s interval.

SOLUTION

Belt motion:

$$v = v_0 + at$$

$$4\text{ ft/s} = 2\text{ ft/s} + a(4\text{ s})$$

$$a = \frac{4\text{ ft/s} - 2\text{ ft/s}}{4\text{ s}} = 0.5\text{ ft/s}^2 = 6\text{ in./s}^2$$

Since the belt does not slip relative to the periphery of the drum, the tangential acceleration at the periphery of the drum is

$$a_t = 6\text{ in./s}^2$$

(*a*) *Angular acceleration of drum B.*

$$\alpha_B = \frac{a_t}{r_B} = \frac{6\text{ in./s}^2}{15\text{ in.}}$$

$\boldsymbol{\alpha}_B = 0.400\text{ rad/s}^2 \circlearrowright$ ◀

(*b*) *Angular displacement of drum B.*

At $t = 0$, $$\omega_0 = \frac{v_0}{r_B} = \frac{24\text{ in./s}}{15\text{ in.}} = 1.6\text{ rad/s}$$

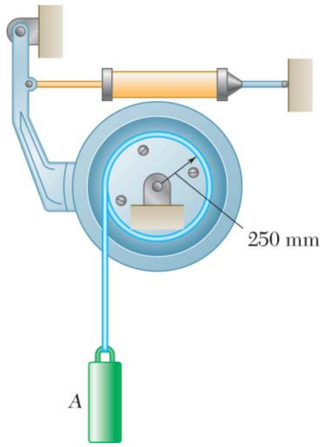
At $t = 4$ s, $$\omega_1 = \frac{v_1}{r_B} = \frac{48\text{ in./s}}{15\text{ in.}} = 3.2\text{ rad/s}$$

$$\omega_1^2 = \omega_0^2 + 2\alpha_B\theta_B$$

$$\theta_B = \frac{\omega_1^2 - \omega_0^2}{2\alpha_B} = \frac{(3.2)^2 - (1.6)^2}{(2)(0.400)} = 9.6\text{ radians}$$

In revolutions, $$\theta_B = \frac{9.6}{2\pi}$$

$\theta_B = 1.528\text{ rev}$ ◀



PROBLEM 15.29

Cylinder A is moving downward with a velocity of 3 m/s when the brake is suddenly applied to the drum. Knowing that the cylinder moves 6 m downward before coming to rest and assuming uniformly accelerated motion, determine (*a*) the angular acceleration of the drum, (*b*) the time required for the cylinder to come to rest.

SOLUTION

Given: $(v_A)_0 = 3 \text{ m/s}, \quad (v_A)_1 = 0, \quad (s_A)_1 - (s_A)_0 = 6 \text{ m}$

Constant Acceleration: $$\frac{(v_A)_1^2}{2} - \frac{(v_A)_0^2}{2} = a_A\left[(s_A)_1 - (s_A)_0\right]$$

$$a_A = \frac{(v_A)_1^2 - (v_A)_0^2}{2\left[(s_A)_1 - (s_A)_0\right]} = \frac{0 - 3^2}{(2)(6)} = -0.75 \text{ m/s}$$

(*a*) At the surface of the drum, $r = 250 \text{ mm} = 0.25 \text{ m}$

$$a_t = r\alpha \qquad \alpha = \frac{a_t}{r} = -\frac{0.75}{0.25} = -3.00 \text{ rad/s}^2$$

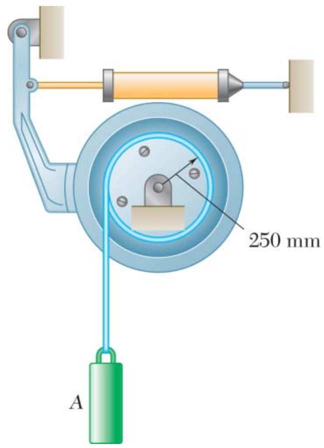
$$\alpha = 3.00 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

(*b*) $$\omega_0 = \frac{(v_A)_0}{r} = \frac{3}{0.25} = 12 \text{ rad/s}$$

$$\omega_1 = 0 = \omega_0 + \alpha t$$

$$t = \frac{\omega_1 - \omega_0}{\alpha} = \frac{0 - 12}{-3.00} = 4.00 \text{ s}$$

$$t = 4.00 \text{ s} \blacktriangleleft$$



PROBLEM 15.30

The system shown is held at rest by the brake-and-drum system shown. After the brake is partially released at $t = 0,$ it is observed that the cylinder moves 5 m in 4.5 s. Assuming uniformly accelerated motion, determine (*a*) the angular acceleration of the drum, (*b*) the angular velocity of the drum at $t = 3.5$ s.

SOLUTION

(*a*) Assume uniformly accelerated motion. $s_A = \frac{1}{2} a_A t^2$

$$a_A = \frac{2s_A}{t^2} = \frac{(2)(5)}{4.5^2} = 0.49383 \text{ m/s}^2 \downarrow$$

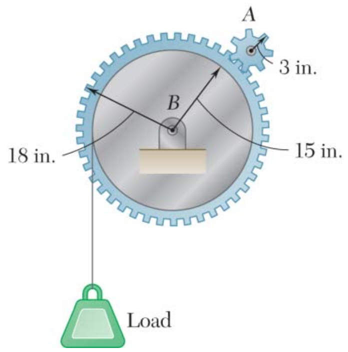
For the drum, $a_t = a_A = 0.49383 \text{ m/s}^2, \quad r = 250 \text{ mm} = 0.25 \text{ m}$

$$a_t = r\alpha, \quad \alpha = \frac{a_t}{r} = \frac{0.49383}{0.25}$$

$$= 1.97531 \text{ rad/s}^2, \qquad \alpha = 1.975 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

(*b*) $\omega = \omega_0 + \alpha t = 0 + (1.97531)(3.5)$

$$= 6.91 \text{ rad/s}, \qquad \omega = 6.91 \text{ rad/s} \circlearrowright \blacktriangleleft$$



PROBLEM 15.31

A load is to be raised 20 ft by the hoisting system shown. Assuming gear *A* is initially at rest, accelerates uniformly to a speed of 120 rpm in 5 s, and then maintains a constant speed of 120 rpm, determine (*a*) the number of revolutions executed by gear *A* in raising the load, (*b*) the time required to raise the load.

SOLUTION

The load is raised a distance $h = 20$ ft $= 240$ in.

For gear-pulley *B*, radius to rope groove is $r_1 = 15$ in.

Required angle change for *B*: $$\theta_B = \frac{h}{r_1} = \frac{240}{15} = 16 \text{ radians}$$

Circumferential travel of gears *A* and *B*:

$$s = r_2\theta_B = r_A\theta_A \qquad \text{where } r_2 = 18 \text{ in. and } r_A = 3 \text{ in.}$$

$$s = (18 \text{ in.})(16 \text{ radians}) = 288 \text{ in.}$$

(*a*) *Angle change of gear A*: $$\theta_A = \frac{s}{r_A} = \frac{288}{3} = 96 \text{ radians}$$

In revolutions, $$\theta_A = \frac{96}{2\pi}$$ $\theta_A = 15.28$ rev ◀

(*b*) *Motion of gear A*. $$\omega_0 = 0, \quad \omega_f = 120 \text{ rpm} = 4\pi \text{ rad/s}$$

Gear *A* is uniformly accelerated over the first 5 seconds.

$$\alpha = \frac{\omega_f - \omega_0}{t} = \frac{4\pi \text{ rad/s}}{5 \text{ s}} = 2.5133 \text{ rad/s}^2$$

$$\theta_A = \frac{1}{2}\alpha t^2 = \frac{1}{2}(2.5133)(5)^2 = 31.416 \text{ radians}$$

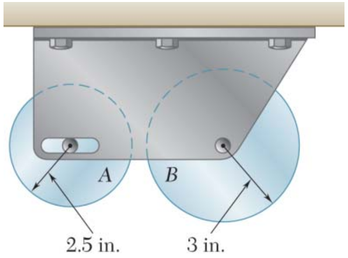
The angle change over the constant speed phase is

$$\Delta\theta = \theta_A - \theta = 96 - 31.416 = 64.584 \text{ radians}$$

For uniform motion, $$\Delta\theta = \omega_f(\Delta t)$$

$$\Delta t = \frac{\Delta\theta}{\omega_f} = \frac{64.584}{4\pi} = 5.139 \text{ s}$$

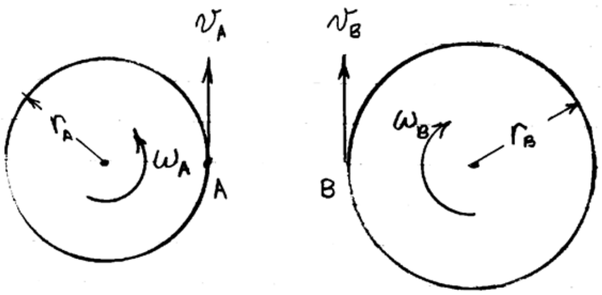
Total time elapsed: $$t_f = 5 \text{ s} + \Delta t$$ $t_f = 10.14$ s ◀



PROBLEM 15.32

A simple friction drive consists of two disks *A* and *B*. Initially, disk *B* has a clockwise angular velocity of 500 rpm, and disk *A* is at rest. It is known that disk *B* will coast to rest in 60 s. However, rather than waiting until both disks are at rest to bring them together, disk *A* is given a constant angular acceleration of 3 rad/s^2 counterclockwise. Determine (*a*) at what time the disks can be brought together if they are not to slip, (*b*) the angular velocity of each disk as contact is made.

SOLUTION



Motion of disk B: $(\omega_B)_0 = 500 \text{ rpm} = 52.360 \text{ rad/s}$

Assume that the angular acceleration of disk *B* is constant.

$$\omega_B = (\omega_B)_0 + \alpha_B t$$

At $t = 60$ s, $\omega_B = 0$

$$\alpha_B = \frac{\omega_B - (\omega_B)_0}{t} = \frac{0 - 52.360}{60} = -0.87266 \text{ rad/s}^2$$

$$v_B = r_B\omega_B = (3)(52.360 - 0.87266\,t) = 157.08 - 2.618t \text{ in./s}$$

Motion of disk A: $(\omega_A)_0 = 0, \ \alpha_A = 3 \text{ rad/s}^2$

$$\omega_A = (\omega_A)_0 + \alpha_A t = 3t$$

$$v_A = r_A\omega_A = (2.5)(3t) = 7.5t \text{ in./s}$$

PROBLEM 15.32 (Continued)

If disks are not to slip, $v_A = v_B$

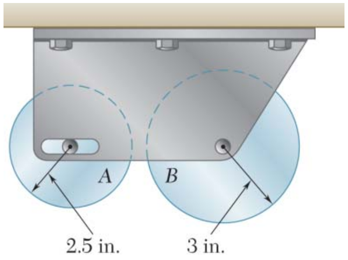
$$7.5t = 157.08 - 2.618t$$

(*a*) $t = 15.52\text{ s}$ ◀

(*b*) $\omega_A = (3)(15.52) = 46.6\text{ rad/s}$ ↺

$\omega_B = 52.360 - (0.87266)(15.52) = 38.8\text{ rad/s}$ ↻

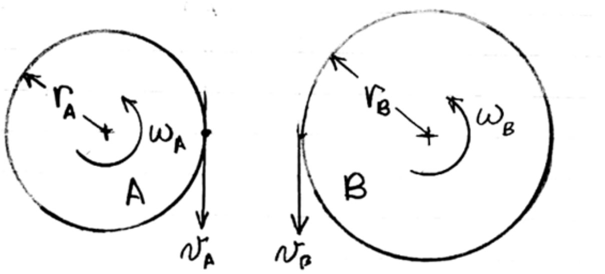
$\boldsymbol{\omega}_A = 445\text{ rpm}$ ↺, $\boldsymbol{\omega}_B = 371\text{ rpm}$ ↻ ◀



PROBLEM 15.33

Two friction wheels *A* and *B* are both rotating freely at 300 rpm counterclockwise when they are brought into contact. After 12 s of slippage, during which time each wheel has a constant angular acceleration, wheel *B* reaches a final angular velocity of 75 rpm counterclockwise. Determine (*a*) the angular acceleration of each wheel during the period of slippage, (*b*) the time at which the angular velocity of wheel *A* is equal to zero.

SOLUTION



Motion of disk B: $(\omega_B)_0 = 300 \text{ rpm} = 31.416 \text{ rad/s}$ ↺

At $t = 12\text{s}, \quad \omega_B = 75 \text{ rpm} = 7.854 \text{ rad/s}$ ↺

Angular acceleration.

$$\alpha_B = \frac{\omega_B - (\omega_B)_0}{t} = \frac{7.854 - 31.416}{12} = -1.9635 \text{ rad/s}$$

Velocity at contact point with disk *A* at $t = 12$ s:

$$v_B = r_B \omega_B = (3)(7.854) = 23.562 \text{ in./s} \downarrow$$

Motion of disk A: $(\omega_A)_0 = 300 \text{ rpm} = 31.416 \text{ rad/s}$ ↺

Assume that slipping ends when $t = 12$ s.

Then, $v_A = 23.562 \text{ in./s} \downarrow$

$$|\omega_A| = \frac{v_A}{r_A} = \frac{23.562}{2.5} = 9.4248 \text{ rad/s}$$

$$\omega_A = 9.4248 \text{ rad/s} \text{ ↻} = -9.4248 \text{ rad/s} \text{ ↺}$$

$$\alpha_A = \frac{\omega_A - (\omega_A)_0}{t} = \frac{-9.4248 - 31.416}{12} = -3.4034 \text{ rad/s}^2 \text{ ↺}$$

(*a*) $\boldsymbol{\alpha}_A = 3.40 \text{ rad/s}^2$ ↻ ◀

$\boldsymbol{\alpha}_B = 1.963 \text{ rad/s}^2$ ↻ ◀

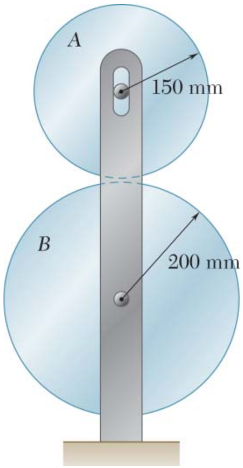
PROBLEM 15.33 (Continued)

(*b*) Time when ω_A is zero.

$$\omega_A = (\omega_A)_0 + \alpha_A t = 0$$

$$t = \frac{\omega_A - (\omega_A)_0}{\alpha_A} = \frac{0 - 31.416}{-3.4034}$$

$t = 9.23$ s ◀



PROBLEM 15.34

Two friction disks *A* and *B* are to be brought into contact without slipping when the angular velocity of disk *A* is 240 rpm counterclockwise. Disk *A* starts from rest at time $t = 0$ and is given a constant angular acceleration of magnitude α. Disk *B* starts from rest at time $t = 2$ s and is given a constant clockwise angular acceleration, also of magnitude α. Determine (*a*) the required angular acceleration magnitude α, (*b*) the time at which the contact occurs.

SOLUTION

When contact is made: $\omega_A = 240 \text{ rpm} = 8\pi \text{ rad/s} \circlearrowleft$

Let *C* be the contact point between the two gears.

$$v_C = r_A\omega_A = (0.15)(8\pi) = 1.2\pi \text{ m/s}^2 \longrightarrow$$

$$\omega_B = \frac{v_C}{r_B} = \frac{1.2\pi}{0.2} = 6\pi \text{ rad/s} \circlearrowright$$

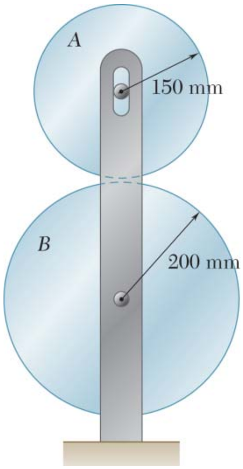
$$\omega_A = 8\pi = \alpha t \text{ rad/s}$$

$$\omega_B = 6\pi = \alpha(t-2) \text{ rad/s}$$

Subtracting, $2\pi = (\alpha)(2) \qquad \alpha = \pi \text{ rad/s}^2$

(*a*) $\alpha = 3.14 \text{ rad/s}^2$ ◀

(*b*) $t = \dfrac{8\pi}{\alpha} = \dfrac{8\pi}{\pi} = 8 \text{ s}$ $\qquad t = 8.00 \text{ s}$ ◀



PROBLEM 15.35

Two friction disks *A* and *B* are brought into contact when the angular velocity of disk *A* is 240 rpm counterclockwise and disk *B* is at rest. A period of slipping follows and disk *B* makes 2 revolutions before reaching its final angular velocity. Assuming that the angular acceleration of each disk is constant and inversely proportional to the cube of its radius, determine (*a*) the angular acceleration of each disk, (*b*) the time during which the disks slip.

SOLUTION

Given:

$$\circlearrowleft \; (\omega_A)_0 = 240 \text{ rpm} = 8\pi \text{ rad/s} \qquad \circlearrowleft \; (\omega_A)_1 = 8\pi - \alpha_A t_1$$

$$\circlearrowright \; \theta_B = 4\pi = \frac{1}{2}\alpha_B t_1^2 = \frac{1}{2}\left(\frac{r_A}{r_B}\right)^3 \alpha_A t_1^2 = \frac{1}{2}\left(\frac{0.15}{0.2}\right)^3 \alpha_A t_1^2$$

$$\alpha_A t_1^2 = (8\pi)\left(\frac{0.2}{0.15}\right)^3 = 59.574 \text{ radians}$$

$$\circlearrowright \; (\omega_B)_1 = \alpha_B t_1 = \left(\frac{0.15}{0.2}\right)^3 \alpha_A t_1 = 0.421875\alpha_A t_1$$

Let v_C be the velocity at the contact point.

$$v_C = r_A\omega_A = (0.15)(8\pi - \alpha_A t_1) = 1.2\pi - 0.15\alpha_A t_1$$

and

$$v_C = r_B\omega_B = (0.2)(0.421875\alpha_A t_1) = 0.084375\alpha_A t_1$$

Equating the two expressions for v_C,

$$1.2\pi - 0.15\alpha_A t_1 = 0.084375\alpha_A t_1 \qquad \text{or} \qquad \alpha_A t_1 = 16.0850 \text{ rad/s}$$

Then,

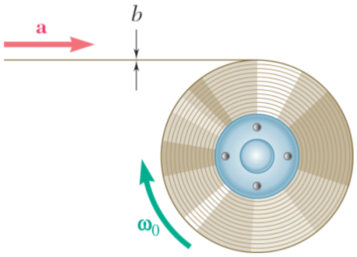
$$t_1 = \frac{\alpha_A t_1^2}{\alpha_A t_1} = \frac{59.574}{16.0850} = 3.7037 \text{ s}$$

(*a*)

$$\alpha_A = \frac{16.0850}{3.7037} = 4.3429 \text{ rad/s}^2 \circlearrowright \qquad \alpha_A = 4.34 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

$$\alpha_B = \left(\frac{0.15}{0.2}\right)^3 (4.3429) = 1.83218 \text{ rad/s}^2 \circlearrowright \qquad \alpha_B = 1.832 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

(*b*) From above, $t_1 = 3.70 \text{ s} \blacktriangleleft$



PROBLEM 15.36*

Steel tape is being wound onto a spool that rotates with a constant angular velocity ω_0. Denoting by r the radius of the spool and tape at any given time and by b the thickness of the tape, derive an expression for the acceleration of the tape as it approaches the spool.

SOLUTION

Let one layer of tape be wound and let v be the tape speed.

$$v\Delta t = 2\pi r \quad \text{and} \quad \Delta r = b$$

$$\frac{\Delta r}{\Delta t} = \frac{bv}{2\pi r} = \frac{b\omega}{2\pi}$$

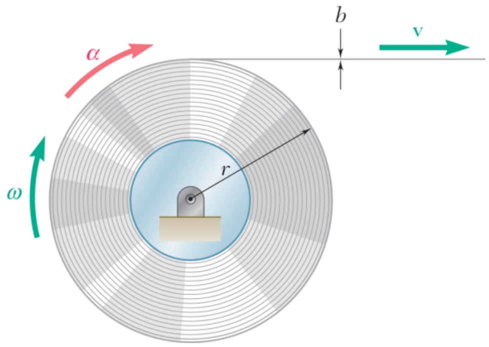
For the spool:

$$\frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{v}{r}\right) = \frac{1}{r}\frac{dv}{dt} + v\frac{d}{dt}\left(\frac{1}{r}\right)$$

$$= \frac{a}{r} - \frac{v}{r^2}\frac{dr}{dt} = \frac{a}{r} - \frac{v}{r^2}\frac{b\omega}{2\pi}$$

$$= \frac{1}{r}\left[a - \frac{b\omega^2}{2\pi}\right] = 0$$

$$\mathbf{a} = \frac{b\omega_0^2}{2\pi} \longrightarrow \blacktriangleleft$$



PROBLEM 15.37*

In a continuous printing process, paper is drawn into the presses at a constant speed v. Denoting by r the radius of the paper roll at any given time and by b the thickness of the paper, derive an expression for the angular acceleration of the paper roll.

SOLUTION

Let one layer of paper be unrolled.

$$v\Delta t = 2\pi r \quad \text{and} \quad \Delta r = -b$$

$$\frac{\Delta r}{\Delta t} = \frac{-bv}{2\pi r} = \frac{dr}{dt}$$

$$\begin{aligned}\alpha &= \frac{d\omega}{dt} \\ &= \frac{d}{dt}\left(\frac{v}{r}\right) \\ &= \frac{1}{r}\frac{dv}{dt} + v\frac{d}{dt}\left(\frac{1}{r}\right) \\ &= 0 - \frac{v}{r^2}\frac{dr}{dt} \\ &= \left(-\frac{v}{r^2}\right)\left(\frac{-bv}{2\pi r}\right) \\ &= \frac{bv^2}{2\pi r^3}\end{aligned}$$

$$\alpha = \frac{bv^2}{2\pi r^3} \circlearrowright \blacktriangleleft$$

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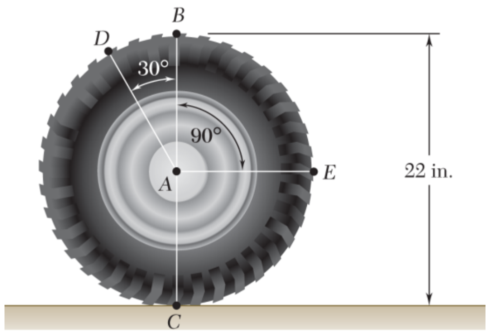


PROBLEM 15.38

An automobile travels to the right at a constant speed of 48 mi/h. If the diameter of a wheel is 22 in., determine the velocities of Points *B*, *C*, *D*, and *E* on the rim of the wheel.

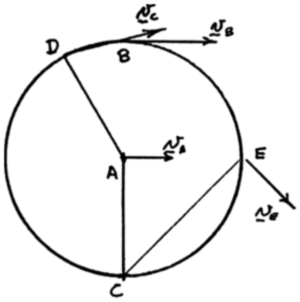
SOLUTION

$$\mathbf{v}_A = 48 \text{ mi/h} = 70.4 \text{ ft/s}$$

$$\mathbf{v}_C = 0$$ ◀

$$d = 22 \text{ in.} \quad r = \frac{d}{2} = 11 \text{ in.} = 0.91667 \text{ ft}$$

$$\omega = \frac{v_A}{r} = \frac{70.4}{0.91667} = 76.8 \text{ rad/s} \circlearrowright$$



$$v_{B/A} = v_{D/A} = v_{E/A} = r\omega$$

$$= (0.91667)(76.8) = 70.4 \text{ ft/s}$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = [70.4 \text{ ft/s} \longrightarrow] + [70.4 \text{ ft/s} \longrightarrow]$$

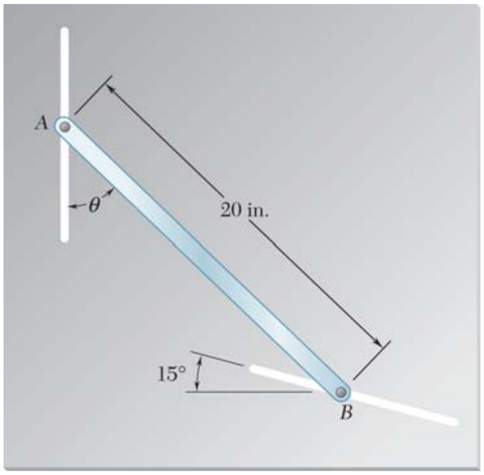
$$\mathbf{v}_B = 140.8 \text{ ft/s} \longrightarrow$$ ◀

$$\mathbf{v}_D = \mathbf{v}_A + \mathbf{v}_{D/A} = [70.4 \text{ ft/s} \longrightarrow] + [70.4 \text{ ft/s} \nearrow 30°]$$

$$\mathbf{v}_D = 136.0 \text{ ft/s} \nearrow 15.0°$$ ◀

$$\mathbf{v}_E = \mathbf{v}_A + \mathbf{v}_{E/A} = [70.4 \text{ ft/s} \longrightarrow] + [70.4 \text{ ft/s} \downarrow]$$

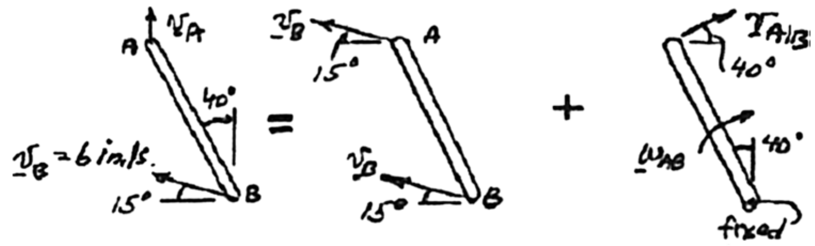
$$\mathbf{v}_E = 99.6 \text{ ft/s} \searrow 45.0°$$ ◀



PROBLEM 15.39

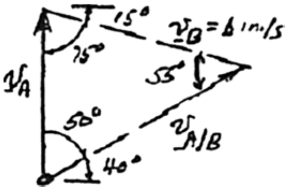
The motion of rod *AB* is guided by pins attached at *A* and *B* which slide in the slots shown. At the instant shown, $\theta = 40°$ and the pin at *B* moves upward to the left with a constant velocity of 6 in./s. Determine (*a*) the angular velocity of the rod, (*b*) the velocity of the pin at end *A*.

SOLUTION


v_A
A
40°
v_B = 6 in./s
15°
B
=
v_B
A
15°
v_B
15°
B
+
$v_{A/B}$
40°
ω_{AB}
40°
fixed

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$[v_A \uparrow] = [6 \text{ in./s} \measuredangle 15°] + [v_{A/B} \measuredangle 40°]$$


15°
v_B = 6 in./s
75°
55°
v_A
50°
$v_{A/B}$
40°

Law of sines.

$$\frac{v_A}{\sin 55°} = \frac{v_{A/B}}{\sin 75°} = \frac{6 \text{ in./s}}{\sin 50°}$$

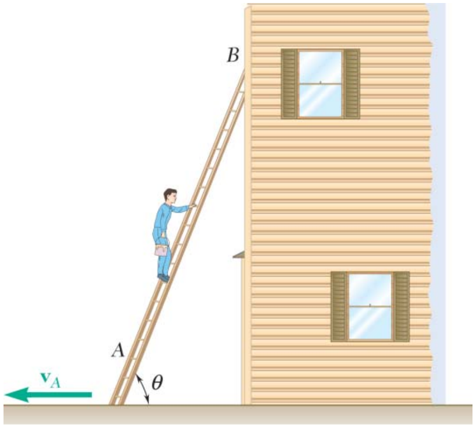
(*b*) $\mathbf{v}_A = 6.42 \text{ in./s} \uparrow$ ◀

$$\mathbf{v}_{A/B} = 7.566 \text{ in./s} \measuredangle 40°$$

$$v_{A/B} = (AB)\omega_{AB} \quad AB = 20 \text{ in.}$$

$$7.566 \text{ in./s} = (20 \text{ in.})\omega_{AB}$$

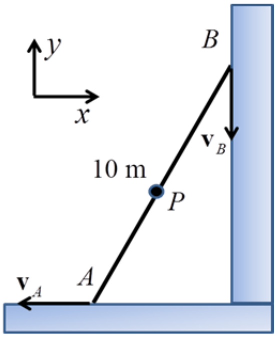
(*a*) $\omega_{AB} = 0.3783 \text{ rad/s}$ $\boldsymbol{\omega}_{AB} = 0.378 \text{ rad/s} \circlearrowright$ ◀



PROBLEM 15.40

A painter is halfway up a 10-m ladder when the bottom starts sliding out from under him. Knowing that point A has a velocity $\mathbf{v}_A = 2$ m/s directed to the left when $\theta = 60°$, determine (*a*) the angular velocity of the ladder, (*b*) the velocity of the painter.

SOLUTION



Given: $\mathbf{v}_A = -2\mathbf{i}$ m/s, $\theta = 60°$

Geometry:

$$\mathbf{r}_{B/A} = 10\cos 60°\mathbf{i} + 10\sin 60°\mathbf{j}$$
$$\mathbf{r}_{P/A} = 5\cos 60°\mathbf{i} + 5\sin 60°\mathbf{j}$$

Relative Velocity:

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \omega_{AB}\mathbf{k} \times \mathbf{r}_{B/\mathrm{A}}$$
$$v_B\mathbf{j} = -2\mathbf{i} + \omega_{AB}\mathbf{k} \times (5\mathbf{i} + 8.660\mathbf{j})$$
$$= -(2 + 8.660\omega_{AB})\mathbf{i} + 5\omega_{AB}\mathbf{j}$$

Equate Components:

(*a*) $\mathbf{i}$:

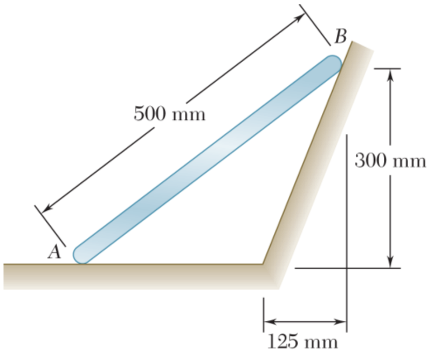
$$0 = -(2 + 8.660\omega_{AB})$$
$$\omega_{AB} = -0.231 \text{ rad/s}$$

$\boldsymbol{\omega}_{AB} = 0.231$ rad/s ↻ ◀

Velocity of Painter (point P):

$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{P/A} = \mathbf{v}_A + \omega_{AB}\mathbf{k} \times \mathbf{r}_{\mathrm{P/A}}$$
$$= -2\mathbf{i} - 0.231\mathbf{k} \times (2.5\mathbf{i} + 4.330\mathbf{j})$$

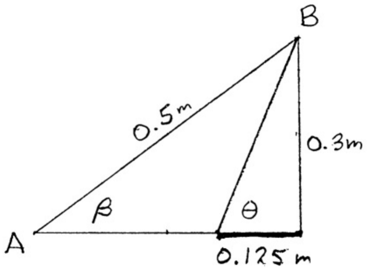
(*b*) $\mathbf{v}_P = -(1.00 \text{ m/s})\mathbf{i} - (0.577 \text{ m/s})\mathbf{j}$ ◀



PROBLEM 15.41

Rod AB can slide freely along the floor and the inclined plane. At the instant shown the velocity of end A is 1.4 m/s to the left. Determine (a) the angular velocity of the rod, (b) the velocity of end B of the rod.

SOLUTION



Geometry:

$$\sin\beta = \frac{0.3}{0.5}, \qquad \beta = 36.87°$$

$$\tan\theta = \frac{0.3}{0.125}, \qquad \theta = 67.38°$$

Velocity analysis:

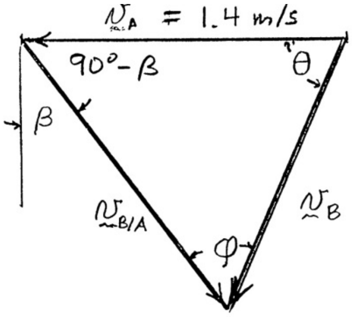
$$\mathbf{v}_A = 1.4 \text{ m/s} \longleftarrow$$

$$\mathbf{v}_{B/A} = r\omega_{AB} = 0.5\omega_{AB} \searrow \beta$$

$$\mathbf{v}_B = v_B \nearrow \theta$$

Plane motion = Translation with A + Rotation about A.

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$



Draw velocity vector diagram.

$$\varphi = 180° - \theta - (90° - \beta) = 59.49°$$

Law of sines:

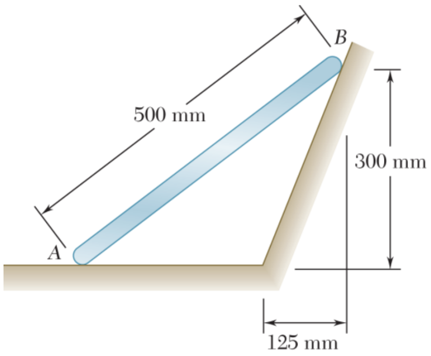
$$\frac{v_{B/A}}{\sin\theta} = \frac{v_B}{\sin(90° - \beta)} = \frac{v_A}{\sin\varphi}$$

$$v_{B/A} = \frac{v_A\sin\theta}{\sin\varphi} = \frac{1.4\sin 67.38°}{\sin 59.49°} = 1.5 \text{ m/s}$$

(a) $$\omega_{AB} = \frac{1.5}{0.5} = 3.00 \text{ rad/s} \qquad \omega_{AB} = 3.00 \text{ rad/s} \circlearrowright \blacktriangleleft$$

(b) $$v_B = \frac{v_A\cos\beta}{\sin\varphi} = \frac{1.4\cos 36.87°}{\sin 59.49°} = 1.3 \text{ m/s}$$

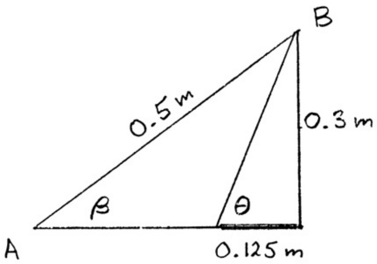
$$\mathbf{v}_B = 1.300 \text{ m/s} \swarrow 67.4° \blacktriangleleft$$



PROBLEM 15.42

Rod *AB* can slide freely along the floor and the inclined plane. At the instant shown the angular velocity of the rod is 4.2 rad/s counterclockwise. Determine (*a*) the velocity of end *A* of the rod, (*b*) the velocity of end *B* of the rod.

SOLUTION



Geometry:

$$\sin\beta = \frac{0.3}{0.5}, \qquad \beta = 36.87^\circ$$

$$\tan\theta = \frac{0.3}{0.125}, \qquad \theta = 67.38^\circ$$

Velocity analysis:

$$\omega_{AB} = 4.2 \text{ rad/s} \circlearrowleft$$

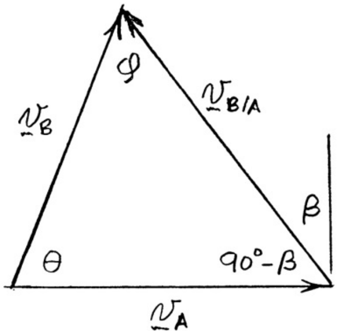
$$\mathbf{v}_{B/A} = r_{B/A}\omega_{AB} = (0.5)(4.2) = 2.1 \text{ m/s} \searrow \beta$$

$$\mathbf{v}_B = v_B \measuredangle \theta$$

$$\mathbf{v}_A = v_A \longrightarrow$$

Plane motion = Translation with *A* + Rotation about *A*.

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$



Draw velocity vector diagram.

$$\varphi = 180^\circ - \theta - (90^\circ - \beta) = 59.49^\circ$$

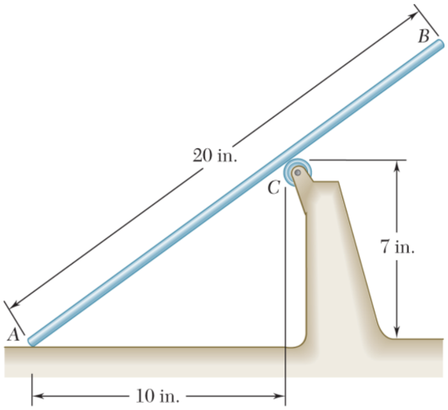
Law of sines: $\dfrac{v_A}{\sin\varphi} = \dfrac{v_B}{\sin(90^\circ - \beta)} = \dfrac{v_{B/A}}{\sin\theta}$

(*a*) $$v_A = \frac{v_{B/A}\sin\varphi}{\sin\theta} = \frac{2.1\sin 59.49^\circ}{\sin 67.38^\circ} = 1.96 \text{ m/s}$$

$$\mathbf{v}_A = 1.960 \text{ m/s} \longrightarrow \blacktriangleleft$$

(*b*) $$v_B = \frac{v_{B/A}\cos\beta}{\sin\theta} = \frac{2.1\cos 36.87^\circ}{\sin 67.38^\circ} = 1.82 \text{ m/s}$$

$$\mathbf{v}_B = 1.820 \text{ m/s} \measuredangle 67.4^\circ \blacktriangleleft$$



PROBLEM 15.43

Rod *AB* moves over a small wheel at *C* while end *A* moves to the right with a constant velocity of 25 in./s. At the instant shown, determine (*a*) the angular velocity of the rod, (*b*) the velocity of end *B* of the rod.

SOLUTION

Slope angle of rod. $\tan\theta = \dfrac{7}{10} = 0.7,$ $\theta = 35°$

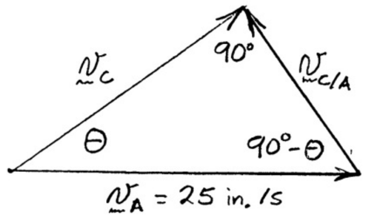
$$\overline{AC} = \frac{10}{\cos\theta} = 12.2066 \text{ in.} \qquad \overline{CB} = 20 - \overline{AC} = 7.7934 \text{ in.}$$

Velocity analysis.

$$\mathbf{v}_A = 25 \text{ in./s} \longrightarrow, \qquad \mathbf{v}_C = v_C \measuredangle\, \theta$$

$$\mathbf{v}_{C/A} = \overline{AC}\omega_{AB} \searrow \theta$$

$$\mathbf{v}_C = \mathbf{v}_A + v_{C/A}$$



Draw corresponding vector diagram.

$$v_{C/A} = v_A \sin\theta = 25\sin 35° = 14.34 \text{ in./s}$$

(*a*) $$\omega_{AB} = \frac{v_{C/A}}{AC} = \frac{14.34}{12.2066} = 1.175 \text{ rad/s}$$

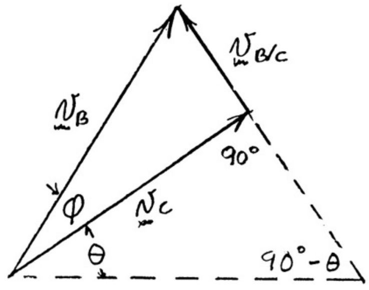
$$\boldsymbol{\omega}_{AB} = 1.175 \text{ rad/s} \circlearrowleft \blacktriangleleft$$

$$v_C = v_A \cos\theta = 25\cos\theta = 20.479 \text{ in./s}$$

$$\mathrm{v}_{B/C} = \overline{CB}\omega_{AB} = (7.7934)(1.175) = 9.1551 \text{ in./s}$$

$v_{B/C}$ has same direction as $v_{C/A}$.

$$\mathbf{v}_B = \mathbf{v}_C + \mathbf{v}_{B/C}$$



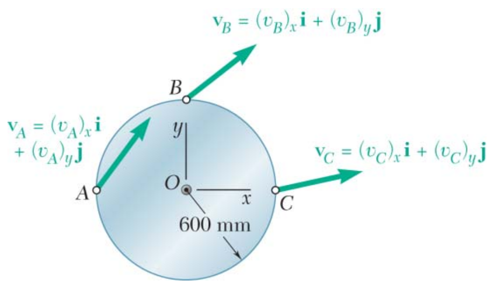
Draw corresponding vector diagram.

$$\tan\phi = \frac{v_{B/C}}{v_C} = \frac{9.1551}{20.479}, \quad \phi = 24.09°$$

(*b*) $$v_B = \frac{v_C}{\cos\phi} = \frac{20.479}{\cos 24.09°} = 22.4 \text{ in./s} = 1.869 \text{ ft/s}$$

$$\phi + \theta = 59.1°$$

$$\mathbf{v}_B = 1.869 \text{ ft/s} \measuredangle\, 59.1° \blacktriangleleft$$


$\mathbf{v}_B = (v_B)_x\mathbf{i} + (v_B)_y\mathbf{j}$
B
$\mathbf{v}_A = (v_A)_x\mathbf{i} + (v_A)_y\mathbf{j}$
y
$\mathbf{v}_C = (v_C)_x\mathbf{i} + (v_C)_y\mathbf{j}$
O
A
x
C
600 mm

PROBLEM 15.44

The disk shown moves in the *xy* plane. Knowing that $(v_A)_y = -7$ m/s, $(v_B)_x = -7.4$ m/s, and $(v_C)_x = -1.4$ m/s, determine (*a*) the angular velocity of the disk, (*b*) the velocity of point *B*.

SOLUTION

In units of m/s,

$$\mathbf{v}_{B/A} = \omega\mathbf{k} \times \mathbf{r}_{B/A} = \omega\mathbf{k} \times (0.6\mathbf{i} + 0.6\mathbf{j}) = -0.6\omega\mathbf{i} + 0.6\omega\mathbf{j}$$

$$\mathbf{v}_{C/A} = \omega\mathbf{k} \times \mathbf{r}_{C/A} = \omega\mathbf{k} \times 1.2\mathbf{i} = 1.2\omega\mathbf{j}$$

$$\mathbf{v}_{\mathbf{B}} = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$-7.4\mathbf{i} + (v_B)_y\,\mathbf{j} = (v_A)_x\,\mathbf{i} - 7\mathbf{j} - 0.6\omega\mathbf{i} + 0.6\omega\mathbf{j}$$

Components.

$$\mathbf{i}:\quad -7.4 = (v_A)_x - 0.6\omega \qquad (1)$$

$$\mathbf{j}:\quad (v_B)_y = -7 + 0.6\omega \qquad (2)$$

$$\mathbf{v}_{\mathbf{C}} = \mathbf{v}_A + \mathbf{v}_{C/A}$$

$$-1.4\mathbf{i} + (v_C)_y\,\mathbf{j} = (v_A)_x\,\mathbf{i} - 7\mathbf{j} + 1.2\omega\mathbf{j}$$

Components.

$$\mathbf{i}:\quad -1.4 = (v_A)_x \qquad (3)$$

$$\mathbf{j}:\quad (v_C)_y = -7 + 1.2\omega \qquad (4)$$

From (3),

$$(v_A)_x = -1.4\ \text{m/s}$$

(*a*) From (1),

$$\omega = \frac{-7.4 - (-1.4)}{-0.6} = 10\ \text{rad/s,}$$

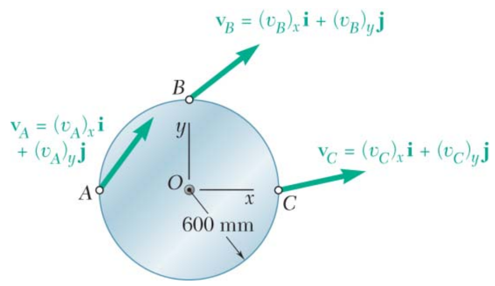
$\boldsymbol{\omega} = 10.00\ \text{rad/s}\ \circlearrowleft$ ◀

From (2),

$$(v_B)_y = -7 + (0.6)(10) = -1\ \text{m/s}$$

(*b*)

$\mathbf{v}_B = -(7.40\ \text{m/s})\mathbf{i} - (1.000\ \text{m/s})\mathbf{j}$ ◀


$\mathbf{v}_B = (v_B)_x\mathbf{i} + (v_B)_y\mathbf{j}$
B
$\mathbf{v}_A = (v_A)_x\mathbf{i} + (v_A)_y\mathbf{j}$
y
O
A
x
C
$\mathbf{v}_C = (v_C)_x\mathbf{i} + (v_C)_y\mathbf{j}$
600 mm

PROBLEM 15.45

The disk shown moves in the *xy* plane. Knowing that $(v_A)_y = -7$ m/s, $(v_B)_x = -7.4$ m/s, and $(v_C)_x = -1.4$ m/s, , determine (*a*) the velocity of point *O*, (*b*) the point of the disk with zero velocity.

SOLUTION

In units of m/s,

$$\mathbf{v}_{B/A} = \omega\mathbf{k} \times \mathbf{r}_{B/A} = \omega\mathbf{k} \times (0.6\mathbf{i} + 0.6\mathbf{j}) = -0.6\omega\mathbf{i} + 0.6\omega\mathbf{j}$$

$$\mathbf{v}_{C/A} = \omega\mathbf{k} \times \mathbf{r}_{C/A} = \omega\mathbf{k} \times 1.2\mathbf{i} = 1.2\omega\mathbf{j}$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$-7.4\mathbf{i} + (v_B)_y\mathbf{j} = (v_A)_x\mathbf{i} - 7\mathbf{j} - 0.6\omega\mathbf{i} + 0.6\omega\mathbf{j}$$

Components.

$$\mathbf{i}: \quad -7.4 = (v_A)_x - 0.6\omega \qquad (1)$$

$$\mathbf{j}: \quad (v_B)_y = -7 + 0.6\omega \qquad (2)$$

$$\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{C/A}$$

$$-1.4\mathbf{i} + (v_C)_y = (v_A)_x\mathbf{i} - 7\mathbf{j} + 1.2\omega\mathbf{j}$$

Components.

$$\mathbf{i}: \quad -1.4 = (v_A)_x \qquad (3)$$

$$\mathbf{j}: \quad (v_C)_y = -7 + 1.2\omega \qquad (4)$$

From (3), $(v_A)_x = -1.4$ m/s, $\quad \mathbf{v}_A = -1.4\mathbf{i} - 7\mathbf{j}$

From (1), $\omega = \dfrac{-7.4 - (-1.4)}{-0.6} = 10$ rad/s, $\quad \boldsymbol{\omega} = (10.00 \text{ rad/s})\mathbf{k}$

(*a*)

$$\mathbf{v}_O = \mathbf{v}_A + \mathbf{v}_{O/A} = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{O/A} = \mathbf{v}_A + 10\mathbf{k} \times (0.6\mathbf{i})$$

$$= -1.4\mathbf{i} - 7\mathbf{j} + 6\mathbf{j} = -1.4\mathbf{i} - 1\mathbf{j}$$

$$\mathbf{v}_O = -(1.400 \text{ m/s})\mathbf{i} - (1.000 \text{ m/s})\mathbf{j} \blacktriangleleft$$

(*b*)

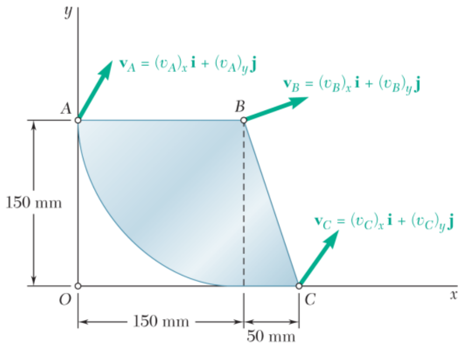
$$0 = \mathbf{v}_O + \boldsymbol{\omega} \times (x\mathbf{i} + y\mathbf{j})$$

$$0 = -1.4\mathbf{i} - 1\mathbf{j} + 10\mathbf{k} \times (x\mathbf{i} + y\mathbf{j}) = -1.4\mathbf{i} - 1\mathbf{j} + 10x\mathbf{j} - 10y\mathbf{j}$$

Components.

$$\mathbf{i}: \quad 0 = -1.4 - 10y, \quad y = -0.14 \text{ m} \qquad y = -140.0 \text{ mm} \blacktriangleleft$$

$$\mathbf{j}: \quad 0 = -1 + 10x, \quad x = 0.1 \text{ m} \qquad x = 100.0 \text{ mm} \blacktriangleleft$$



y
$\mathbf{v}_A = (v_A)_x\,\mathbf{i} + (v_A)_y\,\mathbf{j}$
$\mathbf{v}_B = (v_B)_x\,\mathbf{i} + (v_B)_y\,\mathbf{j}$
A
B
150 mm
$\mathbf{v}_C = (v_C)_x\,\mathbf{i} + (v_C)_y\,\mathbf{j}$
O
C
x
150 mm
50 mm

PROBLEM 15.46

The plate shown moves in the xy plane. Knowing that $(v_A)_x = 250$ mm/s, $(v_B)_x = -450$ mm/s, and $(v_C)_x = -500$ mm/s, determine (*a*) the angular velocity of the plate, (*b*) the velocity of Point A.

SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega\mathbf{k}$

Relative position vectors: $\mathbf{r}_{B/A} = (150\text{ mm})\mathbf{i}$

$$\mathbf{r}_{C/A} = (200\text{ mm})\mathbf{i} + (150\text{ mm})\mathbf{j}$$

Velocity vectors: $\mathbf{v}_A = (250\text{ mm/s})\mathbf{i} + (v_A)_y\,\mathbf{j}$

$$\mathbf{v}_B = (v_B)_x\,\mathbf{i} - (450\text{ mm/s})\mathbf{j}$$

$$\mathbf{v}_C = -(500\text{ mm/s})\mathbf{i} + (v_C)_y\,\mathbf{j}$$

Unknowns are ω, $(v_A)_y$, $(v_B)_x$, and $(v_C)_y$.

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{B/A}$$

$$(v_B)_x\,\mathbf{i} - 450\mathbf{j} = 250\mathbf{i} + (v_A)_y\,\mathbf{j} + \omega\mathbf{k} \times 150\mathbf{i}$$

$$= 250\mathbf{i} + (v_A)_y\,\mathbf{j} + 150\omega\mathbf{j}$$

$\mathbf{i}$: $$(v_B)_x = 250 \qquad (1)$$

$\mathbf{j}$: $$-450 = (v_A)_y + 150\omega \qquad (2)$$

$$\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{C/A} = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{C/A}$$

$$-500\mathbf{i} + (v_C)_y\,\mathbf{j} = 250\mathbf{i} + (v_A)_y\,\mathbf{j} + \omega\mathbf{k} \times (200\mathbf{i} - 150\mathbf{j})$$

$$= 250\mathbf{i} + (v_A)_y\,\mathbf{j} + 200\omega\mathbf{j} + 150\omega\mathbf{i}$$

$\mathbf{i}$: $$-500 = 250 + 150\omega \qquad (3)$$

$\mathbf{j}$: $$(v_C)_y = (v_A)_y + 150\omega \qquad (4)$$

(*a*) *Angular velocity of the plate.*

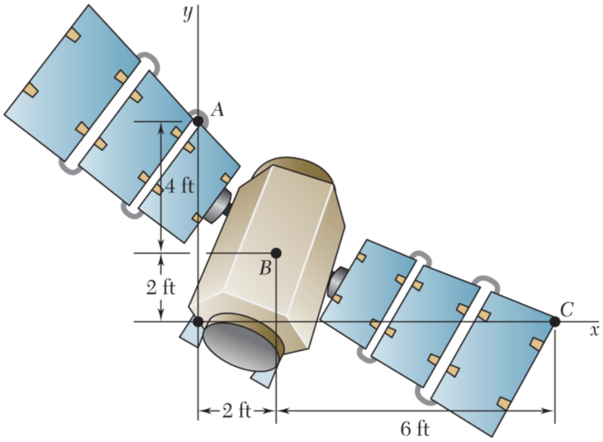
From Eq. (3), $$\omega = -\frac{750}{150} = -5$$

$$\boldsymbol{\omega} = -(5.00\text{ rad/s})\mathbf{k} = 5.00\text{ rad/s} \curvearrowright \blacktriangleleft$$

(*b*) *Velocity of Point A.*

From Eq. (2), $(v_A)_y = -450 - 150\omega = -450 - (150)(-5) = 300$ mm/s

$$\mathbf{v}_A = (250\text{ mm/s})\mathbf{i} + (300\text{ mm/s})\mathbf{j} \blacktriangleleft$$



PROBLEM 15.47

Velocity sensors are placed on a satellite that is moving only in the xy plane. Knowing that at the instant shown the uni-directional sensors measure $(v_A)_x = 2$ ft/s, $(v_B)_x = -0.333$ ft/s, and $(v_C)_y = -2$ ft/s, determine (*a*) the angular velocity of the satellite, (*b*) the velocity of point *B*.

SOLUTION

Given: $(v_A)_x = 2 \text{ ft/s},\ (v_B)_x = -0.333 \text{ ft/s},\ (v_C)_y = -2 \text{ ft/s}$

Geometry:

$$\mathbf{r}_{B/A} = (2\text{ ft})\mathbf{i} - (4\text{ ft})\mathbf{j}$$

$$\mathbf{r}_{B/C} = -(6\text{ ft})\mathbf{i} + (2\text{ ft})\mathbf{j}$$

Relative Velocity:

$$\mathbf{v}_B = \mathbf{v}_A + \omega_{ABC}\mathbf{k} \times \mathbf{r}_{B/\mathrm{A}}$$

$$-0.333\mathbf{i} + (v_B)_y\,\mathbf{j} = 2\mathbf{i} + (v_A)_y\,\mathbf{j} + \omega_{ABC}\mathbf{k} \times (2\mathbf{i} - 4\mathbf{j})$$

Equate Components:

(c) **i**: $-0.333 = 2 + 4\omega_{ABC}$

$$\omega_{ABC} = -0.583 \text{ rad/s}$$

$$\boldsymbol{\omega}_{ABC} = 0.583 \text{ rad/s} \circlearrowright \blacktriangleleft$$

Relative Velocity:

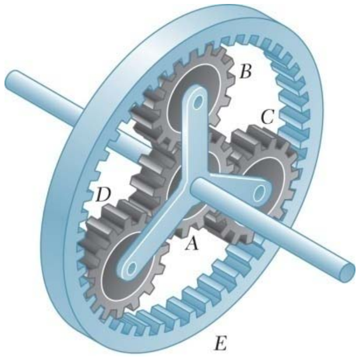
$$\mathbf{v}_B = \mathbf{v}_C + \omega_{ABC}\mathbf{k} \times \mathbf{r}_{\mathrm{B/C}}$$

$$-0.333\mathbf{i} + (v_B)_y\,\mathbf{j} = (v_C)_x\,\mathbf{i} - 2\mathbf{j} - 0.583\mathbf{k} \times (-6\mathbf{i} + 2\mathbf{j})$$

Equate Components:

(d) **j:** $(v_B)_y = 1.500 \text{ ft/s}$

$$\mathbf{v}_B = -(0.333 \text{ ft/s})\mathbf{i} + (1.500 \text{ ft/s})\mathbf{j} \qquad \mathbf{v}_B = 1.537 \text{ ft/s} \measuredangle 77.48^\circ \blacktriangleleft$$



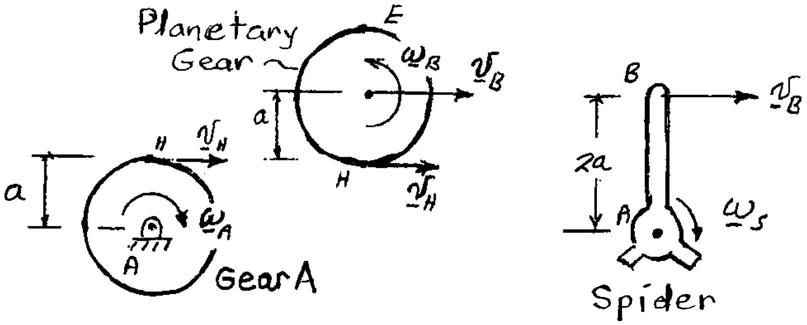
PROBLEM 15.48

In the planetary gear system shown, the radius of gears A, B, C, and D is a and the radius of the outer gear E is $3a$. Knowing that the angular velocity of gear A is ω_A clockwise and that the outer gear E is stationary, determine (*a*) the angular velocity of each planetary gear, (*b*) the angular velocity of the spider connecting the planetary gears.

SOLUTION

Gear E is stationary. $v_E = 0$

Let A be the center of gear A and the spider. Since the motions of gears B, C, and D are similar, only gear B is considered. Let H be the effective contact point between gears A and B.



Gear A: $\mathbf{v}_H = a\omega_A \longrightarrow$

(*a*) *Planetary gears B, C, and D*: $\mathbf{v}_H = \mathbf{v}_E + \mathbf{v}_{H/E}$

$$\xrightarrow{+}: a\omega_A = 0 + (2a)\omega_B \qquad \omega_B = \frac{1}{2}\omega_A$$

$$\boldsymbol{\omega}_B = \boldsymbol{\omega}_C = \boldsymbol{\omega}_D = \frac{1}{2}\omega_A \circlearrowleft \blacktriangleleft$$

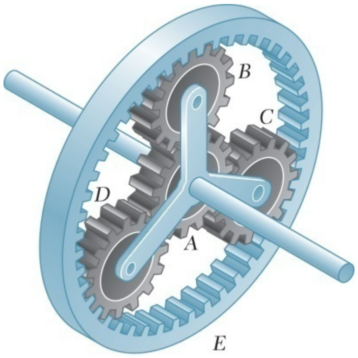
$$\mathbf{v}_B = \mathbf{v}_E + \mathbf{v}_{B/E}$$

$$\xrightarrow{+}: v_B = 0 + a\left(\frac{1}{2}\omega_A\right) \qquad \mathbf{v}_B = \frac{1}{2}a\omega_A \longrightarrow \qquad (1)$$

(*b*) *Spider*. $\mathbf{v}_B = (2a)\omega_s \longrightarrow$ (2)

Equating expressions (1) and (2) for $\mathbf{v}_B$,

$$\frac{1}{2}a\omega_A = (2a)\omega_s \qquad \omega_s = \frac{1}{4}\omega_A \qquad \boldsymbol{\omega}_s = \frac{1}{4}\omega_A \circlearrowright \blacktriangleleft$$



PROBLEM 15.49

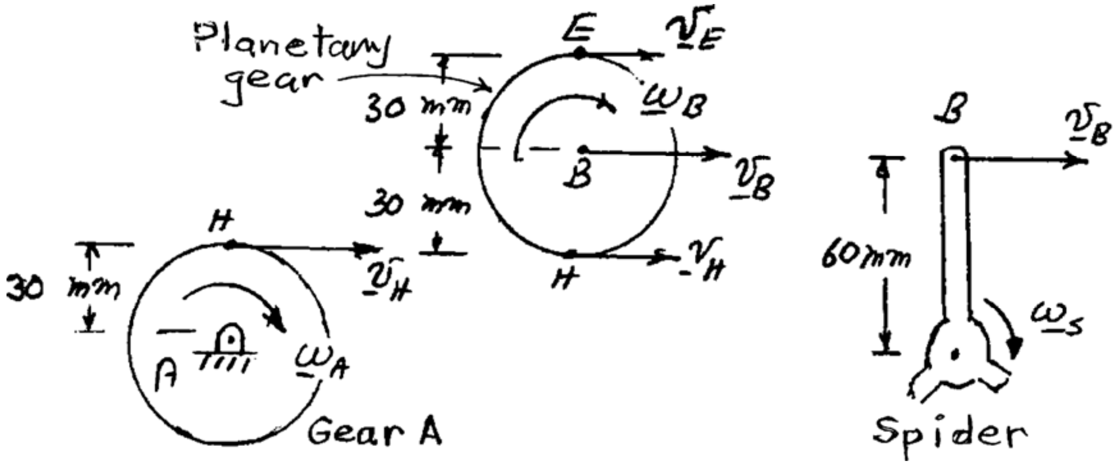
In the planetary gear system shown, the radius of gears *A*, *B*, *C*, and *D* is 30 mm and the radius of the outer gear *E* is 90 mm. Knowing that gear *E* has an angular velocity of 180 rpm clockwise and that the central gear *A* has an angular velocity of 240 rpm clockwise, determine (*a*) the angular velocity of each planetary gear, (*b*) the angular velocity of the spider connecting the planetary gears.

SOLUTION

Since the motions of the planetary gears *B*, *C*, and *D* are similar, only gear *B* is considered. Let Point *H* be the effect contact point between gears *A* and *B* and let Point *E* be the effective contact point between gears *B* and *E*.

Given angular velocities: $\boldsymbol{\omega}_E = 180 \text{ rpm } \circlearrowright = 6\pi \text{ rad/s } \circlearrowright$

$\boldsymbol{\omega}_A = 240 \text{ rpm } \circlearrowright = 8\pi \text{ rad/s } \circlearrowright$


Planetary gear
E
$\underline{v}_E$
30 mm
$\underline{\omega}_B$
B
$\underline{v}_B$
30 mm
H
$\underline{v}_H$
H
$\underline{v}_H$
30 mm
A
$\underline{\omega}_A$
Gear A
B
$\underline{v}_B$
60 mm
$\underline{\omega}_S$
Spider

Outer gear *E*: radius $= r_E = 90$ mm

$$v_E = r_E \omega_E = (90 \text{ mm})(6\pi \text{ rad/s}) = 540\pi \text{ mm/s}$$

$$\mathbf{v}_E = 540\pi \text{ mm/s} \longrightarrow$$

Gear *A*: radius $= r_A = 30$ mm

$$v_H = r_A \omega_A = (30 \text{ mm})(8\pi \text{ rad/s}) = 240\pi \text{ mm/s}$$

$$\mathbf{v}_H = 240\pi \text{ mm/s} \longrightarrow$$

PROBLEM 15.49 (Continued)

Planetary gear *B*: radius $= r_B = 30$ mm, $\boldsymbol{\omega}_B = \omega_B$ ↻

$$\mathbf{v}_H = \mathbf{v}_E + \mathbf{v}_{H/E}$$

$$[(30\text{ mm})\omega_A \longrightarrow] = [540\pi\text{ mm/s} \longrightarrow] + [(60\text{ mm})\omega_B \longleftarrow]$$

$$30\omega_A = 540\pi - 60\omega_B$$

$$\omega_B = \frac{540\pi + 30\omega_A}{60} = 9\pi + \frac{1}{2}\omega_A = 9\pi - \frac{1}{2}(8\pi) = 5\pi\text{ rad/s} ↻$$

(*a*) *Angular velocity of planetary gears*:

$$\boldsymbol{\omega}_B = \boldsymbol{\omega}_C = \boldsymbol{\omega}_D = 5\pi\text{ rad/s} ↻ = 150\text{ rpm} ↻ ◀$$

$$\mathbf{v}_B = \mathbf{v}_H + v_{B/H} = [(30\text{ mm})\omega_A \longrightarrow] + [30\text{ mm }\omega_B \longrightarrow]$$

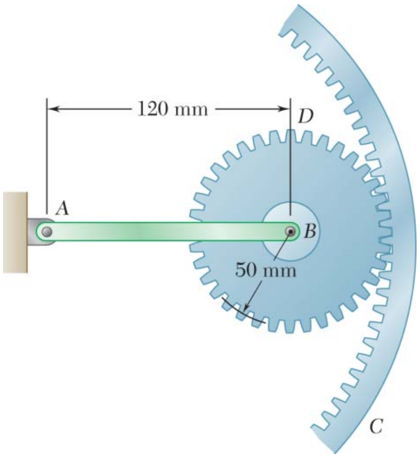
$$v_B = (30\text{ mm})(8\pi\text{ rad/s}) + (30\text{ mm})(5\pi\text{ rad/s}) = 390\pi\text{ mm/s}$$

(*b*) *Spider*: arm $= r_s = 60$ mm, $\boldsymbol{\omega}_s = \omega_s$ ↻

$$v_B = r_s\omega_s$$

$$\omega_s = \frac{v_B}{r_s} = \frac{390\pi\text{ mm/s}}{60\text{ mm}} = 6.5\pi\text{ rad/s}$$

$$\boldsymbol{\omega}_s = 6.5\pi\text{ rad/s} ↻ = 195\text{ rpm} ↻ ◀$$

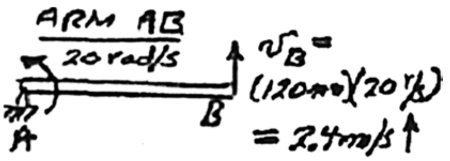

120 mm
D
A
B
50 mm
C

PROBLEM 15.50

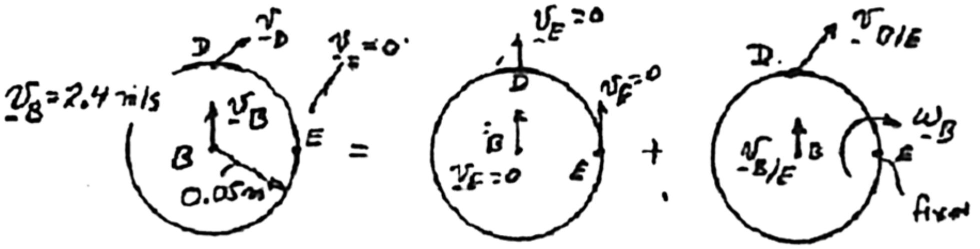
Arm *AB* rotates with an angular velocity of 20 rad/s counterclockwise. Knowing that the outer gear *C* is stationary, determine (*a*) the angular velocity of gear *B*, (*b*) the velocity of the gear tooth located at Point *D*.

SOLUTION

Arm *AB*:


ARM AB
20 rad/s
A
B
v_B =
(120 mm)(20 r/s)
= 2.4 m/s ↑

Gear *B*:


v_B = 2.4 m/s
D
v_D
v_E = 0
B
v_B
0.05 m
E
=
v_E = 0
D
B
v_E = 0
E
v_E = 0
+
D
$v_{D/E}$
$v_{B/E}$
B
E
ω_B
fixed

(*a*) $BE = 0.05\text{ m}$: $\mathbf{v}_B = \mathbf{v}_D + \mathbf{v}_{B/E} = 0 + (BE)\omega_B$

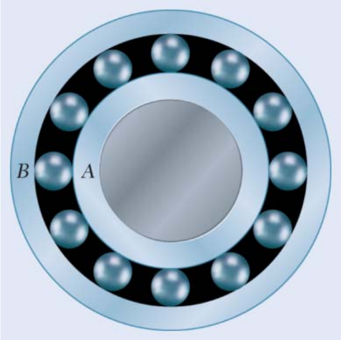
$$2.4\text{ m/s}\uparrow = 0 + (0.05\text{ m})\omega_B\uparrow$$

$\omega_B = 48\text{ rad/s}$ $\boldsymbol{\omega}_B = 48\text{ rad/s}$ ↻ ◀

(*b*) $DE = (0.05\sqrt{2})$: $\mathbf{v}_D = \mathbf{v}_E + \mathbf{v}_{D/E} = 0 + (DE)\omega_B$

$$v_D = 0 + (0.05\sqrt{2})(48)$$

$v_D = 3.39\text{ m/s}$ $\mathbf{v}_D = 3.39\text{ m/s}$ $\measuredangle$ 45° ◀



PROBLEM 15.51

In the simplified sketch of a ball bearing shown, the diameter of the inner race *A* is 60 mm and the diameter of each ball is 12 mm. The outer race *B* is stationary while the inner race has an angular velocity of 3600 rpm. Determine (*a*) the speed of the center of each ball, (*b*) the angular velocity of each ball, (*c*) the number of times per minute each ball describes a complete circle.

SOLUTION

Data: $\omega_A = 3600 \text{ rpm} = 376.99 \text{ rad/s}, \quad \omega_B = 0$

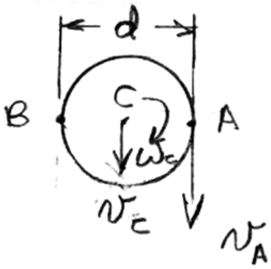
$$r_A = \frac{1}{2} d_A = 30 \text{ mm}$$

$$d = \text{diameter of ball} = 12 \text{ mm}$$

Velocity of point on inner race in contact with a ball.

$$v_A = r_A \omega_A = (30)(376.99) = 11310 \text{ mm/s}$$

Consider a ball with its center at Point *C*.



$$v_A = v_B + v_{A/B}$$

$$v_A = 0 + \omega_C d$$

$$\omega_C = \frac{v_A}{d} = \frac{11310}{12}$$

$$= 942.48 \text{ rad/s}$$

$$v_C = v_B + v_{C/B}$$

$$= 0 + \frac{1}{2} d\omega = (6)(942.48) = 5654.9 \text{ mm/s}$$

(*a*) $v_C = 5.65 \text{ m/s}$ ◀

(*b*) Angular velocity of ball.

$$\omega_C = 942.48 \text{ rad/s}$$

$\omega_C = 9000 \text{ rpm}$ ◀

(*c*) Distance traveled by center of ball in 1 minute.

$$l_C = v_C t = 5654.9(60) = 339290 \text{ mm}$$

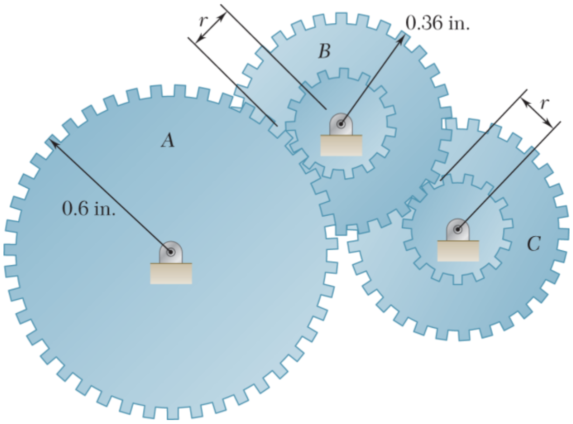
Circumference of circle: $2\pi r = 2\pi(30 + 6)$

$$= 226.19 \text{ mm}$$

Number of circles completed in 1 minute:

$$n = \frac{l}{2\pi r} = \frac{339290}{226.19}$$

$n = 1500$ ◀



PROBLEM 15.52

A simplified gear system for a mechanical watch is shown. Knowing that gear A has a constant angular velocity of 1 rev/h and gear C has a constant angular velocity of 1 rpm, determine (*a*) the radius r, (*b*) the magnitudes of the accelerations of the points on gear B that are in contact with gears A and C.

SOLUTION

Point where A contacts B:

$$v_1 = r_A\omega_A = r\omega_B \qquad \omega_B = \frac{r_A\omega_A}{r} \qquad (1)$$

Point where B contacts C:

$$v_2 = r_B\omega_B = r\omega_C \qquad \omega_C = \frac{r_B}{r}\omega_B \qquad (2)$$

From Eqs. (1) and (2),

$$\omega_C = \frac{r_A r_B}{r^2}\omega_A$$

$$r^2 = r_A r_B \frac{\omega_A}{\omega_C}$$

Data:

$$r_A = 0.6 \text{ in.}, \quad r_B = 0.36 \text{ in.} \quad \frac{\omega_A}{\omega_C} = \frac{1 \text{ rev/h}}{1 \text{ rev/m}} = \frac{1}{60}$$

$$r^2 = \frac{(0.6 \text{ in.})(0.36 \text{ in.})}{60} = 0.0036 \text{ in}^2$$

(*a*) *Radius r*:

$$r = 0.0600 \text{ in.} \blacktriangleleft$$

Angular velocity of B.

$$\omega_C = 1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

$$\omega_B = \frac{r}{r_B}\omega_C = \frac{0.060}{0.36}\frac{2\pi}{60} = 0.017453 \text{ rad/s}$$

(*b*) *Point where B contacts A.*

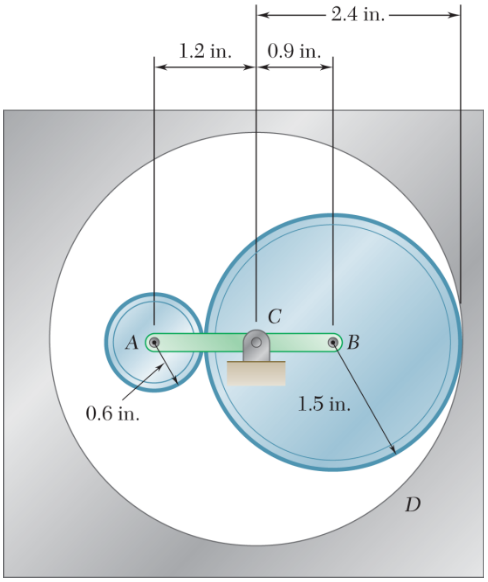
$$a_n = r\omega_B^2 = (0.0600 \text{ in.})(0.017453 \text{ rad/s})^2$$

$$a_n = 18.28\times 10^{-6} \text{ in./s}^2 \blacktriangleleft$$

Point where B contacts C.

$$a_n = r_B\omega_B^2 = (0.36 \text{ in.})(0.017453 \text{ rad/s})^2$$

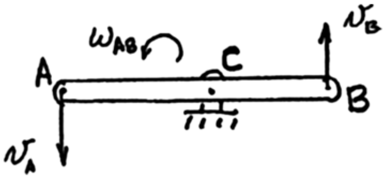
$$a_n = 109.7\times 10^{-6} \text{ in./s}^2 \blacktriangleleft$$

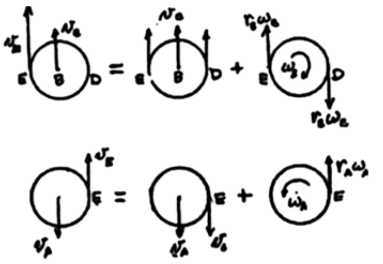


PROBLEM 15.53

Arm *ACB* rotates about Point *C* with an angular velocity of 40 rad/s counterclockwise. Two friction disks *A* and *B* are pinned at their centers to arm *ACB* as shown. Knowing that the disks roll without slipping at surfaces of contact, determine the angular velocity of (*a*) disk *A*, (*b*) disk *B*.

SOLUTION





Arm *ACB*: Fixed axis rotation.

$$r_{A/C} = 24 \text{ mm}, \quad \mathbf{v}_A = r_{A/C}\omega_{AB} = (24)(40) = 960 \text{ mm/s}\downarrow$$

$$r_{B/C} = 18 \text{ mm}, \quad \mathbf{v}_B = r_{B/C}\omega_{AB} = (18)(40) = 720 \text{ mm/s}\uparrow$$

Disk *B*: Plane motion = Translation with *B* + Rotation about *B*.

$$r_B = 30 \text{ mm}, \quad \mathbf{v}_D = \mathbf{v}_B - \mathbf{v}_{D/B}$$

$$0 = 720\uparrow + 30\ \omega_B\downarrow \text{ mm/s}$$

$$\omega_B = \frac{720}{30} = 24 \text{ rad/s} \circlearrowright$$

$$\mathbf{v}_E = \mathbf{v}_B + \mathbf{v}_{E/B}$$

$$= 720\uparrow + (30)(24)\uparrow = 1440 \text{ mm/s}\uparrow$$

Disk *A*: Plane motion = Translation with *A* + Rotation about *A*.

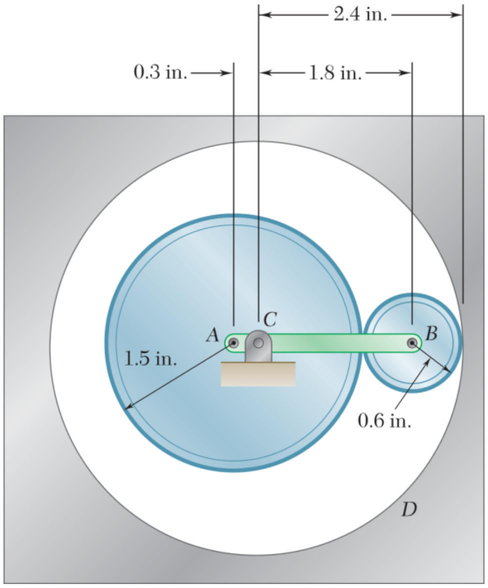
$$r_A = 12 \text{ mm}, \quad \mathbf{v}_E = \mathbf{v}_A - \mathbf{v}_{E/A}$$

$$1440\uparrow = 960\downarrow + 12\omega_A\uparrow$$

$$\omega_A = \frac{1440 + 960}{12} = 200 \text{ rad/s} \circlearrowleft$$

(*a*) $\boldsymbol{\omega}_A = 200 \text{ rad/s} \circlearrowleft$ ◀

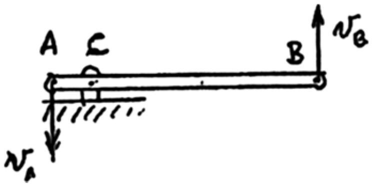
(*b*) $\boldsymbol{\omega}_B = 24.0 \text{ rad/s} \circlearrowright$ ◀

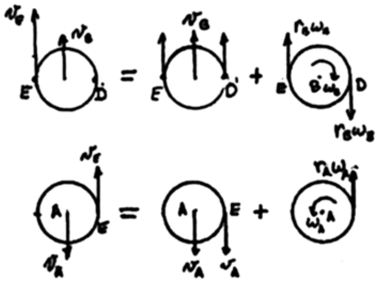


PROBLEM 15.54

Arm ACB rotates about Point C with an angular velocity of 40 rad/s counterclockwise. Two friction disks A and B are pinned at their centers to arm ACB as shown. Knowing that the disks roll without slipping at surfaces of contact, determine the angular velocity of (*a*) disk A, (*b*) disk B.

SOLUTION





Arm ACB: Fixed axis rotation.

$$r_{A/C} = 0.3 \text{ in.}, \quad \mathbf{v}_A = r_{A/C}\omega_{AB}\downarrow = (0.3)(40)\downarrow = 12 \text{ in./s}\downarrow$$

$$r_{B/C} = 1.8 \text{ in.}, \quad \mathbf{v}_B = r_{B/C}\omega_{AB}\uparrow = (1.8)(40)\uparrow = 72 \text{ in./s}\uparrow$$

Disk B: Plane motion = Translation with B + Rotation about B.

$$r_B = 0.6 \text{ in.}, \quad \mathbf{v}_D = \mathbf{v}_B - \mathbf{v}_{B/A}$$

$$0 = 72\uparrow + 0.6\omega_B\downarrow$$

$$\omega_B = \frac{72}{0.6} = 120 \text{ rad/s} \circlearrowright$$

$$\mathbf{v}_E = \mathbf{v}_B + \mathbf{v}_{E/B}$$

$$= 72\uparrow + (0.6)(120)\uparrow = 144 \text{ in./s}\uparrow$$

Disk A: Plane motion = Translation with A + Rotation about A.

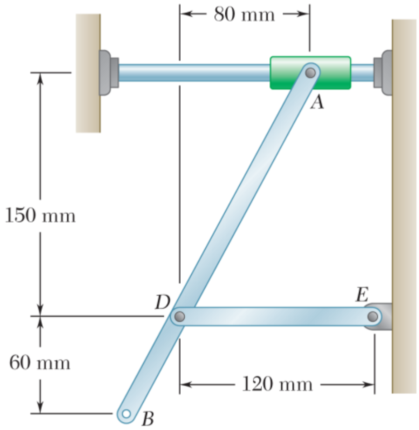
$$r_A = 1.5 \text{ in.}, \quad \mathbf{v}_E = \mathbf{v}_A - \mathbf{v}_{E/A}$$

$$144\uparrow = 12\downarrow \; 1.5\omega_A$$

$$\omega_A = \frac{144 + 12}{1.5} = 104 \text{ rad/s} \circlearrowleft$$

(*a*) $\boldsymbol{\omega}_A = 104.0 \text{ rad/s} \circlearrowleft$ ◀

(*b*) $\boldsymbol{\omega}_B = 120.0 \text{ rad/s} \circlearrowright$ ◀



PROBLEM 15.55

Knowing that at the instant shown the velocity of collar A is 900 mm/s to the left, determine (a) the angular velocity of rod ADB, (b) the velocity of Point B.

SOLUTION

Consider rod ADB.

$$\mathbf{v}_D = v_D\mathbf{j}, \quad \mathbf{v}_A = -(900 \text{ mm/s})\mathbf{i}$$

$$\mathbf{r}_{D/A} = -(80 \text{ mm})\mathbf{i} - (150 \text{ mm})\mathbf{j}$$

$$\mathbf{v}_{D/A} = \boldsymbol{\omega}_{AD} \times \mathbf{r}_{D/A} = \omega_{AD}\mathbf{k} \times (-80\mathbf{i} - 150\mathbf{j})$$

$$= 150\omega_{AD}\mathbf{i} - 80\omega_{AD}\mathbf{j}$$

$$\mathbf{v}_0 = \mathbf{v}_A + \mathbf{v}_{D/A}$$

$$v_D\mathbf{j} = -900\mathbf{i} + 150\omega_{AD}\mathbf{i} - 80\omega_{AD}\mathbf{j}$$

Equate components.

$$\mathbf{i}: \quad 0 = -900 + 150\omega_{AD} \qquad \omega_{AD} = 6 \text{ rad/s}$$

(a) *Angular velocity of ADB.* $\boldsymbol{\omega}_{AD} = (6.00 \text{ rad/s})\mathbf{k} = 6.00 \text{ rad/s} \circlearrowleft$ ◀

By proportions,

$$\mathbf{r}_{B/A} = \frac{150+60}{150}\mathbf{r}_{D/A} = 1.4\,\mathbf{r}_{D/A}$$

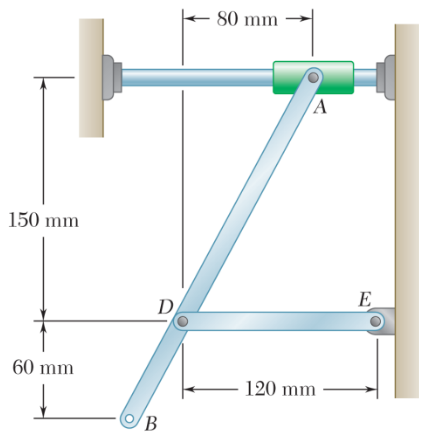
$$= -(112 \text{ mm})\mathbf{i} - (210 \text{ mm})\mathbf{j}$$

$$\mathbf{v}_B = \mathbf{v}_A + \omega_{AD}\mathbf{k} \times \mathbf{r}_{B/A}$$

$$= -900\mathbf{i} + 6\mathbf{k} \times (-112\mathbf{i} - 210\mathbf{j})$$

$$= -900\mathbf{i} - 672\mathbf{j} + 1260\mathbf{i}$$

(b) *Velocity of B.* $\mathbf{v}_B = (360 \text{ mm/s})\mathbf{i} - (672 \text{ mm/s})\mathbf{j} = 762 \text{ mm/s} \searrow 61.8°$ ◀



PROBLEM 15.56

Knowing that at the instant shown the angular velocity of rod *DE* is 2.4 rad/s clockwise, determine (*a*) the velocity *A*, (*b*) the velocity of Point *B*.

SOLUTION

Rod *DE*: Point *E* is fixed. $\boldsymbol{\omega}_{DE} = 2.4 \text{ rad} \circlearrowright$

$$v_D = \omega_{DB} r_{DE} = (2.4 \text{ rad/s})(120 \text{ mm}) = 288 \text{ mm/s}$$

$$\mathbf{v}_D = 288 \text{ mm/s} \uparrow = (288 \text{ mm/s})\mathbf{j}$$

Rod *ADB*: $\mathbf{r}_{A/D} = (80 \text{ mm})\mathbf{i} + (150 \text{ mm})\mathbf{j}, \;\; \boldsymbol{\omega}_{AD} = \boldsymbol{\omega}_{AD}\mathbf{k}, \;\; \mathbf{v}_A = v_A\mathbf{i}$

$$\mathbf{v}_A = \mathbf{v}_D + \mathbf{v}_{D/A} = \mathbf{v}_D + \omega_{AD}\mathbf{k} \times \mathbf{r}_{A/D}$$

$$v_A\mathbf{i} = (288 \text{ mm/s})\mathbf{j} + \omega_{AD}\mathbf{k} \times [(80 \text{ mm})\mathbf{i} + (150 \text{ mm})\mathbf{j}]$$

$$v_A\mathbf{i} = 288\mathbf{j} + 80\omega_{AD}\mathbf{j} - 150\omega_{AD}\mathbf{i}$$

Equate components.

i: $$v_A = -150\omega_{AD} \tag{1}$$

j: $$0 = 288 + 80\omega_{AD} \tag{2}$$

From Eq. (2), $\omega_{AD} = -\dfrac{288}{80}$ $\qquad \boldsymbol{\omega}_{AD} = (-3.6 \text{ rad/s})\mathbf{k}$

From Eq. (1), $v_A = -(150)(-3.6) = 540 \text{ mm/s}$

(*a*) *Velocity of collar A.* $\mathbf{v}_A = 540 \text{ mm/s} \longrightarrow$ ◀

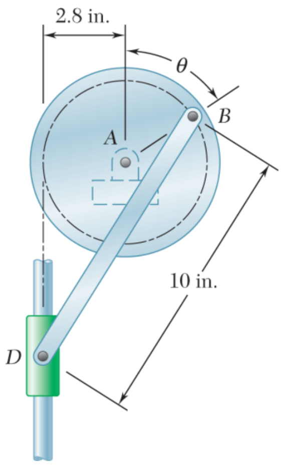
(*b*) *Velocity of Point B.*

By proportions $\mathbf{r}_{B/D} = -\dfrac{60}{150}\mathbf{r}_{A/D} = -(32 \text{ mm})\mathbf{i} - 60 \text{ mm } \mathbf{j}$

$$\begin{aligned}\mathbf{v}_B &= \mathbf{v}_D + \mathbf{v}_{B/D} = \mathbf{v}_D + \boldsymbol{\omega}_{AD} \times \mathbf{r}_{B/D} \\ &= (288 \text{ mm/s})\mathbf{j} + [-(3.6 \text{ rad/s})\mathbf{k}] \times [-(32 \text{ mm})\mathbf{i} - (60 \text{ mm})\mathbf{j}] \\ &= (288 \text{ mm/s})\mathbf{j} + (115.2 \text{ mm/s})\mathbf{j} - (216 \text{ mm/s})\mathbf{i}\end{aligned}$$

$$\mathbf{v}_B = -(216 \text{ mm/s})\mathbf{i} + (403.2 \text{ mm/s})\mathbf{j}$$

$\mathbf{v}_B = 457 \text{ mm/s} \nwarrow 61.8°$ ◀



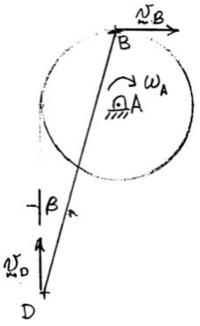
PROBLEM 15.57

Knowing that the disk has a constant angular velocity of 15 rad/s clockwise, determine the angular velocity of bar *BD* and the velocity of collar *D* when (*a*) $\theta = 0$, (*b*) $\theta = 90°$, (*c*) $\theta = 180°$.

SOLUTION

Disk *A*

$$\omega_A = 15 \text{ rad/s} \circlearrowright, \quad AB = 2.8 \text{ in.}$$



Rotation about a fixed axis.

$$v_B = (AB)\,\omega_A = (2.8)(15) = 42 \text{ in./s}$$

(*a*) $\theta = 0°$.

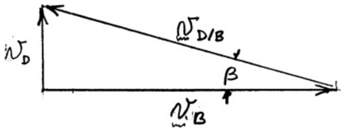
$$\mathbf{v}_B = 42 \text{ in./s} \longrightarrow$$

$$\sin\beta = \frac{2.8}{10}, \quad \beta = 16.260°$$

Bar *BD*:

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$

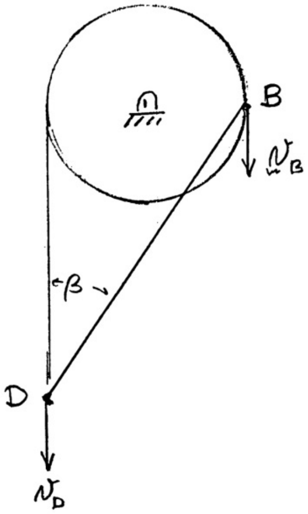
$$v_D \uparrow = [42 \longrightarrow] + [v_{D/B} \searrow \beta]$$



$$v_{D/B} = \frac{42}{\cos\beta} = 43.75 \text{ in./s}$$

$$\omega_{DB} = \frac{v_{D/B}}{DB} = \frac{43.75}{10}, \qquad \boldsymbol{\omega}_{DB} = 4.38 \text{ rad/s} \circlearrowright \blacktriangleleft$$

$$v_D = v_B \tan\beta, \qquad \mathbf{v}_D = 12.25 \text{ in./s} \uparrow \blacktriangleleft$$



(*b*) $\theta = 90°$.

$$\mathbf{v}_B = 42 \text{ in./s} \downarrow$$

$$\sin\beta = \frac{5.6}{10}, \quad \beta = 34.06°$$

Bar *BD*:

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$

$$v_D \downarrow = [42 \downarrow] + [v_{D/B} \searrow \beta]$$

Components:

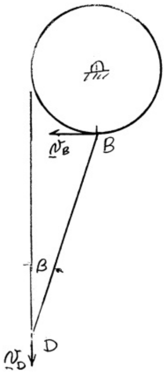
$$\xrightarrow{+}: \quad v_{D/B} = 0$$

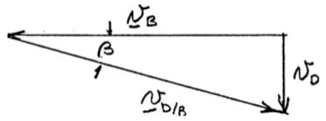
$$+\downarrow: \quad v_D = v_B$$

$$\boldsymbol{\omega}_{DB} = 0 \blacktriangleleft$$

$$\mathbf{v}_D = 42.0 \text{ in./s} \downarrow \blacktriangleleft$$

PROBLEM 15.57 (Continued)





(*c*) $\theta = 180°$. $\qquad v_B = 42 \text{ in./s} \leftarrow$

$$\sin\beta = \frac{2.8}{10}, \quad \beta = 16.26°$$

Bar *BD*: $\qquad \mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$

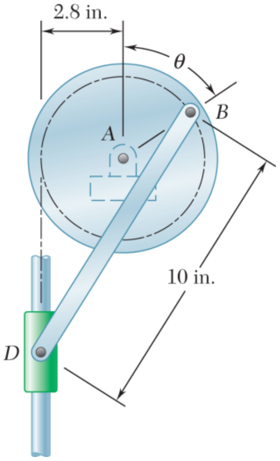
$$v_D \downarrow = [42 \leftarrow] + [v_{D/B} \searrow \beta]$$

$$v_{D/B} = \frac{42}{\cos\beta} = 43.75$$

$$\omega_{DB} = \frac{v_{D/B}}{DB} = \frac{43.75}{10}$$

$$\boldsymbol{\omega}_{DB} = 4.38 \text{ rad/s} \circlearrowleft \blacktriangleleft$$

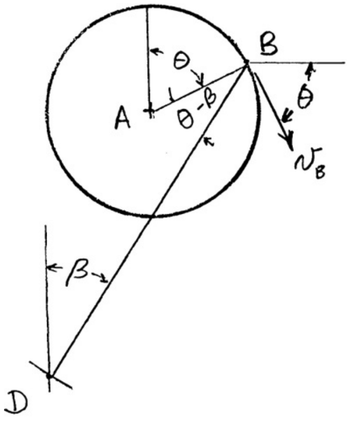
$$v_D = v_B \tan\beta \qquad \mathbf{v}_D = 12.25 \text{ in./s} \downarrow \blacktriangleleft$$



PROBLEM 15.58

The disk has a constant angular velocity of 20 rad/s clockwise. (*a*) Determine the two values of the angle θ for which the velocity of collar *D* is zero. (*b*) For each of these values of θ, determine the corresponding value of the angular velocity of bar *BD*.

SOLUTION



$$r_A = 2.8 \text{ in.}, \qquad l_{BD} = 10 \text{ in.}$$

From geometry, $$l_{BD} \sin\beta = r_A + r_A \sin\theta \tag{1}$$

$\mathbf{v}_B$ is tangent to the circular path of *B*,

thus $$\mathbf{v}_B = r_A\omega_A \searrow \theta$$

For rod *BD* $$\mathbf{v}_{B/D} = l_{BD}\omega_{BD} \searrow \beta$$

$$\mathbf{v}_B = \mathbf{v}_D + \mathbf{v}_{B/D} = 0 + \mathbf{v}_{B/D}$$

$$\mathbf{v}_{B/D} = \mathbf{v}_B$$

(*a*) For matching direction $\theta = \beta$ or $\theta = 180° + \beta$

For $\beta = \theta$, $\sin\beta = \sin\theta$ so that $l_{BD}\sin\beta = r_A + r_A\sin\beta$

$$\sin\beta = \frac{r_A}{l_{BD} - r_A} = \frac{2.8}{10 - 2.8}, \quad \beta = 22.9°, \qquad \theta = 22.9° \blacktriangleleft$$

For $\theta = 180° + \beta$, $\sin\theta = -\sin\beta$,

$$l_{BD}\sin\beta = r_A - r_A\sin\beta$$

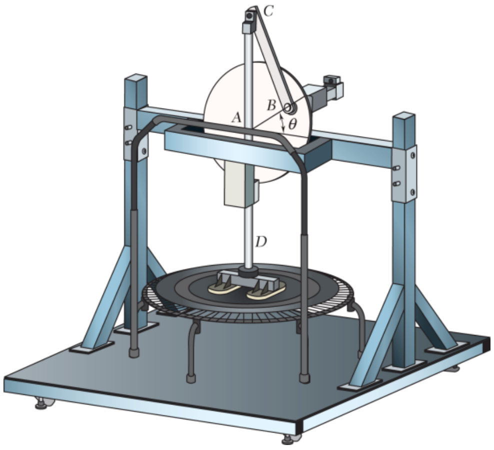
$$\sin\beta = \frac{r_A}{l_{BD} + r_A} = \frac{2.8}{10 + 2.8}, \quad \beta = 12.6° \qquad \theta = 192.6° \blacktriangleleft$$

(*b*) For matching magnitudes $v_{D/B} = v_B$

$$l_{BD}\omega_{BD} = r_A\omega_A, \qquad \omega_{BD} = \frac{r_A\omega_A}{l_{BD}} = \frac{(2.8)(20)}{10} = 5.6 \text{ rad/s}$$

For $\theta = 22.9°$, $$\boldsymbol{\omega}_{BD} = 5.60 \text{ rad/s} \circlearrowright \blacktriangleleft$$

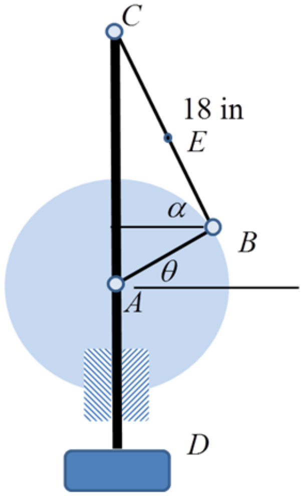
For $\theta = 192.6°$, $$\boldsymbol{\omega}_{BD} = 5.60 \text{ rad/s} \circlearrowleft \blacktriangleleft$$



PROBLEM 15.59

The test rig shown was developed to perform fatigue testing on fitness trampolines. A motor drives the 9-in radius flywheel *AB,* which is pinned at its center point *A*, in a counterclockwise direction. The flywheel is attached to slider *CD* by the 18-in connecting rod *BC*. Knowing that the "feet" at *D* should hit the trampoline twice every second, at the instant when $\theta = 30°$, determine (a) the angular velocity of the connecting rod *BC*, (b) the velocity of *D*, (c) the velocity of midpoint *CB*.

SOLUTION



Given:

$$r_{B/A} = 9 \text{ in, } r_{C/B} = 18 \text{ in, } \theta = 30°$$

$$\omega_{AB} = 2\,\frac{\text{Rev}}{s}\bullet\frac{2\pi \text{ rad}}{\text{Rev}} = 4\pi \text{ rad/s}$$

$$(v_C)_x = 0,\ \mathbf{v}_D = \mathbf{v}_C$$

Geometry:

$$9\cos\theta = 18\cos\alpha \quad\Rightarrow\quad \alpha = 64.341°$$

$$\mathbf{r}_{B/A} = (9\cos\theta \text{ in})\mathbf{i} + (9\sin\theta)\mathbf{j}$$

$$= 7.794\mathbf{i} + 4.5\mathbf{j}$$

$$\mathbf{r}_{C/B} = -(18\cos\alpha \text{ in})\mathbf{i} + (18\sin\alpha \text{ in})\mathbf{j}$$

$$= -7.794\mathbf{i} + 16.225\mathbf{j}$$

$$\mathbf{r}_{E/C} = (9\cos\alpha \text{ in})\mathbf{i} - (9\sin\alpha \text{ in})\mathbf{j}$$

$$= 3.897\mathbf{i} - 8.112\mathbf{j}$$

Relative Velocity:

$$\mathbf{v}_B = \cancel{\mathbf{v}_A} + \omega_{AB}\mathbf{k}\times\mathbf{r}_{B/A} \qquad (1)$$

$$\mathbf{v}_C = \mathbf{v}_B + \omega_{BC}\mathbf{k}\times\mathbf{r}_{C/B} \qquad (2)$$

Sub (1) into (2):

$$\mathbf{v}_C = \omega_{AB}\mathbf{k}\times\mathbf{r}_{B/A} + \omega_{BC}\mathbf{k}\times\mathbf{r}_{C/B}$$

$$(v_C)_y\,\mathbf{j} = 4\pi\mathbf{k}\times(7.794\mathbf{i} + 4.5\mathbf{j}) + \omega_{BC}\mathbf{k}\times(-7.794\mathbf{i} + 16.225\mathbf{j})$$

(a) Equate Components: **i**:

$$0 = -18\pi\mathbf{i} - 16.225\omega_{BC}\mathbf{j}$$

$$\boldsymbol{\omega}_{BC} = 3.485 \text{ rad/s } \circlearrowright \blacktriangleleft$$

(b) **j**:

$$(v_C)_y = 97.945 - 3.485\bullet(-7.794)$$

$$(v_C)_y = 125.110 \text{ in/s}$$

$$\mathbf{v}_D = 125.11 \text{ in/s} \uparrow \blacktriangleleft$$

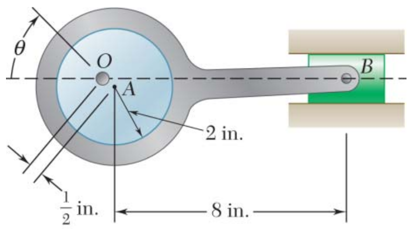
(c) Relative Velocity:

$$\mathbf{v}_E = \mathbf{v}_C + \omega_{BC}\mathbf{k}\times\mathbf{r}_{E/C}$$

$$= 125.11\mathbf{j} - 3.485\mathbf{k}\times(3.897\mathbf{i} - 8.112\mathbf{j})$$

$$= -28.274\mathbf{i} + 111.527\mathbf{j} \text{ in/s}$$

$$\mathbf{v}_E = 115.056 \text{ in/s} \nwarrow 75.77° \blacktriangleleft$$



PROBLEM 15.60

In the eccentric shown, a disk of 2-in.-radius revolves about shaft O that is located 0.5 in. from the center A of the disk. The distance between the center A of the disk and the pin at B is 8 in. Knowing that the angular velocity of the disk is 900 rpm clockwise, determine the velocity of the block when $\theta = 30°$.

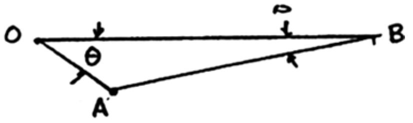
SOLUTION

Geometry.

$$(OA)\sin\theta = (AB)\sin\beta$$

$$\sin\beta = \frac{(OA)\sin\theta}{AB}$$

$$= \frac{0.5\ \sin 30°}{8}, \quad \beta = 1.79°$$

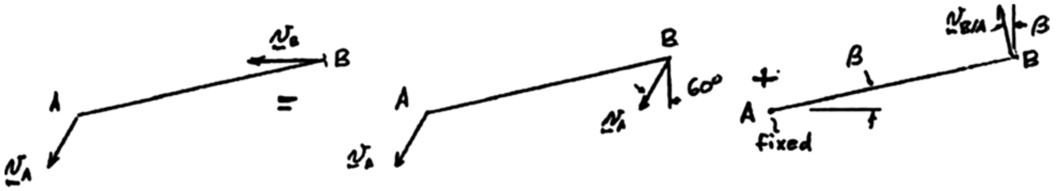


Shaft and eccentric disk. (Rotation about O)

$$\omega_{OA} = 900\ \text{rpm} = 30\pi\ \text{rad/s} \curvearrowright$$

$$\mathbf{v}_A = (OA)\,\omega_{OA} = (0.5)(30\pi) = 15\pi\ \text{in/s} \swarrow$$

Rod AB. (Plane motion = Translation with A + Rotation about A.)



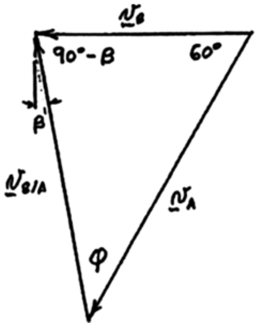
$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \quad [v_B \longleftarrow] = [v_A \swarrow 60°] + [v_{A/B} \searrow \beta]$$

Draw velocity vector diagram.

$$90° - \beta = 88.21°$$

$$\varphi = 180° - 60° - 88.21°$$

$$= 31.79°$$



Law of sines.

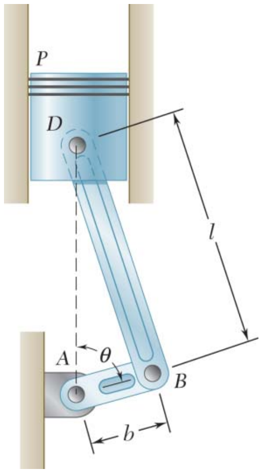
$$\frac{v_B}{\sin\phi} = \frac{v_A}{\sin(90° - \beta)}$$

$$v_B = \frac{v_A \sin\varphi}{\sin(90° - \beta)}$$

$$= \frac{(15\pi)\sin 31.79°}{\sin 88.21°}$$

$$= 24.837\ \text{in./s}$$

$\mathbf{v}_B = 24.8$ in./s ⟵ ◀



PROBLEM 15.61

In the engine system shown, $l = 160$ mm and $b = 60$ mm. Knowing that the crank AB rotates with a constant angular velocity of 1000 rpm clockwise, determine the velocity of the piston P and the angular velocity of the connecting rod when (a) $\theta = 0$, (b) $\theta = 90°$.

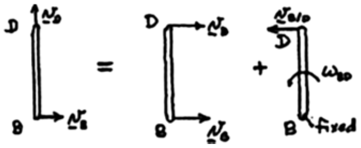
SOLUTION

$$\omega_{AB} = 1000 \text{ rpm} \circlearrowright = \frac{(1000)(2\pi)}{60} = 104.72 \text{ rad/s} \circlearrowright$$

(a) $\theta = 0°$. *Crank AB.* (Rotation about A) $\mathbf{r}_{B/A} = 0.06 \text{ m} \uparrow$

$$\mathbf{v}_B = v_{B/A}\omega_{AB} = (0.06)(104.72) = 6.2832 \text{ m/s} \rightarrow$$

Rod BD. (Plane motion = Translation with B + Rotation about B)



$$\mathbf{v}_D = \mathbf{v}_B + v_{D/B}$$

$$v_D \uparrow = [6.2832 \rightarrow] + [v_{D/B} \leftarrow]$$

$$v_D = 0$$

$$v_{D/B} = 6.2832 \text{ m/s}$$

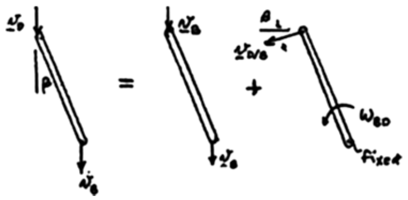
$$\text{v}_P = \text{v}_D \qquad \mathbf{v}_P = 0 \text{ ◀}$$

$$\omega_{BD} = \frac{v_B}{l} = \frac{6.2832}{0.16} \qquad \boldsymbol{\omega}_{BD} = 39.3 \text{ rad/s} \circlearrowleft \text{ ◀}$$

(b) $\theta = 90°$. *Crank AB.* (Rotation about A) $\mathbf{r}_{B/A} = 0.06 \text{ m} \rightarrow$

$$\mathbf{v}_B = r_{B/A}\omega_{AB} = (0.06)(104.72) = 6.2832 \text{ m/s} \downarrow$$

Rod BD. (Plane motion = Translation with B + Rotation about B.)



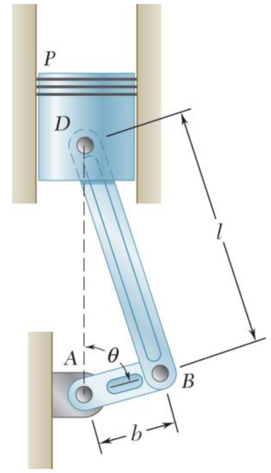
$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$

$$[v_D \downarrow] = [6.2832 \downarrow] + [v_{D/B} \nearrow \beta]$$

$$v_{D/B} = 0, \quad v_D = 6.2832 \text{ m/s}$$

$$\omega_{BD} = \frac{v_{D/B}}{l} \qquad \omega_{BD} = 0 \text{ ◀}$$

$$\mathbf{v}_P = \mathbf{v}_D = 6.2832 \text{ m/s} \downarrow \qquad \mathbf{v}_P = 6.28 \text{ m/s} \downarrow \text{ ◀}$$



PROBLEM 15.62

In the engine system shown $l = 160$ mm and $b = 60$ mm. Knowing that crank AB rotates with a constant angular velocity of 1000 rpm clockwise, determine the velocity of the piston P and the angular velocity of the connecting rod when $\theta = 60°$.

SOLUTION

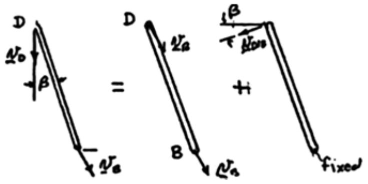
$$\omega_{AB} = 1000 \text{ rpm} = \frac{(1000)(2\pi)}{60} = 104.72 \text{ rad/s} \circlearrowright$$

$\underline{\theta = 60°}$. *Crank AB*. (Rotation about A) $\mathbf{r}_{B/A} = 3$ in. $\measuredangle\, 30°$

$$\mathbf{v}_B = r_{B/A}\omega_{AB} = (0.06)(104.72) = 6.2832 \text{ m/s} \searrow 60°$$

Rod BD. (Plane motion = Translation with B + Rotation about B.)

Geometry. $l\sin\beta = r\sin\theta$



$$\sin\beta = \frac{r}{l}\sin\theta = \frac{0.06}{0.16}\sin 60°$$

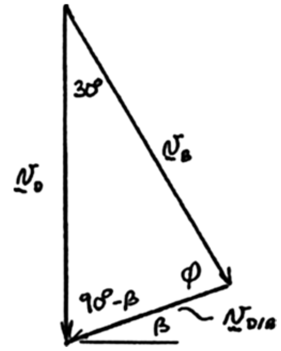
$$\beta = 18.95°$$

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$

$$[v_D \downarrow] = [314.16 \searrow 60°] + [v_{D/B} \swarrow \beta]$$

Draw velocity vector diagram.

$$\varphi = 180° - 30° - (90° - \beta) = 78.95°$$



$\underline{\text{Law of sines}}$.

$$\frac{v_D}{\sin\varphi} = \frac{v_{D/B}}{\sin 30°} = \frac{v_B}{\sin(90° - \beta)}$$

$$v_D = \frac{v_B \sin\varphi}{\cos\beta}$$

$$= \frac{6.2832 \;\sin 78.95°}{\cos 18.95°}$$

$$= 6.52 \text{ m/s}$$

$$v_P = v_D$$

$\mathbf{v}_P = 6.52$ m/s $\downarrow$ ◀

PROBLEM 15.62 (Continued)

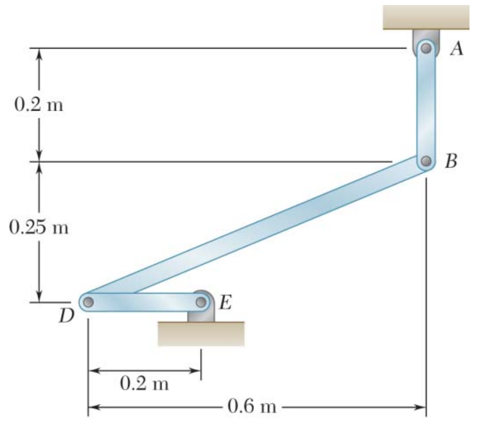
$$v_{D/B} = \frac{v_B \sin 30°}{\cos \beta}$$

$$= \frac{6.2832 \sin 30°}{\cos 18.95°}$$

$$= 3.3216 \text{ m/s}$$

$$\omega_{BD} = \frac{v_{D/B}}{l} = \frac{3.3216}{0.16}$$

$\omega_{BD} = 20.8$ rad/s ↺ ◀



PROBLEM 15.63

Knowing that at the instant shown the angular velocity of rod *AB* is 15 rad/s clockwise, determine (*a*) the angular velocity of rod *BD*, (*b*) the velocity of the midpoint of rod *BD*.

SOLUTION

Rod *AB*: $\boldsymbol{\omega}_{AB} = 15 \text{ rad/s} \circlearrowright$

$$v_B = (AB)\omega_{AB} = (0.200)(15) = 3 \text{ m/s} \qquad \mathbf{v}_B = 3 \text{ m/s} \longleftarrow$$

Rod *BD*: $\mathbf{v}_B = -(3 \text{ m/s})\mathbf{i}, \quad \mathbf{v}_D = v_D\mathbf{j}, \quad \boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k}$

$$\mathbf{r}_{B/D} = (0.6 \text{ m})\mathbf{i} + (0.25 \text{ m})\mathbf{j}$$

$$\mathbf{v}_B = \mathbf{v}_D + \mathbf{v}_{B/D} = \mathbf{v}_D + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{B/D}$$

$$-3\mathbf{i} = v_D\mathbf{j} + \omega_{BD}\mathbf{k} \times (0.6\mathbf{i} + 0.25\mathbf{j})$$

$$= v_D\mathbf{j} + 0.6\omega_{BD}\mathbf{j} - 0.25\omega_{BD}\mathbf{i}$$

Equate components.

$$\mathbf{i}: \quad -3 = -0.25\omega_{BD} \qquad (1)$$

$$\mathbf{j}: \quad 0 = v_D + 0.6\omega_{BD} \qquad (2)$$

(*a*) *Angular velocity of rod BD.*

From Eq. (1), $\omega_{BD} = \dfrac{3}{0.25}$ $\qquad \boldsymbol{\omega}_{BD} = 12.00 \text{ rad/s} \circlearrowleft$ ◀

From Eq. (2), $v_D = -0.6\omega_{BD}$ $\qquad v_D = -7.2 \text{ m/s}$

(*b*) *Velocity of midpoint M of rod BD.*

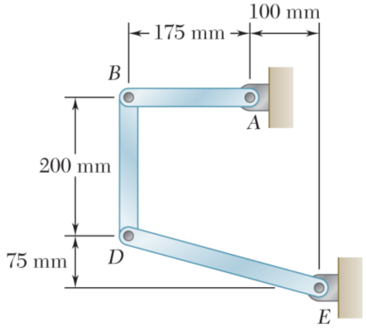
$$\mathbf{r}_{M/D} = \frac{1}{2}\mathbf{r}_{B/D} = (0.3 \text{ m})\mathbf{i} + (0.125 \text{ m})\mathbf{j}$$

$$\mathbf{v}_M = \mathbf{v}_D + \mathbf{v}_{M/D} = v_D\mathbf{j} + \omega_{BD}\mathbf{k} \times \mathbf{r}_{M/D}$$

$$= -7.2\mathbf{j} + 12.00\mathbf{k} \times (0.3\mathbf{i} + 0.125\mathbf{j})$$

$$= -(1.500 \text{ m/s})\mathbf{i} - (3.60 \text{ m/s})\mathbf{j}$$

$\mathbf{v}_M = 3.90 \text{ m/s} \swarrow 67.4°$ ◀



PROBLEM 15.64

In the position shown, bar *AB* has an angular velocity of 4 rad/s clockwise. Determine the angular velocity of bars *BD* and *DE*.

SOLUTION

Bar *AB*: (Rotation about *A*) $\boldsymbol{\omega}_{AB} = 4 \text{ rad/s} \circlearrowright = -(4 \text{ rad/s})\mathbf{k}$

$$\mathbf{r}_{B/A} = -(175 \text{ mm})\mathbf{i} \quad \mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = (-4\mathbf{k}) \times (-175\mathbf{i})$$

$$\mathbf{v}_B = (700 \text{ mm/s})\mathbf{j}$$

Bar *BD*: (Plane motion $=$ Translation with $B +$ Rotation about B.)

$$\boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \quad \mathbf{r}_{D/B} = -(200 \text{ mm})\mathbf{j}$$

$$\mathbf{v}_D = \mathbf{v}_B + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{D/B} = 700\mathbf{j} + (\omega_{BD}\mathbf{k}) \times (-200\mathbf{j})$$

$$\mathbf{v}_D = 700\mathbf{j} + 200\omega_{BD}\mathbf{i}$$

Bar *DE*: (Rotation about *E*) $\boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}$

$$\mathbf{r}_{D/E} = -(275 \text{ mm})\mathbf{i} + (75 \text{ mm})\mathbf{j}$$

$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = (\omega_{DE}\mathbf{k}) \times (-275\mathbf{i} + 75\mathbf{j})$$

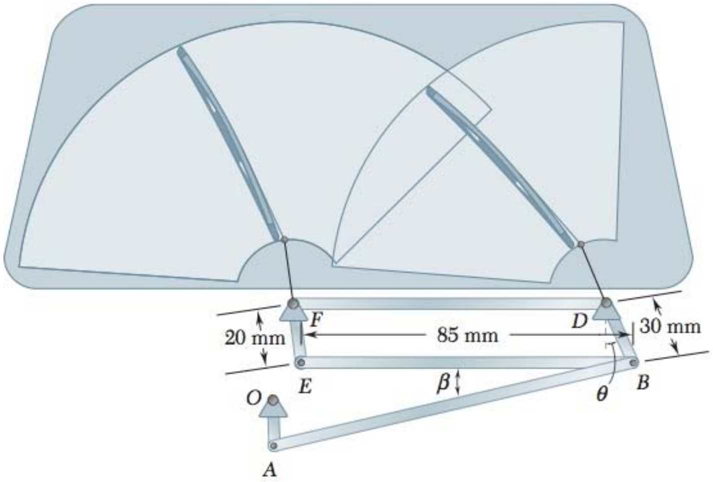
$$\mathbf{v}_D = -275\omega_{DE}\mathbf{j} - 75\omega_{DE}\mathbf{i}$$

Equating components of the two expressions for $\mathbf{v}_D$,

$$\mathbf{j}: \quad 700 = -275\omega_{DE} \qquad \omega_{DE} = -2.5455 \text{ rad/s} \qquad \boldsymbol{\omega}_{DE} = 2.55 \text{ rad/s} \circlearrowright \blacktriangleleft$$

$$\mathbf{i}: \quad 200\omega_{BD} = -75\omega_{DE} \qquad \omega_{BD} = -\frac{3}{8}\omega_{BD}$$

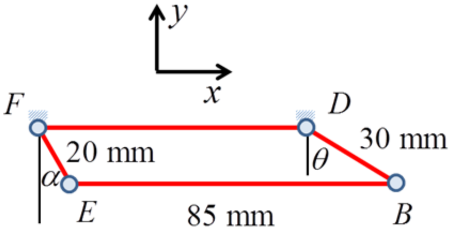
$$\omega_{BD} = -\left(\frac{3}{8}\right)(-2.5455) = 0.95455 \text{ rad/s} \qquad \boldsymbol{\omega}_{BD} = 0.955 \text{ rad/s} \circlearrowleft \blacktriangleleft$$



PROBLEM 15.65

Linkage *DBEF* is part of a windshield wiper mechanism, where points *O*, *F* and *D* are fixed pinned connections. At the position shown, $\theta = 60°$ and link *EB* is horizontal. Knowing that link *EF* has a counterclockwise angular velocity of 4 rad/s at the instant shown, determine the angular velocity of links *EB* and *DB*.

SOLUTION



Given:

$\theta = 60°$

$\omega_{EF} = 4$ rad/s

Geometry:

$30\cos\theta = 20\cos\alpha \quad \Rightarrow \quad \alpha = 41.41°$

$$\mathbf{r}_{E/F} = (20\sin\alpha)\mathbf{i} - (20\cos\alpha)\mathbf{j} = 13.229\mathbf{i} - 15.0\mathbf{j} \text{ mm}$$

$$\mathbf{r}_{B/E} = 85.0\mathbf{i} \text{ mm}$$

$$\mathbf{r}_{B/D} = (30\sin\theta)\mathbf{i} - (30\cos\theta)\mathbf{j} = 25.981\mathbf{i} - 15.0\mathbf{j} \text{ mm}$$

Relative Velocity:

$$\mathbf{v}_E = \cancel{\mathbf{v}_F} + \omega_{EF}\mathbf{k} \times \mathbf{r}_{\mathrm{E/F}} \qquad (1)$$

$$\mathbf{v}_B = \mathbf{v}_E + \omega_{EB}\mathbf{k} \times \mathbf{r}_{\mathrm{B/E}} \qquad (2)$$

$$\mathbf{v}_B = \cancel{\mathbf{v}_D} + \omega_{BD}\mathbf{k} \times \mathbf{r}_{\mathrm{B/D}} \qquad (3)$$

Sub (1) into (2) = (3):

$$\omega_{EF}\mathbf{k} \times \mathbf{r}_{\mathrm{E/F}} + \omega_{EB}\mathbf{k} \times \mathbf{r}_{\mathrm{B/E}} = \omega_{BD}\mathbf{k} \times \mathbf{r}_{\mathrm{B/D}}$$

Put in known values:

$$4\mathbf{k} \times (13.229\mathbf{i} - 15.0\mathbf{j}) + \omega_{EB}\mathbf{k} \times 85.0\mathbf{i} = \omega_{BD}\mathbf{k} \times (25.981\mathbf{i} - 15.0\mathbf{j})$$

Equate Components:

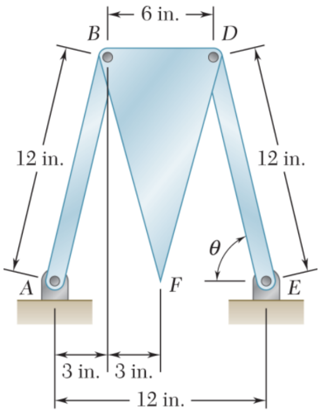
i: $60 = 15\omega_{BD} \quad \Rightarrow \quad \omega_{BD} = 4$ rad/s

$\boldsymbol{\omega}_{BD} = 4.000$ rad/s ↺ ◀

j: $52.915 + 85.0\omega_{EB} = 25.981(4)$

$\omega_{EB} = 0.600$ rad/s

$\boldsymbol{\omega}_{EB} = 0.600$ rad/s ↺ ◀



PROBLEM 15.66

Robert's linkage is named after Richard Robert (1789–1864) and can be used to draw a close approximation to a straight line by locating a pen at Point *F*. The distance *AB* is the same as *BF*, *DF* and *DE*. Knowing that the angular velocity of bar *AB* is 5 rad/s clockwise in the position shown, determine (*a*) the angular velocity of bar *DE*, (*b*) the velocity of Point *F*.

SOLUTION

Bar *AB*: $\boldsymbol{\omega}_{AB} = 5 \text{ rad/s} \circlearrowright = -(5 \text{ rad/s})\mathbf{k}$

In inches,

$$\mathbf{r}_{B/A} = 3\mathbf{i} + \sqrt{12^2 - 3^2}\,\mathbf{j} = 3\mathbf{i} + \sqrt{135}\,\mathbf{j}$$

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = -5\mathbf{k} \times (3\mathbf{i} + \sqrt{135}\,\mathbf{j})$$

$$= 5\sqrt{135}\,\mathbf{i} - 15\mathbf{j}$$

Object *BDF*: $\mathbf{r}_{D/B} = (6 \text{ in.})\mathbf{i}, \quad \mathbf{r}_{F/B} = 3\mathbf{i} - \sqrt{135}\,\mathbf{j} \text{ (in.)}, \quad \boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k}$

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{B/D} = \mathbf{v}_B + \omega_{BD}\mathbf{k} \times \mathbf{r}_{D/B}$$

$$= 5\sqrt{135}\,\mathbf{i} - 15\mathbf{j} + \omega_{BD}\mathbf{k} \times 6\mathbf{i}$$

$$= 5\sqrt{135}\,\mathbf{i} - 15\mathbf{j} + 6\omega_{BD}\mathbf{j} \qquad (1)$$

Bar *DE*: $\boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}, \quad \mathbf{r}_{D/E} = -(3 \text{ in.})\mathbf{i} + (\sqrt{135} \text{ in.})\mathbf{j},$

Point *E* is fixed so $\mathbf{v}_E = 0$

$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = \omega_{DE}\mathbf{k} \times (-3\mathbf{i} + \sqrt{135}\,\mathbf{j})$$

$$= -\sqrt{135}\,\omega_{DE}\mathbf{i} - 3\omega_{DE}\mathbf{j} \qquad (2)$$

Equating like components of $\mathbf{v}_D$ from Eqs. (1) and (2),

i: $5\sqrt{135} = -\sqrt{135}\,\omega_{DE}$ (3)

j: $-15 + 6\omega_{BD} = -3\omega_{DE}$ (4)

(*a*) *Angular velocity of bar DE.*

From Eq. (3), $\omega_{DE} = -5 \text{ rad/s}$ $\qquad \boldsymbol{\omega}_{DE} = 5.00 \text{ rad/s} \circlearrowright$ ◀

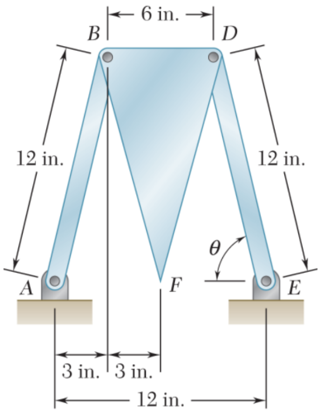
From Eq. (4), $\omega_{BD} = \frac{1}{6}(15 - 3\omega_{DE}) = \frac{1}{6}(15 + 15)$ $\qquad \boldsymbol{\omega}_{BD} = 5.00 \text{ rad/s} \circlearrowleft$

PROBLEM 15.66 (Continued)

(*b*) *Velocity of Point F.* $\mathbf{v}_{F/B} = 3i - \sqrt{135}\mathbf{j}$

$$\mathbf{v}_F = \mathbf{v}_B = \mathbf{v}_{F/B} = \mathbf{v}_B + \omega_{BD}\mathbf{k} \times \mathbf{r}_{F/B}$$
$$= 5\sqrt{135}\mathbf{i} - 15\mathbf{j} + 5\mathbf{k} \times (3\mathbf{i} - \sqrt{135}\mathbf{j})$$
$$= 5\sqrt{135}\mathbf{i} - 15\mathbf{j} + 15\mathbf{j} + 5\sqrt{135}\mathbf{i} = 10\sqrt{135}\mathbf{i}$$

$\mathbf{v}_F = 116.2 \text{ in./s} \longrightarrow$ ◀

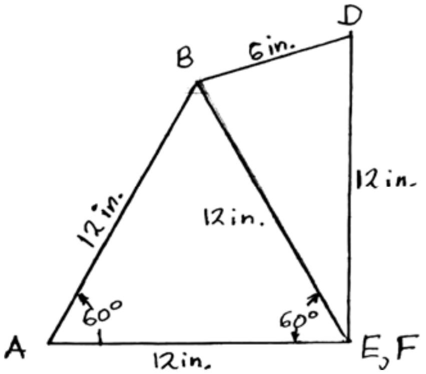


PROBLEM 15.67

Robert's linkage is named after Richard Robert (1789–1864) and can be used to draw a close approximation to a straight line by locating a pen at Point *F*. The distance *AB* is the same as *BF*, *DF* and *DE*. Knowing that the angular velocity of plate *BDF* is 2 rad/s counterclockwise when $\theta = 90°$, determine (*a*) the angular velocities of bars *AB* and *DE*, (*b*) the velocity of Point *F*. When $\theta = 90°$, point F may be assumed to coincide with point E, with negligible error in the velocity analysis.

SOLUTION

When $\theta = 90°$, the configuration of the linkage is close to that shown at the right.



Bar *AB*: $\boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k}$

In inches, $\mathbf{r}_{B/A} = 6\mathbf{i} + 6\sqrt{3}\mathbf{j}$

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \omega_{AB}\mathbf{k} \times (6\mathbf{i} + 6\sqrt{3}\mathbf{j}) = -6\sqrt{3}\omega_{AB}\mathbf{i} + 6\omega_{AB}\mathbf{j}$$

Object *BDF*: $\mathbf{r}_{D/B} = 6\mathbf{i} + 6(2-\sqrt{3})\mathbf{j}$

$\mathbf{r}_{F/B} = 6\mathbf{i} - 6\sqrt{3}\mathbf{j} \qquad \boldsymbol{\omega}_{BD} = (2\text{ rad/s})\mathbf{k}$

$$\begin{aligned}\mathbf{v}_D &= \mathbf{v}_B + \mathbf{v}_{D/B} = \mathbf{v}_B = \omega_{BD}\mathbf{k} \times \mathbf{r}_{D/B}\\ &= -6\sqrt{3}\omega_{AB}\mathbf{i} + 6\omega_{AB}\mathbf{j} + 2\mathbf{k} \times [6\mathbf{i} + 6(2-\sqrt{3})\mathbf{j}]\\ &= -6\sqrt{3}\omega_{AB}\mathbf{i} + 6\omega_{AB}\mathbf{j} + 24\mathbf{i} - 12\sqrt{3}\mathbf{i} + 12\mathbf{j} \qquad (1)\end{aligned}$$

Bar *DE*: $\boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}, \qquad \mathbf{r}_{D/E} = 12\mathbf{j}$

$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = \omega_{DE}\mathbf{k} \times 12\mathbf{j} = -12\omega_{DE}\mathbf{i} \qquad (2)$$

Equating like components of $\mathbf{v}_D$ from Eqs. (1) and (2),

i: $-6\sqrt{3}\omega_{AB} + 12 - 6\sqrt{3} = -12\omega_{DE}$ (3)

j: $6\omega_{AB} + 12 = 0$

$\omega_{AB} = -2\text{ rad/s}$ $\qquad \omega_{AB} = 2.00\text{ rad/s} \circlearrowright$ ◀

From Eq. (3), $-12\omega_{DE} = -(6\sqrt{3})(-2) + 24 - 12\sqrt{3}$

$\omega_{DE} = 2 - 2\sqrt{3} = -1.4641$ $\qquad \boldsymbol{\omega}_{DE} = 1.464\text{ rad/s} \circlearrowright$ ◀

PROBLEM 15.67 (Continued)

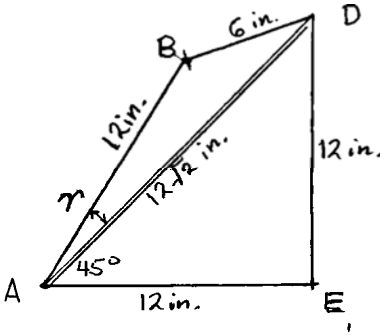
$$\mathbf{v}_D = -(12)(-1.4641)\mathbf{i} = 17.569\mathbf{i}$$

$$\mathbf{v}_F = \mathbf{v}_D + \omega_{BD}\mathbf{k} \times \mathbf{r}_{F/D}$$

$$= 17.569\mathbf{i} + 2\mathbf{k} \times (-12\mathbf{j}) = (41.569 \text{ in./s})\mathbf{i}$$

$$\mathbf{v}_F = 41.6 \text{ in./s} \longrightarrow \blacktriangleleft$$

Note: The exact configuration of the linkage when $\theta = 90°$ may be calculated from trigonometry using the figure given below.



Applying the law of cosines to triangle *ADB* gives

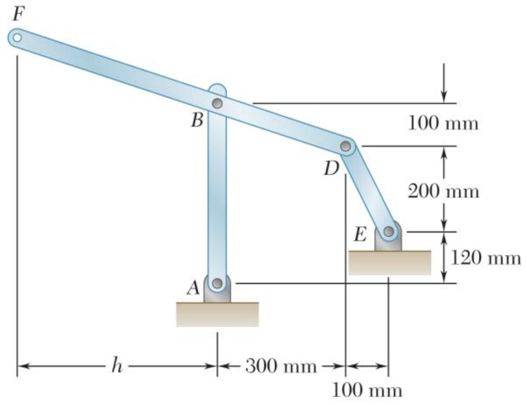
$$\gamma = 13.5°$$

so that angle *EAB* is

$$45° + 13.5° = 58.5°.$$

We used 60° in the approximate analysis.

Point *F* then lies about 0.53 in. to the right of Point *E*.



PROBLEM 15.68

In the position shown, bar *DE* has a constant angular velocity of 10 rad/s clockwise. Knowing that $h = 500$ mm, determine (*a*) the angular velocity of bar *FBD*, (*b*) the velocity of Point *F*.

SOLUTION

Bar *DE*: (Rotation about *E*)

$$\boldsymbol{\omega}_{DE} = 10 \text{ rad/s} \circlearrowright = -(10 \text{ rad/s})\mathbf{k}$$

$$\mathbf{r}_{D/E} = -(0.1 \text{ m})\mathbf{i} + (0.2 \text{ m})\mathbf{j}$$

$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = (-10\mathbf{k}) \times (-0.1\mathbf{i} + 0.2\mathbf{j})$$

$$= (1 \text{ m/s})\mathbf{j} + (2 \text{ m/s})\mathbf{i}$$

Bar *FBD*: (Plane motion = Translation with *D* + Rotation about *D*.)

$$\boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \quad \mathbf{r}_{B/D} = -(0.3 \text{ m})\mathbf{i} + (0.1 \text{ m})\mathbf{j}$$

$$\mathbf{v}_B = \mathbf{v}_D + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{B/D}$$

$$= \mathbf{j} + 2\mathbf{i} + (\omega_{BD}\mathbf{k}) \times (-0.3\mathbf{i} + 0.1\mathbf{j})$$

$$= \mathbf{j} + 2\mathbf{i} - 0.3\omega_{BD}\mathbf{j} - 0.1\omega_{BD}\mathbf{i}$$

Bar *AB*: (Rotation about *A*)

$$\boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k} \quad \mathbf{r}_{B/A} = (0.42 \text{ m})\mathbf{j}$$

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = (\omega_{AB}\mathbf{k}) \times (0.42\mathbf{j}) = -0.42\omega_{AB}\mathbf{i}$$

Equating components of the two expressions for $\mathbf{v}_B$,

(*a*) **j**: $1 - 0.3\omega_{BD} = 0 \qquad \omega_{BD} = 3.3333 \text{ rad/s} \qquad \boldsymbol{\omega}_{BD} = 3.33 \text{ rad/s} \circlearrowleft$ ◀

i: $2 - 0.1\omega_{BD} = -0.42\omega_{AB} \qquad 2 - (0.1)(3.3333) = -0.42\omega_{AB}$

$$\omega_{AB} = -3.9683 \text{ rad/s} \qquad \omega_{AB} = 3.97 \text{ rad/s} \circlearrowright$$

Bar *FBD*: $\mathbf{r}_{F/D} = C\mathbf{r}_{B/D} \quad \text{where} \quad C = \dfrac{h + 0.3}{0.3}$

$$\mathbf{v}_F = \mathbf{v}_B + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{F/D}$$

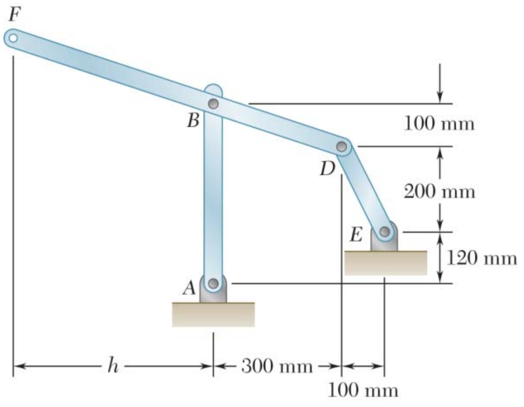
$$= \mathbf{j} + 2\mathbf{i} + C(-0.3\omega_{BD}\mathbf{j} - 0.1\omega_{BD}\mathbf{i})$$

$$= \mathbf{j} + 2\mathbf{i} + C(-\mathbf{j} - 0.33333\mathbf{i})$$

With $h = 500 \text{ mm} = 0.5 \text{ m}, \quad C = \dfrac{0.8}{0.3} = 2.6667$

$$\mathbf{v}_F = \mathbf{j} + 2\mathbf{i} - 2.6667\mathbf{j} - 0.88889\mathbf{i}$$

(*b*) $\mathbf{v}_F = (1.11111 \text{ m/s})\mathbf{i} - (1.66667 \text{ m/s})\mathbf{j} \qquad \mathbf{v}_F = 2.00 \text{ m/s} \searrow 56.3°$ ◀



PROBLEM 15.69

In the position shown, bar *DE* has a constant angular velocity of 10 rad/s clockwise. Determine (*a*) the distance *h* for which the velocity of Point *F* is vertical, (*b*) the corresponding velocity of Point *F*.

SOLUTION

Bar *DE*: (Rotation about *E*)

$$\boldsymbol{\omega}_{DE} = 10 \text{ rad/s} \circlearrowright = -(10 \text{ rad/s})\mathbf{k}$$

$$\mathbf{r}_{D/E} = -(0.1 \text{ m})\mathbf{i} + (0.2 \text{ m})\mathbf{j}$$

$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = (-10\mathbf{k}) \times (-0.1\mathbf{i} + 0.2\mathbf{j})$$
$$= (1 \text{ m/s})\mathbf{j} + (2 \text{ m/s})\mathbf{i}$$

Bar *FBD*: (Plane motion = Translation with *D* + Rotation about *D*.)

$$\boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \quad \mathbf{r}_{B/D} = -(0.3 \text{ m})\mathbf{i} + (0.1 \text{ m})\mathbf{j}$$

$$\mathbf{v}_B = \mathbf{v}_D + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{B/D}$$
$$= \mathbf{j} + 2\mathbf{i} + (\omega_{BD}\mathbf{k}) \times (-0.3\mathbf{i} + 0.1\mathbf{j})$$
$$= \mathbf{j} + 2\mathbf{i} - 0.3\omega_{BD}\mathbf{j} - 0.1\omega_{BD}\mathbf{i}$$

Bar *AB*: (Rotation about *A*)

$$\boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k} \quad \mathbf{r}_{B/A} = (0.42 \text{ m})\mathbf{j}$$

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = (\omega_{AB}\mathbf{k}) \times (0.42\mathbf{j}) = -0.42\omega_{AB}\mathbf{i}$$

Equating components of the two expressions for $\mathbf{v}_B$,

(*a*) **j**: $1 - 0.3\omega_{BD} = 0 \qquad \omega_{BD} = 3.3333 \text{ rad/s}$

i: $2 - 0.1\omega_{BD} = -0.42\omega_{AB} \qquad 2 - (0.1)(3.3333) = -0.42\omega_{AB}$

$$\omega_{AB} = -3.9683 \text{ rad/s} \qquad \omega_{AB} = 3.97 \text{ rad/s} \circlearrowright$$

Bar *FBD*:

$$\mathbf{r}_{F/D} = C\mathbf{r}_{B/D} \quad \text{where} \quad C = \frac{h + 0.3}{0.3}$$

$$\mathbf{v}_F = \mathbf{v}_B + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{F/D}$$
$$= \mathbf{j} + 2\mathbf{i} + C(-0.3\omega_{BD}\mathbf{j} - 0.1\omega_{BD}\mathbf{i})$$
$$= \mathbf{j} + 2\mathbf{i} + C(-\mathbf{j} - 0.33333\mathbf{i})$$

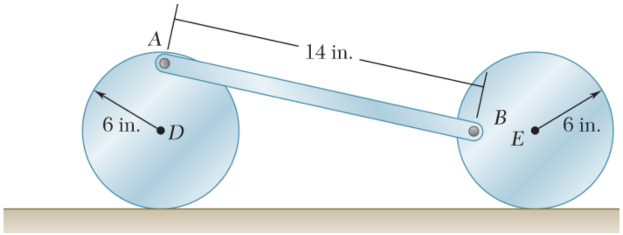
PROBLEM 15.69 (Continued)

But $\mathbf{v}_F = v_F\mathbf{j}$. Equating components of the two expressions for $\mathbf{v}_F$,

$$\mathbf{i}:\quad 0 = 2 - 0.33333C \qquad C = 6$$

(*a*) $$h = 0.3C - 0.3 = (0.3)(6) - 0.3 \qquad h = 1.500 \text{ m} \blacktriangleleft$$

(*b*) $$\mathbf{j}:\quad v_F = \mathbf{j} - C\mathbf{j} = (1-6)\mathbf{j} \qquad \mathbf{v}_F = 5.00 \text{ m/s}\downarrow \blacktriangleleft$$



PROBLEM 15.70

Both 6-in.-radius wheels roll without slipping on the horizontal surface. Knowing that the distance *AD* is 5 in., the distance *BE* is 4 in. and *D* has a velocity of 6 in./s to the right, determine the velocity of Point *E*.

SOLUTION

Disk *D*: Velocity at the contact Point *P* with the ground is zero.

$$\mathbf{v}_0 = 6\text{ in./s} \longrightarrow$$

$$\omega_D = \frac{v_D}{r_{D/P}} = \frac{6\text{ in./s}}{6\text{ in.}} = 1\text{ rad/s} \qquad \boldsymbol{\omega}_D = 1\text{ rad/s} \circlearrowright$$

At Point *A*,

$$v_A = r_{A/P}\omega_D = (6\text{ in.} + 5\text{ in.})(1\text{ rad/s}) = 11\text{ in./s}$$

$$\mathbf{v}_A = 11\text{ in./s} \longrightarrow$$

Disk *E*: Velocity at the contact Point *Q* with the ground is zero. $\boldsymbol{\omega}_E = \omega_E \circlearrowleft = \omega_E\mathbf{k}.$

$$\mathbf{r}_{B/Q} = -(4\text{ in.})\mathbf{i} + (6\text{ in.})\mathbf{j}$$

$$\mathbf{v}_B = \mathbf{v}_{B/Q} = \boldsymbol{\omega}_E \times \mathbf{r}_{B/Q} = \omega_E\mathbf{k} \times (-4\mathbf{i} + 6\mathbf{j})$$

$$\mathbf{v}_B = -6\omega_E\mathbf{i} - 4\omega_E\mathbf{j} \qquad (1)$$

Connecting rod *AB*:

$$\mathbf{r}_{B/A} = (\sqrt{14^2 - 5^2})\mathbf{i} - 5\mathbf{j} \text{ in inches.}$$

$$\mathbf{v}_{B/A} = \sqrt{171}\mathbf{i} - 5\mathbf{j} \qquad \boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k}$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \omega_{AB}\mathbf{k} \times (\sqrt{171}\mathbf{i} - 5\mathbf{j})$$

$$= 11\mathbf{i} + 5\omega_{AB}\mathbf{i} + \sqrt{171}\omega_{AB}\mathbf{j} \qquad (2)$$

Equating expressions (1) and (2) for $\mathbf{v}_B$ gives

$$-6\omega_E\mathbf{i} - 4\omega_E\mathbf{j} = 11\mathbf{i} + 5\omega_{AB}\mathbf{i} + \sqrt{171}\omega_{AB}\mathbf{j}$$

Equating like components and transposing terms,

i: $$5\omega_{AB} + 6\omega_E = -11 \qquad (3)$$

j: $$\sqrt{171}\omega_{AB} + 4\omega_E = 0 \qquad (4)$$

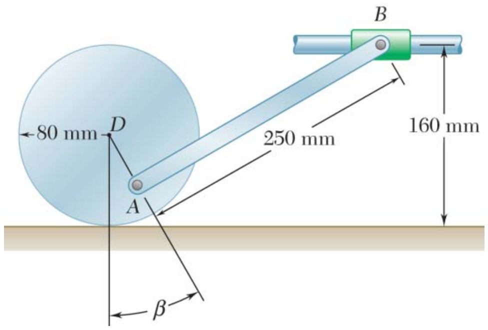
Solving the simultaneous equations (3) and (4),

$$\omega_{AB} = 0.75265\text{ rad/s}, \qquad \omega_E = -2.4605\text{ rad/s}$$

Velocity of Point E.

$$\mathbf{v}_E = \omega_E\mathbf{k} \times \mathbf{r}_{E/Q} = -2.4605\mathbf{k} \times 6\mathbf{j}$$

$$\mathbf{v}_E = 14.76\text{ in./s }\mathbf{i} = 14.76\text{ in./s} \longrightarrow \blacktriangleleft$$



PROBLEM 15.71

The 80-mm-radius wheel shown rolls to the left with a velocity of 900 mm/s. Knowing that the distance AD is 50 mm, determine the velocity of the collar and the angular velocity of rod AB when (a) $\beta = 0$, (b) $\beta = 90°$.

SOLUTION

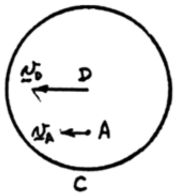
(a) $\beta = 0.$

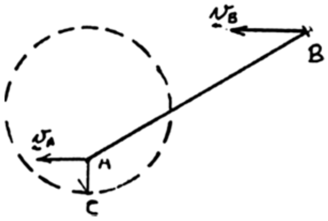
Wheel AD. $\mathbf{v}_C = 0, \qquad \mathbf{v}_D = 45 \text{ in./s} \leftarrow$

$$\omega_{AD} = \frac{v_D}{CD} = \frac{900}{80} = 11.25 \text{ rad/s} \circlearrowleft$$

$$CA = (CD) - (DA) = 80 - 50 = 30 \text{ mm}$$

$$v_A = (CA)\omega_{AD} = (30)(11.25) = 337.5 \text{ mm/s} \leftarrow$$





Rod AB. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$$[v_B \leftarrow] = [337.5 \leftarrow] + [v_{B/A} \nwarrow \varphi] \qquad \mathbf{v}_B = 338 \text{ mm/s} \leftarrow \; ◀$$

$$v_{B/A} = 0 \qquad \omega_{AB} = 0 \; ◀$$

(b) $\beta = 90°.$

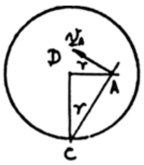
Wheel AD. $\mathbf{v}_C = 0, \qquad \omega_{AD} = 11.25 \text{ rad/s} \circlearrowleft$

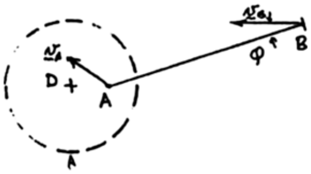
$$\tan\gamma = \frac{DA}{DC} = \frac{50}{80}, \qquad \gamma = 32.005°$$

$$CA = \frac{DC}{\cos\gamma} = 94.34 \text{ mm}$$

$$v_A = (CA)\omega_{AD} = (94.34)(11.25) = 1061.3 \text{ mm/s}$$

$$\mathbf{v}_A = [1061.3 \text{ mm/s} \nwarrow 32.005°]$$



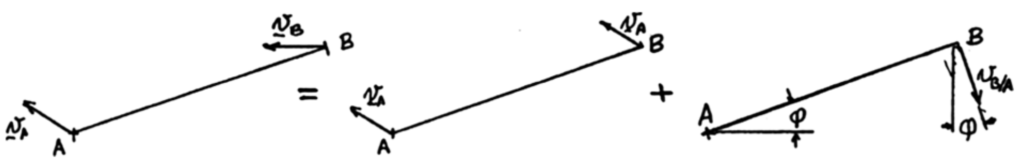


Rod AB. $\mathbf{v}_B = v_B \leftarrow$

$$\sin\varphi = \frac{80}{250}, \quad \varphi = 18.663°$$

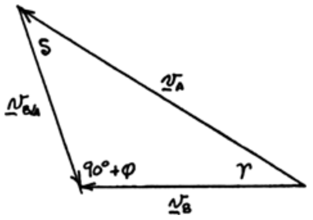
Plane motion = Translation with A + Rotation about A.

PROBLEM 15.71 (Continued)



$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \qquad [v_B \leftarrow] = [v_A \searrow \gamma] + [v_{B/A} \searrow \varphi]$$

Draw velocity vector diagram.



$$\delta = 180° - \gamma - (90° + \varphi)$$

$$= 90° - 32.005° - 18.663° = 39.332°$$

Law of sines.

$$\frac{v_B}{\sin\delta} = \frac{v_{B/A}}{\sin\gamma} = \frac{v_A}{\sin(90° + \varphi)}$$

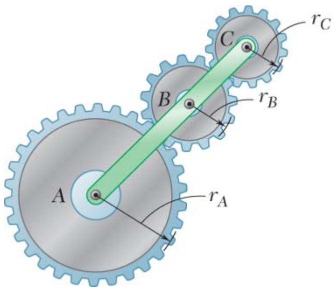
$$v_B = \frac{v_A \sin\delta}{\sin(90° + \varphi)} = \frac{(1061.3)\sin 39.332°}{\sin 108.663°}$$

$$= 710 \text{ mm/s} \qquad \mathbf{v}_B = 710 \text{ mm/s} \leftarrow \blacktriangleleft$$

$$v_{B/A} = \frac{v_A \sin\gamma}{\sin(90° + \varphi)} = \frac{(1061.3)\sin 32.005°}{\sin 108.663°}$$

$$= 593.8 \text{ mm/s}$$

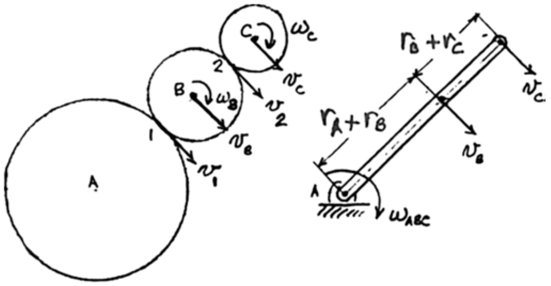
$$\omega_{AB} = \frac{v_{B/A}}{\overline{AB}} = \frac{593.8}{250} = 2.37 \text{ rad/s} \qquad \boldsymbol{\omega}_{AB} = 2.37 \text{ rad/s} \circlearrowright \blacktriangleleft$$



PROBLEM 15.72*

For the gearing shown, derive an expression for the angular velocity ω_C of gear C and show that ω_C is independent of the radius of gear B. Assume that Point A is fixed and denote the angular velocities of rod ABC and gear A by ω_{ABC} and ω_A respectively.

SOLUTION



Label the contact point between gears A and B as 1 and that between gears B and C as 2.

Rod ABC: $\omega_{ABC} = \omega_{ABC}$ Assume $\circlearrowright$ for sketch.

$$v_A = 0$$

$$v_B = (r_A + r_B)\omega_{ABC} \searrow$$

$$v_C = (r_A + 2r_B + r_C)\omega_{ABC} \searrow$$

Gear A: $\omega_A = 0, \quad v_A = 0, \quad v_1 = 0$

Gear B: $v_1 = v_B - r_B\,\omega_B = 0$

$$(r_A + r_B)\omega_{ABC} - r_B\,\omega_B = 0$$

$$\omega_B = \left(\frac{r_A + r_B}{r_B}\right)\omega_{ABC} \circlearrowright$$

$$v_2 = v_B + r_B\,\omega_B$$

$$= 2(r_A + r_B)\omega_{ABC} \searrow$$

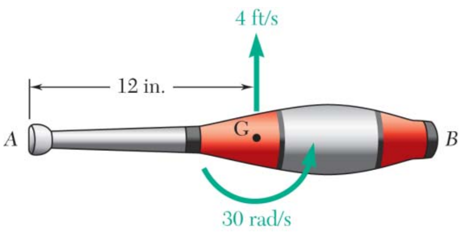
PROBLEM 15.72* (Continued)

Gear C: $$v_2 = v_C - r_C\,\omega_C$$

$$2(r_A + r_B)\omega_{ABC} = (r_A + 2r_B + r_C)\omega_{ABC} - r_C\omega_C$$

$$\omega_C = (r_A - r_C)\omega_{ABC} = -r_C\,\omega_C \qquad \omega_C = \left(1 - \frac{r_A}{r_C}\right)\omega_{ABC} \blacktriangleleft$$

Note that the result is independent of r_B.



PROBLEM 15.73

A juggling club is thrown vertically into the air. The center of gravity G of the 20 in. club is located 12 in. from the knob. Knowing that at the instant shown G has a velocity of 4 ft/s upwards and the club has an angular velocity of 30 rad/s counterclockwise, determine (*a*) the speeds of Point A and B, (*b*) the location of the instantaneous center of rotation.

SOLUTION

Unit vectors: $\mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1 \uparrow, \quad \mathbf{k} = 1 \circlearrowleft$

Relative positions: $\mathbf{r}_{A/G} = -(1\text{ ft})\mathbf{i}, \quad \mathbf{r}_{B/A} = \left(\frac{8}{12}\text{ ft}\right)\mathbf{i}$

Angular velocity: $\boldsymbol{\omega} = 30\text{ rad/s} \circlearrowleft = (30\text{ rad/s})\mathbf{k}$

Velocity at A:

$$\begin{aligned}\mathbf{v}_A &= \mathbf{v}_G + \mathbf{v}_{A/G} = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{A/G}\\ &= (4\text{ ft/s})\mathbf{j} + (30\text{ rad/s})\mathbf{k} \times (-1\text{ ft})\mathbf{i}\\ &= (4\text{ ft/s})\mathbf{j} - (30\text{ ft/s})\mathbf{j} = -(26\text{ ft/s})\mathbf{j}\\ &= 26\text{ ft/s}\downarrow\end{aligned}$$

$v_A = 26.0\text{ ft/s}$ ◀

Velocity at B:

$$\begin{aligned}\mathbf{v}_B &= \mathbf{v}_G + \mathbf{v}_{B/G} = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{B/G}\\ &= (4\text{ ft/s})\mathbf{j} + (30\text{ rad/s})\mathbf{k} \times \left(\frac{8}{12}\text{ ft}\right)\mathbf{i}\\ &= (4\text{ ft/s})\mathbf{j} + (20\text{ ft/s})\mathbf{j} = (24\text{ ft/s})\mathbf{j}\\ &= 24\text{ ft/s}\uparrow\end{aligned}$$

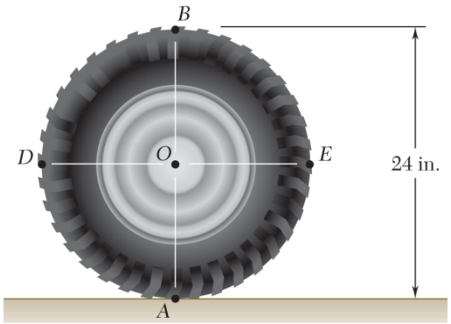
$v_B = 24.0\text{ ft/s}$ ◀

Let $\mathbf{r}_{C/G} = x\mathbf{i}$ bet the position of the instantaneous center C relative to G.

$$\begin{aligned}\mathbf{v}_C &= \mathbf{v}_G + \mathbf{v}_{C/G} = \mathbf{v}_G + \boldsymbol{\omega} \times (x\,\mathbf{i})\\ &= (4\text{ ft/s})\mathbf{j} + (30\text{ rad/s})\mathbf{k} \times (x\,\mathbf{i})\\ &= (4\text{ ft/s})\mathbf{j} + (30\text{ ft/s})\,x\,\mathbf{j} = 0\end{aligned}$$

$$x = -\frac{4\text{ ft/s}}{30\text{ rad/s}} = -\frac{4}{30}\text{ ft} = -1.6\text{ in.}$$

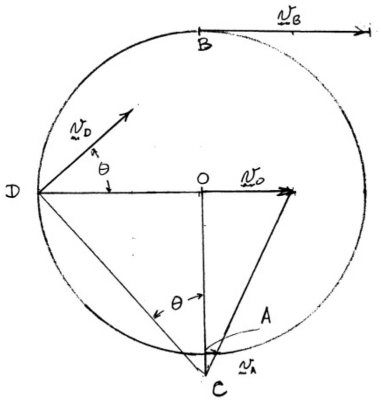
Point C lies 1.6 in. to the left of G. ◀



PROBLEM 15.74

At the instant shown during deceleration, the velocity of an automobile is 40 ft/s to the right. Knowing that the velocity of the contact point *A* of the wheel with the ground is 5 ft/s to the right, determine (*a*) the instantaneous center of rotation of the wheel, (*b*) the velocity of point *B*, (*c*) the velocity of point *D*.

SOLUTION



$$\omega = \frac{v_O - v_A}{l_{AO}} = \frac{40-5}{1} = 35 \text{ rad/s} \circlearrowright$$

(*a*)

$$l_{CA} = \frac{v_A}{\omega} = \frac{5}{35} = \frac{1}{7} \text{ ft}$$

$$= 1.714 \text{ in.}$$

C lies 1.714 in. below *A*. ◀

(*b*)

$$l_{CB} = 2 + \frac{1}{7} = \frac{15}{7} \text{ ft}$$

$$v_B = l_{CB}\omega = \frac{15}{7} \cdot 35 = 75 \text{ ft/s}$$

$\mathbf{v}_B = 75.0$ ft/s ⟶ ◀

(*c*)

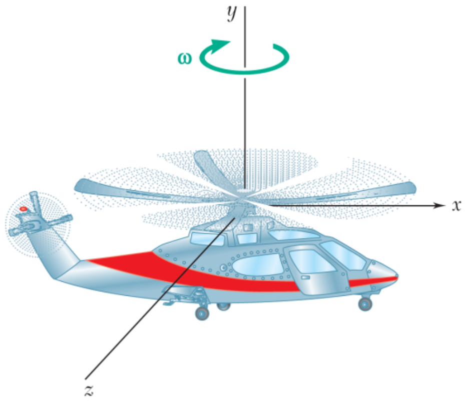
$$l_{OD} = 1 \text{ ft}, \quad l_{CO} = 1 + \frac{1}{7} = \frac{8}{7} \text{ ft}$$

$$l_{CD} = \sqrt{1^2 + \left(\frac{8}{7}\right)^2} = 1.5186 \text{ ft}$$

$$v_D = l_{CD}\omega = (1.5186)(35) = 53.2 \text{ ft/s}$$

$$\tan\theta = \frac{l_{OD}}{l_{CO}} = \frac{7}{8}, \quad \theta = 41.2^\circ$$

$\mathbf{v}_D = 53.2$ ft/s $\measuredangle$ 41.2° ◀



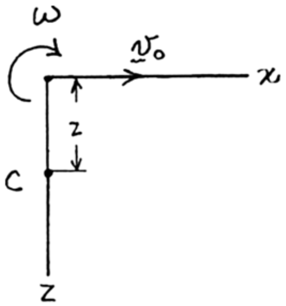
PROBLEM 15.75

A helicopter moves horizontally in the x direction at a speed of 120 mi/h. Knowing that the main blades rotate clockwise with an angular velocity of 180 rpm, determine the instantaneous axis of rotation of the main blades.

SOLUTION

$$\mathbf{v}_0 = 120 \text{ mi/h} = 176 \text{ ft/s} \longrightarrow$$

$$\omega = 180 \text{ rpm} = \frac{(180)(2\pi)}{60} = 18.85 \text{ rad/s}$$

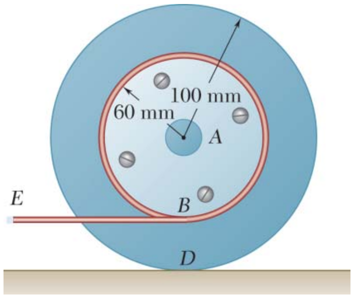


$$v_0 = z\omega$$

$$z = \frac{v_0}{\omega} = \frac{176}{18.85} = 9.34 \text{ ft}$$

Instantaneous axis is parallel to the y axis and passes through the point $x = 0$ ◀

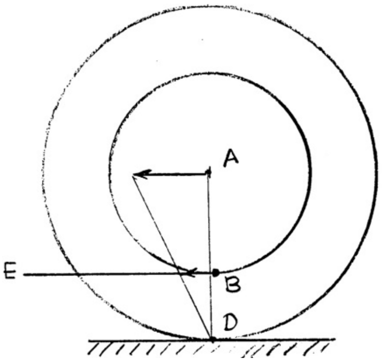
$z = 9.34$ ft ◀



PROBLEM 15.76

A 60-mm-radius drum is rigidly attached to a 100-mm-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that end *E* of the cord is pulled to the left with a velocity of 120 mm/s, determine (*a*) the angular velocity of the drums, (*b*) the velocity of the center of the drums, (*c*) the length of cord wound or unwound per second.

SOLUTION



Since the drum rolls without sliding, its instantaneous center lies at *D*.

$$\mathbf{v}_E = \mathbf{v}_B = 120 \text{ mm/s} \longleftarrow$$

$$v_A = v_{A/D}\omega, \quad v_B = r_{B/D}\omega$$

(*a*)

$$\omega = \frac{v_B}{v_{B/D}} = \frac{120}{100-60} = 3 \text{ rad/s}$$

$$\boldsymbol{\omega} = 3.00 \text{ rad/s} \circlearrowleft \blacktriangleleft$$

(*b*)

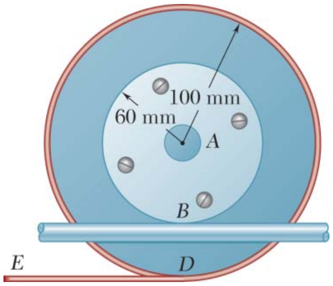
$$v_A = (100)(3) = 300 \text{ mm/s}$$

$$\mathbf{v}_A = 300 \text{ mm/s} \longleftarrow \blacktriangleleft$$

Since v_A is greater than v_B, cord is being wound.

$$v_A - v_B = 300 - 120 = 180 \text{ mm/s}$$

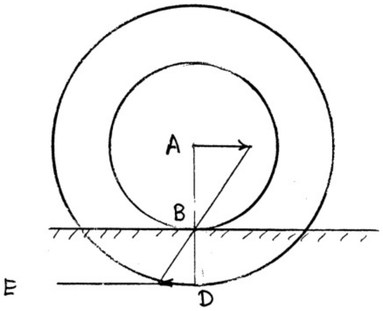
(*c*) Cord wound per second $= 180.0$ mm ◀



PROBLEM 15.77

A 60-mm-radius drum is rigidly attached to a 100-mm-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that end *E* of the cord is pulled to the left with a velocity of 120 mm/s, determine (*a*) the angular velocity of the drums, (*b*) the velocity of the center of the drums, (*c*) the length of cord wound or unwound per second.

SOLUTION



Since the drum rolls without sliding, its instantaneous center lies at *B*.

$$\mathbf{v}_E = \mathbf{v}_D = 120 \text{ mm/s} \longleftarrow$$

$$v_A = r_{A/B}\omega, \; v_D = r_{D/B}\omega$$

(*a*)
$$\omega = \frac{v_D}{r_{D/B}} = \frac{120}{100-60} = 3 \text{ rad/s}$$

$$\boldsymbol{\omega} = 3.00 \text{ rad/s} \circlearrowright \blacktriangleleft$$

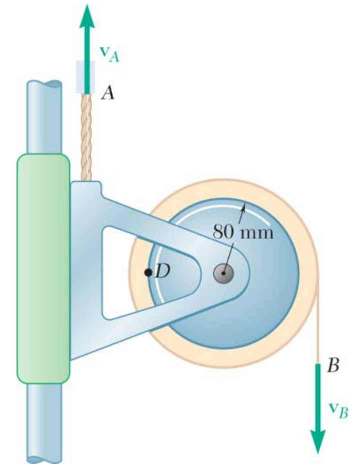
(*b*)
$$v_A = (60)(3.00) = 180 \text{ mm/s}$$

$$\mathbf{v}_A = 180 \text{ mm/s} \longrightarrow \blacktriangleleft$$

Since $\mathbf{v}_A$ is to the right and $\mathbf{v}_D$ is to the left, cord is being unwound.

$$v_A - v_E = 180 + 120 = 300 \text{ mm/s}$$

(*c*) Cord unwound per second $= 300$ mm ◀



PROBLEM 15.78

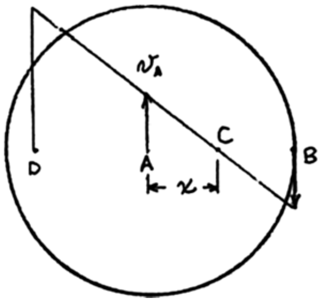
The spool of tape shown and its frame assembly are pulled upward at a speed $v_A = 750$ mm/s. Knowing that the 80-mm-radius spool has an angular velocity of 15 rad/s clockwise and that at the instant shown the total thickness of the tape on the spool is 20 mm, determine (*a*) the instantaneous center of rotation of the spool, (*b*) the velocities of Points *B* and *D*.

SOLUTION

$$\mathbf{v}_A = 750 \text{ mm/s}\uparrow$$

$$\omega = 15 \text{ rad/s} \circlearrowright$$

$$x = \frac{v_A}{\omega} = \frac{750}{15} = 50 \text{ mm}$$



(*a*) The instantaneous center lies 50 mm to the right of the axle. ◀

$$CB = 80 + 20 - 50 = 50 \text{ mm}$$

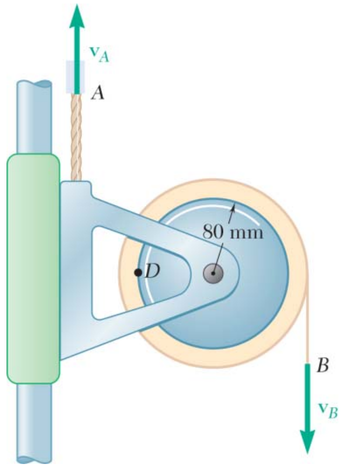
(*b*) $v_B = (CB)\omega = (50)(15) = 750$ mm/s

$$\mathbf{v}_B = 750 \text{ mm/s}\downarrow$$ ◀

$$CD = 80 + 50 = 130 \text{ mm}$$

$$v_D = (CD)\omega = (130)(15) = 1950 \text{ mm/s}$$

$$\mathbf{v}_D = 1.950 \text{ m/s}\uparrow$$ ◀



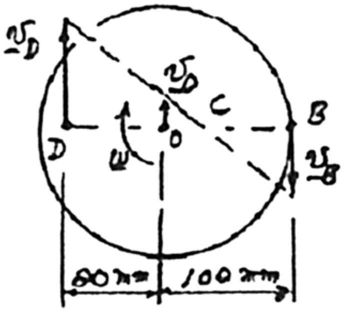
PROBLEM 15.79

The spool of tape shown and its frame assembly are pulled upward at a speed $v_A = 100$ mm/s. Knowing that end B of the tape is pulled downward with a velocity of 300 mm/s and that at the instant shown the total thickness of the tape on the spool is 20 mm, determine (*a*) the instantaneous center of rotation of the spool, (*b*) the velocity of Point D of the spool.

SOLUTION

$$v_D = v_A = 100 \text{ mm/s}$$

(*a*) Since $\mathbf{v}_0$ and $\mathbf{v}_B$ are parallel, instantaneous center C is located at intersection of BC and line joining end points of $\mathbf{v}_D$ and $\mathbf{v}_B$.



Similar triangles.

$$\frac{OC}{v_0} = \frac{BC}{v_B} = \frac{OC + BC}{v_0 + v_B}$$

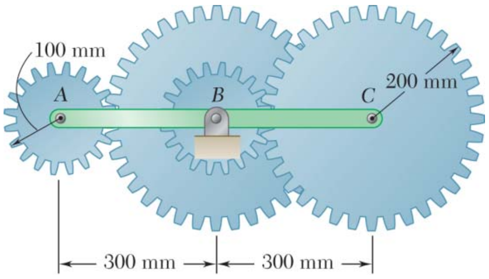
$$OC = \frac{v_0}{v_0 + v_B}(OC + BC)$$

$$OC = \frac{100 \text{ mm/s}}{(100 + 300) \text{ mm/s}}(100 \text{ mm})$$

$$= 25 \text{ mm} \quad \blacktriangleleft$$

(*b*) $$\frac{v_D}{(DO) + (OC)} = \frac{v_0}{(OC)}; \qquad \frac{v_D}{(80 + 25) \text{ mm}} = \frac{100 \text{ mm/s}}{25 \text{ mm}}$$

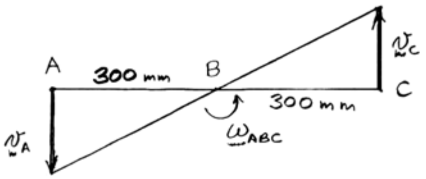
$$\mathbf{v}_D = 420 \text{ mm/s} \uparrow \blacktriangleleft$$


100 mm
A
B
C
200 mm
300 mm
300 mm

PROBLEM 15.80

The arm *ABC* rotates with an angular velocity of 4 rad/s counterclockwise. Knowing that the angular velocity of the intermediate gear *B* is 8 rad/s counterclockwise, determine (*a*) the instantaneous centers of rotation of gears *A* and *C*, (*b*) the angular velocities of gears *A* and *C*.

SOLUTION


A
300 mm
B
v_C
C
300 mm
ω_{ABC}
v_A

Contact points:

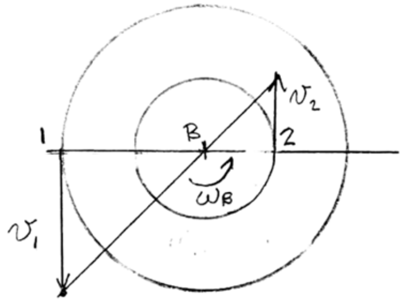
1 between gears *A* and *B*.

2 between gears *B* and *C*.

Arm ABC: $\omega_{ABC} = 4 \text{ rad/s} \circlearrowleft$

$$v_A = (0.300)(4) = 1.2 \text{ m/s} \downarrow$$

$$v_C = (0.300)(4) = 1.2 \text{ m/s} \uparrow$$

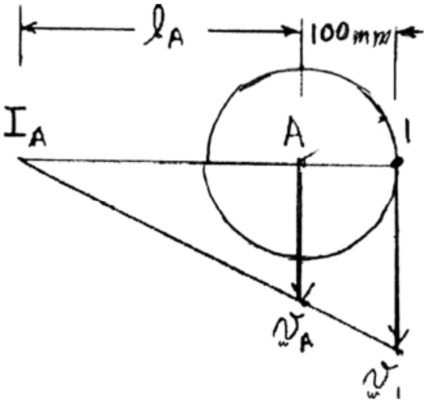

v_2
1
B
2
ω_B
v_1

Gear B: $\omega_B = 8 \text{ rad/s} \circlearrowleft$

$$v_1 = (0.200)(8) = 1.6 \text{ m/s} \downarrow$$

$$v_2 = (0.100)(8) = 0.8 \text{ m/s} \uparrow$$

Gear A:

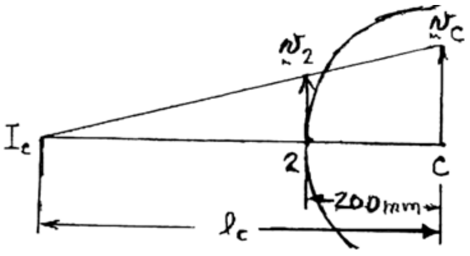

ℓ_A
100 mm
I_A
A
1
v_A
v_1

$$\omega_A = \frac{v_1 - v_A}{0.100} = \frac{1.6 - 1.2}{0.100}$$

$$\omega_A = 4 \text{ rad/s} \circlearrowright$$

$$\ell_A = \frac{v_A}{\omega_A} = \frac{1.2}{4} = 0.3 \text{ m} = 300 \text{ mm}$$

PROBLEM 15.80 (Continued)



Gear C:

$$\omega_C = \frac{v_C - v_2}{0.200} = \frac{1.2 - 0.8}{0.2}$$

$$\omega_C = 2 \text{ rad/s} \circlearrowleft$$

$$\ell_C = \frac{v_C}{\omega_C} = \frac{1.2}{2} = 0.6 \text{ m}$$

(*a*) *Instantaneous centers.*

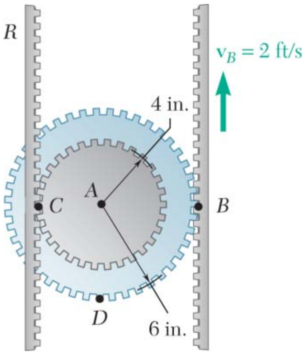
Gear *A*: 300 mm left of *A* ◀

Gear *C*: 600 mm left of *C* ◀

(*b*) *Angular velocities.*

$\boldsymbol{\omega}_A = 4.00$ rad/s $\circlearrowright$ ◀

$\boldsymbol{\omega}_C = 2.00$ rad/s $\circlearrowleft$ ◀



PROBLEM 15.81

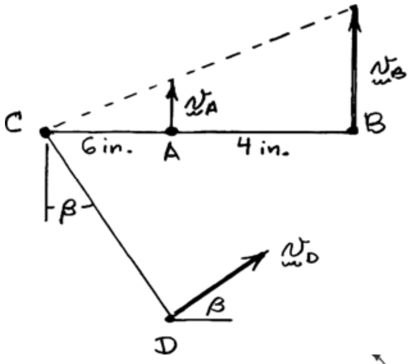
The double gear rolls on the stationary left rack R. Knowing that the rack on the right has a constant velocity of 2 ft/s, determine (a) the angular velocity of the gear, (b) the velocities of Points A and D.

SOLUTION

Since the rack R is stationary, Point C is the instantaneous center of the double gear.

Given: $\mathbf{v}_B = 2 \text{ ft/s}\uparrow = 24 \text{ in./s}\uparrow$

Make a diagram showing the locations of Points A, B, C, and D on the double gear.



$$v_B = \omega l_{CB}$$

$$\omega = \frac{v_B}{l_{CB}} = \frac{24 \text{ in./s}}{10 \text{ in.}} = 2.40 \text{ rad/s}$$

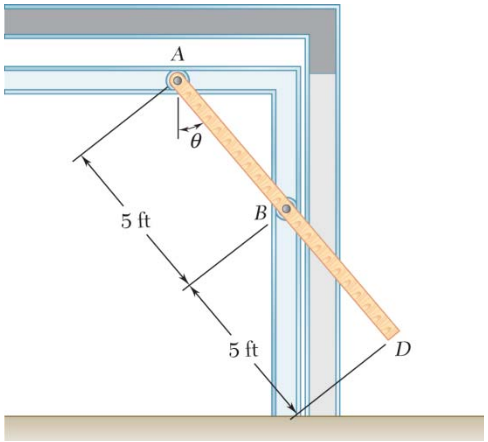
(a) *Angular velocity of the gear.* $\boldsymbol{\omega} = 2.40 \text{ rad/s} \circlearrowleft$ ◀

(b) *Velocity of Point A.* $v_A = l_{AC}\omega = (4 \text{ in.})(2.40 \text{ rad/s})$ $\mathbf{v}_A = 9.60 \text{ in./s} = 0.800 \text{ ft/s}\uparrow$ ◀

Geometry: $l_{CD} = \sqrt{(4 \text{ in.})^2 + (6 \text{ in.})^2} = \sqrt{52} \text{ in.}$

$$\tan\beta = \frac{4 \text{ in.}}{6 \text{ in.}} \qquad \beta = 33.7°$$

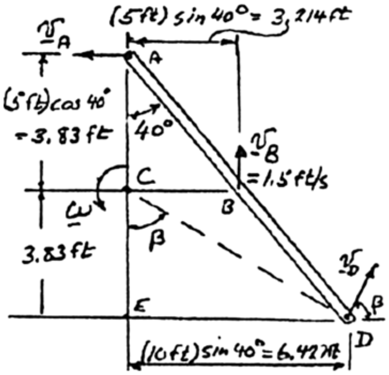
Velocity of Point D. $\mathbf{v}_D = l_{CD}\omega = \sqrt{52}(2.40) = 17.31 \text{ in./s}$ $\mathbf{v}_D = 1.442 \text{ ft/s} \measuredangle 33.7°$ ◀


A
θ
5 ft
B
5 ft
D

PROBLEM 15.82

An overhead door is guided by wheels at A and B that roll in horizontal and vertical tracks. Knowing that when $\theta = 40°$ the velocity of wheel B is 1.5 ft/s upward, determine (a) the angular velocity of the door, (b) the velocity of end D of the door.

SOLUTION


(5 ft) sin 40° = 3.214 ft
v_A
A
(5 ft) cos 40° = 3.83 ft
40°
v_B = 1.5 ft/s
C
ω
B
β
3.83 ft
v_D
β
E
D
(10 ft) sin 40° = 6.427 ft

Locate instantaneous center at intersection of lines drawn perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$.

(a) Angular velocity.

$$v_B = (BC)\omega$$

$$1.5 \text{ ft/s} = (3.214 \text{ ft})\omega$$

$$\omega = 0.4667 \text{ rad/s} \qquad \boldsymbol{\omega} = 0.467 \text{ rad/s} ↺ ◀$$

(b) Velocity of D:

In ΔCDE:

$$\beta = \tan^{-1}\frac{6.427}{3.83} = 59.2°$$

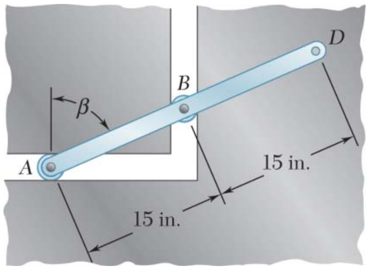
$$CD = \frac{6.427}{\sin\beta} = 7.482 \text{ ft}$$

$$v_D = (CD)\omega$$

$$= (7.482 \text{ ft})(0.4667 \text{ rad/s})$$

$$= 3.49 \text{ ft/s}$$

$$\mathbf{v}_D = 3.49 \text{ ft/s} \measuredangle\ 59.2° ◀$$

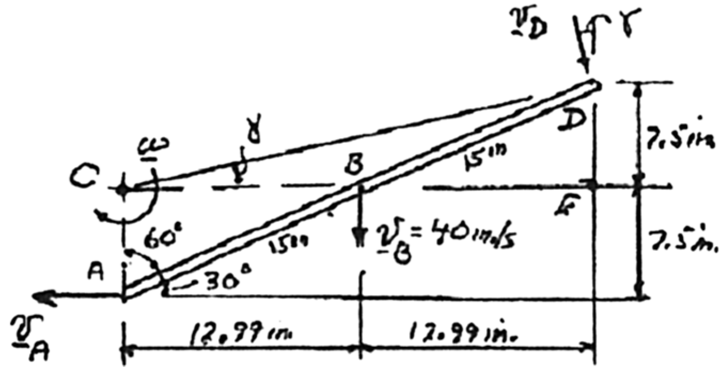

D
B
β
A
15 in.
15 in.

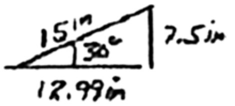
PROBLEM 15.83

Rod *ABD* is guided by wheels at *A* and *B* that roll in horizontal and vertical tracks. Knowing that at the instant shown $\beta = 60°$ and the velocity of wheel *B* is 40 in./s downward, determine (*a*) the angular velocity of the rod, (*b*) the velocity of Point *D*.

SOLUTION

Rod *ABD*:


$\mathbf{v}_D$
γ
ω
C
γ
B
15 in
D
7.5 in
E
60°
15 in
$\mathbf{v}_B = 40$ in/s
7.5 in.
A
30°
$\mathbf{v}_A$
12.99 in
12.99 in.


15 in
30°
7.5 in
12.99 in

We locate the instantaneous center by drawing lines perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_D$.

(*a*) Angular velocity.

$$v_B = (BC)\omega$$

$$40 \text{ in./s} = (12.99 \text{ in.})\omega$$

$$\omega = 3.079 \text{ rad/s}$$

$\boldsymbol{\omega} = 3.08$ rad/s ↻ ◀

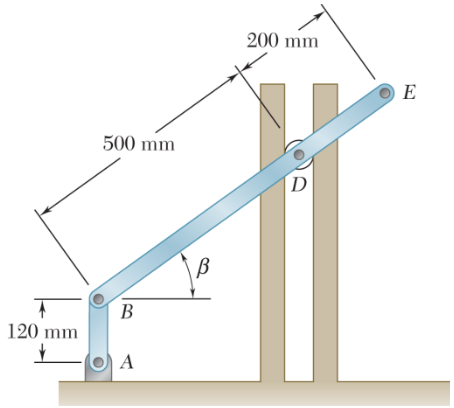
(*b*) Velocity of *D*:

In ΔCDE: $\gamma = \tan^{-1}\dfrac{7.5}{25.98} = 16.1°;\quad CD = \dfrac{25.98}{\cos\gamma} = 27.04 \text{ in.}$

$$v_D = (CD)\omega = (27.04 \text{ in.})(3.079 \text{ rad/s}) = 83.3 \text{ in./s}$$

$\mathbf{v}_D = 83.3$ in./s ↖ 16.1°

$\mathbf{v}_D = 83.3$ in./s ↖ 73.9° ◀


200 mm
E
500 mm
D
β
B
120 mm
A

PROBLEM 15.84

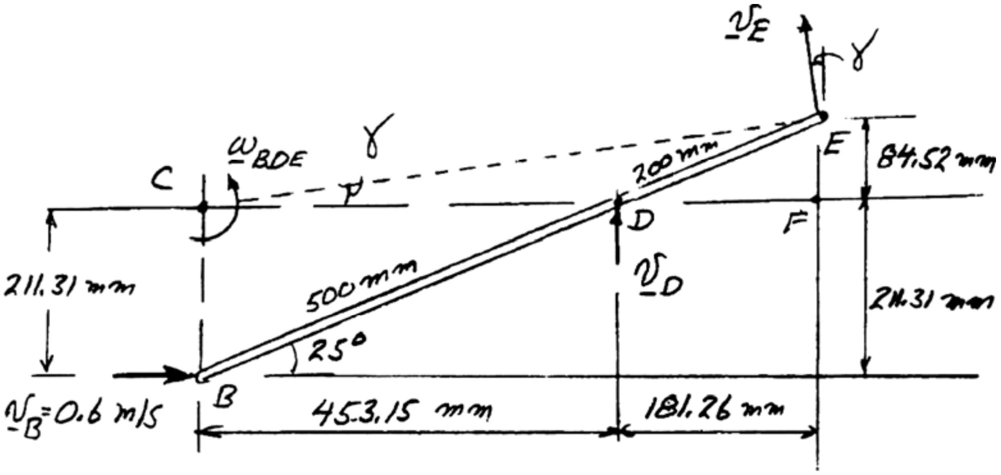
Rod *BDE* is partially guided by a roller at *D* that moves in a vertical track. Knowing that at the instant shown the angular velocity of crank *AB* is 5 rad/s clockwise and that $\beta = 25°$, determine (*a*) the angular velocity of the rod, (*b*) the velocity of Point *E*.

SOLUTION

Crank *AB*: $\boldsymbol{\omega}_{AB} = 5 \text{ rad/s} \circlearrowright \qquad \mathbf{r}_{B/A} = 120 \text{ mm} \uparrow$

$$v_B = \omega_{AB} r_{B/A} = (5)(0.120) \qquad \mathbf{v}_B = 0.6 \text{ m/s} \longrightarrow$$

Rod *BDE*: Draw a diagram of the geometry of the rod and note that $\mathbf{v}_B = 0.6 \text{ m/s} \longrightarrow$ and $\mathbf{v}_D = \mathbf{v}_D \uparrow$.


v_E
γ
ω_{BDE}
γ
C
200 mm
E
84.52 mm
D
F
211.31 mm
500 mm
v_D
211.31 mm
25°
v_B=0.6 m/s
B
453.15 mm
181.26 mm

Locate Point *C*, the instantaneous center, by noting that *BC* is perpendicular to $\mathbf{v}_B$ and *DC* is perpendicular to $\mathbf{v}_D$. Calculate lengths of *BC* and *CD*.

$$l_{BC} = 500 \sin 25° = 211.31 \text{ mm}$$
$$l_{CD} = 500 \cos 25° = 453.15.$$

(*a*) *Angular velocity of the rod.*

$$\omega_{BCD} = \frac{v_B}{l_{BC}} = \frac{0.6 \text{ m/s}}{0.21131 \text{ m}} = 2.8394 \text{ rad/s}$$

$$\boldsymbol{\omega}_{BCD} = 2.84 \text{ rad/s} \circlearrowleft \blacktriangleleft$$

PROBLEM 15.84 (Continued)

(*b*) *Velocity of Point E.*

Locate Point F on the diagram.

$$CF = 700\cos 25°\ \text{mm} \qquad FE = 200\sin 25°$$

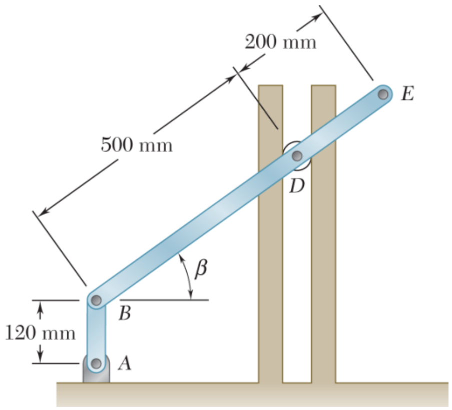
$$\tan\gamma = \frac{FE}{CF} = \frac{200\sin 25°}{700\cos 25°} = \frac{2}{7}\tan 25° = 0.13323$$

$$\gamma = 7.6° \qquad \beta = 90° - \gamma = 82.4°$$

$$l_{CE} = \sqrt{(CF)^2 + (FE)^2} = 640.02\ \text{mm} = 0.64002\ \text{m}$$

$$v_E = l_{CE}\omega = (0.64002)(2.8394)$$

$\mathbf{v}_E = 1.817\ \text{m/s} \searrow 82.4°$ ◀



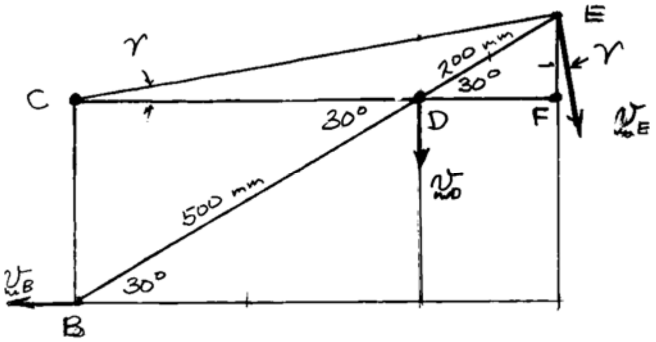
PROBLEM 15.85

Rod *BDE* is partially guided by a roller at *D* which moves in a vertical track. Knowing that at the instant shown $\beta = 30°$, Point *E* has a velocity of 2 m/s down and to the right, determine the angular velocities of rod *BDE* and crank *AB*.

SOLUTION

Crank *AB*: When *AB* is vertical, the velocity $\mathbf{v}_B$ at Point *B* is horizontal.

Rod *BDE*: Draw a diagram of the geometry of the rod and note that $\mathbf{v}_B$ is horizontal and $\mathbf{v}_D$ is vertical.



Locate Point *C*, the instantaneous center *C*, by noting that *CB* is vertical and *CD* is horizontal. From the diagram, with Point *F* added,

$$CF = 700\cos 30° \text{ mm} \qquad FE = 200\sin 30° \text{ mm}$$

$$CE = \sqrt{(CF)^2 + (FE)^2} = 614.41 \text{ mm} = 0.61441 \text{ m}$$

Angular velocity of rod BDE

$$\omega_{BDE} = \frac{v_E}{(CE)} = \frac{2 \text{ m/s}}{0.61441 \text{ m}} = 3.2552 \text{ rad/s}$$

$\boldsymbol{\omega}_{BDE} = 3.26$ rad/s ↻ ◀

Velocity of B. $CB = 500\sin 30° \text{ mm} = 250 \text{ mm} = 0.250 \text{ m}$

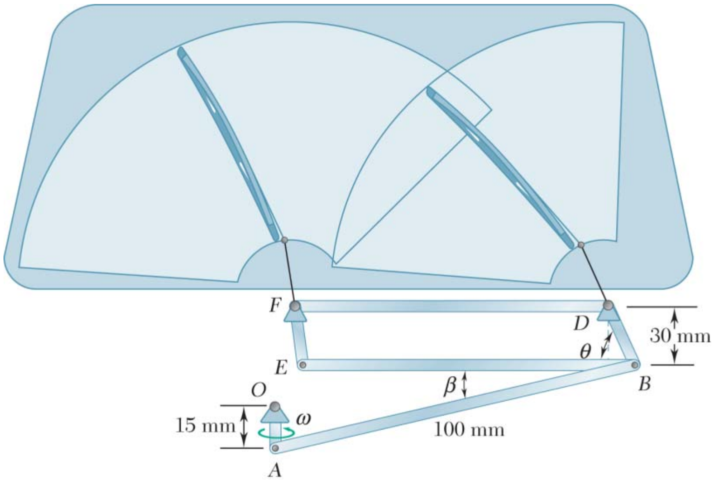
$$v_B = (CB)\omega_{BDE} = (0.250)(3.2552)$$

$$\mathbf{v}_B = 0.81379 \text{ m/s} \longleftarrow$$

Angular velocity of crank AB: $AB = 120 \text{ mm} = 0.120 \text{ m}$

$$\omega_{AB} = \frac{v_B}{(AB)} = \frac{0.81379 \text{ m/s}}{0.120 \text{ m}}$$

$\boldsymbol{\omega}_{AB} = 6.78$ rad/s ↺ ◀

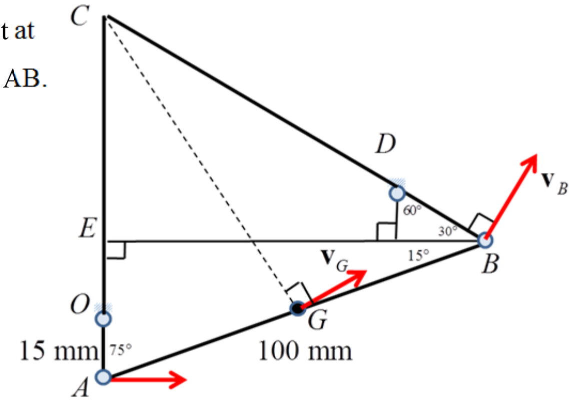


PROBLEM 15.86

A motor at *O* drives the windshield wiper mechanism so that *OA* has a constant angular velocity of 15 rpm. Knowing that at the instant shown linkage *OA* is vertical, θ= 30°, and β= 15°, determine (*a*) the angular velocity of bar *AB*, (*b*) the velocity of the center of the center of bar *AB*.

SOLUTION

Given: θ=30°, β=15°, $\omega_{OA} = 15\frac{\text{rev}}{\text{min}}\frac{\text{min}}{60\text{ s}}\frac{2\pi\text{ rad}}{\text{rev}} = \frac{\pi}{2}\text{rad/s}$



Geometry: Perpendiculars to $\mathbf{v}_A$ and $\mathbf{v}_B$ intersect at the instantaneous center, C, of link AB.

$$l_{BE} = 100\cos\beta = 96.593\text{ mm}$$

$$l_{BE} = l_{BC}\cos 30°$$
$$l_{BC} = 111.536\text{ mm}$$

$$\angle EAB = 75°$$
$$\angle ECB = 60°$$

Therefore:
$$l_{AC} = l_{BC}\cos 60° + l_{AB}\cos 75$$
$$= 81.650\text{ mm}$$

Using Law of Cosines for Triangle ACG

$$l_{CG}^2 = l_{AC}^2 + l_{AG}^2 - 2l_{AC}1_{AG}\cos 75°$$
$$= 81.65^2 + 50^2 - 2(81.65)(50)\cos 75°$$

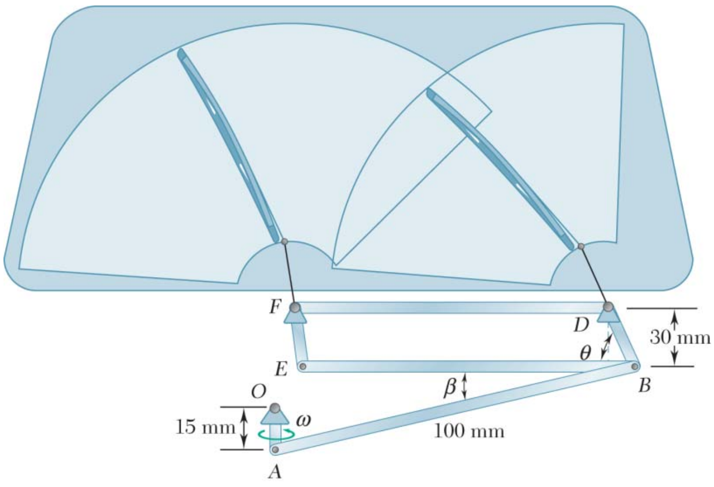
$$l_{CG} = 83.984$$

(*a*) Velocities:
$$\mathbf{v}_A = l_{OA}\omega_{OA} = l_{AC}\omega_{AB}$$
$$\omega_{AB} = \frac{l_{OA}\omega_{OA}}{l_{AC}} \Rightarrow \omega_{AB} = 0.2886\text{ rad/s}$$

$\boldsymbol{\omega}_{AB} = 0.289\text{ rad/s}$ ↻ ◀

(*b*) $v_G = l_{CG}\omega_{AB} \Rightarrow v_G = 24.236\text{ mm/s}$

$\mathbf{v}_G = 24.236\text{ mm/s}\ \angle 30°$ ◀



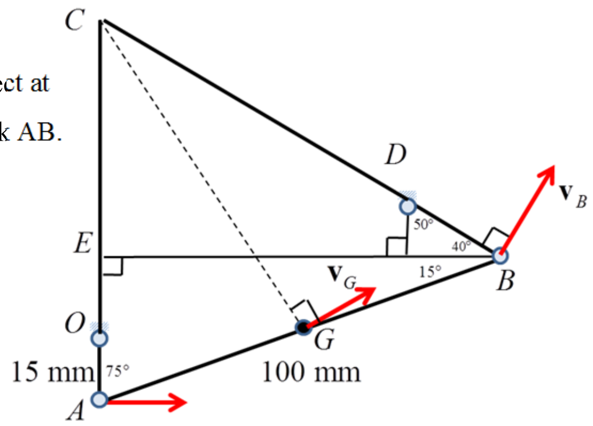
PROBLEM 15.87

A motor at O drives the windshield wiper mechanism so that point B has a speed of 20 mm/s. Knowing that at the instant shown linkage OA is vertical, $\theta = 40°$, and $\beta = 15°$, determine (a) the angular velocity of bar OA, (b) the velocity of the center of the center of bar AB.

SOLUTION

Given: $\theta = 40°$, $\beta = 15°$

$v_B = 20$ mm/s



Geometry: Perpendiculars to $\mathbf{v}_A$ and $\mathbf{v}_B$ intersect at the instantaneous center, C, of link AB.

$$l_{BE} = 100\cos\beta$$
$$= 96.593 \text{ mm}$$

$$l_{BE} = l_{BC}\cos 40°$$
$$l_{BC} = 126.093 \text{ mm}$$

Law of cosines: $l_{CG}^2 = l_{BC}^2 + l_{BG}^2 - 2l_{BC}l_{BG}\cos 65°$

$$l_{CG} = 105.674 \text{ mm}$$

(a) Velocities: $\mathbf{v}_B = l_{BC}\omega_{AB}$

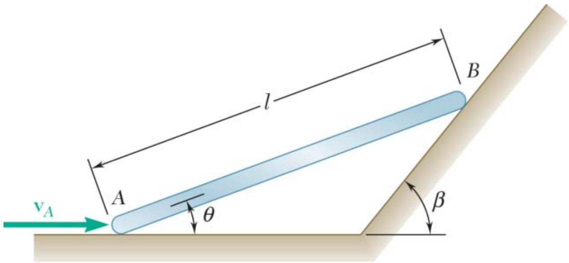
$$\omega_{AB} = \frac{\mathbf{v}_B}{l_{BC}}$$
$$= 0.159 \text{ rad/s}$$

$\boldsymbol{\omega}_{AB} = 0.159$ rad/s ⤸ ◀

(b) $v_G = l_{CG}\omega_{AB}$

$$= 16.761 \text{ mm/s}$$

$\mathbf{v}_G = 16.761$ mm/s ∠ 25° ◀



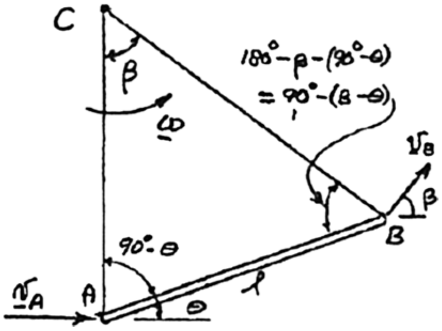
PROBLEM 15.88

Rod AB can slide freely along the floor and the inclined plane. Denoting by $\mathbf{v}_A$ the velocity of Point A, derive an expression for (a) the angular velocity of the rod, (b) the velocity of end B.

SOLUTION

Locate the instantaneous center at intersection of lines drawn perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$.

Law of sines.



$$\frac{AC}{\sin[90^\circ - (\beta - \theta)]} = \frac{BC}{\sin(90^\circ - \theta)}$$

$$= \frac{l}{\sin\beta}$$

$$\frac{AC}{\cos(\beta - \theta)} = \frac{BC}{\cos\theta}$$

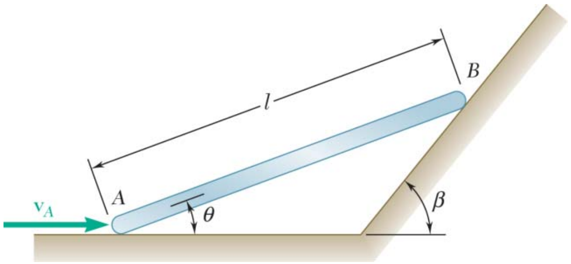
$$= \frac{l}{\sin\beta}$$

$$AC = l\frac{\cos(\beta - \theta)}{\sin\beta}$$

$$BC = l\frac{\cos\theta}{\sin\beta}$$

(a) Angular velocity: $v_A = (AC)\omega = l\dfrac{\cos(\beta - \theta)}{\sin\beta}\omega$ $\qquad \omega = \dfrac{v_A}{l} \cdot \dfrac{\sin\beta}{\cos(\beta - \theta)}$ ◀

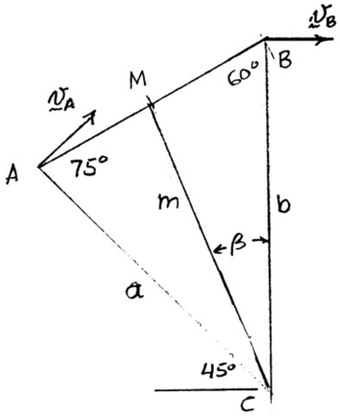
(b) Velocity of B: $v_B = (BC)\omega = l\dfrac{\cos\theta}{\sin\beta} \cdot \left[\dfrac{v_\theta}{l} \cdot \dfrac{\sin\beta}{\cos(\beta - \theta)}\right]$ $\qquad v_B = v_A\dfrac{\cos\theta}{\cos(\beta - \theta)}$ ◀



PROBLEM 15.89

Small wheels have been attached to the ends of bar *AB* and roll freely along the surfaces shown. Knowing that the velocity of wheel *B* is 7.5 ft/s to the right at the instant shown, determine (*a*) the velocity of end *A* of the bar, (*b*) the angular velocity of the bar, (*c*) the velocity of the midpoint of the bar.

SOLUTION



Given: $\mathbf{v}_A = v_A \measuredangle 45°$, $\mathbf{v}_B = 7.5 \text{ ft/s} \longrightarrow$

Locate the instantaneous center (point *C*) of rod *AB* by noting that velocity directions at points *A* and *B* are known. Draw *AC* perpendicular to $\mathbf{v}_A$ and *BC* perpendicular to $\mathbf{v}_B$.

Let $l = AB = 24 \text{ in.} = 2 \text{ ft}$

Law of sines for triangle *ABC*.

$$\frac{b}{\sin 75°} = \frac{a}{\sin 60°} = \frac{l}{\sin 45°} = 2.8284 \text{ ft}$$

$$a = 2.4495 \text{ ft}, \quad b = 2.73205 \text{ ft}$$

$$\omega = \frac{v_B}{b} = \frac{7.5}{2.73205} = 2.7452 \text{ rad/s}$$

(*a*) $v_A = a\omega = (2.4495)(2.7452) = 6.724 \text{ ft/s}$

$$\mathbf{v}_A = 6.72 \text{ ft/s} \measuredangle 45.0° \blacktriangleleft$$

(*b*) $\boldsymbol{\omega} = 2.75 \text{ rad/s} \circlearrowright \blacktriangleleft$

(*c*) Let *M* be the midpoint of *AB*.

Law of cosines for triangle *CMB*.

$$m^2 = b^2 + \left(\frac{l}{2}\right)^2 - 2b\frac{l}{2}\cos 60°$$

$$= (2.73205)^2 + (1)^2 - (2)(2.73205)(1)\cos 60°$$

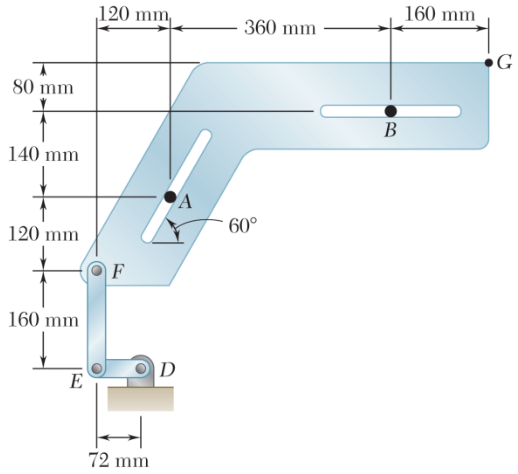
$$m = 2.3942 \text{ ft}$$

Law of sines.

$$\frac{\sin\beta}{\frac{l}{2}} = \frac{\sin 60°}{m}, \quad \sin\beta = \frac{1\sin 60°}{2.3942}, \quad \beta = 21.2°$$

$$v_M = m\omega = (2.3942)(2.7452) = 6.573 \text{ ft/s},$$

$$\mathbf{v}_M = 6.57 \text{ ft/s} \measuredangle 21.2° \blacktriangleleft$$


120 mm
360 mm
160 mm
G
80 mm
B
140 mm
A
60°
120 mm
F
160 mm
D
E
72 mm

PROBLEM 15.90

Two slots have been cut in plate *FG* and the plate has been placed so that the slots fit two fixed pins *A* and *B*. Knowing that at the instant shown the angular velocity of crank *DE* is 6 rad/s clockwise, determine (*a*) the velocity of Point *F*, (*b*) the velocity of Point *G*.

SOLUTION

Crank *DE*: $v_E = (DE)\omega_{DE} = (72 \text{ mm})(6 \text{ rad/s})$

$\mathbf{v}_E = 432 \text{ mm/s} \uparrow$

Rod *EF*: $(v_F)_y = v_E = 432 \text{ mm/s} \uparrow$

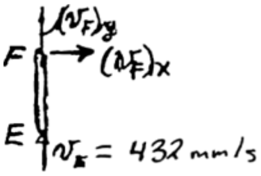
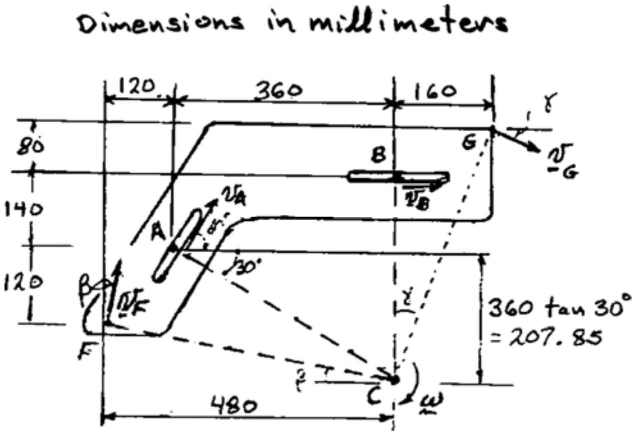

$(v_F)_y$
F
$(v_F)_x$
E
$v_E = 432$ mm/s

Plate *FG*:


Dimensions in millimeters
120
360
160
γ
G
v_G
80
B
v_B
140
v_A
A
β
30°
120
β
v_F
F
γ
360 tan 30°
= 207.85
β
C
ω
480

$\mathbf{v}_A$ and $\mathbf{v}_B$ are velocities of points on the plate next to the pins *A* and *B*. We draw lines perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$ to locate the instantaneous center *C*.

(*a*) Velocity of Point *F*:

$$v_F = (CF)\omega$$

$$(v_F)_y = [(CF)\omega]\cos\beta = [(CF)\cos\beta]\omega$$

But $(v_F)_y = 432 \text{ mm/s}$ and $(CF)\cos\beta = 480 \text{ mm}$:

$$432 \text{ mm/s} = (480 \text{ mm})\omega$$

$$\omega = 0.9 \text{ rad/s}$$

$\boldsymbol{\omega} = 0.9 \text{ rad/s} \circlearrowright$

PROBLEM 15.90 (Continued)

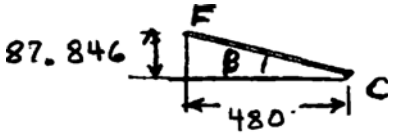
$CF = 487.97\text{ mm}$

$\beta = 10.37°$

$v_F = (CF)\omega$

$= (487.97\text{ mm})(0.9\text{ rad/s})$

$= 439.18\text{ mm/s}$



$\mathbf{v}_F = 439\text{ mm/s}$ ∠ 10.4°

$\mathbf{v}_F = 439\text{ mm/s}$ ∠ 79.6° ◀

(*b*) Velocity of Point *G*:

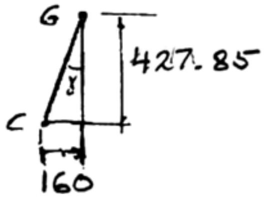
$CG = 456.78\text{ mm}$

$\gamma = 20.50°$

$v_G = (CG)\omega$

$= (456.78\text{ in.})(0.9\text{ rad/s})$

$v_G = 411.11\text{ mm/s}$



$\mathbf{v}_G = 411\text{ mm/s}$ ∠ 20.5° ◀

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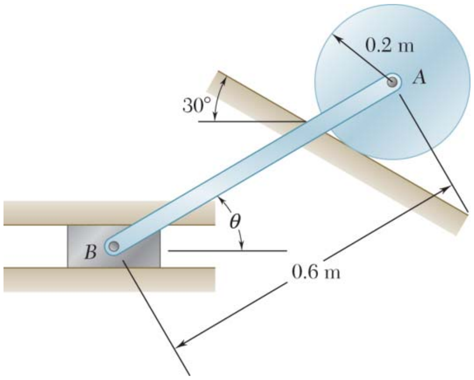


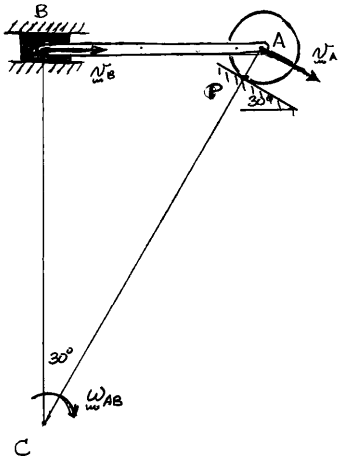
PROBLEM 15.91

The disk is released from rest and rolls down the incline. Knowing that the speed of A is 1.2 m/s when $\theta = 0°$, determine at that instant (*a*) the angular velocity of the rod, (*b*) the velocity of B. Only portions of the two tracks are shown.

SOLUTION

Draw the slider, rod, and disk at $\theta = 0°$.

Let Point P be the contact point between the disk and the incline. It is the instantaneous center of the disk. $\mathbf{v}_A$ is parallel to the incline. So that



$$\mathbf{v}_A = v_A \searrow 30°$$

Constraint of slider:

$$\mathbf{v}_B = v_B \longrightarrow$$

To locate the instantaneous center C of the rod AB, extend the line AP to meet the vertical line through B at Point C.

$$l_{AC} = l_{AB}/\sin 30°$$
$$l_{BC} = l_{AB}/\tan 30°$$

Angular velocity of rod AB.

$$\omega_{AB} = \frac{v_A}{l_{AC}} = \frac{v_A \sin 30°}{l_{AB}} = \frac{(1.2\text{ m/s})\sin 30°}{0.6\text{ m}}$$

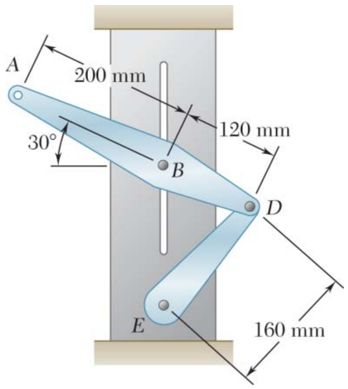
$$\boldsymbol{\omega}_{AB} = 1.000\text{ rad/s} \circlearrowright \blacktriangleleft$$

(*b*) *Velocity of Point B.*

$$v_B = l_{BC}\,\omega_{AB}$$

$$v_B = \frac{l_{AB}}{\tan 30°}\frac{v_A \sin 30°}{l_{AB}} = v_A \cos 30° = 1.2\cos 30°$$

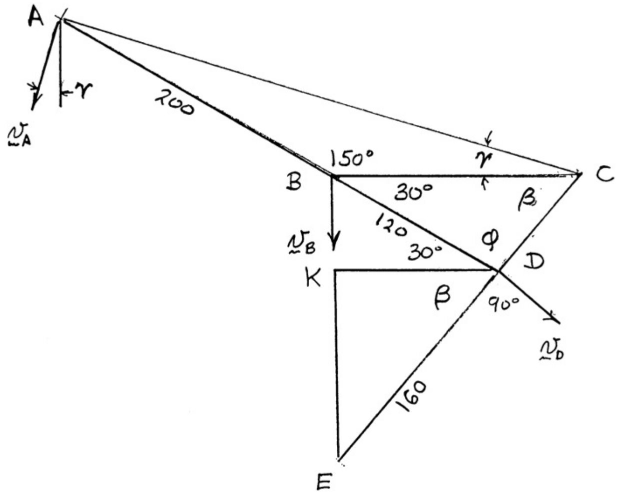
$$\mathbf{v}_B = 1.039\text{ m/s} \longrightarrow \blacktriangleleft$$



PROBLEM 15.92

The pin at *B* is attached to member *ABD* and can slide freely along the slot cut in the fixed plate. Knowing that at the instant shown the angular velocity of arm *DE* is 3 rad/s clockwise, determine (*a*) the angular velocity of member *ABD*, (*b*) the velocity of point *A*.

SOLUTION



$$\omega_{DE} = 3 \text{ rad/s} \circlearrowright$$

$$v_D = (DE)\omega_{DE} = (160)(3)$$
$$= 480 \text{ mm/s}$$

$\mathbf{v}_D$ is perpendicular to *DE*.

$$\mathbf{v}_B = v_B \downarrow$$

Locate the instantaneous center (point *C*) of bar *ABD* by noting that velocity directions at points *B* and *D* are known. Draw *BC* perpendicular to $\mathbf{v}_B$ and *DC* perpendicular to $\mathbf{v}_D$.

$$BD = 120 \text{ mm}, \quad DK = (BD)\cos 30° = 120 \cos 30°$$

$$\cos\beta = \frac{DK}{ED} = \frac{120 \cos 30°}{160}, \quad \beta = 49.495°, \quad \varphi = 180° - 30° - \beta = 100.505°$$

PROBLEM 15.92 (Continued)

Law of sines for triangle *BCD*.

$$\frac{CD}{\sin 30^\circ} = \frac{BC}{\sin \varphi} = \frac{BD}{\sin \beta}$$

$$CD = \frac{(BD)\sin 30^\circ}{\sin \beta} = \frac{120 \sin 30^\circ}{\sin \beta} = 78.911 \text{ mm}$$

$$BC = \frac{(BD)\sin \varphi}{\sin \beta} = \frac{120 \sin \varphi}{\sin \beta} = 155.177 \text{ mm}$$

Law of cosines for triangle *ABC*. $(AC)^2 = (BC)^2 + (AB)^2 - 2(AB)(BC)\cos 150^\circ$

$$(AC)^2 = 155.177^2 + 200^2 - (2)(155.177)(200)\cos 150^\circ, \quad AC = 343.27 \text{ mm}$$

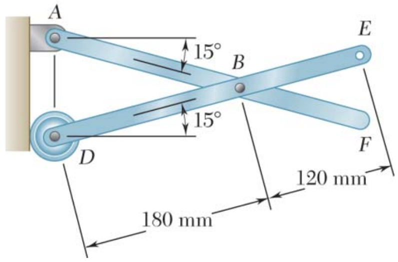
Law of sines. $\dfrac{\sin \gamma}{AB} = \dfrac{\sin 150^\circ}{AC} \qquad \sin \gamma = \dfrac{200 \ \sin 150^\circ}{343.27}, \qquad \gamma = 16.9^\circ$

(*a*) $\omega_{ABD} = \dfrac{v_D}{CD} = \dfrac{480}{78.911} = 6.0828 \text{ rad/s}$ ↺

$\omega_{ABD} = 6.08 \text{ rad/s}$ ↺ ◀

(*b*) $v_A = (AC)\omega_{ABD} = (343.27)(6.0828) = 2088 \text{ mm/s}$

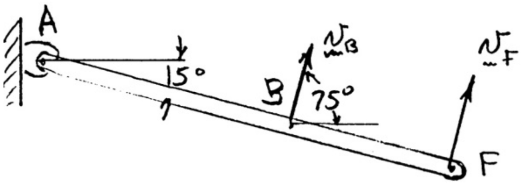
$\mathbf{v}_A = 2.09 \text{ m/s}$ ↙ 73.1° ◀

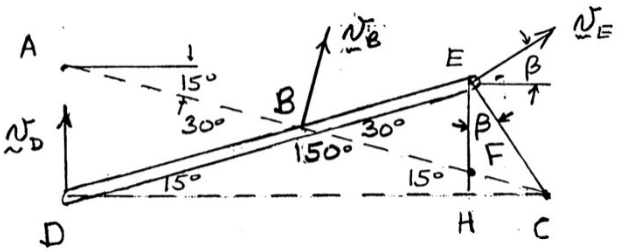


PROBLEM 15.93

Two identical rods *ABF* and *DBE* are connected by a pin at *B*. Knowing that at the instant shown the velocity of point *D* is 200 mm/s upward, determine the velocity of (*a*) point *E*, (*b*) point *F*.

SOLUTION





$$v_B = (AB)\omega_{ABF}, \qquad \mathbf{v}_B = v_B \measuredangle 75^\circ$$

$$\mathbf{v}_D = 200 \text{ mm/s} \uparrow$$

Locate the instantaneous center (point *C*) of bar *DBE* by noting that the velocity directions at points *B* and *D* are known. Draw *BC* perpendicular to $\mathbf{v}_B$ and *DC* perpendicular to $\mathbf{v}_D$.

Law of sines for triangle *BCD*.

$$\frac{CD}{\sin 150^\circ} = \frac{BC}{\sin 15^\circ} = \frac{BD}{\sin 15^\circ}$$

$$BC = BD = 180 \text{ mm} \qquad CD = \frac{180 \sin 150^\circ}{\sin 15^\circ} = 347.73 \text{ mm}$$

$$\omega_{DBE} = \frac{v_D}{CD} = \frac{200}{347.73} = 0.57515 \text{ rad/s} \circlearrowright$$

$$v_B = (BC)\omega_{DBE} = (180)(0.57515) = 103.528 \text{ mm/s}$$

$$\omega_{ABF} = \frac{v_B}{AB} = \frac{103.528}{180} = 0.57515 \text{ rad/s} \circlearrowleft$$

$$v_F = (AF)\omega_{ABF} = (300)(0.57515) = 172.546 \text{ mm/s}$$

Law of cosines for triangle *DCE*. $(CE)^2 = (CD)^2 + (DE)^2 - 2(CD)(DE)\cos 15°$

$$(CE)^2 = 347.73^2 + 300^2 - (2)(347.73)(300)\cos 15°, \quad CE = 96.889 \text{ mm}$$

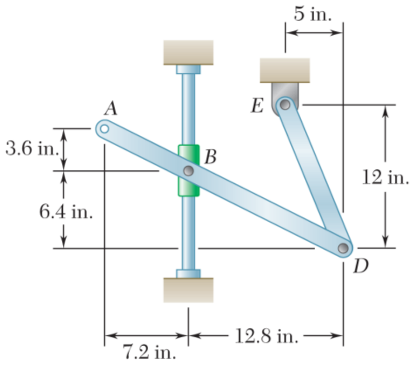
$$EH = DE \ \sin 15° = 300 \ \sin 15°$$

$$\cos\beta = \frac{EH}{CE} = \frac{300\sin 15°}{96.889} \qquad \beta = 36.7°$$

(*a*) $v_E = (CE)\omega_{BCD} = (96.889)(0.57515) = (55.7)$ mm/s,

$\mathbf{v}_E = 55.7$ mm/s ∠ 36.7° ◀

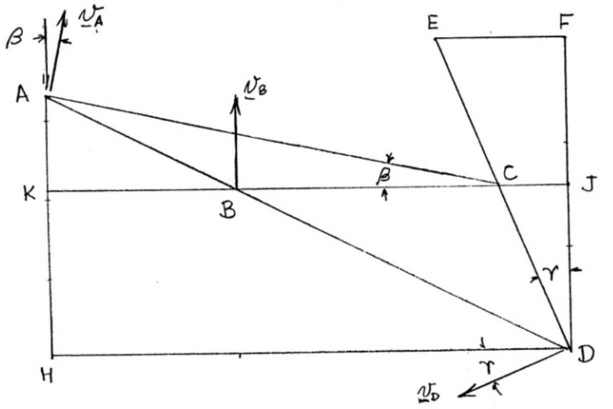
(*b*) $\mathbf{v}_F = 172.5$ mm/s ∠ 75.0° ◀



PROBLEM 15.94

Arm *ABD* is connected by pins to a collar at *B* and to crank *DE*. Knowing that the velocity of collar *B* is 16 in./s upward, determine (*a*) the angular velocity of arm *ABD*, (*b*) the velocity of point *A*.

SOLUTION



$$\mathbf{v}_B = 16 \text{ in./s}\uparrow \qquad \tan\gamma = \frac{EF}{DF} = \frac{5}{12} \qquad \mathbf{v}_D = v_D \swarrow \gamma$$

Locate the instantaneous center (point *C*) of bar *ABD* by noting that velocity directions at points *B* and *D* are known. Draw *BC* perpendicular to $\mathbf{v}_B$ and *DC* perpendicular to $\mathbf{v}_D$.

$$CJ = (DJ)\tan\gamma = (6.4)\left(\frac{5}{12}\right) = 2.6667 \text{ in.},$$

$$CB = JB - CJ = 12.8 - 2.6667 = 10.1333 \text{ in.}$$

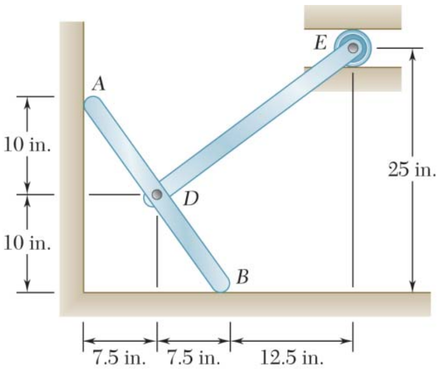
(*a*) $$\omega_{ABD} = \frac{v_B}{CB} = \frac{16}{10.1333} = 1.57895 \text{ rad/s}$$ $\omega_{ABD} = 1.579 \text{ rad/s} \circlearrowright$ ◀

$$CK = CB + BK = 10.1333 + 7.2 = 17.3333 \text{ in.}$$

$$\tan\beta = \frac{KA}{CK} = \frac{3.6}{17.3333}, \quad \beta = 11.733^\circ, \quad 90^\circ - \beta = 78.3^\circ$$

$$AC = \frac{CK}{\cos\beta} = \frac{17.3333}{\cos 11.733^\circ} = 17.7032 \text{ in.}$$

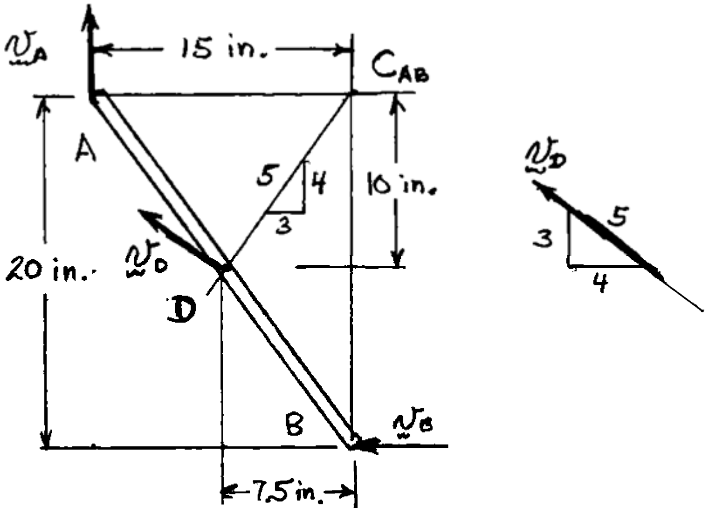
(*b*) $$v_A = (AC)\omega_{ABD} = (17.7032)(1.57895) = 28.0 \text{ in./s},$$ $\mathbf{v}_A = 28.0 \text{ in./s} \measuredangle 78.3^\circ$ ◀


E
A
D
B
10 in.
10 in.
25 in.
7.5 in.
7.5 in.
12.5 in.

PROBLEM 15.95

Two 25-in. rods are pin-connected at D as shown. Knowing that B moves to the left with a constant velocity of 24 in./s, determine at the instant shown (*a*) the angular velocity of each rod, (*b*) the velocity of E.

SOLUTION


v_A
15 in.
C_{AB}
A
5
4
3
10 in.
v_D
3
5
4
20 in.
v_D
D
B
v_B
7.5 in.

Rod AB: Draw lines perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$ to locate instantaneous center C_{AB}.

$$v_B = (BC_{AB})\omega_{AB}$$
$$24 \text{ in./s} = (20 \text{ in.})\omega_{AB}$$

$\boldsymbol{\omega}_{AB} = 1.200$ rad/s ⟳ ◀

Velocity of D:

$$DC_{AB} = 12.5 \text{ in.}$$
$$v_D = (DC_{AB})\omega_{AB}$$
$$= (12.5 \text{ in.})(1.2 \text{ rad/s})$$

$$\mathbf{v}_D = 15 \text{ in./s} \;\; \nwarrow_{3}^{4}$$

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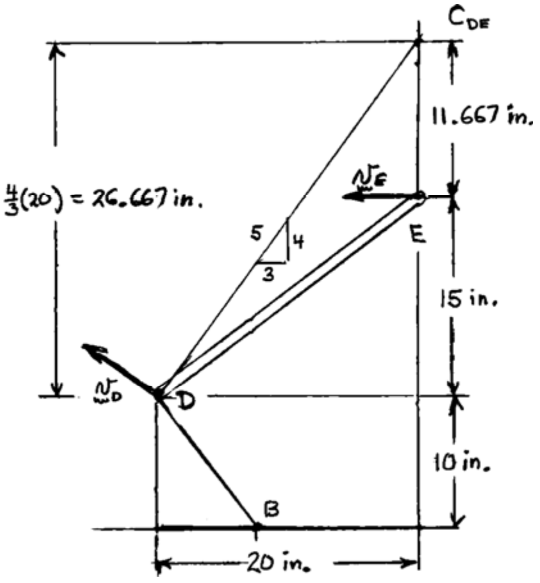
PROBLEM 15.95 (Continued)

Rod *DE*:



C_{DE}
11.667 in.
$\frac{4}{3}(20) = 26.667$ in.
v_E
E
5
4
3
15 in.
v_D
D
10 in.
B
20 in.

Draw lines perpendicular to $\mathbf{v}_D$ and $\mathbf{v}_E$ to locate instantaneous center C_{DE}.

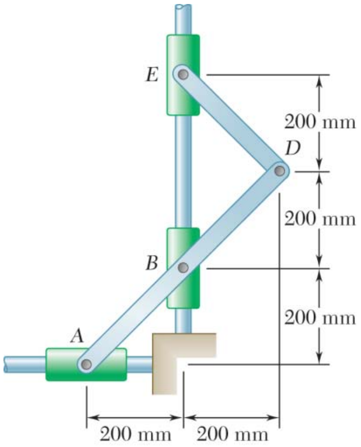
$$DC_{DE} = \sqrt{(20)^2 + (26.667)^2} = 33.333 \text{ in.}$$

$$v_D = (DC_{DE})\omega_{DE}$$

(*a*) $15 \text{ in./s} = (33.333 \text{ in.})\omega_{DE}; \quad \omega_{DE} = 0.45 \text{ rad/s}$ $\boldsymbol{\omega}_{DE} = 0.450 \text{ rad/s} \circlearrowright$ ◀

$$v_E = (EC_{DE})\omega_{DE} = (11.667 \text{ in.})(0.45 \text{ rad/s})$$

(*b*) $v_E = 5.25 \text{ in./s}$ $\mathbf{v}_E = 5.25 \text{ in./s} \longleftarrow$ ◀



PROBLEM 15.96

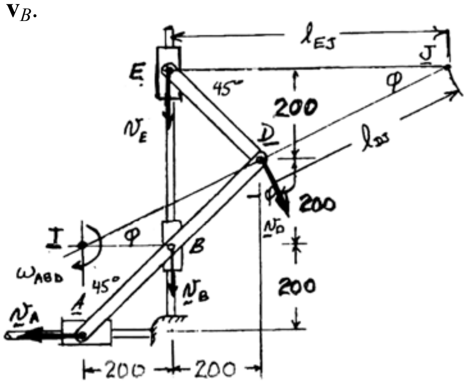
Two rods *ABD* and *DE* are connected to three collars as shown. Knowing that the angular velocity of *ABD* is 5 rad/s clockwise, determine at the instant shown (*a*) the angular velocity of *DE*, (*b*) the velocity of collar *E*.

SOLUTION

$\omega_{ABC} = 5 \text{ rad/s}$ ↻ $\quad \mathbf{v}_A = v_A$ ←

$\mathbf{v}_B = v_B$ ↓ $\quad \mathbf{v}_E = v_E$ ↓

Locate Point *I*, the instantaneous center of rod *ABD* by drawing *IA* perpendicular to $\mathbf{v}_A$ and *IB* perpendicular to $\mathbf{v}_B$.



Dimensions in mm

$$\tan\phi = \frac{200}{400} \qquad \phi = 26.565°$$

$$I_{ID} = \frac{400}{\cos\phi} = 447.21 \text{ mm}$$

$$v_D = \omega_{ABD} l_{ID} = (5)(447.21 \text{ mm})$$

$$v_D = 2236.1 \text{ mm/s} \searrow \phi$$

Locate Point *J*, the instantaneous center of rod *DE* by drawing *JD* perpendicular to $\mathbf{v}_D$ and *JE* perpendicular to $\mathbf{v}_E$.

$$l_{JD} = \frac{400}{\cos\phi} = 447.21 \text{ mm}$$

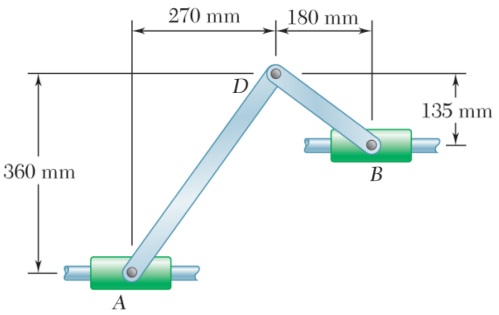
$$\omega_{DE} = \frac{v_D}{l_{JD}} = \frac{2236.1 \text{ mm/s}}{447.21} = 5 \text{ rad/s}$$

(*a*) $\boldsymbol{\omega}_{DE} = 5.00 \text{ rad/s}$ ↺ ◀

$$l_{JE} = 200 + l_{JD}\cos\phi = 600 \text{ mm}$$

$$v_E = l_{JE}\,\omega_{DE} = (600)(5) = 3000 \text{ mm/s}$$

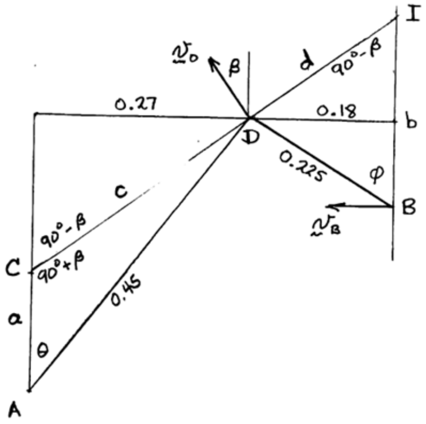
(*b*) $\mathbf{v}_E = 3.00 \text{ m/s}$ ↓ ◀



PROBLEM 15.97

At the instant shown, the velocity of collar A is 0.4 m/s to the right and the velocity of collar B is 1 m/s to the left. Determine (a) the angular velocity of bar AD, (b) the angular velocity of bar BD, (c) the velocity of point D.

SOLUTION



Method 1

Assume $\mathbf{v}_D$ has the direction indicated by the angle β as shown. Draw CDI perpendicular to $\mathbf{v}_D$. Then, point C is the instantaneous center of rod AD and point I is the instantaneous center of rod BD.

Geometry. $$AD = \sqrt{(0.27)^2 + (0.36)^2} = 0.45 \text{ m}$$

$$BD = \sqrt{(0.18)^2 + (0.135)^2} = 0.225 \text{ m}$$

$$\sin\theta = \frac{0.27}{0.45} = 0.6, \qquad \sin\phi = \frac{0.18}{0.225} = 0.8$$

$$\frac{c}{\sin\theta} = \frac{0.45}{\sin(90° + \beta)} = \frac{0.45}{\cos\beta} \qquad \frac{d}{\sin\phi} = \frac{0.225}{\sin(90° - \beta)} = \frac{0.225}{\cos\beta}$$

$$c = \frac{0.45\sin\theta}{\cos\beta} \qquad d = \frac{0.225\sin\phi}{\cos\beta}$$

$$a = 0.36 - 0.27\tan\beta \qquad b = 0.135 + 0.18\tan\beta$$

Kinematics. $v_A = 0.4$ m/s, $v_B = 1$ m/s

$$\omega_{AD} = \frac{v_A}{a} = \frac{v_D}{c}, \qquad \omega_{BD} = \frac{v_B}{b} = \frac{v_D}{d}$$

$$v_D = \frac{cv_A}{a} = \frac{dv_B}{b}$$

$$a = \frac{cv_A}{dv_B}b = \frac{0.45\sin\theta}{\cos\beta} \cdot \frac{\cos\beta}{0.225\sin\phi} \cdot \frac{0.4}{1}b = 0.6b$$

$$0.36 - 0.27\tan\beta = 0.6(0.135 + 0.18\tan\beta)$$

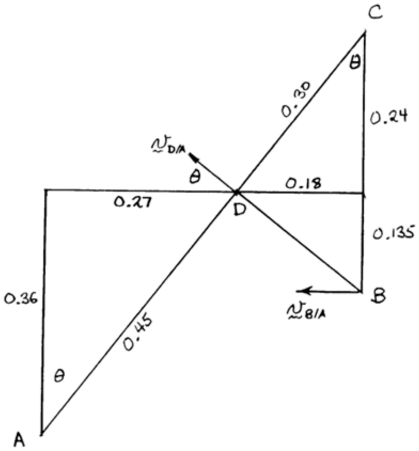
$$0.279 = 0.378\tan\beta, \qquad \tan\beta = 0.73809, \qquad \beta = 36.43°$$

$$a = 0.36 - (0.27)(0.73809) = 0.1607 \text{ m}$$

$$b = 0.135 + (0.18)(0.73809) = 0.2679 \text{ m}$$

$$c = \frac{(0.45)(0.6)}{\cos\beta} = 0.3356 \text{ m} \qquad d = \frac{(0.225)(0.8)}{\cos\beta} = 0.2237 \text{ m}$$

PROBLEM 15.97 (Continued)



(*a*) $\omega_{AD} = \dfrac{v_A}{a} = \dfrac{0.4}{0.1607} = 2.4891 \text{ rad/s}$ $\quad \boldsymbol{\omega}_{AD} = 2.49 \text{ rad/s}$ ↺ ◀

(*b*) $\omega_{BD} = \dfrac{v_B}{b} = \dfrac{1}{0.2679} = 3.733 \text{ rad/s}$ $\quad \boldsymbol{\omega}_{BD} = 3.73 \text{ rad/s}$ ↻ ◀

(*c*) $v_D = c\omega_{AD} = (0.3356)(2.4891) = 0.835 \text{ m/s},$

$\mathbf{v}_D = 0.835 \text{ m/s}$ ∡ 53.6° ◀

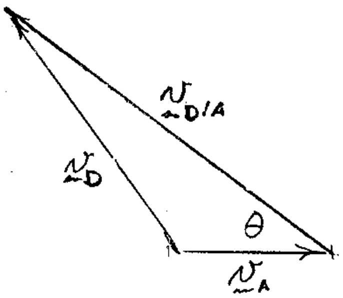
Method 2

Consider the motion using a frame of reference that is translating with collar *A*. For motion relative to this frame.

$$\mathbf{v}_A = 0.4 \text{ m/s} \longrightarrow$$

$$\mathbf{v}_B = 1 \text{ m/s} \longleftarrow$$

$$\mathbf{v}_{A/A} = 0, \qquad \mathbf{v}_{B/A} = 1.4 \text{ m/s} \longleftarrow$$



$$\tan\theta = \frac{0.27}{0.36}, \qquad \theta = 36.87°$$

$$AD = \frac{0.36}{\cos\theta} = 0.45 \text{ m}$$

$$\mathbf{v}_{D/A} = 0.45\omega_{AD} \; ∡ \; \theta$$

Locate the instantaneous center (point *C*) for the relative motion of bar *BD* by noting that the relative velocity directions at points *B* and *D* are known. Draw *BC* perpendicular to $\mathbf{v}_{B/A}$ and *DC* perpendicular to $\mathbf{v}_{D/A}$.

$$CD = \frac{0.18}{\sin\theta} = 0.3 \text{ m}$$

$$BC = (CD)\cos\theta + 0.135 = 0.375 \text{ m}$$

$$\omega_{BD} = \frac{v_{B/A}}{CB} = \frac{1.4}{0.375} = 3.73 \text{ rad/s}$$

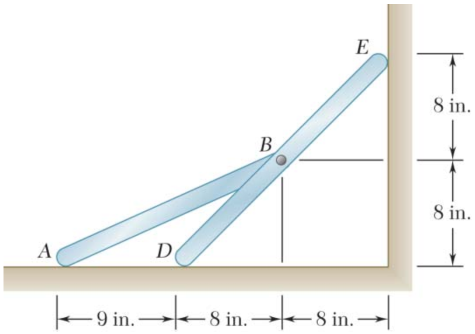
$$v_{D/A} = (CD)\omega_{BD} = (0.3)(3.73) = 1.120 \text{ m/s}$$

(*a*) $\omega_{AD} = \dfrac{v_{D/A}}{AD} = \dfrac{1.120}{0.45}$ $\quad \boldsymbol{\omega}_{AD} = 2.49 \text{ rad/s}$ ↺ ◀

(*b*) $\boldsymbol{\omega}_{BD} = 3.73 \text{ rad/s}$ ↻ ◀

(*c*) $\mathbf{v}_D = \mathbf{v}_A + \mathbf{v}_{D/A} = [0.4 \text{ m/s} \longrightarrow] + [1.120 \; ∡ \; \theta]$

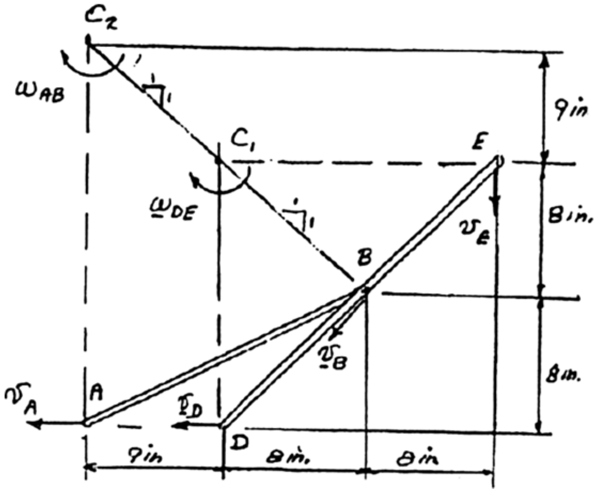
$= 0.835 \text{ m/s}$ ∡ 53.6° $\quad \mathbf{v}_D = 0.835 \text{ m/s}$ ∡ 53.6° ◀



PROBLEM 15.98

Two rods *AB* and *DE* are connected as shown. Knowing that Point *D* moves to the left with a velocity of 40 in./s, determine (*a*) the angular velocity of each rod, (*b*) the velocity of Point *A*.

SOLUTION



We locate two instantaneous centers at intersections of lines drawn as follows:

C_1: For rod *DE*, draw lines perpendicular to $\mathbf{v}_D$ and $\mathbf{v}_E$.

C_2: For rod *AB*, draw lines perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$.

Geometry:

$$BC_1 = (8\text{ in.})\sqrt{2} = 8\sqrt{2}\text{ in.}$$
$$DC_1 = 16\text{ in.}$$
$$BC_2 = (9\text{ in.} + 8\text{ in.})\sqrt{2} = 17\sqrt{2}\text{ in.}$$
$$AC_2 = 25\text{ in.}$$

(*a*) Rod *DE*:

$$v_D = (DC_1)\omega_{DE}$$
$$40\text{ in./s} = (16\text{ in.})\omega_{DE}$$
$$\omega_{DE} = 2.5\text{ rad/s}$$

$\boldsymbol{\omega}_{DE} = 2.5\text{ rad/s}$ ↻ ◀

$$v_B = (BC)\omega_{DE}$$
$$= (8\sqrt{2}\text{ in.})(2.5\text{ rad/s})$$
$$\mathbf{v}_B = 20\sqrt{2}\text{ in./s} \swarrow 45^\circ$$

PROBLEM 15.98 (Continued)

Rod *AB*:

$$v_B = (BC_2)\omega_{AB}$$

$$20\sqrt{2}\text{ in./s} = (17\sqrt{2}\text{ in.})\omega_{AB}$$

$$\omega_{AB} = \frac{20}{17}\text{ rad/s} = 1.1765\text{ rad/s}$$

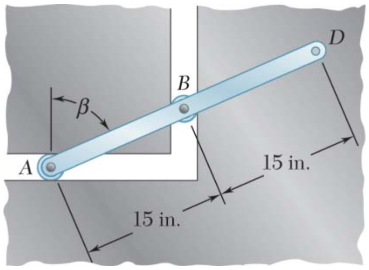
$$\boldsymbol{\omega}_{AB} = 1.177\text{ rad/s} \circlearrowright \blacktriangleleft$$

(*b*)

$$v_A = (AC_2)\omega_{AB}$$

$$= (25\text{ in.})(1.1765\text{ rad/s})$$

$$v_A = 29.41\text{ in./s} \qquad \mathbf{v}_A = 29.4\text{ in./s} \longleftarrow \blacktriangleleft$$

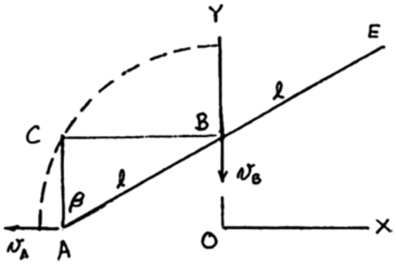


PROBLEM 15.99

Describe the space centrode and the body centrode of rod *ABD* of Problem 15.83. (*Hint:* The body centrode need not lie on a physical portion of the rod.)

PROBLEM 15.83 Rod *ABD* is guided by wheels at *A* and *B* that roll in horizontal and vertical tracks. Knowing that at the instant shown $\beta = 60°$ and the velocity of wheel *B* is 40 in./s downward, determine (*a*) the angular velocity of the rod, (*b*) the velocity of Point *D*.

SOLUTION



Draw *x* and *y* axes as shown with origin at the intersection of the two slots. These axes are fixed in space.

$$\mathbf{v}_A = v_A \longleftarrow, \qquad \mathbf{v}_B = v_B \downarrow$$

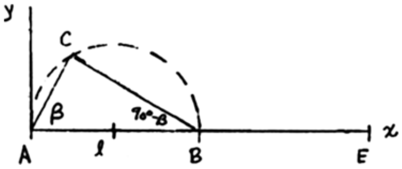
Locate the space centrode (Point *C*) by noting that velocity directions at Points *A* and *B* are known. Draw *AC* perpendicular to $\mathbf{v}_A$ and *BC* perpendicular to $\mathbf{v}_B$.

The coordinates of Point *C* are $x_C = -l \sin\beta$ and $y_C = l\cos\beta$

$$x_C^2 + y_C^2 = l^2 = (15 \text{ in.})^2$$

The *space centrode* is a quarter circle of 15 in. radius centered at *O*. ◀

Redraw the figure, but use axes *x* and *y* that move with the body. Place origin at *A*.



$$x_C = (AC)\cos\beta$$

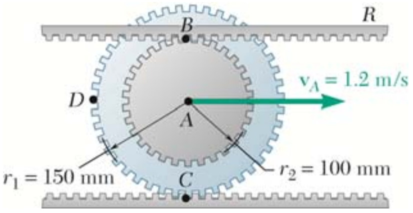
$$= l\cos^2\beta = \frac{l}{2}(1+\cos 2\beta)$$

$$y_C = (AC)\sin\beta$$

$$= l\cos\beta\sin\beta = \frac{l}{2}\sin 2\beta$$

$$\left(x_C - \frac{l}{2}\right)^2 + y_C^2 = \left(\frac{l}{2}\right)^2 = (x_C - 7.5)^2 + y_C^2 = 7.5^2$$

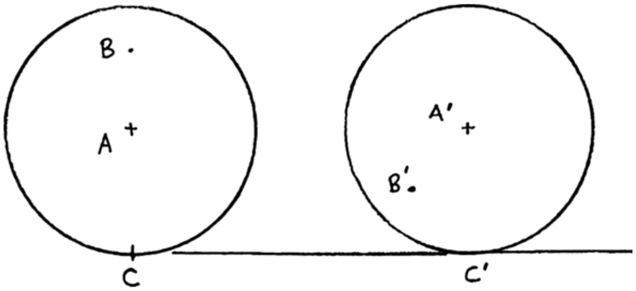
The *body centrode* is a semicircle of 7.5 in. radius centered midway between *A* and *B*. ◀



PROBLEM 15.100

Describe the space centrode and the body centrode of the gear of Sample Problem 15.6 as the gear rolls on the stationary horizontal rack.

SOLUTION



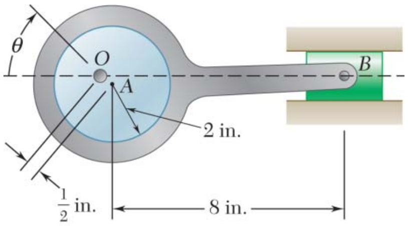
Let Points, *A, B,* and *C* move to *A*′, *B*′, and *C*′ as shown.

Since the instantaneous center always lies on the fixed lower rack, the space centrode is the lower rack.

space centrode: lower rack ◀

Since the point of contact of the gear with the lower rack is always a point on the circumference of the gear, the body centrode is the circumference of the gear.

body centrode: circumference of gear ◀

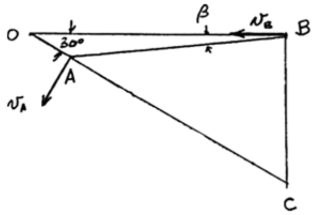


PROBLEM 15.101

Using the method of Section 15.3, solve Problem 15.60.

PROBLEM 15.60 In the eccentric shown, a disk of 2-in.-radius revolves about shaft *O* that is located 0.5 in. from the center *A* of the disk. The distance between the center *A* of the disk and the pin at *B* is 8 in. Knowing that the angular velocity of the disk is 900 rpm clockwise, determine the velocity of the block when $\theta = 30°$.

SOLUTION



$$(OA) = 0.5 \text{ in.} \quad \omega_{OA} = 900 \text{ rpm} = 30\pi \text{ rad/s} \circlearrowright$$

$$\mathbf{v}_A = (OA)\omega_{OA} = (0.5)(30\pi) = 15\pi \text{ in./s}$$

$$\mathbf{v}_A = v_A \swarrow 60°, \qquad \mathbf{v}_B = v_A \longleftarrow$$

Locate the instantaneous center (Point *C*) of bar *BD* by noting that velocity directions at Point *B* and *A* are known. Draw *BC* perpendicular to $\mathbf{v}_B$ and *AC* perpendicular to $\mathbf{v}_A$.

$$\sin\beta = \frac{(OA)\sin 30°}{AB} = \frac{0.5\sin 30°}{8}, \quad \beta = 1.79°$$

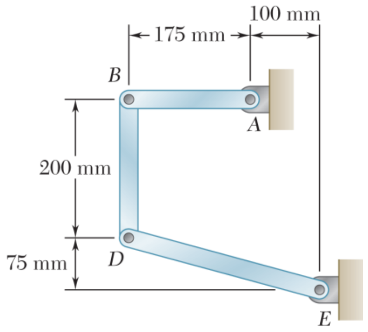
$$OB = (OA)\cos 30° + (AB)\cos\beta = 0.5\cos 30° + 8\cos\beta$$
$$= 8.4291 \text{ in.}$$

$$AC = \frac{OB}{\cos 30°} - OA = \frac{8.4291}{\cos 30°} - 0.5 = 9.2331 \text{ in.}$$

$$BC = (OB)\tan 30° = 4.8665 \text{ in.}$$

$$\omega_{AB} = \frac{v_A}{AC} = \frac{v_B}{BC}$$

$$v_B = \left(\frac{BC}{AC}\right)v_A = \frac{(4.8665)(15\pi)}{9.2331} = 24.84 \text{ in./s} \qquad \mathbf{v}_B = 24.8 \text{ in./s} \longleftarrow \blacktriangleleft$$



PROBLEM 15.102

Using the method of Section 15.3, solve Problem 15.64.

PROBLEM 15.64 In the position shown, bar *AB* has an angular velocity of 4 rad/s clockwise. Determine the angular velocity of bars *BD* and *DE*.

SOLUTION

Bar *AB*: (Rotation about *A*) $\boldsymbol{\omega}_{AB} = 4 \text{ rad/s} \curvearrowright$

$$\overline{AB} = 175 \text{ mm} \qquad v_B = \omega_{AB}(\overline{AB}) = (4)(175)$$

$$\mathbf{v}_B = 700 \text{ mm/s} \uparrow$$

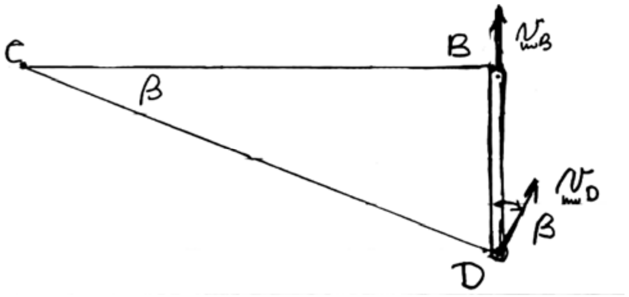
Bar *DE*: (Rotation about *E*) $\boldsymbol{\omega}_{DE} = \omega_{DE} \curvearrowright$

$$\overline{DE} = \sqrt{(275)^2 + (75)^2} = 285.04 \text{ mm}$$

$$\mathbf{v}_D = 285.04\omega_{DE} \measuredangle \beta$$

$$\tan\beta = \frac{75 \text{ mm}}{275 \text{ mm}} = 0.27273$$

Bar *BD*: $\mathbf{v}_B = 700 \text{ mm/s} \uparrow, \qquad \mathbf{v}_D = 285.04\omega_{DE} \measuredangle \beta$



Locate the instantaneous center of bar *BD* by drawing line *BC* perpendicular to $\mathbf{v}_B$ and line *DC* perpendicular to $\mathbf{v}_D$.

$$\overline{BD} = 200 \text{ mm}$$

$$\overline{CB} = \frac{\overline{BD}}{\tan\beta} = \frac{(200)(275)}{75} = 733.3 \text{ mm}$$

$$\overline{CD} = \frac{\overline{BD}}{\sin\beta} = \frac{(200)(285.04)}{75} = 760.11 \text{ mm}$$

PROBLEM 15.102 (Continued)

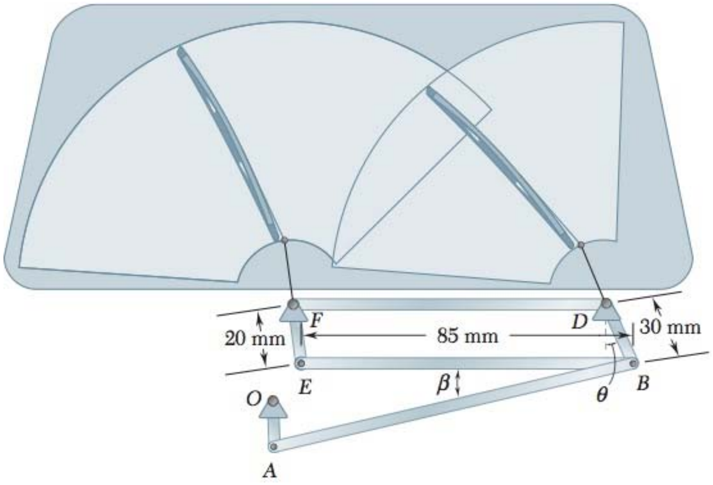
$$\omega_{BD} = \frac{v_B}{CB} = \frac{700\text{ mm/s}}{733.33\text{ mm}} = 0.95455\text{ rad/s}$$

$$v_D = \omega_{BD}(CD) = (0.95455\text{ rad/s})(760.11\text{ mm}) = 725.56\text{ mm/s}$$

$$\omega_{DE} = \frac{v_D}{285.04} = \frac{725.56}{285.04} = 2.5455\text{ rad/s}$$

Angular velocities: $\boldsymbol{\omega}_{BD} = 0.955\text{ rad/s}\ \circlearrowright$ ◀

$\boldsymbol{\omega}_{DE} = 2.55\text{ rad/s}\ \circlearrowleft$ ◀

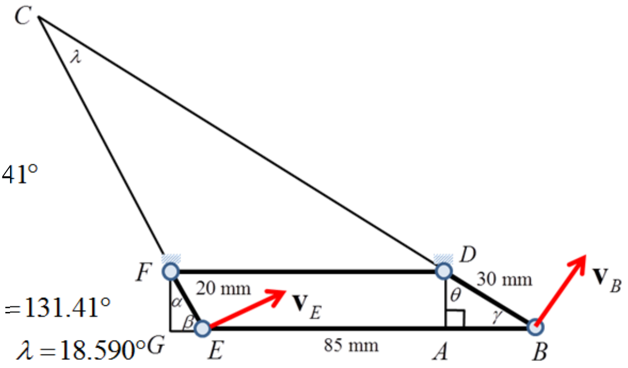


PROBLEM 15.103

Using the method of Section 15.3, solve Problem 15.65.

PROBLEM 15.65 Linkage *DBEF* is part of a windshield wiper mechanism, where points *O*, *F* and *D* are fixed pinned connections. At the position shown, $\theta = 60°$ and link *EB* is horizontal. Knowing that link *EF* has a counterclockwise angular velocity of 4 rad/s at the instant shown, determine the angular velocity of links *EB* and *DB*.

SOLUTION



Given:

$\theta = 60°$

$\omega_{EF} = 4$ rad/s

Geometry:

$30\cos\theta = 20\cos\alpha \quad \Rightarrow \quad \alpha = 41.41°$

$\gamma = 90° - \theta \quad \Rightarrow \quad \gamma = 30.0°$

$\beta = 90° - \alpha \quad \Rightarrow \quad \beta = 48.59°$

$\angle CEB = 180° - \beta \quad \Rightarrow \quad \angle CEB = 131.41°$

$\lambda = 180° - 131.41° - 30.0° \quad \Rightarrow \quad \lambda = 18.590°$

Law of sines:

$$\frac{l_{BC}}{\sin 131.41°} = \frac{85 \text{ mm}}{\sin\lambda} = \frac{l_{EC}}{\sin 30.0°}$$

$l_{BC} = 199.97$ mm

$l_{EC} = 133.31$ mm

Velocity of E:

$\mathbf{v}_E = l_{EF}\omega_{EF} = l_{EC}\omega_{EB}$

$$\omega_{EB} = \frac{l_{EF}\omega_{EF}}{l_{EC}}$$

$= 0.600$ rad/s

$\boldsymbol{\omega}_{EB} = 0.600$ rad/s ↻ ◀

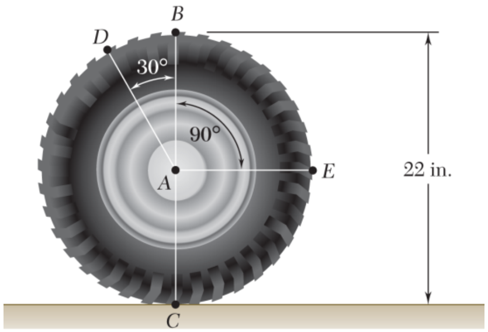
Velocity of B:

$\mathbf{v}_B = l_{BD}\omega_{BD} = l_{BC}\omega_{EB}$

$$\omega_{BD} = \frac{l_{BC}\omega_{EB}}{l_{BD}}$$

$= 4.000$ rad/s

$\boldsymbol{\omega}_{BD} = 4.000$ rad/s ↻ ◀



PROBLEM 15.104

Using the method of section 15.3, solve Problem 15.38.

PROBLEM 15.38 An automobile travels to the right at a constant speed of 48 mi/h. If the diameter of a wheel is 22 in., determine the velocities of Points *B*, *C*, *D*, and *E* on the rim of the wheel.

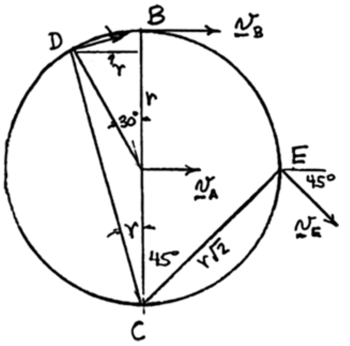
SOLUTION

$$v_A = 48 \text{ mi/h} = 70.4 \text{ ft/s}$$

$$\mathbf{v}_C = 0 \quad ◀$$

$$d = 22 \text{ in.,} \quad r = \frac{1}{2}d = 11 \text{ in.} = 0.91667 \text{ ft}$$

Point *C* is the instantaneous center.



$$\omega = \frac{v_A}{r} = \frac{70.4}{9.1667} = 76.8 \text{ rad/s}$$

$$CB = 2r = 1.8333 \text{ ft}$$

$$v_B = (CB)\omega = (1.8333)(76.8) = 140.8 \text{ ft/s}$$

$$\mathbf{v}_B = 140.8 \text{ ft/s} \longrightarrow \quad ◀$$

$$\gamma = \frac{1}{2}(30°) = 15°$$

$$CD = 2r\cos 15° = (2)(0.91667)\cos 15° = 1.7709 \text{ ft}$$

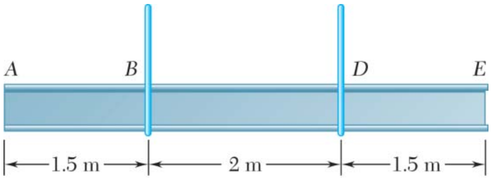
$$v_D = (CD)\omega = (1.7709)(76.8) = 136.0 \text{ ft/s}$$

$$\mathbf{v}_D = 136.0 \text{ ft/s} \nearrow 15.0° \quad ◀$$

$$CE = r\sqrt{2} = 0.91667\sqrt{2} = 1.2964 \text{ ft}$$

$$v_E = (CE)\omega = (1.2964)(76.8) = 99.56 \text{ ft/s}$$

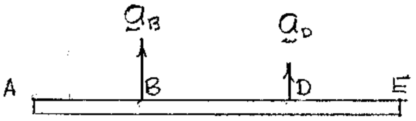
$$\mathbf{v}_E = 99.6 \text{ ft/s} \searrow 45.0° \quad ◀$$



PROBLEM 15.105

A 5-m steel beam is lowered by means of two cables unwinding at the same speed from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow the unwinding motion. At the instant considered the deceleration of the cable attached at *B* is 2.5 m/s^2, while that of the cable attached at *D* is 1.5 m/s^2. Determine (*a*) the angular acceleration of the beam, (*b*) the acceleration of points *A* and *E*.

SOLUTION



$$\boldsymbol{\alpha} = \alpha \circlearrowright, \quad \boldsymbol{\omega} \approx 0, \quad \mathbf{r}_{D/B} = 2 \text{ m} \longrightarrow$$

$$\mathbf{a}_B = 2.5 \text{ m/s}^2 \uparrow, \qquad \mathbf{a}_D = 1.5 \text{ m/s}^2 \uparrow$$

$$\mathbf{a}_D = a_B + \left(\mathbf{a}_{D/B}\right)_t + \left(a_{D/B}\right)_n$$

$$1.5 \uparrow = [2.5 \uparrow] + [2\alpha \downarrow] + \left[(2)(0)^2 \longleftarrow\right]$$

(*a*) $$1.5 = 2.5 - 2\alpha \qquad \alpha = 0.500 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

(*b*) $$\mathbf{a}_A = \mathbf{a}_B + \left(\mathbf{a}_{A/B}\right)_t + \left(\mathbf{a}_{A/B}\right)_n$$

$$= [2.5 \uparrow] + \left[(1.5)(0.5) \uparrow\right] + [0 \longrightarrow]$$

$$= 3.25 \text{ m/s}^2 \uparrow$$

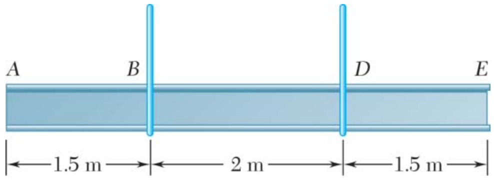
$$\mathbf{a}_A = 3.25 \text{ m/s}^2 \uparrow \blacktriangleleft$$

$$\mathbf{a}_E = \mathbf{a}_B + \left(\mathbf{a}_{E/B}\right)_t + \left(\mathbf{a}_{E/B}\right)_n$$

$$= [2.5 \uparrow] + \left[(2 + 1.5)(0.5) \downarrow\right] + [0 \longleftarrow]$$

$$= 0.75 \text{ m/s}^2 \uparrow$$

$$\mathbf{a}_E = 0.750 \text{ m/s}^2 \uparrow \blacktriangleleft$$



PROBLEM 15.106

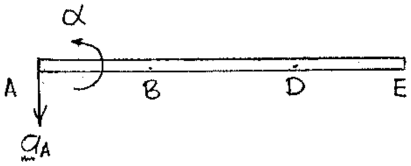
For a 5-m steel beam *AE* the acceleration of point *A* is $2\ \text{m/s}^2$ downward and the angular acceleration of the beam is $1.2\ \text{rad/s}^2$ counterclockwise. Knowing that at the instant considered the angular velocity of the beam is zero, determine the acceleration (*a*) of cable *B*, (*b*) of cable *D*.

SOLUTION

$$\mathbf{a}_A = 2\ \text{m/s}^2 \downarrow, \qquad \alpha = 1.2\ \text{rad/s}^2 \circlearrowleft$$

$$\mathbf{r}_{B/A} = 1.5\ \text{m} \longrightarrow, \qquad \mathbf{r}_{D/A} = 3.5\ \text{m} \longrightarrow$$

$$\omega \approx 0$$



(*a*)

$$\mathbf{a}_B = \mathbf{a}_A + \left(\mathbf{a}_{B/A}\right)_t + \left(\mathbf{a}_{B/A}\right)_n$$

$$= [2 \downarrow] + [(1.5)(1.2) \uparrow] + \left[(1.5)(0)^2 \longleftarrow\right]$$

$$= 0.2\ \text{m/s}^2 \downarrow$$

$$\mathbf{a}_B = 0.200\ \text{m/s}^2 \downarrow \blacktriangleleft$$

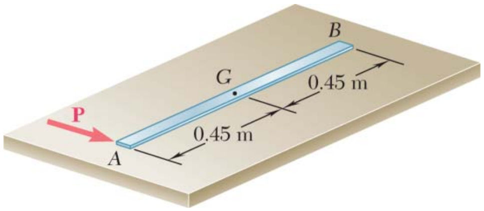
(*b*)

$$\mathbf{a}_D = \mathbf{a}_A + \left(\mathbf{a}_{D/A}\right)_t + \left(\mathbf{a}_{D/A}\right)_n$$

$$= [2 \downarrow] + [(3.5)(1.2) \uparrow] + \left[(3.5)(0)^2 \longleftarrow\right]$$

$$= 2.2\ \text{m/s}^2 \uparrow$$

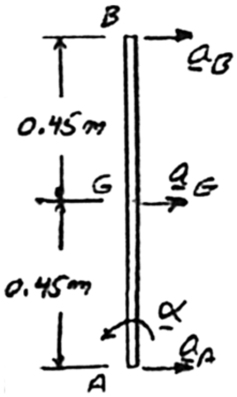
$$\mathbf{a}_D = 2.20\ \text{m/s}^2 \uparrow \blacktriangleleft$$



PROBLEM 15.107

A 900-mm rod rests on a horizontal table. A force **P** applied as shown produces the following accelerations: $\mathbf{a}_A = 3.6\ \text{m/s}^2$ to the right, $\alpha = 6\ \text{rad/s}^2$ counterclockwise as viewed from above. Determine the acceleration (*a*) of Point *G*, (*b*) of Point *B*.

SOLUTION



(*a*)

$$\mathbf{a}_G = \mathbf{a}_A + \mathbf{a}_{G/A} = [a_A \longrightarrow] + [(AG)\alpha \longleftarrow]$$

$$\mathbf{a}_G = [3.6\ \text{m/s}^2 \longrightarrow] + [(0.45\ \text{m})(6\ \text{rad/s}^2) \longleftarrow]$$

$$\mathbf{a}_G = [3.6\ \text{m/s}^2 \longrightarrow] + [2.7\ \text{m/s}^2 \longleftarrow]$$

$$\mathbf{a}_G = 0.9\ \text{m/s}^2 \longrightarrow \blacktriangleleft$$

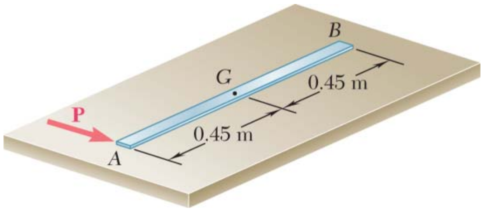
(*b*)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} = [a_A \longrightarrow] + [(AB)\alpha \longleftarrow]$$

$$\mathbf{a}_B = [3.6\ \text{m/s}^2 \longrightarrow] + [(0.9\ \text{m})(6\ \text{rad/s}^2) \longleftarrow]$$

$$\mathbf{a}_B = [3.6\ \text{m/s}^2 \longrightarrow] + [5.4\ \text{m/s}^2 \longleftarrow]$$

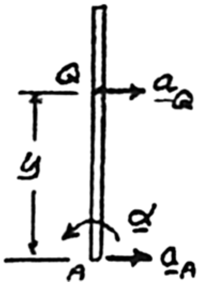
$$\mathbf{a}_B = 1.8\ \text{m/s}^2 \longleftarrow \blacktriangleleft$$



PROBLEM 15.108

In Problem 15.107, determine the point of the rod that (*a*) has no acceleration, (*b*) has an acceleration of 2.4 m/s^2 to the right.

SOLUTION



(*a*) For $a_Q = 0$:

$$\mathbf{a}_Q = \mathbf{a}_A + \mathbf{a}_{Q/A} = \mathbf{a}_A \longrightarrow + (AQ)\alpha \longleftarrow$$

$$0 = 3.6 \text{ m/s}^2 \longrightarrow + (y)(6 \text{ rad/s}^2) \longleftarrow$$

$$y = \frac{3.6 \text{ m/s}^2}{6 \text{ rad/s}} = 0.6 \text{ m}$$

$\mathbf{a} = 0$ at 0.6 m from A ◀

(*b*) For $\mathbf{a}_Q = 2.4 \text{ m/s}^2 \longrightarrow$:

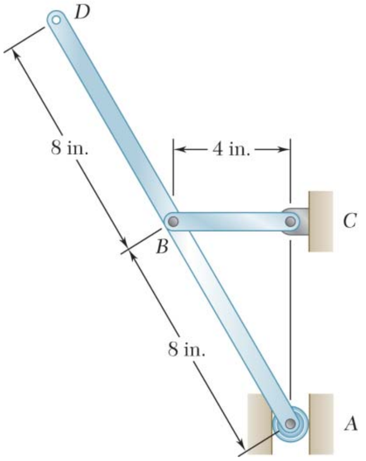
$$\mathbf{a}_Q = \mathbf{a}_A + \mathbf{a}_{Q/A} = [a_A \longrightarrow] + [(AQ)\alpha \longleftarrow]$$

$$2.4 \text{ m/s}^2 \longrightarrow = [3.6 \text{ m/s}^2 \longrightarrow] + [(y)(6 \text{ rad/s}^2) \longleftarrow]$$

$$1.2 \text{ m/s}^2 \longleftarrow = (y)(6 \text{ rad/s}^2) \longleftarrow$$

$$y = 0.2 \text{ m}$$

$\mathbf{a} = 2.4 \text{ m/s}^2 \longrightarrow$ at 0.2 m from A ◀



PROBLEM 15.109

Knowing that at the instant shown crank *BC* has a constant angular velocity of 45 rpm clockwise, determine the acceleration (*a*) of Point *A*, (*b*) of Point *D*.

SOLUTION

Geometry. Let β be angle *BAC*.

$$\sin\beta = \frac{4 \text{ in.}}{8 \text{ in.}} \qquad \beta = 30°$$

Velocity analysis. $\boldsymbol{\omega}_{BC} = 45 \text{ rpm} \circlearrowright = 4.7124 \text{ rad/s} \circlearrowright \qquad \mathbf{v}_A = v_A \uparrow$

$$v_B = (BC)\omega_{BC} = (4)(4.7124) = 18.8496 \text{ in./s} \qquad \mathbf{v}_B = 18.8496 \text{ in./s} \uparrow$$

$\mathbf{v}_A$ and $\mathbf{v}_B$ are parallel; hence, the instantaneous center of rotation of rod *AD* lies at infinity.

$$\boldsymbol{\omega}_{AD} = 0 \qquad \mathbf{v}_A = \mathbf{v}_B = 18.8496 \text{ in./s} \uparrow$$

Acceleration analysis. $\alpha_{BC} = 0$

Crank *BC*: $(a_B)_t = (BC)\alpha = 0$

$$(a_B)_n = (BC)\omega_{BC}^2 = (4)(4.7124)^2$$

$$\mathbf{a}_B = 88.827 \text{ in./s}^2 \rightarrow$$

Rod *ABD*: $\boldsymbol{\alpha}_{AD} = \alpha_{AD} \circlearrowleft \qquad \mathbf{a}_A = a_A \uparrow$

$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_t + (\mathbf{a}_{A/B})_A$$

$$[a_A \uparrow] = [88.827 \rightarrow] + [8\alpha_{AD} \nearrow 30°] + [8\omega_{AD}^2 \searrow 60°]$$

Resolve into components.

$$\xrightarrow{+}: \quad 0 = 88.827 + 8\alpha_{AD}\cos 30° + 0 \qquad \alpha_{AD} = -12.821 \text{ rad/s}^2$$

(*a*) $+\uparrow: \quad a_A = 8\alpha_{AD}\sin 30° = (8)(-12.821)\sin 30° = -51.284 \text{ in./s}^2$

$$\mathbf{a}_A = 51.3 \text{ in./s}^2 \downarrow \blacktriangleleft$$

(*b*)

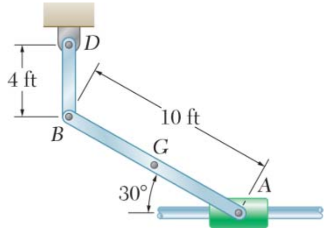
$$\mathbf{a}_D = \mathbf{a}_B + (\mathbf{a}_{D/B})_t + (\mathbf{a}_{D/B})_t$$

$$= [88.827 \longrightarrow] + [8\alpha_{BD} \swarrow 30^\circ] + [8\omega_{BD}^2 [\searrow 60^\circ]$$

$$= [88.827 \longrightarrow] + [(8)(-12.821) \swarrow 30^\circ] + 0$$

$$= [88.827 \longrightarrow] + [102.568 \nearrow 30^\circ] = [177.653 \longrightarrow \; + 51.284 \uparrow]$$

$$\mathbf{a}_D = 184.9 \text{ in./s}^2 \nearrow 16.1^\circ \blacktriangleleft$$



PROBLEM 15.110

End A of rod AB moves to the right with a constant velocity of 6 ft/s. For the position shown, determine (a) the angular acceleration of rod AB, (b) the acceleration of the midpoint G of rod AB.

SOLUTION

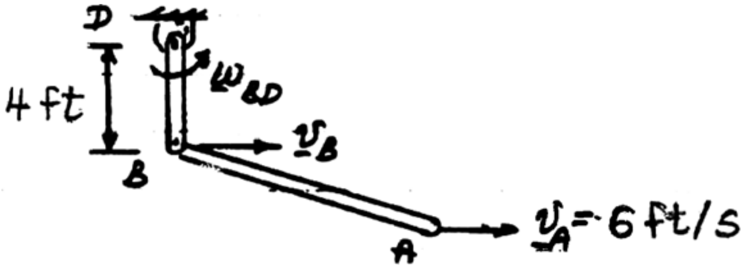
Use units of ft and seconds.

Geometry and unit vectors: $\mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1 \uparrow, \quad \mathbf{k} = 1 \circlearrowleft$

$$\mathbf{r}_{B/A} = -(10\cos 30°)\mathbf{i} + (10\sin 30°)\mathbf{j} \qquad \mathbf{r}_{B/D} = -4\mathbf{j}$$

Velocity analysis.

Rod AB: $\mathbf{v}_A = 6 \text{ ft/s} \longrightarrow, \quad \mathbf{v}_B = v_B \longrightarrow$



Since $\mathbf{v}_A$ and $\mathbf{v}_B$ are parallel, the instantaneous center lies at infinity, so $\omega_{AB} = 0$ and $\mathbf{v}_B = \mathbf{v}_A$.

Acceleration analysis.

Rod AB: $\mathbf{a}_A = 0$ since $\mathbf{v}_A$ is constant. $\qquad \boldsymbol{\alpha}_{AB} = \alpha_{AB} \circlearrowleft = \alpha_{AB}\mathbf{k}$

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} = \mathbf{a}_A + \mathbf{a}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$
$$= 0 + \alpha_{AB}\mathbf{k} \times (-10\cos 30° \mathbf{i} + 5\mathbf{j}) - 0$$
$$\mathbf{a}_B = -(10\cos 30°)\alpha_{AB}\mathbf{j} - 5\alpha_{AB}\mathbf{j} \tag{1}$$

Rod BD: $\mathbf{a}_D = 0, \quad \boldsymbol{\alpha}_{BD} = \alpha_{BD}\mathbf{k}$

$$\mathbf{a}_B = \mathbf{a}_D + \mathbf{a}_{B/D} = 0 + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{B/D} - \omega_{BD}^2 \mathbf{r}_{B/D}$$
$$= \alpha_{BD}\mathbf{k} \times (-4\mathbf{j}) - (1.5)^2(-4\mathbf{j})$$
$$= 4\alpha_{BD}\mathbf{i} + 9\mathbf{j} \tag{2}$$

Equating the coefficients of $\mathbf{j}$ in the expressions (1) and (2) for $\mathbf{a}_B$.

$$-(10\cos 30°)\alpha_{AB} = 9 \qquad \alpha_{AB} = -1.0392$$

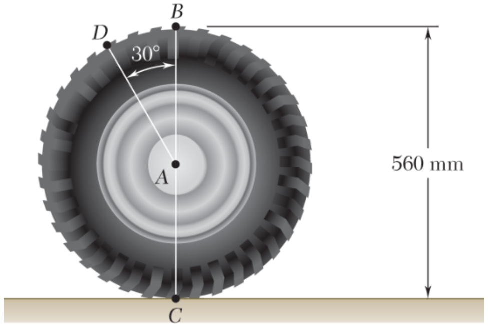
PROBLEM 15.110 (Continued)

(*a*) *Angular acceleration of rod AB*: $\boldsymbol{\alpha}_{AB} = 1.039\ \text{rad/s}^2 \circlearrowright$ ◀

(*b*) *Acceleration of midpoint G of rod AB*.

$$\begin{aligned} \mathbf{a}_G &= \mathbf{a}_A + \mathbf{a}_{G/A} = \mathbf{a}_A + \boldsymbol{\alpha}_{AB}\mathbf{k} \times \mathbf{r}_{G/A} - \omega_{AB}^2 \mathbf{r}_{G/A} \\ &= 0 - 1.0392\mathbf{k} \times (-5\cos 30^\circ \mathbf{i} + 5\sin 30^\circ \mathbf{j}) \end{aligned}$$

$\mathbf{a}_G = (2.60\ \text{ft/s}^2)\mathbf{i} + (4.50\ \text{ft/s}^2)\mathbf{j} = 5.20\ \text{ft/s}^2 \measuredangle\ 60^\circ$ ◀



PROBLEM 15.111

An automobile travels to the left at a constant speed of 72 km/h. Knowing that the diameter of the wheel is 560 mm, determine the acceleration (*a*) of Point *B*, (*b*) of Point *C*, (*c*) of Point *D*.

SOLUTION

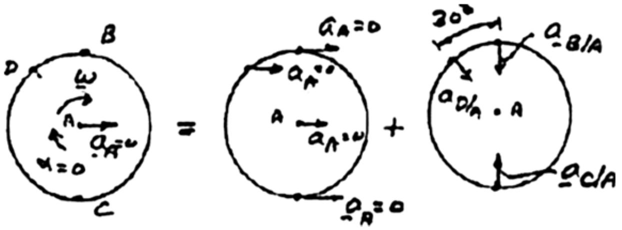
$$\mathbf{v}_A = 72\ \text{km/h} \cdot \frac{\text{h}}{3600\ \text{s}} \cdot \frac{1000\ \text{m}}{\text{km}} = 20\ \text{m/s} \longrightarrow$$

Rolling with no sliding, instantaneous center is at *C*.

$$v_A = (AC)\omega; \quad 20\ \text{m/s} = (0.28\ \text{m})\omega$$

$$\boldsymbol{\omega} = 71.429\ \text{rad/s} \circlearrowright$$

Acceleration.



Plane motion = Trans. with *A* + Rotation about *A*

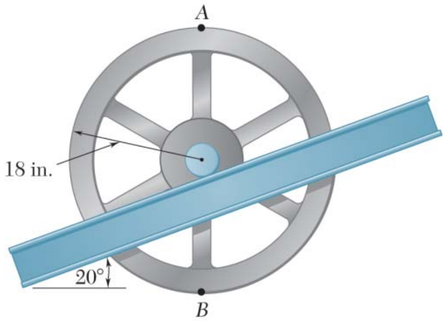
$$a_{B/A} = a_{C/A} = a_{D/A} = r\omega^2 = (0.280\ \text{m})(71.429\ \text{rad/s})^2 = 1428.6\ \text{m/s}^2$$

(*a*) $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} = 0 + 1428.6\ \text{m/s}^2 \downarrow$ $\qquad \mathbf{a}_B = 1430\ \text{m/s}^2 \downarrow$ ◀

(*b*) $\mathbf{a}_C = \mathbf{a}_A + \mathbf{a}_{C/A} = 0 + 1428.6\ \text{m/s}^2 \uparrow$ $\qquad \mathbf{a}_C = 1430\ \text{m/s}^2 \uparrow$ ◀

(*c*) $\mathbf{a}_D = \mathbf{a}_A + \mathbf{a}_{D/A} = 0 + 1428.6\ \text{m/s}^2 \searrow 60°$

$\mathbf{a}_D = 1430\ \text{m/s}^2 \searrow 60°$ ◀



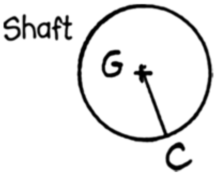
PROBLEM 15.112

The 18-in.-radius flywheel is rigidly attached to a 1.5-in.-radius shaft that can roll along parallel rails. Knowing that at the instant shown the center of the shaft has a velocity of 1.2 in./s and an acceleration of $0.5\ \text{in./s}^2$, both directed down to the left, determine the acceleration (*a*) of Point *A*, (*b*) of Point *B*.

SOLUTION

Velocity analysis.

Let Point *G* be the center of the shaft and Point *C* be the point of contact with the rails. Point *C* is the instantaneous center of the wheel and shaft since that point does not slip on the rails.



$$\mathbf{v}_G = r\omega, \quad \omega = \frac{v_G}{r} = \frac{1.2}{1.5} = 0.8\ \text{rad/s} \circlearrowleft$$

Acceleration analysis.

Since the shaft does not slip on the rails,

$$\mathbf{a}_C = a_C \nwarrow 20^\circ$$

Also, $$\mathbf{a}_G = [0.5\ \text{in./s}^2 \swarrow 20^\circ]$$

$$\mathbf{a}_C = \mathbf{a}_G + (\mathbf{a}_{C/G})_t + (\mathbf{a}_{C/G})_n$$

$$[a_C \nwarrow 20^\circ] = [0.5\ \text{in./s}^2 \swarrow 20^\circ] + [1.5\alpha \nearrow 20^\circ] + [1.5\omega^2 \nwarrow 20^\circ]$$

Components $\swarrow 20^\circ$: $\quad 0.5 = -1.5\alpha \qquad \alpha = 0.33333\ \text{rad/s}^2 \circlearrowleft$

(*a*) *Acceleration of Point A.*

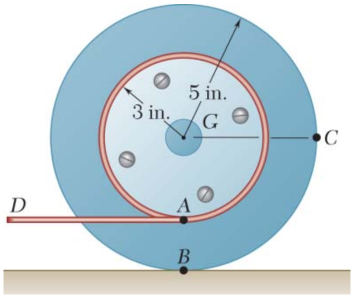
$$\begin{aligned}
\mathbf{a}_A &= \mathbf{a}_G + (\mathbf{a}_{A/G})_t + (\mathbf{a}_{A/G})_n \\
&= [0.5 \swarrow 20^\circ] + [18\alpha \leftarrow] + [18\omega^2 \downarrow] \\
&= [0.4698 \leftarrow] + [0.1710 \downarrow] + [6 \leftarrow] + [11.52 \downarrow] \\
&= [6.4698 \leftarrow] + [11.670 \downarrow]
\end{aligned}$$

$\mathbf{a}_A = 13.35\ \text{in./s}^2 \swarrow 61.0^\circ$ ◀

(*b*) *Acceleration of Point B.*

$$\begin{aligned}
\mathbf{a}_B &= \mathbf{a}_G + (\mathbf{a}_{B/G})_t + (\mathbf{a}_{B/G})_n \\
&= [0.5 \swarrow 20^\circ] + [18\alpha \rightarrow] + [18\omega^2 \uparrow] \\
&= [0.4698 \leftarrow] + [0.1710 \downarrow] + [6 \rightarrow] + [11.52 \uparrow] \\
&= [5.5302 \rightarrow] + [11.349 \uparrow]
\end{aligned}$$

$\mathbf{a}_B = 12.62\ \text{in./s}^2 \nearrow 64.0^\circ$ ◀



PROBLEM 15.113

A 3-in.-radius drum is rigidly attached to a 5-in.-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that at the instant shown end D of the cord has a velocity of 8 in./s and an acceleration of 30 in./s^2, both directed to the left, determine the accelerations of Points A, B, and C of the drums.

SOLUTION

Velocity analysis. $\quad \text{v}_D = \text{v}_A = 8\text{ in./s}$

Instantaneous center is at Point B. $\quad v_A = (AB)\omega, \quad 8 = (5-3)\omega$

$$\omega = 4\text{ rad/s}\circlearrowleft$$

Acceleration analysis. $\quad \mathbf{a}_B = [a_B\uparrow]$ for no slipping.

$$\alpha = \alpha\circlearrowleft$$

$$\mathbf{a}_A = [30\text{ in./s}^2 \longleftarrow] + [(a_A)_n\uparrow]$$

$$\mathbf{a}_G = [a_G \longleftarrow]$$

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

$$[a_B\uparrow] = [30 \longleftarrow] + [(a_A)_n\uparrow] + [(5-3)\alpha \longrightarrow] + [5-3)\omega^2\uparrow]$$

Components $\xrightarrow{+}$: $\quad 0 = -30 + 2\alpha \qquad \alpha = 15\text{ rad/s}^2\circlearrowleft$

$$\mathbf{a}_B = \mathbf{a}_G + (\mathbf{a}_{B/G})_t + (\mathbf{a}_{B/G})_n$$

$$[a_B\uparrow] = [a_G \longleftarrow] + [5\alpha \longrightarrow] + [5\omega^2\uparrow]$$

Components $\xrightarrow{+}$: $\quad 0 = -a_G + 5\alpha \qquad a_G = 5\alpha = 75\text{ in./s}^2$

$+\uparrow$: $\quad a_B = (5)(4)^2 = 80\text{ in./s}^2 \qquad \mathbf{a}_B = 80.0\text{ in./s}^2\uparrow$ ◀

$$\mathbf{a}_A = \mathbf{a}_G + (\mathbf{a}_{A/G})_t + (\mathbf{a}_{A/G})_n$$

$$= [75 \longleftarrow] + [3\alpha \longrightarrow] + [3\omega^2\uparrow]$$

$$= [75 \longleftarrow] + [45 \longrightarrow] + [48\uparrow]$$

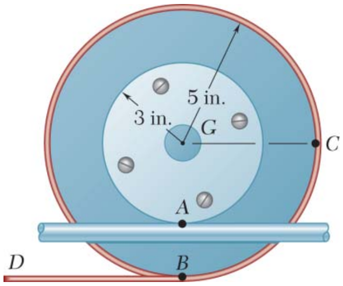
$$= [30\text{ in./s}^2 \longleftarrow] + [48\text{ in./s}^2\uparrow]$$

$\mathbf{a}_A = 56.6\text{ in./s}^2 \searrow 58.0^\circ$ ◀

PROBLEM 15.113 (Continued)

$$\begin{aligned}
\mathbf{a}_C &= \mathbf{a}_G + (\mathbf{a}_{C/G})_t + (\mathbf{a}_{C/G})_n \\
&= [75 \leftarrow] + [5\alpha \uparrow] + [5\omega^2 \leftarrow] \\
&= [75 \leftarrow] + [75 \uparrow] + [80 \leftarrow] \\
&= [155 \text{ in./s}^2 \leftarrow] + [75 \text{ in./s}^2 \uparrow]
\end{aligned}$$

$\mathbf{a}_C = 172.2 \text{ in./s}^2 \nwarrow 25.8°$ ◀



PROBLEM 15.114

A 3-in.-radius drum is rigidly attached to a 5-in.-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that at the instant shown end D of the cord has a velocity of 8 in./s and an acceleration of 30 in./s^2, both directed to the left, determine the accelerations of Points A, B, and C of the drums.

SOLUTION

Velocity analysis. $\quad \mathbf{v}_D = \mathbf{v}_B = 8\text{ in./s}$

Instantaneous center is at Point A. $\quad v_B = (AB)\omega, \quad 8 = (5-3)\omega$

$$\omega = 4\text{ rad/s} \circlearrowright$$

Acceleration analysis. $\quad \mathbf{a}_A = [a_A \uparrow]$ for no slipping. $\alpha = \alpha \circlearrowright$

$$\mathbf{a}_B = [30\text{ in./s}^2 \longleftarrow] + [(a_B)_n \uparrow]$$

$$\mathbf{a}_G = [a_G \longrightarrow]$$

$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_t + (\mathbf{a}_{A/B})_n$$

$$[a_A \uparrow] = [30 \longleftarrow] + [(a_B)_n \uparrow] + [(5-3)\alpha \longrightarrow] + [(5-3)]\omega^2 \downarrow]$$

Components $\xrightarrow{+}$: $\quad 0 = -30 + 2\alpha \qquad \alpha = 15\text{ rad/s}^2 \circlearrowright$

$$\mathbf{a}_A = \mathbf{a}_G + (\mathbf{a}_{A/G})_t + (\mathbf{a}_{A/G})_n$$

$$a_A \uparrow = [a_G \longrightarrow] + [3\alpha \longleftarrow] + [3\omega^2 \uparrow]$$

Components $\xrightarrow{+}$: $\quad 0 = a_G - 3\alpha \qquad a_G = 3\alpha = 45\text{ in./s}^2$

$+\uparrow$: $\quad a_A = 3\omega^2 = (3)(4)^2 = 48\text{ in./s}^2 \qquad \mathbf{a}_A = 48.0\text{ in./s}^2 \uparrow$ ◀

$$\mathbf{a}_B = \mathbf{a}_G + (\mathbf{a}_{B/G})_t + (\mathbf{a}_{B/G})_n$$

$$= [45 \longrightarrow] + [5\alpha \longleftarrow] + [5\omega^2 \uparrow]$$

$$= [45 \longrightarrow] + [75 \longleftarrow] + [80 \uparrow]$$

$$= [30\text{ in./s}^2 \longleftarrow] + [80\text{ in./s}^2 \uparrow]$$

$\mathbf{a}_B = 85.4\text{ in./s}^2 \nwarrow 69.4^\circ$ ◀

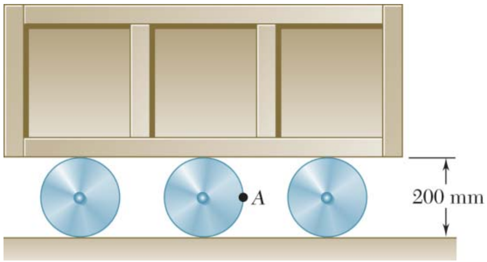
PROBLEM 15.114 (Continued)

$$\mathbf{a}_C = \mathbf{a}_G + (\mathbf{a}_{C/G})_t + (\mathbf{a}_{C/G})_n$$

$$= [45 \longrightarrow] + [5\alpha \downarrow] + [5\omega^2 \longleftarrow]$$

$$= [45 \longrightarrow] + [75 \downarrow] + [80 \longleftarrow]$$

$$= [35 \text{ in./s}^2 \longleftarrow] + [75 \text{ in./s}^2 \downarrow] \qquad \mathbf{a}_C = 82.8 \text{ in./s}^2 \swarrow 65.0^\circ \blacktriangleleft$$

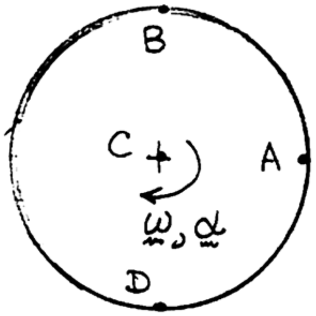


PROBLEM 15.115

A heavy crate is being moved a short distance using three identical cylinders as rollers. Knowing that at the instant shown the crate has a velocity of 200 mm/s and an acceleration of 400 mm/s^2, both directed to the right, determine (*a*) the angular acceleration of the center cylinder, (*b*) the acceleration of point *A* on the center cylinder.

SOLUTION

Geometry. Let point *C* be the center of the center cylinder, *B* its contact point with the crate, and *D* its contact point with the ground. Let *r* be the radius of the cylinder. $r = 100$ mm.



Velocity analysis. Since the contacts at *B* and *D* are rolling contacts without slipping, $v_B = 200 \text{ mm/s} \rightarrow$ and $\mathbf{v}_D = 0$. Point *D* is the instantaneous center of rotation.

$$\omega = \frac{v_B}{2r} = \frac{200 \text{ mm/s}}{200 \text{ mm}} = 1 \text{ rad/s} \curvearrowright$$

Point *C* moves on a horizontal line.

Acceleration analysis.

$$\mathbf{a}_C = [a_C \rightarrow]$$

$$\mathbf{a}_D = \mathbf{a}_C + (\mathbf{a}_{D/C})_t + (\mathbf{a}_{D/C})_n = [a_C \rightarrow] + [r\alpha \leftarrow] + [r\omega^2 \uparrow]$$

Component →:

$$0 = a_C - r\alpha \qquad (1)$$

$$\mathbf{a}_B = \mathbf{a}_C + (\mathbf{a}_{B/C})_t + (\mathbf{a}_{B/C})_n = [a_C \rightarrow] + [r\alpha \rightarrow] + [r\omega^2 \downarrow]$$

Component →:

$$400 \text{ mm/s}^2 = a_C + r\alpha \qquad (2)$$

Solving (1) and (2) simultaneously,

$$a_C = 200 \text{ mm/s}^2$$

$$r\alpha = 200 \text{ mm/s}^2$$

(*a*) $$\alpha = \frac{200 \text{ mm/s}^2}{100 \text{ mm}}$$

$\alpha = 2.00 \text{ rad/s}^2 \curvearrowright$ ◀

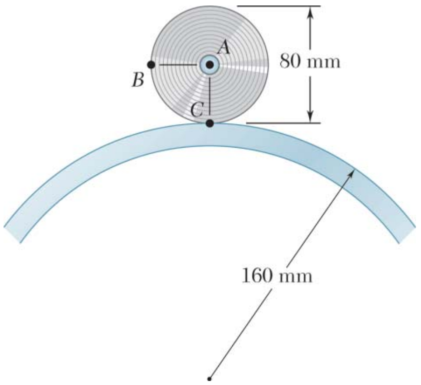
PROBLEM 15.115 (Continued)

$$\mathbf{a}_A = \mathbf{a}_C + (\mathbf{a}_{A/C})_t + (\mathbf{a}_{A/C})_n = [a_C \longrightarrow] + [r\alpha \downarrow] + [r\omega^2 \longleftarrow]$$

$$= [200 \text{ mm/s}^2 \longrightarrow] + [200 \text{ mm/s}^2 \downarrow] + [(100 \text{ mm})\ (1 \text{ rad/s})^2 \longleftarrow]$$

$$= [100 \text{ mm/s}^2 \longrightarrow] + [200 \text{ mm/s}^2 \downarrow]$$

(*b*) $a_A = 223.6$ mm/s $\qquad \mathbf{a}_A = 0.224 \text{ m/s}^2 \searrow 63.4^\circ$ ◀



PROBLEM 15.116

A wheel rolls without slipping on a fixed cylinder. Knowing that at the instant shown the angular velocity of the wheel is 10 rad/s clockwise and its angular acceleration is 30 rad/s^2 counterclockwise, determine the acceleration of (*a*) Point *A*, (*b*) Point *B*, (*c*) Point *C*.

SOLUTION

Velocity analysis. $r = 0.04 \text{ m} \qquad \boldsymbol{\omega} = 10 \text{ rad/s} \circlearrowright$

Point *C* is the instantaneous center of the wheel.

$$\mathbf{v}_A = [(r\omega) \longrightarrow] = [(0.04)(10) \longrightarrow] = 0.4 \text{ m/s} \longrightarrow]$$

Acceleration analysis. $\alpha = 30 \text{ rad/s}^2 \circlearrowleft$

Point *A* moves on a circle of radius $\rho = R + r = 0.16 + 0.04 = 0.2$ m.

Since the wheel does not slip, $\mathbf{a}_C = a_C \uparrow$

$$\mathbf{a}_C = \mathbf{a}_A + (\mathbf{a}_{C/A})_t + (\mathbf{a}_{C/A})_n$$

$$[a_C \uparrow] = [(a_A)_t \longleftarrow] + \left[\frac{v_A^2}{\rho} \downarrow\right] + [r\alpha \longrightarrow] + [r\omega^2 \uparrow]$$

$$= [(a_A)_t \longleftarrow] + \left[\frac{(0.4)^2}{0.2} \downarrow\right] + [(0.04)(30) \longrightarrow] + [(0.04)(10)^2 \uparrow]$$

$$= [(a_A)_t \longleftarrow] + [0.8 \downarrow] + [1.2 \longrightarrow] + [4 \uparrow]$$

Components. $\xrightarrow{+}$: $-(a_A)_t + 1.2 = 0 \qquad (a_A)_t = 1.2 \text{ m/s}^2$

$+\uparrow$: $a_C = -0.8 + 4.0 \qquad a_C = 3.2 \text{ m/s}^2$

(*a*) *Acceleration of Point A.*

$$\mathbf{a}_A = [1.2 \text{ m/s}^2 \longleftarrow] + [0.8 \text{ m/s}^2 \downarrow] \qquad \mathbf{a}_A = 1.442 \text{ m/s}^2 \swarrow 33.7° \blacktriangleleft$$

PROBLEM 15.116 (Continued)

(*b*) *Acceleration of Point B.* $\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$

$$\mathbf{a}_B = [1.2 \longleftarrow] + [0.8 \downarrow] + [r\alpha \downarrow] + [r\omega^2 \longrightarrow]$$

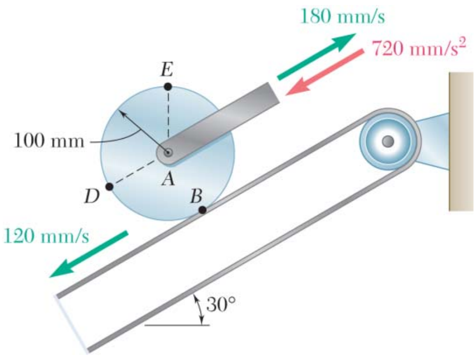
$$= [1.2 \longleftarrow] + [0.8 \downarrow] + [(0.04)(30) \downarrow] + [(0.04)(10)^2 \longrightarrow]$$

$$= [2.8 \text{ m/s}^2 \longrightarrow] + [2 \text{ m/s}^2 \downarrow]$$

$\mathbf{a}_B = 3.44 \text{ m/s}^2 \searrow 35.5°$ ◀

(*c*) *Acceleration of Point C.* $\mathbf{a}_C = a_C \uparrow$

$\mathbf{a}_C = 3.20 \text{ m/s}^2 \uparrow$ ◀



PROBLEM 15.117

The 100 mm radius drum rolls without slipping on a portion of a belt that moves downward to the left with a constant velocity of 120 mm/s. Knowing that at a given instant the velocity and acceleration of the center A of the drum are as shown, determine the acceleration of Point D.

SOLUTION

Velocity analysis.

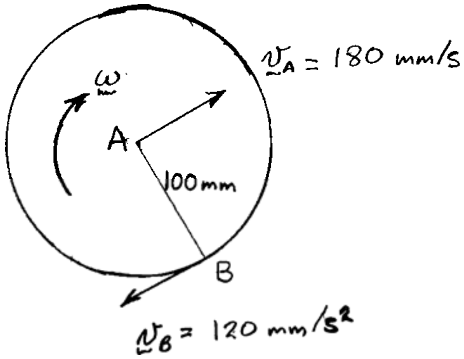
$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$[180\text{ mm/s} \nearrow] = [120\text{ mm/s} \swarrow] + [(100\text{ mm})\omega \nearrow]$$

Components $\nearrow$:

$$180 = -120 = 100\omega$$

$$\omega = 3\text{ rad/s} \circlearrowright$$



Acceleration analysis.

Point A moves on a path parallel to the belt. The path is assumed to be straight.

$$\mathbf{a}_A = 720\text{ mm/s}^2 \swarrow 30^\circ$$

Since the drum rolls without slipping on the belt, the component of acceleration of Point B on the drum parallel to the belt is the same as the belt acceleration. Since the belt moves at constant velocity, this component of acceleration is zero. Thus

$$\mathbf{a}_B = a_B \nwarrow 60^\circ$$

Let the angular acceleration of the drum be $\alpha \circlearrowleft$.

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

$$[a_B \nwarrow] = [720 \swarrow] + [r\alpha \nearrow] + [r\omega^2 \nwarrow]$$

Components $\swarrow$: $\quad 0 = 720 - 100\alpha$

$$\alpha = 7.2\text{ rad/s} \circlearrowleft$$

PROBLEM 15.117 (Continued)

Acceleration of Point D.

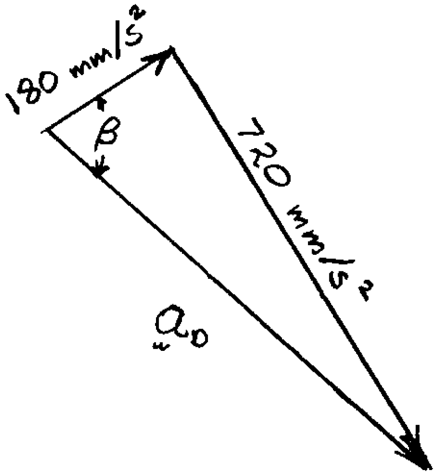
$$\mathbf{a}_D = \mathbf{a}_A + (\mathbf{a}_{D/A})_t + (\mathbf{a}_{D/A})_n$$

$$= [a_A \swarrow 30°] + [r\alpha \nwarrow 60°] + [r\omega^2 \nearrow 30°]$$

$$= [720 \swarrow 30°] + [(100)(7.2) \nwarrow 60°] + [(100)(3)^2 \nearrow 30°]$$

Components: $\nearrow$ 30°: $-720 + 900 = 180 \text{ mm/s}^2$

Components: $\nwarrow$ 60°: 720 mm/s^2

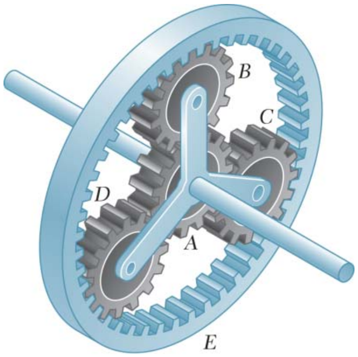


$$a_D = \sqrt{180^2 + 720^2} = 742.16 \text{ mm/s}^2$$

$$\tan\beta = \frac{720}{180} \qquad \beta = 76.0°$$

$$\beta - 30° = 46.0°$$

$\mathbf{a}_D = 742 \text{ mm/s}^2 \nwarrow 46.0°$ ◀



PROBLEM 15.118

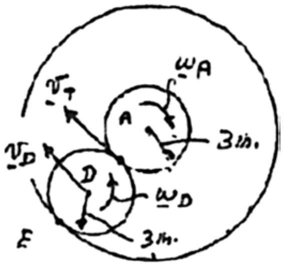
In the planetary gear system shown the radius of gears *A*, *B*, *C*, and *D* is 3 in. and the radius of the outer gear *E* is 9 in. Knowing that gear *A* has a constant angular velocity of 150 rpm clockwise and that the outer gear *E* is stationary, determine the magnitude of the acceleration of the tooth of gear *D* that is in contact with (*a*) gear *A*, (*b*) gear *E*.

SOLUTION

Velocity. $T =$ Tooth of gear D in contact with gear A

Gears: $v_T = r\omega_A = (3\text{ in.})\omega_A$

Since $v_E = 0$, E is instantaneous center of gear D.



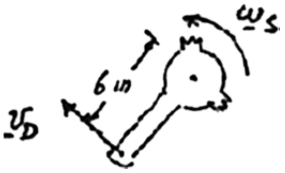
$$v_T = 2r\omega_D$$

$$(3\text{ in.})\omega_A = 2(3\text{ in.})\omega_D$$

$$\omega_D = \frac{1}{2}\omega_A$$

$$v_D = r\omega_D = (3\text{ in.})\frac{1}{2}\omega_A = (1.5\text{ in.})\omega_\text{A}$$

Spider:



$$v_D = (6\text{ in.})\omega_S$$

$$(1.5\text{ in.})\omega_\text{A} = (6\text{ in.})\omega_S$$

$$\omega_S = \frac{1}{4}\omega_A$$

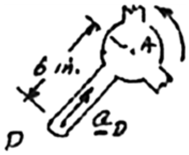
$$\boldsymbol{\omega}_A = 150\text{ rpm} = 15.708\text{ rad/s} \circlearrowright$$

$$\boldsymbol{\omega}_D = \frac{1}{2}\omega_A = 7.854\text{ rad/s} \circlearrowleft$$

$$\boldsymbol{\omega}_S = \frac{1}{4}\omega_A = 3.927\text{ rad/s} \circlearrowright$$

Acceleration.

Spider: $\omega_S = 3.927\text{ rad/s}$

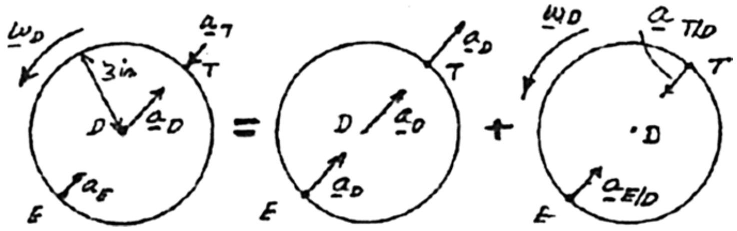


$$a_D = (AD)\omega_S^2 = (6\text{ in.})(3.927\text{ rad/s})^2$$

$$\mathbf{a}_D = 92.53\text{ in./s}^2 \nearrow$$

PROBLEM 15.118 (Continued)

Gear *D*:



Plane motion = Trans. with *D* + Rotation about *D*

(*a*) Tooth *T* in contact with gear *A*.

$$\mathbf{a}_T = \mathbf{a}_D + \mathbf{a}_{T/D} = a_D + (DT)\omega_D^2$$

$$= 92.53 \text{ in./s}^2 \nearrow + (3 \text{ in.})(7.854 \text{ rad/s})^2 \swarrow$$

$$= 92.53 \text{ in./s}^2 \nearrow + 185.06 \text{ in./s}^2 \swarrow$$

$$\mathbf{a}_T = 92.53 \text{ in./s}^2 \swarrow$$

$a_T = 92.5 \text{ in./s}^2$ ◀

(*b*) Tooth *E* in contact with gear *E*.

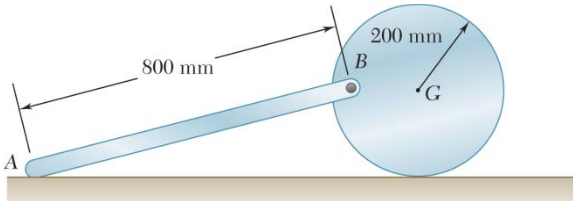
$$\mathbf{a}_E = \mathbf{a}_D + \mathbf{a}_{E/D} = \mathbf{a}_D + (ED)\omega_D^2$$

$$= 92.53 \text{ in./s}^2 \nearrow + (3 \text{ in.})(7.854 \text{ rad/s})^2 \nearrow$$

$$= 92.53 \text{ in./s}^2 \nearrow + 185.06 \text{ in./s}^2 \nearrow$$

$$\mathbf{a}_E = 277.6 \text{ in./s}^2 \nearrow$$

$a_E = 278 \text{ in./s}^2$ ◀



PROBLEM 15.119

The 200-mm-radius disk rolls without sliding on the surface shown. Knowing that the distance *BG* is 160 mm and that at the instant shown the disk has an angular velocity of 8 rad/s counterclockwise and an angular acceleration of 2 rad/s^2 clockwise, determine the acceleration of *A*.

SOLUTION

Units: m, m/s, m/s^2

Unit vectors: $\mathbf{i} = 1 \longrightarrow$, $\mathbf{j} = 1 \uparrow$, $\mathbf{k} = 1 \circlearrowleft$

Geometric analysis. Let *P* be the point where the disk contacts the flat surface.

$$\mathbf{r}_{G/A} = 0.200\mathbf{j} \qquad \mathbf{r}_{B/G} = -0.16\mathbf{i}$$

$$\mathbf{r}_{A/B} = -\sqrt{0.600}\mathbf{i} - 0.200\mathbf{j}$$

Velocity analysis. $\boldsymbol{\omega}_G = (8 \text{ rad/s})\mathbf{k}, \quad \mathbf{v}_P = 0, \ \mathbf{v}_A = v_A\mathbf{i}$

$$\mathbf{v}_B = \mathbf{v}_P + \mathbf{v}_{G/P} = \mathbf{v}_P + \boldsymbol{\omega}_G \times \mathbf{r}_{G/P}$$

$$= 0 + 8\mathbf{k} \times (-0.160\mathbf{i} + 0.200\mathbf{j}) = -1.6\mathbf{i} - 1.28\mathbf{j}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} = \mathbf{v}_B + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

$$v_A\mathbf{i} = -1.6\mathbf{i} - 1.28\mathbf{j} + \omega_{AB}\mathbf{k} \times (-\sqrt{0.600}\mathbf{i} - 0.200\mathbf{j})$$

$$= -1.6\mathbf{i} - 1.28\mathbf{j} - 0.77460\omega_{AB}\mathbf{j} + 0.2\omega_{AB}\mathbf{i}$$

Resolve into components and transpose terms.

$\mathbf{j}$: $\quad 0.77460\omega_{AB} = -1.28 \qquad \omega_{AB} = -1.6525 \text{ rad/s}$ ◀

Acceleration analysis: $\mathbf{a}_A = \mathbf{a}_A\mathbf{j}, \quad \mathbf{a}_G = -2 \text{ rad/s}^2\mathbf{k}$

$$\mathbf{a}_P = (\omega_G^2 r)\mathbf{j} = (8)^2(0.2)\mathbf{j} = (12.8 \text{ m/s}^2)\mathbf{j}$$

$$\mathbf{a}_B = \mathbf{a}_P + \mathbf{a}_{B/P} = \mathbf{a}_P + \mathbf{a}_G \times \mathbf{r}_{B/P} - \omega_G^2\mathbf{r}_{B/P}$$

$$= 12.8\mathbf{j} + (-2\mathbf{k}) \times (-0.160\mathbf{i} + 0.200\mathbf{j}) - (8)^2(-0.160\mathbf{i} + 0.200\mathbf{j})$$

$$= 12.8\mathbf{j} + 0.32\mathbf{j} + 0.4\mathbf{i} + 10.24\mathbf{i} - 12.8\mathbf{j}$$

$$= 10.64\mathbf{i} + 0.32\mathbf{j}$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} = \mathbf{a}_B + \alpha_{AB}\mathbf{k} \times \mathbf{r}_{A/B} - \omega_{AB}^2\mathbf{r}_{A/B}$$

$$= 10.64\mathbf{i} + 0.32\mathbf{j} + \alpha_{AB}\mathbf{k} \times (-\sqrt{0.600}\mathbf{i} - 0.200\mathbf{j}) - (1.6525)^2(-\sqrt{0.600}\mathbf{i} - 0.200\mathbf{j})$$

$$= 10.64\mathbf{i} + 0.32\mathbf{j} - 0.77460\alpha_{AB}\mathbf{j} + 0.2\alpha_{AB}\mathbf{i} + 2.115\mathbf{i} + 0.54615\mathbf{j}$$

$$a_A\mathbf{i} = 12.755\mathbf{i} + 0.86615\mathbf{j} + 0.2\alpha_{AB}\mathbf{i} - 0.77460\alpha_{AB}\mathbf{j}$$

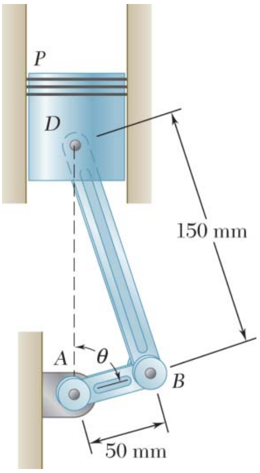
PROBLEM 15.119 (Continued)

Resolve into components and transpose terms.

j: $0 = 0.86615 - 0.77460\alpha_{AB} \qquad \alpha_{AB} = 1.1182$

i: $a_A = 12.755 + 0.2\alpha_{AB} = 12.755 + (0.2)(1.1182) = 12.98$

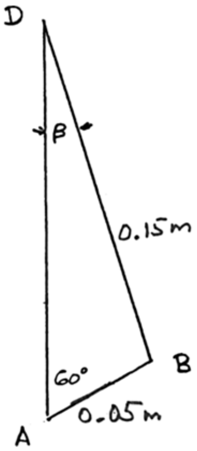
$\mathbf{a}_A = (12.98 \text{ m/s}^2)\mathbf{i} = 12.98 \text{ m/s}^2 \longrightarrow$ ◀



PROBLEM 15.120

Knowing that crank AB rotates about Point A with a constant angular velocity of 900 rpm clockwise, determine the acceleration of the piston P when $\theta = 60°$.

SOLUTION



Law of sines. $\dfrac{\sin\beta}{0.05} = \dfrac{\sin 60°}{0.15} \qquad \beta = 16.779°$

Velocity analysis. $\omega_{AB} = 900 \text{ rpm} = 30\pi \text{ rad/s} \circlearrowright$

$\mathbf{v}_B = 0.05\omega_{AB} = 1.5\pi \text{ m/s} \nwarrow 60°$

$\mathbf{v}_D = v_D \downarrow \qquad \omega_{BD} = \omega_{BD} \circlearrowleft$

$\mathbf{v}_{D/B} = 0.15\,\omega_{BD} \swarrow \beta$

$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$

$[v_D \downarrow] = [1.5\pi \nwarrow 60°] + [0.15\omega_{BD} \swarrow \beta]$

Components $\xrightarrow{+}$: $0 = 1.5\pi\cos 60° - 0.15\omega_{BD}\cos\beta$

$$\omega_{BD} = \frac{1.5\pi \cos 60°}{0.15 \cos \beta} = 16.4065 \text{ rad/s} \circlearrowleft$$

Acceleration analysis. $\alpha_{AB} = 0$

$\mathbf{a}_B = 0.05\omega_{AB}^2 = (0.05)(30\pi)^2 = 444.13 \text{ m/s}^2 \swarrow 30°$

$\mathbf{a}_D = a_D \downarrow \qquad \alpha_{BD} = \alpha_{BD} \circlearrowright$

$\mathbf{a}_{D/B} = [0.15\alpha_{AB} \nearrow \beta] + [0.15\omega_{BD}^2 \searrow \beta]$

$= [0.15\alpha_{BD} \nearrow \beta] + [40.376 \searrow \beta]$

$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B}$ Resolve into components.

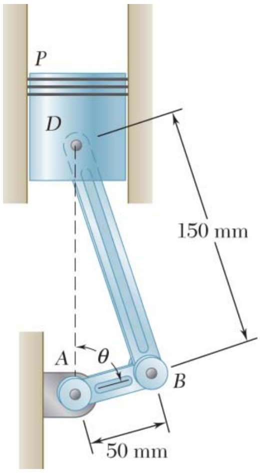
PROBLEM 15.120 (Continued)

$\xrightarrow{+}$: $0 = -444.13 \cos 30° + 0.15\alpha_{BD} \cos\beta + 40.376 \sin\beta$

$\alpha_{BD} = 2597.0 \text{ rad/s}^2$

$+\downarrow$: $a_D = 444.13 \sin 30° - (0.15)(2597.0)\sin\beta + 40.376 \cos\beta$

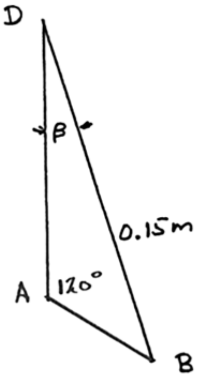
$= 148.27 \text{ m/s}^2$ $\quad \mathbf{a}_P = \mathbf{a}_D$ $\quad \mathbf{a}_P = 148.3 \text{ m/s}^2 \downarrow$ ◀



PROBLEM 15.121

Knowing that crank *AB* rotates about Point *A* with a constant angular velocity of 900 rpm clockwise, determine the acceleration of the piston *P* when $\theta = 120°$.

SOLUTION



Law of sines. $\dfrac{\sin\beta}{0.05} = \dfrac{\sin 120°}{0.15}, \qquad \beta = 16.779°$

Velocity analysis. $\omega_{AB} = 900 \text{ rpm} = 30\pi \text{ rad/s} \circlearrowright$

$$\mathbf{v}_B = 0.05\omega_{AB} = 1.5\pi \text{ m/s} \swarrow 60°$$

$$\mathbf{v}_D = v_D \downarrow \qquad \omega_{BD} = \omega_{BD} \circlearrowleft$$

$$\mathbf{v}_{D/B} = 0.15\omega_{BD} \swarrow \beta$$

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$

$$[v_D \downarrow] = [1.5\pi \swarrow 60°] + [0.15\omega_{BD} \swarrow \beta]$$

Components $\xrightarrow{+}$: $0 = -1.5\pi \cos 60° - 0.15\omega_{BD} \cos\beta$

$$\omega_{BD} = -\frac{1.5\pi \cos 60°}{0.15 \cos\beta} = 16.4065 \text{ rad/s} \circlearrowright$$

Acceleration analysis. $\alpha_{AB} = 0$

$$\mathbf{a}_B = 0.05\omega_{AB}^2 = (0.05)(30\pi)^2 = 444.13 \text{ m/s}^2 \nwarrow 30°$$

$$\mathbf{a}_D = a_D \downarrow \qquad \alpha_{BD} = \alpha_{BD} \circlearrowright$$

$$\mathbf{a}_{D/B} = [0.15\alpha_{AB} \nearrow \beta] + [0.15\omega_{BD}^2 \searrow \beta]$$

$$= [6\alpha_{BD} \nearrow \beta] + [40.376 \searrow \beta]$$

$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B}$ Resolve into components.

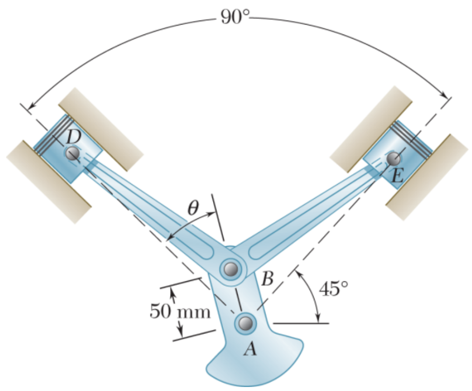
PROBLEM 15.121 (Continued)

$\xrightarrow{+}$: $0 = -444.13\cos 30° + 0.15\alpha_{BD}\cos\beta + 40.376\sin\beta$

$\alpha_{BD} = 2597.0 \text{ rad/s}^2$

$+\downarrow$: $a_D = -444.13\sin 30° - (0.15)(2597.0)\sin\beta + 40.376\cos\beta$

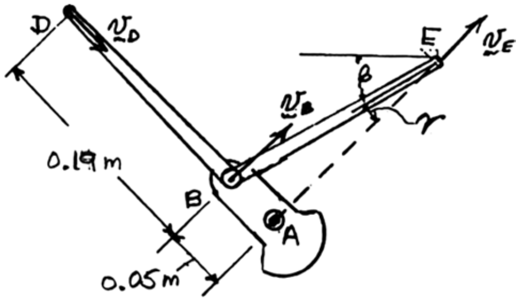
$= -296 \text{ m/s}^2 \qquad \mathbf{a}_P = \mathbf{a}_D \qquad\qquad \mathbf{a}_P = 296 \text{ m/s}^2 \uparrow$ ◀



PROBLEM 15.122

In the two-cylinder air compressor shown, the connecting rods *BD* and *BE* are each 190 mm long and crank *AB* rotates about the fixed Point *A* with a constant angular velocity of 1500 rpm clockwise. Determine the acceleration of each piston when $\theta = 0$.

SOLUTION



Crank AB.

$$\mathbf{v}_A = 0, \quad \mathbf{a}_A = 0, \quad \omega_{AB} = 1500 \text{ rpm} = 157.08 \text{ rad/s} \circlearrowright, \quad \alpha_{AB} = 0$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = 0 + [0.05\,\omega_{AB} \nearrow 45°] = [7.854 \text{ m/s} \nearrow 45°]$$

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

$$= 0 + [0.05\alpha_{AB} \swarrow 45°] + [0.05\omega_{AB}^2 \searrow 45°]$$

$$= [(0.05)(157.08)^2 \searrow 45°] = 1233.7 \text{ m/s}^2 \searrow 45°$$

Rod BD.

$$\mathbf{v}_D = v_D \searrow 45° \qquad \boldsymbol{\omega}_{BD} = \omega_{BD} \circlearrowleft$$

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{B/D}$$

$$v_D \searrow 45° = [7.854 \nearrow 45°] + [0.19\omega_{BD} \swarrow 45°]$$

Components $\nearrow$ 45°: $\quad 0 = 7.854 - 0.19\omega_{BD} \qquad \omega_{BD} = 41.337 \text{ rad/s}$

$$\mathbf{a}_D = a_D \searrow 45°$$

$$\mathbf{a}_D = \mathbf{a}_B + (\mathbf{a}_{D/B})_t + (\mathbf{a}_{D/B})_n$$

$$[a_D \searrow 45°] = [1233.7 \searrow 45°] + [0.19\,\alpha_{BD} \swarrow 45°] + [0.19\,\omega_{BD}^2 \searrow 45°]$$

PROBLEM 15.122 (Continued)

Components $\searrow 45°$: $$a_D = 1233.7 + (0.19)(41.337)^2 = 1558.4 \text{ m/s}^2$$

$$\mathbf{a}_D = 1558 \text{ m/s}^2 \searrow 45° \blacktriangleleft$$

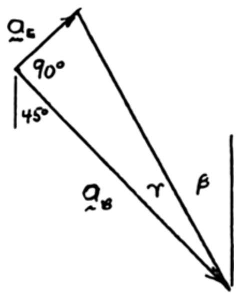
Rod BE. $$\sin\gamma = \frac{0.05}{0.19}, \quad \gamma = 15.258°, \quad \beta = 45° - \gamma = 29.742°$$

$$\mathbf{v}_E = v_E \measuredangle 45°$$

Since $\mathbf{v}_E$ is parallel to $\mathbf{v}_B$, $\omega_{BE} = 0.$

$$\mathbf{a}_E = a_E \measuredangle 45° \quad (a_{B/E})_n = 0.19\omega_{BE}^2 = 0$$

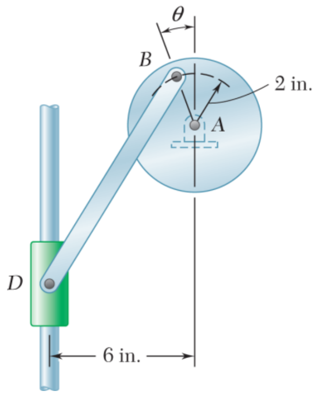
$$(\mathbf{a}_{E/B})_t = (a_{E/B})_t \searrow \beta \qquad \mathbf{a}_E = \mathbf{a}_B + (\mathbf{a}_{B/E})_t$$



Draw vector addition diagram.

$$\begin{aligned} \gamma &= 45° - \beta \\ &= 15.258° \\ a_E &= a_B \tan\gamma \\ &= 1233.7 \tan\gamma \\ &= 336.52 \text{ m/s}^2 \end{aligned}$$

$$\mathbf{a}_E = 337 \text{ m/s}^2 \measuredangle 45° \blacktriangleleft$$



PROBLEM 15.123

The disk shown has a constant angular velocity of 500 rpm counter-clockwise. Knowing that rod BD is 10 in. long, determine the acceleration of collar D when (*a*) $\theta = 90°$, (*b*) $\theta = 180°$.

SOLUTION

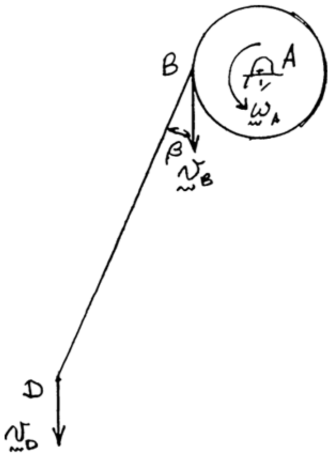
Disk A.

$$\boldsymbol{\omega}_A = 500 \text{ rpm} \circlearrowleft = 52.36 \text{ rad/s} \circlearrowleft$$

$$\boldsymbol{\alpha}_A = 0, \quad (AB) = 2 \text{ in.}$$

$$v_B = (AB)\omega_A = (2)(52.36) = 104.72 \text{ in./s}$$

$$a_B = (AB)\omega_A^2 = (2)(52.36)^2 = 5483.1 \text{ in./s}^2$$



(*a*) $\theta = 90°$.

$$\mathbf{v}_B = 104.72 \text{ m/s}\downarrow, \quad \mathbf{v}_D = v_D\downarrow$$

$$\sin\beta = \frac{2 \text{ in.}}{5 \text{ in.}} = 0.4 \qquad \beta = 23.58°$$

$\mathbf{v}_D$ and $\mathbf{v}_B$ are parallel.

$$\omega_{BD} = 0$$

$$\mathbf{a}_B = 5483.1 \text{ in./s}^2 \longrightarrow, \quad \mathbf{a}_D = a_D\uparrow, \quad \boldsymbol{\alpha}_{BD} = \alpha_{BD} \circlearrowleft$$

$$\mathbf{a}_{D/B} = [(BD)\alpha_{BD} \nwarrow \beta] + [(BD)\omega_{BD}^2 \nearrow \beta]$$

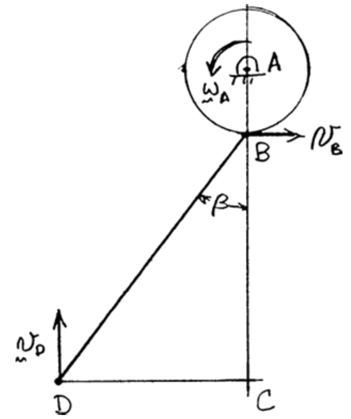
$$= [10\,\alpha_{BD} \nwarrow \beta] + 0$$

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} \quad \text{Resolve into components.}$$

$\xrightarrow{+}$: $\quad 0 = 5483.1 + (10\cos\beta)\alpha_{BD} \qquad \alpha_{BD} = -598.26 \text{ rad/s}^2$

$+\uparrow$: $\quad a_D = 0 - (10\sin\beta)(-598.26) + 0 = 2393.0 \text{ in./s}^2 \qquad \mathbf{a}_D = 199.4 \text{ ft/s}^2\uparrow$ ◀

PROBLEM 15.123 (Continued)



(*b*) $\theta = 180°$. $\mathbf{v}_B = 104.72 \text{ in./s} \longrightarrow, \quad \mathbf{v}_D = v_D \uparrow$

$$\sin\beta = \frac{6 \text{ in.}}{10 \text{ in.}} = 0.6 \qquad \beta = 36.87$$

$$\mathbf{v}_B = 104.72 \text{ in./s} \longrightarrow, \quad \mathbf{v}_D = v_D \uparrow$$

Instantaneous center of bar *BD* lies at Point *C*.

$$\omega_{BD} = \frac{v_B}{(BD)} = \frac{104.72}{10 \cos\beta} = 13.09 \text{ rad/s}$$

$$\mathbf{a}_B = 5483.1 \text{ in./s}^2 \uparrow, \quad \mathbf{a}_D = a_D \uparrow, \quad \boldsymbol{\alpha}_{BD} = \alpha_{BD} \circlearrowleft$$

$$\mathbf{a}_{D/B} = [(BD)\alpha_{BD} \nwarrow \beta] + [(BD)\omega_{BD}^2 \nearrow \beta]$$

$$= [10\alpha_{BD} \nwarrow \beta] + [1713.5 \nearrow \beta]$$

$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B}$ Resolve in components.

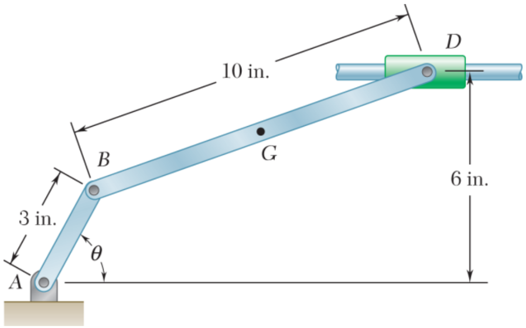
$\xrightarrow{+}$: $0 = 0 + (10 \cos\beta)\alpha_{BD} + 1713.5 \sin\beta$

$$\alpha_{BD} = -128.51 \text{ rad/s}^2$$

$+\uparrow$: $a_D = 5483.1 - (10 \sin\beta)(-128.51) + 1713.5 \cos\beta$

$$= 7625.0 \text{ in./s}^2$$

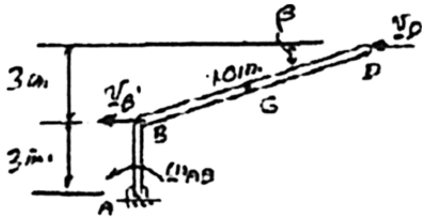
$\mathbf{a}_D = 635 \text{ ft/s}^2 \uparrow$ ◀


D
10 in.
B
G
6 in.
3 in.
θ
A

PROBLEM 15.124

Arm AB has a constant angular velocity of 16 rad/s counterclockwise. At the instant when $\theta = 90°$, determine the acceleration (*a*) of collar D, (*b*) of the midpoint G of bar BD.

SOLUTION


β
v_D
3 in.
10 in.
D
v_B'
G
B
3 in.
ω_{AB}
A

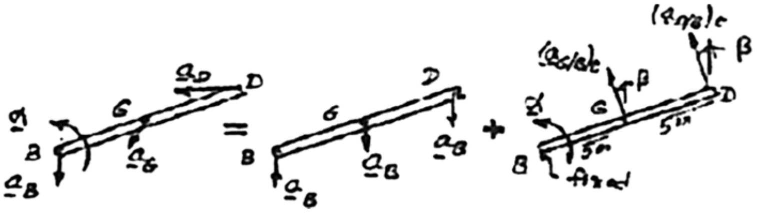
Rod AB:

$$a_B = (AB)\omega_{AB}^2$$
$$= (3\text{ in.})(16\text{ rad/s})^2$$
$$\mathbf{a}_B = 768\text{ in./s}^2 \downarrow$$

Rod BD: instantaneous center is at OD; $\boldsymbol{\omega}_{BD} = 0$

$$\sin\beta = (3\text{ in.})/(10\text{ in.}) = 0.3; \quad \beta = 17.46°$$

Acceleration.

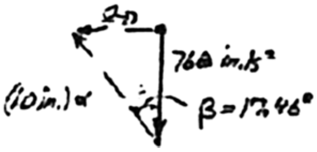

α
B
a_B
G
a_G
a_D
D
B
a_B
G
a_B
D
a_B
$(a_{D/B})_t$
$(a_{G/B})_t$
β
β
α
G
D
5 in
5 in
B
fixed

Plane motion = Trans. with *B* + Rotation about *B*

(*a*)

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} = a_B + (\mathbf{a}_{D/B})_t + (\mathbf{a}_{D/B})_n$$
$$= [a_B \downarrow] + [(BD)\alpha \searrow\!| \beta] + [(BD)\omega_{BD}^2 \measuredangle \beta]$$
$$= [768\text{ in./s}^2 \downarrow] + [(10\text{ in.})\ \alpha \searrow\!| \beta] + [(10\text{ in.})(0)^2 \measuredangle \beta]$$
$$\mathbf{a}_D \leftrightarrow = [768\text{ in./s} \downarrow] + [(10\text{ in.})\alpha \searrow\!| \beta]$$

Vector diagram:


a_D
768 in./s²
(10 in.)α
β = 17.46°

$$\mathbf{a}_D = (768\text{ in./s}^2)\tan 17.46°$$
$$= 241.62\text{ in./s}^2$$

$$\mathbf{a}_D = 242\text{ in./s}^2 \longleftarrow \blacktriangleleft$$

$$(10\text{ in.})\alpha = (768\text{ in./s}^2)/\cos 17.46°$$
$$(10\text{ in.})\alpha = 805.08\text{ in./s}^2$$
$$\boldsymbol{\alpha} = 80.5\text{ rad/s}^2 \circlearrowleft$$

PROBLEM 15.124 (Continued)

(*b*)

$$\mathbf{a}_G = \mathbf{a}_B + \mathbf{a}_{G/B} = \mathbf{a}_B + (\mathbf{a}_{G/B})_t + (\mathbf{a}_{G/B})_n$$

$$= [\mathbf{a}_B \downarrow] + [(BG)\alpha \searrow \beta] + [(BG)\omega_{BD}^2 \measuredangle \beta]$$

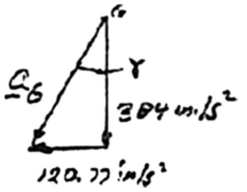
$$= [768 \text{ in./s}^2 \downarrow] + [(5 \text{ in.})(80.5 \text{ rad/s}^2) \searrow \beta] + [(BG)(0)^2]$$

$$\mathbf{a}_G = [768 \text{ in./s}^2 \downarrow] + [402.5 \text{ in./s}^2 \searrow 17.46°]$$

$\xleftarrow{+}$ components:

$$(a_G)_x = (402.5 \text{ in./s}^2)\sin 17.46°$$

$$(\mathbf{a}_G)_x = 120.77 \text{ in./s}^2 \longleftarrow$$



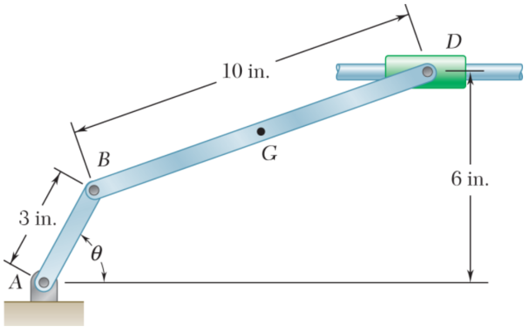
$+\downarrow$ components:

$$(a_G)_y = 768 \text{ in./s}^2 - (402.5 \text{ in./s}^2)\cos 17.46°$$

$$= 768 \text{ in./s}^2 - 384 \text{ in./s}^2$$

$$(\mathbf{a}_G)_y = 384 \text{ in./s}^2 \downarrow$$

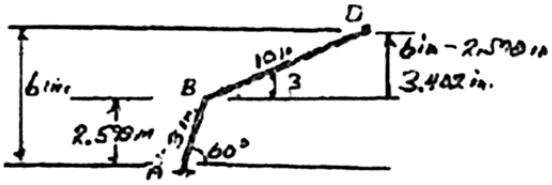
$$\mathbf{a}_G = 403 \text{ in./s}^2 \swarrow 72.5° \blacktriangleleft$$


10 in.
D
B
G
6 in.
3 in.
θ
A

PROBLEM 15.125

Arm *AB* has a constant angular velocity of 16 rad/s counterclockwise. At the instant when $\theta = 60°$, determine the acceleration of collar *D*.

SOLUTION

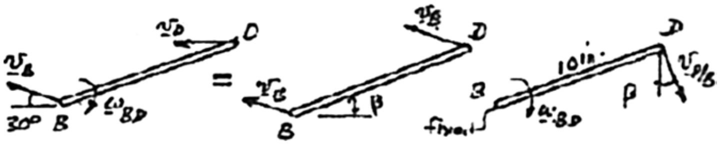

D
6 in − 2.598 in
3.402 in.
10 in
B
3
6 in.
2.598 in
3 in
60°
A

$$\beta = \sin^{-1}\frac{3.403\text{ in.}}{10\text{ in.}}$$

$$\beta = 19.89°$$

Velocity. $\mathbf{v}_B = (AB)\omega_{AB} = (3\text{ in.})(16\text{ rad/s}) = 48\text{ in./s} \measuredangle 30°$

Rod *BD*:


v_D
D
v_B
30°
B
ω_{BD}
=
v_B
D
v_B
β
B
10 in.
D
$v_{D/B}$
3
β
fixed
ω_{BD}

Plane motion = Trans. with *B* + Rotation about *B*

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B} = \mathbf{v}_B + [(BD)\omega_{BD} \measuredangle \beta]$$

$$\mathbf{v}_D \leftrightarrow = [48\text{ in./s} \measuredangle 30°] + [(10\text{ in.})\omega_{BD} \measuredangle 19.89°]$$

$+\uparrow$ components: $(48\text{ in./s})\sin 30° - (10\text{ in.})\omega_{BD}\cos 19.89°$

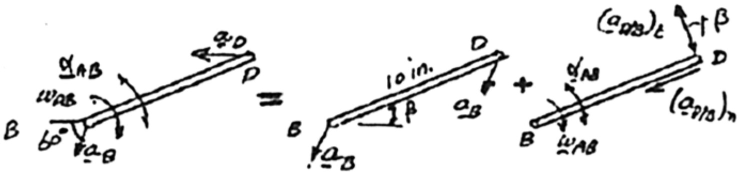
$$\boldsymbol{\omega}_{BD} = \frac{(48\text{ in./s})\sin 30°}{(10\text{ in.})\cos 19.890°} = 2.552\text{ rad/s} \circlearrowright$$

Acceleration.

Rod *AB*: $\mathbf{a}_B = [(AB)\omega_{AB}^2 \measuredangle 60°] = [(3\text{ in.})(16\text{ rad/s})^2 \measuredangle 60°]$

$$\mathbf{a}_B = 768\text{ in./s}^2 \measuredangle 60°$$

Rod *BD*:


α_{AB}
a_D
ω_{AB}
D
B
60°
a_B
=
10 in.
D
β
a_B
B
a_B
+
$(a_{D/B})_t$
β
D
α_{AB}
$(a_{D/B})_n$
B
ω_{AB}

Plane motion = Trans. with *B* + Rotation about *B*

PROBLEM 15.125 (Continued)

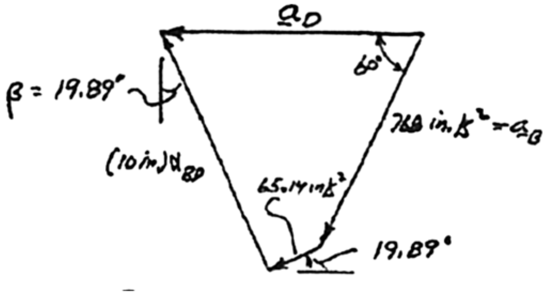
$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{B/D} = \mathbf{a}_B + (\mathbf{a}_{D/B})_t + (\mathbf{a}_{D/B})_n$$

$$a_D \leftrightarrow = [a_B \swarrow 60°] + [(BD)\alpha_{BD} \nwarrow \beta] + [(BD)\omega_{DB}^2 \swarrow \beta]$$

$$= [768 \text{ in./s}^2 \swarrow 60°] + [(10 \text{ in.})\alpha_{BD} \nwarrow \beta] + [(10 \text{ in.})(2.552 \text{ rad/s}^2) \swarrow \beta]$$

$$a_D \leftrightarrow = [768 \text{ in./s}^2 \swarrow] + [(10 \text{ in.})\alpha_{BD} \nwarrow 19.89°] + [65.14 \text{ in./s}^2 \swarrow 19.89°]$$

Vector diagram.



y components:

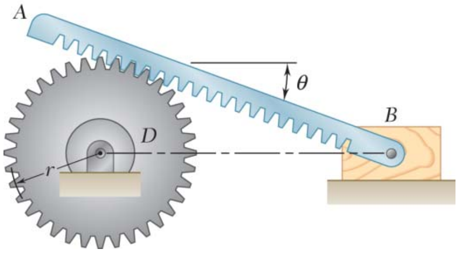
$+\downarrow$: $768 \sin 60° + 65.14 \sin 19.89° - 10\alpha_{BD} \cos 19.89° = 0$

$\alpha_{BD} = 73.09 \text{ rad/s}^2$

x components:

$\xleftarrow{+}$: $a_D = 768 \cos 60° + 65.14 \cos 19.89° + (10)(73.09) \sin 19.89°$

$a_D = 693.9 \text{ in./s}^2$ $\mathbf{a}_D = 694 \text{ in./s}^2 \longleftarrow$ ◀



PROBLEM 15.126

A straight rack rests on a gear of radius $r = 3$ in. and is attached to a block B as shown. Knowing that at the instant shown $\theta = 20°$, the angular velocity of gear D is 3 rad/s clockwise, and it is speeding up at a rate of 2 rad/s^2, determine (*a*) the angular acceleration of *AB,* (*b*) the acceleration of block *B*.

SOLUTION

Let Point P on the gear and Point Q on the rack be located at the contact point between them.

Units: inches, in./s, in./s^2

Unit vectors: $\mathbf{i} = 1 \rightarrow,\ \mathbf{j} = 1 \uparrow,\ \mathbf{k} = 1 \circlearrowleft$

Geometry: $\mathbf{r}_{P/D} = 3(\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$

$$\mathbf{r}_{B/Q} = \frac{3}{\tan\theta}(\cos\theta\mathbf{i} - \sin\theta\mathbf{j})$$

$$\theta = 20°$$

Gear D: $\boldsymbol{\omega}_{\text{gear}} = 3 \text{ rad/s} \circlearrowright$ $\qquad$ $\boldsymbol{\alpha}_{\text{gear}} = 2 \text{ rad/s}^2 \circlearrowright$

$v_P = \omega_{\text{gear}} r = (3)(3) = 9 \text{ in./s}$ $\qquad$ $\mathbf{v}_P = 9 \text{ in./s} \searrow \theta$

$(a_P)_n = \omega_{\text{gear}}^2 r = (3)^2(3) = 27 \text{ in./s}^2$ $\qquad$ $(\mathbf{a}_P)_n = 27 \text{ in./s}^2 \swarrow \theta$

$(a_P)_t = \alpha_{\text{gear}} r = (2)(3) = 6 \text{ in./s}^2$ $\qquad$ $(\mathbf{a}_P)_t = 6 \text{ in./s}^2 \searrow \theta$

Velocity analysis.

Gear to rack contact: $\mathbf{v}_Q = \mathbf{v}_P = 9 \text{ in./s} \searrow \theta$

Rack *AQB*: $\boldsymbol{\omega}_{AB} = \omega_{AB} \circlearrowleft, \qquad \boldsymbol{\alpha}_{AB} = \alpha_{AB} \circlearrowleft$

$$\mathbf{v}_B = v_B \rightarrow, \qquad \boldsymbol{\alpha}_B = \alpha_B \rightarrow$$

$$\mathbf{v}_B = \mathbf{v}_Q + \mathbf{v}_{B/Q} = \mathbf{v}_Q + \omega_{AB}\mathbf{k} \times \mathbf{r}_{B/Q}$$

$$v_B\mathbf{i} = 9(\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) + \omega_{AB}\mathbf{k} \times (7.74535\mathbf{i} - 2.81908\mathbf{j})$$

$$= (9\cos\theta + 2.81907\omega_{AB})\mathbf{i} + (-9\sin\theta + 2.81907\omega_{AB})\mathbf{j}$$

Equating like components,

$\mathbf{j}$: $\qquad 0 = -9\sin\theta + 7.74535\omega_{AB}$

$\omega_{AB} = 0.39742$ $\qquad$ $\omega_{AB} = 0.39742 \text{ rad/s} \circlearrowleft$

PROBLEM 15.126 (Continued)

Acceleration analysis.

Gear to rack contact: $(\mathbf{a}_Q)_t = (\mathbf{a}_P)_t = 6 \text{ in./s}^2 \searrow \theta$

$$(\mathbf{a}_Q)_n = (\mathbf{a}_P)_n + r\omega_{rd}^2 \angle\!\nearrow \theta$$

where $\boldsymbol{\omega}_{rd} = \boldsymbol{\omega}_{AB} - \boldsymbol{\omega}_D = 3.39742 \text{ rad/s} \circlearrowright$

$$(\mathbf{a}_Q)_n = 27 \text{ in./s}^2 \swarrow 20^\circ + (3)(3.39742)^2 \angle\!\nearrow 20^\circ$$

$$= 7.6274 \text{ in./s}^2 \angle\!\nearrow 20^\circ$$

Then, $\mathbf{a}_Q = 6(\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j}) + 7.6274(\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$

$$= (8.2469 \text{ in./s}^2)\mathbf{i} + (5.1153 \text{ in./s}^2)\mathbf{j}$$

$$\mathbf{a}_B = \mathbf{a}_Q + \mathbf{a}_{B/Q} = \mathbf{a}_Q + \alpha_{AB}\mathbf{k} \times \mathbf{r}_{B/Q} - \omega_{AB}^2 \mathbf{r}_{B/Q}$$

$$a_B\mathbf{i} = 8.2469\mathbf{i} + 5.1153\mathbf{j} + \alpha_{AB}\mathbf{k} \times (7.74535\mathbf{i} - 2.81908\mathbf{j})$$

$$-(0.39742)^2(7.74535\mathbf{i} - 2.81908\mathbf{j})$$

$$= (8.2469 + 2.81908\alpha_{AB} - 1.22332)\mathbf{i}$$

$$+ (5.1153 + 7.74535\alpha_{AB} + 0.44526)\mathbf{j}$$

Equating like components of $\mathbf{a}_B$,

j: $0 = 5.1153 + 7.74535\alpha_{AB} + 0.44526$

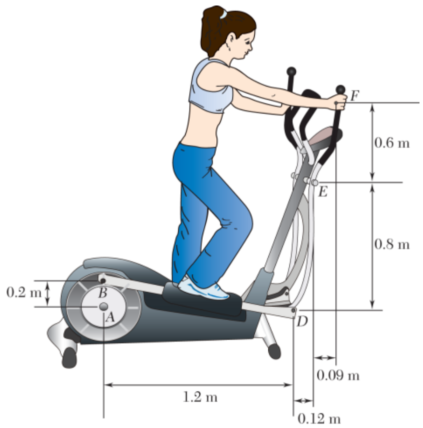
$$\alpha_{AB} = -0.71792 \text{ rad/s}^2$$

i: $a_B = 8.2469 + (2.81908)(-0.71792) - 1.22332$

$$a_B = 5.00 \text{ in./s}^2$$

(*a*) *Angular acceleration of AB*: $\boldsymbol{\alpha}_{AB} = 0.718 \text{ rad/s}^2 \circlearrowright$ ◀

(*b*) *Acceleration of block B*: $\mathbf{a}_B = 5.00 \text{ in./s}^2 \longrightarrow$ ◀



PROBLEM 15.127

The elliptical exercise machine has fixed axes of rotation at points *A* and *E*. Knowing that at the instant shown the flywheel *AB* has a constant angular velocity of 6 rad/s clockwise, determine the acceleration of point *D*.

SOLUTION

Given: $\boldsymbol{\omega}_{AB} = -6\mathbf{k}$ rad/s, $\boldsymbol{\alpha}_{AB} = 0$ rad/s^2

Geometry:

$\mathbf{r}_{B/A} = 0.2\mathbf{j}$ m

$\mathbf{r}_{D/B} = 1.2\mathbf{i} - 0.2\mathbf{j}$ m

$\mathbf{r}_{D/E} = -0.12\mathbf{i} - 0.8\mathbf{j}$ m

Relative Velocity:

$\mathbf{v}_B = \cancel{\mathbf{v}_A} + \omega_{AB}\mathbf{k} \times \mathbf{r}_{B/A}$ (1)

$\mathbf{v}_D = \mathbf{v}_B + \omega_{BD}\mathbf{k} \times \mathbf{r}_{D/B}$ (2)

$\mathbf{v}_D = \cancel{\mathbf{v}_E} + \omega_{DEF}\mathbf{k} \times \mathbf{r}_{D/E}$ (3)

Sub (1) into (2) = (3): $\omega_{AB}\mathbf{k} \times \mathbf{r}_{B/A} + \omega_{BD}\mathbf{k} \times \mathbf{r}_{D/B} = \omega_{DEF}\mathbf{k} \times \mathbf{r}_{D/E}$

Put in known values: $-6\mathbf{k} \times 0.2\mathbf{j} + \omega_{BD}\mathbf{k} \times (1.2\mathbf{i} - 0.2\mathbf{j}) = \omega_{DEF}\mathbf{k} \times (-0.12\mathbf{i} - 0.8\mathbf{j})$

Equate components:

i: $1.2 + 0.2\omega_{BD} = 0.8\omega_{DEF}$

j: $1.2\omega_{BD} = -0.12\omega_{DEF}$

Solve simultaneously: $\omega_{BD} = -0.1463$ rad/s, $\omega_{DEF} = 1.463$ rad/s

Relative acceleration:

$\mathbf{a}_B = \cancel{\mathbf{a}_A} + \cancel{\alpha_{AB}}\mathbf{k} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A}$ (4)

$\mathbf{a}_D = \mathbf{a}_B + \alpha_{BD}\mathbf{k} \times \mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B}$ (5)

$\mathbf{a}_D = \cancel{\mathbf{a}_E} + \alpha_{DEF}\mathbf{k} \times \mathbf{r}_{D/E} - \omega_{DEF}^2\mathbf{r}_{D/E}$ (6)

Sub (4) into (5) = (6): $-\omega_{AB}^2\mathbf{r}_{B/A} + \alpha_{BD}\mathbf{k} \times \mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B} = \alpha_{DEF}\mathbf{k} \times \mathbf{r}_{D/E} - \omega_{DEF}^2\mathbf{r}_{D/E}$

Put in known values:

$$-(-6)^2(0.2\mathbf{j}) + \alpha_{BD}\mathbf{k} \times (1.2\mathbf{i} - 0.2\mathbf{j}) - (-0.1463)^2(1.2\mathbf{i} - 0.2\mathbf{j}) = \alpha_{DEF}\mathbf{k} \times (-0.12\mathbf{i} - 0.8\mathbf{j}) - (1.463)^2(-0.12\mathbf{i} - 0.8\mathbf{j})$$

Equate components:

i: $0.2\alpha_{BD} - 0.02570 = 0.8\alpha_{DEF} + 0.2570$

j: $-7.2 + 1.2\alpha_{BD} + 0.004283 = -0.12\alpha_{DEF} + 1.7132$

Solve simultaneously: $\alpha_{BD} = 7.278$ rad/s^2

PROBLEM 15.127 (Continued)

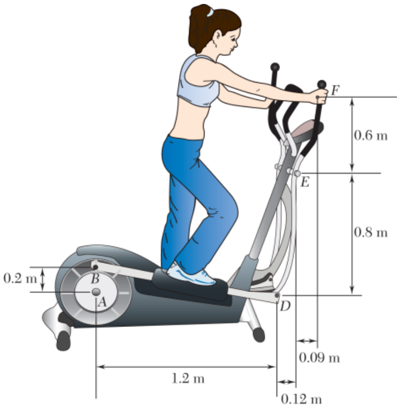
Relative acceleration:

$$\mathbf{a}_D = \cancel{\mathbf{a}_E} + \alpha_{DEF}\mathbf{k} \times \mathbf{r}_{\mathrm{D}/E} - \omega_{DEF}^2 \mathbf{r}_{\mathrm{D}/E}$$

$$= 1.466\mathbf{k} \times (-0.12\mathbf{i} - 0.8\mathbf{j}) - (1.463)^2 (-0.12\mathbf{i} - 0.8\mathbf{j})$$

$$\mathbf{a}_D = 1.430\mathbf{i} + 1.537\mathbf{j} \text{ m/s}^2$$

$\mathbf{a}_F = 2.099 \text{ m/s}^2 \measuredangle 47.08°$ ◀



PROBLEM 15.128

The elliptical exercise machine has fixed axes of rotation at points *A* and *E*. Knowing that at the instant shown the flywheel *AB* has a constant angular velocity of 6 rad/s clockwise, determine (*a*) the angular acceleration of bar *DEF*, (*b*) the acceleration of point *F*.

SOLUTION

Given: $\boldsymbol{\omega}_{AB} = -6\mathbf{k}$ rad/s, $\boldsymbol{\alpha}_{AB} = 0$ rad/s^2

Geometry:

$$\mathbf{r}_{B/A} = 0.2\mathbf{j}\text{ m}$$
$$\mathbf{r}_{D/B} = 1.2\mathbf{i} - 0.2\mathbf{j}\text{ m}$$
$$\mathbf{r}_{D/E} = -0.12\mathbf{i} - 0.8\mathbf{j}\text{ m}$$
$$\mathbf{r}_{F/E} = 0.09\mathbf{i} + 0.6\mathbf{j}\text{ m}$$

Relative Velocity:

$$\mathbf{v}_B = \cancel{\mathbf{v}_A} + \omega_{AB}\mathbf{k}\times\mathbf{r}_{B/A} \qquad (1)$$
$$\mathbf{v}_D = \mathbf{v}_B + \omega_{BD}\mathbf{k}\times\mathbf{r}_{D/B} \qquad (2)$$
$$\mathbf{v}_D = \cancel{\mathbf{v}_E} + \omega_{DEF}\mathbf{k}\times\mathbf{r}_{D/E} \qquad (3)$$

Sub (1) into (2) = (3):

$$\omega_{AB}\mathbf{k}\times\mathbf{r}_{B/A} + \omega_{BD}\mathbf{k}\times\mathbf{r}_{D/B} = \omega_{DEF}\mathbf{k}\times\mathbf{r}_{D/E}$$

Put in known values:

$$-6\mathbf{k}\times 0.2\mathbf{j} + \omega_{BD}\mathbf{k}\times(1.2\mathbf{i} - 0.2\mathbf{j}) = \omega_{DEF}\mathbf{k}\times(-0.12\mathbf{i} - 0.8\mathbf{j})$$

Equate components:

i: $1.2 + 0.2\omega_{BD} = 0.8\omega_{DEF}$

j: $1.2\omega_{BD} = -0.12\omega_{DEF}$

Solve simultaneously: $\omega_{BD} = -0.1463$ rad/s, $\omega_{DEF} = 1.463$ rad/s

Relative acceleration:

$$\mathbf{a}_B = \cancel{\mathbf{a}_A} + \cancel{\alpha_{AB}}\mathbf{k}\times\mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A} \qquad (4)$$
$$\mathbf{a}_D = \mathbf{a}_B + \alpha_{BD}\mathbf{k}\times\mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B} \qquad (5)$$
$$\mathbf{a}_D = \cancel{\mathbf{a}_E} + \alpha_{DEF}\mathbf{k}\times\mathbf{r}_{D/E} - \omega_{DEF}^2\mathbf{r}_{D/E} \qquad (6)$$

Sub (4) into (5) = (6):

$$-\omega_{AB}^2\mathbf{r}_{B/A} + \alpha_{BD}\mathbf{k}\times\mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B} = \alpha_{DEF}\mathbf{k}\times\mathbf{r}_{D/E} - \omega_{DEF}^2\mathbf{r}_{D/E}$$

Put in known values:

$$-(-6)^2(0.2\mathbf{j}) + \alpha_{BD}\mathbf{k}\times(1.2\mathbf{i} - 0.2\mathbf{j}) - (-0.1463)^2(1.2\mathbf{i} - 0.2\mathbf{j}) =$$
$$\alpha_{DEF}\mathbf{k}\times(-0.12\mathbf{i} - 0.8\mathbf{j}) - (1.463)^2(-0.12\mathbf{i} - 0.8\mathbf{j})$$

PROBLEM 15.128 (Continued)

Equate components: **i**: $0.2\alpha_{BD} - 0.02570 = 0.8\alpha_{DEF} + 0.2570$

j: $-7.2 + 1.2\alpha_{BD} + 0.004283 = -0.12\alpha_{DEF} + 1.7132$

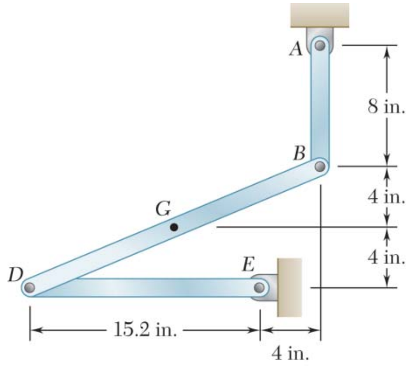
(a) Solve simultaneously: $\alpha_{DEF} = 1.466 \text{ rad/s}^2$ $\boldsymbol{\alpha}_{DEF} = 1.466 \text{ rad/s}^2$ ↺ ◀

(b) Relative acceleration:

$$\mathbf{a}_F = \cancel{\mathbf{a}_E} + \alpha_{DEF}\mathbf{k} \times \mathbf{r}_{F/E} - \omega_{DEF}^2 \mathbf{r}_{F/E}$$

$$= 1.466\mathbf{k} \times (0.09\mathbf{i} + 0.6\mathbf{j}) - (1.463)^2 (0.09\mathbf{i} + 0.6\mathbf{j})$$

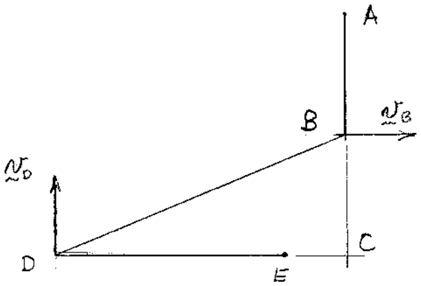
$\mathbf{a}_F = -1.0724\mathbf{i} - 1.153\mathbf{j} \text{ m/s}^2$ $\mathbf{a}_F = 1.575 \text{ m/s}^2$ ↙ 47.08° ◀



PROBLEM 15.129

Knowing that at the instant shown bar *AB* has a constant angular velocity of 19 rad/s clockwise, determine (*a*) the angular acceleration of bar *BGD*, (*b*) the angular acceleration of bar *DE*.

SOLUTION



Velocity analysis. $\omega_{AB} = 19 \text{ rad/s} \circlearrowright$

$$v_B = (AB)\omega_{AB} = (8)(19) = 152 \text{ in./s}$$

$$\mathbf{v}_B = v_B \longrightarrow, \quad \mathbf{v}_D = v_D \uparrow$$

Instantaneous center of bar *BD* lies at *C*.

$$\omega_{BD} = \frac{v_B}{BC} = \frac{152}{8} = 19 \text{ rad/s} \circlearrowleft$$

$$v_D = (CD)\omega_{BD} = (19.2)(19) = 364.8 \text{ in./s}$$

$$\omega_{DE} = \frac{v_D}{DE} = \frac{364.8}{15.2} = 24 \text{ rad/s}^2 \circlearrowright$$

Acceleration analysis. $\alpha_{AB} = 0.$

$$\mathbf{a}_B = [(AB)\omega_{AB}^2 \uparrow] = [(8)(19)^2 \uparrow] = 2888 \text{ in./s} \uparrow$$

$$\mathbf{a}_D = [(DE)\alpha_{DE} \downarrow] + [(DE)\omega_{DE}^2 \longrightarrow]$$

$$= [15.2\alpha_{DE} \downarrow] + [8755.2 \text{ in./s}^2 \longrightarrow]$$

$$(\mathbf{a}_{D/B})_t = [19.2\alpha_{BD} \downarrow] + [8\alpha_{DB} \longrightarrow]$$

$$(\mathbf{a}_{D/B})_n = [19.2\omega_{BD}^2 \longrightarrow] + [8\omega_{BD}^2 \uparrow]$$

$$= [6931.2 \text{ in./s}^2 \longrightarrow] + [2888 \text{ in./s}^2 \uparrow]$$

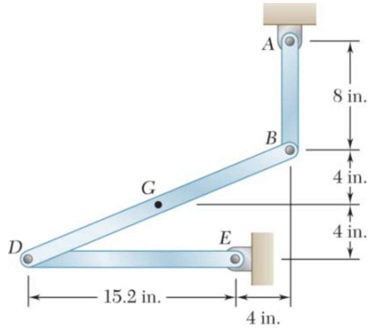
$$\mathbf{a}_D = \mathbf{a}_B + (\mathbf{a}_{D/B})_t + (a_{D/B})_n$$ Resolve into components.

$\xrightarrow{+}$: $8755.2 = 0 + 8\alpha_{BD} + 6931.2$

(*a*) $\boldsymbol{\alpha}_{BD} = 228 \text{ rad/s}^2 \circlearrowleft$ ◀

$+\downarrow$: $15.2\alpha_{DE} = -2888 + (19.2)(228) - 2888$

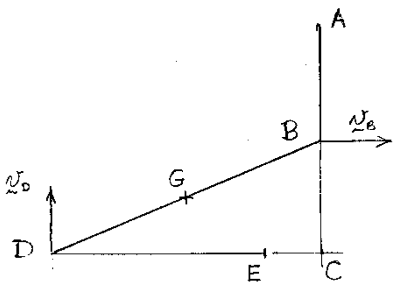
(*b*) $\alpha_{DE} = -92 \text{ rad/s}$ $\boldsymbol{\alpha}_{DE} = 92.0 \text{ rad/s}^2 \circlearrowright$ ◀



PROBLEM 15.130

Knowing that at the instant shown bar *DE* has a constant angular velocity of 18 rad/s clockwise, determine (*a*) the acceleration of Point *B*, (*b*) the acceleration of Point *G*.

SOLUTION



Velocity analysis. $\omega_{DE} = 18 \text{ rad/s} \circlearrowright$

$$\mathbf{v}_D = (DE)\omega_{DE} = (15.2)(18) = 273.6 \text{ in./s}$$

$$\mathbf{v}_D = v_D \uparrow, \quad \mathbf{v}_B = v_B \longrightarrow$$

Point *C* is the instantaneous center of bar *BD*.

$$\omega_{BD} = \frac{v_D}{CD} = \frac{273.6}{19.2} = 14.25 \text{ rad/s} \circlearrowright$$

$$v_B = (CB)\omega_{BD} = (8)(14.25) = 114 \text{ in./s}$$

$$\omega_{AB} = \frac{v_B}{AB} = \frac{114}{8} = 14.25 \text{ rad/s} \circlearrowleft$$

Acceleration analysis. $\alpha_{DE} = 0$

$$\mathbf{a}_D = [(DE)\omega_{DE}^2 \longrightarrow] = [(15.2)(18)^2 \longrightarrow] = [4924.8 \text{ in./s}^2 \longrightarrow]$$

$$\mathbf{a}_B = [(AB)\alpha_{AB} \longrightarrow] + [(AB)\omega_{AB}^2 \uparrow]$$

$$= [8\alpha_{AB} \longrightarrow] + [1624.5 \text{ in./s}^2 \uparrow]$$

$$(\mathbf{a}_{D/B})_t = [19.2\alpha_{BD} \downarrow] + [8\alpha_{BD} \longrightarrow]$$

$$(\mathbf{a}_{D/B})_n = [19.2\omega_{BD}^2 \longrightarrow] + [8\omega_{BD}^2 \uparrow]$$

$$\mathbf{a}_D = \mathbf{a}_B + (\mathbf{a}_{D/B})_t + (\mathbf{a}_{D/B})_n$$ Resolve into components.

PROBLEM 15.130 (Continued)

$+\downarrow$: $0 = 1624.5 - 19.2\alpha_{BD} + 1624.5,$

$$\alpha_{BD} = 169.21875 \text{ rad/s}^2$$

$\xrightarrow{+}$: $4924.8 = 8\alpha_{AB} + (8)(169.21875) + 3898.8$

$$\alpha_{AB} = -40.96875 \text{ rad/s}^2$$

(*a*) $\mathbf{a}_B = [(8)(-40.96875) \longrightarrow] + [1624.5 \text{ in./s}^2 \uparrow]$

$= [327.75 \text{ in./s}^2 \longleftarrow] + [1624.5 \text{ in./s}^2 \uparrow],$

$\mathbf{a}_B = 138.1 \text{ ft/s}^2 \nwarrow 78.6°$ ◀

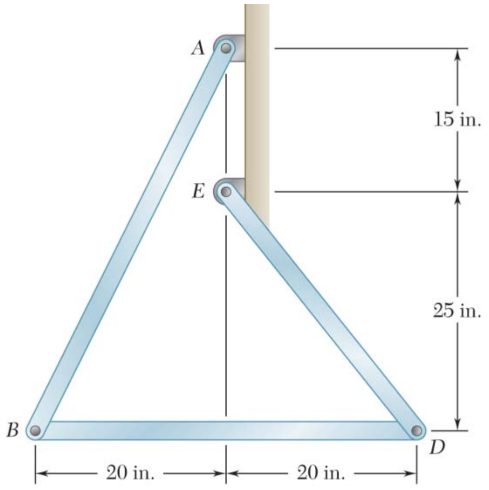
(*b*) $\mathbf{a}_G = \mathbf{a}_B + \mathbf{a}_{G/B} = \mathbf{a}_B + \frac{1}{2}\mathbf{a}_{D/B}$

$= \mathbf{a}_B + \frac{1}{2}(\mathbf{a}_D - \mathbf{a}_B) = \frac{1}{2}(\mathbf{a}_B + \mathbf{a}_D)$

$= \left[\frac{-327.75 + 4924.8}{2} \longrightarrow\right] + \left[\frac{1624.5}{2} \uparrow\right]$

$= [2298.5 \text{ in./s}^2] \longrightarrow + [812.25 \text{ in./s}^2 \uparrow]$

$\mathbf{a}_G = 203 \text{ ft/s}^2 \nearrow 19.5°$ ◀



PROBLEM 15.131

Knowing that at the instant shown bar *AB* has a constant angular velocity of 4 rad/s clockwise, determine the angular acceleration (*a*) of bar *BD*, (*b*) of bar *DE*.

SOLUTION

Relative position vectors.

$$\mathbf{r}_{B/A} = -(20 \text{ in.})\mathbf{i} - (40 \text{ in.})\mathbf{j}$$
$$\mathbf{r}_{D/B} = (40 \text{ in.})\mathbf{i}$$
$$\mathbf{r}_{D/E} = (20 \text{ in.})\mathbf{i} - (25 \text{ in.})\mathbf{j}$$

Velocity analysis.

Bar *AB* (Rotation about *A*):

$$\boldsymbol{\omega}_{AB} = 4 \text{ rad/s} \circlearrowright = -(4 \text{ rad/s})\mathbf{k}$$

$$\mathbf{r}_{B/A} = -(20 \text{ in.})\mathbf{i} - (40 \text{ in.})\mathbf{j} \qquad \mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = (-4\mathbf{k}) \times (-20\mathbf{i} - 40\mathbf{j})$$
$$\mathbf{v}_B = -(160 \text{ in./s})\mathbf{i} + (80 \text{ in./s})\mathbf{j}$$

Bar *BD* (Plane motion = Translation with *B* + Rotation about *B*):

$$\boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \qquad \mathbf{r}_{D/B} = (40 \text{ in.})\mathbf{i}$$
$$\mathbf{v}_D = \mathbf{v}_B + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{D/B} = \mathbf{v}_B + (\omega_{BD}\mathbf{k}) \times (40\mathbf{i})$$
$$\mathbf{v}_D = -(160 \text{ in/s})\mathbf{i} + (40\omega_{BD} + 80 \text{ in./s})\mathbf{j}$$

Bar *DE* (Rotation about *E*):

$$\boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}$$
$$\mathbf{r}_{D/E} = (20 \text{ in.})\mathbf{i} - (25 \text{ in.})\mathbf{j}$$
$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = (\omega_{DE}\mathbf{k}) \times (20\mathbf{i} - 25\mathbf{j})$$
$$\mathbf{v}_D = 20\omega_{DE}\mathbf{j} + 25\omega_{DE}\mathbf{i}$$

Equating components of the two expression for $\mathbf{v}_D$,

$\mathbf{i}$: $\quad -160 = 25\omega_{DE} \qquad \omega_{DE} = -6.4 \text{ rad/s}$

$\mathbf{j}$: $\quad 40\omega_{BE} + 80 = 20\omega_{DE} \qquad 40\omega_{BD} + 80 = 20(-6.4) \qquad \omega_{BD} = -5.2 \text{ rad/s}$

PROBLEM 15.131 (Continued)

Summary of angular velocities: $\boldsymbol{\omega}_{AB} = 4 \text{ rad/s}$ ↻ $\boldsymbol{\omega}_{DE} = 6.4 \text{ rad/s}$ ↻ $\boldsymbol{\omega}_{BD} = 5.2 \text{ rad/s}$ ↻

Acceleration analysis. $\boldsymbol{\alpha}_{AB} = 0, \quad \boldsymbol{\alpha}_{BD} = \alpha_{BD}\mathbf{k}, \quad \boldsymbol{\alpha}_{DE} = \alpha_{DE}\mathbf{k}$

Bar *AB* (Rotation about *A*):

$$\begin{aligned}\mathbf{a}_B &= \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}\\ &= 0 - (4)^2(-20\mathbf{i} - 40\mathbf{j})\\ &= (320 \text{ in./s}^2)\mathbf{i} + (640 \text{ in./s}^2)\mathbf{j}\end{aligned}$$

Bar *BD* (Translation with *B* + Rotation about *B*):

$$\begin{aligned}\mathbf{a}_D &= \mathbf{a}_B + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{D/B} - \omega_{BD}^2 \mathbf{r}_{D/B}\\ &= 320\mathbf{i} + 640\mathbf{j} + \alpha_{BD}\mathbf{k} \times (40\mathbf{i}) - (5.2)^2(40)\mathbf{i}\\ &= 320\mathbf{i} + 640\mathbf{j} + 40\alpha_{BD}\mathbf{j} - 1081.6\mathbf{i}\\ &= -761.60\mathbf{i} + (640 + 40\alpha_{BD})\mathbf{j} \qquad (1)\end{aligned}$$

Bar *DE* (Rotation about *E*):

$$\begin{aligned}\mathbf{a}_D &= \boldsymbol{\alpha}_{DE} \times \mathbf{r}_{D/E} - \omega_{DE}^2 \mathbf{r}_{D/E}\\ &= \alpha_{DE}\mathbf{k} \times (20\mathbf{i} - 25\mathbf{j}) - (6.4)^2(20\mathbf{i} - 25\mathbf{j})\\ &= -20\alpha_{DE}\mathbf{j} + 25\alpha_{DE}\mathbf{i} - 819.20\mathbf{i} + 1024\mathbf{j}\\ &= (25\alpha_{DE} - 819.20)\mathbf{i} + (20\alpha_{DE} + 1024)\mathbf{j} \qquad (2)\end{aligned}$$

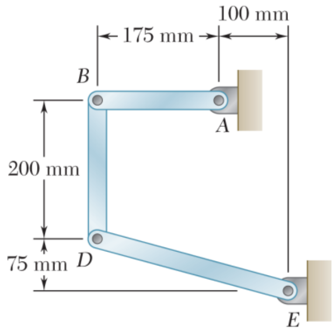
Equate like components of $\mathbf{a}_D$ expressed by Eqs. (1) and (2).

i: $-761.60 = 25\alpha_{DE} - 819.20 \qquad \alpha_{DE} = 2.3040 \text{ rad/s}^2$

j: $640 + 40\alpha_{BD} = (20)(2.304) + 1024 \qquad \alpha_{BD} = 10.752 \text{ rad/s}^2$

(*a*) Angular acceleration of bar *BD*. $\boldsymbol{\alpha}_{BD} = 10.75 \text{ rad/s}^2$ ↺ ◀

(*b*) Angular acceleration of bar *DE*. $\boldsymbol{\alpha}_{DE} = 2.30 \text{ rad/s}^2$ ↺ ◀



PROBLEM 15.132

Knowing that at the instant shown bar *AB* has a constant angular velocity of 4 rad/s clockwise, determine the angular acceleration (*a*) of bar *BD*, (*b*) of bar *DE*.

SOLUTION

Velocity analysis.

Bar *AB* (Rotation about *A*): $\boldsymbol{\omega}_{AB} = 4 \text{ rad/s} \circlearrowright = -(4 \text{ rad/s})\mathbf{k}$

$\mathbf{r}_{B/A} = -(175 \text{ mm})\mathbf{i} \qquad \mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = (-4\mathbf{k}) \times (-175\mathbf{i})$

$\mathbf{v}_B = (700 \text{ mm/s})\mathbf{j}$

Bar *BD* (Plane motion = Translation with B + Rotation about B):

$\boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \qquad \mathbf{r}_{D/B} = -(200 \text{ mm})\mathbf{j}$

$\mathbf{v}_D = \mathbf{v}_B + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{D/B} = 700\mathbf{j} + (\omega_{BD}\mathbf{k}) \times (-200\mathbf{j})$

$\mathbf{v}_D = 700\mathbf{j} + 200\omega_{BD}\mathbf{i}$

Bar *DE* (Rotation about E): $\boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}$

$\mathbf{r}_{D/E} = -(275 \text{ mm})\mathbf{i} + (75 \text{ mm})\mathbf{j}$

$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = (\omega_{DE}\mathbf{k}) \times (-275\mathbf{i} + 75\mathbf{j})$

$\mathbf{v}_D = -275\omega_{DE}\mathbf{j} - 75\omega_{DE}\mathbf{i}$

Equating components of the two expressions for $\mathbf{v}_D$,

$\mathbf{j}$: $\quad 700 = -275\omega_{DE} \qquad \omega_{DE} = -2.5455 \text{ rad/s} \qquad\qquad \boldsymbol{\omega}_{DE} = 2.55 \text{ rad/s} \circlearrowright$

$\mathbf{i}$: $\quad 200\omega_{BD} = -75\omega_{BD} \qquad \omega_{DE} = -\frac{3}{8}\omega_{BD}$

$$\omega_{BD} = -\left(\frac{3}{8}\right)(-2.5455) = 0.95455 \text{ rad/s} \qquad\qquad \boldsymbol{\omega}_{BD} = 0.955 \text{ rad/s} \circlearrowleft$$

Acceleration analysis. $\quad \boldsymbol{\alpha}_{AB} = 0$

Bar *AB*: $\quad \mathbf{a}_B = -\omega_{AB}^2\mathbf{r}_{B/A} = -(4)^2(-175\mathbf{i}) = (2800 \text{ mm/s}^2)\mathbf{i}$

Bar *BD*: $\quad \boldsymbol{\alpha}_{BD} = \alpha_{BD}\mathbf{k}$

$$\begin{aligned}\mathbf{a}_D &= \mathbf{a}_B + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B} \\ &= 2800\mathbf{i} + \alpha_{BD}\mathbf{k} \times (-200\mathbf{j}) - (0.95455)^2(-200\mathbf{j}) \\ &= (2800 + 200\,\alpha_{BD})\mathbf{i} + 182.23\mathbf{j} \qquad (1)\end{aligned}$$

PROBLEM 15.132 (Continued)

Bar *DE*:

$$
\begin{aligned}
\alpha_{DE} &= \alpha_{DE}\mathbf{k}\\
\mathbf{a}_D &= \boldsymbol{\alpha}_{DE}\times\mathbf{r}_{D/E}-\omega_{DE}^2 r_{D/E}\\
&= \alpha_{DE}\mathbf{k}\times(-275\mathbf{i}+75\mathbf{j})-(2.5455)^2(-275\mathbf{i}+75\mathbf{j})\\
&= -275\alpha_{DE}\mathbf{j}-75\alpha_{DE}\mathbf{i}+1781.8\mathbf{i}-485.95\mathbf{j}\\
&= (-75\alpha_{DE}+1781.8)\mathbf{i}-(275\alpha_{DE}+485.95)\mathbf{j} \qquad (2)
\end{aligned}
$$

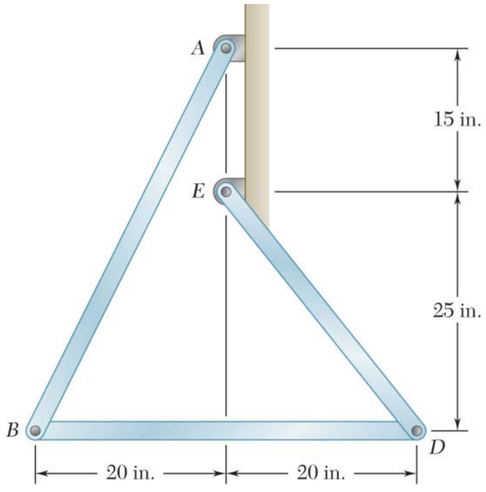
Equate like components of $\mathbf{a}_D$ expressed by Eqs. (1) and (2).

$\mathbf{j}$: $182.23 = -(275\alpha_{DE}+485.95)$ $\qquad \alpha_{DE} = -2.4298\ \text{rad/s}^2$

$\mathbf{i}$: $(2800+200\alpha_{BD}) = [-(75)(-2.4298)+1781.8]$ $\qquad \alpha_{BD} = -4.1795\ \text{rad/s}^2$

(*a*) Angular acceleration of bar *BD*. $\boldsymbol{\alpha}_{BD} = 4.18\ \text{rad/s}^2$ ↻ ◀

(*b*) Angular acceleration of bar *DE*. $\boldsymbol{\alpha}_{DE} = 2.43\ \text{rad/s}^2$ ↻ ◀



PROBLEM 15.133

Knowing that at the instant shown bar *AB* has an angular velocity of 4 rad/s and an angular acceleration of 2 rad/s^2, both clockwise, determine the angular acceleration (*a*) of bar *BD*, (*b*) of bar *DE* by using the vector approach as is done in Sample Problem 15.16.

SOLUTION

Relative position vectors.

$$\mathbf{r}_{B/A} = -(20\text{ in.})\mathbf{i} - (40\text{ in.})\mathbf{j}$$
$$\mathbf{r}_{D/B} = (40\text{ in.})\mathbf{i}$$
$$\mathbf{r}_{D/E} = (20\text{ in.})\mathbf{i} - (25\text{ in.})\mathbf{j}$$

Velocity analysis.

Bar *AB* (Rotation about *A*):

$$\boldsymbol{\omega}_{AB} = 4\text{ rad/s} \circlearrowright = -(4\text{ rad/s})\mathbf{k}$$

$$\mathbf{r}_{B/A} = -(20\text{ in.})\mathbf{i} - (40\text{ in.})\mathbf{j} \qquad \mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = (-4\mathbf{k}) \times (-20\mathbf{i} - 40\mathbf{j})$$
$$\mathbf{v}_B = -(160\text{ in./s})\mathbf{i} + (80\text{ in./s})\mathbf{j}$$

Bar *BD* (Plane motion = Translation with *B* + Rotation about *B*):

$$\boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \qquad \mathbf{r}_{D/B} = (40\text{ in.})\mathbf{i}$$
$$\mathbf{v}_D = \mathbf{v}_B + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{D/B} = \mathbf{v}_B + (\omega_{BD}\mathbf{k}) \times (40\mathbf{i})$$
$$\mathbf{v}_D = -(160\text{ in/s})\mathbf{i} + (40\omega_{BD} + 80\text{ in./s})\mathbf{j}$$

Bar *DE* (Rotation about *E*):

$$\boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}$$
$$\mathbf{r}_{D/E} = (20\text{ in.})\mathbf{i} - (25\text{in.})\mathbf{j}$$
$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = (\omega_{DE}\mathbf{k}) \times (20\mathbf{i} - 25\mathbf{j})$$
$$\mathbf{v}_D = 20\omega_{DE}\mathbf{j} + 25\omega_{DE}\mathbf{i}$$

Equating components of the two expression for $\mathbf{v}_D$,

$\mathbf{i}$: $\quad -160 = 25\omega_{DE} \qquad \omega_{DE} = -6.4\text{ rad/s}$

$\mathbf{j}$: $\quad 40\omega_{BE} + 80 = 20\omega_{DE} \qquad 40\omega_{BD} + 80 = 20(-6.4) \qquad \omega_{BD} = -5.2\text{ rad/s}$

Summary of angular velocities: $\boldsymbol{\omega}_{AB} = 4 \text{ rad/s} \circlearrowright \quad \boldsymbol{\omega}_{DE} = 6.4 \text{ rad/s} \circlearrowright \quad \boldsymbol{\omega}_{BD} = 5.2 \text{ rad/s} \circlearrowright$

Acceleration analysis. $\boldsymbol{\alpha}_{AB} = -(2 \text{ rad/s}^2)\mathbf{k}, \quad \boldsymbol{\alpha}_{BD} = \alpha_{BD}\mathbf{k}, \quad \boldsymbol{\alpha}_{DE} = \alpha_{DE}\mathbf{k}$

Bar *AB* (Rotation about *A*)

$$\begin{aligned}
\mathbf{a}_B &= \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A} \\
&= (-2\mathbf{k}) \times (-20\mathbf{i} - 40\mathbf{j}) - (4)^2(-20\mathbf{i} - 40\mathbf{j}) \\
&= -(80 \text{ in./s}^2)\mathbf{i} + (40 \text{ in./s}^2)\mathbf{j} + (320 \text{ in./s}^2)\mathbf{i} + (640 \text{ in./s}^2)\mathbf{j} \\
&= (240 \text{ in./s}^2)\mathbf{i} + (680 \text{ in./s}^2)\mathbf{j}
\end{aligned}$$

Bar *BD* (Translation with *B* + Rotation about *B*):

$$\begin{aligned}
\mathbf{a}_D &= \mathbf{a}_B + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{D/B} - \omega_{BD}^2 \mathbf{r}_{D/B} \\
&= 240\mathbf{i} + 680\mathbf{j} + \alpha_{BD}\mathbf{k} \times (40\mathbf{i}) - (5.2)^2(40)\mathbf{i} \\
&= 240\mathbf{i} + 680\mathbf{j} + 40\alpha_{BD}\mathbf{j} - 1081.6\mathbf{i} \\
&= -841.60\mathbf{i} + (680 + 40\alpha_{BD})\mathbf{j} \qquad (1)
\end{aligned}$$

Bar *DE* (Rotation about *E*):

$$\begin{aligned}
\mathbf{a}_D &= \boldsymbol{\alpha}_{DE} \times \mathbf{r}_{D/E} - \omega_{DE}^2 \mathbf{r}_{D/E} \\
&= \alpha_{DE}\mathbf{k} \times (20\mathbf{i} - 25\mathbf{j}) - (6.4)^2(20\mathbf{i} - 25\mathbf{j}) \\
&= 20\alpha_{DE}\mathbf{j} + 25\alpha_{DE}\mathbf{i} - 819.20\mathbf{i} + 1024\mathbf{j} \\
&= (25\alpha_{DE} - 819.20)\mathbf{i} + (20\alpha_{DE} + 1024)\mathbf{j} \qquad (2)
\end{aligned}$$

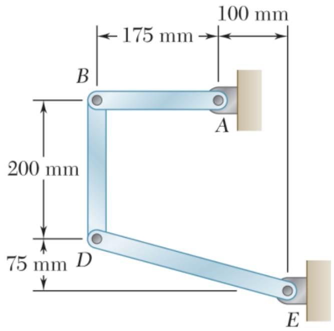
Equate like components of $\mathbf{a}_D$ expressed by Eqs. (1) and (2).

i: $-841.60 = 25\alpha_{DE} - 819.20 \qquad \alpha_{DE} = -0.896 \text{ rad/s}^2$

j: $680 + 40\alpha_{BD} = (20)(-0.896) + 1024 \qquad \alpha_{BD} = 8.152 \text{ rad/s}^2$

(*a*) Angular acceleration of bar *BD*. $\boldsymbol{\alpha}_{BD} = 8.15 \text{ rad/s}^2 \circlearrowleft$ ◀

(*b*) Angular acceleration of bar *DE*. $\boldsymbol{\alpha}_{DE} = 0.896 \text{ rad/s}^2 \circlearrowright$ ◀



PROBLEM 15.134

Knowing that at the instant shown bar *AB* has an angular velocity of 4 rad/s and an angular acceleration of 2 rad/s^2, both clockwise, determine the angular acceleration (*a*) of bar BD, (*b*) of bar *DE* by using the vector approach as is done in Sample Problem 15.16.

SOLUTION

Velocity analysis.

Bar *AB* (Rotation about *A*):

$$\boldsymbol{\omega}_{AB} = 4 \text{ rad/s} \circlearrowright = -(4 \text{ rad/s})\mathbf{k}$$

$$\mathbf{r}_{B/A} = -(175 \text{ mm})\mathbf{i} \qquad \mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = (-4\mathbf{k}) \times (-175\mathbf{i})$$

$$\mathbf{v}_B = (700 \text{ mm/s})\mathbf{j}$$

Bar *BD* (Plane motion = Translation with $B +$ Rotation about B):

$$\boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \qquad \mathbf{r}_{D/B} = -(200 \text{ mm})\mathbf{j}$$

$$\mathbf{v}_D = \mathbf{v}_B + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{D/B} = 700\mathbf{j} + (\omega_{BD}\mathbf{k}) \times (-200\mathbf{j})$$

$$\mathbf{v}_D = 700\mathbf{j} + 200\omega_{BD}\mathbf{i}$$

Bar *DE* (Rotation about *E*):

$$\boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}$$

$$\mathbf{r}_{D/E} = -(275 \text{ mm})\mathbf{i} + (75 \text{ mm})\mathbf{j}$$

$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = (\omega_{DE}\mathbf{k}) \times (-275\mathbf{i} + 75\mathbf{j})$$

$$\mathbf{v}_D = -275\omega_{DE}\mathbf{j} - 75\omega_{DE}\mathbf{i}$$

Equating components of the two expressions for $\mathbf{v}_D$,

$$\mathbf{j}: \quad 700 = -275\omega_{DE} \quad \omega_{DE} = -2.5455 \text{ rad/s} \qquad \boldsymbol{\omega}_{DE} = 2.55 \text{ rad/s} \circlearrowright$$

$$\mathbf{i}: \quad 200\omega_{BD} = -75\omega_{DE} \quad \omega_{BD} = -\frac{3}{8}\omega_{DE}$$

$$\omega_{BD} = -\left(\frac{3}{8}\right)(-2.5455) = 0.95455 \text{ rad/s} \qquad \boldsymbol{\omega}_{BD} = 0.955 \text{ rad/s} \circlearrowleft$$

Acceleration analysis.

$$\boldsymbol{\alpha}_{AB} = 2 \text{ rad/s} \circlearrowright = -(2 \text{ rad/s}^2)\mathbf{k}$$

Bar *AB*:

$$\mathbf{a}_B = \alpha_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 r_{B/A}$$

$$= (-2\mathbf{k}) \times (-175\mathbf{i}) - (4)^2(-175\mathbf{i}) = 2800 \text{ mm/s}^2\mathbf{i} + 350 \text{ mm/s}^2\mathbf{j}$$

Bar *BD*:

$$\boldsymbol{\alpha}_{BD} = \alpha_{BD}\mathbf{k}$$

$$\mathbf{a}_D = \mathbf{a}_B + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B}$$

$$= 2800\mathbf{i} + 350\mathbf{j} + \alpha_{BD}\mathbf{k} \times (-200\mathbf{j}) - (0.95455)^2(-200\mathbf{j})$$

$$= (2800 + 200\,\alpha_{BD})\mathbf{i} + 532.23\mathbf{j} \tag{1}$$

PROBLEM 15.134 (Continued)

Bar *DE*:

$$\alpha_{DE} = \alpha_{DE}\mathbf{k}$$

$$\begin{aligned}\mathbf{a}_D &= \boldsymbol{\alpha}_{DE} \times \mathbf{r}_{D/E} - \omega_{DE}^2 r_{D/E} \\ &= \alpha_{DE}\mathbf{k} \times (-275\mathbf{i} + 75\mathbf{j}) - (2.5455)^2(-275\mathbf{i} + 75\mathbf{j}) \\ &= -275\alpha_{DE}\mathbf{j} - 75\alpha_{DE}\mathbf{i} + 1781.8\mathbf{i} - 485.95\mathbf{j} \\ &= (-75\alpha_{DE} + 1781.8)\mathbf{i} - (275\alpha_{DE} + 485.95)\mathbf{j} \qquad (2)\end{aligned}$$

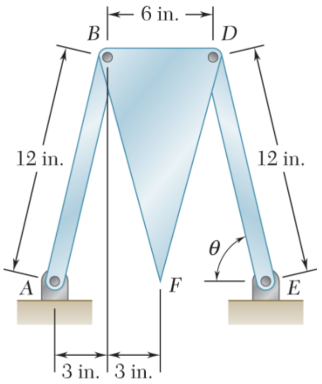
Equate like components of $\mathbf{a}_D$ expressed by Eqs. (1) and (2).

$\mathbf{j}$: $\quad 532.23 = -(275\alpha_{DE} + 485.95) \qquad \alpha_{DE} = -3.7025 \text{ rad/s}^2$

$\mathbf{i}$: $\quad (2800 + 200\alpha_{BD}) = [-(75)(-3.7025) + 1781.8] \qquad \alpha_{BD} = -3.7025 \text{ rad/s}^2$

(*a*) Angular acceleration of bar *BD*. $\boldsymbol{\alpha}_{BD} = 3.70 \text{ rad/s}^2 \circlearrowright$ ◀

(*b*) Angular acceleration of bar *DE*. $\boldsymbol{\alpha}_{DE} = 3.70 \text{ rad/s}^2 \circlearrowright$ ◀



PROBLEM 15.135

Robert's linkage is named after Richard Robert (1789–1864) and can be used to draw a close approximation to a straight line by locating a pen at Point *F*. The distance *AB* is the same as *BF*, *DF* and *DE*. Knowing that at the instant shown bar *AB* has a constant angular velocity of 4 rad/s clockwise, determine (*a*) the angular acceleration of bar *DE*, (*b*) the acceleration of Point *F*.

SOLUTION

Units: inches, in./s, in./s^2

Unit vectors: $\mathbf{i}=1\longrightarrow,\ \mathbf{j}=1\uparrow,\ \mathbf{k}=1\circlearrowleft.$

Geometry:

$$\mathbf{r}_{B/A}=3\mathbf{i}+\sqrt{12^2-3^2}\,\mathbf{j}=3\mathbf{i}+\sqrt{135}\mathbf{j}$$

$$\mathbf{r}_{D/B}=6\mathbf{i}\qquad \mathbf{r}_{F/B}=3\mathbf{i}-\sqrt{135}\mathbf{j}$$

$$\mathbf{r}_{D/E}=-3\mathbf{i}+\sqrt{135}\mathbf{j}$$

Velocity analysis: $\boldsymbol{\omega}_{AB}=4\text{ rad/s}^2\circlearrowright=-4\,\mathbf{k}$

Bar *AB*: $\mathbf{v}_B=\boldsymbol{\omega}_{AB}\times\mathbf{r}_{B/A}=-4\mathbf{k}\times(3\mathbf{i}+\sqrt{135}\mathbf{j})=4\sqrt{135}\mathbf{i}-12\mathbf{j}$

Object *BDF*:

$$\mathbf{v}_D=\mathbf{v}_B+\mathbf{v}_{D/B}=\mathbf{v}_B+\boldsymbol{\omega}_{BD}\times\mathbf{r}_{D/B}$$

$$=4\sqrt{135}\mathbf{i}-12\mathbf{j}+\omega_{BD}\mathbf{k}\times6\mathbf{i}$$

$$=4\sqrt{135}\mathbf{i}-12\mathbf{j}+6\omega_{BD}\mathbf{j}\qquad(1)$$

Bar *DE*:

$$\mathbf{v}_D=\omega_{DE}\times\mathbf{r}_{D/E}=\omega_{DE}\mathbf{k}\times(-3\mathbf{i}+\sqrt{135}\mathbf{j})$$

$$=-\sqrt{135}\omega_{DE}\mathbf{i}-3\omega_{DE}\mathbf{j}\qquad(2)$$

Equating like components of $\mathbf{v}_D$ from Eqs. (1) and (2),

$$\mathbf{i}:\quad 4\sqrt{135}=-\sqrt{135}\omega_{DE}\qquad(3)$$

$$\mathbf{j}:\quad -12+6\omega_{BD}=-3\omega_{DE}\qquad(4)$$

From Eq. (3), $\omega_{DE}=-4$ $\qquad\omega_{DE}=5\text{ rad/s}\circlearrowright$

From Eq. (4), $\omega_{BD}=\frac{1}{6}(12-3\omega_{DE})=5$ $\qquad\omega_{BD}=4\text{ rad/s}\circlearrowright$

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PROBLEM 15.135 (Continued)

Acceleration Analysis: $\boldsymbol{\alpha}_{AB} = 0$

Bar *AB*:

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$
$$= 0 - (4)^2(3\mathbf{i} + \sqrt{135}\mathbf{j})$$
$$= -48\mathbf{i} - 16\sqrt{135}\mathbf{j}$$

Object *BDF*:

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} = \mathbf{a}_B + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{D/B} - \omega_{BD}^2 \mathbf{r}_{D/B}$$

$$\mathbf{a}_D = -48\mathbf{i} - 16\sqrt{135}\mathbf{j} + \alpha_{BD}\mathbf{k} \times (6\mathbf{i}) - (4)^2(6\mathbf{i})$$
$$= -144\mathbf{i} - 16\sqrt{135}\mathbf{j} + 6\alpha_{BD}\mathbf{j} \tag{5}$$

Bar *DE*:

$$\mathbf{a}_D = \boldsymbol{\alpha}_{DE} \times \mathbf{r}_{D/E} - \omega_{DE}^2 \mathbf{r}_{D/E}$$

$$\mathbf{a}_D = \alpha_{DE}\mathbf{k} \times (-3\mathbf{i} + \sqrt{135}\mathbf{j}) - (4)^2(-3\mathbf{i} + \sqrt{135}\mathbf{j})$$
$$= -\sqrt{135}\alpha_{DE}\mathbf{i} - 3\alpha_{DE}\mathbf{j} + 48\mathbf{i} - 16\sqrt{135}\mathbf{j} \tag{6}$$

Equating like components of $\mathbf{a}_D$ from Eqs. (5) and (6),

$$\mathbf{i}: \quad -144 = -\sqrt{135}\alpha_{DE} + 48 \tag{7}$$

$$\mathbf{j}: \quad -16\sqrt{135} + 6\alpha_{BD} = -3\alpha_{DE} - 16\sqrt{135} \tag{8}$$

From Eq. (7), $\alpha_{DE} = \dfrac{192}{\sqrt{135}}$

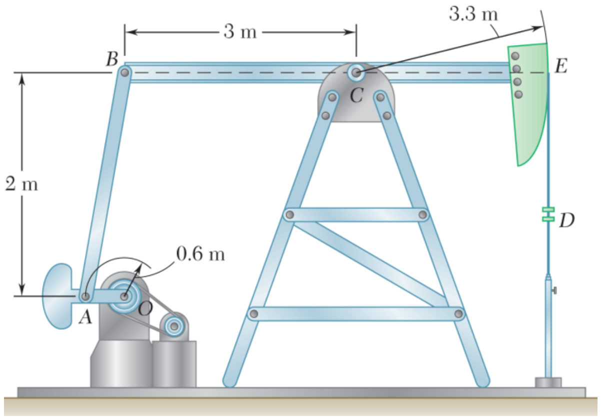
From Eq. (8), $\alpha_{BD} = -\dfrac{1}{2}\alpha_{BD} = -\dfrac{96}{\sqrt{135}}$

(*a*) Angular acceleration of bar *DE*: $\boldsymbol{\alpha}_{DE} = 16.53 \text{ rad/s}^2$ ↺ ◀

(*b*) Acceleration of Point *F*:

$$\mathbf{a}_F = \mathbf{a}_B + \mathbf{a}_{F/B} = \mathbf{a}_B + \alpha_{BD} \times \mathbf{r}_{F/B} - \omega_{BD}^2 \mathbf{r}_{F/B}$$
$$= -48\mathbf{i} - 16\sqrt{135}\mathbf{j} + \left(-\frac{96}{\sqrt{135}}\mathbf{k}\right) \times (3\mathbf{i} - \sqrt{135}\mathbf{j}) - (4)^2(3\mathbf{i} - \sqrt{135}\mathbf{j})$$
$$= -48\mathbf{i} - 16\sqrt{135}\mathbf{j} - \frac{288}{\sqrt{135}}\mathbf{j} - 96\mathbf{i} - 48\mathbf{i} + 16\sqrt{135}\mathbf{j}$$
$$= -192\mathbf{i} - \frac{288}{\sqrt{135}}\mathbf{j}$$

$\mathbf{a}_F = -(192.0 \text{ in./s}^2)\mathbf{i} - (24.8 \text{ in./s}^2)\mathbf{j} = 193.6 \text{ in./s}^2$ ↙ 7.36° ◀



PROBLEM 15.136

For the oil pump rig shown, link *AB* causes the beam *BCE* to oscillate as the crank *OA* revolves. Knowing that *OA* has a radius of 0.6 m and a constant clockwise angular velocity of 20 rpm, determine the velocity and acceleration of Point *D* at the instant shown.

SOLUTION

Units: meters, m/s, m/s^2

Unit vectors: $\mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1 \uparrow, \quad \mathbf{k} = 1 \circlearrowleft.$

Crank *OA*: $\mathbf{r}_{OA} = 0.6 \text{ m}, \quad \boldsymbol{\omega}_{OA} = 20 \text{ rpm} \circlearrowright \; = 2.0944 \text{ rad/s} \circlearrowright$

$$\mathbf{v}_A = \omega_{OA} r_{OA} = (2.0944)(0.6) \qquad \mathbf{v}_A = 1.25664 \text{ m/s} \uparrow$$

$$\boldsymbol{\alpha}_{OA} = 0 \qquad\qquad (a_A)_t = 0$$

$$(\mathbf{a}_A)_n = \omega_{OA}^2 r_{OA} = (2.0944)^2(0.6) = 2.6319 \text{ m/s}^2$$

$$\mathbf{a}_A = 2.6319 \text{ m/s}^2 \longrightarrow$$

Rod *AB*: $\mathbf{v}_B = v_A \uparrow,$

Since $\mathbf{v}_B$ and $\mathbf{v}_A$ are parallel, $\mathbf{v}_A = \mathbf{v}_B$ and $\omega_{AB} = 0.$

$$\mathbf{v}_B = 1.25664 \text{ m/s} \uparrow$$

$$\begin{aligned}\mathbf{a}_B &= \mathbf{a}_A + \mathbf{a}_{B/A} = \mathbf{a}_A + \alpha_{AB}\mathbf{k} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A} \\ &= 2.6319\mathbf{i} + \alpha_{AB}\mathbf{k} \times (0.6\mathbf{i} + 2\mathbf{j}) - 0 \\ &= (2.6319 - 2\alpha_{AB})\mathbf{i} + 0.6\alpha_{AB}\mathbf{j} \end{aligned} \qquad (1)$$

Beam *BCE*: Point *C* is a pivot.

$$v_B = \omega_{BCE} r_{BC} \qquad \omega_{BCE} = \frac{v_B}{r_{BC}} = \frac{1.25664}{3} = 0.41888$$

$$v_E = \omega_{BCE} r_{CE} = (0.41888)(3.3) = 1.38230$$

$$\boldsymbol{\omega}_{BCE} = 0.41888 \text{ rad/s} \circlearrowright \qquad \mathbf{v}_E = 1.38230 \text{ m/s} \downarrow$$

$$\begin{aligned}\mathbf{a}_B &= \boldsymbol{\alpha}_{BCE} \times \mathbf{r}_{B/C} - \omega_{BCE}^2 r_{B/C} \\ &= \alpha_{BCE}\mathbf{k} \times (-3\mathbf{i}) - (0.41888)^2(-3\mathbf{i}) \\ &= 0.52638\mathbf{i} - 3\alpha_{BCE}\mathbf{j} \end{aligned} \qquad (2)$$

PROBLEM 15.136 (Continued)

Equating like components of $\boldsymbol{\alpha}_B$ expressed by Eqs. (1) and (2),

i: $2.6319 - 2\alpha_{AB} = 0.52638$ $\alpha_{AB} = 1.05276 \text{ rad/s}$

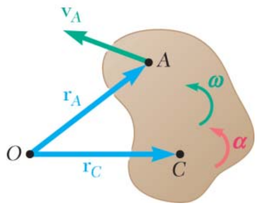
j: $0.6\alpha_{AB} = -3\alpha_{BCE}$ $\alpha_{BCE} = -0.21055 \text{ rad/s}^2$

$\boldsymbol{\alpha}_{AB} = 1.055276 \text{ rad/s}$ ↺ $\boldsymbol{\alpha}_{BCE} = 0.21055 \text{ rad/s}^2$ ↻

$$\mathbf{a}_E = (\mathbf{a}_{E/C})_t + (\mathbf{a}_{E/C})_n$$
$$(\mathbf{a}_{E/C})_t = \boldsymbol{\alpha}_{BCE} \times \mathbf{r}_{E/C} = (-0.21055\mathbf{k}) \times (3.3\mathbf{i})$$
$$= -(0.69482 \text{ m/s}^2)\mathbf{j}$$

String *ED*: $\mathbf{v}_D = \mathbf{v}_E$ $\mathbf{v}_D = 1.382 \text{ m/s}$ ↓ ◀

$\mathbf{a}_D = (\mathbf{a}_{E/C})_t = -(0.69482 \text{ m/s}^2)\mathbf{j}$ $\mathbf{a}_D = 0.695 \text{ m/s}^2$ ↓ ◀



PROBLEM 15.137

Denoting by $\mathbf{r}_A$ the position vector of Point A of a rigid slab that is in plane motion, show that (*a*) the position vector $\mathbf{r}_C$ of the instantaneous center of rotation is

$$\mathbf{r}_C = \mathbf{r}_A + \frac{\boldsymbol{\omega} \times \mathbf{v}_A}{\omega^2}$$

where $\boldsymbol{\omega}$ is the angular velocity of the slab and $\mathbf{v}_A$ is the velocity of Point A, (*b*) the acceleration of the instantaneous center of rotation is zero if, and only if,

$$\mathbf{a}_A = \frac{\alpha}{\omega}\mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{v}_A$$

where $\boldsymbol{\alpha} = \alpha\mathbf{k}$ is the angular acceleration of the slab.

SOLUTION

(*a*) At the instantaneous center C, $\mathbf{v}_C = 0$

$$\mathbf{v}_A = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{A/C} = \boldsymbol{\omega} \times \mathbf{r}_{A/C}$$

$$\boldsymbol{\omega} \times \mathbf{v}_A = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/C}) = -\omega^2 \mathbf{r}_{A/C}$$

$$\mathbf{r}_{A/C} = -\frac{\boldsymbol{\omega} \times \mathbf{v}_A}{\omega^2} = -\mathbf{r}_{C/A} \qquad \text{or} \qquad \mathbf{r}_{C/A} = \frac{\boldsymbol{\omega} \times \mathbf{v}_A}{\omega^2}$$

$$\mathbf{r}_C - \mathbf{r}_A = -\frac{\boldsymbol{\omega} \times \mathbf{v}_A}{\omega^2}$$

$$\mathbf{r}_C = \mathbf{r}_A + \frac{\boldsymbol{\omega} \times \mathbf{v}_A}{\omega^2} \blacktriangleleft$$

(*b*)

$$\mathbf{a}_A = \mathbf{a}_C + \boldsymbol{\alpha} \times \mathbf{r}_{A/C} + \boldsymbol{\omega} \times \mathbf{v}_{A/C}$$

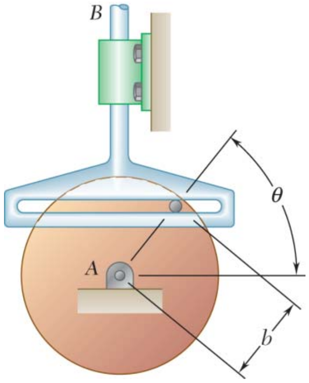
$$= \mathbf{a}_C - \alpha\mathbf{k} \times \frac{\boldsymbol{\omega} \times \mathbf{v}_A}{\omega^2} + \boldsymbol{\omega} \times (\mathbf{v}_A - \mathbf{v}_C)$$

$$= \mathbf{a}_C - \frac{\alpha\omega}{\omega^2}\mathbf{k} \times (\mathbf{k} \times \mathbf{v}_A) + \boldsymbol{\omega} \times \mathbf{v}_A$$

$$= \mathbf{a}_C + \frac{\alpha}{\omega}\mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{v}_A$$

Set $\mathbf{a}_C = 0$.

$$\mathbf{a}_A = \frac{\alpha}{\omega}\mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{v}_A \blacktriangleleft$$

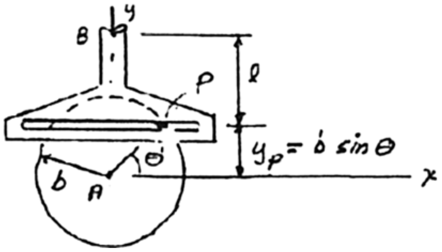


PROBLEM 15.138*

The drive disk of the scotch crosshead mechanism shown has an angular velocity ω and an angular acceleration α, both directed counterclockwise. Using the method of Section 15.4 B, derive expressions for the velocity and acceleration of Point *B*.

SOLUTION

Origin at *A*.



$$y_B = l + y_P = l + b\sin\theta$$

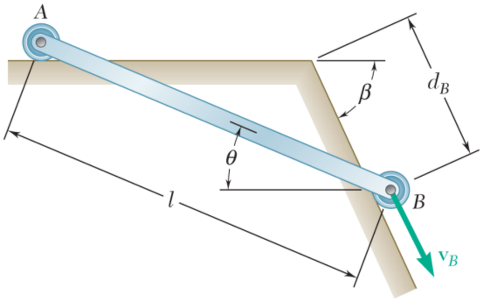
$$v_B = \dot{y}_B = b\cos\theta\dot{\theta} = b\cos\theta\,\omega \qquad v_B = b\omega\cos\theta \blacktriangleleft$$

$$a_B = \ddot{y}_B$$

$$= \frac{d}{dt}v_B$$

$$= \frac{d}{dt}(b\cos\theta\dot{\theta})$$

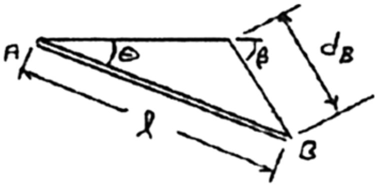
$$a_B = -b\sin\theta\dot{\theta}^2 + b\cos\theta\ddot{\theta} \qquad a_B = b\alpha\cos\theta - b\omega^2\sin\theta \blacktriangleleft$$



PROBLEM 15.139*

The wheels attached to the ends of rod *AB* roll along the surfaces shown. Using the method of Section 15.4 B, derive an expression for the angular velocity of the rod in terms of v_B, θ, l, and β.

SOLUTION



Law of sines.

$$\frac{d_B}{\sin\theta} = \frac{l}{\sin\beta}$$

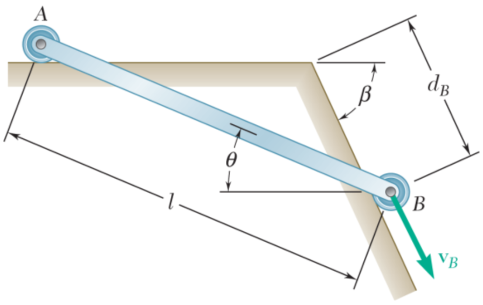
$$d_B = \frac{l}{\sin\beta}\sin\theta$$

$$v_B = \frac{d}{dt}(d_B)$$

$$= \frac{l}{\sin\beta}\cos\theta\frac{d\theta}{dt}$$

$$= \frac{l}{\sin\beta}\cos\theta\,\omega$$

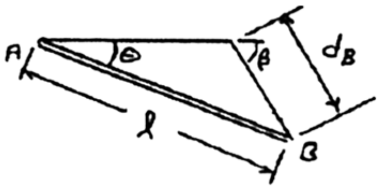
$$\omega = \frac{v_B\sin\beta}{l\cos\theta} \blacktriangleleft$$



PROBLEM 15.140*

The wheels attached to the ends of rod *AB* roll along the surfaces shown. Using the method of Section 15.4 B and knowing that the acceleration of wheel *B* is zero, derive an expression for the angular acceleration of the rod in terms of $v_B, \theta, l,$ and β.

SOLUTION



Law of sines.

$$\frac{d_B}{\sin\theta} = \frac{l}{\sin\beta}$$

$$d_B = \frac{l}{\sin\beta}\sin\theta$$

$$v_B = \frac{d}{dt}(d_B)$$

$$= \frac{l}{\sin\beta}\cos\theta\frac{d\theta}{dt}$$

$$= \frac{l}{\sin\beta}\cos\theta\,\omega$$

$$\omega = \frac{v_B\sin\beta}{l\cos\theta}$$

Note that

$$a_B = \frac{dv_B}{dt} = 0.$$

$$\alpha = \frac{d\omega}{dt} = \frac{v_B\sin\beta}{l}\cdot\frac{\sin\theta}{\cos^2\theta}\cdot\frac{d\theta}{dt}$$

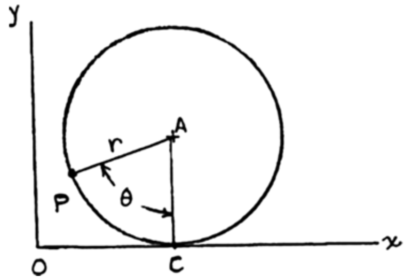
$$\alpha = \frac{v_B\sin\beta\sin\theta}{l\cos^2\theta}\cdot\frac{v_B\sin\beta}{l\cos\theta}$$

$$\alpha = \left[\frac{v_B\sin\beta}{l}\right]^2\frac{\sin\theta}{\cos^3\theta} \quad ◀$$

PROBLEM 15.141*

A disk of radius r rolls to the right with a constant velocity **v**. Denoting by P the point of the rim in contact with the ground at $t = 0,$ derive expressions for the horizontal and vertical components of the velocity of P at any time t.

SOLUTION



$$x_A = r\theta, \quad y_A = r$$

$$x_P = x_A - r\sin\theta$$

$$= r\theta - r\sin\theta$$

$$y_P = y_A - r\cos\theta$$

$$= r - r\cos\theta$$

$$\theta = \frac{x_A}{r}$$

$$\dot{x}_A = v, \quad \dot{y}_A = 0, \quad \dot{\theta} = \frac{v}{r}$$

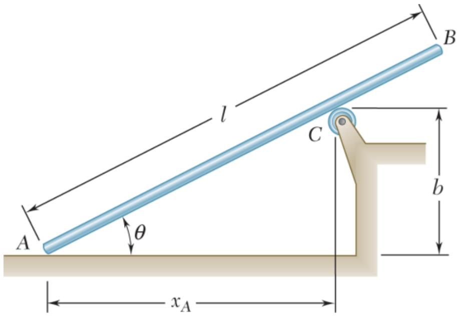
$$x_A = vt, \quad \theta = \frac{vt}{r}$$

$$\dot{x}_P = v_x = r\dot{\theta} - r\cos\theta\dot{\theta} = r\left(1 - \cos\frac{vt}{r}\right)\frac{v}{r}$$

$$v_x = v\left(1 - \cos\frac{vt}{r}\right) \blacktriangleleft$$

$$\dot{y}_P = v_y = r\sin\theta\dot{\theta} = r\left(\sin\frac{vt}{r}\right)\frac{v}{r}$$

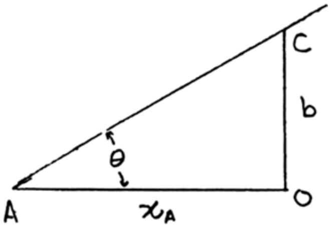
$$v_y = v\sin\frac{vt}{r} \blacktriangleleft$$



PROBLEM 15.142*

Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity $\mathbf{v}_A$. Using the method of Section 15.4 B, derive expressions for the angular velocity and angular acceleration of the rod.

SOLUTION



$$\tan\theta = \frac{b}{x_A} \qquad \cot\theta = \frac{x_A}{b} = u$$

$$\theta = \cot^{-1} u$$

$$\dot{\theta} = -\frac{\dot{u}}{1+u^2}$$

$$\ddot{\theta} = \frac{(2u\dot{u})\dot{u}}{(1+u^2)^2} - \frac{\ddot{u}}{1+u^2}$$

But $\qquad \omega = \dot{\theta} \quad \text{and} \quad \alpha = \ddot{\theta}$

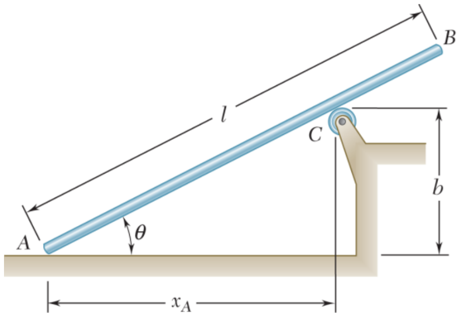
$$u = \frac{x_A}{b}, \quad \dot{u} = \frac{\dot{x}_A}{b} = -\frac{v_A}{b}, \quad \ddot{u} = -\frac{\dot{v}_A}{b} = 0$$

Then $\qquad \omega = \frac{\frac{v_A}{b}}{1+\left(\frac{x_A}{b}\right)^2} = \frac{bv_A}{b^2+x_A^2},$

$$\omega = \frac{bv_A}{b^2+x_A^2} \circlearrowleft \blacktriangleleft$$

$$\alpha = \frac{2\left(\frac{x_A}{b}\right)\left(\frac{v_A}{b}\right)^2}{\left[1+\left(\frac{x_A}{b}\right)^2\right]^2} - 0 = \frac{2bx_Av_A^2}{\left(b^2+x_A^2\right)^2},$$

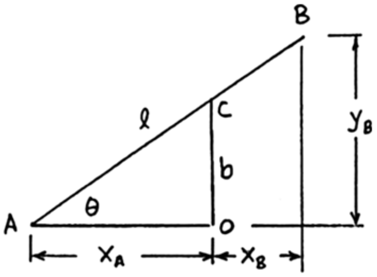
$$\alpha = \frac{2bx_Av_A^2}{\left(b^2+x_A^2\right)^2} \circlearrowleft \blacktriangleleft$$



PROBLEM 15.143*

Rod *AB* moves over a small wheel at *C* while end *A* moves to the right with a constant velocity $\mathbf{v}_A$. Using the method of Section 15.4 B, derive expressions for the horizontal and vertical components of the velocity of Point *B*.

SOLUTION



$$\sin\theta = \frac{b}{\left(b^2 + x_A^2\right)^{1/2}}, \qquad \cos\theta = \frac{x_A}{\left(b^2 + x_A^2\right)^{1/2}}$$

$$x_B = l\cos\theta - x_A$$

$$= \frac{l x_A}{\left(b^2 + x_A^2\right)^{1/2}} - x_A$$

$$y_B = l\sin\theta = \frac{lb}{\left(b^2 + x_A^2\right)^{1/2}}$$

$$\dot{x}_B = \frac{l\dot{x}_A}{\left(b^2 + x_A^2\right)^{1/2}} - \frac{l x_A x_A \dot{x}_A}{\left(b^2 + x_A^2\right)^{3/2}} - \dot{x}_A$$

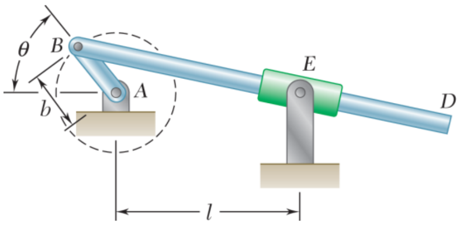
$$= \frac{l b^2 \dot{x}_A}{\left(b^2 + x_A^2\right)^{3/2}} - \dot{x}_A$$

$$\dot{y}_B = -\frac{l b x_A \dot{x}_A}{\left(b^2 + x_A^2\right)^{3/2}}$$

But $\dot{x}_A = -v_A, \qquad \dot{x}_B = (v_B)_x, \qquad \dot{y}_B = (v_B)_y$

$$(v_B)_x = v_A - \frac{l b^2 v_A}{\left(b^2 + x_A^2\right)^{3/2}} \longrightarrow \blacktriangleleft$$

$$(v_B)_y = \frac{l b x_A v_A}{\left(b^2 + x_A^2\right)^{3/2}} \uparrow \blacktriangleleft$$

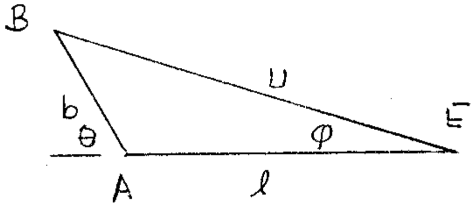


PROBLEM 15.144

Crank AB rotates with a constant clockwise angular velocity $\boldsymbol{\omega}$. Using the method of Section 15.4 B, derive expressions for the angular velocity of rod BD and the velocity of the point on the rod coinciding with Point E in terms of θ, ω, b, and l.

SOLUTION

Law of cosines for triangle ABE.



$$u^2 = l^2 + b^2 - 2bl\cos(180° - \theta)$$
$$= l^2 + b^2 + 2bl\cos\theta$$
$$\cos\varphi = \frac{l + b\cos\theta}{u}$$
$$\tan\varphi = \frac{b\sin\theta}{l + b\cos\theta}$$

$$\frac{d}{dt}(\tan\varphi) = \sec^2\varphi\dot{\varphi} = \frac{(l + b\cos\theta)(b\cos\theta)\dot{\theta} + (b\sin\theta)(b\cos\theta)\dot{\theta}}{(l + b\cos\theta)^2}$$

$$\dot{\varphi} = \frac{(\cos^2\varphi)[bl\cos\theta + b^2(\cos^2\theta + \sin^2\theta)]\dot{\theta}}{(l + b\cos\theta)^2}$$

$$= \frac{bl\cos\theta + b^2}{u^2}\dot{\theta} = \frac{b(b + l\cos\theta)}{l^2 + b^2 + 2bl\cos\theta}\dot{\theta}$$

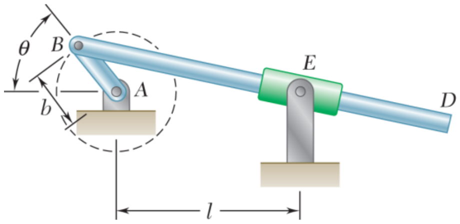
But, $\dot{\theta} = \omega$, $\dot{\varphi} = \omega_{BD}$, and $v_E = -\dot{u}$

Hence,
$$\omega_{BD} = \frac{b(b + l\cos\theta)}{l^2 + b^2 + 2bl\cos\theta}\omega \;\circlearrowright \blacktriangleleft$$

Differentiate the expression for u^2.

$$2u\dot{u} = -2bl\sin\theta\dot{\theta}$$
$$v_E = -\dot{u} = \frac{bl\sin\theta}{l^2 + b^2 + 2bl\cos\theta}\omega$$

$$\mathbf{v}_E = \frac{bl\sin\theta}{l^2 + b^2 + 2bl\cos\theta}\omega \;\nwarrow\; \tan^{-1}\left(\frac{b\sin\theta}{l + b\cos\theta}\right) \blacktriangleleft$$

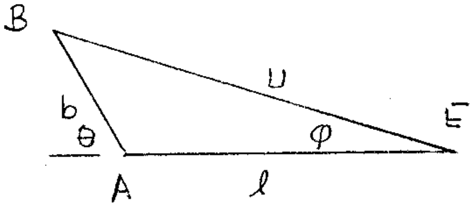


PROBLEM 15.145

Crank *AB* rotates with a constant clockwise angular velocity **ω**. Using the method of Section 15.4 B, derive an expression for the angular acceleration of rod *BD* in terms of θ, ω, b, and l.

SOLUTION

Law of cosines for triangle *ABE*.



$$u^2 = l^2 + b^2 - 2bl\cos(180° - \theta)$$

$$= l^2 + b^2 + 2bl\cos\theta$$

$$\cos\varphi = \frac{l + b\cos\theta}{u}$$

$$\tan\varphi = \frac{b\sin\theta}{l + b\cos\theta}$$

$$\frac{d}{dt}(\tan\varphi) = \sec^2\varphi\dot{\varphi} = \frac{(l + b\cos\theta)(b\cos\theta)\dot{\theta} + (b\sin\theta)(b\cos\theta)\dot{\theta}}{(l + b\cos\theta)^2}$$

$$\dot{\varphi} = \frac{(\cos^2\varphi)[bl\cos\theta + b^2(\cos^2\theta + \sin^2\theta)]\dot{\theta}}{(l + b\cos\theta)^2}$$

$$= \frac{bl\cos\theta + b^2}{u^2}\dot{\theta} = \frac{b(b + l\cos\theta)}{l^2 + b^2 + 2bl\cos\theta}\dot{\theta}$$

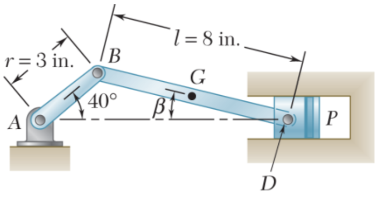
$$\ddot{\varphi} = \frac{b(b + l\cos\theta)}{l^2 + b^2 + 2bl\cos\theta}\ddot{\theta}$$

$$+ \frac{(l^2 + b^2 + 2bl\cos\theta)(-bl\sin\theta) - b(b + l\cos\theta)(-2bl\sin\theta)}{(l^2 + b^2 + 2bl\cos\theta)^2}\dot{\theta}^2$$

$$= \frac{b(b + l\cos\theta)}{l^2 + b^2 + 2bl\cos\theta}\ddot{\theta} - \frac{bl(l^2 - b^2)\sin\theta}{(l^2 + b^2 + 2bl\cos\theta)^2}\dot{\theta}^2$$

But, $\dot{\theta} = \omega,\qquad \ddot{\theta} = \dot{\omega} = 0,\qquad \ddot{\varphi} = \alpha_{BD}$

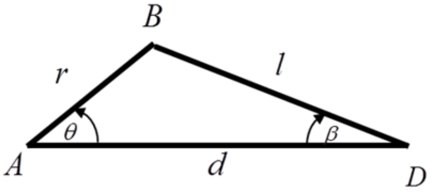
$$\alpha_{BD} = \frac{bl(l^2 - b^2)\sin\theta}{l^2 + b^2 + 2bl\cos\theta}\omega^2 \circlearrowright \blacktriangleleft$$



PROBLEM 15.146

Solve the engine system from Sample Prob. 15.15 using the methods of Section 15.4B. Hint: Define the angle between the horizontal and the crank AB as θ and derive the motion in terms of this parameter.

SOLUTION



Given: $r = 3 \text{ in}, \; l = 8 \text{ in}, \; \theta = 40°$

$$\dot{\theta} = 2000 \text{ rpm} = 209.4 \text{ rad/s} \circlearrowright$$

$$\ddot{\theta} = 0$$

Law of sines: $$\frac{\sin\theta}{l} = \frac{\sin\beta}{r} \qquad (1)$$

At $\theta = 40°$ $$\beta = \sin^{-1}\left(\frac{r\sin\theta}{l}\right) = 13.948°$$

Differentiate (1) with respect to t: $$\frac{\cos\theta\dot{\theta}}{l} = \frac{\cos\beta\dot{\beta}}{r} \quad \Rightarrow \quad \dot{\beta} = \frac{r\cos\theta\dot{\theta}}{l\cos\beta} \qquad (2)$$

At $\theta = 40°$ $$\dot{\beta} = \frac{3\cos 40°(-209.4)}{8\cos 13.948°} = -61.99 \text{ rad/s} \text{ or } \dot{\beta} = 61.99 \text{ rad/s} \circlearrowleft$$

Differentiate (2) with respect to t: $$\ddot{\beta} = \frac{r}{l}\left[\frac{\left(-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}\right)\cos\beta + \cos\theta\dot{\theta}\sin\beta\dot{\beta}}{\cos^2\beta}\right]$$

At $\theta = 40°$ $$\ddot{\beta} = -9940.2 \text{ rad/s}^2 \qquad \boldsymbol{\alpha}_{BD} = 9940.2 \text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$

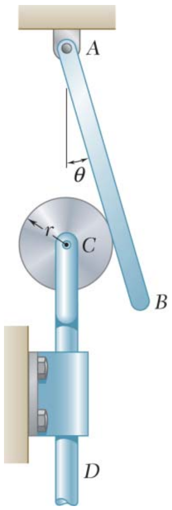
Write an equation for d: $$d = r\cos\theta + l\cos\beta$$

Differentiate with respect to t: $$\dot{d} = -r\sin\theta\dot{\theta} - l\sin\beta\dot{\beta}$$

Differentiate again: $$\ddot{d} = -r\cos\theta\dot{\theta}^2 - r\sin\theta\cancel{\ddot{\theta}} - l\cos\beta\dot{\beta}^2 - l\sin\beta\ddot{\beta}$$

At $\theta = 40°$ $$\ddot{d} = -3\cos 40°(-209.4)^2 - 8\cos 13.95°(-61.99)^2 - 8\sin 13.95°(-9940.2)$$
$$= -111{,}477.5 \text{ in/s}$$

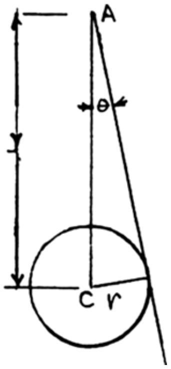
$$\mathbf{a}_D = 9289.8 \text{ ft/s}^2 \longleftarrow \blacktriangleleft$$



PROBLEM 15.147*

The position of rod *AB* is controlled by a disk of radius *r* that is attached to yoke *CD*. Knowing that the yoke moves vertically upward with a constant velocity $\mathbf{v}_0$, derive expression for the angular velocity and angular acceleration of rod *AB*.

SOLUTION



From geometry,

$$y = \frac{r}{\sin\theta}$$

$$\frac{dy}{dt} = -\frac{r\cos\theta}{\sin^2\theta}\frac{d\theta}{dt}$$

But,

$$\frac{dy}{dt} = -v_0 \quad \text{and} \quad \frac{d\theta}{dt} = \omega$$

$$v_0 = \frac{r\cos\theta}{\sin^2\theta}\omega$$

$$\omega = \frac{v_0}{r}\frac{\sin^2\theta}{\cos\theta}$$

$$\omega = \frac{v_0}{r}\frac{\sin^2\theta}{\cos\theta} \circlearrowleft \blacktriangleleft$$

From geometry,

$$y = \frac{r}{\sin\theta}$$

$$\frac{dy}{dt} = -\frac{r\cos\theta}{\sin^2\theta}\frac{d\theta}{dt}$$

But,

$$\frac{dy}{dt} = -v_0 \quad \text{and} \quad \frac{d\theta}{dt} = \omega$$

$$v_0 = \frac{r\cos\theta}{\sin^2\theta}\omega$$

$$\omega = \frac{v_0}{r}\frac{\sin^2\theta}{\cos\theta}$$

PROBLEM 15.147* (Continued)

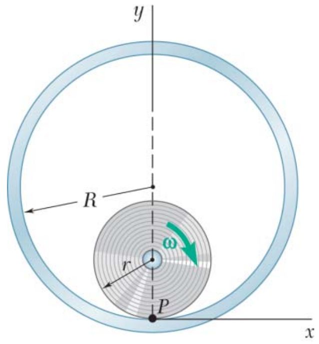
Angular acceleration.

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega$$

$$= \frac{v_0}{r}\frac{(2\cos^2\theta\sin\theta + \sin^3\theta)}{\cos^2\theta}\left(\frac{v_0}{r}\frac{\sin^2\theta}{\cos\theta}\right)$$

$$= \left(\frac{v_0}{r}\right)^2 \frac{(1+\cos^2\theta)\sin^3\theta}{\cos^3\theta}$$

$$\boldsymbol{\alpha} = \left(\frac{v_0}{r}\right)^2 (1+\cos^2\theta)\tan^3\theta \;\circlearrowleft \quad \blacktriangleleft$$



PROBLEM 15.148*

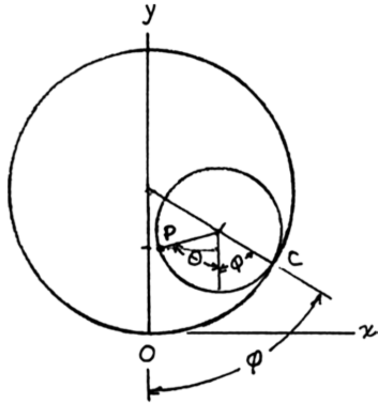
A wheel of radius r rolls without slipping along the inside of a fixed cylinder of radius R with a constant angular velocity $\boldsymbol{\omega}$. Denoting by P the point of the wheel in contact with the cylinder at $t = 0$, derive expressions for the horizontal and vertical components of the velocity of P at any time t. (The curve described by Point P is a *hypocycloid*.)

SOLUTION

Define angles θ and φ as shown.

$$\dot{\theta} = \omega, \qquad \theta = \omega t$$

Since the wheel rolls without slipping, the arc OC is equal to arc PC.



$$r(\varphi + \theta) = R\varphi$$

$$\varphi = \frac{r\theta}{R - r}$$

$$\dot{\varphi} = \frac{r\dot{\theta}}{R - r} = \frac{r\omega}{R - r}$$

$$\varphi = \frac{r\omega t}{R - r}$$

$$x_P = (R - r)\sin\varphi - r\sin\theta$$

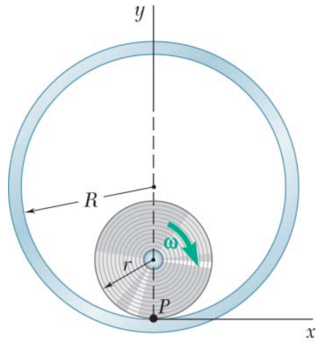
$$\begin{aligned}(v_P)_x &= \dot{x}_P \\ &= (R - r)\cos\varphi\dot{\varphi} - r\cos\theta\dot{\theta} \\ &= (R - r)\left(\cos\frac{r\omega t}{R - r}\right)\left(\frac{r\omega}{R - r}\right) - r(\cos\omega t)(\omega)\end{aligned}$$

$$(v_P)_x = r\omega\left(\cos\frac{r\omega t}{R - r} - \cos\omega t\right) \blacktriangleleft$$

$$y_P = R - (R - r)\cos\varphi - r\cos\theta$$

$$\begin{aligned}(v_P)_y &= \dot{y}_P \\ &= (R - r)\sin\varphi\dot{\varphi} + r\sin\theta\dot{\theta} \\ &= (R - r)\left(\sin\frac{r\omega t}{R - r}\right)\left(\frac{r\omega}{R - r}\right) + r(\sin\omega t)(\omega)\end{aligned}$$

$$(v_P)_y = r\omega\left(\sin\frac{r\omega t}{R - r} + \sin\omega t\right) \blacktriangleleft$$

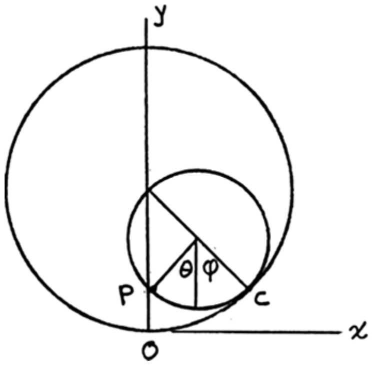


PROBLEM 15.149*

In Problem 15.148, show that the path of P is a vertical straight line when $r = R/2$. Derive expressions for the corresponding velocity and acceleration of P at any time t.

SOLUTION

Define angles θ and φ as shown.



$$\dot{\theta} = \omega, \quad \theta = \omega t, \quad \ddot{\theta} = 0$$

Since the wheel rolls without slipping, the arc OC is equal to arc PC.

$$r(\varphi + \theta) = R\theta$$
$$= 2r\theta$$
$$\varphi = 0$$
$$\dot{\varphi} = \dot{\theta} = \omega$$
$$\ddot{\varphi} = \ddot{\theta} = 0$$

$$x_P = (R - r)\sin\varphi - r\sin\theta$$
$$= r\sin\theta - r\sin\theta$$
$$= 0$$

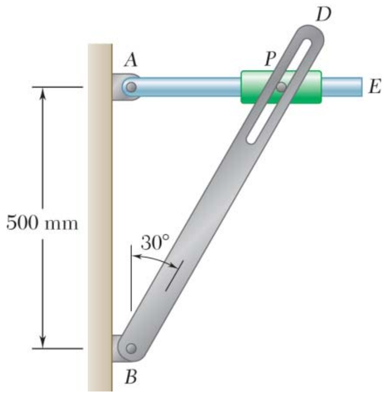
The path is the y axis. ◀

$$y_P = R - (R - r)\cos\varphi - r\cos\theta$$
$$= R - r\cos\theta - r\cos\theta$$
$$= R(1 - \cos\theta)$$
$$v = \dot{y}_P = R\sin\theta\dot{\theta}$$

$\mathbf{v} = (R\omega\sin\omega t)\mathbf{j}$ ◀

$$a = \dot{v}$$
$$= (R\cos\theta\dot{\theta}^2 - \sin\theta\ddot{\theta})$$
$$= R\omega^2\cos\theta$$

$\mathbf{a} = (R\omega^2\cos\omega t)\mathbf{j}$ ◀



PROBLEM 15.150

Pin P is attached to the collar shown; the motion of the pin is guided by a slot cut in rod BD and by the collar that slides on rod AE. Knowing that at the instant considered the rods rotate clockwise with constant angular velocities, determine for the given data the velocity of pin P.

$\omega_{AE} = 8$ rad/s, $\omega_{BD} = 3$ rad/s

SOLUTION

$$AB = 500\text{ mm} = 0.5\text{ m},\quad AP = 0.5\tan 30°,\quad BP = \frac{0.5}{\cos 30°}$$

$$\boldsymbol{\omega}_{AE} = 8\text{ rad/s} \circlearrowright,\qquad \boldsymbol{\omega}_{BD} = 3\text{ rad/s} \circlearrowright$$

Let P' be the coinciding point on AE and u_1 be the outward velocity of the collar along the rod AE.

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/AE} = [(AP)\omega_{AE} \downarrow] + [u_1 \longrightarrow]$$

Let P'' be the coinciding point on BD and u_2 be the outward speed along the slot in rod BD.

$$\mathbf{v}_P = \mathbf{v}_{P''} + \mathbf{v}_{P/BD} = [(BP)\omega_{BD} \nwarrow 30°] + [u_2 \measuredangle 60°]$$

Equate the two expressions for $\mathbf{v}_P$ and resolve into components.

$$\xrightarrow{+}:\quad u_1 = \left(\frac{0.5}{\cos 30°}\right)(3)(\cos 30°) + u_2\cos 60°$$

or

$$u_1 = 1.5 + 0.5u_2 \tag{1}$$

$$+\uparrow:\quad -(0.5\tan 30°)(8) = -\left(\frac{0.5}{\cos 30°}\right)(3)\sin 30° + u_2\sin 60°$$

$$u_2 = \frac{1}{\sin 60°}[1.5\tan 30° - 4\tan 30°] = -1.66667\text{ m/s}$$

From (1),

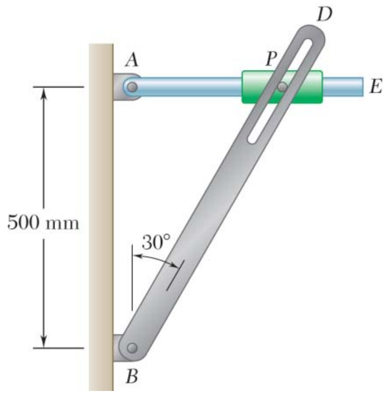
$$u_1 = 1.5 + (0.5)(-1.66667) = 0.66667\text{ m/s}$$

$$\mathbf{v}_P = [(0.5\tan 30°)(8) \downarrow] + [0.66667 \longrightarrow] = [2.3094\text{ m/s} \downarrow] + [0.66667\text{ m/s} \longrightarrow]$$

$$v_P = -\sqrt{2.3094^2 + 0.66667^2} = 2.4037\text{ m/s}$$

$$\tan\beta = \frac{2.3094}{0.66667}\qquad \beta = 73.9°$$

$$\mathbf{v}_P = 2.40\text{ m/s} \nwarrow 73.9° \blacktriangleleft$$



PROBLEM 15.151

Pin P is attached to the collar shown; the motion of the pin is guided by a slot cut in rod BD and by the collar that slides on rod AE. Knowing that at the instant considered the rods rotate clockwise with constant angular velocities, determine for the given data the velocity of pin P.

$\omega_{AE} = 7 \text{ rad/s}, \ \omega_{BD} = 4.8 \text{ rad/s}$

SOLUTION

$$AB = 500 \text{ mm} = 0.5 \text{ m}, \quad AP = 0.5\tan 30^\circ, \quad BP = \frac{0.5}{\cos 30^\circ}$$

$$\boldsymbol{\omega}_{AE} = 7 \text{ rad/s} \circlearrowright, \qquad \boldsymbol{\omega}_{BD} = 4.8 \text{ rad/s} \circlearrowright$$

Let P' be the coinciding point on AE and u_1 be the outward velocity of the collar along the rod AE.

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/AE} = [(AP)\omega_{AE} \downarrow] + [u_1 \longrightarrow]$$

Let P'' be the coinciding point on BD and u_2 be the outward speed along the slot in rod BD.

$$\mathbf{v}_P = \mathbf{v}_{P''} + \mathbf{v}_{P/BD} = [(BP)\omega_{BD} \nwarrow 30^\circ] + [u_2 \measuredangle\ 60^\circ]$$

Equate the two expressions for $\mathbf{v}_P$ and resolve into components.

$$\xrightarrow{+}: \quad u_1 = \left(\frac{0.5}{\cos 30^\circ}\right)(4.8)(\cos 30^\circ) + u_2 \cos 60^\circ$$

or $\qquad u_1 = 2.4 + 0.5u_2 \qquad (1)$

$$+\uparrow: \quad -(0.5\tan 30^\circ)(7) = -\left(\frac{0.5}{\cos 30^\circ}\right)(4.8)\sin 30^\circ + u_2 \sin 60^\circ$$

$$u_2 = \frac{1}{\sin 60^\circ}[2.4\tan 30^\circ - 3.5\tan 30^\circ] = -0.73333 \text{ m/s}$$

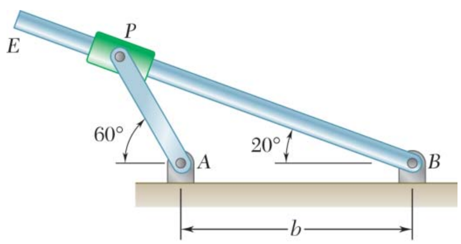
From (1), $\quad u_1 = 2.4 + (0.5)(-0.73333) = 2.0333 \text{ m/s}$

$$\mathbf{v}_P = [(0.5\tan 30^\circ)(7) \downarrow] + [2.0333 \longrightarrow] = [2.0207 \text{ m/s} \downarrow] + [2.0333 \text{ m/s} \longrightarrow]$$

$$v_P = \sqrt{(2.0333)^2 + (2.0207)^2} = 2.87 \text{ m/s}$$

$$\tan\beta = -\frac{2.0207}{2.0333}, \qquad \beta = -44.8^\circ$$

$$\mathbf{v}_P = 2.87 \text{ m/s} \nwarrow 44.8^\circ \blacktriangleleft$$



PROBLEM 15.152

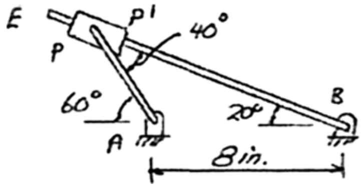
Two rotating rods are connected by slider block P. The rod attached at A rotates with a constant angular velocity ω_A. For the given data, determine for the position shown (*a*) the angular velocity of the rod attached at B, (*b*) the relative velocity of slider block P with respect to the rod on which it slides.

$b = 8$ in., $\omega_A = 6$ rad/s.

SOLUTION

Dimensions:

Law of sines.



$$\frac{AP}{\sin 20^\circ} = \frac{BP}{\sin 120^\circ} = \frac{8 \text{ in.}}{\sin 40^\circ}$$

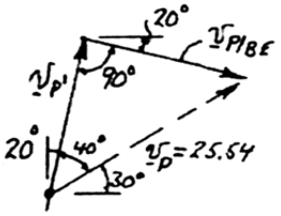
$$AP = 4.2567 \text{ in.}$$

$$BP = 10.7784 \text{ in.}$$

$$\boldsymbol{\omega}_{AP} = 6 \text{ rad/s} \circlearrowright$$

Velocities.

Note: $P' =$ Point of BE coinciding with P.



$$v_P = (AP)\omega_{AP}$$

$$= (4.2567 \text{ in.})(6 \text{ rad/s})$$

$$= 25.540 \text{ in./s} \measuredangle 30^\circ$$

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/BE}$$

$$[25.540 \measuredangle 30^\circ] = [v_{P'} \measuredangle 70^\circ] + [v_{P/BE} \searrow 30^\circ]$$

(*a*)

$$v_{P'} = (25.54)\cos 40^\circ$$

$$= 19.565 \text{ in./s}$$

$$\omega_{BE} = \frac{v_{P'}}{BP}$$

$$= \frac{19.565 \text{ in./s}}{10.7784 \text{ in.}}$$

$$= 1.8152 \text{ rad/s}$$

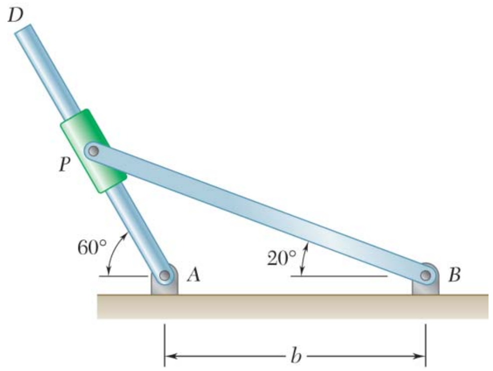
$\boldsymbol{\omega}_{BE} = 1.815$ rad/s $\circlearrowright$ ◀

(*b*)

$$v_{P/BE} = (25.54)\sin 40^\circ$$

$$= 16.417 \text{ in./s}$$

$\mathbf{v}_{P/BE} = 16.42$ in./s $\searrow$ 20° ◀



PROBLEM 15.153

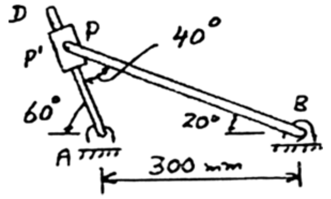
Two rotating rods are connected by slider block P. The rod attached at A rotates with a constant angular velocity ω_A. For the given data, determine for the position shown (a) the angular velocity of the rod attached at B, (b) the relative velocity of slider block P with respect to the rod on which it slides.

$b = 300$ mm, $\omega_A = 10$ rad/s.

SOLUTION

Dimensions:

Law of sines.



$$\frac{AP}{\sin 20°} = \frac{BP}{\sin 120°} = \frac{300 \text{ mm}}{\sin 40°}$$

$$AP = 159.63 \text{ mm}$$

$$BP = 404.19 \text{ mm}$$

$$\boldsymbol{\omega}_{AD} = 10 \text{ rad/s} \circlearrowright$$

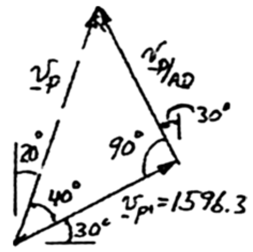
Velocities.

Note: $P' =$ Point of AD coinciding with P.

$$\mathbf{v}_{P'} = (AP')\omega_{AD}$$

$$= (159.63 \text{ mm})(10 \text{ rad/s})$$

$$= 1596.3 \text{ mm/s} \measuredangle 30°$$



$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/AD}$$

$$[v_P \measuredangle 70°] = [1596.3 \measuredangle 30°] + [v_{P/AD} \searrow 60°]$$

(a)

$$v_P = (1596.3)/\cos 40°$$

$$= 2083.8 \text{ mm/s}$$

$$\omega_{BP} = \frac{v_P}{BP}$$

$$= \frac{2083.8 \text{ mm/s}}{404.19 \text{ mm}}$$

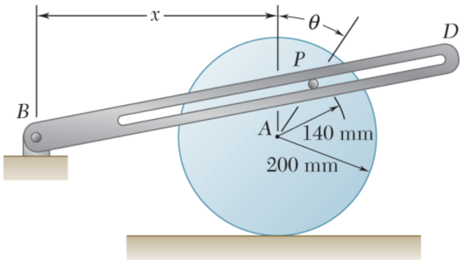
$$= 5.155 \text{ rad/s}$$

$$\boldsymbol{\omega}_{BD} = 5.16 \text{ rad/s} \circlearrowright \blacktriangleleft$$

(b)

$$v_{P/AD} = (1596.3)\tan 40° = 1339.5 \text{ mm/s}$$

$$\mathbf{v}_{P/AD} = 1.339 \text{ m/s} \searrow 60° \blacktriangleleft$$

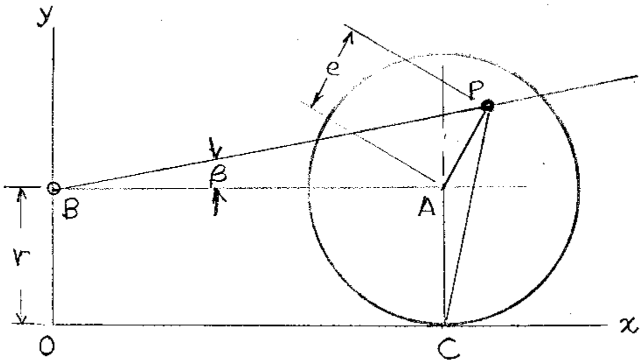


PROBLEM 15.154

Pin P is attached to the wheel shown and slides in a slot cut in bar BD. The wheel rolls to the right without slipping with a constant angular velocity of 20 rad/s. Knowing that $x = 480$ mm when $\theta = 0$, determine the angular velocity of the bar and the relative velocity of pin P with respect to the rod for the given data.

(a) $\theta = 0$, (b) $\theta = 90°$.

SOLUTION



Coordinates.

$$x_A = (x_A)_0 + r\theta, \quad y_A = r$$
$$x_B = 0, \quad y_B = r$$
$$x_C = x_A, \quad y_C = 0$$
$$x_P = x_A + e\sin\theta$$
$$y_P = r + e\cos\theta$$

Data:

$$(x_A)_0 = 480 \text{ mm} = 0.48 \text{ m}$$
$$r = 200 \text{ mm} = 0.20 \text{ m}$$
$$e = 140 \text{ mm} = 0.14 \text{ m}$$

Velocity analysis.

$$\boldsymbol{\omega}_{AC} = \omega_{AC} \circlearrowright, \quad \boldsymbol{\omega}_{BD} = \omega_{BD} \circlearrowright,$$

$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{P/A} = [r\omega_{AC} \longrightarrow] + [e\omega_{AC} \searrow \theta]$$

$$\mathbf{v}_{P'} = [x_P\omega_{BD} \downarrow] + [(e\cos\theta)]\omega_{BD} \longrightarrow]$$

$$\mathbf{v}_{P/F} = [u\cos\beta \longrightarrow] + [u\sin\beta \uparrow]$$

Use $\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/F}$ and resolve into components.

$$\xrightarrow{+}: \quad (r + e\cos\theta)\omega_{AC} = (e\cos\theta)\omega_{BD} + (\cos\beta)u \qquad (1)$$

$$+\downarrow: \quad (e\sin\theta)\omega_{AC} = x_P\omega_{BD} - (\sin\beta)u \qquad (2)$$

(a) $\theta = 0$. $\quad x_A = 0.48 \text{ m}, \quad x_P = 0.48 \text{ m}, \quad \omega_{AC} = 20 \text{ rad/s}$

$$\tan\beta = \frac{e\cos\theta}{x_P} = \frac{0.14}{0.48}, \quad \beta = 16.26°$$

Substituting into Eqs. (1) and (2),

$$(0.20 + 0.14)(20) = 0.14\omega_{BD} + (\cos 16.26°)u \qquad (1)$$

$$0 = 0.48\omega_{BD} - (\sin 16.26°)u \qquad (2)$$

Solving simultaneously, $\omega_{BD} = 3.81 \text{ rad/s}$, $\quad \boldsymbol{\omega}_{BD} = 3.81 \text{ rad/s} \circlearrowright$ ◀

$u = 6.53 \text{ m/s}$, $\quad \mathbf{v}_{P/F} = 6.53 \text{ m/s} \measuredangle 16.26°$ ◀

(b) $\theta = 90°$. $\quad x_P = 0.48 + (0.20)\left(\frac{\pi}{2}\right) + 0.14 = 0.93416 \text{ m}$

$\beta = 0$

Substituting into Eqs. (1) and (2),

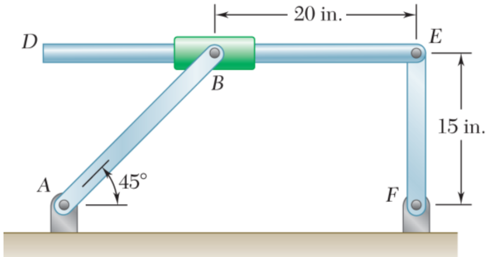
$$(0.20)(20) = u \qquad (1)$$

$$u = 4 \text{ m/s}$$

$$(0.14)(20) = 0.93416\omega_{BD} \qquad (2)$$

$\omega_{BD} = 2.9973 \text{ rad/s}$, $\quad \boldsymbol{\omega}_{BD} = 3.00 \text{ rad/s} \circlearrowright$ ◀

$\mathbf{v}_{P/F} = 4.00 \text{ m/s} \longrightarrow$ ◀



PROBLEM 15.155

Knowing that at the instant shown the angular velocity of bar *AB* is 15 rad/s clockwise and the angular velocity of bar *EF* is 10 rad/s clockwise, determine (*a*) the angular velocity of rod *DE*, (*b*) the relative velocity of collar *B* with respect to rod *DE*.

SOLUTION

$$\omega_{EF} = 10 \text{ rad/s} \circlearrowright, \quad \mathbf{v}_E = \left[r_{E/F}\omega_{EF} \longrightarrow\right] = (15)(10) = 150 \text{ in./s} \longrightarrow$$

$$\mathbf{v}_{B'} = \mathbf{v}_E + \mathbf{v}_{B/E} = [150 \text{ in./s} \longrightarrow] + [20\omega_{DE} \downarrow]$$

$$\mathbf{v}_{B/ED} = u \longrightarrow$$

$$\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/ED} = [150 \text{ in./s} \longrightarrow] + [20\omega_{DE} \downarrow] + [u \longrightarrow]$$

$$\mathbf{v}_B = \left[(AB)\omega_{AB} \searrow 45^\circ\right] = \left[\left(\frac{15}{\cos 45^\circ}\right)(15) \searrow 45^\circ\right] = [225 \text{ in./s} \downarrow] + [225 \text{ in./s} \longrightarrow]$$

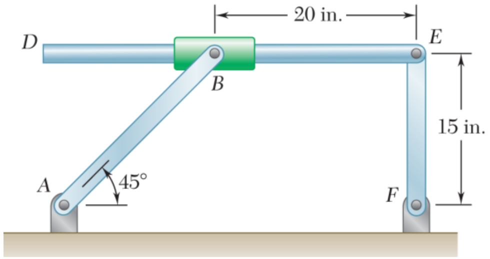
Equate the two expressions for $\mathbf{v}_B$ and resolve into components.

$$\xrightarrow{+}: \quad 150 + u = 225, \qquad u = 75 \text{ in./s}$$

$$+\downarrow: \quad 20\omega_{DE} = 225, \qquad \omega_{DE} = 11.25 \text{ rad/s}$$

(*a*) $\omega_{DE} = 11.25 \text{ rad/s} \circlearrowleft$ ◀

(*b*) $\mathbf{v}_{B/DE} = 75.0 \text{ in./s} \longrightarrow$ ◀



PROBLEM 15.156

Knowing that at the instant shown the angular velocity of rod *DE* is 10 rad/s clockwise and the angular velocity of bar *EF* is 15 rad/s counterclockwise, determine(*a*) the angular velocity of bar *AB*, (*b*) the relative velocity of collar *B* with respect to rod *DE*.

SOLUTION

$$\omega_{EF} = 15 \text{ rad/s} \qquad \mathbf{v}_E = \left[r_{E/F}\,\omega_{EF} \longleftarrow \right] = (15)(15) = 225 \text{ in./s} \longleftarrow$$

$$\mathbf{v}_{B'} = \mathbf{v}_E + \mathbf{v}_{B/ED} = \left[225 \text{ in./s} \longleftarrow\right] + \left[20\omega_{DE} \downarrow\right] = \left[225 \text{ in./s} \longleftarrow\right] + \left[200 \text{ in./s} \downarrow\right]$$

$$\mathbf{v}_{B/ED} = u \longrightarrow$$

$$\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/ED} = \left[225 \text{ in./s} \longleftarrow\right] + \left[200 \text{ in./s} \downarrow\right] + \left[u \longrightarrow\right]$$

$$\mathbf{v}_B = \left[(AB)\omega_{AB} \searrow 45^\circ\right] = \left[\frac{15}{\cos 45^\circ}\omega_{AB} \searrow 45^\circ\right] = \left[15\omega_{AB} \uparrow\right] + \left[15\omega_{AB} \longleftarrow\right]$$

Equate the two expressions for $\mathbf{v}_B$ and resolve into components.

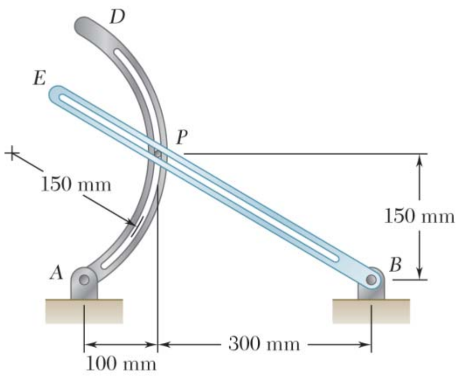
$$\xrightarrow{+}: \quad -225 + u = -15\omega_{AB} \qquad (1)$$

$$+\downarrow: \quad 200 = 15\omega_{AB} \qquad \omega_{AB} = \frac{200}{15} = 13.333 \text{ rad/s}$$

(*a*) $\omega_{AB} = 13.33$ rad/s $\circlearrowright$ ◀

From (1), $u = 225 - (15)(13.333) = 25$ in./s

(*b*) $\mathbf{v}_{B/ED} = 25.0$ in./s $\longrightarrow$ ◀



PROBLEM 15.157

The motion of pin P is guided by slots cut in rods AD and BE. Knowing that bar AD has a constant angular velocity of 4 rad/s clockwise and bar BE has an angular velocity of 5 rad/s counterclockwise and is slowing down at a rate of 2 rad/s^2, determine the velocity of P for the position shown.

SOLUTION

Units: meters, m/s, m/s^2

Unit vectors: $\mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1 \uparrow, \quad \mathbf{k} = 1 \circlearrowleft.$

Geometry: Slope angle θ of rod BE.

$$\tan\theta = \frac{0.15}{0.3} = 0.5 \qquad \theta = 26.565^\circ \measuredangle$$

$$\mathbf{r}_{P/A} = 0.1\mathbf{i} + 0.15\mathbf{j} \qquad \mathbf{r}_{P/B} = -0.3\mathbf{i} + 0.15\mathbf{j}$$

Angular velocities: $\boldsymbol{\omega}_{AD} = -(4 \text{ rad/s})\mathbf{k} \qquad \boldsymbol{\omega}_{BE} = (5 \text{ rad/s})\mathbf{k}$

Angular accelerations: $\boldsymbol{\alpha}_{AD} = 0 \qquad \boldsymbol{\alpha}_{BE} = -(2 \text{ rad/s}^2)\mathbf{k}$

Velocity of Point P' on rod AD coinciding with the pin:

$$\mathbf{v}_{P'} = \boldsymbol{\omega}_{AD} \times \mathbf{r}_{P/A} = (-4\mathbf{k}) \times (0.1\mathbf{i} + 0.15\mathbf{j}) = 0.6\mathbf{i} - 0.4\mathbf{j}$$

Velocity of the pin relative to rod AD:

$$\mathbf{v}_{P/AD} = u_1 \uparrow = u_1\mathbf{j}$$

Velocity of P: $\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/AD} = 0.6\mathbf{i} - 0.4\mathbf{j} + u_1\mathbf{j}$

Velocity of Point P'' on rod BE coinciding with the pin:

$$\mathbf{v}_{P''} = \boldsymbol{\omega}_{BE} \times \mathbf{r}_{P/B} = 5\mathbf{k} \times (-0.3\mathbf{i} + 0.15\mathbf{j}) = -0.75\mathbf{i} - 1.5\mathbf{j}$$

Velocity of the pin relative to rod BE:

$$\mathbf{v}_{P/BE} = u_2 \nwarrow \theta = -u_2\cos\theta\,\mathbf{i} + u_2\sin\theta\,\mathbf{j}$$

Velocity of P:

$$\begin{aligned}\mathbf{v}_P &= \mathbf{v}_{P''} + \mathbf{v}_{P/BE} \\ &= -0.75\mathbf{i} - 1.5\mathbf{j} - u_2\cos\theta\,\mathbf{i} + u_2\sin\theta\,\mathbf{j}\end{aligned}$$

PROBLEM 15.157 (Continued)

Equating the two expressions for $\mathbf{v}_P$ and resolving into components,

i: $$0.6 = -0.75 - u_2 \cos\theta$$

$$u_2 = -\frac{1.35}{\cos 26.565^\circ} = -1.50965$$

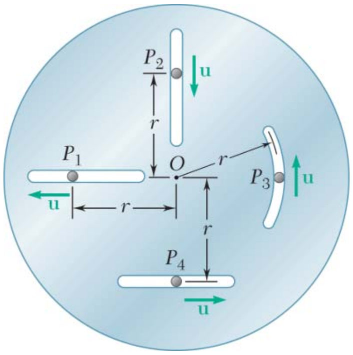
j: $$-0.4 + u_1 = -1.5 + u_2 \sin\theta$$

$$u_1 = -1.1 + (-1.50935)\sin 26.535^\circ = -1.77500$$

Velocity of *P*:

$$\mathbf{v}_P = 0.6\mathbf{i} - 0.4\mathbf{j} - 1.775\mathbf{j} = 0.6\mathbf{i} - 2.175\mathbf{j}$$

$\mathbf{v}_P = 2.26$ m/s ↙ 74.6° ◀



PROBLEM 15.158

Four pins slide in four separate slots cut in a circular plate as shown. When the plate is at rest, each pin has a velocity directed as shown and of the same constant magnitude u. If each pin maintains the same velocity relative to the plate when the plate rotates about O with a constant counterclockwise angular velocity ω, determine the acceleration of each pin.

SOLUTION

For each pin: $\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/F} + \mathbf{a}_C$

Acceleration of the coinciding Point P′ of the plate.

For each pin $\mathbf{a}_{P'} = r\omega^2$ towards the center O.

Acceleration of the pin relative to the plate.

For pins P_1, P_2 and P_4, $\mathbf{a}_{P/F} = 0$

For pin P_3, $\mathbf{a}_{P/F} = \dfrac{u^2}{r} \leftarrow$

Coriolis acceleration $\mathbf{a}_C$.

For each pin $a_C = 2\omega u$ with $\mathbf{a}_C$ in a direction obtained by rotating **u** through 90° in the sense of ω, i.e., $\circlearrowleft$.

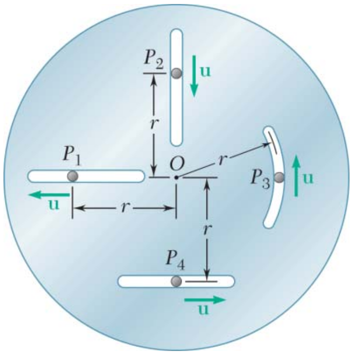
Then

$$\mathbf{a}_1 = [r\omega^2 \rightarrow] + [2\omega u \downarrow] \qquad \mathbf{a}_1 = r\omega^2\mathbf{i} - 2\omega u\mathbf{j} \blacktriangleleft$$

$$\mathbf{a}_2 = [r\omega^2 \downarrow] + [2\omega u \rightarrow] \qquad \mathbf{a}_2 = 2\omega u\mathbf{i} - r\omega^2\mathbf{j} \blacktriangleleft$$

$$\mathbf{a}_3 = [r\omega^2 \leftarrow] + \left[\frac{u^2}{r} \leftarrow\right] + [2\omega u \leftarrow] \qquad \mathbf{a}_3 = -\left(r\omega^2 + \frac{u^2}{r} + 2\omega u\right)\mathbf{i} \blacktriangleleft$$

$$\mathbf{a}_4 = [r\omega^2 \uparrow] + [2\omega u \uparrow] \qquad \mathbf{a}_4 = (r\omega^2 + 2\omega u)\mathbf{j} \blacktriangleleft$$



PROBLEM 15.159

Solve Problem 15.158, assuming that the plate rotates about O with a constant clockwise angular velocity $\boldsymbol{\omega}$.

PROBLEM 15.158 Four pins slide in four separate slots cut in a circular plate as shown. When the plate is at rest, each pin has a velocity directed as shown and of the same constant magnitude u. If each pin maintains the same velocity relative to the plate when the plate rotates about O with a constant counterclockwise angular velocity ω, determine the acceleration of each pin.

SOLUTION

For each pin: $\qquad \mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/F} + \mathbf{a}_C$

Acceleration of the coinciding Point P′ of the plate.

For each pin $\mathbf{a}_{P'} = r\omega^2$ towards the center O.

Acceleration of the pin relative to the plate.

For pins P_1, P_2 and P_4, $\qquad \mathbf{a}_{P/F} = 0$

For pin P_3, $\qquad \mathbf{a}_{P/F} = \dfrac{u^2}{r} \leftarrow$

Coriolis acceleration $\mathbf{a}_C$.

For each pin $a_C = 2\omega u$ with $\mathbf{a}_C$ in a direction obtained by rotating $\mathbf{u}$ through $90°$ in the sense of $\boldsymbol{\omega}$.

Then $\qquad \mathbf{a}_1 = [r\omega^2 \rightarrow] + [2\omega u \uparrow] \qquad \mathbf{a}_1 = r\omega^2\mathbf{i} + 2\omega u\mathbf{j}$ ◀

$\mathbf{a}_2 = [r\omega^2 \downarrow] + [2\omega u \leftarrow] \qquad \mathbf{a}_2 = -2\omega u\mathbf{i} - r\omega^2\mathbf{j}$ ◀

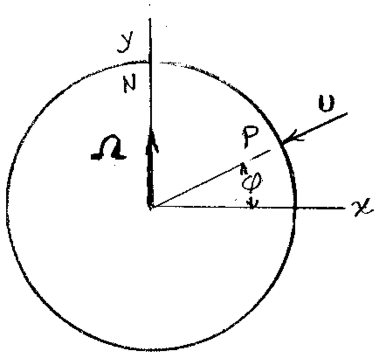
$\mathbf{a}_3 = [r\omega^2 \leftarrow] + \left[\dfrac{u^2}{r} \leftarrow\right] + [2\omega u \rightarrow] \qquad \mathbf{a}_3 = \left(2\omega u - r\omega^2 - \dfrac{u^2}{r}\right)\mathbf{i}$ ◀

$\mathbf{a}_4 = [r\omega^2 \uparrow] + [2\omega u \downarrow] \qquad \mathbf{a}_4 = (r\omega^2 - 2\omega u)\mathbf{j}$ ◀

PROBLEM 15.160

The cage of a mine elevator moves downward at a constant speed of 12.2 m/s. Determine the magnitude and direction of the Coriolis acceleration of the cage if the elevator is located (*a*) at the equator, (*b*) at latitude $40°$ north, (*c*) at latitude $40°$ south.

SOLUTION



Earth makes one revolution $(2\pi \text{ radians})$ in 23.933 h (86160 s).

$$\mathbf{\Omega} = \frac{2\pi}{86160}\mathbf{j} = \left(72.926 \times 10^{-6} \text{ rad/s}\right)\mathbf{j}$$

Velocity relative to the Earth at latitude angle ϕ.

$$\mathbf{v}_{P/\text{earth}} = 12.2(-\cos\phi\mathbf{i} - \sin\phi\mathbf{j})$$

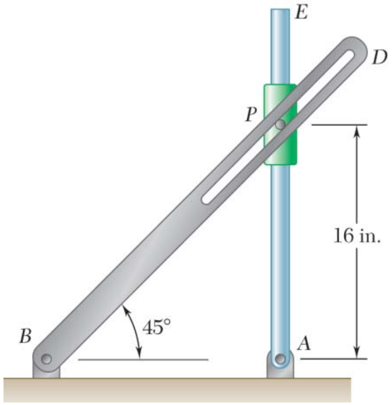
Coriolis acceleration $\mathbf{a}_c$.

$$\begin{aligned}\mathbf{a}_c &= 2\mathbf{\Omega} \times \mathbf{v}_{P/\text{earth}}\\ &= (2)\left(72.926 \times 10^{-6}\mathbf{j}\right) \times \left[12.2(-\cos\phi\mathbf{i} - \sin\phi\mathbf{j})\right]\\ &= \left(1.7794 \times 10^{-3}\cos\phi\right)\mathbf{k}\end{aligned}$$

(*a*) $\phi = 0°, \ \cos\phi = 1.000$ $\quad \mathbf{a}_c = 1.779 \times 10^{-3} \text{ m/s}^2$ west ◀

(*b*) $\phi = 40°, \ \cos\phi = 0.76604$ $\quad \mathbf{a}_c = 1.363 \times 10^{-3} \text{ m/s}^2$ west ◀

(*c*) $\phi = -40°, \ \cos\phi = 0.76604$ $\quad \mathbf{a}_c = 1.363 \times 10^{-3} \text{ m/s}^2$ west ◀



PROBLEM 15.161

Pin P is attached to the collar shown; the motion of the pin is guided by a slot cut in bar BD and by the collar that slides on rod AE. Rod AE rotates with a constant angular velocity of 6 rad/s clockwise and the distance from A to P increases at a constant rate of 8 ft/s. Determine at the instant shown (*a*) the angular acceleration of bar BD, (*b*) the relative acceleration of pin P with respect to bar BD.

SOLUTION

$$AP = 16 \text{ in.} = 1.3333 \text{ ft}, \qquad BP = 16\sqrt{2} \text{ in.} = 1.3333\sqrt{2} \text{ ft}$$

Given: $\boldsymbol{\omega}_{AE} = 6 \text{ rad/s} \circlearrowright, \quad \boldsymbol{\alpha}_{AE} = 0, \quad \mathbf{v}_{P/AE} = 8 \text{ ft/s} \uparrow, \quad \mathbf{a}_{P/AE} = 0.$

Find: $\boldsymbol{\alpha}_{BD}$ and $\mathbf{a}_{P/BD}$.

Velocity of coinciding point P' on rod AE.

$$\mathbf{v}_{P'} = (AP)\omega_{AE} = (1.3333)(6) = 8 \text{ ft/s} \longrightarrow = (8 \text{ ft/s})\mathbf{i}$$

Velocity of P relative to rod AE. $\mathbf{v}_{P/AE} = (8 \text{ ft/s})\mathbf{j}$

Velocity of point P. $\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/AE} = (8 \text{ ft/s})\mathbf{i} + (8 \text{ ft/s})\mathbf{j}$

Velocity of coinciding point P'' on rod BD.

$$\mathbf{v}_{P''} = (BP)\omega_{BD} \nwarrow 45^\circ = 1.3333\sqrt{2}\omega_{BD} \nwarrow 45^\circ = -1.3333\omega_{BD}\mathbf{i} + 1.3333\omega_{BD}\mathbf{j}$$

Velocity of P relative to rod BD. $\mathbf{v}_{P/BD} = (\cos 45^\circ)u\mathbf{i} + (\sin 45^\circ)u\mathbf{j}$

Velocity of point P. $\mathbf{v}_P = \mathbf{v}_{P''} + \mathbf{v}_{P/BD}$

$$\mathbf{v}_P = -1.3333\omega_{BD}\mathbf{i} + 1.3333\omega_{BD}\mathbf{j} + (\cos 45^\circ)u\mathbf{i} + (\sin 45^\circ)u\mathbf{j}$$

Equating the two expressions for $\mathbf{v}_P$ and resolving into components.

$$\mathbf{i}: \quad 8 = -1.3333\omega_{BD} + (\cos 45^\circ)u \qquad (1)$$

$$\mathbf{j}: \quad 8 = 1.3333\omega_{BD} + (\sin 45^\circ)u \qquad (2)$$

Solving (1) and (2), $\omega_{BD} = 0, \quad u = 8\sqrt{2} \text{ ft/s}, \quad \mathbf{v}_{P/BD} = (8 \text{ ft/s})\mathbf{i} + (8 \text{ ft/s})\mathbf{j}$

Acceleration of coinciding point P' on rod AE.

$$\mathbf{a}_{P'} = (AP)\alpha_{AE}\mathbf{i} - (AP)\omega_{AE}^2\mathbf{j} = 0 - (1.3333)(6)^2\mathbf{j} = -(48 \text{ ft/s}^2)\mathbf{j}$$

PROBLEM 15.161 (Continued)

Acceleration of P relative to rod AE. $\mathbf{a}_{P/AE} = 0$

Coriolis acceleration. $2\boldsymbol{\omega}_{AE} \times \mathbf{v}_{P/AE} = (2)(-6\mathbf{k}) \times 8\mathbf{j} = (96 \text{ ft/s}^2)\mathbf{i}$

Acceleration of point P.

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/AE} + 2\boldsymbol{\omega}_{AE} \times \mathbf{v}_{P/AE} = (96 \text{ ft/s}^2)\mathbf{i} - (48 \text{ ft/s}^2)\mathbf{j}$$

Acceleration of coinciding point P'' on rod BD.

$$\mathbf{a}_{P''} = \alpha_{BD}\mathbf{k} \times \mathbf{r}_{P/B} - \omega_{BD}^2\mathbf{r}_{P/B} = -1.3333\alpha_{BD}\mathbf{i} + 1.3333\alpha_{BD}\mathbf{j} + 0$$

Acceleration of P relative to rod BD. $\mathbf{a}_{P/BD} = (\cos 45°)a_r\mathbf{i} + (\sin 45°)a_r\mathbf{j}$

Coriolis acceleration. $2\boldsymbol{\omega}_{BD} \times \mathbf{v}_{P/BD} = 0$

Acceleration of point P.

$$\mathbf{a}_P = \mathbf{a}_{P''} + \mathbf{a}_{P/AE} + 2\boldsymbol{\omega}_{BD} \times \mathbf{v}_{P/BD}$$

$$= -1.3333\alpha_{BD}\mathbf{i} + 1.3333\alpha_{BD}\mathbf{j} + (\cos 45°)a_r\mathbf{i} + (\sin 45°)a_r\mathbf{j}$$

Equating the two expressions for $\mathbf{a}_P$ and resolving into components.

$$\mathbf{i}: \quad 96 = -1.3333\alpha_{BD} + (\cos 45°)a_r \qquad (3)$$

$$\mathbf{j}: \quad -48 = 1.3333\alpha_{BD} + (\sin 45°)a_r \qquad (4)$$

Solving (3) and (4), $\alpha_{BD} = -54 \text{ rad/s}^2, \quad a_r = 24\sqrt{2} \text{ ft/s}$

(*a*) $\boldsymbol{\alpha}_{BD} = 54 \text{ rad/s}^2$ ↻ ◀

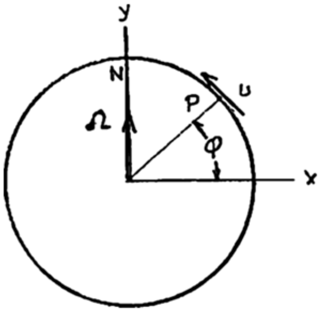
(*b*) $\mathbf{a}_{P/BD} = 33.9 \text{ ft/s}^2$ ∠ 45° ◀

PROBLEM 15.162

A rocket sled is tested on a straight track that is built along a meridian. Knowing that the track is located at latitude 40° north, determine the Coriolis acceleration of the sled when it is moving north at a speed of 900 km/h.

SOLUTION

Earth makes one revolution $(2\pi \text{ radians})$ in $23.933 \text{ h} = 86{,}160 \text{ s}$.



$$\Omega = \frac{2\pi}{86{,}160}$$
$$= (72.926 \times 10^{-6} \text{ rad/s})\mathbf{j}$$

Speed of sled.

$$u = 900 \text{ km/h}$$
$$= 250 \text{ m/s}$$

Velocity of sled relative to the Earth.

$$\mathbf{v}_{P/\text{earth}} = 250(-\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j})$$

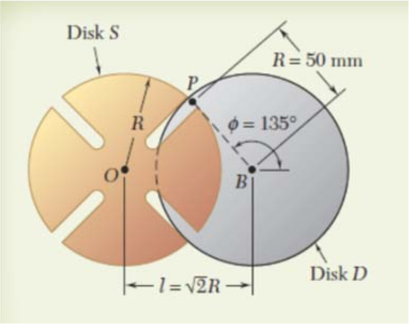
Coriolis acceleration.

$$\mathbf{a}_C = 2\mathbf{\Omega} \times \mathbf{v}_{P/\text{earth}}$$
$$\mathbf{a}_C = (2)(72.926 \times 10^{-6}\,\mathbf{j}) \times [250(-\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j})]$$
$$= 0.036463 \sin\varphi\,\mathbf{k}$$

At latitude $\varphi = 40°$,

$$\mathbf{a}_C = 0.036463 \sin 40°\mathbf{k}$$
$$= (0.0234 \text{ m/s}^2)\mathbf{k}$$

$\mathbf{a}_C = 0.0234 \text{ m/s}^2$ west ◀



PROBLEM 15.163

Solve the Geneva mechanism Sample Prob. 15.20 using vector algebra.

SAMPLE PROBLEM 15.20 In the Geneva mechanism of Sample Prob. 15.9, disk D rotates with a constant counterclockwise angular velocity $\boldsymbol{\omega}_D$ of 10 rad/s. At the instant when $\phi = 150°$, determine the angular acceleration of S.

SOLUTION

Given: $\boldsymbol{\omega}_D = 10\mathbf{k}$ rad/s, $\boldsymbol{\alpha}_D = 0$ rad/s^2

$R = 50$ mm, $\phi = 150°$

Law of cosines:

$$r^2 = \left(\sqrt{2}R\right)^2 + R^2 - 2\left(\sqrt{2}R\right)R\cos 30°$$

$$r = 37.1 \text{ mm}$$

Law of sines:

$$\frac{\sin\beta}{R} = \frac{\sin 30°}{r} \quad \Rightarrow \quad \beta = 42.37°$$

Let P´ be a point on disk S that is coincident with point P on disk D.

Geometry:

$$\mathbf{r}_{P/B} = -R\cos 30°\mathbf{i} + R\sin 30°\mathbf{j} \text{ mm}$$
$$= -43.3\mathbf{i} + 25.0\mathbf{j} \text{ mm}$$
$$\mathbf{r}_{P'/O} = r\cos\beta\mathbf{i} + r\sin\beta\mathbf{j} \text{ mm}$$
$$= 27.41\mathbf{i} + 25.0\mathbf{j} \text{ mm}$$

Relative Velocity:

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/S} \qquad (1)$$

where:

$$\mathbf{v}_P = \cancel{\mathbf{v}_B} + \boldsymbol{\omega}_D \times \mathbf{r}_{P/B}$$
$$= -250.0\mathbf{i} - 433.01\mathbf{j}$$
$$\mathbf{v}_{P'} = \cancel{\mathbf{v}_O} + \boldsymbol{\omega}_S \times \mathbf{r}_{P'/O}$$
$$= -25.0\omega_S\mathbf{i} + 27.41\omega_S\mathbf{j}$$
$$\mathbf{v}_{P/S} = -v_{P/S}\cos\beta\mathbf{i} - v_{P/S}\sin\beta\mathbf{j}$$

Substitution into eqn. (1) and equating components

$$\mathbf{i}: \quad -250.0 = -25.0\omega_S - v_{P/S}\cos\beta$$

$$\mathbf{j}: \quad -433.01 = 27.41\omega_S - v_{P/S}\sin\beta$$

Solve simultaneously: $\omega_S = -4.0825$ rad/s, $v_{P/S} = 476.51$ mm/s

PROBLEM 15.163 (Continued)

Relative acceleration: $\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/S} + \mathbf{a}_C \qquad (2)$

where:

$$\mathbf{a}_P = \cancel{\mathbf{a}_B} - \omega_D^2 \times \mathbf{r}_{P/B} + \cancel{\alpha_D} \times \mathbf{r}_{P/\mathrm{B}}$$

$$= 4330.1\mathbf{i} - 2500.0\mathbf{j}$$

$$\mathbf{a}_{P'} = \cancel{\mathbf{a}_O} + -\omega_S^2 \times \mathbf{r}_{P'/\mathrm{O}} + \alpha_S \times \mathbf{r}_{P'/\mathrm{O}}$$

$$= (-456.82 - 25.0\alpha_S)\mathbf{i} + (-416.67 + 27.41\alpha_S)\mathbf{j}$$

$$\mathbf{a}_{P/\mathrm{S}} = -a_{P/\mathrm{S}}\cos\beta\mathbf{i} - a_{P/\mathrm{S}}\sin\beta\mathbf{j}$$

$$\mathbf{a}_C = 2\omega_S \times \mathbf{v}_{P/\mathrm{S}}$$

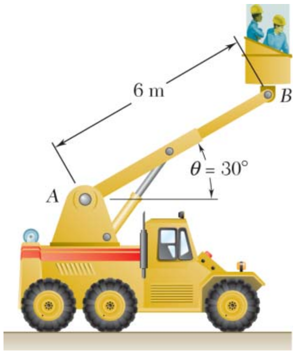
$$= -2621.89\mathbf{i} + 2874.6\mathbf{j}$$

Substitution into eqn. (2) and equating components

$$\mathbf{i}: \quad 4330.1 = -456.82 - 25\alpha_S - 0.7388a_{P/\mathrm{S}} - 2621.89$$

$$\mathbf{j}: \quad -2500.0 = -416.67 + 27.41\alpha_S - 0.6739a_{P/\mathrm{S}} + 2874.6$$

Solve simultaneously: $\alpha_S = -233.3 \text{ rad/s}^2$ $\qquad \alpha_S = 233 \text{ rad/s}^2 \circlearrowright$ ◀



PROBLEM 15.164

At the instant shown the length of the boom *AB* is being *decreased* at the constant rate of 0.2 m/s and the boom is being lowered at the constant rate of 0.08 rad/s. Determine (*a*) the velocity of Point *B*, (*b*) the acceleration of Point *B*.

SOLUTION

Velocity of coinciding Point B′ on boom.

$$\mathbf{v}_{B'} = r\omega = (6)(0.08) = 0.48 \text{ m/s} \searrow 60^\circ$$

Velocity of Point B relative to the boom.

$$\mathbf{v}_{B/\text{boom}} = 0.2 \text{ m/s} \swarrow 30^\circ$$

(*a*) *Velocity of Point B.*

$$\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/\text{boom}}$$

$$\xrightarrow{+}: \quad (v_B)_x = 0.48\cos 60^\circ - 0.2\cos 30^\circ = 0.06680 \text{ m/s}$$

$$+\uparrow: \quad (v_B)_y = -0.48\sin 60^\circ - 0.2\sin 30^\circ = -0.51569 \text{ m/s}$$

$$v_B = \sqrt{0.06680^2 + 0.51569^2}$$

$$= 0.520 \text{ m/s}$$

$$\tan\beta = \frac{0.51569}{0.06680}, \quad \beta = 82.6^\circ \qquad \mathbf{v}_B = 0.520 \text{ m/s} \searrow 82.6^\circ \blacktriangleleft$$

Acceleration of coinciding Point B′ on boom.

$$\mathbf{a}_{B'} = r\omega^2 = (6)(0.08)^2 = 0.0384 \text{ m/s}^2 \swarrow 30^\circ$$

Acceleration of B relative to the boom.

$$\mathbf{a}_{B/\text{boom}} = 0$$

Coriolis acceleration.

$$2\omega u = (2)(0.08)(0.2) = 0.032 \text{ m/s}^2 \nwarrow 60^\circ$$

PROBLEM 15.164 (Continued)

(*b*) *Acceleration of Point B.*

$$\mathbf{a}_B = \mathbf{a}_B + \mathbf{a}_{B/\text{boom}} + 2\omega u$$

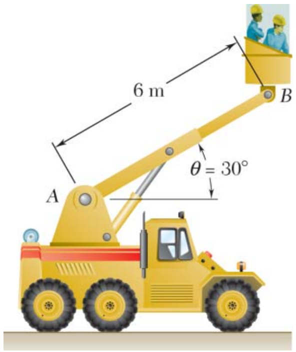
$$\xrightarrow{+}: \quad (a_B)_x = -0.0384\cos 30° + 0 - 0.032\cos 60° = -0.04926 \text{ m/s}^2$$

$$+\uparrow: \quad (a_B)_y = -0.0384\sin 30° + 0 + 0.032\sin 60° = 0.008513 \text{ m/s}^2$$

$$a_B = \sqrt{(0.04926)^2 + (0.008513)^2}$$

$$= 0.0500 \text{ m/s}^2$$

$$\tan\beta = \frac{0.008513}{0.04926}, \quad \beta = 9.8° \qquad \mathbf{a}_B = 50.0 \text{ mm/s}^2 \nwarrow 9.8° \blacktriangleleft$$



PROBLEM 15.165

At the instant shown the length of the boom *AB* is being *increased* at the constant rate of 0.2 m/s and the boom is being lowered at the constant rate of 0.08 rad/s. Determine (*a*) the velocity of Point *B*, (*b*) the acceleration of Point *B*.

SOLUTION

Velocity of coinciding Point B′ on boom.

$$\mathbf{v}_{B'} = r\omega = (6)(0.08) = 0.48 \text{ m/s} \searrow 60^\circ$$

Velocity of Point B relative to the boom.

$$\mathbf{v}_{B/\text{boom}} = 0.2 \text{ m/s} \nearrow 30^\circ$$

(*a*) *Velocity of Point B.*

$$\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/\text{boom}}$$

$$\xrightarrow{+}: \quad (v_B)_x = 0.48 \cos 60^\circ + 0.2 \cos 30^\circ = 0.4132 \text{ m/s}$$

$$+\uparrow: \quad (v_B)_y = -0.48 \sin 60^\circ + 0.2 \sin 30^\circ = -0.3157 \text{ m/s}$$

$$v_B = \sqrt{(0.4132)^2 + (0.3157)^2}$$
$$= 0.520 \text{ m/s}$$

$$\tan\beta = -\frac{0.3157}{0.4132}, \quad \beta = -37.4^\circ \qquad \mathbf{v}_B = 0.520 \text{ m/s} \searrow 37.4^\circ \blacktriangleleft$$

Acceleration of coinciding Point B′ on boom.

$$\mathbf{a}_{B'} = r\omega^2 = (6)(0.08)^2 = 0.0384 \text{ m/s}^2 \swarrow 30^\circ$$

Acceleration of B relative to the boom.

$$\mathbf{a}_{B/\text{boom}} = 0$$

Coriolis acceleration.

$$2\omega u = (2)(0.08)(2) = 0.032 \text{ m/s}^2 \searrow 60^\circ$$

PROBLEM 15.165 (Continued)

(*b*) *Acceleration of Point B.*

$$\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/\text{boom}} + 2\omega u$$

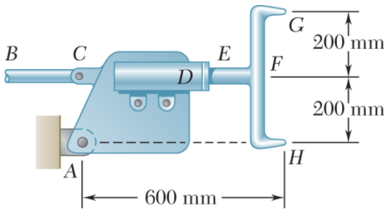
$$\xrightarrow{+}: \quad (a_B)_x = -0.0384 \cos 30° + 0.032 \cos 60° = -0.017255 \text{ m/s}^2$$

$$+\uparrow: \quad (a_B)_y = -0.0384 \sin 30° - 0.032 \sin 60° = -0.04691 \text{ m/s}^2$$

$$a_B = \sqrt{(0.017255)^2 + (0.04691)^2}$$

$$= 0.0500 \text{ m/s}^2$$

$$\tan\beta = \frac{0.04691}{0.017255}, \quad \beta = 69.8° \qquad \mathbf{a}_B = 50.0 \text{ mm/s}^2 \swarrow 69.8° \blacktriangleleft$$



PROBLEM 15.166

In the automated welding setup shown, the position of the two welding tips *G* and *H* is controlled by the hydraulic cylinder *D* and rod *BC*. The cylinder is bolted to the vertical plate which at the instant shown rotates counterclockwise about *A* with a constant angular velocity of 1.6 rad/s. Knowing that at the same instant the length *EF* of the welding assembly is increasing at the constant rate of 300 mm/s, determine (*a*) the velocity of tip *H*, (*b*) the acceleration of tip *H*.

SOLUTION

Body *ACD* rotates about point *A* with angular velocity

$$\boldsymbol{\omega}_A = 1.6 \text{ rad/s} \circlearrowleft = (1.6 \text{ rad/s})\mathbf{k}$$

If the rod *EF* were not moving relative to body *ACD*, the velocity of point H', currently at point *H*, would be

$$\mathbf{v}_{H'} = \boldsymbol{\omega}_A \times \mathbf{r}_{H/A} = 1.6\mathbf{k} \times 600\mathbf{i} = (0.96 \text{ mm/s})\mathbf{j}$$

The velocity of point *H* relative to H' is

$$\mathbf{v}_{H/H'} = 300 \text{ mm/s} \longrightarrow = (300 \text{ mm/s})\mathbf{i}$$

(*a*) *Velocity of tip H.*

$$\mathbf{v}_H = \mathbf{v}_{H'} + \mathbf{v}_{H/H'} = (960 \text{ mm/s})\mathbf{j} + (300 \text{ mm/s})\mathbf{i}$$

$$\mathbf{v}_{\text{H}} = 1006 \text{ mm/s} \measuredangle\ 72.6° \blacktriangleleft$$

Angular acceleration of body *ACD*: $\boldsymbol{\alpha}_A = 0$

$$\mathbf{a}_{H'} = \boldsymbol{\alpha}_A \times \mathbf{r}_{H/A} - \omega_A^2 \mathbf{r}_{H/A} = 0 - (1.6)^2(600\mathbf{i})$$

$$= -(1536 \text{ mm/s}^2)\mathbf{i}$$

Since the sliding motion of *H* relative to H' occurs at constant velocity,

$$\mathbf{a}_{H/H'} = 0$$

Coriolis acceleration. $\mathbf{a}_c = 2\boldsymbol{\omega}_A \times \mathbf{v}_{H/H'}$

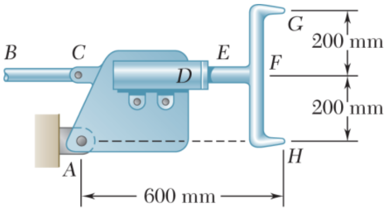
$$a_c = (2)(1.6\mathbf{k}) \times (300\mathbf{i}) = (960 \text{ mm/s}^2)\mathbf{j}$$

(*b*) *Acceleration of tip H.*

$$\mathbf{a}_H = \mathbf{a}_{H'} + \mathbf{a}_{H/H'} + \mathbf{a}_c$$

$$= -(1536 \text{ mm/s}^2)\mathbf{i} + 0 + (960 \text{ mm/s}^2)\mathbf{j}$$

$$\mathbf{a}_H = 1811 \text{ mm/s}^2 \measuredangle\ 32.0° \blacktriangleleft$$



PROBLEM 15.167

In the automated welding setup shown, the position of the two welding tips *G* and *H* is controlled by the hydraulic cylinder *D* and rod *BC*. The cylinder is bolted to the vertical plate which at the instant shown rotates counterclockwise about *A* with a constant angular velocity of 1.6 rad/s. Knowing that at the same instant the length *EF* of the welding assembly is increasing at the constant rate of 300 mm/s, determine (*a*) the velocity of tip *G*, (*b*) the acceleration of tip *G*.

SOLUTION

Body *ACD* rotates about point *A* with angular velocity

$$\boldsymbol{\omega}_A = 1.6 \text{ rad/s} \circlearrowleft = (1.6 \text{ rad/s})\mathbf{k}$$

If the rod *EF* were not moving relative to body *ACD*, the velocity of point G', currently at point *G*, would be

$$\mathbf{v}_G^1 = \boldsymbol{\omega}_A \times \mathbf{r}_{G/A} = 1.6\mathbf{k} \times (600\mathbf{i} + 400\mathbf{j})$$

$$= -(640 \text{ mm/s})\mathbf{i} + (960 \text{ mm/s})\mathbf{j}$$

The velocity of point *G* relative to G' is

$$\mathbf{v}_{G/G'} = 300 \text{ mm/s} \longrightarrow = (300 \text{ mm/s})\mathbf{i}$$

(*a*) *Velocity of tip G.*

$$\mathbf{v}_G = \mathbf{v}_{G'} + \mathbf{v}_{G/G'} = -640\mathbf{i} + 960\mathbf{j} + 300\mathbf{i}$$

$$= -(340 \text{ mm/s})\mathbf{i} + (960 \text{ mm/s})\mathbf{j}$$

$$\mathbf{v}_G = 1018 \text{ mm/s} \measuredangle 70.5° \blacktriangleleft$$

Angular acceleration of body *ACD* is $\boldsymbol{\alpha}_A = 0$.

$$\mathbf{a}_{G'} = \boldsymbol{\alpha}_A \times \mathbf{r}_{G/A} - \omega_A^2 \mathbf{r}_{G/A}$$

$$= 0 - (1.6)^2 (600\mathbf{i} + 400\mathbf{j})$$

$$= -(1536 \text{ mm/s}^2)\mathbf{i} - (1024 \text{ mm/s}^2)\mathbf{j}$$

Since the sliding motion of *G* relative to G' occurs at constant velocity,

$$\mathbf{a}_{G/G'} = 0$$

Coriolis acceleration. $\mathbf{a}_c = 2\boldsymbol{\omega}_A \times \mathbf{v}_{G/G'}$

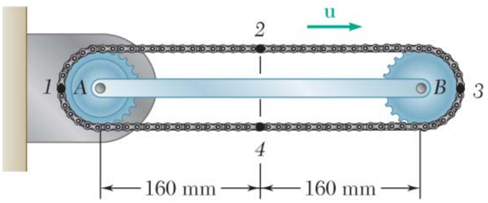
$$a_c = (2)(1.6\mathbf{k}) \times (300\mathbf{i}) = (960 \text{ mm/s}^2)\mathbf{j}$$

PROBLEM 15.167 (Continued)

(*b*) *Acceleration of tip G.*

$$\begin{aligned}\mathbf{a}_G &= \mathbf{a}_{G'} + \mathbf{a}_{G/G'} + \mathbf{a}_c \\ &= -1536\mathbf{i} - 1024\mathbf{j} + 0 + 960\mathbf{j} \\ &= -\left(1536\ \text{mm/s}^2\right)\mathbf{i} - \left(64\ \text{mm/s}^2\right)\mathbf{j}\end{aligned}$$

$\mathbf{a}_G = 1537\ \text{mm/s}^2 \swarrow 2.4°$ ◀



PROBLEM 15.168

A chain is looped around two gears of radius 40 mm that can rotate freely with respect to the 320-mm arm *AB*. The chain moves about arm *AB* in a clockwise direction at the constant rate of 80 mm/s relative to the arm. Knowing that in the position shown arm *AB* rotates clockwise about *A* at the constant rate $\omega = 0.75$ rad/s, determine the acceleration of each of the chain links indicated.

Links 1 and 2.

SOLUTION

Let the arm *AB* be a rotating frame of reference. $\Omega = 0.75 \text{ rad/s} \circlearrowright = -(0.75 \text{ rad/s})\mathbf{k}$:

Link 1:

$$\mathbf{r}_1 = -(40 \text{ mm})\mathbf{i}, \quad \mathbf{v}_{1/AB} = u \uparrow = (80 \text{ mm/s})\mathbf{j}$$

$$\mathbf{a}_1' = -\Omega^2 \mathbf{r}_1 = -(0.75)^2(-40) = (22.5 \text{ mm/s})\mathbf{i}$$

$$\mathbf{a}_{1/AB} = \frac{u^2}{\rho} = \frac{80^2}{40} 160 \text{ mm/s} \rightarrow = (160 \text{ mm/s}^2)\mathbf{i}$$

$$2\Omega \times \mathbf{v}_{P/AB} = (2)(-0.75\mathbf{k}) \times (80\mathbf{j})$$

$$= (120 \text{ mm/s})\mathbf{i}$$

$$\mathbf{a}_1 = \mathbf{a}_1' + \mathbf{a}_{1/AB} + 2\Omega \times \mathbf{v}_{1/AB}$$

$$= (302.5 \text{ mm/s}^2)\mathbf{i}$$

$\mathbf{a}_1 = 303 \text{ mm/s}^2 \longrightarrow$ ◀

Link 2:

$$\mathbf{r}_2 = (160 \text{ mm})\mathbf{i} + (40 \text{ mm})\mathbf{j}$$

$$\mathbf{v}_{2/AB} = u \rightarrow = (80 \text{ mm/s})\mathbf{i}$$

$$\mathbf{a}_2' = -\Omega^2 \mathbf{r}_2$$

$$= -(0.75)^2(160\mathbf{i} + 40\mathbf{j})$$

$$= -(90 \text{ mm/s}^2)\mathbf{i} - (22.5 \text{ mm/s}^2)\mathbf{j}$$

$$\mathbf{a}_{2/AB} = 0$$

$$2\Omega \times \mathbf{v}_{2/AB} = (2)(-0.75\mathbf{k}) \times (80\mathbf{i})$$

$$= -(120 \text{ mm/s}^2)\mathbf{j}$$

$$\mathbf{a}^2 = \mathbf{a}_2' + \mathbf{a}_{2/AB} + 2\Omega \times \mathbf{v}_{2/AB}$$

$$= -90\mathbf{i} - 22.5\mathbf{j} - 120\mathbf{j}$$

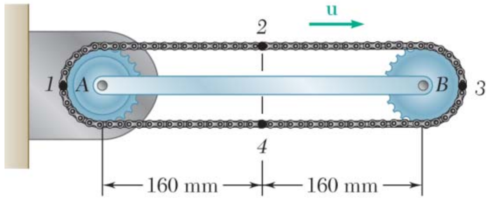
$$= -(90 \text{ mm/s}^2)\mathbf{i} - (142.5 \text{ mm/s}^2)\mathbf{j}$$

$$\mathbf{a}_2 = \sqrt{(90)^2 + (142.5)^2}$$

$$= 168.5 \text{ mm/s}^2$$

$$\tan\beta = \frac{142.5}{90}, \quad \beta = 57.7°$$

$\mathbf{a}_2 = 168.5 \text{ mm/s}^2 \swarrow 57.7°$ ◀



PROBLEM 15.169

A chain is looped around two gears of radius 40 mm that can rotate freely with respect to the 320-mm arm *AB*. The chain moves about arm *AB* in a clockwise direction at the constant rate 80 mm/s relative to the arm. Knowing that in the position shown arm *AB* rotates clockwise about *A* at the constant rate $\omega = 0.75$ rad/s, determine the acceleration of each of the chain links indicated.

Links 3 and 4.

SOLUTION

Let arm *AB* be a rotating frame of reference. $\quad \mathbf{\Omega} = 0.75 \text{ rad/s} \circlearrowright = -(0.75 \text{ rad/s})\mathbf{k}$

Link 3:

$$\mathbf{r}_3 = (360 \text{ mm})\mathbf{i} \quad \mathbf{v}_{3/AB} = u \downarrow = -(80 \text{ mm/s})\mathbf{j}$$

$$\mathbf{a}_{3'} = -\Omega^2 \mathbf{r}_3 = -(0.75)^2(360) = -(202.5 \text{ mm/s}^2)\mathbf{i}$$

$$\mathbf{a}_{3/AB} = \frac{u^2}{\rho} = \frac{(80)^2}{40} = 160 \text{ mm/s}^2\mathbf{i} \leftarrow = -(160 \text{ mm/s}^2)\mathbf{i}$$

$$2\mathbf{\Omega} \times \mathbf{v}_{3/AB} = (2)(-0.75\mathbf{k}) \times (-80\mathbf{j}) = -(120 \text{ mm/s}^2)\mathbf{i}$$

$$\mathbf{a}_3 = \mathbf{a}_{3'} + \mathbf{a}_{3/AB} + 2\mathbf{\Omega} \times \mathbf{v}_{3/AB}$$

$$= -(482.5 \text{ mm/s}^2)\mathbf{i}$$

$\mathbf{a}_3 = 483 \text{ mm/s}^2 \longleftarrow$ ◀

Link 4:

$$\mathbf{r}_4 = (160 \text{ mm})\mathbf{i} - (40 \text{ mm})\mathbf{j}$$

$$\mathbf{v}_{4/AB} = u \leftarrow = -(80 \text{ mm/s}^2)\mathbf{i}$$

$$\mathbf{a}_{4'} = -\Omega^2 \mathbf{r}_4$$

$$= -(0.75)^2(160\mathbf{i} - 40\mathbf{j})$$

$$= -(90 \text{ mm/s}^2)\mathbf{i} + (22.5 \text{ mm/s}^2)\mathbf{j}$$

$$\mathbf{a}_{4/AB} = 0$$

$$2\mathbf{\Omega} \times \mathbf{v}_{4/AB} = (2)(-0.75\mathbf{k}) \times (-80\mathbf{i})$$

$$= (120 \text{ mm/s}^2)\mathbf{j}$$

$$\mathbf{a}_4 = \mathbf{a}_{4'} + \mathbf{a}_{4/AB} + 2\mathbf{\Omega} \times \mathbf{v}_{4/AB}$$

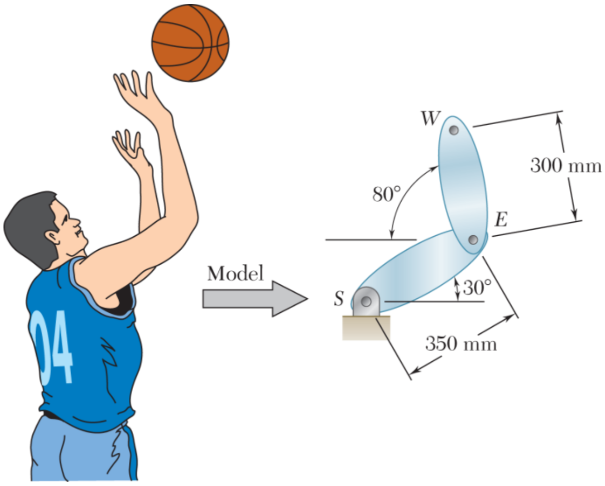
$$= -(90 \text{ mm/s}^2)\mathbf{i} + (142.5 \text{ mm/s}^2)\mathbf{j}$$

$$\mathbf{a}_4 = \sqrt{(90)^2 + (142.5)^2}$$

$$= 168.5 \text{ mm/s}^2$$

$$\tan\beta = \frac{142.5}{90}, \quad \beta = 57.7°$$

$\mathbf{a}_4 = 168.5 \text{ mm/s}^2 \nwarrow 57.7°$ ◀



PROBLEM 15.170

A basketball player shoots a free throw in such a way that his shoulder can be considered a pin joint at the moment of release as shown. Knowing that at the instant shown the upper arm *SE* has a constant angular velocity of 2 rad/s counterclockwise and the forearm *EW* has a constant clockwise angular velocity of 4 rad/s with respect to *SE*, determine the velocity and acceleration of the wrist *W*.

SOLUTION

Units: meters, m/s, m/s^2

Unit vectors: $\mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1 \uparrow, \quad \mathbf{k} = 1 \circlearrowleft$

Relative positions:

$$\mathbf{r}_{E/S} = (0.35\cos 30°)\mathbf{i} + (0.35\sin 30°)\mathbf{j} = 0.30311\mathbf{i} + 0.175\mathbf{j}$$

$$\mathbf{r}_{W/E} = -(0.3\cos 80°)\mathbf{i} + (0.3\sin 80°)\mathbf{j} = -0.05209\mathbf{i} + 0.29544\mathbf{j}$$

$$\mathbf{r}_{W/S} = \mathbf{r}_{E/S} + \mathbf{r}_{W/E} = 0.25101\mathbf{i} + 0.47044\mathbf{j}$$

Use a frame of reference rotating with the upper arm *SE* with angular velocity

$$\mathbf{\Omega} = (2 \text{ rad/s})\mathbf{k} \quad (\dot{\mathbf{\Omega}} = 0)$$

The motion of the wrist *W* relative to this frame is a rotation about the elbow *E* with angular velocity

$$\boldsymbol{\omega} = -(4 \text{ rad/s})\mathbf{k} \quad (\dot{\boldsymbol{\omega}} = 0)$$

Motion of Point *W′* in the frame coinciding with *W*.

$$\mathbf{v}_{W'} = \mathbf{\Omega} \times \mathbf{r}_{W/S} = (2\mathbf{k}) \times (0.25101\mathbf{i} + 0.47044\mathbf{j})$$
$$= -0.94088\mathbf{i} + 0.50204\mathbf{j}$$

$$\mathbf{a}_{W'} = -\Omega^2\mathbf{r}_{W/S} = -(2)^2(0.25101\mathbf{i} + 0.47044\mathbf{j})$$
$$= -1.00408\mathbf{i} - 1.88176\mathbf{j}$$

Motion of *W* relative to the frame.

$$\mathbf{v}_{W/SE} = \boldsymbol{\omega} \times \mathbf{r}_{W/E} = (-4\mathbf{k}) \times (-0.05210\mathbf{i} + 0.29544\mathbf{j})$$
$$= 1.18176\mathbf{i} + 0.2084\mathbf{j}$$

$$\mathbf{a}_{W/SE} = -\omega^2\mathbf{r}_{W/E} = -(4)^2(-0.05210\mathbf{i} + 0.29544\mathbf{j})$$
$$= 0.8336\mathbf{i} - 4.72708\mathbf{j}$$

Velocity of *W*: $\mathbf{v}_W = \mathbf{v}_{W'} + \mathbf{v}_{W/SE} = 0.24088\mathbf{i} + 0.71044\mathbf{j}$

$$\mathbf{v}_W = 0.750 \text{ m/s} \measuredangle 71.3° \blacktriangleleft$$

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PROBLEM 15.170 (Continued)

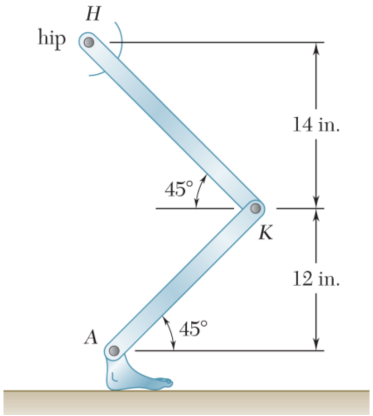
Coriolis acceleration:

$$\begin{aligned} 2\boldsymbol{\Omega} \times \mathbf{v}_{W/SE} &= (2)(2\mathbf{k}) \times (1.18176\mathbf{i} + 0.2084\mathbf{j}) \\ &= -0.8336\mathbf{i} + 4.72704\mathbf{j} \end{aligned}$$

Acceleration of *W*:

$$\begin{aligned} \mathbf{a}_W &= \mathbf{a}_{W'} + \mathbf{a}_{W/SE} + 2\boldsymbol{\Omega} \times \mathbf{v}_{W/SE} \\ &= -1.00408\mathbf{i} - 1.88176\mathbf{j} \end{aligned}$$

$\mathbf{a}_W = 2.13 \text{ m/s}^2$ ↙ 61.9° ◀



PROBLEM 15.171

The human leg can be crudely approximated as two rigid bars (the femur and the tibia) connected with a pin joint. At the instant shown the veolcity and acceleration of the ankle is zero. During a jump, the velocity of the ankle *A* is zero, the tibia *AK* has an angular velocity of 1.5 rad/s counterclockwise and an angular acceleration of 1 rad/s^2 counterclockwise. Determine the relative angular velocity and angular acceleration of the femur *KH* with respect to *AK* so that the velocity and acceleration of *H* are both straight up at the instant shown.

SOLUTION

Units: inches, in./s, in./s^2

Unit vectors: $\mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1 \uparrow, \quad \mathbf{k} = 1 \circlearrowleft$

Relative positions:

$$\mathbf{r}_{K/A} = (12\text{ in.})\mathbf{i} + (12\text{ in.})\mathbf{j}$$

$$\mathbf{r}_{H/K} = -(14\text{ in.})\mathbf{i} + (14\text{ in.})\mathbf{j}$$

$$\mathbf{r}_{H/A} = \mathbf{r}_{K/A} + \mathbf{r}_{H/K} = -(2\text{ in.})\mathbf{i} + (26\text{ in.})\mathbf{j}$$

Use a frame of reference moving with the lower leg *AK* with angular velocity

$$\boldsymbol{\Omega} = 1.5\text{ rad/s} \circlearrowleft = (1.5\text{ rad/s})\mathbf{k}$$

and angular acceleration $\dot{\boldsymbol{\Omega}} = 1.0\text{ rad/s} \circlearrowleft = (1.0\text{ rad/s}^2)\mathbf{k}$

The motion of the hip *H* relative to this frame is a rotation about the knee *K* with angular velocity

$$\boldsymbol{\omega} = \omega\mathbf{k}$$

and angular acceleration $\boldsymbol{\alpha} = \alpha\mathbf{k}$

Both $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are measured relative to the lower leg *AK*.

Motion of Point H' in the frame coinciding with *H*.

$$\mathbf{v}_{H'} = \boldsymbol{\Omega} \times \mathbf{r}_{H/A} = 1.5\mathbf{k} \times (-2\mathbf{i} + 26\mathbf{j})$$

$$= -(39\text{ in./s})\mathbf{i} - (3\text{ in./s})\mathbf{j}$$

$$\mathbf{a}_{H'} = \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{H/A} - \Omega^2\mathbf{r}_{H/A}$$

$$= (1.0\mathbf{k}) \times (-2\mathbf{i} + 26\mathbf{j}) - (1.5)^2(-2\mathbf{i} + 26\mathbf{j})$$

$$= -26\mathbf{i} - 2\mathbf{j} + 4.5\mathbf{i} - 58.5\mathbf{j}$$

$$= -(21.5\text{ in./s}^2)\mathbf{i} - (60.5\text{ in./s}^2)\mathbf{j}$$

PROBLEM 15.171 (Continued)

Motion of H relative to the frame.

$$\mathbf{v}_{H/AK} = \omega\mathbf{k}\times\mathbf{r}_{H/K} = \omega\mathbf{k}\times(-14\mathbf{i}+14\mathbf{j})$$
$$= -14\omega\mathbf{i} - 14\omega\mathbf{j}$$
$$\mathbf{a}_{H/AK} = \alpha\mathbf{k}\times\mathbf{r}_{H/K} - \omega^2\mathbf{r}_{H/K}$$
$$= \alpha\mathbf{k}\times(-14\mathbf{i}+14\mathbf{j}) - \omega^2(-14\mathbf{i}+14\mathbf{j})$$
$$= -14\alpha\,\mathbf{i} - 14\alpha\,\mathbf{j} + 14\omega^2\mathbf{i} - 14\omega^2\mathbf{j}$$

Velocity of H. $\quad \mathbf{v}_H = \mathbf{v}_H \uparrow = \mathbf{v}_H\mathbf{j}$

$$\mathbf{v}_H = \mathbf{v}_{H'} + \mathbf{v}_{H/AK}$$
$$v_H\mathbf{j} = -39\mathbf{i} - 3\mathbf{j} - 14\omega\mathbf{i} - 14\omega\mathbf{j}$$

Resolve into components.

i: $\quad 0 = -39 - 14\omega \qquad \omega = -\frac{39}{14} = -2.7857 \text{ rad/s}$

j: $\quad v_H = -3 - 14\omega \qquad v_H = -36 \text{ in./s}$

Relative angular velocity: $\quad \boldsymbol{\omega} = -(2.79 \text{ rad/s})\mathbf{k} = 2.79 \text{ rad/s} \circlearrowright$ ◀

Coriolis acceleration: $\quad 2\boldsymbol{\Omega}\times\mathbf{v}_{H/AK} = (2)(1.5\mathbf{k})\times(-14\omega\mathbf{i} - 14\omega\mathbf{j})$

$$= 42\omega\mathbf{i} - 42\omega\mathbf{j} = -(117 \text{ in./s}^2)\mathbf{i} + (117 \text{ in./s}^2)\mathbf{j}$$

Acceleration of H. $\quad \mathbf{a}_H = a_H \uparrow = a_H\mathbf{j}$

$$\mathbf{a}_H = \mathbf{a}_{H'} + \mathbf{a}_{H/AK} + 2\boldsymbol{\Omega}\times\mathbf{v}_{H/AK}$$
$$a_H\mathbf{j} = -21.5\mathbf{i} - 60.5\mathbf{j} - 14\alpha\mathbf{i} - 14\alpha\,\mathbf{j} + 14\omega^2\mathbf{i} - 14\omega^2\mathbf{j} - 117\mathbf{i} + 117\mathbf{j}$$

Resolve into components.

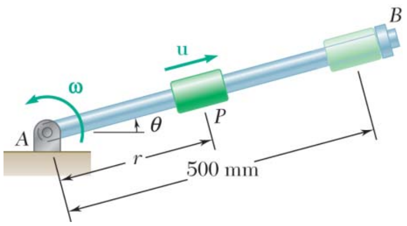
i: $\quad 0 = -21.5 - 14\alpha + (14)(-2.7857)^2 - 117$

$$\alpha = -2.1327 \text{ rad/s}^2$$

j: $\quad a_H = -60.5 - (14)(-2.1327) - (14)(-2.7857)^2 + 117$

$$= -22.284 \text{ in./s}^2$$

Relative angular acceleration: $\quad \boldsymbol{\alpha} = -(2.13 \text{ rad/s}^2)\mathbf{k} = 2.13 \text{ rad/s}^2 \circlearrowright$ ◀



PROBLEM 15.172

The collar P slides outward at a constant relative speed u along rod AB, which rotates counterclockwise with a constant angular velocity of 20 rpm. Knowing that $r = 250$ mm when $\theta = 0$ and that the collar reaches B when $\theta = 90°$, determine the magnitude of the acceleration of the collar P just as it reaches B.

SOLUTION

$$\omega = 20 \text{ rpm} = \frac{(20)(2\pi)}{60} = \frac{2\pi}{3} \text{ rad/s}$$

$$\alpha = 0$$

$$\theta = 90° = \frac{\pi}{2} \text{ radians}$$

Uniform rotational motion.

$$\theta = \theta_0 + \omega t$$

$$t = \frac{\theta - \theta_0}{\omega} = \frac{\frac{\pi}{2}}{\frac{2\pi}{3}} = 0.75 \text{ s}$$

Uniform motion along rod.

$$r = r_0 + ut$$

$$u = \frac{r - r_0}{t} = \frac{0.5 - 0.25}{0.75} = \frac{1}{3} \text{ m/s},$$

$$\mathbf{v}_{P/AB} = \frac{1}{3} \text{ m/s} \uparrow$$

Acceleration of coinciding Point P' on the rod. $(r = 0.5 \text{ m})$

$$\mathbf{a}_{P'} = r\omega^2$$

$$= (0.5)\left(\frac{2\pi}{3}\right)^2$$

$$= \frac{2\pi^2}{9} \text{ m/s}^2 \downarrow$$

$$= 2.1932 \text{ m/s}^2 \downarrow$$

Acceleration of collar P relative to the rod. $\mathbf{a}_{P/AB} = 0$

Coriolis acceleration. $2\boldsymbol{\omega} \times \mathbf{v}_{P/AB} = 2\omega u = (2)\left(\frac{2\pi}{3}\right)\left(\frac{1}{3}\right) = 1.3963 \text{ m/s}^2 \leftarrow$

Acceleration of collar P.

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/AB} + 2\boldsymbol{\omega} \times \mathbf{v}_{P/AB}$$

$$\mathbf{a}_P = [2.1932 \text{ m/s}^2 \downarrow] + [1.3963 \text{ m/s}^2 \leftarrow]$$

$$\mathbf{a}_P = 2.60 \text{ m/s}^2 \swarrow 57.5°$$

$a_P = 2.60 \text{ m/s}^2$ ◀

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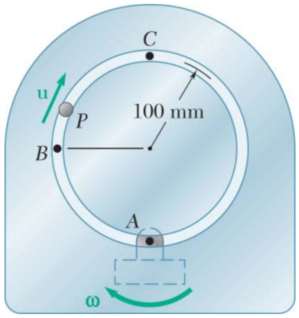


PROBLEM 15.173

Pin P slides in a circular slot cut in the plate shown at a constant relative speed $u = 90$ mm/s. Knowing that at the instant shown the plate rotates clockwise about A at the constant rate $\omega = 3$ rad/s, determine the acceleration of the pin if it is located at (*a*) Point A, (*b*) Point B, (*c*) Point C.

SOLUTION

$$\omega = 3 \text{ rad/s} \circlearrowright, \quad \alpha = 0, \quad u = 90 \text{ mm/s} = 0.09 \text{ m/s}, \quad \dot{u} = 0$$

$$\rho = 100 \text{ mm}$$

$$\frac{u^2}{\rho} = \frac{(90)^2}{100} = 81 \text{ mm/s}^2 = 0.081 \text{ m/s}^2$$

$$\omega^2 = 36 \text{ rad}^2/\text{s}^2$$

$$2\omega u = (2)(3)(90) = 540 \text{ mm/s}^2 = 0.54 \text{ m/s}^2$$

(*a*) *Point A.*

$$\mathbf{r}_A = 0, \qquad \mathbf{v}_{A/F} = 0.09 \text{ m/s} \leftarrow$$

$$\mathbf{a}_{A'} = 0, \qquad \mathbf{a}_{A/F} = \frac{u^2}{\rho}\uparrow = 0.081 \text{ m/s}^2 \uparrow$$

Coriolis acceleration. $2\omega u \uparrow = 0.54 \text{ m/s}^2 \uparrow$

$$\mathbf{a}_A = \mathbf{a}_{A'} + \mathbf{a}_{A/F} + [2\omega u \uparrow] = 0.621 \text{ m/s}^2 \uparrow$$

$\mathbf{a}_A = 0.621 \text{ m/s}^2 \uparrow$ ◀

(*b*) *Point B.*

$$\mathbf{r}_B = 0.1\sqrt{2} \text{ m} \nwarrow 45^\circ, \qquad \mathbf{v}_{B/F} = 0.09 \text{ m/s} \uparrow$$

$$\mathbf{a}_{B'} = -\omega^2 \mathbf{r}_B = -(9)(0.1\sqrt{2}) \nwarrow 45^\circ = 0.9\sqrt{2} \text{ m/s}^2 \searrow 45^\circ$$

$$\mathbf{a}_{B/F} = \frac{u^2}{\rho}$$

$$= 0.081 \text{ m/s}^2 \rightarrow$$

Coriolis acceleration. $2\omega u = 0.54 \text{ m/s}^2 \rightarrow$

$$\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/F} + [2\omega u \rightarrow]$$

$$= [1.521 \text{ m/s}^2 \rightarrow] + [0.9 \text{ m/s}^2 \downarrow]$$

$\mathbf{a}_B = 1.767 \text{ m/s}^2 \searrow 30.6^\circ$ ◀

PROBLEM 15.173 (Continued)

(*c*) *Point C.*

$$\mathbf{r}_C = 0.2 \text{ m}\uparrow$$

$$\mathbf{v}_{C/F} = 0.09 \text{ m/s} \rightarrow$$

$$\mathbf{a}_{C'} = -\omega^2 \mathbf{r}_C = -(9)(0.2\uparrow) = 1.8 \text{ m/s}^2 \downarrow$$

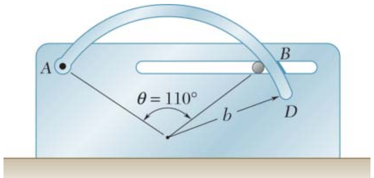
$$\mathbf{a}_{C/F} = \frac{u^2}{\rho} = 0.081 \text{ m/s}^2 \downarrow$$

Coriolis acceleration. $2\omega u = 0.54 \text{ m/s}^2 \downarrow$

$$\mathbf{a}_C = \mathbf{a}_{C'} + \mathbf{a}_{C/F} + [2\omega u \downarrow]$$

$$= 2.421 \text{ m/s}^2 \downarrow$$

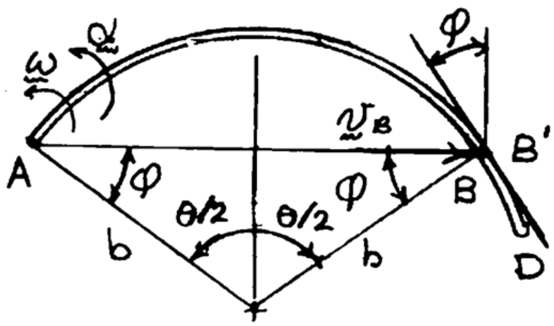
$\mathbf{a}_C = 2.42 \text{ m/s}^2 \downarrow$ ◀



PROBLEM 15.174

Rod *AD* is bent in the shape of an arc of a circle of radius b = 150 mm. The position of the rod is controlled by pin *B* which slides in a horizontal slot and also slides along the rod. Knowing that at the instant shown pin *B* moves to the right at a constant speed of 75 mm/s, determine (*a*) the angular velocity of the rod, (*b*) the angular acceleration of the rod.

SOLUTION



Given:

$$\mathbf{v}_B = 75\text{ mm/s} \longrightarrow$$

$$\mathbf{a}_B = 0$$

$$b = 150\text{ mm}$$

Definitions:

$$\boldsymbol{\omega} = \omega \circlearrowleft, \quad \boldsymbol{\alpha} = \alpha \circlearrowleft$$

$$\phi = 90^\circ - \frac{\theta}{2}$$

$$AB = 2b\cos\phi$$

Point B' is point on rod *AD* that currently coincides with pin *B*.

Velocity analysis $\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/AD}$

$$\mathbf{v}_B = v_B \longrightarrow$$

$$\mathbf{v}_{B'} = (AB)\omega = (2b\cos\phi)\omega \uparrow$$

$$\mathbf{v}_{B/AD} = \mathbf{v}_{B/AD} \searrow \phi$$

$$[v_B \longrightarrow] = [(2b\cos\phi)\omega \uparrow] + [v_{B/AD} \searrow \phi]$$

Components.

$$\nearrow \phi: \quad v_B\cos\phi = \left[(2b\cos\phi)\omega\right]\sin\phi + 0$$

$$\boldsymbol{\omega} = \frac{v_B}{2b\sin\phi} \circlearrowleft \tag{1}$$

$$\uparrow: \quad 0 = (2b\cos\phi)\omega - v_{B/AD}\cos\phi$$

$$\mathbf{v}_{B/AD} = 2b\omega = \frac{v_B}{\sin\phi} \searrow \phi \tag{2}$$

PROBLEM 15.174 (Continued)

Acceleration analysis. $\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/AD} + \mathbf{a}_c$

$$\mathbf{a}_B = 0$$

$$\mathbf{a}_{B'} = [(AB)\alpha \uparrow] + (AB)[\omega^2 \longleftarrow] = [(2b\cos\phi)\alpha \uparrow] + [(2b\cos\phi)\omega^2 \longleftarrow]$$

$$\mathbf{a}_{B/AD} = [(a_{B/AD})_t \searrow \phi] + [(a_{B/AD})_n \swarrow \phi] = [(a_{B/AD})_t \searrow \phi] + \left[\frac{v_{B/AD}^2}{\rho} \swarrow \phi\right]$$

$$= [(a_{B/AD})_t \searrow \phi] + \left[\frac{v_B^2}{b\sin^2\phi} \swarrow \quad\right]$$

$$\mathbf{a}_c = [2\omega v_{B/AD} \measuredangle \phi] = \left[\frac{v_B^2}{b\sin^2\phi} \measuredangle \phi\right]$$

Components.

$$\measuredangle \phi:\quad 0 = (2b\cos\phi)\alpha\sin\phi - (2b\cos\phi)\omega^2\cos\phi + 0 - \frac{v_B^2}{b\sin^2\phi} + \frac{v_B^2}{b\sin^2\phi}$$

$$\alpha = \omega^2\frac{\cos\phi}{\sin\phi} \qquad (3)$$

Data: $v_B = 75$ mm/s, $b = 150$ mm, $\theta = 110°$

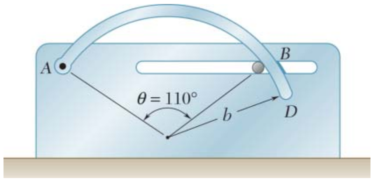
$$\phi = 90° - 55° = 35°$$

(*a*) From (1), $\omega = \dfrac{75}{(2)(150)\sin 35°} = 0.43586$ rad/s

$$\boldsymbol{\omega} = 0.436 \text{ rad/s} \circlearrowleft \blacktriangleleft$$

(*b*) From (3), $\alpha = (0.43586)^2\dfrac{\cos 35°}{\sin 35°} = 0.271$ rad/s^2

$$\boldsymbol{\alpha} = 0.271 \text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$

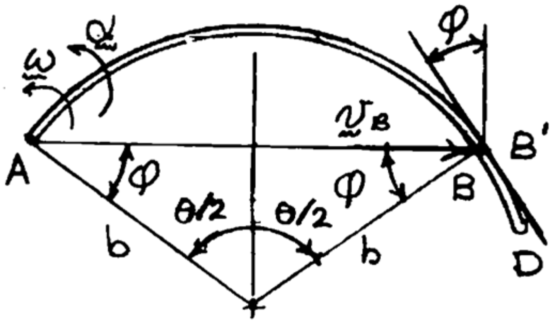


PROBLEM 15.175

Solve Prob. 15.174 when $\theta = 90°$.

Problem 15.174: Rod *AD* is bent in the shape of an arc of a circle of radius $b = 150$ mm. The position of the rod is controlled by pin *B* which slides in a horizontal slot and also slides along the rod. Knowing that at the instant shown pin *B* moves to the right at a constant speed of 75 mm/s, determine (*a*) the angular velocity of the rod, (*b*) the angular acceleration of the rod.

SOLUTION



Given:

$\mathbf{v}_B = 75 \text{ mm/s} \longrightarrow$

$\mathbf{a}_B = 0$

$b = 150 \text{ mm}$

Definitions:

$\boldsymbol{\omega} = \omega \circlearrowleft, \quad \boldsymbol{\alpha} = \alpha \circlearrowleft$

$$\phi = 90° - \frac{\theta}{2}$$

$AB = 2b\cos\phi$

Point B' is point on rod *AD* that currently coincides with pin *B*.

Velocity analysis $\quad \mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/AD}$

$\mathbf{v}_B = v_B \longrightarrow$

$\mathbf{v}_{B'} = (AB)\omega = (2b\cos\phi)\omega \uparrow$

$\mathbf{v}_{B/AD} = \mathbf{v}_{B/AD} \searrow \phi$

$[v_B \longrightarrow] = [(2b\cos\phi)\omega \uparrow] + [v_{B/AD} \searrow \phi]$

Components.

$\measuredangle \phi: \quad v_B\cos\phi = \left[(2b\cos\phi)\omega\right]\sin\phi + 0$

$$\boldsymbol{\omega} = \frac{v_B}{2b\sin\phi} \circlearrowleft \qquad (1)$$

$\uparrow: \quad 0 = (2b\cos\phi)\omega - v_{B/AD}\cos\phi$

$$\mathbf{v}_{B/AD} = 2b\omega = \frac{v_B}{\sin\phi} \searrow \phi \qquad (2)$$

PROBLEM 15.175 (Continued)

Acceleration analysis. $\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/AD} + \mathbf{a}_c$

$$\mathbf{a}_B = 0$$

$$\mathbf{a}_{B'} = [(AB)\alpha \uparrow] + (AB)[\omega^2 \leftarrow] = [(2b\cos\phi)\alpha \uparrow] + [(2b\cos\phi)\omega^2 \leftarrow]$$

$$\mathbf{a}_{B/AD} = [(a_{B/AD})_t \searrow \phi] + [(a_{B/AD})_n \swarrow \phi] = [(a_{B/AD})_t \searrow \phi] + \left[\frac{v_{B/AD}^2}{\rho} \swarrow \phi\right]$$

$$= [(a_{B/AD})_t \searrow \phi] + \left[\frac{v_B^2}{b\sin^2\phi} \swarrow \right]$$

$$\mathbf{a}_c = [2\omega v_{B/AD} \nearrow \phi] = \left[\frac{v_B^2}{b\sin^2\phi} \nearrow \phi\right]$$

Components.

$$\nearrow \phi: \quad 0 = (2b\cos\phi)\alpha\sin\phi - (2b\cos\phi)\omega^2\cos\phi + 0 - \frac{v_B^2}{b\sin^2\phi} + \frac{v_B^2}{b\sin^2\phi}$$

$$\alpha = \omega^2 \frac{\cos\phi}{\sin\phi} \qquad (3)$$

Data: $v_B = 75$ mm/s, $b = 150$ mm, $\theta = 90°$

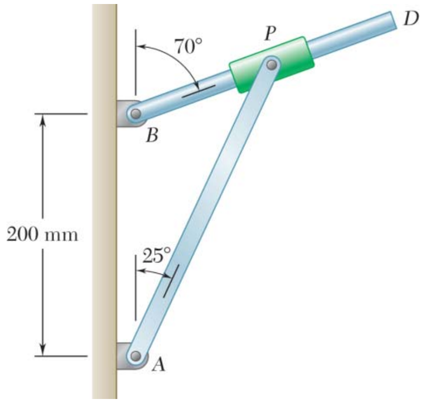
$$\phi = 90° - 45° = 45°$$

(*a*) From (1), $\omega = \dfrac{75}{(2)(150)\sin 45°} = 0.35355$ rad/s

$$\boldsymbol{\omega} = 0.354 \text{ rad/s} \circlearrowleft \blacktriangleleft$$

(*b*) From (3), $\alpha = (0.35355)^2 \dfrac{\cos 45°}{\sin 45°} = 0.1250$ rad/s^2

$$\boldsymbol{\alpha} = 0.1250 \text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$



PROBLEM 15.176

Knowing that at the instant shown the rod attached at *A* has an angular velocity of 5 rad/s counterclockwise and an angular acceleration of 2 rad/s^2 clockwise, determine the angular velocity and the angular acceleration of the rod attached at *B*.

SOLUTION

Geometry: Apply the law of sines to the triangle *ABP* to determine the lengths $\overline{AP}$ and $\overline{BP}$.

$$\text{Angle } PBA = 180° - 70° = 110° \qquad \text{Angle } APB = 180° - 25° - 110° = 45°$$

$$\overline{AB} = 200 \text{ mm} = 0.2 \text{ m} \qquad \frac{0.2}{\sin 45°} = \frac{\overline{AP}}{\sin 110°} = \frac{\overline{BP}}{\sin 25°}$$

$$\overline{AP} = 0.265785 \text{ m} \qquad \overline{BP} = 0.119534$$

Unit vectors: $\mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1 \uparrow, \quad \mathbf{k} = 1 \circlearrowleft$

Relative position vectors:

$$\mathbf{r}_{P/A} = 0.265785(\sin 25°\mathbf{i} + \cos 25°\mathbf{j}) = 0.11233\mathbf{i} - 0.24088\mathbf{j}$$
$$\mathbf{r}_{P/B} = 0.119534(\sin 70°\mathbf{i} + \cos 70°\mathbf{j}) = 0.11233\mathbf{i} + 0.04088\mathbf{j}$$

Angular velocities: $\boldsymbol{\omega}_{AP} = 5 \text{ rad/s} \circlearrowleft = (5 \text{ rad/s})\mathbf{k}$

$$\boldsymbol{\omega}_{BP} = \omega_{BP}\mathbf{k}$$

Angular accelerations: $\boldsymbol{\alpha}_{AP} = 2 \text{ rad/s}^2 \circlearrowright = -(2 \text{ rad/s}^2)\mathbf{k}$

$$\boldsymbol{\alpha}_{BP} = \alpha_{BP}\mathbf{k}$$

Velocity of *P*:

$$\mathbf{v}_P = \boldsymbol{\omega}_{AP} \times \mathbf{r}_{P/A} = 5\mathbf{k} \times (0.11233\mathbf{i} + 0.24088\mathbf{j})$$
$$= -(1.2044 \text{ m/s})\mathbf{i} + (0.56165 \text{ m/s})\mathbf{j}$$

Acceleration of *P*:

$$\mathbf{a}_P = \boldsymbol{\alpha}_{AP} \times \mathbf{r}_{P/A} - \omega_{AP}^2\mathbf{r}_{P/A}$$
$$= (-2\mathbf{k}) \times (0.11233\mathbf{i} + 0.24088\mathbf{j}) - (5)^2(0.11233\mathbf{i} + 0.24088\mathbf{j})$$
$$= -(2.3265 \text{ m/s}^2)\mathbf{i} - (6.2467 \text{ m/s}^2)\mathbf{j}$$

Consider the slider *P* as a particle sliding along the rotating rod *BP* with a relative velocity

$$\mathbf{v}_{\text{rel}} = u \measuredangle 20° = u(\cos 20°\mathbf{i} + \sin 20°\mathbf{j})$$

PROBLEM 15.176 (Continued)

and a relative acceleration

$$\mathbf{a}_{\text{rel}} = \dot{u} \measuredangle 20° = \dot{u}(\cos 20°\mathbf{i} + \sin 20°\mathbf{j})$$

Consider the rod BP as a rotating frame of reference.

Motion of Point P' on the rod currently at P.

$$\mathbf{v}_{P'} = \boldsymbol{\omega}_{BP} \times \mathbf{r}_{P/B} = \omega_{BP}\mathbf{k} \times (0.11233\mathbf{i} + 0.04088\mathbf{j})$$
$$= -0.04088\omega_{BP}\mathbf{i} + 0.11233\omega_{BP}\mathbf{j}$$

$$\mathbf{a}_{P'} = \boldsymbol{\alpha}_{BP} \times \mathbf{r}_{P/B} - \omega_{BP}^2\mathbf{r}_{P/B}$$
$$= \alpha_{BP}\mathbf{k} \times (0.11233\mathbf{i} + 0.04088\mathbf{j}) - \omega_{BP}^2(0.11233\mathbf{i} + 0.04088\mathbf{j})$$
$$= -0.04088\alpha_{BP}\mathbf{i} + 0.11233\alpha_{BP}\mathbf{j} - 0.11233\omega_{BP}^2\mathbf{i} - 0.04088\omega_{BP}^2\mathbf{j}$$

Velocity of P: $\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{\text{rel}}$

Resolve into components.

i: $-1.2044 = -0.04088\omega_{BP} + u\cos 20°$

j: $0.56165 = 0.11233\omega_{BP} + u\sin 20°$

Solving the simultaneous equations for ω_{BP} and u,

$$\omega_{BP} = 7.8612 \text{ rad/s} \qquad u = -0.93971 \text{ m/s}$$

Angular velocity of BP: $\boldsymbol{\omega}_{BP} = 7.86 \text{ rad/s} \circlearrowleft$ ◀

Relative velocity: $\mathbf{v}_{\text{rel}} = -0.93971(\cos 20°\mathbf{i} + \sin 20°\mathbf{j})$

Coriolis acceleration:

$$2\boldsymbol{\omega}_{BP} \times \mathbf{v}_{\text{rel}} = (2)(7.8612\mathbf{k}) \times (-0.93971\cos 20°\mathbf{i} - 0.93971\sin 20°\mathbf{j})$$
$$= (5.0532 \text{ m/s}^2)\mathbf{i} - (13.8835 \text{ m/s}^2)\mathbf{j}$$

Acceleration of P: $\mathbf{a}_{P'} = \mathbf{a}_{P'} + \mathbf{a}_{\text{rel}} + 2\boldsymbol{\omega}_{BP} \times \mathbf{v}_{\text{rel}}$

Resolve into components.

i: $-2.3265 = -0.04088\alpha_{BP} - 0.11233\omega_{BP}^2 + \dot{u}\cos 20° + 5.0532$

$$-0.04088\alpha_{BP} + \dot{u}\cos 20° = -0.43788 \qquad (1)$$

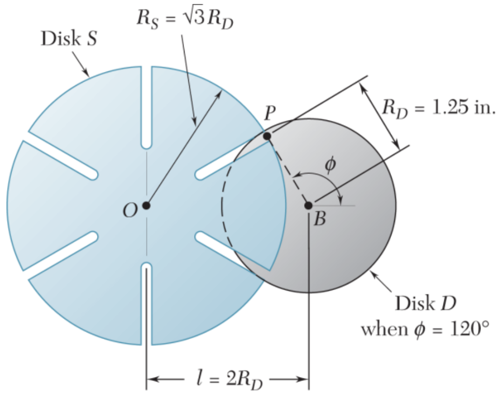
j: $-6.2467 = 0.11233\alpha_{BP} - 0.04088\omega_{BP}^2 + \dot{u}\sin 20° - 13.8835$

$$0.11233\alpha_{BP} + \dot{u}\sin 20° = 10.1631 \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously,

$$\alpha_{BP} = 81.146 \text{ rad/s}^2 \quad \dot{u} = 3.0641 \text{ m/s}^2$$

Angular acceleration of BP: $\boldsymbol{\alpha}_{BP} = 81.1 \text{ rad/s}^2 \circlearrowleft$ ◀

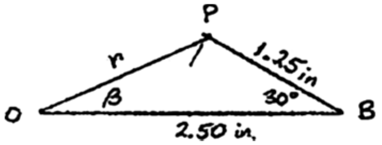


PROBLEM 15.177

The Geneva mechanism shown is used to provide an intermittent rotary motion of disk S. Disk D rotates with a constant counterclockwise angular velocity ω_D of 8 rad/s. A pin P is attached to disk D and can slide in one of the six equally spaced slots cut in disk S. It is desirable that the angular velocity of disk S be zero as the pin enters and leaves each of the six slots; this will occur if the distance between the centers of the disks and the radii of the disks are related as shown. Determine the angular velocity and angular acceleration of disk S at the instant when $\phi = 150°$.

SOLUTION

Geometry:



Law of cosines.
$$r^2 = 1.25^2 + 2.50^2 - (2)(1.25)(2.50)\cos 30°$$
$$r = 1.54914 \text{ in.}$$

Law of sines.
$$\frac{\sin\beta}{1.25} = \frac{\sin 30°}{r}$$
$$\beta = 23.794°$$

Let disk S be a rotating frame of reference.

$$\mathbf{\Omega} = \omega_S \circlearrowright, \quad \dot{\mathbf{\Omega}} = \alpha_S \circlearrowright$$

Motion of coinciding Point P′ on the disk.

$$\mathbf{v}_{P'} = r\omega_S = 1.54914\omega_S \searrow \beta$$

$$\mathbf{a}_{P'} = -\alpha_S \mathrm{k} \times \mathrm{r}_{P/O} - \omega_S^2 \mathrm{r}_{P/O} = [1.54914\alpha_S \searrow \beta] + [1.54914\omega_S^2 \swarrow \beta]$$

Motion relative to the frame.

$$\mathbf{v}_{P/S} = u \swarrow \beta \quad \mathrm{a}_{P/S} = \dot{u} \swarrow \beta$$

Coriolis acceleration.
$$2\omega_S u \nwarrow \beta$$

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/S} = [1.54914\omega_S \searrow \beta] + [u \swarrow \beta]$$

$$\mathbf{a}_P = \mathbf{a}_P + \mathbf{a}_{P/S} + 2\omega_S u \nwarrow$$

$$= [1.54914\alpha_S \searrow \beta] + [1.54914\omega_S^2 \swarrow \beta] + [\dot{u} \swarrow \beta] + [2\omega_S u \nwarrow \beta]$$

Motion of disk D. (Rotation about B)

$$\mathbf{v}_P = (BP)\omega_D = (1.25)(8) = 10 \text{ in./s} \swarrow 30°$$

$$\mathbf{a}_P = [(BP)\alpha_D \swarrow 60°] + [(BP)\omega_S^2 \searrow 30°] = 0 + [(1.25)(8)^2 \searrow 30°]$$

$$= 80 \text{ in./s}^2 \searrow 30°$$

PROBLEM 15.177 (Continued)

Equate the two expressions for $\mathbf{v}_P$ and resolve into components.

$$\searrow \beta: \quad 1.54914\omega_S = 10\cos(30° + \beta)$$

$$\omega_S = \frac{10\cos 53.794°}{1.54914}$$

$$= 3.8130 \text{ rad/s}$$

$\boldsymbol{\omega}_S = 3.81 \text{ rad/s} \circlearrowright$ ◀

$$\swarrow \beta: \quad u = 10\sin(30° + \beta) = 10\sin 53.794° = 8.0690 \text{ in./s}$$

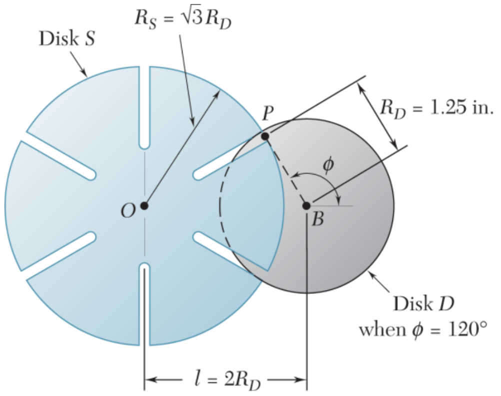
Equate the two expressions for $\mathbf{a}_P$ and resolve into components.

$$\searrow \beta: \quad 1.54914\alpha_S - 2\omega_S u = 80\sin(30° + \beta)$$

$$\alpha_S = \frac{80\sin 53.794° + (2)(3.8130)(8.0690)}{1.54914}$$

$$= 81.4 \text{ rad/s}^2$$

$\boldsymbol{\alpha}_S = 81.4 \text{ rad/s}^2 \circlearrowright$ ◀



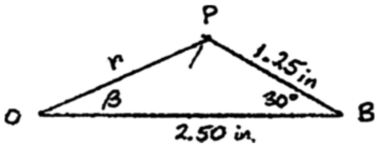
PROBLEM 15.178

In Problem 15.177, determine the angular velocity and angular acceleration of disk S at the instant when $\phi = 135°$.

PROBLEM 15.177 The Geneva mechanism shown is used to provide an intermittent rotary motion of disk S. Disk D rotates with a constant counterclockwise angular velocity ω_D of 8 rad/s. A pin P is attached to disk D and can slide in one of the six equally spaced slots cut in disk S. It is desirable that the angular velocity of disk S be zero as the pin enters and leaves each of the six slots; this will occur if the distance between the centers of the disks and the radii of the disks are related as shown. Determine the angular velocity and angular acceleration of disk S at the instant when $\phi = 150°$.

SOLUTION

Geometry:



Law of cosines.

$$r^2 = 1.25^2 + 2.50^2 - (2)(1.25)(2.50)\cos 45°$$

$$r = 1.84203 \text{ in.}$$

Law of sines.

$$\frac{\sin\beta}{1.25} = \frac{\sin 45°}{r}$$

$$\beta = 28.675°$$

Let disk S be a rotating frame of reference.

$$\boldsymbol{\Omega} = \omega_S \circlearrowright, \quad \dot{\boldsymbol{\Omega}} = \alpha_S \circlearrowright$$

Motion of coinciding Point P′ on the disk.

$$\mathbf{v}_{P'} = r\omega_s = 1.84203\omega_s \searrow \beta$$

$$\mathbf{a}_{P'} = -\alpha_S \mathbf{k} \times \mathbf{r}_{P/O} - \omega_S^2 \mathbf{r}_{P/O} = [1.84203\alpha_S \searrow \beta] + [1.84203\omega_S^2 \swarrow \beta]$$

Motion relative to the frame.

$$\mathbf{v}_{P/S} = u \swarrow \beta \qquad \mathbf{a}_{P/S} = \dot{u} \swarrow \beta$$

Coriolis acceleration. $2\omega_S u \nwarrow \beta$

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/S} = [1.84203\omega_S \searrow \beta] + [u \swarrow \beta]$$

$$\mathbf{a}_P = \mathbf{a}_P + \mathbf{a}_{P/S} + 2\omega_S u \nwarrow$$

$$= [1.84203\alpha_S \searrow \beta] + [1.84203\omega_S^2 \swarrow \beta] + [\dot{u} \swarrow \beta] + [2\omega_S u \nwarrow \beta]$$

PROBLEM 15.178 (Continued)

Motion of disk D. (Rotation about *B*)

$$\mathbf{v}_P = (BP)\omega_D = (1.25)(8) = 10 \text{ in./s} \swarrow 30^\circ$$

$$\mathbf{a}_P = [(BP)\alpha_D \searrow 45^\circ] + [(BP)\omega_S^2 \searrow 45^\circ] = 0 + [(1.25)(8)^2 \searrow 45^\circ]$$

$$= 80 \text{ in./s}^2 \searrow 45^\circ$$

Equate the two expressions for $\mathbf{v}_P$ and resolve into components.

$$\searrow \beta: \quad 1.84203\omega_S = 10\cos(45^\circ + \beta)$$

$$\omega_S = \frac{10\cos 73.675^\circ}{1.84203}$$

$$= 1.52595 \text{ rad/s} \qquad\qquad \boldsymbol{\omega}_S = 1.526 \text{ rad/s} \circlearrowright \blacktriangleleft$$

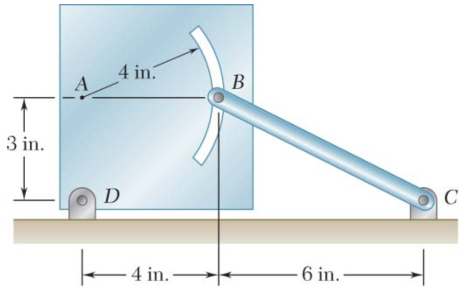
$$\swarrow \beta: \quad u = 10\sin(45^\circ + \beta) = 10\sin 73.675^\circ = 9.5968 \text{ in./s}$$

Equate the two expressions for $\mathbf{a}_P$ and resolve into components.

$$\searrow \beta: \quad 1.84203\alpha_S - 2\omega_S u = 80\sin(45^\circ + \beta)$$

$$\alpha_S = \frac{80\sin 73.675^\circ + (2)(1.52595)(9.5968)}{1.84203}$$

$$= 57.6 \text{ rad/s}^2 \qquad\qquad \boldsymbol{\alpha}_S = 57.6 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$



PROBLEM 15.179

At the instant shown, bar *BC* has an angular velocity of 3 rad/s and an angular acceleration of $2\ \text{rad/s}^2$, both counterclockwise. Determine the angular acceleration of the plate.

SOLUTION

Relative position vectors.

$$\mathbf{r}_{B/D} = (4\text{ in.})\mathbf{i} + (3\text{ in.})\mathbf{j}$$
$$\mathbf{r}_{B/C} = -(6\text{ in.})\mathbf{i} + (3\text{ in.})\mathbf{j}$$

Velocity analysis.

$$\boldsymbol{\omega}_{BC} = 3\text{ rad/s}\ \circlearrowleft$$

Bar *BC* (Rotation about *C*):

$$\boldsymbol{\omega}_{BC} = (3\text{ rad/s})\mathbf{k}$$
$$\mathbf{v}_B = \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C} = 3\mathbf{k} \times (-6\mathbf{i} + 3\mathbf{j})$$
$$= -(9\text{ in./s})\mathbf{i} - (18\text{ in./s})\mathbf{j}$$

Plate (Rotation about *D*):

$$\omega_P = \omega_P\mathbf{k}$$

Let Point B' be the point in the plate coinciding with *B*.

$$\mathbf{v}_{B'} = \boldsymbol{\omega}_P \times \mathbf{r}_{B/D} = \omega_P\mathbf{k} \times (4\mathbf{i} + 3\mathbf{j})$$
$$= -3\omega_P\mathbf{i} + 4\omega_P\mathbf{j}$$

Let plate be a rotating frame.

$$\mathbf{v}_{B/F} = v_{\text{rel}}\mathbf{j}$$
$$\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/F}$$
$$= -3\omega_P\mathbf{i} + (4\omega_P + v_{\text{rel}})\mathbf{j}$$

Equate like components of $\mathbf{v}_B$.

$$\mathbf{i}: \quad -9 = -3\omega_P \qquad \boldsymbol{\omega}_P = (3\text{ rad/s})\mathbf{k}$$
$$\mathbf{j}: \quad -18 = (4)(3) + v_{\text{rel}} \qquad \mathbf{v}_{\text{rel}} = -(30\text{ in./s})\mathbf{j}$$

Acceleration analysis.

$$\boldsymbol{\alpha}_{BC} = 2\text{ rad/s}^2\ \circlearrowleft$$

Bar *BC*:

$$\boldsymbol{\alpha}_{BC} = (2\text{ rad/s}^2)\mathbf{k}$$
$$\mathbf{a}_B = \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} - \omega_{BC}^2\,\mathbf{r}_{B/C}$$
$$= 2\mathbf{k} \times (-6\mathbf{i} + 3\mathbf{j}) - (3)^2(-6\mathbf{i} + 3\mathbf{j})$$
$$= -12\mathbf{j} - 6\mathbf{i} + 54\mathbf{i} - 27\mathbf{j} = (48\text{ in./s}^2)\mathbf{i} - (39\text{ in./s}^2)\mathbf{j}$$

Plate:

$$\boldsymbol{\alpha}_P = \alpha_P\mathbf{k}$$
$$\mathbf{a}_{B'} = \boldsymbol{\alpha}_P \times \mathbf{r}_{B/D} - \omega_P^2\,\mathbf{r}_{B/D}$$
$$= \alpha_P\mathbf{k} \times (4\mathbf{i} + 3\mathbf{j}) - (3)^2(4\mathbf{i} + 3\mathbf{j})$$
$$= -3\alpha_P\mathbf{i} + 4\alpha_P\mathbf{j} - 36\mathbf{i} - 27\mathbf{j}$$

PROBLEM 15.179 (Continued)

Relative to the frame (plate), the acceleration of pin *B* is

$$\mathbf{a}_{B/F} = (a_{\text{rel}})_t\,\mathbf{j} - \frac{v_{\text{rel}}^2}{\rho}\mathbf{i} = (a_{\text{rel}})_t\,\mathbf{j} - \frac{30^2}{4}\mathbf{i}$$

$$= -(225\text{ in./s}^2)\mathbf{i} + (a_{\text{rel}})_t\,\mathbf{j}$$

Coriolis acceleration. $2\boldsymbol{\omega}_P \times \mathbf{v}_{P/F}$

$$\mathbf{a}_c = 2(3\mathbf{k}) \times (-30\mathbf{j}) = (180\text{ in./s}^2)\mathbf{i}$$

Then

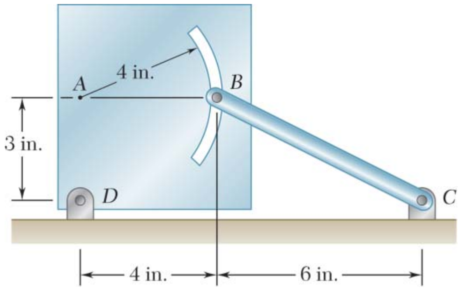
$$\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/F} + \mathbf{a}_c$$

$$\mathbf{a}_B = -(3\alpha_P + 36)\mathbf{i} + (4\alpha_P - 27)\mathbf{j} - 225\mathbf{i} + (a_{\text{rel}})_t\,\mathbf{j} + 180\mathbf{i}$$

$$\mathbf{a}_B = -(3\alpha_P + 81)\mathbf{i} + [4\alpha_P + (a_{\text{rel}})_t - 27]\mathbf{j}$$

Equate like components of $\mathbf{a}_B$.

i: $48 = -(3\alpha_P + 81)$ $\alpha_P = -43\text{ rad/s}^2$ $\boldsymbol{\alpha}_P = 43.0\text{ rad/s}^2 \circlearrowright$ ◀



PROBLEM 15.180

At the instant shown bar *BC* has an angular velocity of 3 rad/s and an angular acceleration of $2\ \text{rad/s}^2$, both clockwise. Determine the angular acceleration of the plate.

SOLUTION

Relative position vectors.

$$\mathbf{r}_{B/D} = (4\text{ in.})\mathbf{i} + (3\text{ in.})\mathbf{j}$$
$$\mathbf{r}_{B/C} = -(6\text{ in.})\mathbf{i} + (3\text{ in.})\mathbf{j}$$

Velocity analysis.

$$\boldsymbol{\omega}_{BC} = 3\text{ rad/s} \circlearrowright$$

Bar *BC* (Rotation about *C*):

$$\begin{aligned}\boldsymbol{\omega}_{BC} &= -(3\text{ rad/s})\mathbf{k}\\ \mathbf{v}_B &= \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C}\\ &= (-3\mathbf{k}) \times (-6\mathbf{i} + 3\mathbf{j})\\ &= (9\text{ in./s})\mathbf{i} + (18\text{ in./s})\mathbf{j}\end{aligned}$$

Plate (Rotation about *D*):

$$\omega_P = \omega_P\mathbf{k}$$

Let Point B' be the point in the plate coinciding with *B*:

$$\begin{aligned}\mathbf{v}_{B'} &= \boldsymbol{\omega}_P \times \mathbf{r}_{B/D}\\ &= \omega_P\mathbf{k} \times (4\mathbf{i} + 3\mathbf{j})\\ &= -3\omega_P\mathbf{i} + 4\omega_P\mathbf{j}\end{aligned}$$

Let the plate be a rotating frame.

$$\begin{aligned}\mathbf{v}_{B/F} &= v_{\text{rel}}\mathbf{j}\\ \mathbf{v}_B &= \mathbf{v}_{B'} + \mathbf{v}_{B/F}\\ &= -3\omega_P\mathbf{i} + (4\omega_P + v_{\text{rel}})\mathbf{j}\end{aligned}$$

Equate like components of $\mathbf{v}_B$.

$$\mathbf{i}:\quad 9 = -3\omega_P \qquad \boldsymbol{\omega}_P = -(3\text{ rad/s})\mathbf{k}$$

$$\mathbf{j}:\quad 18 = (4)(3) + v_{\text{rel}} \qquad \mathbf{v}_{\text{rel}} = (30\text{ in./s})\mathbf{j}$$

Acceleration analysis.

$$\boldsymbol{\alpha}_{BC} = 2\text{ rad/s}^2 \circlearrowright$$

Bar *BC*:

$$\begin{aligned}\boldsymbol{\alpha}_{BC} &= -(2\text{ rad/s}^2)\mathbf{k}\\ \mathbf{a}_B &= \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} - \omega_{BC}^2\mathbf{r}_{B/C}\\ &= (-2\mathbf{k}) \times (-6\mathbf{i} + 3\mathbf{j}) - (3)^2(-6\mathbf{i} + 3\mathbf{j})\\ &= 12\mathbf{j} + 6\mathbf{i} + 54\mathbf{i} - 27\mathbf{j}\\ &= (60\text{ in./s}^2)\mathbf{i} - (15\text{ in./s}^2)\mathbf{j}\end{aligned}$$

PROBLEM 15.180 (Continued)

Plate:

$$\boldsymbol{\alpha}_P = \alpha_P \mathbf{k}$$

$$\begin{aligned}\mathbf{a}_{B'} &= \boldsymbol{\alpha}_P \times \mathbf{r}_{B/D} - \omega_P^2 \mathbf{r}_{B/D} \\ &= \alpha_P \mathbf{k} \times (4\mathbf{i} + 3\mathbf{j}) - (3)^2 (4\mathbf{i} + 3\mathbf{j}) \\ &= -3\alpha_P \mathbf{i} + 4\alpha_P \mathbf{j} - 36\mathbf{i} - 27\mathbf{j}\end{aligned}$$

Relative to the frame (plate), the acceleration of pin *B* is

$$\begin{aligned}\mathbf{a}_{B/F} &= (a_{\text{rel}})_t \mathbf{j} - \frac{v_{\text{rel}}^2}{\rho} \mathbf{i} \\ &= (a_{\text{rel}})_t \mathbf{j} - \frac{30^2}{4} \mathbf{i} \\ &= -(225 \text{ in./s}^2)\mathbf{i} + (a_{\text{rel}})_t \mathbf{j}\end{aligned}$$

Coriolis acceleration.

$$2\boldsymbol{\omega}_P \times \mathbf{v}_{P/F}$$

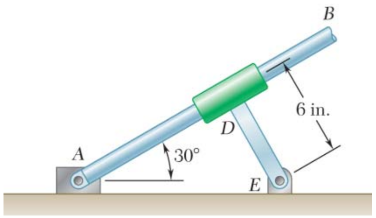
$$\mathbf{a}_c = 2(-3\mathbf{k}) \times (30\mathbf{j}) = (180 \text{ in./s}^2)\mathbf{i}$$

Then

$$\begin{aligned}\mathbf{a}_B &= \mathbf{a}_{B'} + \mathbf{a}_{B/F} + \mathbf{a}_c \\ \mathbf{a}_B &= -(3\alpha_P + 36)\mathbf{i} + (4\alpha_P - 27)\mathbf{j} - 225\mathbf{i} + (a_{\text{rel}})_t \mathbf{j} + 180\mathbf{i} \\ \mathbf{a}_B &= -(3\alpha_P + 81)\mathbf{i} + [4\alpha_P + (a_{\text{rel}})_t - 27]\mathbf{j}\end{aligned}$$

Equate like components of $\mathbf{a}_B$.

$$\mathbf{i}\text{:} \quad 60 = -(3\alpha_P + 81) \qquad \alpha_P = -47 \text{ rad/s}^2 \qquad \boldsymbol{\alpha}_P = 47.0 \text{ rad/s}^2 \curvearrowright \blacktriangleleft$$



PROBLEM 15.181*

Rod *AB* passes through a collar that is welded to link *DE*. Knowing that at the instant shown block *A* moves to the right at a constant speed of 75 in./s, determine (*a*) the angular velocity of rod *AB*, (*b*) the velocity relative to the collar of the point of the rod in contact with the collar, (*c*) the acceleration of the point of the rod in contact with the collar. (*Hint:* Rod *AB* and link *DE* have the same $\boldsymbol{\omega}$ and the same $\boldsymbol{\alpha}$.)

SOLUTION

Let $\boldsymbol{\omega} = \omega \circlearrowleft$ and $\boldsymbol{\alpha} = \alpha \circlearrowleft$ be the angular velocity and angular acceleration of the link *DE* and collar rigid body. Let *F* be a frame of reference moving with this body. The rod *AB* slides in the collar relative to the frame of reference with relative velocity $\mathbf{u} = u \measuredangle 30^\circ$ and relative acceleration $\dot{\mathbf{u}} = \dot{u} \measuredangle 30^\circ$. Note that this relative motion is a translation that applies to all points along the rod. Let Point *A* be moving with the end of the rod and A' be moving with the frame. Point *E* is a fixed point.

Geometry.

$$\mathbf{r}_{A'/E} = \frac{6 \text{ in.}}{\sin 30^\circ} = 12 \text{ in.} \longleftarrow$$

Velocity analysis.

$$\mathbf{v}_A = 75 \text{ in./s} \longrightarrow$$

$$\mathbf{v}_{A'} = 12\omega \downarrow$$

$$\mathbf{v}_A = \mathbf{v}_{A'} + \mathbf{u} \quad \text{Resolve into components.}$$

$$\xrightarrow{+}: \quad 75 = 0 + u\cos 30^\circ \quad u = \frac{75}{\cos 30^\circ} = 86.603 \text{ in./s}$$

$$+\uparrow: \quad 0 = -12\omega + u\sin 30^\circ \quad \omega = \frac{u\sin 30^\circ}{12} = 3.6085 \text{ rad/s}$$

(*a*) Angular velocity. $\boldsymbol{\omega} = 3.61 \text{ rad/s} \circlearrowleft$ ◀

(*b*) Velocity of rod *AB* relative to the collar. $\mathbf{u} = 86.6 \text{ in./s} \measuredangle 30^\circ$ ◀

(*c*) Acceleration analysis.

$$\mathbf{a}_A = 0$$

$$\mathbf{a}_{A'} = [12\alpha \downarrow] + [12\omega^2 \longrightarrow] = [12\alpha \downarrow] + [156.25 \longrightarrow]$$

Coriolis acceleration.

$$\mathbf{a}_c = 2\omega u \searrow 60^\circ = 625.01 \text{ in./s}^2 \searrow 60^\circ$$

$$\mathbf{a}_A = \mathbf{a}_{A'} + \dot{\mathbf{u}} + \mathbf{a}_c \quad \text{Resolve into components.}$$

$$\xrightarrow{+}: \quad 0 = 156.25 + \dot{u}\cos 30^\circ - 625.01\cos 60^\circ$$

$$\dot{u} = 180.43 \text{ in./s}^2$$

$$+\uparrow: \quad 0 = -12\alpha + \dot{u}\sin 30^\circ + 625.01\sin 60^\circ$$

$$\alpha = 52.624 \text{ rad/s}^2$$

PROBLEM 15.181* (Continued)

For rod *AB*, $\boldsymbol{\omega}_{AB} = 3.6085 \text{ rad/s} \circlearrowleft$

$$\boldsymbol{\alpha}_{AB} = 52.624 \text{ rad/s}^2 \circlearrowleft$$

Let *P* be the point on *AB* coinciding with collar *D*.

$$\mathbf{r}_{P/A} = 12\cos 30° \nearrow 30° = 10.392 \text{ in.} \nearrow 30°.$$

$$\mathbf{a}_P = \mathbf{a}_A + (\mathbf{a}_{P/A})_t + (\mathbf{a}_{P/A})_n$$

$$= 0 + [(10.392)(52.624) \nwarrow 60°] + [(10.392)(3.6085)^2 \swarrow 30°]$$

$$= [546.87 \nwarrow 60°] + [135.32 \swarrow 30°] = [390.63 \longleftarrow] + [405.94 \downarrow]$$

$$\mathbf{a}_P = 563 \text{ in./s}^2 \swarrow 46.1° \blacktriangleleft$$

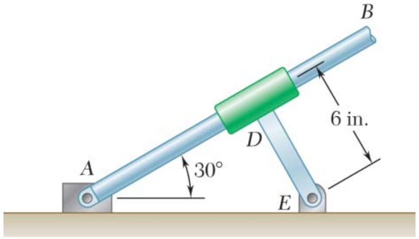
$\mathbf{a}_P$ may also be determined from $\mathbf{a}_P = \mathbf{a}_{P'} + \dot{\mathbf{u}} + \mathbf{a}_c$ using the rotating frame. The already calculated vectors $\dot{\mathbf{u}}$ and $\mathbf{a}_c$ also apply at Points P' and P.

$$\mathbf{a}_{P'} = \mathbf{a}_D = 6\alpha \swarrow 30° + 6\omega \searrow 60°$$

$$= [315.74 \text{ in./s}^2 \swarrow 30°] + [78.13 \text{ in./s}^2 \swarrow 60°]$$

Then $\mathbf{a}_P = [315.74 \swarrow 30°] + [78.13 \searrow 60°] + (180.43 \nearrow 30°] + [625.01 \nwarrow 60°]$

$$= (135.31 \swarrow 30°] + [546.88 \nwarrow 60°]$$



PROBLEM 15.182*

Solve Problem 15.181, assuming that block *A* moves to the left at a constant speed of 75 in./s.

PROBLEM 15.181 Rod *AB* passes through a collar which is welded to link *DE*. Knowing that at the instant shown block *A* moves to the right at a constant speed of 75 in./s, determine (*a*) the angular velocity of rod *AB*, (*b*) the velocity relative to the collar of the point of the rod in contact with the collar, (*c*) the acceleration of the point of the rod in contact with the collar. (*Hint:* Rod *AB* and link *DE* have the same $\boldsymbol{\omega}$ and the same $\boldsymbol{\alpha}$.)

SOLUTION

Let $\boldsymbol{\omega} = \omega \circlearrowleft$ and $\boldsymbol{\alpha} = \alpha \circlearrowleft$ be the angular velocity and angular acceleration of the link *DE* and collar rigid body. Let *F* be a frame of reference moving with this body. The rod *AB* slides in the collar relative to the frame of reference with relative velocity $\mathbf{u} = u \measuredangle 30^\circ$ and relative acceleration $\dot{\mathbf{u}} = \dot{u} \measuredangle 30^\circ$. Note that this relative motion is a translation that applies to all points along the rod. Let Point *A* be moving with the end of the rod and A' be moving with the frame. Point *E* is a fixed point.

Geometry. $\quad \mathbf{r}_{A'/E} = \dfrac{6 \text{ in.}}{\sin 30^\circ} = 12 \text{ in.} \leftarrow$

Velocity analysis. $\quad \mathbf{v}_A = 75 \text{ in./s} \leftarrow$

$$\mathbf{v}_{A'} = 12\omega \downarrow$$

$$\mathbf{v}_A = \mathbf{v}_{A'} + \mathbf{u} \quad \text{Resolve into components.}$$

$$\xrightarrow{+}: \quad -75 = 0 + u\cos 30^\circ \quad u = \frac{75}{\cos 30^\circ} = -86.603 \text{ in./s}$$

$$+\uparrow: \quad 0 = -12\omega + u\sin 30^\circ \quad \omega = \frac{u\sin 30^\circ}{12} = -3.6085 \text{ rad/s}$$

(*a*) Angular velocity. $\quad \boldsymbol{\omega} = 3.61 \text{ rad/s} \circlearrowright$ ◀

(*b*) Velocity of rod *AB* relative to the collar. $\quad \mathbf{u} = 86.6 \text{ in./s} \swarrow 30^\circ$ ◀

Acceleration analysis. $\quad \mathbf{a}_A = 0$

$$\mathbf{a}_{A'} = [12\alpha \downarrow] + [12\omega^2 \rightarrow] = [12\alpha \downarrow] + [156.25 \rightarrow]$$

Coriolis acceleration. $\quad \mathbf{a}_c = 2\omega u \searrow 60^\circ = 625.01 \text{ in./s}^2 \searrow 60^\circ$

$$\mathbf{a}_A = \tilde{\mathbf{a}}_{A'} + \dot{\mathbf{u}} + \mathbf{a}_c \quad \text{Resolve into components.}$$

$$\xrightarrow{+}: \quad 0 = 156.25 + \dot{u}\cos 30^\circ - 625.01\cos 60^\circ$$

$$\dot{u} = 180.43 \text{ in./s}^2$$

$$+\uparrow: \quad 0 = -12\alpha + \dot{u}\sin 30^\circ + 625.01\sin 60^\circ$$

$$\alpha = 52.624 \text{ rad/s}^2$$

PROBLEM 15.182* (Continued)

For rod *AB*, $\boldsymbol{\omega}_{AB} = 3.6085 \text{ rad/s} \circlearrowright$

$$\boldsymbol{\alpha}_{AB} = 52.624 \text{ rad/s}^2 \circlearrowright$$

Let *P* be the point on *AB* coinciding with collar *D*.

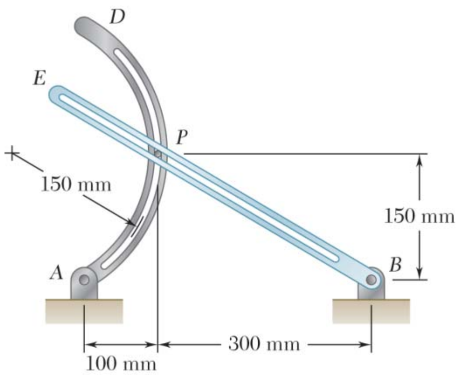
$$\mathbf{r}_{P/A} = 12\cos 30^\circ \measuredangle 30^\circ = 10.392 \text{ in.} \measuredangle 30^\circ.$$

$$\mathbf{a}_P = \mathbf{a}_A + (\mathbf{a}_{P/A})_t + (\mathbf{a}_{P/A})_n$$

$$= 0 + [(10.392)(52.624) \nwarrow 60^\circ] + [(10.392)(3.6085)^2 \swarrow 30^\circ]$$

$$= [546.87 \nwarrow 60^\circ] + [135.32 \swarrow 30^\circ] = [390.63 \longleftarrow] + [405.94 \downarrow]$$

$$\mathbf{a}_P = 563 \text{ in./s}^2 \swarrow 46.1^\circ \blacktriangleleft$$



PROBLEM 15.183*

In Problem 15.157, determine the acceleration of pin *P*.

PROBLEM 15.157 The motion of pin *P* is guided by slots cut in rods *AD* and *BE*. Knowing that bar *AD* has a constant angular velocity of 4 rad/s clockwise and bar *BE* has an angular velocity of 5 rad/s counterclockwise and is slowing down at a rate of 2 rad/s^2, determine the velocity of *P* for the position shown.

SOLUTION

Units: meters, m/s, m/s^2

Unit vectors: $\mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1 \uparrow, \quad \mathbf{k} = 1 \circlearrowleft$

From the solution of Problem 15.157,

$$\theta = 26.565^\circ \qquad R = 0.100 \text{ m}$$

$$\mathbf{r}_{P/A} = 0.1\mathbf{i} + 0.15\mathbf{j} \qquad \mathbf{r}_{P/B} = -0.3\mathbf{i} + 0.15\mathbf{j}$$

$$\boldsymbol{\omega}_{AD} = -(4 \text{ rad/s})\mathbf{k} \qquad \boldsymbol{\omega}_{BE} = (5 \text{ rad/s}^2)\mathbf{k}$$

$$\boldsymbol{\alpha}_{AD} = 0 \qquad \boldsymbol{\alpha}_{BE} = -(2 \text{ rad/s}^2)\mathbf{k}$$

$$\mathbf{v}_{P/AD} = u_1\mathbf{j} = -(1.775 \text{ m/s})\mathbf{j}$$

$$\begin{aligned}\mathbf{v}_{P/BE} &= -u_2\cos\theta\mathbf{i} + u_2\sin\theta\mathbf{j}\\ &= (-1.50935)(-\cos 26.565^\circ\mathbf{i} + \sin 26.565^\circ)\mathbf{j}\\ &= (1.35 \text{ m/s})\mathbf{i} + (0.675 \text{ m/s})\mathbf{j}\end{aligned}$$

Acceleration of Point *P*′ on rod *AD* coinciding with the pin:

$$\begin{aligned}\mathbf{a}_{P'} &= \boldsymbol{\alpha}_{AD} \times \mathbf{r}_{P/A} - \omega_{AD}^2\mathbf{r}_{P/A}\\ &= 0 - (4)^2(0.1\mathbf{i} + 0.15\mathbf{j}) = -1.6\mathbf{i} - 2.4\mathbf{j}\end{aligned}$$

Acceleration of the pin relative to rod *AD*:

$$\begin{aligned}\mathbf{a}_{P/AD} &= \dot{u}_1\mathbf{j} - \frac{u_1^2}{R}\mathbf{i} = \dot{u}_1\mathbf{j} - \frac{(1.775)^2}{0.15}\mathbf{i}\\ &= \dot{u}_1\mathbf{j} - 21.004\mathbf{i}\end{aligned}$$

Coriolis acceleration:

$$\mathbf{a}_1 = 2\boldsymbol{\omega}_{AD} \times \mathbf{v}_{P/AD}$$

$$\mathbf{a}_1 = 2(-4\mathbf{k}) \times (-1.775\mathbf{j}) = -14.2\mathbf{i}$$

Acceleration of *P*:

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/AD} + \mathbf{a}_1$$

$$\mathbf{a}_P = -36.804\mathbf{i} - 2.4\mathbf{j} + \dot{u}_1\mathbf{j}$$

PROBLEM 15.183* (Continued)

Acceleration of Point P'' on rod BE coinciding with the pin

$$\begin{aligned}\mathbf{a}_{P''} &= \boldsymbol{\alpha}_{BE} \times \mathbf{r}_{P/B} - \omega_{BE}^2 \mathbf{r}_{P/E} \\ &= (-2\mathbf{k}) \times (-0.3\mathbf{i} + 0.15\mathbf{j}) - (5)^2(-0.3\mathbf{i} + 0.15\mathbf{j}) \\ &= 0.6\mathbf{j} + 0.3\mathbf{i} + 7.5\mathbf{i} - 3.75\mathbf{j} = 7.8\mathbf{i} - 3.15\mathbf{j}\end{aligned}$$

Acceleration of the pin relative to the rod BE:

$$\mathbf{a}_{P/BE} = \dot{u}_2(-\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

Coriolis acceleration:

$$\begin{aligned}\mathbf{a}_2 &= 2\boldsymbol{\omega}_{BE} \times \mathbf{v}_{P/BE} \\ &= (2)(5\mathbf{k}) \times (1.35\mathbf{i} - 0.675\mathbf{j}) = -13.5\mathbf{j} + 6.75\mathbf{i}\end{aligned}$$

Acceleration of P:

$$\mathbf{a}_P = \mathbf{a}_{P''} + \mathbf{a}_{P/BE} + \mathbf{a}_2$$

$$\begin{aligned}\mathbf{a}_P &= 7.8\mathbf{i} - 3.15\mathbf{j} + \dot{u}_2(-\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + 13.5\mathbf{j} + 6.75\mathbf{i} \\ &= 14.55\mathbf{i} + 10.35\mathbf{j} - \dot{u}_2\cos\theta\mathbf{i} + \dot{u}_2\sin\theta\mathbf{j}\end{aligned}$$

Equating the two expressions for $\mathbf{a}_P$ and resolving into components,

i:

$$-36.804 = 14.55 - \dot{u}_2\cos\theta$$

$$\dot{u}_2 = \frac{36.804 + 14.55}{\cos 26.565^\circ} = 57.415\ \text{m/s}^2$$

j:

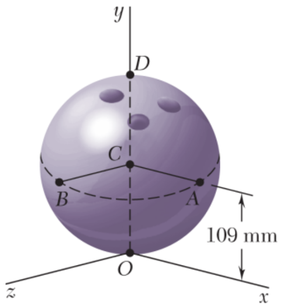
$$-2.4 + \dot{u}_1 = 10.35 + \dot{u}_2\sin\theta$$

$$\dot{u}_1 = 12.75 + 57.415\sin 26.565^\circ = 38.426\ \text{m/s}^2$$

Acceleration of the pin.

$$\begin{aligned}\mathbf{a}_P &= -36.804\mathbf{i} - 2.4\mathbf{j} + 38.427\mathbf{j} \\ &= -36.804\mathbf{i} + 36.027\mathbf{j}\end{aligned}$$

$\mathbf{a}_P - (36.8\ \text{m/s}^2)\mathbf{i} + (36.0\ \text{m/s}^2)\mathbf{j} = 51.5\ \text{m/s}^2 \measuredangle 44.4^\circ$ ◀



PROBLEM 15.184

The bowling ball shown rolls without slipping on the horizontal *xz* plane with an angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$. Knowing that $\mathbf{v}_A = (4.8\text{ m/s})\mathbf{i} - (4.8\text{ m/s})\mathbf{j} + (3.6\text{ m/s})\mathbf{k}$ and $\mathbf{v}_D = (9.6\text{ m/s})\mathbf{i} + (7.2\text{ m/s})\mathbf{k}$, determine (*a*) the angular velocity of the bowling ball, (*b*) the velocity of its center *C*.

SOLUTION

At the given instant, the origin is not moving.

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_A: \quad 4.8\mathbf{i} - 4.8\mathbf{j} + 3.6\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0.109 & 0.109 & 0 \end{vmatrix}$$

$$4.8\mathbf{i} - 4.8\mathbf{j} + 3.6\mathbf{k} = -0.109\omega_z\mathbf{i} + 0.109\omega_z\mathbf{j} + 0.109\left(\omega_x - \omega_y\right)\mathbf{k}$$

$\mathbf{i}$: $-0.109\omega_z = 4.8 \qquad \omega_z = -44.037$ rad/s

$\mathbf{j}$: $0.109\omega_z = -4.8 \qquad \omega_z = -44.037$ rad/s

$\mathbf{k}$: $0.109\left(\omega_x - \omega_y\right) = 3.6 \qquad \omega_x - \omega_y = 33.028$ rad/s

$$\mathbf{v}_D = \boldsymbol{\omega} \times \mathbf{r}_D: \quad 9.6\mathbf{i} + 7.2\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & 0.218 & 0 \end{vmatrix}$$

$$9.6\mathbf{i} + 7.2\mathbf{k} = -0.218\omega_z\mathbf{i} + 0.218\omega_x\mathbf{k}$$

$\mathbf{i}$: $-0.218\omega_z = 9.6 \qquad \omega_z = -44.037$ rad/s

$\mathbf{j}$: $0.218\omega_x = 7.2 \qquad \omega_x = 33.028$ rad/s

$$\omega_y = \omega_x - 33.028 = 0$$

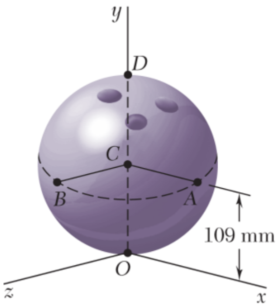
(*a*) *Angular velocity.*

$$\boldsymbol{\omega} = \left(33.0\text{ rad/s}\right)\mathbf{i} - \left(44.0\text{ rad/s}\right)\mathbf{k} \blacktriangleleft$$

(*b*) *Velocity of point C.*

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_C = \left(33.028\mathbf{i} - 44.037\mathbf{k}\right) \times 0.109\mathbf{j}$$
$$= 4.8\mathbf{i} + 3.6\mathbf{k}$$

$$\mathbf{v}_C = \left(4.80\text{ m/s}\right)\mathbf{i} + \left(3.60\text{ m/s}\right)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.185

The bowling ball shown rolls without slipping on the horizontal *xz* plane with an angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$. Knowing that $\mathbf{v}_B = (3.6\text{ m/s})\mathbf{i} - (4.8\text{ m/s})\mathbf{j} + (4.8\text{ m/s})\mathbf{k}$ and $\mathbf{v}_D = (7.2\text{ m/s})\mathbf{i} + (9.6\text{ m/s})\mathbf{k}$, determine (*a*) the angular velocity of the bowling ball, (*b*) the velocity of its center *C*.

SOLUTION

At the given instant, the origin is not moving.

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_B: \quad 3.6\mathbf{i} - 4.8\mathbf{j} + 4.8\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & 0.109 & 0.109 \end{vmatrix}$$

$$3.6\mathbf{i} - 4.8\mathbf{j} + 4.8\mathbf{k} = 0.109\left(\omega_y - \omega_z\right)\mathbf{i} - 0.109\omega_x\mathbf{j} + 0.109\omega_x\mathbf{k}$$

$$\mathbf{i}: \quad 0.109\left(\omega_y - \omega_z\right) = 3.6 \qquad \omega_y - \omega_z = 33.028\text{ rad/s}$$

$$\mathbf{j}: \quad -0.109\omega_x = -4.8 \qquad \omega_x = 44.037\text{ rad/s}$$

$$\mathbf{k}: \quad 0.109\omega_x = 4.8 \qquad \omega_x = 44.037\text{ rad/s}$$

$$\mathbf{v}_D = \boldsymbol{\omega} \times \mathbf{r}_D: \quad 7.2\mathbf{i} + 9.6\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & 0.218 & 0 \end{vmatrix}$$

$$7.2\mathbf{i} + 9.6\mathbf{k} = -0.218\omega_z\mathbf{i} + 0.218\omega_x\mathbf{k}$$

$$\mathbf{i}: \quad -0.218\omega_z = 7.2 \qquad \omega_z = -33.028\text{ rad/s}$$

$$\mathbf{k}: \quad 0.218\omega_x = 9.6 \qquad \omega_x = 44.037\text{ rad/s}$$

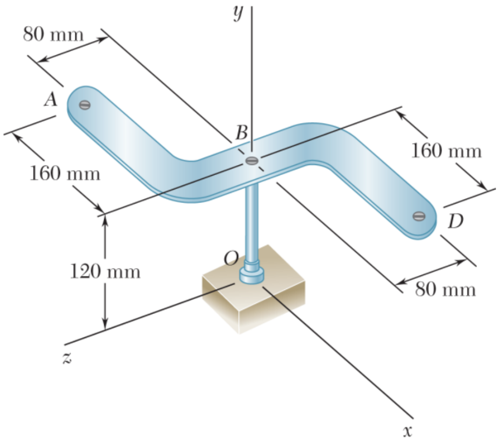
$$\omega_y = 33.028 + \omega_z = 0$$

(*a*) *Angular velocity.* $\boldsymbol{\omega} = \left(44.0\text{ rad/s}\right)\mathbf{i} - \left(33.0\text{ rad/s}\right)\mathbf{k}$ ◀

(*b*) *Velocity of point C.*

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_C = \left(44.037\mathbf{i} - 33.028\mathbf{k}\right) \times \left(0.109\mathbf{j}\right)$$
$$= 3.6\mathbf{i} + 4.8\mathbf{k}$$

$\mathbf{v}_C = \left(3.60\text{ m/s}\right)\mathbf{i} + \left(4.80\text{ m/s}\right)\mathbf{k}$ ◀



PROBLEM 15.186

Plate *ABD* and rod *OB* are rigidly connected and rotate about the ball-and-socket joint *O* with an angular velocity $\boldsymbol{\omega} = \omega_x\,\mathbf{i} + \omega_x\,\mathbf{j} + \omega_z\,\mathbf{k}$. Knowing that $\mathbf{v}_A = (80\text{ mm/s})\mathbf{i} + (360\text{ mm/s})\mathbf{j} + (v_A)_z\,\mathbf{k}$ and $\omega_x = 1.5\text{ rad/s}$, determine (*a*) the angular velocity of the assembly, (*b*) the velocity of Point *D*.

SOLUTION

$$\omega_x = 1.5\text{ rad/s} \qquad \boldsymbol{\omega} = (1.5\text{ rad/s})\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

$$\mathbf{r}_A = -(160\text{ mm})\mathbf{i} + (120\text{ mm})\mathbf{j} + (80\text{ mm})\mathbf{k}$$

$$\mathbf{r}_D = +(160\text{ mm})\mathbf{i} + (120\text{ mm})\mathbf{j} - (80\text{ mm})\mathbf{k}$$

(*a*)

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_A$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.5 & \omega_y & \omega_z \\ -160 & +120 & +80 \end{vmatrix}$$

$$\mathbf{v}_A = (80\omega_y - 120\omega_z)\mathbf{i} + (-160\omega_z - 120)\mathbf{j} + (180 + 160\omega_y)\mathbf{k}$$

But we are given:

$$\mathbf{v}_A = (80\text{ mm/s})\mathbf{i} + (360\text{ mm/s})\mathbf{j} + (v_A)_z\mathbf{k}$$

$$(v_A)_x:\quad 80\omega_y - 120\omega_z = 80 \tag{1}$$

$$(v_A)_y:\quad -160\omega_x - 120 = 360 \qquad \omega_z = -3\text{ rad/s} \tag{2}$$

$$(v_A)_z:\quad 180 + 160\omega_y = (v_A)_z \tag{3}$$

Substitute $\omega_z = -3.0\text{ rad/s}$ into Eq. (1):

$$80\omega_y - 120(-3) = 80$$

$$\omega_y = -3.5\text{ rad/s}$$

Substitute $\omega_y = -3.5\text{ rad/s}$ into Eq. (3):

$$180 + 160(-3.5) = (v_A)_z$$

$$(v_A)_z = -380\text{ in./s}$$

We have: $\boldsymbol{\omega} = (1.5\text{ rad/s})\mathbf{i} - (3.5\text{ rad/s})\mathbf{j} - (3.0\text{ rad/s})\mathbf{k}$ ◀

PROBLEM 15.186 (Continued)

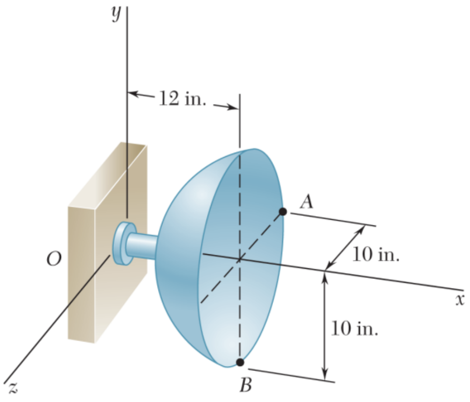
(*b*) Velocity of *D*.

$$\mathbf{v}_D = \boldsymbol{\omega} \times \mathbf{r}_D$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.5 & -3.5 & -3.0 \\ +160 & +120 & -80 \end{vmatrix}$$

$$= (360 + 280)\mathbf{i} + (-480 + 120)\mathbf{j} + (180 + 560)\mathbf{k}$$

$$\mathbf{v}_D = (640 \text{ mm/s})\mathbf{i} - (360 \text{ mm/s})\mathbf{j} + (740 \text{ mm/s})\mathbf{k} \blacktriangleleft$$



PROBLEM 15.187

At the instant considered the radar antenna shown rotates about the origin of coordinates with an angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$. Knowing that $(v_A)_y = 15$ in./s, $(v_B)_y = 9$ in./s, and $(v_B)_z = 18$ in./s, determine (*a*) the angular velocity of the antenna, (*b*) the velocity of point *A*.

SOLUTION

$$\mathbf{r}_A = (12\text{ in.})\mathbf{i} - (10\text{ in.})\mathbf{k} \qquad \mathbf{v}_A = (v_A)_x\mathbf{i} + 15\mathbf{j} + (v_A)_z\mathbf{k}$$

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_A: \quad (v_A)_x\mathbf{i} + 15\mathbf{j} + (v_A)_z\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 12 & 0 & -10 \end{vmatrix}$$

$$(v_A)_x\mathbf{i} + 15\mathbf{j} + (v_A)_z\mathbf{k} = -10\omega_y\mathbf{i} + (12\omega_z + 10\omega_x)\mathbf{j} - 12\omega_z\mathbf{k}$$

$$\mathbf{i}: \quad (v_A)_x = -10\omega_y \tag{1}$$

$$\mathbf{j}: \quad 15 = 12\omega_z + 10\omega_x \tag{2}$$

$$\mathbf{k}: \quad (v_A)_z = -12\omega_y \tag{3}$$

$$\mathbf{r}_B = (12\text{ in.})\mathbf{i} - (10\text{ in.})\mathbf{j} \qquad \mathbf{v}_B = (v_B)_x\mathbf{i} + 9\mathbf{j} + 18\mathbf{k}$$

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_B: \quad (v_B)_x\mathbf{i} + 9\mathbf{j} + 18\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 12 & -10 & 0 \end{vmatrix}$$

$$(v_B)_x\mathbf{i} + 9\mathbf{j} + 18\mathbf{k} = 10\omega_z\mathbf{i} + 12\omega_z\mathbf{j} - (10\omega_x + 12\omega_y)\mathbf{k}$$

$$\mathbf{i}: \quad (v_B)_x = 12\omega_z \tag{4}$$

$$\mathbf{j}: \quad 9 = 12\omega_z \tag{5}$$

$$\mathbf{k}: \quad 18 = -10\omega_x - 12\omega_y \tag{6}$$

From (5),
$$\omega_z = 0.75\text{ rad/s}$$

From (2),
$$\omega_x = \frac{1}{10}(15 - 12\omega_z) = 0.600\text{ rad/s}$$

From (6),
$$\omega_y = -\frac{1}{12}(18 + 10\omega_x) = -2.00\text{ rad/s}$$

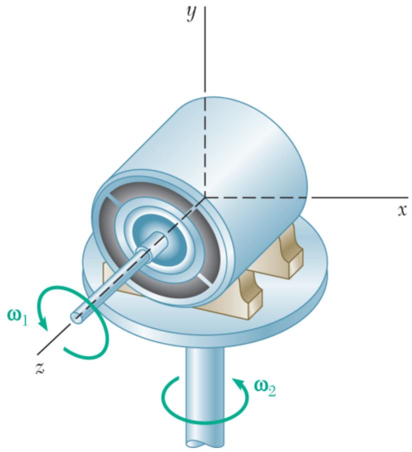
PROBLEM 15.187 (Continued)

(*a*) *Angular velocity.* $\boldsymbol{\omega} = (0.600 \text{ rad/s})\mathbf{i} - (2.00 \text{ rad/s})\mathbf{j} + (0.750 \text{ rad/s})\mathbf{k}$ ◀

From (1), $(v_A)_x = -10\omega_y = 20.0 \text{ in./s}$

From (3), $(v_A)_z = -12\omega_y = 24.0 \text{ in./s}$

(*b*) *Velocity of point A.* $v_A = (20.0 \text{ in./s})\mathbf{i} + (15.00 \text{ in./s})\mathbf{j} + (24.0 \text{ in./s})\mathbf{k}$ ◀



PROBLEM 15.188

The rotor of an electric motor rotates at the constant rate $\omega_1 = 1800$ rpm. Determine the angular acceleration of the rotor as the motor is rotated about the *y* axis with a constant angular velocity $\boldsymbol{\omega}_2$ of 6 rpm counterclockwise when viewed from the positive *y* axis.

SOLUTION

$$\omega_1 = 1800 \text{ rpm}$$
$$= 60\pi \text{ rad/s}$$
$$\omega_2 = 6 \text{ rpm}$$
$$= 0.2\pi \text{ rad/s}$$

Total angular velocity.

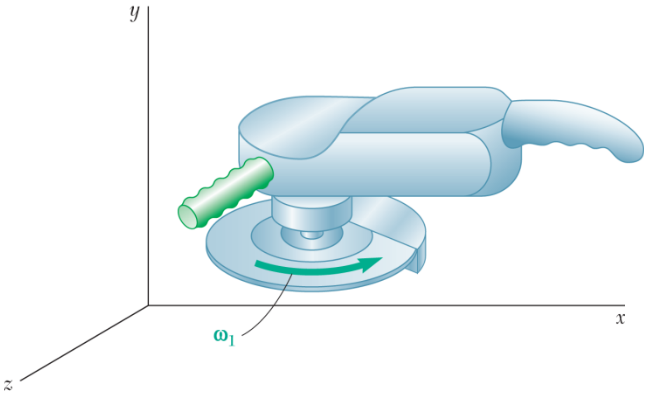
$$\boldsymbol{\omega} = \omega_2\mathbf{j} + \omega_1\mathbf{k}$$
$$\boldsymbol{\omega} = (0.2\pi \text{ rad/s})\mathbf{j} + (60\pi \text{ rad/s})\mathbf{k}$$

Angular acceleration.

Frame *Oxyz* is rotating with angular velocity $\boldsymbol{\Omega} = \omega_2\mathbf{j}$.

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$$
$$= \dot{\boldsymbol{\omega}}_{Oxyz} + \boldsymbol{\Omega} \times \boldsymbol{\omega}$$
$$= 0 + \omega_2\mathbf{j} \times (\omega_2\mathbf{j} + \omega_1\mathbf{k})$$
$$= \omega_2\omega_1\mathbf{i}$$
$$\boldsymbol{\alpha} = (0.2\pi)(60\pi)\mathbf{i}$$
$$= (12\pi^2 \text{ rad/s}^2)\mathbf{i}$$

$\boldsymbol{\alpha} = (118.4 \text{ rad/s}^2)\mathbf{i}$ ◀



PROBLEM 15.189

The disk of a portable sander rotates at the constant rate $\omega_1 = 4400$ rpm as shown. Determine the angular acceleration of the disk as a worker rotates the sander about the z axis with an angular velocity of 0.5 rad/s and an angular acceleration of 2.5 rad/s^2, both clockwise when viewed from the positive z axis.

SOLUTION

Spin rate: $\omega_1 = 4400 \text{ rpm} = 460.77 \text{ rad/s}$

Angular velocity of disk relative to the housing:

$$\boldsymbol{\omega}_1 = (460.77 \text{ rad/s})\,\mathbf{j}$$

Angular motion of the housing:

$$\boldsymbol{\omega}_2 = -(0.5 \text{ rad/s})\mathbf{k} \qquad \dot{\boldsymbol{\omega}}_2 = -(2.5 \text{ rad/s}^2)\mathbf{k}$$

Consider a frame of reference rotating with angular velocity

$$\boldsymbol{\Omega} = \omega_2\mathbf{k} = -(0.5 \text{ rad/s})\mathbf{k}$$

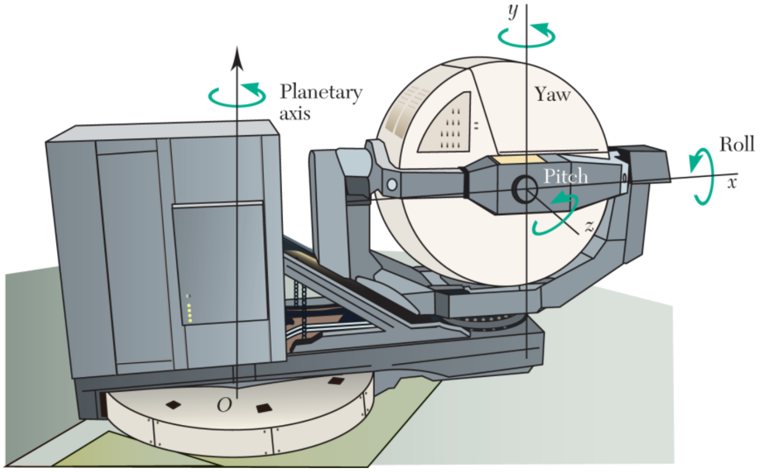
Angular velocity of the disk:

$$\begin{aligned}\boldsymbol{\omega} &= \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2\\ &= (460.77 \text{ rad/s})\mathbf{j} - (0.5 \text{ rad/s})\mathbf{k}\end{aligned}$$

Angular acceleration of the disk:

$$\begin{aligned}\boldsymbol{\alpha} &= \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\Omega} \times (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\\ &= 0 - 2.5\mathbf{k} + (-0.5\mathbf{k}) \times (460.77\mathbf{j} - 0.5\mathbf{k})\\ &= (230.38 \text{ rad/s}^2)\mathbf{i} - (2.5 \text{ rad/s}^2)\mathbf{k}\end{aligned}$$

$$\boldsymbol{\alpha} = (230 \text{ rad/s}^2)\mathbf{i} - (2.5 \text{ rad/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.190

A flight simulator is used to train pilots on how to recognize spatial disorientation. It has four degrees of freedom, and can rotate around a planetary axis as well as in yaw, pitch, and roll. Knowing that the simulator is rotating around the planetary axis with a constant angular velocity of 20 rpm counterclockwise as seen from above, determine the angular acceleration of the cab if (*a*) the cab has a constant pitch angular velocity of +3 **k** rad/s, (*b*) the cab has a constant roll angular velocity of –4 **i** rad/s.

SOLUTION

Given:

$$\boldsymbol{\omega}_{OA} = 20\mathbf{j} \text{ rpm} = 2.094\,\mathbf{j} \text{ rad/s}$$

$$\boldsymbol{\alpha}_{OA} = 0 \text{ rad/s}^2$$

Angular Acceleration:

$$\boldsymbol{\alpha} = (\dot{\boldsymbol{\omega}})_{OXYZ} + \boldsymbol{\Omega} \times \boldsymbol{\omega}$$

(a)

$$\boldsymbol{\omega}_{PI} = 3\mathbf{k} \text{ rad/s}, \; \dot{\boldsymbol{\omega}}_{PI} = 0 \text{ rad/s}^2, \; \boldsymbol{\Omega} = \boldsymbol{\omega}_{OA}$$

$$\boldsymbol{\alpha}_{AB} = \dot{\boldsymbol{\omega}}_{PL} + \boldsymbol{\omega}_{OA} \times \boldsymbol{\omega}_{PL}$$

$$= 0 + (2.094\,\mathbf{j}) \times 3\mathbf{k}$$

$$= 6.283\mathbf{i} \text{ rad/s}^2$$

$$\boldsymbol{\alpha}_{AB} = 6.283\mathbf{i} \text{ rad/s}^2 \blacktriangleleft$$

(b)

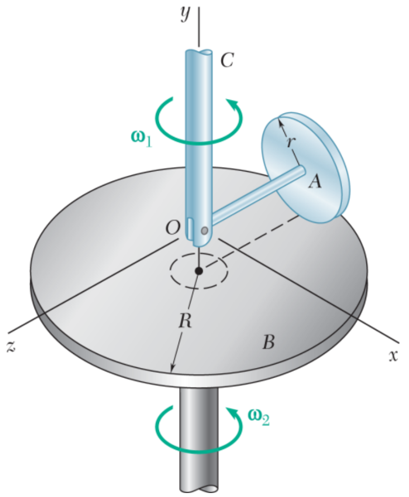
$$\boldsymbol{\omega}_{PI} = -4\mathbf{i} \text{ rad/s}, \; \dot{\boldsymbol{\omega}}_{PI} = 0 \text{ rad/s}^2, \; \boldsymbol{\Omega} = \boldsymbol{\omega}_{OA}$$

$$\boldsymbol{\alpha}_{AB} = \dot{\boldsymbol{\omega}}_{PL} + \boldsymbol{\omega}_{OA} \times \boldsymbol{\omega}_{PL}$$

$$= 0 + (2.094\,\mathbf{j}) \times -4\mathbf{i}$$

$$= 8.378\mathbf{k} \text{ rad/s}^2$$

$$\boldsymbol{\alpha}_{AB} = 8.378\mathbf{k} \text{ rad/s}^2 \blacktriangleleft$$



PROBLEM 15.191

In the system shown, disk A is free to rotate about the horizontal rod OA. Assuming that disk B is stationary $(\omega_2 = 0)$, and that shaft OC rotates with a constant angular velocity ω_1, determine (*a*) the angular velocity of disk A, (*b*) the angular acceleration of disk A.

SOLUTION

Disk A (In rotation about O):

Since

$$\omega_y = \omega_1,$$

$$\boldsymbol{\omega}_A = \omega_x \mathbf{i} + \omega_1 \mathbf{j} + \omega_z \mathbf{k}$$

Point D is point of contact of wheel and disk.

$$\mathbf{r}_{D/O} = -r\mathbf{j} - R\mathbf{k}$$

$$\mathbf{v}_D = \boldsymbol{\omega}_A \times \mathbf{r}_{D/O}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_1 & \omega_z \\ 0 & -r & -R \end{vmatrix}$$

$$\mathbf{v}_D = (-R\omega_1 + r\omega_z)\mathbf{i} + R\omega_1 \mathbf{j} - r\omega_x \mathbf{k}$$

Since $\omega_2 = 0,\ \mathbf{v}_D = 0.$

Each component of $\mathbf{v}_D$ is zero.

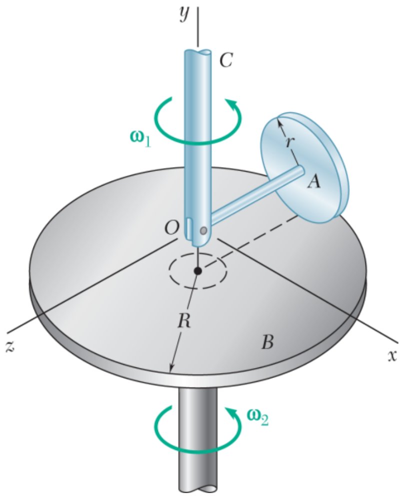
$$(v_D)_z = r\omega_x = 0; \quad \omega_x = 0$$

$$(v_D)_x = -R\omega_1 + r\omega_z = 0; \quad \omega_z = \left(\frac{R}{r}\right)\omega_1$$

(*a*) Angular velocity. $$\boldsymbol{\omega}_A = \omega_1 \mathbf{j} + \left(\frac{R}{r}\right)\omega_1 \mathbf{k} \blacktriangleleft$$

(*b*) Angular acceleration. Disk A rotates about y axis at rate ω_1.

$$\boldsymbol{\alpha}_A = \frac{d\boldsymbol{\omega}_A}{dt} = \boldsymbol{\omega}_y \times \boldsymbol{\omega}_A = \omega_1 \mathbf{j} \times \left(\omega_1 \mathbf{j} + \frac{R}{r}\omega_1 \mathbf{k}\right) \qquad \boldsymbol{\alpha}_A = \frac{R}{r}\omega_1^2 \mathbf{i} \blacktriangleleft$$



PROBLEM 15.192

In the system shown, disk *A* is free to rotate about the horizontal rod *OA*. Assuming that shaft *OC* and disk *B* rotate with constant angular velocities ω_1 and ω_2, respectively, both counterclockwise, determine (*a*) the angular velocity of disk *A*, (*b*) the angular acceleration of disk *A*.

SOLUTION

Disk *A* (in rotation about *O*):

Since $\omega_y = \omega_1$,

$$\boldsymbol{\omega}_A = \omega_x\mathbf{i} + \omega_1\mathbf{j} + \omega_z\mathbf{k}$$

Point *D* is point of contact of wheel and disk.

$$\mathbf{r}_{D/O} = -r\mathbf{j} - R\mathbf{k}$$

$$\mathbf{v}_D = \boldsymbol{\omega}_A \times \mathbf{r}_{D/O} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_1 & \omega_z \\ 0 & -r & -R \end{vmatrix}$$

$$\mathbf{v}_D = (-R\omega_1 + r\omega_z)\mathbf{i} + R\omega_x\mathbf{j} - r\omega_x\mathbf{k} \tag{1}$$

Disk *B*:

$$\boldsymbol{\omega}_B = \omega_2\mathbf{j}$$

$$\mathbf{v}_D = \boldsymbol{\omega}_B \times \mathbf{r}_{D/O} = \omega_2\mathbf{j} \times (-r\mathbf{j} - R\mathbf{k}) = -R\omega_2\mathbf{i} \tag{2}$$

From Eqs. 1 and 2: $\mathbf{v}_D = \mathbf{v}_D\!: \quad (-R\omega_1 - r\omega_z)\mathbf{i} + R\omega_x\mathbf{j} - r\omega_x\mathbf{k} = -R\omega_2\mathbf{i}$

Coefficients of **k**: $-r\omega_x = 0; \qquad \omega_x = 0$

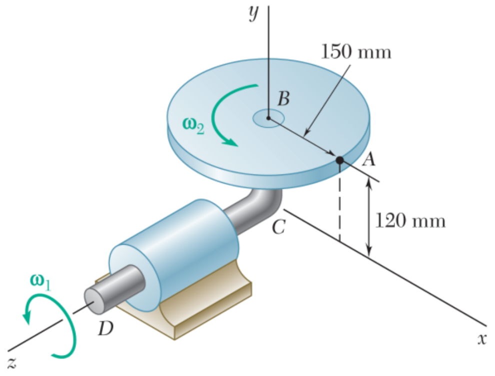
Coefficients of **i**: $(-R\omega_1 + r\omega_z) = -R\omega_2; \quad \omega_z = \dfrac{R}{r}(\omega_1 - \omega_2)$

(*a*) Angular velocity.

$$\boldsymbol{\omega}_A = \omega_1\mathbf{j} + \frac{R}{r}(\omega_1 - \omega_2)\mathbf{k} \blacktriangleleft$$

(*b*) Angular acceleration. Disk *A* rotates about *y* axis at rate ω_1.

$$\boldsymbol{\alpha}_A = \frac{d\boldsymbol{\omega}_A}{dt} = \boldsymbol{\omega}_y \times \boldsymbol{\omega}_A = \omega_1\mathbf{j} \times \left[\omega_1\mathbf{j} + \frac{R}{r}(\omega_1 - \omega_2)\mathbf{k}\right] \qquad \boldsymbol{\alpha}_A = \frac{R}{r}\omega_1(\omega_1 - \omega_2)\mathbf{i} \blacktriangleleft$$



PROBLEM 15.193

The L-shaped arm *BCD* rotates about the *z* axis with a constant angular velocity ω_1 of 5 rad/s. Knowing that the 150-mm-radius disk rotates about *BC* with a constant angular velocity ω_2 of 4 rad/s, determine (*a*) the velocity of Point *A*, (*b*) the acceleration of Point *A*.

SOLUTION

Total angular velocity.

$$\boldsymbol{\omega} = \omega_2\mathbf{j} + \omega_1\mathbf{k}$$

$$\boldsymbol{\omega} = (4\text{ rad/s})\mathbf{j} + (5\text{ rad/s})\mathbf{k}$$

Angular acceleration.

Frame *Oxyz* is rotating with angular velocity $\boldsymbol{\Omega} = \omega_1\mathbf{k}$.

$$\begin{aligned}\alpha &= \dot{\boldsymbol{\omega}}\\ &= \dot{\boldsymbol{\omega}}_{Oxyz} + \boldsymbol{\Omega}\times\boldsymbol{\omega}\\ &= 0 + \omega_1\mathbf{k}\times(\omega_2\mathbf{j} + \omega_1\mathbf{k})\\ &= -\omega_1\omega_2\mathbf{i}\\ \alpha &= -(5)(4)\mathbf{i}\\ &= -20\mathbf{i}\end{aligned}$$

$$\boldsymbol{\alpha} = -(20.0\text{ rad/s}^2)\mathbf{i}$$

(*a*) *Velocity of Point A.*

$$\mathbf{r}_A = (0.15\text{ m})\mathbf{i} + (0.12\text{ m})\mathbf{j}$$

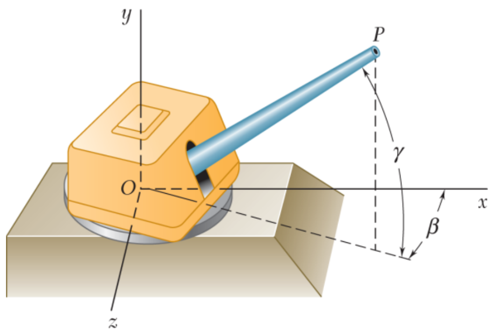
$$\begin{aligned}\mathbf{v}_A &= \boldsymbol{\omega}\times\mathbf{r}_A\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 4 & 5\\ 0.15 & 0.12 & 0\end{vmatrix}\\ &= -0.6\mathbf{i} + 0.75\mathbf{j} - 0.6\mathbf{k}\end{aligned}$$

$$\mathbf{v}_A = -(0.600\text{ m/s})\mathbf{i} + (0.750\text{ m/s})\mathbf{j} - (0.600\text{ m/s})\mathbf{k}\ ◀$$

(*b*) *Acceleration of Point A.*

$$\begin{aligned}\mathbf{a}_A &= \boldsymbol{\alpha}\times\mathbf{r}_A + \boldsymbol{\omega}\times\mathbf{v}_A\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -20 & 0 & 0\\ 0.15 & 0.6 & 0\end{vmatrix} + \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 4 & 5\\ -0.6 & 0.75 & -0.6\end{vmatrix}\\ &= -2.4\mathbf{k} - 6.15\mathbf{i} - 3\mathbf{j} + 2.4\mathbf{k}\\ &= -6.15\mathbf{i} - 3\mathbf{j}\end{aligned}$$

$$\mathbf{a}_A = -(6.15\text{ m/s}^2)\mathbf{i} - (3.00\text{ m/s}^2)\mathbf{j}\ ◀$$



PROBLEM 15.194

A gun barrel of length $OP = 4$ m is mounted on a turret as shown. To keep the gun aimed at a moving target, the azimuth angle β is being increased at the rate $d\beta/dt = 30°/\text{s}$ and the elevation angle γ is being increased at the rate $d\gamma/dt = 10°/\text{s}$. For the position $\beta = 90°$ and $\gamma = 30°$, determine (*a*) the angular velocity of the barrel, (*b*) the angular acceleration of the barrel, (*c*) the velocity and acceleration of Point *P*.

SOLUTION

Let

$$\boldsymbol{\omega}_1 = -\frac{d\beta}{dt}\mathbf{j} = -\left(\frac{30\pi}{180}\text{ rad/s}\right)\mathbf{j} = -\left(\frac{\pi}{6}\text{ rad/s}\right)\mathbf{j}$$

$$\boldsymbol{\omega}_2 = -\frac{d\gamma}{dt}\mathbf{i} = -\left(\frac{10\pi}{180}\text{ rad/s}\right)\mathbf{i} = -\left(\frac{\pi}{18}\text{ rad/s}\right)\mathbf{i}$$

(*a*) *Angular velocity* $\quad \boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$

$$\boldsymbol{\omega} = -(0.1745\text{ rad/s})\mathbf{i} - (0.524\text{ rad/s})\mathbf{j} \blacktriangleleft$$

(*b*) *Angular acceleration.*

Frame *Oxyz* is rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}_1$

$$\begin{aligned}\alpha &= \dot{\boldsymbol{\omega}}\\ &= \dot{\boldsymbol{\omega}}_{Oxyz} + \boldsymbol{\Omega}\times\boldsymbol{\omega}_1\\ &= 0 + \boldsymbol{\omega}_1\times(\boldsymbol{\omega}_1+\boldsymbol{\omega}_2)\\ &= \boldsymbol{\omega}_1\times\boldsymbol{\omega}_2\\ &= \left(-\frac{\pi}{6}\mathbf{j}\right)\times\left(-\frac{\pi}{18}\mathbf{i}\right)\\ &= -\frac{\pi^2}{108}\mathbf{k}\end{aligned}$$

$$\boldsymbol{\alpha} = -(0.0914\text{ rad/s}^2)\mathbf{k} \blacktriangleleft$$

(*c*) *Velocity and acceleration of Point P.*

For $\beta = 90°$ and $\gamma = 30°$, $\quad \mathbf{r}_P = (4\text{ m})(\sin 30°\mathbf{j} + \cos 30°\mathbf{k})$

$$\mathbf{v}_P = \boldsymbol{\omega}\times\mathbf{r}_C = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\left(\frac{\pi}{18}\right) & -\left(\frac{\pi}{6}\right) & 0\\ 0 & 4\sin 30° & 4\cos 30°\end{vmatrix}$$

$$= -1.8138\mathbf{i} + 0.6046\mathbf{j} - 3.4907\mathbf{k}$$

$$\mathbf{v}_P = -(1.818\text{ m/s})\mathbf{i} + (0.605\text{ m/s})\mathbf{j} - (3.49\text{ m/s})\mathbf{k} \blacktriangleleft$$

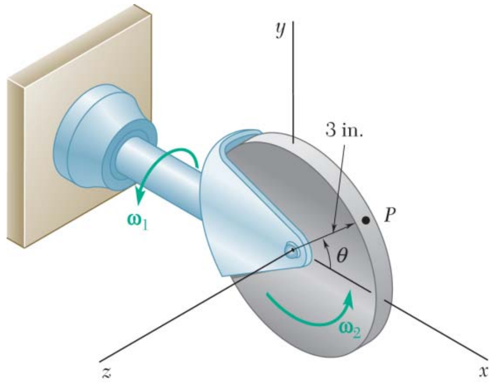
PROBLEM 15.194 (Continued)

$$\mathbf{a}_P = \alpha \times \mathbf{r}_P + \boldsymbol{\omega} \times \mathbf{v}_P$$

$$= -\frac{\pi^2}{108}\mathbf{k} \times (2\mathbf{j} + 3.4641\mathbf{k}) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\left(\frac{\pi}{18}\right) & -\left(\frac{\pi}{6}\right) & 0 \\ -1.8138 & 0.6046 & 3.4907 \end{vmatrix}$$

$$= 0.18277\mathbf{i} + 0\mathbf{j} + 0.18277\mathbf{i} - 0.06092\mathbf{j} - 1.0552\mathbf{k}$$

$$\mathbf{a}_P = (0.366 \text{ m/s}^2)\mathbf{i} - (0.0609 \text{ m/s}^2)\mathbf{j} - (1.055 \text{ m/s}^2)\mathbf{k}$$ ◀



PROBLEM 15.195

A 3-in.-radius disk spins at the constant rate $\omega_2 = 4$ rad/s about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $\omega_1 = 5$ rad/s. For the position shown, determine (*a*) the angular acceleration of the disk, (*b*) the acceleration of Point *P* on the rim of the disk if $\theta = 0$, (*c*) the acceleration of Point *P* on the rim of the disk if $\theta = 90°$.

SOLUTION

Angular velocity.

$$\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{k}$$
$$\boldsymbol{\omega} = (5\text{ rad/s})\mathbf{i} + (4\text{ rad/s})\mathbf{k}$$

(*a*) *Angular acceleration.*

Frame *Oxyz* is rotating with angular velocity $\boldsymbol{\Omega} = \omega_1\mathbf{i}.$

$$\begin{aligned}\alpha &= \dot{\boldsymbol{\omega}}\\ &= \dot{\boldsymbol{\omega}}_{Oxyz} + \boldsymbol{\Omega}\times\boldsymbol{\omega}\\ &= 0 + \omega_1\mathbf{i}\times(\omega_1\mathbf{i} + \omega_2\mathbf{k})\\ &= -\omega_1\omega_2\mathbf{j}\\ &= -(4)(5)\mathbf{j}\\ &= -20\mathbf{j}\end{aligned}$$

$\boldsymbol{\alpha} = -(20.0\text{ rad/s}^2)\mathbf{j}$ ◀

(*b*) $\theta = 0$. *Acceleration at Point P.*

$$\begin{aligned}\mathbf{r}_P &= (3\text{ in.})\mathbf{i}\\ &= (0.25\text{ ft})\mathbf{i}\\ \mathbf{v}_P &= \boldsymbol{\omega}\times\mathbf{r}_P\\ &= (5\mathbf{i} + 4\mathbf{k})\times 0.25\mathbf{i}\\ &= (1\text{ ft/s})\mathbf{j}\\ \mathbf{a}_P &= \boldsymbol{\alpha}\times\mathbf{r}_P + \boldsymbol{\omega}\times\mathbf{v}_P\\ &= -20\mathbf{j}\times 0.25\mathbf{i} + (5\mathbf{i} + 4\mathbf{k})\times(1\text{ ft/s})\mathbf{j}\\ &= 5\mathbf{k} + 5\mathbf{k} - 4\mathbf{i}\\ &= -4\mathbf{i} + 10\mathbf{k}\end{aligned}$$

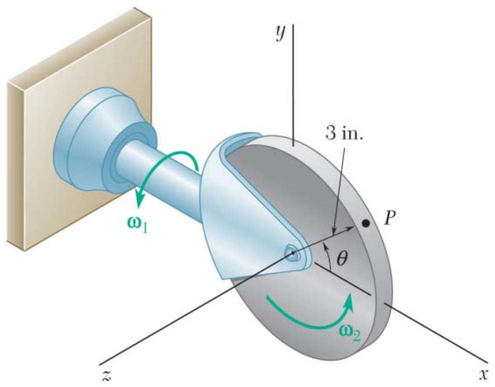
$\mathbf{a}_P - (4.00\text{ ft/s}^2)\mathbf{i} + (10.00\text{ ft/s}^2)\mathbf{k}$ ◀

PROBLEM 15.195 (Continued)

(*c*) $\theta = 90°$. *Acceleration at Point P.*

$$
\begin{aligned}
\mathbf{r}_P &= (0.25\text{ ft})\mathbf{j}\\
\mathbf{v}_P &= \boldsymbol{\omega}\times\mathbf{r}_P\\
&= (5\mathbf{i}+4\mathbf{k})\times 0.25\mathbf{j}\\
&= -(1.25\text{ ft/s})\mathbf{i}+(1\text{ ft/s})\mathbf{j}\\
\mathbf{a}_P &= \boldsymbol{\alpha}\times\mathbf{r}_P+\boldsymbol{\omega}\times\mathbf{v}_P\\
&= -20\mathbf{j}\times 0.25\mathbf{j}+(5\mathbf{i}+4\mathbf{k})\times(-1.25\mathbf{i}+\mathbf{j})\\
&= 0+0-6.25\mathbf{j}-4\mathbf{j}+0\\
&= -10.25\mathbf{j}
\end{aligned}
$$

$\mathbf{a}_P = -(10.25\text{ ft/s}^2)\mathbf{j}$ ◀



PROBLEM 15.196

A 3-in.-radius disk spins at the constant rate $\omega_2 = 4$ rad/s about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $\omega_1 = 5$ rad/s. Knowing that $\theta = 30°$, determine the acceleration of Point P on the rim of the disk.

SOLUTION

Angular velocity.

$$\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{k}$$

$$\boldsymbol{\omega} = (5 \text{ rad/s})\mathbf{i} + (4 \text{ rad/s})\mathbf{k}$$

Angular acceleration. Frame *Oxyz* is rotating with angular velocity $\Omega = \omega_1\mathbf{i}$.

$$\begin{aligned}\boldsymbol{\alpha} &= \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_{Oxyz} + \Omega \times \boldsymbol{\omega}\\ &= 0 + \omega_1\mathbf{i} \times (\omega_1\mathbf{i} + \omega_2\mathbf{k})\\ &= -\omega_1\omega_2\mathbf{j}\\ &= -(4)(5)\mathbf{j} = -20\mathbf{j}\end{aligned}$$

$$\boldsymbol{\alpha} = -(20.0 \text{ rad/s}^2)\mathbf{j}$$

Geometry. $\theta = 30°$,

$$\begin{aligned}\mathbf{r}_P &= (3 \text{ in.})(\cos 30°\mathbf{i} + \sin 30°\mathbf{j})\\ &= (0.25 \text{ ft})(\cos 30°\mathbf{i} + \sin 30°\mathbf{j})\end{aligned}$$

Velocity of Point *P*.

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}_P$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 4 \\ 0.25\cos 30° & 0.25\sin 30° & 0 \end{vmatrix}$$

$$= -(0.5 \text{ ft/s})\mathbf{i} + (0.86603 \text{ ft/s})\mathbf{j} + (0.625 \text{ ft/s})\mathbf{k}$$

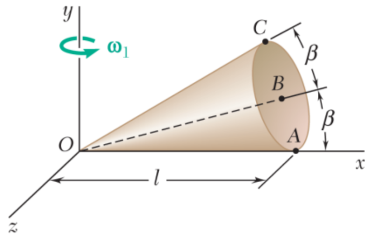
Acceleration of Point P.

$$\mathbf{a}_P = \boldsymbol{\alpha} \times \mathbf{r}_P + \boldsymbol{\omega} \times \mathbf{v}_P$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -20 & 0 \\ 0.25\cos 30° & 0.25\sin 30° & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 4 \\ -0.5 & 0.86603 & 0.625 \end{vmatrix}$$

$$= 4.3301\mathbf{k} - 3.4641\mathbf{i} - 5.125\mathbf{j} + 4.3301\mathbf{k}$$

$$\mathbf{a}_P = -(3.46 \text{ ft/s}^2)\mathbf{i} - (5.13 \text{ ft/s}^2)\mathbf{j} + (8.66 \text{ ft/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.197

The cone shown rolls on the *zx* plane with its apex at the origin of coordinates. Denoting by $\boldsymbol{\omega}_1$ the constant angular velocity of the axis *OB* of the cone about the *y* axis, determine (*a*) the rate of spin of the cone about the axis *OB*, (*b*) the total angular velocity of the cone, (*c*) the angular acceleration of the cone.

SOLUTION

Since the cone rolls on the *xz* plane, $\mathbf{v}_0 = \mathbf{v}_A = 0$

The instantaneous axis of rotation lies along the *x*-axis.

Total angular velocity. $\boldsymbol{\omega} = \omega\mathbf{i}$

(*a*) *Rate of spin*.

$$\boldsymbol{\omega}_s = \omega_s\left(\cos\beta\mathbf{i} + \sin\beta\mathbf{j}\right)$$

$$\boldsymbol{\omega} = \omega_1\mathbf{j} + \boldsymbol{\omega}_{spin}$$

$$\omega\mathbf{i} = \left(\omega_s\cos\beta\right)\mathbf{i} + \left(\omega_s\sin\beta\right)\mathbf{j} + \omega_1\mathbf{j}$$

Components.

$$\mathbf{j}: \quad 0 = \omega_s\sin\beta + \omega_1 \qquad \omega_s = -\frac{\omega_1}{\sin\beta}$$

$$\mathbf{i}: \quad \omega = \omega_s\cos\beta = -\frac{\omega_1\cos\beta}{\sin\beta} = -\frac{\omega_1}{\tan\beta}$$

$$|\omega_s| = \frac{\omega_1}{\sin\beta} \blacktriangleleft$$

(*b*) *Angular velocity*.

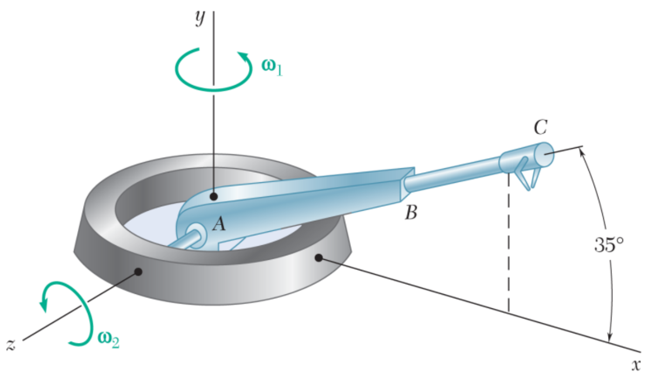
$$\boldsymbol{\omega} = -\frac{\omega_1}{\tan\beta}\mathbf{i} \blacktriangleleft$$

(*c*) *Angular acceleration*. The vector $\boldsymbol{\omega}$ rotates about the *y*-axis at a rate ω_1.

$$\boldsymbol{\Omega} = \omega_1\mathbf{j}$$

$$\boldsymbol{\alpha} = \boldsymbol{\Omega}\times\boldsymbol{\omega} = \omega_1\mathbf{j}\times\left(-\frac{\omega_1}{\tan\beta}\mathbf{i}\right)$$

$$\boldsymbol{\alpha} = \frac{\omega_1^2}{\tan\beta}\mathbf{k} \blacktriangleleft$$



PROBLEM 15.198

At the instant shown, the robotic arm *ABC* is being rotated simultaneously at the constant rate $\omega_1 = 0.15$ rad/s about the *y* axis, and at the constant rate $\omega_2 = 0.25$ rad/s about the *z* axis. Knowing that the length of arm *ABC* is 1 m, determine (*a*) the angular acceleration of the arm, (*b*) the velocity of Point *C*, (*c*) the acceleration of Point *C*.

SOLUTION

Angular velocity:

$$\boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k}$$
$$= (0.15 \text{ rad/s})\mathbf{j} + (0.25 \text{ rad/s})\mathbf{k}$$

Consider a frame of reference rotating with angular velocity

$$\boldsymbol{\Omega} = \omega_1 \mathbf{j} = (0.15 \text{ rad/s})\mathbf{j}$$

(*a*) *Angular acceleration of the arm.*

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\Omega} \times (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)$$
$$= 0 + 0 + (0.15\mathbf{j}) \times (0.15\mathbf{j} + 0.25\mathbf{k})$$

$$\boldsymbol{\alpha} = (0.0375 \text{ rad/s}^2)\mathbf{i} \blacktriangleleft$$

Arm *ABC* rotates about the fixed Point *A*.

$$\mathbf{r}_{C/A} = (1 \text{ m})(\cos 35°\mathbf{i} + \sin 35°)\mathbf{j}$$
$$= (0.81915 \text{ m})\mathbf{i} + (0.57358 \text{ m})\mathbf{j}$$

(*b*) *Velocity of Point C.*

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_{C/A}$$

$$\mathbf{v}_C = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.15 & 0.25 \\ 0.81915 & 0.57358 & 0 \end{vmatrix}$$
$$= -0.14340\mathbf{i} + 0.20479\mathbf{j} - 0.12287\mathbf{k}$$

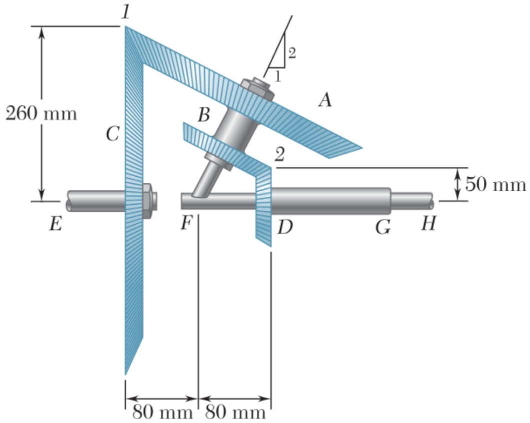
$$\mathbf{v}_C = -(0.1434 \text{ m/s})\mathbf{i} + (0.204 \text{ m/s})\mathbf{j} - (0.1229 \text{ m/s})\mathbf{k} \blacktriangleleft$$

(*c*) *Acceleration of C*:

$$\mathbf{a}_C = \boldsymbol{\alpha} \times \mathbf{r}_{C/A} + \boldsymbol{\omega} \times \mathbf{v}_C$$

$$\mathbf{a}_C = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.0375 & 0 & 0 \\ 0.81915 & 0.57358 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.15 & 0.25 \\ -0.14340 & 0.20479 & -0.12287 \end{vmatrix}$$
$$= 0.021509\mathbf{k} + 0.06962\mathbf{i} + 0.03585\mathbf{j} + 0.02151\mathbf{k}$$

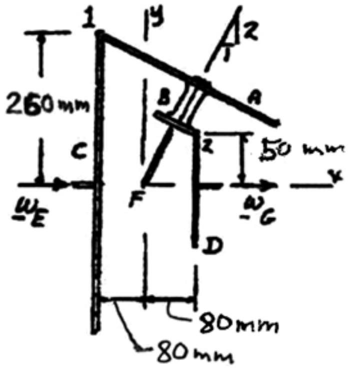
$$\mathbf{a}_C = -(0.0696 \text{ m/s}^2)\mathbf{i} + (0.0359 \text{ m/s}^2)\mathbf{j} + (0.0430 \text{ m/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.199

In the planetary gear system shown, gears *A* and *B* are rigidly connected to each other and rotate as a unit about the inclined shaft. Gears *C* and *D* rotate with constant angular velocities of 30 rad/s and 20 rad/s, respectively (both counterclockwise when viewed from the right). Choosing the *x* axis to the right, the *y* axis upward, and the *z* axis pointing out of the plane of the figure, determine (*a*) the common angular velocity of gears *A* and *B*, (*b*) the angular velocity of shaft *FH*, which is rigidly attached to the inclined shaft.

SOLUTION



Place origin at *F*.

Point 1: $$\mathbf{r}_1 = -(80\text{ mm})\mathbf{i} + (260\text{ mm})\mathbf{j}$$

Point 2: $$\mathbf{r}_2 = +(80\text{ mm})\mathbf{i} + (50\text{ mm})\mathbf{j}$$

$$\boldsymbol{\omega}_E = +(30\text{ rad/s})\mathbf{i}$$

$$\boldsymbol{\omega}_G = +(20\text{ rad/s})\mathbf{i}$$

$$\mathbf{v}_1 = \boldsymbol{\omega}_E \times \mathbf{r}_1$$

$$= (30\mathbf{i}) \times (-80\mathbf{i} + 260\mathbf{j})$$

$$\mathbf{v}_1 = (7800\text{ mm/s})\mathbf{k} \qquad (1)$$

$$\mathbf{v}_2 = \boldsymbol{\omega}_G \times \mathbf{r}_2$$

$$= (20\mathbf{i}) \times (80\mathbf{i} + 50\mathbf{j})$$

$$\mathbf{v}_2 = (1000\text{ mm/s})\mathbf{k} \qquad (2)$$

Motion of gear unit *AB*: $$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

$$\mathbf{v}_1 = \boldsymbol{\omega} \times \mathbf{r}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ -80 & +260 & 0 \end{vmatrix}$$

$$= -260\omega_z\mathbf{i} - 80\omega_z\mathbf{j} + (260\omega_x + 80\omega_y)\mathbf{k}$$

Recall from Eq. (1) that $\mathbf{v} = 7800\mathbf{k}$.

$$7800\mathbf{k} = -260\omega_z\mathbf{i} - 80\omega_z\mathbf{j} + (260\omega_x + 80\omega_y)\mathbf{k}$$

Equate coefficients of unit vectors.

$$\omega_z = 0$$

$$7800 = 260\omega_x + 80\omega_y \qquad (3)$$

$$\mathbf{v}_2 = \boldsymbol{\omega} \times \mathbf{r}_2$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & 0 \\ 80 & 50 & 0 \end{vmatrix}$$

$$= (50\omega_x - 80\omega_y)\mathbf{k}$$

Recall from Eq. (2) that $\mathbf{v}_2 = 1000\mathbf{k}$, and write

$$1000 = 50\omega_x - 80\omega_y \qquad (4)$$

Add Eqs. (3) and (4): $8800 = 310\omega_x \qquad \omega_x = 28.387 \text{ rad/s}$

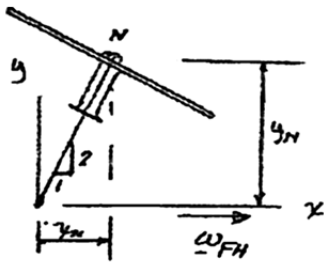
Eq. (4): $1000 = 50(28.387) - 80\omega_y \qquad \omega_y = 5.242 \text{ rad/s}$

(*a*) Common angular velocity of unit *AB*. $\boldsymbol{\omega} = (28.387 \text{ rad/s})\mathbf{i} + (5.242 \text{ rad/s})\mathbf{j}$

$\boldsymbol{\omega} = (28.4 \text{ rad/s})\mathbf{i} + (5.24 \text{ rad/s})\mathbf{j}$ ◀

(*b*) Angular velocity of shaft *FH*. (See figure in text.)

Point *N* is at nut, which is a part of unit *AB* and also is a part of shaft *GH*.



$$x_N = \frac{1}{2}y_N$$

$$\mathbf{r}_N = x_N\mathbf{i} + y_N\mathbf{j}$$

$$\mathbf{r}_N = \frac{1}{2}y_N\mathbf{i} + y_N\mathbf{j}$$

Nut *N* as a part of unit *AB*: $\boldsymbol{\omega} = (28.387 \text{ rad/s})\mathbf{i} + (5.242 \text{ rad/s})\mathbf{j}$

$$\mathbf{v}_N = \boldsymbol{\omega} \times \mathbf{r}_N$$

$$= (28.387\mathbf{i} + 5.242\mathbf{j}) \times \left(\frac{1}{2}y_N\mathbf{i} + y_N\mathbf{j}\right)$$

$$\mathbf{v}_N = (28.387y_N - 2.621y_N)\mathbf{k}$$

$$= +(25.766y_N)\mathbf{k} \qquad (5)$$

Nut *N* as a part of shaft *FH*. $\boldsymbol{\omega}_{FH} = \omega_{FH}\mathbf{i}$

$$\mathbf{v}_N = \boldsymbol{\omega}_{FH} \times \mathbf{r}_N$$

$$= (\omega_{FH}\mathbf{i}) \times \left(\frac{1}{2}y_N\mathbf{i} + y_N\mathbf{j}\right)$$

$$= \omega_{FH}\, y_N\mathbf{k} \qquad (6)$$

Equating expressions for $\mathbf{v}_N$ from Eqs. (5) and (6),

$$+(25.766y_N)\mathbf{k} = \omega_{FH}\, y_N\mathbf{k}$$

$$\boldsymbol{\omega}_{FH} = (25.766 \text{ rad/s})\mathbf{i}$$

$\boldsymbol{\omega}_{FH} = (25.8 \text{ rad/s})\mathbf{i}$ ◀

PROBLEM 15.200

In Problem 15.199, determine (*a*) the common angular acceleration of gears *A* and *B*, (*b*) the acceleration of the tooth of gear *A* which is in contact with gear *C* at Point *I*.

SOLUTION

See the solution to part (*a*) of Problem 15.199 for the calculation of the common angular velocity of unit *AB*.

$$\boldsymbol{\omega} = (28.387 \text{ rad/s})\mathbf{i} + (5.242 \text{ rad/s})\mathbf{j}$$

The angular velocity vector ω rotates about the *x*-axis with angular velocity $\boldsymbol{\omega}_{FH}$. See part (*b*) of Problem 15.199 for the calculation of $\boldsymbol{\omega}_{FH}$.

$$\boldsymbol{\omega}_{FH} = (25.776 \text{ rad/s})\mathbf{i}$$

(*a*) Common angular acceleration of unit *AB*.

$$\begin{aligned}\boldsymbol{\alpha} &= \boldsymbol{\omega}_{FH} \times \boldsymbol{\omega}\\ &= (25.776\mathbf{i}) \times (28.387\mathbf{i} + 5.242\mathbf{j})\\ &= 135.12\mathbf{k}\end{aligned}$$

$$\boldsymbol{\alpha} = 135.1 \text{ rad/s}^2\mathbf{k} \blacktriangleleft$$

The position and velocity vectors of a tooth at the contact Point 1 of gears *A* and *C* are

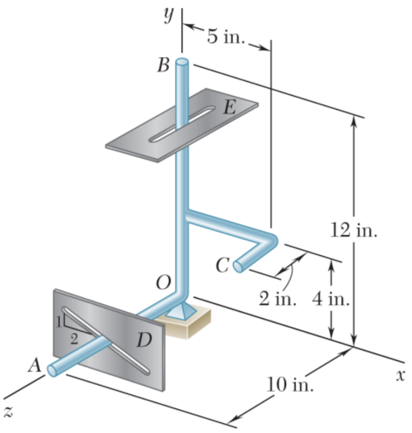
$$\mathbf{r}_1 = -(80 \text{ mm})\mathbf{i} + (260 \text{ mm})\mathbf{j}$$
$$\mathbf{v}_1 = (7800 \text{ mm/s})\mathbf{k}$$

as determined in part (*a*) of Problem 15.199.

(*b*) Acceleration of the tooth at Point 1 of gear *A*.

$$\begin{aligned}\mathbf{a}_1 &= \boldsymbol{\alpha} \times \mathbf{r}_1 + \boldsymbol{\omega} \times \mathbf{v}_1\\ &= (135.12\mathbf{k}) \times (-80\mathbf{i} + 260\mathbf{j}) + (28.387\mathbf{i} + 5.242\mathbf{j}) \times 7800\mathbf{k}\\ &= -10810\mathbf{j} - 35131\mathbf{i} - 221419\mathbf{j} + 40888\mathbf{i}\\ &= (5757 \text{ mm/s}^2)\mathbf{i} - (232229 \text{ mm/s}^2)\mathbf{j}\end{aligned}$$

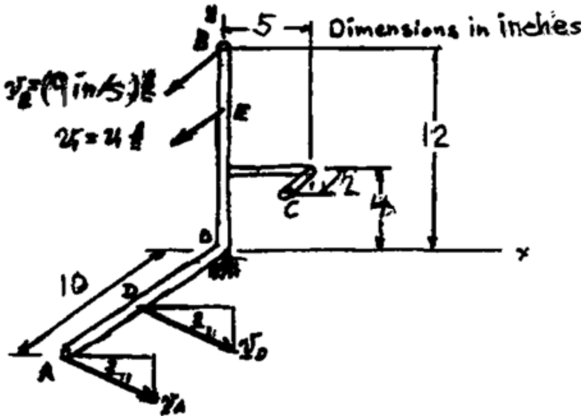
$$\mathbf{a}_1 = (5.8 \text{ m/s}^2)\mathbf{i} - (232 \text{ m/s}^2)\mathbf{j} \blacktriangleleft$$



PROBLEM 15.201

Several rods are brazed together to form the robotic guide arm shown which is attached to a ball-and-socket joint at *O*. Rod *OA* slides in a straight inclined slot while rod *OB* slides in a slot parallel to the *z* axis. Knowing that at the instant shown $\mathbf{v}_B = (9 \text{ in./s})\mathbf{k}$, determine (*a*) the angular velocity of the guide arm, (*b*) the velocity of Point *A*, (*c*) the velocity of Point *C*.

SOLUTION



Since rod at *D* slides in slot which is of slope 1:2,

$$(v_D)_x = -2(v_D)_y$$

and $$(v_A)_x = -2(v_A)_y$$

(*a*) Angular velocity.

$$\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$$

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_B: \quad (9 \text{ in./s})\mathbf{k} = (\omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}) \times (12 \text{ in.})\mathbf{j}$$

$$9\mathbf{k} = 12\omega_x \mathbf{k} - 12\omega_z \mathbf{i}$$

Coefficients of **k**: $$9 = 12\omega_x \qquad \omega_x = 0.75 \text{ rad/s}$$

Coefficients of **i**: $$0 = -12\omega_z \qquad \omega_z = 0$$

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_A$$

$$\mathbf{v}_A = (0.75\mathbf{i} + \omega_y \mathbf{j}) \times (10\mathbf{k})$$

$$(v_A)_x \mathbf{i} + (v_A)_y \mathbf{j} + (v_A)_z \mathbf{k} = -7.5\mathbf{j} + 10\omega_y \mathbf{i}$$

PROBLEM 15.201 (Continued)

Coefficients of **j**: $(v_A)_y = -7.5$

Coefficients of **i**: $(v_A)_x = 10\omega_y$

Coefficients of **k**: $(v_A)_z = 0$

Recall the Equations $(v_A)_x = -2(v_A)_y$

and $10\omega_y = -2(-7.5)$

So, $\omega_y = 1.5 \text{ rad/s}$ and $(v_A)_x = 15 \text{ in./s}$

$$\boldsymbol{\omega} = (0.75 \text{ rad/s})\mathbf{i} + (1.5 \text{ rad/s})\mathbf{j} \blacktriangleleft$$

(*b*) Velocity of *A*:

$$\mathbf{v}_A = (15 \text{ in./s})\mathbf{i} - (7.5 \text{ in./s})\mathbf{j} \blacktriangleleft$$

(*c*) Velocity of *C*:

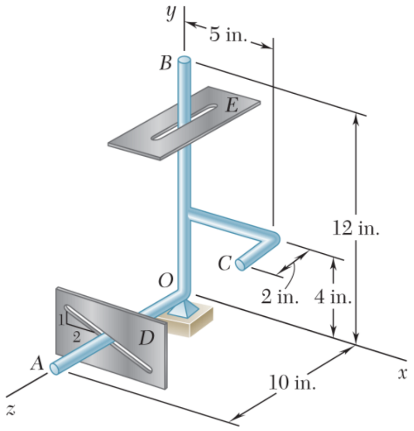
$$\mathbf{r}_C = 5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_C$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.75 & 1.5 & 0 \\ 5 & 4 & 2 \end{vmatrix}$$

$$= 3\mathbf{i} - 1.5\mathbf{j} + (3 - 7.5)\mathbf{k}$$

$$\mathbf{v}_C = (3 \text{ in./s})\mathbf{i} - (1.5 \text{ in./s})\mathbf{j} - (4.5 \text{ in./s})\mathbf{k} \blacktriangleleft$$

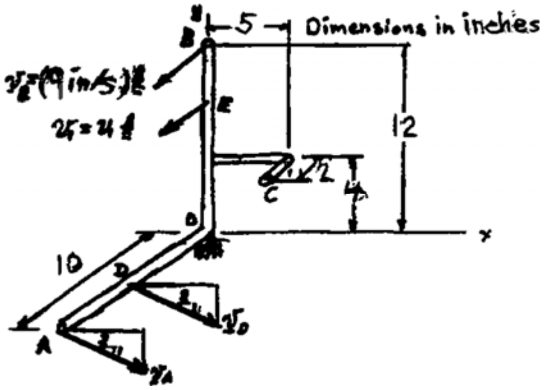


PROBLEM 15.202

In Problem 15.201 the speed of Point *B* is known to be constant. For the position shown, determine (*a*) the angular acceleration of the guide arm, (*b*) the acceleration of Point *C*.

PROBLEM 15.201 Several rods are brazed together to form the robotic guide arm shown, which is attached to a ball-and-socket joint at *O*. Rod *OA* slides in a straight inclined slot while rod *OB* slides in a slot parallel to the *z*-axis. Knowing that at the instant shown $\mathbf{v}_B = (9 \text{ in./s})\mathbf{k}$, determine (*a*) the angular velocity of the guide arm, (*b*) the velocity of Point *A*, (*c*) the velocity of Point *C*.

SOLUTION



Since rod at *D* slides in slot which is of slope 1:2,

$$(v_D)_x = -2(v_D)_y$$

and

$$(v_A)_x = -2(v_A)_y$$

Angular velocity.

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_B: \quad (9 \text{ in./s})\mathbf{k} = (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times (12 \text{ in.})\mathbf{j}$$

$$9\mathbf{k} = 12\omega_x\mathbf{k} - 12\omega_z\mathbf{i}$$

Coefficients of **k**: $9 = 12\omega_x \quad \omega_x = 0.75 \text{ rad/s}$

Coefficients of **i**: $0 = -12\omega_z \quad \omega_z = 0$

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_A$$

$$\mathbf{v}_A = (0.75\mathbf{i} + \omega_y\mathbf{j}) \times (10\mathbf{k})$$

$$(v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} + (v_A)_z\mathbf{k} = -7.5\mathbf{j} + 10\omega_y\mathbf{i}$$

Coefficients of **j**: $(v_A)_y = -7.5$

Coefficients of **i**: $(v_A)_x = 10\omega_y$

Coefficients of **k**: $(v_A)_z = 0$

PROBLEM 15.202 (Continued)

Recall the Equations $(v_A)_x = -2(v_A)_y$

and $10\omega_y = -2(-7.5)$

So, $\omega_y = 1.5 \text{ rad/s} \quad \text{and} \quad (v_A)_x = 15 \text{ in./s}$

$$\boldsymbol{\omega} = (0.75 \text{ rad/s})\mathbf{i} + (1.5 \text{ rad/s})\mathbf{j}$$

Velocity of *A*: $\mathbf{v}_A = (15 \text{ in./s})\mathbf{i} - (7.5 \text{ in./s})\mathbf{j}$

Velocity of *C*: $\mathbf{r}_C = 5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_C$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.75 & 1.5 & 0 \\ 5 & 4 & 2 \end{vmatrix}$$

$$= 3\mathbf{i} - 1.5\mathbf{j} + (3 - 7.5)\mathbf{k}$$

$$\mathbf{v}_C = (3 \text{ in./s})\mathbf{i} - (1.5 \text{ in./s})\mathbf{j} - (4.5 \text{ in./s})\mathbf{k}$$

$$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_B + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_C)$$

$$= \boldsymbol{\alpha} \times \mathbf{r}_C + \boldsymbol{\omega} \times \mathbf{v}_B$$

$$\mathbf{a}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha_x & \alpha_y & \alpha_z \\ 0 & 12 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.75 & 1.5 & 0 \\ 0 & 0 & 9 \end{vmatrix}$$

$$= -12\alpha_z\mathbf{i} + 12\alpha_x\mathbf{k} + 13.5\mathbf{i} - 6.75\mathbf{j}$$

$$\mathbf{a}_B = (13.5 - 12\alpha_z)\mathbf{i} - 6.75\mathbf{j} + 12\alpha_x\mathbf{k} \qquad (1)$$

$$(a_B)_x = 13.5 - 12\alpha_z = 0 \qquad \alpha_z = 1.125 \text{ rad/s}^2$$

$$(a_B)_y = -6.75 \qquad (a_B)_y = -6.75 \text{ in./s}^2$$

$$(a_B)_z = 12\alpha_x = 0 \qquad \alpha_x = 0$$

$$\mathbf{a}_A = \boldsymbol{\alpha} \times \mathbf{r}_A + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_A)$$

$$= \boldsymbol{\alpha} \times \mathbf{r}_A + \boldsymbol{\omega} \times \mathbf{v}_A$$

$$\mathbf{a}_A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \alpha_y & 1.125 \\ 0 & 0 & 10 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.75 & 1.5 & 0 \\ 15 & -7.5 & 0 \end{vmatrix}$$

$$= 10\alpha_y\mathbf{i} + (-5.625 - 22.5)\mathbf{k}$$

$$\mathbf{a}_A = 10\alpha_y\mathbf{i} - 28.125\mathbf{k}$$

Thus, $(a_A)_x = 10\alpha_y \quad (a_A)_y = 0 \quad (a_A)_z = -28.125 \text{ in./s}^2$

But $(a_A)_x = -2(a_A)_y = 0$

Therefore, $(a_A)_x = 10\alpha_y = 0 \quad \alpha_y = 0$

PROBLEM 15.202 (Continued)

(*a*) Angular acceleration: $\boldsymbol{\alpha} = (1.125 \text{ rad/s}^2)\mathbf{k}$ ◀

(*b*) Acceleration of *C*:

$$\mathbf{a}_C = \boldsymbol{\alpha} \times \mathbf{r}_C + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_C)$$

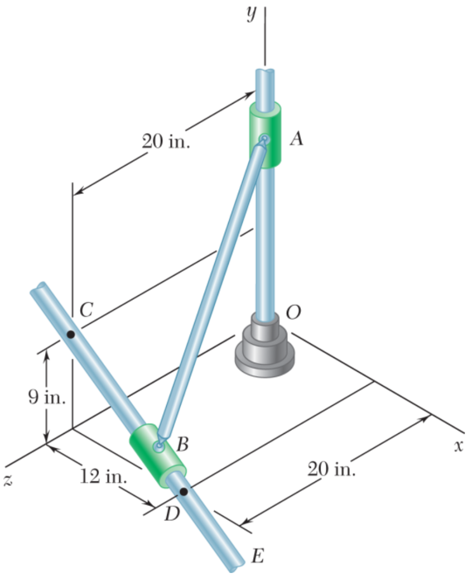
$$= \boldsymbol{\alpha} \times \mathbf{r}_C + \boldsymbol{\omega} \times \mathbf{v}_C$$

$$\mathbf{r}_C = (5 \text{ in.})\mathbf{i} + (4 \text{ in.})\mathbf{j} + (2 \text{ in.})\mathbf{k}$$

$$\mathbf{a}_C = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1.125 \\ 5 & 4 & 2 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.75 & 1.5 & 0 \\ 3 & -1.5 & -4.5 \end{vmatrix}$$

$$= -4.5\mathbf{i} + 5.625\mathbf{j} - 6.75\mathbf{i} + 3.375\mathbf{j} + (-1.125 - 4.5)\mathbf{k}$$

$$\mathbf{a}_C = -(11.3 \text{ in./s}^2)\mathbf{i} + (9 \text{ in./s}^2)\mathbf{j} - (5.63 \text{ in./s}^2)\mathbf{k}$$ ◀



PROBLEM 15.203

Rod *AB* of length 25 in. is connected by ball-and-socket joints to collars *A* and *B*, which slide along the two rods shown. Knowing that collar *B* moves toward Point *E* at a constant speed of 20 in./s, determine the velocity of collar *A* as collar *B* passes through Point *D*.

SOLUTION

Geometry.

$$l_{AB}^2 = x_{A/B}^2 + y_{A/B}^2 + z_{A/B}^2: \quad 25^2 = (-12)^2 + y_{A/B}^2 + (-20)^2$$

$$y_{A/B} = 9 \text{ in.}$$

$$\mathbf{r}_{A/B} = (-12 \text{ in.})\mathbf{i} + (9 \text{ in.})\mathbf{j} - (20 \text{ in.})\mathbf{k}$$

$$\mathbf{r}_{D/C} = (12 \text{ in.})\mathbf{i} - (9 \text{ in.})\mathbf{j},$$

$$l_{CD} = \sqrt{(12)^2 + (-9)^2} = 15 \text{ in.}$$

Velocity of collar B.

$$\mathbf{v}_B = v_B \frac{r_{D/C}}{l_{CD}}$$

$$\mathbf{v}_B = (20)\frac{(12\mathbf{i} - 9\mathbf{j})}{15} = (16 \text{ in./s})\mathbf{i} - (12 \text{ in./s})\mathbf{j}$$

Velocity of collar A.

$$\mathbf{v}_A = v_A\mathbf{j}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

where

$$\mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0.$

Forming $\mathbf{r}_{A/B} \cdot \mathbf{v}_A$, we get

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot (\mathbf{v}_B + \mathbf{v}_{A/B})$$

$$= \mathbf{r}_{A/B} \cdot \mathbf{v}_B + \mathbf{r}_{A/B} \cdot \mathbf{v}_{A/B}$$

or

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot \mathbf{v}_B \qquad (1)$$

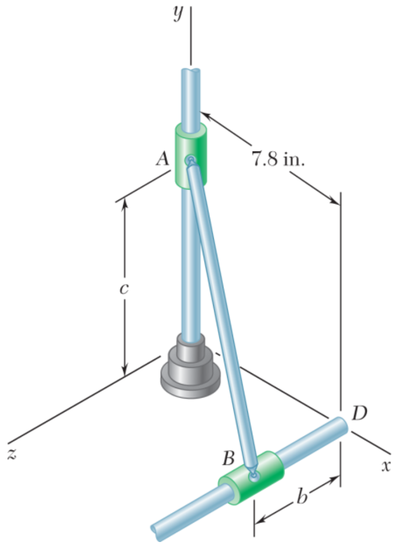
From Eq. (1),

$$(-12\mathbf{i} + 9\mathbf{j} - 20\mathbf{k}) \cdot (v_A\mathbf{j}) = (-12\mathbf{i} + 9\mathbf{j} - 80\mathbf{k}) \cdot (16\mathbf{i} - 12\mathbf{j})$$

$$9v_A = (-12)(16) + (9)(-12)$$

or

$$v_A = -33.333 \text{ in./s} \qquad \mathbf{v}_A = -(33.3 \text{ in./s})\mathbf{j} \blacktriangleleft$$



PROBLEM 15.204

Rod *AB* of length 13 in. is connected by ball-and-socket joints to collars *A* and *B*, which slide along the two rods shown. Knowing that collar *B* moves toward point *D* at a constant speed of 36 in./s, determine the velocity of collar *A* when $b = 4$ in.

SOLUTION

Geometry: $l_{AB}^2 = x_{A/B}^2 + y_{A/B}^2 + z_{A/B}^2$: $13^2 = (-7.8)^2 + c^2 + (-4)^2$, $c = 9.6$ in.

$$\mathbf{r}_{A/B} = (-7.8\text{ in.})\mathbf{i} + (9.6\text{ in.})\mathbf{j} - (4\text{ in.})\mathbf{k}$$

Velocity of collar B. $\mathbf{v}_B = (-36\text{ in./s})\mathbf{k}$

Velocity of collar A. $\mathbf{v}_A = v_A\mathbf{j}$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}, \quad \text{where} \quad \mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

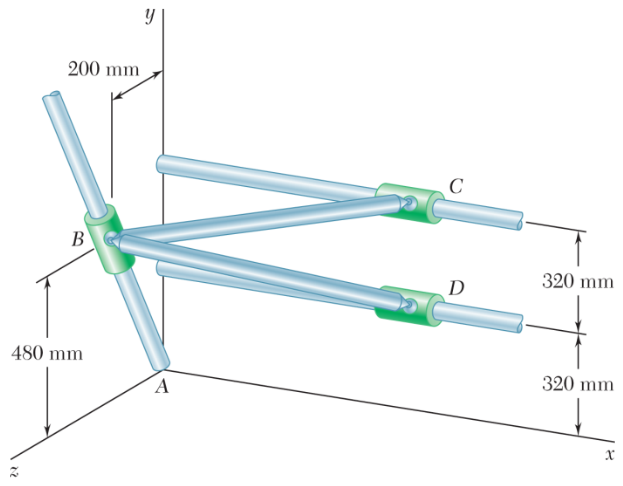
Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0.$

Forming $\mathbf{r}_{A/B} \cdot \mathbf{v}_A$, we get $\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot (\mathbf{v}_B + \mathbf{v}_{A/B}) = \mathbf{r}_{A/B} \cdot \mathbf{v}_B + \mathbf{r}_{A/B} \cdot \mathbf{v}_{A/B}$

or $\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot \mathbf{v}_B$ (1)

From (1), $(-7.8\mathbf{i} + 9.6\mathbf{j} - 4\mathbf{k}) \cdot (v_A\mathbf{j}) = (-7.8\mathbf{i} + 9.6\mathbf{j} - 4\mathbf{k}) \cdot (-36\mathbf{k})$

$9.6v_A = 144$ or $v_A = 15$ in./s $\mathbf{v}_A = (15.00\text{ in./s})\mathbf{j}$ ◀



PROBLEM 15.205

Rods *BC* and *BD* are each 840 mm long and are connected by ball-and-socket joints to collars that may slide on the fixed rods shown. Knowing that collar *B* moves toward *A* at a constant speed of 390 mm/s, determine the velocity of collar *C* for the position shown.

SOLUTION

Geometry.

$$l_{BC}^2 = x_{C/B}^2 + y_{C/B}^2 + z_{C/B}^2$$

$$(840)^2 = c^2 + (640-480)^2 + (200)^2 \qquad c = 800\text{ mm}$$

$$\mathbf{r}_{C/B} = (800\text{ mm})\mathbf{i} + (160\text{ mm})\mathbf{j} - (200\text{ mm})\mathbf{k}$$

Velocity of B.

$$\mathbf{v}_B = v_B\left(-\frac{12}{13}\mathbf{j} - \frac{5}{13}\mathbf{k}\right)$$

$$= (390\text{ mm/s})\left(-\frac{12}{13}\mathbf{j} - \frac{5}{13}\mathbf{k}\right)$$

$$= -(360\text{ mm/s})\mathbf{j} - (150\text{ mm/s})\mathbf{k}$$

Velocity of C.

$$\mathbf{v}_C = v_C\mathbf{i}$$

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{C/B}$$

where $\mathbf{v}_{C/B} = \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$

Noting that $\mathbf{v}_{C/B}$ is perpendicular to $\mathbf{r}_{C/B}$, we get $\mathbf{r}_{C/B} \cdot \mathbf{v}_{C/B} = 0$

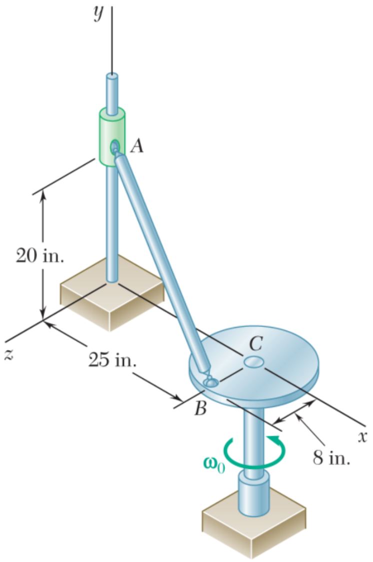
So that

$$\mathbf{r}_{C/B} \cdot \mathbf{v}_C = \mathbf{r}_{C/B} \cdot \mathbf{v}_B$$

$$(800\mathbf{i} + 160\mathbf{j} - 200\mathbf{k}) \cdot (-360\mathbf{j} - 150\mathbf{k}) = (800\mathbf{i} + 160\mathbf{j} - 200\mathbf{k}) \cdot (v_C\mathbf{i})$$

$$(160)(-360) + (-200)(-150) = 800\, v_C$$

$$v_C = -34.5\text{ mm/s} \qquad \mathbf{v}_C = -(34.5\text{ mm/s})\mathbf{i} \blacktriangleleft$$



PROBLEM 15.206

Rod AB is connected by ball-and-socket joints to collar A and to the 16-in.-diameter disk C. Knowing that disk C rotates counterclockwise at the constant rate $\omega_0 = 3$ rad/s in the zx plane, determine the velocity of collar A for the position shown.

SOLUTION

Geometry.

$$\mathbf{r}_{B/C} = (-8\text{ in.})\mathbf{k}$$

$$\mathbf{r}_{A/B} = -(25\text{ in.})\mathbf{i} + (20\text{ in.})\mathbf{j} - (8\text{ in.})\mathbf{k}$$

Velocity at B.

$$\mathbf{v}_B = \omega_0 \mathbf{j} \times \mathbf{r}_{B/C}$$

$$= 3\mathbf{j} \times (-8\mathbf{k})$$

$$= -(24\text{ in./s})\mathbf{i}$$

Velocity of collar A.

$$\mathbf{v}_A = v_A \mathbf{j}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

where $$\mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0$.

Forming $\mathbf{r}_{A/B} \cdot \mathbf{v}_A$, we get

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot (\mathbf{v}_B + \mathbf{v}_{A/B})$$

$$= \mathbf{r}_{A/B} \cdot \mathbf{v}_B + \mathbf{r}_{A/B} \cdot \mathbf{v}_{A/B}$$

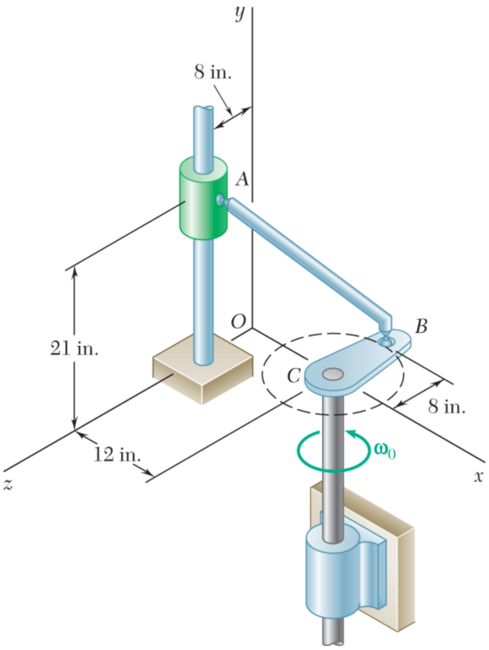
or $$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot \mathbf{v}_B \qquad (1)$$

From Eq. (1), $$(-25\mathbf{i} + 20\mathbf{j} - 8\mathbf{k}) \cdot (v_A \mathbf{j}) = (-25\mathbf{i} + 20\mathbf{j} - 8\mathbf{k}) \cdot (24\mathbf{i})$$

$$20 v_A = -600$$

or $$v_A = -30\text{ in./s}$$

$$\mathbf{v}_A = -(30.0\text{ in./s})\mathbf{j} \blacktriangleleft$$



PROBLEM 15.207

Rod *AB* of length 29 in. is connected by ball-and-socket joints to the rotating crank *BC* and to the collar *A*. Crank *BC* is of length 8 in. and rotates in the horizontal *xz* plane at the constant rate $\omega_0 = 10$ rad/s. At the instant shown, when crank *BC* is parallel to the *z* axis, determine the velocity of collar *A*.

SOLUTION

Geometry.

$$\mathbf{r}_{B/C} = (-8 \text{ in.})\mathbf{k},$$

$$\mathbf{r}_{A/B} = (-12 \text{ in.})\mathbf{i} + (21 \text{ in.})\mathbf{j} + (16 \text{ in.})\mathbf{k}$$

Velocity at B.

$$\mathbf{v}_B = \omega_0 \mathbf{j} \times \mathbf{r}_{B/C}$$

$$= 10\mathbf{j} \times (-8\mathbf{k})$$

$$= (-80 \text{ in./s})\mathbf{i}$$

Velocity of collar A.

$$\mathbf{v}_A = v_A \mathbf{j}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

where

$$\mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0.$

Forming $\mathbf{r}_{A/B} \cdot \mathbf{v}_A$, we get

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot (\mathbf{v}_B + \mathbf{v}_{A/B})$$

$$= \mathbf{r}_{A/B} \cdot \mathbf{v}_B + \mathbf{r}_{A/B} \cdot \mathbf{v}_{A/B}$$

or

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot \mathbf{v}_B \qquad (1)$$

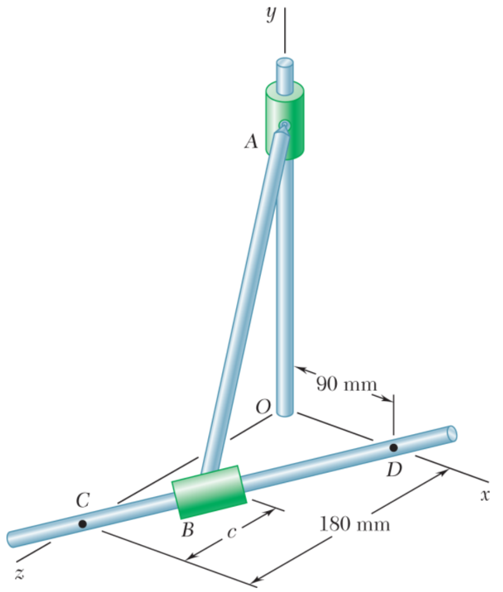
From Eq. (1),

$$(-12\mathbf{i} + 21\mathbf{j} + 16\mathbf{k}) \cdot (v_A \mathbf{j}) = (-12\mathbf{i} + 21\mathbf{j} + 16\mathbf{k}) \cdot (-80\mathbf{i})$$

$$21 v_A = 960$$

or

$$v_A = 45.714 \text{ in./s} \qquad \mathbf{v}_A = (45.7 \text{ in./s})\mathbf{j} \blacktriangleleft$$



PROBLEM 15.208

Rod *AB* of length 300 mm is connected by ball-and-socket joints to collars *A* and *B*, which slide along the two rods shown. Knowing that collar *B* moves toward Point *D* at a constant speed of 50 mm/s, determine the velocity of collar *A* when $c = 80$ mm.

SOLUTION

Geometry.

$$\mathbf{r}_A = y\mathbf{j},$$

$$\mathbf{r}_D = (90\text{ mm})\mathbf{i}$$

$$\mathbf{r}_C = (180\text{ mm})\mathbf{k}$$

$$\mathbf{r}_{D/C} = \mathbf{r}_D - \mathbf{r}_C$$

$$= (40\text{ mm})\mathbf{i} - (180\text{ mm})\mathbf{k}$$

$$l_{CD} = \sqrt{(90)^2 + (180)^2}$$

$$= 201.246\text{ mm}$$

$$\mathbf{r}_{B/C} = \frac{c(\mathbf{r}_{D/C})}{180}$$

$$= \frac{80(90\mathbf{i} - 180\mathbf{k})}{180}$$

$$= (40\text{ mm})\mathbf{i} - (80\text{ mm})\mathbf{k}$$

$$\mathbf{r}_B = \mathbf{r}_C + \mathbf{r}_{B/C}$$

$$= 180\mathbf{k} + 40\mathbf{i} - 80\mathbf{k}$$

$$= (40\text{ mm})\mathbf{i} + (100\text{ mm})\mathbf{k}$$

$$\mathbf{r}_{A/B} = \mathbf{r}_A - \mathbf{r}_B$$

$$= -(40\text{ mm})\mathbf{i} + (y\text{ mm})\mathbf{j} - (100\text{ mm})\mathbf{k}$$

$$l_{AB}^2 = x_{A/B}^2 + y^2 + z_{A/B}^2: \quad 300^2 = (-40)^2 + y^2 + (-100)^2$$

$$y = 280\text{ mm},$$

$$\mathbf{r}_{A/B} = (-40\text{ mm})\mathbf{i} + (280\text{ mm})\mathbf{j} - (100\text{ mm})\mathbf{k}$$

Velocity of collar B. $$\mathbf{v}_B = v_B \frac{\mathbf{r}_{D/C}}{l_{CD}}$$

$$\mathbf{v}_B = \frac{(50)(90\mathbf{i} - 180\mathbf{k})}{201.246}$$

$$= (22.3607 \text{ mm/s})\mathbf{i} - (44.7214 \text{ mm/s})\mathbf{k}$$

Velocity of collar A. $$\mathbf{v}_A = v_A\mathbf{j}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

where $$\mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0.$

Forming $\mathbf{r}_{A/B} \cdot \mathbf{v}_A$, we get $$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot (\mathbf{v}_B + \mathbf{v}_{A/B})$$

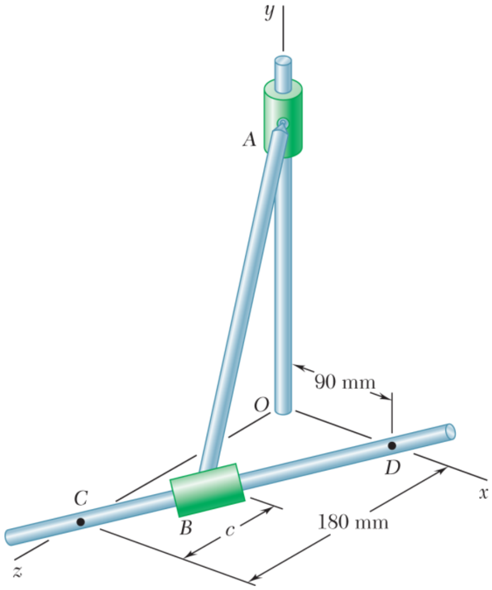
$$= \mathbf{r}_{A/B} \cdot \mathbf{v}_B + \mathbf{r}_{A/B} \cdot \mathbf{v}_{A/B}$$

or $$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot \mathbf{v}_B \qquad (1)$$

From Eq. (1), $$(-40\mathbf{i} + 280\mathbf{j} - 100\mathbf{k}) \cdot (v_A\mathbf{j}) = (-40\mathbf{i} + 280\mathbf{j} - 100\mathbf{k}) \cdot (22.3607\mathbf{i} - 44.7214\mathbf{k})$$

$$280v_A = (-40)(22.3607) + (-100)(-44.7214)$$

or $$v_A = 12.7775 \text{ mm/s} \qquad \mathbf{v}_A = (12.78 \text{ mm/s})\mathbf{j} ◀$$



PROBLEM 15.209

Rod *AB* of length 300 mm is connected by ball-and-socket joints to collars *A* and *B*, which slide along the two rods shown. Knowing that collar *B* moves toward Point *D* at a constant speed of 50 mm/s, determine the velocity of collar *A* when $c = 120$ mm.

SOLUTION

Geometry.

$$\mathbf{r}_A = y\mathbf{j},$$

$$\mathbf{r}_D = (90\text{ mm})\mathbf{i}$$

$$\mathbf{r}_C = (180\text{ mm})\mathbf{k}$$

$$\mathbf{r}_{D/C} = \mathbf{r}_D - \mathbf{r}_C$$

$$= (90\text{ mm})\mathbf{i} - (180\text{ mm})\mathbf{k}$$

$$l_{CD} = \sqrt{(90)^2 + (-180)^2}$$

$$= 201.246\text{ mm}$$

$$\mathbf{r}_{B/C} = \frac{c(\mathbf{r}_{D/C})}{180}$$

$$= \frac{120(90\mathbf{i} - 180\mathbf{k})}{180}$$

$$= (60\text{ mm})\mathbf{i} - (120\text{ mm})\mathbf{k}$$

$$\mathbf{r}_B = \mathbf{r}_C + \mathbf{r}_{B/C}$$

$$= 180\mathbf{k} + 60\mathbf{i} - 120\mathbf{k}$$

$$= (60\text{ mm})\mathbf{i} + (60\text{ mm})\mathbf{k}$$

$$\mathbf{r}_{A/B} = \mathbf{r}_A - \mathbf{r}_B$$

$$= -60\mathbf{i} + y\mathbf{j} - 60\mathbf{k}$$

$$l_{AB}^2 = x_{A/B}^2 + y^2 + z_{A/B}^2: \quad 300^2 = 60^2 + y^2 + 60^2$$

$$y = 287.75\text{ mm},$$

$$\mathbf{r}_{A/B} = (-60\text{ mm})\mathbf{i} + (287.75\text{ mm})\mathbf{j} - (60\text{ mm})\mathbf{k}$$

PROBLEM 15.209 (Continued)

Velocity of collar B.

$$\mathbf{v}_B = v_B \frac{\mathbf{r}_{D/C}}{l_{CD}}$$

$$\mathbf{v}_B = \frac{(50)(90\mathbf{i} - 180\mathbf{k})}{201.246}$$

$$= (22.3607 \text{ mm/s})\mathbf{i} - (44.7214 \text{ mm/s})\mathbf{k}$$

Velocity of collar A.

$$\mathbf{v}_A = v_A \mathbf{j}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

where $\mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$

Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0$.

Forming $\mathbf{r}_{A/B} \cdot \mathbf{v}_A$, we get

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot (\mathbf{v}_B + \mathbf{v}_{A/B})$$

$$= \mathbf{r}_{A/B} \cdot \mathbf{v}_B + \mathbf{r}_{A/B} \cdot \mathbf{v}_{A/B}$$

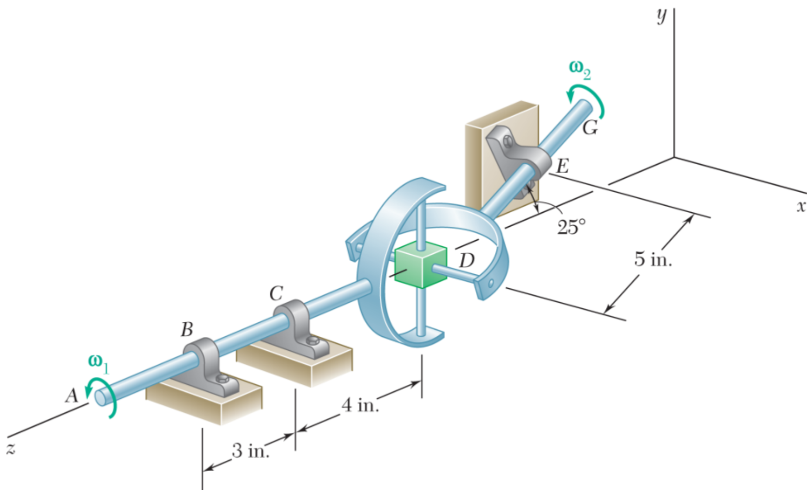
or

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot \mathbf{v}_B \qquad (1)$$

From Eq. (1), $(-60\mathbf{i} + 287.75\mathbf{j} - 60\mathbf{k}) \cdot (v_A \mathbf{j}) = (-60\mathbf{i} + 287.75\mathbf{j} - 60\mathbf{k}) \cdot (22.3607\mathbf{i} - 44.7214\mathbf{j})$

$$287.75 v_A = (-60)(22.3607) + (-60)(-44.7214)$$

or $v_A = 4.6626 \text{ mm/s}$ $\qquad \mathbf{v}_A = (4.66 \text{ mm/s})\mathbf{j}$ ◀



PROBLEM 15.210

Two shafts *AC* and *EG*, which lie in the vertical *yz* plane, are connected by a universal joint at *D*. Shaft *AC* rotates with a constant angular velocity ω_1 as shown. At a time when the arm of the crosspiece attached to shaft *AC* is vertical, determine the angular velocity of shaft *EG*.

SOLUTION

Angular velocity of shaft AC. $\boldsymbol{\omega}_{AC} = \omega_1 \mathbf{k}$

Let $\omega_3 \mathbf{j}$ be the angular velocity of body *D* relative to shaft *AD*.

Angular velocity of body D. $\boldsymbol{\omega}_D = \omega_1 \mathrm{k} + \omega_3 \mathbf{j}$

Angular velocity of shaft EG. $\boldsymbol{\omega}_{EG} = \omega_2(\cos 25° \mathbf{k} - \sin 25° \mathbf{j})$

Let $\omega_4 \mathbf{i}$ be the angular velocity of body *D* relative to shaft *EG*.

Angular velocity of body D. $\boldsymbol{\omega}_D = \omega_2(\cos 25° \mathbf{k} - \sin 25° \mathbf{j}) + \omega_4 \mathbf{i}$

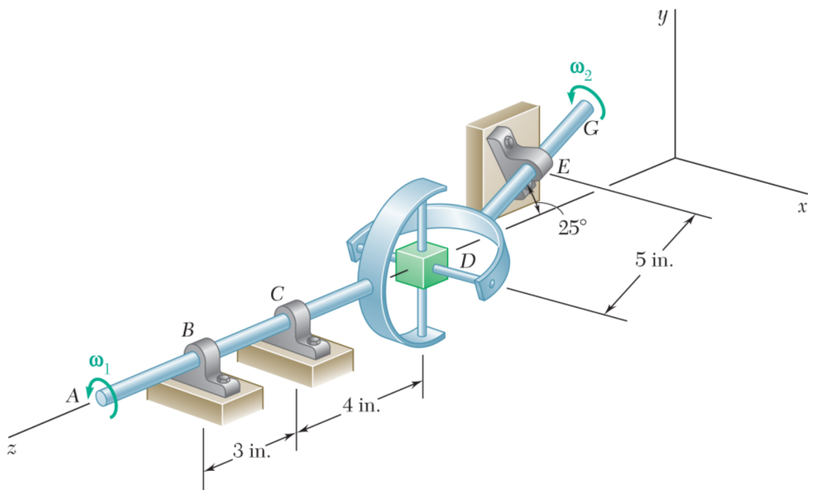
Equate the two expressions for ω_D and resolve into components.

$$\mathbf{i}: \quad 0 = \omega_4 \tag{1}$$

$$\mathbf{j}: \quad \omega_3 = -\omega_2 \sin 25° \tag{2}$$

$$\mathbf{k}: \quad \omega_1 = \omega_2 \cos 25° \tag{3}$$

From Eq. (3), $\omega_2 = \dfrac{\omega_1}{\cos 25°}$ $\qquad \boldsymbol{\omega}_{EG} = \dfrac{\omega_2}{\cos 25°}(-\sin 25° \mathbf{j} + \cos 25° \mathbf{k})$ ◀



PROBLEM 15.211

Solve Problem 15.210, assuming that the arm of the crosspiece attached to the shaft *AC* is horizontal.

PROBLEM 15.210 Two shafts *AC* and *EG*, which lie in the vertical *yz* plane, are connected by a universal joint at *D*. Shaft *AC* rotates with a constant angular velocity ω_1 as shown. At a time when the arm of the crosspiece attached to shaft *AC* is vertical, determine the angular velocity of shaft *EG*.

SOLUTION

Angular velocity of shaft AC. $\boldsymbol{\omega}_{AC} = \omega_1 \mathbf{k}$

Let $\omega_3 \mathbf{i}$ be the angular velocity of body *D* relative to shaft *AD*.

Angular velocity of body D. $\boldsymbol{\omega}_D = \omega_1 \mathbf{k} + \omega_3 \mathbf{i}$

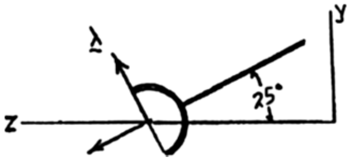
Angular velocity of shaft EG. $\boldsymbol{\omega}_{EG} = \omega_2 (\cos 25° \mathbf{k} - \sin 25° \mathbf{j})$

Let $\omega_4 \boldsymbol{\lambda}$ be the angular velocity of body *D* relative to shaft *EG*,

where $\boldsymbol{\lambda}$ is a unit vector along the clevis axle attached to shaft *EG*.

$$\boldsymbol{\lambda} = \cos 25° \mathbf{j} + \sin 25° \mathbf{k}$$

$$\omega_4 \boldsymbol{\lambda} = \omega_4 \cos 25° \mathbf{j} + \omega_4 \sin 25° \mathbf{k}$$



Angular velocity of body D.

$$\boldsymbol{\omega}_D = \boldsymbol{\omega}_{EG} + \omega_4 \boldsymbol{\lambda}$$

$$\boldsymbol{\omega}_D = (\omega_4 \cos 25° - \omega_2 \sin 25°)\mathbf{j} + (\omega_4 \sin 25° + \omega_2 \cos 25°)\mathbf{k}$$

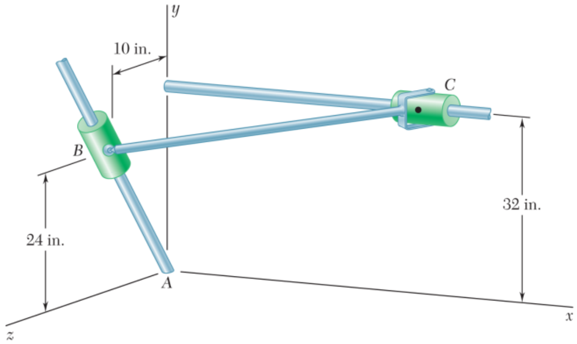
Equate the two expressions for $\boldsymbol{\omega}_D$ and resolve into components.

$$\mathbf{i}: \quad \omega_3 = 0 \tag{1}$$

$$\mathbf{j}: \quad 0 = \omega_4 \cos 25° - \omega_2 \sin 25° \tag{2}$$

$$\mathbf{k}: \quad \omega_1 = \omega_4 \sin 25° + \omega_2 \cos 25° \tag{3}$$

From Eqs. (2) and (3), $\omega_2 = \omega_1 \cos 25°$ $\boldsymbol{\omega}_{EG} = \omega_1 \cos 25°(-\sin 25° \mathbf{j} + \cos 25° \mathbf{k})$ ◀



PROBLEM 15.212

Rod *BC* has a length of 42 in.and is connected by a ball-and-socket joint to collar *B* and by a clevis connection to collar *C*. Knowing that collar *B* moves toward *A* at a constant speed of 19.5 in./s, determine at the instant shown (*a*) the angular velocity of the rod, (*b*) the velocity of collar *C*.

SOLUTION

Geometry:
$$\mathbf{r}_C = x_C\mathbf{i} + (640 \text{ mm})\mathbf{j},\ \mathbf{r}_B = (24 \text{ in.})\mathbf{j} + (10 \text{ in.})\mathbf{k}$$

$$l_{AB} = \sqrt{24^2 + 10^2} = 26 \text{ in.}$$

$$\mathbf{r}_{C/B} = x_C\mathbf{i} + (8 \text{ in.})\mathbf{j} - (10 \text{ in.})\mathbf{k}$$

Length of rod *BC*.
$$l_{BC}^2 = 42^2 = x_C^2 + 8^2 + 10^2$$

Solving for x_C,
$$x_C = 40 \text{ in.}$$

$$\mathbf{r}_{C/B} = (40 \text{ in.})\mathbf{i} + (8 \text{ in.})\mathbf{j} - (10 \text{ in.})\mathbf{k}$$

Velocity:
$$\mathbf{v}_B = \frac{19.5}{26}(-24\mathbf{j} - 10\mathbf{k}) = (-18 \text{ in./s})\mathbf{j} - (7.5 \text{ in./s})\mathbf{k}$$

$$\mathbf{v}_C = v_C\mathbf{i}$$

Angular velocity of collar C:
$$\boldsymbol{\omega}_C = \omega_C\mathbf{i}$$

The axle of the clevis at *C* is perpendicular to the *x*-axis and to the rod *BC*.

A vector along this axle is
$$\mathbf{p} = \mathbf{i} \times \mathbf{r}_{C/B}$$

$$\mathbf{p} = \mathbf{i} \times (40\mathbf{i} + 8\mathbf{j} - 10\mathbf{k}) = (10 \text{ in.})\mathbf{j} + (8 \text{ in.})\mathbf{k}$$

$$p = \sqrt{10^2 + 8^2} = 12.806 \text{ in.}$$

Let $\boldsymbol{\lambda}$ be a unit vector along the axle.
$$\boldsymbol{\lambda} = \frac{\mathbf{p}}{p} = 0.78087\mathbf{j} + 0.62470\mathbf{k}$$

Let $\boldsymbol{\omega}_s = \omega_s\boldsymbol{\lambda}$ be the angular velocity of rod *BC* relative to collar *C*.

$$\boldsymbol{\omega}_s = 0.78087\omega_s\mathbf{j} + 0.62470\omega_s\mathbf{k}$$

Angular velocity of rod BC:
$$\boldsymbol{\omega}_{BC} = \boldsymbol{\omega}_C + \boldsymbol{\omega}_s$$

$$\boldsymbol{\omega}_{BC} = \omega_C\mathbf{i} + 0.78087\omega_s\mathbf{j} + 0.62470\omega_s\mathbf{k}$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$v_C\mathbf{i} = -18\mathbf{j} - 7.5\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_C & 0.78087\omega_s & 0.62470\omega_s \\ 40 & 8 & -10 \end{vmatrix}$$

PROBLEM 15.212 (Continued)

Resolving into components,

$$\mathbf{i}:\quad v_C = -12.806\omega_s \tag{1}$$

$$\mathbf{j}:\quad 0 = -18 + 10\omega_C + 24.988\omega_s \tag{2}$$

$$\mathbf{k}:\quad 0 = -7.5 + 8\omega_C - 31.235\omega_s \tag{3}$$

Solving the simultaneous equations (1), (2), and (3),

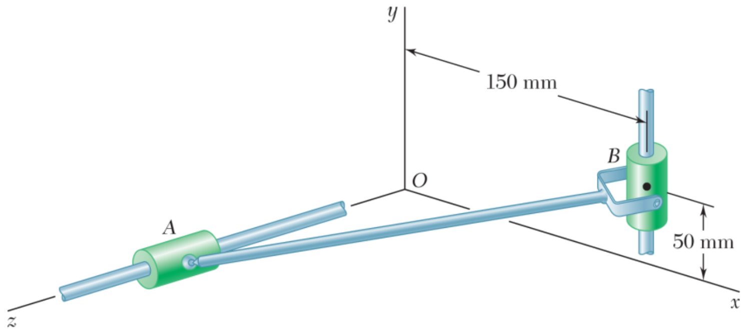
$$\omega_C = 1.4634 \text{ rad/s},\quad \omega_s = 0.13470 \text{ rad/s},\quad v_C = -1.725 \text{ in./s}$$

(*a*) Angular velocity of rod BC:

$$\boldsymbol{\omega}_{BC} = 1.4634\mathbf{i} + (0.78087)(0.13470)\mathbf{j} + (0.62470)(0.13470)\mathbf{k}$$

$$\boldsymbol{\omega}_{BC} = (1.463 \text{ rad/s})\mathbf{i} + (0.1052 \text{ rad/s})\mathbf{j} + (0.0841 \text{ rad/s})\mathbf{k} \blacktriangleleft$$

(*b*) Velocity of collar C: $\mathbf{v}_C = -(1.725 \text{ in./s})\mathbf{i} \blacktriangleleft$



PROBLEM 15.213

Rod *AB* has a length of 275 mm and is connected by a ball-and-socket joint to collar *A* and by a clevis connection to collar *B*. Knowing that collar *B* moves down at a constant speed of 1.35 m/s, determine at the instant shown (*a*) the angular velocity of the rod, (*b*) the velocity of collar *A*.

SOLUTION

Geometry. Determine the position of collar *A*.

$$\mathbf{r}_A = z_A\mathbf{k}, \quad \mathbf{r}_B = x_B\mathbf{i} + y_B\mathbf{j} = (150\text{ mm})\mathbf{i} + (50\text{ mm})\mathbf{j}$$

$$\mathbf{r}_{A/B} = z_A\mathbf{k} - 150\mathbf{i} - 50\mathbf{j}$$

Length of rod *AB*: $l_{AB}^2 = 275^2 = z_A^2 - 150^2 - 50^2$

Solving for z_A, $z_A = 225\text{ mm}$

$$\mathbf{r}_{A/B} = (-150\text{ mm})\mathbf{i} - (50\text{ mm})\mathbf{j} + (225\text{ mm})\mathbf{k} = (-0.15\text{ m})\mathbf{i} - (0.05\text{ m})\mathbf{j} + (0.225\text{ m})\mathbf{k}$$

Velocity. $\mathbf{v}_B = (-1.35\text{ m/s})\mathbf{j}, \quad \mathbf{v}_A = v_A\mathbf{k}$

Angular velocity of collar B. $\boldsymbol{\omega}_B = \omega_B\mathbf{j}$

The axle of the clevis at *B* is perpendicular to both the *y*-axis and the rod *AB*.

A vector along this axle is $\mathbf{p} = \mathbf{j} \times \mathbf{r}_{A/B}$

$$\mathbf{p} = \mathbf{j} \times r_{A/B} = \mathbf{j} \times (-150\mathbf{i} - 50\mathbf{j} + 225\mathbf{k}) = (225\text{ mm})\mathbf{i} - (150\text{ mm})\mathbf{k}$$

$$\mathbf{p} = \sqrt{225^2 + 150^2} = 270.42\text{ mm}$$

Let $\boldsymbol{\lambda}$ be a unit vector along the axle. $\boldsymbol{\lambda} = \dfrac{\mathbf{p}}{p} = 0.83205\mathbf{i} + 0.55470\mathbf{k}$

Let $\boldsymbol{\omega}_s = \omega_s\boldsymbol{\lambda}$ be the angular velocity of rod *AB* relative to collar *B*.

$$\boldsymbol{\omega}_s = 0.83205\omega_s\mathbf{i} + 0.55470\omega_s\mathbf{k}$$

Angular velocity of rod AB. $\boldsymbol{\omega}_{AB} = \boldsymbol{\omega}_B + \boldsymbol{\omega}_s$

$$\boldsymbol{\omega}_{AB} = 0.83205\omega_s\mathbf{i} + \omega_B\mathbf{j} + 0.55470\omega_s\mathbf{k}$$

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

$$v_A\mathbf{k} = -1.35\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.83205\omega_s & \omega_B & 0.55470\omega_s \\ -0.15 & -0.05 & 0.225 \end{vmatrix}$$

Resolving into components,

$$\mathbf{i}: \quad 0 = 0 + 0.225\omega_B + 0.027735\omega_s \qquad (1)$$

$$\mathbf{j}: \quad 0 = -1.35 + -0.270416\omega_s \qquad (2)$$

$$\mathbf{k}: \quad v_A = 0 + 0.15\omega_B - 0.041603\omega_s \qquad (3)$$

From (2), $\omega_s = -4.9923$ rad/s

From (1), $\omega_B = 0.61539$ rad/s

(*a*) *Angular velocity of rod AB.*

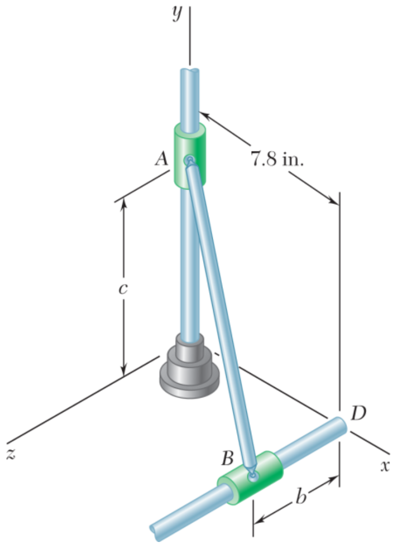
$$\boldsymbol{\omega}_{AB} = (0.83205)(-4.9923)\mathbf{i} + 0.61539\mathbf{j} + (0.55470)(-4.9923)\mathbf{k}$$

$$\boldsymbol{\omega}_{AB} = -(4.15 \text{ rad/s})\mathbf{i} + (0.615 \text{ rad/s})\mathbf{j} - (2.77 \text{ rad/s})\mathbf{k} \blacktriangleleft$$

(*b*) *Velocity of collar A*.

From (3), $v_A = (0.15)(0.61539) - (0.041603)(-4.9923) = 0.3000$ m/s

$$\mathbf{v}_A = (0.300 \text{ m/s})\mathbf{k} \blacktriangleleft$$



PROBLEM 15.214

For the mechanism of the Prob.15.204, determine the acceleration of collar A.

PROBLEM 15.204 Rod AB of length 13 in. is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves toward point D at a constant speed of 36 in./s, determine the velocity of collar A when $b = 4$ in.

SOLUTION

Geometry.

$$l_{AB}^2 = x_{A/B}^2 + y_{A/B}^2 + z_{A/B}^2 : \quad 13^2 = (-7.8)^2 + c^2 + (-4)^2, \quad c = 9.6 \text{ in.}$$

$$\mathbf{r}_{A/B} = (-7.8 \text{ in.})\mathbf{i} + (9.6 \text{ in.})\mathbf{j} - (4 \text{ in.})\mathbf{k}$$

Velocity of collar B.

$$\mathbf{v}_B = (-36 \text{ in./s})\mathbf{k}$$

Velocity of collar A.

$$\mathbf{v}_A = v_A\mathbf{j}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}, \qquad \text{where} \qquad \mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get

$$\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0$$

Forming $\mathbf{r}_{A/B} \cdot \mathbf{v}_A$, we get

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot (\mathbf{v}_B + \mathbf{v}_{A/B}) = \mathbf{r}_{A/B} \cdot \mathbf{v}_B + \mathbf{r}_{A/B} \cdot \mathbf{v}_{A/B}$$

or

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot \mathbf{v}_B \tag{1}$$

From (1),

$$(-7.8\mathbf{i} + 9.6\mathbf{j} - 4\mathbf{k}) \cdot (v_A\mathbf{j}) = (-7.8\mathbf{i} + 9.6\mathbf{j} - 4\mathbf{k}) \cdot (-36\mathbf{k})$$

$$9.6 v_A = 144 \qquad \text{or} \qquad v_A = 15 \text{ in./s}$$

Relative velocity.

$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B$$

$$\mathbf{v}_{A/B} = (15 \text{ in./s})\mathbf{j} + (36 \text{ in./s})\mathbf{k} \qquad (v_{A/B})^2 = 15^2 + 36^2 = 1521 (\text{in./s})^2$$

Acceleration of collar B.

$$\mathbf{a}_B = 0$$

Acceleration of collar A.

$$\mathbf{a}_A = a_A\mathbf{j}$$

PROBLEM 15.214 (Continued)

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}, \qquad \text{where} \qquad \mathbf{a}_{A/B} = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} + \boldsymbol{\omega}_{AB} \times \mathbf{v}_{A/B}$$

Noting that $\boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{A/B} \cdot \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} = 0$

We note also that $\mathbf{r}_{A/B} \cdot \boldsymbol{\omega}_{AB} \times \mathbf{v}_{A/B} = \mathbf{v}_{A/B} \cdot \mathbf{r}_{A/B} \times \boldsymbol{\omega}_{A/B}$

$$= -\mathbf{v}_{A/B} \cdot \mathbf{v}_{A/B} = -\left(v_{A/B}\right)^2$$

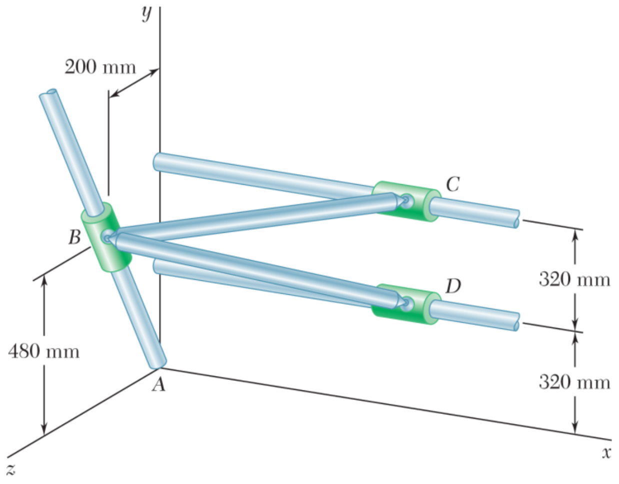
Then, $\mathbf{r}_{A/B} \cdot \mathbf{a}_{A/B} = 0 - \left(v_{A/B}\right)^2 = -\left(v_{A/B}\right)^2$

Forming $\mathbf{r}_{A/B} \cdot \mathbf{a}_A$, we get $\mathbf{r}_{A/B} \cdot \mathbf{a}_A = \mathbf{r}_{A/B} \cdot \left(\mathbf{a}_A + \mathbf{a}_{A/B}\right) = \mathbf{r}_{A/B} \cdot \mathbf{a}_B + \mathbf{r}_{A/B} \cdot \mathbf{a}_{A/B}$

or $$\mathbf{r}_{A/B} \cdot \mathbf{a}_A = \mathbf{r}_{A/B} \cdot \mathbf{a}_B - \left(v_{A/B}\right)^2 \qquad (2)$$

From (2), $\left(-7.8\mathbf{i} + 9.6\mathbf{j} - 4\mathbf{k}\right) \cdot \left(a_A \mathbf{j}\right) = 0 - 1521$

$$9.6 a_A = -1521 \qquad \mathbf{a}_A = -\left(158.4 \text{ in./s}^2\right)\mathbf{j} \blacktriangleleft$$



PROBLEM 15.215

In Problem 15.205, determine the acceleration of collar *C*.

PROBLEM 15.205 Rod *BC* and *BD* are each 840 mm long and are connected by ball-and-socket joints to collars which may slide on the fixed rods shown. Knowing that collar *B* moves toward *A* at a constant speed of 390 mm/s, determine the velocity of collar *C* for the position shown.

SOLUTION

Geometry.

$$l_{BC}^2 = x_{C/B}^2 + y_{C/B}^2 + z_{C/B}^2$$

$$(840)^2 = c^2 + (640-480)^2 + (200)^2 \qquad c = 800m$$

$$\mathbf{r}_{C/B} = (800\text{ mm})\mathbf{i} + (160\text{ mm})\mathbf{j} - (200\text{ mm})\mathbf{k}$$

Velocity of B.

$$\mathbf{v}_B = v_B\left(-\frac{12}{13}\mathbf{j} - \frac{5}{13}\mathbf{k}\right)$$

$$= (390\text{ mm/s})\left(-\frac{12}{13}\mathbf{j} - \frac{5}{13}\mathbf{k}\right)$$

$$= -(360\text{ mm/s})\mathbf{j} - (150\text{ mm/s})\mathbf{k}$$

Velocity of C.

$$\mathbf{v}_C = v_C\mathbf{i}$$

where

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{C/B}$$

$$\mathbf{v}_{C/B} = \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

Noting that $\mathbf{v}_{C/B}$ is perpendicular to $\mathbf{r}_{C/B}$, we get $\mathbf{r}_{C/B} \cdot \mathbf{v}_C = \mathbf{r}_{C/B} \cdot \mathbf{v}_B$

$$(800\mathbf{i} + 160\mathbf{j} - 200\mathbf{k}) \cdot (-360\mathbf{j} + 150\mathbf{k}) = (800\mathbf{i} + 160\mathbf{j} - 200\mathbf{k}) \cdot (v_C\mathbf{i})$$

$$(160)(-360) + (-200)(-150) = 800v_C$$

$$v_C = -34.5\text{ mm/s}$$

$$\mathbf{v}_C = -(34.5\text{ mm/s})\mathbf{i}$$

Relative velocity:

$$\mathbf{v}_{C/B} = \mathbf{v}_C - \mathbf{v}_B$$

$$\mathbf{v}_{C/B} = -(34.5\text{ mm/s})\mathbf{i} + (360\text{ mm/s})\mathbf{j} + (150\text{ mm/s})\mathbf{k}$$

$$v_{C/B}^2 = (34.5)^2 + (360)^2 + (150)^2 = 153290\text{ mm}^2/\text{s}^2$$

Acceleration of collar B:

$$\mathbf{a}_B = 0$$

Acceleration of collar C:

$$\mathbf{a}_C = \mathbf{a}_C\mathbf{i}$$

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}$$

PROBLEM 15.215 (Continued)

where $\mathbf{a}_{C/B} = \boldsymbol{\alpha}_{CB} \times \mathbf{r}_{C/B} + \boldsymbol{\omega}_{CB} \times \mathbf{v}_{C/B}$

Noting that $\boldsymbol{\alpha}_{CB} \times \mathbf{r}_{C/B}$ is perpendicular to $\mathbf{r}_{C/B}$, we get $\mathbf{r}_{C/B} \cdot \boldsymbol{\alpha}_{CB} \times \mathbf{r}_{C/B} = 0$

We note also that $\mathbf{r}_{B/C} \cdot \boldsymbol{\omega}_{CB} \times \mathbf{v}_{C/B} = \mathbf{v}_{C/B} \cdot \mathbf{r}_{C/B} \times \boldsymbol{\omega}_{C/B}$

$$= -\mathbf{v}_{C/B} \cdot \mathbf{v}_{C/B} = -(v_{C/B})^2$$

Then, $\mathbf{r}_{C/B} \cdot \mathbf{a}_C = 0 - (v_{C/B})^2 = -(v_{C/B})^2$

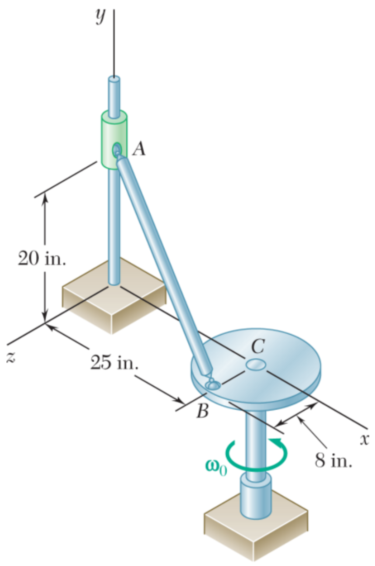
Forming $\mathbf{r}_{C/B} \cdot \mathbf{a}_C$, we get $\mathbf{r}_{C/B} \cdot \mathbf{a}_C = \mathbf{r}_{C/B} \cdot (\mathbf{a}_B + \mathbf{a}_{C/B}) = \mathbf{r}_{B/C} \cdot \mathbf{a}_B + \mathbf{r}_{B/C} \cdot \mathbf{a}_{C/B}$

so that $\mathbf{r}_{C/B} \cdot \mathbf{a}_C = \mathbf{r}_{C/B} \cdot \mathbf{a}_B - (v_{C/B})^2$ (2)

From Equation (2), $(800\mathbf{i} + 160\mathbf{j} - 200\mathbf{k}) \cdot a_C\mathbf{i} = 0 - 153290$

$$800a_C = -153290$$

$a_C = -191.6 \text{ mm/s}^2$ $\mathbf{a}_C = -(191.6 \text{ mm/s}^2)\mathbf{i}$ ◀



PROBLEM 15.216

In Problem 15.206, determine the acceleration of collar *A*.

PROBLEM 15.206 Rod *AB* is connected by ball-and-socket joints to collar *A* and to the 16-in.-diameter disk *C*. Knowing that disk *C* rotates counterclockwise at the constant rate $\omega_0 = 3$ rad/s in the *zx* plane, determine the velocity of collar *A* for the position shown.

SOLUTION

Geometry.

$$\mathbf{r}_{B/C} = (-8\text{ in.})\mathbf{k}$$

$$\mathbf{r}_{A/B} = -(25\text{ in.})\mathbf{i} + (20\text{ in.})\mathbf{j} - (8\text{ in.})\mathbf{k}$$

Velocity at B.

$$\mathbf{v}_B = \omega_0\mathbf{j} \times \mathbf{r}_{B/C}$$

$$= 3\mathbf{j} \times (-8\mathbf{k})$$

$$= (24\text{ in./s})\mathbf{i}$$

Velocity of collar A.

$$\mathbf{v}_A = v_A\mathbf{j}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

where $\mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$

Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0$.

Forming $\mathbf{r}_{A/B} \cdot \mathbf{v}_A$, we get

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot (\mathbf{v}_B + \mathbf{v}_{A/B})$$

$$= \mathbf{r}_{A/B} \cdot \mathbf{v}_B + \mathbf{r}_{A/B} \cdot \mathbf{v}_{A/B}$$

or

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot \mathbf{v}_B \tag{1}$$

From Eq. (1)

$$(-25\mathbf{i} + 20\mathbf{j} - 8\mathbf{k}) \cdot (v_A\mathbf{j}) = (-25\mathbf{i} + 20\mathbf{j} - 8\mathbf{k}) \cdot (24\mathbf{i})$$

$$20v_A = -600$$

or

$$v_A = -30\text{ in./s}$$

Relative velocity.

$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B$$

$$\mathbf{v}_{A/B} = (-30\text{ in./s})\mathbf{j} + (24\text{ in./s})\mathbf{i}$$

$$(v_{A/B})^2 = (-30)^2 + (24)^2$$

$$= 1476\text{ in}^2/\text{s}^2$$

PROBLEM 15.216 (Continued)

Acceleration at B.

$$\begin{aligned}\mathbf{a}_B &= \omega_0\mathbf{j}\times\mathbf{v}_B\\ &= 3\mathbf{j}\times 24\mathbf{i}\\ &= -(72\text{ in./s}^2)\mathbf{k}\end{aligned}$$

Acceleration of collar A.

$$\mathbf{a}_A = a_A\mathbf{j}$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

where

$$\mathbf{a}_{A/B} = \boldsymbol{\alpha}_{AB}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}_{AB}\times\mathbf{v}_{A/B}$$

Noting that $\boldsymbol{\alpha}_{AB}\times\mathbf{r}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{A/B}\cdot\boldsymbol{\alpha}_{AB}\times\mathbf{r}_{A/B} = 0$

We note also that

$$\begin{aligned}\mathbf{r}_{A/B}\cdot\boldsymbol{\omega}_{AB}\times\mathbf{v}_{A/B} &= \mathbf{v}_{A/B}\cdot\mathbf{r}_{A/B}\times\boldsymbol{\omega}_{A/B}\\ &= -\mathbf{v}_{A/B}\cdot\mathbf{v}_{A/B}\\ &= -(v_{A/B})^2\end{aligned}$$

Then

$$\begin{aligned}\mathbf{r}_{A/B}\cdot\mathbf{a}_{A/B} &= 0-(v_{A/B})^2\\ &= -(v_{A/B})^2\end{aligned}$$

Forming $\mathbf{r}_{A/B}\cdot\mathbf{a}_A$, we get

$$\begin{aligned}\mathbf{r}_{A/B}\cdot\mathbf{a}_A &= \mathbf{r}_{A/B}\cdot(\mathbf{a}_A+\mathbf{a}_{A/B})\\ &= \mathbf{r}_{A/B}\cdot\mathbf{a}_B + \mathbf{r}_{A/B}\cdot\mathbf{a}_{A/B}\end{aligned}$$

or

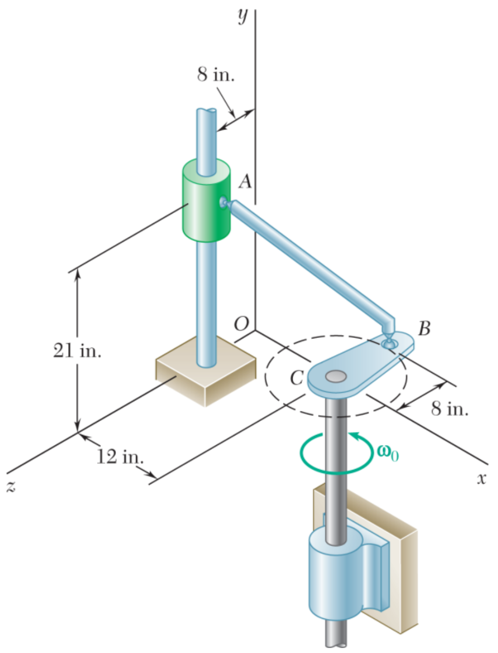
$$\mathbf{r}_{A/B}\cdot\mathbf{a}_A = \mathbf{r}_{A/B}\cdot\mathbf{a}_B - (v_{A/B})^2 \qquad (2)$$

From Eq. (2),

$$(-25\mathbf{i}+20\mathbf{j}-8\mathbf{k})\cdot(a_A\mathbf{j}) = (-25\mathbf{i}+20\mathbf{j}-8\mathbf{k})\cdot(-72\mathbf{k})-1476$$

$$\begin{aligned}20a_A &= 576-1476\\ &= -45\text{ in./s}^2\end{aligned}$$

$\mathbf{a}_A = -(45.0\text{ in./s}^2)\mathbf{j}$ ◀



PROBLEM 15.217

In Problem 15.207, determine the acceleration of collar *A*.

PROBLEM 15.207 Rod *AB* of length 29 in. is connected by ball-and-socket joints to the rotating crank *BC* and to the collar *A*. Crank *BC* is of length 8 in. and rotates in the horizontal *xz* plane at the constant rate $\omega_0 = 10$ rad/s. At the instant shown, when crank *BC* is parallel to the *z* axis, determine the velocity of collar *A*.

SOLUTION

Geometry.

$$\mathbf{r}_{B/C} = (-8 \text{ in.})\mathbf{k},$$
$$\mathbf{r}_{A/B} = (-12 \text{ in.})\mathbf{i} + (21 \text{ in.})\mathbf{j} + (16 \text{ in.})\mathbf{k}$$

Velocity at B.

$$\begin{aligned}\mathbf{v}_B &= \omega_0 \mathbf{j} \times \mathbf{r}_{B/C}\\ &= 10\mathbf{j} \times (-8\mathbf{k})\\ &= (-80 \text{ in./s})\mathbf{i}\end{aligned}$$

Velocity of collar A.

$$\mathbf{v}_A = v_A \mathbf{j}$$
$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

where

$$\mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0.$

Forming $\mathbf{r}_{A/B} \cdot \mathbf{v}_A$, we get

$$\begin{aligned}\mathbf{r}_{A/B} \cdot \mathbf{v}_A &= \mathbf{r}_{A/B} \cdot (\mathbf{v}_B + \mathbf{v}_{A/B})\\ &= \mathbf{r}_{A/B} \cdot \mathbf{v}_B + \mathbf{r}_{A/B} \cdot \mathbf{v}_{A/B}\end{aligned}$$

or

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot \mathbf{v}_B \qquad (1)$$

From Eq. (1)

$$(-12\mathbf{i} + 21\mathbf{j} + 16\mathbf{k}) \cdot (v_A \mathbf{j}) = (-12\mathbf{i} + 21\mathbf{j} - 16\mathbf{k}) \cdot (-80\mathbf{i})$$
$$21 v_A = 960$$

or

$$v_A = 45.714 \text{ in./s} \qquad \mathbf{v}_A = (45.7 \text{ in./s})\mathbf{j} \blacktriangleleft$$

Relative velocity.

$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B$$
$$\mathbf{v}_{A/B} = (45.714 \text{ in./s})\mathbf{j} + (80 \text{ in./s})\mathbf{i}$$
$$(v_{A/B})^2 = (45.714)^2 + (80)^2$$
$$= 8489.8 \text{ (in./s}^2)$$

Acceleration of Point B.

$$\mathbf{a}_B = \omega_0 \mathbf{j} \times \mathbf{v}_B$$
$$= 10\mathbf{j} \times (-80)\mathbf{i}$$
$$= (800 \text{ in./s}^2)\mathbf{k}$$

Acceleration of collar A.

$$\mathbf{a}_A = a_A \mathbf{j}$$
$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

where

$$\mathbf{a}_{A/B} = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} + \boldsymbol{\omega}_{AB} \times \mathbf{v}_{A/B}$$

Noting that $\boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{A/B} \cdot \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} = 0$.

We note also that

$$\mathbf{r}_{A/B} \cdot \boldsymbol{\omega}_{AB} \times \mathbf{v}_{A/B} = \mathbf{v}_{A/B} \cdot \mathbf{r}_{A/B} \times \boldsymbol{\omega}_{A/B}$$
$$= -\mathbf{v}_{A/B} \cdot \mathbf{v}_{A/B}$$
$$= -(v_{A/B})^2$$

Then

$$\mathbf{r}_{A/B} \cdot \mathbf{a}_{A/B} = 0 - (v_{A/B})^2$$
$$= -(v_{A/B})^2$$

Forming $\mathbf{r}_{A/B} \cdot \mathbf{a}_A$, we get

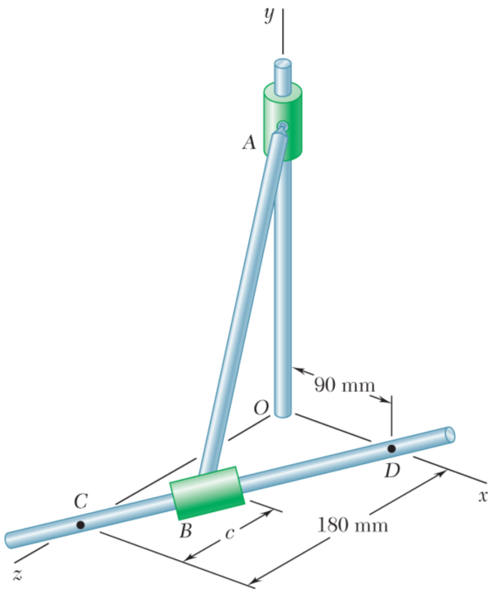
$$\mathbf{r}_{A/B} \cdot \mathbf{a}_A = \mathbf{r}_{A/B} \cdot (\mathbf{a}_A + \mathbf{a}_{A/B})$$
$$= \mathbf{r}_{A/B} \cdot \mathbf{a}_B + \mathbf{r}_{A/B} - \mathbf{a}_{A/B}$$

or

$$\mathbf{r}_{A/B} \cdot \mathbf{a}_A = \mathbf{r}_{A/B} \cdot \mathbf{a}_B - (v_{A/B})^2 \qquad (2)$$

From Eq. (2) $\quad (-12\mathbf{i} + 21\mathbf{j} + 16\mathbf{k}) \cdot (a_A \mathbf{j}) = (-12\mathbf{i} + 21\mathbf{j} + 16\mathbf{k}) \cdot (800\mathbf{k}) - 8489.8$

$$21 a_A = 12{,}800 - 8489.8 \qquad \mathbf{a}_A = (205 \text{ in./s}^2)\mathbf{j} \blacktriangleleft$$



PROBLEM 15.218

In Problem 15.208, determine the acceleration of collar *A*.

PROBLEM 15.208 Rod *AB* of length 300 mm is connected by ball-and-socket joints to collars *A* and *B*, which slide along the two rods shown. Knowing that collar *B* moves toward Point *D* at a constant speed of 50 mm/s, determine the velocity of collar *A* when $c = 80$ mm.

SOLUTION

Geometry.

$$\mathbf{r}_A = y\mathbf{j}, \quad \mathbf{r}_D = (90\text{ mm})\mathbf{i} \quad \mathbf{r}_C = (180\text{ mm})\mathbf{k}$$

$$\mathbf{r}_{D/C} = \mathbf{r}_D - \mathbf{r}_C = (90\text{ mm})\mathbf{i} - (180\text{ mm})\mathbf{k}$$

$$l_{CD} = \sqrt{(90)^2 + (180)^2} = 201.246\text{ mm}$$

$$\mathbf{r}_{B/C} = \frac{c(\mathbf{r}_{D/C})}{180} = \frac{80(90\mathbf{i} - 180\mathbf{k})}{180} = (40\text{ mm})\mathbf{i} - (80\text{ mm})\mathbf{k}$$

$$\mathbf{r}_B = \mathbf{r}_C + \mathbf{r}_{B/C} = 180\mathbf{k} + 40\mathbf{i} - 80\mathbf{k} = (40\text{ mm})\mathbf{i} + (100\text{ mm})\mathbf{k}$$

$$\mathbf{r}_{A/B} = \mathbf{r}_A - \mathbf{r}_B = -(40\text{ mm})\mathbf{i} + (y\text{ mm})\mathbf{j} - (100\text{ mm})\mathbf{k}$$

$$l_{AB}^2 = x_{A/B}^2 + y^2 + z_{A/B}^2: \quad 300^2 = (-40)^2 + y^2 + (-100)^2$$

$$y = 280\text{ mm},$$

$$\mathbf{r}_{A/B} = (-40\text{ mm})\mathbf{i} + (280\text{ mm})\mathbf{j} - (100\text{ mm})\mathbf{k}$$

Velocity of collar B.

$$\mathbf{v}_B = v_B \frac{\mathbf{r}_{D/C}}{l_{CD}}$$

$$\mathbf{v}_B = \frac{(50)(90\mathbf{i} - 180\mathbf{k})}{201.246}$$

$$= (22.3607\text{ mm/s})\mathbf{i} - (44.7214\text{ mm/s})\mathbf{k}$$

Velocity of collar A.

$$\mathbf{v}_A = v_A\mathbf{j}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

where

$$\mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0.$

PROBLEM 15.218 (Continued)

Forming $\mathbf{r}_{A/B} \cdot \mathbf{v}_A$, we get

$$\begin{aligned}\mathbf{r}_{A/B} \cdot \mathbf{v}_A &= \mathbf{r}_{A/B} \cdot (\mathbf{v}_B + \mathbf{v}_{A/B})\\ &= \mathbf{r}_{A/B} \cdot \mathbf{v}_B + \mathbf{r}_{A/B} \cdot \mathbf{v}_{A/B}\end{aligned}$$

or

$$\mathbf{r}_{A/B} \cdot \mathbf{v}_A = \mathbf{r}_{A/B} \cdot \mathbf{v}_B \qquad (1)$$

From (1)

$$\begin{aligned}(-40\mathbf{i} + 280\mathbf{j} - 100\mathbf{k}) \cdot (v_A\mathbf{j}) &= (-40\mathbf{i} + 280\mathbf{j} - 100\mathbf{k}) \cdot (22.3607\mathbf{i} - 44.7214\mathbf{k})\\ 280v_A &= (-40)(22.3607) + (-100)(-44.7214)\end{aligned}$$

or

$$v_A = 12.7775 \text{ mm/s} \qquad \mathbf{v}_A = (12.7775 \text{ mm/s})\,\mathbf{j}$$

Relative velocity.

$$\begin{aligned}\mathbf{v}_{A/B} &= \mathbf{v}_A - \mathbf{v}_B\\ &= (12.7775 \text{ mm/s})\mathbf{i} - (22.3607 \text{ mm/s})\mathbf{j} + (44.7214 \text{ mm/s})\mathbf{k}\\ \mathbf{v}_{A/B} \cdot \mathbf{v}_{A/B} &= (12.7775)^2 + (22.3607)^2 + (44.7214)^2\\ &= 2663.3 \text{ mm}^2/\text{s}^2\end{aligned}$$

Acceleration of collar B.

$$\mathbf{a}_B = 0$$

Acceleration of collar A.

$$\begin{aligned}\mathbf{a}_A &= a_A\mathbf{j}\\ \mathbf{a}_A &= \mathbf{a}_B + \mathbf{a}_{A/B}\end{aligned}$$

where

$$\mathbf{a}_{A/B} = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} + \boldsymbol{\omega}_{AB} \times \mathbf{v}_{A/B}$$

Noting that $\boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{A/B} \cdot \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} = 0$

We note also that

$$\begin{aligned}\mathbf{r}_{A/B} \cdot \boldsymbol{\omega}_{AB} \times \mathbf{v}_{A/B} &= \mathbf{v}_{A/B} \cdot \mathbf{r}_{A/B} \times \boldsymbol{\omega}_{A/B}\\ &= -\mathbf{v}_{A/B} \cdot \mathbf{v}_{A/B}\\ &= -(v_{A/B})^2\end{aligned}$$

Then

$$\begin{aligned}\mathbf{r}_{A/B} \cdot \mathbf{a}_{A/B} &= 0 - (v_{A/B})^2\\ &= -(v_{A/B})^2\end{aligned}$$

Forming $\mathbf{r}_{A/B} \cdot \mathbf{a}_A$, we get

$$\begin{aligned}\mathbf{r}_{A/B} \cdot \mathbf{a}_A &= \mathbf{r}_{A/B} \cdot (\mathbf{a}_A + \mathbf{a}_{A/B})\\ &= \mathbf{r}_{A/B} \cdot \mathbf{a}_B + \mathbf{r}_{A/B} \cdot \mathbf{a}_{A/B}\end{aligned}$$

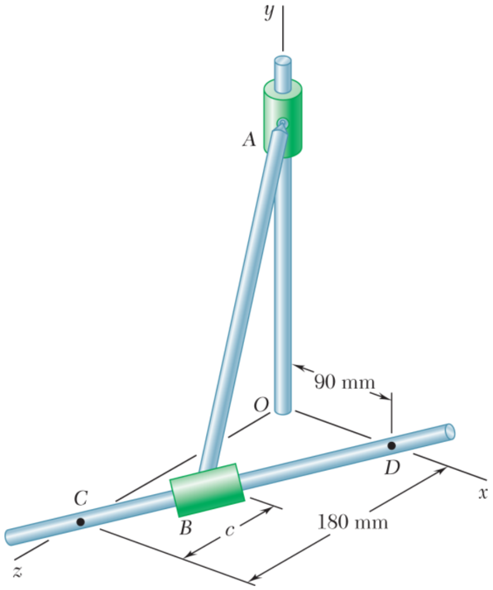
or

$$\mathbf{r}_{A/B} \cdot \mathbf{a}_A = \mathbf{r}_{A/B} \cdot \mathbf{a}_B - (v_{A/B})^2 \qquad (2)$$

From Eq. (2)

$$\begin{aligned}(-40\mathbf{i} + 280\mathbf{j} - 100\mathbf{k}) \cdot (a_A\mathbf{j}) &= 0 - 2663.3\\ 280a_A &= -2663.3 \qquad a_A = -9.512 \text{ mm/s}^2\end{aligned}$$

$$\mathbf{a}_A = -(9.51 \text{ mm/s}^2)\mathbf{j} \blacktriangleleft$$



PROBLEM 15.219

In Problem 15.209, determine the acceleration of collar *A*.

PROBLEM 15.209 Rod *AB* of length 300 mm is connected by ball-and-socket joints to collars *A* and *B*, which slide along the two rods shown. Knowing that collar *B* moves toward Point *D* at a constant speed of 50 mm/s, determine the velocity of collar *A* when $c = 120$ mm.

SOLUTION

Geometry.

$$\mathbf{r}_A = y\mathbf{j}, \quad \mathbf{r}_D = (90\text{ mm})\mathbf{i} \quad \mathbf{r}_C = (180\text{ mm})\mathbf{k}$$

$$\mathbf{r}_{D/C} = \mathbf{r}_D - \mathbf{r}_C = (40\text{ mm})\mathbf{i} - (180\text{ mm})\mathbf{k}$$

$$l_{CD} = \sqrt{(90)^2 + (-180)^2} = 201.246$$

$$\mathbf{r}_{B/C} = \frac{c(\mathbf{r}_{D/C})}{180} = \frac{120(90\mathbf{i} - 180\mathbf{k})}{180} = (60\text{ mm})\mathbf{i} - (120\text{ mm})\mathbf{k}$$

$$\mathbf{r}_B = \mathbf{r}_C + \mathbf{r}_{B/C} = 180\mathbf{k} + 60\mathbf{i} - 120\mathbf{k} = (60\text{ mm})\mathbf{i} + (60\text{ mm})\mathbf{k}$$

$$\mathbf{r}_{A/B} = \mathbf{r}_A - \mathbf{r}_B = -60\mathbf{i} + y\,\mathbf{j} - 60\mathbf{k}$$

$$l_{AB}^2 = x_{A/B}^2 + y^2 + z_{A/B}^2: \quad 300^2 = 60^2 + y^2 + 60^2$$

$$y = 287.75\text{ mm},$$

$$\mathbf{r}_{A/B} = (-60\text{ mm})\mathbf{i} + (287.75\text{ mm})\mathbf{j} - (60\text{ mm})\mathbf{k}$$

Velocity of collar B.

$$\mathbf{v}_B = v_B \frac{\mathbf{r}_{D/C}}{l_{CD}}$$

$$\mathbf{v}_B = \frac{(50)(90\mathbf{i} - 180\mathbf{k})}{201.246}$$

$$= (22.3607\text{ mm/s})\mathbf{i} - (44.7214\text{ mm/s})\mathbf{k}$$

Velocity of collar A.

$$\mathbf{v}_A = v_A\mathbf{j}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

where

$$\mathbf{v}_{A/B} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

Noting that $\mathbf{v}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0$.

Forming $\mathbf{r}_{A/B}\cdot\mathbf{v}_A$, we get $\quad \mathbf{r}_{A/B}\cdot\mathbf{v}_A = \mathbf{r}_{A/B}\cdot(\mathbf{v}_B+\mathbf{v}_{A/B})$

$$= \mathbf{r}_{A/B}\cdot\mathbf{v}_B + \mathbf{r}_{A/B}\cdot\mathbf{v}_{A/B}$$

or $\quad \mathbf{r}_{A/B}\cdot\mathbf{v}_A = \mathbf{r}_{A/B}\cdot\mathbf{v}_B \qquad (1)$

From Eq. (1) $\quad (-60\mathbf{i}+287.75\mathbf{j}-60\mathbf{k})\cdot(v_A\mathbf{j}) = (-60\mathbf{i}+287.75\mathbf{j}-60\mathbf{k})\cdot(1.11803\mathbf{i}-2.23607\mathbf{j})$

$$287.75v_A = (-60)(22.3607)+(-60)(-44.7214)$$

or $\quad v_A = 4.6626\text{ mm/s} \qquad \mathbf{v}_A = (4.6626\text{ mm/s})\mathbf{j}$

Relative velocity. $\quad \mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B$

$$= -(22.3607\text{ mm/s})\mathbf{i} + (4.6626\text{ mm/s})\mathbf{j} + (44.7214\text{ mm/s})\mathbf{k}$$

$$\mathbf{v}_{A/B}\cdot\mathbf{v}_{A/B} = (22.3607)^2+(4.6626)^2+(44.7214)^2$$

$$= 2521.7$$

Acceleration of collar B. $\quad \mathbf{a}_B = 0$

Acceleration of collar A. $\quad \mathbf{a}_A = a_A\mathbf{j}$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

where $\quad \mathbf{a}_{A/B} = \boldsymbol{\alpha}_{AB}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}_{AB}\times\mathbf{v}_{A/B}$

Noting that $\boldsymbol{\alpha}_{AB}\times\mathbf{r}_{A/B}$ is perpendicular to $\mathbf{r}_{A/B}$, we get $\mathbf{r}_{A/B}\cdot\boldsymbol{\alpha}_{AB}\times\mathbf{r}_{A/B} = 0.$

We note also that $\quad \mathbf{r}_{A/B}\cdot\boldsymbol{\omega}_{AB}\times\mathbf{v}_{A/B} = \mathbf{v}_{A/B}\cdot\mathbf{r}_{A/B}\times\boldsymbol{\omega}_{A/B}$

$$= -\mathbf{v}_{A/B}\cdot\mathbf{v}_{A/B}$$

$$= -(v_{A/B})^2$$

Then $\quad \mathbf{r}_{A/B}\cdot\mathbf{a}_{A/B} = 0-(v_{A/B})^2$

$$= -(v_{A/B})^2$$

Forming $\mathbf{r}_{A/B}\cdot\mathbf{a}_A$, we get $\quad \mathbf{r}_{A/B}\cdot\mathbf{a}_A = \mathbf{r}_{A/B}\cdot(\mathbf{a}_A+\mathbf{a}_{A/B})$

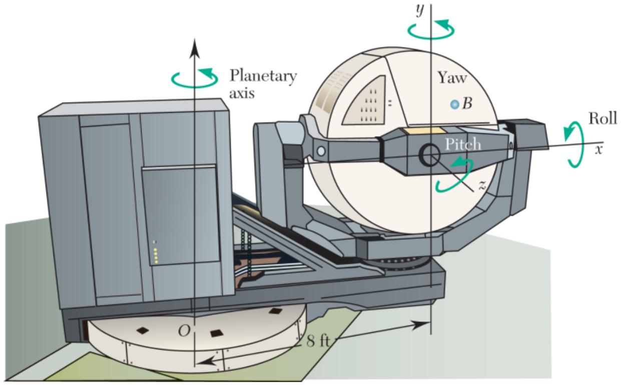
$$= \mathbf{r}_{A/B}\cdot\mathbf{a}_B + \mathbf{r}_{A/B}\cdot\mathbf{a}_{A/B}$$

or $\quad \mathbf{r}_{A/B}\cdot\mathbf{a}_A = \mathbf{r}_{A/B}\cdot\mathbf{a}_B - (v_{A/B})^2 \qquad (2)$

From Eq. (2), $\quad (-60\mathbf{i}+287.75\mathbf{j}-60\mathbf{k})\cdot(a_A\mathbf{j}) = 0-2521.7$

$$287.75a_A = -2521.7 \qquad a_A = -8.764\text{ mm/s}^2$$

$$\mathbf{a}_A = -(8.76\text{ mm/s}^2)\mathbf{j} \blacktriangleleft$$



PROBLEM 15.220

A flight simulator is used to train pilots on how to recognize spatial disorientation. It has four degrees of freedom, and can rotate around a planetary axis as well as in yaw, pitch, and roll. The pilot is seated so that her head B is located at $\mathbf{r} = 2\mathbf{i} + 1\mathbf{j}$ ft with respect to the center of the cab A. Knowing that the cab is rotating about the planetary axis with a constant angular velocity of 20 rpm counterclockwise as seen from above, and pitches with a constant angular velocity of +3 $\mathbf{k}$ rad/s, determine (*a*) the velocity of the pilot's head, (*b*) the angular acceleration of the cab, (*c*) the acceleration of the pilot's head.

SOLUTION

Given:

$$\boldsymbol{\omega}_{OA} = 20\mathbf{j} \text{ rpm} = 2.094\,\mathbf{j} \text{ rad/s}$$

$$\boldsymbol{\alpha}_{OA} = 0 \text{ rad/s}^2$$

$$\mathbf{r}_{B/A} = 2\mathbf{i} + 1\mathbf{j} \text{ ft}$$

$$\boldsymbol{\omega}_{PI} = 3\mathbf{k} \text{ rad/s}, \; \dot{\boldsymbol{\omega}}_{PI} = 0 \text{ rad/s}^2$$

Angular Velocity:

$$\boldsymbol{\omega}_{AB} = \boldsymbol{\omega}_{OA} + \boldsymbol{\omega}_{PI} = 2.094\,\mathbf{j} + 3\mathbf{k} \text{ rad/s}$$

Let O' be the point on the planetary axis at the same height as A, the center of the cab.

(*a*) Velocity of Head (point B)

$$\begin{aligned}\mathbf{v}_B &= \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}\\ &= \boldsymbol{\omega}_{OA} \times \mathbf{r}_{A/O'} + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}\\ &= 2.094\,\mathbf{j} \times (8\mathbf{i}) + (2.094\,\mathbf{j} + 3\mathbf{k}) \times (2\mathbf{i} + 1\mathbf{j})\\ &= -3.00\mathbf{i} + 6.00\mathbf{j} - 20.944\mathbf{k} \text{ ft/s}\end{aligned}$$

$$\mathbf{v}_B = (-3.000 \text{ ft/s})\mathbf{i} + (6.000 \text{ ft/s})\mathbf{j} - (20.944 \text{ ft/s})\mathbf{k} \; ◀$$

(*b*) Angular Acceleration:

$$\boldsymbol{\alpha} = (\dot{\boldsymbol{\omega}})_{OXYZ} + \boldsymbol{\Omega} \times \boldsymbol{\omega}$$

$$\begin{aligned}\boldsymbol{\alpha}_{AB} &= \dot{\boldsymbol{\omega}}_{PI} + \boldsymbol{\omega}_{OA} \times \boldsymbol{\omega}_{PI}\\ &= 0 + (2.094\,\mathbf{j}) \times 3\mathbf{k}\\ &= 6.283\mathbf{i} \text{ rad/s}^2\end{aligned}$$

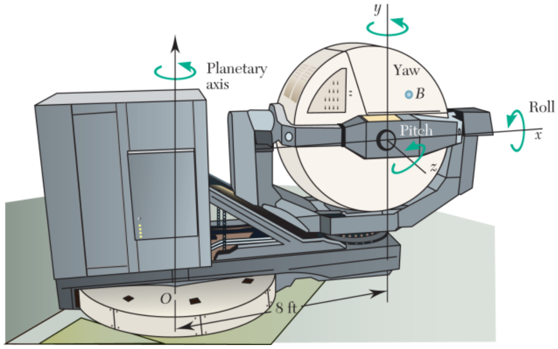
$$\boldsymbol{\alpha}_{AB} = (6.283 \text{ rad/s}^2)\mathbf{i} \; ◀$$

(*c*) Acceleration of the Head (point B):

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

$$\begin{aligned}\mathbf{a}_B &= \boldsymbol{\omega}_{OA} \times (\boldsymbol{\omega}_{OA} \times \mathbf{r}_{A/O'}) + \boldsymbol{\omega}_{AB} \times (\boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}) + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A}\\ &= 2.094\,\mathbf{j} \times (2.094\,\mathbf{j} \times 8\mathbf{i}) + (2.094\,\mathbf{j} + 3\mathbf{k}) \times \left[(2.094\,\mathbf{j} + 3\mathbf{k}) \times (2\mathbf{i} + 1\mathbf{j})\right] + 6.283\mathbf{i} \times (2\mathbf{i} + 1\mathbf{j})\\ &= -35.092\mathbf{i} - 26.1772\mathbf{i} - 9.00\mathbf{j} + 6.283\mathbf{k} + 6.283\mathbf{k}\end{aligned}$$

$$\mathbf{a}_B = (-62.865 \text{ ft/s}^2)\mathbf{i} - (9.000 \text{ ft/s}^2)\mathbf{j} + (12.566 \text{ ft/s}^2)\mathbf{k} \; ◀$$



PROBLEM 15.221

A flight simulator is used to train pilots on how to recognize spatial disorientation. It has four degrees of freedom, and can rotate around a planetary axis as well as in yaw, pitch, and roll. The pilot is seated so that her head *B* is located at $\boldsymbol{r} = 2\mathbf{i} + 1\mathbf{j}$ ft with respect to the center of the cab *A*. The cab is rotating about the planetary axis with an angular velocity of 20 rpm counterclockwise as seen from above and is increasing this by 1 rad/s^2. Knowing that the cab rolls with a constant angular velocity of -4 **i** rad/s, determine *(a)* the velocity of the pilot's head, *(b)* the angular acceleration of the cab, *(c)* the acceleration of the pilot's head.

SOLUTION

Given:

$$\boldsymbol{\omega}_{OA} = 20\mathbf{j}\text{ rpm} = 2.094\mathbf{j}\text{ rad/s}$$

$$\boldsymbol{\alpha}_{OA} = \dot{\boldsymbol{\omega}}_{OA} = 1\mathbf{j}\text{ rad/s}^2$$

$$\mathbf{r}_{B/A} = 2\mathbf{i} + 1\mathbf{j}\text{ ft}$$

$$\boldsymbol{\omega}_R = -4\mathbf{i}\text{ rad/s},\ \dot{\boldsymbol{\omega}}_R = 0\text{ rad/s}^2$$

Angular Velocity:

$$\boldsymbol{\omega}_{AB} = \boldsymbol{\omega}_{OA} + \boldsymbol{\omega}_R = -4.0\mathbf{i} + 2.094\mathbf{j}\text{ rad/s}$$

Let O' be the point on the planetary axis at the same height as A, the center of the cab.

(d) Velocity of Head (point B):

$$\begin{aligned}\mathbf{v}_B &= \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}\\ &= \boldsymbol{\omega}_{OA} \times \mathbf{r}_{A/O'} + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}\\ &= 2.094\,\mathbf{j} \times (8\mathbf{i}) + (-4.0\mathbf{i} + 2.094\mathbf{j}) \times (2\mathbf{i} + 1\mathbf{j})\\ &= -24.944\mathbf{k}\text{ ft/s}\end{aligned}$$

$$\mathbf{v}_B = -(24.94\text{ ft/s})\mathbf{k} \blacktriangleleft$$

(e) Angular Acceleration:

$$\boldsymbol{\alpha} = (\dot{\boldsymbol{\omega}})_{OXYZ} + \boldsymbol{\Omega} \times \boldsymbol{\omega}$$

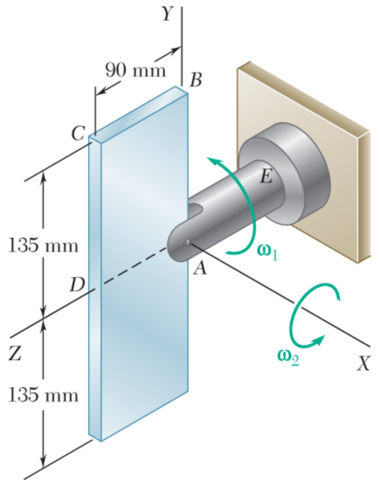
$$\begin{aligned}\boldsymbol{\alpha}_{AB} &= \dot{\boldsymbol{\omega}}_{OA} + \dot{\boldsymbol{\omega}}_R + \boldsymbol{\omega}_{OA} \times \boldsymbol{\omega}_R\\ &= 1\mathbf{j} + 0 + (2.094\,\mathbf{j}) \times (-4\mathbf{i})\\ &= 1.0\mathbf{j} + 8.378\mathbf{k}\text{ rad/s}^2\end{aligned}$$

$$\boldsymbol{\alpha}_{AB} = (1.00\text{ rad/s}^2)\mathbf{j} + (8.378\text{ rad/s}^2)\mathbf{k} \blacktriangleleft$$

(f) Acceleration of the Head (point B): $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$

$$\begin{aligned}\mathbf{a}_B &= \boldsymbol{\omega}_{OA} \times (\boldsymbol{\omega}_{OA} \times \mathbf{r}_{A/O'}) + \boldsymbol{\alpha}_{OA} \times \mathbf{r}_{A/O'} + \boldsymbol{\omega}_{AB} \times (\boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}) + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A}\\ &= 2.094\,\mathbf{j} \times (2.094\,\mathbf{j} \times 8\mathbf{i}) + 1\mathbf{j} \times 8\mathbf{i} + (-4.0\mathbf{i} + 2.094\mathbf{j}) \times \left[(-4.0\mathbf{i} + 2.094\mathbf{j}) \times (2\mathbf{i} + 1\mathbf{j})\right] + (1.0\mathbf{j} + 8.378\mathbf{k}) \times (2\mathbf{i} + 1\mathbf{j})\\ &= -35.092\mathbf{i} - 8\mathbf{k} - 17.151\mathbf{i} - 32.755\mathbf{j} - 8.378\mathbf{i} + 16.755\mathbf{j} - 2.0\mathbf{k}\\ &= -60.620\mathbf{i} - 16.000\mathbf{j} - 10.00\mathbf{k}\text{ ft/s}^2\end{aligned}$$

$$\mathbf{a}_B = (-60.62\text{ ft/s}^2)\mathbf{i} - (16.00\text{ ft/s}^2)\mathbf{j} - (10.00\text{ ft/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.222

The rectangular plate shown rotates at the constant rate $\omega_2 = 12$ rad/s with respect to arm *AE*, which itself rotates at the constant rate $w_1 = 9$ rad/s about the *Z* axis. For the position shown, determine the velocity and acceleration of the point of the plate indicated.

Corner *B*.

SOLUTION

Geometry. With the origin at *A*, $\mathbf{r}_B = (0.135 \text{ m})\mathbf{j}$

Let frame *AXYZ* rotate about the *Y* axis with constant angular velocity $\mathbf{\Omega} = \omega_1\mathbf{k} = (9 \text{ rad/s})\mathbf{k}$. Then the motion relative to the frame consists of rotation about the *X* axis with constant angular velocity $\boldsymbol{\omega}_2 = \omega_2\mathbf{i} = (12 \text{ rad/s})\mathbf{i}$.

Motion of coinciding Point B′.

$$\begin{aligned}\mathbf{v}_{B'} &= \mathbf{\Omega} \times \mathbf{r}_B\\ &= 9\mathbf{k} \times 0.135\mathbf{j}\\ &= -(1.215 \text{ m/s})\mathbf{i}\\ \mathbf{a}_{B'} &= \boldsymbol{\alpha} \times \mathbf{r}_B + \mathbf{\Omega} \times \mathbf{v}_{B'}\\ &= 0 + 9\mathbf{k} \times (-1.215\mathbf{i})\\ &= -(10.935 \text{ m/s}^2)\mathbf{j}\end{aligned}$$

Motion relative to the frame.

$$\begin{aligned}\mathbf{v}_{B/F} &= \boldsymbol{\omega}_2 \times \mathbf{r}_B = 12\mathbf{i} \times 0.135\mathbf{j}\\ &= (1.62 \text{ m/s})\mathbf{k}\\ \mathbf{a}_{B/F} &= \boldsymbol{\alpha}_2 \times \mathbf{r}_B + \boldsymbol{\omega}_2 \times \mathbf{v}_{B/F}\\ &= 0 + 12\mathbf{i} \times 1.62\mathbf{k}\\ &= -(19.44 \text{ m/s}^2)\mathbf{j}\end{aligned}$$

Velocity of Point B. $\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/F}$

$\mathbf{v}_B = -(1.215 \text{ m/s})\mathbf{i} + (1.620 \text{ m/s})\mathbf{k}$ ◀

Coriolis acceleration. $2\mathbf{\Omega} \times \mathbf{v}_{B/F}$

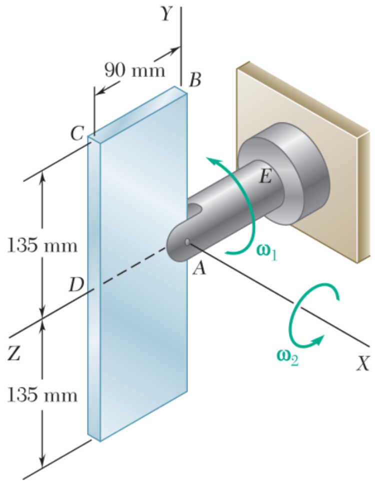
$$2\mathbf{\Omega} \times \mathbf{v}_{B/F} = (2)(9\mathbf{k}) \times 1.62\mathbf{k} = 0$$

Acceleration of Point B.

$$\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/F} + 2\mathbf{\Omega} \times \mathbf{v}_{B/F}$$

$$\mathbf{a}_B = -(30.375 \text{ in./s}^2)\mathbf{j}$$

$\mathbf{a}_B = -(30.4 \text{ m/s}^2)\mathbf{j}$ ◀



PROBLEM 15.223

The rectangular plate shown rotates at the constant rate $\omega_2 = 12$ rad/s with respect to arm AE, which itself rotates at the constant rate $\omega_1 = 9$ rad/s about the Z axis. For the position shown, determine the velocity and acceleration of the point of the plate indicated.

Corner C.

SOLUTION

Geometry. With the origin at A, $\mathbf{r}_C = (0.135 \text{ m})\mathbf{j} + (0.09 \text{ m})\mathbf{k}$

Let frame $AXYZ$ rotate about the Y axis with constant angular velocity $\mathbf{\Omega} = \omega_1\mathbf{k} = (9 \text{ rad/s})\mathbf{k}$. Then the motion relative to the frame consists of rotation about the X axis with constant angular velocity $\boldsymbol{\omega}_2 = \omega_2\mathbf{i} = (12 \text{ rad/s})\mathbf{i}$.

Motion of coinciding Point C' in the frame.

$$\begin{aligned}
\mathbf{v}_{C'} &= \mathbf{\Omega} \times \mathbf{r}_C \\
&= 9\mathbf{k} \times (0.135\mathbf{j} + 0.09\mathbf{k}) \\
&= -(1.215 \text{ m/s})\mathbf{i} \\
\mathbf{a}_{C'} &= \boldsymbol{\alpha} \times \mathbf{r}_C + \mathbf{\Omega} \times \mathbf{v}_{C'} \\
&= 0 + 9\mathbf{k} \times (1.215\mathbf{i}) \\
&= -(10.935 \text{ m/s}^2)\mathbf{j}
\end{aligned}$$

Motion relative to the frame.

$$\begin{aligned}
\mathbf{v}_{C/F} &= \boldsymbol{\omega}_2 \times \mathbf{r}_C \\
&= 12\mathbf{i} \times (0.135\mathbf{j} + 0.09\mathbf{k}) \\
&= -(1.08 \text{ m/s})\mathbf{j} + (1.62 \text{ m/s})\mathbf{k} \\
\mathbf{a}_{C/F} &= \boldsymbol{\alpha}_2 \times \mathbf{r}_C + \boldsymbol{\omega}_2 \times \mathbf{v}_{C/F} \\
&= 0 + 12\mathbf{i} \times (-1.08\mathbf{j} + 1.62\mathbf{k}) \\
&= -(19.44 \text{ m/s}^2)\mathbf{j} - (12.96 \text{ m/s}^2)\mathbf{k}
\end{aligned}$$

Velocity of Point C. $\mathbf{v}_C = \mathbf{v}_{C'} + \mathbf{v}_{C/F}$

$$\mathbf{v}_C = -(1.215 \text{ m/s})\mathbf{i} - (1.080 \text{ m/s})\mathbf{j} + (1.620 \text{ m/s})\mathbf{k} \blacktriangleleft$$

PROBLEM 15.223 (Continued)

Coriolis acceleration. $2\boldsymbol{\Omega}\times\mathbf{v}_{C/F}$

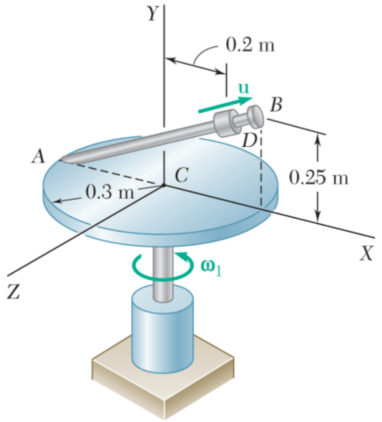
$$2\boldsymbol{\Omega}\times\mathbf{v}_{C/F} = (2)(9\mathbf{k})\times(-1.08\mathbf{j}+1.62\mathbf{k})$$

$$= (19.44\ \text{m/s}^2)\mathbf{i}$$

Acceleration of Point C. $\mathbf{a}_C = \mathbf{a}_{C'} + \mathbf{a}_{C/F} + 2\boldsymbol{\Omega}\times\mathbf{v}_{C/F}$

$$\mathbf{a}_C = (19.44\ \text{m/s}^2)\mathbf{i} - (30.375\ \text{m/s}^2)\mathbf{j} - (12.96\ \text{m/s}^2)\mathbf{k}$$

$$\mathbf{a}_C = (19.44\ \text{m/s}^2)\mathbf{i} - (30.4\ \text{m/s}^2)\mathbf{j} - (12.96\ \text{m/s}^2)\mathbf{k}\ \blacktriangleleft$$



PROBLEM 15.224

Rod *AB* is welded to the 0.3-m-radius plate, which rotates at the constant rate $\omega_1 = 6$ rad/s. Knowing that collar *D* moves toward end *B* of the rod at a constant speed $u = 1.3$ m, determine, for the position shown, (*a*) the velocity of *D*, (*b*) the acceleration of *D*.

SOLUTION

Geometry.

$$\mathbf{r}_{B/A} = (0.6\text{ m})\mathbf{i} + (0.25\text{ m})\mathbf{j} \qquad \mathbf{r}_{C/A} = (0.3\text{ m})\mathbf{i}$$

$$\mathbf{r}_{D/A} = \frac{0.5}{0.6}\mathbf{r}_{B/A}$$

$$\mathbf{r}_{D/C} = \mathbf{r}_{D/A} - \mathbf{r}_{C/A}$$

$$= (0.2\text{ m})\mathbf{i} + (0.20833\text{ m})\mathbf{j}$$

$$l_{AB} = \sqrt{0.6^2 + 0.25^2}$$

$$= 0.65\text{ m}$$

Unit vector along *AB*:

$$\boldsymbol{\lambda}_{AB} = \frac{\mathbf{r}_{B/A}}{l_{AB}}$$

$$= \frac{12}{13}\mathbf{i} + \frac{5}{12}\mathbf{j}$$

Let *Oxyz* be a frame of reference currently coinciding with *OXYZ*, but rotating with angular velocity

$$\boldsymbol{\Omega} = \omega_1\mathbf{j} = (6\text{ rad/s})\mathbf{j}$$

(*a*) *Velocity of D.*

$$\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/AB}$$

$$\mathbf{v}_{D'} = \boldsymbol{\Omega} \times \mathbf{r}_{D/C}$$

$$= 6\mathbf{j} \times (0.2\mathbf{i} + 0.20833\mathbf{j})$$

$$= -(1.2\text{ m/s})\mathbf{k}$$

$$\mathbf{v}_{D/AB} = u\boldsymbol{\lambda}_{AB}$$

$$= 1.3\left(\frac{12}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}\right)$$

$$= (1.2\text{ m/s})\mathbf{i} + (0.5\text{ m/s})\mathbf{j}$$

$$\mathbf{v}_D = (1.2\text{ m/s})\mathbf{i} + (0.5\text{ m/s})\mathbf{j} - (1.2\text{ m/s})\mathbf{k} \quad ◀$$

PROBLEM 15.224 (Continued)

(*b*) *Acceleration of D.*

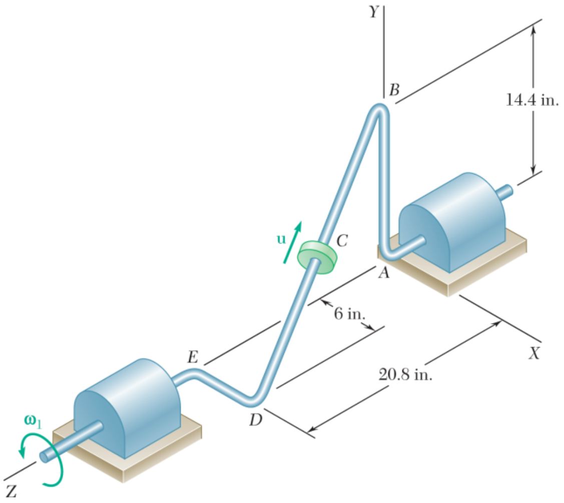
$$\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/AB} + 2\mathbf{\Omega} \times \mathbf{v}_{D/AB}$$

$$\begin{aligned}\mathbf{a}_{D'} &= \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_{D/C}) \\ &= 6\mathbf{j} \times (6\mathbf{j} \times (0.2\mathbf{i} + 0.20833\mathbf{j})) \\ &= -(7.2 \text{ m/s}^2)\mathbf{i}\end{aligned}$$

$$\mathbf{a}_{D/AB} = 0$$

$$\begin{aligned}2\mathbf{\Omega} \times \mathbf{v}_{D/F} &= (2)(6\mathbf{j}) \times ((1.2)\mathbf{i} + (0.5)\mathbf{j}) \\ &= -(14.4 \text{ m/s}^2)\mathbf{k}\end{aligned}$$

$$\mathbf{a}_D = -(7.2 \text{ m/s}^2)\mathbf{i} - (14.4 \text{ m/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.225

The bent rod shown rotates at the constant rate $\omega_1 = 5$ rad/s and collar C moves toward point B at a constant relative speed $u = 39$ in./s. Knowing that collar C is halfway between points B and D at the instant shown, determine its velocity and acceleration.

SOLUTION

Geometry: $\mathbf{r}_B = (14.4\text{ in.})\mathbf{j}, \qquad \mathbf{r}_D = (6\text{ in.})\mathbf{i} + (20.8\text{ in.})\mathbf{k}$

$$\mathbf{r}_{B/D} = -(6\text{ in.})\mathbf{i} + (14.4\text{ in.})\mathbf{j} - (20.8\text{ in.})\mathbf{k} \qquad l_{DB} = \sqrt{6^2 + 14.4^2 + 20.8^2} = 26\text{ in.}$$

$$\mathbf{r}_C = \frac{1}{2}(\mathbf{r}_B + \mathbf{r}_D) = (3\text{ in.})\mathbf{i} + (7.2\text{ in.})\mathbf{j} + (10.4\text{ in.})\mathbf{k}$$

Let frame $Axyz$ rotate with angular velocity $\quad \boldsymbol{\Omega} = \omega_1\mathbf{k} = (5\text{ rad/s})\mathbf{k}$

Velocity Analysis: $\quad u = 39\text{ in./s}$

$$\mathbf{v}_{C'} = \boldsymbol{\Omega} \times \mathbf{r}_C = 5\mathbf{k} \times (3\mathbf{i} + 7.2\mathbf{j} + 10.4\mathbf{k}) = -(36\text{ in./s})\mathbf{i} + (15\text{ in./s})\mathbf{j}$$

$$\mathbf{v}_{C/F} = \frac{u}{l_{DB}} r_{D/B} = \frac{39}{26}(-6\mathbf{i} + 14.4\mathbf{j} - 20.8\mathbf{k}) = -(9\text{ in./s})\mathbf{i} + (21.6\text{ in./s})\mathbf{j} - (31.2\text{ in./s})\mathbf{k}$$

$$\mathbf{v}_C = \mathbf{v}_{C'} + \mathbf{v}_{C/F}$$

$$\mathbf{v}_C = -(45.0\text{ in./s})\mathbf{i} + (36.6\text{ in./s})\mathbf{j} - (31.2\text{ in./s})\mathbf{k} \blacktriangleleft$$

Acceleration Analysis: $\quad \mathbf{a}_{C/F} = 0$

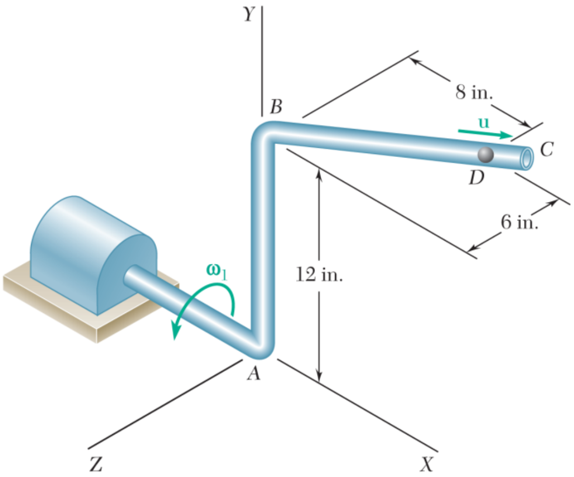
$$\mathbf{a}_{C'} = \boldsymbol{\Omega} \times \mathbf{v}_{C'} = 5\mathbf{k} \times (-36\mathbf{i} + 15\mathbf{j}) = -(75\text{ in./s}^2)\mathbf{i} - (180\text{ in./s}^2)\mathbf{j}$$

$$\mathbf{a}_{C/F} = 0$$

$$2\boldsymbol{\Omega} \times \mathbf{v}_{C/F} = (2)(5\mathbf{k}) \times (-9\mathbf{i} + 21.6\mathbf{j} - 31.2\mathbf{k}) = -(216\text{ in./s}^2)\mathbf{i} - (90\text{ in./s}^2)\mathbf{j}$$

$$\mathbf{a}_C = \mathbf{a}_{C'} + \mathbf{a}_{C/F} + 2\boldsymbol{\Omega} \times \mathbf{v}_{C/F}$$

$$\mathbf{a}_C = -(291\text{ in./s}^2)\mathbf{i} - (270\text{ in./s}^2)\mathbf{j} \blacktriangleleft$$



PROBLEM 15.226

The bent pipe shown rotates at the constant rate $\omega_1 = 10$ rad/s. Knowing that a ball bearing D moves in portion BC of the pipe toward end C at a constant relative speed $u = 2$ ft/s, determine at the instant shown (*a*) the velocity of D, (*b*) the acceleration of D.

SOLUTION

With the origin at Point A,

$$\mathbf{r}_D = (8\text{ in.})\mathbf{i} + (12\text{ in.})\mathbf{j} - (6\text{ in.})\mathbf{k}$$

$$\mathbf{r}_{C/B} = (8\text{ in.})\mathbf{i} - (6\text{ in.})\mathbf{k},$$

$$l_{BC} = \sqrt{8^2 + 6^2} = 10\text{ in.}$$

Let the frame $Axyz$ rotate with angular velocity $\boldsymbol{\Omega} = \omega_1\mathbf{i} = (10\text{ rad/s})\mathbf{i}$

(*a*) *Velocity of D.*

$$\begin{aligned}\mathbf{v}_{D'} &= \boldsymbol{\Omega} \times \mathbf{r}_D \\ &= 10\mathbf{i} \times (8\mathbf{i} + 12\mathbf{j} - 6\mathbf{k}) \\ &= (60\text{ in./s})\mathbf{j} + (120\text{ in./s})\mathbf{k}\end{aligned}$$

$$u = 2\text{ ft/s} = 24\text{ in./s},$$

$$\begin{aligned}\mathbf{v}_{D/F} &= \frac{24}{10}(8\mathbf{i} - 6\mathbf{k}) \\ &= (19.2\text{ in./s})\mathbf{i} - (14.4\text{ in./s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{v}_D &= \mathbf{v}_{D'} + \mathbf{v}_{D/F} \\ &= (19.2\text{ in./s})\mathbf{i} + (60\text{ in./s})\mathbf{j} + (105.6\text{ in./s})\mathbf{k}\end{aligned}$$

$$\mathbf{v}_D = (1.600\text{ ft/s})\mathbf{i} + (5.00\text{ ft/s})\mathbf{j} + (8.80\text{ ft/s})\mathbf{k} \blacktriangleleft$$

(*b*) *Acceleration of D.*

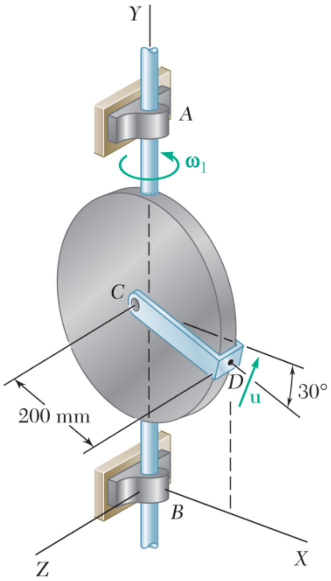
$$\mathbf{a}_{D'} = \boldsymbol{\Omega} \times \mathbf{v}_{D'} = 10\mathbf{i} \times (60\mathbf{j} + 120\mathbf{k}) = -(1200\text{ in./s}^2)\mathbf{j} + (600\text{ in./s}^2)\mathbf{k}$$

$$\mathbf{a}_{D/F} = 0$$

$$2\boldsymbol{\Omega} \times \mathbf{v}_{D/F} = (2)(10\mathbf{i}) \times (19.2\mathbf{i} - 14.4\mathbf{k}) = (288\text{ in./s}^2)\mathbf{j}$$

$$\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/F} + 2\boldsymbol{\Omega} \times \mathbf{v}_{D/F} = -(912\text{ in./s}^2)\mathbf{j} + (600\text{ in./s}^2)\mathbf{k}$$

$$\mathbf{a}_D = -(76.0\text{ ft/s}^2)\mathbf{j} + (50.0\text{ ft/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.227

The circular plate shown rotates about its vertical diameter at the constant rate $\omega_1 = 10$ rad/s. Knowing that in the position shown the disk lies in the XY plane and Point D of strap CD moves upward at a constant relative speed $u = 1.5$ m/s, determine (a) the velocity of D, (b) the acceleration of D.

SOLUTION

Geometry.

$$\begin{aligned}\mathbf{r}_{D/C} &= (0.2\text{ m})(\cos 30°\mathbf{i} - \sin 30°\mathbf{j})\\ &= (0.1\sqrt{3}\text{ m})\mathbf{i} - (0.1\text{ m})\mathbf{j}\end{aligned}$$

Let frame $Cxyz$, which at the instant shown coincides with $CXYZ$, rotate with angular velocity

$$\boldsymbol{\Omega} = \omega_1\mathbf{j} = (10\text{ rad/s})\mathbf{j}.$$

Motion of coinciding Point D' in the frame.

$$\begin{aligned}\mathbf{v}_{D'} &= \boldsymbol{\Omega}\times\mathbf{r}_{D/C}\\ &= 10\mathbf{j}\times(0.1\sqrt{3}\mathbf{i} + 0.1\mathbf{j})\\ &= -(\sqrt{3}\text{ m/s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{D'} &= -\Omega^2(r\cos 30°)\mathbf{i}\\ &= -10^2(0.1\sqrt{3})\mathbf{i}\\ &= -(10\sqrt{3}\text{ m/s}^2)\mathbf{i}\end{aligned}$$

Motion of Point D relative to the frame. $\quad u = 1.5$ m/s

$$\begin{aligned}\mathbf{v}_{D/F} &= u(\sin 30°\mathbf{i} + \cos 30°\mathbf{j})\\ &= (0.75\text{ m/s})\mathbf{i} + (0.75\sqrt{3}\text{ m/s})\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{D/F} &= \frac{u^2}{\rho}\cdot(-\cos 30°\mathbf{i} + \sin 30°\mathbf{j})\\ &= \frac{1.5^2}{0.2}(-\cos 30°\mathbf{i} + \sin 30°\mathbf{j})\\ &= -(5.625\sqrt{3}\text{ m/s}^2)\mathbf{i} + (5.625\text{ m/s}^2)\mathbf{j}\end{aligned}$$

(*a*) *Velocity of Point D.*

$$\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/F}$$

$$\mathbf{v}_D = (0.75 \text{ m/s})\mathbf{i} + (0.75\sqrt{3} \text{ m/s})\mathbf{j} - (\sqrt{3} \text{ m/s})\mathbf{k}$$

$$\mathbf{v}_D = (0.750 \text{ m/s})\mathbf{i} + (1.299 \text{ m/s})\mathbf{j} - (1.732 \text{ m/s})\mathbf{k} \blacktriangleleft$$

Coriolis acceleration. $2\mathbf{\Omega} \times \mathbf{v}_{D/F}$

$$2\mathbf{\Omega} \times \mathbf{v}_{D/F} = (2)(10\mathbf{j}) \times (0.75\mathbf{i} + 0.75\sqrt{3}\mathbf{j})$$

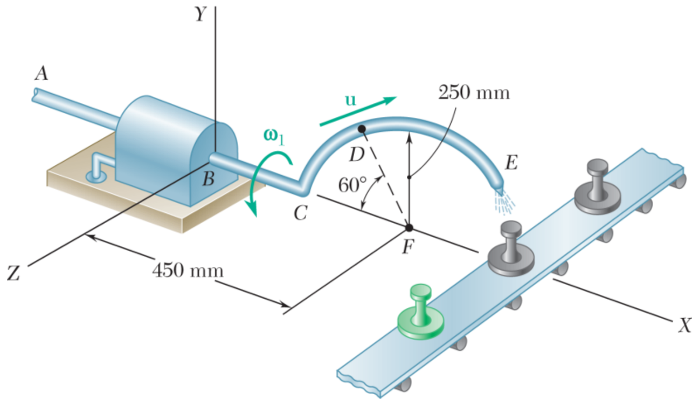
$$= -(15 \text{ m/s}^2)\mathbf{k}$$

(*b*) *Acceleration of Point D.*

$$\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/F} + 2\mathbf{\Omega} \times \mathbf{v}_{D/F}$$

$$\mathbf{a}_D = -(15.625\sqrt{3} \text{ m/s}^2)\mathbf{i} + (5.625 \text{ m/s}^2)\mathbf{j} - (15 \text{ m/s}^2)\mathbf{k}$$

$$\mathbf{a}_D = (27.1 \text{ m/s}^2)\mathbf{i} + (5.63 \text{ m/s}^2)\mathbf{j} - (15.00 \text{ m/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.228

Manufactured items are spray-painted as they pass through the automated work station shown. Knowing that the bent pipe *ACE* rotates at the constant rate $\omega_1 = 0.4$ rad/s and that at Point *D* the paint moves through the pipe at a constant relative speed $u = 150$ mm/s, determine, for the position shown, (*a*) the velocity of the paint at *D*, (*b*) the acceleration of the paint at *D*.

SOLUTION

Use a frame of reference *CE* rotating about the *x*-axis with angular velocity

$$\mathbf{\Omega} = \boldsymbol{\omega}_1 = (0.4 \text{ rad/s})\mathbf{i}$$

Geometry:

$$\begin{aligned}\mathbf{r}_{D/F} &= (250 \text{ mm})(-\cos 60°\mathbf{i} + \sin 60°\mathbf{j})\\ &= -(125 \text{ mm})\mathbf{i} + (216.51 \text{ mm})\mathbf{j}\end{aligned}$$

Motion of Point *D*′ fixed in the frame *CE* but coinciding with Point *D* at the instant considered.

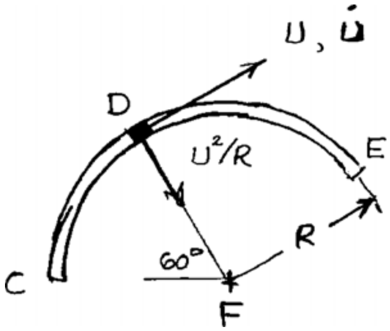
$$\mathbf{v}_{D'} = \Omega \times \mathbf{r}_{D/F} = (0.4\mathbf{i}) \times (-125\mathbf{i} + 216.51\mathbf{j}) = (86.603 \text{ mm/s})\mathbf{k}$$

$$\begin{aligned}\mathbf{a}_{D'} &= \dot{\Omega}\mathbf{i} \times \mathbf{r}_{D/F} + \mathbf{\Omega} \times \mathbf{v}_{D'}\\ &= 0 + (0.4\mathbf{i}) \times (86.603\mathbf{k}) = -(34.641 \text{ mm/s}^2)\mathbf{j}\end{aligned}$$

Motion of *D* relative to the frame *CE*.

$$\begin{aligned}\mathbf{v}_{D/CE} &= u(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) = (150 \text{ mm/s})(\cos 30°\mathbf{i} + \sin 30°\mathbf{j})\\ &= (129.90 \text{ mm/s})\mathbf{i} + (75 \text{ mm/s})\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{D/CE} &= \dot{u}(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) + \frac{u^2}{R}(\sin 30°\mathbf{i} - \cos 30°\mathbf{j})\\ &= 0 + \frac{(150)^2}{250}(\sin 30°\mathbf{i} - \cos 30°\mathbf{j})\\ &= (45 \text{ mm/s}^2)\mathbf{i} - (77.94 \text{ mm/s}^2)\mathbf{j}\end{aligned}$$



(*a*) *Velocity of D.* $\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/CE}$

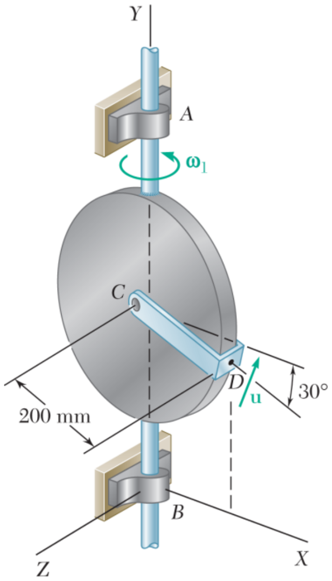
$$\mathbf{v}_D = (129.9 \text{ mm/s})\mathbf{i} + (75.0 \text{ mm/s})\mathbf{j} + (86.6 \text{ mm/s})\mathbf{k} \blacktriangleleft$$

Coriolis acceleration $2\mathbf{\Omega} \times \mathbf{v}_{D/CE}$

$$2\mathbf{\Omega} \times \mathbf{v}_{D/CE} = (2)(0.4\mathbf{i}) \times (129.90\mathbf{i} + 75\mathbf{j}) = (60 \text{ mm/s}^2)\mathbf{k}$$

(*b*) *Acceleration of D.* $\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/CE} + 2\mathbf{\Omega} \times \mathbf{v}_{D/CE}$

$$\mathbf{a}_D = (45.0 \text{ mm/s}^2)\mathbf{i} - (112.6 \text{ mm/s}^2)\mathbf{j} + (60.0 \text{ mm/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.229

Solve Problem 15.227, assuming that at the instant shown the angular velocity $\boldsymbol{\omega}_1$ of the plate is 10 rad/s and is decreasing at the rate of 25 rad/s^2, while the relative speed u of Point D of strap CD is 1.5 m/s and is decreasing at the rate of 3 m/s^2.

PROBLEM 15.227 The circular plate shown rotates about its vertical diameter at the constant rate $\omega_1 = 10$ rad/s. Knowing that in the position shown the disk lies in the XY plane and Point D of strap CD moves upward at a constant relative speed $u = 1.5$ m/s, determine (*a*) the velocity of D, (*b*) the acceleration of D.

SOLUTION

Geometry.

$$\mathbf{r}_{D/C} = (0.2\text{ m})(\cos 30°\mathbf{i} - \sin 30°\mathbf{j})$$
$$= (0.1\sqrt{3}\text{ m})\mathbf{i} - (0.1\text{ m})\mathbf{j}$$

Let frame $Cxyz$, which at the instant shown coincides with $CXYZ$, rotate with angular velocity and angular acceleration

$$\boldsymbol{\Omega} = \omega_1\mathbf{j} = (10\text{ rad/s})\mathbf{j}.$$

$$\dot{\boldsymbol{\Omega}} = -(25\text{ rad/s}^2)\mathbf{j}.$$

Motion of coinciding Point D' in the frame

$$\mathbf{v}_{D'} = \boldsymbol{\Omega} \times \mathbf{r}_{D/C}$$
$$= 10\mathbf{j} \times (0.1\sqrt{3}\mathbf{i} + 0.1\mathbf{j}) = -(\sqrt{3}\text{ m/s})\mathbf{k}$$
$$\mathbf{a}_{D'} = -\Omega^2(r\cos 30°)\mathbf{i} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{D/C}$$
$$= -10^2(0.1\sqrt{3})\mathbf{i} - 25\mathbf{j} \times (0.1\sqrt{3}\mathbf{i} + 0.1\mathbf{j})$$
$$= -(10\sqrt{3}\text{ m/s}^2)\mathbf{i} + (2.5\sqrt{3}\text{ m/s}^2)\mathbf{k}$$

Motion of Point D relative to the frame. $u = 1.5$ m/s $\quad \dot{u} = -3$ m/s^2

$$\mathbf{v}_{D/F} = u(\sin 30°\mathbf{i} + \cos 30°\mathbf{j})$$
$$= (0.75\text{ m/s})\mathbf{i} + (0.75\sqrt{3}\text{ m/s})\mathbf{j}$$

$$\mathbf{a}_{D/F} = \frac{u^2}{\rho}\cdot(-\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) + \dot{u}(\sin 30°\mathbf{i} + \cos 30°\mathbf{j})$$
$$= \frac{1.5^2}{0.2}(-\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) - 3(\sin 30°\mathbf{i} + \cos 30°\mathbf{j})$$
$$= -(5.625\sqrt{3}\text{ m/s}^2)\mathbf{i} + (5.625\text{ m/s}^2)\mathbf{j} - (1.5\text{ m/s}^2)\mathbf{i} - (1.5\sqrt{3}\text{ m/s}^2)\mathbf{j}$$

PROBLEM 15.229 (Continued)

(*a*) *Velocity of Point D.*

$$\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/F}$$

$$\mathbf{v}_D = (0.75\text{ m/s})\mathbf{i} + (0.75\sqrt{3}\text{ m/s})\mathbf{j} - (\sqrt{3}\text{ m/s})\mathbf{k}$$

$$\mathbf{v}_D = (0.750\text{ m/s})\mathbf{i} + (1.299\text{ m/s})\mathbf{j} - (1.732\text{ m/s})\mathbf{k} \blacktriangleleft$$

Coriolis acceleration. $2\boldsymbol{\Omega} \times \mathbf{v}_{D/F}$

$$2\boldsymbol{\Omega} \times \mathbf{v}_{D/F} = (2)(10\mathbf{j}) \times (0.75\mathbf{i} + 0.75\sqrt{3}\mathbf{j})$$

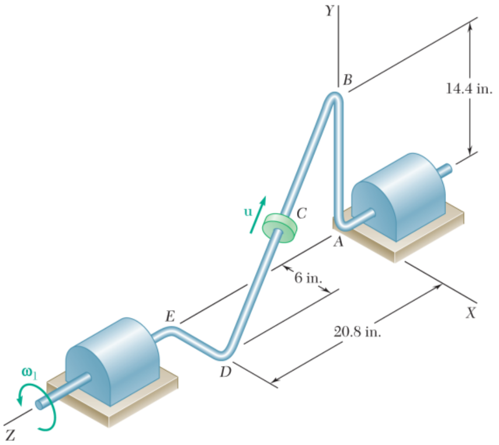
$$= -(15\text{ m/s}^2)\mathbf{k}$$

(*b*) *Acceleration of Point D.*

$$\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/F} + 2\boldsymbol{\Omega} \times \mathbf{v}_{D/F}$$

$$\begin{aligned}\mathbf{a}_D = &-(10\sqrt{3}\text{ m/s}^2)\mathbf{i} + (2.5\sqrt{3}\text{ m/s}^2)\mathbf{k}\\ &-(5.625\sqrt{3}\text{ m/s}^2)\mathbf{i} + (5.625\text{ m/s}^2)\mathbf{j}\\ &-(1.5\text{ m/s}^2)\mathbf{i} - (1.5\sqrt{3}\text{ m/s}^2)\mathbf{j} - (15\text{ m/s}^2)\mathbf{k}\end{aligned}$$

$$\mathbf{a}_D = -(28.6\text{ m/s}^2)\mathbf{i} + (3.03\text{ m/s}^2)\mathbf{j} - (10.67\text{ m/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.230

Solve Prob. 15.225, assuming that at the instant shown the angular velocity ω_1 of the rod is 5 rad/s and is increasing at the rate of 10 rad/s^2, while the relative speed u of the collar C is 39 in./s and is decreasing at the rate of 260 in./s^2.

SOLUTION

Geometry: $\mathbf{r}_B = (14.4 \text{ in.})\mathbf{j}, \quad \mathbf{r}_D = (6 \text{ in.})\mathbf{i} + (20.8 \text{ in.})\mathbf{k}$

$$\mathbf{r}_{B/D} = -(6 \text{ in.})\mathbf{i} + (14.4 \text{ in.})\mathbf{j} - (20.8 \text{ in.})\mathbf{k} \qquad l_{DB} = \sqrt{6^2 + 14.4^2 + 20.8^2} = 26 \text{ in.}$$

$$\mathbf{r}_C = \frac{1}{2}(\mathbf{r}_B + \mathbf{r}_D) = (3 \text{ in.})\mathbf{i} + (7.2 \text{ in.})\mathbf{j} + (10.4 \text{ in.})\mathbf{k}$$

Let frame *Axyz* rotate with angular velocity $\quad \mathbf{\Omega} = \omega_1\mathbf{k} = (5 \text{ rad/s})\mathbf{k}.$

Velocity analysis: $\quad u = 39 \text{ in./s}$

$$\mathbf{v}_{C'} = \mathbf{\Omega} \times \mathbf{r}_C = 5\mathbf{k} \times (3\mathbf{i} + 7.2\mathbf{j} + 10.4\mathbf{k}) = -(36 \text{ in./s})\mathbf{i} + (15 \text{ in./s})\mathbf{j}$$

$$\mathbf{v}_{C/F} = \frac{u}{l_{DB}} r_{D/B} = \frac{39}{26}(-6\mathbf{i} + 14.4\mathbf{j} - 20.8\mathbf{k}) = -(9 \text{ in./s})\mathbf{i} + (21.6 \text{ in./s})\mathbf{j} - (31.2 \text{ in./s})\mathbf{k}$$

$$\mathbf{v}_C = \mathbf{v}_{C'} + \mathbf{v}_{C/F}$$

$$\mathbf{v}_C = -(45.0 \text{ in/s})\mathbf{i} + (36.6 \text{ in/s})\mathbf{j} - (31.2 \text{ in/s})\mathbf{k} \blacktriangleleft$$

Acceleration analysis: $\quad \alpha = 10\mathbf{k} \text{ rad/s}, \quad \dot{u} = -260 \text{ in./s}^2$

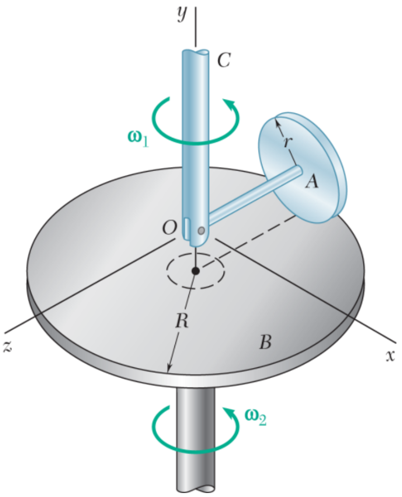
$$\mathbf{a}_{C'} = \mathbf{\Omega} \times \mathbf{v}_{C'} = 5\mathbf{k} \times (-36\mathbf{i} + 15\mathbf{j}) = -(75 \text{ in./s}^2)\mathbf{i} - (180 \text{ in./s}^2)\mathbf{j}$$

$$\begin{aligned}\mathbf{a}_{C/F} &= \boldsymbol{\alpha} \times \mathbf{r}_C + \dot{\mathbf{u}} \\ &= 10\mathbf{k} \times (3\mathbf{i} + 7.2\mathbf{j} + 10.4\mathbf{k}) + \frac{-260}{26}(-6\mathbf{i} + 14.4\mathbf{j} - 20.8\mathbf{k}) \\ &= -(72 \text{ in./s}^2)\mathbf{i} + (30 \text{ in./s}^2)\mathbf{j} + (60 \text{ in./s}^2)\mathbf{i} - (144 \text{ in./s}^2)\mathbf{j} + (208 \text{ in./s}^2)\mathbf{k}\end{aligned}$$

$$2\mathbf{\Omega} \times \mathbf{v}_{C/F} = (2)(5\mathbf{k}) \times (-9\mathbf{i} + 21.6\mathbf{j} - 31.2\mathbf{k}) = -(216 \text{ in./s}^2)\mathbf{i} - (90 \text{ in./s}^2)\mathbf{j}$$

$$\mathbf{a}_C = \mathbf{a}_{C'} + \mathbf{a}_{C/F} + 2\mathbf{\Omega} \times \mathbf{v}_{C/F}$$

$$\mathbf{a}_C = -(303 \text{ in/s}^2)\mathbf{i} - (384 \text{ in/s}^2)\mathbf{j} + (208 \text{ in/s}^2)\mathbf{k} \blacktriangleleft$$

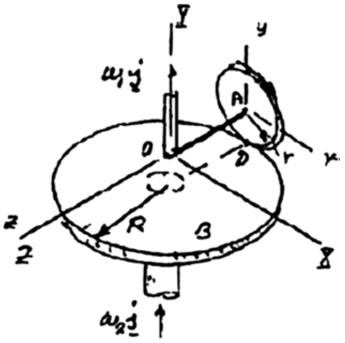


PROBLEM 15.231

Using the method of Section 15.7A, solve Problem 15.192.

PROBLEM 15.192 In the system shown, disk *A* is free to rotate about the horizontal rod *OA*. Assuming that shaft *OC* and disk *B* rotate with constant angular velocities ω_1 and ω_2, respectively, both counterclockwise, determine (*a*) the angular velocity of *A*, (*b*) the angular acceleration of disk *A*.

SOLUTION



Moving frame *Axyz* rotates with angular velocity

$$\boldsymbol{\Omega} = \omega_1 \mathbf{j}$$

$$\boldsymbol{\omega}_{\text{disk}/F} = \omega_x \mathbf{i} + \omega_z \mathbf{k}$$

$$\mathbf{r}_{D/A} = -r\mathbf{j} - R\mathbf{k}$$

(*a*) Total angular velocity of disk *A*:

$$\begin{aligned} \boldsymbol{\omega} &= \omega_1 \mathbf{j} + \boldsymbol{\omega}_{\text{disk}/F} \\ &= \omega_x \mathbf{i} + \omega_1 \mathbf{j} + \omega_z \mathbf{k} \end{aligned} \tag{1}$$

Denote by *D* point of contact of disks

Consider disk *B*: $\mathbf{v}_D = \omega_2 \mathbf{j} \times (-R\mathbf{k}) = -R\omega_2 \mathbf{i}$ (2)

Consider system *OC*, *OA* and disk *A*.

$$\begin{aligned} \mathbf{v}_{D'} &= \boldsymbol{\Omega} \times \mathbf{r}_{B/A} \\ &= \omega_1 \mathbf{j} \times (-r\mathbf{j} - R\mathbf{k}) \\ &= -R\omega_1 \mathbf{i} \\ \mathbf{v}_{B/F} &= \omega_{\text{disk}/F} \times \mathbf{r}_{D/A} \\ &= (\omega_2 \mathbf{i} + \omega_z \mathbf{k}) \times (-r\mathbf{j} - R\mathbf{k}) \\ &= -r\omega_x \mathbf{k} + R\omega_x \mathbf{j} + r\omega_z \mathbf{i} \\ \mathbf{v}_D &= \mathbf{v}_{D'} + \mathbf{v}_{D/F} \\ &= -R\omega_1 \mathbf{i} - r\omega_x \mathbf{k} + R\omega_x \mathbf{j} + r\omega_2 \mathbf{i} \end{aligned} \tag{3}$$

Equate $\mathbf{v}_D = \mathbf{v}_D$ from Eq. (2) and Eq. (3).

$$-R\omega_2 \mathbf{i} = -R\omega_1 + r\omega_2 + R\omega_x \mathbf{j} - r\omega_x \mathbf{k}$$

PROBLEM 15.231 (Continued)

Coefficient of **j**: $0 = R\omega_x \rightarrow \omega_x = 0$

Coefficient of **i**: $-R\omega_2 = -R\omega_1 + r\omega_z;$

$$\omega_z = \frac{R}{r}(\omega_1 - \omega_2)$$

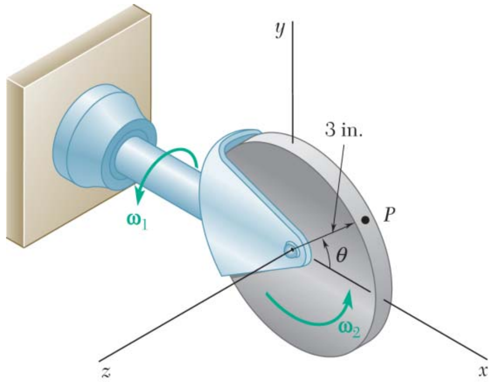
Eq. (3): $$\boldsymbol{\omega} = \omega_1 \mathbf{j} + \frac{R}{r}(\omega_1 - \omega_2)\mathbf{k} \blacktriangleleft$$

(*b*) Disk *A* rotates about *y* axis at rate $\boldsymbol{\omega}_1$.

$$\boldsymbol{\alpha} = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}$$

$$= \omega_1 \mathbf{j} \times \left[\omega \mathbf{j} + \frac{R}{r}(\omega_1 - \omega_2) \right] \mathbf{k}$$

$$\boldsymbol{\alpha} = \omega_1(\omega_1 - \omega_2)\frac{R}{r}\mathbf{j} \blacktriangleleft$$



PROBLEM 15.232

Using the method of Section 15.7A, solve Problem 15.196.

PROBLEM 15.196 A 3-in.-radius disk spins at the constant rate $\omega_2 = 4$ rad/s about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $\omega_1 = 5$ rad/s. Knowing that $\theta = 30°$, determine the acceleration of Point P on the rim of the disk.

SOLUTION

Let frame $Oxyz$ rotate with angular velocity

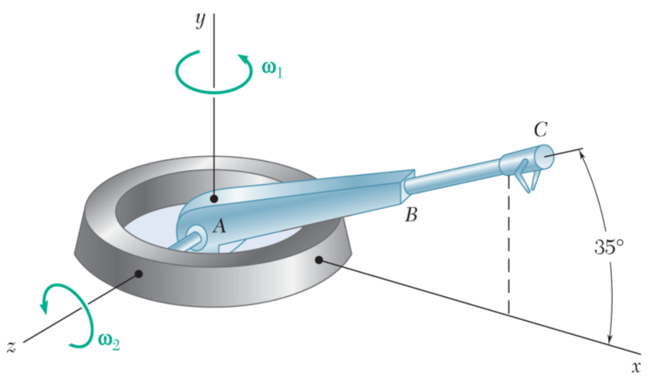
$$\mathbf{\Omega} = \omega_1\mathbf{i} = (5 \text{ rad/s})\mathbf{i}$$

The motion relative to the frame is the spin

$$\begin{aligned}
\omega_2\mathbf{k} &= (4 \text{ rad/s})\mathbf{k} \\
\theta &= 30° \\
\mathbf{r}_P &= (3 \text{ in.})(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) \\
\mathbf{v}_{P'} &= \omega_1\mathbf{i} \times \mathbf{r}_P \\
&= 5\mathbf{i} \times (3\cos 30°\mathbf{i} + 3\sin 30°\mathbf{j}) \\
&= (7.5 \text{ in./s})\mathbf{k} \\
\mathbf{v}_{P/F} &= \omega_2\mathbf{k} \times \mathbf{r}_P \\
&= 4\mathbf{k} \times (3\cos 30°\mathbf{i} + 3\sin 30°\mathbf{j}) \\
&= -(6 \text{ in./s})\mathbf{i} + (10.392 \text{ in./s})\mathbf{j} \\
\mathbf{a}_{P'} &= \dot{\omega}_1\mathbf{i} \times \mathbf{r}_P + \omega_1\mathbf{i} \times \mathbf{v}_{P'} \\
&= 0 + 5\mathbf{i} \times 7.5\mathbf{k} \\
&= -(37.5 \text{ in./s}^2)\mathbf{j} \\
\mathbf{a}_{P/F} &= \dot{\omega}_2\mathbf{k} \times \mathbf{r}_P \\
&= \omega_2\mathbf{k} \times \mathbf{v}_{P/F} \\
&= 0 + 4\mathbf{k} \times (-6\mathbf{i} + 10.392\mathbf{j}) \\
&= -(41.569 \text{ in./s}^2)\mathbf{i} - (24 \text{ in./s}^2)\mathbf{j} \\
\mathbf{a}_c &= 2\mathbf{\Omega} \times \mathbf{v}_{P/F} \\
&= (2)(5\mathbf{i}) \times (-6\mathbf{i} + 10.392\mathbf{j}) \\
&= (103.92 \text{ in./s}^2)\mathbf{k}
\end{aligned}$$

Acceleration at Point P. $\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/F} + \mathbf{a}_c$

$$\mathbf{a}_P = -(41.6 \text{ in./s}^2)\mathbf{i} - (61.5 \text{ in./s}^2)\mathbf{j} + (103.9 \text{ in./s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.233

Using the method of Section 15.7A, solve Problem 15.198.

PROBLEM 15.198 At the instant shown, the robotic arm *ABC* is being rotated simultaneously at the constant rate $\omega_1 = 0.15$ rad/s about the y axis, and at the constant rate $\omega_2 = 0.25$ rad/s about the z axis. Knowing that the length of arm *ABC* is 1 m, determine (*a*) the angular acceleration of the arm, (*b*) the velocity of Point *C*, (*c*) the acceleration of Point *C*.

SOLUTION

Geometry: Dimensions in meters.

$$\mathbf{r}_{C/A} = (1.0\cos 35°)\mathbf{i} + (1.0\sin 35°)\mathbf{j} = 0.81915\mathbf{i} + 0.57358\mathbf{j}$$

Angular velocities:

$$\boldsymbol{\omega}_1 = \omega_1\mathbf{j} = (0.15 \text{ rad/s})\mathbf{j} \qquad (\dot{\omega}_1 = 0)$$

$$\boldsymbol{\omega}_2 = \omega_2\mathbf{k} = (0.15 \text{ rad/s})\mathbf{k} \qquad (\dot{\omega}_2 = 0)$$

Use a frame of reference rotating about the y-axis.

Its angular velocity is $\boldsymbol{\Omega} = \omega_1\mathbf{j} = (0.15 \text{ rad/s})\mathbf{j}$

(*a*) *Angular acceleration*:

$$\begin{aligned}\boldsymbol{\alpha} &= \dot{\omega}_1\mathbf{j} + \dot{\omega}_2\mathbf{k} + \boldsymbol{\Omega}\times\boldsymbol{\omega}_2 \\ &= 0 + 0 + (0.15\mathbf{j})(0.25\mathbf{k})\end{aligned}$$

$\boldsymbol{\alpha} = (0.0375 \text{ rad/s}^2)\mathbf{i}$ ◀

Motion of coinciding Point C.

$$\begin{aligned}\mathbf{v}_{C'} &= \boldsymbol{\Omega}\times\mathbf{r}_{C/A} = 0.15\mathbf{j}\times(0.81915\mathbf{i} + 0.57358\mathbf{j}) \\ &= -(0.12287 \text{ m/s})\mathbf{k} \\ \mathbf{a}_{C'} &= \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}_{C/A}) = (0.15\mathbf{j})\times(-0.12287\mathbf{k}) \\ &= -(0.018431 \text{ m/s}^2)\mathbf{i}\end{aligned}$$

Motion of C relative to the frame.

$$\begin{aligned}\mathbf{v}_{C/F} &= \omega_2\mathbf{k}\times\mathbf{r}_{C/A} = 0.25\mathbf{k}(0.81915\mathbf{i} + 0.57358\mathbf{j}) \\ &= -(0.14339 \text{ m/s})\mathbf{i} + (0.20479 \text{ m/s})\mathbf{j} \\ \mathbf{a}_{C/F} &= \omega_1\mathbf{k}\times\mathbf{v}_{C/F} = 0.25\mathbf{k}\times(-0.14339\mathbf{i} + 0.20479\mathbf{j}) \\ &= -(0.051198 \text{ m/s}^2)\mathbf{i} - (0.035848 \text{ m/s}^2)\mathbf{j}\end{aligned}$$

PROBLEM 15.233 (Continued)

(*b*) *Velocity of C.* $\mathbf{v}_C = \mathbf{v}_{C'} + \mathbf{v}_{C/F}$

$$\mathbf{v}_C - (0.143\text{ m/s})\mathbf{i} + (0.205\text{ m/s})\mathbf{j} - (0.123\text{ m/s})\mathbf{k} \blacktriangleleft$$

Coriolis acceleration. $2\mathbf{\Omega} \times \mathbf{v}_{C/F}$

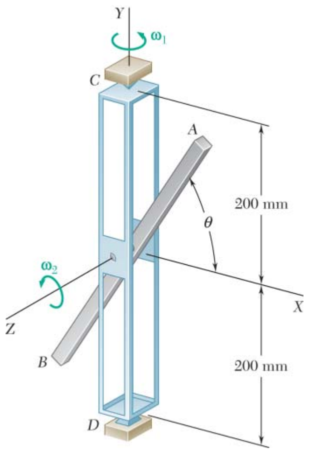
$$2\mathbf{\Omega} \times \mathbf{v}_{C/F} = (2)(0.15\mathbf{j}) \times (-0.14339\mathbf{i} + 0.20479\mathbf{j})$$

$$= (0.043017\text{ m/s}^2)\mathbf{k}$$

(*c*) *Acceleration of C.* $\mathbf{a}_C = \mathbf{a}_{C'} + \mathbf{a}_{C/F} + 2\mathbf{\Omega} \times \mathbf{v}_{C/F}$

$$\mathbf{a}_C = -0.01843\mathbf{i} - 0.051198\mathbf{i} - 0.035848\mathbf{j} + 0.043017\mathbf{k}$$

$$\mathbf{a}_C = -(0.0696\text{ m/s}^2)\mathbf{i} - (0.0358\text{ m/s}^2)\mathbf{j} + (0.0430\text{ m/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.234

The 400-mm bar AB is made to rotate at the constant rate $\omega_2 = d\theta/dt = 8$ rad/s with respect to the frame CD which itself rotates at the constant rate $\omega_1 = 12$ rad/s about the Y axis. Knowing that $\theta = 60°$ at the instant shown, determine the velocity and acceleration of point A.

SOLUTION

Geometry: $\theta = 60°, \quad \mathbf{r}_A = (0.200\text{ m})(\cos 60°\mathbf{i} + \sin 60°\mathbf{j}) = (0.1\text{ m})\mathbf{i} + \left(0.1\sqrt{3}\text{ m}\right)\mathbf{j}$

Let the frame $Oxyz$ rotate about the y axis with constant angular velocity $\mathbf{\Omega} = \omega_1\mathbf{j} = (12\text{ rad/s})\mathbf{j}$. Then, the motion relative to the frame consists of rotation about the z axis with constant angular velocity $\boldsymbol{\omega}_2 = \omega_2\mathbf{k} = (8\text{ rad/s})\mathbf{k}$.

Motion of the coinciding point A' in the frame.

$$\mathbf{v}_{A'} = \omega_1\mathbf{j}\times\mathbf{r}_A = 12\mathbf{j}\times\left(0.1\mathbf{i} + 0.1\sqrt{3}\mathbf{j}\right) = -(1.2\text{ m/s})\mathbf{k}$$

$$\mathbf{a}_{A'} = \left(\frac{d\omega_1}{dt}\right)\mathbf{j}\times\mathbf{r}_A + \omega_1\mathbf{j}\times\mathbf{v}_{A'}$$

$$= 0 + 12\mathbf{j}\times(-1.2\mathbf{k}) = -\left(14.4\text{ m/s}^2\right)\mathbf{i}$$

Motion relative to the frame.

$$\mathbf{v}_{A/F} = \boldsymbol{\omega}_2\mathbf{k}\times\mathbf{r}_A = 8\mathbf{k}\times\left(0.1\mathbf{i} + 0.1\sqrt{3}\mathbf{j}\right) = -\left(0.8\sqrt{3}\text{ m/s}\right)\mathbf{i} + (0.8\text{ m/s})\mathbf{j}$$

$$\mathbf{a}_{A/F} = \left(\frac{d\omega_2}{dt}\right)\mathbf{k}\times\mathbf{r}_A + \omega_2\mathbf{k}\times\mathbf{v}_{A/F}$$

$$= 0 + 8\mathbf{k}\times\left(-0.8\sqrt{3}\mathbf{i} + 0.8\mathbf{j}\right) = -\left(6.4\text{ m/s}^2\right)\mathbf{i} - \left(6.4\sqrt{3}\text{ m/s}^2\right)\mathbf{j}$$

Velocity of point A: $\mathbf{v}_A = \mathbf{v}_{A'} + \mathbf{v}_{A/F}$

$$\mathbf{v}_A = -\left(0.8\sqrt{3}\text{ m/s}\right)\mathbf{i} + (0.8\text{ m/s})\mathbf{j} - (1.2\text{ m/s})\mathbf{k}$$

$$\mathbf{v}_A = -(1.386\text{ m/s})\mathbf{i} + (0.800\text{ m/s})\mathbf{j} - (1.200\text{ m/s})\mathbf{k} \blacktriangleleft$$

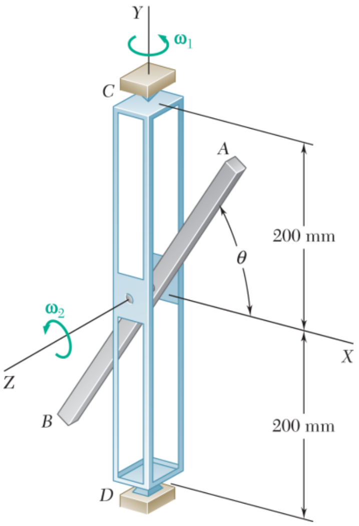
Coriolis acceleration:

$$2\mathbf{\Omega}\times\mathbf{v}_{A/F} = (2)(12\mathbf{j})\times\left(-0.8\sqrt{3}\mathbf{i} + 0.8\mathbf{j}\right) = \left(19.2\sqrt{3}\text{ m/s}^2\right)\mathbf{k}$$

Acceleration of point A: $\mathbf{a}_A = \mathbf{a}_{A'} + \mathbf{a}_{A/F} + 2\boldsymbol{\Omega} \times \mathbf{v}_{A/F}$

$$\mathbf{a}_A = -\left(20.8\text{ m/s}^2\right)\mathbf{i} - \left(6.4\sqrt{3}\text{ m/s}^2\right)\mathbf{j} + \left(19.2\sqrt{3}\text{ m/s}^2\right)\mathbf{k}$$

$$\mathbf{a}_A = -\left(20.8\text{ m/s}^2\right)\mathbf{i} - \left(11.09\text{ m/s}^2\right)\mathbf{j} + \left(33.3\text{ m/s}^2\right)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.235

The 400-mm bar AB is made to rotate at the rate $\omega_2 = d\theta/dt$ with respect to the frame CD which itself rotates at the rate ω_1 about the Y axis. At the instant shown $\omega_1 = 12$ rad/s, $d\omega_1/dt = -16$ rad/s^2, $\omega_2 = 8$ rad/s, $d\omega_2/dt = 10$ rad/s^2, and $\theta = 60°$. Determine the velocity and acceleration of point A at this instant.

SOLUTION

Geometry. $\theta = 60°$, $\mathbf{r}_A = (0.200\text{ m})(\cos 60°\mathbf{i} + \sin 60°\mathbf{j}) = (0.1\text{ m})\mathbf{i} + (0.1\sqrt{3}\text{ m})\mathbf{j}$

Let the frame $Oxyz$ rotate about the y axis with angular velocity $\boldsymbol{\Omega} = \omega_1\mathbf{j} = (12\text{ rad/s})\mathbf{j}$ and angular acceleration $\boldsymbol{\alpha} = \left(\frac{d\omega_1}{dt}\right)\mathbf{j} = -(16\text{ rad/s}^2)\mathbf{j}$. Then, the motion relative to the frame consists of rotation about the z axis with constant angular velocity $\boldsymbol{\omega}_2 = \omega_2\mathbf{k} = (8\text{ rad/s})\mathbf{k}$ and angular acceleration

$$\boldsymbol{\alpha}_2 = \left(\frac{d\omega_2}{dt}\right)\mathbf{k} = (10\text{ rad/s}^2)\mathbf{k}.$$

Motion of the coinciding point A′ in the frame.

$$\mathbf{v}_{A'} = \omega_1\mathbf{j} \times \mathbf{r}_A = 12\mathbf{j} \times (0.1\mathbf{i} + 0.1\sqrt{3}\mathbf{j}) = -(1.2\text{ m/s})\mathbf{k}$$

$$\mathbf{a}_{A'} = \left(\frac{d\omega_1}{dt}\right)\mathbf{j} \times \mathbf{r}_A + \omega_1\mathbf{j} \times \mathbf{v}_{A'}$$

$$= (-16\mathbf{j}) \times (0.1\mathbf{i} + 0.\sqrt{3}\mathbf{j}) + 12\mathbf{j} \times (-1.2)\mathbf{k} = (1.6\text{ m/s}^2)\mathbf{k} - (14.4\text{ m/s}^2)\mathbf{i}$$

Motion relative to the frame.

$$\mathbf{v}_{A/F} = \omega_2\mathbf{k} \times \mathbf{r}_A = 8\mathbf{k} \times (0.1\mathbf{i} + 0.1\sqrt{3}\mathbf{j}) = -(0.8\sqrt{3}\text{ m/s})\mathbf{i} + (0.8\text{ m/s})\mathbf{j}$$

$$\mathbf{a}_{A/F} = \left(\frac{d\omega_2}{dt}\right)\mathbf{k} \times \mathbf{r}_A + \omega_2\mathbf{k} \times \mathbf{v}_{A/F}$$

$$= 10\mathbf{k} \times (0.1\mathbf{i} + 0.1\sqrt{3}\mathbf{j}) + 8\mathbf{k} \times (-0.8\sqrt{3}\mathbf{i} + 0.8\mathbf{j})$$

$$= -\sqrt{3}\mathbf{i} + \mathbf{j} - 6.4\mathbf{i} - 6.4\sqrt{3}\mathbf{j} = (6.4 + \sqrt{3})\mathbf{i} - (6.4\sqrt{3} - 1)\mathbf{j}$$

Velocity of point A: $\mathbf{v}_A = \mathbf{v}_{A'} + \mathbf{v}_{A/F}$

$$\mathbf{v}_A = -(0.8\sqrt{3}\text{ m/s})\mathbf{i} + (0.8\text{ m/s})\mathbf{j} - (1.2\text{ m/s})\mathbf{k}$$

$$\mathbf{v}_A = -(1.386\text{ m/s})\mathbf{i} + (0.800\text{ m/s})\mathbf{j} - (1.200\text{ m/s})\mathbf{k}$$ ◀

PROBLEM 15.235 (Continued)

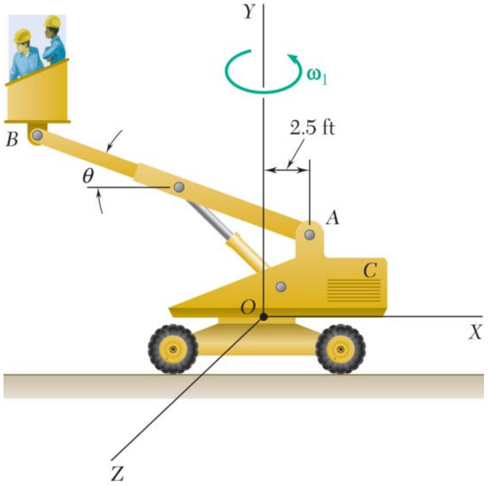
Coriolis acceleration:

$$2\boldsymbol{\Omega} \times \mathbf{v}_{A/F} = (2)(12\mathbf{j}) \times \left(-0.8\sqrt{3}\mathbf{i} + 0.8\mathbf{j}\right) = \left(19.2\sqrt{3}\ \text{m/s}^2\right)\mathbf{k}$$

Acceleration of point A: $\mathbf{a}_A = \mathbf{a}_{A'} + \mathbf{a}_{A/F} + 2\boldsymbol{\Omega} \times \mathbf{v}_{A/F}$

$$\mathbf{a}_A = -\left(20.8 + \sqrt{3}\right)\mathbf{i} - \left(6.4\sqrt{3} - 1.0\right)\mathbf{j} + \left(19.2\sqrt{3} + 1.2\right)\mathbf{k}$$

$$\mathbf{a}_A = -\left(22.5\ \text{m/s}^2\right)\mathbf{i} - \left(10.09\ \text{m/s}^2\right)\mathbf{j} + \left(34.9\ \text{m/s}^2\right)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.236

The arm AB of length 16 ft is used to provide an elevated platform for construction workers. In the position shown, arm AB is being raised at the constant rate $d\theta/dt = 0.25$ rad/s; simultaneously, the unit is being rotated about the Y axis at the constant rate $\omega_1 = 0.15$ rad/s. Knowing that $\theta = 20°$, determine the velocity and acceleration of Point B.

SOLUTION

Frame of reference. Let moving frame $Axyz$ rotate about the Y axis with angular velocity

$$\begin{aligned}\mathbf{\Omega} &= \omega_1\mathbf{j}\\ &= (0.15 \text{ rad/s})\mathbf{j}.\end{aligned}$$

Geometry.

$$\begin{aligned}\mathbf{r}_{B/A} &= -16\cos 20°\mathbf{i} + 16\sin 20°\mathbf{j}\\ &= -(15.035 \text{ ft})\mathbf{i} + (5.4723 \text{ ft})\mathbf{j}\end{aligned}$$

Place Point O on Y axis at same level as Point A.

$$\begin{aligned}\mathbf{r}_{B/O} &= \mathbf{r}_{B/A} + \mathbf{r}_{A/O}\\ &= \mathbf{r}_{B/A} + (2.5 \text{ ft})\mathbf{i}\\ &= -(12.535 \text{ ft})\mathbf{i} + (5.4723 \text{ ft})\mathbf{j}\end{aligned}$$

Motion of corresponding Point B′ in the frame.

$$\begin{aligned}\mathbf{v}_{B'} &= \mathbf{\Omega} \times \mathbf{r}_{B/O}\\ &= (0.15\mathbf{j}) \times -(12.535\mathbf{i} + 5.4723\mathbf{j})\\ &= (1.8803 \text{ ft/s})\mathbf{k}\\ \mathbf{a}_{B'} &= \mathbf{\Omega} \times \mathbf{v}_{B'}\\ &= (0.15\mathbf{j}) \times (1.8803\mathbf{k})\\ &= (0.28204 \text{ ft/s}^2)\mathbf{i}\end{aligned}$$

PROBLEM 15.236 (Continued)

Motion of Point B relative to the frame.

$$\boldsymbol{\omega}_2 = -\frac{d\theta}{dt}\mathbf{k}$$

$$= -(0.25\text{ rad/s})\mathbf{k}$$

$$\mathbf{v}_{B/F} = \boldsymbol{\omega}_2 \times \mathbf{r}_{B/A}$$

$$= (-0.25)\mathbf{k} \times (-15.035\mathbf{i} + 5.4723\mathbf{j})$$

$$= (1.36808\text{ ft/s})\mathbf{i} + (3.7588\text{ ft/s})\mathbf{j}$$

$$\mathbf{a}_{B/F} = \boldsymbol{\omega}_2 \times \mathbf{v}_{B/F}$$

$$= (-0.25\mathbf{k}) \times (1.36808\mathbf{i} + 3.7588\mathbf{j})$$

$$= (0.93969\text{ ft/s}^2)\mathbf{i} - (0.34202\text{ ft/s}^2)\mathbf{j}$$

Velocity of Point B. $\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/F}$

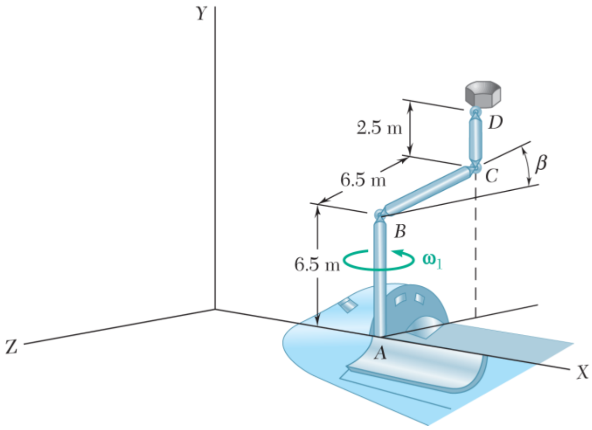
$$\mathbf{v}_B = -(1.37\text{ ft/s})\mathbf{i} + (3.76\text{ ft/s})\mathbf{j} + (1.88\text{ ft/s})\mathbf{k} \blacktriangleleft$$

Coriolis acceleration. $2\boldsymbol{\Omega} \times \mathbf{v}_{B/F} = (2)(0.15\mathbf{j}) \times (1.36808\mathbf{i} + 3.7588\mathbf{j})$

$$= -(0.41042\text{ ft/s}^2)\mathbf{k}$$

Acceleration of Point B. $\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/F} + 2\boldsymbol{\Omega} \times \mathbf{v}_{B/F}$

$$\mathbf{a}_B = (1.22\text{ ft/s}^2)\mathbf{i} - (0.342\text{ ft/s}^2)\mathbf{j} - (0.410\text{ ft/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.237

The remote manipulator system (RMS) shown is used to deploy payloads from the cargo bay of space shuttles. At the instant shown, the whole RMS is rotating at the constant rate $\omega_1 = 0.03 \text{ rad/s}$ about the axis *AB*. At the same time, portion *BCD* rotates as a rigid body at the constant rate $\omega_2 = d\beta/dt = 0.04 \text{ rad/s}$ about an axis through *B* parallel to the *X* axis. Knowing that $\beta = 30°$, determine (*a*) the angular acceleration of *BCD*, (*b*) the velocity of *D*, (*c*) the acceleration of *D*.

SOLUTION

At the instant given, Points *A*, *B*, *C*, and *D* lie in a plane which is parallel to the *YZ* plane. The plane *ABCD* is rotating with angular velocity.

$$\mathbf{\Omega} = \omega_1 \mathbf{j}$$
$$= (0.03 \text{ rad/s})\mathbf{j} \qquad (\dot{\omega}_1 = 0)$$

Body *BCD* is rotating about an axis through *B* parallel to the *x*-axis at angular velocity.

$$\boldsymbol{\omega}_2 = \frac{d\beta}{dt}\mathbf{i} = (0.04 \text{ rad/s})\mathbf{i} \qquad (\dot{\omega}_2 = 0)$$

(*a*) *Angular acceleration of BCD.*

$$\boldsymbol{\alpha}_{BCD} = \dot{\omega}_1 \mathbf{j} + \dot{\omega}_2 \mathbf{i} + \mathbf{\Omega} \times \boldsymbol{\omega}_2$$
$$= 0 + 0 + (0.03\mathbf{j}) \times (0.04\mathbf{i})$$

$$\boldsymbol{\alpha}_{BCD} = -(0.0012 \text{ rad/s}^2)\mathbf{k} \blacktriangleleft$$

Let the plane of *BCD* be a rotating frame of reference rotating about *AB* with angular velocity $\mathbf{\Omega}$.

Geometry: $\beta = 30°$

$$\mathbf{r}_{D/B} = (6.5 \text{ m})(\sin\beta\mathbf{j} - \cos\beta\mathbf{k}) + (2.5 \text{ m})\mathbf{j} = (5.75 \text{ m})\mathbf{j} - (5.6292 \text{ m})\mathbf{k}$$

Motion of Point *D′* in the frame.

$$\mathbf{v}_{D'} = \mathbf{\Omega} \times \mathbf{r}_{D/B} = 0.03\mathbf{j} \times (3.25\mathbf{j} - 5.6292\mathbf{k})$$
$$= -(0.168875 \text{ m/s})\mathbf{i}$$
$$\mathbf{a}_{D'} = \mathbf{\Omega} \times \mathbf{v}_{D'} = (0.03\mathbf{j}) \times (-0.168875\mathbf{i})$$
$$= (0.0050662 \text{ m/s}^2)\mathbf{k}$$

Motion of *D* relative to the frame: This motion is a rotation about *B* with angular velocity.

$$\boldsymbol{\omega}_2 = (0.04\text{ rad/s})\mathbf{i}$$

$$\begin{aligned}\mathbf{v}_{D/\text{frame}} &= \boldsymbol{\omega}_2 \times \mathbf{r}_{D/B}\\ &= (0.04\mathbf{i}) \times (5.75\mathbf{j} - 5.6292\mathbf{k})\\ &= (0.22517\text{ m/s})\mathbf{j} + (0.23\text{ m/s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{D/\text{frame}} &= \boldsymbol{\omega}_2 \times r_{D/\text{frame}}\\ &= (0.04\mathbf{i}) \times (0.22517\mathbf{j} + 0.23\mathbf{k})\\ &= -(0.0092\text{ m/s}^2)\mathbf{j} + (0.009007\text{ m/s}^2)\mathbf{k}\end{aligned}$$

(*b*) *Velocity of D.* $\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/\text{frame}}$

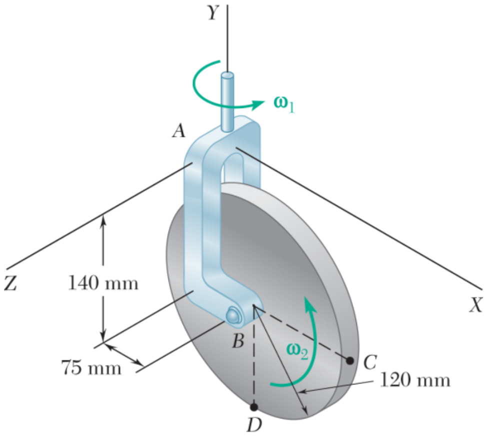
$$\mathbf{v}_D = -(0.169\text{ m/s})\mathbf{i} + (0.225\text{ m/s})\mathbf{j} + (0.230\text{ m/s})\mathbf{k} \blacktriangleleft$$

Coriolis acceleration: $2\boldsymbol{\Omega} \times \mathbf{v}_{D/\text{frame}}$

$$\begin{aligned}2\boldsymbol{\Omega} \times \mathbf{v}_{D/\text{frame}} &= (2)(0.03\mathbf{j}) \times (0.22517\mathbf{j} + 0.23\mathbf{k})\\ &= (0.0138\text{ m/s}^2)\mathbf{i}\end{aligned}$$

(*c*) *Acceleration of D.* $\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/\text{frame}} + 2\boldsymbol{\Omega} \times \mathbf{v}_{D/\text{frame}}$

$$\mathbf{a}_D = (0.0138\text{ m/s}^2)\mathbf{i} - (0.0092\text{ m/s}^2)\mathbf{j} + (0.0141\text{ m/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.238

A disk of radius 120 mm rotates at the constant rate $\omega_2 = 5$ rad/s with respect to the arm AB, which itself rotates at the constant rate $\omega_1 = 3$ rad/s. For the position shown, determine the velocity and acceleration of Point C.

SOLUTION

Geometry.

$$\mathbf{r}_{C/A} = (0.195\text{ m})\mathbf{i} - (0.1)\mathbf{j}$$

$$\mathbf{r}_{C/B} = (0.12\text{ m})\mathbf{i}$$

Let frame $Axyz$, which coincides with the fixed frame $AXYZ$ at the instant shown, be rotating about the y axis with constant angular velocity $\boldsymbol{\Omega} = \omega_1\mathbf{j} = (3\text{ rad/s})\mathbf{j}$. Then the motion relative to the frame consists of rotation about the axle B with a constant angular velocity $\boldsymbol{\omega}_2 = \omega_2\mathbf{k} = (5\text{ rad/s})\mathbf{k}$.

Motion of the coinciding Point C′ in the frame.

$$\begin{aligned}\mathbf{v}_{C'} &= \boldsymbol{\Omega}\times\mathbf{r}_{C/A}\\ &= 3\mathbf{j}\times(0.195\mathbf{i} - 0.14\mathbf{j})\\ &= -(0.585\text{ m/s})\mathbf{k}\\ \mathbf{a}_{C'} &= \boldsymbol{\Omega}\times\mathbf{v}_{C'}\\ &= 3\mathbf{j}\times(-0.585\mathbf{k})\\ &= -(1.755\text{ m/s}^2)\mathbf{i}\end{aligned}$$

Motion relative to the frame.

$$\begin{aligned}\mathbf{v}_{C/F} &= \boldsymbol{\omega}_2\times\mathbf{r}_{C/B}\\ &= 5\mathbf{k}\times 0.12\mathbf{i}\\ &= (0.6\text{ m/s})\mathbf{j}\\ \mathbf{a}_{C/F} &= \boldsymbol{\omega}_2\times\mathbf{v}_{C/F}\\ &= 5\mathbf{k}\times(-0.6\mathbf{j})\\ &= -(3\text{ m/s}^2)\mathbf{i}\end{aligned}$$

Velocity of Point C.

$$\mathbf{v}_C = \mathbf{v}_{C'} + \mathbf{v}_{C/F}$$

$$\mathbf{v}_C = (0.600\text{ m/s})\mathbf{j} - (0.585\text{ m/s})\mathbf{k} \quad \blacktriangleleft$$

Coriolis acceleration.

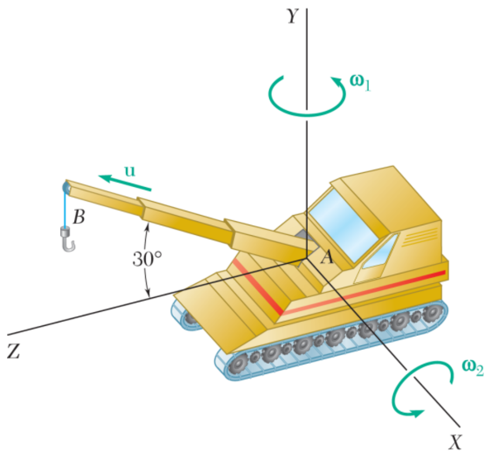
$$2\boldsymbol{\Omega}\times\mathbf{v}_{C/F} = (2)(3\mathbf{j})\times(0.6\mathbf{j}) = 0$$

Acceleration of Point C.

$$\mathbf{a}_C = \mathbf{a}_{C'} + \mathbf{a}_{C/F} + 2\boldsymbol{\Omega}\times\mathbf{v}_{C/F}$$

$$\mathbf{a}_C = -1.7551\mathbf{i} - 3\mathbf{i} + 0$$

$$\mathbf{a}_C = -(4.76\text{ m/s}^2)\mathbf{i} \quad \blacktriangleleft$$



PROBLEM 15.239

The crane shown rotates at the constant rate $\omega_1 = 0.25$ rad/s; simultaneously, the telescoping boom is being lowered at the constant rate $\omega_2 = 0.40$ rad/s. Knowing that at the instant shown the length of the boom is 20 ft and is increasing at the constant rate $u = 1.5$ ft/s, determine the velocity and acceleration of Point *B*.

SOLUTION

Geometry.

$$\begin{aligned}\mathbf{r}_{B/A} &= \mathbf{r}_B \\ &= (20\text{ ft})(\sin 30°\mathbf{j} + \cos 30°\mathbf{k}) \\ &= (10\text{ ft})\mathbf{j} + (10\sqrt{3}\text{ ft})\mathbf{k}\end{aligned}$$

Method 1

Let the unextending portion of the boom *AB* be a rotating frame of reference.

Its angular velocity is

$$\begin{aligned}\mathbf{\Omega} &= \omega_2\mathbf{i} + \omega_1\mathbf{j} \\ &= (0.40\text{ rad/s})\mathbf{i} + (0.25\text{ rad/s})\mathbf{j}.\end{aligned}$$

Its angular acceleration is

$$\begin{aligned}\boldsymbol{\alpha} &= \omega_1\mathbf{j} \times \omega_2\mathbf{i} \\ &= -\omega_1\omega_2\mathbf{k} \\ &= -(0.10\text{ rad/s}^2)\mathbf{k}.\end{aligned}$$

Motion of the coinciding Point B′ in the frame.

$$\begin{aligned}\mathbf{v}_{B'} &= \mathbf{\Omega} \times \mathbf{r}_B \\ &= (0.40\mathbf{i} + 0.25\mathbf{j}) \times (10\mathbf{j} + 10\sqrt{3}\mathbf{k}) \\ &= (2.5\sqrt{3}\text{ ft/s})\mathbf{i} - (4\sqrt{3}\text{ ft/s})\mathbf{j} + (4\text{ ft/s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{B'} &= \boldsymbol{\alpha} \times \mathbf{r}_B + \mathbf{\Omega} \times \mathbf{v}_{B'} \\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -0.10 \\ 0 & 10 & 10\sqrt{3}\end{vmatrix} + \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.40 & 0.25 & 0 \\ 2.5\sqrt{3} & -4\sqrt{3} & 4\end{vmatrix}\end{aligned}$$

$$\mathbf{i} + \mathbf{i} - 1.6\mathbf{j} - 3.8538\mathbf{k} = (2\text{ ft/s}^2)\mathbf{i} - (1.6\text{ ft/s}^2)\mathbf{j} - (3.8538\text{ ft/s}^2)\mathbf{k}$$

Motion relative to the frame.

$$\begin{aligned}\mathbf{v}_{B/F} &= u(\sin 30°\mathbf{j} + \cos 30°\mathbf{k}) \\ &= (1.5\text{ ft/s})\sin 30°\mathbf{j} + (1.5\text{ ft/s})\cos 30°\mathbf{k}\end{aligned}$$

$$\mathbf{a}_{B/F} = 0$$

Velocity of Point B.

$$\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/F}$$

$$\mathbf{v}_B = 2.5\sqrt{3}\mathbf{i} - 4\sqrt{3}\mathbf{j} + 4\mathbf{k} + 1.5\sin 30°\mathbf{j} + 1.5\cos 30°\mathbf{k}$$

$$\mathbf{v}_B = (4.33 \text{ ft/s})\mathbf{i} - (6.18 \text{ ft/s})\mathbf{j} + (5.30 \text{ ft/s})\mathbf{k} \blacktriangleleft$$

Coriolis acceleration. $2\mathbf{\Omega} \times \mathbf{v}_{B/F}$

$$2\mathbf{\Omega} \times \mathbf{v}_{B/F} = (2)(0.40\mathbf{i} + 0.25\mathbf{j}) \times (1.5\sin 30°\mathbf{j} + 1.5\cos 30°\mathbf{k})$$

$$= (0.64952 \text{ ft/s}^2)\mathbf{i} - (1.03923 \text{ ft/s}^2)\mathbf{j} + (0.6 \text{ ft/s}^2)\mathbf{k}$$

Acceleration of Point B.

$$\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/F} + 2\mathbf{\Omega} \times \mathbf{v}_{B/F}$$

$$\mathbf{a}_B = (2 + 0.64952)\mathbf{i} - (1.6 + 1.03923\mathbf{j}) + (-3.8538 + 0.6)\mathbf{k}$$

$$\mathbf{a}_B = (2.65 \text{ ft/s}^2)\mathbf{i} - (2.64 \text{ ft/s}^2)\mathbf{j} - (3.25 \text{ ft/s}^2)\mathbf{k} \blacktriangleleft$$

Method 2

Let frame *Axyz*, which at the instant shown coincides with *AXYZ*, rotate with an angular velocity $\mathbf{\Omega}_1 = \omega_1\mathbf{j} = (0.25 \text{ rad/s})\mathbf{j}$. Then the motion relative to this frame consists of turning the boom relative to the cab and extending the boom.

Motion of the coinciding Point B′ in the frame.

$$\mathbf{v}_{B'} = \mathbf{\Omega} \times \mathbf{r}_B$$

$$= 0.25\mathbf{j} \times (10\mathbf{j} + 10\sqrt{3}\mathbf{k})$$

$$= (2.5\sqrt{3} \text{ m/s})\mathbf{i}$$

$$\mathbf{a}_{B'} = \mathbf{\Omega} \times \mathbf{v}_{B'}$$

$$= 0.25\mathbf{j} \times (2.5\sqrt{3}\mathbf{i})$$

$$= -(0.625\sqrt{3} \text{ m/s}^2)\mathbf{k}$$

Motion of Point B relative to the frame.

Let the unextending portion of the boom be a rotating frame with constant angular velocity $\mathbf{\Omega}_2 = \omega_2\mathbf{i} = (0.40 \text{ rad/s})\mathbf{i}$. The motion relative to this frame is the extensional motion with speed u.

$$\mathbf{v}_{B''} = \mathbf{\Omega}_2 \times \mathbf{r}_B$$

$$= 0.40\mathbf{i} \times (10\mathbf{j} + 10\sqrt{3}\mathbf{k})$$

$$= -(4\sqrt{3} \text{ ft/s})\mathbf{j} + (4 \text{ ft/s})\mathbf{k}$$

$$\mathbf{a}_{B''} = \mathbf{\Omega}_2 \times \mathbf{v}_{B''}$$

$$= 0.40\mathbf{i} \times (-4\sqrt{3}\mathbf{j} + 4\mathbf{k})$$

$$= -(1.6 \text{ ft/s}^2)\mathbf{j} - (1.6\sqrt{3} \text{ ft/s}^2)\mathbf{k}$$

$$\mathbf{v}_{B/\text{boom}} = u(\sin 30°\mathbf{j} + \cos 30°\mathbf{k})$$

$$= (1.5 \text{ ft/s})\sin 30°\mathbf{j} + (1.5 \text{ ft/s})\cos 30°\mathbf{k}$$

$$\mathbf{a}_{B/\text{boom}} = 0$$

PROBLEM 15.239 (Continued)

$$
\begin{aligned}
2\boldsymbol{\Omega}_2 \times \mathbf{v}_{B/\text{boom}} &= (2)(0.40\mathbf{i}) \times (1.5\sin 30^\circ\mathbf{j} + 1.5\cos 30^\circ\mathbf{k}) \\
&= -(1.03923\ \text{ft/s}^2)\mathbf{j} + (0.6\ \text{ft/s}^2)\mathbf{k} \\
\mathbf{v}_{B/F} &= \mathbf{v}_{B''} + \mathbf{v}_{B/\text{boom}} \\
&= -4\sqrt{3}\mathbf{j} + 4\mathbf{k} + 1.5\sin 30^\circ\mathbf{j} + 1.5\cos 30^\circ\mathbf{k} \\
&= -(6.1782\ \text{ft/s})\mathbf{j} + (5.299\ \text{ft/s})\mathbf{k} \\
\mathbf{a}_{B/F} &= \mathbf{a}_{B''} + \mathbf{a}_{B/\text{boom}} + 2\boldsymbol{\Omega}_2 \times \mathbf{v}_{B/\text{boom}} \\
&= -1.6\mathbf{j} - 1.6\sqrt{3}\mathbf{k} + 0 - 1.03923\mathbf{j} + 0.6\mathbf{k} \\
&= -(2.6392\ \text{ft/s}^2)\mathbf{j} - (2.1713\ \text{ft/s}^2)\mathbf{k}
\end{aligned}
$$

Velocity of Point B.

$$\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/F}$$

$$\mathbf{v}_B = 2.5\sqrt{3}\mathbf{i} - 6.1782\mathbf{j} + 5.299\mathbf{k}$$

$$\mathbf{v}_B = (4.33\ \text{ft/s})\mathbf{i} - (6.18\ \text{ft/s})\mathbf{j} + (5.30\ \text{ft/s})\mathbf{k} \quad \blacktriangleleft$$

Coriolis acceleration.

$$2\boldsymbol{\Omega}_1 \times \mathbf{v}_{B/F}$$

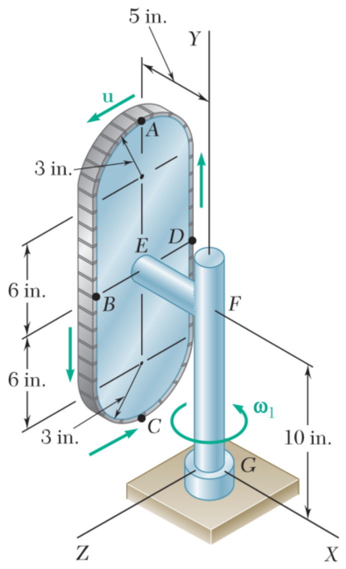
$$
\begin{aligned}
2\boldsymbol{\Omega}_1 \times \mathbf{v}_{B/F} &= (2)(0.25\mathbf{j}) \times (-6.1782\mathbf{j} + 5.299\mathbf{k}) \\
&= (2.6495\ \text{ft/s}^2)\mathbf{i}
\end{aligned}
$$

Acceleration of Point B.

$$\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/F} + 2\boldsymbol{\Omega}_1 \times \mathbf{v}_{B/F}$$

$$\mathbf{a}_B = -0.625\sqrt{3}\mathbf{k} - 2.6392\mathbf{j} - 2.1713\mathbf{k} + 2.6495\mathbf{i}$$

$$\mathbf{a}_B = (2.65\ \text{ft/s}^2)\mathbf{i} - (2.64\ \text{ft/s}^2)\mathbf{j} - (3.25\ \text{ft/s}^2)\mathbf{k} \quad \blacktriangleleft$$



PROBLEM 15.240

The vertical plate shown is welded to arm *EFG*, and the entire unit rotates at the constant rate $\omega_1 = 1.6$ rad/s about the *Y* axis. At the same time, a continuous link belt moves around the perimeter of the plate at a constant speed $u = 4.5$ in./s. For the position shown, determine the acceleration of the link of the belt located (*a*) at Point *A*, (*b*) at Point *B*.

SOLUTION

Let the moving frame of reference be the unit, less the pulleys and belt. It rotates about the *Y* axis with constant angular velocity $\boldsymbol{\Omega} = \omega_1 \mathbf{j} = (1.6 \text{ rad/s})\mathbf{j}$. The relative motion is that of the pulleys and belt with speed $u = 90$ mm/s.

(*a*) *Acceleration at Point A.*

$$\mathbf{r}_A = -(5 \text{ in.})\mathbf{i} + (19 \text{ in.})\mathbf{j}$$

$$\begin{aligned}\mathbf{v}_{A'} &= \boldsymbol{\Omega} \times \mathbf{r}_A \\ &= 1.6\mathbf{j} \times (-5\mathbf{i} + 19\mathbf{j}) \\ &= (8 \text{ in./s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{A'} &= \boldsymbol{\Omega} \times \mathbf{v}_{A'} \\ &= 1.6\mathbf{j} \times 8\mathbf{k} \\ &= (12.8 \text{ in./s}^2)\mathbf{i}\end{aligned}$$

$$\mathbf{v}_{A/F} = u\mathbf{k} = (4.5 \text{ in./s})\mathbf{k}$$

$$\begin{aligned}\mathbf{a}_{A/F} &= -\left(\frac{u^2}{\rho}\right)\mathbf{j} \\ &= -\left(\frac{4.5^2}{3}\right)\mathbf{j} \\ &= -\left(6.75 \text{ in./s}^2\right)\mathbf{j}\end{aligned}$$

$$\begin{aligned}2\boldsymbol{\Omega} \times \mathbf{v}_{A/F} &= (2)(1.6\mathbf{j}) \times (4.5\mathbf{k}) \\ &= (14.4 \text{ in./s}^2)\mathbf{i}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_A &= \mathbf{a}_{A'} + \mathbf{a}_{A/F} + 2\boldsymbol{\Omega} \times \mathbf{v}_{A/F} \\ &= 12.8\mathbf{i} - 6.75\mathbf{j} + 14.4\mathbf{i}\end{aligned}$$

$$\mathbf{a}_A = (27.2 \text{ in./s}^2)\mathbf{i} - (6.75 \text{ in./s}^2)\mathbf{j} \blacktriangleleft$$

(*b*) *Acceleration of Point B.*

$$\mathbf{r}_B = -(5\text{ in.})\mathbf{i} + (10\text{ in.})\mathbf{j} + (3\text{ in.})\mathbf{k}$$

$$\mathbf{v}_{B'} = \mathbf{\Omega} \times \mathbf{r}_B$$

$$= 1.6\mathbf{j} \times (-5\mathbf{i} + 10\mathbf{j} + 3\mathbf{k})$$

$$= (4.8\text{ in./s})\mathbf{i} + (8\text{ in./s})\mathbf{k}$$

$$\mathbf{a}_{B'} = \mathbf{\Omega} \times \mathbf{v}_{B'}$$

$$= 1.6\mathbf{j} \times (4.8\mathbf{i} + 8\mathbf{k})$$

$$= (12.8\text{ in./s}^2)\mathbf{i} - (7.68\text{ in./s}^2)\mathbf{k}$$

$$\mathbf{v}_{B/F} = -u\mathbf{j}$$

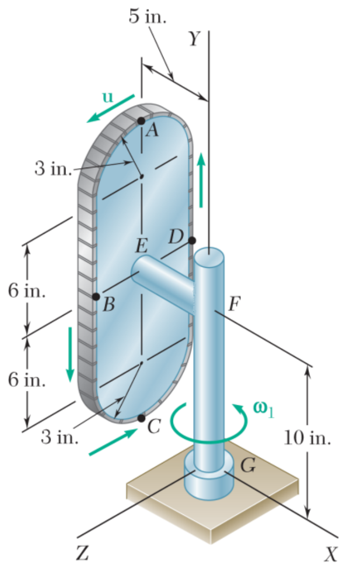
$$= -(3\text{ in./s})\mathbf{j}$$

$$\mathbf{a}_{B/F} = 0$$

$$2\mathbf{\Omega} \times \mathbf{v}_{B/F} = (2)(1.6\mathbf{j}) \times (4.5\mathbf{j}) = 0$$

$$\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/F} + 2\Omega \times \mathbf{v}_{B/F}$$

$$= 12.8\mathbf{i} - 7.68\mathbf{k} + 0 + 0 \qquad \mathbf{a}_B = (12.80\text{ in./s}^2)\mathbf{i} - (7.68\text{ in./s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.241

The vertical plate shown is welded to arm *EFG*, and the entire unit rotates at the constant rate $\omega_1 = 1.6$ rad/s about the *Y* axis. At the same time, a continuous link belt moves around the perimeter of the plate at a constant speed $u = 4.5$ in./s. For the position shown, determine the acceleration of the link of the belt located (*a*) at Point *C*, (*b*) at Point *D*.

SOLUTION

Let the moving frame of reference be the unit, less the pulleys and belt. It rotates about the *Y* axis with constant angular velocity $\mathbf{\Omega} = \omega_1\mathbf{j} = (1.6 \text{ rad/s})\mathbf{j}$. The relative motion is that of the pulleys and belt with speed $u = 90$ mm/s.

(*a*) *Acceleration at Point C.*

$$\begin{aligned}
\mathbf{r}_C &= -(5 \text{ in.})\mathbf{i} + (4 \text{ in.})\mathbf{j}\\
\mathbf{v}_{C'} &= \mathbf{\Omega} \times \mathbf{r}_A\\
&= 1.6\mathbf{j} \times (-5\mathbf{i} + 4\mathbf{j})\\
&= (8 \text{ in./s})\mathbf{k}\\
\mathbf{a}_{C'} &= \mathbf{\Omega} \times \mathbf{v}_{A'}\\
&= 1.6\mathbf{j} \times 8\mathbf{k}\\
&= (12.8 \text{ in./s}^2)\mathbf{i}\\
\mathbf{v}_{C/F} &= -u\mathbf{k} = -(4.5 \text{ in./s})\mathbf{k}\\
\mathbf{a}_{C/F} &= \left(\frac{u^2}{\rho}\right)\mathbf{j}\\
&= \left(\frac{4.5^2}{3}\right)\mathbf{j}\\
&= (6.75 \text{ in./s}^2)\mathbf{j}\\
2\mathbf{\Omega} \times \mathbf{v}_{C/F} &= (2)(1.6\mathbf{j}) \times (-4.5\mathbf{k})\\
&= -(14.4 \text{ in./s}^2)\mathbf{i}\\
\mathbf{a}_C &= \mathbf{a}_{C'} + \mathbf{a}_{C/F} + 2\mathbf{\Omega} \times \mathbf{v}_{C/F}\\
&= 12.8\mathbf{i} + 6.75\mathbf{j} - 14.4\mathbf{i}
\end{aligned}$$

$\mathbf{a}_C = -(1.600 \text{ in./s}^2)\mathbf{i} + (6.75 \text{ in./s}^2)\mathbf{j}$ ◀

(*b*) *Acceleration at Point D.*

$$\mathbf{r}_D = -(5\text{ in.})\mathbf{i} + (10\text{ in.})\mathbf{j} - (3\text{ in.})\mathbf{k}$$

$$\begin{aligned}\mathbf{v}_{D'} &= \boldsymbol{\Omega} \times \mathbf{r}_B \\ &= 1.6\mathbf{j} \times (-5\mathbf{i} + 10\mathbf{j} - 3\mathbf{k}) \\ &= -(4.8\text{ in./s})\mathbf{i} + (8\text{ in./s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{D'} &= \boldsymbol{\Omega} \times \mathbf{v}_{B'} \\ &= 1.6\mathbf{j} \times (-4.8\mathbf{i} + 8\mathbf{k}) \\ &= (12.8\text{ in./s}^2)\mathbf{i} + (7.68\text{ in./s}^2)\mathbf{k}\end{aligned}$$

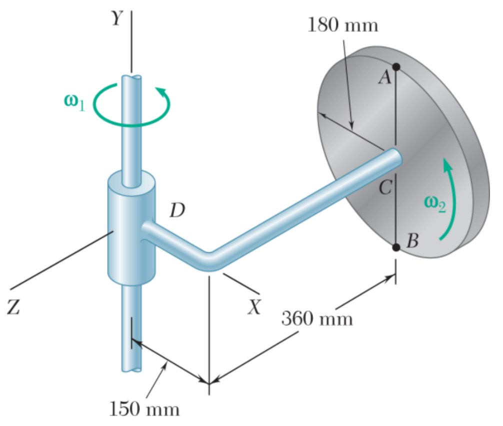
$$\mathbf{v}_{D/F} = u\mathbf{j} = (4.5\text{ in./s})\mathbf{j}$$

$$\mathbf{a}_{D/F} = 0$$

$$2\boldsymbol{\Omega} \times \mathbf{v}_{D/F} = (2)(1.6) \times (-4.5\mathbf{j}) = 0$$

$$\begin{aligned}\mathbf{a}_D &= \mathbf{a}_{D'} + \mathbf{a}_{D/F} + 2\boldsymbol{\Omega} \times \mathbf{v}_{D/F} \\ &= 12.8\mathbf{i} + 7.68\mathbf{k} + 0 + 0\end{aligned}$$

$$\mathbf{a}_D = (12.80\text{ in./s}^2)\mathbf{i} + (7.68\text{ in./s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.242

A disk of 180-mm radius rotates at the constant rate $\omega_2 = 12$ rad/s with respect to arm *CD*, which itself rotates at the constant rate $\omega_1 = 8$ rad/s about the *Y* axis. Determine at the instant shown the velocity and acceleration of Point *A* on the rim of the disk.

SOLUTION

Geometry.

$$\mathbf{r}_{A/D} = (0.15\text{ m})\mathbf{i} + (0.18\text{ m})\mathbf{j} - (0.36\text{ m})\mathbf{k}$$

$$\mathbf{r}_{A/C} = (0.18\text{ m})\mathbf{j}$$

Let frame *Dxyz*, which coincides with the fixed frame *DXYZ* at the instant shown, be rotating about the *y* axis with constant angular velocity $\mathbf{\Omega} = \omega_1\mathbf{j} = (8\text{ rad/s})\mathbf{j}$. Then the motion relative to the frame consists of a rotation of the disk *AB* about the bent axle *CD* with constant angular velocity $\boldsymbol{\omega}_2 = \omega_2\mathbf{k} = (12\text{ rad/s})\mathbf{k}$.

Motion of the coinciding Point A′ in the frame.

$$\begin{aligned}\mathbf{v}_{A'} &= \mathbf{\Omega} \times \mathbf{r}_{A/D}\\ &= 8\mathbf{j} \times (0.15\mathbf{i} + 0.18\mathbf{j} - 0.36\mathbf{k})\\ &= -(2.88\text{ m/s})\mathbf{i} - (1.2\text{ m/s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{A'} &= \mathbf{\Omega} \times \mathbf{v}_{A'}\\ &= 8\mathbf{j} \times (-2.88\mathbf{i} - 1.2\mathbf{k})\\ &= -(9.6\text{ m/s}^2)\mathbf{i} + (23.04\text{ m/s}^2)\mathbf{k}\end{aligned}$$

Motion of Point A relative to the frame.

$$\begin{aligned}\mathbf{v}_{A/F} &= \boldsymbol{\omega}_2 \times \mathbf{r}_{A/D}\\ &= 12\mathbf{k} \times 0.18\mathbf{j}\\ &= -(2.16\text{ m/s})\mathbf{i}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{A/F} &= \boldsymbol{\omega}_2 \times \mathbf{v}_{A/F}\\ &= 12\mathbf{k} \times (-2.16\mathbf{i})\\ &= -(25.92\text{ m/s}^2)\mathbf{j}\end{aligned}$$

Velocity of Point A.

$$\mathbf{v}_A = \mathbf{v}_{A'} + \mathbf{v}_{A/F}$$

$$\mathbf{v}_A = -2.88\mathbf{i} - 1.2\mathbf{k} - 2.16\mathbf{i}$$

$$\mathbf{v}_A = -(5.04\text{ m/s})\mathbf{i} - (1.200\text{ m/s})\mathbf{k} \quad ◀$$

PROBLEM 15.242 (Continued)

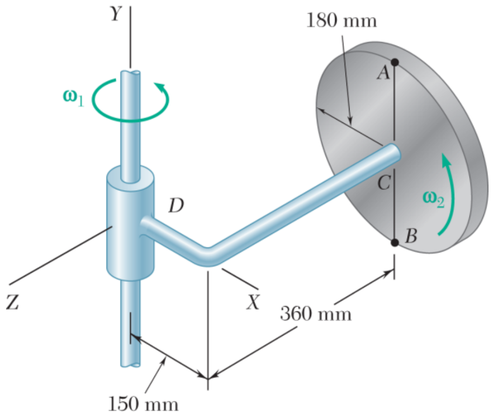
Coriolis acceleration. $2\mathbf{\Omega}\times\mathbf{v}_{A/F}$

$$2\mathbf{\Omega}\times\mathbf{v}_{A/F} = (2)(8\mathbf{j})\times(-2.16\mathbf{i})$$
$$= (34.56\text{ m/s}^2)\mathbf{k}$$

Acceleration of Point A.

$$\mathbf{a}_A = \mathbf{a}_{A'} + \mathbf{a}_{A/F} + 2\mathbf{\Omega}\times\mathbf{v}_{A/F}$$
$$\mathbf{a}_A = -9.6\mathbf{i} + 23.04\mathbf{k} - 25.92\mathbf{j} + 34.56\mathbf{k}$$

$$\mathbf{a}_A = -(9.60\text{ m/s}^2)\mathbf{i} - (25.9\text{ m/s}^2)\mathbf{j} + (57.6\text{ m/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.243

A disk of 180-mm radius rotates at the constant rate $\omega_2 = 12$ rad/s with respect to arm *CD*, which itself rotates at the constant rate $\omega_1 = 8$ rad/s about the *Y* axis. Determine at the instant shown the velocity and acceleration of Point *B* on the rim of the disk.

SOLUTION

Geometry.

$$\mathbf{r}_{B/D} = (0.15\text{ m})\mathbf{i} - (0.18\text{ m})\mathbf{j} - (0.36\text{ m})\mathbf{k}$$

$$\mathbf{r}_{B/C} = -(0.18\text{ m})\mathbf{j}$$

Let frame *Dxyz*, which coincides with the fixed frame *DXYZ* at the instant shown, be rotating about the *Y* axis with constant angular velocity $\boldsymbol{\Omega} = \omega_1\mathbf{j} = (8\text{ rad/s})\mathbf{j}$. Then the motion relative to the frame consists of a rotation of the disk *AB* about the bent axle *CD* with constant angular velocity $\boldsymbol{\omega}_2 = \omega_2\mathbf{k} = (12\text{ rad/s})\mathbf{k}$.

Motion of the coinciding Point B′ in the frame.

$$\begin{aligned}\mathbf{v}_{B'} &= \boldsymbol{\Omega} \times \mathbf{r}_{B/D}\\ &= 8\mathbf{j} \times (0.15\mathbf{i} - 0.18\mathbf{j} - 0.36\mathbf{k})\\ &= -(2.88\text{ m/s})\mathbf{i} - (1.2\text{ m/s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{B'} &= \boldsymbol{\Omega} \times \mathbf{v}_{B'}\\ &= 8\mathbf{j} \times (-2.88\mathbf{i} - 1.2\mathbf{k})\\ &= -(9.6\text{ m/s}^2)\mathbf{i} + (23.04\text{ m/s}^2)\mathbf{k}\end{aligned}$$

Motion of Point B relative to the frame.

$$\begin{aligned}\mathbf{v}_{B/F} &= \boldsymbol{\omega}_2 \times \mathbf{r}_{B/D}\\ &= 12\mathbf{k} \times (-0.18\mathbf{j})\\ &= (2.16\text{ m/s})\mathbf{i}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{B/F} &= \boldsymbol{\omega}_2 \times \mathbf{v}_{B/F}\\ &= 12\mathbf{k} \times 2.16\mathbf{i}\\ &= (25.92\text{ m/s}^2)\mathbf{j}\end{aligned}$$

Velocity of Point B.

$$\mathbf{v}_B = \mathbf{v}_{B'} + \mathbf{v}_{B/F}$$

$$\mathbf{v}_B = -2.88\mathbf{i} - 1.2\mathbf{k} + 2.16\mathbf{i}$$

$$\mathbf{v}_B = -(0.720\text{ m/s})\mathbf{i} - (1.200\text{ m/s})\mathbf{k} \quad \blacktriangleleft$$

PROBLEM 15.243 (Continued)

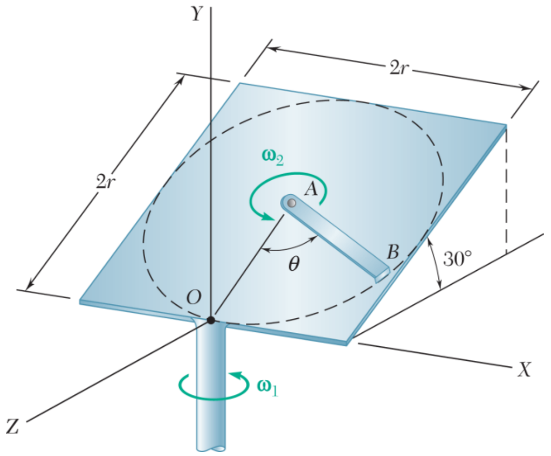
Coriolis acceleration. $2\boldsymbol{\Omega}\times\mathbf{v}_{B/F}$

$$2\boldsymbol{\Omega}\times\mathbf{v}_{B/F} = (2)(8\mathbf{j})\times(2.16\mathbf{i})$$
$$= -(34.56)\mathbf{k}$$

Acceleration of Point B. $\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/F} + 2\boldsymbol{\Omega}\times\mathbf{v}_{B/F}$

$$\mathbf{a}_B = -9.6\mathbf{i} + 23.04\mathbf{k} + 25.92\mathbf{j} - 34.56\mathbf{k}$$

$$\mathbf{a}_B = -(9.60\text{ m/s}^2)\mathbf{i} + (25.9\text{ m/s}^2)\mathbf{j} - (11.52\text{ m/s}^2)\mathbf{k}$$ ◀



PROBLEM 15.244

A square plate of side $2r$ is welded to a vertical shaft that rotates with a constant angular velocity $\boldsymbol{\omega}_1$. At the same time, rod AB of length r rotates about the center of the plate with a constant angular velocity $\boldsymbol{\omega}_2$ with respect to the plate. For the position of the plate shown, determine the acceleration of end B of the rod if (*a*) $\theta = 0$, (*b*) $\theta = 90°$, (*c*) $\theta = 180°$.

SOLUTION

Use a frame of reference moving with the plate.

Its angular velocity is $\boldsymbol{\Omega} = \omega_1 \mathbf{j} \qquad (\dot{\boldsymbol{\Omega}} = 0)$

Geometry:

$$\mathbf{r}_{A/O} = r(\sin 30° \mathbf{j} - \cos 30° \mathbf{k})$$

$$\mathbf{r}_{B/O} = \mathbf{r}_{A/O} + \mathbf{r}_{B/A}$$

Acceleration of coinciding Point B′ in the frame.

$$\mathbf{a}_{B'} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/O})$$

Motion relative to the frame. (Rotation about A with angular velocity $\boldsymbol{\omega}_2$).

$$\boldsymbol{\omega}_2 = \omega_2(\cos 30° \mathbf{j} + \sin 30° \mathbf{k}) \qquad (\dot{\omega}_2 = 0)$$

$$\mathbf{v}_{B/F} = \boldsymbol{\omega}_2 \times \mathbf{r}_{B/A}$$

$$\mathbf{a}_{B/F} = \boldsymbol{\omega}_2 \times \mathbf{v}_{B/F} = -\omega_2^2 \mathbf{r}_{B/A}$$

Coriolis acceleration: $\quad 2\boldsymbol{\Omega} \times \mathbf{v}_{B/F}$

Acceleration of B. $\quad \mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/F} + 2\boldsymbol{\Omega} \times \mathbf{v}_{B/F}$

(*a*) $\quad \theta = 0$

$$\mathbf{r}_{B/A} = r(-\sin 30° \mathbf{j} + \cos 30° \mathbf{k})$$

$$\mathbf{r}_{B/O} = 0$$

$$\mathbf{a}_{B'} = 0$$

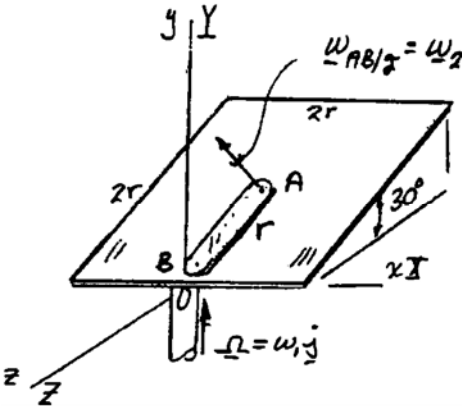
$$\mathbf{v}_{B/F} = r\omega_2 \mathbf{i}$$

$$\mathbf{a}_{B/F} = -\omega_2^2 \mathbf{r}_{B/A} = r\omega_2^2(\sin 30° \mathbf{j} - \cos 30° \mathbf{k})$$

$$2\boldsymbol{\Omega} \times \mathbf{v}_{B/F} = 2(\omega_1 \mathbf{j}) \times (r\omega_2 \mathbf{i}) = -2r\omega_1\omega_2 \mathbf{k}$$

$$\mathbf{a}_B = \mathbf{a}_{B'} + \mathbf{a}_{B/F} + \mathbf{a}_C$$

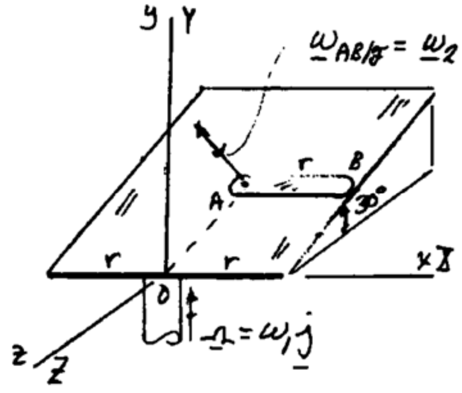
$$= 0 + r\omega_2^2(\sin 30° \mathbf{j} - \cos 30° \mathbf{k}) - 2r\omega_1\omega_2 \mathbf{k}$$



$$\mathbf{a}_B = r\omega_2^2 \sin 30° \mathbf{j} - (r\omega_2^2 \cos 30° + 2r\omega_1\omega_2)\mathbf{k} \blacktriangleleft$$

PROBLEM 15.244 (Continued)

(*b*) $\theta = 90°$

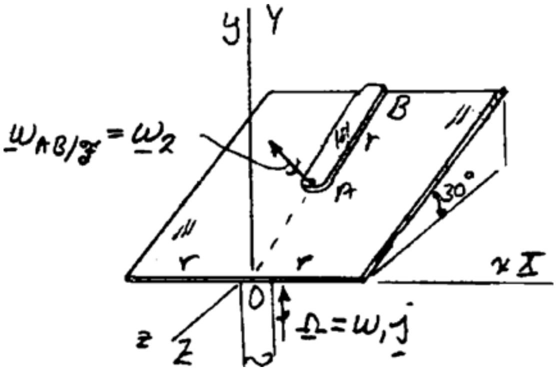

y
Y
$\underline{\omega}_{AB/F} = \underline{\omega}_2$
r
B
A
30°
r
r
O
x X
z Z
$\underline{\Omega} = \omega_1 \underline{j}$

$$\begin{aligned}
\mathbf{r}_{B/A} &= r\mathbf{i} \\
\mathbf{r}_{B/O} &= r\mathbf{i} + r\sin 30°\mathbf{j} - r\cos 30°\mathbf{k} \\
a_{B'} &= \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/O}) \\
&= \omega_1\mathbf{j} \times [\omega_1\mathbf{j} \times (r\mathbf{i} + r\sin 30°\mathbf{j} - r\cos 30°\mathbf{k})] \\
&= -r\omega_1^2\mathbf{i} + r\omega_1^2\cos 30°\mathbf{k} \\
\mathbf{v}_{B/F} &= r\omega_2(\sin 30°\mathbf{j} - \cos 30°\mathbf{k}) \\
\mathbf{a}_{B/F} &= -r\omega_2^2\mathbf{i} \\
2\mathbf{\Omega} \times \mathbf{v}_{B/F} &= 2(\omega_1\mathbf{j}) \times (r\omega_2)(\sin 30°\mathbf{j} - \cos 30°\mathbf{k}) \\
&= -2r\omega_1\omega_2\cos 30°\mathbf{i}
\end{aligned}$$

$$\begin{aligned}
\mathbf{a}_B &= \mathbf{a}_{B'} + \mathbf{a}_{B/F} + \mathbf{a}_C \\
&= [-r\omega_1^2\mathbf{i} + r\omega_1^2\cos 30°\mathbf{k}] - r\omega_2^2\mathbf{i} - 2r\omega_1\omega_2\cos 30°\mathbf{i}
\end{aligned}$$

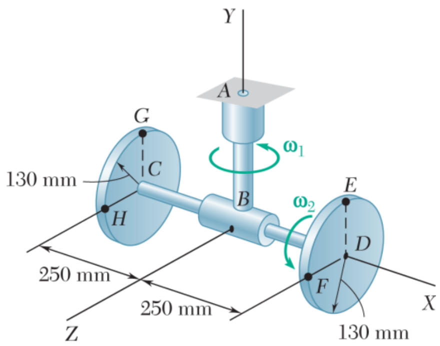
$$\mathbf{a}_B = -r(\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2\cos 30°)\mathbf{i} + r\omega_1^2\cos 30°\mathbf{k} \blacktriangleleft$$

(*c*) $\theta = 180°$


y
Y
$\underline{\omega}_{AB/F} = \underline{\omega}_2$
B
r
A
30°
r
r
O
x X
z Z
$\underline{\Omega} = \omega_1 \underline{j}$

$$\begin{aligned}
\mathbf{r}_{B/A} &= r(\sin 30°\mathbf{j} - \cos 30°\mathbf{k}) \\
\mathbf{r}_{B/O} &= 2r(\sin 30°\mathbf{j} - \cos 30°\mathbf{k}) \\
\mathbf{a}_{B'} &= +(2r\cos 30°)\omega_1^2\mathbf{k} = +2r\omega_1^2\cos 30°\mathbf{k} \\
\mathbf{v}_{B/F} &= -r\omega_2\mathbf{i} \\
\mathbf{a}_{B/F} &= r\omega_2^2(-\sin 30°\mathbf{j} + \cos 30°\mathbf{k}) \\
2\mathbf{\Omega} \times \mathbf{v}_{B/F} &= 2(\omega_1\mathbf{j}) \times (-r\omega_2\mathbf{i}) = 2r\omega_1\omega_2\mathbf{k} \\
\mathbf{a}_B &= \mathbf{a}_{B'} + \mathbf{a}_{B/F} + 2\mathbf{\Omega} \times \mathbf{v}_{B/F} \\
&= 2r\omega_1^2\cos 30°\mathbf{k} + r\omega_2^2(-\sin 30°\mathbf{j} + \cos 30°\mathbf{k}) + 2r\omega_1\omega_2\mathbf{k}
\end{aligned}$$

$$\mathbf{a}_B = -r\omega_2^2\sin 30°\mathbf{j} + r(2\omega_1^2\cos 30° + \omega_2^2\cos 30° + 2\omega_1\omega_2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.245

Two disks, each of 130-mm radius, are welded to the 500-mm rod *CD*. The rod-and-disks unit rotates at the constant rate $\omega_2 = 3$ rad/s with respect to arm *AB*. Knowing that at the instant shown $\omega_1 = 4$ rad/s, determine the velocity and acceleration of (*a*) Point *E*, (*b*) Point *F*.

SOLUTION

Let the frame of reference *BXYZ* be rotating about the *Y* axis with angular velocity $\mathbf{\Omega} = \omega_2 \mathbf{j} = (4 \text{ rad/s})\mathbf{j}$. The motion relative to this frame is a rotation about the *X* axis with angular velocity $\omega_x \mathbf{i} = (3 \text{ rad/s})\mathbf{i}$.

(*a*) *Point E.*

$$\mathbf{r}_{E/B} = (0.25 \text{ m})\mathbf{i} + (0.13 \text{ m})\mathbf{j}$$

$$\mathbf{r}_{E/D} = (0.13 \text{ m})\mathbf{j}$$

Motion of Point E′ in the frame.

$$\begin{aligned}\mathbf{v}_{E'} &= \mathbf{\Omega} \times \mathbf{r}_{E/B}\\ &= 4\mathbf{j} \times (0.25\mathbf{i} + 0.13\mathbf{j})\\ &= -(1 \text{ m/s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{E'} &= \mathbf{\Omega} \times \mathbf{v}_{E'}\\ &= 4\mathbf{j} \times (-\mathbf{k})\\ &= -(4 \text{ m/s}^2)\mathbf{i}\end{aligned}$$

Motion of Point E relative to the frame.

$$\begin{aligned}\mathbf{v}_{E/F} &= \omega_x \mathbf{i} \times \mathbf{r}_{E/D}\\ &= 3\mathbf{i} \times 0.13\mathbf{j}\\ &= (0.39 \text{ m/s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{E/F} &= \omega_x \mathbf{i} \times \mathbf{v}_{E/F}\\ &= 3\mathbf{i} \times (0.39\mathbf{k})\\ &= -(1.17 \text{ m/s}^2)\mathbf{j}\end{aligned}$$

Coriolis acceleration.

$$\mathbf{a}_c = 2\mathbf{\Omega} \times \mathbf{v}_{E/F}$$

$$\mathbf{a}_c = (2)(4\mathbf{j}) \times (0.39\mathbf{k}) = (3.12 \text{ m/s}^2)\mathbf{i}$$

Velocity of Point E.

$$\mathbf{v}_E = \mathbf{v}_{E'} + \mathbf{v}_{E/F}$$

$$\mathbf{v}_E = -\mathbf{k} + 0.39\mathbf{k} = 0.61\mathbf{k}$$

$$\mathbf{v}_E = (0.610 \text{ m/s})\mathbf{k}$$ ◀

Acceleration of Point E.

$$\mathbf{a}_E = \mathbf{a}_{E'} + \mathbf{a}_{E/F} + \mathbf{a}_c$$

$$\mathbf{a}_E = -4\mathbf{i} - 1.17\mathbf{j} + 3.12\mathbf{i}$$

$$\mathbf{a}_E = -(0.880 \text{ m/s}^2)\mathbf{i} + (1.170 \text{ m/s}^2)\mathbf{j}$$ ◀

PROBLEM 15.245 (Continued)

(*b*) *Point F.*

$$\mathbf{r}_{F/B} = (0.25\text{ m})\mathbf{i} + (0.13\text{ m})\mathbf{k}$$

$$\mathbf{r}_{F/D} = (0.13\text{ m})\mathbf{k}$$

Motion of Point F′ in the frame.

$$\begin{aligned}\mathbf{v}_{F'} &= \boldsymbol{\Omega} \times \mathbf{r}_{F/B}\\ &= 4\mathbf{j} \times (0.25\mathbf{i} + 0.13\mathbf{k})\\ &= (0.52\text{ m/s})\mathbf{i} - (1\text{ m/s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{F'} &= \boldsymbol{\Omega} \times \mathbf{v}_{F'}\\ &= (4\mathbf{j}) \times (0.52\mathbf{i} - \mathbf{k})\\ &= -(4\text{ m/s}^2)\mathbf{i} - (2.08\text{ m/s}^2)\mathbf{k}\end{aligned}$$

Motion of Point F relative to the frame.

$$\begin{aligned}\mathbf{v}_{F/F} &= \omega_x\mathbf{i} \times \mathbf{r}_{F/D}\\ &= 3\mathbf{i} \times (0.13\mathbf{k})\\ &= -(0.39\text{ m/s})\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{F/F} &= \omega_x\mathbf{i} \times \mathbf{v}_{F/F}\\ &= 3\mathbf{i} \times (-0.39\mathbf{j})\\ &= -(1.17\text{ m/s}^2)\mathbf{k}\end{aligned}$$

Coriolis acceleration.

$$\mathbf{a}_c = 2\boldsymbol{\Omega} \times \mathbf{v}_{F/F}$$

$$\mathbf{a}_c = (2)(4\mathbf{j}) \times (-0.39\mathbf{j}) = 0$$

Velocity of Point F.

$$\mathbf{v}_F = \mathbf{v}_{F'} + \mathbf{v}_{F/F}$$

$$\mathbf{v}_F = 0.52\mathbf{i} - \mathbf{k} - 0.39\mathbf{j}$$

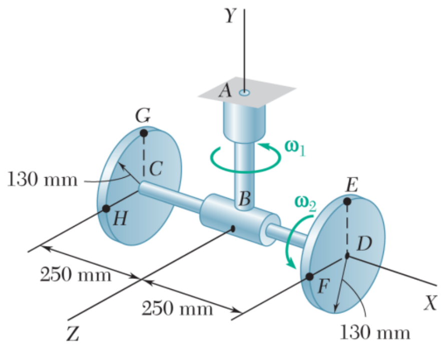
$$\mathbf{v}_F = (0.520\text{ m/s})\mathbf{i} - (0.390\text{ m/s})\mathbf{j} - (1.000\text{ m/s})\mathbf{k} \quad \blacktriangleleft$$

Acceleration of Point F.

$$\mathbf{a}_F = \mathbf{a}_{F'} + \mathbf{a}_{F/F} + \mathbf{a}_c$$

$$\mathbf{a}_F = -4\mathbf{i} - 2.08\mathbf{k} - 1.17\mathbf{k} + 0$$

$$\mathbf{a}_F = -(4.00\text{ m/s}^2)\mathbf{i} - (3.25\text{ m/s}^2)\mathbf{k} \quad \blacktriangleleft$$



PROBLEM 15.246

In Problem 15.245, determine the velocity and acceleration of (*a*) Point *G*, (*b*) Point *H*.

PROBLEM 15.245 Two disks, each of 130-mm radius, are welded to the 500-mm rod *CD*. The rod-and-disks unit rotates at the constant rate $\omega_2 = 3$ rad/s with respect to arm *AB*. Knowing that at the instant shown $\omega_1 = 4$ rad/s, determine the velocity and acceleration of (*a*) Point *E*, (*b*) Point *F*.

SOLUTION

Let the frame of reference *BXYZ* be rotating about the *Y* axis with angular velocity $\mathbf{\Omega} = \omega_2\mathbf{j} = (4 \text{ rad/s})\mathbf{j}$. The motion relative to this frame is a rotation about the *X* axis with angular velocity $\omega_x\mathbf{i} = (3 \text{ rad/s})\mathbf{i}$.

(*a*) *Point G.*

$$\mathbf{r}_{G/B} = -(0.25 \text{ m})\mathbf{i} + (0.13 \text{ m})\mathbf{j}$$

$$\mathbf{r}_{G/C} = (0.13 \text{ m})\mathbf{j}$$

Motion of Point G′ in the frame.

$$\begin{aligned}\mathbf{v}_{G'} &= \mathbf{\Omega} \times \mathbf{r}_{G/B}\\ &= 4\mathbf{j} \times (-0.25\mathbf{i} + 0.13\mathbf{j})\\ &= (1 \text{ m/s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{G'} &= \mathbf{\Omega} \times \mathbf{v}_{G'}\\ &= 4\mathbf{j} \times \mathbf{k}\\ &= (4 \text{ m/s}^2)\mathbf{i}\end{aligned}$$

Motion of Point G relative to the frame.

$$\begin{aligned}\mathbf{v}_{G/F} &= \omega_x\mathbf{i} \times \mathbf{r}_{G/C}\\ &= 3\mathbf{i} \times 0.13\mathbf{j}\\ &= (0.39 \text{ m/s})\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{G/F} &= \omega_x\mathbf{i} \times \mathbf{v}_{G/F}\\ &= 3\mathbf{i} \times (0.39\mathbf{k})\\ &= -(1.17 \text{ m/s}^2)\mathbf{j}\end{aligned}$$

Coriolis acceleration.

$$\mathbf{a}_c = 2\mathbf{\Omega} \times \mathbf{v}_{G/F}$$

$$\begin{aligned}\mathbf{a}_c &= (2)(4\mathbf{j}) \times (0.39\mathbf{k})\\ &= (3.12 \text{ m/s}^2)\mathbf{i}\end{aligned}$$

Velocity of Point G.

$$\mathbf{v}_G = \mathbf{v}_{G'} + \mathbf{v}_{G/F}$$

$$\mathbf{v}_G = \mathbf{k} + 0.39\mathbf{k}$$

$$\mathbf{v}_G = (1.390 \text{ m/s})\mathbf{k} \blacktriangleleft$$

PROBLEM 15.246 (Continued)

Acceleration of Point G. $\mathbf{a}_G = \mathbf{a}_{G'} + \mathbf{a}_{G/F} + \mathbf{a}_c$

$\mathbf{a}_G = 4\mathbf{i} - 1.17\mathbf{j} + 3.12\mathbf{i}$ $\qquad \mathbf{a}_G = (7.12 \text{ m/s}^2)\mathbf{i} - (1.170 \text{ m/s}^2)\mathbf{j}$ ◀

(*b*) *Point H.* $\mathbf{r}_{H/B} = -(0.25 \text{ m})\mathbf{i} + (0.13 \text{ m})\mathbf{k}$

$\mathbf{r}_{H/C} = (0.13 \text{ m})\mathbf{k}$

Motion of Point H′ in the frame.

$$\begin{aligned} \mathbf{v}_{H'} &= \mathbf{\Omega} \times \mathbf{r}_{H/B} \\ &= 4\mathbf{j} \times (-0.25\mathbf{i} + 0.13\mathbf{k}) \\ &= (0.52 \text{ m/s})\mathbf{i} + (1 \text{ m/s})\mathbf{k} \\ \mathbf{a}_{H'} &= \mathbf{\Omega} \times \mathbf{v}_{H'} \\ &= 4\mathbf{j} \times (0.52\mathbf{i} + \mathbf{k}) \\ &= (4 \text{ m/s}^2)\mathbf{i} - (2.08 \text{ m/s}^2)\mathbf{k} \end{aligned}$$

Motion of Point H relative to the frame.

$$\begin{aligned} \mathbf{v}_{H/F} &= \omega_x \mathbf{i} \times \mathbf{r}_{H/C} \\ &= 3\mathbf{i} \times (0.13\mathbf{k}) \\ &= -(0.39 \text{ m/s})\mathbf{j} \\ \mathbf{a}_{H/F} &= \omega_x \mathbf{i} \times \mathbf{v}_{H/F} \\ &= 3\mathbf{i} \times (-0.39\mathbf{j}) \\ &= -(1.17 \text{ m/s}^2)\mathbf{k} \end{aligned}$$

Coriolis acceleration. $\mathbf{a}_c = 2\mathbf{\Omega} \times \mathbf{v}_{H/F}$

$\mathbf{a}_c = (2)(4\mathbf{j}) \times (0.39\mathbf{j}) = 0$

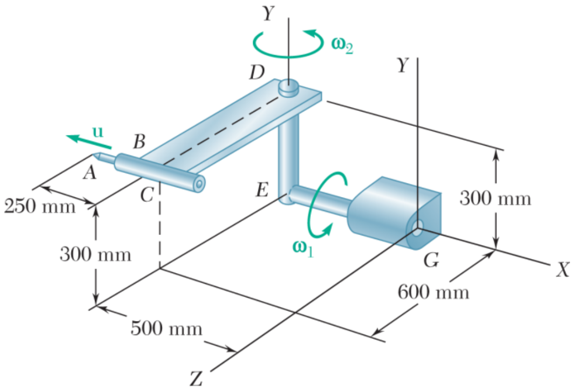
Velocity of Point H. $\mathbf{v}_H = \mathbf{v}_{H'} + \mathbf{v}_{H/F}$

$\mathbf{v}_H = 0.52\mathbf{i} + \mathbf{k} - 0.39\mathbf{j}$

$\mathbf{v}_H = (0.520 \text{ m/s})\mathbf{i} - (0.390 \text{ m/s})\mathbf{j} + (1.000 \text{ m/s})\mathbf{k}$ ◀

Acceleration of Point H. $\mathbf{a}_H = \mathbf{a}_{H'} + \mathbf{a}_{H/F} + \mathbf{a}_c$

$\mathbf{a}_H = 4\mathbf{i} - 2.08\mathbf{k} - 1.17\mathbf{k} + 0$ $\qquad \mathbf{a}_H = (4.00 \text{ m/s}^2)\mathbf{i} - (3.25 \text{ m/s}^2)\mathbf{k}$ ◀



PROBLEM 15.247

The position of the stylus tip A is controlled by the robot shown. In the position shown the stylus moves at a constant speed $u = 180$ mm/s relative to the solenoid BC. At the same time, arm CD rotates at the constant rate $\omega_2 = 1.6$ rad/s with respect to component DEG. Knowing that the entire robot rotates about the X axis at the constant rate $\omega_1 = 1.2$ rad/s, determine (*a*) the velocity of A, (*b*) the acceleration of A.

SOLUTION

Geometry:

$$\mathbf{r}_{D/G} = -(500\text{ mm})\mathbf{i} + (300\text{ mm})\mathbf{j} + (600\text{ mm})\mathbf{k}$$

$$\mathbf{r}_{A/D} = -(250\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{k}$$

$$\mathbf{r}_{A/G} = \mathbf{r}_{A/D} + \mathbf{r}_{D/G} = -(750\text{ mm})\mathbf{i} + (300\text{ mm})\mathbf{j} + (600\text{ mm})\mathbf{k}$$

Angular velocities:

$$\boldsymbol{\omega}_1 = \omega_1\mathbf{i} = (1.2\text{ rad/s})\mathbf{i} \qquad (\dot{\omega}_1 = 0)$$

$$\boldsymbol{\omega}_2 = \omega_2\mathbf{j} = (1.6\text{ rad/s})\mathbf{j} \qquad (\dot{\omega}_2 = 0)$$

Stylus motion:

$$\mathbf{u} = -u\mathbf{i} = -(180\text{ mm/s})\mathbf{i} \qquad (\dot{u} = 0)$$

Method 1

Let the rigid body BCD be a rotating frame of reference.

Its angular velocity is $\boldsymbol{\omega}_{CD} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = (1.2\text{ rad/s})\mathbf{i} + (1.6\text{ rad/s})\mathbf{j}$

Its angular acceleration is

$$\boldsymbol{\alpha}_{CD} = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_{CD} = (1.2\text{ rad/s})\mathbf{i} \times [(1.2\text{ rad/s})\mathbf{i} + (1.6\text{ rad/s})\mathbf{j}]$$
$$= (1.92\text{ rad/s}^2)\mathbf{k}$$

Motion of the coinciding Point A' in the frame.

$$\mathbf{v}_{A'} = \mathbf{v}_D + \mathbf{v}_{A'/D}$$

$$\begin{aligned}\mathbf{v}_{A'} &= \boldsymbol{\omega}_1 \times \mathbf{r}_{D/G} + (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \mathbf{r}_{A/D}\\ &= (1.2\text{ rad/s})\mathbf{i} \times [-(500\text{ mm})\mathbf{i} + (300\text{ mm})\mathbf{j}]\\ &\quad + [(1.2\text{ rad/s})\mathbf{i} + (1.6\text{ rad/s})\mathbf{j}] \times [-(250\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{k}]\\ &= (360\text{ mm/s})\mathbf{k} - (720\text{ mm/s})\mathbf{j} + (400\text{ mm/s})\mathbf{k} + (960\text{ mm/s})\mathbf{i}\end{aligned}$$

$$\mathbf{v}_{A'} = (960\text{ mm/s})\mathbf{i} - (720\text{ mm/s})\mathbf{j} + (760\text{ mm/s})\mathbf{k}$$

$$\begin{aligned}\mathbf{a}_D &= \omega_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_{D/G})\\ &= (1.2\text{ rad/s})\mathbf{i} \times \{(1.2\text{ rad/s})\mathbf{i} \times [-(500\text{ mm})\mathbf{i} + (300\text{ mm})\mathbf{j}]\}\\ &= -(432\text{ mm/s}^2)\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_{A'/D} &= \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{A/D} + \boldsymbol{\omega}_{CD} \times (\boldsymbol{\omega}_{CD} \times \mathbf{r}_{A/D})\\ &= (1.92\text{ rad/s}^2)\mathbf{k} \times [-(250\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{k}]\\ &\quad + \boldsymbol{\omega}_{CD} \times \{[(1.2\text{ rad/s})\mathbf{i} + (1.6\text{ rad/s})\mathbf{j}] \times [-(250\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{k}]\}\end{aligned}$$

$$\mathbf{a}_{A'/D} = -(480\text{ mm/s}^2)\mathbf{j} + \omega_{CD} \times [-(720\text{ mm/s})\mathbf{j} + (400\text{ mm/s})\mathbf{k} + (960\text{ mm/s})\mathbf{i}]$$

$$= -480\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.2 & 1.6 & 0 \\ 960 & -720 & 400 \end{vmatrix}$$

$$= -480\mathbf{j} + 640\mathbf{i} - 480\mathbf{j} - 864\mathbf{k} - 1536\mathbf{k}$$

$$\mathbf{a}_{A'/D} = (640\text{ mm/s}^2)\mathbf{i} - (960\text{ mm/s}^2)\mathbf{j} - (2400\text{ mm/s}^2)\mathbf{k}$$

$$\mathbf{a}_{A'} = \mathbf{a}_D + \mathbf{a}_{A'/D}$$

$$\mathbf{a}_{A'} = (640\text{ mm/s}^2)\mathbf{i} - (1392\text{ mm/s}^2)\mathbf{j} - (2400\text{ mm/s}^2)\mathbf{k}$$

Motion of Point A relative to the frame.

$$\mathbf{v}_{A/F} = \mathbf{u} = -(180\text{ mm/s})\mathbf{i}$$

$$\mathbf{a}_{A/F} = 0$$

(*a*) *Velocity of A.* $\mathbf{v}_A = \mathbf{v}_{A'} + \mathbf{v}_{A/F}$

$$\mathbf{v}_A = (960\text{ mm/s})\mathbf{i} - (720\text{ mm/s})\mathbf{j} + (760\text{ mm/s})\mathbf{k} - (180\text{ mm/s})\mathbf{i}$$

$$\mathbf{v}_A = (0.78\text{ m/s})\mathbf{i} - (0.72\text{ m/s})\mathbf{j} + (0.76\text{ m/s})\mathbf{k} \blacktriangleleft$$

Coriolis acceleration: $\mathbf{a}_c = 2\boldsymbol{\omega}_{CD} \times \mathbf{v}_{A/F}$

$$\mathbf{a}_c = 2[(1.2\text{ rad/s})\mathbf{i} + (1.6\text{ rad/s})\mathbf{j}] \times (-180\text{ mm/s})\mathbf{i}$$

$$= +(576\text{ mm/s}^2)\mathbf{k}$$

(*b*) *Acceleration of A.* $\mathbf{a}_A = \mathbf{a}_{A'} + \mathbf{a}_{A/F} + \mathbf{a}_c$

$$\mathbf{a}_A = (640\text{ mm/s}^2)\mathbf{i} - (1392\text{ mm/s}^2)\mathbf{j} - (2400\text{ mm/s}^2)\mathbf{k} + (576\text{ mm/s}^2)\mathbf{k}$$

$$\mathbf{a}_A = (0.64\text{ m/s}^2)\mathbf{i} - (1.392\text{ m/s}^2)\mathbf{j} - (1.824\text{ m/s}^2)\mathbf{k} \blacktriangleleft$$

Method 2

Use a frame of reference rotating about the x axis with angular velocity.

$$\boldsymbol{\omega}_1 = \omega_1\mathbf{i} = (1.2\text{ rad/s})\mathbf{i} \qquad (\dot{\omega}_1 = 0)$$

Motion of coinciding Point A' in the frame.

$$\mathbf{v}_{A'} = \boldsymbol{\omega}_1 \times \mathbf{r}_{A/G} = (1.2\text{ rad/s})\mathbf{i} \times [-(500\text{ mm})\mathbf{i} + (300\text{ mm})\mathbf{j} + (600\text{ mm})\mathbf{k}]$$

$$= (360\text{ mm/s})\mathbf{k} - (720\text{ mm/s})\mathbf{j}$$

$$\mathbf{a}_{A'} = \omega_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_{A/G}) = \boldsymbol{\omega}_1 \times \mathbf{v}_{A'}$$

$$= (1.2\text{ rad/s})\mathbf{i} \times [(360\text{ mm/s})\mathbf{k} - (720\text{ mm/s})\mathbf{j}]$$

$$= -(432\text{ mm/s}^2)\mathbf{j} - (864\text{ mm/s}^2)\mathbf{k}$$

Motion of Point A relative to the frame.

$$\begin{aligned}\mathbf{v}_{A/F} &= \boldsymbol{\omega}_2 \times \mathbf{r}_{A/D} + \mathbf{u}\\ &= (1.6\text{ rad/s})\mathbf{j} \times [-(250\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{k}] - (180\text{ mm/s})\mathbf{i}\\ &= (400\text{ mm/s})\mathbf{k} + (960\text{ mm/s})\mathbf{i} - (180\text{ mm/s})\mathbf{i}\end{aligned}$$

$\mathbf{a}_{A/F}$: (Since *A* moves on *CD*, which rotates at rate $\boldsymbol{\omega}_2$, we have a Coriolis term here).

$$\begin{aligned}\mathbf{a}_{A/F} &= \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{r}_{A/D}) + 2\boldsymbol{\omega}_2 \times \mathbf{u}\\ &= \boldsymbol{\omega}_2 \times \{(1.6\text{ rad/s})\mathbf{j} \times [-(250\text{ mm})\mathbf{i} + (600\text{ mm})\mathbf{k}]\} + 2\boldsymbol{\omega}_2 \times \mathbf{u}\\ &= (1.6\text{ rad/s})\mathbf{j} \times [(400\text{ mm/s})\mathbf{k} + (960\text{ mm/s})\mathbf{i}] + 2(1.6\text{ rad/s})\mathbf{j} \times (-180\text{ mm})\mathbf{i}\\ &= (640\text{ mm/s}^2)\mathbf{i} - (1536\text{ mm/s}^2)\mathbf{k} + (576\text{ mm/s}^2)\mathbf{k}\\ &= (640\text{ mm/s}^2)\mathbf{i} - (960\text{ mm/s}^2)\mathbf{k}\end{aligned}$$

(*a*) *Velocity of A.*

$$\mathbf{v}_A = \mathbf{v}_{A'} + \mathbf{v}_{A/F}$$

$$\mathbf{v}_A = 360\mathbf{k} - 720\mathbf{j} + 400\mathbf{k} + 960\mathbf{i} - 180\mathbf{i}$$

$$\mathbf{v}_A = (0.78\text{ m/s})\mathbf{i} - (0.72\text{ m/s})\mathbf{j} + (0.76\text{ m/s})\mathbf{k} \blacktriangleleft$$

Coriolis acceleration:

$$\mathbf{a}_c = 2\boldsymbol{\omega}_1 \times \mathbf{v}_{A/F}$$

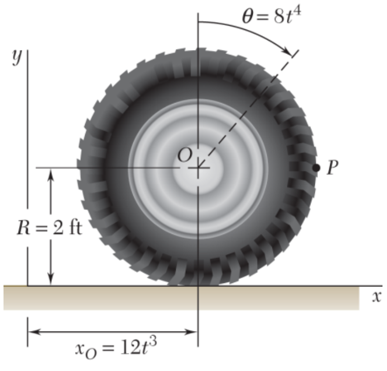
$$\begin{aligned}\mathbf{a}_c &= 2(1.2\text{ rad/s})\mathbf{i} \times [(400\text{ mm})\mathbf{k} + (780\text{ mm/s})\mathbf{i}]\\ &= -(960\text{ mm/s}^2)\mathbf{j}\end{aligned}$$

(*b*) *Acceleration of A.*

$$\mathbf{a}_A = \mathbf{a}_{A'} + \mathbf{a}_{A/F} + \mathbf{a}_c$$

$$\mathbf{a}_A = -432\mathbf{j} - 864\mathbf{k} + 640\mathbf{i} - 960\mathbf{k} - 960\mathbf{j}$$

$$\mathbf{a}_A = (0.64\text{ m/s}^2)\mathbf{i} - (1.392\text{ m/s}^2)\mathbf{j} - (1.824\text{ m/s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.248

A wheel moves in the xy plane in such a way that the location of its center is given by the equations $x_o = 12t^3$ and $y_o = R = 2$, where x_O and y_O are measured in feet and t is measured in seconds. The angular displacement of a radial line measured from a vertical reference line is $\theta = 8t^4$, where θ is measured in radians. Determine the velocity of the point P located on the horizontal diameter of the wheel at $t = 1$ s.

SOLUTION

Given: $x_O = 12t^3$ ft, $y_O = R = 2$ ft

$\theta = -8t^4$ rad

$\mathbf{r}_{\mathrm{P/O}} = 2\mathbf{i}$ ft

Angular Velocity of wheel:

$$\boldsymbol{\omega}_{OP} = \frac{d\theta}{dt}\mathbf{k}$$
$$= -32t^3\mathbf{k} \text{ rad/s}$$

Velocity of Point O:

$$\mathbf{v}_O = \frac{dx_O}{dt}\mathbf{i}$$
$$= 36t^2\mathbf{i} \text{ ft/s}$$

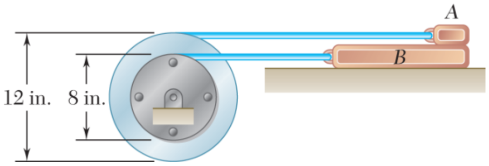
At t=1 s: $\boldsymbol{\omega}_{OP} = -32\mathbf{k}$ rad/s, $\mathbf{v}_O = 36\mathbf{i}$ ft/s

Velocity of Point P:

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega}_{OP} \times \mathbf{r}_{\mathrm{P/O}}$$

$$\mathbf{v}_P = 36\mathbf{i} - 32\mathbf{k} \times 2\mathbf{i} \text{ ft/s}$$
$$= 36\mathbf{i} - 64\mathbf{j} \text{ ft/s}$$

$\mathbf{v}_P = (36.0 \text{ ft/s})\mathbf{i} - (64.0 \text{ ft/s})\mathbf{j}$ ◀



PROBLEM 15.249

Two blocks and a pulley are connected by inextensible cords as shown. The relative velocity of block *A* with respect to block *B* is 2.5 ft/s to the left at time $t = 0$ and 1.25 ft/s to the left when $t = 0.25$ s. Knowing that the angular acceleration of the pulley is constant, find (*a*) the relative acceleration of block *A* with respect to block *B*, (*b*) the distance block *A* moves relative to block *B* during the interval $0 \le t \le 0.25$ s.

SOLUTION

(*a*) For the pulley,

$$r = \frac{1}{2}d, \qquad r_A = \frac{1}{2}(12) = 6 \text{ in.} = \frac{1}{2} \text{ ft}$$

$$r_B = \frac{1}{2}(8) = 8 \text{ in.} = \frac{1}{3} \text{ ft}$$

$$v_A = r_A\omega, \quad v_B = r_B\omega \qquad v_{A/B} = v_A - v_B = (r_A - r_B)\omega$$

$$\omega = \frac{v_{A/B}}{r_A - r_B}$$

At $t = 0$,

$$\omega_0 = \frac{2.5}{\frac{1}{2} - \frac{1}{3}} = 15 \text{ rad/s} \circlearrowleft$$

At $t = 0.25$ s,

$$\omega_1 = \frac{1.25}{\frac{1}{2} - \frac{1}{3}} = 7.5 \text{ rad/s} \circlearrowleft$$

$$\alpha = \frac{\omega_1 - \omega_2}{t} = \frac{7.5 - 15}{0.25} = -30 \text{ rad/s}^2 \circlearrowleft = 30 \text{ rad/s}^2 \circlearrowright$$

$$\mathbf{a}_A = r_A\alpha = \left(\frac{1}{2}\right)(30) = 15 \text{ ft/s}^2 \longrightarrow$$

$$\mathbf{a}_B = r_B\alpha = \left(\frac{1}{3}\right)(30) = 10 \text{ ft/s}^2 \longleftarrow$$

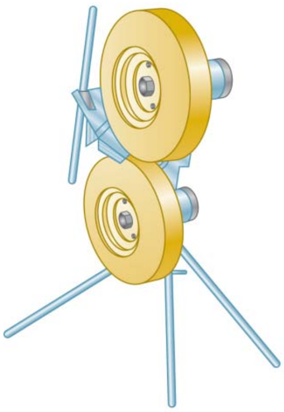
$$\mathbf{a}_{A/B} = a_A - a_B = 5 \text{ ft/s}^2 \longrightarrow \qquad a_{A/B} = 5.00 \text{ ft/s}^2 \longrightarrow \blacktriangleleft$$

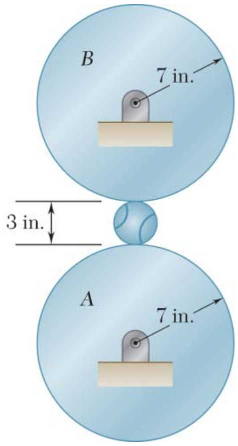
(*b*)

$$\mathbf{x}_{A/B} = (\mathbf{v}_{A/B})_0 t + \frac{1}{2}(\mathbf{a}_{A/B})t^2$$

$$= (-2.5)(0.25) + \frac{1}{2}(5)(0.25)^2 = -0.46875 \text{ ft} \longleftarrow$$

$$\mathbf{x}_{A/B} = 5.63 \text{ in.} \longleftarrow \blacktriangleleft$$




B
7 in.
3 in.
A
7 in.

PROBLEM 15.250

A baseball pitching machine is designed to deliver a baseball with a ball speed of 70 mph and a ball rotation of 300 rpm clockwise. Knowing that there is no slipping between the wheels and the baseball during the ball launch, determine the angular velocities of wheels *A* and *B*.

SOLUTION

Let Point *G* be the center of the ball, *A* its contact point with wheel *A*, and *B* its contact point with wheel *B*.

Given: $$v_G = \left(70\frac{\text{m}}{\text{h}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right)\left(\frac{5280\text{ ft}}{\text{mi}}\right) = 102.667\text{ ft/s} = 1232\text{ in./s}$$

$$\omega_{\text{ball}} = (300\text{ rpm})\left(\frac{2\pi\text{ rad}}{\text{rev}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 10\pi\text{ rad/s}$$

$$\mathbf{v}_G = 1232\text{ in./s} \longrightarrow \qquad \boldsymbol{\omega}_{\text{ball}} = 10\pi\text{ rad/s} \circlearrowright$$

Unit vectors: $\mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1\uparrow, \quad \mathbf{k} = 1 \circlearrowleft$

Relative positions: $$\mathbf{r}_{A/G} = \frac{1}{2}(-3\mathbf{j}) = -(1.5\text{ in.})\mathbf{j}$$

$$\mathbf{r}_{B/G} = \frac{1}{2}(3\mathbf{j}) = (1.5\text{ in.})\mathbf{j}$$

Velocities at *A* and *B*.
$$\begin{aligned}\mathbf{v}_A &= \mathbf{v}_G + \boldsymbol{\omega}_{\text{ball}} \times \mathbf{r}_{A/G} = 1232\mathbf{i} + (-10\pi\mathbf{k}) \times (1.5\mathbf{j}) \\ &= (1232 + 47.12)\mathbf{i} = 1279.12\text{ in./s} \longrightarrow\end{aligned}$$

$$\begin{aligned}\mathbf{v}_B &= \mathbf{v}_G + \boldsymbol{\omega}_{\text{ball}} \times \mathbf{r}_{B/G} = 1232\mathbf{i} + (-10\pi\mathbf{k}) \times (1.5\pi\mathbf{j}) \\ &= (1232 - 47.12)\mathbf{i} = 1184.88\text{ in./s} \longrightarrow\end{aligned}$$

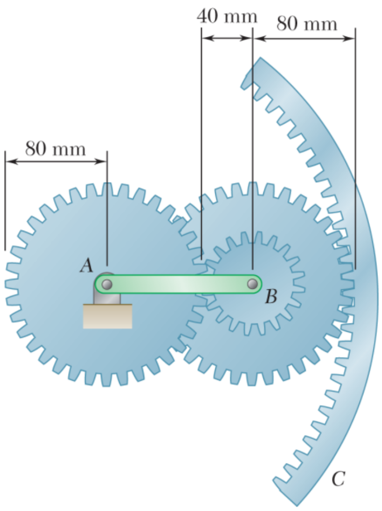
Angular velocity of A. $$\omega_A = \frac{v_A}{r_A} = \frac{1184.88}{7} = 169.27\text{ rad/s}$$

$$\boldsymbol{\omega}_A = 1616\text{ rpm} \circlearrowright \blacktriangleleft$$

Angular velocity of B.

$$\omega_B = \frac{v_B}{r_B} = \frac{1279.12}{7} = 182.73\text{ rad/s} \circlearrowleft$$

$$\boldsymbol{\omega}_B = 1745\text{ rpm} \circlearrowleft \blacktriangleleft$$



PROBLEM 15.251

Knowing that inner gear *A* is stationary and outer gear *C* starts from rest and has a constant angular acceleration of 4 rad/s^2 clockwise, determine at $t = 5$ s (*a*) the angular velocity of arm *AB*, (*b*) the angular velocity of gear *B*, (*c*) the acceleration of the point on gear *B* that is in contact with gear *A*.

SOLUTION

Angular velocity of gear *C* at $t = 5$ s.

$$\omega_c = \alpha_c t = (4\text{ rad/s}^2)(5\text{ s}) = 20\text{ rad/s} \qquad \boldsymbol{\omega}_c = 20\text{ rad/s} \circlearrowright$$

Let Point 1 be the contact point between gears *A* and *B*. Let Point 2 be the contact point between gears *B* and *C*. Points *A*, *B*, and *C* are the centers, respectively, of gears *A*, *B*, and *C*.

Positions: Take *x* axis along the straight line A1 B2.

$$x_A = x_C = 0$$
$$x_1 = r_A = 80\text{ mm}$$
$$x_B = (80\text{ mm} + 40\text{ mm}) = 120\text{ mm}$$
$$x_2 = 120\text{ mm} + 80\text{ mm} = 200\text{ mm}$$

Velocity at 1. Since gear *A* is stationary, $v_1 = 0$.

Velocity at 2. $v_2 = x_2\omega_C = (200)(20)$

$$\mathbf{v}_2 = 4000\text{ mm/s}\downarrow$$

Point 1 is the instantaneous center of gear *B*.

$$v_2 = (x_2 - x_1)\omega_B = (120\text{ mm})\omega_B$$
$$\omega_B = \frac{v_2}{120} = \frac{4000}{120} = 33.333\text{ rad/s}$$
$$v_B = (x_B - x_1)\omega_B = (40\text{ mm})(33.333\text{ rad/s}) = 1333.33\text{ mm/s}$$
$$\omega_{AB} = \frac{v_B}{x_B} = \frac{1333.33}{120} = 11.111\text{ rad/s}$$

(*a*) *Angular velocity of arm AB*: $\boldsymbol{\omega}_{AB} = 11.11\text{ rad/s} \circlearrowright$ ◀

(*b*) *Angular velocity of gear B*: $\boldsymbol{\omega}_B = 33.3\text{ rad/s} \circlearrowright$ ◀

PROBLEM 15.251 (Continued)

Calculate tangential accelerations:

$$(a_1)_t = 0$$

$$(a_2)_t = (a_C)_t = r_C\alpha_2 = (200 \text{ mm})(4 \text{ rad/s}^2)$$

$$(a_2)_t = 800 \text{ mm/s}^2 \downarrow$$

$$(a_2)_t = (x_2 - x_1)\alpha_B = (120 \text{ mm})\alpha_B$$

$$\alpha_B = \frac{(a_2)_t}{120} = \frac{800}{120} = 6.667 \text{ rad/s}^2$$

$$(a_B)_t = (x_B - x_1)\alpha_B = (40 \text{ mm})(6.667 \text{ rad/s}^2)$$

$$= 266.67 \text{ mm/s}^2$$

$$\alpha_{AB} = \frac{(a_B)_t}{r_B} = \frac{266.67}{120} = 2.2222 \text{ rad/s}^2$$

Angular accelerations: $\boldsymbol{\alpha}_{AB} = 2.2222 \text{ rad/s}^2 \circlearrowright$ $\boldsymbol{\alpha}_B = 6.667 \text{ rad/s} \circlearrowright$

Acceleration of Point *B*.

$$\mathbf{a}_B = [\alpha_{AB}\, x_B \downarrow] + [\omega_{AB}^2\, x_B \longleftarrow]$$

$$= [(2.2222)(120) \downarrow] + [(11.11)^2(120) \longleftarrow]$$

$$= (266.67 \text{ mm/s}^2) \downarrow + (14815 \text{ mm/s}^2) \longleftarrow$$

(*c*) *Acceleration of Point 1 on gear B.*

$$\mathbf{a}_1 = \mathbf{a}_B + [\alpha_B(x_B - x_1) \uparrow] + [\omega_B^2(x_B - x_1) \longrightarrow]$$

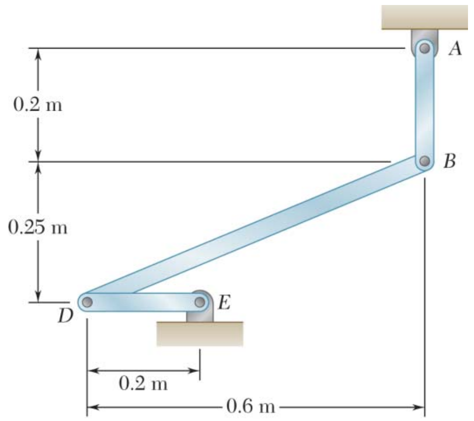
$$= [266.67 \downarrow] + [14815 \longleftarrow] + [(6.667)(40) \uparrow] + [(33.333)^2(40) \longrightarrow]$$

$$= [266.67 \downarrow] + [14815 \longleftarrow] + [266.67 \uparrow] + [44444 \longrightarrow]$$

$$= 29629 \text{ mm/s} \longrightarrow$$

$$\mathbf{a}_1 = 29.6 \text{ m/s}^2 \longrightarrow \blacktriangleleft$$

Note that the tangential component of acceleration is zero as expected.

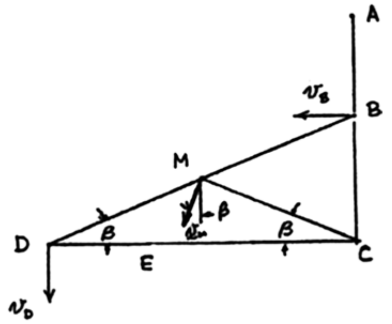


PROBLEM 15.252

Knowing that at the instant shown bar *AB* has an angular velocity of 10 rad/s clockwise and it is slowing down at a rate of 2 rad/s^2, determine the angular accelerations of bar *BD* and bar *DE*.

SOLUTION

Velocity Analysis $\quad \boldsymbol{\omega}_{AB} = 10 \text{ rad/s} ↻$



$$v_B = (AB)\omega_{AB}$$
$$= (0.200)(10)$$
$$= 2 \text{ m/s}$$

$$\mathbf{v}_B = v_B \longleftarrow$$

$$\mathbf{v}_D = v_D \downarrow$$

Locate the instantaneous center (Point *C*) of bar *BD* by noting that velocity directions at Points *B* and *D* are known. Draw *BC* perpendicular to $\mathbf{v}_B$ and *DC* perpendicular to $\mathbf{v}_D$.

$$\omega_{BD} = \frac{v_B}{BC} = \frac{2}{0.25} = 8 \text{ rad/s} \qquad \boldsymbol{\omega}_{BD} = 8.00 \text{ rad/s} ↺$$

$$v_D = (CE)\omega_{BD} = (0.6)(8) = 4.8 \text{ m/s} \qquad \mathbf{v}_D = 4.8 \text{ m/s} \downarrow$$

$$\omega_{DE} = \frac{v_D}{DE} = \frac{4.8}{0.2} = 24 \text{ rad/s} \qquad \boldsymbol{\omega}_{DE} = 24 \text{ rad/s} ↺$$

Acceleration Analysis: $\quad \alpha_{AB} = 2 \text{ rad/s}^2 ↺, \qquad \omega_{AB} = 10 \text{ rad/s} ↻$

Unit vectors: $\quad \mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1 \uparrow, \quad \mathbf{k} = 1 ↺$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$
$$= 0 + (2\mathbf{k}) \times (-0.2)\mathbf{j} - (10)^2(-0.2\mathbf{j})$$
$$= (0.4 \text{ m/s}^2)\mathbf{i} + (20 \text{ m/s}^2)\mathbf{j}$$

PROBLEM 15.252 (Continued)

$$\mathbf{a}_D = \mathbf{a}_B + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{D/B} - \omega_{BD}^2 \mathbf{r}_{D/B}$$

$$= 0.4\mathbf{i} + 20\mathbf{j} + \alpha_{BD}\mathbf{k} \times (-0.6\mathbf{i} - 0.25\mathbf{j}) - (8)^2(-0.6\mathbf{i} - 0.25\mathbf{j})$$

$$\mathbf{a}_D = (38.8 + 0.25\alpha_{BD})\mathbf{i} + (36 - 0.6\alpha_{BD})\mathbf{j} \qquad (1)$$

$$\mathbf{a}_D = \mathbf{a}_E + \boldsymbol{\alpha}_{DE} \times \mathbf{r}_{D/E} - \omega_{DE}^2 \mathbf{r}_{D/E}$$

$$= 0 + \alpha_{DE}\mathbf{k} \times (-0.2\mathbf{i}) - (24)^2(-0.2\mathbf{i})$$

$$= 115.2\mathbf{i} - 0.2\alpha_{DE}\mathbf{j} \qquad (2)$$

Equate like components of $\mathbf{a}_D$ from Equations (1) and (2).

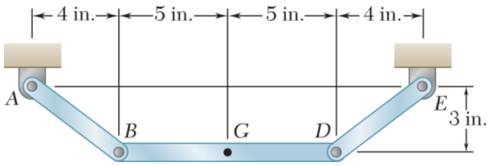
i: $38.8 + 0.25\alpha_{BD} = 115.2 \qquad \alpha_{BD} = 305.6 \text{ rad/s}^2$

j: $36 - 0.6\alpha_{BD} = -0.2\alpha_{DE}$

$$\alpha_{DE} = 3\alpha_{BD} - 180 = 736.8 \text{ rad/s}^2$$

Angular acceleration: $\boldsymbol{\alpha}_{BD} = 306 \text{ rad/s}^2 \circlearrowleft$ ◀

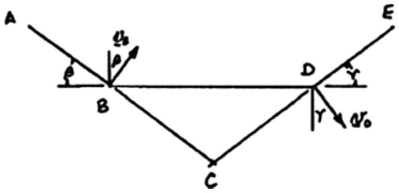
$\boldsymbol{\alpha}_{DE} = 737 \text{ rad/s}^2 \circlearrowleft$ ◀



PROBLEM 15.253

Knowing that at the instant shown rod AB has zero angular acceleration and an angular velocity of 15 rad/s counterclockwise, determine (*a*) the angular acceleration of arm DE, (*b*) the acceleration of Point D.

SOLUTION



$$\tan\beta = \frac{3}{4}, \quad \beta = 36.87°$$

$$AB = \frac{4}{\cos\beta} = 5 \text{ in.}$$

$$DE = \frac{4}{\cos\beta} = 5 \text{ in.}$$

$$v_B = (AB)\omega_{AB} = (5)(15)$$

$$= 75 \text{ in./s}$$

$$\mathbf{v}_B = v_B \nearrow \beta, \quad \mathbf{v}_D = v_D \searrow \beta$$

Point C is the instantaneous center of bar BD.

$$CB = \frac{5}{\cos\beta} = 6.25 \text{ in.} \quad \omega_{BD} = \frac{v_B}{CB} = \frac{75}{6.25} = 12 \text{ rad/s} \circlearrowright$$

$$CD = \frac{5}{\cos\beta} = 6.25 \text{ in.} \quad v_D = (CD)\omega_{BD} = (6.25)(12) = 75 \text{ in./s}$$

$$\omega_{DE} = \frac{v_D}{DE} = \frac{75}{5} = 15 \text{ rad/s} \circlearrowleft$$

Acceleration analysis. $\alpha_{AB} = 0$

$$\mathbf{a}_B = [(AB)\alpha_{AB} \nearrow \beta] + [(AB)\omega_{AB}^2 \nwarrow \beta]$$

$$= 0 + [(5)(15)^2 \nwarrow \beta] = 1125 \text{ in./s}^2 \nwarrow \beta$$

$$\mathbf{a}_{D/B} = [(BD)\alpha_{BD} \uparrow] + [(BD)\omega_{BD}^2 \longleftarrow]$$

$$= [10\alpha_{BD} \uparrow] + [(10)(12)^2 \longleftarrow]$$

$$= [10\alpha_{BD} \uparrow] + [1440 \text{ in./s}^2 \longleftarrow]$$

$$\mathbf{a}_D = [(DE)\alpha_{DE} \searrow \beta] + [(DE)\omega_{DE}^2 \nearrow \beta]$$

$$= [5\alpha_{DE} \searrow \beta] + [(5)(15)^2 \nearrow \beta]$$

$$= [5\alpha_{DE} \searrow \beta] + [1125 \text{ in./s}^2 \nearrow \beta]$$

$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B}$ Resolve into components.

PROBLEM 15.253 (Continued)

(*a*) $\xrightarrow{+}$: $5\alpha_{DE}\sin\beta + 1125\cos\beta = -1125\cos\beta - 1440$

$$\alpha_{DE} = -1080 \text{ rad/s}^2$$

$\boldsymbol{\alpha}_{DE} = 1080 \text{ rad/s}^2 \circlearrowright$ ◀

(*b*)

$$\mathbf{a}_D = [(5)(-1080) \searrow \beta] + [1125 \text{ in./s}^2 \measuredangle \beta]$$

$$= [5400 \text{ in./s}^2 \nwarrow \beta] + [1125 \text{ in./s}^2 \measuredangle \beta]$$

$$\tan\gamma = \frac{1125}{5400}$$

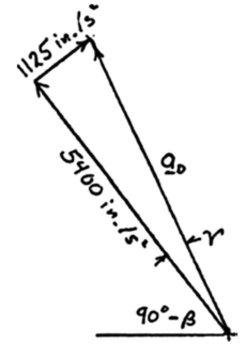
$$\gamma = 11.77°$$

$$a_D = \sqrt{5400^2 + 1125^2}$$

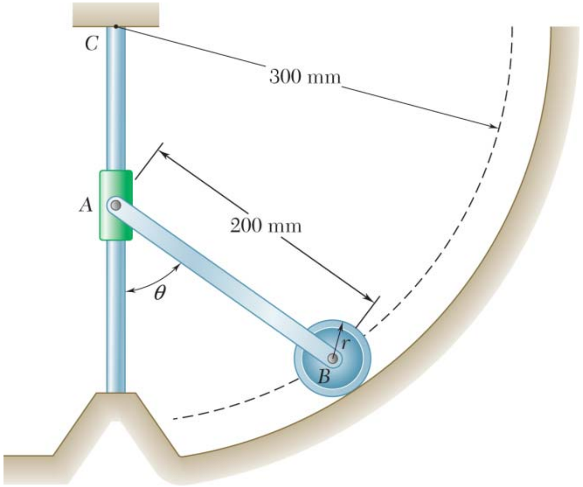
$$= 5516 \text{ in./s}^2$$

$$= 460 \text{ ft/s}^2$$

$$90° - \beta + \gamma = 64.9°$$



$\mathbf{a}_D = 460 \text{ ft/s}^2 \searrow 64.9°$ ◀

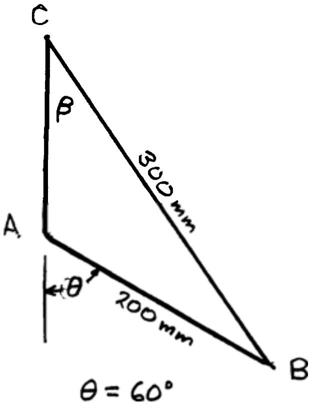


PROBLEM 15.254

Rod AB is attached to a collar at A and is fitted with a wheel at B that has a radius r = 15 mm. Knowing that when $\theta = 60°$ the collar has velocity of 250 mm/s upward and the speed of the collar is decreasing at a rate of 150 mm/s^2, determine (*a*) the angular acceleration of rod AB, (*b*) the angular acceleration of the wheel.

SOLUTION

Geometry. We note that Point B moves on a circle of radius R = 300 mm centered at C. It is useful to use the angle β, angle ACB of the triangle ABC, which indicates the motion of B along this curve. Applying the law of sines to the triangle ABC,



$$\frac{\overrightarrow{AB}}{\sin(\pi-\theta)} = \frac{\overrightarrow{BC}}{\sin\beta} \qquad (\theta = 60°)$$

or

$$\sin\beta = \frac{\overrightarrow{AB}}{\overrightarrow{BC}}\sin\theta = \frac{200\text{ mm}}{300\text{ mm}}\sin 60°$$

$$\beta = 35.264°$$

With meters as the length unit and the unit vectors defined as

$$\mathbf{i} = 1 \longrightarrow, \quad \mathbf{j} = 1 \uparrow, \quad \text{and } \mathbf{k} = 1 \circlearrowleft$$

The relative position vectors are

$$\mathbf{r}_{B/A} = (0.2\text{ m})(\sin 60°\mathbf{i} - \cos 60°\mathbf{j})$$
$$\mathbf{r}_{B/C} = (0.3\text{ m})(\sin\beta\mathbf{i} - \cos\beta\mathbf{j})$$

Let Point P be the contact point where the wheel rolls on the fixed surface.

$$\mathbf{r}_{P/B} = (0.015\text{ m})(\sin\beta\mathbf{i} - \cos\beta\mathbf{j})$$

Velocity analysis.

$$\mathbf{v}_A = 0.250\text{ m/s}\uparrow \qquad \mathbf{v}_P = 0$$
$$\mathbf{v}_B = v_B \measuredangle \beta \qquad \boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k} \qquad \boldsymbol{\omega}_{BP} = \omega_{BP}\mathbf{k}$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$
$$= 0.250\mathbf{j} + (\omega_{AB})\mathbf{k} \times (0.2\sin 60°\mathbf{i} - 0.2\cos 60°\mathbf{j})$$
$$v_B(\cos\beta\mathbf{i} + \sin\beta\mathbf{j}) = 0.25\mathbf{j} + 0.1\omega_{AB}\mathbf{i} + 0.173205\omega_{AB}\mathbf{j}$$

Resolve into components:

i: $v_B \cos\beta = +0.1\omega_{AB}$ (1)

j: $v_B \sin\beta = 0.25 - 0.173205\omega_{AB}$ (2)

Solving the simultaneous Equations (1) and (2),

$$v_B = -0.29873 \text{ m/s} \qquad \omega_{AB} = -2.43913$$

$$\mathbf{v}_B = 0.29873 \text{ m/s} \swarrow \beta \qquad \boldsymbol{\omega}_{AB} = 2.43913 \text{ rad/s} \circlearrowright$$

Since the wheel rolls without slipping, Point P is its instantaneous center.

$$|v_B| = r\omega_{BP} \qquad \omega_{PB} = \frac{0.29873}{0.15} \qquad \boldsymbol{\omega}_{BP} = 1.992 \text{ rad/s} \circlearrowleft$$

Acceleration analysis $\mathbf{a}_A = 150 \text{ mm/s}^2 \downarrow$

$(\mathbf{a}_P)_t = 0$

$$\begin{aligned}
\mathbf{a}_B &= \mathbf{a}_A + \mathbf{a}_{B/A} = \mathbf{a}_A + \boldsymbol{\alpha}_{AB}\mathbf{k} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A} \\
&= -0.150\mathbf{j} + \alpha_{AB}\mathbf{k} \times (0.2\sin 60^\circ \mathbf{i} - 0.2\cos 60^\circ \mathbf{j}) \\
&\quad - (2.43913)^2 (0.2\sin 60^\circ \mathbf{i} - 0.2\cos 60^\circ \mathbf{j}) \\
&= -0.150\mathbf{j} + (0.2\sin 60^\circ)\alpha_{AB}\mathbf{j} + (0.2\cos 60^\circ)\alpha_{AB}\mathbf{i} \\
&\quad - 1.03046\mathbf{i} + 0.59494\mathbf{j} \\
&= (0.2\cos 60^\circ)\alpha_{AB}\mathbf{i} + (0.2\sin 60^\circ)\alpha_{AB}\mathbf{j} - 1.03046\mathbf{i} + 0.44494\mathbf{j}
\end{aligned}$$

Consider the motion of B along its circular path.

$$\begin{aligned}
\mathbf{a}_B &= [(a_B)_t \nearrow \beta] + \left[\frac{v_B^2}{R} \nwarrow \beta\right] \\
&= (a_B)_t(\cos\beta\mathbf{i} + \sin\beta\mathbf{j}) + \frac{v_B^2}{R}(-\sin\beta\mathbf{i} + \cos\beta\mathbf{j}) \\
&= (a_B)_t \cos\beta\mathbf{i} + (a_B)_t \sin\beta\mathbf{j} - \frac{(0.29873)^2}{0.3}(-\sin\beta\mathbf{i} + \cos\beta\mathbf{j}) \\
&= (a_B)_t \cos\beta\mathbf{i} + (a_B)_t \sin\beta\mathbf{j} - 0.17174\mathbf{i} + 0.24288\mathbf{j}
\end{aligned}$$

Equate the two expression for $\mathbf{a}_B$ and resolve into components.

i: $(a_B)_t \cos\beta - 0.17174 = (0.2\cos 60^\circ)\alpha_{AB} - 1.03046$ (3)

j: $(a_B)_t \sin\beta + 0.24288 = 0.2\sin 60^\circ \alpha_{AB} + 0.44494$ (4)

Solving Eqs. (3) and (4) simultaneously,

$$(a_B)_t = -2.0187 \text{ m/s}^2 \qquad \alpha_{AB} = -7.8956 \text{ rad/s}^2$$

PROBLEM 15.254 (Continued)

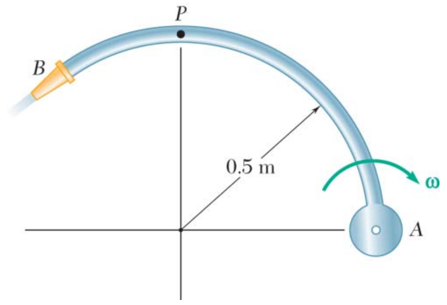
(*a*) *Angular acceleration of AB.* $\alpha_{AB} = 7.90 \text{ rad/s}^2 \circlearrowright$ ◀

(*b*) *Angular acceleration of the wheel.*

$$(a_P)_t = (a_B)_t + r\alpha_{BP} = 0$$

$$\alpha_{BP} = \frac{(a_B)_t}{r} = -\frac{(-2.0187)}{0.015} = 134.6 \text{ rad/s}^2$$

$\boldsymbol{\alpha}_{BP} = 134.6 \text{ rad/s}^2 \circlearrowleft$ ◀



PROBLEM 15.255

Water flows through a curved pipe *AB* that rotates with a constant clockwise angular velocity of 90 rpm. If the velocity of the water relative to the pipe is 8 m/s, determine the total acceleration of a particle of water at Point *P*.

SOLUTION

Let the curved pipe be a rotating frame of reference. Its angular velocity is $90 \text{ rpm} \circlearrowright$ or $\boldsymbol{\omega} = 9.4248 \text{ rad/s} \circlearrowright$.

Motion of the frame of reference at Point P′.

$$\mathbf{v}_{P'} = (AP)\omega \nearrow 45^\circ = (0.5\sqrt{2})(9.4248) \nearrow 45^\circ = 6.6643 \text{ m/s} \nearrow 45^\circ$$

$$\mathbf{a}_{P'} = (AP)\omega^2 \searrow 45^\circ = (0.5\sqrt{2})(9.4248)^2 \searrow 45^\circ = 62.81 \text{ m/s}^2 \searrow 45^\circ$$

Motion of water relative to the frame at Point P.

$$\mathbf{v}_{P/F} = 8 \text{ m/s} \leftarrow$$

$$\begin{aligned}\mathbf{a}_{P/F} &= \frac{(v_{P/F})^2}{\rho} \downarrow \\ &= \frac{(8 \text{ m/s})^2}{0.5 \text{ m}} \downarrow \\ &= 128 \text{ m/s}^2 \downarrow\end{aligned}$$

Coriolis acceleration.

$$\begin{aligned}a_c &= 2\omega v_{P/F} \\ &= (2)(9.4248)(8) \\ &= 150.797 \text{ m/s}^2\end{aligned}$$

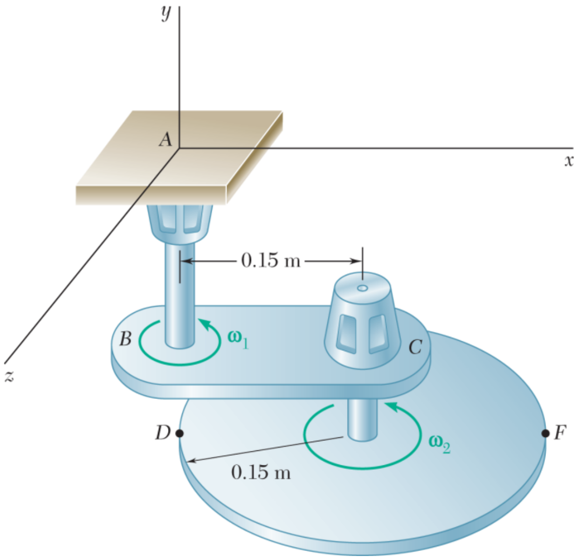
$$\mathbf{a}_c = 150.797 \text{ m/s}^2 \uparrow$$

Acceleration of water at Point P.

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/F} + \mathbf{a}_c$$

$$\begin{aligned}\mathbf{a}_P &= [62.81 \searrow 45^\circ] + [128 \downarrow] + [150.797 \uparrow] \\ &= [44.413 \text{ m/s}^2 \rightarrow] + [21.616 \text{ m/s}^2 \downarrow]\end{aligned}$$

$$\mathbf{a}_P = 49.4 \text{ m/s}^2 \searrow 26.0^\circ \blacktriangleleft$$



PROBLEM 15.256

A disk of 0.15-m radius rotates at the constant rate ω_2 with respect to plate BC, which itself rotates at the constant rate ω_1 about the y axis. Knowing that $\omega_1 = \omega_2 = 3$ rad/s, determine, for the position shown, the velocity and acceleration (*a*) of Point *D*, (*b*) of Point *F*.

SOLUTION

Frame *AXYZ* is fixed.

Moving frame, *Exyz*, rotates about $\overline{y}$ axis at

$$\mathbf{\Omega} = \omega_1 \mathbf{j} = (3 \text{ rad/s})\mathbf{j}$$

(*a*) *Point D*:

$$\boldsymbol{\omega}_2 = \omega_2 \mathbf{j} = (3 \text{ rad/s})\mathbf{j}$$

$$\mathbf{r}_{D/A} = 0$$

$$\mathbf{r}_{D/E} = -(0.15 \text{ m})\mathbf{i}$$

$$\mathbf{v}_{D'} = \mathbf{\Omega} \times \mathbf{r}_{D/A} = 0$$

$$\begin{aligned} \mathbf{v}_{D/F} &= \boldsymbol{\omega}_A \times \mathbf{r}_{D/E} \\ &= (3 \text{ rad/s})\mathbf{j} \times (-0.15 \text{ m})\mathbf{i} \\ &= (0.45 \text{ m/s})\mathbf{k} \end{aligned}$$

$$\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/F}$$

$\mathbf{v}_D = (0.45 \text{ m/s})\mathbf{k}$ ◀

$$\mathbf{a}_{D'} = \mathbf{\Omega} \times \mathbf{v}_{D'} = 0$$

$$\begin{aligned} \mathbf{a}_{D/F} &= \boldsymbol{\omega}_1 \times \mathbf{v}_{D/F} \\ &= (3 \text{ rad/s})\mathbf{j} \times (0.45 \text{ m/s})\mathbf{k} \\ &= (1.35 \text{ m/s}^2)\mathbf{i} \end{aligned}$$

$$\begin{aligned} \mathbf{a}_c &= 2\mathbf{\Omega} \times \mathbf{v}_{D/F} \\ &= 2(3 \text{ rad/s})\mathbf{j} \times (0.45 \text{ m/s})\mathbf{k} = (2.70 \text{ m/s}^2)\mathbf{i} \end{aligned}$$

$$\begin{aligned} \mathbf{a}_D &= \mathbf{a}_{D'} + \mathbf{a}_{D/F} + \mathbf{a}_c \\ &= 0 + (1.35 \text{ m/s}^2)\mathbf{i} + (2.70 \text{ m/s}^2)\mathbf{i} \end{aligned}$$

$\mathbf{a}_D = (4.05 \text{ m/s}^2)\mathbf{i}$ ◀

PROBLEM 15.256 (Continued)

(*b*) *Point F*:

$$\boldsymbol{\omega}_2 = \omega_2\mathbf{j} = (3\text{ rad/s})\mathbf{j}$$

$$\mathbf{r}_{F/A} = (0.3\text{ m})\mathbf{i};$$

$$\mathbf{r}_{F/E} = (0.15\text{ m})\mathbf{i}$$

$$\mathbf{v}_{F'} = \boldsymbol{\Omega}\times\mathbf{r}_{F/A}$$

$$= (3\text{ rad/s})\mathbf{j}\times(0.3\text{ m})\mathbf{i}$$

$$= -(0.9\text{ m/s})\mathbf{k}$$

$$\mathbf{v}_{F/F} = \boldsymbol{\omega}_2\times\mathbf{r}_{F/E}$$

$$= (3\text{ rad/s})\mathbf{j}\times(0.15\text{ m})\mathbf{i}$$

$$= -(0.45\text{ m/s})\mathbf{k}$$

$$\mathbf{v}_F = \mathbf{v}_{F'} + \mathbf{v}_{F/F}$$

$$= -(0.9\text{ m/s})\mathbf{k} - (0.45\text{ m/s})\mathbf{k}$$

$\mathbf{v}_F = -(1.35\text{ m/s})\mathbf{k}$ ◀

$$\mathbf{a}_{F'} = \boldsymbol{\Omega}\times\mathbf{v}_{F'}$$

$$= (3\text{ rad/s})\mathbf{j}\times(-0.9\text{ m/s})\mathbf{k}$$

$$= -(2.7\text{ m/s}^2)\mathbf{i}$$

$$\mathbf{a}_{F/F} = \boldsymbol{\omega}_2\times\mathbf{v}_{F/F}$$

$$= (3\text{ rad/s})\mathbf{j}\times(-0.45\text{ m/s})\mathbf{k}$$

$$= -(1.35\text{ m/s}^2)\mathbf{i}$$

$$\mathbf{a}_c = 2\boldsymbol{\Omega}\times\mathbf{v}_{F/F}$$

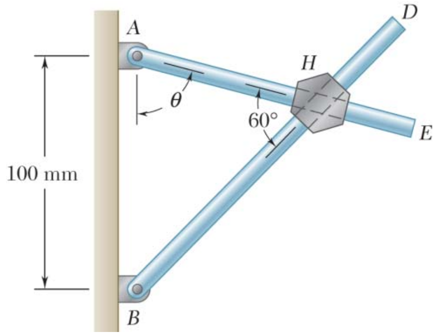
$$= 2(3\text{ rad/s})\mathbf{j}\times(-0.45\text{ m/s})\mathbf{k}$$

$$= -(2.7\text{ m/s}^2)\mathbf{i}$$

$$\mathbf{a}_F = \mathbf{a}_{F'} + \mathbf{a}_{F/F} + \mathbf{a}_c$$

$$= -(2.7\text{ m/s}^2)\mathbf{i} - (1.35\text{ m/s}^2)\mathbf{i} - (2.7\text{ m/s}^2)\mathbf{i}$$

$\mathbf{a}_F = -(6.75\text{ m/s}^2)\mathbf{i}$ ◀



PROBLEM 15.257

Two rods *AE* and *BD* pass through holes drilled into a hexagonal block. (The holes are drilled in different planes so that the rods will not touch each other.) Knowing that rod *AE* has an angular velocity of 20 rad/s clockwise and an angular acceleration of 4 rad/s^2 counterclockwise when $\theta = 90°$, determine, (*a*) the relative velocity of the block with respect to each rod, (*b*) the relative acceleration of the block with respect to each rod.

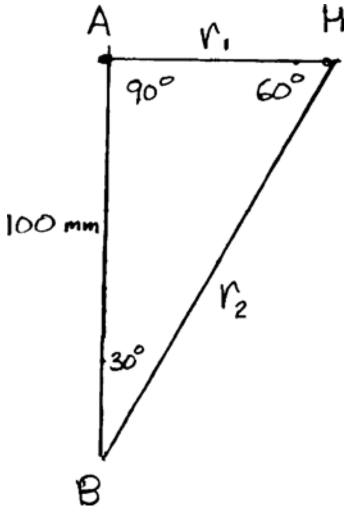
SOLUTION

Geometry: When $\theta = 90°$, Point *H* is located as shown in the sketch. Apply the law of sines to the triangle *ABH*.

$$\frac{100\text{ mm}}{\sin 60°} = \frac{r_1}{\sin 30°} = \frac{r_2}{\sin 90°}$$

$$r_1 = 57.735\text{ mm}$$

$$r_2 = 115.470\text{ mm}$$



The angle at *H* remains at 60° so that rods *AE* and *BD* have a common angular velocity $\boldsymbol{\omega} = \omega \circlearrowleft$ and a common angular acceleration $\boldsymbol{\alpha} = \alpha \circlearrowleft$, where

$$\omega = -20\text{ rad/s}$$

and

$$\alpha = 4\text{ rad/s}$$

Consider the double slider *H* as a particle sliding along the rotating rod *AH* with relative velocity $u_1 \longrightarrow$ and relative acceleration $\dot{u}_1 \longrightarrow$.

Let H' be the point on rod *AE* that coincides with *H*.

$$\mathbf{v}_{H'} = [r_1\omega \uparrow] = [(57.735\text{ mm})(-20\text{ rad/s}) \uparrow] = [1154.7\text{ mm/s} \downarrow]$$

$$\mathbf{a}_{H'} = [r_1\alpha \uparrow] + [r_1\omega^2 \longleftarrow]$$

$$= [(57.735\text{ mm})(4\text{ rad/s}^2) \uparrow] + [(57.735\text{ mm})(20\text{ rad/s})^2 \longleftarrow]$$

$$= [230.9\text{ mm/s}^2 \uparrow] + [23094\text{ mm/s}^2 \longleftarrow]$$

PROBLEM 15.257 (Continued)

The corresponding Coriolis acceleration is

$$\mathbf{a}_1 = [2\omega u_1 \uparrow] = [(2)(-20)u_1 \uparrow] = 40u_1 \downarrow$$

$$\mathbf{v}_H = \mathbf{v}_{H'} + u_1 \longrightarrow = [1154.7 \text{ mm} \downarrow] + [u_1 \longrightarrow] \qquad (1)$$

$$\mathbf{a}_H = a_{H'} + [\dot{u}_1 \longrightarrow] + [40u_1 \downarrow]$$

$$= [230.9 \text{ mm/s} \uparrow] + [23094 \text{ mm/s}^2 \longleftarrow] + [\dot{u}_1 \longrightarrow] + [40u_1 \downarrow] \qquad (2)$$

Now consider the double slider H as a particle sliding along the rotating rod BD with relative velocity $u_2 \nearrow 60°$ and relative acceleration $\dot{u}_2 \nearrow 60°$.

Let H'' be the point on rod BD that coincides with H.

$$\mathbf{v}_{H''} = r_2\omega \nwarrow 30° = (115.47 \text{ mm})(-20 \text{ rad/s}) \nwarrow 30° = 2309.4 \text{ mm/s} \searrow 30°$$

$$\mathbf{a}_{H''} = r_2\alpha \nwarrow 30° + r_2\omega^2 \swarrow 60°$$

$$= (115.47 \text{ mm})(4 \text{ rad/s}^2) \nwarrow 30° + (115.47 \text{ mm})(20 \text{ rad/s})^2 \swarrow 60°$$

$$= 461.9 \text{ mm/s}^2 \nwarrow 30° + 46188 \text{ mm/s}^2 \swarrow 60°$$

The corresponding Coriolis acceleration is

$$\mathbf{a}_2 = 2\omega u_2 \nwarrow 30° = (2)(-20 \text{ rad/s})u_2 \nwarrow 60° = 40u_2 \searrow 30°$$

$$\mathbf{v}_H = \mathbf{v}_{H''} + u_2 \nearrow 60° = 2309.4 \text{ mm/s} \searrow 30° + u_2 \nearrow 60° \qquad (3)$$

$$\mathbf{a}_H = \mathbf{a}_{H''} + \dot{u}_2 \nearrow 60° + 40u_2 \searrow 30°$$

$$= [(461.9 \text{ mm/s}^2) \nwarrow 30°] + [(46188 \text{ mm/s}^2) \swarrow 60°] + [\dot{u}_2 \nearrow 60°] + [40u_2 \searrow 30°] \qquad (4)$$

Equate expression (1) and (3) for $\mathbf{v}_H$ and resolve into components.

$+\uparrow$ $\quad 1154.7 \text{ mm/s} = -(2309.4 \text{ mm/s})\sin 30° + u_2 \sin 60°$

$$u_2 = \frac{2309.4\sin 30° - 1154.7}{\sin 60°} \qquad u_2 = 0$$

$\xrightarrow{+}$: $\quad u_1 = 2309.4\cos 30° + 0 \qquad u_1 = 2000 \text{ mm/s}$

Substitute the values for u_1 and u_2 into the Coriolis acceleration terms, and equate expressions (2) and (4) for $\mathbf{a}_H$, and resolve into components.

$+\uparrow$: $\quad 230.9 \text{ mm/s}^2 - (40 \text{ rad/s})(2000 \text{ mm/s})$

$$= (461.9 \text{ mm/s}^2)\sin 30° + (46188 \text{ mm/s}^2)\sin 60°$$

$$+ \dot{u}_2 \sin 60° - (40)(0)\sin 30°$$

$$\dot{u}_2 = \frac{230.9 - 80000 - 230.9 + 40000}{\sin 60°} \qquad \dot{u}_2 = -46188 \text{ mm/s}^2$$

PROBLEM 15.257 (Continued)

$$\xrightarrow{+}: \qquad \dot{u}_1 - 23094\ \text{mm/s}^2 = -(461.4\ \text{mm/s}^2)\cos 30^\circ - (46188\ \text{mm/s}^2)\cos 60^\circ$$
$$-(46188\ \text{mm/s}^2)\cos 60^\circ + 0$$

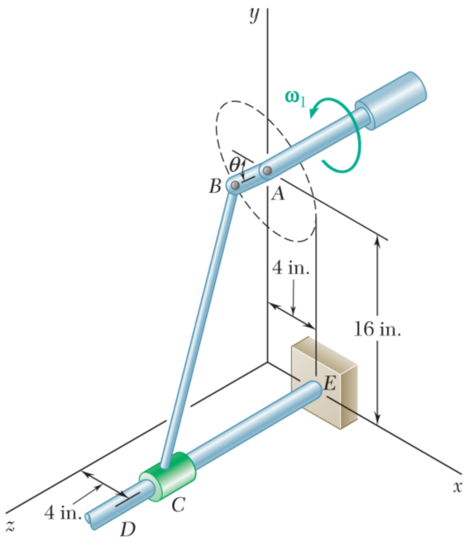
$$\dot{u}_1 = -23494\ \text{mm/s}^2$$

(*a*) *Relative velocities*: AE: $u_1 \longrightarrow = 2.00\ \text{m/s} \longrightarrow$ ◀

BD: $u_2 \measuredangle 60^\circ = 0$ ◀

(*b*) *Relative accelerations*: AE: $\dot{u}_1 = 23.5\ \text{m/s}^2 \longleftarrow$ ◀

BD: $\dot{u}_2 \measuredangle 60^\circ = 46.2\ \text{m/s}^2 \swarrow 60^\circ$ ◀



PROBLEM 15.258

Rod *BC* of length 24 in. is connected by ball-and-socket joints to a rotating arm *AB* and to a collar *C* that slides on the fixed rod *DE*. Knowing that the length of arm *AB* is 4 in. and that it rotates at the constant rate $\omega_1 = 10$ rad/s, determine the velocity of collar *C* when $\theta = 0$.

SOLUTION

Geometry.

$$l_{BC} = 24 \text{ in.}$$
$$\mathbf{r}_{C/B} = (8 \text{ in.})\mathbf{i} - (16 \text{ in.})\mathbf{j} + z_{C/B}\mathbf{k}$$
$$24^2 = 8^2 + 16^2 + z_{C/B}^2$$
$$z_{C/B} = 16 \text{ in.}$$
$$\mathbf{r}_{C/B} = (8 \text{ in.})\mathbf{i} - (16 \text{ in.})\mathbf{j} + (16 \text{ in.})\mathbf{k}$$
$$\mathbf{r}_{B/A} = -(4 \text{ in.})\mathbf{i}$$

Velocity at *B*.

$$\begin{aligned}\mathbf{v}_B &= \omega_1\mathbf{k} \times \mathbf{r}_{B/A}\\ &= 10\mathbf{k} \times (-4\mathbf{i})\\ &= -(40 \text{ in./s})\mathbf{j}\end{aligned}$$

Velocity of collar *C*.

$$\mathbf{v}_C = v_C\mathbf{k}$$
$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{C/B}$$

where

$$\mathbf{v}_{C/B} = \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C}$$

Noting that $\mathbf{v}_{C/B}$ is perpendicular to $\mathbf{r}_{C/B}$, we get $\mathbf{r}_{C/B} \cdot \mathbf{v}_{C/B} = 0$

Forming $\mathbf{r}_{C/B} \cdot \mathbf{v}_C$, we get

$$\begin{aligned}\mathbf{r}_{C/B} \cdot \mathbf{v}_C &= \mathbf{r}_{C/B} \cdot (\mathbf{v}_B + \mathbf{v}_{C/B})\\ &= \mathbf{r}_{C/B} \cdot \mathbf{v}_B + \mathbf{r}_{C/B} \cdot \mathbf{v}_{C/B}\end{aligned}$$

or

$$\mathbf{r}_{C/B} \cdot \mathbf{v}_C = \mathbf{r}_{C/B}\mathbf{v}_B \qquad (1)$$

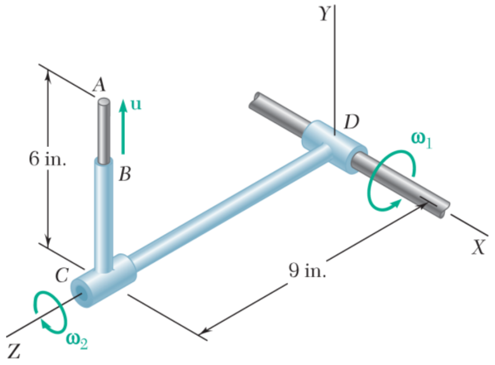
From Eq. (1)

$$(8\mathbf{i} - 16\mathbf{j} + 16\mathbf{k}) \cdot (v_C\mathbf{k}) = (8\mathbf{i} - 16\mathbf{j} + 16\mathbf{k}) \cdot (-40\mathbf{j})$$
$$16v_C = (-16)(-40)$$

or

$$v_C = 40 \text{ in./s}$$

$\mathbf{v}_C = (40.0 \text{ in./s})\mathbf{k}$ ◀

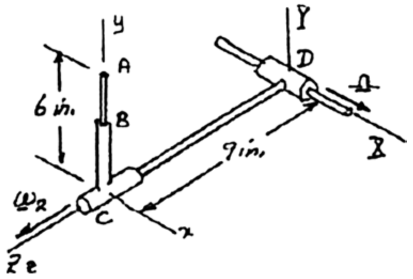


PROBLEM 15.259

In the position shown, the thin rod moves at a constant speed $u = 3$ in./s out of the tube *BC*. At the same time, tube *BC* rotates at the constant rate $\omega_2 = 1.5$ rad/s with respect to arm *CD*. Knowing that the entire assembly rotates about the *X* axis at the constant rate $\omega_1 = 1.2$ rad/s, determine the velocity and acceleration of end *A* of the rod.

SOLUTION

Geometry. $\mathbf{r}_A = (6\text{ in.})\mathbf{j} + (9\text{ in.})\mathbf{k}$



Method 1

Let the rigid body *DCB* be a rotating frame of reference.

Its angular velocity is $\mathbf{\Omega} = \omega_1\mathbf{i} + \omega_2\mathbf{k}$

$$= (1.2\text{ rad/s})\mathbf{i} - (1.5\text{ rad/s})\mathbf{k}.$$

Its angular acceleration is $\boldsymbol{\alpha} = \omega_1\mathbf{i} \times \omega_2\mathbf{k}$

$$= -\omega_1\omega_2\mathbf{j}$$

$$= (1.8\text{ rad/s}^2)\mathbf{j}.$$

Motion of the coinciding Point A′ in the frame.

$$\mathbf{v}_{A'} = \mathbf{\Omega} \times \mathbf{r}_A$$

$$= (1.2\mathbf{i} - 1.5\mathbf{k}) \times (6\mathbf{j} + 9\mathbf{k})$$

$$= 7.2\mathbf{k} - 10.8\mathbf{j} + 9\mathbf{i}$$

$$= (9\text{ in./s})\mathbf{i} - (10.8\text{ in./s})\mathbf{j} + (7.2\text{ in./s})\mathbf{k}$$

$$\mathbf{a}_{A'} = \boldsymbol{\alpha} \times \mathbf{r}_A + \mathbf{\Omega} \times \mathbf{v}_{A'}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1.8 & 0 \\ 0 & 6 & 9 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.2 & 0 & -1.5 \\ 9 & -10.8 & 7.2 \end{vmatrix}$$

$$= 16.2\mathbf{i} - 16.2\mathbf{i} - 22.14\mathbf{j} - 12.96\mathbf{k}$$

$$= -(22.14\text{ in./s}^2)\mathbf{j} - (12.96\text{ in./s}^2)\mathbf{k}$$

Motion of Point A relative to the frame.

$$\mathbf{v}_{A/F} = u\mathbf{j} = (3\text{ in./s})\mathbf{j},$$

$$\mathbf{a}_{A/F} = 0$$

PROBLEM 15.259 (Continued)

Velocity of Point A.

$$\mathbf{v}_A = \mathbf{v}_{A'} + \mathbf{v}_{A/F}$$
$$\mathbf{v}_A = 9\mathbf{i} - 10.8\mathbf{j} + 7.2\mathbf{k} + 3\mathbf{j}$$

$$\mathbf{v}_A = (9.00 \text{ in./s})\mathbf{i} - (7.80 \text{ in./s})\mathbf{j} + (7.20 \text{ in./s})\mathbf{k} \blacktriangleleft$$

Coriolis acceleration.

$$\begin{aligned} 2\mathbf{\Omega} \times \mathbf{v}_{A/F} &= (2)(1.2\mathbf{i} - 1.5\mathbf{k}) \times 3\mathbf{j} \\ &= (9 \text{ in./s}^2)\mathbf{i} + (7.2 \text{ in./s}^2)\mathbf{k} \end{aligned}$$

Acceleration of Point A.

$$\mathbf{a}_A = \mathbf{a}_{A'} + \mathbf{a}_{A/F} + 2\mathbf{\Omega} \times \mathbf{v}_{A/F}$$
$$\mathbf{a}_A = -22.14\mathbf{j} - 12.92\mathbf{k} + 9\mathbf{i} + 7.2\mathbf{k}$$

$$\mathbf{a}_A = (9.00 \text{ in./s}^2)\mathbf{i} - (22.1 \text{ in./s}^2)\mathbf{j} - (5.76 \text{ in./s}^2)\mathbf{k} \blacktriangleleft$$

Method 2

Let frame *Dxyz*, which at instant shown coincides with *DXYZ*, rotate with an angular velocity $\mathbf{\Omega} = \omega_1\mathbf{i} = 1.2\mathbf{i}$ rad/s. Then the motion relative to the frame consists of the rotation of body *DCB* about the *Z* axis with angular velocity $\omega_2\mathbf{k} = -(1.5 \text{ rad/s})\mathbf{k}$ plus the sliding motion $\mathbf{u} = u\mathbf{i} = (3 \text{ in./s})\mathbf{j}$ of the rod *AB* relative to the body *DCB*.

Motion of the coinciding Point A′ in the frame.

$$\begin{aligned} \mathbf{v}_{A'} &= \mathbf{\Omega} \times \mathbf{r}_A \\ &= 1.2\mathbf{i} \times (6\mathbf{j} + 9\mathbf{k}) \\ &= -(10.8 \text{ in./s})\mathbf{j} + (7.2 \text{ in./s})\mathbf{k} \\ \mathbf{a}_{A'} &= \mathbf{\Omega} \times \mathbf{v}_{A'} \\ &= 1.2\mathbf{i} \times (-10.8\mathbf{j} + 7.2\mathbf{k}) \\ &= -(8.64 \text{ in./s}^2)\mathbf{j} - (12.96 \text{ in./s}^2)\mathbf{k} \end{aligned}$$

Motion of Point A relative to the frame.

$$\begin{aligned} \mathbf{v}_{A/F} &= \omega_2\mathbf{k} \times \mathbf{r}_A + u\mathbf{j} \\ &= (-1.5\mathbf{k}) \times (6\mathbf{j} + 9\mathbf{k}) + 3\mathbf{j} \\ &= (9 \text{ in./s})\mathbf{i} + (3 \text{ in./s})\mathbf{j} \\ \mathbf{a}_{A/F} &= \alpha_2\mathbf{k} \times \mathbf{r}_A + \omega_2\mathbf{k} \times (\omega_2\mathbf{k} \times \mathbf{r}_A) + \dot{u}\mathbf{j} + 2\omega_2\mathbf{k} \times (u\mathbf{j}) \\ &= 0 + (-1.5\mathbf{k}) \times (9\mathbf{i}) + 0 + (2)(-1.5\mathbf{k}) \times (3\mathbf{j}) \\ &= -13.5\mathbf{j} + 9\mathbf{i} \\ &= (9 \text{ in./s}^2)\mathbf{i} - (13.5 \text{ in./s}^2)\mathbf{j} \end{aligned}$$

Velocity of Point A.

$$\mathbf{v}_A = \mathbf{v}_{A'} + \mathbf{v}_{A/F}$$
$$\mathbf{v}_A = -10.8\mathbf{j} + 7.2\mathbf{k} + 9\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{v}_A = (9.00 \text{ in./s})\mathbf{i} - (7.80 \text{ in./s})\mathbf{j} + (7.20 \text{ in./s})\mathbf{k} \blacktriangleleft$$

PROBLEM 15.259 (Continued)

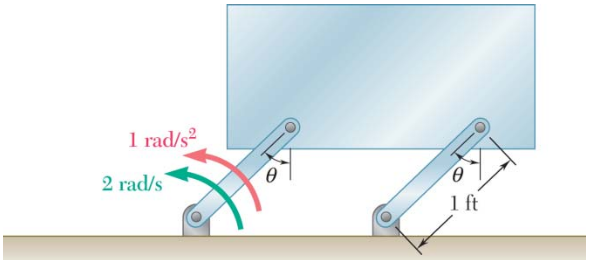
Coriolis acceleration.

$$2\boldsymbol{\Omega} \times \mathbf{v}_{A/F} = (2)(1.2\mathbf{i}) \times (-9\mathbf{i} + 3\mathbf{j})$$
$$= (7.2 \text{ in./s}^2)\mathbf{k}$$

Acceleration of Point A.

$$\mathbf{a}_A = \mathbf{a}_{A'} + \mathbf{a}_{A/F} + 2\boldsymbol{\Omega} \times \mathbf{v}_{A/F}$$
$$\mathbf{a}_A = -8.64\mathbf{j} - 12.96\mathbf{k} + 9\mathbf{i} - 13.5\mathbf{j} + 7.2\mathbf{k}$$

$$\mathbf{a}_A = (9.00 \text{ in./s}^2)\mathbf{i} - (22.1 \text{ in./s}^2)\mathbf{j} - (5.76 \text{ in./s}^2)\mathbf{k} \blacktriangleleft$$



PROBLEM 15.CQ1

A rectangular plate swings from arms of equal length as shown below. What is the magnitude of the angular velocity of the plate?

(*a*) 0 rad/s

(*b*) 1 rad/s

(*c*) 2 rad/s

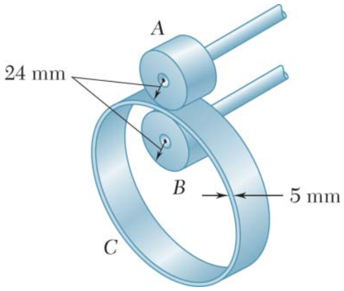
(*d*) 3 rad/s

(*e*) Need to know the location of the center of gravity

SOLUTION

Since the arms are of equal length, each point of the arms that is attached to the plate follows the same circular path; therefore, the plate is undergoing *curvilinear translation* and has no angular velocity.

Answer: (*a*) ◀



PROBLEM 15.CQ2

Knowing that wheel *A* rotates with a constant angular velocity and that no slipping occurs between ring *C* and wheel *A* and wheel *B*, which of the following statements concerning the angular speeds are true?

(*a*) $\omega_a = \omega_b$

(*b*) $\omega_a > \omega_b$

(*c*) $\omega_a < \omega_b$

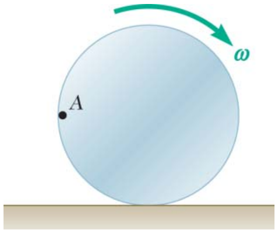
(*d*) $\omega_a = \omega_c$

(*e*) the contact points between *A* and *C* have the same acceleration

SOLUTION

Since there is no slipping and hoop C has a 5 mm thickness, the contact point between A and C will have a greater speed than the contact point between B and C. The radii of A and B are the same; therefore, the angular velocity of A must be higher than B in order for the contact point on A to be moving at a greater speed than the contact point on B.

Answer: (*b*) ◀



PROBLEM 15.CQ3

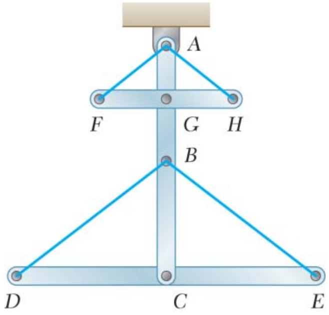
The ball rolls without slipping on the fixed surface as shown. What is the direction of the velocity of Point *A*?

(*a*) → (*b*) ↗ (*c*) ↑ (*d*) ↓ (*e*) ↘

SOLUTION

The velocity of point A can be found by using relative velocity between point A and the contact point of the ball with the ground. The contact point has zero velocity if the ball rolls without slipping. The relative motion term is thus equal to the velocity of A. The direction of this term is perpendicular to the vector from the contact point to point A.

Answer: (*b*) ◀



PROBLEM 15.CQ4

Three uniform rods, *ABC*, *DCE* and *FGH* are connected as shown. Which of the following statements are true?

(*a*) $\omega_{ABC} = \omega_{DCE} = \omega_{FGH}$

(*b*) $\omega_{DCE} > \omega_{ABC} > \omega_{FGH}$

(*c*) $\omega_{DCE} < \omega_{ABC} < \omega_{FGH}$

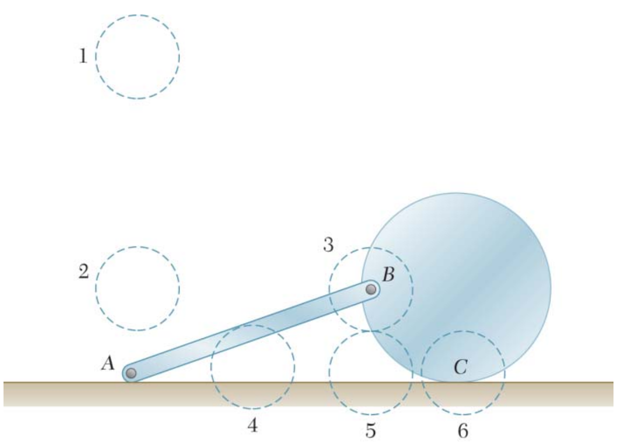
(*d*) $\omega_{ABC} > \omega_{DCE} > \omega_{FGH}$

(*e*) $\omega_{FGH} = \omega_{DCE} < \omega_{ABC}$

SOLUTION

Assuming that the cables cannot stretch, the entire assembly moves as a single rigid body. Since rigid bodies have only a single angular velocity, the angular velocity of each piece must be the same.

Answer: (*a*) ◀



PROBLEM 15.CQ5

The disk rolls without sliding on the fixed horizontal surface. At the instant shown, the instantaneous center of zero velocity for rod *AB* would be located in which region?

(*a*) region 1

(*b*) region 2

(*c*) region 3

(*d*) region 4

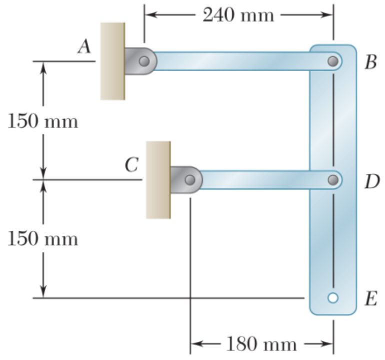
(*e*) region 5

(*f*) region 6

SOLUTION

The instantaneous center of *AB* can be found by locating the intersection of lines perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$. Since $\mathbf{v}_A$ is horizontal, the instantaneous center of AB must be in either region 1 or 2. Since the wheel rolls without sliding, $\mathbf{v}_B$ is perpendicular to the line BC and the instantaneous center of AB must lie along the line that goes through BC.

Answer: (*a*) ◀



PROBLEM 15.CQ6

Bar *BDE* is pinned to two links, *AB* and *CD*. At the instant shown the angular velocities of link *AB*, link *CD* and bar *BDE* are ω_{AB}, ω_{CD}, and ω_{BDE}, respectively. Which of the following statements concerning the angular speeds of the three objects is true at this instant?

(*a*) $\omega_{AB} = \omega_{CD} = \omega_{BDE}$

(*b*) $\omega_{BDE} > \omega_{AB} > \omega_{CD}$

(*c*) $\omega_{AB} = \omega_{CD} > \omega_{BDE}$

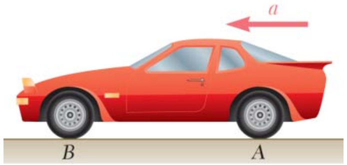
(*d*) $\omega_{AB} > \omega_{CD} > \omega_{BDE}$

(*e*) $\omega_{CD} > \omega_{AB} > \omega_{BDE}$

SOLUTION

Since bars AB and CD are horizontal and undergoing fixed axis rotation, the velocities of points B and D are in the vertical direction and equal; therefore $\omega_{BDE} = 0$. ω_{AB} can be found by dividing the velocity of B by the length of link AB and ω_{CD} can be found by dividing the velocity of D by the length of link CD. Since CD is shorter than link AB, $\omega_{CD} > \omega_{AB}$.

Answer: (*e*) ◀



PROBLEM 15.CQ7

A rear wheel drive car starts from rest and accelerates to the left so that the tires do not slip on the road. What is the direction of the acceleration of the point on the tire in contact with the road, that is, Point *A*?

(*a*) ← (*b*) ↖ (*c*) ↑ (*d*) ↓ (*e*) ↙

SOLUTION

The tangential acceleration will be zero since the tires do not slip, but there will be an acceleration component perpendicular to the ground.

Answer: (*c*) ◀

PROBLEM 15.CQ8

A person walks radially inward on a platform that is rotating counterclockwise about its center. Knowing that the platform has a constant angular velocity $\boldsymbol{\omega}$ and the person walks with a constant speed $\mathbf{u}$ relative to the platform, what is the direction of the acceleration of the person at the instant shown?

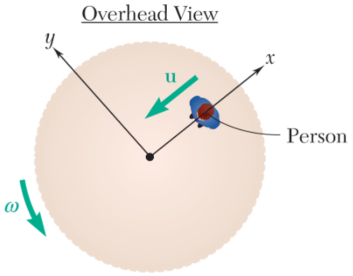
(*a*) Negative *x*

(*b*) Negative *y*

(*c*) Negative *x* and positive *y*

(*d*) Positive *x* and positive *y*

(*e*) Negative *x* and negative *y*

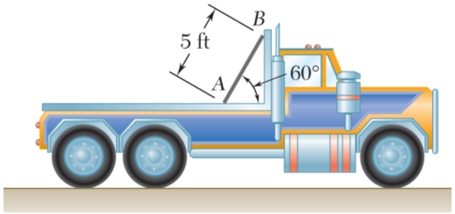


SOLUTION

The $\omega^2 r$ term will be in the negative *x*-direction and the Coriolis acceleration will be in the negative *y*-direction.

Answer: (*e*) ◀

CHAPTER 16



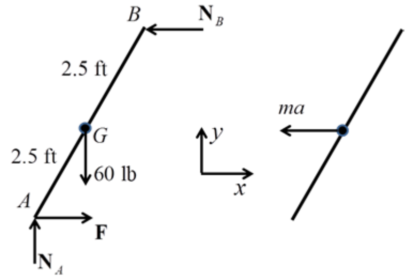
PROBLEM 16.1

A 60-lb uniform thin panel is placed in a truck with end *A* resting on a rough horizontal surface and end *B* supported by a smooth vertical surface. Knowing that the deceleration of the truck is 12 ft/s^2, determine (*a*) the reactions at ends *A* and *B*, (*b*) the minimum required coefficient of static friction at end *A*.

SOLUTION

Given: $m = \dfrac{W}{g} = \dfrac{60 \text{ lb}}{32.2 \text{ ft/s}^2} = 1.8633 \text{ slugs}, \quad a = 12 \text{ ft/s}^2 \leftarrow$

Free Body Diagram:



Kinetics: $\sum F_y = 0 \;\Rightarrow\; N_A = 60 \text{ lb}$

$$+\circlearrowleft\ \Sigma M_A = N_B(5 \text{ ft})(0.866) - (60 \text{ lb})(2.5 \text{ ft})(0.5)$$

$$= \frac{(60 \text{ lb})}{32.2 \text{ ft/s}^2}(12 \text{ ft/s}^2)(0.866)$$

$$N_B = 28.501 \text{ lb}$$

$$\overset{+}{\leftarrow}\ \Sigma F_x = N_B - F = \frac{60 \text{ lb}}{32 \text{ ft/s}^2}(12 \text{ ft/s}^2)$$

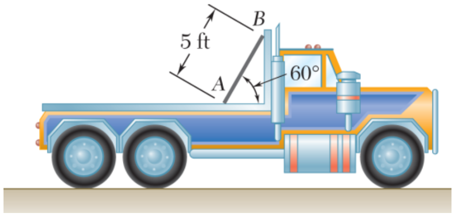
$$F = 6.1404 \text{ lb}$$

(*a*) $$R_A = \sqrt{N_A^2 + F^2} = 60.3133 \text{ lb}$$

$$\alpha = \tan^{-1}\frac{60.3133}{6.1404} = 84.18° \qquad R_A = 60.31 \text{ lb} \measuredangle 84.2° \blacktriangleleft$$

and $N_B = 28.5 \text{ lb} \leftarrow \blacktriangleleft$

(*b*) $$\mu = \frac{F}{N_A} = \frac{6.1404}{60} = 0.10234 \qquad \mu = 0.1023 \blacktriangleleft$$



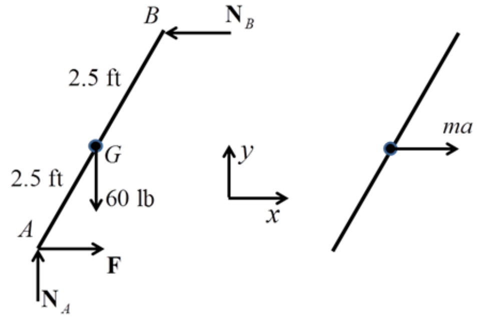
PROBLEM 16.2

A 60-lb uniform thin panel is placed in a truck with end *A* resting on a rough horizontal surface and end *B* supported by a smooth vertical surface. Knowing that the panel remains in the position shown, determine (*a*) the maximum allowable acceleration of the truck, (*b*) the corresponding minimum required coefficient of static friction at end *A*.

SOLUTION

Given: $$m = \frac{W}{g} = \frac{60 \text{ lb}}{32.2 \text{ ft/s}^2} = 1.8633 \text{ slugs}, \quad N_B = 0$$

Free Body Diagram:



Kinetics: $$\sum F_y = 0 \Rightarrow N_A = 60 \text{ lb}$$

$$+\circlearrowright \Sigma M_A = (60 \text{ lb})(2.5 \text{ ft})(0.5) = \frac{(60 \text{ lb})(a)}{32.2 \text{ ft/s}^2}(2.5 \text{ ft})(\sin 60°)$$

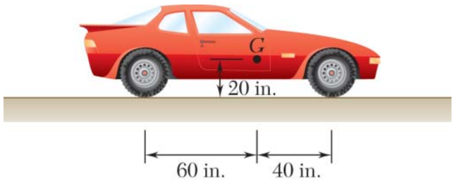
$$a = 18.5907 \text{ ft/s}^2$$

(*a*) or $a = 18.59 \text{ ft/s}^2 \longrightarrow$ ◀

$$\xrightarrow{+} \Sigma F_x = F = \frac{(60 \text{ lb})}{32.2 \text{ ft/s}^2}\left(18.5912 \text{ ft/s}^2\right) = 34.6419 \text{ lb}$$

(*b*) $$\mu = \frac{F}{N_A} = \frac{34.6419 \text{ lb}}{60 \text{ lb}} = 0.5773$$

or $\mu = 0.577$ ◀

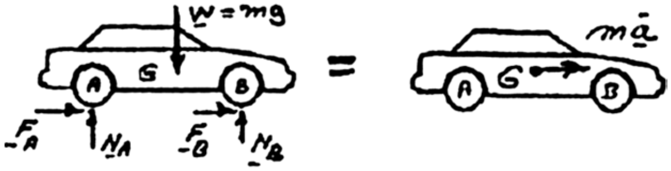

G
20 in.
60 in.
40 in.

PROBLEM 16.3

Knowing that the coefficient of static friction between the tires and the road is 0.80 for the automobile shown, determine the maximum possible acceleration on a level road, assuming (*a*) four-wheel drive, (*b*) rear-wheel drive, (*c*) front-wheel drive.

SOLUTION

(*a*) Four-wheel drive:



$$+\uparrow \Sigma F_y = 0: \quad N_A + N_B - W = 0 \qquad N_A + N_B = W = mg$$

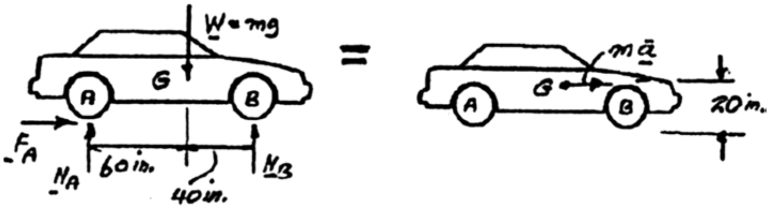
Thus: $F_A + F_B = \mu_k N_A + \mu_k N_B = \mu_k (N_A + N_B) = \mu_k W = 0.80mg$

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad F_A + F_B = m\bar{a}$$

$$0.80mg = m\bar{a}$$

$$\bar{a} = 0.80g = 0.80(32.2\ \text{ft/s}^2) \qquad \mathbf{\bar{a}} = 25.8\ \text{ft/s}^2 \longrightarrow \blacktriangleleft$$

(*b*) Rear-wheel drive:



$$+\circlearrowleft \Sigma M_B = \Sigma (M_B)_{\text{eff}}: \quad (40\ \text{in.})W - (100\ \text{in.})N_A = -(20\ \text{in.})m\bar{a}$$

$$N_A = 0.4W + 0.2m\bar{a}$$

Thus: $F_A = \mu_k N_B = 0.80(0.4W + 0.2m\bar{a}) = 0.32mg + 0.16m\bar{a}$

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad F_A = m\bar{a}$$

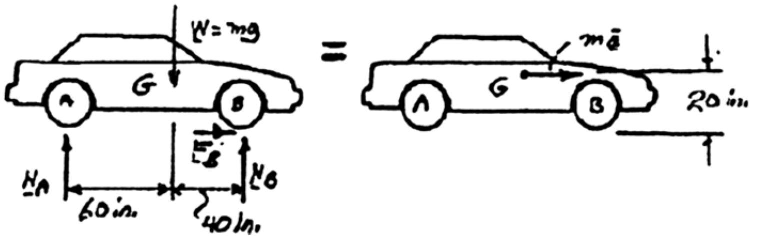
$$0.32mg + 0.16m\bar{a} = m\bar{a}$$

$$0.32g = 0.84\bar{a}$$

$$\bar{a} = \frac{0.32}{0.84}(32.2\ \text{ft/s}^2) \qquad \mathbf{\bar{a}} = 12.27\ \text{ft/s}^2 \longrightarrow \blacktriangleleft$$

PROBLEM 16.3 (Continued)

(*c*) Front-wheel drive:



$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad (100 \text{ in.})N_B - (60 \text{ in.})W = -(20 \text{ in.})m\overline{a}$$

$$N_B = 0.6W - 0.2m\overline{a}$$

Thus: $$F_B = \mu_k N_B = 0.80(0.6W - 0.2m\overline{a}) = 0.48mg - 0.16m\overline{a}$$

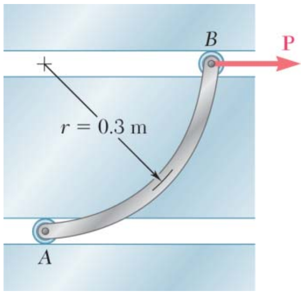
$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad F_B = m\overline{a}$$

$$0.48mg - 0.16m\overline{a} = m\overline{a}$$

$$0.48g = 1.16\overline{a}$$

$$\overline{a} = \frac{0.48}{1.16}(32.2 \text{ ft/s}^2)$$

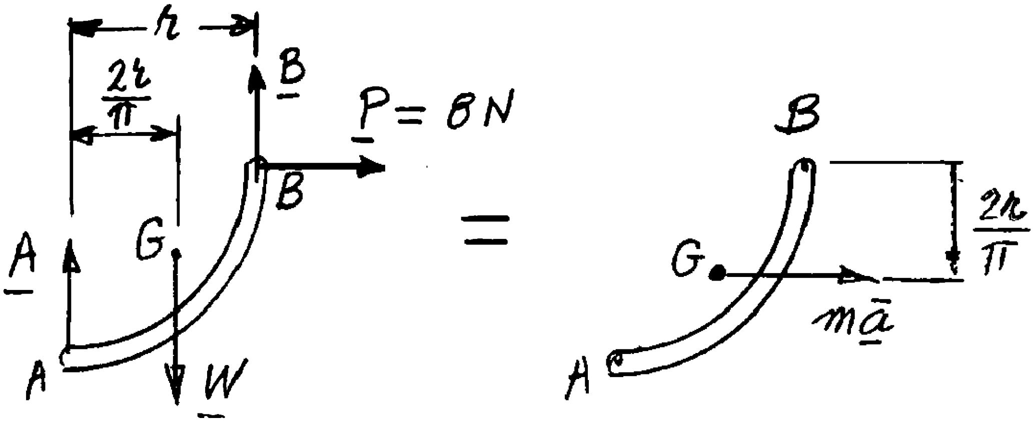
$$\overline{\mathbf{a}} = 13.32 \text{ ft/s}^2 \longrightarrow \blacktriangleleft$$



PROBLEM 16.4

The motion of the 2.5-kg rod *AB* is guided by two small wheels which roll freely in horizontal slots. If a force **P** of magnitude 8 N is applied at *B*, determine (*a*) the acceleration of the rod, (*b*) the reactions at *A* and *B*.

SOLUTION



(*a*) $\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad P = m\bar{a}$

$$\bar{a} = \frac{P}{m} = \frac{8\text{ N}}{2.5\text{ kg}} = 3.20\text{ m/s}^2$$

$\bar{\mathbf{a}} = 3.20\text{ m/s}^2 \longrightarrow$ ◀

(*b*) $+\uparrow \Sigma M_B = \Sigma (M_B)_{\text{eff}}: \quad W\left(r - \dfrac{2r}{\pi}\right) - Ar = m\bar{a}\left(\dfrac{2r}{\pi}\right)$

$$A = W\left(1 - \frac{2}{\pi}\right) - m\bar{a}\left(\frac{2}{\pi}\right) = mg\left(1 - \frac{2}{\pi}\right) - P\left(\frac{2}{\pi}\right)$$

$$= (2.5\text{ kg})(9.81\text{ m/s}^2)\left(1 - \frac{2}{\pi}\right) - (8\text{ N})\left(\frac{2}{\pi}\right)$$

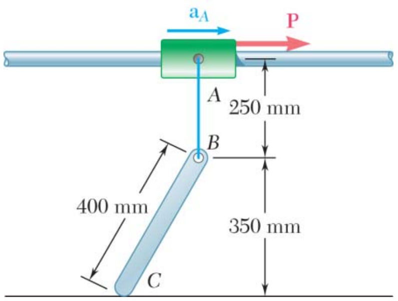
$$= 8.912\text{ N} - 5.093\text{ N} = 3.819\text{ N}$$

$\mathbf{A} = 3.82\text{ N}\uparrow$ ◀

$+\uparrow \Sigma F_y = 0: \quad A + B - W = 0$

$B = W - A = (2.5)(9.81) - 3.819,$

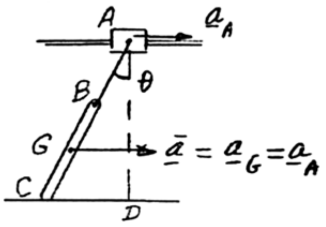
$\mathbf{B} = 20.71\text{ N}\uparrow$ ◀



PROBLEM 16.5

A uniform rod BC of mass 4 kg is connected to a collar A by a 250-mm cord AB. Neglecting the mass of the collar and cord, determine (*a*) the smallest constant acceleration $\mathbf{a}_A$ for which the cord and the rod lie in a straight line, (*b*) the corresponding tension in the cord.

SOLUTION



Geometry and kinematics:

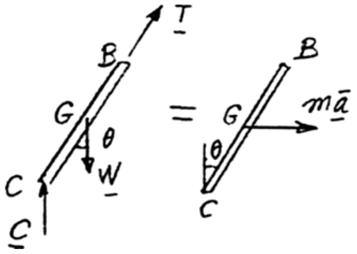
Distance between collar and floor $= AD = 250 \text{ mm} + 350 \text{ mm} = 600 \text{ mm}$

When cord and rod lie in a straight line:

$$AC = AB + BC = 250 \text{ mm} + 400 \text{ mm} = 650 \text{ mm}$$

$$\cos\theta = \frac{AD}{AC} = \frac{600 \text{ mm}}{650 \text{ mm}}$$

$$\theta = 22.62°$$



Kinetics

(*a*) Acceleration at *A*.

$$+\circlearrowleft \Sigma M_C = \Sigma (M_C)_{\text{eff}}: \quad W(CG)\sin\theta = m\bar{a}(CG)\cos\theta$$

$$m\bar{a} = mg\tan\theta$$

$$\bar{a} = g\tan\theta = (9.81 \text{ m/s}^2)\tan 22.62°$$

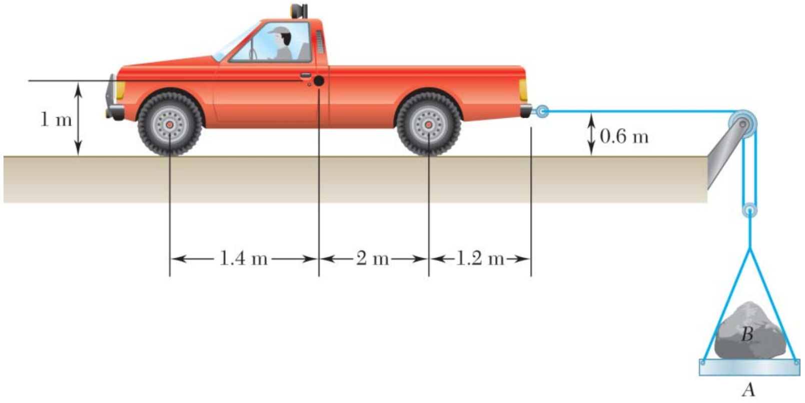
$$\mathbf{a}_A = \bar{\mathbf{a}} = 4.09 \text{ m/s}^2 \longrightarrow \blacktriangleleft$$

(*b*) Tension in the cord.

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad T\sin\theta = m\bar{a} = mg\tan\theta$$

$$T = \frac{mg}{\cos\theta} = \frac{(4 \text{ kg})(9.81)}{\cos 22.62°}$$

$$T = 42.5 \text{ N} \blacktriangleleft$$



PROBLEM 16.6

A 2000-kg truck is being used to lift a 400-kg boulder *B* that is on a 50-kg pallet *A*. Knowing the acceleration of the rear-wheel drive truck is 1 m/s^2, determine (*a*) the reaction at each of the front wheels, (*b*) the force between the boulder and the pallet.

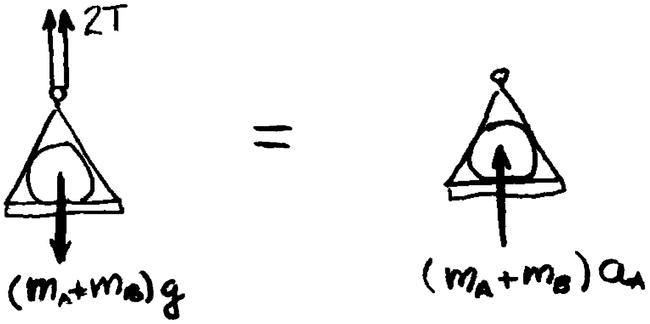
SOLUTION

Kinematics: Acceleration of truck: $\mathbf{a}_T = 1 \text{ m/s}^2 \longleftarrow$.

When the truck moves 1 m to the left, the boulder *B* and pallet *A* are raised 0.5 m.

Then, $\mathbf{a}_A = 0.5 \text{ m/s}^2 \uparrow$ $\quad\quad$ $\mathbf{a}_B = 0.5 \text{ m/s}^2 \uparrow$

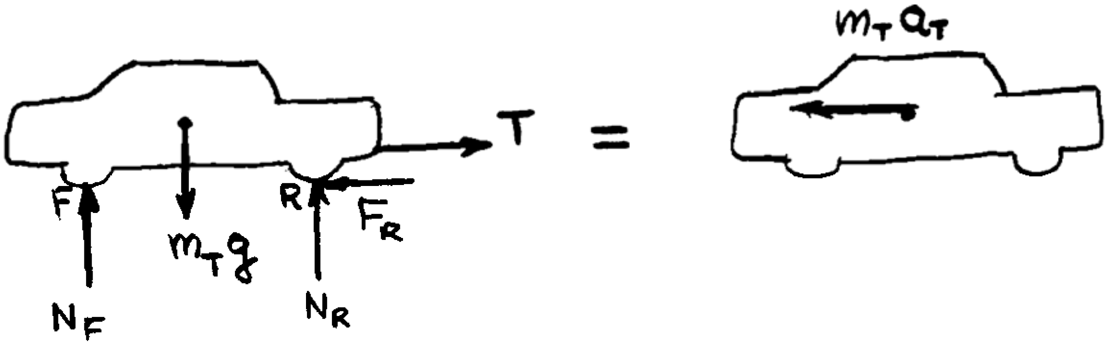
Kinetics: Let *T* be the tension in the cable.



Pallet and boulder: $+\!\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad 2T-(m_A+m_B)_g = (m_A+m_B)a_B$

$$2T-(450 \text{ kg})(9.81 \text{ m/s}^2) = (450 \text{ kg})(0.5 \text{ m/s}^2)$$

$$T = 2320 \text{ N}$$



PROBLEM 16.6 (Continued)

Truck: $+\circlearrowleft M_R = \Sigma(M_R)_{\text{eff}}: \quad -N_F(3.4\text{ m}) + m_T g(2.0\text{ m}) - T(0.6\text{ m}) = m_T a_T(1.0\text{ m})$

$$N_F = \frac{(2.0\text{ m})(2000\text{ kg})(9.81\text{ m/s}^2)}{3.4\text{ m}} - \frac{(0.6\text{ m})(2320\text{ N})}{3.4\text{ m}} + \frac{(1.0\text{ m})(2000\text{ kg})(1.0\text{ m/s})}{3.4\text{ m}}$$

$$= 11541.2\text{ N} - 409.4\text{ N} - 588.2\text{ N}$$

$$= 10544\text{ N}$$

$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N_F + N_R - m_T g = 0$

$$10544\text{ N} + N_R - (2000\text{ kg})(9.81\text{ m/s}^2) = 0$$

$$N_R = 9076\text{ N}$$

$\xleftarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad F_R - T = m_T a_T$

$$F_R = 2320\text{ N} + (2000\text{ kg})(1.0\text{ m/s}^2)$$

$$= 4320\text{ N}$$

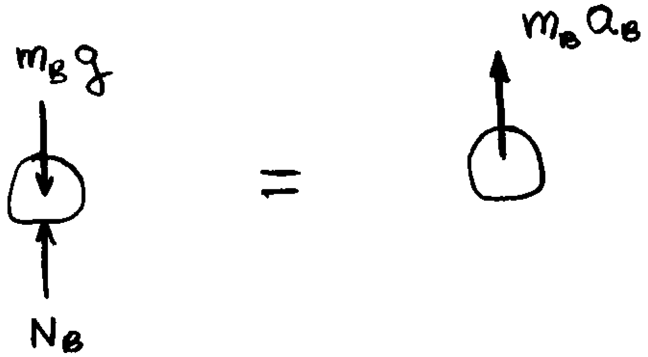
(*a*) *Reaction at each front wheel*:

$\frac{1}{2}N_F \uparrow$: $\qquad 5270\text{ N}\uparrow$ ◀

Reaction at each rear wheel:

$\frac{1}{2}F_R \longleftarrow + \frac{1}{2}N_R \uparrow \qquad 5030\text{ N} \measuredangle 64.5^\circ$

(*b*) *Force between boulder and pallet*.

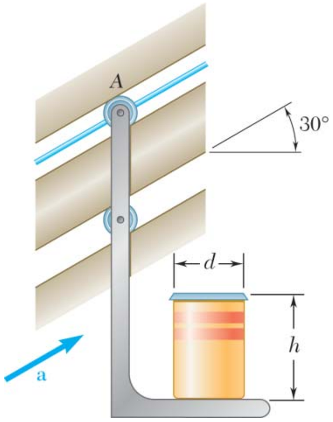


Boulder $\quad +\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N_B + M_B g - m_B a_B$

$$N_B = (400\text{ kg})(9.81\text{ m/s}^2) + (400\text{ kg})(0.5\text{ m/s}^2)$$

$$= 4124\text{ N}$$

4120 N (compression) ◀

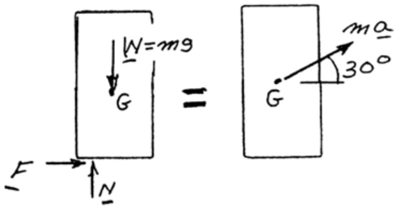


PROBLEM 16.7

The support bracket shown is used to transport a cylindrical can from one elevation to another. Knowing that $\mu_s = 0.25$ between the can and the bracket, determine (*a*) the magnitude of the upward acceleration **a** for which the can will slide on the bracket, (*b*) the smallest ratio *h*/*d* for which the can will tip before it slides.

SOLUTION

(*a*) Sliding impends



$$\xrightarrow{+} \Sigma F_y = \Sigma(F_x)_{\text{eff}}: \qquad F = ma\cos 30°$$

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N - mg = ma\sin 30°$$

$$N = m(g + a\sin 30°)$$

$$\mu_s = \frac{F}{N}$$

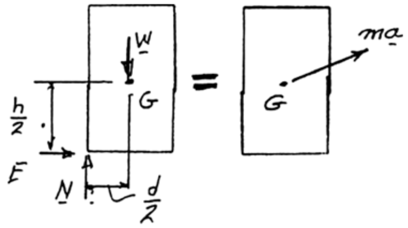
$$0.25 = \frac{ma\cos 30°}{m(g + a\sin 30°)}$$

$$g + a\sin 30° = 4a\cos 30°$$

$$\frac{a}{g} = \frac{1}{4\cos 30° - \sin 30°}$$

$\mathbf{a} = 0.337g \measuredangle 30°$ ◀

(*b*) Tipping impends

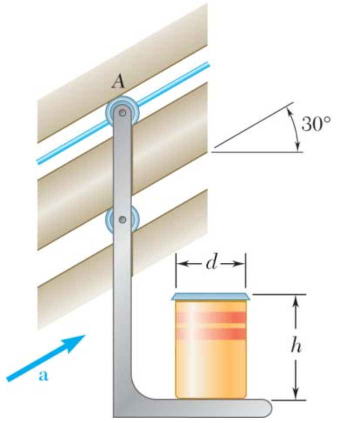


$$\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad F\left(\frac{h}{2}\right) - N\left(\frac{d}{2}\right) = 0$$

$$\frac{F}{N} = \frac{d}{h}$$

$$\mu = \frac{F}{N}; \quad 0.25 = \frac{d}{h};$$

$\frac{h}{d} = 4.00$ ◀



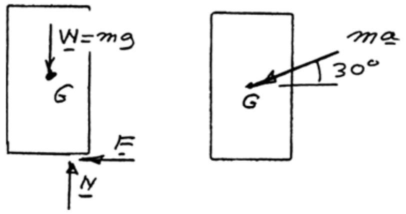
PROBLEM 16.8

Solve Problem 16.7, assuming that the acceleration **a** of the bracket is directed downward.

PROBLEM 16.7 The support bracket shown is used to transport a cylindrical can from one elevation to another. Knowing that $\mu_s = 0.25$ between the can and the bracket, determine (*a*) the magnitude of the upward acceleration **a** for which the can will slide on the bracket, (*b*) the smallest ratio *h*/*d* for which the can will tip before it slides.

SOLUTION

(*a*) Sliding impends:



$$\overset{+}{\leftarrow} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \qquad F = ma\cos 30°$$

$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad N - mg = -ma\sin 30°$$

$$N = m(g - a\sin 30°)$$

$$\mu_s = \frac{F}{N}$$

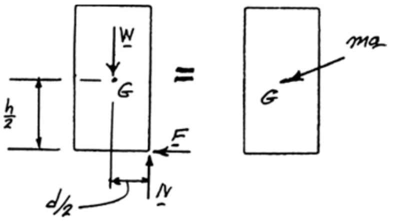
$$0.25 = \frac{ma\cos 30°}{m(g - a\sin 30°)}$$

$$g - a\sin 30° = 4a\cos 30°$$

$$\frac{a}{g} = \frac{1}{4\cos 30° + \sin 30°} = 0.25226$$

$\mathbf{a} = 0.252g$ ↙ 30° ◀

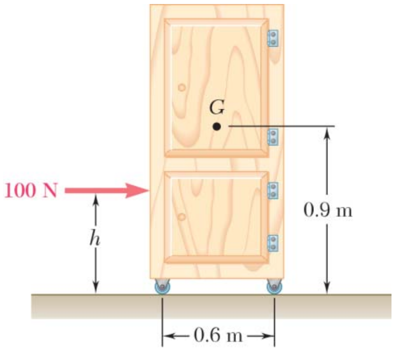
(*b*) Tipping impends:



$$\overset{+}{\circlearrowleft} \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad F\left(\frac{h}{2}\right) = W\left(\frac{d}{2}\right) \quad \frac{F}{N} = \frac{d}{h}$$

$$\mu = \frac{F}{N}; \qquad 0.25 = \frac{d}{h};$$

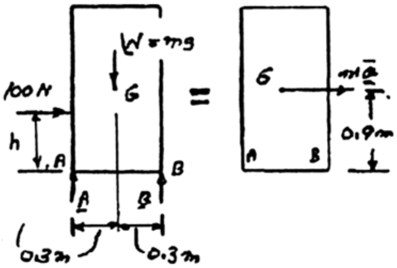
$\frac{h}{d} = 4.00$ ◀



PROBLEM 16.9

A 20-kg cabinet is mounted on casters that allow it to move freely $(\mu = 0)$ on the floor. If a 100-N force is applied as shown, determine (*a*) the acceleration of the cabinet, (*b*) the range of values of h for which the cabinet will not tip.

SOLUTION



(*a*) Acceleration

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad 100 \text{ N} = m\bar{a}$$

$$100 \text{ N} = (20 \text{ kg})\bar{a}$$

$$\bar{\mathbf{a}} = 5.00 \text{ m/s}^2 \longrightarrow \blacktriangleleft$$

(*b*) For tipping to impend $\circlearrowright$: $A = 0$

$$+\circlearrowright \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad (100 \text{ N})h - mg(0.3 \text{ m}) = m\bar{a}(0.9 \text{ m})$$

$$(100 \text{ N})h - (20 \text{ kg})(9.81 \text{ m/s}^2)(0.3 \text{ m}) = (100 \text{ N})(0.9 \text{ m})$$

$$h = 1.489 \text{ m}$$

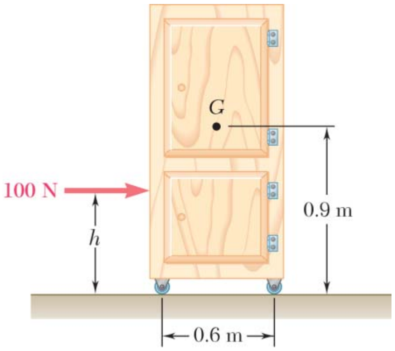
For tipping to impend $\circlearrowleft$: $B = 0$

$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad (100 \text{ N})h + mg(0.3 \text{ m}) = m\bar{a}(0.9 \text{ m})$$

$$(100 \text{ N})h + (20 \text{ kg})(9.81 \text{ m/s}^2)(0.3 \text{ m}) = (100 \text{ N})(0.9 \text{ m})$$

$$h = 0.311 \text{ m}$$

Cabinet will not tip: $0.311 \text{ m} \le h \le 1.489 \text{ m} \blacktriangleleft$

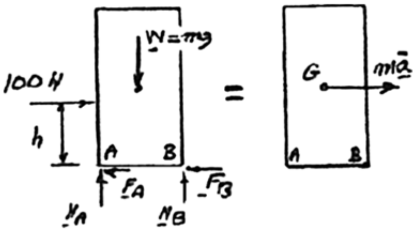


PROBLEM 16.10

Solve Problem 16.9, assuming that the casters are locked and slide on the rough floor $(\mu_k = 0.25)$.

PROBLEM 16.9 A 20-kg cabinet is mounted on casters that allow it to move freely $(\mu = 0)$ on the floor. If a 100-N force is applied as shown, determine (*a*) the acceleration of the cabinet, (*b*) the range of values of h for which the cabinet will not tip.

SOLUTION



(*a*) Acceleration.

$$+\uparrow \Sigma F_y = 0: \quad N_A + N_B - W = 0$$

$$N_A + N_B = mg$$

But, $F = \mu N$, thus $F_A + F_B = \mu(mg)$

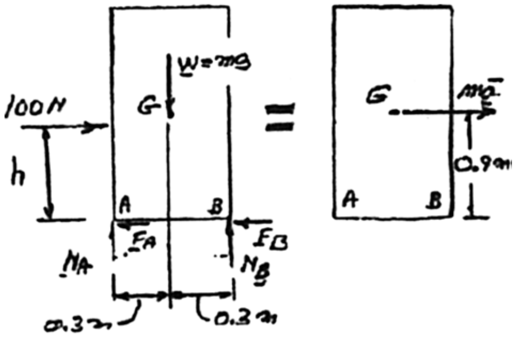
$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad 100 \text{ N} - (F_A + F_B) = m\bar{a}$$

$$100 \text{ N} - \mu m g = m\bar{a}$$

$$100 \text{ N} - 0.25(20 \text{ kg})(9.81 \text{ m/s}^2) = (20 \text{ kg})\bar{a}$$

$$\bar{a} = 2.548 \text{ m/s}^2 \qquad \bar{\mathbf{a}} = 2.55 \text{ m/s}^2 \longrightarrow \blacktriangleleft$$

(*b*) Tipping of cabinet.



$$W = mg = (20 \text{ kg})(9.81 \text{ m/s}^2)$$

$$W = 196.2 \text{ N}$$

For tipping to impend $\circlearrowright$: $\quad N_A = 0.$

$$+\circlearrowright \Sigma M_B = \Sigma (M_B)_{\text{eff}}: \quad (100 \text{ N})h - W(0.3 \text{ m}) = m\bar{a}(0.9 \text{ m})$$

$$(100 \text{ N})h - (196.2 \text{ N})(0.3 \text{ m}) = (20 \text{ kg})(2.548 \text{ m/s}^2)(0.9 \text{ m})$$

$$h = 1.047 \text{ m}$$

PROBLEM 16.10 (Continued)

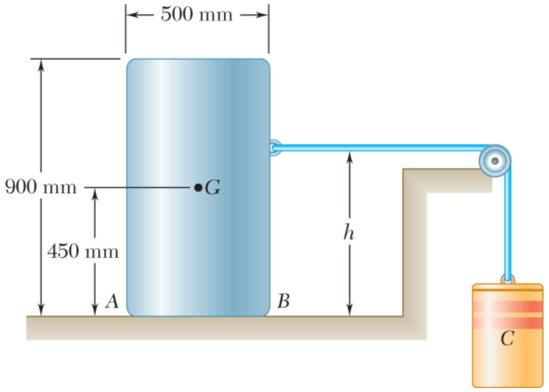
For tipping to impend ↺: $N_B = 0$

$$+\circlearrowright \Sigma M_A = \Sigma (M_A)_{\text{eff}}: \quad (100\text{ N})h + W(0.3\text{ m}) = m\overline{a}(0.9\text{ m})$$

$$(100\text{ N})h + (196.2\text{ N})(0.3\text{ m}) = (20\text{ kg})(2.548\text{ m/s}^2)(0.9\text{ m})$$

$$h = -0.130\text{ m} \quad \text{(impossible)}$$

Cabinet will not tip: $h \leq 1.047$ m ◀



PROBLEM 16.11

A completely filled barrel and its contents have a combined mass of 90 kg. A cylinder C is connected to the barrel at a height h = 550 mm as shown. Knowing $\mu_s = 0.40$ and $\mu_k = 0.35,$ determine the maximum mass of C so the barrel will not tip.

SOLUTION

Kinematics: Assume that the barrel is sliding, but not tipping.

$$\alpha = 0 \qquad \mathbf{a}_G = a \longrightarrow$$

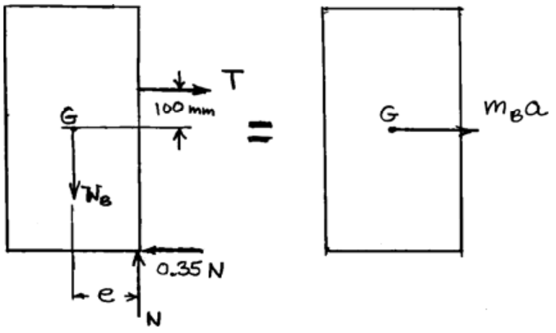
Since the cord is inextensible, $\mathbf{a}_C = a \downarrow$

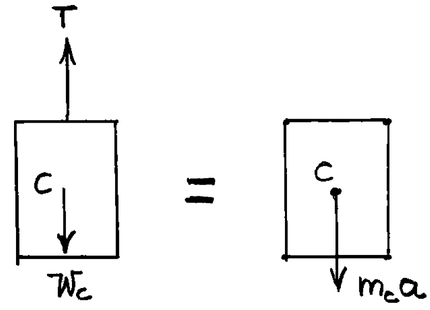
Kinetics: Draw the free body diagrams of the barrel and the cylinder. Let T be the tension in the cord.

The barrel is sliding. $F_F = \mu_k N = 0.35\, N$

Assume that tipping is impending, so that the line of action of the reaction on the bottom of the barrel passes through Point B.

$$e = 250 \text{ mm}$$





For the barrel. $+\!\uparrow \Sigma F_y = 0: \quad N - W_B = 0 \qquad N = W_B = m_B g = 882.90 \text{ N}$

$$+\circlearrowleft \Sigma M_G = 0: \quad Ne - 100T - (450)(0.35\, N) = 0$$

$$T = \frac{e - (450)(0.35)}{100} N = \frac{250 - 157.5}{100}(882.90) = 816.68 \text{ N}$$

$$\xrightarrow{+} \Sigma F_x = m_B a: \quad T - 0.35\, N = m_B a$$

$$a = \frac{816.68}{90} - 0.35\frac{882.90}{90} = 5.6407 \text{ m/s}^2$$

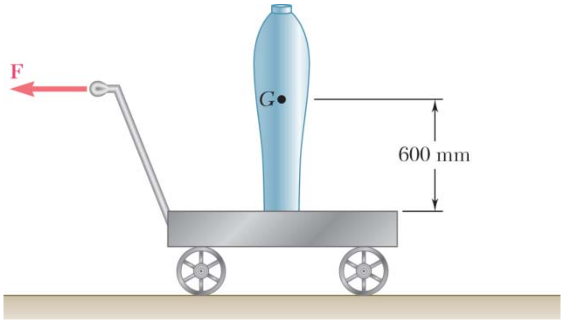
PROBLEM 16.11 (Continued)

For the cylinder: $\overset{+}{\downarrow}\Sigma F = m_C a: \quad W_C - T = m_C a$

$$m_C g - T = m_C a$$

$$m_C = \frac{T}{g-a} = \frac{816.68}{9.81 - 5.6407} = 195.88 \text{ kg}$$

$m_C = 195.9 \text{ kg}$ ◀

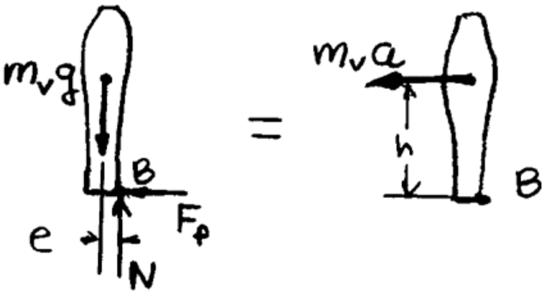


PROBLEM 16.12

A 40-kg vase has a 200-mm-diameter base and is being moved using a 100-kg utility cart as shown. The cart moves freely ($\mu = 0$) on the ground. Knowing the coefficient of static friction between the vase and the cart is $\mu_s = 0.4$, determine the maximum force **F** that can be applied if the vase is not to slide or tip.

SOLUTION

Vase:



$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad N - m_V g = 0$$

$$N = m_V g = (40 \text{ kg})(9.81 \text{ m/s}^2) = 392.4 \text{ N}$$

For impending sliding, $F_f = \mu_S N$

$$F_f = (0.4)(392.4 \text{ N}) = 156.96 \text{ N}$$

$$\xleftarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad F_f = m_V a$$

$$a = \frac{F_f}{m_V} = \frac{156.96 \text{ N}}{40} = 3.924 \text{ m/s}^2$$

This is the limiting value of *a* for sliding.

$$+\circlearrowright M_B = \Sigma (M_B)_{\text{eff}}: \quad m_V g e = m_V a h$$

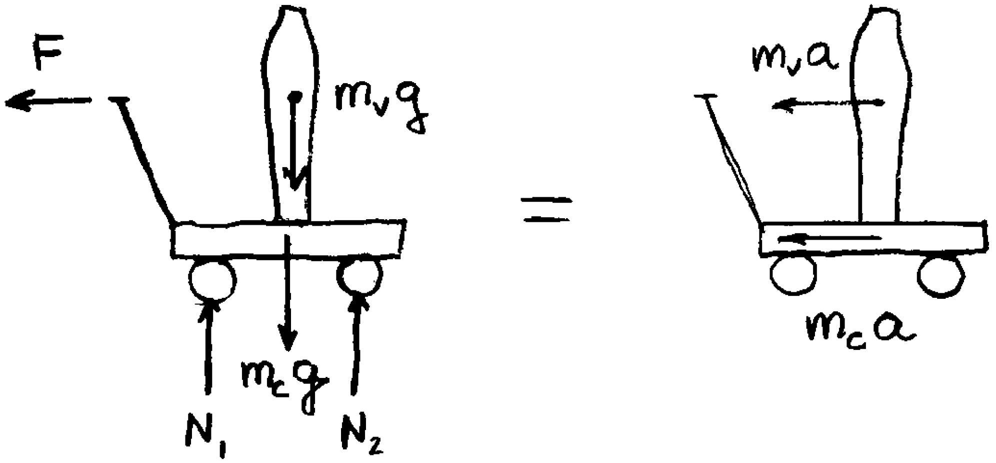
$$a = \frac{e}{h} g = \frac{100 \text{ mm}}{600 \text{ mm}} (9.81 \text{ m/s}^2) = 1.635 \text{ m/s}^2$$

This is the limiting value of *a* for tipping.

The smaller value governs. $a = 1.635 \text{ m/s}^2$

PROBLEM 16.12 (Continued)

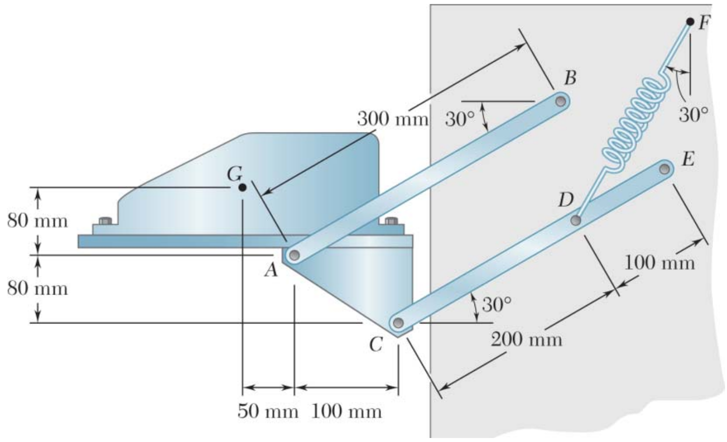
Cart and vase:



$$\overset{+}{\leftarrow}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad F = m_C a + m_V a$$

$$F = (100\text{ kg})(1.635\text{ m/s}^2) + (40\text{ kg})(1.635\text{ m/s}^2)$$

$$F = 229\text{ N} \blacktriangleleft$$



PROBLEM 16.13

The retractable shelf shown is supported by two identical linkage-and-spring systems; only one of the systems is shown. A 20-kg machine is placed on the shelf so that half of its weight is supported by the system shown. If the springs are removed and the system is released from rest, determine (*a*) the acceleration of the machine, (*b*) the tension in link *AB*. Neglect the weight of the shelf and links.

SOLUTION

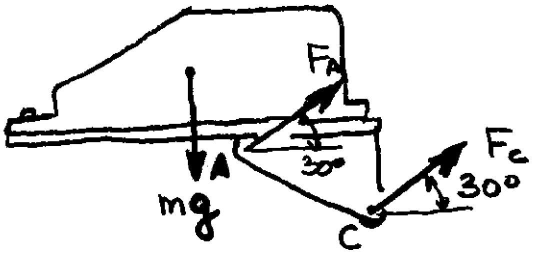
The links *AB* and *CE* keep the line *AC* on the shelf parallel to the fixed line *BE*. Thus the shelf moves in curvilinear translation.

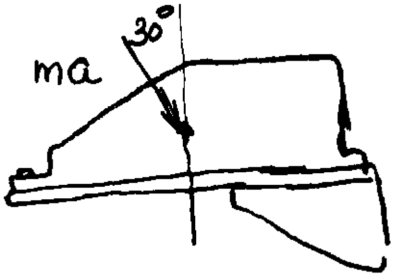
$$\alpha = 0 \qquad \text{and} \qquad \mathbf{a}_G \text{ is perpendicular to } \overrightarrow{AB}$$

Since link *AB* is a massless two-force member, the force at *A* is along link *AB*.

Since link *CDE* is massless and the spring *DF* is removed, the force at *C* is directed along the link *CDE*.

Consider the kinetics of the shelf.





Force perpendicular to link *AB*. $\measuredangle$ 30°

$$mg\cos 30° = m\bar{a}$$

$$\bar{a} = g\cos 30° = (9.81 \text{ m/s}) \cos 30° = 8.4957 \text{ m/s}^2$$

(*a*) *Acceleration of the machine*: $\bar{\mathbf{a}} = 8.50 \text{ m/s}^2 \nwarrow 60°$ ◀

$$\curvearrowleft + \ \Sigma M_C = \Sigma (M_C)_{\text{eff}}: \quad (F_A\cos 30°)(0.080) + (F_A\sin 30°)(0.100) - mg(0.150)$$

$$= (ma\sin 30°)(0.160) + (ma\cos 30°)(0.150)$$

$$0.11928F_A - 0.150\,mg = -0.04990\,mg\cos 30°$$

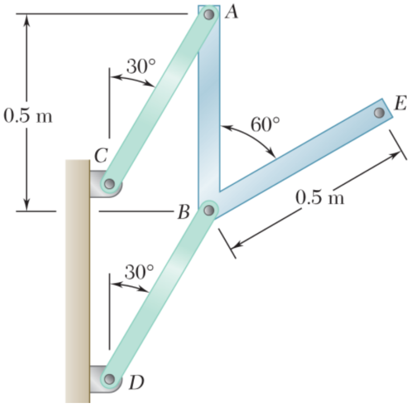
PROBLEM 16.13 (Continued)

(*b*) *Tension in link AB.* $F_A = 89525\, mg$

Taking *mg* to be half the weight of the machine,

$$mg = \frac{1}{2}(20 \text{ kg})(9.81 \text{ m/s}^2) = 98.1 \text{ N}$$

$$F_A = (0.89522)(98.1 \text{ N}) \qquad F = 87.8 \text{ N} \blacktriangleleft$$


A
30°
0.5 m
C
60°
E
0.5 m
B
30°
D

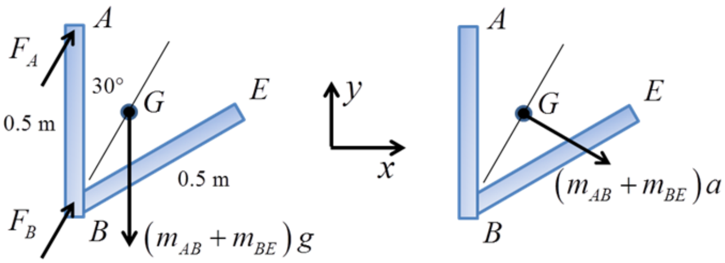
PROBLEM 16.14

Bars *AB* and *BE*, each of mass 4 kg, are welded together and are pin-connected to two links *AC* and *BD*. Knowing that the assembly is released from rest in the position shown and neglecting the masses of the links, determine (*a*) the acceleration of the assembly, (*b*) the forces in the links.

SOLUTION

Given: $m_{AB} = m_{BE} = 14\text{ kg}$

Free Body Diagram:


F_A
A
30°
G
E
0.5 m
0.5 m
F_B
B
$(m_{AB} + m_{BE})g$
y
x
A
G
E
$(m_{AB} + m_{BE})a$
B

Kinetics: $\Sigma M_G = 0 \Rightarrow F_A = 0$

$$\searrow^{+}\ \Sigma F_{\perp BG} = 8(9.81)(0.5) = (8\text{ kg})(a)$$

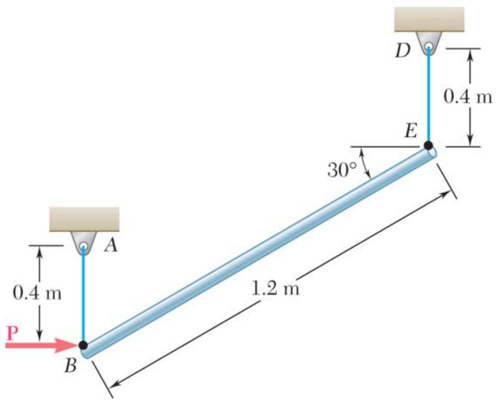
$$a = 4.905\text{ m/s}^2 \qquad \text{or} \qquad a = \frac{g}{2}$$

$$+\nearrow\ \Sigma F_{\parallel BG} = F_B - (78.48\text{ N})(0.866) = 0$$

$$F_B = 67.966\text{ N}$$

(*a*) $a = 4.91\text{ m/s}^2 \searrow 30°$ ◀

(*b*) $F_A = 0,\quad F_B = 68.0\text{ N compression}$ ◀


D
0.4 m
E
30°
A
0.4 m
1.2 m
P
B

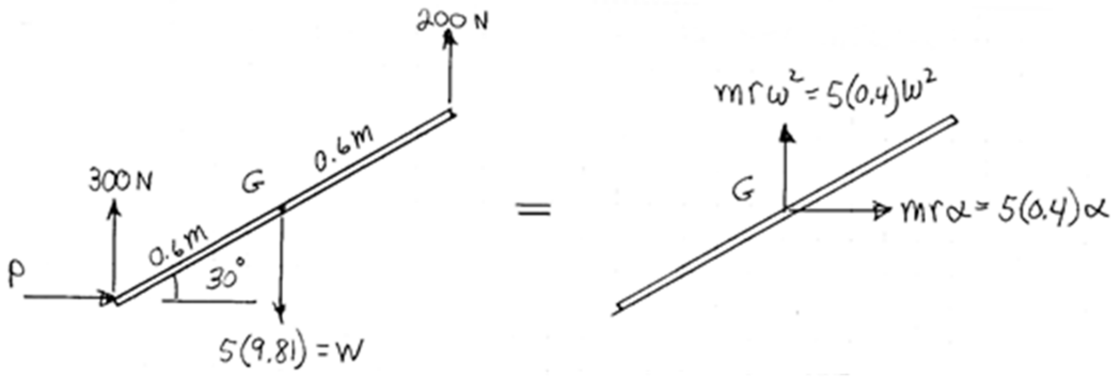
PROBLEM 16.15

At the instant shown the tensions in the vertical ropes *AB* and *DE* are 300 N and 200 N, respectively. Knowing that the mass of the uniform bar *BE* is 5 kg, determine, at this instant, (*a*) the force **P**, (*b*) the magnitude of the angular velocity of each rope, (*c*) the angular acceleration of each rope.

SOLUTION

Given: $T_{AB} = 300\text{ N},\ T_{DE} = 200\text{ N},\ m_{BE} = 5\text{ kg}$

Free Body Diagram:


200 N
300 N
G
0.6 m
0.6 m
30°
P
5(9.81) = W
=
$mr\omega^2 = 5(0.4)\omega^2$
G
$mr\alpha = 5(0.4)\alpha$

Kinetics:

$$\circlearrowleft + \sum M_G = 0 = 200(0.6)(0.866) - 300(0.6)(0.866) + P(0.6)(0.5)$$

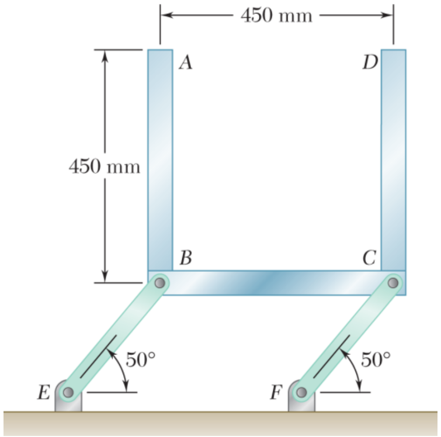
(*a*) **P** = 173.2 N → ◀

$$+\uparrow \sum F_y = 300 - 5(9.81) + 200 = 5(0.4)\omega^2$$

(*b*) $|\omega| = 15.02$ rad/s ◀

$$\xrightarrow{+} \sum F_x = P = 5(0.4)\alpha = 173.2$$

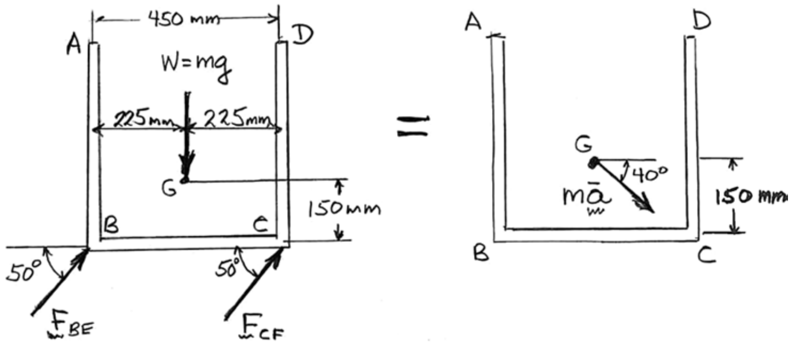
(*c*) $\alpha = 86.6\ \text{rad/s}^2 \circlearrowright$ ◀



PROBLEM 16.16

Three bars, each of mass 3 kg, are welded together and are pin-connected to two links *BE* and *CF*. Neglecting the weight of the links, determine the force in each link immediately after the system is released from rest.

SOLUTION



Mass center of *ABCD* is at *G*

$$\bar{y} = \frac{\Sigma m_i \bar{y}_i}{\Sigma m_i} = \frac{3(0.225) + 3(0.225) + 3(0)}{9} = 0.15 \text{ m}$$

$$W = mg$$

$\searrow 40^\circ \quad \Sigma F = \Sigma F_{\text{eff}}: \quad mg\cos 50^\circ = m\bar{a}$

$$(9.81 \text{ m/s}^2)\cos 50^\circ = \bar{a}$$

$$\bar{\mathbf{a}} = 6.3057 \text{ m/s}^2 \searrow 40^\circ$$

$$+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}:$$

$$(F_{CF}\sin 50^\circ)(0.450 \text{ m}) - (9 \text{ kg})(9.81 \text{ m/s}^2)(0.225 \text{ m}) = -m\bar{a}\sin 40^\circ(0.225 \text{ m}) - m\bar{a}\cos 40^\circ(0.150 \text{ m})$$

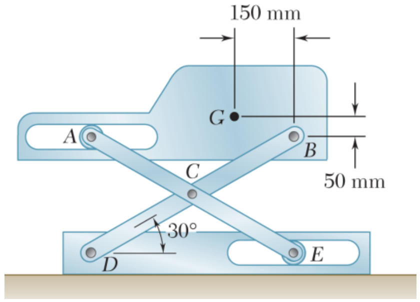
$$0.34472\, F_{CF} - 19.8653 = -m\bar{a}(0.14463 + 0.11491)$$

$$0.34472\, F_{CF} - 19.8653 = -9(6.3057)(0.25953)$$

$F_{CF} = +14.8998 \text{ N}$ $\qquad F_{CF} = +14.90 \text{ N}$ compression ◀

$\nearrow 50^\circ \quad \Sigma F = \Sigma F_{\text{eff}}: \quad F_{BE} + 14.9 \text{ lb} - (9 \text{ kg})(9.81)\sin 50^\circ = 0$

$F_{BF} = +52.734 \text{ N}$ $\qquad F_{BE} = +52.7 \text{ N}$ compression ◀


150 mm
G
A
B
C
50 mm
30°
D
E

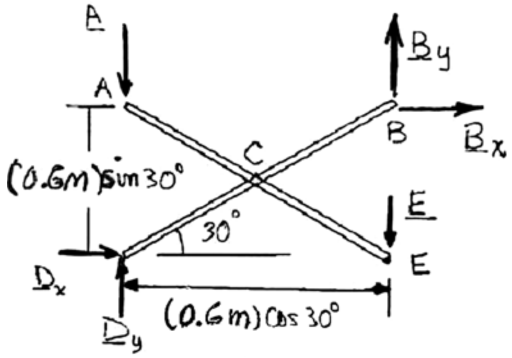
PROBLEM 16.17

Members *ACE* and *DCB* are each 600 mm long and are connected by a pin at *C*. The mass center of the 10-kg member *AB* is located at *G*. Determine (*a*) the acceleration of *AB* immediately after the system has been released from rest in the position shown, (*b*) the corresponding force exerted by roller *A* on member *AB*. Neglect the weight of members *ACE* and *DCB*.

SOLUTION

Analysis of linkage

Since members *ACE* and *DCB* are of negligible mass, their inertial terms may also be neglected and the methods of statics may be applied to their analysis.

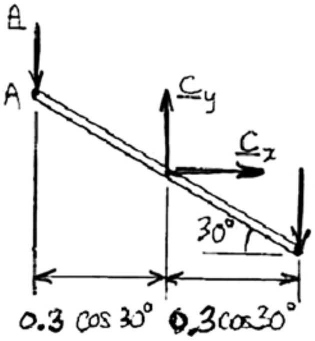

A
A
B_y
B
B_x
C
(0.6m)sin30°
30°
E
E
D_x
D_y
(0.6m)cos30°

Free body: *Entire linkage*:

$$+\circlearrowleft \Sigma M_D = 0:$$

$$(B_y - E)(0.6\cos 30°) - B_x(0.6\sin 30°) = 0$$

$$(B_y - E)\cos 30° - B_x \sin 30° = 0 \qquad (1)$$


A
A
C_y
C_x
E
30°
0.3 cos 30°
0.3cos30°

Free body: *member ACE*

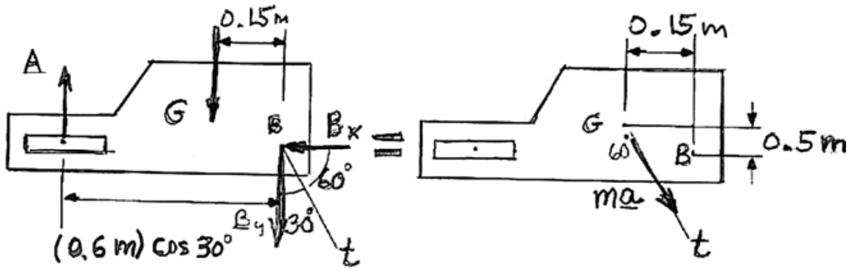
$$+\circlearrowleft \Sigma M_C = 0:$$

$$A(0.3\cos 30°) - E(0.3\cos 30°) = 0; \quad E = A$$

Carrying into Eq. (1):

$$(B_y - A)\cos 30° - B_x \sin 30° = 0 \qquad (2)$$

Equations of Motion for Member AB


A
0.15m
G
B
B_x
60°
B_y
30°
t
(0.6 m) cos 30°
=
0.15m
G
60°
B
0.5m
ma
t

PROBLEM 16.17 (Continued)

(*a*) $+\searrow 60° \quad \Sigma F_t - \Sigma(F_t)_{\text{eff}}:$

$$(B_y - A)\cos 30° - B_x \sin 30° + W\cos 30° = m\bar{a}$$

Recalling equation (2), we have,

$$W\cos 30° = m\bar{a} \qquad \bar{a} = \frac{W}{m}\cos 30° = g\cos 30°$$

$\bar{a} = (9.81\ \text{m/s}^2)\cos 30°$ $\qquad\qquad$ $\bar{\mathbf{a}} = 8.50\ \text{m/s}^2 \searrow 60°$ ◀

(*b*) $+\circlearrowright \Sigma M_B = \Sigma(M_B)_{\text{eff}}:$

$$W(0.15\ \text{m}) - A(0.6\ \text{m})\cos 30° = (m\bar{a}\sin 60°)(0.15\ \text{m}) - (m\bar{a}\cos 60°)(0.05\ \text{m})$$

But, $\quad m\bar{a} = W\cos 30°$

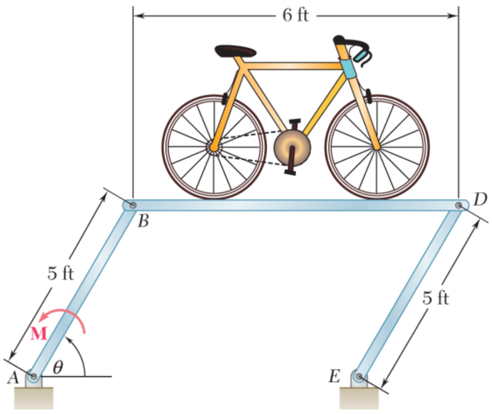
$$0.15\,W - 0.6\,A\cos 30° = W\cos 30°(0.15\sin 60° - 0.05\cos 60°)$$

$$A = W\left(\frac{1}{4\cos 30°} - \frac{1}{4}\sin 60° + \frac{1}{12}\cos 60°\right) = 0.11384\,W = 0.11384\,mg$$

Recalling that $m = 10$ kg so that $mg = 98.1$ N,

$$A = 0.11384(98.1) = 11.168\ \text{N}$$

$\mathbf{A} = 11.17\ \text{N}\uparrow$ ◀



PROBLEM 16.18

A prototype rotating bicycle rack is designed to save space at a train station. The combined weight of platform *BD* and the bicycle is 40 lbs and is centered at 1 ft. above the midpoint of the platform. The motor at *A* causes the support beam *AB* to have an angular velocity of 10 rpm and zero angular acceleration at θ = 30°. At this instant, determine the vertical components of the forces exerted on platform *BD* by the pins at *B* and *D*.

SOLUTION

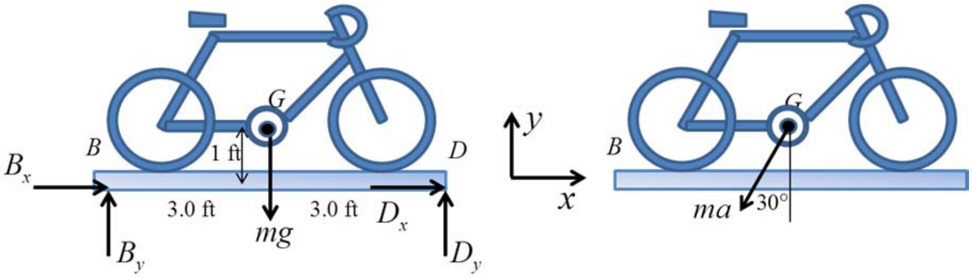
Given:

$$m = \frac{40 \text{ lbs}}{32.2 \text{ ft/s}^2} = 1.2422 \text{ slugs}$$

$$\omega_{AB} = 10\left(\frac{2\pi}{60}\right) = 1.0472 \text{ rad/s}$$

$$\alpha_{AB} = 0, \quad \theta = 30°, \quad r_{B/A} = 5 \text{ ft}$$

Free Body Diagram:



Kinematics:

$$a = r_{B/A}\omega_{AB}^2 \rightarrow a = 5.483 \text{ ft/s}$$

$$\overline{\mathbf{a}} = \text{-}a\sin 30°\mathbf{i} - a\cos 30°\mathbf{j}$$

Kinetics:

$$\sum \mathbf{M}_B = I_G \cancel{\alpha_{AB}} + \mathbf{r}_{G/B} \times m\mathbf{a}$$

$$\left(-3mg + 6D_y\right)\mathbf{k} = m\left(3\mathbf{i} + \mathbf{j}\right) \times \left(-a\sin 30°\mathbf{i} - a\cos 30°\mathbf{j}\right)$$

$$-3mg + 6D_y = \left(-3\cos 30° + \sin 30°\right)ma$$

$$D_y = \frac{m}{6}\left[\left(-3\cos 30° + \sin 30°\right)a + 3g\right]$$

$$\sum F_y = ma_y$$

$$B_y + D_y - mg = -a\cos 30°$$

$$B_y = m(g - a\cos 30°) - \text{D}_y$$

Solve simultaneously:

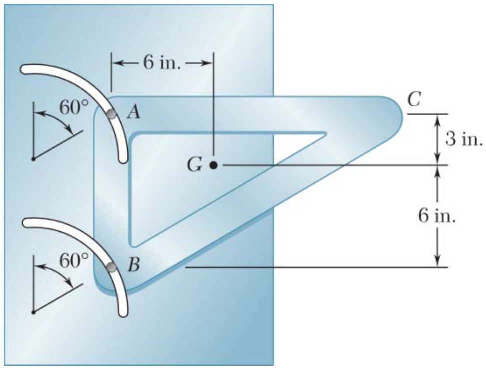
$$D_y = \frac{1.2422}{6}\left[\left(-3\cos 30° + \sin 30°\right)\bullet 5.483 + 3\bullet 32.2\right]$$

$$= 17.618 \text{ lb}$$

$$B_y = 1.2422(32.2 - 5.483\cos 30°) - 17.618$$

$$= 16.483 \text{ lb}$$

$B_y = 16.48$ lb and $D_y = 17.62$ lb ◀

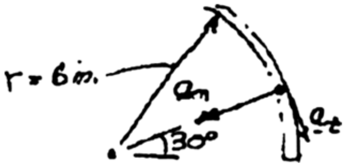

6 in.
C
3 in.
60°
A
G
6 in.
60°
B

PROBLEM 16.19

The triangular weldment *ABC* is guided by two pins that slide freely in parallel curved slots of radius 6 in. cut in a vertical plate. The weldment weighs 16 lb and its mass center is located at Point *G*. Knowing that at the instant shown the velocity of each pin is 30 in./s downward along the slots, determine (*a*) the acceleration of the weldment, (*b*) the reactions at *A* and *B*.

SOLUTION

Slot: $v = 30\text{ in./s}$ ↘

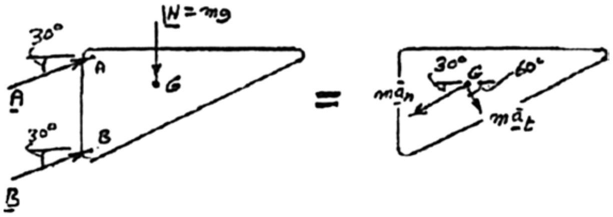

r = 6 in.
a_n
30°
a_t

$$a_n = \frac{v^2}{r} = \frac{(30\text{ in./s})^2}{6\text{ in.}} = 150\text{ in./s}^2$$

$$\mathbf{a}_n = 12.5\text{ ft/s}^2 \nearrow 30°$$

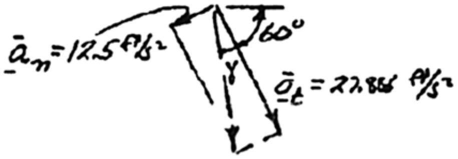
$$\mathbf{a}_t = a_t \searrow 60°$$

Weldment is in translation $\overline{\mathbf{a}}_n = 12.5\text{ ft/s}^2$


W = mg
30°
A
A
G
30°
B
B
=
$m\bar{a}_n$
30°
G
60°
$m\bar{a}_t$

$$\searrow 60°\ \Sigma F = \Sigma F_{\text{eff}}:\quad mg\cos 30° = ma_t$$

$$\overline{\mathbf{a}}_t = 27.886\text{ ft/s}^2 \searrow 60°$$


$\bar{a}_n = 12.5$ ft/s²
60°
γ
$\bar{a}_t = 27.886$ ft/s²

(*a*) Acceleration

$$\beta = \tan^{-1}\frac{\bar{a}_n}{\bar{a}_t} = \tan^{-1}\frac{12.5}{27.886} = 24.14°$$

$$\bar{a}^2 = a_t^2 + a_n^2$$

$$= (27.886)^2 + (12.5)^2$$

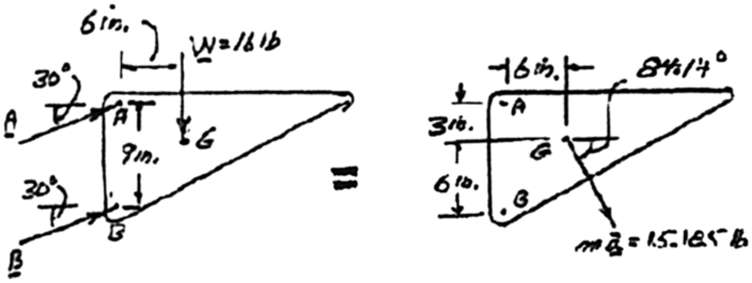
$$\overline{\mathbf{a}} = 30.56\text{ ft/s}^2 \searrow 84.1°$$

$$\overline{\mathbf{a}} = 30.6\text{ ft/s}^2 \searrow 84.1° ◀$$

PROBLEM 16.19 (Continued)

(*b*) Reactions

$$m\bar{a} = \frac{16 \text{ lb}}{32.2 \text{ ft/s}^2}(30.56 \text{ ft/s}^2) = 15.185 \text{ lb}$$



$$\circlearrowleft + \Sigma M_A = \Sigma(M_A)_{\text{eff}}:$$

$$B\cos 30°(9 \text{ in.}) - (16 \text{ lb})(6 \text{ in.}) = (15.185 \text{ lb})(\cos 84.14°)(3 \text{ in.}) - (15.185 \text{ lb})(\sin 84.14°)(6 \text{ in.})$$

$$7.794B - 96 = +4.651 - 90.634$$

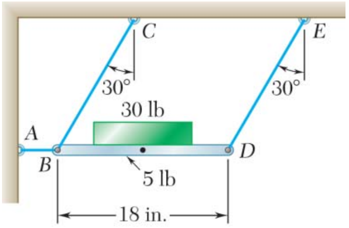
$$B = +1.285 \text{ lb} \qquad \mathbf{B} = 1.285 \text{ lb} \measuredangle 30° \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad A\cos 30° + B\cos 30° = m\bar{a}\cos 84.14°$$

$$A\cos 30° + (1.285 \text{ lb})\cos 30° = (15.185 \text{ lb})\cos 84.14°$$

$$A\cos 30° + 1.113 \text{ lb} = 1.550 \text{ lb}$$

$$A = +0.505 \text{ lb} \qquad \mathbf{A} = 0.505 \text{ lb} \measuredangle 30° \blacktriangleleft$$

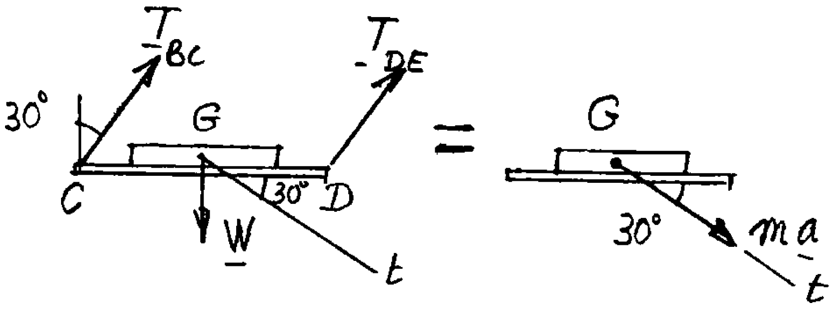

C
E
30°
30°
30 lb
A
B
D
5 lb
18 in.

PROBLEM 16.20

The coefficients of friction between the 30-lb block and the 5-lb platform *BD* are $\mu_s = 0.50$ and $\mu_k = 0.40$. Determine the accelerations of the block and of the platform immediately after wire *AB* has been cut.

SOLUTION

Assume that the block does not slide relative to the platform. Draw the free body diagram of the platform and block.

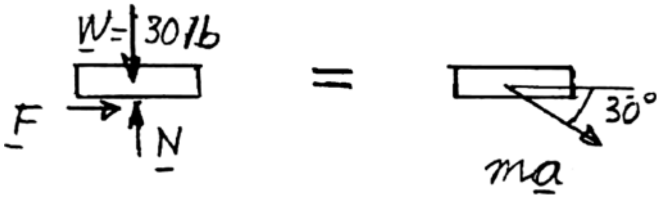

$\mathbf{T}_{BC}$
$\mathbf{T}_{DE}$
30°
G
C
30°
D
$\mathbf{W}$
t
=
G
30°
$m\mathbf{a}$
t

$$+\searrow\ \Sigma F_t = \Sigma(F_t)_{\text{eff}}:\quad W\sin 30° = ma$$

$$a = \frac{W}{m}\sin 30° = \frac{1}{2}g = 16.1\text{ ft/s}^2$$

$$\mathbf{a} = 16.1\text{ ft/s}^2 \searrow 30°$$

Check whether or not the block will slide relative to the platform. Draw the free body diagram of the block alone.


$\mathbf{W}$=30 lb
$\mathbf{F}$
$\mathbf{N}$
=
30°
$m\mathbf{a}$

$$ma = m\left(\frac{1}{2}g\right) = \frac{1}{2}W = 15\text{ lb}$$

$$\xrightarrow{+}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}:\quad F = (15\text{ lb})\cos 30° = 12.99\text{ lb}$$

$$+\!\downarrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}:\quad 30\text{ lb} - N = (15\text{ lb})\sin 30°$$

$$N = 30 - 7.5 = 22.5\text{ lb}$$

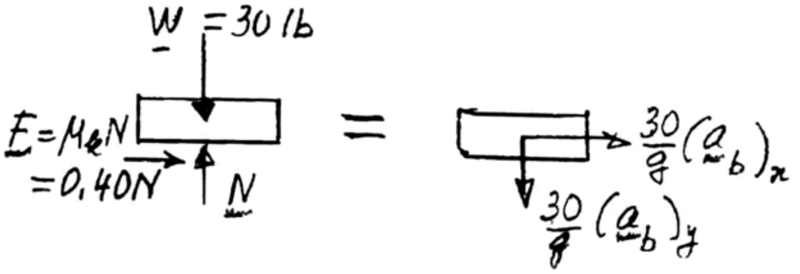
PROBLEM 16.20 (Continued)

Thus, $F_m = \mu_s N = 0.50\ (22.5\text{ lb}) = 11.25\text{ lb}$

Since $F > F_m$, the block will slide

Now assume that the block slides relative to the platform.

Equations of motion for block: (assuming sliding)


W = 30 lb
F = μ_k N
= 0.40N
N
=
30/g (a_b)_x
30/g (a_b)_y

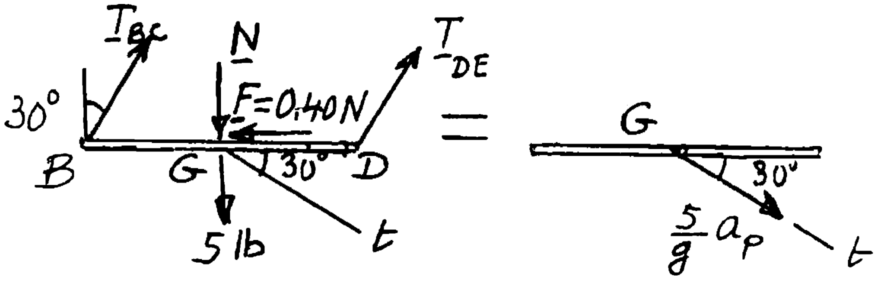
$$+\!\downarrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad 30 - N = \frac{30}{g}(a_b)_y$$

$$(a_b)_y = g\left(1 - \frac{N}{30}\right) \tag{1}$$

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad 0.40\text{ N} = \frac{30}{g}(a_b)_x$$

$$(a_b)_x = g\left(\frac{N}{75}\right) \tag{2}$$

Equations of motion for platform.


T_BC
N
T_DE
30°
F=0.40N
B
G
30°
D
5 lb
t
=
G
30°
5/g a_P
t

$$+\!\searrow \Sigma F_t = \Sigma (F_t)_{\text{eff}}: \quad (N - 5)\sin 30° - (0.40\text{ N})\cos 30° = \frac{5}{g}a_P$$

$$a_p = g(0.5 + 0.030718\text{ N}) \tag{3}$$

If contact is maintained between block and platform, we must have

$$(a_b)_y = (a_p)_y = a_p \sin 30° \tag{4}$$

PROBLEM 16.20 (Continued)

Substituting from (1) and (3) into (4):

$$g\left(1-\frac{N}{30}\right)=g(0.5+0.030718\text{ N})\sin 30^\circ$$

$$(0.015359+0.033333)N=0.75$$

$$N=15.403\text{ lb}$$

Substituting for N in (2) and (1):

$$(a_b)_x=(32.2\text{ ft/s}^2)\,\frac{15.403}{75},\quad (\mathbf{a}_b)_x=6.61\text{ ft/s}^2\rightarrow$$

$$(a_b)_y=(32.2\text{ ft/s}^2)\left(1-\frac{15.403}{30}\right),\quad (\mathbf{a}_b)_y=15.67\text{ ft/s}^2\downarrow$$

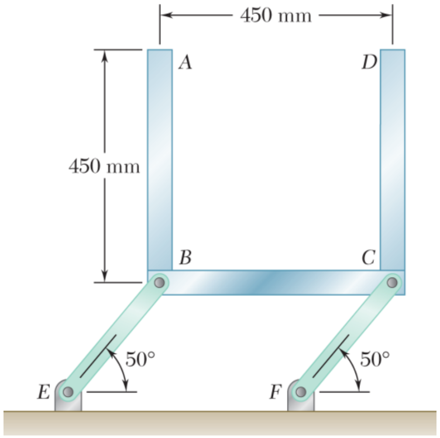
$$\mathbf{a}_b=17.01\text{ ft/s}^2\searrow 67.1^\circ$$ ◀

Substituting for N in (3):

$$a_P=(32.2\text{ ft/s}^2)(0.5+0.030718\times 15.403)=31.335\text{ ft/s}^2$$

$$\mathbf{a}_P=31.3\text{ ft/s}^2\searrow 30^\circ$$ ◀

Note: Since $N>0$, we check that contact between block and platform is maintained.



PROBLEM 16.21

Draw the shear and bending-moment diagrams for the vertical rod *AB* of Problem 16.16.

PROBLEM 16.16 Three bars, each of mass 3 kg, are welded together and are pin-connected to two links *BE* and *CF*. Neglecting the weight of the links, determine the force in each link immediately after the system is released from rest.

SOLUTION

From the solution of Problem 16.16, the acceleration of all points of vertical rod *AB* is

$$\mathbf{a} = 6.3057\ \text{m/s}^2 \searrow 40^\circ$$

or

$$\mathbf{a} = 4.8304\ \text{m/s}^2 \longrightarrow + 4.0532\ \text{m/s}^2 \downarrow$$

Mass of rod *AB*: $m = 3\ \text{kg}$

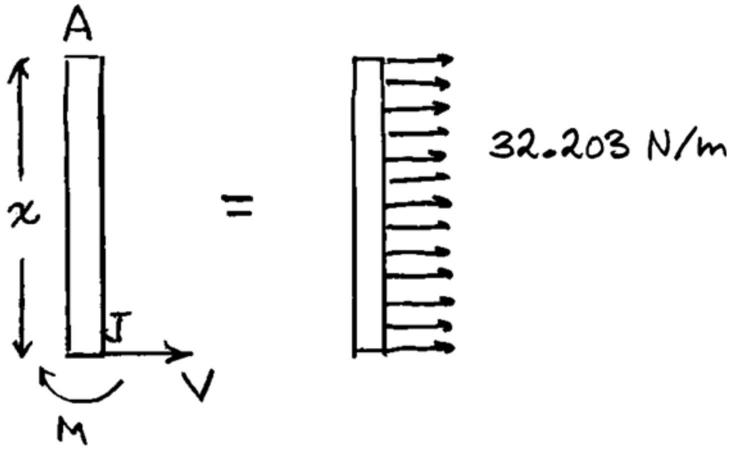
Mass per unit length: $\dfrac{m}{l} = \dfrac{3\ \text{kg}}{0.450\ \text{m}} = 6.6667\ \text{kg/m}$

Effective force per length: $\dfrac{m}{l}\mathbf{a}$

$$(6.6667\ \text{kg/m})(4.8304\ \text{m/s}^2) \longrightarrow + (6.6667\ \text{kg/m})(4.0532\ \text{m/s}^2)$$

$$32.203\ \text{N/m} \longrightarrow + 27.021\ \text{N/m} \downarrow$$

Only the horizontal component contributes to the shear and bending moment. Let *x* be a vertical coordinate (positive down) with its origin at *A*. Draw the free body diagram of the portion of the rod *AB* lying above the section defined by *x*.



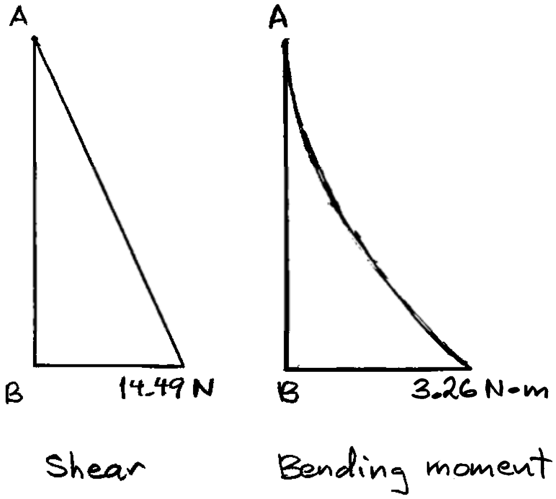
PROBLEM 16.21 (Continued)

$$\xrightarrow{+} \Sigma F = \Sigma (F_j)_{\text{eff}}: \quad V = 32.203x$$

$$\circlearrowleft + \Sigma M_J = \Sigma (M_j)_{\text{eff}}: \quad M = (32.203x)\frac{x}{2}$$

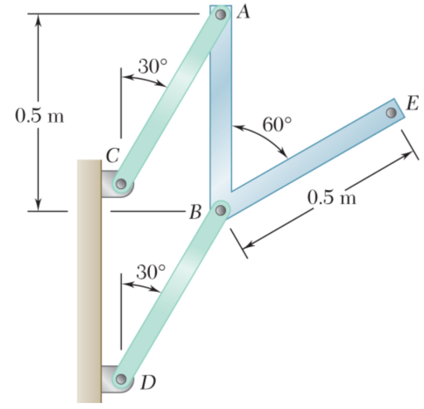
$$= 16.101x^2$$

Shear and bending moment diagrams.


A
B
14.49 N
Shear
A
B
3.26 N-m
Bending moment

$$V_{\max} = (32.203 \text{ N/m})(0.450 \text{ m}) \qquad V_{\max} = 14.49 \text{ N} \blacktriangleleft$$

$$M_{\max} = (16.101 \text{ N/m})(0.450 \text{ m})^2 \qquad M_{\max} = 3.26 \text{ N} \cdot \text{m} \blacktriangleleft$$



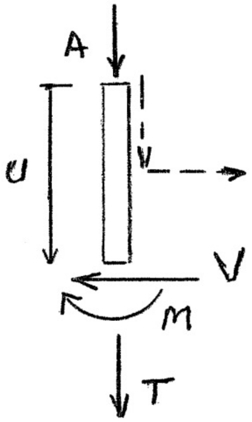
PROBLEM 16.22

Draw the shear and bending-moment diagrams for each of the bars *AB* and *BE* of Prob. 16.14.

PROBLEM 16.14

Bars *AB* and *BE*, each of mass 4 kg, are welded together and are pin-connected to two links *AC* and *BD*. Knowing that the assembly is released from rest in the position shown and neglecting the masses of the links, determine (*a*) the acceleration of the assembly, (*b*) the forces in the links.

SOLUTION



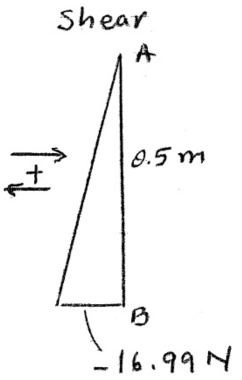
$$ma_n = \left[\left(\frac{4}{0.5}\right)u\left(\frac{g}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\right] = 2\sqrt{3}gu$$

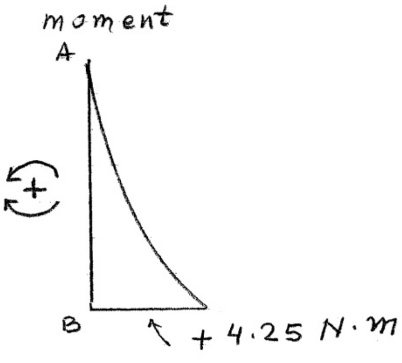
$$\xrightarrow{+}\Sigma F_n = 0;\ -V = ma_n$$

$-V = 33.983u,$ at $u = 0.5\text{ m}\ \ V = -16.99\text{ N}$ $\circlearrowleft + \Sigma M = M = ma_n\left(\frac{u}{2}\right)$

$$M = 33.983\frac{u^2}{2} = 16.991u^2$$

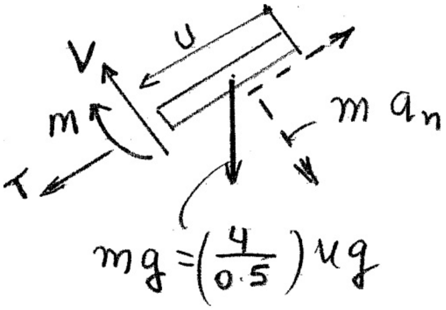
at $u = 0.5\text{ m},\ M = 4.25\text{ N}\cdot\text{m}$





◀

PROBLEM 16.22 (Continued)

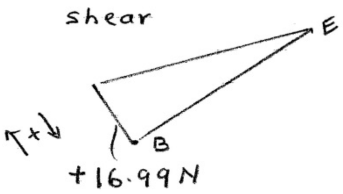

V
u
m
T
$m a_n$
$mg = \left(\frac{4}{0.5}\right)ug$

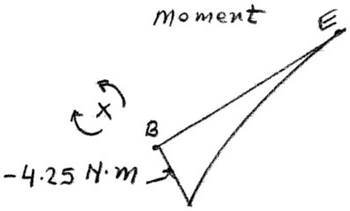
$$ma_n = \left[\left(\frac{4}{0.5}\right)u\right]\left[\frac{g}{2}\left(\frac{\sqrt{3}}{2}\right)\right] = 2\sqrt{3}gu$$

$$+\searrow\ \Sigma F_n = \frac{4ug}{0.5}\left(\frac{\sqrt{3}}{2}\right) - V = \left(2\sqrt{3}g\right)u$$

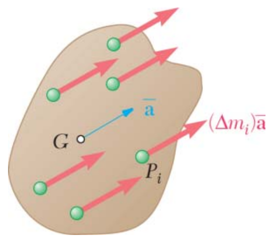
$$V = 2\sqrt{3}gu$$

$$+\circlearrowleft\ \Sigma M = M + \left(8ug\tfrac{\sqrt{3}}{2}\right)\frac{u}{2} = 2\sqrt{3}gu^2/2 \qquad M = -\sqrt{3}gu^2$$


shear
E
B
↑+↓
+16.99 N


moment
E
B
↶+↷
−4.25 N·m

◀

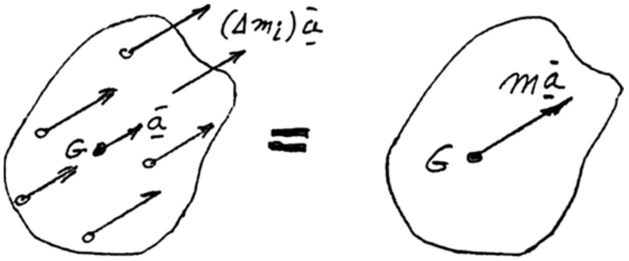


PROBLEM 16.23

For a rigid body in translation, show that the system of the inertial terms consists of vectors $(\Delta m_i)\bar{\mathbf{a}}$ attached to the various particles of the body, where $\bar{\mathbf{a}}$ is the acceleration of the mass center *G* of the body. Further show, by computing their sum and the sum of their moments about *G*, that the inertial terms reduce to a single vector $m\bar{\mathbf{a}}$ attached at *G*.

SOLUTION

Since body is in translation, each particle has same acceleration as *G*, namely $\bar{\mathbf{a}}$. The inertial terms consist of $(\Delta m_i i)\bar{\mathbf{a}}$.



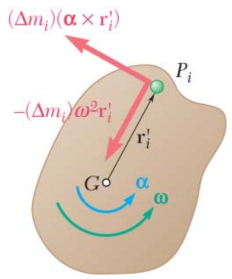
The sum of these vectors is: $\Sigma(\Delta m_i)\bar{\mathbf{a}} = (\Sigma\Delta m_i)\bar{\mathbf{a}}$

or since $\Sigma\Delta m_i = m,$

$$\Sigma(\Delta m_i)\bar{\mathbf{a}} = m\bar{\mathbf{a}}$$

The sum of the moments about *G* is: $\Sigma r_i' \times (\Delta m_i)\bar{\mathbf{a}} = (\Sigma\Delta m_i r_i') \times \bar{\mathbf{a}}$ (1)

But, $\Sigma\Delta m_i r_i' = m\bar{r}' = 0,$ because *G* is the mass center. It follows that the right-hand member of Eq. (1) is zero. Thus, the moment about *G* of $m\bar{a}$ must also be zero, which means that its line of action passes through *G* and that it may be attached at *G*.


$(\Delta m_i)(\boldsymbol{\alpha} \times \mathbf{r}'_i)$
P_i
$-(\Delta m_i)\omega^2\mathbf{r}'_i$
$\mathbf{r}'_i$
G
$\boldsymbol{\alpha}$
$\boldsymbol{\omega}$

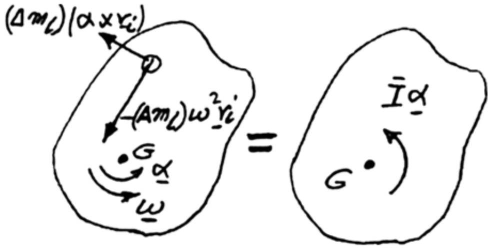
PROBLEM 16.24

For a rigid body in centroidal rotation, show that the system of the inertial terms consists of vectors $-(\Delta m_i)\boldsymbol{\omega}^2\mathbf{r}'_i$ and $(\Delta m_i)(\boldsymbol{\alpha} \times \mathbf{r}'_i)$ attached to the various particles P_i of the body, where $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are the angular velocity and angular acceleration of the body, and where $\mathbf{r}'_i$ denotes the position vector of the particle P_i relative to the mass center G of the body. Further show, by computing their sum and the sum of their moments about G, that the inertial terms reduce to a couple $\bar{I}\boldsymbol{\alpha}$.

SOLUTION

For centroidal rotation: $\mathbf{a}_i = (\mathbf{a}_i)_t + (\mathbf{a}_i)_n = \boldsymbol{\alpha} \times \mathbf{r}'_i - \omega^2 \mathbf{r}'_i$

Inertial terms are: $(\Delta m_i)a_i = (\Delta m_i)(\boldsymbol{\alpha} \times \mathbf{r}_i) - (\Delta m_i)\omega^2 r'_i(\Delta m_i)(\alpha \times \mathbf{r}_i)$


$(\Delta m_i)(\alpha \times r_i)$
$-(\Delta m_i)\omega^2 r_i$
G
α
ω
=
$\bar{I}\alpha$
G

$$\Sigma(\Delta m_i)\mathbf{a}_i = \Sigma(\Delta m_i)(\boldsymbol{\alpha} \times \mathbf{r}'_i) - \Sigma(\Delta m_i)\omega^2 r'_i$$

$$= \alpha \times \Sigma(\Delta m_i)\mathbf{r}'_i - \omega^2 \Sigma(\Delta m_i) r'_i$$

Since G is the mass center, $\Sigma(\Delta m_i) r'_i = 0$

inertial terms reduce to a couple, Summing moments about G,

$$\Sigma(\mathbf{r}'_1 \times \Delta m_i \mathbf{a}_i) = \Sigma[\mathbf{r}'_i \times (\Delta m_i)(\boldsymbol{\alpha} \times \mathbf{r}'_i)] - \Sigma \mathbf{r}'_i \times (\Delta m_i)\omega^2 r'_i$$

But, $\mathbf{r}'_i \times (\Delta m_i)\omega^2 \mathbf{r}'_i = \omega^2 (\Delta m_i)(\mathbf{r}'_i \times \mathbf{r}'_i) = 0$

and, $\mathbf{r}'_i \times (\Delta m_i)(\boldsymbol{\alpha} \times \mathbf{r}'_i) = (\Delta m_i) r_1'^2 \boldsymbol{\alpha}$

Thus, $\Sigma(\mathbf{r}'_i \times \Delta m_i \mathbf{a}_i) = \Sigma(\Delta m_i) r_1^2 \alpha = \left[\Sigma(\Delta m_i) r_i'^2\right] \alpha$

Since $\Sigma(\Delta m) r_i'^2 = \bar{I}$, the moment of the couple is $\bar{I}\boldsymbol{\alpha}$.

PROBLEM 16.25

It takes 10 min for a 2.4-Mg flywheel to coast to rest from an angular velocity of 300 rpm. Knowing that the radius of gyration of the flywheel is 1 m, determine the average magnitude of the couple due to kinetic friction in the bearing.

SOLUTION

$$\bar{I} = m\bar{k}^2 = 2400 \text{ kg}(1 \text{ m})^2 = 2400 \text{ kg}\cdot\text{m}^2$$

$$\omega_0 = 300 \text{ rpm}\left(\frac{2\pi}{60}\right) = 10\pi \text{ rad/s}$$

$$\omega = \omega_0 + \alpha t;$$

$$0 = 10\pi \text{ rad/s} + \alpha\left(600 \text{ s}\right)$$

$$\alpha = -0.05236 \text{ rad/s}^2$$

$$M = \bar{I}\alpha = \left(2400 \text{ kg}\cdot\text{m}^2\right)\left(-0.05236 \text{ rad/s}^2\right) = 125.66 \text{ N}\cdot\text{m}$$

or $M = 125.7 \text{ N}\cdot\text{m}$ ◀

PROBLEM 16.26

The rotor of an electric motor has an angular velocity of 3600 rpm when the load and power are cut off. The 120-lb rotor, which has a centroidal radius of gyration of 9 in., then coasts to rest. Knowing that kinetic friction results in a couple of magnitude 2.5 lb·ft exerted on the rotor, determine the number of revolutions that the rotor executes before coming to rest.

SOLUTION

$$\bar{I} = m\bar{k}^2 = \left(\frac{120 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{9 \text{ in.}}{12}\right)^2 = 2.0963 \text{ lb ft}\cdot\text{s}^2$$

$$M = \bar{I}\alpha; \quad 2.5 \text{ lb ft} = \left(2.0963 \text{ lb ft}\cdot\text{s}^2\right)\alpha$$

$$\alpha = 1.1926 \text{ rad/s}^2 \ (\text{deceleration})$$

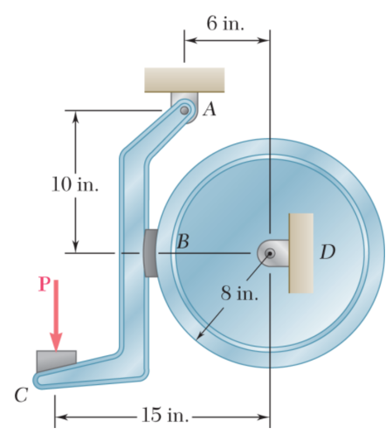
$$\omega_0 = 3600 \text{ rpm}\left(\frac{2\pi}{60}\right) = 120\pi \text{ rad/s}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta;$$

$$0 = \left(120\pi\frac{\text{rad}}{\text{s}}\right)^2 + 2\left(-1.1926 \text{ rad/s}^2\right)\theta$$

$$\theta = 59{,}585.07 \text{ rad}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 9{,}483.26 \text{ rev}$$

or $\theta = 9480$ rev ◀



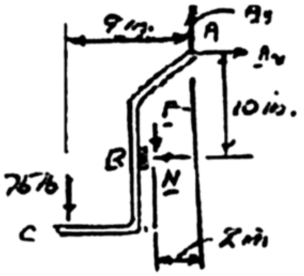
PROBLEM 16.27

The 8-in.-radius brake drum is attached to a larger flywheel that is not shown. The total mass moment of inertia of the drum and the flywheel is $14 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ and the coefficient of kinetic friction between the drum and the brake shoe is 0.35. Knowing that the angular velocity of the flywheel is 360 rpm counterclockwise when a force **P** of magnitude 75 lb is applied to the pedal *C*, determine the number of revolutions executed by the flywheel before it comes to rest.

SOLUTION

Lever *ABC*: Static equilibrium (friction force ↓)

$$F = \mu_k N = 0.35N$$



$$+\circlearrowleft \Sigma M_A = 0: \quad N(10 \text{ in.}) - F(2 \text{ in.}) - (75 \text{ lb})(9 \text{ in.}) = 0$$

$$10N - 2(0.35N) - 675 = 0$$

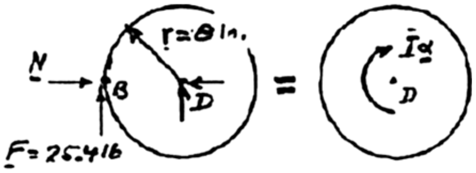
$$N = 72.58 \text{ lb}$$

$$F = \mu_k N$$

$$= 0.35(72.58 \text{ lb})$$

$$= 25.40 \text{ lb}$$

Drum:



$$r = 8 \text{ in.} = \frac{2}{3} \text{ ft}$$

$$\omega_0 = 360 \text{ rpm}\left(\frac{2\pi}{60}\right)$$

$$\omega_0 = 12\pi \text{ rad/s}$$

$$+\circlearrowleft \Sigma M_D = \Sigma (M_D)_{\text{eff}}: \quad Fr = \bar{I}\alpha$$

$$(25.4 \text{ lb})\left(\frac{2}{3} \text{ ft}\right) = (14 \text{ lb} \cdot \text{ft} \cdot \text{s}^2)\alpha$$

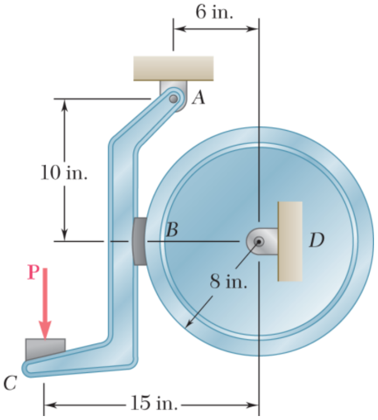
$$\alpha = 1.2097 \text{ rad/s}^2 \text{ (deceleration)}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta: \quad 0 = (12\pi \text{ rad/s})^2 + 2(-1.2097 \text{ rad/s}^2)\theta$$

$$\theta = 587.4 \text{ rad}$$

$$\theta = 587.4 \text{ rad}\left(\frac{1}{2\pi}\right) = 93.49 \text{ rev}$$

$\theta = 93.5$ rev ◀



PROBLEM 16.28

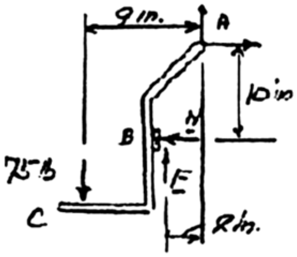
Solve Problem 16.27, assuming that the initial angular velocity of the flywheel is 360 rpm clockwise.

PROBLEM 16.27 The 8-in.-radius brake drum is attached to a larger flywheel that is not shown. The total mass moment of inertia of the drum and the flywheel is 14 lb · ft · s^2 and the coefficient of kinetic friction between the drum and the brake shoe is 0.35. Knowing that the angular velocity of the flywheel is 360 rpm counterclockwise when a force **P** of magnitude 75 lb is applied to the pedal *C*, determine the number of revolutions executed by the flywheel before it comes to rest.

SOLUTION

Lever *ABC*: Static equilibrium (friction force ↑)

$$F = \mu_k N = 0.35N$$



$$+\circlearrowright \Sigma M_A = 0: \quad N(10 \text{ in.}) + F(2 \text{ in.}) - (75 \text{ lb})(9 \text{ in.}) = 0$$

$$10N + 2(0.35N) - 675 = 0$$

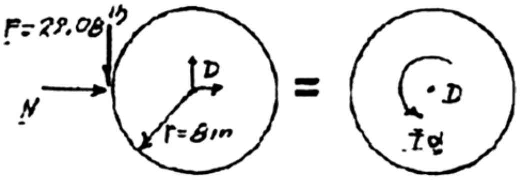
$$N = 63.08 \text{ lb}$$

$$F = \mu_k N$$

$$= 0.35(63.08 \text{ lb})$$

$$= 22.08 \text{ lb}$$

Drum:



$$r = 8 \text{ in.} = \frac{2}{3} \text{ ft}$$

$$\omega_0 = 360 \text{ rpm}\left(\frac{2\pi}{60}\right)$$

$$\omega_0 = 12\pi \text{ rad/s}$$

$$+\circlearrowright \Sigma M_D = \Sigma (M_D)_{\text{eff}} = 0: \quad Fr = \bar{I}\alpha$$

$$(22.08 \text{ lb})\left(\frac{2}{3} \text{ ft}\right) = (14 \text{ lb} \cdot \text{ft} \cdot \text{s}^2)\alpha$$

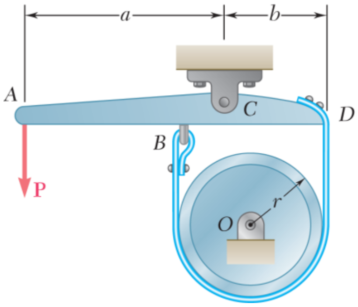
$$\alpha = 1.0515 \text{ rad/s}^2 \quad \text{(deceleration)}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta: \quad 0 = (12\pi \text{ rad/s})^2 + 2(-1.0515 \text{ rad/s}^2)\theta$$

$$\theta = 675.8 \text{ rad}$$

$$\theta = 675.8 \text{ rad} = 107.56 \text{ rev}$$

$$\theta = 107.6 \text{ rev} \blacktriangleleft$$

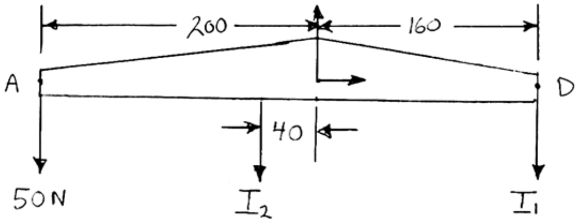


PROBLEM 16.29

The 100-mm-radius brake drum is attached to a flywheel which is not shown. The drum and flywheel together have a mass of 300 kg and a radius of gyration of 600 mm. The coefficient of kinetic friction between the brake band and the drum is 0.30. Knowing that a force **P** of magnitude 50 N is applied at *A* when the angular velocity is 180 rpm counterclockwise, determine the time required to stop the flywheel when *a* = 200 mm and *b* = 160 mm.

SOLUTION

Equilibrium of lever *AD*



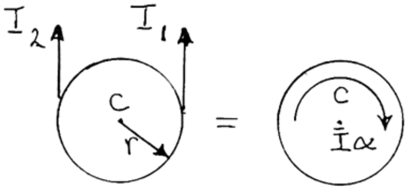
(Dimensions in mm)

$$+\circlearrowleft \Sigma M_C = 0;$$

$$(50\text{ N})(200) + T_2(40) - T_1(160) = 0$$

$$4T_1 - T_2 = 250\text{ N} \qquad (1)$$

Equation of Motion for flywheel and drum



$$\bar{I} = m\bar{k}^2$$

$$= (300\text{ kg})(0.600\text{ m})^2$$

$$= 108\text{ kg}\cdot\text{m}^2$$

$$r = 0.100\text{ m}$$

$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad T_2 r - T_1 r = \bar{I}\alpha$$

$$(T_2 - T_1)(0.100\text{ m}) = 108\,\alpha$$

$$\alpha = (925.93 \times 10^{-6})(T_2 - T_1) \qquad (2)$$

Belt Friction

Using μ_k instead of μ_s since the brake band is slipping:

$$\frac{T_2}{T_1} = e^{\mu_k \beta} \quad \text{or} \quad T_2 = T_1 e^{\mu_k \beta} \qquad (3)$$

Making $\mu_k = 0.30$ and $\beta = 180° = \pi$ rad in (3):

$$T_2 = T_1 e^{0.30\pi} \qquad T_2 = 2.5663 T_1 \qquad (4)$$

PROBLEM 16.29 (Continued)

Substituting for T_2 from (4) into (1):

$$4T_1 - 2.5663T_1 = 250 \text{ N} \qquad T_1 = 174.38 \text{ N}$$

From (1):

$$T_2 = 4(174.38) - 250 \qquad T_2 = 447.51 \text{ N}$$

Substituting for T_1 and T_2 into (2):

$$\alpha = (925.93 \times 10^{-6})(447.51 - 174.38), \quad \boldsymbol{\alpha} = 0.2529 \text{ rad/s}^2 \circlearrowright$$

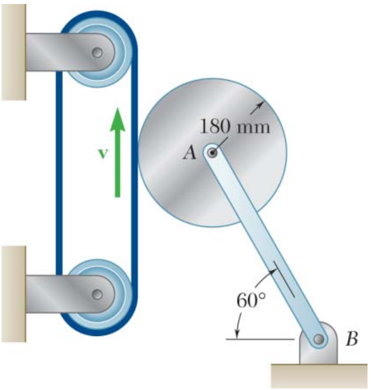
Kinematics

$$\boldsymbol{\omega}_0 = 180 \text{ rpm} \circlearrowleft \qquad \omega_0 = +18.850 \text{ rad/s}$$

$$\boldsymbol{\alpha} = 0.2529 \text{ rad/s}^2 \circlearrowright \qquad \alpha = -0.2529 \text{ rad/s}^2$$

$$\omega = \omega_0 + \alpha t: \quad 0 = 18.850 - 0.2529t$$

$t = 74.5$ s ◀

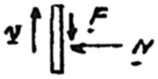


PROBLEM 16.30

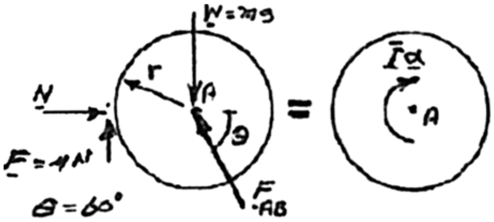
The 180-mm radius disk is at rest when it is placed in contact with a belt moving at a constant speed. Neglecting the weight of the link *AB* and knowing that the coefficient of kinetic friction between the disk and the belt is 0.40, determine the angular acceleration of the disk while slipping occurs.

SOLUTION

Belt: $F = \mu_k N$



Disk:



$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad N - F_{AB}\cos\theta = 0$$

$$F_{AB}\cos\theta = N \qquad (1)$$

$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad \mu_k N + F_{AB}\sin\theta - mg = 0$$

$$F_{AB}\sin\theta = mg - \mu_k N \qquad (2)$$

$$\frac{\text{Eq. (2)}}{\text{Eq. (1)}}: \quad \tan\theta = \frac{mg - \mu_k N}{N}$$

$$N\tan\theta = mg - \mu_k N$$

$$N = \frac{mg}{\tan\theta + \mu_k}$$

$$F = \mu_k N = \frac{mg\mu_k}{\tan\theta + \mu_k}$$

$$\overset{+}{\circlearrowright}\Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad Fr = \bar{I}\alpha$$

$$\alpha = \frac{r}{I}F$$

$$= \frac{r}{\frac{1}{2}mr^2} \cdot \frac{mg\mu_k}{\tan\theta + \mu_k}$$

$$= \frac{2g}{r} \cdot \frac{\mu_k}{\tan\theta + \mu_k}$$

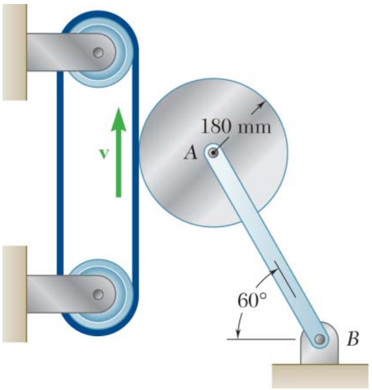
Data:

$$r = 0.18 \text{ m}$$

$$\theta = 60°$$

$$\mu_k = 0.40$$

$$\alpha = \frac{2(9.81 \text{ m/s}^2)}{0.18 \text{ m}} \cdot \frac{0.40}{\tan 60° + 0.40} \qquad \boldsymbol{\alpha} = 20.4 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$



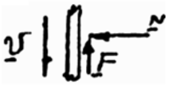
PROBLEM 16.31

Solve Problem 16.30, assuming that the direction of motion of the belt is reversed.

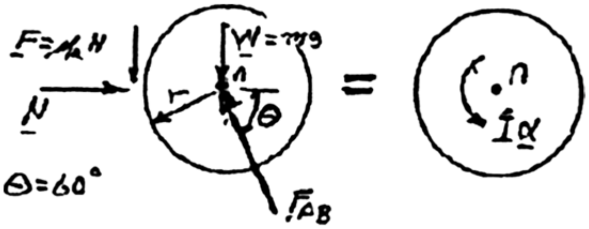
PROBLEM 16.30 The 180-mm disk is at rest when it is placed in contact with a belt moving at a constant speed. Neglecting the weight of the link *AB* and knowing that the coefficient of kinetic friction between the disk and the belt is 0.40, determine the angular acceleration of the disk while slipping occurs.

SOLUTION

Belt: $F = \mu_k N$



Disk:



$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad N - F_{AB}\cos\theta; \quad F_{AB}\cos\theta = N \qquad (1)$$

$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad F_{AB}\sin\theta - mg - \mu_k N = 0$$

$$F_{AB}\sin\theta = mg + \mu_k N \qquad (2)$$

$$\frac{\text{Eq. (2)}}{\text{Eq. (1)}}: \quad \tan\theta = \frac{mg + \mu_k N}{N}$$

$$N\tan\theta = mg + \mu_k N; \quad N = \frac{mg}{\tan\theta - \mu_k}$$

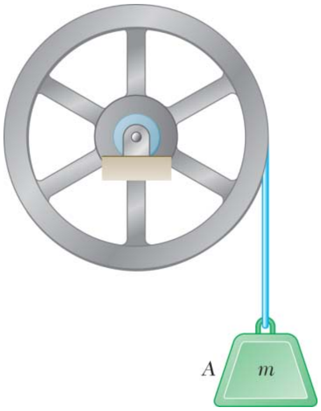
$$+\circlearrowright \Sigma M_A = \Sigma (M_A)_{\text{eff}}: \quad Fr = \bar{I}\alpha$$

$$\alpha = \frac{Fr}{r} = \frac{\mu_k N r}{I} = \frac{\mu_k r}{\frac{1}{2}mr^2} \cdot \frac{mg}{\tan\theta - \mu_k} = \frac{2g}{r} \cdot \frac{\mu_k}{\tan\theta - \mu_k}$$

Data: $r = 0.18\ \text{m}, \quad \theta = 60°, \quad \mu_k = 0.40$

$$\alpha = \frac{2(9.81\ \text{m/s}^2)}{0.18\ \text{m}} \cdot \frac{0.40}{\tan 60° - 0.40}$$

$\boldsymbol{\alpha} = 32.7\ \text{rad/s}^2$ ↺ ◀

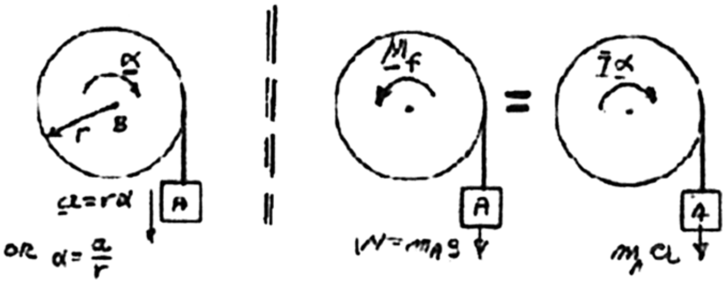


PROBLEM 16.32

In order to determine the mass moment of inertia of a flywheel of radius 600 mm, a 12-kg block is attached to a wire that is wrapped around the flywheel. The block is released and is observed to fall 3 m in 4.6 s. To eliminate bearing friction from the computation, a second block of mass 24 kg is used and is observed to fall 3 m in 3.1 s. Assuming that the moment of the couple due to friction remains constant, determine the mass moment of inertia of the flywheel.

SOLUTION

Kinematics: Kinetics:



$$+\circlearrowright \Sigma M_B = \Sigma (M_B)_{\text{eff}}: \quad (m_A y)r - M_f = \bar{I}\alpha + (m_A a)r$$

$$m_A gr - M_f = \bar{I}\frac{a}{r} + m_A ar \qquad (1)$$

Case 1:

$$y = 3 \text{ m}$$

$$t = 4.6 \text{ s}$$

$$y = \frac{1}{2}at^2$$

$$3 \text{ m} = \frac{1}{2}a(4.6 \text{ s})^2$$

$$a = 0.2836 \text{ m/s}^2$$

$$m_A = 12 \text{ kg}$$

Substitute into Eq. (1)

$$(12 \text{ kg})(9.81 \text{ m/s}^2)(0.6 \text{ m}) - M_f = \bar{I}\left(\frac{0.2836 \text{ m/s}^2}{0.6 \text{ m}}\right) + (12 \text{ kg})(0.2836 \text{ m/s}^2)(0.6 \text{ m})$$

$$70.632 - M_f = \bar{I}(0.4727) + 2.0419 \qquad (2)$$

PROBLEM 16.32 (Continued)

Case 2:

$$y = 3 \text{ m}$$
$$t = 3.1 \text{ s}$$
$$y = \frac{1}{2}at^2$$
$$3 \text{ m} = \frac{1}{2}a(3.1 \text{ s})^2$$
$$a = 0.6243 \text{ m/s}^2$$
$$m_A = 24 \text{ kg}$$

Substitute into Eq. (1):

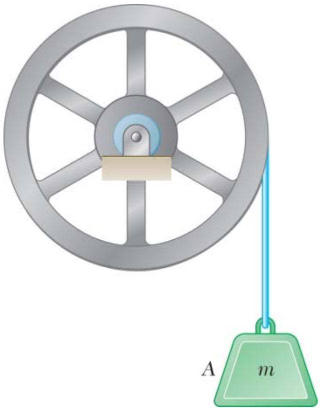
$$(24 \text{ kg})(9.81 \text{ m/s}^2)(0.6 \text{ m}) - M_f = \overline{I}\left(\frac{0.6243 \text{ m/s}^2}{0.6 \text{ m}}\right) + (24 \text{ kg})(0.6243 \text{ m/s}^2)(0.6 \text{ m})$$

$$141.264 - M_f = \overline{I}(1.0406) + 8.9899 \qquad (3)$$

Subtract Eq. (1) from Eq. (2) to eliminate M_f

$$70.632 = \overline{I}(1.0406 - 0.4727) + 6.948$$
$$63.684 = \overline{I}(0.5679)$$
$$\overline{I} = 112.14 \text{ kg}\cdot\text{m}^2$$

$\overline{I} = 112.1 \text{ kg}\cdot\text{m}^2$ ◀

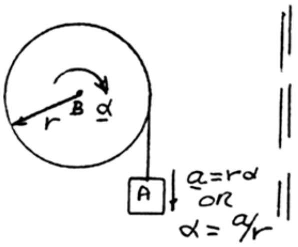


PROBLEM 16.33

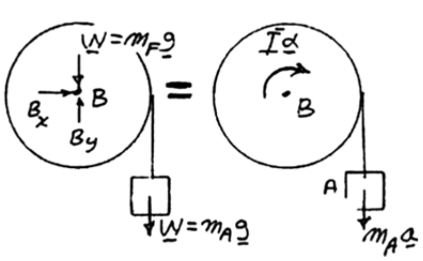
The flywheel shown has a radius of 20 in. a weight of 250 lbs, and a radius of gyration of 15 in. A 30-lb block *A* is attached to a wire that is wrapped around the flywheel, and the system is released from rest. Neglecting the effect of friction, determine (*a*) the acceleration of block *A*, (*b*) the speed of block *A* after it has moved 5 ft.

SOLUTION

Kinematics:



Kinetics:



$$+\circlearrowright \Sigma M_B = \Sigma (M_B)_{\text{eff}}: \quad (m_A g)r = \bar{I}\alpha + (m_A a)r$$

$$m_A g r = m_F k^2 \left(\frac{a}{r}\right) + m_A a r$$

$$a = \frac{W_A}{m_A + m_F \left(\frac{k}{r}\right)^2}$$

(*a*) Acceleration of *A*

$$a = \frac{(30 \text{ lb})}{\left(\frac{30 \text{ lb}}{32.2 \text{ ft/s}^2}\right) + \left(\frac{250 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{15 \text{ in.}}{20 \text{ in.}}\right)^2} = 5.6615 \text{ ft/s}^2$$

or $\mathbf{a}_A = 5.66 \text{ ft/s}^2 \downarrow$ ◀

(*b*) Velocity of *A*

$$v_A^2 = (v_A)_0^2 + 2a_A s$$

For $s = 5$ ft

$$v_A^2 = 0 + 2(5.6615 \text{ in./s}^2)(5 \text{ ft})$$

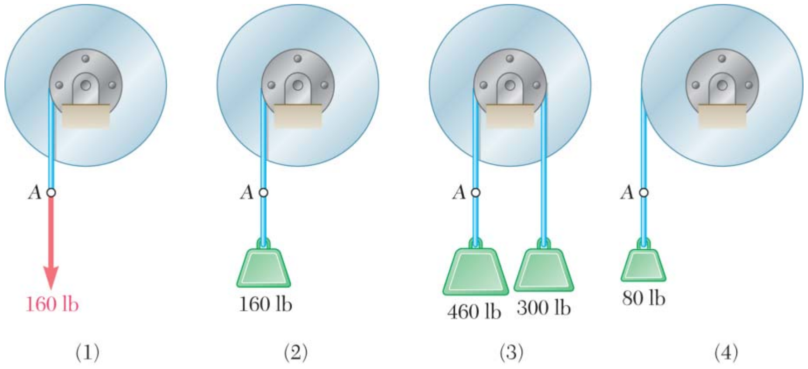
$$= 56.6154 \text{ ft}^2/\text{s}^2$$

$$v_A = 7.5243 \text{ ft/s}$$

or $v_A = 7.52$ ft/s ◀

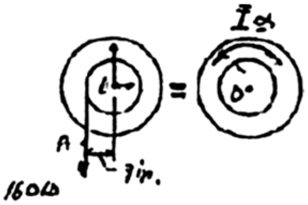
PROBLEM 16.34

Each of the double pulleys shown has a mass moment of inertia of 15 $\text{lb}\cdot\text{ft}\cdot\text{s}^2$ and is initially at rest. The outside radius is 18 in., and the inner radius is 9 in. Determine (*a*) the angular acceleration of each pulley, (*b*) the angular velocity of each pulley after Point *A* on the cord has moved 10 ft.



SOLUTION

Case 1:



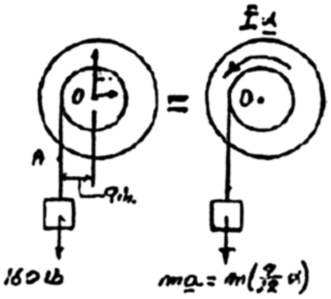
(*a*) $\stackrel{\curvearrowleft}{+}\Sigma M_0 = \Sigma(M_0)_{\text{eff}}:\quad (160\text{ lb})\left(\frac{9}{12}\text{ft}\right) = (15\text{ lb}\cdot\text{ft}\cdot\text{s}^2)\alpha$

$\boldsymbol{\alpha} = 8\text{ rad/s}^2 \circlearrowright$ ◀

(*b*) $\theta = \dfrac{10\text{ ft}}{\left(\frac{9}{12}\text{ft}\right)} = 13.333\text{ rad}$

$\omega^2 = 2\alpha\theta = 2(8\text{ rad/s}^2)(13.33\text{ rad})$ $\qquad \boldsymbol{\omega} = 14.61\text{ rad/s} \circlearrowright$ ◀

Case 2:



(*a*) $\stackrel{\curvearrowleft}{+}\Sigma M_0 = \Sigma(M_0)_{\text{eff}}:\quad (160)\left(\frac{9}{12}\right) = 15\alpha + ma\left(\frac{9}{12}\right)$

$$120 = 15\alpha + \frac{160}{32.2}\left(\frac{9}{12}\alpha\right)\left(\frac{9}{12}\right)$$

$$120 = (15 + 2.795)\alpha$$

$$\alpha = 6.7435\text{ rad/s}^2$$

$\boldsymbol{\alpha} = 6.74\text{ rad/s}^2 \circlearrowright$ ◀

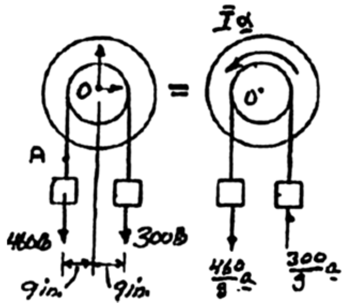
(*b*) $\theta = \dfrac{10\text{ ft}}{\left(\frac{9}{12}\text{ft}\right)} = 13.333\text{ rad}$

$$\omega^2 = 2\alpha\theta$$

$= 2(6.7435\text{ rad/s}^2)(13.333\text{ rad})$ $\qquad \boldsymbol{\omega} = 13.41\text{ rad/s} \circlearrowright$ ◀

PROBLEM 16.34 (Continued)

Case 3:



$\bar{I}\alpha$
O
=
O'
A
460lb
300lb
9in.
9in.
$\frac{460}{g}a$
$\frac{300}{g}a$

(*a*) $$+\circlearrowright \Sigma M_0 = \Sigma (M_0)_{\text{eff}}:$$

$$(460)\left(\frac{9}{12}\right) - (300)\left(\frac{9}{12}\right) = 15\alpha + \frac{460}{32.2}a\left(\frac{9}{12}\right) + \frac{300}{32.2}a\left(\frac{9}{12}\right)$$

$$120 = 15\alpha + \frac{460}{32.2}\left(\frac{9}{12}\right)^2 \alpha + \frac{300}{32.2}\left(\frac{9}{12}\right)^2 \alpha$$

$$\alpha = 4.2437 \text{ rad/s}^2$$

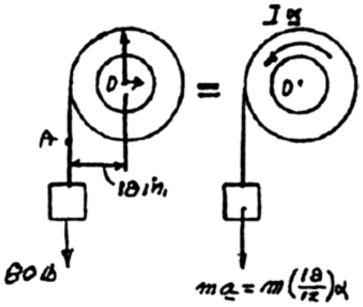
$\boldsymbol{\alpha} = 4.24 \text{ rad/s}^2 \circlearrowright$ ◀

(*b*) $$\theta = \frac{10 \text{ ft}}{\frac{9}{12}} = 13.333 \text{ rad}$$

$$\omega^2 = 2\alpha\theta = 2(4.2437)(13.333)$$

$\boldsymbol{\omega} = 10.64 \text{ rad/s}^2 \circlearrowright$ ◀

Case 4:



$\bar{I}\alpha$
O
=
O'
A
18 in.
80 lb
$ma = m\left(\frac{18}{12}\right)\alpha$

(*a*) $$+\circlearrowright \Sigma M_0 = \Sigma (M_0)_{\text{eff}}: \quad (80)\left(\frac{18}{12}\right) = 15\alpha + \frac{80}{32.2}a\left(\frac{18}{12}\right)$$

$$120 = 15\alpha + \frac{80}{32.2}\left(\frac{18}{12}\right)^2 \alpha$$

$$120 = (15 + 5.5901)\alpha$$

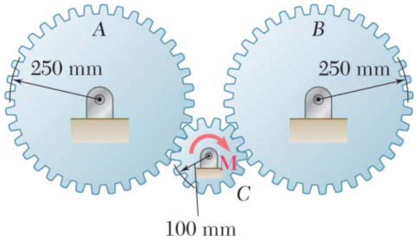
$$\alpha = 5.828 \text{ rad/s}^2$$

$\boldsymbol{\alpha} = 5.83 \text{ rad/s}^2 \circlearrowright$ ◀

(*b*) $$\theta = \frac{10 \text{ ft}}{\left(\frac{18}{12} \text{ ft}\right)} = 6.6667 \text{ rad}$$

$$\omega^2 = 2\alpha\theta = 2(5.828 \text{ rad/s}^2)(6.6667 \text{ rad})$$

$\boldsymbol{\omega} = 8.82 \text{ rad/s} \circlearrowright$ ◀



PROBLEM 16.35

Each of the gears *A* and *B* has a mass of 9 kg and has a radius of gyration of 200 mm; gear *C* has a mass of 3 kg and has a radius of gyration of 75 mm. If a couple **M** of constant magnitude 5 N-m is applied to gear *C*, determine (*a*) the angular acceleration of gear *A*, (*b*) the tangential force which gear *C* exerts on gear *A*.

SOLUTION

Kinematics:

We express that the tangential components of the accelerations of the gear teeth are equal:

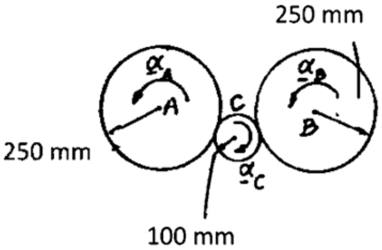
$$a_t = 0.25\alpha_A$$

$$= 0.25\alpha_B$$

$$= 0.1\alpha_C$$

$$\alpha_B = \alpha_A$$

$$\alpha_C = 2.5\alpha_A \qquad (1)$$



Kinetics:

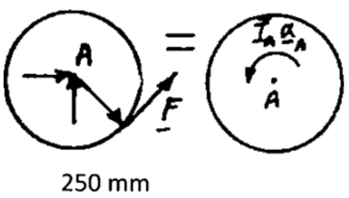
Gear *A*:

$$\bar{I}_A = m_A\bar{k}_A^2 = 9(0.20)^2$$

$$= 0.36\ \text{kg}\cdot\text{m}^2$$

$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad F(0.25) = 0.36\alpha_A$$

$$F = 1.44\alpha_A \qquad (2)$$



Because of symmetry, gear *C* exerts an equal force *F* on gear *B*.

Gear *C*:

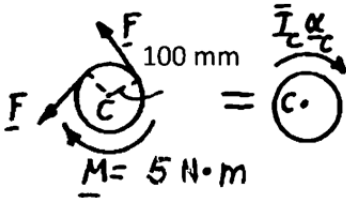
$$\bar{I}_C = m_C\bar{k}_C^2$$

$$= 3(0.075)^2$$

$$= 0.016875\ \text{kg}\cdot\text{m}^2$$

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad M - 2Fr_C = \bar{I}_C\alpha_C$$

$$5\ \text{N}\cdot\text{m} - 2F(0.1\ \text{m}) = 0.016875\alpha_C$$



(*a*) Angular acceleration of gear *A*.

Substituting for α_C from (1) and for *F* from (2):

$$5 - 2(1.44\alpha_A)(0.1) = 0.016875(2.5\alpha_A)$$

$$5 - 0.288\alpha_A = 0.04219\alpha_A$$

$$5 = 0.3302\alpha_A$$

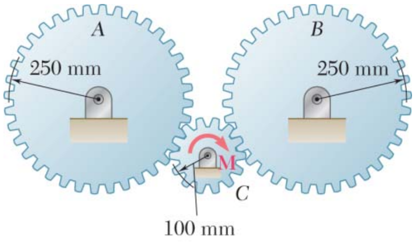
$$\alpha_A = 15.143\ \text{rad/s}^2$$

$\boldsymbol{\alpha}_A = 15.14\ \text{rad/s}^2$ ↺ ◀

(*b*) Tangential force *F*.

Substituting for α_A into (2): $F = 1.44(15.143)$

$\mathbf{F}_A = 21.8\ \text{N}$ ↗ ◀



PROBLEM 16.36

Solve Problem 16.35, assuming that the couple **M** is applied to disk *A*.

PROBLEM 16.35 Each of the gears *A* and *B* has a mass of 9 kg and has a radius of gyration of 200 mm; gear *C* has a mass of 3 kg and has a radius of gyration of 75 mm. If a couple **M** of constant magnitude 5 N-m is applied to gear *C*, determine (*a*) the angular acceleration of gear *A*, (*b*) the tangential force which gear *C* exerts on gear *A*.

SOLUTION

Kinematics: $\boldsymbol{\alpha}_A = \alpha_A \circlearrowright \qquad \boldsymbol{\alpha}_A = \alpha_A \circlearrowleft \qquad \boldsymbol{\alpha}_B = \alpha_B \circlearrowright$

At the contact point between gears *A* and *C*,

$$a_t = r_A\alpha_A = r_C\alpha_C$$

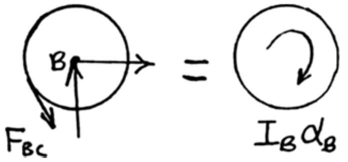
$$\alpha_C = \frac{r_A}{r_C}\alpha_A = \frac{0.25}{0.1}\alpha_A$$

At the contact point between gears *B* and *C*,

$$a_t = r_B\alpha_B = r_C\alpha_C$$

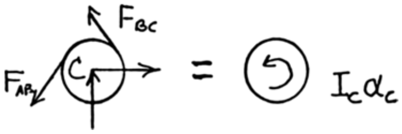
$$\alpha_B = \frac{r_C}{r_A}\alpha_C = \frac{0.1}{0.25}\cdot\frac{0.25}{0.1}\alpha_A = \alpha_A$$

Kinetics:



Gear *B*: $+\circlearrowright \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad F_{BC}r_B = I_B\alpha_B = I_B\alpha_A$

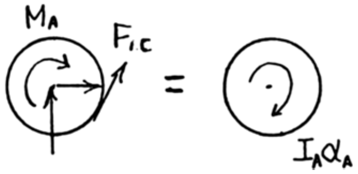
$$F_{BC} = \frac{I_B}{r_B}\alpha_A$$



Gear *C*: $+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad F_{BC}r_C + F_{AB}r_C = I_C\alpha_C$

$$\frac{r_C}{r_B}I_B\alpha_A + F_{AB}r_C = \frac{0.25}{0.1}I_C\alpha_A$$

$$F_{AC} = \frac{1}{r_C}\left(\frac{0.1}{0.25}I_B + \frac{0.25}{0.1}I_C\right)\alpha_A \tag{1}$$



Gear *A*: $\circlearrowleft + M_A = \Sigma(M_A)_{\text{eff}}: \quad M_A - r_AF_{AB} = I_A\alpha_A$

$$M_A - \frac{r_A}{r_C}\left(\frac{0.1}{0.25}I_B + \frac{0.25}{0.1}I_C\right)\alpha_A = I_A\alpha_A$$

$$M_A = \left(I_A + I_B + \left(\frac{0.25}{0.1}\right)^2 I_C\right)\alpha_A \tag{2}$$

PROBLEM 16.36 (Continued)

Data: From Eq. (2)

$$m_A = m_B = 9 \text{ kg}$$
$$k_A = k_B = 0.2 \text{ m}$$
$$I_A = I_B = m_A k_A^2 = 9(0.2)^2$$
$$= 0.36 \text{ kg} \cdot \text{m}^2$$

$$m_C = 3 \text{ kg}$$
$$k_C = 0.075 \text{ m}$$
$$I_C = 3(0.075)^2 = 0.016875 \text{ kg} \cdot \text{m}^2$$
$$M_A = 5 \text{ N} \cdot \text{m}$$
$$5 = \left[0.36 + 0.36 + \left(\frac{0.25}{0.1}\right)^2 (0.016875)\right]\alpha_A$$

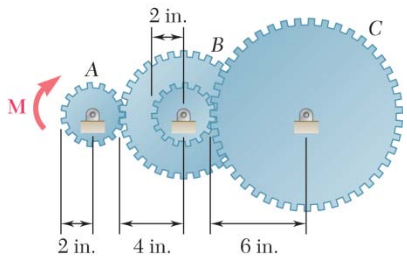
(*a*) Angular acceleration. $\alpha_A = 6.0572 \text{ rad/s}^2$ $\boldsymbol{\alpha}_A = 6.06 \text{ rad/s}^2 \circlearrowright$ ◀

(*b*) Tangential gear force.

From Eq. (1)

$$F_{AC} = \frac{1}{0.1}\left[\left(\frac{0.1}{0.25}\right)(0.36) + \left(\frac{0.25}{0.1}\right)(0.016875)\right](6.0572)$$
$$= 11.278$$

$\mathbf{F}_{AC} = 11.28 \text{ N} \nearrow$ ◀



PROBLEM 16.37

Gear *A* weighs 1 lb and has a radius of gyration of 1.3 in; Gear *B* weighs 6 lb and has a radius of gyration of 3 in.; gear *C* weighs 9 lb and has a radius of gyration of 4.3 in. Knowing a couple **M** of constant magnitude of 40 lb · in is applied to gear *A*, determine (*a*) the angular acceleration of gear *C*, (*b*) the tangential force which gear *B* exerts on gear *C*.

SOLUTION

Masses and moments of inertia.

$$m_A = \frac{1 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.031056 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_B = \frac{6 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.18634 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_C = \frac{9 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.27950 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$\bar{I}_A = m_A\bar{k}_A^2 = (0.031056 \text{ lb} \cdot \text{s}^2/\text{ft})\left(\frac{1.3 \text{ in.}}{12 \text{ in./ft}}\right)^2$$

$$= 0.36448 \times 10^{-3} \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

$$\bar{I}_B = m_B\bar{k}_B^2 = (0.18634 \text{ lb} \cdot \text{s}^2/\text{ft})\left(\frac{3 \text{ in.}}{12 \text{ in./ft}}\right)^2$$

$$= 11.646 \times 10^{-3} \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

$$\bar{I}_C = m_C\bar{k}_C^2 = (0.27950 \text{ lb} \cdot \text{s}^2/\text{ft})\left(\frac{4.3 \text{ in.}}{12 \text{ in./ft}}\right)^2$$

$$= 35.889 \times 10^{-3} \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

Kinematics. Gear *A*: $r_A = 2 \text{ in.}$

Gear *B*: $r_1 = 4 \text{ in.}, \quad r_2 = 2 \text{ in.}$

Gear *C*: $r_C = 6 \text{ in.}$

Point of contact between *A* and *B*.

$$a_t = r_A\alpha_A = r_1\alpha_B$$

$$\alpha_A = \frac{r_1}{r_A}\alpha_B = \frac{4 \text{ in.}}{2 \text{ in.}}\alpha_B$$

PROBLEM 16.37 (Continued)

Point of contact between *B* and *C*.

$$a_t = r_2\alpha_B = r_C\alpha_C$$

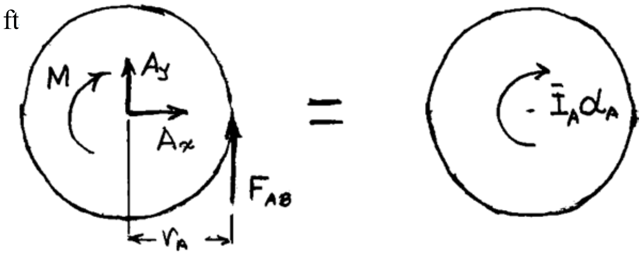
$$\alpha_B = \frac{r_C}{r_2}\alpha_C = \frac{6\text{ in.}}{2\text{ in.}}\alpha_C$$

Summary. $$\alpha_B = 3\alpha_C \qquad (1)$$

$$\alpha_A = 2\alpha_B = 6\alpha_C \qquad (2)$$

Kinetics. Applied couple: $M = 40\text{ lb}\cdot\text{in.} = 3.3333\text{ lb}\cdot\text{ft}$

Gear *A*:



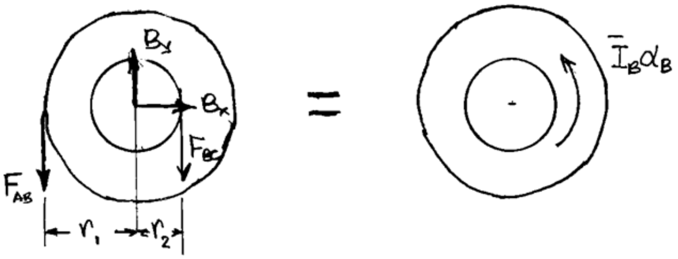
$$\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad M - F_{AB}r_A = \bar{I}_A\alpha_A$$

$$F_{AB} = \frac{M}{r_A} - \frac{\bar{I}_A(6\alpha_C)}{r_A}$$

$$= \frac{3.3333\text{ lb}\cdot\text{ft}}{(2/12)\text{ ft}} - \frac{(0.36448\times10^{-3})(6)}{(2/12)}\alpha_C$$

$$= 20\text{ lb} - 0.013121\alpha_C \qquad (3)$$

Gear *B*:



$$+\circlearrowright \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad F_{AB}r_1 - F_{BC}r_2 = \bar{I}_B\alpha_C$$

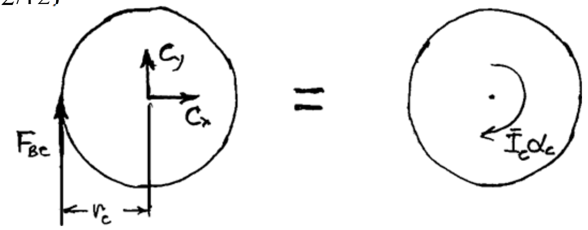
$$F_{BC} = F_{AB} = \frac{r_1}{r_2} - \frac{\bar{I}_B\alpha_B}{r_2}$$

$$= 2F_{AB} - \frac{3\bar{I}_B}{r_2}\alpha_C$$

$$= 2[20\text{ lb} - 0.013121\alpha_C] - \frac{(3)(11.646\times10^{-3})}{(2/12)}\alpha_C$$

$$= 40\text{ lb} - 0.23587\alpha_C$$

Gear *C*:



$$\circlearrowleft + \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad F_{BC}r_C = \bar{I}_C\alpha_C$$

$$(40 - 0.23587\alpha_C)\left(\frac{6}{12}\right) = (35.889\times10^{-3})\alpha_C$$

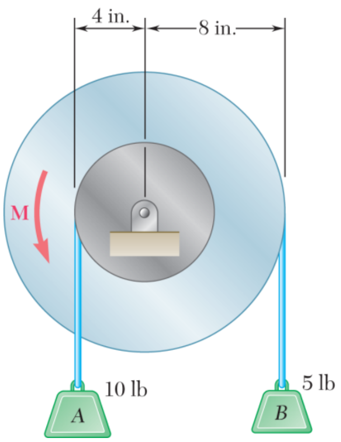
$$20 = 0.153824\alpha_C$$

(*a*) *Angular acceleration of gear C.* $\boldsymbol{\alpha}_C = 130.0\text{ rad/s}^2 \circlearrowright$ ◀

(*b*) *Tangential force which gear B exerts on gear C.*

$$F_{BC} = 40\text{ lb} - (0.23587)(130.0) = 9.33\text{ lb}$$

9.33 lb ↑ ◀



PROBLEM 16.38

The 25-lb double pulley shown is at rest and in equilibrium when a constant $3.5\ \text{lb}\cdot\text{ft}$ couple **M** is applied. Neglecting the effect of friction and knowing that the radius of gyration of the double pulley is 6 in., determine (*a*) the angular acceleration of the double pulley, (*b*) the tension in each rope.

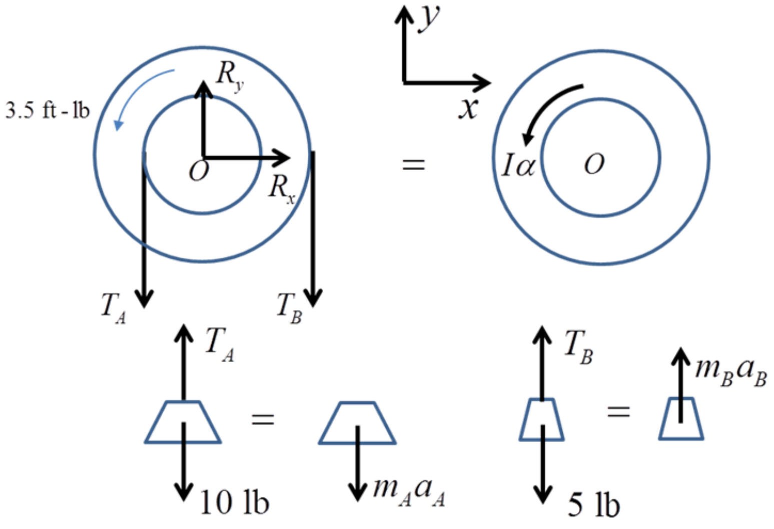
SOLUTION

Given:

$$m_p = \frac{25}{32.2}\ \text{slugs},\ \text{m}_A = \frac{10}{32.2}\ \text{slugs},\ \text{m}_B = \frac{5}{32.2}\ \text{slugs}$$

$$M = 3.5\ \text{ft-lbs},\ k_P = 6\ \text{in}$$

Free Body Diagrams:



Kinetics:

$$+\circlearrowleft\ \Sigma M_0 = 3.5 + T_A\left(\frac{4}{12}\right) - T_B\left(\frac{8}{12}\right)$$

$$= \bar{I}\,\alpha = \left(\frac{25}{32.2}\right)\left(\frac{6}{12}\right)^2 \alpha$$

Kinematics:

$$a_A = r_A\alpha = \left(\frac{4}{12}\right)\alpha \downarrow$$

$$a_B = r_B\alpha = \left(\frac{8}{12}\right)\alpha \uparrow$$

PROBLEM 16.38 (Continued)

Kinetics:

$$A:\ +\!\downarrow\ \Sigma F = 10 - T_A = \left(\frac{10}{32.2}\right)a_A$$

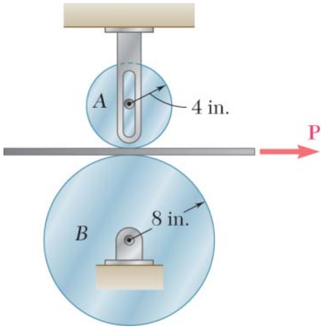
$$B:\ +\!\uparrow\ \Sigma F = T_B - 5 = \left(\frac{5}{32.2}\right)a_B$$

$$\text{Elimination: } 3.5 + \left(10 - \frac{10a_A}{g}\right)\frac{4}{12} - \left(5 + \frac{5a_B}{g}\right)\frac{8}{12} = \frac{25}{g}(0.5)^2\alpha$$

$$3.5 + \cancel{3.333} - 1.111\left(\frac{\alpha}{g}\right) - \cancel{3.333} - 2.222\left(\frac{\alpha}{g}\right) = 6.25\left(\frac{\alpha}{g}\right)$$

(*a*) $\boldsymbol{\alpha} = 11.76\ \text{rad/s}^2 \circlearrowright$ ◀

(*b*) $T_A = 8.78\ \text{lb}\ ,\ T_B = 6.22\ \text{lb}$ ◀



PROBLEM 16.39

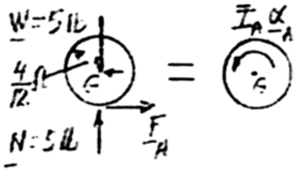
A belt of negligible mass passes between cylinders *A* and *B* and is pulled to the right with a force **P**. Cylinders *A* and *B* weigh, respectively, 5 and 20 lb. The shaft of cylinder *A* is free to slide in a vertical slot and the coefficients of friction between the belt and each of the cylinders are $\mu_s = 0.50$ and $\mu_k = 0.40$. For $P = 3.6$ lb, determine (*a*) whether slipping occurs between the belt and either cylinder, (*b*) the angular acceleration of each cylinder.

SOLUTION

Assume that no slipping occurs.

Then: $a_{\text{belt}} = (4\text{ in.})\alpha_A = (8\text{ in.})\alpha_B \qquad \alpha_B = \frac{1}{2}\alpha_A \qquad (1)$

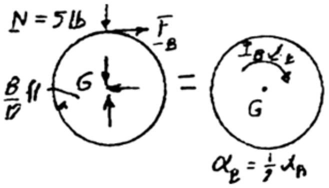
Cylinder *A*. $+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad F_A\left(\frac{4}{12}\text{ft}\right) = \bar{I}_A\alpha_A$



$$F_A\left(\frac{4}{12}\right) = \frac{1}{2}\frac{5}{g}\left(\frac{4}{12}\right)^2\alpha_A$$

$$F_A = \frac{5}{6}\frac{\alpha_A}{g} \qquad (2)$$

Cylinder *B*. $+\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad F_B\left(\frac{8}{12}\text{ft}\right) = \bar{I}_B\alpha_B$

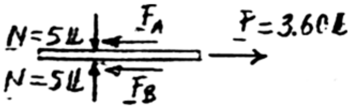


$$F_B\left(\frac{8}{12}\right) = \frac{1}{2}\frac{20}{g}\left(\frac{8}{12}\right)^2\left(\frac{1}{2}\alpha_A\right)$$

$$F_B = \frac{20}{6}\frac{\alpha_A}{g} \qquad (3)$$

Belt $\quad \xrightarrow{+} \Sigma F_A = 0: \quad P - F_A - F_B = 0 \qquad (4)$

$$3.60 - \frac{5}{6}\frac{\alpha_A}{g} - \frac{20}{6}\frac{\alpha_A}{g} = 0$$



$$\alpha_A = \frac{(3.60)6}{25}g$$

$$= 0.864g$$

$$\boldsymbol{\alpha}_A = 27.82\text{ rad/s}^2 \circlearrowleft$$

PROBLEM 16.39 (Continued)

Check that belt does not slip.

From (2): $F_A = \frac{5}{6}(0.864) = 0.720 \text{ lb}$

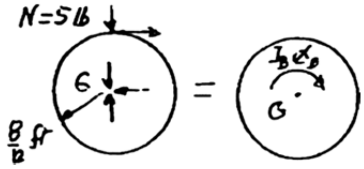
From (4): $F_e = P - F_A = 3.60 - 0.720 = 2.88 \text{ lb}$

But $F_m = \mu_s N = 0.50(5 \text{ lb}) = 2.50 \text{ lb}$

Since $F_e > F_m$, assumption is wrong.

Slipping occurs between disk *B* and the belt. ◀

We redo analysis of *B*, assuming slipping. $\left(\alpha_B \neq \frac{1}{2}\alpha_A\right)$



$$F_B = \mu_k N = 0.40(5 \text{ lb}) = 2 \text{ lb}$$

$$\overset{+}{\circlearrowright}\Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad (2 \text{ lb})\left(\frac{8}{12}\text{ ft}\right) = \frac{1}{2}\frac{20}{g}\left(\frac{8}{12}\right)^2 \alpha_B$$

$$\alpha_B = 0.3g, \qquad \boldsymbol{\alpha}_B = 9.66 \text{ rad/s}^2 \circlearrowright$$ ◀

Belt Eq. (4):

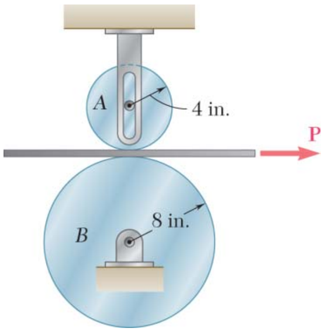
$$\begin{aligned} P - F_A - F_B &= 0 \\ F_A &= P - F_B \\ &= 3.60 - 2 \\ &= 1.60 \text{ lb} \end{aligned}$$

Since $F_A < F_m$, There is no slipping between *A* and the belt. ◀

Our analysis of disk *A*, therefore is valid. Using Eq. (2),

We have $1.60 \text{ lb} = \frac{5}{6}\frac{\alpha_A}{g}$

$$\alpha_A = 1.92g \qquad \boldsymbol{\alpha}_A = 61.8 \text{ rad/s}^2 \circlearrowleft$$ ◀



PROBLEM 16.40

Solve Problem 16.39 for $P = 2.00$ lb.

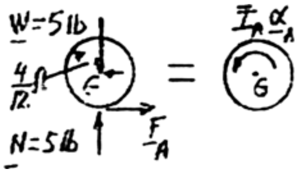
PROBLEM 16.39 A belt of negligible mass passes between cylinders *A* and *B* and is pulled to the right with a force **P**. Cylinders *A* and *B* weigh, respectively, 5 and 20 lb. The shaft of cylinder *A* is free to slide in a vertical slot and the coefficients of friction between the belt and each of the cylinders are $\mu_s = 0.50$ and $\mu_k = 0.40$. For $P = 3.60$ lb, determine (*a*) whether slipping occurs between the belt and either of the cylinders, (*b*) the angular acceleration of each cylinder.

SOLUTION

Assume that no slipping occurs.

Then: $a_{\text{belt}} = (4\text{ in.})\alpha_A = (8\text{ in.})\alpha_B \qquad \alpha_B = \frac{1}{2}\alpha_A \qquad (1)$

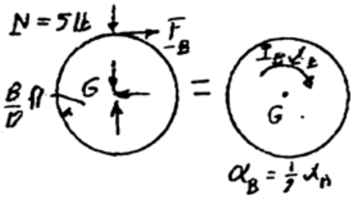
Cylinder *A* $\quad +\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad F_A\left(\frac{4}{12}\text{ft}\right) = \bar{I}_A\alpha_A$



$$F_A\left(\frac{4}{12}\right) = \frac{1}{2}\frac{5}{g}\left(\frac{4}{12}\right)^2 \alpha_A$$

$$F_A = \frac{5}{6}\frac{\alpha_A}{g} \qquad (2)$$

Cylinder *B* $\quad +\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad F_B\left(\frac{4}{12}\text{ft}\right) = \bar{I}_B\alpha_B$

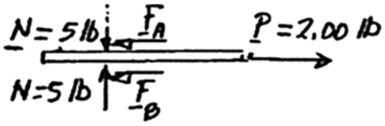


$$F_B\left(\frac{8}{12}\right) = \frac{1}{2}\frac{20}{g}\left(\frac{8}{12}\right)^2\left(\frac{1}{2}\alpha_A\right)$$

$$F_B = \frac{20}{6}\frac{\alpha_A}{g} \qquad (3)$$

Belt $\quad \xrightarrow{+} \Sigma F_A = 0: \quad P - F_A - F_B = 0$

$$2.00 - \frac{5}{6}\frac{\alpha_A}{g} - \frac{20}{6}\frac{\alpha_A}{g} = 0$$



$$\alpha_A = \frac{(2.00)6}{25}g$$

$$= 0.480g$$

$$\alpha_A = 15.46\text{ rad/s}^2 \circlearrowleft$$

PROBLEM 16.40 (Continued)

From (2):
$$F_A = \frac{5}{6}(0.480) = 0.400 \text{ lb}$$

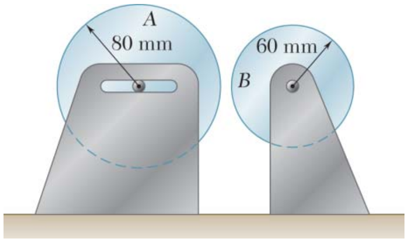
From (4):
$$F_B = \frac{20}{6}(0.480) = 1.600 \text{ lb}$$

But
$$F_M = \mu_s N = 0.50(5 \text{ lb}) = 2.50 \text{ lb}$$

Thus, F_A and F_B are both $< F_m$. Our assumption was right:

There is no slipping between cylinders and belt $\quad \boldsymbol{\alpha}_A = 15.46 \text{ rad/s}^2 \circlearrowleft$ ◀

From (1): $\quad \boldsymbol{\alpha}_B = 7.73 \text{ rad/s}^2 \circlearrowright$ ◀



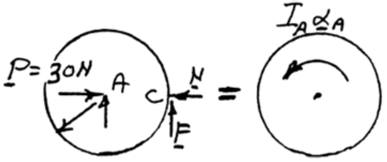
PROBLEM 16.41

Disk *A* has a mass of 6 kg and an initial angular velocity of 360 rpm clockwise; disk *B* has a mass of 3 kg and is initially at rest. The disks are brought together by applying a horizontal force of magnitude 20 N to the axle of disk *A*. Knowing that $\mu_k = 0.15$ between the disks and neglecting bearing friction, determine (*a*) the angular acceleration of each disk, (*b*) the final angular velocity of each disk.

SOLUTION

While slipping occurs, a friction force $F\uparrow$ is applied to disk *A*, and $F\downarrow$ to disk *B*.

Disk *A*:



$$I_A = \frac{1}{2}m_A r_A^2$$

$$= \frac{1}{2}(6\text{ kg})(0.08\text{ m})^2$$

$$= 0.0192\text{ kg}\cdot\text{m}^2$$

$$\Sigma F:\quad N = P = 20\text{ N}$$

$$F = \mu N = 0.15(20) = 3\text{ N}$$

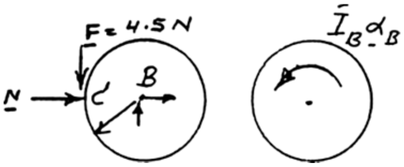
$$+\curvearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}:\quad Fr_A = \bar{I}_A\alpha_A$$

$$(3\text{ N})(0.08\text{ m}) = (0.0192\text{ kg}\cdot\text{m}^2)\alpha_A$$

$$\alpha_A = 10.227$$

$\boldsymbol{\alpha}_A = 12.50\text{ rad/s}^2 \curvearrowright$ ◀

Disk *B*:



$$\bar{I}_B = \frac{1}{2}m_B r_B^2$$

$$= \frac{1}{2}(3\text{ kg})(0.06\text{ m})^2$$

$$= 0.0054\text{ kg}\cdot\text{m}^2$$

$$+\curvearrowright \Sigma M_B = \Sigma(M_B)_{\text{eff}}:\quad Fr_B = \bar{I}_B\alpha_B$$

$$(3\text{ N})(0.06\text{ m}) = (0.0054\text{ kg}\cdot\text{m}^2)\alpha_B$$

$$\alpha_B = 33.333\text{ rad/s}^2$$

$\boldsymbol{\alpha}_B = 33.3\text{ rad/s}^2 \curvearrowright$ ◀

$$(\omega_A)_0 = 360\text{ rpm} = 12\pi\text{ rad/s} \curvearrowright$$

PROBLEM 16.41 (Continued)

Sliding stops when $\mathbf{v}_C = \mathbf{v}_{C'}$ or $\omega_A r_A = \omega_B r_B$

$$r_A[(\omega_A)_0 - \alpha_A t] = r_B \alpha_B t$$

$$(0.08\text{ m})[12\pi\text{ rad/s} - (12.5\text{ rad/s}^2)t] = (0.06\text{ m})(33.333\text{ rad/s}^2)t$$

$$t = 1.00531\text{ s}$$

$$+\circlearrowright\ \omega_A = (\omega_A)_0 - \alpha_A t$$

$$= 12\pi\text{ rad/s} - (12.5\text{ rad/s}^2)(1.00531\text{ s})$$

$$= 25.132\text{ rad/s}$$

$$\omega_A = (25.132\text{ rad/s})$$

$$= 240.00\text{ rpm}$$

or $\boldsymbol{\omega}_A = 240\text{ rpm} \circlearrowright$ ◀

$$+\circlearrowleft\ \omega_B = \alpha_B t$$

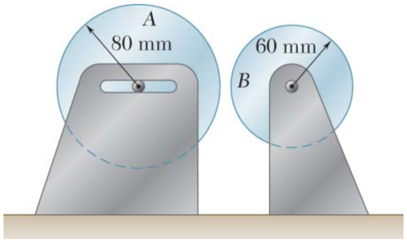
$$= 33.333\text{ rad/s}^2(1.00531\text{ s})$$

$$= 33.510\text{ rad/s}$$

$$\omega_B = (33.510\text{ rad/s})$$

$$= 320.00\text{ rpm}$$

or $\boldsymbol{\omega}_B = 320\text{ rpm} \circlearrowleft$ ◀



PROBLEM 16.42

Solve Problem 16.41, assuming that initially disk *A* is at rest and disk *B* has an angular velocity of 360 rpm clockwise.

PROBLEM 16.41 Disk *A* has a mass of 6 kg and an initial angular velocity of 360 rpm clockwise; disk *B* has a mass of 3 kg and is initially at rest. The disks are brought together by applying a horizontal force of magnitude 20 N to the axle of disk *A*. Knowing that $\mu_k = 0.15$ between the disks and neglecting bearing friction, determine (*a*) the angular acceleration of each disk, (*b*) the final angular velocity of each disk.

SOLUTION

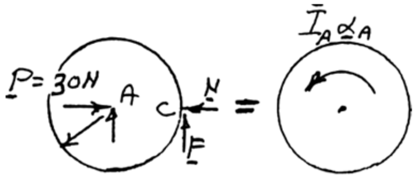
While slipping occurs, a friction force $F\uparrow$ is applied to disk *A*, and $F\downarrow$ to disk *B*.

Disk A:

$$I_A = \frac{1}{2} m_A r_A^2$$

$$= \frac{1}{2}(6\text{ kg})(0.08\text{ m})^2$$

$$= 0.0192\text{ kg}\cdot\text{m}^2$$



$$\Sigma F: \quad N = P = 20\text{ N}$$

$$F = \mu N = 0.15(20) = 3\text{ N}$$

$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad Fr_A = \bar{I}_A \alpha_A$$

$$(3\text{ N})(0.08\text{ m}) = (0.0192\text{ kg}\cdot\text{m}^2)\alpha_A$$

$$\alpha_A = 10.227$$

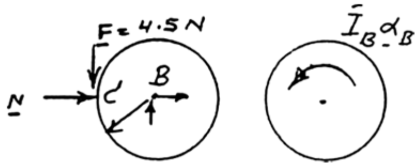
$\boldsymbol{\alpha}_A = 12.50\text{ rad/s}^2 \circlearrowleft$ ◀

Disk B:

$$\bar{I}_B = \frac{1}{2} m_B r_B^2$$

$$= \frac{1}{2}(3\text{ kg})(0.06\text{ m})^2$$

$$= 0.0054\text{ kg}\cdot\text{m}^2$$



$$+\circlearrowleft \Sigma M_A = \Sigma(M_B)_{\text{eff}}: \quad Fr_B = \bar{I}_B \alpha_B$$

$$(3\text{ N})(0.06\text{ m}) = (0.0054\text{ kg}\cdot\text{m}^2)\alpha_B$$

$$\alpha_B = 33.333\text{ rad/s}^2$$

$\boldsymbol{\alpha}_B = 33.3\text{ rad/s}^2 \circlearrowleft$ ◀

PROBLEM 16.42 (Continued)

Sliding starts when $\mathbf{v}_C = \mathbf{v}_{C'}$. That is when

$$\omega_A r_A = \omega_B r_B$$

$$(\alpha_A t) r_A = [(\omega_B)_0 - \alpha_B t] r_B$$

$$[(12.5 \text{ rad/s}^2)t](0.08 \text{ m}) = [12\pi \text{ rad/s} - (33.333 \text{ rad/s}^2)(t)](0.06 \text{ m})$$

$$t = 4.02124 \qquad t = 0.75398 \text{ s}$$

$$\circlearrowleft \; \omega_A = \alpha_A t = (12.5 \text{ rad/s}^2)(0.75398 \text{ s}) = 9.4248 \text{ rad/s}$$

$$+\circlearrowleft \; \omega_A = (9.4248 \text{ rad/s}) = 90.00 \text{ rpm} \circlearrowleft$$

or $\boldsymbol{\omega}_A = 90.0 \text{ rpm} \circlearrowleft$ ◀

$$+\circlearrowright \; \omega_B = (\omega_B)_0 - \alpha_B t$$

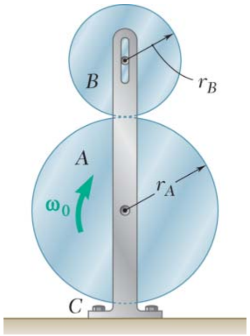
$$= 12\pi \text{ rad/s} - (33.333 \text{ rad/s}^2)(0.75398 \text{ s})$$

$$= 12.566 \text{ rad/s} \circlearrowleft$$

$$\omega_B = (12.566 \text{ rad/s})$$

$$= 120.00 \text{ rpm}$$

or $\boldsymbol{\omega}_B = 120.0 \text{ rpm} \circlearrowright$ ◀

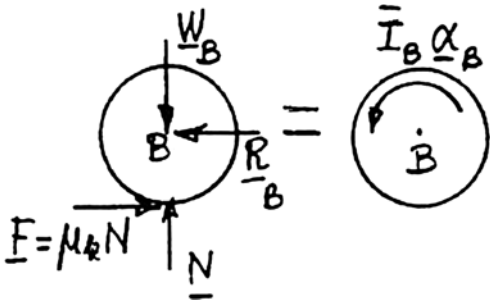

B
r_B
A
r_A
ω_0
C

PROBLEM 16.43

Disk A has a mass $m_A = 4$ kg, a radius $r_A = 300$ mm, and an initial angular velocity $\omega_0 = 300$ rpm clockwise. Disk B has a mass $m_B = 1.6$ kg, a radius $r_B = 180$ mm, and is at rest when it is brought into contact with disk A. Knowing that $\mu_k = 0.35$ between the disks and neglecting bearing friction, determine (a) the angular acceleration of each disk, (b) the reaction at the support C.

SOLUTION

(a) Disk B.


$\underline{W}_B$
B
$\underline{R}_B$
$\underline{F} = \mu_k N$
$\underline{N}$
=
$\bar{I}_B \underline{\alpha}_B$
B

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N - W_B = 0$$

$$N = W_B = m_B g$$

Thus, $$F = \mu_k N = \mu_k m_B g$$

$$+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad F r_B = \bar{I}_B \alpha_B$$

$$\mu_k m_B g r_B = \frac{1}{2} m_B r_B^2 \alpha_B$$

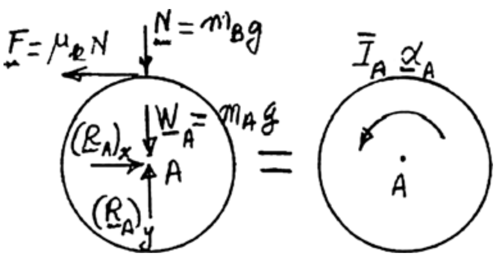
$$\boldsymbol{\alpha}_B = \frac{2\mu_k g}{r_B} \circlearrowleft \qquad (1)$$

For the given data: $\alpha_B = \dfrac{2(0.35)(9.81 \text{ m/s}^2)}{0.180 \text{ m}} = 38.15 \text{ rad/s}^2$ $\qquad \boldsymbol{\alpha}_B = 38.2 \text{ rad/s}^2 \circlearrowleft$ ◀

$$W_B = m_B g = (1.6 \text{ kg})(9.81 \text{ m/s}^2) = 15.696 \text{ N}$$
$$F = \mu_k m_B g = (0.35)(15.696) = 5.4936 \text{ N}$$

$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad F - R_B = 0$ $\qquad \mathbf{R}_B = 5.4936 \text{ N} \longleftarrow$

Disk A:


$\underline{F} = \mu_k N$
$\underline{N} = m_B g$
$(R_A)_x$
$\underline{W}_A = m_A g$
A
$(R_A)_y$
=
$\bar{I}_A \underline{\alpha}_A$
A

$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad F r_A = \bar{I}_A \alpha_A$$

$$\mu_k m_B g r_A = \frac{1}{2} m_A r_A^2 \alpha_A$$

$$\boldsymbol{\alpha}_A = \frac{2\mu_k g}{r_A} \frac{m_B}{m_A} \circlearrowleft \qquad (2)$$

PROBLEM 16.43 (Continued)

For the given data: $\alpha_A = \dfrac{2(0.35)(9.81\ \text{m/s}^2)}{0.300\ \text{m}} \dfrac{1.6\ \text{kg}}{4\ \text{kg}} = 9.156\ \text{rad/s}^2$ $\qquad \boldsymbol{\alpha}_A = 9.16\ \text{rad/s}^2 \circlearrowleft$ ◀

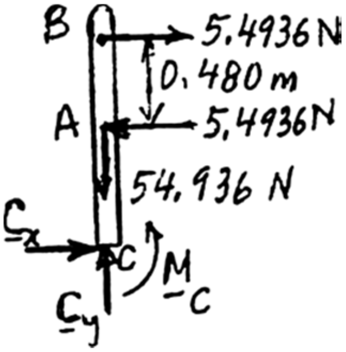
$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad (R_A)_x - F = 0$ $\qquad (\mathbf{R}_A)_x = 5.4936\ \text{N} \longrightarrow$

$+\!\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad (R_A)_y - m_A g - m_B g = 0$

$$(R_A)_y = (m_A + m_B)g = (4\ \text{kg} + 1.6\ \text{kg})(9.81\ \text{m/s}^2)$$
$$= 54.936\ \text{N}$$

$(\mathbf{R}_A)_y = 54.936\ \text{N} \uparrow$

(*b*) *Reaction at C.*

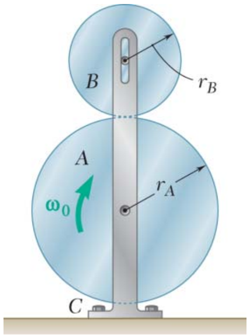


$\xrightarrow{+} \Sigma F_x = 0: \quad C_x = 0$

$+\!\uparrow \Sigma F_y = 0: \quad C_y - 54.936\ \text{N} = 0$

$C_y = 54.936\ \text{N}$ $\qquad \mathbf{C} = 54.9\ \text{N} \uparrow$ ◀

$+\circlearrowleft \Sigma M_C = 0: \quad M_C - (5.4936\ \text{N})(0.480\ \text{m}) = 0$ $\qquad \mathbf{M}_C = 2.64\ \text{N} \cdot \text{m} \circlearrowleft$ ◀



PROBLEM 16.44

Disk B is at rest when it is brought into contact with disk A, which has an initial angular velocity ω_0. (*a*) Show that the final angular velocities of the disks are independent of the coefficient of friction μ_k between the disks as long as $\mu_k \neq 0$. (*b*) Express the final angular velocity of disk A in terms of ω_0 and the ratio m_A/m_B of the masses of the two disks.

SOLUTION

(*a*) Disk *B*.

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N - w_B = 0$$

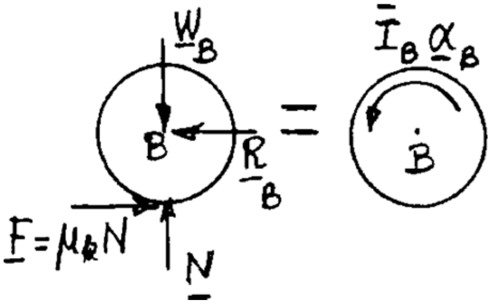
$$N = w_B = m_B g$$

Thus, $$F = \mu_k N = \mu_k m_B g$$

$$+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad Fr_B = \bar{I}_B \alpha_B$$

$$\mu_k m_B g r_B = \frac{1}{2} m_B r_B^2 \alpha_B$$

$$\boldsymbol{\alpha}_B = \frac{2\mu_k g}{r_B} \circlearrowleft \qquad (1)$$

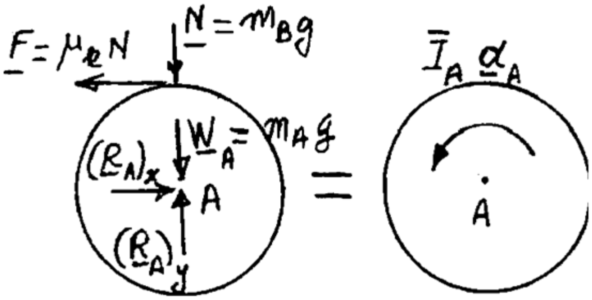


Disk *A*.

$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad Fr_A = \bar{I}_A \alpha_A$$

$$\mu_k m_B g r_A = \frac{1}{2} m_A r_A^2 \alpha_A$$

$$\boldsymbol{\alpha}_A = \frac{2\mu_k g}{r_A} \frac{m_B}{m_A} \circlearrowleft \qquad (2)$$

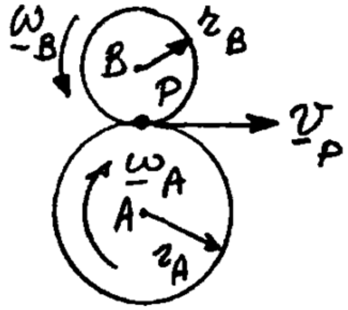


Disk *A*. $$\omega_A = \omega_0 - \alpha_A t = \omega_0 - \frac{2\mu_k g}{r_A} \frac{m_B}{m_A} t \circlearrowright \qquad (3)$$

Disk *B*. $$\omega_B = \alpha_B t = \frac{2\mu_k g}{r_B} t \circlearrowleft \qquad (4)$$

PROBLEM 16.44 (Continued)

When disks have stopped slipping:



$$v_P = \omega_A r_A = \omega_B r_B$$

$$\omega_0 r_A - (2\mu_k g)\frac{m_B}{m_A}t = 2\mu_k g t$$

$$t = \frac{\omega_0 r_A}{2\mu_k g}\frac{1}{1+\frac{m_B}{m_A}}$$

$$t = \frac{\omega_0 r_A}{2\mu_k g}\frac{m_A}{m_A + m_B} \qquad (5)$$

Substituting for t from (5) into (3) and (4):

$$\omega_A = \omega_0 - \frac{2\mu_k g}{r_A}\frac{m_B}{m_A}\frac{\omega_0 r_A}{2\mu_k g}\frac{m_A}{m_A+m_B} = \omega_0 - \frac{\omega_0 m_B}{m_A+m_B}$$

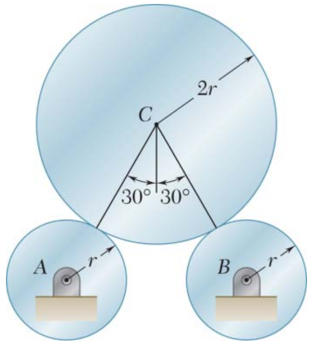
$$\omega_A = \omega_0 \frac{m_A + m_B - m_B}{m_A + m_B} \qquad \omega_A = \omega_0 \frac{m_A}{m_A+m_B} \circlearrowright \qquad (6)$$

$$\omega_B = \frac{2\mu_k g}{r_A}\frac{\omega_0 r_A}{2\mu_k g}\frac{m_A}{m_A+m_B} \qquad \omega_B = \omega_0 \frac{r_A}{r_B}\frac{m_A}{m_A+m_B} \circlearrowleft \qquad (7)$$

(*a*) From Eqs. (6) and (7), it is apparent that ω_A and ω_B are independent of μ_k. However, if $\mu_k = 0$, we have from Eqs. (3) and (4) $\omega_A = \omega_0 \circlearrowright$ and $\omega_B = 0$.

(*b*) We can write (6) in the form $\omega_A = \omega_0/(1 + m_B/m_A) \circlearrowright$ ◀

which shows that ω_A depends only upon ω_0 and m_B/m_A.



PROBLEM 16.45

Cylinder *A* has an initial angular velocity of 720 rpm clockwise, and cylinders *B* and *C* are initially at rest. Disks *A* and *B* each weigh 5 lb and have a radius $r = 4$ in. Disk *C* weighs 20 lb and has a radius of 8 in. The disks are brought together when *C* is placed gently onto *A* and *B*. Knowing that $\mu_k = 0.25$ between *A* and *C* and no slipping occurs between *B* and *C*, determine (*a*) the angular acceleration of each disk, (*b*) the final angular velocity of each disk.

SOLUTION

Assume Point *C*, the center of cylinder *C*, does not move. This is true provided the cylinders remain in contact as shown. Slipping occurs initially between disks *A* and *C* and ceases when the tangential velocities at their contact point are equal. We first determine the angular accelerations of each disk while slipping occurs.

Masses and moments of inertia:

$$m_A = m_B = \frac{W_A}{g} = \frac{5\ \text{lb}}{32.2\ \text{ft/s}^2} = 0.15528\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$m_C = \frac{W_C}{g} = \frac{20\ \text{lb}}{32.2\ \text{ft/s}^2} = 0.62112\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$I_A = I_B = \frac{1}{2}m_A r^2 = \frac{1}{2}(0.15528)\left(\frac{4}{12}\right)^2 = 0.0086266\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

$$I_C = \frac{1}{2}m_C(2r)^2 = \frac{1}{2}(0.62112)\left(\frac{8}{12}\right)^2 = 0.138027\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

Kinematics: No slipping at contact *BC*.

$$(\mathbf{a}_t)_{BC} = (a_t)_{BC} \measuredangle 30°$$

$$(a_t)_{BC} = r\alpha_B = 2r\alpha_C \qquad \alpha_B = 2\alpha_C \qquad (1)$$

Friction condition:

$$F_{AC} = \mu_k N_{AC} \qquad (2)$$

Kinetics:

Disk *B*:

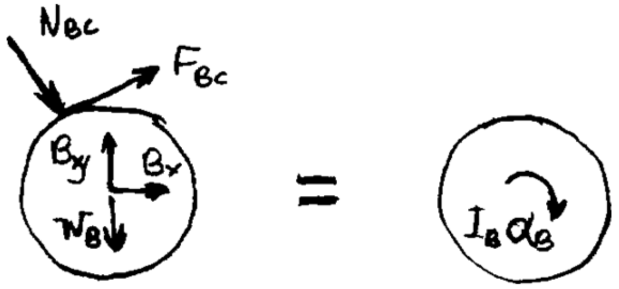
$$\circlearrowright\!+\ \Sigma F_B = \Sigma(F_B)_{\text{eff}}:$$

$$F_{BC}r = I_B\alpha_B$$

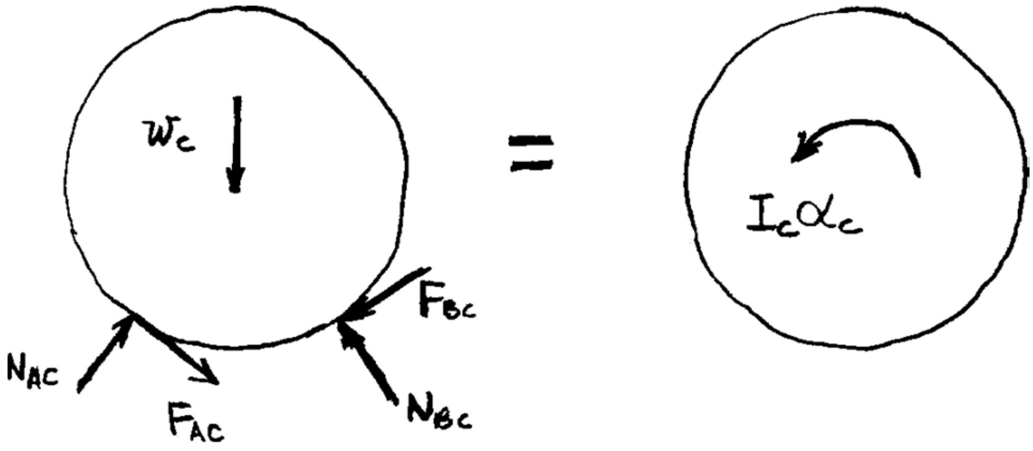
$$F_{BC} = \frac{I_B}{r}\alpha_B = \frac{2I_B}{r}\alpha_C$$

$$F_{BC} = \frac{(2)(0.0086266)}{4/12}\alpha_C$$

$$= 0.051760\alpha_C \qquad (3)$$



PROBLEM 16.45 (Continued)



Disk *C*:

$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad F_{AC}(2r) - F_{BC}(2r) = I_C\alpha_C$$

$$F_{AC} = F_{BC} + \frac{I_C}{2r}\alpha_C$$

$$= 0.051760\alpha_C + \frac{0.138027}{8/12}\alpha_C$$

$$F_{AC} = 0.25880\alpha_C \qquad (4)$$

From Eq. (2), $N_{AC} = \dfrac{F_{AC}}{0.25} = 1.03520\alpha_C \qquad (5)$

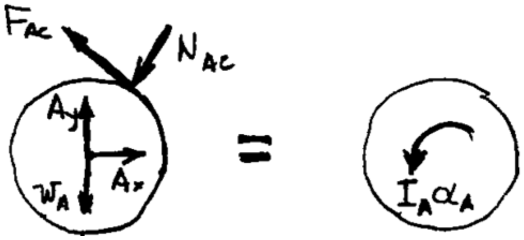
$$+\nearrow 30^\circ \Sigma F = \Sigma F_{\text{eff}}: \quad W_B \sin 30^\circ + F_{BC} - F_{AC}\cos 60^\circ - N_{AC}\sin 60^\circ = 0$$

$$(20)\sin 30^\circ + (0.051760 - 0.25880\cos 60^\circ - 1.03520\sin 60^\circ)\alpha_C = 0$$

$$10 - 0.97415\alpha_C = 0$$

$$\alpha_C = 10.2654 \text{ rad/s}^2$$

$$F_{BC} = (0.051760)(10.2654) = 0.53134 \text{ lb.}$$
$$F_{AC} = (0.25880)(10.2654) = 2.6567 \text{ lb.}$$
$$N_{AC} = (1.03520)(10.2654) = 10.6267 \text{ lb.}$$



Check that $N_{BC} > 0$.

$$+\searrow 60^\circ \Sigma F = \Sigma F_{\text{eff}}: \quad N_{BC} - W_C\cos 30^\circ + N_{AC}\cos 60^\circ - F_{AC}\sin 60^\circ = 0$$

$$N_{BC} = (20 \text{ lb})\cos 30^\circ - (10.6267 \text{ lb})\cos 60^\circ + (2.6567 \text{ lb})\sin 60^\circ = 0$$
$$N_{BC} = 14.3079 \text{ lb.}$$

PROBLEM 16.45 (Continued)

Disk *A*:

$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad F_{AC}r = I_A\alpha_A$$

$$\alpha_A = \frac{F_{AC}r}{I_A} = \frac{(2.6567)(4/12)}{0.0086266} = 102.69 \text{ rad/s}^2$$

(*a*) *Angular accelerations of disks.* $\boldsymbol{\alpha}_A = 102.7 \text{ rad/s}^2 \circlearrowleft$ ◀

From Eq. (1), $\boldsymbol{\alpha}_B = 20.5 \text{ rad/s}^2 \circlearrowright$ ◀

$\boldsymbol{\alpha}_C = 10.27 \text{ rad/s}^2 \circlearrowleft$ ◀

(*b*) *Final angular velocities.*

Disk *A*: $\omega_0 = 720 \text{ rpm} = 24\pi \text{ rad/s} \circlearrowright$

$$\omega_A = \omega_0 - \alpha_A t \circlearrowright$$
$$= 24\pi - 102.69t \circlearrowright$$

$$(v_t)_{AC} = r\omega_A = \frac{4}{12}(24\pi - 102.69t)$$

$$(\mathbf{v}_t)_{AC} = (8\pi - 34.23) \text{ ft/s} \searrow 30^\circ$$

Disk *C*: $\omega_C = \alpha_C t \circlearrowleft = 10.2654t \circlearrowleft$

$$(\mathbf{v}_t)_{CA} = 2r\omega_C = \left(\frac{8}{12}\right)(10.2654)t$$

$$(\mathbf{v}_t)_{CA} = 6.8436t \searrow 30^\circ$$

Time when tangential velocities are equal.

$$8\pi - 34.23t = 6.8436t \qquad t = 0.6119 \text{ s}$$

$\omega_A = 24\pi - (102.69)(0.6119) = 12.562 \text{ rad/s}$ $\qquad \boldsymbol{\omega}_A = 120.0 \text{ rpm} \circlearrowright$ ◀

$\omega_C = (10.2654)(0.6119) = 6.2813 \text{ rad/s}$ $\qquad \boldsymbol{\omega}_C = 60.0 \text{ rpm} \circlearrowleft$ ◀

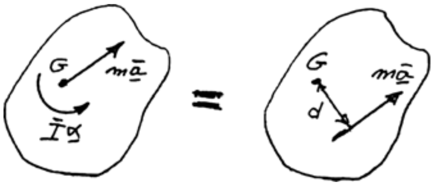
$\boldsymbol{\omega}_B = 120.0 \text{ rpm} \circlearrowright$ ◀

PROBLEM 16.46

Show that the system of the inertial terms for a rigid body in plane motion reduces to a single vector, and express the distance from the mass center G of the body to the line of action of this vector in terms of the centroidal radius of gyration $\overline{k}$ of the body, the magnitude $\overline{a}$ of the acceleration of G, and the angular acceleration α.

SOLUTION

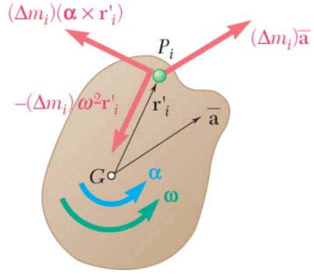
We know that the system of inertial terms can be reduced to the vector $m\overline{\mathbf{a}}$ at G and the couple $\overline{I}\boldsymbol{\alpha}$. We further know from Chapter 3 on statics that a force-couple system in a plane can be further reduced to a single force.



The perpendicular distance d from G to the line of action of the single vector $m\overline{\mathbf{a}}$ is expressed by writing

$$+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad \overline{I}\alpha = (m\overline{a})d$$

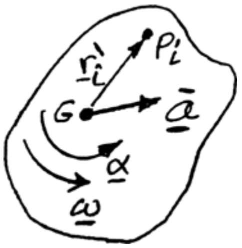
$$d = \frac{\overline{I}\alpha}{m\overline{a}} = \frac{m\overline{k}^2\alpha}{m\overline{a}} \qquad d = \frac{\overline{k}^2\alpha}{\overline{a}} \blacktriangleleft$$


$(\Delta m_i)(\boldsymbol{\alpha}\times \mathbf{r}'_i)$
P_i
$(\Delta m_i)\overline{\mathbf{a}}$
$-(\Delta m_i)\omega^2\mathbf{r}'_i$
$\mathbf{r}'_i$
$\overline{\mathbf{a}}$
G
$\boldsymbol{\alpha}$
$\boldsymbol{\omega}$

PROBLEM 16.47

For a rigid body in plane motion, show that the system of the inertial terms consists of vectors $(\Delta m_i)\,\overline{\mathbf{a}}$, $-(\Delta m_i)\omega^2\mathbf{r}_i'$, and $(\Delta m_i)(\boldsymbol{\alpha}\times\mathbf{r}_i')$ attached to the various particles P_i of the body, where $\overline{\mathbf{a}}$ is the acceleration of the mass center G of the body, $\boldsymbol{\omega}$ is the angular velocity of the body, $\boldsymbol{\alpha}$ is its angular acceleration, and $\mathbf{r}_i'$ denotes the position vector of the particle P_i, relative to G. Further show, by computing their sum and the sum of their moments about G, that the inertial terms reduce to a vector $m\overline{\mathbf{a}}$ attached at G and a couple $\overline{I}\boldsymbol{\alpha}$.

SOLUTION



Kinematics: The acceleration of P_L is

$$\mathbf{a}_i = \overline{\mathbf{a}} + \mathbf{a}_{P_i/G}$$

$$\mathbf{a}_i = \overline{\mathbf{a}} + \boldsymbol{\alpha}\times\mathbf{r}_i' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_i')$$

$$= \overline{\mathbf{a}} + \boldsymbol{\alpha}\times\mathbf{r}_i' - \omega^2\mathbf{r}_i'$$

Note: that $\boldsymbol{\alpha}\times\mathbf{r}_i'$ is $\perp$ to $\mathbf{r}_i'$

Thus, the inertial terms are as shown in Figure P16.47 (also shown above). We write

$$(\Delta m_i)\mathbf{a}_i = (\Delta m_i)\overline{\mathbf{a}} + (\Delta m_i)(\boldsymbol{\alpha}\times\mathbf{r}_i') - (\Delta m_i)\omega^2 r_i'$$

The sum of the inertial terms is

$$\Sigma(\Delta m_i)\mathbf{a}_i = \Sigma(\Delta m_i)\overline{\mathbf{a}} + \Sigma(\Delta m_i)(\boldsymbol{\alpha}\times\mathbf{r}_i') - \Sigma(\Delta m_i)\omega^2\mathbf{r}_i'$$

$$\Sigma(\Delta m_i)\mathbf{a}_i = \overline{\mathbf{a}}\,\Sigma(\Delta m_i) + \boldsymbol{\alpha}\times\Sigma(\Delta m_i)\,\mathbf{r}_i' - \omega^2\Sigma(\Delta m_i)\mathbf{r}_i'$$

We note that $\Sigma(\Delta m_i) = m.$ And since G is the mass center,

$$\Sigma(\Delta m_i)\mathbf{r}_i' = m\,\overline{r}_i' = 0$$

Thus, $$\Sigma(\Delta m_i)\mathbf{a}_i = m\overline{\mathbf{a}} \qquad (1)$$

The sum of the moments about G of the inertial terms is:

$$\Sigma(\mathbf{r}_i'\times\Delta m_i\mathbf{a}_i) = \Sigma\mathbf{r}_i'\times\Delta m_i\overline{\mathbf{a}} + \Sigma\mathbf{r}_i'\times(\Delta m_i)(\boldsymbol{\alpha}\times\mathbf{r}_i') - \Sigma\mathbf{r}_i'\times(\Delta m_i)\omega^2\mathbf{r}_i'$$

$$\Sigma(\mathbf{r}_i'\times\Delta m_i\mathbf{a}_i) = (\Sigma\mathbf{r}_i'\Delta m_i)\,\overline{\mathbf{a}} + \Sigma[\mathbf{r}_i'\times(\boldsymbol{\alpha}\times\mathbf{r}_i')\Delta m_i] - \omega^2\Sigma(\mathbf{r}_i'\times\mathbf{r}_i')\Delta m_i$$

Since G is the mass center, $\Sigma\mathbf{r}_i'\Delta m_i = 0$

Also, for each particle, $\mathbf{r}_i'\times\mathbf{r}_i' = 0$

Thus, $$\Sigma(\mathbf{r}_i'\times\Delta m_i\mathbf{a}_i) = \Sigma[\mathbf{r}_i'\times(\boldsymbol{\alpha}_i\times\mathbf{r}_i')\Delta m_i]$$

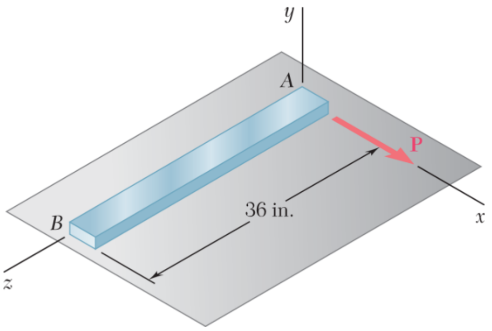
PROBLEM 16.47 (Continued)

Since $\alpha \perp \mathbf{r}_i'$, we have $\mathbf{r}_i' \times (\boldsymbol{\alpha} \times \mathbf{r}_i') = r_i^2 \alpha$ and

$$\begin{aligned}\Sigma(\mathbf{r}_{ii}' \times \Delta m_i \mathbf{a}_i) &= \Sigma r_i'^{\,2}\ (\Delta m_i)\alpha \\ &= (\Sigma r_i'^{\,2}\, \Delta m_i)\boldsymbol{\alpha}\end{aligned}$$

Since $\Sigma r_i'^{\,2}\ \Delta m_i = \bar{I}$ $\qquad \Sigma(\mathbf{r}_i' \times \Delta m_i \mathbf{a}_i) = \bar{I}\,\alpha$ (2)

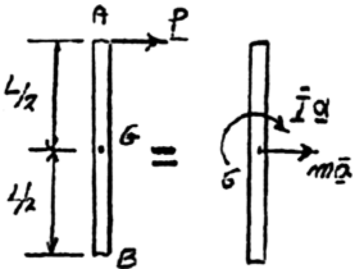
From Eqs. (1) and (2) we conclude that system of inertial terms reduce to $m\bar{\mathbf{a}}$ attached at G and a couple $\bar{I}\boldsymbol{\alpha}$.



PROBLEM 16.48

A uniform slender rod *AB* rests on a frictionless horizontal surface, and a force **P** of magnitude 0.25 lb is applied at *A* in a direction perpendicular to the rod. Knowing that the rod weighs 1.75 lb, determine (*a*) the acceleration of Point *A*, (*b*) the acceleration of Point *B*, (*c*) the location of the point on the bar that has zero acceleration.

SOLUTION



$$m = \frac{W}{g}$$

$$I = \frac{1}{12}\frac{W}{g}L^2$$

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad P = m\bar{a} = \frac{W}{g}\bar{a}$$

$$\bar{a} = \frac{P}{W}g = \frac{0.25\text{ lb}}{1.75\text{ lb}}g = \frac{1}{7}g \qquad \bar{\mathbf{a}} = \frac{1}{7}g \longrightarrow$$

$$+\circlearrowright \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad P\frac{L}{2} = \bar{I}\alpha = \frac{1}{12}\frac{W}{g}L^2\alpha$$

$$\alpha = 6\frac{P}{W}\frac{5}{L} = 6\frac{0.25\text{ lb}}{1.75\text{ lb}}\cdot\frac{g}{L} = \frac{6}{7}\frac{g}{L} \qquad \boldsymbol{\alpha} = \frac{6}{7}\frac{g}{L}\circlearrowright$$

We calculate the accelerations immediately after the force is applied. After the rod acquires angular velocity, there will be additional normal accelerations.

(*a*) Acceleration of Point *A*.

$$\xrightarrow{+} \mathbf{a}_A = \bar{\mathbf{a}} + \frac{L}{2}\alpha = \frac{1}{7}g + \frac{L}{2}\cdot\frac{6}{7}g = \frac{4}{7}g = \frac{4}{7}(32.2\text{ ft/s}^2) \qquad \mathbf{a}_A = 18.40\text{ ft/s}^2 \longrightarrow \blacktriangleleft$$

(*b*) Acceleration of Point *B*.

$$\xrightarrow{+} \mathbf{a}_B = \bar{\mathbf{a}} - \frac{L}{2}\alpha = \frac{1}{7}g - \frac{L}{2}\cdot\frac{6}{7}g = -\frac{2}{7}g = -\frac{2}{7}(32.2\text{ ft/s}^2) \qquad \mathbf{a}_B = 9.20\text{ ft/s}^2 \longleftarrow \blacktriangleleft$$

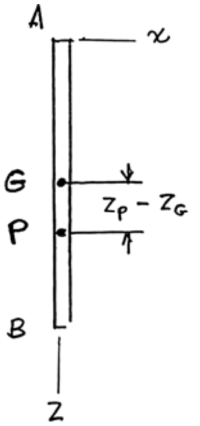
PROBLEM 16.48 (Continued)

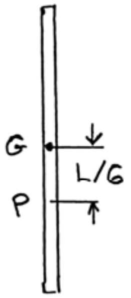
(*c*) Point of zero acceleration.

$$a_P = 0$$

$$\overline{a} - (z_P - z_G)\alpha = 0$$

$$z_P - z_G = \frac{\overline{a}}{\alpha} = \frac{\frac{1}{7}g}{\frac{6}{7}\cdot\frac{g}{L}} = \frac{1}{6}L$$



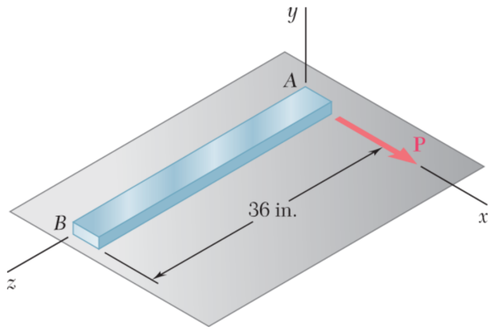


Since $z_G = \dfrac{1}{2}L$

$$z_P = \frac{1}{2}L + \frac{1}{6}L = \frac{2}{3}L$$

$$z_P = \frac{2}{3}(36 \text{ in.})$$

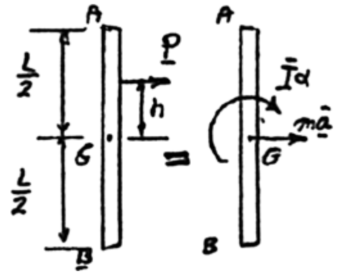
$z_P = 24.0$ in. ◀



PROBLEM 16.49

(*a*) In Problem 16.48, determine the point of the rod *AB* at which the force **P** should be applied if the acceleration of Point *B* is to be zero. (*b*) Knowing that $P = 0.25$ lb, determine the corresponding acceleration of Point *A*.

SOLUTION



$$\xrightarrow{+} \Sigma F_x = \Sigma (F_X)_{\text{eff}}: \quad P = m\bar{a} = \frac{W}{g}\bar{a}$$

$$\mathbf{a} = \frac{P}{W} g \longrightarrow$$

$$\circlearrowright^{+} \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad Ph = \bar{I}\alpha = \frac{1}{12}\frac{W}{g}L^2\alpha$$

$$\boldsymbol{\alpha} = \frac{12Ph}{WL^2} g \circlearrowright$$

(*a*) Position of force *P*.

$$\xrightarrow{+} \mathbf{a}_B = \bar{\mathbf{a}} - \frac{L}{2}\alpha$$

$$0 = \frac{P}{W}g - \frac{L}{2}\cdot\frac{12Ph}{WL^2}g$$

$$h = \frac{L}{6} = \frac{36 \text{ in.}}{6} = 6 \text{ in.}$$

Thus, *P* is located 12 in. from end *A*. ◀

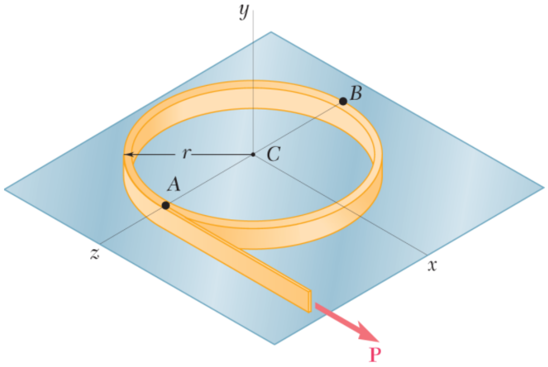
For $h = \frac{L}{6}$: $\alpha = \frac{12P\left(\frac{L}{6}\right)}{WL^2}g = 2\frac{P}{W}\cdot\frac{g}{L} \circlearrowright$

(*b*) Acceleration of Point *A*.

$$\xrightarrow{+} a_A = \bar{a} + \frac{L}{2}\alpha = \frac{P}{W}g + \frac{L}{2}\cdot 2\frac{P}{W}\frac{g}{L} = 2\frac{P}{W}g$$

$$a_A = 2\frac{0.25 \text{ lb}}{1.75 \text{ lb}}(32.2 \text{ ft/s}^2)$$

$\mathbf{a}_A = 9.20 \text{ ft/s}^2 \longrightarrow$ ◀

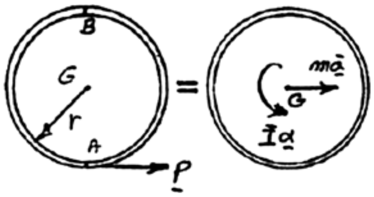


PROBLEM 16.50

A force **P** of magnitude 3 N is applied to a tape wrapped around the body indicated. Knowing that the body rests on a frictionless horizontal surface, determine the acceleration of (*a*) Point *A*, (*b*) Point *B*.

A thin hoop of mass 2.4 kg.

SOLUTION



Hoop: $\bar{I} = mr^2$

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad P = m\bar{a}$$

$$\bar{\mathbf{a}} = \frac{P}{m} \longrightarrow$$

$$+\circlearrowleft \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad Pr = \bar{I}\alpha = mr^2\alpha$$

$$\boldsymbol{\alpha} = \frac{P}{mr} \circlearrowleft$$

(*a*) Acceleration of Point *A*.

$$\xrightarrow{+} a_A = \bar{a} + r\alpha = \frac{P}{m} + r\left(\frac{P}{mr}\right) = 2\frac{P}{m}$$

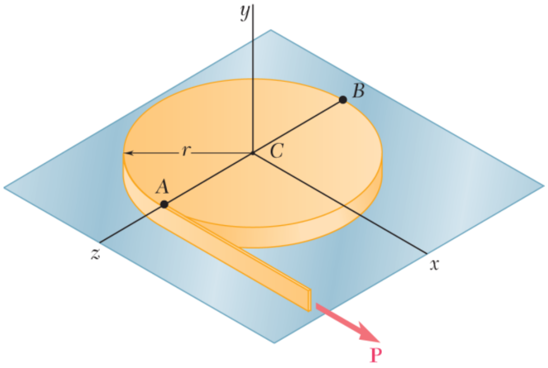
$$a_A = 2\frac{3\text{ N}}{2.4\text{ kg}} = 2.5\text{ m/s}^2$$

$\mathbf{a}_A = 2.50\text{ m/s}^2 \longrightarrow$ ◀

(*b*) Acceleration of Point *B*.

$$\xrightarrow{+} a_B = \bar{a} - r\alpha = \frac{P}{m} - r\left(\frac{P}{mr}\right) = 0$$

$\mathbf{a}_B = 0$ ◀

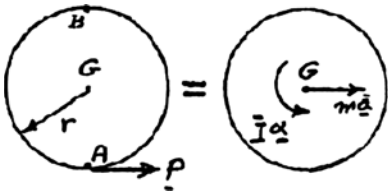


PROBLEM 16.51

A force **P** of magnitude 3 N is applied to a tape wrapped around the body indicated. Knowing that the body rests on a frictionless horizontal surface, determine the acceleration of (*a*) Point *A*, (*b*) Point *B*.

A uniform disk of mass 2.4 kg.

SOLUTION



Disk: $\bar{I} = \frac{1}{2}mr^2$

$$\xrightarrow{+} \Sigma F_X = \Sigma(F_X)_{\text{eff}}: \quad P = m\bar{a}$$

$$\bar{\mathbf{a}} = \frac{P}{m} \longrightarrow$$

$$+\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad Pr = \bar{I}\alpha$$

$$Pr = \frac{1}{2}mr^2\alpha$$

$$\alpha = \frac{2P}{mr} \circlearrowright$$

(*a*) Acceleration of Point *A*.

$$\xrightarrow{+} a_A = \bar{a} + r\alpha = \frac{P}{m} + r\cdot\frac{2P}{mr} = 3\frac{P}{m}$$

$$a_A = 3\frac{3\text{ N}}{2.4\text{ kg}} = 3.75\text{ m/s}^2$$

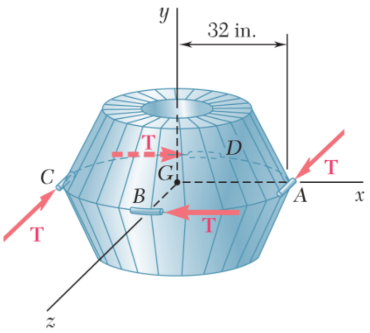
$$\mathbf{a}_A = 3.75\text{ m/s}^2 \longrightarrow \blacktriangleleft$$

(*b*) Acceleration of Point *B*.

$$\xrightarrow{+} a_B = \bar{a} - r\alpha = \frac{P}{m} - r\cdot\frac{2P}{mr} = -\frac{P}{m}$$

$$a_B = -\frac{3\text{ N}}{2.4\text{ kg}} = -1.25\text{ m/s}^2$$

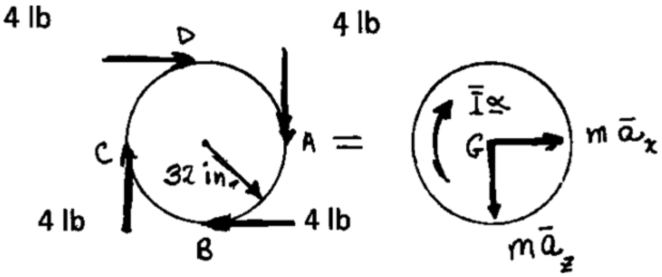
$$\mathbf{a}_B = 1.25\text{ m/s}^2 \longleftarrow \blacktriangleleft$$



PROBLEM 16.52

A 250-lb satellite has a radius of gyration of 24 in. with respect to the *y* axis and is symmetrical with respect to the *zx* plane. Its orientation is changed by firing four small rockets *A*, *B*, *C*, and *D*, each of which produces a 4-lb thrust **T** directed as shown. Determine the angular acceleration of the satellite and the acceleration of its mass center *G* (*a*) when all four rockets are fired, (*b*) when all rockets except *D* are fired.

SOLUTION



$$W = 250\text{ lb},\qquad m = \frac{W}{g}\qquad \bar{I} = mk_y^2 = \left(\frac{250\text{ lb}}{32.2\text{ ft/s}^2}\right)\left(\frac{24\text{ in.}}{12}\right)^2 = 31.056\text{ slug}\cdot\text{ft}^2$$

(*a*) With all four rockets fired:

$$\Sigma F = \Sigma F_{\text{eff}}:\qquad 0 = m\bar{a}\qquad\qquad \bar{\mathbf{a}} = 0 \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}:\quad 4Tr = \bar{I}\alpha$$

$$-4(4\text{ lb})\left(\frac{32\text{ in.}}{12}\right) = 31.056\alpha\qquad\qquad \boldsymbol{\alpha} = -1.3739\text{ rad/s}^2 \circlearrowright$$

$$\boldsymbol{\alpha} = -(1.374\text{ rad/s}^2)\mathbf{j} \blacktriangleleft$$

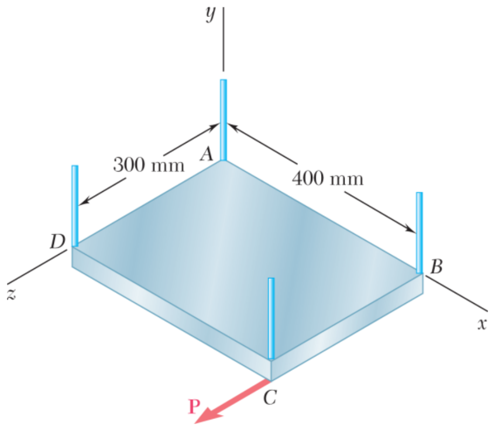
(*b*) With all rockets except *D*:

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}:\quad -4\text{ lb} = \frac{250}{32.2}\bar{a}_x\qquad \bar{a}_x = -0.51520\text{ ft/s}^2$$

$$+\downarrow \Sigma F_z = \Sigma(F_z)_{\text{eff}}:\qquad 0 = \frac{250}{32.2}\bar{a}\qquad \bar{a}_z = 0\qquad\qquad \bar{\mathbf{a}} = -(0.515\text{ ft/s}^2)\mathbf{i} \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}:\quad 3Tr = \bar{I}\alpha$$

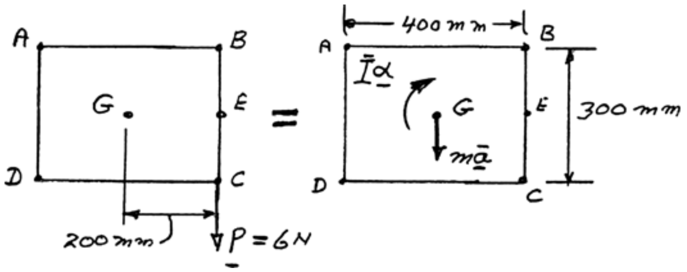
$$-3(4\text{ lb})\left(\frac{32\text{ in.}}{12}\right) = 31.056\alpha\qquad\qquad \boldsymbol{\alpha} = -(1.030\text{ rad/s}^2)\mathbf{j} \blacktriangleleft$$


y
300 mm
A
400 mm
D
B
z
x
P
C

PROBLEM 16.53

A rectangular plate of mass 5 kg is suspended from four vertical wires, and a force **P** of magnitude 6 N is applied to corner *C* as shown. Immediately after **P** is applied, determine the acceleration of (*a*) the midpoint of edge *BC*, (*b*) corner *B*.

SOLUTION


A
B
G
E
=
D
C
200 mm
P = 6N
400 mm
A
B
$\bar{I}\alpha$
G
E
300 mm
$m\bar{a}$
D
C

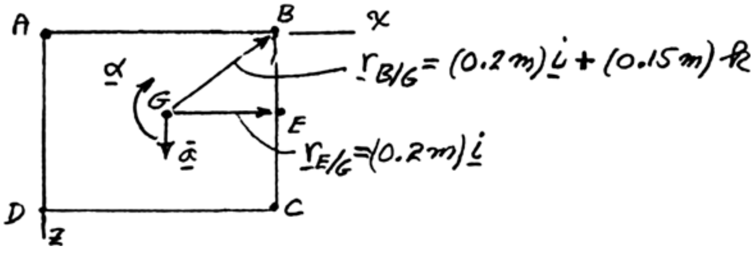
$$\bar{I} = \frac{1}{12}m(b^2 + h^2) = \frac{1}{12}(5\text{ kg})[(0.4\text{ m})^2 + (0.3\text{ m})^2] = 0.10417\text{ kg}\cdot\text{m}^2$$

$$+\downarrow \Sigma F = \Sigma F_{\text{eff}}: \qquad P = m\bar{a}$$

$$6N = (5\text{ kg})\bar{a} \qquad \bar{a} = +(1.2\text{ m/s}^2)\mathbf{k}$$

$$+\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad P(0.2\text{ m}) = \bar{I}\alpha$$

$$(6\text{ N})(0.2\text{ m}) = (0.10417\text{ kg}\cdot\text{m}^2)\alpha \qquad \boldsymbol{\alpha} = -(11.52\text{ rad/s}^2)\mathbf{j}$$


A
B
x
$\mathbf{r}_{B/G} = (0.2\text{ m})\mathbf{i} + (0.15\text{ m})\mathbf{k}$
α
G
E
$\bar{a}$
$\mathbf{r}_{E/G} = (0.2\text{ m})\mathbf{i}$
D
C
z

(*a*)

$$\mathbf{a}_E = \bar{\mathbf{a}} + \boldsymbol{\alpha} \times \mathbf{r}_{E/G}$$

$$= +(1.2\text{ m/s}^2)\mathbf{k} - (11.52\text{ rad/s}^2)\mathbf{j} \times (0.2\text{ m})\mathbf{i}$$

$$= +(1.2\text{ m/s}^2)\mathbf{k} + (2.304\text{ m/s}^2)\mathbf{k} \qquad \mathbf{a}_E = (3.50\text{ m/s}^2)\mathbf{k} \blacktriangleleft$$

PROBLEM 16.53 (Continued)

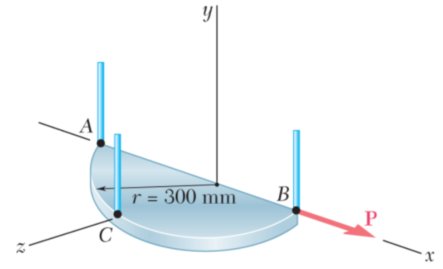
(*b*)

$$\mathbf{a}_B = \overline{\mathbf{a}} + \boldsymbol{\alpha} \times \mathbf{r}_{B/G}$$

$$= +(1.2 \text{ m/s}^2)\mathbf{k} - (11.52 \text{ rad/s})\mathbf{j} \times [(0.2 \text{ m})\mathbf{i} + (0.15 \text{ m})\mathbf{k}]$$

$$= +(1.2 \text{ m/s}^2)\mathbf{k} + (2.304 \text{ m/s}^2)\mathbf{k} + (1.728 \text{ m/s}^2)\mathbf{i}$$

$$\mathbf{a}_B = (1.728 \text{ m/s}^2)\mathbf{i} + (3.5 \text{ m/s}^2)\mathbf{k} \blacktriangleleft$$

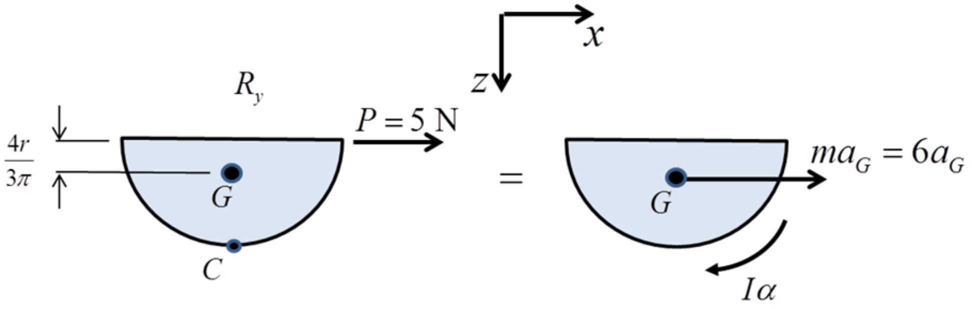


PROBLEM 16.54

A uniform semicircular plate of mass 6 kg is suspended from three vertical wires at points *A*, *B*, and *C*, and a force **P** of magnitude 5 N is applied to point *B*. Immediately after **P** is applied, determine the acceleration of (*a*) the mass center of the plate, (*b*) point *C*.

SOLUTION

Free Body Diagram:



$$\bar{I}\alpha = \left[\left(\frac{1}{2}mr^2\right) - m\left(\frac{4r}{3\pi}\right)^2\right]\alpha = mr^2\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)\alpha$$

$$= (6\text{ kg})(0.3\text{ m})^2\left(0.5 - \frac{1.7777}{\pi^2}\right)\alpha = \left(0.17273\text{ kg}\cdot\text{m}^2\right)\alpha$$

(*a*) $\Sigma F_x = 5\text{ N} = 6\text{ kg}\,a_G, \quad a_G = 0.8333\text{ m/s}$

or $a_G = 0.833\mathbf{i}\text{ m/s}^2$ ◀

$$\Sigma M_G = 5N\left(\frac{0.4}{\pi}\text{ m}\right) = \left(0.17273\text{ kg m}^2\right)\alpha$$

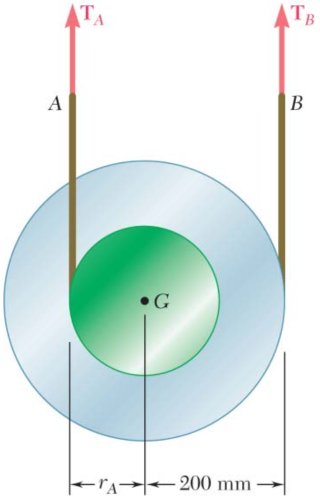
$$\alpha = 3.6856\text{ rad/s}^2 \curvearrowright$$

$$\mathbf{a}_C = \mathbf{a}_G + \mathbf{a}_{C/G} = \mathbf{a}_C = 0.8333 \longrightarrow + \longleftarrow \left(0.3 - \frac{0.4}{\pi}\right)\alpha$$

$$\mathbf{a}_C = 0.8333 \longrightarrow + \longleftarrow 0.63641$$

$$= 0.1969 \longrightarrow \text{m/s}^2$$

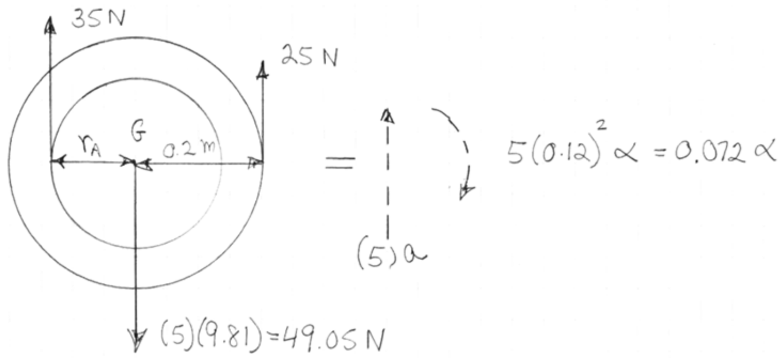
or $\mathbf{a}_C = 0.1969\mathbf{i}\text{ m/s}^2$ ◀


$\mathbf{T}_A$
$\mathbf{T}_B$
A
B
G
r_A
200 mm

PROBLEM 16.55

A drum of 200-mm radius is attached to a disk of radius r_A = 150 mm. The disk and drum have a combined mass of 5 kg and a combined radius of gyration of 120 mm and are suspended by two cords. Knowing that T_A = 35 N and T_B = 25 N, determine the accelerations of points A and B on the cords.

SOLUTION


35 N
25 N
r_A
G
0.2 m
=
$5(0.12)^2 \alpha = 0.072\alpha$
(5)a
(5)(9.81)=49.05 N

$$+\uparrow \sum F_y = (5\text{ kg})a = (35 + 25 - 49.05)\ N; \quad a = 2.19\text{ m/s}^2 \uparrow$$

$$+\circlearrowright \sum M_G = 35\ N(0.15\text{ m}) - 25\ N(0.2\text{ m}) = 0.072\alpha$$

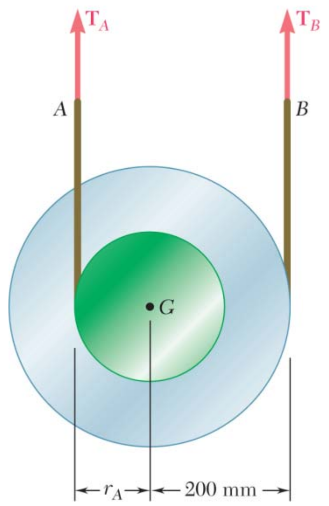
$$\alpha = 3.4722\text{ rad/s}^2$$

$$\mathbf{a}_A = \uparrow 2.19\text{ m/s}^2 + \uparrow 3.4722\text{ rad/s}^2 (0.15\text{ m}) = 2.711\text{ m/s}^2$$

or $\mathbf{a}_A = 2.71\text{ m/s}^2 \uparrow$ ◀

$$\mathbf{a}_B = \uparrow 2.19\text{ m/s}^2 + \downarrow 3.4722\text{ rad/s}^2 (0.2\text{ m}) = 1.496\text{ m/s}^2 \uparrow$$

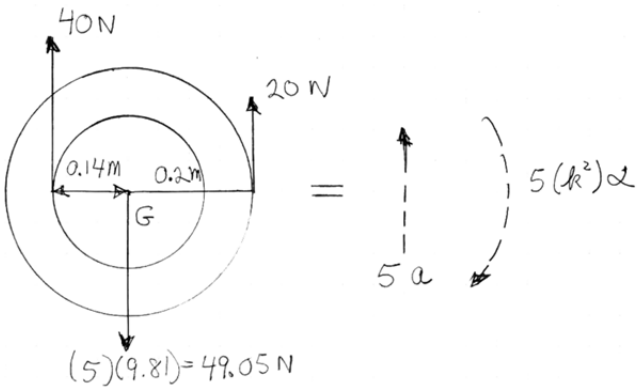
or $\mathbf{a}_B = 1.496\text{ m/s}^2 \uparrow$ ◀



PROBLEM 16.56

A drum of 200-mm radius is attached to a disk of radius r_A = 140 mm. The disk and drum have a combined mass of 5 kg and are suspended by two cords. Knowing that the acceleration of point B on the cord is zero, T_A = 40 N, and T_B = 20 N, determine the combined radius of gyration of the disk and drum.

SOLUTION



$$a_B = 0: \quad a = 0.2\alpha$$

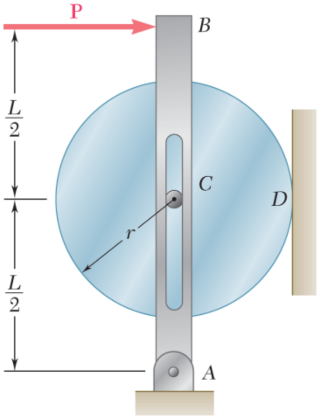
$$+\uparrow \; \Sigma F_y = 5(a) = (40 + 20 - 49.05)\ N$$

$$a = 2.19\ \text{m/s}^2 \uparrow \qquad \alpha = 10.95\ \text{rad/s}^2 \circlearrowright$$

$$+\circlearrowright \; \Sigma M_G = 40\ N(0.14\ \text{m}) - 20\ N(0.2\ \text{m}) = (5\ \text{kg})k^2(10.95\ \text{rad/s}^2)$$

$$k^2 = 0.02922\ \text{m}^2 \qquad k = 0.17095\ \text{m}$$

or $k = 170.9$ mm ◀

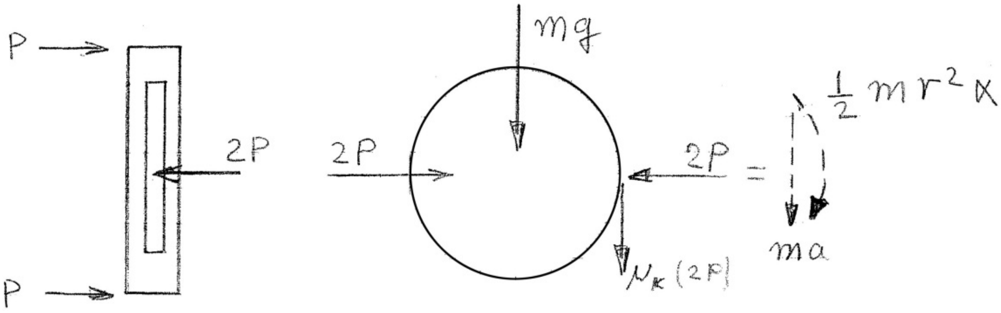


PROBLEM 16.57

The 12-lb uniform disk shown, of radius $r = 3.2$ in., rotates counterclockwise. Its center *C* is constrained to move in a slot cut in the vertical member *AB* and a 11-lb horizontal force **P** is applied at *B* to maintain contact at *D* between the disk and the vertical wall. The disk moves downward under the influence of gravity and the friction at *D*. Knowing that the coefficient of kinetic friction between the disk and the wall is 0.12 and neglecting friction in the vertical slot, determine (*a*) the angular acceleration of the disk, (*b*) the acceleration of the center *C* of the disk.

SOLUTION

Free Body Diagram:



$$+\circlearrowright\ \Sigma M_G = \mu_k(2P)r = \frac{1}{2}mr^2\alpha$$

$$\alpha = \frac{4\mu_k P}{mr}$$

(*a*)

$$\alpha = \frac{4(0.12)(11\text{ lb})}{\left(\frac{12\text{ lb}}{32.2\text{ ft/s}^2}\right)\left(\frac{3.2}{12}\text{ft}\right)}$$

or $\boldsymbol{\alpha} = 53.1\text{ rad/s}^2 \circlearrowright$ ◀

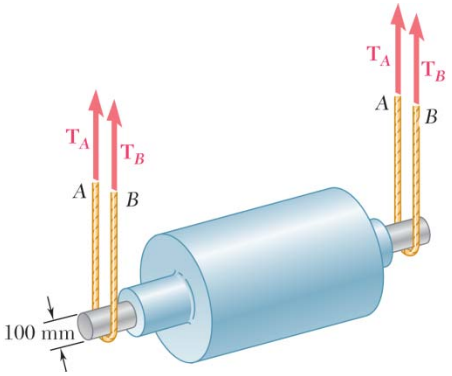
$$+\downarrow\ \Sigma F_y = mg + \mu_k(2P) = ma$$

$$a = g + \frac{2\mu_k P}{m}$$

(*b*)

$$a = 32.2\text{ ft/s}^2 + \frac{2(0.12)(11\text{ lb})}{\left(\frac{12\text{ lb}}{32.2\text{ ft/s}^2}\right)}$$

or $\mathbf{a} = 39.3\text{ ft/s}^2 \downarrow$ ◀



PROBLEM 16.58

The steel roll shown has a mass of 1200 kg, a centroidal radius of gyration of 150 mm, and is lifted by two cables looped around its shaft. Knowing that for each cable $T_A = 3100$ N and $T_B = 3300$ N, determine (*a*) the angular acceleration of the roll, (*b*) the acceleration of its mass center.

SOLUTION

Data:

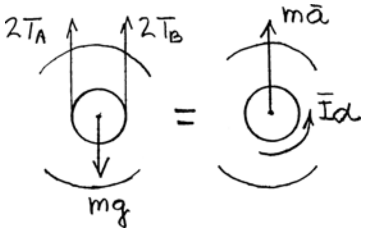
$$m = 1200 \text{ kg}$$

$$\bar{I} = mk^2 = (1200)(0.150)^2 = 27 \text{ kg}\cdot\text{m}^2$$

$$r = \frac{1}{2}d = \frac{1}{2}(0.100) = 0.050 \text{ m}$$

$$T_A = 3100 \text{ N}$$

$$T_B = 3300 \text{ N}$$



(*a*) Angular acceleration.

$$+\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad 2T_B r - 2T_A r = \bar{I}\alpha$$

$$\alpha = \frac{2(T_B - T_A)r}{\bar{I}}$$

$$= \frac{(2)(3300 - 3100)(0.050)}{27}$$

$\boldsymbol{\alpha} = 0.741 \text{ rad/s}^2 \circlearrowright$ ◀

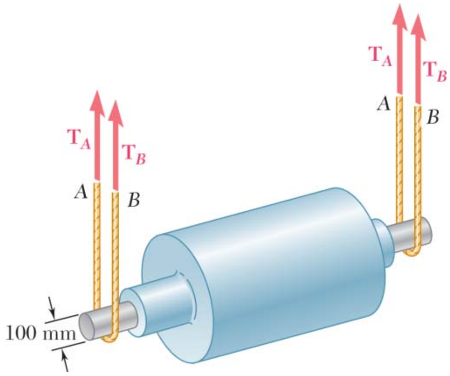
(*b*) Acceleration of mass center.

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad 2T_A + 2T_B - mg = m\bar{a}$$

$$\bar{a} = \frac{2(T_A + T_B)}{m} - g$$

$$= \frac{2(3100 + 3300)}{1200} - 9.81$$

$\bar{\mathbf{a}} = 0.857 \text{ m/s}^2 \uparrow$ ◀



PROBLEM 16.59

The steel roll shown has a mass of 1200 kg, has a centriodal radius of gyration of 150 mm, and is lifted by two cables looped around its shaft. Knowing that at the instant shown the acceleration of the roll is 150 mm/s^2 downward and that for each cable $T_A = 3000$ N, determine (*a*) the corresponding tension T_B, (*b*) the angular acceleration of the roll.

SOLUTION

Data:

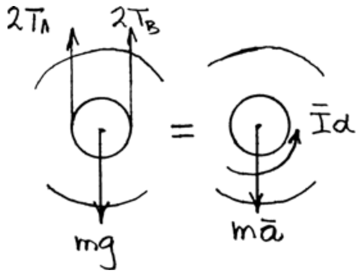
$$m = 1200 \text{ kg}$$

$$\bar{I} = mk^2 = (1200)(0.150)^2 = 27 \text{ kg}\cdot\text{m}^2$$

$$r = \frac{1}{2}d = \frac{1}{2}(0.100) = 0.050 \text{ m}$$

$$T_A = 3000 \text{ N}$$

$$\bar{\mathbf{a}} = 0.150 \text{ m/s}^2 \downarrow$$



(*a*) Tension in cable *B*.

$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad 2T_A + 2T_B - mg = -m\bar{a}$$

$$T_B = \frac{mg - ma}{2} - T_A$$

$$= \frac{m(g - \bar{a})}{2} - T_A$$

$$= \frac{(1200)(9.81 - 0.150)}{2} - 3000$$

$$= 2796 \text{ N}$$

$T_B = 2800$ N ◀

(*b*) Angular acceleration.

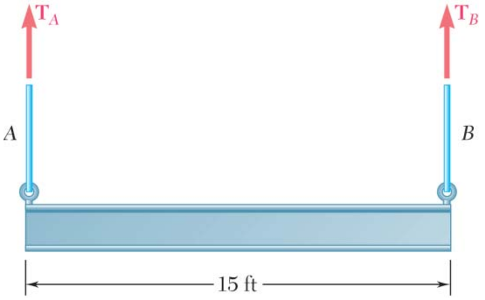
$$+\circlearrowleft \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad 2T_B r - 2T_A r = \bar{I}\alpha$$

$$\alpha = \frac{2(T_B - T_A)r}{\bar{I}}$$

$$= \frac{(2)(2796 - 3000)}{27}$$

$$= -15.11 \text{ rad/s}^2$$

$\boldsymbol{\alpha} = 15.11 \text{ rad/s}^2 \circlearrowright$ ◀



PROBLEM 16.60

A 15-ft beam weighing 500 lb is lowered by means of two cables unwinding from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow the unwinding motion. Knowing that the deceleration of cable *A* is 20 ft/s^2 and the deceleration of cable *B* is 2 ft/s^2, determine the tension in each cable.

SOLUTION

Kinematics:

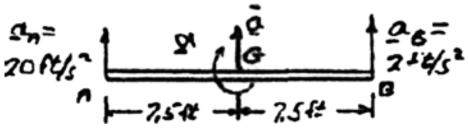
$$a_B = a_A + (15\text{ ft})\alpha$$

$$2\uparrow = 20\uparrow + 15\alpha$$

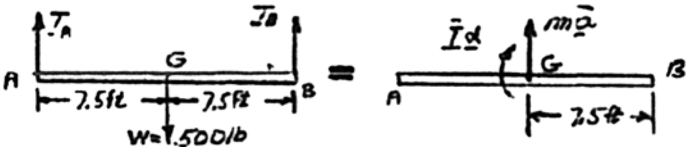
$$\alpha = 1.2\text{ rad/s}^2 \circlearrowright$$

$$\bar{a} = \frac{1}{2}(a_A + a_B) = \frac{1}{2}(2+20)$$

$$\bar{\mathbf{a}} = 11\text{ ft/s}^2\uparrow$$



Kinetics:



$$\bar{I} = \frac{1}{12}mL^2$$

$$= \frac{1}{12}\frac{500}{32.2\text{ ft/s}^2}(15\text{ ft})^2$$

$$= 291.15\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$+\circlearrowright \Sigma M_B = \Sigma(M_B)_{\text{eff}}:\quad T_A(15\text{ ft}) - W(2.5\text{ ft}) = m\bar{a}(7.5\text{ ft}) + \bar{I}\alpha$$

$$T_A(15\text{ ft}) - (500\text{ lb})(7.5\text{ ft}) = \frac{500\text{ lb}}{32.2\text{ ft/s}^2}(11\text{ ft/s}^2)(7.5\text{ ft})$$

$$+(291.15\text{ lb}\cdot\text{ft/s})(1.2\text{ rad/s}^2)$$

$$15T_A - 3750 = 1281 + 349.3$$

$$T_A = 358.7\text{ lb}$$

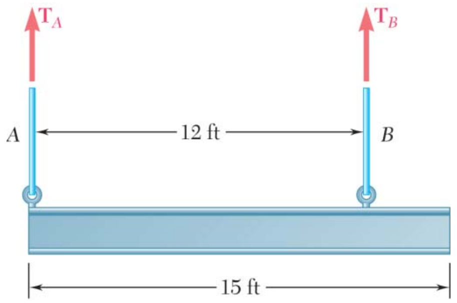
$T_A = 359\text{ lb}$ ◀

$$+\uparrow\ \Sigma F = \Sigma F_{\text{eff}}:\quad T_A + T_B - W = m\bar{a}$$

$$358.7\text{ lb} + T_B - 500\text{ lb} = \frac{500\text{ lb}}{32.2\text{ ft/s}^2}(11\text{ ft/s}^2)$$

$$T_B = 312.2\text{ lb}$$

$T_B = 312\text{ lb}$ ◀


T_A
T_B
A
12 ft
B
15 ft

PROBLEM 16.61

A 15-ft beam weighing 500 lb is lowered by means of two cables unwinding from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow the unwinding motion. Knowing that the deceleration of cable A is 20 ft/s^2 and the deceleration of cable B is 2 ft/s^2, determine the tension in each cable.

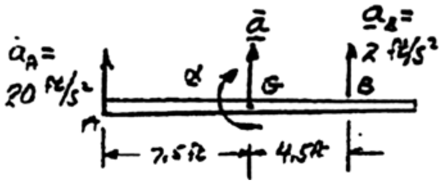
SOLUTION

Kinematics:

$$a_B = a_A + 12\alpha$$

$$2\uparrow = 20\uparrow + 12\alpha$$

$$\alpha = 1.5\ \text{rad/s}^2 \circlearrowright$$


$a_A =$ 20 ft/s²
$\bar a$
$a_B =$ 2 ft/s²
α
G
B
A
7.5 ft
4.5 ft

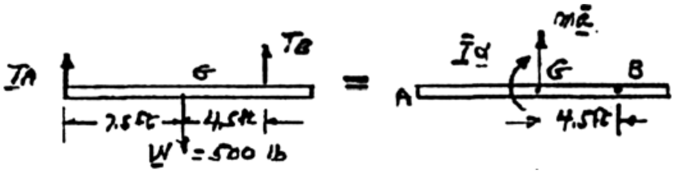
$$\bar{\mathbf{a}} = a_A + 7.5\alpha$$

$$= 20\uparrow + (7.5)(1.5)\downarrow$$

$$\bar{\mathbf{a}} = 8.75\ \text{ft/s}^2\uparrow$$

$$\bar I = \frac{1}{2}mL^2 = \frac{1}{12}\frac{500}{32.2}(15)^2 = 291.15\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

Kinetics:


T_A
G
T_B
7.5 ft
4.5 ft
W = 500 lb
=
$\bar I\alpha$
$m\bar a$
A
G
B
4.5 ft

$$+\circlearrowright \Sigma M_B = \Sigma(M_B)_{\text{eff}}:\quad T_A(12\ \text{ft}) + W(4.5\ \text{ft}) = m\bar a(4.5\ \text{ft}) + \bar I\alpha$$

$$T_A(12\ \text{ft}) - (500\ \text{lb})(4.5\ \text{ft}) = \frac{500\ \text{lb}}{32.2\ \text{ft/s}^2}(8.75\ \text{ft/s}^2)(4.5\ \text{ft}) + (291.15)(1.5\ \text{rad/s}^2)$$

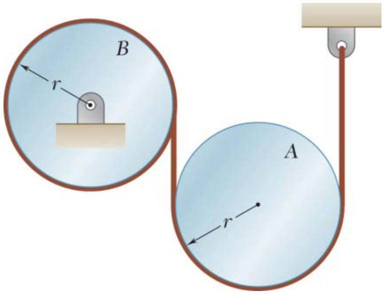
$$12T_A - 2250 = 611.4 + 436.7$$

$T_A = 275$ lb ◀

$$+\uparrow\ \Sigma F = \Sigma F_{\text{eff}}:\quad T_A + T_B - W = m\bar a$$

$$275\ \text{lb} + T_B - 500 = \frac{500}{32.2}(8.75)$$

$T_B = 361$ lb ◀



PROBLEM 16.62

Two uniform cylinders, each of weight $W = 14$ lb and radius $r = 5$ in., are connected by a belt as shown. If the system is released from rest, determine (*a*) the angular acceleration of each cylinder, (*b*) the tension in the portion of belt connecting the two cylinders, (*c*) the velocity of the center of the cylinder *A* after it has moved through 3 ft.

SOLUTION

Kinematics

Let $\mathbf{a}_A = a_A \downarrow$ be the acceleration of the center of cylinder *A*, $\mathbf{a}_{AB} = a_{AB} \downarrow$ be acceleration of the cord between the disks, $\boldsymbol{\alpha}_A = \alpha_A \circlearrowleft$ be the angular acceleration of disk *A*, and $\boldsymbol{\alpha}_B = \alpha_B \circlearrowright$ be the angular acceleration of disk *B*.

$$a_A = r\alpha_A \tag{1}$$

$$a_{AB} = a_A + r\alpha_A = 2r\alpha_A = r\alpha_B \tag{2}$$

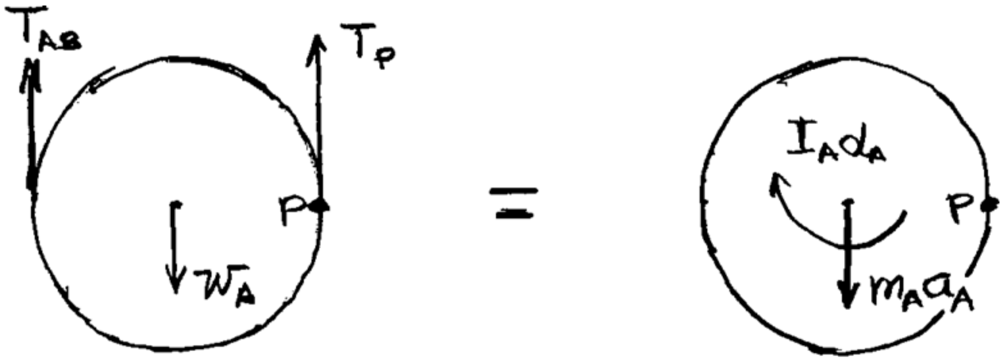
Masses and moments of inertia

$$m_A = m_B = m \tag{3}$$

$$\bar{I}_A = \bar{I}_B = \frac{1}{2}mr^2 \tag{4}$$

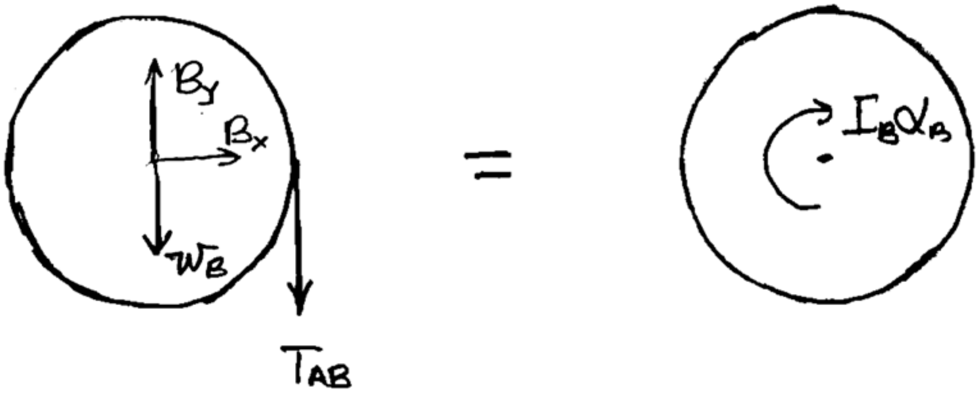
Kinetics: Let T_{AB} be the tension in the portion of the cable between disks *A* and *B*.

Disk *A*: $\circlearrowleft \Sigma M_P = \Sigma(M_P)_{\text{eff}}: \quad rW_A - 2rT_{AB} = rm_A a_A + \bar{I}_A \alpha_A$ (5)



PROBLEM 16.62 (Continued)

Disk B: $\circlearrowright \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad rT_{AB} = \bar{I}_B\alpha_B$ (6)



Add $2\times$ Eq. (6) to Eq. (5) to eliminate T_{AB}.

$$rW_A = rm_Aa_A + \bar{I}_A\alpha_A + 2\bar{I}_B\alpha_B \tag{7}$$

Use Eqs. (1) and (2) to eliminate a_A and α_B

$$\begin{aligned} rW_A &= rm_A(r\alpha_A) + \bar{I}_A\alpha_A + 2\bar{I}_B\cdot(2\alpha_A) \\ &= (m_Ar^2 + \bar{I}_A + 4\bar{I}_B)\alpha_A \\ &= \left[mr^2 + \frac{1}{2}mr^2 + 4\left(\frac{1}{2}mr^2\right)\right]\alpha_A \\ &= 3.5\,mr^2\alpha_A \end{aligned}$$

$$\alpha_A = \frac{rW_A}{3.5\,mr^2} = \frac{1g}{3.5r}$$

$$\alpha_B = 2\alpha_A = \frac{2g}{3.5r}$$

From Eq. (2),

$$T_{AB} = \frac{\bar{I}_B\alpha_B}{r} = \frac{\left(\frac{1}{2}mr^2\right)(2g)}{3.5r^2}$$

$$= \frac{mg}{3.5} = \frac{W}{3.5}$$

Data: $W = 14$ lb, $g = 32.2\text{ ft/s}^2$,

$$r = 5\text{ in.} = \frac{5}{12}\text{ ft}$$

PROBLEM 16.62 (Continued)

(*a*) *Angular accelerations.*

$$\alpha_A = \frac{32.2\text{ ft/s}^2}{(3.5)\left(\frac{5}{12}\text{ ft}\right)} = 22.08\text{ rad/s} \qquad \boldsymbol{\alpha}_A = 22.1\text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$

$$\alpha_B = 2\alpha_A \qquad \boldsymbol{\alpha}_B = 44.2\text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

(*b*) *Tension T_{AB}.*

$$T_{AB} = \frac{W}{3.5} = \frac{14\text{ lb}}{3.5} \qquad T_{AB} = 4.00\text{ lb} \blacktriangleleft$$

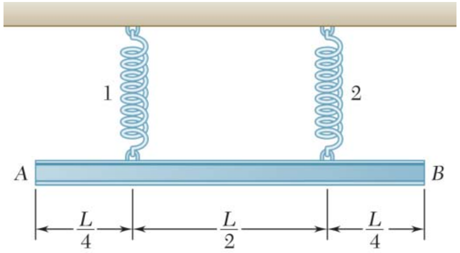
(*c*) *Velocity of the center of A.*

$$a_A = \frac{5}{12}\alpha_A = \frac{5}{12}(22.08) = 9.20\text{ ft/s}^2$$

$$v_A^2 = [(v_A)_0]^2 + 2a_A d_A$$

$$= 0 + (2)(9.20\text{ ft/s})(3\text{ ft}) = 55.2\text{ ft}^2/\text{s}^2$$

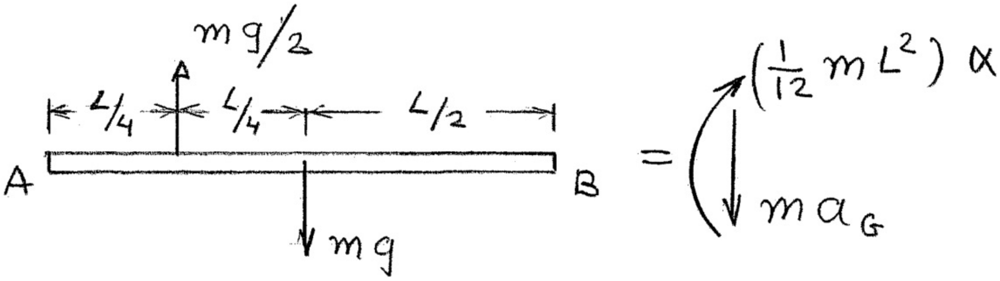
$$v_A = 7.43\text{ ft/s} \qquad \mathbf{v}_A = 7.43\text{ ft/s}\downarrow \blacktriangleleft$$


1
2
A
B
$\frac{L}{4}$
$\frac{L}{2}$
$\frac{L}{4}$

PROBLEM 16.63

A beam *AB* of mass *m* and of uniform cross section is suspended from two springs as shown. If spring *2* breaks, determine at that instant (*a*) the angular acceleration of the beam, (*b*) the acceleration of point *A*, (*c*) the acceleration of point *B*.

SOLUTION


mg/2
L/4
L/4
L/2
A
B
mg
=
$\left(\frac{1}{12}mL^2\right)\alpha$
ma_G

$$+\downarrow\ \Sigma F_y = \frac{mg}{2} = ma_G$$

$$+\circlearrowright\ M_G = \left(\frac{mg}{2}\right)\frac{L}{4} = \frac{1}{12}mL^2\alpha$$

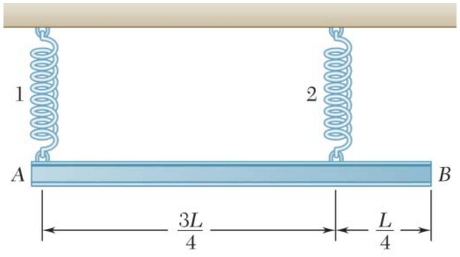
(*a*) $a_G = \frac{g}{2}\downarrow, \qquad \alpha = \frac{3g}{2L}\circlearrowright$ ◀

(*b*) $$\mathbf{a}_A = \downarrow\frac{g}{2} + \uparrow\alpha\frac{L}{2} = \frac{g}{4}\uparrow$$

$\mathbf{a}_A = \frac{g}{4}\uparrow$ ◀

(*c*) $$\mathbf{a}_B = \downarrow\frac{g}{2} + \downarrow 0.75g$$

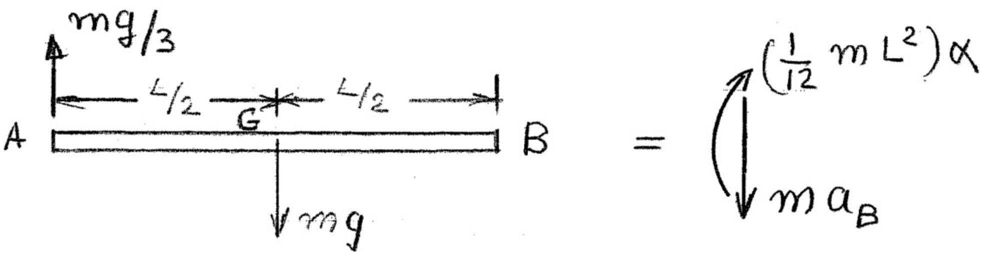
$\mathbf{a}_B = \frac{5g}{4}\downarrow$ ◀



PROBLEM 16.64

A beam *AB* of mass *m* and of uniform cross section is suspended from two springs as shown. If spring *2* breaks, determine at that instant (*a*) the angular acceleration of the beam, (*b*) the acceleration of point *A*, (*c*) the acceleration of point *B*.

SOLUTION



$$+\!\downarrow \Sigma F_y = \frac{2mg}{3} = ma_G$$

$$+\!\circlearrowright \Sigma M_G = \left(\frac{mg}{3}\right)\frac{L}{2} = \frac{1}{12}mL^2\alpha$$

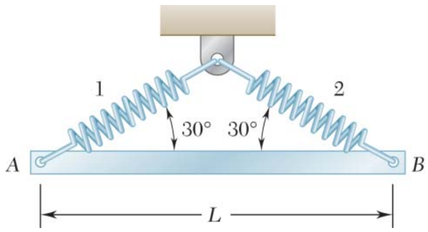
(*a*) $a_G = \frac{2g}{3}\downarrow, \qquad \alpha = \frac{2g}{L}\circlearrowright$ ◀

(*b*) $$\mathbf{a}_A = \downarrow\frac{2g}{3} + \frac{2g}{L}\frac{L}{2}\uparrow = \frac{g}{3}\uparrow,$$

$\mathbf{a}_A = \frac{g}{3}\uparrow$ ◀

(*c*) $$\mathbf{a}_B = \downarrow\frac{2g}{3} + \frac{2g}{L}\frac{L}{2}\downarrow = \frac{5g}{3}\downarrow$$

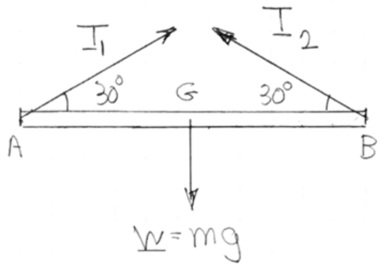
$\mathbf{a}_B = \frac{5g}{3}\downarrow$ ◀



PROBLEM 16.65

A uniform slender bar *AB* of mass *m* is suspended from two springs as shown. If spring 2 breaks, determine at that instant (*a*) the angular acceleration of the bar, (*b*) the acceleration of point *A*, (*c*) the acceleration of point *B*.

SOLUTION



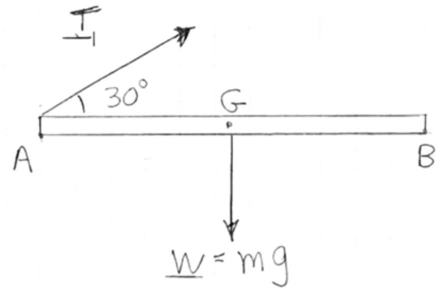
Before the spring breaks:

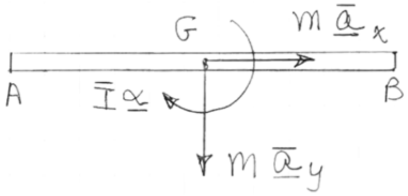
$$+\uparrow \sum F_y = 0: T_1 \sin 30° + T_2 \sin 30° - W = 0$$

$$2T_1 \sin 30° = W = mg$$

$$T_1 = mg \measuredangle 30°$$

Immediately after spring 2 breaks, elongation of spring 1 is unchanged. Thus we still have $T_1 = mg \measuredangle 30°$





$$(a)\quad +\circlearrowright \sum M_G = \sum (M_G)_{\text{eff}}: (T_1 \sin 30°)\frac{L}{2} = \bar{I}\alpha$$

$$(mg \sin 30°)\frac{L}{2} = \frac{1}{12} mL^2 \alpha$$

$$\alpha = \frac{3g}{L} \circlearrowright \blacktriangleleft$$

$$\xrightarrow{+} \sum F_x = \sum (F_x)_{\text{eff}}: T_1 \cos 30° = m\bar{a}_x$$

$$\bar{\mathbf{a}}_x = 0.866\, g \longrightarrow$$

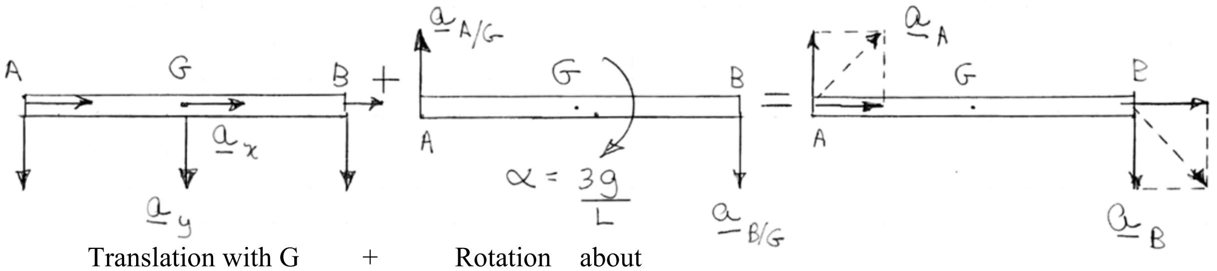
$$+\downarrow \sum F_y = \sum (F_y)_{\text{eff}}: W - T \sin 30° = m\bar{a}_y$$

$$mg - 0.5\, mg = m\bar{a}_y$$

$$\bar{\mathbf{a}}_y = \frac{1}{2} g \downarrow$$

PROBLEM 16.65 (Continued)

Accelerations of A and B


A
G
B
$\underline{a}_x$
$\underline{a}_y$
+
$\underline{a}_{A/G}$
G
B
A
$\alpha = \frac{3g}{L}$
$\underline{a}_{B/G}$
=
$\underline{a}_A$
G
B
A
$\underline{a}_B$

Translation with G + Rotation about G

(*b*) *Acceleration of A*:

$$\mathbf{a}_A = \mathbf{a}_G + \mathbf{a}_{A/G} = \left[0.866\,g \rightarrow +\ 0.5\,g \downarrow\right] + \left(\frac{3g}{L}\right)\left(\frac{L}{2}\right)\uparrow$$

$$= \left(0.866\,g \rightarrow\right) + \left(g \uparrow\right)$$

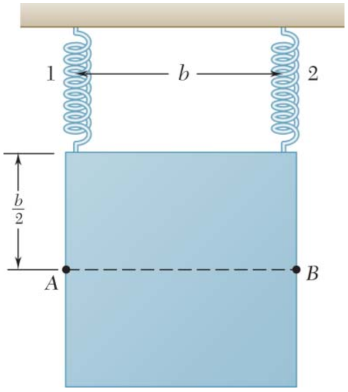
$\mathbf{a}_A = 1.323\,g \measuredangle\ 49.1°$ ◀

(*c*) *Acceleration of B*:

$$\mathbf{a}_B = \mathbf{a}_G + \mathbf{a}_{B/G} = \left[0.866\,g \rightarrow +\ 0.5\,g \downarrow\right] + \left(\frac{3g}{L}\right)\left(\frac{L}{2}\right)\downarrow$$

$$= \left(0.866\,g \rightarrow\right) + \left(2g \downarrow\right)$$

$\mathbf{a}_B = 2.18\,g \searrow\ 66.6°$ ◀


1
b
2
$\frac{b}{2}$
A
B

PROBLEM 16.66

A thin plate of the shape indicated and of mass m is suspended from two springs as shown. If spring 2 breaks, determine the acceleration at that instant (*a*) of Point *A*, (*b*) of Point *B*.

A square plate of side *b*.

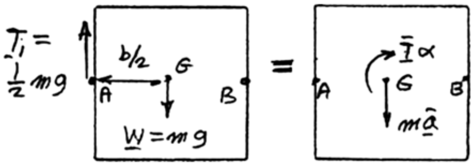
SOLUTION

$$\bar{I} = \frac{1}{12}m(b^2 + b^2)$$

$$\bar{I} = \frac{1}{6}mb^2$$

Statics: $$T_1 = T_2 = \frac{1}{2}W = \frac{1}{2}mg$$

Kinetics: $$+\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad T\frac{b}{2} = \bar{I}\alpha$$


$T_1 = \frac{1}{2}mg$
$b/2$
G
A
B
$\underline{W} = mg$
=
$\bar{\underline{I}}\alpha$
G
A
B
$m\underline{\bar{a}}$

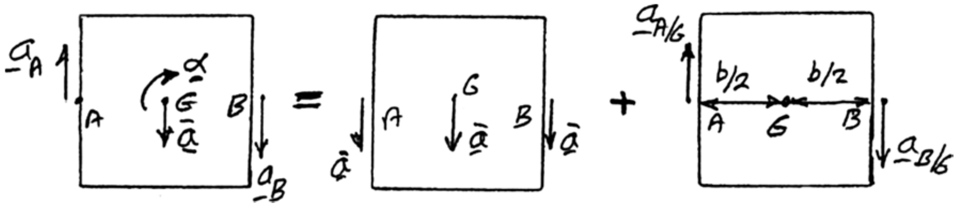
$$\frac{1}{2}mg\left(\frac{b}{2}\right) = \frac{1}{6}mb^2\alpha$$

$$\boldsymbol{\alpha} = \frac{3g}{2b} \circlearrowright$$

$$+\downarrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad W - T_1 = m\bar{a}$$

$$mg - \frac{1}{2}mg = m\bar{a} \qquad \bar{\mathbf{a}} = \frac{1}{2}g\downarrow$$

Kinematics:


$\underline{a}_A$
$\underline{\alpha}$
G
$\underline{\bar{a}}$
A
B
$\underline{a}_B$
=
$\underline{\bar{a}}$
A
G
$\underline{\bar{a}}$
B
$\underline{\bar{a}}$
+
$\underline{a}_{A/G}$
b/2
b/2
A
G
B
$\underline{a}_{B/G}$

Plane motion = Translation + Rotation

PROBLEM 16.66 (Continued)

(*a*)

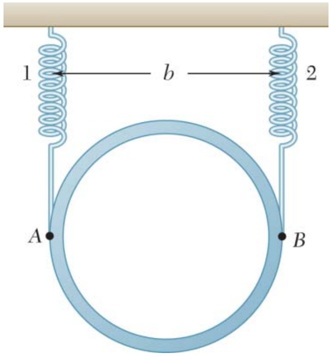
$$\mathbf{a}_A = \mathbf{a}_G + \mathbf{a}_{A/G} = \bar{\mathbf{a}}\downarrow + \frac{b}{2}\alpha\uparrow$$

$$\mathbf{a}_A = \frac{g}{2}\downarrow + \frac{b}{2}\left(\frac{3g}{2b}\right)\uparrow = \frac{g}{4}\uparrow \qquad \mathbf{a}_A = \frac{1}{4}g\uparrow \blacktriangleleft$$

(*b*)

$$\mathbf{a}_B = \mathbf{a}_G + \mathbf{a}_{B/G} = \bar{\mathbf{a}}\downarrow + \frac{b}{2}\alpha\downarrow$$

$$\mathbf{a}_B = \frac{1}{2}g\downarrow + \frac{b}{2}\left(\frac{3g}{2b}\right)\downarrow = \frac{5}{4}g\downarrow \qquad \mathbf{a}_B = \frac{5}{4}g\downarrow \blacktriangleleft$$



PROBLEM 16.67

A thin plate of the shape indicated and of mass m is suspended from two springs as shown. If spring 2 breaks, determine the acceleration at that instant (*a*) of point A, (*b*) of point B.

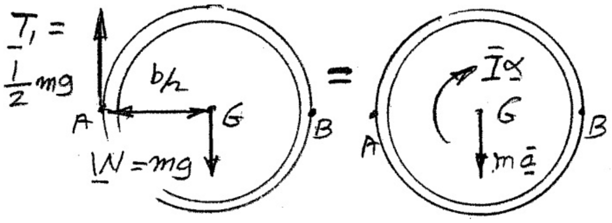
A thin hoop of diameter b.

SOLUTION

$$\bar{I} = m\left(\frac{b}{2}\right)^2 = \frac{1}{4}mb^2$$

Statics:
$$T_1 = T_2 = \frac{1}{2}W = \frac{1}{2}mg$$

Kinetics:



$$+\circlearrowright\ \Sigma M_G = \Sigma(M_G)_{\text{eff}}:$$

$$T_1\left(\frac{b}{2}\right) = \bar{I}\alpha$$

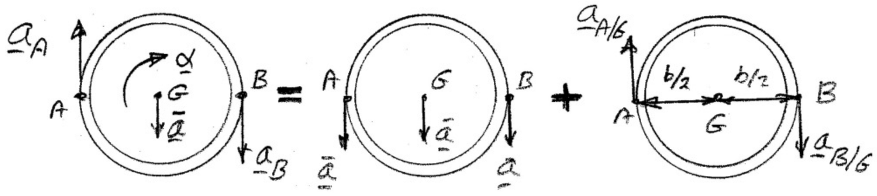
$$\frac{1}{2}mg\left(\frac{b}{2}\right) = \frac{1}{4}mb^2$$

$$\boldsymbol{\alpha} = \frac{g}{b}\ \circlearrowright$$

$$+\downarrow\ \Sigma F_y = \Sigma(F_y)_{\text{eff}}:\quad W - T = m\bar{a}$$

$$mg - \frac{1}{2}mg = m\bar{a} \qquad \bar{\mathbf{a}} = \frac{1}{2}g\downarrow$$

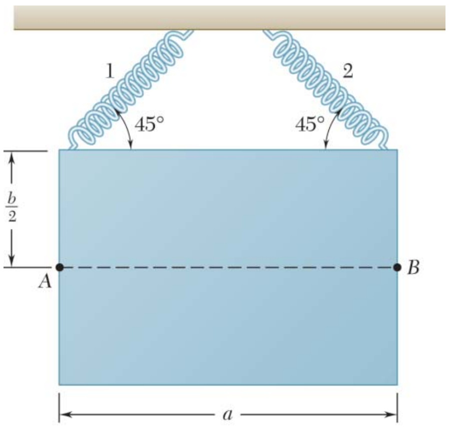
Kinematics:



Plane motion = *Translation* + *Rotation*

(*a*) $$\mathbf{a}_A = \mathbf{a}_G + \mathbf{a}_{A/G} = \bar{\mathbf{a}}\downarrow + \frac{b}{2}\alpha\uparrow = \frac{1}{2}g\downarrow + \frac{b}{2}\left(\frac{g}{b}\right)\uparrow; \qquad \mathbf{a}_A = 0 \blacktriangleleft$$

(*b*) $$\mathbf{a}_B = \mathbf{a}_G + \mathbf{a}_{B/G} = \bar{\mathbf{a}}\downarrow + \frac{b}{2}\alpha\downarrow = \frac{1}{2}g\downarrow + \frac{b}{2}\left(\frac{g}{b}\right)\downarrow; \qquad \mathbf{a}_B = g\downarrow \blacktriangleleft$$


1
2
45°
45°
$\frac{b}{2}$
A
B
a

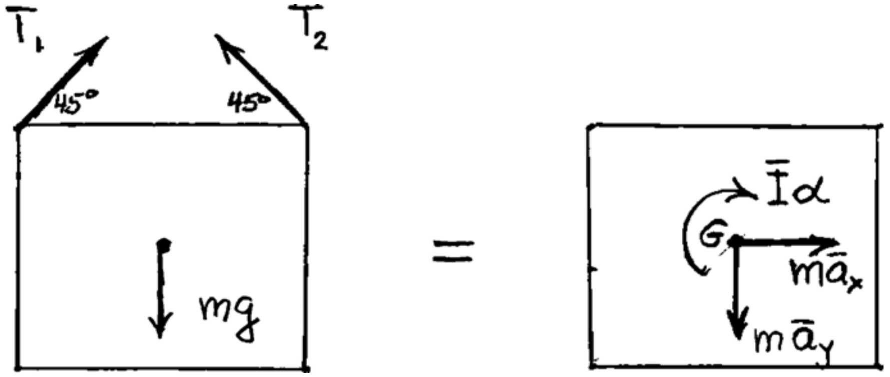
PROBLEM 16.68

A thin plate of the shape indicated and of mass m is suspended from two springs as shown. If spring 2 breaks, determine the acceleration at that instant (*a*) of Point A, (*b*) of Point B.

A rectangular plate of height b and width a.

SOLUTION

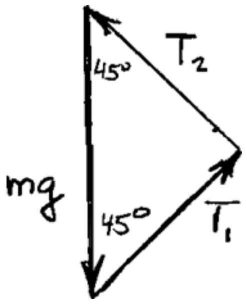
Moment of inertia. $$I = \frac{1}{12}m(a^2 + b^2)$$


T_1
T_2
45°
45°
mg
=
$\bar{I}\alpha$
G
$m\bar{a}_x$
$m\bar{a}_y$

Statics: $$a_x = 0, \quad a_y = 0, \quad \alpha = 0$$

Draw the force triangle showing equilibrium.

$$T_1 = T_2 = mg\sin 45°$$


T_2
45°
mg
45°
T_1

Kinetics: $T_2 = 0$

Since there is no time for displacements to occur, the tension in spring 1 remains equal to

$$T_1 = mg\sin 45°$$

Then $$\mathbf{T}_1 = \left[\frac{1}{2}mg \longrightarrow\right] + \left[\frac{1}{2}mg \uparrow\right]$$

PROBLEM 16.68 (Continued)

$$+\Sigma\mathbf{F} = m\overline{\mathbf{a}}: \quad [mg \downarrow] + \left[\frac{1}{2}mg \longrightarrow\right] + \left[\frac{1}{2}mg \uparrow = m\overline{\mathbf{a}}\right]$$

$$\overline{\mathbf{a}} = \left[\frac{1}{2}g \longrightarrow\right] + \left[\frac{1}{2}g \downarrow\right]$$

$$\circlearrowleft + \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad \frac{1}{2}mg\left(\frac{a}{2}\right) + \frac{1}{2}mg\left(\frac{b}{2}\right) = \bar{I}\alpha$$

$$\frac{1}{4}mg(a+b) = \frac{1}{12}m(a^2+b^2)\alpha \qquad \boldsymbol{\alpha} = \frac{3g(a+b)}{a^2+b^2} \circlearrowright$$

In vector notation, $\overline{\mathbf{a}} = \frac{1}{2}g(\mathbf{i} - \mathbf{j})$

$$\boldsymbol{\alpha} = -\frac{3g(a+b)}{a^2+b^2}\mathbf{k}$$

Kinematics. $\mathbf{a}_P = \overline{\mathbf{a}} + \boldsymbol{\alpha} \times \mathbf{r}_{P/G} - \omega^2\mathbf{r}_{P/G}$

Since there is no time to acquire angular velocity, $\omega^2 = 0$

(*a*) *Acceleration at A.* $\mathbf{r}_{A/G} = -\frac{1}{2}a\mathbf{i}$

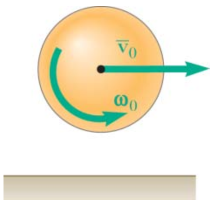
$$\mathbf{a}_A = \frac{1}{2}g(\mathbf{i} - \mathbf{j}) + \left[-\frac{3g(a+b)}{a^2+b^2}\mathbf{k}\right] \times \left(-\frac{1}{2}a\mathbf{i}\right)$$

$$\mathbf{a}_A = \frac{1}{2}g(\mathbf{i} - \mathbf{j}) + \frac{3g(a+b)a}{a^2+b^2}\mathbf{j} \blacktriangleleft$$

(*b*) *Acceleration at B.* $\mathbf{r}_{B/G} = \frac{1}{2}a\mathbf{i}$

$$\mathbf{a}_B = \frac{1}{2}g(\mathbf{i} - \mathbf{j}) + \left[-\frac{3g(a+b)}{a^2+b^2}\mathbf{k}\right] \times \left(\frac{1}{2}a\mathbf{i}\right)$$

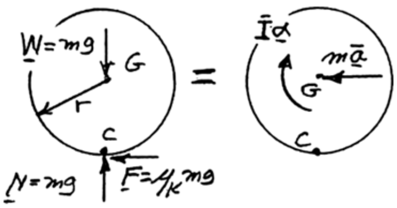
$$\mathbf{a}_B = \frac{1}{2}g(\mathbf{i} - \mathbf{j}) - \frac{3g(a+b)a}{a^2+b^2}\mathbf{j} \blacktriangleleft$$



PROBLEM 16.69

A sphere of radius r and mass m is projected along a rough horizontal surface with the initial velocities indicated. If the final velocity of the sphere is to be zero, express, in terms of v_0, r, and μ_k, (*a*) the required magnitude of $\boldsymbol{\omega}_0$, (*b*) the time t_1 required for the sphere to come to rest, (*c*) the distance the sphere will move before coming to rest.

SOLUTION



Kinetics: $I = m\bar{k}^2$

$\overset{+}{\leftarrow} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad F = m\bar{a}$

$\mu_k mg = m\bar{a} \qquad \bar{\mathbf{a}} = \mu_k g \longleftarrow$

$+\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad Fr = \bar{I}\alpha$

$(\mu_k mg)r = m\bar{k}^2\alpha$

$$\boldsymbol{\alpha} = \frac{\mu_k gr}{\bar{k}^2} \circlearrowright$$

Kinematics: $\xrightarrow{+} v = v_0 - \bar{a}t$

$v = v_0 - \mu_k gt$

For $v = 0$ when $t = t_1$ $\qquad 0 = v_0 - \mu_k g t_1; \qquad t_1 = \dfrac{v_0}{\mu_k g} \qquad (1)$

$+\circlearrowright\ \omega = \omega_0 - \alpha t$

$$\omega = \omega_0 - \frac{\mu_k gr}{\bar{k}^2}t$$

For $\omega = 0$ when $t = t_1$ $\qquad 0 = \omega_0 - \dfrac{\mu_k gr}{\bar{k}^2}t_1; \qquad t_1 = \dfrac{\bar{k}^2}{\mu_k gr}\omega_0 \qquad (2)$

Set Eq. (1) = Eq. (2) $\qquad \dfrac{v_0}{\mu_k g} = \dfrac{\bar{k}^2}{\mu_k gr}\omega_0; \qquad \omega_0 = \dfrac{r}{\bar{k}^2}v_0 \qquad (3)$

Distance traveled: $\qquad s_1 = v_0 t_1 - \dfrac{1}{2}\bar{a}t_1^2$

$$s_1 = v_0\left(\frac{v_0}{\mu_k g}\right) - \frac{1}{2}(\mu_k g)\left(\frac{v_0}{\mu_k g}\right)^2; \qquad s_1 = \frac{v_0^2}{2\mu_k g} \qquad (4)$$

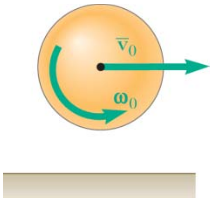
PROBLEM 16.69 (Continued)

For a solid sphere $\bar{k}^2 = \frac{2}{5}r^2$

(*a*) Eq. (3): $\omega_0 = \frac{r}{\frac{2}{5}r^2}v_0 = \frac{5}{2}\frac{v_0}{r}$ $\qquad \boldsymbol{\omega}_0 = \frac{5}{2}\frac{v_0}{r} \circlearrowleft$ ◀

(*b*) Eq. (1) $\qquad t_1 = \frac{v_0}{\mu_k g}$ ◀

(*c*) Eq. (4) $\qquad s_1 = \frac{v_0^2}{2\mu_k g}$ ◀



PROBLEM 16.70

Solve Problem 16.69, assuming that the sphere is replaced by a uniform thin hoop of radius r and mass m.

PROBLEM 16.69 A sphere of radius r and mass m is projected along a rough horizontal surface with the initial velocities indicated. If the final velocity of the sphere is to be zero, express, in terms of v_0, r, and μ_k, (*a*) the required magnitude of $\boldsymbol{\omega}_0$, (*b*) the time t_1 required for the sphere to come to rest, (*c*) the distance the sphere will move before coming to rest.

SOLUTION

Kinetics:

$$I = m\bar{k}^2$$

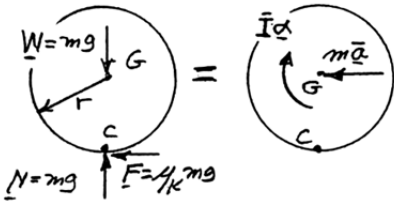
$$\xleftarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad F = m\bar{a}$$

$$\mu_k mg = m\bar{a} \qquad \bar{\mathbf{a}} = \mu_k g \longleftarrow$$

$$+\circlearrowright \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad Fr = \bar{I}\alpha$$

$$(\mu_k mg)r = m\bar{k}^2\alpha$$

$$\boldsymbol{\alpha} = \frac{\mu_k g r}{\bar{k}^2} \circlearrowright$$



Kinematics:

$$\xrightarrow{+} v = v_0 - \bar{a}t$$

$$v = v_0 - \mu_k g t$$

For $v = 0$ when $t = t_1$

$$0 = v_0 - \mu_k g t_1; \qquad t_1 = \frac{v_0}{\mu_k g} \tag{1}$$

$$+\circlearrowleft \omega = \omega_0 - \alpha t$$

$$\omega = \omega_0 - \frac{\mu_k g r}{\bar{k}^2} t$$

For $\omega = 0$ when $t = t_1$

$$0 = \omega_0 - \frac{\mu_k g r}{\bar{k}^2} t_1; \qquad t_1 = \frac{\bar{k}^2}{\mu_k g r}\omega_0 \tag{2}$$

Set Eq. (1) = Eq. (2)

$$\frac{v_0}{\mu_k g} = \frac{\bar{k}^2}{\mu_k g r}\omega_0; \qquad \omega_0 = \frac{r}{\bar{k}^2} v_0 \tag{3}$$

Distance traveled:

$$s_1 = v_0 t_1 - \frac{1}{2}\bar{a}t_1^2$$

$$s_1 = v_0\left(\frac{v_0}{\mu_k g}\right) - \frac{1}{2}(\mu_k g)\left(\frac{v_0}{\mu_k g}\right)^2; \qquad s_1 = \frac{v_0^2}{2\mu_k g} \tag{4}$$

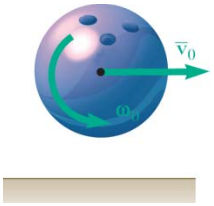
PROBLEM 16.70 (Continued)

For a hoop, $\bar{k} = r$

(*a*) Eq. (3): $\omega_0 = \frac{r}{r^2} r_0 = \frac{v_0}{r}$ $\boldsymbol{\omega}_0 = \frac{v_0}{r} \circlearrowright$ ◀

(*b*) Eq. (1): $t_1 = \frac{v_0}{\mu_k g}$ ◀

(*c*) Eq. (4): $s_1 = \frac{v_0^2}{2\mu_k g}$ ◀

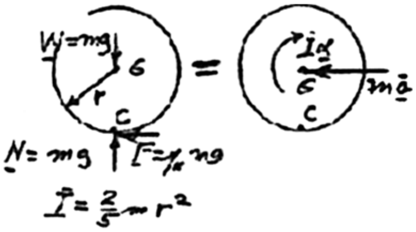


PROBLEM 16.71

A bowler projects an 8-in.-diameter ball weighing 12 lb along an alley with a forward velocity $\mathbf{v}_0$ of 15 ft/s and a backspin $\boldsymbol{\omega}_0$ of 9 rad/s. Knowing that the coefficient of kinetic friction between the ball and the alley is 0.10, determine (*a*) the time t_1 at which the ball will start rolling without sliding, (*b*) the speed of the ball at time t_1, (*c*) the distance the ball will have traveled at time t_1.

SOLUTION

Kinetics:



$$\overset{+}{\longleftarrow} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad \mu_k mg = m\bar{a}$$

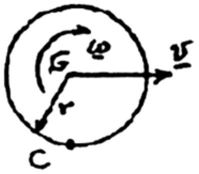
$$\bar{\mathbf{a}} = \mu_k g \longleftarrow$$

$$+\circlearrowright \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad Fr = \bar{I}\alpha$$

$$(\mu_k mg)r = \frac{2}{5}mr^2\alpha$$

$$\boldsymbol{\alpha} = \frac{5}{2}\frac{\mu_k g}{r} \circlearrowright$$

Kinematics: When the ball rolls, the instant center of rotation is at C, and when



$$t = t_1 \quad v = r\omega \tag{1}$$

$$\bar{v} = \bar{v}_0 - \bar{a}t = \bar{v}_0 - \mu_k g t \tag{2}$$

$$\omega = -\omega_0 + \alpha t = -\omega_0 + \frac{5}{2}\frac{\mu_k g}{r}t$$

When $t = t_1$:

Eq. (1): $v = r\omega$:

$$\bar{v}_0 - \mu_k g t_1 = \left(-\omega_0 + \frac{5}{2}\frac{\mu_k g}{r}t_1\right)r$$

$$\bar{v}_0 - \mu_k g t_1 = -\omega_0 r + \frac{5}{2}\mu_k g t_1$$

$$t_1 = \frac{2}{g}\frac{(\bar{v}_0 + r\omega_0)}{\mu_k g} \tag{3}$$

$$\bar{v}_0 = 15 \text{ ft/s}, \quad \omega_0 = 9 \text{ rad/s}, \quad r = 4 \text{ in.} = \frac{1}{3} \text{ ft}$$

(*a*)

$$t_1 = \frac{2}{7}\frac{\left(15 + \frac{1}{3}(9)\right)}{0.1(32.2)} = 1.5972 \text{ s}$$

$t_1 = 1.597$ s ◀

PROBLEM 16.71 (Continued)

(*b*) Eq. (2):

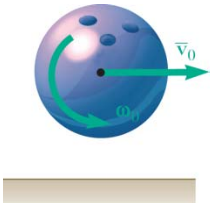
$$\bar{v}_1 = v_0 - \mu_k g t_1$$
$$= 15 - 0.1(32.2)(1.5972)$$
$$\bar{v}_1 = 15 - 5.1429$$
$$= 9.857 \text{ ft/s} \qquad \bar{v}_1 = 9.86 \text{ ft/s} \blacktriangleleft$$

(*c*)

$$\bar{a} = \mu_k g = 0.1(32.2 \text{ ft/s}^2) = 3.22 \text{ ft/s}^2 \longleftarrow$$

$$\xrightarrow{+} s_1 = \bar{v}_0 t_1 - \frac{1}{2}\bar{a}t_1^2$$
$$= (15 \text{ ft/s})(1.597 \text{ s}) - \frac{1}{2}(3.22 \text{ ft/s}^2)(1.597 \text{ s})^2$$
$$= 23.96 - 4.11 = 19.85 \text{ ft} \qquad s_1 = 19.85 \text{ ft} \blacktriangleleft$$

PROBLEM 16.72

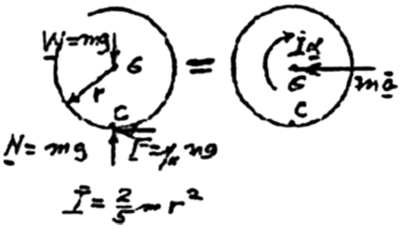

$\mathbf{v}_0$
ω_0

Solve Problem 16.71, assuming that the bowler projects the ball with the same forward velocity but with a backspin of 18 rad/s.

PROBLEM 16.71 A bowler projects an 8-in.-diameter ball weighing 12 lb along an alley with a forward velocity $\mathbf{v}_0$ of 15 ft/s and a backspin $\boldsymbol{\omega}_0$ of 9 rad/s. Knowing that the coefficient of kinetic friction between the ball and the alley is 0.10, determine (*a*) the time t_1 at which the ball will start rolling without sliding, (*b*) the speed of the ball at time t_1, (*c*) the distance the ball will have traveled at time t_1.

SOLUTION

Kinetics:


$\underline{W} = mg$
G
r
C
$\underline{N} = mg$
$\underline{F} = \mu_k mg$
$=$
$\bar{I}\underline{\alpha}$
G
$m\underline{\bar{a}}$
C
$\bar{I} = \frac{2}{5}mr^2$

$$\xleftarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad \mu_k mg = m\bar{a}$$

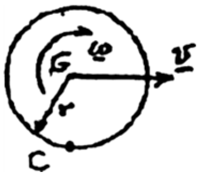
$$\bar{\mathbf{a}} = \mu_k g \longleftarrow$$

$$+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad Fr = \bar{I}\alpha$$

$$(\mu_k mg)r = \frac{2}{5}mr^2\alpha$$

$$\alpha = \frac{5}{2}\frac{\mu_k g}{r} \circlearrowright$$

Kinematics: When the ball rolls, the instant center of rotation is at C, and when


G
ω
$\bar{v}$
r
C

$$t = t_1 \quad v = r\omega \tag{1}$$

$$\bar{v} = \bar{v}_0 - \bar{a}t = \bar{v}_0 - \mu_k g^t \tag{2}$$

$$\omega = -\omega_0 + \alpha t = -\omega_0 + \frac{5}{2}\frac{\mu_k g}{r}t$$

When $t = t_1$:

Eq. (1) $\quad v = r\omega$:

$$\bar{v}_0 - \mu_k g t_1 = \left(-\omega_0 + \frac{5}{2}\frac{\mu_k g}{r}t_1\right)r$$

$$\bar{v}_0 - \mu_k g t_1 = -\omega_0 r + \frac{5}{2}\mu_k g t_1$$

$$t_1 = \frac{2}{7}\frac{(\bar{v}_0 + r\omega_0)}{\mu_k g} \tag{3}$$

$$\bar{v}_0 = 15 \text{ ft/s}, \quad \omega_0 = 18 \text{ rad/s}, \quad r = \frac{1}{3} \text{ ft}$$

PROBLEM 16.72 (Continued)

(*a*) Eq. (3):

$$t_1 = \frac{2}{7}\frac{\left(15 + \frac{1}{3}(18)\right)}{0.1(32.2)} = 1.8634\text{ s} \qquad t_1 = 1.863\text{ s} \blacktriangleleft$$

(*b*) Eq. (2):

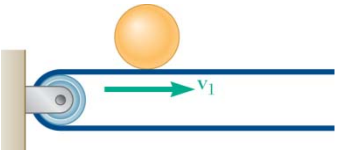
$$v_1 = v_0 - \mu_k g^t$$
$$= 15 - 0.1(32.2)(1.8634)$$

$$v_1 = 15 - 6.000$$
$$= 9\text{ ft/s} \qquad \overline{v}_1 = 9\text{ ft/s} \blacktriangleleft$$

(*c*)

$$\overline{a} = \mu_k g = 0.1(32.2\text{ ft/s}^2) = 3.22\text{ ft/s}^2 \longleftarrow$$

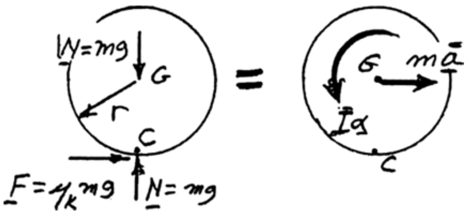
$$\xrightarrow{+} s_1 = \overline{v}_0 t_1 - \frac{1}{2}\overline{a}t_1^2$$
$$= (15\text{ ft/s})(1.8634\text{ s}) - \frac{1}{2}(3.22\text{ ft/s}^2)(1.8634\text{ s})^2$$
$$= 27.95 - 5.59$$
$$= 22.36\text{ ft} \qquad s_1 = 22.4\text{ ft} \blacktriangleleft$$



PROBLEM 16.73

A uniform sphere of radius r and mass m is placed with no initial velocity on a belt that moves to the right with a constant velocity $\mathbf{v}_1$. Denoting by μ_k the coefficient of kinetic friction between the sphere and the belt, determine (a) the time t_1 at which the sphere will start rolling without sliding, (b) the linear and angular velocities of the sphere at time t_1.

SOLUTION



Kinetics: $\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad F = m\bar{a}$

$$\mu_k mg = m\bar{a}$$

$$\bar{\mathbf{a}} = \mu_k g \longrightarrow$$

$$+\circlearrowright \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad Fr = \bar{I}\alpha$$

$$(\mu_k mg)r = \frac{2}{5}mr^2\alpha$$

$$\boldsymbol{\alpha} = \frac{5}{2}\frac{\mu_k g}{r} \circlearrowright$$

Kinematics: $\xrightarrow{+} \bar{v} = \bar{a}t = \mu_k gt \qquad (1)$

$$+\circlearrowright \omega = \alpha t = \frac{5}{2}\frac{\mu_k g}{r}t \qquad (2)$$

Point C is the point of contact with belt.

$$\xrightarrow{+} v_C = \bar{v} + \omega r = \mu_k gt + \left(\frac{5}{2}\frac{\mu_k g}{r}t\right)r$$

$$v_C = \frac{7}{2}\mu_k gt$$

(a) When sphere starts rolling $(t = t_1)$, we have

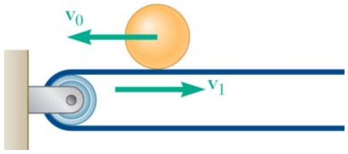
$$v_C = v_1$$

$$v_1 = \frac{7}{2}\mu_k gt_1 \qquad t_1 = \frac{2}{7}\frac{v_1}{\mu_k g} \blacktriangleleft$$

(b) Velocities when $t = t_1$

Eq. (1): $\bar{v} = \mu g\left(\frac{2}{7}\frac{v_1}{\mu_k g}\right) \qquad \bar{\mathbf{v}} = \frac{2}{7}v_1 \longrightarrow \blacktriangleleft$

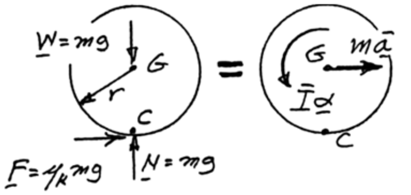
Eq. (2): $\omega = \left(\frac{5}{2}\frac{\mu_k g}{r}\right)\left(\frac{2}{7}\frac{v_1}{\mu_k g}\right) \qquad \boldsymbol{\omega} = \frac{5}{7}\frac{v_1}{r} \circlearrowright \blacktriangleleft$



PROBLEM 16.74

A sphere of radius r and m has a linear velocity $\mathbf{v}_0$ directed to the left and no angular velocity as it is placed on a belt moving to the right with a constant velocity $\mathbf{v}_1$. If after first sliding on the belt the sphere is to have no linear velocity relative to the ground as it starts rolling on the belt without sliding, determine in terms of v_1 and the coefficient of kinetic friction μ_k between the sphere and the belt (*a*) the required value of v_0, (*b*) time t_1 at which the sphere will start rolling on the belt, (*c*) the distance the sphere will have moved relative to the ground at time t_1.

SOLUTION



Kinetics: $\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad F = m\bar{a}$

$$\mu_k mg = m\bar{a}$$

$$\bar{\mathbf{a}} = \mu_k g \longrightarrow$$

$$+\circlearrowright \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad F_r = \bar{I}\alpha$$

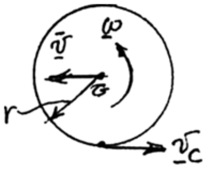
$$(\mu_k mg)r = \frac{5}{2} mr^2 \alpha$$

$$\boldsymbol{\alpha} = \frac{5}{2}\frac{\mu_k g}{r} \circlearrowright$$

Kinematics: $\xleftarrow{+} \bar{v} = v_0 - \bar{a}t = v_0 - \mu_k g t \qquad (1)$

$$+\circlearrowright \omega = \alpha t = \frac{5}{2}\frac{\mu_k g}{r} t \qquad (2)$$

Point C is the point of contact with belt.



$$\xrightarrow{+} v_C = -\bar{v} + r\omega$$

$$v_C = -\bar{v} + r\frac{5}{2}\frac{\mu_k g}{r} t$$

$$v_C = -\bar{v} + \frac{5\mu_k g}{2} t \qquad (3)$$

But, when $t = t_1$, $\bar{v} = 0$ and $v_c = v_1$

Eq. (3): $v_1 = \dfrac{5\mu_k g}{2} t_1$ $\qquad t_1 = \dfrac{2v_1}{5\mu_k g}$ ◀

Eq. (1): $\bar{v} = v_0 - \mu_k g t$

When $t = t_1$, $\bar{v} = 0$, $\quad 0 = v_0 - \mu_k g\left(\dfrac{2v_1}{5\mu_k g}\right)$ $\qquad v_0 = \dfrac{2}{5} v_1$ ◀

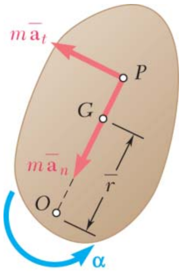
PROBLEM 16.74 (Continued)

Distance when $t = t_1$: $\xleftarrow{+}\ s = v_0 t_1 - \frac{1}{2} a t_1^2$

$$s = \left(\frac{2}{5} v_1\right)\left(\frac{2v_1}{5\mu_k g}\right) - \frac{1}{2}(\mu_k g)\left(\frac{2v_1}{5\mu_k g}\right)^2$$

$$s = \frac{v_1^2}{\mu_k g}\left(\frac{4}{25} - \frac{2}{25}\right);$$

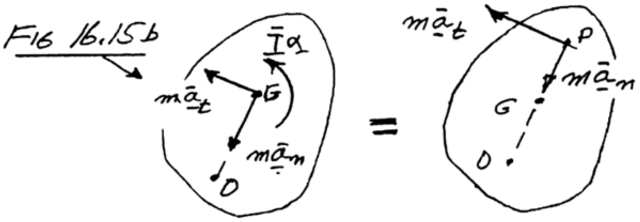
$$\mathbf{s} = \frac{2}{25}\frac{v_1^2}{\mu_k g} \longleftarrow \quad ◀$$



PROBLEM 16.75

Show that the couple $\bar{I}\boldsymbol{\alpha}$ of Figure 16.15 can be eliminated by attaching the vectors $m\bar{\mathbf{a}}_t$ and $m\bar{\mathbf{a}}_n$ at a Point P called the *center of percussion,* located on line OG at a distance $GP = \bar{k}^2/\bar{r}$ from the mass center of the body.

SOLUTION



$$OG = \bar{r} \qquad \bar{\mathbf{a}}_t = \bar{r}\alpha$$

We first observe that the sum of the vectors is the same in both figures. To have the same sum of moments about G, we must have

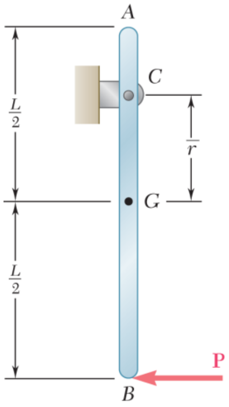
$$+\circlearrowleft \Sigma M_G = \Sigma M_G{:} \quad \bar{I}\alpha = (m\bar{a}_t)(GP)$$

$$m\bar{k}^2\alpha = m\bar{r}\alpha(GP) \qquad\qquad GP = \frac{\bar{k}^2}{\bar{r}} \quad \text{(Q.E.D.)} \blacktriangleleft$$

Note: The center of rotation and the center of percussion are interchangeable. Indeed, since $OG = \bar{r}$, we may write

$$GP = \frac{\bar{k}^2}{GO} \quad \text{or} \quad GO = \frac{\bar{k}^2}{GP}$$

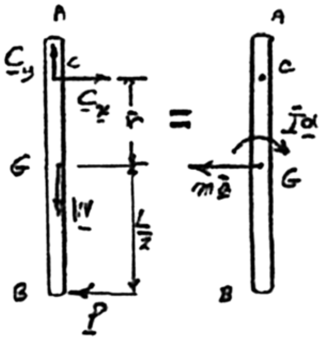
Thus, if Point P is selected as center of rotation, then Point O is the center of percussion.



PROBLEM 16.76

A uniform slender rod of length L = 900 mm and mass m = 4 kg is suspended from a hinge at C. A horizontal force **P** of magnitude 75 N is applied at end B. Knowing that $\bar{r} = 225$ mm, determine (a) the angular acceleration of the rod, (b) the components of the reaction at C.

SOLUTION



(a) Angular acceleration.

$$\bar{a} = \bar{r}\alpha \qquad \bar{I} = \frac{1}{12}mL^2$$

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad P\left(\bar{r} + \frac{L}{2}\right) = (m\bar{a})\bar{r} + \bar{I}\alpha$$

$$= (m\bar{r}\alpha)\bar{r} + \frac{1}{12}mL^2\alpha$$

$$P\left(\bar{r} + \frac{L}{2}\right) = m\left(\bar{r}^2 + \frac{1}{12}L^2\right)\alpha$$

Substitute data: $$(75\text{ N})\left[0.225\text{ m} + \frac{0.9\text{ m}}{2}\right] = (4\text{ kg})\left[(0.225\text{ m})^2 + \frac{1}{12}(0.9\text{ m})^2\right]\alpha$$

$$50.625 = 0.4725\alpha$$

$$\alpha = 107.14\text{ rad/s}^2$$

$\boldsymbol{\alpha} = 107.1\text{ rad/s}^2 \circlearrowright$ ◀

(b) Components of reaction at C.

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad C_y - W = 0$$

$$C_y = W = mg = (4\text{ kg})(9.81\text{ m/s}^2)$$

$\mathbf{C}_y = 39.2\text{ N}\uparrow$ ◀

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad C_x - P = -m\bar{a}$$

$$C_x = P - m\bar{a} = P - m(\bar{r}\alpha)$$

$$= 75\text{ N} - (4\text{ kg})(0.225\text{ m})(107.14\text{ rad/s}^2)$$

$$C_x = 75\text{ N} - 96.4\text{ N}$$

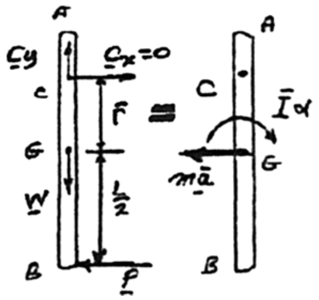
$$= -21.4\text{ N}$$

$\mathbf{C}_x = 21.4\text{ N} \longleftarrow$ ◀

PROBLEM 16.77

In Problem 16.76, determine (*a*) the distance $\overline{r}$ for which the horizontal component of the reaction at *C* is zero, (*b*) the corresponding angular acceleration of the rod.

SOLUTION



(*a*) Distance $\overline{r}$.

$$\overline{a} = \overline{r}\alpha$$

$$\overline{I} = \frac{1}{12}mL^2$$

$$\xleftarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad P = m\overline{a}$$

$$P = m(\overline{r}\alpha)$$

$$\alpha = \frac{P}{m\overline{r}} \qquad (1)$$

$$+\circlearrowright \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad P\frac{L}{2} = \overline{I}\alpha$$

$$P\frac{L}{2} = \frac{1}{12}mL^2\alpha$$

$$P\frac{L}{2} = \frac{1}{12}mL^2\left(\frac{P}{m\overline{r}}\right)$$

$$\frac{L}{2} = \frac{L^2}{12\overline{r}}$$

$$\overline{r} = \frac{1}{6}L \quad \overline{r} = \frac{900\text{ mm}}{6}$$

$\overline{r} = 150\text{ mm}$ ◀

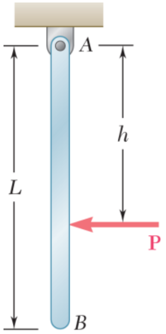
(*b*) Angular acceleration.

Eq. (1):

$$\alpha = \frac{P}{m\overline{r}} = \frac{P}{m\left(\frac{L}{6}\right)} = \frac{6P}{mL}$$

$$\alpha = \frac{6(75\text{ N})}{(4\text{ kg})(0.9\text{ m})} = 125\text{ rad/s}^2$$

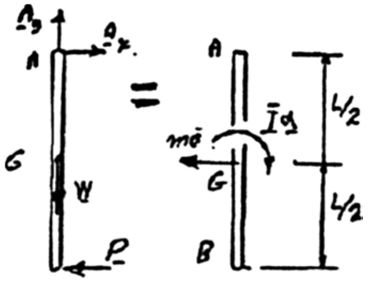
$\alpha = 125\text{ rad/s}^2 \circlearrowright$ ◀



PROBLEM 16.78

A uniform slender rod of length $L = 36$ in. and weight $W = 4$ lb hangs freely from a hinge at A. If a force **P** of magnitude 1.5 lb is applied at B horizontally to the left ($h = L$), determine (a) the angular acceleration of the rod, (b) the components of the reaction at A.

SOLUTION



$$\bar{a} = \frac{1}{2}\alpha \quad \bar{I} = \frac{1}{12}mL^2$$

$$+\circlearrowright \Sigma M_A = \Sigma (M_A)_{\text{eff}}: \quad PL = (m\bar{a})\frac{L}{2} + \bar{I}\alpha$$

$$= \left(m\frac{L}{2}\alpha\right)\frac{L}{2} + \frac{1}{12}mL^2\alpha$$

$$PL = \frac{1}{3}mL^2\alpha$$

(a) Angular acceleration.

$$\alpha = \frac{3P}{mL}$$

$$= \frac{3(1.5\text{ lb})}{\left(\frac{4\text{ lb}}{32.2\text{ ft/s}^2}\right)(3\text{ ft})}$$

$$= 12.08\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 12.08\text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

(b) Components of the reaction at A.

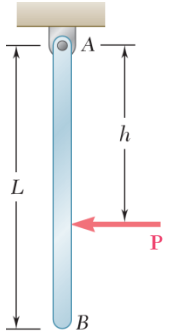
$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad A_y - W = 0$$

$$A_y = W = 4\text{ lb} \qquad \mathbf{A}_y = 4.00\text{ lb}\uparrow \blacktriangleleft$$

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad A_x - P = -m\bar{a}$$

$$A_x = P - m\left(\frac{L}{2}\alpha\right) = P - m\frac{L}{2}\left(\frac{3P}{mL}\right) = -\frac{P}{2}$$

$$A_x = -\frac{P}{2} = -\frac{1.5\text{ lb}}{2} = -0.75\text{ lb} \qquad \mathbf{A}_x = 0.750\text{ lb} \leftarrow \blacktriangleleft$$

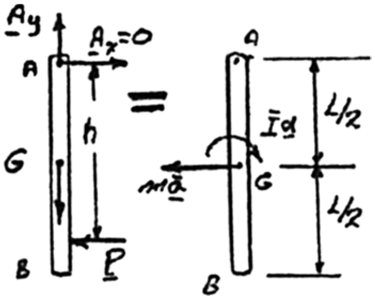


PROBLEM 16.79

In Problem 16.78, determine (*a*) the distance *h* for which the horizontal component of the reaction at *A* is zero, (*b*) the corresponding angular acceleration of the rod.

PROBLEM 16.78 A uniform slender rod of length $L = 36$ in. and weight $W = 4$ lb hangs freely from a hinge at *A*. If a force **P** of magnitude 1.5 lb is applied at *B* horizontally to the left ($h = L$), determine (*a*) the angular acceleration of the rod, (*b*) the components of the reaction at *A*.

SOLUTION



$$\bar{a} = \frac{L}{2}\alpha$$

$$\bar{I} = \frac{1}{12}mL^2$$

$\xleftarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}:$ $\quad P = m\bar{a}$

$$P = m\left(\frac{L}{2}\alpha\right)$$

(*b*) Angular acceleration. $\quad \alpha = \dfrac{2P}{mL}$

$$\alpha = \frac{2(1.5 \text{ lb})}{\left(\frac{4 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(3 \text{ ft})}$$

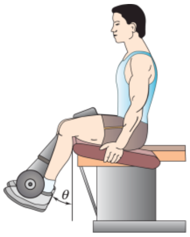
$\alpha = 8.05 \text{ rad/s}^2 \circlearrowright$ ◀

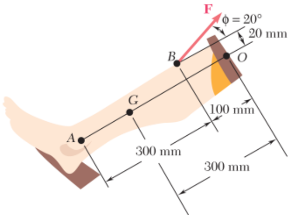
$+\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}:$ $\quad P\left(h - \dfrac{L}{2}\right) = \bar{I}\alpha: \quad P(h-L)$

$$P\left(h - \frac{L}{2}\right) = \frac{1}{12}mL^2\left(\frac{2P}{mL}\right) = \frac{PL}{6}$$

$$\left(h - \frac{L}{2}\right) = \frac{L}{6}; \quad h = \frac{L}{2} + \frac{L}{6} = \frac{2}{3}L$$

(*a*) Distance *h*. $\quad h = 24$ in. ◀

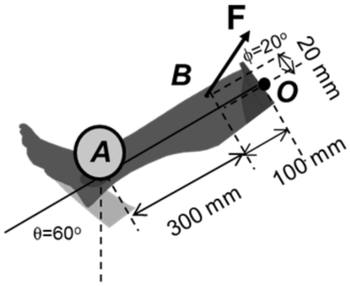




PROBLEM 16.80

An athlete performs a leg extension on a machine using a 20 kg mass at *A* located 400 mm away from the knee joint center *O*. Biomechanical studies show that the patella tendon inserts at *B*, 100 mm below point *O* and 20 mm from the center line of the tibia (see figure). The mass of the lower leg and foot is 5 kg, the CG of this segment is 300 mm from the knee, and the radius of gyration about the knee is 350 mm. Knowing that the leg is moving at a constant angular velocity of 30 degrees per second when θ = 60°, determine (*a*) the force ***F*** in the patella tendon, (*b*) the magnitude of the joint force at the knee joint center *O*.

SOLUTION



Given:

$$m_A = 20 \text{ kg, } m_l = 5 \text{ kg}$$

$$\omega_{OA} = 30\left(\frac{\pi}{180}\right) = 0.5236 \text{ rad/s}, \; \alpha_{OA} = 0$$

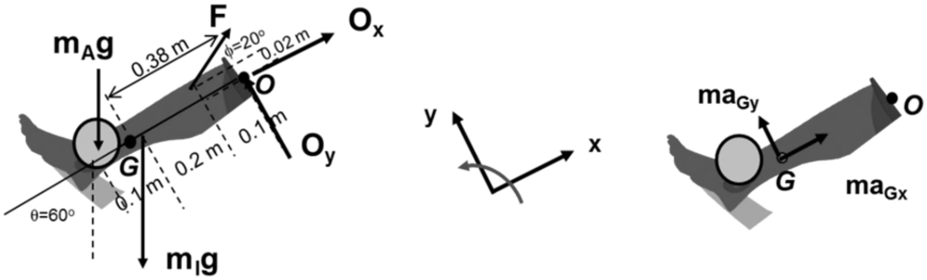
$$k_O = 0.35 \text{ m}, \;\; \theta = 60°, \;\; \phi = 30°$$

Location of Mass Center:

$$\bar{x} = \frac{\sum m_i \bar{x}_i}{\sum m_i} = \frac{m_A(-0.4) + m_l(-0.3)}{m_A + m_l}$$

$$= -0.38 \text{ or } 0.38\text{m from point O towards A}$$

Free Body Diagram:



Kinetics:

$$\sum F_x = ma_{Gx}$$

$$-20(9.81)\cos 60° - 5(9.81)\cos 60° + O_x + F\cos 20° = ma_{Gx}$$

$$O_x = 122.63 - F\cos 20° + ma_{Gx} \qquad (1)$$

$$\sum F_y = ma_{Gy}$$

$$-20(9.81)\sin 60° - 5(9.81)\sin 60° + O_y + F\sin 20° = ma_{Gy}$$

$$O_y = 212.39 - F\sin 20° + ma_{Gy} \qquad (2)$$

(a) $\sum M_O = I_O \cancel{\alpha_{OA}}$

$$20(9.81)\sin 60°(0.4) + 5(9.81)\sin 60°(0.3) - F\cos 20°(0.02) - F\sin 20°(0.1) = 0$$

$F = 1522.9$ N ◀

PROBLEM 16.80 (Continued)

Kinematics: $\mathbf{a}_G = \cancel{\mathbf{a}_O} + \cancel{\boldsymbol{\alpha}_{OA}} \times \mathbf{r}_{G/O} - \omega_{OA}^2 \mathbf{r}_{G/O}$ therefore: $a_{Gx} = 0.1042$ m/s

$a_{Gy} = 0$

$$= -(0.5236)^2(-0.38\mathbf{i})$$
$$= 0.1042\mathbf{i} \text{ m/s}$$

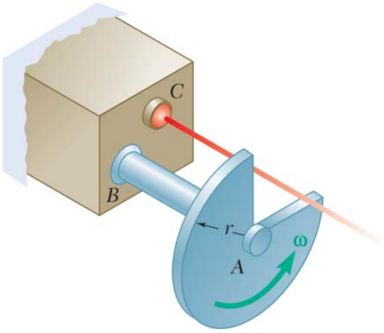
Substitute into (1):
$$O_x = 122.63 - 1522.9\cos 20° + 25(0.1042)$$
$$= -1305.85 \text{ N}$$

Substitute into (2):
$$O_y = 212.39 - 1522.9\sin 20° + 25(0)$$
$$= -308.47 \text{ N}$$

(b) Magnitude of Force at O: $O = \sqrt{O_x^2 + O_y^2}$

$$= \sqrt{(-1305.85)^2 + (-308.47)^2}$$

$O = 1341.8$ N ◀

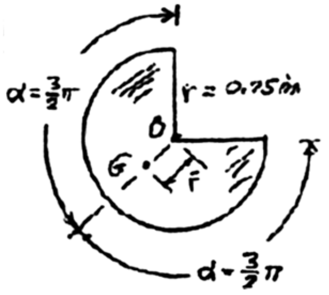


PROBLEM 16.81

The shutter shown was formed by removing one quarter of a disk of 0.75-in. radius and is used to interrupt a beam of light emanating from a lens at *C*. Knowing that the shutter weighs 0.125 lb and rotates at the constant rate of 24 cycles per second, determine the magnitude of the force exerted by the shutter on the shaft at *A*.

SOLUTION

See inside front cover for centroid of a circular sector.



$$\bar{r} = \frac{2r\sin\alpha}{3\alpha}$$

$$\bar{r} = \frac{2(0.75\text{ in.})\sin\left(\frac{3}{4}\pi\right)}{3\left(\frac{3}{4}\pi\right)}$$

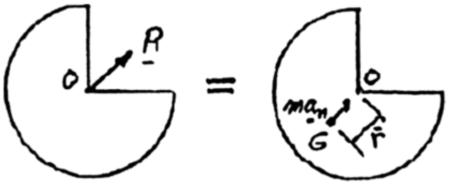
$$\bar{r} = 0.15005\text{ in.}$$

$$a_n = r\omega^2$$

$$\omega = 24\text{ rad/s}$$

$$= 24(2\pi)\text{ rad/s}$$

$$\omega = 150.8\text{ rad/s}$$



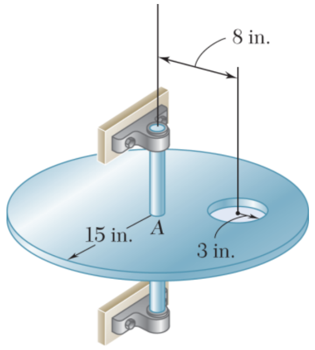
$$+\nearrow\ \Sigma F = \Sigma F_{\text{eff}}: \quad R = ma_n = m\bar{r}\omega^2$$

$$= \frac{(0.125\text{ lb})}{32.2\text{ ft/s}^2}\left(\frac{0.15005}{12}\text{ ft}\right)(150.8\text{ rad/s})^2$$

$$R = 1.1038\text{ lb} \nearrow$$

Force on shaft is $R = 1.104\text{ lb} \swarrow$

Magnitude: $R = 1.104\text{ lb}$ ◀


8 in.
15 in.
A
3 in.

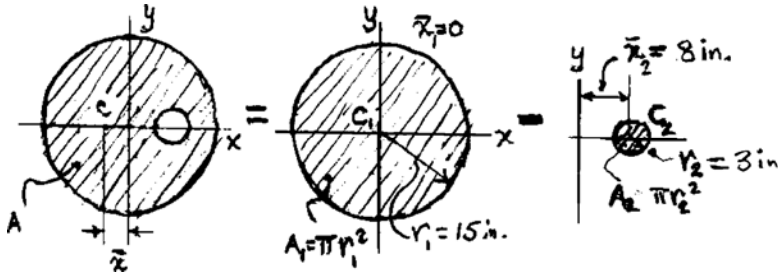
PROBLEM 16.82

A 6-in.-diameter hole is cut as shown in a thin disk of 15-in.-diameter. The disk rotates in a horizontal plane about its geometric center *A* at the constant rate of 480 rpm. Knowing that the disk has a mass of 60 lb after the hole has been cut, determine the horizontal component of the force exerted by the shaft on the disk at *A*.

SOLUTION

Determination of mass center of disk

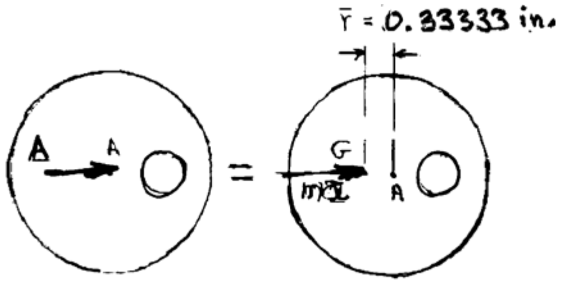
We determine the centroid of the composite area:


y
C
x
A
$\bar{x}$
=
y
$\bar{x}_1=0$
C_1
x
$A_1=\pi r_1^2$
$r_1=15$ in.
−
y
$\bar{x}_2=8$ in.
C_2
$r_2=3$ in
A_2 πr_2^2

$$\bar{x}A = \bar{x}_1A_1 - \bar{x}_2A_2 \quad \text{or} \quad \bar{x}(A_1 - A_2) = \bar{x}_1A_1 - \bar{x}_2A_2$$

$$\bar{x} = \frac{\bar{x}A_1 - \bar{x}_2A_2}{A_1 - A_2} = \frac{0 - (8)\pi(3)^2}{\pi(15)^2 - \pi(3)^2} = -\frac{(8)(3)^2}{(15)^2 - (3)^2} = -0.33333 \text{ in.}$$

Kinetics


$\bar{r}$ = 0.33333 in.
A
A
=
G
$m\bar{a}$
A

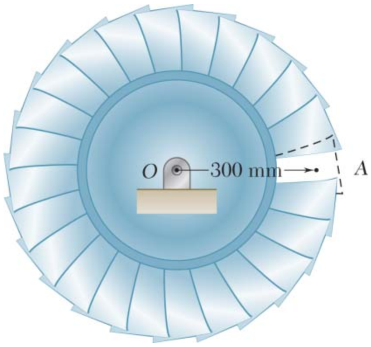
Mass center *G* coincides with centroid *C*

ω = 480 rpm = 50.265 rad/s

$$\bar{a} = \bar{r}\omega^2 = \left(\frac{0.33333}{12}\right)(50.265 \text{ rad/s})^2 = 70.183 \text{ ft/s}^2$$

$$\Sigma F = \Sigma(F)_{\text{eff}}: \ \mathbf{A} = m\bar{\mathbf{a}} = \left(\frac{60 \text{ lb}}{32.2}\right)(70.183 \text{ ft/s}^2)$$

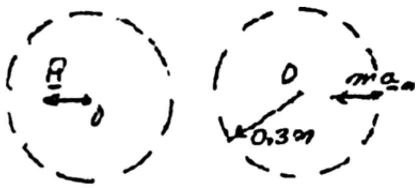
A = 130.8 N ⟶ ◀



PROBLEM 16.83

A turbine disk of mass 26 kg rotates at a constant rate of 9600 rpm. Knowing that the mass center of the disk coincides with the center of rotation O, determine the reaction at O immediately after a single blade at A, of mass 45 g, becomes loose and is thrown off.

SOLUTION



$$\omega = 9600 \text{ rpm}\left(\frac{2\pi}{60}\right)$$

$$\omega = 320\pi \text{ rad/s}$$

Consider before it is thrown off.

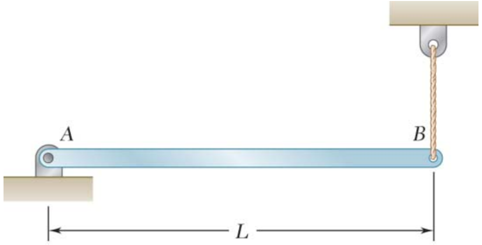
$$\xleftarrow{+} \Sigma F = \Sigma F_{\text{eff}}: \quad R = m a_n = m F \omega^2$$

$$= (45 \times 10^{-3} \text{ kg})(0.3 \text{ m})(320\,\pi)^2$$

$$R = 13.64 \text{ kN}$$

Before blade was thrown off, the disk was balanced $(R = 0)$. Removing vane at A also removes its reaction, so disk is unbalanced and reaction is

$$\mathbf{R} = 13.64 \text{ kN} \longrightarrow \blacktriangleleft$$

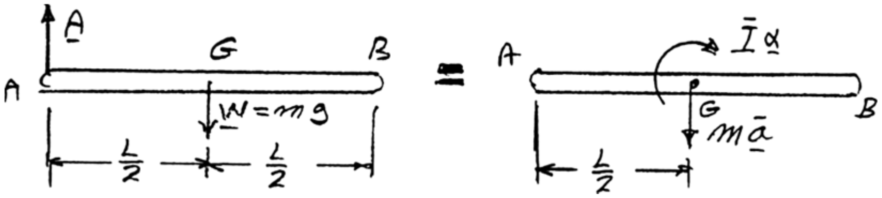


PROBLEM 16.84

A uniform rod of length L and mass m is supported as shown. If the cable attached at end B suddenly breaks, determine (*a*) the acceleration of end B, (*b*) the reaction at the pin support.

SOLUTION

$$w = 0 \quad \bar{a} = \frac{L}{2}\alpha$$



$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad W\frac{L}{2} = \bar{I}\alpha + m\bar{a}\frac{L}{2}$$

$$mg\frac{L}{2} = \frac{1}{12}mL^2\alpha + m\left(\frac{L}{2}\alpha\right)\frac{L}{2}$$

$$mg\frac{L}{2} = \frac{1}{3}mL^2\alpha \quad \alpha = \frac{3}{2}\frac{g}{L} \circlearrowright$$

(*b*) Reaction at *A*.

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad A - mg = -m\bar{a} = -m\frac{L}{2}\alpha$$

$$A - mg = -m\left(\frac{L}{2}\right)\left(\frac{3}{2}\frac{g}{L}\right)$$

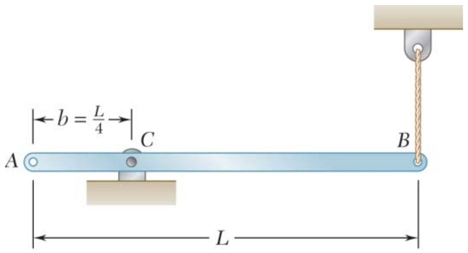
$$A - mg = -\frac{3}{4}mg$$

$$A = \frac{1}{4}mg \qquad \mathbf{A} = \frac{1}{4}mg\uparrow \blacktriangleleft$$

(*a*) Acceleration of *B*.

$$\mathbf{a}_B = \mathbf{a}_n + \mathbf{a}_{B/A} = 0 + L\alpha\downarrow$$

$$\mathbf{a}_B = L\left(\frac{3}{2}\frac{g}{L}\right) = \frac{3}{2}g\downarrow \qquad \mathbf{a}_B = \frac{3}{2}g\downarrow \blacktriangleleft$$

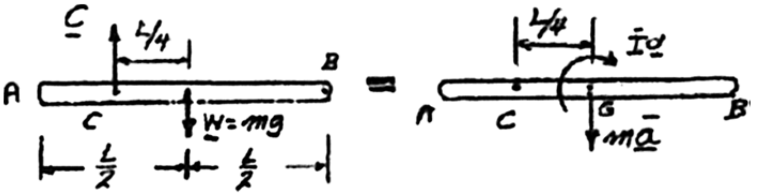


PROBLEM 16.85

A uniform rod of length L and mass m is supported as shown. If the cable attached at end B suddenly breaks, determine (*a*) the acceleration of end B, (*b*) the reaction at the pin support.

SOLUTION

$$\omega = 0 \qquad \bar{a} = \frac{L}{4}\alpha$$



$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad W\frac{L}{4} = \bar{I}\alpha + m\bar{a}\frac{L}{4}$$

$$mg\frac{L}{4} = \frac{1}{12}mL^2\alpha + m\left(\frac{L}{4}\alpha\right)\frac{L}{4}$$

$$mg\frac{L}{4} = \frac{7}{48}mL^2\alpha \qquad \boldsymbol{\alpha} = \frac{12g}{7L} \circlearrowright$$

(*b*) Reaction at *C*.

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad C - mg = -m\bar{a} = -m\frac{L}{4}\alpha$$

$$C - mg = -m\left(\frac{L}{4}\right)\left(\frac{12g}{7L}\right)$$

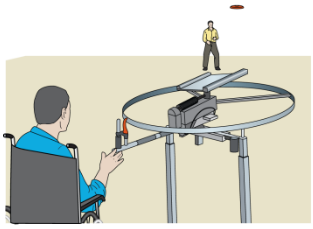
$$C - mg = -\frac{3}{7}mg$$

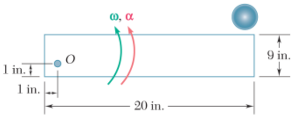
$$C = \frac{4}{7}mg \qquad \mathbf{C} = \frac{4}{7}mg\uparrow \blacktriangleleft$$

(*a*) Acceleration of *B*.

$$a_B = a_C + a_{C/B} = 0 + \frac{3L}{4}\alpha$$

$$a_B = \frac{3L}{4}\left(\frac{12g}{7L}\right) = \frac{9}{7}g \qquad \mathbf{a}_B = \frac{9}{7}g\downarrow \blacktriangleleft$$

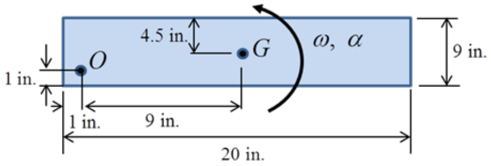



ω, α
1 in.
O
1 in.
20 in.
9 in.

PROBLEM 16.86

An adapted launcher uses a torsional spring about point *O* to help people with mobility impairments throw a Frisbee. Just after the Frisbee leaves the arm, the angular velocity of the throwing arm is 200 rad/s and its acceleration is 10 rad/s^2, both counterclockwise. The rotation point *O* is located 1 in. from the two sides. Assume that you can model the 2-lb throwing arm as a uniform rectangle. Just after the Frisbee leaves the arm, determine *(a)* the moment about *O* caused by the spring, *(b)* the forces on the pin at *O*.

SOLUTION



Given:

$$m = \frac{2 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.06211 \text{ slugs}$$

$$\omega = 200 \text{ rad/s}, \ \alpha = 10 \text{ rad/s}^2$$

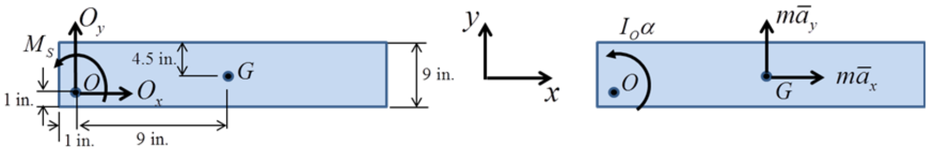
Mass Moment of Inertia:

$$I_O = I_G + md_{G/O}^2$$

$$= \frac{1}{12}(0.06211)\left[\left(\frac{9}{12}\right)^2 + \left(\frac{20}{12}\right)^2\right] + (0.06211)\sqrt{\left(\frac{9}{12}\right)^2 + \left(\frac{3.5}{12}\right)^2}$$

$$= 0.06727 \text{ slug•ft}^2$$

Free Body Diagram:



Kinetics:

$$\sum F_x = m\bar{a}_x \qquad O_x = 0.06211\bar{a}_x$$

$$\sum F_y = m\bar{a}_y \qquad O_y = 0.06211\bar{a}_y$$

$$\sum M_O = I_O\alpha \qquad M_s = 0.06727(10) = 0.6727 \text{ ft•lbs}$$

(a) $M_S = 0.6727$ ft•lbs ◀

Kinematics:

$$\mathbf{a}_G = \cancel{\mathbf{a}_O} + \boldsymbol{\alpha} \times \mathbf{r}_{G/O} - \omega^2 \mathbf{r}_{G/O}$$

$$= 10\mathbf{k} \times \left(\frac{9}{12}\mathbf{i} + \frac{3.5}{12}\mathbf{j}\right) - (200)^2\left(\frac{9}{12}\mathbf{i} + \frac{3.5}{12}\mathbf{j}\right)$$

$$= -30002.9\mathbf{i} - 11659.2\mathbf{j} \text{ ft/s}$$

$$\bar{a}_x = -30002.9 \text{ ft/s and } \bar{a}_y = -11659.2 \text{ ft/s}$$

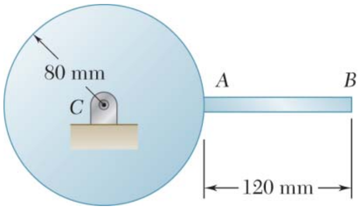
Therefore:

$$O_x = 0.06211(-30002.9) = -1863.5 \text{ lbs} \qquad O_y = 0.06211(-11659.2) = -724.0 \text{ lbs}$$

(b) Magnitude of Force at O:

$$O = \sqrt{O_x^2 + O_y^2} = \sqrt{(-1863.5)^2 + (-724.0)^2}$$

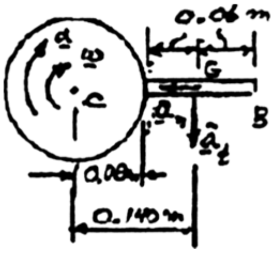
$O = 1999.2$ lbs ◀


80 mm
C
A
B
120 mm

PROBLEM 16.87

A 1.5-kg slender rod is welded to a 5-kg uniform disk as shown. The assembly swings freely about *C* in a vertical plane. Knowing that in the position shown the assembly has an angular velocity of 10 rad/s clockwise, determine (*a*) the angular acceleration of the assembly, (*b*) the components of the reaction at *C*.

SOLUTION


0.06 m
α
ω
C
G
B
$\bar{a}_n$
$\bar{a}_t$
0.08 m
0.140 m

Kinematics:

$$\bar{a}_n = (CG)\omega^2 = (0.14\text{ m})(10\text{ rad/s}^2)$$

$$\bar{\mathbf{a}}_n = 14\text{ m/s}^2 \longleftarrow$$

$$\bar{\mathbf{a}}_t = (CG)\alpha = (0.14\text{ m})\alpha \downarrow$$

Kinetics:

$$\bar{I}_{\text{disk}} = \frac{1}{2}m_{\text{disk}}(CG)^2$$

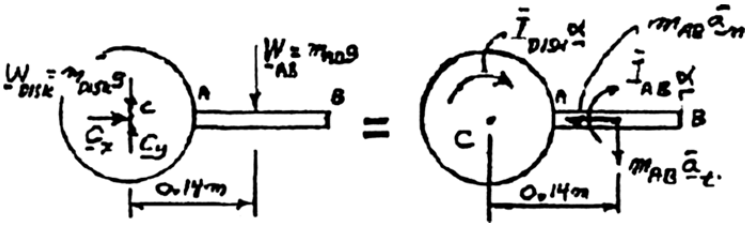
$$= \frac{1}{2}(5\text{ kg})(0.08\text{ m})^2$$

$$= 16\times10^{-3}\text{kg}\cdot\text{m}^2$$

$$\bar{I}_{AB} = \frac{1}{12}m_{AB}(AB)^2$$

$$= \frac{1}{12}(1.5\text{ kg})(0.12\text{ m})^2$$

$$= 1.8\times10^{-3}\text{kg}\cdot\text{m}^2$$


$W_{DISK} = m_{DISK}g$
C
C_x
C_y
A
$W_{AB} = m_{AB}g$
B
0.14 m
=
$\bar{I}_{DISK}\alpha$
$m_{AB}\bar{a}_n$
$\bar{I}_{AB}\alpha$
C
A
B
$m_{AB}\bar{a}_t$.
0.14 m

(*a*) Angular acceleration.

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}:$$

$$W_{AB}(0.14\text{ m}) = \bar{I}_{\text{disk}}\alpha + m_{AB}\bar{a}_t(0.14\text{ m}) + \bar{I}_{AB}\alpha$$

$$(1.5\text{ kg})(9.81\text{ m/s}^2)(0.14\text{ m}) = \bar{I}_{\text{disk}}\alpha + (1.5\text{ kg})(0.14\text{ m})^2\alpha + \bar{I}_{AB}\alpha$$

$$2.060\text{ N}\cdot\text{m} = (16\times10^{-3} + 29.4\times10^{-3} + 1.8\times10^{-3})\alpha$$

$$2.060\text{ N}\cdot\text{m} = (47.2\times10^{-3}\text{kg}\cdot\text{m}^2)\alpha$$

$$\alpha = 43.64\text{ rad/s}^2$$

$$\boldsymbol{\alpha} = 43.6\text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

PROBLEM 16.87 (Continued)

(*b*) Components of reaction of *C*.

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad C_x = -m_{AB}\bar{a}_n = -(1.5\text{ kg})(14\text{ m/s}^2)$$

$$C_x = -21.0\text{ N} \qquad\qquad \mathbf{C}_x = 21.0\text{ N} \longleftarrow \blacktriangleleft$$

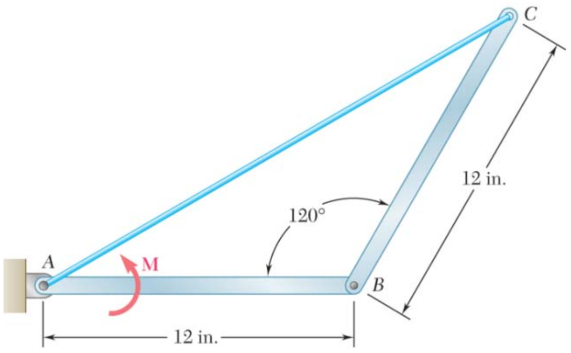
$$+\!\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad a_t = (0.14\text{ m})(\alpha)$$

$$C_y - m_{\text{disk}}g - m_{AB}g = -m_{AB}\bar{a}_t$$

$$C_y - (5\text{ kg})9.81 - (1.5\text{ kg})\,9.81 = -(1.5\text{ kg})(0.14\text{ m})(43.64\text{ rad/s}^2)$$

$$C_y - 49.05\text{ N} - 14.715\text{ N} = -9.164\text{ N}$$

$$C_y = +54.6\text{ N} \qquad\qquad \mathbf{C}_y = 54.6\text{ N} \uparrow \blacktriangleleft$$

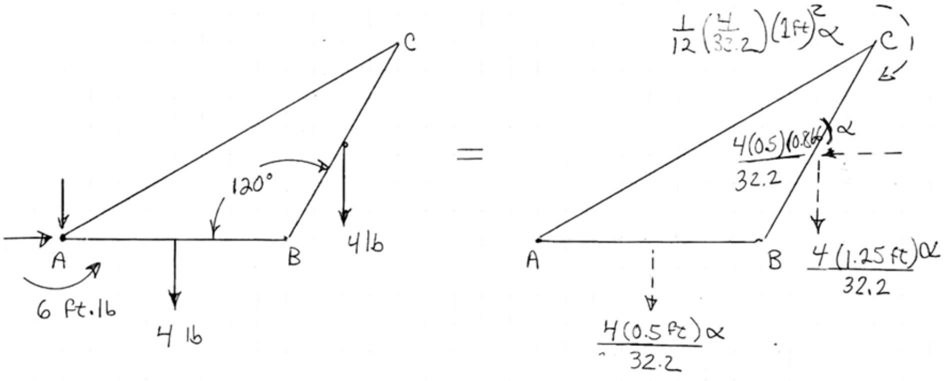

C
12 in.
120°
A
M
B
12 in.

PROBLEM 16.88

Two identical 4-lb slender rods *AB* and *BC* are connected by a pin at *B* and by the cord *AC*. The assembly rotates in a vertical plane under the combined effect of gravity and a 6 lb·ft couple **M** applied to rod *AB*. Knowing that in the position shown the angular velocity of the assembly is zero, determine (*a*) the angular acceleration of the assembly, (*b*) the tension in cord *AC*.

SOLUTION

(*a*)


C
120°
A
B
4 lb
6 ft.lb
4 lb
=
$\frac{1}{12}\left(\frac{4}{32.2}\right)(1\text{ft})^2\alpha$
C
$\frac{4(0.5)(0.866)}{32.2}\alpha$
A
B
$\frac{4(1.25\text{ ft})\alpha}{32.2}$
$\frac{4(0.5\text{ ft})\alpha}{32.2}$

$$\overset{+}{\circlearrowright}\ \Sigma M_A = (4\text{ lb})(0.5\text{ ft}) + (4\text{ lb})(1.25\text{ ft}) - 6\text{ lb}\cdot\text{ft}$$

$$= 2\left(\frac{1}{12}\right)\left(\frac{4\text{ lb}}{32.2\text{ ft/s}^2}\right)(1\text{ ft})^2\alpha + \left(\frac{4\text{ lb}}{32.2\text{ ft/s}^2}\right)(0.5\text{ ft})^2\alpha$$

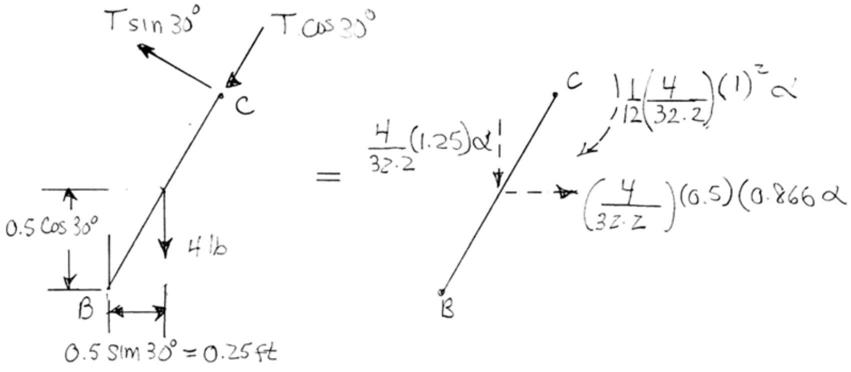
$$+\left(\frac{4\text{ lb}}{32.2\text{ ft/s}^2}\right)(1.25\text{ ft})^2\alpha + \left(\frac{4\text{ lb}}{32.2\text{ ft/s}^2}\right)[0.5\text{ ft}(0.866)]^2\alpha$$

$$(7-6)\text{ lb}\cdot\text{ft} = (0.26915)\alpha$$

$$\alpha = 3.7154\text{ rad/s}^2$$

or $\boldsymbol{\alpha} = 3.72\text{ rad/s}^2 \circlearrowright$ ◀

(*b*)


T sin 30°
T cos 30°
C
0.5 cos 30°
4 lb
B
0.5 sin 30° = 0.25 ft
=
C
$\frac{1}{12}\left(\frac{4}{32.2}\right)(1)^2\alpha$
$\frac{4}{32.2}(1.25)\alpha$
$\left(\frac{4}{32.2}\right)(0.5)(0.866\alpha$
B

PROBLEM 16.88 (Continued)

$$\overset{+}{\circlearrowright}\ \Sigma M_B = (4\text{ lb})(0.25\text{ ft}) - T(1\text{ ft})(\sin 30°)$$

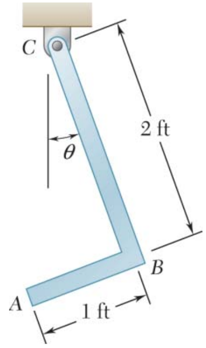
$$= \frac{1}{12}\left(\frac{4\text{ lb}}{32.2\text{ ft/s}^2}\right)(1\text{ ft})^2\alpha + \left(\frac{4\text{ lb}}{32.2\text{ ft/s}^2}\right)(1.25\text{ ft})(0.25\text{ ft})\alpha$$

$$+\left(\frac{4\text{ lb}}{32.2\text{ ft/s}^2}\right)(0.5\text{ ft})(\cos .30°)(0.433\text{ ft})\alpha$$

$$= 0.07246\alpha = (0.07246)(3.7154) = 1 - T(0.5) = 0.26922$$

$$T = 1.462\text{ lb}$$

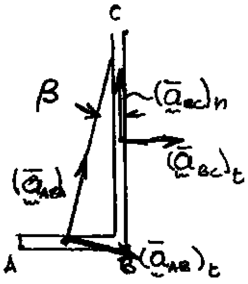
or $T = 1.462\text{ lb}$ ◀


C
2 ft
θ
B
A
1 ft

PROBLEM 16.89

The object *ABC* consists of two slender rods welded together at Point *B*. Rod *AB* has a weight of 2 lb and bar *BC* has a weight of 4 lb. Knowing the magnitude of the angular velocity of *ABC* is 10 rad/s when $\theta = 0$, determine the components of the reaction at Point *C* at this location.

SOLUTION


C
β
$(\bar{a}_{BC})_n$
$(\bar{a}_{BC})_t$
$(\bar{a}_{AB})$
A
B
$(\bar{a}_{AB})_t$

Masses and lengths: $W_{AB} = 2\text{ lb}, \quad L_{AB} = 1\text{ ft}, \quad W_{BC} = 4\text{ lb}, \quad L_{BC} = 2\text{ ft}$

Moments of inertia:

$$\bar{I}_{AB} = \frac{1}{12}m_{AB}L_{AB}^2 = \frac{1}{12}\left(\frac{2}{32.2}\right)(1)^2 = 5.1760\times10^{-3}\text{ slug}\cdot\text{ft}^2$$

$$\bar{I}_{BC} = \frac{1}{12}m_{BC}L_{BC}^2 = \frac{1}{12}\left(\frac{4}{32.2}\right)(2)^2 = 41.408\times10^{-3}\text{ slug}\cdot\text{ft}^2$$

Geometry:

$$r_{CD} = \sqrt{2^2 + 0.5^2} = 2.0616\text{ ft}$$

$$r_{CE} = \frac{1}{2}L_{AB} = 1\text{ ft}$$

$$\tan\beta = \frac{0.5}{2}$$

$$\beta = 14.036°$$

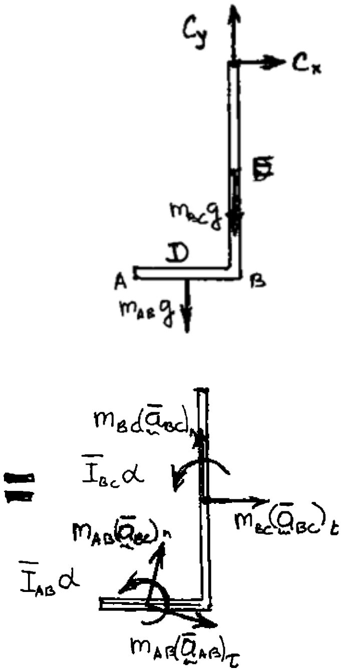
Kinematics: Let $\boldsymbol{\alpha} = \alpha \circlearrowleft$ be the angular acceleration of object *ABC*.

$$(\bar{\mathbf{a}}_{AB})_t = r_{CD}\alpha \searrow \beta$$

$$(\bar{\mathbf{a}}_{AB})_n = r_{CD}\omega^2 \nearrow \beta$$

$$(\bar{\mathbf{a}}_{BC})_t = r_{CE}\alpha \longrightarrow$$

$$(\bar{\mathbf{a}}_{BC})_n = r_{CE}\omega^2 \uparrow$$


C_y
C_x
E
$m_{BC}g$
D
A
B
$m_{AB}g$
=
$m_{BC}(\bar{a}_{BC})_n$
$\bar{I}_{BC}\alpha$
$m_{AB}(\bar{a}_{AB})_n$
$m_{BC}(\bar{a}_{BC})_t$
$\bar{I}_{AB}\alpha$
$m_{AB}(\bar{a}_{AB})_t$

Kinetics: $+\circlearrowleft\Sigma M_C = +\circlearrowleft\Sigma(M_C)_{\text{eff}}$:

$$W_{AB}\frac{L_{AB}}{2} = \bar{I}_{AB}\alpha + r_{CD}m_{AB}(\bar{a}_{AB})_t + \bar{I}_{BC}\alpha + r_{CE}m_{BC}(\bar{a}_{BC})_t = \left(\bar{I}_{AB} + m_{AB}r_{CD}^2 + \bar{I}_{BC} + m_{BC}r_{CE}^2\right)\alpha$$

$$(2)(0.5) = \left[(5.1760\times10^{-3}) + \left(\frac{2}{32.2}\right)(2.0616)^2 + (41.408\times10^{-3}) + \left(\frac{4}{32.2}\right)(1)^2\right]\alpha$$

$\boldsymbol{\alpha} = 2.3\text{ rad/s}^2 \circlearrowleft$

PROBLEM 16.89 (Continued)

$$\xrightarrow{+} \Sigma F_x = \xrightarrow{+} \Sigma(F_x)_{\text{eff}}: \quad C_x = m_{AB}(\overline{a}_{AB})_t \cos\beta + m_{AB}(\overline{a}_{AB})_n \sin\beta + m_{BC}(a_{BC})_t$$

$$= m_{AB} r_{CD}(\alpha\cos\beta + \omega^2 \sin\beta) + m_{BC} r_{CE}\alpha$$

$$= \left(\frac{2}{32.2}\right)(2.0616)(2.3\cos 14.035^\circ + 10^2 \sin 14.035^\circ) + \left(\frac{4}{32.2}\right)(1)(2.3)$$

$$= 3.6770 \text{ lb}$$

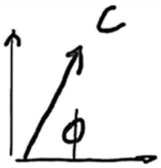
$$+\!\uparrow \Sigma F_y = +\!\uparrow \Sigma(F_y)_{\text{eff}}: \quad C_y - m_{AB}g - m_{BC}g = -m_{AB}(\overline{a}_{AB})_t \sin\beta + m_{AB}(\overline{a}_{AB})_n \cos\beta + m_{BC}(\overline{a}_{BC})_n$$

$$C_y = (W_{AB} + W_{BC}) + m_{AB} r_{CD}(\omega^2 \cos\beta - \alpha\sin\beta) + m_{BC} r_{CE}\,\omega$$

$$C_y = 6 + \left(\frac{2}{32.2}\right)(2.0616)(10^2 \cos 14.035^\circ - 2.3\sin 14.035^\circ)$$

$$+ \left(\frac{4}{32.2}\right)(1)(10)^2$$

$$C_y = 30.773 \text{ lb}$$



Reaction at *C*.

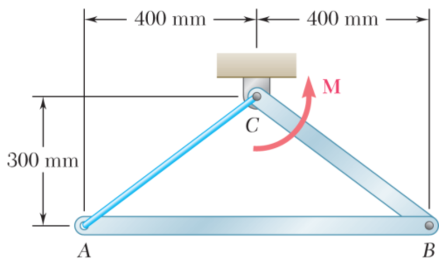
$$C = \sqrt{3.6770^2 + 30.773^2}$$

$$= 30.992 \text{ lb}$$

$$\tan\phi = \frac{30.773}{3.6770}$$

$$\phi = 83.186^\circ$$

$\mathbf{C} = 31.0 \text{ lb} \measuredangle 83.2^\circ$ ◀



PROBLEM 16.90

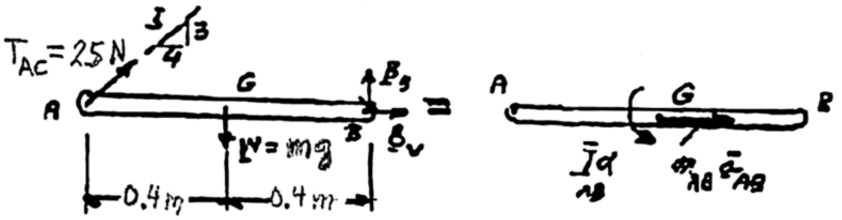
A 3.5-kg slender rod *AB* and a 2-kg slender rod *BC* are connected by a pin at *B* and by the cord *AC*. The assembly can rotate in a vertical plane under the combined effect of gravity and a couple **M** applied to rod *BC*. Knowing that in the position shown the angular velocity of the assembly is zero and the tension in cord *AC* is equal to 25 N, determine (*a*) the angular acceleration of the assembly, (*b*) the magnitude of the couple **M**.

SOLUTION

(*a*) Angular acceleration.

Rod *AB*:

$$\bar{I}_{AB} = \frac{1}{12} m_{AB} L_{AB}^2$$
$$= \frac{1}{12} 3.5\ \text{kg}(0.8\ \text{m})^2$$
$$= 0.18667\ \text{kg} \cdot \text{m}^2$$



$$+\circlearrowright \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \ \frac{3}{5}(25\ \text{N})(0.8\ \text{m}) - (3.5\ \text{kg})(9.81\ \text{m/s}^2)(0.4\ \text{m}) = \bar{I}_{AB}\alpha$$

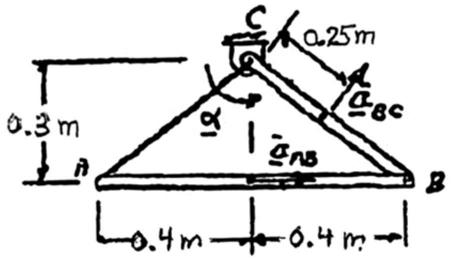
$$\bar{I}_{AB}\alpha = -1.7340\ \text{N} \cdot \text{m} \qquad (1)$$

$$(0.18667\ \text{kg} \cdot \text{m}^2)\alpha = -1.7340\ \text{N} \cdot \text{m}$$

$$\boldsymbol{\alpha} = -9.2893\ \text{rad/s}^2 \qquad \boldsymbol{\alpha} = 9.29\ \text{rad/s}^2 \circlearrowright \blacktriangleleft$$

Entire assembly: Since *AC* is taut, assembly rotates about *C* as a rigid body.

Kinematics:



$$CB = \sqrt{(0.3)^2 + (0.4)^2} = 0.5\ \text{m}$$

$$CG_{BC} = \frac{1}{2} CB = 0.25\ \text{m}$$

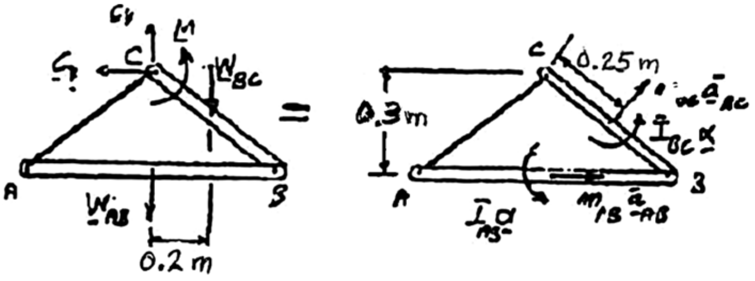
$$\bar{\mathbf{a}}_{BC} = (0.25\ \text{m})\alpha \nearrow$$

$$\bar{\mathbf{a}}_{AB} = (0.3\ \text{m})\alpha \longrightarrow$$

PROBLEM 16.90 (Continued)

Kinetics:

$$\bar{I}_{BC} = \frac{1}{12} m_{BC} (CB)^2$$
$$= \frac{1}{12}(2\ \text{kg})(0.5\ \text{m})^2$$
$$= 0.041667\ \text{kg}\cdot\text{m}^2$$



(*b*) Couple **M**.

$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}:$$

$$M - m_{BC} g(0.2\ \text{m}) = m_{BC}\bar{a}_{BC}(0.25\ \text{m}) + \bar{I}_{BC}\alpha + m_{AB}\bar{a}_{AB}(0.3\ \text{m}) + \bar{I}_{AB}\alpha$$
$$M - (2\ \text{kg})(9.81\ \text{m/s}^2)(0.2\ \text{m}) = 2\ \text{kg}(0.25\ \text{m})^2\alpha + (0.041667\ \text{kg}\cdot\text{m}^2)\alpha + 3.5\ \text{kg}(0.3\ \text{m})^2\alpha + \bar{I}_{AB}\alpha$$

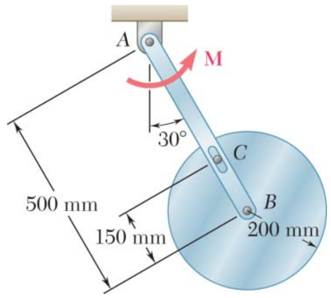
Substitute $\alpha = 9.29\ \text{rad/s}^2$ $\quad \bar{I}_{AB}\alpha = -1.7340\ \text{N}\cdot\text{m}$

$$M - 3.9240 = (0.125)(9.2893) + (0.041667)(9.2893)$$
$$+ (0.315)(9.2893) + (0.18667)(9.2893)$$
$$M - 3.9240 = 1.1612 + 0.3871 + 2.9261 + 1.7340$$
$$M - 3.9240 = 6.2084$$
$$M = +10.132\ \text{N}\cdot\text{m}$$

$\mathbf{M} = 10.13\ \text{N}\cdot\text{m}\ \circlearrowleft$ ◀

$$M - m_{BC} g(0.2\ \text{m}) = I_C\alpha$$

Since C is fixed, we could also use: $M - (3.5\ \text{kg})(9.81\ \text{m/s}^2)(0.2\ \text{m}) = \left(\frac{1}{3} m_{CB}\overline{CB}^2 + \bar{I}_{AB} + 3.5\ \text{kg}(0.3\ \text{m})^2\right)\alpha$



PROBLEM 16.91

A 9-kg uniform disk is attached to the 5-kg slender rod *AB* by means of frictionless pins at *B* and *C*. The assembly rotates in a vertical plane under the combined effect of gravity and of a couple **M** which is applied to rod *AB*. Knowing that at the instant shown the assembly has an angular velocity of 6 rad/s and an angular acceleration of 25 rad/s^2, both counterclockwise, determine (*a*) the couple **M**, (*b*) the force exerted by pin *C* on member *AB*.

SOLUTION

We first consider the entire system and express that the external forces are equivalent to the inertial terms of the disk and the rod.

$$m_R = 5\text{ kg},\qquad m_D = 9\text{ kg}$$

$$W_R = m_D g = (5\text{ kg})(9.81\text{ m/s}^2) = 49.05\text{ N}$$

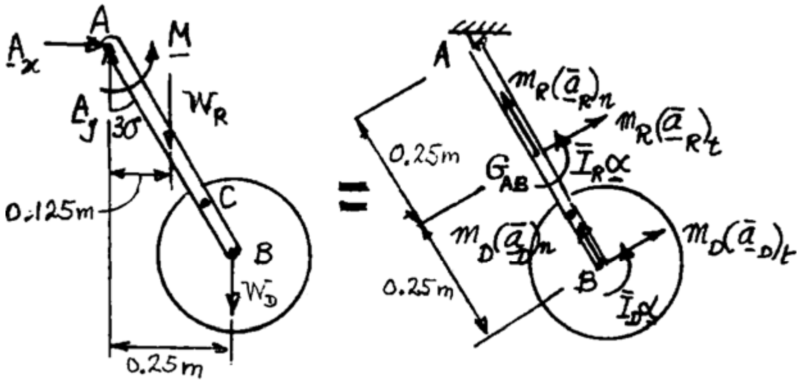
$$W_D = m_D g = (9\text{ kg})(9.81\text{ m/s}^2) = 88.29\text{ N}$$

$$I_R = \frac{1}{12}m_R L_{AB}^2 = \frac{1}{12}(5\text{ kg})(0.5\text{ m})^2 = 0.104167\text{ kg}\cdot\text{m}^2$$

$$I_D = \frac{1}{2}m_D r_D^2 = \frac{1}{2}(9\text{ kg})(0.2\text{ m})^2 = 0.18\text{ kg}\cdot\text{m}^2$$

$$\bar{a}_R = (0.25\text{ m})\alpha$$

$$\bar{a}_D = (0.5\text{ m})\alpha$$



(*a*) $\Sigma M_A = \Sigma(M_A)_{\text{eff}}:\quad M - W_R(0.125) - W_D(0.25) = \bar{I}_R\alpha + m_R(a_R)_t(0.25) + \bar{I}_D\alpha + m_D(\bar{a}_D)_t(0.5)$

$$M - (49.05)(0.125) - (88.29)(0.25) = 0.104167\alpha + (5)(0.25\alpha)(0.25) + 0.18\alpha + (9)(0.5\alpha)(0.5)$$

$$M - 6.1312 - 22.073 = (0.10417 + 0.3125 + 0.18 + 2.25)\alpha$$

$$M - 28.204 = (2.8467)(25)$$

$$M = 99.370\text{ N-m}$$

$M = 99.4$ N-m ↺ ◀

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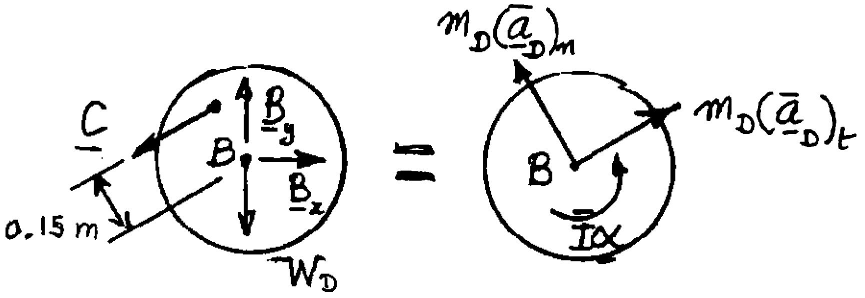
PROBLEM 16.91 (Continued)

(*b*) Consider now the disk alone:


C
B
B_y
B_x
0.15 m
W_D
=
$m_D(\bar{a}_D)_n$
$m_D(\bar{a}_D)_t$
B
$\bar{I}\alpha$

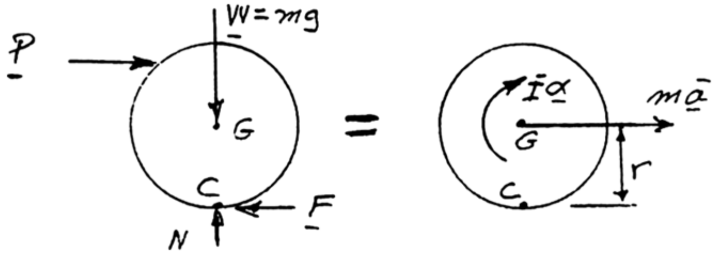
$$+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad C(0.15) = \bar{I}_D\alpha = (0.18)(25)$$

$\mathbf{C} = 30.0\text{ N} \measuredangle 30°$ ◀

PROBLEM 16.92

Derive the equation $\Sigma M_C = I_C\alpha$ for the rolling disk of Figure 16.17, where ΣM_C represents the sum of the moments of the external forces about the instantaneous center *C*, and I_C is the moment of inertia of the disk about *C*.

SOLUTION



$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad \Sigma M_C = (m\bar{a})r + \bar{I}\alpha = (mr\alpha)\bar{r} + \bar{I}\alpha$$

$$= (mr\alpha)\bar{r} + \bar{I}\alpha$$

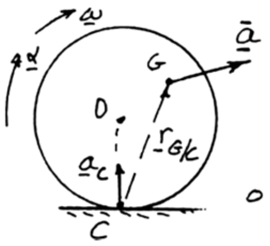
But, we know that $I_C = mr^2 + \bar{I}$

Thus: $\Sigma M_C = I_C\alpha$ (Q.E.D.) ◀

PROBLEM 16.93

Show that in the case of an unbalanced disk, the equation derived in Problem 16.92 is valid only when the mass center *G*, the geometric center *O*, and the instantaneous center *C* happen to lie in a straight line.

SOLUTION



Kinematics:

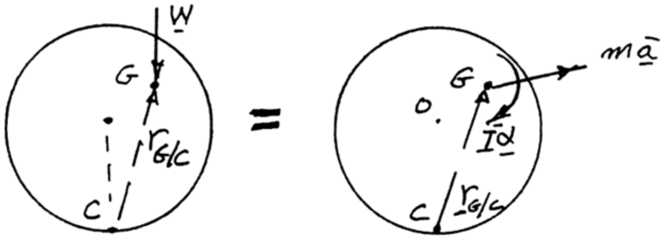
$$\bar{\mathbf{a}} = \mathbf{a}_C + \mathbf{a}_{G/C}$$

$$= \mathbf{a}_C + \boldsymbol{\alpha} \times \mathbf{r}_{G/C} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{G/C})$$

or, since $\boldsymbol{\omega} \perp \mathbf{r}_{G/C}$

$$\bar{\mathbf{a}} = \mathbf{a}_C + \boldsymbol{\alpha} \times \mathbf{r}_{G/C} - \omega^2 \mathbf{r}_{G/C} \quad (1)$$

Kinetics:



$$\Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad \Sigma \mathrm{M}_C = \bar{I}\boldsymbol{\alpha} + \mathbf{r}_{G/C} \times m\bar{\mathbf{a}}$$

Recall Eq. (1):

$$\Sigma \mathbf{M}_C = \bar{I}\alpha + \mathbf{r}_{G/C} \times m(\mathbf{a}_C + \boldsymbol{\alpha} \times \mathbf{r}_{G/C} - \omega^2 r_{G/C})$$

$$\Sigma \mathbf{M}_C = \bar{I}\alpha + \mathbf{r}_{G/C} \times m\mathbf{a}_C + m\mathbf{r}_{G/C} \times (\boldsymbol{\alpha} \times \mathbf{r}_{G/C}) - m\omega^2 \mathbf{r}_{G/C} \times \mathbf{r}_{G/C}$$

But: $\mathbf{r}_{G/C} \times \mathbf{r}_{G/C} = 0$ and $\boldsymbol{\alpha} \perp \mathbf{r}_{G/C}$

$$\mathbf{r}_{G/C} \times m(\boldsymbol{\alpha} \times \mathbf{r}_{G/C}) = mr_{G/C}^2 \alpha$$

Thus:

$$\Sigma \mathbf{M}_C = \left(\bar{I} + mr_{G/C}^2\right)\alpha + \mathbf{r}_{G/C} \times m\mathbf{a}_C$$

Since

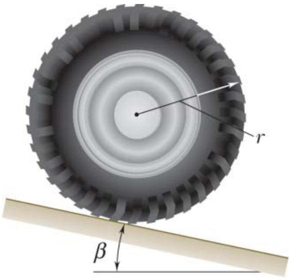
$$I_C = \bar{I} + mr_{G/C}^2$$

$$\Sigma \mathbf{M}_C = I_C \alpha + \mathbf{r}_{G/C} \times m\mathbf{a}_C \quad (2)$$

Eq. (2) reduces to $\Sigma M_C = I_C \alpha$ when $\mathbf{r}_{G/C} \times m\mathbf{a}_C = 0$; that is, when $\mathbf{r}_{G/C}$ and $\mathbf{a}_C$ are collinear.

Referring to the first diagram, we note that this will occur only when Points *G*, *O*, and *C* lie in a straight line.

(Q.E.D.) ◀



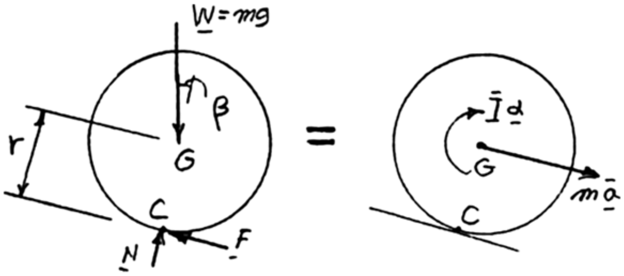
PROBLEM 16.94

A wheel of radius r and centroidal radius of gyration $\bar{k}$ is released from rest on the incline and rolls without sliding. Derive an expression for the acceleration of the center of the wheel in terms of r, $\bar{k}$, β, and g.

SOLUTION

$$\bar{I}\alpha = m\bar{k}^2\alpha$$

$$m\bar{a} = mr\alpha$$



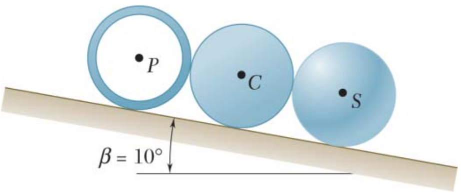
$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad (W\sin\beta)r = (m\bar{a})r + \bar{I}\alpha$$

$$(mg\sin\beta)r = (mr\alpha)r + m\bar{k}^2\alpha$$

$$rg\sin\beta = (r^2 + \bar{k}^2)\alpha$$

$$\alpha = \frac{rg\sin\beta}{r^2 + \bar{k}^2}$$

$$\bar{a} = r\alpha = r\frac{rg\sin\beta}{r^2 + k^2} \qquad \bar{a} = \frac{r^2}{r^2 + \bar{k}^2} g\sin\beta \blacktriangleleft$$

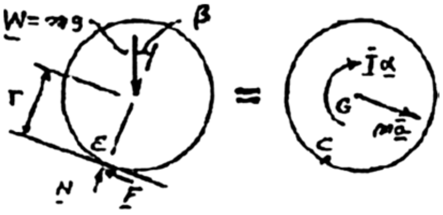


PROBLEM 16.95

A homogeneous sphere S, a uniform cylinder C, and a thin pipe P are in contact when they are released from rest on the incline shown. Knowing that all three objects roll without slipping, determine, after 4 s of motion, the clear distance between (*a*) the pipe and the cylinder, (*b*) the cylinder and the sphere.

SOLUTION

General case: $\bar{I} = m\bar{k}^2 \quad \bar{a} = r\alpha$



$$\overset{+}{\circlearrowright}\Sigma M_C = \Sigma (M_C)_{\text{eff}}: \quad (w\sin\beta)r = \bar{I}\alpha + m\bar{a}r$$

$$mg\sin\beta r = m\bar{k}^2\alpha + mr^2\alpha$$

$$\alpha = \frac{r\theta\sin\beta}{r^2 + \bar{k}^2}$$

$$\bar{a} = r\alpha = r\frac{rg\sin\beta}{r^2 + \bar{k}^2} \qquad \bar{a} = \frac{r^2}{r^2 + \bar{k}^2}g\sin\beta \tag{1}$$

For pipe: $\bar{k} = r \qquad \bar{a}_P = \frac{r^2}{r^2 + r^2}g\sin\beta = \frac{1}{2}g\sin\beta$

For cylinder: $\bar{k}^2 = \frac{1}{2} \qquad a_C = \frac{r^2}{r^2 + \frac{r^2}{2}}g\sin\beta = \frac{2}{3}g\sin\beta$

For sphere: $\bar{k}^2 = \frac{2}{5} \qquad a_S = \frac{r^2}{r^2 + \frac{2}{5}r^2}g\sin\beta = \frac{5}{7}g\sin\beta$

(*a*) Between pipe and cylinder.

$$a_{C/P} = a_C - a_P = \left(\frac{2}{3} - \frac{1}{2}\right)g\sin\beta = \frac{1}{6}g\sin\beta$$

$$x_{C/P} = \frac{1}{2}a_{C/P}t^2 = \frac{1}{2}\left(\frac{1}{6}g\sin\beta\right)t^2$$

SI units: $x_{C/P} = \frac{1}{2}\left(\frac{1}{6}9.81\text{ m/s}^2\right)\sin 10°(4\text{ s})^2 = 2.27\text{ m}$ ◀

US units: $x_{C/P} = \frac{1}{2}\left(\frac{1}{6}32.2\text{ ft/s}^2\right)\sin 10°(4\text{ s})^2 = 7.46\text{ ft}$ ◀

PROBLEM 16.95 (Continued)

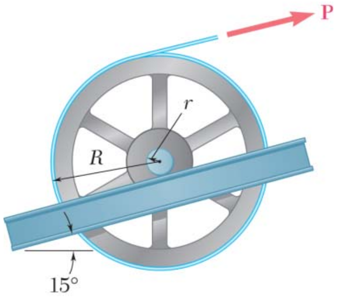
(*b*) Between sphere and cylinder.

$$a_{S/C} = a_S - a_C = \left(\frac{5}{7} - \frac{2}{3}\right) g \sin \beta = \frac{1}{21} g \sin \beta$$

$$x_{S/C} = \frac{1}{2} a_{S/C} t^2 = \frac{1}{2}\left(\frac{1}{21} g \sin \beta\right) t^2$$

SI units: $x_{S/C} = \frac{1}{2}\left(\frac{1}{21} 9.81\text{ m/s}^2\right) \sin 10°(4\text{ s})^2 = 0.649\text{ m}$ ◀

US units: $x_{S/C} = \frac{1}{2}\left(\frac{1}{21} 32.2\text{ ft/s}^2\right) \sin 10°(4\text{ s})^2 = 2.13\text{ ft}$ ◀



PROBLEM 16.96

A 40-kg flywheel of radius $R = 0.5$ m is rigidly attached to a shaft of radius $r = 0.05$ m that can roll along parallel rails. A cord is attached as shown and pulled with a force **P** of magnitude 150 N. Knowing the centroidal radius of gyration is $\bar{k} = 0.4\text{ m}$, determine (*a*) the angular acceleration of the flywheel, (*b*) the velocity of the center of gravity after 5 s.

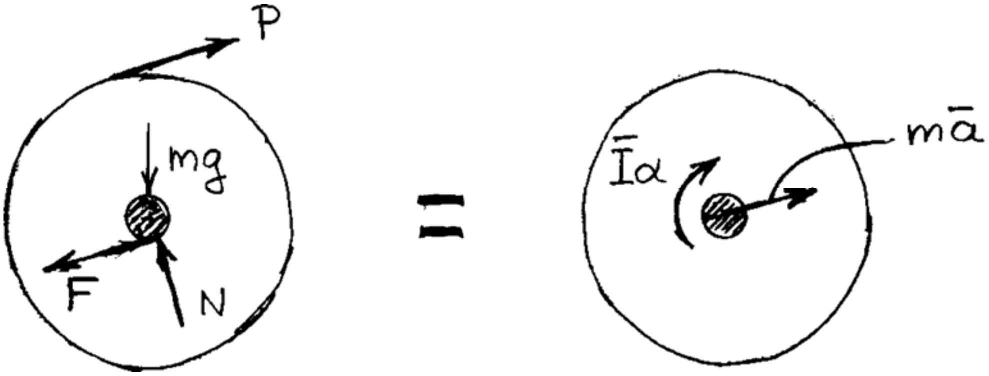
SOLUTION

Mass and moment of inertia: $m = 40\text{ kg}$

$$\bar{I} = m\bar{k}^2 = (40\text{ kg})(0.4\text{ m})^2 = 6.4\text{ kg}\cdot\text{m}^2$$

Kinematics: (no slipping) $\bar{\mathbf{a}} = r\alpha \measuredangle 15^\circ$

Kinetics:



Let Point *C* be the contact point between the flywheel and the rails.

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad P(R+r) - mgr\sin 15^\circ = \bar{I}\alpha + (m\bar{a})r$$

$$P(R+r) - mgr\sin 15^\circ = (\bar{I} + mr^2)\alpha$$

(*a*) *Angular acceleration of the flywheel.*

$$\alpha = \frac{P(R+r) - mgr\sin 15^\circ}{\bar{I} + mr^2}$$

$$= \frac{(150\text{ N})(0.55\text{ m}) - (40\text{ kg})(9.81\text{ m/s}^2)(0.05\text{ m})\sin 15^\circ}{(6.4\text{ kg}\cdot\text{m}^2) + (40\text{ kg})(0.05\text{ m})^2}$$

$$= 11.911\text{ rad/s}^2$$

$\boldsymbol{\alpha} = 11.91\text{ rad/s}^2 \circlearrowright$ ◀

PROBLEM 16.96 (Continued)

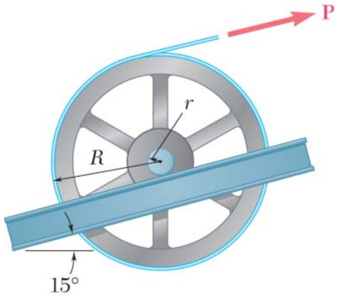
(*b*) *Velocity of center of gravity after* 5 s.

$$\bar{a} = r\alpha = (0.05\text{ m})(11.911\text{ rad/s}^2) = 0.59555\text{ m/s}^2$$

$$\bar{\mathbf{a}} = 0.59555\text{ m/s}^2 \measuredangle\, 15^\circ$$

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}_0 + \bar{\mathbf{a}}t = 0 + (0.59555\text{ m/s}^2)(5\text{ s})$$

$$\bar{\mathbf{v}} = 2.98\text{ m/s} \measuredangle\, 15^\circ \blacktriangleleft$$



PROBLEM 16.97

A 40-kg flywheel of radius $R = 0.5$ m is rigidly attached to a shaft of radius $r = 0.05$ m that can roll along parallel rails. A cord is attached as shown and pulled with a force **P**. Knowing the centroidal radius of gyration is $\bar{k} = 0.4\text{ m}$ and the coefficient of static friction is $\mu_s = 0.4$, determine the largest magnitude of force **P** for which no slipping will occur.

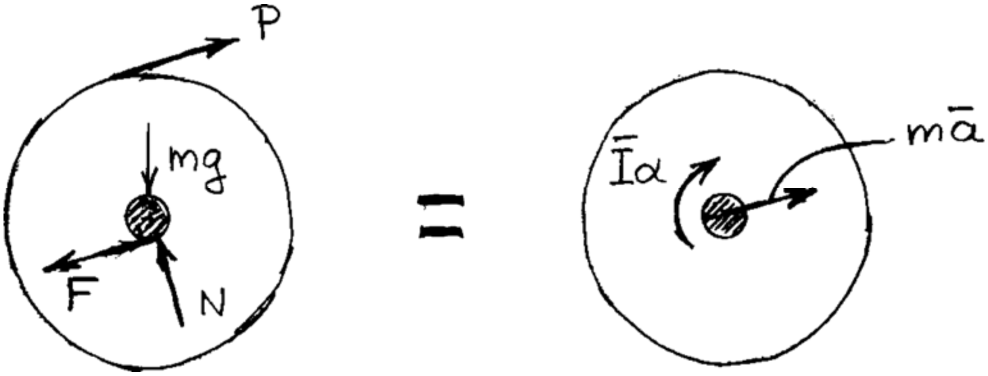
SOLUTION

Mass and moment of inertia: $m = 40\text{ kg}$

$$\bar{I} = m\bar{k}^2 = (40\text{ kg})(0.4\text{ m})^2 = 6.4\text{ kg}\cdot\text{m}^2$$

Kinematics: (no slipping) $\bar{\mathbf{a}} = r\alpha \measuredangle 15°$

Kinetics:



$$+\Sigma F_n = \Sigma(F_n)_{\text{eff}}: \quad N - mg\cos 15° = 0$$

$$N = (40\text{ kg})(9.81\text{ m/s}^2)\cos 15° = 379.03\text{ N}$$

For impending slipping,

$$F = \mu_s N = (0.4)(379.03)$$
$$= 151.61\text{ N}$$

$$+\measuredangle 15° \;\; \Sigma F = \Sigma(F)_{\text{eff}}: \quad P - F - mg\sin 15° = ma$$

$$P - 151.61\text{ N} = (40\text{ kg})(9.81\text{ m/s}^2)\sin 15° = (40\text{ kg})(0.05\text{ m})\alpha$$

$$P - 253.17\text{ N} = (2\text{ kg}\cdot\text{m})\alpha \qquad (1)$$

PROBLEM 16.97 (Continued)

Let C be the contact point between the flywheel and the rails.

$$+\circlearrowleft \Sigma M_C = \Sigma (M_C)_{\text{eff}}: \quad P(R+r) - mgr\sin 15° = \bar{I}a + (m\bar{a})r$$

$$P(R+r) - mgr\sin 15° = (\bar{I} + mr^2)\alpha$$

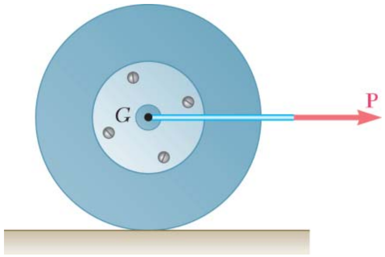
$$(0.55\text{ m})P - (40\text{ kg})(9.81\text{ m/s}^2)(0.05\text{ m})\sin 15°$$

$$= [6.4\text{ kg}\cdot\text{m}^2 + (40\text{ kg})(0.05\text{ m})^2]\alpha \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously,

$$P = 303\text{ N} \qquad \alpha = 24.8\text{ rad/s}^2$$

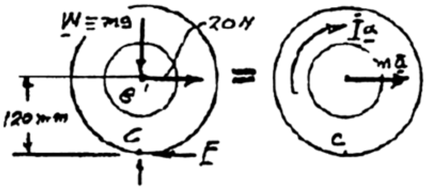
$P = 303\text{ N}$ ◀



PROBLEM 16.98

A drum of 60-mm radius is attached to a disk of 120-mm radius. The disk and drum have a total mass of 6 kg and a combined radius of gyration of 90 mm. A cord is attached as shown and pulled with a force **P** of magnitude 20 N. Knowing that the disk rolls without sliding, determine (*a*) the angular acceleration of the disk and the acceleration of *G*, (*b*) the minimum value of the coefficient of static friction compatible with this motion.

SOLUTION



$$\bar{a} = r\alpha = (0.12\text{ m})\alpha$$

$$\bar{I} = m\bar{k}^2$$

$$= (6\text{ kg})(0.09\text{ m})^2$$

$$\bar{I} = 48.6\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad (20\text{ N})(0.12\text{ m}) = (m\bar{a})r + \bar{I}\alpha$$

$$2.4\text{ N}\cdot\text{m} = (6\text{ kg})(0.12\text{ m})^2\alpha + 48.6\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$2.4 = 135.0\times 10^{-3}\alpha$$

(*a*) $\alpha = 17.778\text{ rad/s}^2$ $\qquad \boldsymbol{\alpha} = 17.78\text{ rad/s}^2 \circlearrowright$ ◀

$$\bar{a} = r\alpha = (0.12\text{ m})(17.778\text{ rad/s}^2)$$

$= 2.133\text{ m/s}^2$ $\qquad \bar{\mathbf{a}} = 2.13\text{ m/s}^2 \longrightarrow$ ◀

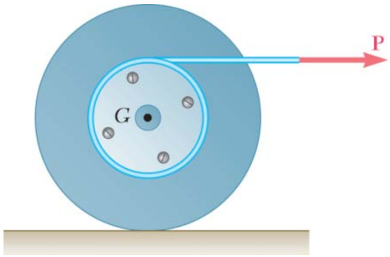
(*b*) $+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N - mg = 0$

$N = (6\text{ kg})(9.81\text{ m/s}^2) \qquad \mathbf{N} = 58.86\text{ N}\uparrow$

$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad 20\text{ N} - F = m\bar{a}$

$20\text{ N} - F = (6\text{ kg})(2.133\text{ m/s}^2) \qquad \mathbf{F} = 7.20\text{ N} \longleftarrow$

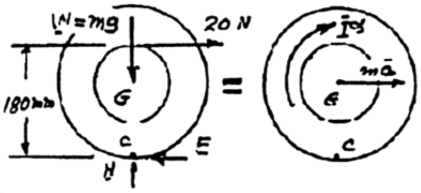
$(\mu_s)_{\min} = \dfrac{F}{N} = \dfrac{7.20\text{ N}}{58.86\text{ N}}$ $\qquad (\mu_s)_{\min} = 0.122$ ◀



PROBLEM 16.99

A drum of 60-mm radius is attached to a disk of 120-mm radius. The disk and drum have a total mass of 6 kg and a combined radius of gyration of 90 mm. A cord is attached as shown and pulled with a force **P** of magnitude 20 N. Knowing that the disk rolls without sliding, determine (*a*) the angular acceleration of the disk and the acceleration of *G*, (*b*) the minimum value of the coefficient of static friction compatible with this motion.

SOLUTION



$$\bar{a} = r\alpha = (0.12\text{ m})\alpha$$

$$\bar{I} = m\bar{k}^2$$

$$= (6\text{ kg})(0.09\text{ m})^2$$

$$\bar{I} = 48.6\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad (20\text{ N})(0.18\text{ m}) = (m\bar{a})r + \bar{I}\alpha$$

$$3.6\text{ N}\cdot\text{m} = (6\text{ kg})(0.12\text{ m})^2\alpha + 48.6\text{ kg}\cdot\text{m}^2$$

$$3.6 = 135\times 10^{-3}\alpha$$

(*a*) $\alpha = 26.667\text{ rad/s}^2$ $\quad \boldsymbol{\alpha} = 26.7\text{ rad/s}^2 \circlearrowright$ ◀

$$\bar{a} = r\alpha = (0.12\text{ m})(26.667\text{ rad/s}^2)$$

$= 3.2\text{ m/s}^2$ $\quad \bar{\mathbf{a}} = 3.20\text{ m/s}^2 \longrightarrow$ ◀

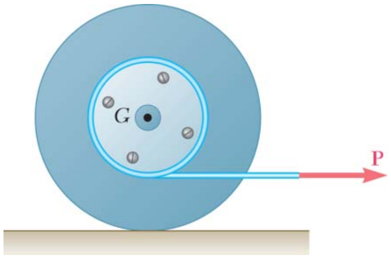
(*b*) $+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N - mg = 0$

$N = (6\text{ kg})(9.81\text{ m/s}^2)\uparrow \qquad \mathbf{N} = 58.86\text{ N}\uparrow$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad 20\text{ N} - F = m\bar{a}$$

$20\text{ N} - F = (6\text{ kg})(3.2\text{ m/s}^2) \qquad \mathbf{F} = 0.8\text{ N} \longleftarrow$

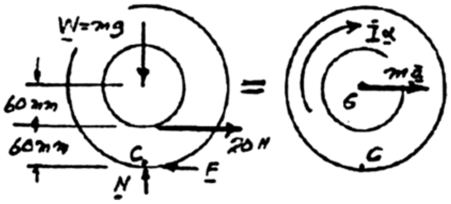
$(\mu_s)_{\min} = \dfrac{F}{N} = \dfrac{0.8\text{ N}}{58.86\text{ N}}$ $\quad (\mu_s)_{\min} = 0.0136$ ◀



PROBLEM 16.100

A drum of 60-mm radius is attached to a disk of 120-mm radius. The disk and drum have a total mass of 6 kg and a combined radius of gyration of 90 mm. A cord is attached as shown and pulled with a force **P** of magnitude 20 N. Knowing that the disk rolls without sliding, determine (*a*) the angular acceleration of the disk and the acceleration of *G*, (*b*) the minimum value of the coefficient of static friction compatible with this motion.

SOLUTION



$$\bar{a} = r\alpha = (0.12\text{ m})\alpha$$

$$\bar{I} = m\bar{k}^2$$

$$= (6\text{ kg})(0.09\text{ m})^2$$

$$\bar{I} = 48.6\text{ kg}\cdot\text{m}^2$$

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad (20\text{ N})(0.06\text{ m}) = (m\bar{a})r + \bar{I}\alpha$$

$$1.2\text{ N}\cdot\text{m} = (6\text{ kg})(0.12\text{ m})^2\alpha + 48.6\times10^{-3}\text{ kg}\cdot\text{m}^2$$

$$1.2 = 135\times10^{-3}\alpha$$

(*a*) $\alpha = 8.889\text{ rad/s}^2$ $\qquad$ $\boldsymbol{\alpha} = 8.89\text{ rad/s}^2 \circlearrowright$ ◀

$$\bar{a} = r\alpha = (0.12\text{ m})(8.889\text{ rad/s}^2)$$

$= 1.0667\text{ m/s}^2$ $\qquad$ $\bar{\mathbf{a}} = 1.067\text{ m/s}^2 \longrightarrow$ ◀

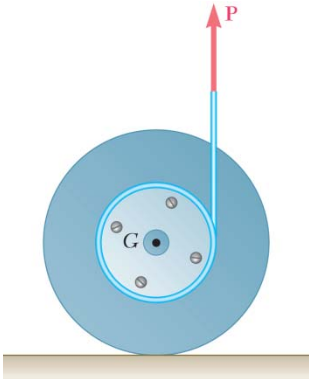
(*b*) $+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N - mg = 0$

$N = (6\text{ kg})(9.81\text{ m/s}^2)$ $\qquad$ $\mathbf{N} = 58.86\text{ N}\uparrow$

$\xrightarrow{+} \Sigma F_k = \Sigma(F_x)_{\text{eff}}: \quad 20\text{ N} - F = m\bar{a}$

$(20\text{ N}) - F = (6\text{ kg})(1.0667\text{ m/s}^2)$ $\qquad$ $\mathbf{F} = 13.6\text{ N} \longleftarrow$

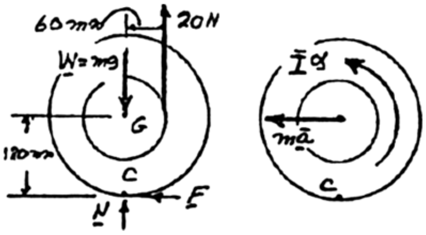
$(\mu_s)_{\min} = \dfrac{F}{N} = \dfrac{13.6\text{ N}}{58.86\text{ N}}$ $\qquad$ $(\mu_s)_{\min} = 0.231$ ◀



PROBLEM 16.101

A drum of 60-mm radius is attached to a disk of 120-mm radius. The disk and drum have a total mass of 6 kg and a combined radius of gyration of 90 mm. A cord is attached as shown and pulled with a force **P** of magnitude 20 N. Knowing that the disk rolls without sliding, determine (*a*) the angular acceleration of the disk and the acceleration of *G*, (*b*) the minimum value of the coefficient of static friction compatible with this motion.

SOLUTION



$$\bar{a} = r\alpha = (0.12\text{ m})\alpha$$

$$\bar{I} = m\bar{k}^2$$

$$= (6\text{ kg})(0.09\text{ m})^2$$

$$\bar{I} = 48.6\text{ kg}\cdot\text{m}^2$$

$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad (20\text{ N})(0.06\text{ m}) = (m\bar{a})r + \bar{I}\alpha$$

$$1.2\text{ N}\cdot\text{m} = (6\text{ kg})(0.12\text{ m})^2\alpha + (48.6\times 10^{-3}\text{ kg}\cdot\text{m}^2)\alpha$$

$$1.2 = 135\times 10^{-3}\alpha$$

(*a*) $\alpha = 8.889\text{ rad/s}^2$ $\quad\boldsymbol{\alpha} = 8.89\text{ rad/s}^2 \circlearrowleft$ ◀

$$\bar{a} = r\alpha = (0.12\text{ m})(8.889\text{ rad/s}^2)$$

$$= 1.0667\text{ m/s}^2$$

$\bar{\mathbf{a}} = 1.067\text{ m/s}^2 \longleftarrow$ ◀

(*b*) $+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N + 20\text{ N} - mg = 0$

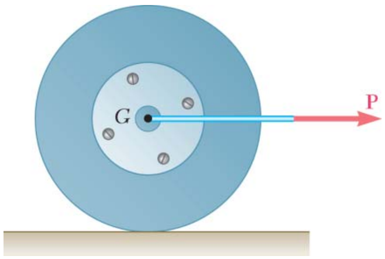
$$N + 20\text{ N} - (6\text{ kg})(9.81\text{ m/s}^2) \qquad \mathbf{N} = 38.86\text{ N}\uparrow$$

$$\overset{+}{\longleftarrow} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad F = m\bar{a}$$

$$F = (6\text{ kg})(1.0667\text{ m/s}^2) \qquad \mathbf{F} = 6.4\text{ N}\longleftarrow$$

$$(\mu_s)_{\min} = \frac{F}{N} = \frac{6.4\text{ N}}{38.86\text{ N}}$$

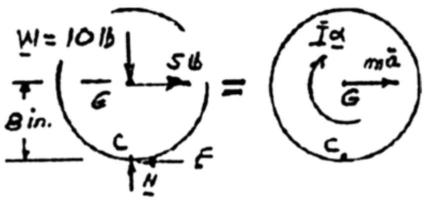
$(\mu_s)_{\min} = 0.165$ ◀



PROBLEM 16.102

A drum of 4-in. radius is attached to a disk of 8-in. radius. The disk and drum have a combined weight of 10 lb and a combined radius of gyration of 6 in. A cord is attached as shown and pulled with a force **P** of magnitude 5 lb. Knowing that the coefficients of static and kinetic friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, respectively, determine (*a*) whether or not the disk slides, (*b*) the angular acceleration of the disk and the acceleration of *G*.

SOLUTION



Assume disk rolls:

$$\bar{a} = r\alpha = \left(\frac{8}{12}\text{ ft}\right)\alpha$$

$$\bar{I} = m\bar{k}^2 = \frac{10\text{ lb}}{32.2}\left(\frac{6}{12}\text{ ft}\right)^2$$

$$\bar{I} = 0.07764\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}:\quad (5\text{ lb})\left(\frac{8}{12}\text{ ft}\right) = (m\bar{a})r + \bar{I}\alpha$$

$$3.333\text{ lb}\cdot\text{ft} = \frac{10\text{ lb}}{32.2}\left(\frac{8}{12}\text{ ft}\right)^2\alpha + 0.07764\alpha$$

$$3.333 = 0.21566\alpha$$

$$\alpha = 15.456\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 15.46\text{ rad/s}^2 \circlearrowright$$

$$\bar{a} = r\alpha = \left(\frac{8}{12}\text{ ft}\right)(15.456\text{ rad/s}^2) \qquad \bar{\mathbf{a}} = 10.30\text{ ft/s}^2 \longrightarrow$$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}:\quad -F + 5\text{ lb} = m\bar{a}$$

$$-F + 5\text{ lb} = \frac{10\text{ lb}}{32.2}(10.30\text{ ft/s}^2)$$

$$F = 1.80\text{ lb}$$

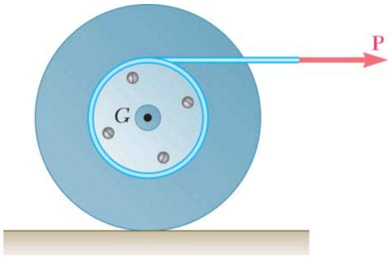
$$+\!\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}:\quad N - 10\text{ lb} = 0 \qquad N = 10\text{ lb}$$

$$F_m = \mu_s N = 0.25(10\text{ lb}) = 2.5\text{ lb}$$

(*a*) Since $F < F_m$, Disk rolls without sliding ◀

(*b*) Angular acceleration of the disk. $\boldsymbol{\alpha} = 15.46\text{ rad/s}^2 \circlearrowright$ ◀

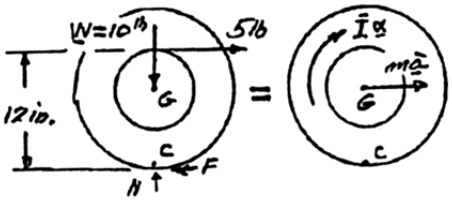
Acceleration of *G*. $\bar{\mathbf{a}} = 10.30\text{ ft/s}^2 \longrightarrow$ ◀



PROBLEM 16.103

A drum of 4-in. radius is attached to a disk of 8-in. radius. The disk and drum have a combined weight of 10 lb and a combined radius of gyration of 6 in. A cord is attached as shown and pulled with a force **P** of magnitude 5 lb. Knowing that the coefficients of static and kinetic friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, respectively, determine (*a*) whether or not the disk slides, (*b*) the angular acceleration of the disk and the acceleration of *G*.

SOLUTION



Assume disk rolls: $\bar{a} = r\alpha = \left(\frac{8}{12}\text{ ft}\right)\alpha$

$$\bar{I} = m\bar{k}^2$$

$$= \frac{10\text{ lb}}{32.2}\left(\frac{6}{12}\text{ ft}\right)^2$$

$$\bar{I} = 0.07764\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}$: $(5\text{ lb})(1\text{ ft}) = (m\bar{a})r + \bar{I}\alpha$

$$5\text{ lb}\cdot\text{ft} = \frac{10\text{ lb}}{32.2}\left(\frac{8}{12}\text{ ft}\right)^2\alpha + 0.07764\alpha$$

$$5 = 0.21566\alpha$$

$$\alpha = 23.184\text{ rad/s}^2$$ $\boldsymbol{\alpha} = 23.2\text{ rad/s}^2 \circlearrowright$

$$\bar{a} = r\alpha = \left(\frac{8}{12}\text{ ft}\right)(23.184\text{ rad/s}^2)$$ $\bar{\mathbf{a}} = 15.46\text{ ft/s}^2 \longrightarrow$

$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $-F + 5\text{ lb} = m\bar{a}$

$$-F + 5\text{ lb} = \frac{10\text{ lb}}{32.2}(15.46\text{ ft/s}^2); \quad F = 0.20\text{ lb}$$

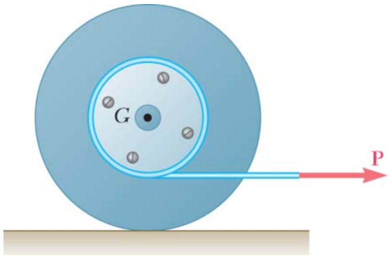
$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $N - 10\text{ lb} = 0$ $\qquad N = 10\text{ lb}$

$$F_m = \mu_s N = 0.25(10\text{ lb}) = 2.5\text{ lb}$$

(*a*) Since $F < F_m$, Disk rolls without sliding ◀

(*b*) Angular acceleration of the disk. $\boldsymbol{\alpha} = 23.2\text{ rad/s}^2 \circlearrowright$ ◀

Acceleration of *G*. $\bar{\mathbf{a}} = 15.46\text{ ft/s}^2 \longrightarrow$ ◀

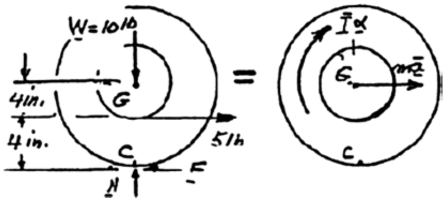


PROBLEM 16.104

A drum of 4-in. radius is attached to a disk of 8-in. radius. The disk and drum have a combined weight of 10 lb and a combined radius of gyration of 6 in. A cord is attached as shown and pulled with a force **P** of magnitude 5 lb. Knowing that the coefficients of static and kinetic friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, respectively, determine (*a*) whether or not the disk slides, (*b*) the angular acceleration of the disk and the acceleration of *G*.

SOLUTION

Assume disk rolls: $\bar{a} = r\alpha = \left(\frac{8}{12}\text{ ft}\right)\alpha$



$$\bar{I} = m\bar{k}^2 = \frac{10\text{ lb}}{32.2}\left(\frac{6}{12}\text{ ft}\right)^2$$

$$\bar{I} = 0.07764\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad (5\text{ lb})\left(\frac{4}{12}\text{ ft}\right) = (m\bar{a})r + \bar{I}\alpha$$

$$1.6667\text{ lb}\cdot\text{ft} = \frac{10\text{ lb}}{32.2}\left(\frac{8}{12}\text{ ft}\right)^2\alpha + 0.07764\alpha$$

$$1.6667 = 0.21566\alpha$$

$$\alpha = 7.728\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 7.73\text{ rad/s}^2 \circlearrowright$$

$$\bar{a} = r\alpha = \left(\frac{8}{12}\text{ ft}\right)7.728\text{ rad/s}^2 \qquad \bar{\mathbf{a}} = 5.153\text{ ft/s}^2 \longrightarrow$$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad -F + 5\text{ lb} = m\bar{a}$$

$$-F + 5\text{ lb} = \frac{10\text{ lb}}{32.2}(5.153\text{ ft/s}^2)$$

$$F = 3.40\text{ lb}$$

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N - 10\text{ lb} = 0 \qquad N = 10\text{ lb}$$

$$F_m = \mu_s N = 0.25(10\text{ lb}) = 2.5\text{ lb}$$

(*a*) Since $F < F_m$, Disk slides ◀

Knowing that disk slides

$$F = \mu_k N = 0.20(10\text{ lb}) = 2\text{ lb}$$

PROBLEM 16.104 (Continued)

(*b*) Angular acceleration.

$$\overset{+}{\circlearrowright}\Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad F\left(\frac{8}{12}\text{ ft}\right) - (5\text{ lb})\left(\frac{4}{12}\text{ ft}\right) = \bar{I}\alpha$$

$$(2\text{ lb})\left(\frac{8}{12}\text{ ft}\right) - 1.6667\text{ lb}\cdot\text{ft} = (0.07764\text{ lb}\cdot\text{ft}\cdot\text{s}^2)\alpha$$

$$-0.3333 = 0.07764\alpha$$

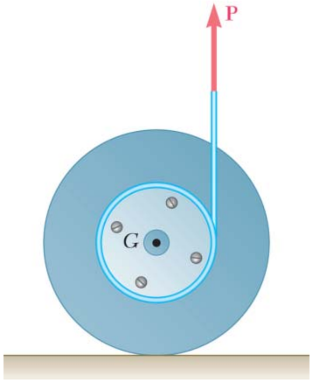
$$\alpha = -4.29\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 4.29\text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$

(*c*) Acceleration of *G*.

$$\xrightarrow{+}\Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad -F + 5\text{ lb} = m\bar{a}$$

$$-2\text{ lb} + 5\text{ lb} = \frac{10\text{ lb}}{32.2}\bar{a}$$

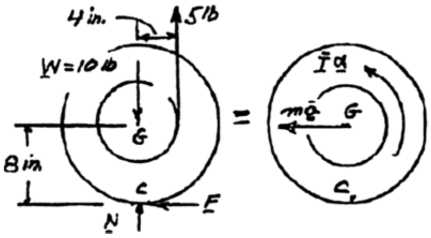
$$\bar{a} = 9.66\text{ ft/s}^2 \qquad \bar{\mathbf{a}} = 9.66\text{ ft/s}^2 \longrightarrow \blacktriangleleft$$



PROBLEM 16.105

A drum of 4-in. radius is attached to a disk of 8-in. radius. The disk and drum have a combined weight of 10 lb and a combined radius of gyration of 6 in. A cord is attached as shown and pulled with a force **P** of magnitude 5 lb. Knowing that the coefficients of static and kinetic friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, respectively, determine (*a*) whether or not the disk slides, (*b*) the angular acceleration of the disk and the acceleration of *G*.

SOLUTION



Assume disk rolls:
$$\bar{a} = r\alpha = \left(\frac{8}{12}\text{ ft}\right)\alpha$$

$$\bar{I} = m\bar{k}^2 = \frac{10\text{ lb}}{32.2}\left(\frac{6}{12}\text{ ft}\right)^2$$

$$\bar{I} = 0.07764\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \qquad (5\text{ lb})\left(\frac{4}{12}\text{ ft}\right) = (m\bar{a})r + \bar{I}\alpha$$

$$1.6667\text{ lb}\cdot\text{ft} = \frac{10\text{ lb}}{32.2}\left(\frac{8}{12}\text{ ft}\right)^2\alpha + 0.07764\alpha$$

$$1.6667 = 0.21566\alpha$$

$$\alpha = 7.728\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 7.73\text{ rad/s}^2 \circlearrowleft$$

$$\bar{a} = r\alpha = \left(\frac{8}{12}\text{ ft}\right)(7.728\text{ rad/s}^2) \qquad \bar{\mathbf{a}} = 5.153\text{ ft/s}^2 \longleftarrow$$

$$\overset{+}{\longleftarrow} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad F = m\bar{a}$$

$$F = \frac{10\text{ lb}}{32.2}(5.153\text{ ft/s}^2); \quad F = 1.60\text{ lb}$$

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad N - 10\text{ lb} + 5\text{ lb} = 0 \qquad N = 5\text{ lb}$$

$$F_m = \mu_s N = 0.25(5\text{ lb}) = 1.25\text{ lb}$$

(*a*) Since $F > F_m$, Disk slides ◀

Knowing that disk slides

$$F = \mu_k N = 0.2(5)$$
$$F = 1.00\text{ lb}$$

PROBLEM 16.105 (Continued)

(*b*) Angular acceleration.

$$+\circlearrowleft \Sigma M_S = \Sigma(M_S)_{\text{eff}}: \quad (5\text{ lb})\left(\frac{4}{12}\text{ ft}\right) - F\left(\frac{8}{12}\text{ ft}\right) = \bar{I}\alpha$$

$$(5\text{ lb})\left(\frac{4}{12}\text{ ft}\right) - (1.00\text{ lb})\left(\frac{8}{12}\text{ ft}\right) = 0.07764\alpha$$

$$1.000 = 0.07764\alpha$$

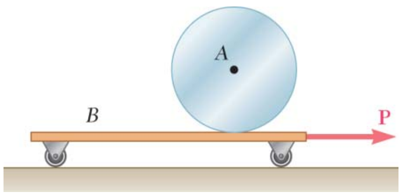
$$\alpha = 12.88\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 12.88\text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$

(*c*) Acceleration of *G*.

$$\xleftarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad F = m\bar{a}$$

$$1.00\text{ lb} = \frac{10\text{ lb}}{32.2}\bar{a}$$

$$\bar{a} = 3.22\text{ ft/s}^2 \qquad \bar{\mathbf{a}} = 3.22\text{ ft/s}^2 \longleftarrow \blacktriangleleft$$



PROBLEM 16.106

A 12-in.-radius cylinder of weight 16 lb rests on a 6-lb carriage. The system is at rest when a force **P** of magnitude 4 lb is applied. Knowing that the cylinder rolls without sliding on the carriage and neglecting the mass of the wheels of the carriage, determine (*a*) the acceleration of the carriage, (*b*) the acceleration of Point *A*, (*c*) the distance the cylinder has rolled with respect to the carriage after 0.5 s.

SOLUTION

Masses and moments of inertia.

$$m_A = \frac{16\text{ lb}}{32.2\text{ ft/s}^2} = 0.49689\text{ lb}\cdot\text{s}^2/\text{ft} \qquad m_B = \frac{6\text{ lb}}{32.2\text{ ft/s}^2} = 0.18634\text{ lb}\cdot\text{s}^2/\text{ft}$$

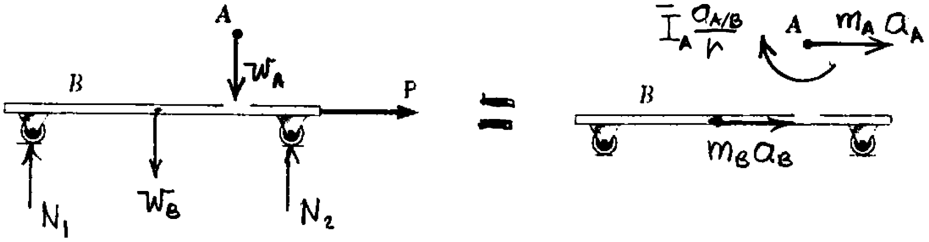
$$\bar{I}_A = \frac{1}{2}m_A r^2 = \frac{1}{2}(0.49689\text{ lb}\cdot\text{s}^2/\text{ft})(1\text{ ft})^2 = 0.24895\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Kinematics: Let $\mathbf{a}_A = a_A \longrightarrow,\ \mathbf{a}_B = a_B \longrightarrow$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{B/A}$$

$$\mathbf{a}_{A/B} = (a_A - a_B) \longrightarrow \qquad \boldsymbol{\alpha} = \frac{a_{A/B}}{r} \circlearrowright \tag{1}$$

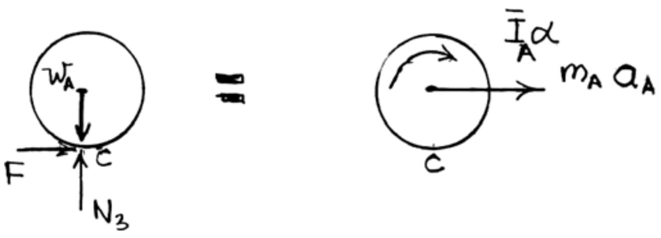
Kinetics: Carriage and cylinder



$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad P = m_A a_A + m_B a_B$$

$$4\text{ lb} = (0.49689\text{ lb}\cdot\text{s}^2/\text{ft})\ a_A + (0.18634\text{ lb}\cdot\text{s}^2/\text{ft})\ a_B \tag{2}$$

Cylinder alone. Point *C* is contact point.



PROBLEM 16.106 (Continued)

$$\circlearrowleft + \Sigma M_C = \Sigma (M_C)_{\text{eff}}: \quad 0 = \bar{I}_A \alpha + m_A a_A r$$

Substituting from Eqs. (1) and (2),

$$0 = \bar{I}_A \frac{a_A - a_B}{r} + m_A a_A r$$

$$0 = \left(\frac{\bar{I}_A}{r} + m_A r \right) a_A - \frac{\bar{I}_A}{r} a_B$$

Data: $$\frac{I_A}{r} = \frac{0.24895 \text{ lb} \cdot \text{s}^2 \cdot \text{ft}}{1 \text{ ft}} = 0.24895 \text{ lb} \cdot \text{s}^2$$

$$m_A r = (0.49689 \text{ lb} \cdot \text{s}^2/\text{ft})(1 \text{ ft}) = 0.49689 \text{ lb} \cdot \text{s}^2$$

$$0 = 0.74584 \, a_A - 0.24895 \, a_B \tag{3}$$

Solving Eqs. (2) and (3) simultaneously,

$$a_A = 3.7909 \text{ ft/s}^2 \quad a_B = 11.3574 \text{ ft/s}^2$$

(*a*) *Acceleration of the carriage.* $\mathbf{a}_B = 11.36 \text{ ft/s}^2 \longrightarrow$ ◀

(*b*) *Acceleration of Point A.* $\mathbf{a}_A = 3.79 \text{ ft/s}^2 \longrightarrow$ ◀

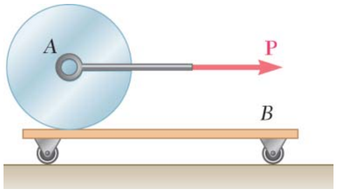
(*c*) *Relative displacement after* 0.5 s.

$$a_{A/B} = 11.3574 \text{ ft/s}^2 - 3.7909 \text{ ft/s}^2 = 7.5665 \text{ ft/s}^2$$

$$x_{A/B} = \frac{1}{2}(a_{A/B}) t^2$$

$$= \frac{1}{2}(7.5665 \text{ ft/s})^2 (0.5 \text{ s})^2$$

$\mathbf{x}_{B/A} = 0.946 \text{ ft} \longrightarrow$ ◀



PROBLEM 16.107

A 12-in.-radius cylinder of weight 16 lb rests on a 6-lb carriage. The system is at rest when a force **P** of magnitude 4 lb is applied. Knowing that the cylinder rolls without sliding on the carriage and neglecting the mass of the wheels of the carriage, determine (*a*) the acceleration of the carriage, (*b*) the acceleration of Point *A*, (*c*) the distance the cylinder has rolled with respect to the carriage after 0.5 s.

SOLUTION

Masses and moments of inertia.

$$m_A = \frac{16 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.49689 \text{ lb}\cdot\text{s}^2/\text{ft} \qquad m_B = \frac{6 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.18634 \text{ lb}\cdot\text{s}^2/\text{ft}$$

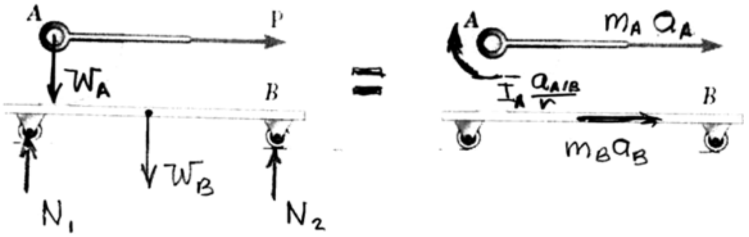
$$\bar{I}_A = \frac{1}{2}m_A r^2 = \frac{1}{2}(0.49689 \text{ lb}\cdot\text{s}^2/\text{ft})(1 \text{ ft})^2 = 0.24895 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Kinematics: Let $\mathbf{a}_A = a_A \longrightarrow$, $\mathbf{a}_B = a_B \longrightarrow$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{B/A}$$

$$\mathbf{a}_{A/B} = (a_B - a_B) \longrightarrow \qquad \alpha = \frac{a_{A/B}}{r} \circlearrowright \tag{1}$$

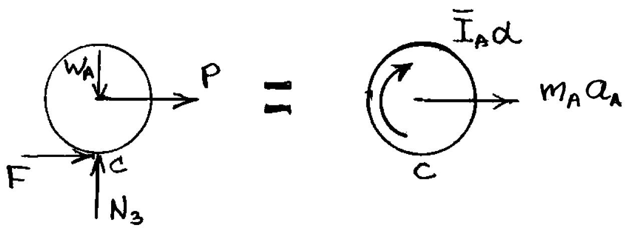
Kinetics: Carriage and cylinder



$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad P = m_A a_A + m_B a_B$$

$$4 \text{ lb} = (0.49689 \text{ lb}\cdot\text{s}^2/\text{ft})\, a_A + (0.18634 \text{ lb}\cdot\text{s}^2/\text{ft}) a_B \tag{2}$$

Cylinder alone. Point *C* is contact point.



PROBLEM 16.107 (Continued)

$$\circlearrowleft + \Sigma M_C = \Sigma(M_B)_{\text{eff}}: \quad Pr = \bar{I}_A \alpha + m_A a_A r$$

Substituting from Eqs. (1) and (2),

$$Pr = \bar{I}_A \frac{a_A - a_B}{r} + m_A a_A r$$

$$Pr = \left(\frac{\bar{I}_A}{r} + m_A r\right) a_A - \frac{\bar{I}_A}{r} a_B$$

Data: $$\frac{\bar{I}_A}{r} = \frac{0.24895 \text{ lb} \cdot \text{s}^2 \cdot \text{ft}}{1 \text{ ft}} = 0.24895 \text{ lb} \cdot \text{s}^2$$

$$m_A r = (0.49689 \text{ lb} \cdot \text{s}^2/\text{ft})(1 \text{ ft}) = 0.49689 \text{ lb} \cdot \text{s}^2$$

$$(4 \text{ lb})(1 \text{ ft}) = 0.74584\, a_A - 0.24895\, a_B \qquad (3)$$

Solving Eqs. (2) and (3) simultaneously,

$$a_A = 6.6284 \text{ ft/s}^2 \qquad a_B = 3.7909 \text{ ft/s}^2$$

(*a*) *Acceleration of the carriage.* $\mathbf{a}_B = 3.79 \text{ ft/s}^2 \longrightarrow$ ◀

(*b*) *Acceleration of Point A.* $\mathbf{a}_A = 6.63 \text{ ft/s}^2 \longrightarrow$ ◀

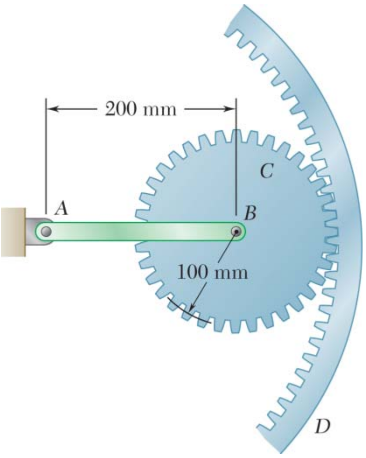
(*c*) *Relative displacement after* 0.5 s.

$$a_{A/B} = 6.6284 \text{ ft/s}^2 - 3.7909 \text{ ft/s}^2 = 2.8375 \text{ ft/s}^2$$

$$x_{A/B} = \frac{1}{2}(a_{A/B})t^2$$

$$= \frac{1}{2}(2.8375 \text{ ft/s}^2)(0.5 \text{ s})^2$$

$\mathbf{x}_{A/B} = 0.355 \text{ ft} \longrightarrow$ ◀

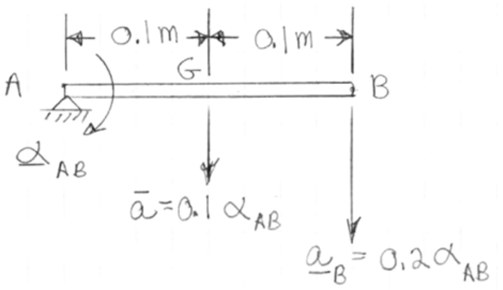

200 mm
C
A
B
100 mm
D

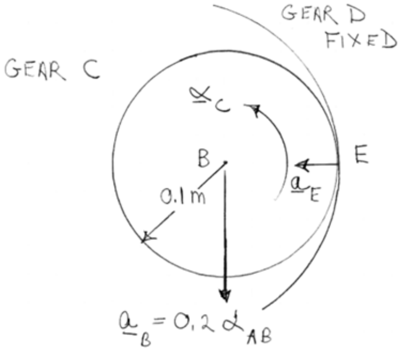
PROBLEM 16.108

Gear *C* has a mass of 5 kg and a centroidal radius of gyration of 75 mm. The uniform bar *AB* has a mass of 3 kg and gear *D* is stationary. If the system is released from rest in the position shown, determine (*a*) the angular acceleration of gear *C*, (*b*) the acceleration of Point *B*.

SOLUTION

Kinematics:


0.1 m
0.1 m
A
G
B
α_{AB}
$\bar{a} = 0.1\,\alpha_{AB}$
$a_B = 0.2\,\alpha_{AB}$


GEAR D FIXED
GEAR C
α_C
B
E
0.1 m
a_E
$a_B = 0.2\,\alpha_{AB}$

Since gear *D* is fixed, we have for Point *E* of gear *C*: $\downarrow (a_E)_t = 0$

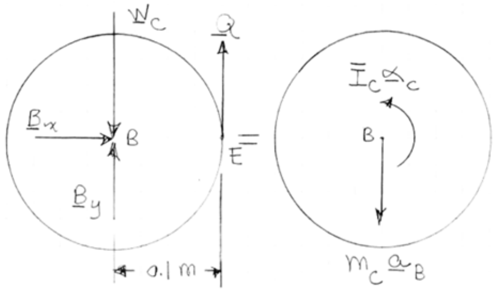
But $\mathbf{a}_E = \mathbf{a}_B + \mathbf{a}_{E/B}$

$+(a_E)_t = (a_B)_t + (\mathbf{a}_{E/B})_t$

$0 = 0.2\,\alpha_{AB} - 0.1\,\alpha_C$

$$\alpha_{AB} = \frac{1}{2}\alpha_C \qquad (1)$$

Gear C


W_C
Q
B_x
B
B_y
E
=
0.1 m
$\bar{I}_C\alpha_C$
B
$m_C a_B$

$+\circlearrowright \Sigma M_B = \Sigma(M_B)_{\text{eff}}:$

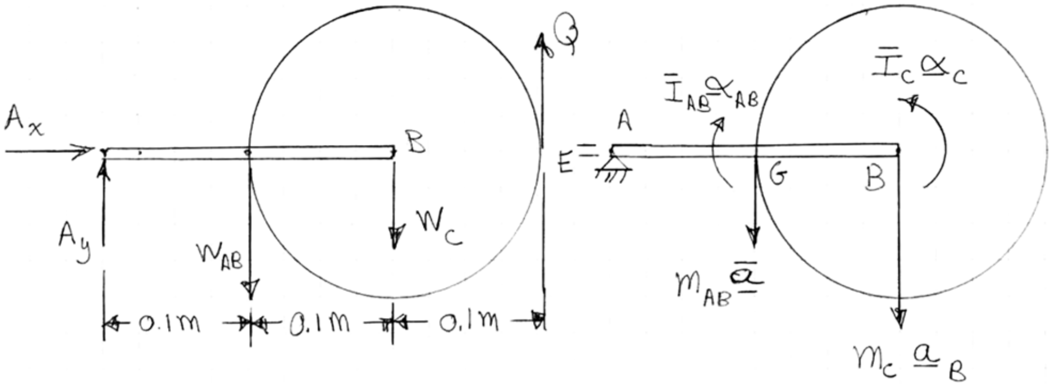
$Q\,(0.1\,\text{m}) = \bar{I}_C\alpha_C$

$= (5\text{ kg})(0.075\text{ m})^2\,\alpha_C$

$$Q = 0.28125\,\alpha_C \qquad (2)$$

PROBLEM 16.108 (Continued)

Bar AB and gear C



Q
A_x
B
W_C
A_y
W_{AB}
0.1 m
0.1 m
0.1 m
E =
$\bar{I}_{AB}\alpha_{AB}$
$\bar{I}_C\alpha_C$
A
G
B
$m_{AB}\bar{a}$
$m_C\, a_B$

(*a*) $+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}:$

$$W_{AB}(0.1) + W_C(0.2) - Q(0.3) = (m_{AB}\,\bar{a})\,0.1 + \bar{I}_{AB}\alpha_{AB} + (m_C\, a_B)0.2 - \bar{I}_C\alpha_C$$

$$(3)g(0.1) + (5)g(0.2) - Q(0.3) = 3(0.1\alpha_{AB})0.1 + \frac{1}{12}(3)(0.2)^2\alpha_{AB}$$

$$+\,5(0.2\,\alpha_{AB})0.2 - 5(0.075)^2\alpha_C$$

$$(1.3)g - 0.3Q = 0.24\alpha_{AB} - 0.028125\alpha_C$$

Substituting for α_{AB} and Q from (2) and (1):

$$1.3g - 0.3(0.28125\alpha_C) = 0.24\left(\frac{1}{2}\alpha_C\right) - 0.028125\,\alpha_C$$

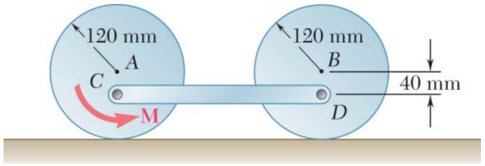
$1.3g = 0.17625\,\alpha_C$ $\qquad \alpha_C = 7.3759(9.81)$

$\alpha_C = 72.36$

$\boldsymbol{\alpha}_C = 72.4 \text{ rad/s}^2 \circlearrowleft$ ◀

(*b*) $a_B = 0.2\alpha_{AB} = 0.2\left(\frac{1}{2}\alpha_C\right) = 0.1\alpha_C = 0.1(72.36)$

$\mathbf{a}_B = 7.24 \text{ m/s}^2\downarrow$ ◀



PROBLEM 16.109

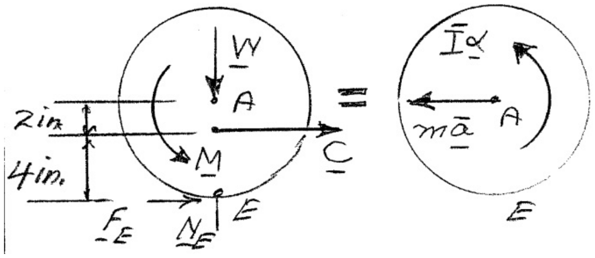
Two uniform disks *A* and *B*, each of mass 2 kg, are connected by a 2.5-kg rod *CD* as shown. A counterclockwise couple **M** of moment 2.25 N·m is applied to disk *A*. Knowing that the disks roll without sliding, determine (*a*) the acceleration of the center of each disk, (*b*) the horizontal component of the force exerted on disk *B* by pin *D*.

SOLUTION

Acting on Disk B

$$\boldsymbol{\alpha}_A = \boldsymbol{\alpha}_B = \boldsymbol{\alpha} \qquad \bar{\mathbf{a}}_A = \bar{\mathbf{a}}_B = \bar{\mathbf{a}} = r\alpha$$

Disk A:



$W = 4$ lb

$\bar{I} = \frac{1}{2}mr^2$

$$+\circlearrowleft \Sigma M_E = \Sigma(M_E)_{\text{eff}}: \quad M - C(0.08\text{ m}) = (m\bar{a})r + \bar{I}\alpha$$

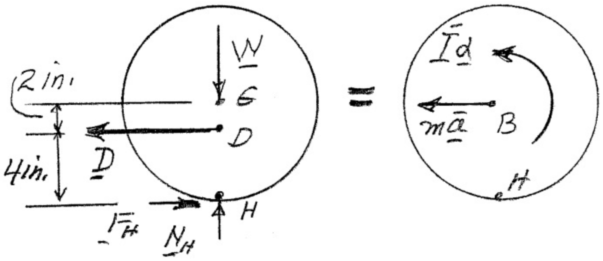
$$2.25\text{ N}\cdot\text{m} - (0.08\text{ m})C = (mr\alpha)r + \frac{1}{2}mr^2\alpha$$

$$= \frac{3}{2}mr^2\alpha$$

$$2.25\text{ N}\cdot\text{m} - (0.08\text{ m})C = \frac{3}{2}(2\text{ kg})(0.12\text{ m})^2\alpha$$

$$2.25\text{ N}\cdot\text{m} - (0.08\text{ m})C = (0.0432\text{ k}\cdot\text{m}^2)\alpha \qquad (1)$$

Disk B:



$\bar{I} = \frac{1}{2}mr^2$

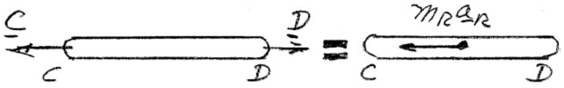
PROBLEM 16.109 (Continued)

$$\overset{+}{\circlearrowright}\ \Sigma M_H = \Sigma(M_H)_{\text{eff}}:\quad D(0.08\text{ m}) = (m\bar{a})r + \bar{I}\alpha = mr^2\alpha + \frac{1}{2}mr^2\alpha$$

$$(0.08\text{ m})D = \frac{3}{2}(2\text{ kg})(0.12\text{ m})^2\alpha$$

$$(0.08\text{ m})D = \left(0.0432\text{ kg}\cdot\text{m}^2\right)\alpha \qquad (2)$$

Rod CD:



$$\xrightarrow{+}\ \bar{\mathbf{a}}_R = \mathbf{a}_C = \mathbf{a}_A + \mathbf{a}_{C/A}$$

$$\bar{a}_R = (0.12\text{ m})\alpha - (0.04\text{ m})\alpha = (0.08\text{ m})\alpha$$

$$\xleftarrow{+}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}:$$

$$C - D = m_R\cdot\bar{a}_R = (2.5\text{ kg})(0.08\text{ m})\alpha = (0.20\text{ kg}\cdot\text{m})\alpha$$

Use (1) : (2)

$$\left[\left(-0.0432\text{ kg}\cdot\text{m}^2\right)\alpha + (2.25\text{ N}\cdot\text{m})\right]\left(\frac{1}{0.08\text{ m}}\right) - \left(\frac{0.0432\text{ kg}\cdot\text{m}^2}{0.08\text{ m}}\right)\alpha = 0.20\text{ kg}\cdot\text{m}\alpha$$

$$28.125\text{ N} = \left[(0.20 + 0.54 + 0.54)\text{kg}\cdot\text{m}\right]\alpha$$

$$\alpha = 21.9726\text{ rad/s}^2\ \circlearrowright$$

(*a*)

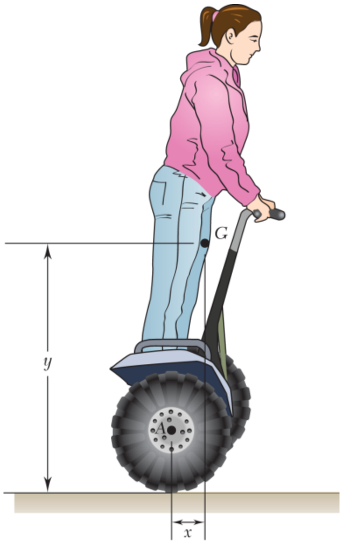
$$\bar{a}_A = \bar{a}_B = r\alpha = (0.12\text{ m})\left(21.9726\text{ rad/s}^2\right) = 2.6367\text{ m/s}^2$$

$$\bar{\mathbf{a}}_A = \bar{\mathbf{a}}_B = 2.64\text{ m/s}^2 \longleftarrow \blacktriangleleft$$

(*b*)

$$D = \left(\frac{0.0432\text{ kg}\cdot\text{m}^2}{0.08\text{ m}}\right)\left(21.9726\text{ rad/s}^2\right) = 11.8652\text{ N}$$

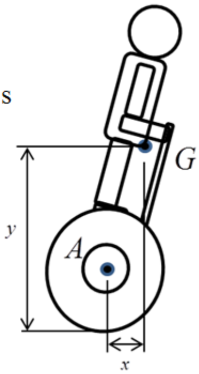
$$\text{or}\quad \boldsymbol{D} = 11.87\text{ N} \longleftarrow \blacktriangleleft$$



PROBLEM 16.110

A single axis personal transport device starts from rest with the rider leaning slightly forward. Together, the two wheels weigh 25 lbs, and each has a radius of 10 in. The mass moment of inertia of the wheels about the axle is 0.15 slug ft^2. The combined weight of the rest of the device and the rider (excluding the wheels) is 200 lbs, and the center of gravity G of this weight is located at $x = 4$ in. in front of the axle A and $y = 36$ in. above the ground. An initial clockwise torque **M** is applied by the motor to the wheels. Knowing that the coefficients of static and kinetic friction are 0.7 and 0.6 respectively, determine (*a*) the torque **M** that will keep the rider in the same angular position, (*b*) the corresponding linear acceleration of the rider.

SOLUTION



Given:

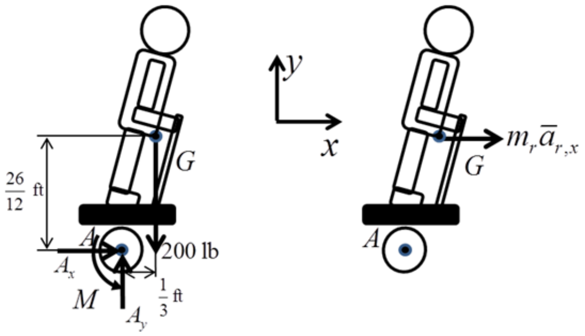
$$m_r = \frac{200 \text{ lb}}{32.2 \text{ ft/s}^2} = 6.211 \text{ slugs}, \; m_w = \frac{25 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.7764 \text{ slugs}$$

$$r_w = \frac{5}{6} \text{ ft, y} = 3 \text{ ft, x} = \frac{1}{3} \text{ ft}$$

$$\mu_s = 0.7, \; \mu_k = 0.6$$

$$\alpha = 0, \; \bar{I}_w = 0.15 \text{ slug}\bullet\text{ft}^2$$

Free Body Diagram of Rider and device: Kinetics:



$$\sum F_x = m_r\bar{a}_{r,x}$$

$$A_x = m_r\bar{a}_{r,x} \qquad (1)$$

$$\sum F_y = m_r \cancel{\bar{a}_{r,y}}$$

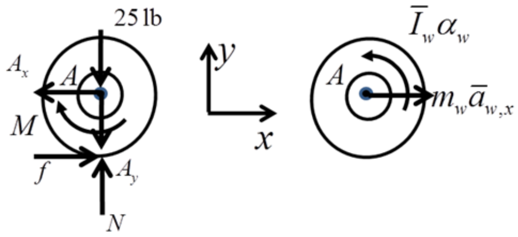
$$A_y - 200 = 0 \Rightarrow A_y = 200 \text{ lb} \qquad (2)$$

$$\sum M_G = \bar{I}\,\cancel{\alpha_r}$$

$$M + \frac{26}{12}A_x - \frac{1}{3}A_y = 0$$

$$A_x = \frac{12}{26}\left(\frac{200}{3} - M\right) \qquad (3)$$

Free Body Diagram of the Wheels: Kinetics:



$$\sum F_x = m_w\bar{a}_{w,x}$$

$$f - A_x = m_w\bar{a}_{w,x} \qquad (4)$$

$$\sum F_y = m_w \cancel{\bar{a}_{w,y}}$$

$$N - A_y - 25 = 0 \Rightarrow \text{N} = 225 \text{ lb} \qquad (5)$$

$$\sum M_A = \bar{I}_w\alpha_w$$

$$fr_w - M = \bar{I}_w\alpha_w \qquad (6)$$

Kinematics:

$$\bar{a}_{r,x} = \bar{a}_{w,x} \qquad (7)$$

PROBLEM 16.110 (Continued)

Assume no slip: $\bar{a}_{w,x} = -r_w\alpha_w \;\Rightarrow\; \alpha_w = -\dfrac{\bar{a}_{w,x}}{r_w}$ (8)

Substitute into (7),(8)→(6): $M = \bar{I}_w\dfrac{\bar{a}_{r,x}}{r_w} + fr_w$ (9)

Substitute into (3),(7)→(4): $f = \dfrac{12}{26}\left(\dfrac{200}{3} - M\right) + m_w\bar{a}_{r,x}$ (10)

Substitute into (3)→(1): $\bar{a}_{r,x} = \dfrac{12}{26m_r}\left(\dfrac{200}{3} - M\right)$ (11)

Substitute (11)-(10): $f = \left(\dfrac{12}{26} + \dfrac{12m_w}{26m_r}\right)\left(\dfrac{200}{3} - M\right)$ (12)

Substitute (11),(12)→(9):

$$M = \frac{\bar{I}_w}{r_w}\frac{12}{26m_R}\left(\frac{200}{3} - M\right) + \left(\frac{12}{26} + \frac{12m_w}{26m_R}\right)\left(\frac{200}{3} - M\right)r_w$$

$$= \frac{0.15}{0.833}\frac{12}{26(6.211)}\left(\frac{200}{3} - M\right) + \left(\frac{12}{26} + \frac{12(0.7764)}{26(6.211)}\right)\left(\frac{200}{3} - M\right)(0.833)$$

(a) Solve for M:

$$M = \frac{\bar{I}_w}{r_w}\frac{12}{26m_r}\left(\frac{200}{3} - M\right) + \left(\frac{12}{26} + \frac{12m_w}{26m_R}\right)\left(\frac{200}{3} - M\right)r_w$$

$$= \frac{0.15}{0.833}\frac{12}{26(6.211)}\left(\frac{200}{3} - M\right) + \left(\frac{12}{26} + \frac{12(0.7764)}{26(6.211)}\right)\left(\frac{200}{3} - M\right)(0.833)$$

$$1.4461M = 29.738$$

$$M = 20.565 \text{ ft-lb}$$

Check for slipping:

$$f = \left(\frac{12}{26} + \frac{12m_w}{26m_r}\right)\left(\frac{200}{3} - 20.5646\right)$$

$$= 23.94 \text{ lb}$$

$$\mu_s N = 0.7(225)$$

$$= 157.5 \text{ lb}$$

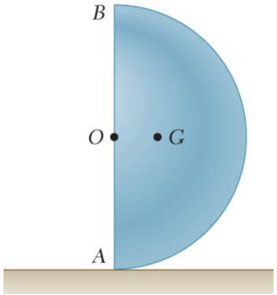
Since $f < \mu_s N$ the tires do not slip: **M** = 20.565 ft-lb ↻ ◀

(b) Using (11):

$$\bar{a}_{r,x} = \frac{12}{26(6.211)}\left(\frac{200}{3} - 20.565\right)$$

$$3.426 \text{ m/s}^2$$

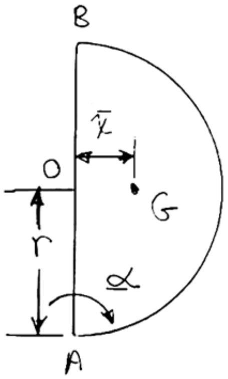
$\mathbf{a}_x = 3.426 \text{ m/s}^2$ → ◀



PROBLEM 16.111

A hemisphere of weight W and radius r is released from rest in the position shown. Determine (*a*) the minimum value of μ_s for which the hemisphere starts to roll without sliding, (*b*) the corresponding acceleration of Point B. [*Hint*: Note that $OG = \frac{3}{8}r$ and that, by the parallel-axis theorem, $\overline{I} = \frac{2}{5}mr^2 - m(OG)^2$.]

SOLUTION



Kinematics: $\omega = 0$

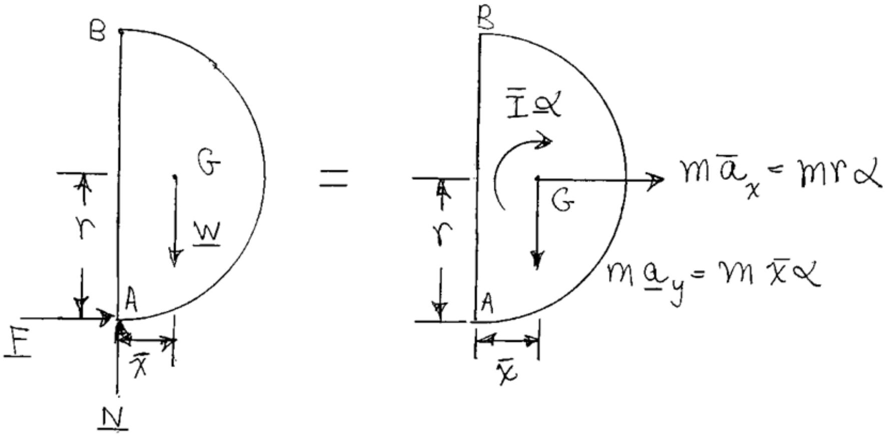
$$\mathbf{a}_O = \mathbf{a}_A + \mathbf{a}_{O/A} = 0 + [r\alpha \longrightarrow]$$

$$\overline{\mathbf{a}} = \mathbf{a}_G = \mathbf{a}_O + \mathbf{a}_{G/O}$$

$$= [r\alpha \longrightarrow] + [\bar{x}\alpha \downarrow]$$

Thus, $\overline{\mathbf{a}}_x = r\alpha \longrightarrow, \qquad \overline{\mathbf{a}}_y = \bar{x}\alpha \downarrow$ (1)

Kinetics:



$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \qquad W\bar{x} = (m\bar{a}_x)r + (m\bar{a}_y)\,\bar{x} + \bar{I}\alpha$$

$$mg\bar{x} = (mr\alpha)\,r + (m\bar{x}\alpha)\,\bar{x} + m\bar{k}^2\alpha$$

$$\alpha = \frac{g\bar{x}}{r^2 + \bar{x}^2 + \bar{k}^2} \tag{2}$$

PROBLEM 16.111 (Continued)

$$\xrightarrow{+}\Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad F = ma_x \qquad F = mr\alpha$$

$$+\!\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad N - W = -m\bar{a}_y \qquad N = mg - m\bar{x}\alpha$$

$$\mu_{\min} = \frac{F}{N} = \frac{mr\alpha}{mg - \bar{x}\alpha} \qquad \mu_{\min} = \frac{r\alpha}{g - \bar{x}\alpha} \tag{3}$$

For a hemisphere:

$$\bar{x} = OG = \frac{3}{8}r \qquad \bar{I} = I_O - m\bar{x}^2 = \frac{2}{5}mr^2 - m\left(\frac{3}{8}r\right)^2$$

$$\bar{I} = \frac{2}{5}mr^2 - \frac{9}{64}mr^2 \qquad \bar{k}^2 = \frac{\bar{I}}{m} = \left(\frac{2}{5} - \frac{9}{64}\right)r^2$$

Substituting into (2)

$$\alpha = \frac{g\left(\frac{3}{8}r\right)}{r^2 + \frac{9}{64}r^2 + \left(\frac{2}{5} - \frac{9}{64}\right)r^2} = \frac{\frac{3}{8}}{\frac{7}{5}}\frac{g}{r} \qquad \alpha = \frac{15}{56}\frac{g}{r}$$

(*a*) Substituting into (3)

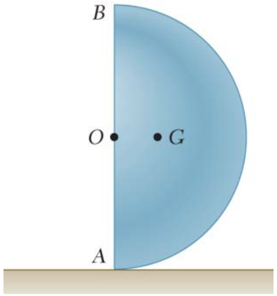
$$\mu_{\min} = \frac{\frac{15}{56}}{1 - \left(\frac{3}{8}\right)\left(\frac{15}{56}\right)} = \frac{0.26786}{0.89955}$$

$\mu_{\min} = 0.298$ ◀

(*b*) $a_B = (2r)\,\alpha = (2r)\left(\dfrac{15g}{56r}\right) = \dfrac{30g}{56}$

$\mathbf{a}_B = 0.536g \longrightarrow$ ◀

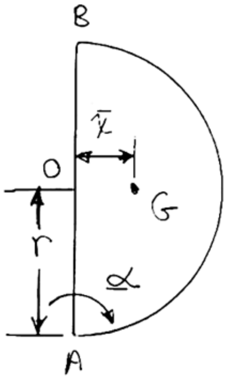
Note: In this problem we *cannot* use the equation $\Sigma M_A = I_A\alpha$, since Points *A*, *O*, and *G* are not aligned.



PROBLEM 16.112

Solve Problem 16.111, considering a half cylinder instead of a hemisphere. [*Hint*. Note that $OG = 4r/3\pi$ and that, by the parallel-axis theorem, $\bar{I} = \frac{1}{2}mr^2 - m(OG)^2$.]

SOLUTION


B
$\bar{x}$
O
G
r
α
A

Kinematics: $\omega = 0$

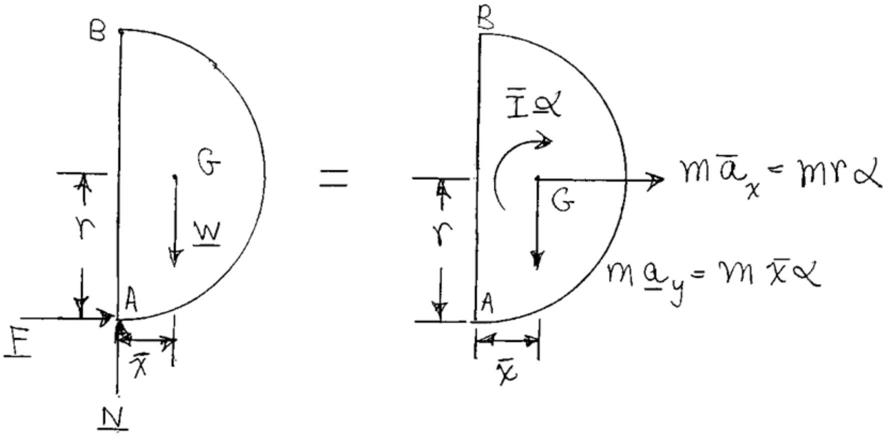
$$\mathbf{a}_O = \mathbf{a}_A + \mathbf{a}_{O/A} = 0 + [r\alpha \longrightarrow]$$

$$\bar{\mathbf{a}} = \mathbf{a}_G = \mathbf{a}_O + \mathbf{a}_{G/O}$$

$$= [r\alpha \longrightarrow] + [\bar{x}\alpha \downarrow]$$

Thus, $\mathbf{a}_x = r\alpha \longrightarrow, \quad \mathbf{a}_y = \bar{x}\alpha \downarrow$ (1)

Kinetics:


B
G
r
W
A
F
$\bar{x}$
N
=
B
$\bar{I}\alpha$
$m\bar{a}_x = mr\alpha$
G
r
$m\bar{a}_y = m\bar{x}\alpha$
A
$\bar{x}$

$$+\circlearrowright \Sigma M_A = \Sigma (M_A)_{\text{eff}}: \quad W\bar{x} = (m\bar{a}_x)r + (m\bar{a}_y)\,\bar{x} + \bar{I}\alpha$$

$$mg\bar{x} = (mr\alpha)\,r + (m\bar{x}\alpha)\,\bar{x} + m\bar{k}^2\alpha$$

$$\alpha = \frac{g\bar{x}}{r^2 + \bar{x}^2 + \bar{k}^2} \qquad (2)$$

PROBLEM 16.112 (Continued)

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \qquad F = ma_x \qquad F = mr\alpha$$

$$+\!\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \qquad N - W = -m\bar{a}_y \qquad N = mg - m\bar{x}\alpha$$

$$\mu_{\min} = \frac{F}{N} = \frac{mr\alpha}{mg - \bar{x}\alpha} \qquad \mu_{\min} = \frac{r\alpha}{g - \bar{x}\alpha} \qquad (3)$$

For a half cylinder:

$$\bar{x} = OG = \frac{4r}{3\pi} \qquad \bar{I} = I_O - m\bar{x}^2 = \frac{1}{2} mr^2 - m\left(\frac{4r}{3\pi}\right)^2$$

$$\bar{k}^2 = \frac{\bar{I}}{m} = \frac{1}{2} r^2 - \left(\frac{4r}{3\pi}\right)^2$$

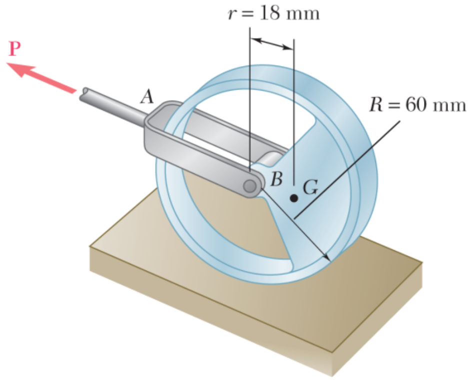
Substituting into (2):

$$\alpha = \frac{g\left(\frac{4r}{3\pi}\right)}{r^2 + \left(\frac{4r}{3\pi}\right)^2 + \frac{1}{2} r^2 - \left(\frac{4r}{3\pi}\right)^2} = \frac{\frac{4}{3\pi}}{\frac{3}{2}} \frac{g}{r} \qquad \alpha = \frac{8}{9\pi} \frac{g}{r}$$

(*a*) Substituting into (3):

$$\mu_{\min} = \frac{\frac{8}{9\pi}}{1 - \left(\frac{4}{3\pi}\right)\left(\frac{8}{9\pi}\right)} = \frac{0.28294}{0.87992} \qquad \mu_{\min} = 0.322 \blacktriangleleft$$

(*b*) $a_B = (2r)\,\alpha = (2r)\left(\frac{8g}{9\pi r}\right) = \frac{16g}{9\pi}$ $\qquad a_B = 0.566g \blacktriangleleft$

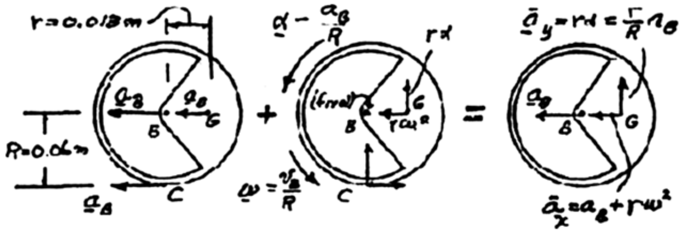

$r = 18$ mm
P
A
$R = 60$ mm
B
G

PROBLEM 16.113

The center of gravity G of a 1.5-kg unbalanced tracking wheel is located at a distance $r = 18$ mm from its geometric center B. The radius of the wheel is $R = 60$ mm and its centroidal radius of gyration is 44 mm. At the instant shown the center B of the wheel has a velocity of 0.35 m/s and an acceleration of 1.2 m/s^2, both directed to the left. Knowing that the wheel rolls without sliding and neglecting the mass of the driving yoke AB, determine the horizontal force **P** applied to the yoke.

SOLUTION

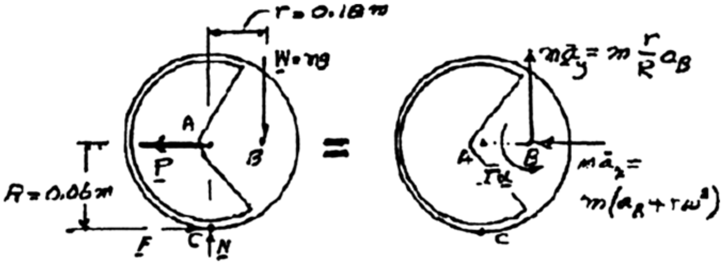
Kinematics: Choose positive $\mathbf{v}_B$ and $\mathbf{a}_B$ to left.


r=0.018m
R=0.06m
a_B
a_B
B
G
C
a_B
+
$\alpha = \frac{a_B}{R}$
$r\alpha$
$(r\alpha)$
G
B
$r\omega^2$
$\omega = \frac{v_B}{R}$
C
=
$\bar{a}_y = r\alpha = \frac{r}{R}a_B$
a_B
B
G
$\bar{a}_x = a_B + r\omega^2$

Trans. with B + Rotation about B = Rolling motion

$$\bar{\mathbf{a}} = [a_B + r\omega^2] \longleftarrow + \left[\frac{r}{R}a_B\right] \uparrow$$

Kinetics:


r=0.018m
W=mg
A
P
B
R=0.06m
F
C
N
=
$m\bar{a}_y = m\frac{r}{R}a_B$
A
B
$\bar{I}\alpha$
$m\bar{a}_x =$
$m(a_B + r\omega^2)$
C

$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad PR - Wr = (m\bar{a}_y)r + (m\bar{a}_x)R + \bar{I}\alpha$$

$$PR - mgr = m\left(\frac{r}{R}a_B\right)r + m(a_B + r\omega^2)R + m\bar{k}^2\frac{a_B}{R}$$

$$= ma_B\left(\frac{r^2}{R} + R + \frac{\bar{k}^2}{R}\right) + mr\left(\frac{v_B}{r}\right)^2 R$$

$$P = mg\left(\frac{r}{R}\right) + ma_B\left(1 + \frac{r^2 + \bar{k}^2}{R^2}\right) + m\frac{r}{R^2}v_B^2 \qquad (1)$$

PROBLEM 16.113 (Continued)

Substitute:

$$m = 1.5 \text{ kg}$$

$$r = 0.018 \text{ m}$$

$$R = 0.06 \text{ m}$$

$$\bar{k} = 0.044 \text{ mm} \quad \text{and} \quad g = 9.81 \text{ m/s}^2 \text{ in Eq. (1)}$$

$$P = 1.5(9.81)\frac{0.018}{0.06} + 1.5(a_B)\left(1 + \frac{0.018^2 + 0.044^2}{0.06^2}\right) + 1.5\frac{0.018}{0.06^2}v_B^2$$

$$P = 4.4145 + 2.4417a_B + 7.5v_B^2 \qquad (2)$$

Data: $\mathbf{v}_B = 0.35 \text{ m/s} \longleftarrow; \quad v_B = +0.35 \text{ m/s}$

$\mathbf{a}_B = 1.2 \text{ m/s}^2 \longleftarrow; \quad a_B = +1.2 \text{ m/s}^2$

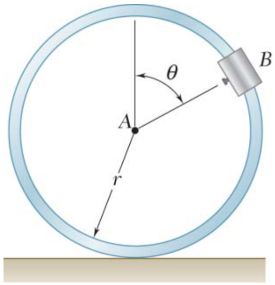
Substitute in Eq. (2):

$$P = 4.4145 + 2.4417(+1.2) + 7.5(+0.35)^2$$

$$= 4.4145 + 2.9300 + 0.9188$$

$$= +8.263 \text{ N}$$

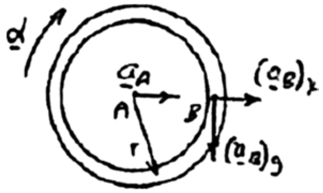
$\mathbf{P} = 8.26 \text{ N} \longleftarrow$ ◀



PROBLEM 16.114

A small clamp of mass m_B is attached at B to a hoop of mass m_h. The system is released from rest when $\theta = 90°$ and rolls without sliding. Knowing that $m_h = 3m_B$, determine (*a*) the angular acceleration of the hoop, (*b*) the horizontal and vertical components of the acceleration of B.

SOLUTION



Kinematics:

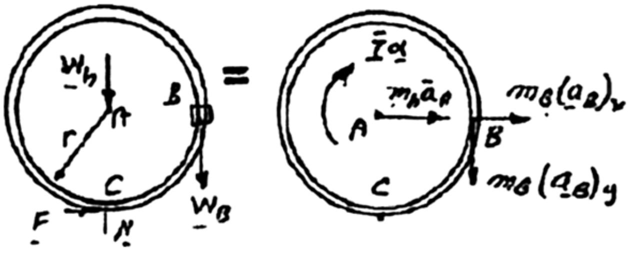
$$\mathbf{a}_A = r\alpha \longrightarrow, \quad a_{B/A} = r\alpha \downarrow$$

$$\mathbf{a}_B = \mathbf{a}_A \longrightarrow + a_{B/A} \downarrow$$

$$\mathbf{a}_B = r\alpha \longrightarrow + r\alpha \downarrow$$

$$(\mathbf{a}_B)_x = r\alpha \longrightarrow \qquad (a_B)_y = r\alpha \downarrow$$

Kinetics:



$$m_h = 3m_B$$

$$\bar{I} = m_h r^2 = 3m_B r^2$$

(*a*) Angular acceleration.

$$+\circlearrowright \Sigma M_C = \Sigma (M_C)_{\text{eff}}: \quad W_B r = \bar{I}\alpha + m_h \bar{a}_A r + m_B (a_B)_x r + m_B (a_B)_y r$$

$$m_B g r = 3m_B r^2 \alpha + (3m_B) r^2 \alpha + m_B r^2 \alpha + m_B r^2 \alpha$$

$$g r = 8r^2 \alpha$$

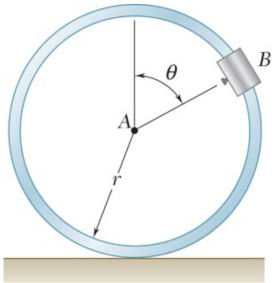
$$\alpha = \frac{1}{8}\frac{g}{r} \circlearrowright \blacktriangleleft$$

(*b*) Components of acceleration of *B*.

$$(\mathbf{a}_B)_x = r\alpha = \frac{1}{8} g \longrightarrow \quad (\mathbf{a}_B)_y = r\alpha = \frac{1}{8} g \downarrow$$

$$(\mathbf{a}_B)_x = \frac{1}{8} g \longrightarrow \blacktriangleleft$$

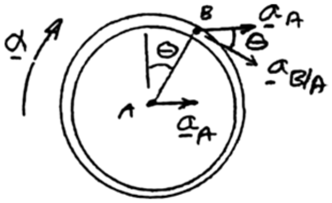
$$(\mathbf{a}_B)_y = \frac{1}{8} g \downarrow \blacktriangleleft$$



PROBLEM 16.115

A small clamp of mass m_B is attached at B to a hoop of mass m_h. Knowing that the system is released from rest and rolls without sliding, derive an expression for the angular acceleration of the hoop in terms of m_B, m_h, r, and θ.

SOLUTION



Kinematics:

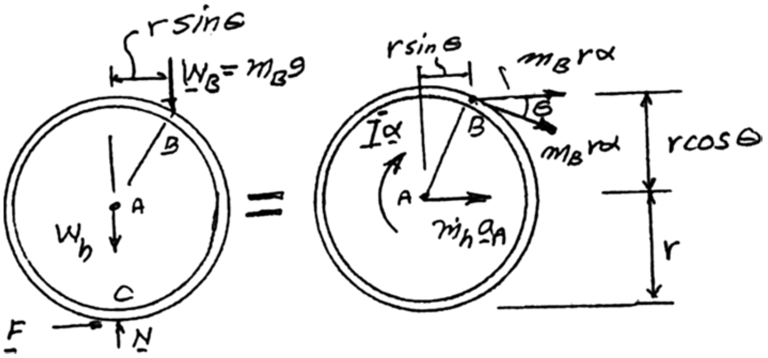
$$\mathbf{a}_A = r\alpha \longrightarrow \quad a_{B/A} = r\alpha \searrow \theta$$

$$\mathbf{a}_B = a_A \longrightarrow + a_{B/A} \searrow \theta$$

$$\mathbf{a}_B = r\alpha \longrightarrow + r\alpha \searrow \theta$$

Kinetics:

$$\bar{I} = m_h r^2$$

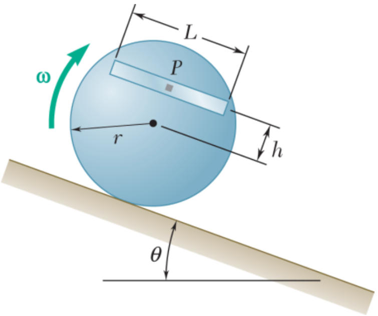


$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad W_B r\sin\theta = \bar{I}\alpha + m_h a_A r + m_B r\alpha(r + r\cos\theta) + m_B r\alpha\sin\theta(r\sin\theta)$$
$$+ m_B r\alpha\cos\theta(r + r\cos\theta)$$

$$m_B gr\sin\theta = m_h r^2\alpha + m_h(r\alpha)r + m_B r\alpha(1+\cos\theta)(r + r\cos\theta)$$
$$+ m_B r\alpha\sin\theta(r\sin\theta)$$

$$m_B gr\sin\theta = 2m_h r^2\alpha + m_B r^2\alpha[(1+\cos\theta)^2 + \sin^2\theta]$$
$$= 2m_h r^2\alpha + m_B r^2\alpha[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

$$m_B gr\sin\theta = r^2\alpha[2m_h + m_B(2 + 2\cos\theta)] \qquad \alpha = \frac{g}{2r}\,\frac{m_B\sin\theta}{m_h + m_B(1+\cos\theta)} \blacktriangleleft$$



PROBLEM 16.116

A 4-lb bar is attached to a 10-lb uniform cylinder by a square pin, *P*, as shown. Knowing that $r = 16$ in., $h = 8$ in., $\theta = 20°$, $L = 20$ in. and $\omega = 2$ rad/s at the instant shown, determine the reactions at *P* at this instant assuming that the cylinder rolls without sliding down the incline.

SOLUTION

Masses and moments of inertia.

Bar: $$m_B = \frac{W_B}{g} = \frac{4\text{lb}}{32.2} = 0.12422 \text{ slug}$$

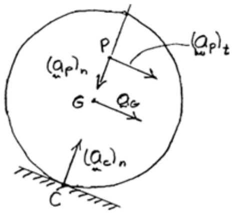
$$\bar{I}_B = \frac{1}{12}m_B L^2 = \frac{1}{12}\left(\frac{4\text{ lb}}{32.2}\right)\left(\frac{20\text{ in.}}{12}\right)^2 = 0.028755 \text{ slug}\cdot\text{ft}^2$$

Disk: $$m_D = \frac{W_D}{g} = \frac{10\text{ lb}}{32.2} = 0.31056 \text{ slug}$$

$$\bar{I}_D = \frac{1}{2}m_C r^2 = \frac{1}{2}\left(\frac{10\text{ lb}}{32.2}\right)\left(\frac{16\text{ in}}{12}\right)^2 = 0.27605 \text{ slug}\cdot\text{ft}^2$$

Kinematics. Let Point *C* be the point of contact between the cylinder and the incline and Point *G* be the mass center of the cylinder without the bar. Assume that the mass center of the bar lies at Point *P*.

$$\boldsymbol{\alpha} = \alpha \curvearrowright. \quad \mathbf{a}_G = a_G \searrow 20^\circ$$



For rolling without slipping, $(a_C)_t = a_G - r\alpha = 0 \quad a_G = r\alpha$

$$(a_P)_t = a_G + h\alpha$$

$$(a_P)_t = (r+h)\alpha$$

$$(a_P)_h = 0 + h\omega^2 = h\omega^2$$

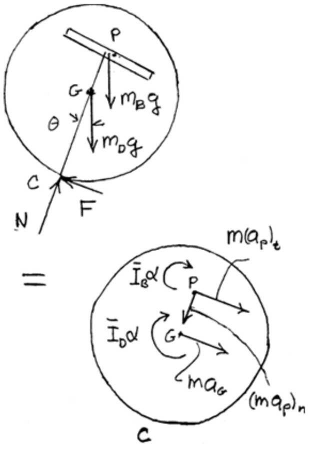
Kinetics: Using the cylinder plus the bar as a free body,

$$+\!\curvearrowright \Sigma M_C = +\Sigma(M_C)_{\text{eff}}: \quad m_D g r\sin\theta + m_B g(r+h)\sin\theta$$

$$= \bar{I}_D\alpha + m_D a_G r + \bar{I}_B\alpha + m_B(a_P)_t(r+h)$$

$$= [\bar{I}_D + m_D r^2 + \bar{I}_B + m_B(r+h)^2]\alpha$$

$$\alpha = \frac{[m_D g r + m_B g(r+h)]\sin\theta}{\bar{I}_D + m_D r^2 + \bar{I}_B + m_B(r+h)^2}$$



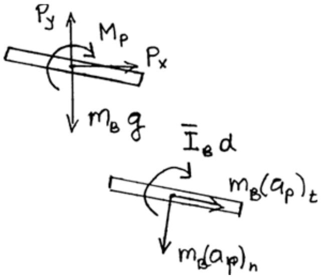
$$= \frac{\left[(10)\left(\frac{16}{12}\right)+(4)\left(\frac{24}{12}\right)\right]\sin 20^\circ}{0.27605+\left(\frac{10}{32.2}\right)\left(\frac{16}{12}\right)^2+(0.028755)+\left(\frac{4}{32.2}\right)\left(\frac{24}{12}\right)^2}$$

$$= 5.3896 \text{ rad/s}^2$$

$$(a_P)_t = \left(\tfrac{24}{12}\right)(5.3896) = 10.7791 \text{ ft/s}^2$$

$$(a_P)_n = \left(\tfrac{8}{12}\right)(2)^2 = 2.6667 \text{ ft/s}^2$$

Using the bar alone as a free body,



$$\underset{+}{\longrightarrow} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad P_x = m_B(a_P)_t \cos 20^\circ - m_B(a_P)_n \sin 20^\circ$$

$$= \left(\frac{4}{32.2}\right)(10.779)\cos 20^\circ - \left(\frac{4}{32.2}\right)(2.6667)\sin 20^\circ$$

$$P_x = 1.1450 \text{ lb}$$

$$\pm\!\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad P_y - m_B g = -m_B(a_P)_t \sin 20^\circ - m_B(a_P)_n \cos 20^\circ$$

$$P_y = (4) - \left(\frac{4}{32.2}\right)(10.779)\sin 20^\circ - \left(\frac{4}{32.2}\right)(2.6667)\cos 20^\circ$$

$$P_y = 3.2307 \text{ lb}$$

$$P = \sqrt{1.145^2 + 3.2307^2} = 3.4276 \text{ lb}$$

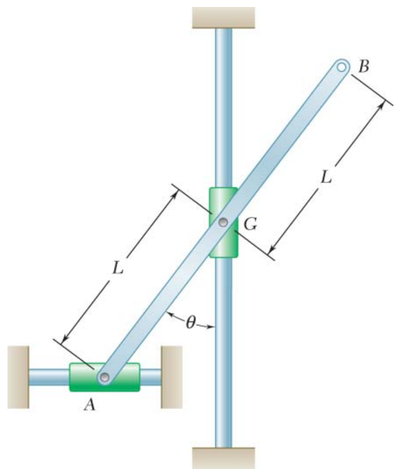
$$\tan\beta = \frac{3.2307}{1.145}$$

$$\beta = 70.5^\circ \qquad \mathbf{P} = 3.43 \text{ lb} \measuredangle 70.5^\circ \blacktriangleleft$$

Recognizing that P is the CG of the bar.

$$\pm\!\circlearrowright \Sigma M_P = \Sigma(M_P)_{\text{eff}}: \qquad M_P = \bar{I}_B \alpha$$

$$= (0.028755)(5.3896) \qquad \mathbf{M}_P = 0.1550 \text{ ft}\cdot\text{lb} \circlearrowright \blacktriangleleft$$

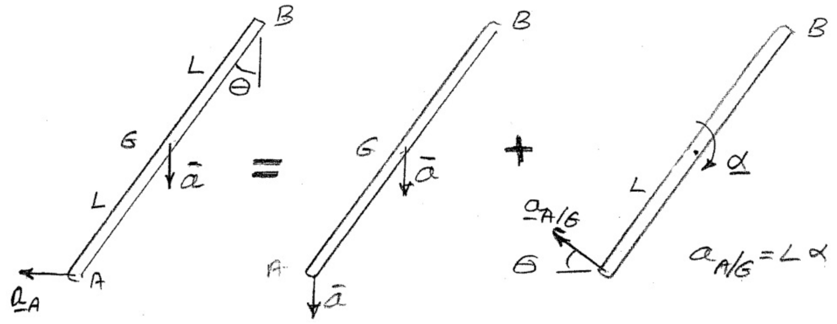


PROBLEM 16.117

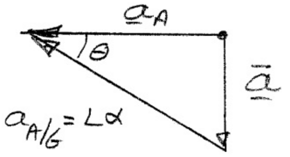
The uniform rod *AB* of mass *m* and length 2*L* is attached to collars of negligible mass that slide without friction along fixed rods. If the rod is released from rest in the position shown, derive an expression for (*a*) the angular acceleration of the rod, (*b*) the reaction at *A*.

SOLUTION

Kinematics: $\omega = 0$

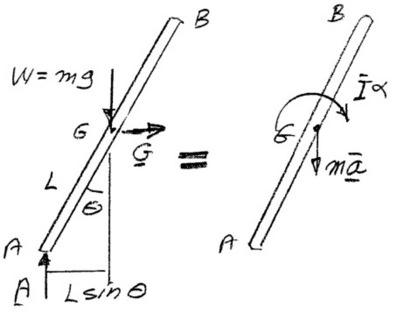


$$\mathbf{a}_A = \mathbf{a}_G + \mathbf{a}_{A/G}: \quad [a_A \longleftarrow] = [\bar{a} \downarrow] + [L\alpha \searrow \theta]$$



$$\bar{a} = L\alpha \sin\theta \downarrow$$

Kinetics:



$$\bar{I} = \frac{1}{12}m(2L)^2 = \frac{1}{3}mL^2$$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}$$

$$G = 0$$

$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}$$

$$mg(L\sin\theta) = \bar{I}\alpha + m\bar{a}(L\sin\theta)$$

$$mgL\sin\theta = \frac{1}{3}mL^2\alpha + m(L\alpha\sin\theta)(L\sin\theta)$$

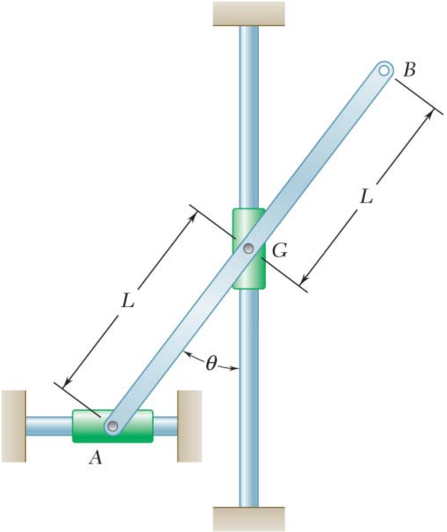
$$\boldsymbol{\alpha} = \frac{g}{L}\left[\frac{\sin\theta}{\frac{1}{3} + \sin^2\theta}\right] \circlearrowright \blacktriangleleft$$

$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad A - mg = -m\bar{a} = -mL\alpha \sin\theta$$

$$A = mg - mL\frac{g}{L}\left[\frac{\sin\theta}{\frac{1}{3} + \sin^2\theta}\right]\sin\theta$$

$$A = mg\frac{\frac{1}{3} + \cancel{\sin^2\theta} - \cancel{\sin^2\theta}}{\frac{1}{3} + \sin^2\theta}$$

$$A = \frac{mg}{1 + 3\sin^2\theta}\uparrow \blacktriangleleft$$



PROBLEM 16.118

The 10-lb-uniform rod AB has a total length $2L = 2$ ft and is attached to collars of negligible mass that slide without friction along fixed rods. If rod AB is released from rest when $\theta = 30°$, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the reaction at A.

SOLUTION

$$m = \frac{W}{g} = \frac{10 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.3106 \text{ slugs}$$

$$L = 1 \text{ ft}$$

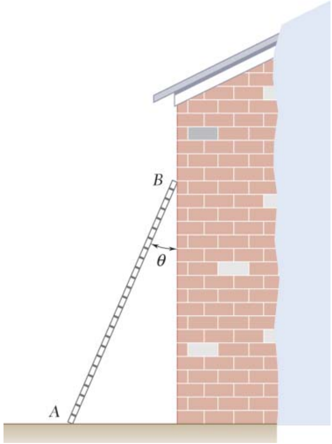
From P16.117

$$\alpha = \frac{32.2 \text{ ft/s}^2}{1.0 \text{ ft}}\left(\frac{\sin 30°}{\frac{1}{3} + \sin^2 30°}\right) = 27.600 \text{ rad/s}^2$$

$$\text{or} \quad \alpha = 27.6 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

$$A = \frac{(0.3106 \text{ slugs})(32.2 \text{ ft/s}^2)}{1 + 3\sin^2 30°} = 5.714 \text{ lbs}$$

$$\text{or} \quad \mathbf{A} = 5.714 \text{ lb} \uparrow \blacktriangleleft$$



PROBLEM 16.119

A 40-lb ladder rests against a wall when the bottom begins to slide out. The ladder is 30 ft long and the coefficient of kinetic friction between the ladder and all surfaces is 0.2. For θ = 40°, determine (*a*) the angular acceleration of the ladder, (*b*) the forces at *A* and *B*.

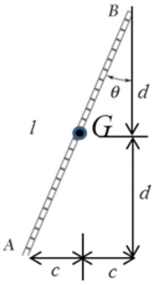
SOLUTION

Given:

$$m = \frac{40\text{ lb}}{32.2\text{ ft/s}^2} = 1.242\text{ slugs}$$

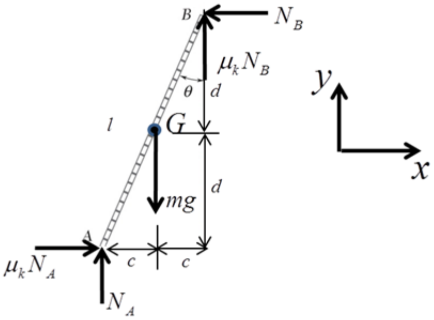
$$l = 30\text{ ft},\ \mu_k = 0.2,\ \theta = 40°$$

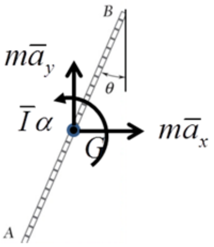
$$\omega = 0,\ \bar{I} = \frac{1}{12}ml^2$$



Geometry: $c = \frac{l}{2}\cos 40° = 5.745\text{ ft},\ \text{d} = \frac{l}{2}\sin 40° = 4.821\text{ ft}$

Free Body Diagram:





Kinetics: $\sum F_x = m\bar{a}_x$ $\qquad$ $\sum F_y = m\bar{a}_y$

$$\mu_k N_A - N_B = m\bar{a}_x \quad (1) \qquad N_A + \mu_k N_B - mg = m\bar{a}_y \quad (2)$$

$$\sum M_G = \bar{I}\alpha$$

$$\mu_k N_B c - N_A c + \mu_k N_A d + N_B d = \bar{I}\alpha$$

$$(\mu_k d - c)N_A + (\mu_k c + d)N_B = \bar{I}\alpha \quad (3)$$

Kinematics:

$$\bar{\mathbf{a}} = \mathbf{a}_A - \cancel{\omega}^2\mathbf{r}_{G/A} + \boldsymbol{\alpha}\times\mathbf{r}_{G/A} \qquad \mathbf{a}_A = \mathbf{a}_B - \cancel{\omega}^2\mathbf{r}_{A/B} + \boldsymbol{\alpha}\times\mathbf{r}_{A/B}$$

$$= a_A\mathbf{i} + \alpha\mathbf{k}\times(c\mathbf{i} + d\mathbf{j}) \qquad a_A\mathbf{i} = a_B\mathbf{j} + \alpha\mathbf{k}\times(-2c\mathbf{i} - 2d\mathbf{j})$$

$$= (a_A - d\alpha)\mathbf{i} + c\alpha\mathbf{j} \quad (4) \qquad a_A\mathbf{i} = 2d\alpha\mathbf{i} + (a_B - 2c\alpha)\mathbf{j} \quad (5)$$

PROBLEM 16.119 (Continued)

Solving the **i** components of (5): $a_A = 2d\alpha$ (6)

Substitute into (6)→(5): $\overline{\mathbf{a}} = d\alpha\,\mathbf{i} + c\alpha\,\mathbf{j}$ (7)

Substitute into (7)→(1): $N_B = \mu_k N_A - md\alpha$ (8)

Substitute into (7), (8)→(2): $N_A + \mu_k\left(\mu_k N_A - md\alpha\right) - mg = mc\alpha$ (9)

Solve (9) for N$_A$):

$$N_A = \frac{m\left(c\alpha + g + \mu_k d\alpha\right)}{1+\mu_k^2} \qquad (10)$$

Substitute (10)-(8):

$$N_B = m\left[\mu_k\,\frac{\left(c\alpha + g + \mu_k d\alpha\right)}{1+\mu_k^2} - d\alpha\right] \qquad (11)$$

Substitute into (10), (11)→(3):

$$\left(\mu_k d - c\right)\frac{m\left(c\alpha + g + \mu_k d\alpha\right)}{1+\mu_k^2} + \left(\mu_k c + d\right)m\left[\mu_k\,\frac{\left(c\alpha + g + \mu_k d\alpha\right)}{1+\mu_k^2} - d\alpha\right] = \overline{I}\alpha$$

Solve for α:

$$\alpha = \frac{\left(\mu_k d - c\right)\dfrac{mg}{1+\mu_k^2} + \left(\mu_k c + d\right)\left[\mu_k\,\dfrac{mg}{1+\mu_k^2}\right]}{\overline{I} - \left(\mu_k d - c\right)\dfrac{m\left(c + \mu_k d\right)}{1+\mu_k^2} - \left(\mu_k c + d\right)m\left[\mu_k\,\dfrac{\left(c + \mu_k d\right)}{1+\mu_k^2} - d\right]}$$

(a) Substitute in known values: $\alpha = -0.510\ \text{rad/s}^2$ ◀

Substitute known values into (10): $N_A = 31.18$ lb.

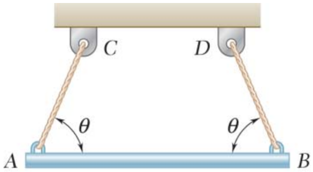
$\mu_k N_A = 6.24$ lb.

Substitute known values into (11): $N_B = 13.52$ lb.

$\mu_k N_B = 2.70$ lb.

(b) $F_A = 31.80$ lb ∠ 78.7° ◀

$F_B = 13.79$ lb ⦩ 11.3° ◀



PROBLEM 16.120

A beam AB of length L and mass m is supported by two cables as shown. If cable BD breaks, determine at that instant the tension in the remaining cable as a function of its initial angular orientation θ.

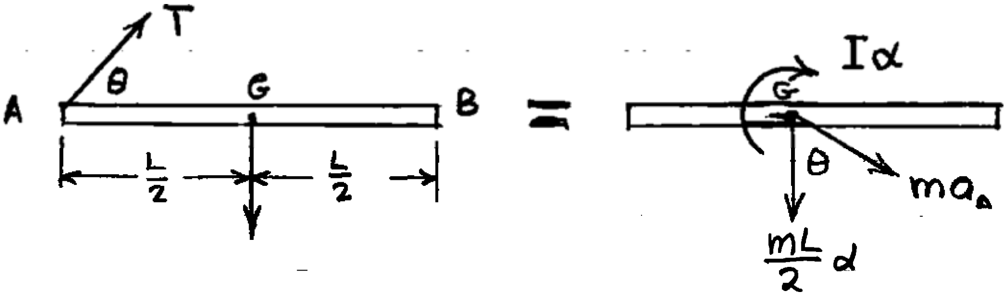
SOLUTION

Kinematics: At the instant just after the cable break,

$$\omega = 0 \qquad \boldsymbol{\alpha} = \alpha \circlearrowright$$

$$\mathbf{a}_G = \mathbf{a}_A + \frac{L}{2}\alpha \downarrow = \alpha_A \searrow \theta + \frac{L}{2}\alpha \downarrow$$

Kinetics: $I = \dfrac{1}{12}mL^2$



$$\circlearrowleft + \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad (T\sin\theta)\frac{L}{2} = I\alpha = \frac{1}{12}mL^2\alpha$$

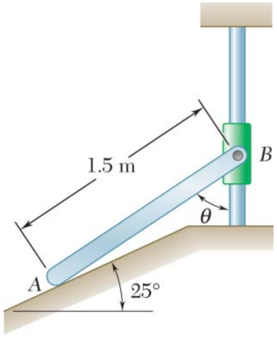
$$\alpha = \frac{6T\sin\theta}{mL}$$

$$\measuredangle \theta \ \Sigma F = \Sigma F_{\text{eff}}: \quad T - mg\sin\theta = -\frac{mL}{2}\alpha$$

$$= -\frac{mL}{2}\cdot\frac{6T\sin\theta}{mL}$$

$$= -3T\sin\theta$$

Solving for T, $$T = \frac{mg\sin\theta}{1+3\sin\theta} \blacktriangleleft$$

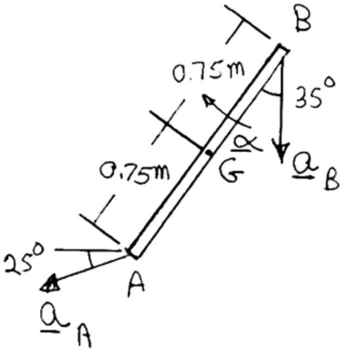


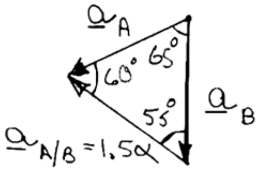
PROBLEM 16.121

End *A* of the 6-kg uniform rod *AB* rests on the inclined surface, while end *B* is attached to a collar of negligible mass which can slide along the vertical rod shown. Knowing that the rod is released from rest when $\theta = 35°$ and neglecting the effect of friction, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the reaction at *B*.

SOLUTION

Kinematics: Assume $\boldsymbol{\alpha} \circlearrowright$ $(\omega = 0)$


B
0.75m
35°
α
G
$\underline{a}_B$
0.75m
25°
A
$\underline{a}_A$


$\underline{a}_A$
60°
65°
55°
$\underline{a}_B$
$\underline{a}_{A/B} = 1.5\alpha$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

$$a_A \measuredangle 25° = a_B \uparrow + 1.5\,\alpha \searrow 35°$$

Law of sines:

$$\frac{a_B}{\sin 60°} = \frac{1.5\,\alpha}{\sin 65°}$$

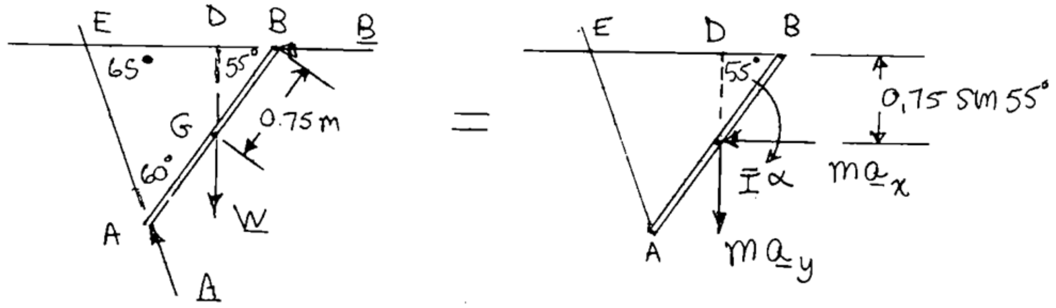
$$a_B = 1.4333\,\alpha$$

$$\bar{\mathbf{a}} = \mathbf{a}_G = \mathbf{a}_B + \mathbf{a}_{G/B} = 1.4333\,\alpha \downarrow + 0.75\,\alpha \searrow 35°$$

$$\bar{\mathbf{a}}_x = 0.75\,\alpha \cos 35° \longleftarrow = 0.61436\,\alpha \longleftarrow$$

$$\bar{\mathbf{a}}_y = -1.4333\,\alpha \downarrow + 0.75\,\alpha \sin 35° \uparrow = 1.0031\,\alpha \downarrow$$

Kinetics:


E
D
B
$\underline{B}$
65°
55°
G
0.75 m
60°
A
W
$\underline{A}$
=
E
D
B
55°
0.75 sin 55°
$\bar{I}\alpha$
$m\underline{a}_x$
A
$m\underline{a}_y$

Law of sines: $\dfrac{EB}{\sin 60°} = \dfrac{AB}{\sin 65°}$, $\quad EB = \dfrac{\sin 60°}{\sin 65°}(1.5\text{ m}) = 1.4333\text{ m}$

$$ED = EB - DB = 1.4333 \text{ m} - (0.75 \text{ m})\cos 55° = 1.0031 \text{ m}$$

$$DG = 0.75 \sin 55° = 0.61436 \text{ m}$$

(*a*) $\overset{+}{\circlearrowright}\ \Sigma M_E = \Sigma (M_E)_{\text{eff}}: \quad W(ED) = \bar{I}\alpha + m\bar{a}_x(DG) + m\bar{a}_y(ED)$

$$(6)(9.81)(1.0031) = \frac{1}{12}(6)(1.5)^2\alpha + (6)(0.61436\,\alpha)(0.61436)$$
$$+ (6)(1.0031\,\alpha)(1.0031)$$

$$59.04 = (1.125 + 2.2646 + 6.3778)\,\alpha$$

$59.04 = 9.427\,\alpha \qquad \alpha = +\,6.263,$ $\qquad$ $\boldsymbol{\alpha} = 6.26 \text{ rad/s}^2 \circlearrowright$ ◀

$$\bar{a}_x = 0.61436\,(6.263) = 3.848 \text{ m/s}^2$$

$$\bar{a}_y = 1.0031\,(6.263) = 6.282 \text{ m/s}^2$$

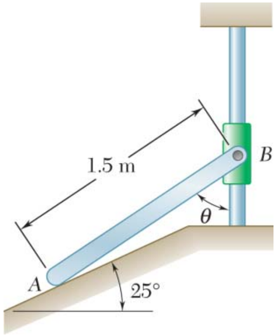
(*b*) $\Sigma M_A = \Sigma (M_A)_{\text{eff}}:$

$$W(0.75 \cos 55°) - B(1.5 \sin 55°)$$
$$= \bar{I}\ \alpha + m\bar{a}_y(0.75 \cos 55°) - m\bar{a}_x(0.75 \sin 55°)$$

$$6(9.81)(0.75 \cos 55°) - B(1.5 \sin 55°)$$
$$= \frac{1}{12}(6)(1.5)^2(6.263) + 6(6.282)(0.75 \cos 55°) - 6(3.848)(0.75 \sin 55°)$$

$$25.321 - 1.2287B = 7.046 + 16.214 - 14.184$$

$16.245 = 1.2287B$ $\qquad$ $\mathbf{B} = 13.22 \text{ N} \longleftarrow$ ◀

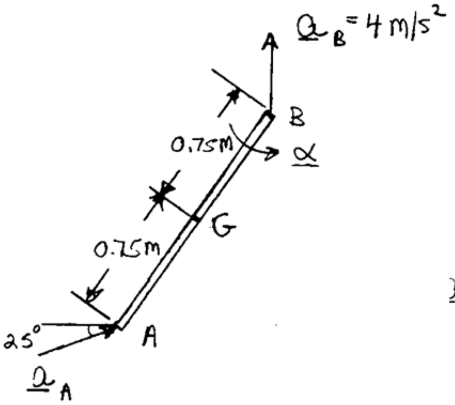


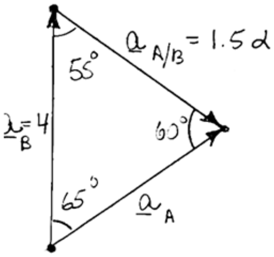
PROBLEM 16.122

End *A* of the 6-kg uniform rod *AB* rests on the inclined surface, while end *B* is attached to a collar of negligible mass which can slide along the vertical rod shown. When the rod is at rest a vertical force **P** is applied at *B*, causing end *B* of the rod to start moving upward with an acceleration of 4 m/s^2. Knowing that $\theta = 35°$, determine the force **P**.

SOLUTION

Kinematics: $\mathbf{a}_B = 4 \text{ m/s}^2 \uparrow \qquad (\omega = 0)$


a_B = 4 m/s²
B
α
0.75m
G
0.75m
25°
A
a_A


$a_{A/B}$ = 1.5α
55°
60°
a_B=4
65°
a_A

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

$$\mathbf{a}_A \measuredangle 25° = 4 \uparrow + 1.5\,\alpha \;\searrow 35°$$

Law of sines:

$$\frac{1.5\,\alpha}{\sin 65°} = \frac{4}{\sin 60°}$$

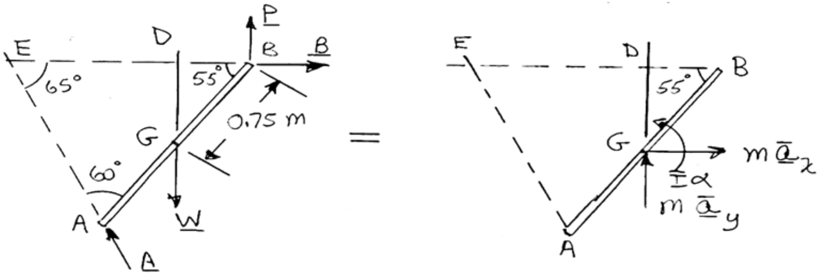
$$\alpha = 2.7907 \text{ rad/s}^2 \circlearrowright$$

$$\bar{\mathbf{a}} = \mathbf{a}_G = \mathbf{a}_B + \mathbf{a}_{A/B} = 4 \uparrow + 0.75\,(2.7907) \;\searrow 35°$$

$$\bar{\mathbf{a}}_x = 0.75\,(2.7907)\cos 35° \longrightarrow = 1.7145 \text{ m/s}^2 \longrightarrow$$

$$\bar{\mathbf{a}}_y = 4 \uparrow + 0.75\,(2.7907)\sin 35° \downarrow = 2.7995 \text{ m/s}^2 \uparrow$$

Kinetics: $\bar{I} = \frac{1}{12} m\ell^2 = \frac{1}{12}(6 \text{ kg})(1.5 \text{ m})^2 = 1.125 \text{ kg}\cdot\text{m}^2$


E
D
P
B
B
65°
55°
0.75 m
G
60°
A
W
A
=
E
D
B
55°
G
$m\bar{a}_x$
$\bar{I}\alpha$
$m\bar{a}_y$
A

Law of sines:

$$\frac{EB}{\sin 60°} = \frac{AB}{\sin 65°}, \qquad EB = \frac{\sin 60°(1.5)}{\sin 65°} = 1.4333 \text{ m}$$

PROBLEM 16.122 (Continued)

$$ED = EB - DB = 1.4333 - 0.75\cos 55° = 1.0031 \text{ m}$$

$$DG = 0.75\sin 55° = 0.61436 \text{ m}$$

$$+\circlearrowleft\ \Sigma M_E = \Sigma (M_E)_{\text{eff}}:\ P(EB) - W(ED) = \bar{I}\alpha + m\bar{a}_x(DG) + m\bar{a}_y(ED)$$

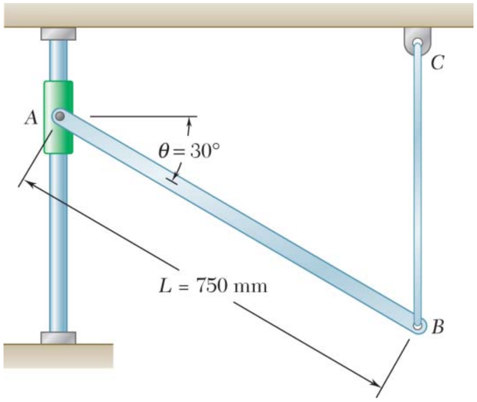
$$P(1.4333) - 6(9.81)(1.0031) = 1.125\,(2.7907) + (6)(1.7145)(0.61436)$$

$$+ (6)(2.7995)(1.0031)$$

$$1.4333P - 59.0425 = 3.1395 + 6.3199 + 16.8491$$

$$1.4333P = 85.531$$

$\mathbf{P} = 59.5 \text{ N} \uparrow$ ◀

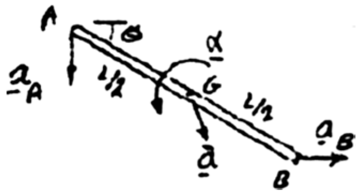

C
A
$\theta = 30°$
$L = 750$ mm
B

PROBLEM 16.123

End *A* of the 8-kg uniform rod *AB* is attached to a collar that can slide without friction on a vertical rod. End *B* of the rod is attached to a vertical cable *BC*. If the rod is released from rest in the position shown, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the reaction at *A*.

SOLUTION

Kinematics:


A
θ
α
a_A
L/2
G
L/2
a_B
$\bar{a}$
B

$$\omega = 0$$

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}; \quad [a_B \longrightarrow] = [a_A \downarrow] + [L\alpha \nearrow \theta]$$

$$+\!\downarrow 0 = a_A - L\alpha\cos\theta$$

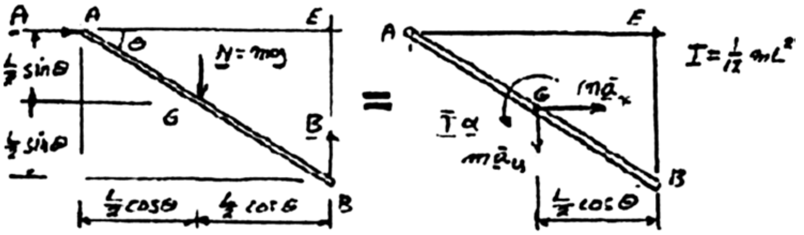
$$\mathbf{a}_A = L\alpha\cos\theta \downarrow$$

$$\bar{\mathbf{a}} = \mathbf{a}_A + \mathbf{a}_{G/A}$$

$$\bar{\mathbf{a}} = [L\alpha\cos\theta\downarrow] + \left[\frac{L}{2}\alpha \nearrow \theta\right]$$

$$\bar{\mathbf{a}}_x = \frac{L}{2}\alpha\sin\theta \longrightarrow; \quad \mathbf{a}_y = \frac{L}{2}\alpha\cos\theta \uparrow$$

Kinetics:


A
A
θ
E
W = mg
$\frac{L}{2}\sin\theta$
G
B
$\frac{L}{2}\sin\theta$
$\frac{L}{2}\cos\theta$
$\frac{L}{2}\cos\theta$
B
=
A
E
$I = \frac{1}{12}mL^2$
G
$m\bar{a}_x$
$\bar{I}\alpha$
$m\bar{a}_y$
B
$\frac{L}{2}\cos\theta$

$$+\circlearrowright \Sigma M_E = \Sigma(M_E)_{\text{eff}}: \quad mg\frac{L}{2}\cos\theta = I\alpha + m\bar{a}_x\left(\frac{L}{2}\sin\theta\right) + m\bar{a}_y\left(\frac{L}{2}\cos\theta\right)$$

$$mg\frac{L}{2}\cos\theta = \frac{1}{12}mL^2\alpha + m\left(\frac{L}{2}\sin\theta\right)^2\alpha + m\left(\frac{L}{2}\cos\theta\right)^2\alpha$$

$$mg\frac{L}{2}\cos\theta = \frac{1}{3}mL^2\alpha \qquad \boldsymbol{\alpha} = \frac{3}{2}\frac{g}{L}\cos\theta \circlearrowright$$

PROBLEM 16.123 (Continued)

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad A = m\bar{a}_x = m\frac{L}{2}\alpha \sin\theta$$

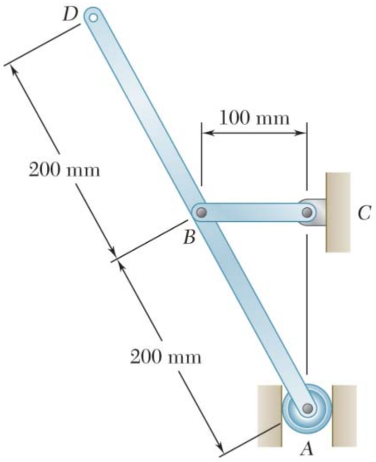
$$A = m\frac{L}{2}\left(\frac{3}{2}\frac{g}{L}\cos\theta\right)\sin\theta$$

$$\mathbf{A} = \frac{3}{4}mg\sin\theta\cos\theta \longrightarrow$$

Data: $m = 8\text{ kg}, \quad \theta = 30°, \quad L = 0.75\text{ m}$

(*a*) Angular acceleration. $\alpha = \frac{3}{2}\frac{9.81\text{ m/s}^2}{0.75\text{ m}}\cos 30°$ $\quad \boldsymbol{\alpha} = 16.99\text{ rad/s}^2 \circlearrowright$ ◀

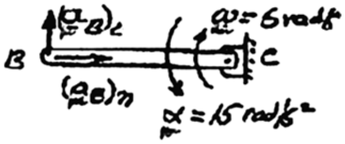
(*b*) Reaction at *A*. $A = \frac{3}{4}(8\text{ kg})(9.81\text{ m/s}^2)\sin 30°\cos 30°$ $\quad \mathbf{A} = 25.5\text{ N} \longrightarrow$ ◀


D
200 mm
100 mm
C
B
200 mm
A

PROBLEM 16.124

The 4-kg uniform rod *ABD* is attached to the crank *BC* and is fitted with a small wheel that can roll without friction along a vertical slot. Knowing that at the instant shown crank *BC* rotates with an angular velocity of 6 rad/s clockwise and an angular acceleration of 15 rad/s^2 counterclockwise, determine the reaction at *A*.

SOLUTION

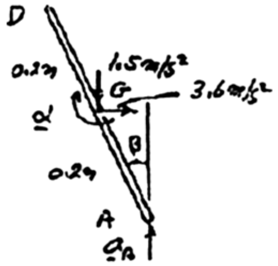

$(a_B)_t$
$\omega = 6$ rad/s
B
C
$(a_B)_n$
$\alpha = 15$ rad/s^2

Crank *BC*:

$$BC = 0.1\ \text{m}$$

$$(a_B)_t = (BC)\alpha = (0.1\ \text{m})(15\ \text{rad/s}^2) = 1.5\ \text{m/s}^2 \qquad (\mathbf{a}_B)_t = 1.5\ \text{rad/s}^2 \downarrow$$

$$(a_B)_n = (BC)\omega^2 = (0.1\ \text{m})(6\ \text{rad/s})^2 = 3.6\ \text{m/s}^2 \qquad (\mathbf{a}_B)_n = 3.6\ \text{m/s}^2 \longrightarrow$$


D
0.2 m
1.5 m/s^2
G
3.6 m/s^2
α
β
0.2 m
A
a_A

Rod *ABD*:

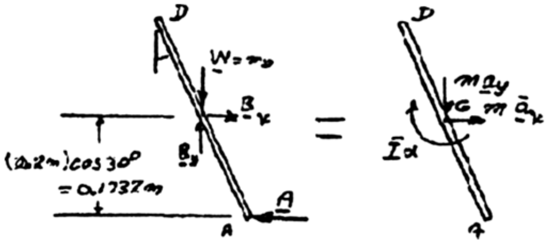
$$\theta = \sin^{-1}\frac{BC}{AB} = \sin^{-1}\frac{0.1\ \text{m}}{0.2\ \text{m}} = 30^\circ$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/G}$$

$$[a_A \uparrow] = [1.5 \downarrow + 3.6 \longrightarrow] + [0.2\alpha \nearrow \beta]$$

$$\xrightarrow{+}\ 0 = 3.6 - (0.2\alpha)\cos\beta$$

$$\alpha = \frac{3.6}{0.2\cos\beta} = \frac{18}{\cos 30^\circ} = 20.78\ \text{rad/s}^2 \qquad \boldsymbol{\alpha} = 20.78\ \text{rad/s}^2 \circlearrowright$$

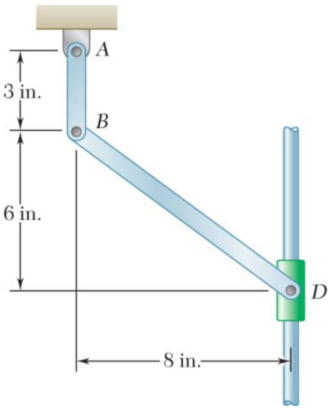

D
W = mg
B_x
B_y
(0.2 m)cos 30°
= 0.1732 m
A
A
=
D
$m\bar{a}_y$
G
$m\bar{a}_x$
$\bar{I}\alpha$
A

Kinetics:

$$+\circlearrowright\ \Sigma M_G = \Sigma(M_G)_{\text{eff}}:\quad A(0.1732\ \text{m}) = I\alpha = \frac{1}{12}mL^2\alpha$$

$$= \frac{1}{12}(4\ \text{kg})(0.4\ \text{m})^2(20.78\ \text{rad/s}^2)$$

$$A = 6.399\ \text{N} \qquad \mathbf{A} = 6.40\ \text{N} \longleftarrow \blacktriangleleft$$

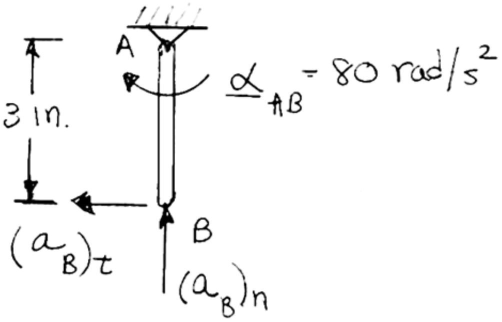

A
3 in.
B
6 in.
D
8 in.

PROBLEM 16.125

The 3-lb uniform rod *BD* is connected to crank *AB* and to a collar of negligible weight. A couple (not shown) is applied to crank *AB* causing it to rotate with an angular velocity of 12 rad/s counterclockwise and an angular acceleration of 80 rad/s^2 clockwise at the instant shown. Neglecting the effect of friction, determine the reaction at *D*.

SOLUTION

Kinematics:


A
$\alpha_{AB} = 80\ \text{rad/s}^2$
3 in.
B
$(a_B)_t$
$(a_B)_n$

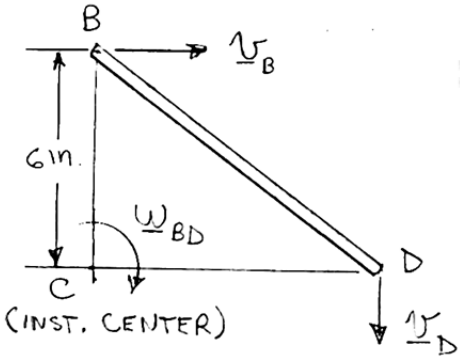
Crank AB:

$$\boldsymbol{\omega}_{AB} = 12\ \text{rad/s}\ \circlearrowleft, \qquad \alpha_{AB} = 80\ \text{rad/s}^2\ \circlearrowright$$

$$(a_B)_n = r\omega^2 = \left(\frac{3}{12}\ \text{ft}\right)(12\ \text{rad/s})^2 = 36.0\ \text{ft/s}^2$$

$$(a_B)_t = r\alpha = \left(\frac{3}{12}\ \text{ft}\right)(80\ \text{rad/s})^2 = 20.0\ \text{ft/s}^2$$

Rod BD


B
v_B
6 in.
ω_{BD}
C
(INST. CENTER)
D
v_D

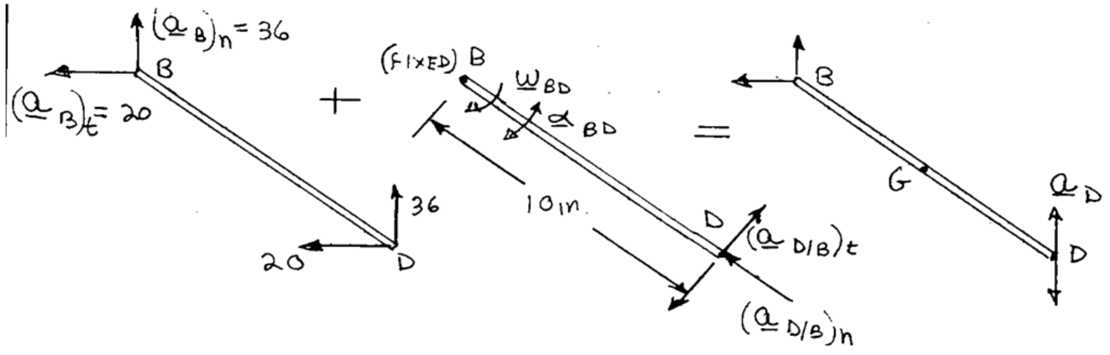
Velocities:

$$\mathbf{v}_B = (AB)\,\boldsymbol{\omega}_{AB} = \frac{3}{12}(12) = 3\ \text{ft/s}$$

$$\boldsymbol{\omega}_{BD} = \frac{v_B}{BC} = \frac{3\ \text{ft/s}}{0.5\ \text{ft}}$$

$$\boldsymbol{\omega}_{BD} = 6\ \text{rad/s}\ \circlearrowright$$

Accelerations:


$(a_B)_n = 36$
B
$(a_B)_t = 20$
36
20
D
+
(FIXED) B
ω_{BD}
α_{BD}
10 in.
D
$(a_{D/B})_t$
$(a_{D/B})_n$
=
B
G
a_D
D

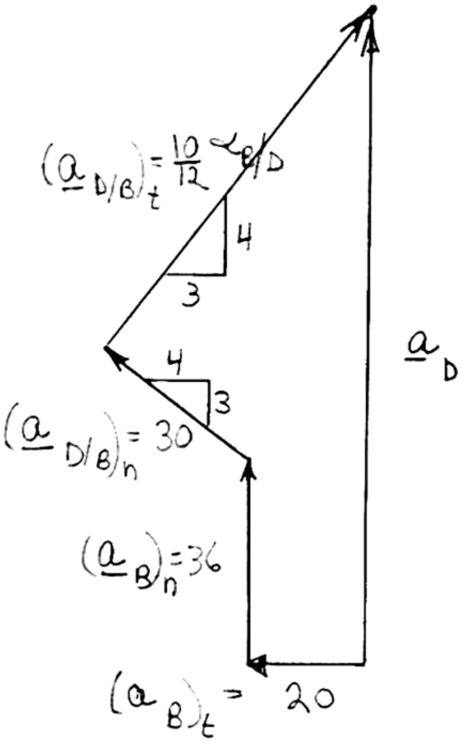
Translation with B and rotation about D = Planar motion

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B}$$

$$a_D \updownarrow = 20 \longleftarrow + 36 \uparrow + \left(a_{D/B}\right)_n \; {}_{4}\overset{5}{\nwarrow}_{3} \;\; + \left(a_{D/B}\right)_t \; {}^{5}\overset{\nearrow}{\swarrow}{}^{4}_{3} \qquad (1)$$

$$\text{Where } \left(a_{D/B}\right)_n = (BD)\,\omega_{BD}^2 = \frac{10}{12}(6)^2 = 30 \text{ ft/s}^2$$

$$\text{And } \left(a_{D/B}\right)_t = (BD)\,\alpha_{BD} = \frac{10}{12}\,\alpha_{BD}$$



$(\underline{a}_{D/B})_t = \frac{10}{12}\alpha_{B/D}$
4
3
$\underline{a}_D$
4
3
$(\underline{a}_{D/B})_n = 30$
$(\underline{a}_B)_n = 36$
$(a_B)_t = 20$

$\xrightarrow{+}$ x components:

$$-20 - \frac{4}{5}(30) + \frac{3}{5}\left(\frac{10}{12}\alpha_{BD}\right) = 0$$

$$\frac{1}{2}\alpha_{BD} = 20 + 24 = 44.0$$

$$\alpha_{BD} = 88 \text{ rad/s}^2$$

Since $\left(\mathbf{a}_{B/D}\right)_t$ is directed $\nearrow$:

$$\boldsymbol{\alpha}_{BD} = 88.0 \text{ rad/s}^2 \circlearrowleft$$

$$\bar{\mathbf{a}} = \mathbf{a}_G = \mathbf{a}_B + \mathbf{a}_{G/B} = \mathbf{a}_B + \frac{1}{2}\mathbf{a}_{D/B}$$

$$\bar{\mathbf{a}} = 20 \longleftarrow + 36 \uparrow + \frac{1}{2}(30)\; {}^{3}\nwarrow_{4} \;\; + \frac{1}{2}\left(\frac{10}{12}\right)(88) \nearrow^{4}_{3}$$

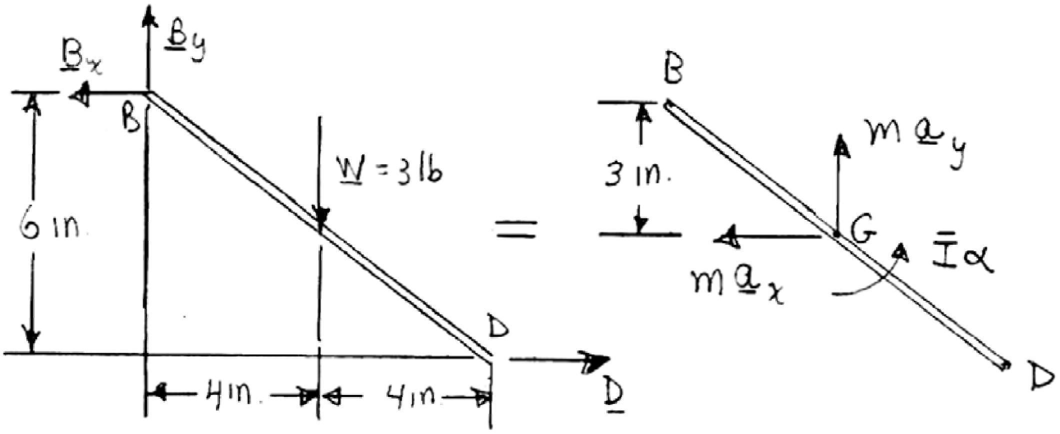
$$\xrightarrow{+} \;\bar{a}_x = -20 - \frac{4}{5}(15) + \frac{3}{5}(36.667) = -10.00$$

$$\bar{\mathbf{a}}_x = 10.00 \text{ ft/s}^2 \longleftarrow$$

$$+\!\uparrow \;\bar{a}_y = 36 + \frac{3}{5}(15) + \frac{4}{5}(36.667) = 74.33, \qquad \mathbf{a}_y = 74.33 \text{ ft/s}^2 \uparrow$$

PROBLEM 16.125 (Continued)

Kinetics: $$\bar{I} = \frac{1}{12}\frac{3}{g}\left(\frac{10}{12}\right)^2 = \frac{0.17361}{g}$$



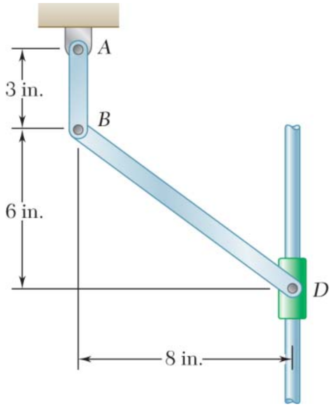
$\underline{B}_x$
$\underline{B}_y$
B
$\underline{W}$ = 3 lb
6 in.
4 in.
4 in.
D
$\underline{D}$
=
B
3 in.
$m\underline{a}_y$
G
$\bar{I}\alpha$
$m\underline{a}_x$
D

$$+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}: D\left(\frac{6}{12}\text{ ft}\right) - W\left(\frac{4}{12}\text{ ft}\right) = \bar{I}\alpha + m\bar{a}_y\left(\frac{4}{12}\right) - m\bar{a}_x\left(\frac{3}{12}\right)$$

$$0.5D - 3\left(\frac{1}{3}\right) = \frac{0.17361}{g}(88) + \frac{3}{g}(74.33)\left(\frac{1}{3}\right) - \frac{3}{g}(10)\left(\frac{1}{4}\right)$$

$$0.5D - 1 = \frac{1}{32.2}(82.108), \qquad D = 7.10\text{ lb}$$

D = 7.10 lb ⟶ ◀



PROBLEM 16.126

The 3-lb uniform rod *BD* is connected to crank *AB* and to a collar of negligible weight. A couple (not shown) is applied to crank *AB* causing it to rotate. At the instant shown, crank *AB* has an angular velocity of 12 rad/s and an angular acceleration of 80 rad/s^2, both counterclockwise. Neglecting the effect of friction, determine the reaction at *D*.

SOLUTION

Kinematics: $\boldsymbol{\omega}_{AB} = 12 \text{ rad/s} \circlearrowleft, \qquad \boldsymbol{\alpha}_{AB} = 80 \text{ rad/s}^2 \circlearrowleft$

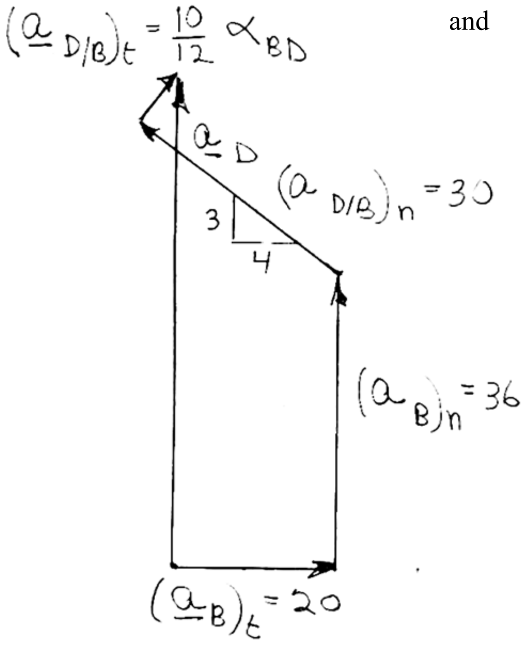
Same analysis as for Prob. 16.117 except that $\boldsymbol{\alpha}_{AB} \circlearrowleft$,

Resulting in $(\mathbf{a}_B)_t \longrightarrow$ Eq. (1) must be replaced by,

$$a_D \updownarrow = 20 \longrightarrow + 36 \uparrow + (\mathbf{a}_{D/B})_n \underset{4}{\overset{3\;\;5}{\nwarrow}} + (\mathbf{a}_{D/B})_n \underset{3}{\overset{5\;\;4}{\nearrow}} \qquad (1)$$

Where we have again $(a_{D/B})_n = 30 \text{ ft/s}^2$

and $(a_{D/B})_t = \dfrac{10}{12}\alpha_{B/D}$



$(\underline{a}_{D/B})_t = \frac{10}{12}\alpha_{BD}$
$\underline{a}_D$
$(a_{D/B})_n = 30$
3
4
$(a_B)_n = 36$
$(\underline{a}_B)_t = 20$

$\xrightarrow{+}$ *x* components:

$$20 - \frac{4}{5}(30) + \frac{3}{5}\left(\frac{10}{12}\alpha_{BD}\right) = 0$$

$$\frac{1}{2}\alpha_{BD} = -20 + 24 = 4.00$$

$$\alpha_{BD} = 8.00 \text{ rad/s}^2$$

Since $(\mathbf{a}_{D/B})_t$ is directed $\nearrow$:

$$\boldsymbol{\alpha}_{BD} = 8.00 \text{ rad/s}^2 \circlearrowleft$$

$$\bar{\mathbf{a}} = \mathbf{a}_G = \mathbf{a}_B + \mathbf{a}_{G/B} = \mathbf{a}_B + \frac{1}{2}\mathbf{a}_{D/B}$$

$$\bar{\mathbf{a}} = 20 \longrightarrow + 36 \uparrow + \frac{1}{2}(30) \underset{4}{\overset{3}{\nwarrow}} + \frac{1}{2}\left(\frac{10}{12}\right)(8.00) \underset{3}{\overset{4}{\nearrow}}$$

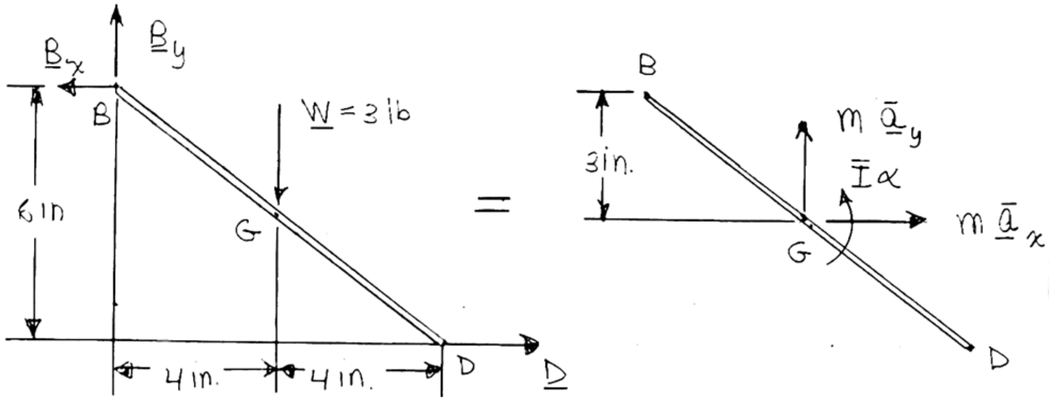
PROBLEM 16.126 (Continued)

$$\xrightarrow{+}\ \bar{a}_x = 20 - \frac{4}{5}(15) + \frac{3}{5}(3.333) = +10.00\ \text{ft/s}^2$$

$$+\!\uparrow\ \bar{a}_y = 36 + \frac{3}{5}(15) + \frac{4}{5}(3.333) = +47.67\ \text{ft/s}^2$$

Thus: $\bar{\mathbf{a}}_x = 10.00\ \text{ft/s}^2 \longrightarrow$, $\bar{\mathbf{a}}_y = 47.67\ \text{ft/s}^2 \uparrow$

Kinetics:



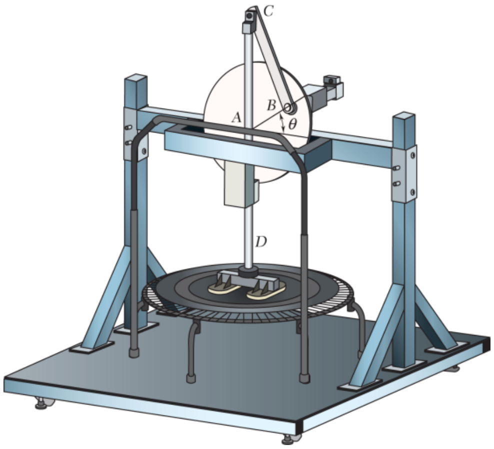
$$\bar{I} = \frac{1}{12} + \frac{3}{g}\left(\frac{10}{12}\right)^2 = \frac{0.17361}{g}$$

$$+\!\circlearrowleft\ \Sigma M_B = \Sigma(M_B)_{\text{eff}} : D\left(\frac{6}{12}\ \text{ft}\right) + W\left(\frac{4}{12}\ \text{ft}\right) = \bar{I}\alpha + m\bar{a}_y\left(\frac{3}{12}\ \text{ft}\right) + m\bar{a}_y\left(\frac{4}{12}\ \text{ft}\right)$$

$$0.5D - 3\left(\frac{1}{3}\right) = \frac{0.17361}{g}(8.00) + \frac{3}{g}(10)\left(\frac{1}{4}\right) + \frac{3}{g}(4.767)\left(\frac{1}{3}\right)$$

$$0.5D - 1 = \frac{1}{32.2}(56.556), \qquad D = 5.513\ \text{lb}$$

$\mathbf{D} = 5.51\ \text{lb} \longrightarrow$ ◀



PROBLEM 16.127

The test rig shown was developed to perform fatigue testing on fitness trampolines. A motor drives the 200-mm radius flywheel *AB,* which is pinned at its center point, in a counterclockwise direction with a constant angular velocity of 120 rpm. The flywheel is attached to slider *CD* by the 400-mm connecting rod *BC*. The mass of the connecting rod *BC* is 5 kg and the mass of the link *CD* and foot is 2 kg. At the instant when $\theta = 0°$ and the foot is just above the trampoline, determine the force exerted by pin *C* on rod *BC*.

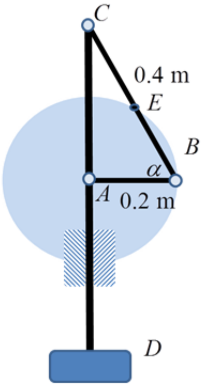
SOLUTION

Given:

$$r = 0.2 \text{ m},\ \omega_{AB} = 120\frac{2\pi}{60} = 4\pi \text{ rad/s}$$

$$m_{BC} = 5 \text{ kg},\ m_{CD} = 2 \text{ kg},$$

$$l_{BC} = 0.4 \text{ m},\ \theta = 0°$$


C
0.4 m
E
B
α
A
0.2 m
D

Geometry:

$$\alpha = \cos^{-1}\left(\frac{0.2}{0.4}\right) = 60°$$

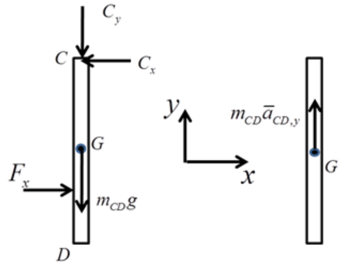
$$\vec{r}_{B/A} = 0.2\mathbf{i}$$

$$\vec{r}_{\text{C/B}} = -0.4\cos\alpha\,\mathbf{i} + 0.4\sin\alpha\,\mathbf{j}$$

$$= -0.2\mathbf{i} + 0.2\sqrt{3}\mathbf{j}$$

$$\bar{I}_{BC} = \frac{1}{12}m_{BC}l_{BC}^2 = \frac{1}{15} \text{ kg}\bullet\text{m}^2$$

Free Body Diagram rod CD:

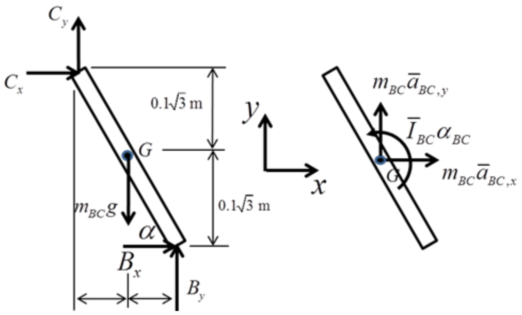

C_y
C
C_x
G
F_x
$m_{CD}g$
D
y
x
$m_{CD}\bar{a}_{CD,y}$
G

Kinetics:

$$\sum F_y = m_{CD}\bar{a}_{CD,y}$$

$$-2g - C_y = 2\bar{a}_{CD,y} \qquad (1)$$

Free Body Diagram rod BC:


C_y
C_x
$0.1\sqrt{3}$ m
G
$0.1\sqrt{3}$ m
$m_{BC}g$
α
B_x
B_y
y
x
$m_{BC}\bar{a}_{BC,y}$
$\bar{I}_{BC}\alpha_{BC}$
G
$m_{BC}\bar{a}_{BC,x}$

Kinetics:

$$\sum F_x = m_{BC}\bar{a}_{BC,x}$$

$$C_x + B_x = 5\bar{a}_{BC,x} \Rightarrow B_x = 5\bar{a}_{BC,x} - C_x \qquad (2)$$

$$\sum F_y = m_{BC}\bar{a}_{BC,y}$$

$$C_y + B_y - 5g = 5\bar{a}_{BC,y} \qquad (3)$$

$$\sum M_G = \bar{I}_{BC}\alpha_{BC}$$

$$-0.1\sqrt{3}C_x - 0.1C_y + 0.1\sqrt{3}B_x + 0.1B_y = \frac{\alpha_{BC}}{15} \qquad (4)$$

PROBLEM 16.127 (Continued)

Kinematics:

$$\mathbf{v}_B = \cancel{\mathbf{v}_A} + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{\text{B/A}}$$
$$= \omega_{AB}\mathbf{k} \times (0.2\mathbf{i})$$
$$= 0.8\pi\,\mathbf{j}$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{\text{C/B}}$$
$$v_C\mathbf{j} = 0.8\pi\,\mathbf{j} + \omega_{BC}\mathbf{k} \times \left(\text{-}0.2\mathbf{i} + 0.2\sqrt{3}\mathbf{j}\right)$$
$$v_C\mathbf{j} = -0.2\sqrt{3}\omega_{BC}\mathbf{i} + \left(0.8\pi - 0.2\omega_{BC}\right)\mathbf{j}$$

Equating **i**-terms:

$$\omega_{BC} = 0$$

Relative Acceleration:

$$\mathbf{a}_B = \cancel{\mathbf{a}_A} - \omega_{AB}^2\mathbf{r}_{B/\text{A}} + \cancel{\boldsymbol{\alpha}_{AB}} \times \mathbf{r}_{B/\text{A}}$$
$$= -3.2\pi^2\mathbf{i}$$

$$\mathbf{a}_C = \mathbf{a}_B - \cancel{\omega_{BC}^2}\,\mathbf{r}_{\text{C/B}} + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{\text{C/B}}$$
$$a_c\mathbf{j} = -3.2\pi^2\mathbf{i} + \alpha_{BC}\mathbf{k} \times \left(-0.2\mathbf{i} + 0.2\sqrt{3}\mathbf{j}\right)$$
$$a_c\mathbf{j} = -\left(3.2\pi^2 + 0.2\sqrt{3}\alpha_{BC}\right)\mathbf{i} \text{ - } 0.2\alpha_{BC}\mathbf{j}$$

Equating **i** components:

$$0 = -\left(3.2\pi^2 + 0.2\sqrt{3}\alpha_{BC}\right)$$

$$\alpha_{BC} = -\frac{16\pi^2}{\sqrt{3}} \text{ rad/s}^2 \qquad (5)$$

Equating **j** components:

$$a_c = \bar{a}_{CD,y} = \frac{3.2\pi^2}{\sqrt{3}} \text{ m/s}^2 \qquad (6)$$

Relative Acceleration:

$$\bar{\mathbf{a}}_{BC} = \mathbf{a}_B - \cancel{\omega_{BC}^2}\,\mathbf{r}_{G/\text{B}} + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{G/\text{B}}$$
$$\bar{a}_{BC,x}\mathbf{i} + \bar{a}_{BC,y}\mathbf{j} = -3.2\pi^2\mathbf{i} - \frac{16\pi^2}{\sqrt{3}}\mathbf{k} \times \left(-0.1\mathbf{i} + 0.1\sqrt{3}\mathbf{j}\right)$$

$$\bar{a}_{BC,x} = -3.2\pi^2 + 1.6\pi^2 \qquad (7)$$

$$\bar{a}_{BC,y} = \frac{1.6\pi^2}{\sqrt{3}} \qquad (8)$$

Substitute into (6)→(1):

$$-2g - C_y = \frac{6.4\pi^2}{\sqrt{3}} \quad \Rightarrow \quad C_y = -56.089 \text{ N} \qquad (9)$$

Substitute into (8)(9)→(3):

$$B_y = 5\frac{1.6\pi^2}{\sqrt{3}} + 5g + 56.0 \quad \Rightarrow \quad B_y = 150.724 \text{ N} \qquad (10)$$

Substitute into (2)(5)(7)(9)(10)→(4):

$$-0.1\sqrt{3}C_x + 0.1(56.089) + 0.1\sqrt{3}\left(-16\pi^2 + 8\pi^2 - C_x\right) + 0.1(150.724) = -\frac{16\pi^2}{15\sqrt{3}} \qquad (11)$$

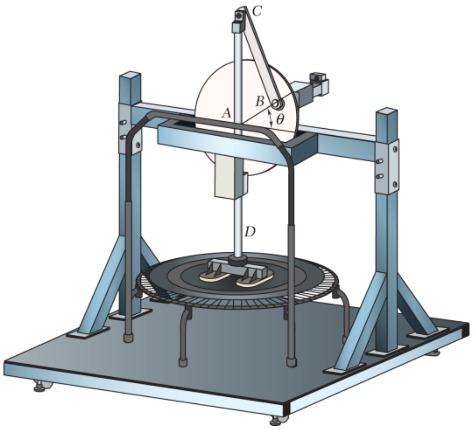
PROBLEM 16.127 (Continued)

Solve (11) for C_x:

$$-0.2\sqrt{3}C_x = -0.1(56.089) + 0.8\sqrt{3}\pi^2 - 0.1(150.724) - \frac{16\pi^2}{15\sqrt{3}}$$

$$C_x = 37.769 \text{ N}$$

C = 67.62 N ↘ 56.0° ◀

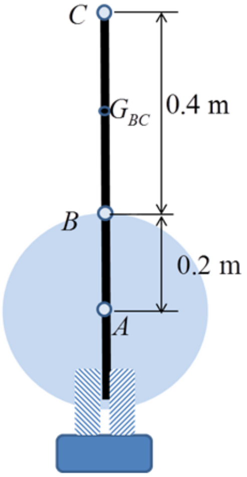


PROBLEM 16.128

Solve Prob. 16.127 for $\theta = 90°$.

PROBLEM 16.127 The test rig shown was developed to perform fatigue testing on fitness trampolines. A motor drives the 200-mm radius flywheel *AB,* which is pinned at its center point, in a counterclockwise direction with a constant angular velocity of 120 rpm. The flywheel is attached to slider *CD* by the 400-mm connecting rod *BC*. The mass of the connecting rod *BC* is 5 kg and the mass of the link *CD* and foot is 2 kg. At the instant when $\theta = 0°$ and the foot is just above the trampoline, determine the force exerted by pin *C* on rod *BC*.

SOLUTION


C
G_{BC}
0.4 m
B
0.2 m
A

Given:

$$r = 0.2\text{ m},\ \omega_{AB} = 120\frac{2\pi}{60} = 4\pi\text{ rad/s}$$

$$m_{BC} = 5\text{ kg},\ m_{CD} = 2\text{ kg},$$

$$l_{BC} = 0.4\text{ m},\ \theta = 90°$$

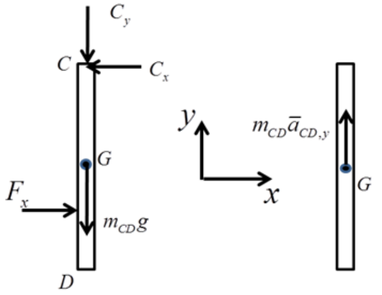
Geometry:

$$\vec{r}_{B/A} = 0.2\mathbf{j}$$

$$\vec{r}_{C/B} = 0.4\mathbf{j}$$

$$\bar{I}_{BC} = \frac{1}{12}m_{BC}l_{BC}^2 = \frac{1}{15}\text{ kg}\cdot\text{m}^2$$

Free Body Diagram rod CD:

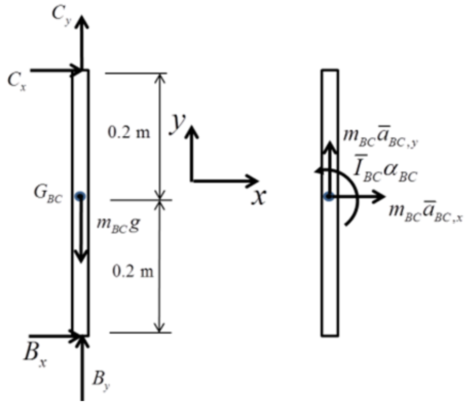

C_y
C
C_x
G
F_x
$m_{CD}g$
D
y
x
$m_{CD}\bar{a}_{CD,y}$
G

Kinetics:

$$\sum F_y = m_{CD}\bar{a}_{CD,y}$$

$$-2g - C_y = 2\bar{a}_{CD,y} \qquad (1)$$

Free Body Diagram rod BC:


C_y
C_x
0.2 m
y
x
G_{BC}
$m_{BC}g$
0.2 m
B_x
B_y
$m_{BC}\bar{a}_{BC,y}$
$\bar{I}_{BC}\alpha_{BC}$
$m_{BC}\bar{a}_{BC,x}$

Kinetics:

$$\sum F_x = m_{BC}\bar{a}_{BC,x}$$

$$C_x + B_x = 5\bar{a}_{BC,x} \ \Rightarrow\ B_x = 5\bar{a}_{BC,x} - C_x \qquad (2)$$

$$\sum F_y = m_{BC}\bar{a}_{BC,y}$$

$$C_y + B_y - 5g = 5\bar{a}_{BC,y} \qquad (3)$$

$$\sum M_G = \bar{I}_{BC}\alpha_{BC}$$

$$-0.2C_x + 0.2B_x = \frac{\alpha_{BC}}{15} \qquad (4)$$

PROBLEM 16.128 (Continued)

Kinematics:

$$\mathbf{v}_B = \cancel{\mathbf{v}_A} + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{\mathrm{B/A}}$$
$$= \omega_{AB}\mathbf{k} \times (0.2\mathbf{j})$$
$$= -0.8\pi\mathbf{i}$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{\mathrm{C/B}}$$
$$v_C\mathbf{j} = -0.8\pi\mathbf{i} + \omega_{BC}\mathbf{k} \times 0.4\mathbf{j}$$
$$v_C\mathbf{j} = -(0.8\pi + 0.4\omega_{BC})\mathbf{i}$$

Equating **i**-terms:

$$\omega_{BC} = -2\pi$$

Relative Acceleration:

$$\mathbf{a}_B = \cancel{\mathbf{a}_A} - \omega_{AB}^2\mathbf{r}_{B/\mathrm{A}} + \cancel{\boldsymbol{\alpha}_{AB}} \times \mathbf{r}_{B/\mathrm{A}}$$
$$= -3.2\pi^2\mathbf{j}$$

$$\mathbf{a}_C = \mathbf{a}_B - \omega_{BC}^2\mathbf{r}_{C/\mathrm{B}} + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/\mathrm{B}}$$
$$a_c\mathbf{j} = -3.2\pi^2\mathbf{j} - 1.6\,\pi^2\mathbf{j} + \alpha_{BC}\mathbf{k} \times 0.4\mathbf{j}$$
$$a_c\mathbf{j} = -0.4\alpha_{BC}\mathbf{i} - \left(1.6\pi^2 + 3.2\pi^2\right)\mathbf{j}$$

Equating **i** components:

$$\alpha_{BC} = 0 \tag{5}$$

Equating **j** components:

$$a_c = \overline{a}_{CD,y} = -4.8\pi^2 \text{ m/s}^2 \tag{6}$$

Relative Acceleration:

$$\overline{\mathbf{a}}_{BC} = \mathbf{a}_B - \omega_{BC}^2\mathbf{r}_{G/\mathrm{B}} + \cancel{\boldsymbol{\alpha}_{BC}} \times \mathbf{r}_{G/\mathrm{B}}$$
$$\overline{a}_{BC,x}\mathbf{i} + \overline{a}_{BC,y}\mathbf{j} = -3.2\pi^2\mathbf{j} - 0.8\pi^2\mathbf{j}$$

$$\overline{a}_{BC,x} = 0 \tag{7}$$

$$\overline{a}_{BC,y} = -4.0\pi^2 \tag{8}$$

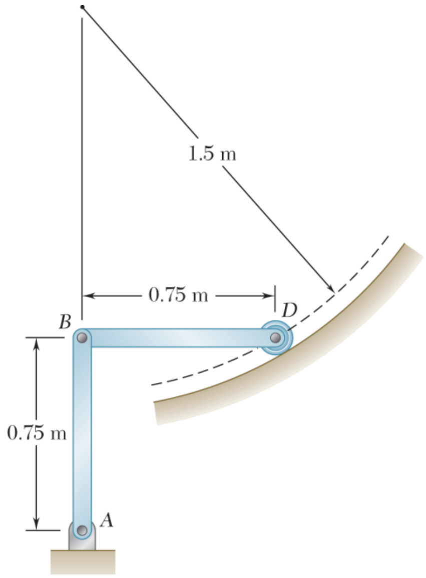
Substitute into (6)→(1):

$$-2g - C_y = -9.6\pi^2 \;\Rightarrow\; C_y = 75.128 \text{ N} \tag{9}$$

Substitute into (2)(5)(7)→(4):

$$-0.2C_x + 0.2\left(5\,\cancel{\overline{a}_{BC,x}} - C_x\right) = \frac{\cancel{\alpha_{BC}}}{15}$$
$$C_x = 0$$

C = 75.13 N ↑ ◀

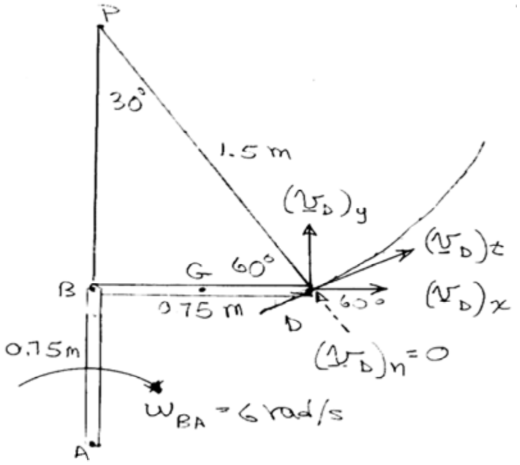


PROBLEM 16.129

The 4-kg uniform slender bar *BD* is attached to bar *AB* and a wheel of negligible mass which rolls on a circular surface. Knowing that at the instant shown bar *AB* has an angular velocity of 6 rad/s and no angular acceleration, determine the reaction at Point *D*.

SOLUTION

Kinematics:



Since $(\mathbf{v}_D)_n = 0$,

$$(v_D)_x \cos 60^\circ = (v_D)_y \cos 30^\circ$$

$$(v_D)_x = (v_B)_x = (AB)\,\omega_{BA} = (0.75 \text{ m})(6 \text{ rad/s})$$

$$(v_D)_y = (v_D)_x \frac{\cos 60^\circ}{\cos 30^\circ}$$

$$= (0.75)(6)\frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{0.75\,(6)}{\sqrt{3}}$$

But

$$(v_D)_y = \omega_{BD}\,(BD)$$

$$\omega_{BD} = \frac{(v_D)_y}{BD} = \frac{0.75\,(6)}{0.75\,\sqrt{3}} = \frac{6}{\sqrt{3}} \text{ rad/s}$$

$$\mathbf{a}_B = AB\,\omega_{BA}^2 = (0.75 \text{ m})(6 \text{ rad/s})^2 = 27 \text{ m/s}^2 \downarrow$$

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B}$$

$$[(\mathbf{a}_D)_t \measuredangle 30^\circ] + [(1.5)(6/\sqrt{3})^2 \searrow 60^\circ] = [27\downarrow] + [0.75\,\alpha_{AB}\uparrow] + [(0.75)(6/\sqrt{3})^2 \longleftarrow]$$

$$60^\circ \searrow: \qquad 0 + 18 = -27\sin 60^\circ + (0.75\sin 60^\circ)\alpha_{BD} + 9\sin 30^\circ$$

$$\alpha_{BD} = \frac{18 + 27\sin 60^\circ - 9\sin 30^\circ}{0.75\sin 60^\circ} = 56.7846 \text{ rad/s}$$

$$\mathbf{a}_G = \mathbf{a}_B + \mathbf{a}_{G/B}$$

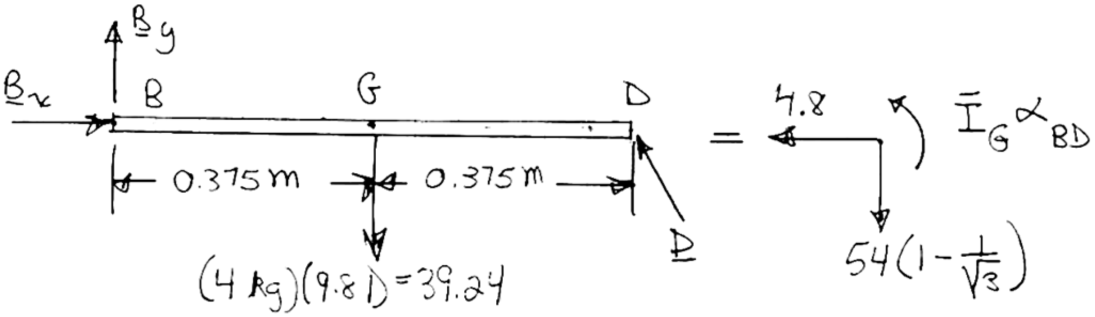
$$= [27\downarrow] + [0.375\alpha_{BD}\uparrow] + [(0.375)(6/\sqrt{3})^2 \longleftarrow]$$

$$= [5.7058 \text{ m/s}^2\downarrow] + [4.5 \text{ m/s}^2 \longleftarrow]$$

PROBLEM 16.129 (Continued)

Kinetics:

$$m = 4\ \text{kg} \qquad I_G = \frac{1}{12}mL^2 = \frac{1}{12}(4)(0.75)^2 = 0.1875\ \text{kg}\cdot\text{m}^2$$



B_y
B_x
B
G
D
0.375 m
0.375 m
(4 kg)(9.81)=39.24
D
=
4.8
$\bar{I}_G\alpha_{BD}$
$54\left(1-\frac{1}{\sqrt{3}}\right)$

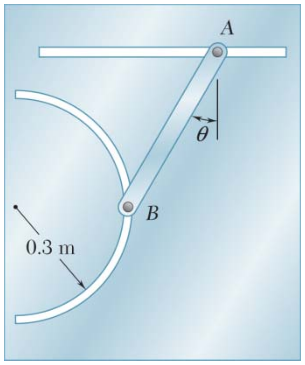
$$+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}:$$

$$-(39.24\ \text{N})(0.375\ \text{m}) + D\cos 30°\ (0.75\ \text{m}) = \left[(0.1875)(56.7846) - (54)(0.375)\left(1-\frac{1}{\sqrt{3}}\right)\right]\ \text{N}\cdot\text{m}$$

$$0.64952\ D = (14.715 + 10.6471 - 8.5587)\ \text{N}\cdot\text{m}$$

$$\mathbf{D} = 25.87\ \text{N} \searrow 60°$$

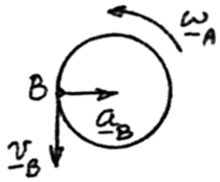
or $\mathbf{D} = 25.9\ \text{N} \searrow 60°$ ◀



PROBLEM 16.130

The motion of the uniform slender rod of length $L = 0.5$ m and mass $m = 3$ kg is guided by pins at A and B that slide freely in frictionless slots, circular and horizontal, cut into a vertical plate as shown. Knowing that at the instant shown the rod has an angular velocity of 3 rad/s counter-clockwise and $\theta = 30°$, determine the reactions at Points A and B.

SOLUTION



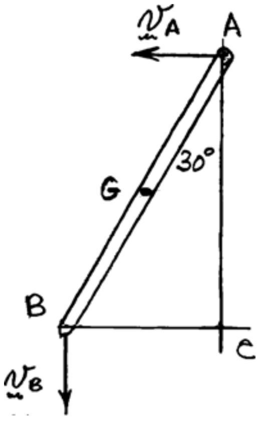
Mass and moment of inertia: $m = 3\text{ kg}$

$$\bar{I} = \frac{1}{12}mL^2 = \frac{1}{12}(3\text{ kg})(0.5\text{ m})^2 = 0.0625\text{ kg}\cdot\text{m}^2$$

Kinematics: $\boldsymbol{\omega} = 3\text{ rad/s} \circlearrowleft$

$$\mathbf{v}_A = v_A \leftarrow \qquad \mathbf{v}_B = v_B \downarrow$$

Locate the instantaneous center C by drawing line $\overline{AC}$ perpendicular to $\mathbf{v}_A$ and line $\overline{BC}$ perpendicular to $\mathbf{v}_B$.



$$\begin{aligned} v_A &= (L\cos 30°)\omega \\ &= (0.5\text{ m}\cos 30°)(3\text{ rad/s}) \\ &= 1.29904\text{ m/s} \end{aligned}$$

$$\begin{aligned} v_B &= (L\sin 30°)\omega \\ &= (0.5\text{ m}\sin 30°)(3\text{ rad/s}) \\ &= 0.75\text{ m/s} \end{aligned}$$

$$\alpha = \alpha \circlearrowleft \qquad \mathbf{a}_A = a_A \rightarrow$$

$$\mathbf{a}_B = \frac{v_B^2}{R} \leftarrow + (a_B)_y \uparrow = \frac{(0.75\text{ m/s})^2}{0.3\text{ m}} \leftarrow (a_B)_y \uparrow$$

$$= 1.875\text{ m/s}^2 \leftarrow + (a_B)_y \uparrow$$

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

where $(\mathbf{a}_{B/A})_t = L\alpha \nwarrow 30° = 0.5\alpha \nwarrow 30°$

and $(\mathbf{a}_{B/A})_n = L\omega^2 \measuredangle 60° = (0.5)(3)^2 \measuredangle 60° = 4.5\text{ m/s}^2 \measuredangle 60°$

Equating the two expressions for $\mathbf{a}_B$ gives

$$1.875\ \text{m/s}^2 \longleftarrow + (a_B)_y \uparrow = \mathbf{a}_A + 0.5\alpha \measuredangle 30^\circ + 4.5\ \text{m/s}^2 \measuredangle 60^\circ$$

$$\longrightarrow: \quad -1.875 = a_A + 0.5\alpha\cos 30^\circ + 2.25$$

$$a_A = -0.5\alpha\cos 30^\circ - 4.125$$

$$\mathbf{a}_G = \mathbf{a}_A + (\mathbf{a}_{G/A})_t + (\mathbf{a}_{G/A})_n$$

$$= (0.5\alpha\cos 30^\circ + 4.125) \longleftarrow + \frac{L}{2}\alpha \searrow 30^\circ + \frac{L}{2}\omega^2 \measuredangle 60^\circ$$

$$= (0.5\alpha\cos 30^\circ + 4.125) \longleftarrow + \frac{0.5}{2}\alpha \searrow 30^\circ + \frac{0.5}{2}(3)^2 \measuredangle 60^\circ$$

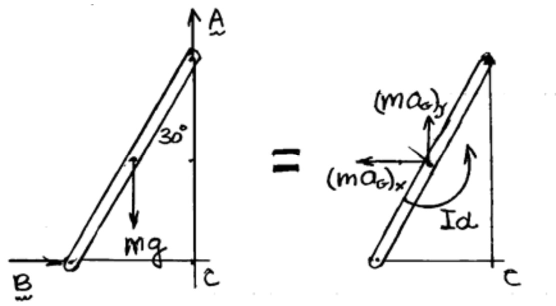
$$= (0.21651\alpha + 3.00) \longleftarrow + (1.94856 - 0.125\alpha) \uparrow$$

Kinetics:

$$m\mathbf{a}_G = (3\ \text{kg})\,\mathbf{a}_G$$

$$= [0.64952\alpha + 9.00] \longleftarrow + [5.8457 - 0.375\alpha] \uparrow$$

$$I\alpha = 0.0625\alpha$$



$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad mg\frac{L}{2}\sin 30^\circ = I\alpha + (ma_G)_x\frac{L}{2}\cos 30^\circ - (ma_G)_y\frac{L}{2}\sin 30^\circ$$

$$(3)(9.81)\frac{0.5}{2}\sin 30^\circ = 0.0625\alpha$$

$$+ (0.64952\alpha + 9.00)\frac{0.5}{2}\cos 30^\circ$$

$$- (5.8457 - 0.375\alpha)\frac{0.5}{2}\sin 30^\circ$$

$$3.67875 = 0.25\alpha + 1.21784$$

$$\alpha = 9.8436\ \text{rad/s}^2$$

$$m\mathbf{a}_G = [(0.64952)(9.8436) + 9.00] \longleftarrow + [5.8457 - (0.375)(9.8436)] \uparrow$$

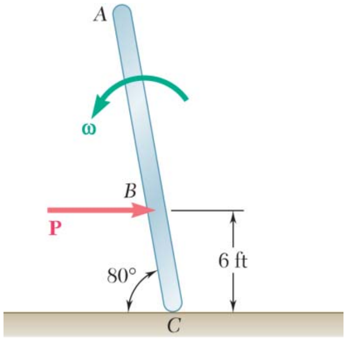
$$= [15.394\ \text{N}] \longleftarrow [2.1544\ \text{N}] \uparrow$$

PROBLEM 16.130 (Continued)

$$\Sigma \mathbf{F} = \Sigma \mathbf{F}_{\text{eff}} = m\mathbf{a}_G: \quad A\uparrow + B \longrightarrow + mg\downarrow = m\mathbf{a}_G$$

$+\uparrow: \quad A - (3 \text{ kg})(9.81 \text{ m/s}) = 2.1544 \text{ N}$ $\qquad \mathbf{A} = 31.6 \text{ N}\uparrow$ ◀

$+\longrightarrow: \quad B = -15.894 \text{ N}$ $\qquad \mathbf{B} = 15.89 \text{ N} \longleftarrow$ ◀



PROBLEM 16.131

At the instant shown, the 20 ft long, uniform 100-lb pole *ABC* has an angular velocity of 1 rad/s counterclockwise and Point *C* is sliding to the right. A 120-lb horizontal force **P** acts at *B*. Knowing the coefficient of kinetic friction between the pole and the ground is 0.3, determine at this instant (*a*) the acceleration of the center of gravity, (*b*) the normal force between the pole and the ground.

SOLUTION

Data: $l = 20 \text{ ft}, \quad W = 100 \text{ lb}, \quad P = 120 \text{ lb}, \quad \mu_k = 0.3$

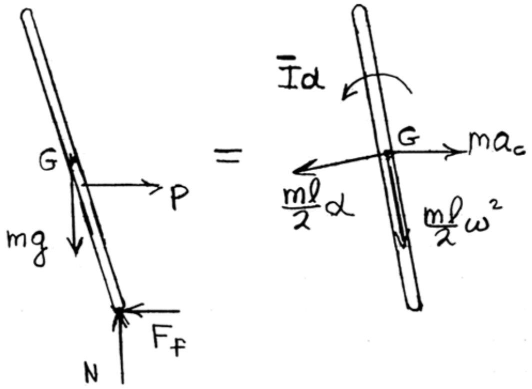
$$\bar{I} = \frac{1}{12}ml^2 = \frac{1}{12}\left(\frac{100}{32.2}\right)(20)^2 = 103.52 \text{ slug}\cdot\text{ft}^2$$

Kinematics: $\boldsymbol{\alpha} = \alpha \circlearrowleft \qquad \mathbf{a}_C = a_C \longrightarrow$

$$\mathbf{a}_G = \mathbf{a}_C + (\mathbf{a}_{G/C})_t + (\mathbf{a}_{C/G})_n$$

$$= [a_C \longrightarrow] + \left[\frac{l}{2}\alpha \swarrow 10°\right] + \left[\frac{l}{2}\omega^2 \searrow 80°\right] \tag{1}$$

Kinetics: Sliding to the right: $F_f = \mu_k N$



$$+\circlearrowleft\Sigma M_G = +\circlearrowleft\Sigma(M_G)_{\text{eff}}: \quad N\frac{l}{2}\sin 10° - F_f\frac{l}{2}\cos 10° + P\left(\frac{l}{2}\cos 10° - h\right) = \bar{I}\alpha$$

$$\bar{I}\alpha - N\frac{l}{2}\sin 10° + \mu_k N\frac{l}{2}\cos 10° = P\left(\frac{l}{2}\cos 10° - h\right)$$

$$103.519\alpha - (10)(\sin 10° - 0.3\cos 10°)N = 120(10\cos 10° - 6) \tag{2}$$

PROBLEM 16.131 (Continued)

$$+\uparrow \Sigma F_y = +\uparrow \Sigma(F_y)_{\text{eff}}: \quad N - mg = -\left(\frac{ml}{2}\alpha\right)\sin 10° - \frac{ml}{2}\omega^2 \cos 10°$$

$$\left(\frac{ml}{2}\sin 10°\right)\alpha + N = mg - \frac{ml}{2}\omega^2 \cos 10°$$

$$\left[\left(\frac{100}{32.2}\right)(10)\sin 10°\right]\alpha + N = (100) - \left(\frac{100}{32.2}\right)(10)(1)^2 \cos 10° \qquad (3)$$

Solving Eqs. (2) and (3) simultaneously,

$$\alpha = 3.8909 \text{ rad/s}^2 \qquad N = 48.433 \text{ lb}$$

$$\xrightarrow{+} \Sigma F_x = \xrightarrow{+} \Sigma(F_x)_{\text{eff}}: \quad P - F_f = ma_C - \frac{ml}{2}\alpha\cos 10° + \frac{ml}{2}\omega^2 \sin 10°$$

$$ma_C = P - \mu_k N + \frac{ml}{2}\alpha\cos 10° - \frac{ml}{2}\omega^2 \sin 10°$$

$$\frac{100}{32.2}a_C = 120 - (0.3)(48.433) + \left(\frac{100}{32.2}\right)(10)(3.8909)\cos 10° - \left(\frac{100}{32.2}\right)(10)(1)^2 \sin 10°$$

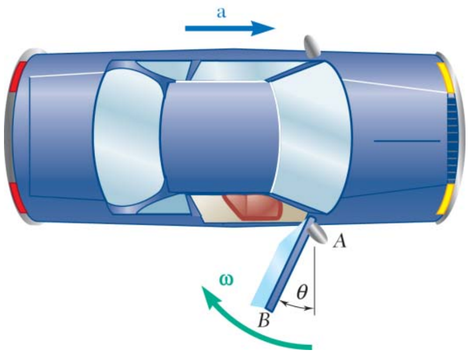
$$a_C = 70.542 \text{ ft/s}^2$$

Using Eq. (1), $\mathbf{a}_G = 70.542 + [(10)(3.8909) \swarrow 10°] + [(10)(1)^2 \searrow 80°]$

$$(a_G)_x = 70.542 - 38.909\cos 10° + 10\cos 80°$$
$$= 33.961 \text{ ft/s}^2$$
$$(a_G)_y = -38.909\sin 10° - 10\sin 80°$$
$$= -16.605 \text{ ft/s}^2$$
$$a_G = \sqrt{(33.961)^2 + (16.605)^2}$$
$$= 37.803 \text{ ft/s}^2$$
$$\tan\beta = \frac{16.605}{33.961} \qquad \beta = 26.055°$$

(*a*) Acceleration at Point *G*. $\mathbf{a}_G = 37.8 \text{ ft/s}^2 \searrow 26.1°$ ◀

(*b*) Normal force. $\mathbf{N} = 48.4 \text{ lb} \uparrow$ ◀

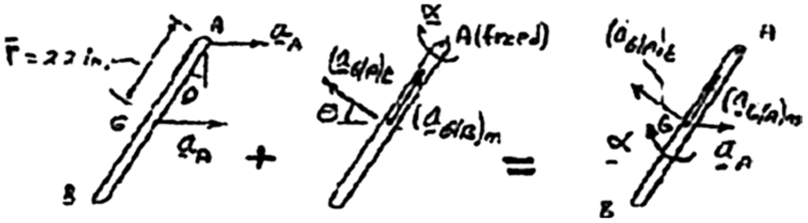


PROBLEM 16.132

A driver starts his car with the door on the passenger's side wide open $(\theta = 0)$. The 80-lb door has a centroidal radius of gyration $\bar{k} = 12.5$ in., and its mass center is located at a distance $r = 22$ in. from its vertical axis of rotation. Knowing that the driver maintains a constant acceleration of 6 ft/s^2, determine the angular velocity of the door as it slams shut $(\theta = 90°)$.

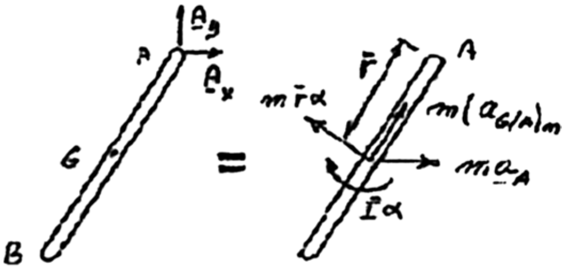
SOLUTION

Kinematics: $$\mathbf{a} = a_A$$


$\bar{r} = 22$ in.
A
a_A
θ
G
a_A
B
+
α
A(fixed)
$(a_{G/A})_t$
θ
$(a_{G/B})_n$
=
$(a_{G/A})_t$
A
$(a_{G/A})_n$
G
α
a_A
B

$$(\mathbf{a}_{G/A})_t = \bar{r}\alpha \searrow \theta$$

Kinetics:


A_y
A
A_x
G
B
=
$\bar{r}$
A
$m\bar{r}\alpha$
$m(a_{G/A})_n$
$m a_A$
$\bar{I}\alpha$

$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad 0 = \bar{I}\alpha + (m\bar{r}\alpha)\bar{r} - m a_A(\bar{r}\cos\theta)$$

$$m\bar{k}^2\alpha + m\bar{r}^2\alpha = m a_A \bar{r}\cos\theta$$

$$\alpha = \frac{a_A\bar{r}}{\bar{k}^2 + \bar{r}^2}\cos\theta$$

Setting $\alpha = \omega\frac{d\omega}{d\theta}$, and using $\bar{r} = \frac{22}{12}$ ft, $\bar{k} = \frac{12.5}{12}$ ft

PROBLEM 16.132 (Continued)

$$\omega\frac{d\omega}{d\theta}=\frac{\left(\frac{22}{12}\text{ ft}\right)a_A}{\left[\left(\frac{12.5}{12}\text{ ft}\right)^2+\left(\frac{22}{12}\text{ ft}\right)^2\right]}\cos\theta$$

$$=0.41234a_A\cos\theta$$

$$\omega d\omega=0.41234a_A\cos\theta\,d\theta$$

$$\int_0^{\omega_f}\omega d\omega=\int_0^{\pi/2}(0.4124a_A)\cos\theta\,d\theta$$

$$\left|\frac{1}{2}\omega^2\right|_0^{\omega_f}=0.41234a_A|\sin\theta|_0^{\pi/2}$$

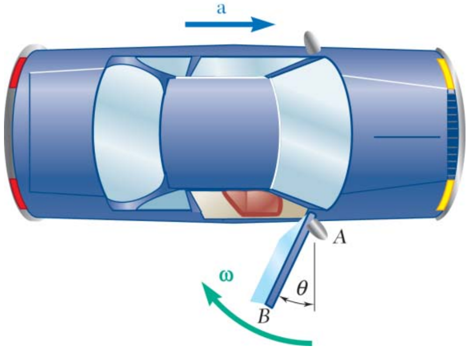
$$\omega_f^2=0.82468a_A \qquad (1)$$

Given data: $\mathbf{a}_A=6\text{ ft/s}^2\longrightarrow$

$$\omega_f^2=0.82468(6)$$

$$=4.948\text{ rad}^2/\text{s}^2$$

$\omega_f=2.22$ rad/s ◀



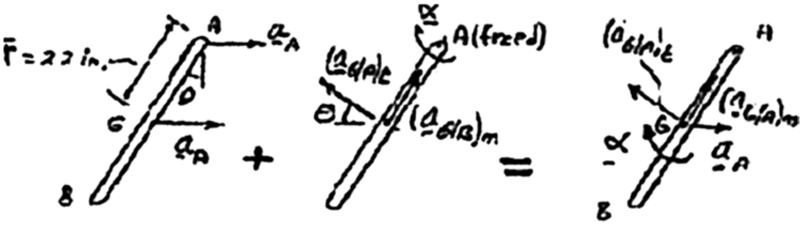
PROBLEM 16.133

For the car of Problem 16.132, determine the smallest constant acceleration that the driver can maintain if the door is to close and latch, knowing that as the door hits the frame its angular velocity must be at least 2 rad/s for the latching mechanism to operate.

SOLUTION

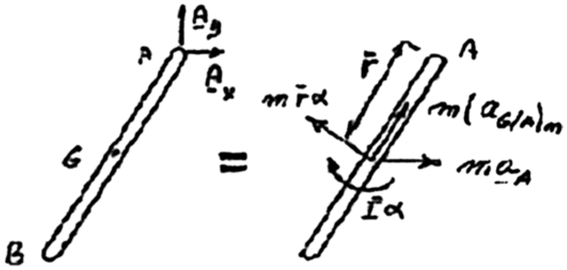
Kinematics:

$$\mathbf{a} = a_A$$



$$(\mathbf{a}_{G/A})_t = \bar{r}\alpha \searrow \theta$$

Kinetics:



$$\overset{+}{\circlearrowright}\Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad 0 = \bar{I}\alpha + (m\bar{r}\alpha)\bar{r} - ma_A(\bar{r}\cos\theta)$$

$$m\bar{k}^2\alpha + m\bar{r}^2\alpha = ma_A\bar{r}\cos\theta$$

$$\alpha = \frac{a_A\bar{r}}{\bar{k}^2 + \bar{r}^2}\cos\theta$$

Setting $\alpha = \omega\frac{d\omega}{d\theta}$, and using $\bar{r} = \frac{22}{12}$ ft, $\bar{k} = \frac{12.5}{12}$ ft

$$\omega\frac{d\omega}{d\theta} = \frac{\left(\frac{22}{12}\text{ ft}\right)a_A}{\left[\left(\frac{12.5}{12}\text{ ft}\right)^2 + \left(\frac{22}{12}\text{ ft}\right)^2\right]}\cos\theta$$

$$= 0.41234 a_A \cos\theta$$

$$\omega\, d\omega = 0.41234 a_A \cos\theta\, d\theta$$

PROBLEM 16.133 (Continued)

$$\int_0^{\omega_f} \omega d\omega = \int_0^{\pi/2} (0.4124 a_A) \cos\theta \, d\theta$$

$$\left| \frac{1}{2}\omega^2 \right|_0^{\omega_f} = 0.41234 a_A \,|\sin\theta|_0^{\pi/2}$$

$$\omega_f^2 = 0.82468 a_A \qquad (1)$$

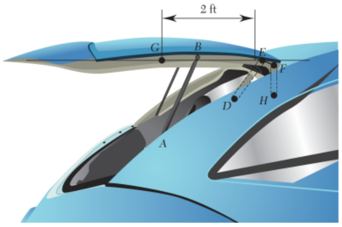
Given data: $\omega_f = 2$ rad/s

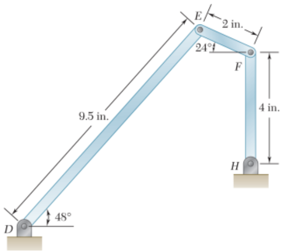
Eq. (1): $\omega_f^2 = 0.82468 a_A$

$$(2)^2 = 0.82468 a_A$$

$$a_A = 4.85 \text{ ft/s}^2$$

$\mathbf{a}_A = 4.85 \text{ ft/s}^2 \longrightarrow$ ◀





PROBLEM 16.134

The hatchback of a car is positioned as shown to help determine the appropriate size for a damping mechanism *AB*. The weight of the door is 40 lbs, and its mass moment of inertia about the center of gravity *G* is 15 lb·ft·s^2. The linkage DEFH controls the motion of the hatch and is shown in more detail in Fig. b. Assume that the mass of the links *DE*, *EF*, and *FH* are negligible compared to the mass of the door. With *AB* removed, determine (a) the initial angular acceleration of the 40 lb door as it is released from rest, (b) the force on link *FH*.

SOLUTION

Given:

$$m = \frac{40 \text{ lb}}{32.2 \text{ ft/s}^2} = 1.242 \text{ slugs}$$

$$\bar{I} = 15 \text{ lb•ft•s}^2$$

$$\omega_{DE} = \omega_{EF} = \omega_{FH} = 0$$

Geometry (see Free Body):

$$a = \frac{2}{12}\cos 24° \Rightarrow 0.1523 \text{ ft}$$

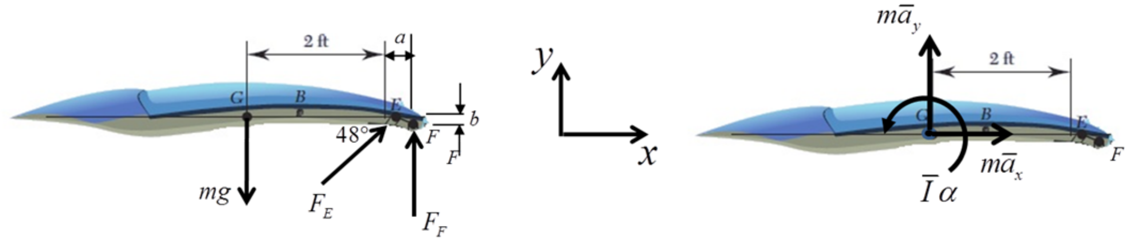
$$b = \frac{2}{12}\sin 24° \Rightarrow 0.06779 \text{ ft}$$

$$\mathbf{r}_{F/H} = \frac{1}{3}\mathbf{j}$$

$$\mathbf{r}_{\text{E/F}} = -a\mathbf{i} + b\mathbf{j} \Rightarrow \mathbf{r}_{\text{E/F}} = -0.1523\mathbf{i} + 0.06779\mathbf{j}$$

$$\mathbf{r}_{\text{E/D}} = \frac{9.5}{12}\cos 48°\mathbf{i} + \frac{9.5}{12}\sin 48°\mathbf{j} \Rightarrow \mathbf{r}_{\text{E/D}} = 0.5297\mathbf{i} + 0.5883\mathbf{j}$$

Free Body Diagram of Door:



Kinetics:

$$\sum F_x = m\bar{a}_x$$

$$F_E \cos 48° = m\bar{a}_x \qquad (1)$$

$$\sum F_y = m\bar{a}_y$$

$$-mg + F_E \sin 48° + F_F = m\bar{a}_y \qquad (2)$$

$$\sum M_G = \bar{I}\alpha_{EF}$$

$$2.1523F_F + 0.06779F_E \cos 48° + 2F_E \sin 48° = 15\alpha_{EF} \qquad (3)$$

PROBLEM 16.134 (Continued)

Kinematics:

$$\mathbf{a}_F = \cancel{\mathbf{a}_H} - \cancel{\omega^2_{FH}}\,\mathbf{r}_{F/\mathrm{H}} + \boldsymbol{\alpha}_{FH} \times \mathbf{r}_{F/\mathrm{H}}$$

$$= \alpha_{FH}\mathbf{k} \times \frac{1}{3}\mathbf{j} \;\Rightarrow \mathbf{a}_F = -\frac{1}{3}\alpha_{FH}\mathbf{i}$$

$$\mathbf{a}_E = \mathbf{a}_F - \cancel{\omega^2_{EF}}\,\mathbf{r}_{E/F} + \boldsymbol{\alpha}_{EF} \times \mathbf{r}_{E/F}$$

$$= -\frac{1}{3}\alpha_{FH}\mathbf{i} + \alpha_{EF}\mathbf{k} \times (-a\mathbf{i} + b\mathbf{j})$$

$$= -\left(\frac{1}{3}\alpha_{FH} + b\alpha_{EF}\right)\mathbf{i} - a\alpha_{EF}\mathbf{j} \tag{4}$$

$$\mathbf{a}_E = \cancel{\mathbf{a}_D} - \cancel{\omega^2_{DE}}\,\mathbf{r}_{E/\mathrm{D}} + \boldsymbol{\alpha}_{DE} \times \mathbf{r}_{E/\mathrm{D}}$$

$$= \alpha_{DE}\mathbf{k} \times (0.5297\mathbf{i} + 0.5883\mathbf{j})$$

$$= -0.5883\alpha_{DE}\mathbf{i} + 0.5297\alpha_{DE}\mathbf{j} \tag{5}$$

Equating (4) and (5):

$$-\left(\frac{1}{3}\alpha_{FH} + b\alpha_{EF}\right)\mathbf{i} - a\alpha_{EF}\mathbf{j} = -0.5883\alpha_{DE}\mathbf{i} + 0.5297\alpha_{DE}\mathbf{j}$$

Equating **i** components:

$$\frac{1}{3}\alpha_{FH} + b\alpha_{EF} = 0.5883\alpha_{DE} \tag{6}$$

Equating **j** components:

$$\alpha_{DE} = -\frac{a}{0.5297}\alpha_{EF} \tag{7}$$

Relative Acceleration:

$$\bar{\mathbf{a}}_G = \mathbf{a}_E - \cancel{\omega^2_{EF}}\,\mathbf{r}_{G/\mathrm{E}} + \boldsymbol{\alpha}_{EF} \times \mathbf{r}_{G/\mathrm{E}}$$

$$\bar{a}_x\mathbf{i} + \bar{a}_y\mathbf{j} = -0.5883\alpha_{DE}\mathbf{i} + 0.5297\alpha_{DE}\mathbf{j} + \alpha_{EF}\mathbf{k} \times (-2\mathbf{i})$$

Equating **i** components:

$$\bar{a}_x = -0.5883\alpha_{DE} \tag{8}$$

Equating **j** components:

$$\bar{a}_y = 0.5297\alpha_{DE} - 2\alpha_{EF} \tag{9}$$

Substitute (1)→(2):

$$-40 + m\bar{a}_x \tan 48° + F_F = m\bar{a}_y \tag{10}$$

Substitute (8)(9)→(10):

$$-40 - 0.5883m\alpha_{DE} \tan 48° + F_F = m(0.5297\alpha_{DE} - 2\alpha_{EF}) \tag{11}$$

Rearrange (11):

$$F_F = 1.1831m\alpha_{DE} - 2m\alpha_{EF} + 40 \tag{12}$$

Substitute (7)→(12):

$$F_F = -2.3401m\alpha_{EF} + 40 \tag{13}$$

Substitute (8)→(1):

$$F_E = -\frac{m0.5883\alpha_{DE}}{\cos 48°} \tag{14}$$

Substitute (7)→(14):

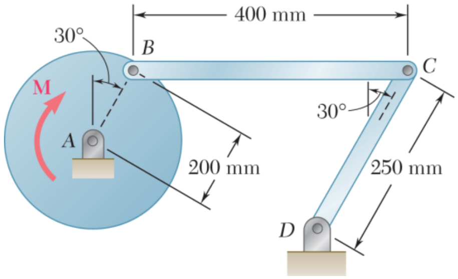
$$F_E = 0.2527m\alpha_{EF} \tag{15}$$

Substitute (13)(15)→(3):

$$2.1523(-2.3401m\alpha_{EF} + 40) + 0.06779(0.2527m\alpha_{EF})\cos 48° + 2(0.2527m\alpha_{EF})\sin 48° = 15\alpha_{EF} \tag{16}$$

(a) Solving (16): $\boldsymbol{\alpha}_{EF} = 4.144$ rad/s ↺ ◀

Substitute into (13): $F_F = -2.3401m(4.144) + 40$ $F_F = 27.954$ lb ◀



PROBLEM 16.135

The 6-kg rod BC connects a 10-kg disk centered at A to a 5-kg rod CD. The motion of the system is controlled by the couple $\mathbf{M}$ applied to disk A. Knowing that at the instant shown disk A has an angular velocity of 36 rad/s clockwise and no angular acceleration, determine (*a*) the couple $\mathbf{M}$, (*b*) the components of the force exerted at C on rod BC.

SOLUTION

Kinematics: *Velocity analysis*. $\boldsymbol{\omega}_{AB} = 36 \text{ rad/s} \circlearrowright$

Disk AB: $\mathbf{v}_B = \overline{AB}\ \omega_{AB} \nwarrow 30^\circ = (0.200)(36) \nwarrow 30^\circ = 7.2 \text{ m/s} \nwarrow 30^\circ$

Rod BC: $\mathbf{v}_C = v_C \searrow 30^\circ$

Since $\mathbf{v}_C$ is parallel to $\mathbf{v}_B$, bar BC is in translation.

$$\mathbf{v}_C = 7.2 \text{ m/s} \nwarrow 30^\circ \qquad \boldsymbol{\omega}_{BC} = 0$$

Rod CD:

$$\omega_{CD} = \frac{v_C}{l_{CD}} = \frac{7.2 \text{ m/s}}{0.25 \text{ m}} = 28.8 \text{ rad/s}$$

$$\boldsymbol{\omega}_{CD} = 28.8 \text{ rad/s} \circlearrowright$$

Acceleration analysis: $\boldsymbol{\alpha}_{AB} = 0$

Disk AB:

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$

$$= 0 - (36)^2(0.2) \swarrow 60^\circ = 259.2 \text{ m/s}^2 \swarrow 60^\circ$$

Rod BC:

$$\boldsymbol{\alpha}_{BC} = \alpha_{BC} \circlearrowleft$$

$$\mathbf{a}_C = \mathbf{a}_B + (\mathbf{a}_{C/B})_t - \omega_{BC}^2 \mathbf{r}_{C/B}$$

$$\mathbf{a}_C = 259.2 \swarrow 60^\circ + 0.4\alpha_{AB} \uparrow + 0 \qquad (1)$$

Rod CD:

$$\boldsymbol{\alpha}_{CD} = \alpha_{CD} \circlearrowleft$$

$$\boldsymbol{\alpha}_C = (\mathbf{a}_{C/D})_t - \omega_{CD}^2 \mathbf{r}_{C/D} = [0.25\alpha_{CD} \swarrow 30^\circ] - [(28.8)^2(0.25) \swarrow 60^\circ]$$

$$\mathbf{a}_C = [0.25\alpha_{CD} \swarrow 30^\circ] - [207.36 \swarrow 60^\circ] \qquad (2)$$

Equate components of two expressions (1) and (2) for $\mathbf{a}_C$.

$$\xrightarrow{+}: \quad -259.2\cos 60^\circ = -0.25\alpha_{CD} \cos 30^\circ - 207.36 \cos 60^\circ$$

$$\alpha_{CD} = 119.719 \text{ rad/s}^2 \qquad \boldsymbol{\alpha}_{CD} = 119.719 \text{ rad/s}^2 \circlearrowleft$$

$$+\uparrow: \quad -259.2\sin 60^\circ + 0.4\alpha_{BC} = 0.25\alpha_{CD} \sin 30^\circ - 207.36\sin 60^\circ$$

$$\alpha_{BC} = 149.649 \text{ rad/s}^2 \qquad \boldsymbol{\alpha}_{BC} = 149.649 \text{ rad/s}^2 \circlearrowleft$$

PROBLEM 16.135 (Continued)

Accelerations of the mass centers.

Disk *AB*: $\bar{\mathbf{a}}_{AB} = \mathbf{a}_A = 0$

Rod *BC*: Mass center at Point *P*. $\mathbf{r}_{P/B} = (0.2\text{ m}) \rightarrow$

$$\mathbf{a}_P = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{P/B} - \omega_{BC}^2 \mathbf{r}_{P/B}$$

$$= [259.2 \swarrow 60°] + [0.2\alpha_{BC} \uparrow] + 0 \quad = [259.2 \swarrow 60°] + [29.9298 \uparrow]$$

Rod *CD*: Mass center at Point *Q*. $\mathbf{r}_{Q/D} = 0.125\text{ m} \nearrow 60°$

$$\mathbf{a}_Q = \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{Q/D} - \omega_{CD}^2 \mathbf{r}_{Q/D}$$

$$= [0.125\ \alpha_{CD} \nwarrow 30°] - [(28.8)^2 (0.125) \swarrow 60°]$$

$$= [14.964875 \nwarrow 30°] - [103.60 \swarrow 60°]$$

Masses: $m_{AB} = 10\text{ kg}, \quad m_{BC} = 6\text{ kg}, \quad m_{CD} = 5\text{ kg}$

Inertial terms at mass centers.

Disk *AB*: $m_{AB}\mathbf{a}_A = 0$

Rod *BC*: $m_{BC}\mathbf{a}_P = 6\mathbf{a}_P = [1555.2\text{ N} \swarrow 60°] + [179.58\text{ N} \uparrow]$

Rod *CD*: $m_{CD}\mathbf{a}_a = 5\mathbf{a}_Q = [74.82\text{ N} \nwarrow 30°] + [518\text{ N} \swarrow 60°]$

Moments of inertia:

Disk *AB*: $\bar{I}_{AB} = \frac{1}{2} m_{AB} r_{AB}^2 = \frac{1}{2}(10)(0.2)^2 = 0.2\text{ kg}\cdot\text{m}^2$

Rod *BC*: $\bar{I}_{BC} = \frac{1}{12} m_{BC} l_{BC}^2 = \frac{1}{12}(6)(0.4)^2 = 0.08\text{ kg}\cdot\text{m}^2$

Rod *CD*: $\bar{I}_{CD} = \frac{1}{12} m_{CD} l_{CD}^2 = \frac{1}{12}(5)(0.25)^2 = 0.0260417\text{ kg}\cdot\text{m}^2$

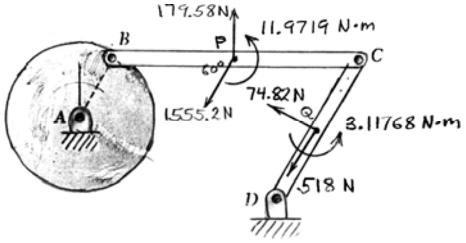
Effective couples at mass centers.

Disk *AB*: $\bar{I}_{AB}\boldsymbol{\alpha}_{AB} = 0$

Rod *BC*: $\bar{I}_{BC}\boldsymbol{\alpha}_{BC} = (0.08)(149.649) \circlearrowleft = 11.97192\text{ N}\cdot\text{m} \circlearrowleft$

Rod *CD*: $\bar{I}_{CD}\boldsymbol{\alpha}_{CD} = (0.0260417)(119.719) \circlearrowleft = 3.11768\text{ N}\cdot\text{m} \circlearrowleft$

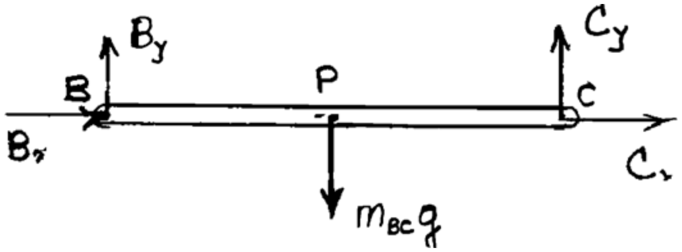
Summary of effective forces and couples



PROBLEM 16.135 (Continued)

Kinetics

Rod *BC*:


B_y
C_y
B
P
C
B_x
C_x
m_BC g

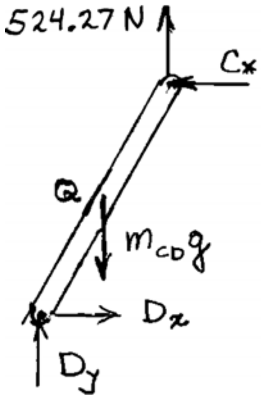
$$+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad l_{BC}C_y - \frac{1}{2}l_{BC}mg = \Sigma(M_B)_{\text{eff}}$$

$$(0.4\text{ m})\,C_y - (0.2\text{ m})(6\text{ kg})(9.81\text{ m/s}^2)$$

$$= 11.9719\text{ N}\cdot\text{m} + (0.2\text{ m})(179.58\text{ N})$$

$$-(0.2\text{ m})(1555.2\text{ N})\sin 60°$$

$$C_y = -524.27\text{ N}$$


524.27 N
C_x
Q
m_CD g
D_x
D_y

Rod *CD*: $+\circlearrowleft \Sigma M_D = \Sigma(M_D)_{\text{eff}}: \quad C_x l_{CD}\cos 30° + (524.27\text{ N})(0.125\text{ m})$

$$-m_{CD}g(0.0625\text{ m}) = \Sigma(M_D)_{\text{eff}}:$$

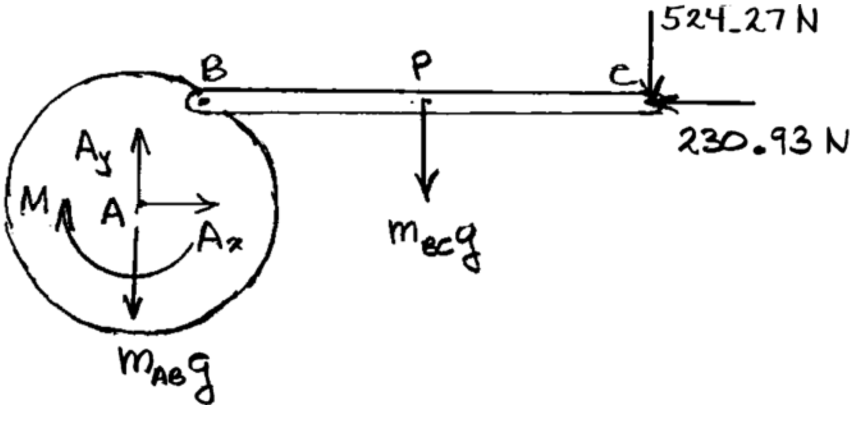
$$(0.25\text{ m})C_x\cos 30° + 65.534\text{ N}\cdot\text{m}$$

$$-(5\text{ kg})(9.81\text{ m/s}^2)(0.0625\text{ m})$$

$$= 3.11768\text{ N}\cdot\text{m} + (0.125\text{ m})(74.82\text{ N})$$

$$C_x = -230.93\text{ N}$$

Disk *AB* and rod *BC*:


524.27 N
B
P
C
230.93 N
A_y
M
A
A_x
m_BC g
m_AB g

PROBLEM 16.135 (Continued)

$$+\circlearrowleft \Sigma M_A = \Sigma (M_A)_{\text{eff}}: \quad -M - (524.27)(0.4 + 0.2\sin 30^\circ) + (230.93)(0.2\cos 30^\circ)$$

$$- (6\text{ kg})(9.81\text{ m/s}^2)(0.2 + 0.2\sin 30^\circ)$$

$$= 11.9719 + (179.58)(0.2 + 0.2\sin 30^\circ)$$

$$- (1555.2\cos 30^\circ)(0.2)$$

$$-M - 262.135 + 40.0 - 17.658$$

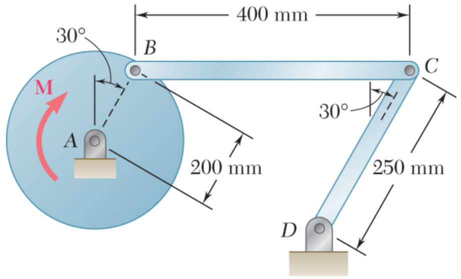
$$= 11.9719 + 53.874 - 269.369$$

$$-M = 36.27\text{ N}\cdot\text{m}$$

(*a*) *Couple applied to disk A.* $\mathbf{M} = 36.3\text{ N}\cdot\text{m} \circlearrowleft$ ◀

(*b*) *Components of force exerted at C* on rod *BC.*

$\mathbf{C} = 231\text{ N} \longleftarrow + 524\text{ N} \downarrow$ ◀



PROBLEM 16.136

The 6-kg rod *BC* connects a 10-kg disk centered at *A* to a 5-kg rod *CD*. The motion of the system is controlled by the couple **M** applied to disk *A*. Knowing that at the instant shown disk *A* has an angular velocity of 36 rad/s clockwise and an angular acceleration of 150 rad/s^2 counterclockwise, determine (*a*) the couple **M**, (*b*) the components of the force exerted at *C* on rod *BC*.

SOLUTION

Kinematics: *Velocity analysis.* $\boldsymbol{\omega}_{AB} = 36 \text{ rad/s} \circlearrowright$

Disk *AB*: $\mathbf{v}_B = \overline{AB}\omega_{AB} \searrow 30^\circ = (0.200)(36) \searrow 30^\circ = 7.2 \text{ m/s} \searrow 30^\circ$

Rod *BC*: $\mathbf{v}_C = v_C \searrow 30^\circ$

Since $\mathbf{v}_C$ is parallel to $\mathbf{v}_B$, bar *BC* is in translation

$$\mathbf{v}_C = 7.2 \text{ m/s} \searrow 30^\circ \qquad \boldsymbol{\omega}_{BC} = 0$$

Rod *CD*: $$\omega_{CD} = \frac{v_C}{l_{CD}} = \frac{7.2 \text{ m/s}}{0.25 \text{ m}} = 28.8 \text{ rad/s}$$

$$\boldsymbol{\omega}_{CD} = 28.8 \text{ rad/s} \circlearrowright$$

Acceleration analysis. $\boldsymbol{\alpha}_{AB} = 0$

Disk *AB*: $$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$

$$= [(150)(0.2) \nwarrow 30^\circ] - [(36)^2(0.2) \swarrow 60^\circ]$$

$$= [30 \text{ m/s}^2 \nwarrow 30^\circ] + [259.2 \text{ m/s}^2 \swarrow 60^\circ]$$

Rod *BC*: $\boldsymbol{\alpha}_{BC} = 150 \text{ rad/s}^2 \circlearrowleft$

$$\mathbf{a}_C = \mathbf{a}_B + (\mathbf{a}_{C/B})_t - \omega_{BC}^2 \mathbf{r}_{C/B}$$

$$\mathbf{a}_C = [30 \nwarrow] + [259.2 \swarrow 60^\circ] + [0.4\alpha_{AB}] + 0 \qquad (1)$$

Rod *CD*: $\boldsymbol{\alpha}_{CD} = \alpha_{CD} \circlearrowleft$

$$\mathbf{a}_C = (\mathbf{a}_{C/D})_t - \omega_{CD}^2 \mathbf{r}_{C/D} = [0.25\alpha_{CD} \nwarrow 30^\circ] - [(28.8)^2(0.25) \swarrow 60^\circ]$$

$$\mathbf{a}_C = [0.25\alpha_{CD} \nwarrow 30^\circ] - [207.36 \swarrow 60^\circ] \qquad (2)$$

PROBLEM 16.136 (Continued)

Equate components of two expressions (1) and (2) for $\mathbf{a}_C$.

$$\xrightarrow{+}: \quad -30\cos 30° - 259.2\cos 60° = -0.25\alpha_{CD}\cos 30° - 207.36\cos 60°$$

$$\alpha_{CD} = 239.719 \text{ rad/s}^2 \qquad \boldsymbol{\alpha}_{CD} = 239.719 \text{ rad/s}^2 \circlearrowleft$$

$$+\!\uparrow: \quad 30\cos 30° - 259.2\sin 60° + 0.4\alpha_{BC} = 0.25\alpha_{CD}\sin 30° - 207.36\sin 60°$$

$$\alpha_{BC} = 149.649 \text{ rad/s}^2 \qquad \boldsymbol{\alpha}_{BC} = 149.649 \text{ rad/s}^2 \circlearrowleft$$

Accelerations of the mass centers.

Disk *AB*: $\bar{\mathbf{a}}_{AB} = \mathbf{a}_A = 0$

Rod *BC*: Mass center at Point *P*. $\mathbf{r}_{P/B} = (0.2 \text{ m}) \longrightarrow$

$$\mathbf{a}_P = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{P/B} - \omega_{BC}^2 \mathbf{r}_{P/B}$$

$$= [30 \nwarrow 30°] + [259.2 \swarrow 60°] + [0.2\alpha_{BC} \uparrow] + 0$$

$$= [30 \nwarrow 30°] + [259.2 \swarrow 60°] + [29.9298 \uparrow]$$

Rod *CD*: Mass center at Point *Q*. $\mathbf{r}_{Q/D} = 0.125 \text{ m} \nearrow 60°$

$$\mathbf{a}_Q = \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{Q/D} - \omega_{CD}^2 \mathbf{r}_{Q/D}$$

$$= [0.125\,\alpha_{CD} \nwarrow 30°] - [(28.8)^2(0.125) \swarrow 60°]$$

$$= [29.964875 \nwarrow 30°] - [103.60 \swarrow 60°]$$

Masses: $m_{AB} = 10 \text{ kg}, \quad m_{BC} = 6 \text{ kg}, \quad m_{CD} = 5 \text{ kg}$

Inertial terms at mass centers.

Disk *AB*: $m_{AB}\mathbf{a}_A = 0$

Rod *BC*: $m_{BC}\mathbf{a}_P = 6\mathbf{a}_P = [180 \text{ N} \nwarrow 30°] + [1555.2 \text{ N} \swarrow 60°] + [179.58 \text{ N} \uparrow]$

Rod *CD*: $m_{CD}\mathbf{a}_a = 5\mathbf{a}_Q = [149.82 \text{ N} \nwarrow 30°] + [518 \text{ N} \swarrow 60°]$

Moments of inertia:

Disk *AB*: $\bar{I}_{AB} = \frac{1}{2} m_{AB} r_{AB}^2 = \frac{1}{2}(10)(0.2)^2 = 0.2 \text{ kg}\cdot\text{m}^2$

Rod *BC*: $\bar{I}_{BC} = \frac{1}{12} m_{BC} l_{BC}^2 = \frac{1}{12}(6)(0.4)^2 = 0.08 \text{ kg}\cdot\text{m}^2$

Rod *CD*: $\bar{I}_{CD} = \frac{1}{12} m_{CD} l_{CD}^2 = \frac{1}{12}(5)(0.25)^2 = 0.0260417 \text{ kg}\cdot\text{m}^2$

PROBLEM 16.136 (Continued)

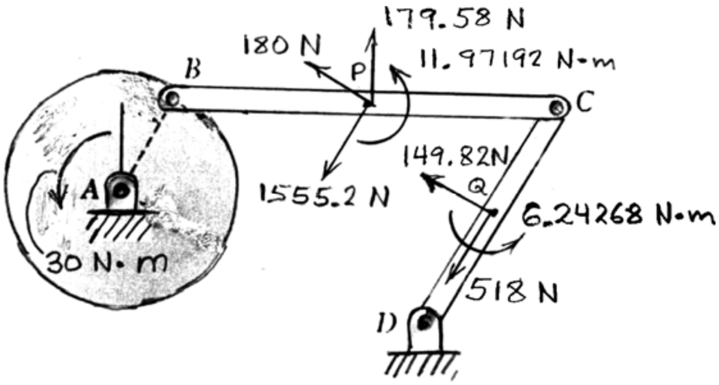
Effective couples at mass centers.

Disk AB: $\bar{I}_{AB}\boldsymbol{\alpha}_{AB} = (0.2)(150) \circlearrowleft = 30\ \text{N}\cdot\text{m} \circlearrowleft$

Rod BC: $\bar{I}_{BC}\boldsymbol{\alpha}_{BC} = (0.08)(149.649) \circlearrowleft = 11.97192\ \text{N}\cdot\text{m} \circlearrowleft$

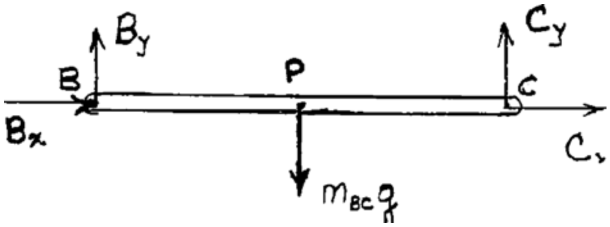
Rod CD: $\bar{I}_{CD}\boldsymbol{\alpha}_{CD} = (0.0260417)(239.719) \circlearrowleft = 6.24268\ \text{N}\cdot\text{m} \circlearrowleft$

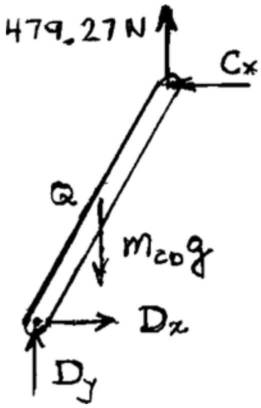
Summary of effective forces and couples.


179.58 N
180 N
P
11.97192 N-m
B
C
149.82N
Q
A
1555.2 N
6.24268 N•m
30 N• m
518 N
D

Kinetics

Rod BC:


B_y
B
P
C_y
C
B_x
C_x
$m_{BC}g$


479.27 N
C_x
Q
$m_{CD}g$
D_x
D_y

$$+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}:\quad l_{BC}C_y - \frac{1}{2}l_{BC}mg = \Sigma(M_B)_{\text{eff}}$$

$$(0.4\ \text{m})C_y - (0.2\ \text{m})(6\ \text{kg})(9.81\ \text{m/s}^2)$$

$$= (0.2\ \text{m})(180\ \text{N}\sin 30°) + 11.9719\ \text{N}\cdot\text{m} + (0.2\ \text{m})(179.58\ \text{N})$$

$$-(0.2\ \text{m})(1555.2\ \text{N})\sin 60°$$

$$C_y = -479.27\ \text{N}$$

Rod CD: $+\circlearrowleft \Sigma M_D = \Sigma(M_D)_{\text{eff}}:\quad C_x l_{CD}\cos 30° + (479.27\ \text{N})(0.125\ \text{m})$

$$-m_{CD}g(0.0625\ \text{m}) = \Sigma(M_D)_{\text{eff}}:$$

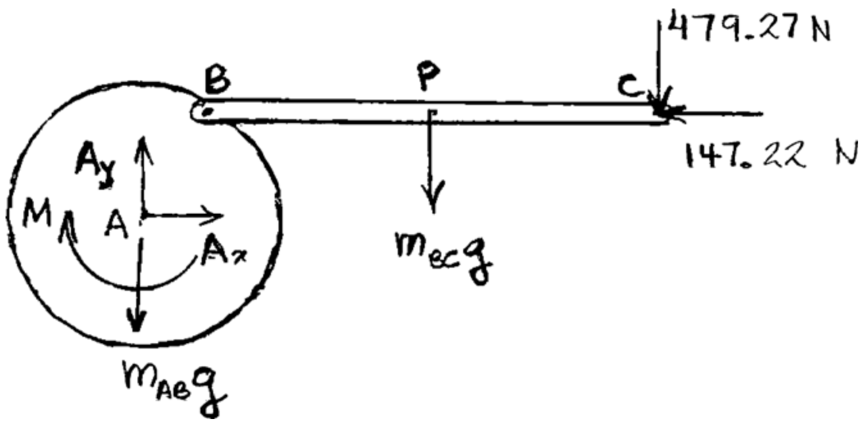
$$(0.25\ \text{m})C_x\cos 30° + 59.909\ \text{N}\cdot\text{m} - (5\ \text{kg})(9.81\ \text{m/s}^2)(0.0625\ \text{m})$$

$$= 6.24268\ \text{N}\cdot\text{m} + (0.125\ \text{m})(149.82\ \text{N})$$

$$C_x = -147.22\ \text{N}$$

PROBLEM 16.136 (Continued)

Disk *AB* and rod *BC*:



$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad -M - (479.27)(0.4 + 0.2\sin 30°) + (147.22)(0.2\cos 30°)$$

$$- (6\text{ kg})(9.81\text{ m/s}^2)(0.2 + 0.2\sin 30°)$$

$$= (180)(0.2 + 0.2\sin 30°) + 11.9719 + (179.58)(0.2 + 0.2\sin 30°)$$

$$- (1555.2\cos 30°)(0.2)$$

$$-M - 239.635 + 25.50 - 17.658$$

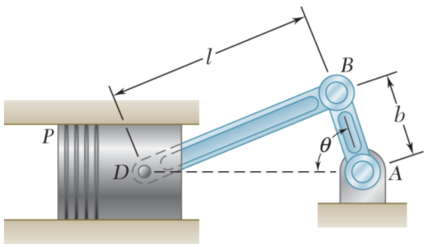
$$= 54 + 11.9719 + 53.874 - 269.369$$

$$-M = 82.27\text{ N}\cdot\text{m}$$

(*a*) *Couple applied to disk A.* $\mathbf{M} = 82.3\text{ N}\cdot\text{m} \circlearrowleft$ ◀

(*b*) *Components of force exerted at C* on rod *BC*.

$\mathbf{C} = 147.2\text{ N} \longleftarrow + 479\text{ N}\downarrow$ ◀

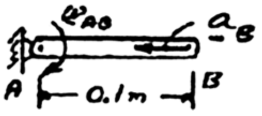


PROBLEM 16.137

In the engine system shown $l = 250$ mm and $b = 100$ mm. The connecting rod BD is assumed to be a 1.2-kg uniform slender rod and is attached to the 1.8-kg piston P. During a test of the system, crank AB is made to rotate with a constant angular velocity of 600 rpm clockwise with no force applied to the face of the piston. Determine the forces exerted on the connecting rod at B and D when $\theta = 180°$. (Neglect the effect of the weight of the rod.)

SOLUTION

Kinematics: Crank AB:



$$\omega_{AB} = 600\text{ rpm}\left(\frac{2\pi}{60}\right) = 62.832\text{ rad/s} \circlearrowright$$

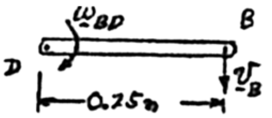
$$a_B = (AB)\omega_{AB}^2 = (0.1\text{ m})(62.832\text{ rad/s}^2)$$

$$\mathbf{a}_B = 394.78\text{ m/s}^2 \longleftarrow$$

Also: $v_B = (AB)\omega_{AB} = (0.1\text{ m})(62.832\text{ rad/s}) = 6.2832\text{ m/s}\downarrow$

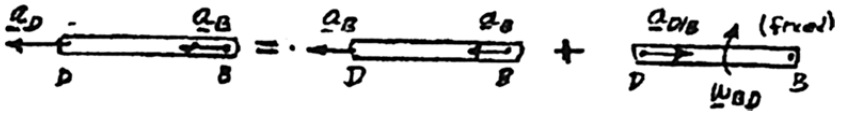
Connecting rod BD:

Velocity Instantaneous center at D.



$$\omega_{BD} = \frac{v_B}{BD} = \frac{6.2832\text{ m/s}}{0.25\text{ m}} = 25.133\text{ rad/s}$$

Acceleration:



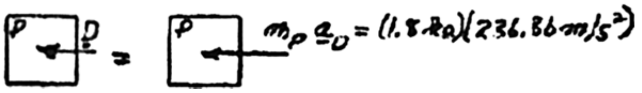
$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} = [a_B \longleftarrow] + [(BD)\omega_{BD}^2 \longrightarrow]$$

$$\mathbf{a}_D = [394.78\text{ m/s}^2 \longleftarrow] + [(0.25\text{ m})(25.133\text{ rad/s})^2 \longrightarrow]$$

$$\mathbf{a}_D = [397.78\text{ m/s}^2 \longleftarrow] + [157.92\text{ m/s}^2 \longleftarrow] = 236.86\text{ m/s}^2 \longleftarrow$$

$$\bar{\mathbf{a}}_{BD} = \frac{1}{2}(\mathbf{a}_B + \mathbf{a}_D) = \frac{1}{2}(394.78 \longleftarrow + 236.86 \longleftarrow) = 315.82\text{ m/s}^2 \longleftarrow$$

Kinetics of piston



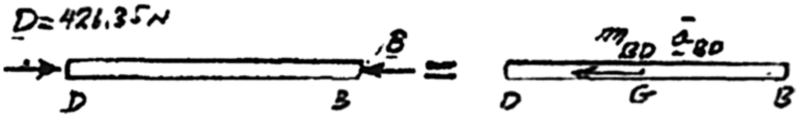
$$\mathbf{D} = 426.35\text{ N} \longleftarrow$$

PROBLEM 16.137 (Continued)

Force exerted on connecting rod at *D* is:

D = 426.35 ⟶

Kinetics of connecting rod: (neglect weight)



$$\xleftarrow{+}\ \Sigma F_x = \Sigma (F_x)_{\text{eff}}:\quad B - D = m_{BD}\bar{a}_{BD}$$

$$B - 426.34\ \text{N} = (1.2\ \text{kg})(315.82\ \text{m/s}^2)$$

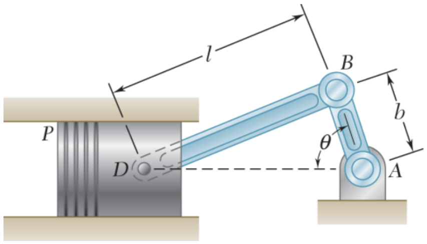
$$B = 426.35\ \text{N} + 378.48\ \text{N}$$

$$= 805.33\ \text{N}$$

Forces exerted on connecting rod.

B = 805 N ⟵ ◀

D = 426 N ⟶ ◀



PROBLEM 16.138

Solve Problem 16.137 when $\theta = 90°$.

PROBLEM 16.137 In the engine system shown $l = 250$ mm and $b = 100$ mm. The connecting rod BD is assumed to be a 1.2-kg uniform slender rod and is attached to the 1.8-kg piston P. During a test of the system, crank AB is made to rotate with a constant angular velocity of 600 rpm clockwise with no force applied to the face of the piston. Determine the forces exerted on the connecting rod at B and D when $\theta = 180°$. (Neglect the effect of the weight of the rod.)

SOLUTION

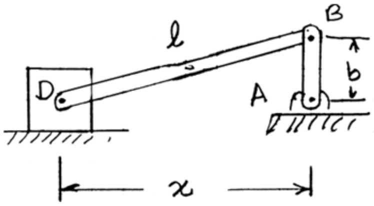
Geometry:

$$l = 0.25 \text{ m}, \quad b = 0.1 \text{ m}, \quad x = \sqrt{l^2 - b^2} = 0.2291 \text{ m}$$

$$\mathbf{r}_{B/A} = (0.1 \text{ m})\mathbf{j}, \quad \mathbf{r}_{D/B} = -(0.2291 \text{ m})\mathbf{i} - (0.1 \text{ m})\mathbf{j}$$

Kinematics:

$$\omega_{AB} = 600 \text{ rev/min} = 62.832 \text{ rad/s}$$



Velocity:

$$\boldsymbol{\omega}_{AB} = -(62.832 \text{ rad/s})\mathbf{k}$$

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$

$$= -(62.832\mathbf{k}) \times (0.1\mathbf{i})$$

$$= (6.2832 \text{ m/s})\mathbf{i}$$

$$\mathbf{v}_D = \mathbf{v}_B + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{D/B}$$

$$v_D\mathbf{i} = 6.2832\mathbf{i} + \omega_{BD}\mathbf{k} \times (-0.2291\mathbf{i} - 0.1\mathbf{j})$$

$$= 6.2832\mathbf{i} + 0.1\omega_{BD}\mathbf{j} - 0.2291\omega_{BD}\mathbf{i}$$

Equating components gives

$$v_D = 6.2832 \text{ m/s}, \quad \omega_{BD} = 0.$$

Acceleration:

$$\boldsymbol{\alpha}_{AB} = 0, \qquad \boldsymbol{\alpha}_{BD} = \alpha_{BD}\mathbf{k} \qquad \mathbf{a}_D = a_D\mathbf{i}$$

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A} = 0 - (62.832)^2(0.1\mathbf{j}) = -(394.78 \text{ m/s}^2)\mathbf{j}$$

$$\mathbf{a}_D = \mathbf{a}_B + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{B/D} - \omega_{BD}^2\mathbf{r}_{D/B}$$

$$a_D\mathbf{i} = -394.78\mathbf{j} + \alpha_{AB}\mathbf{k} \times (-0.2291\mathbf{i} - 0.1\mathbf{j}) - 0$$

$$= -394.78\mathbf{j} - 0.2291\alpha_{AB}\mathbf{j} + 0.1\alpha_{AB}\mathbf{i}$$

Equate like components.

$$\mathbf{j}: \qquad 0 = -394.78 - 0.2291\alpha_{AB} \qquad \alpha_{AB} = -1723 \text{ rad/s}^2$$

$$\mathbf{i}: \qquad a_D = (0.1)(-1723) = -172.3 \text{ m/s}^2$$

$$\boldsymbol{\alpha}_{AB} = -(1723 \text{ rad/s}^2)\mathbf{k} \qquad\qquad \mathbf{a}_D = -(172.3 \text{ m/s}^2)\mathbf{i}$$

PROBLEM 16.138 (Continued)

Acceleration of mass center *G* of bar *BD*

$$\mathbf{r}_{G/D} = \frac{1}{2}(0.2291\mathbf{i} + 0.1\mathbf{j})$$

$$= (0.11455\ \text{m})\mathbf{i} + (0.05\ \text{m})\mathbf{j}$$

$$\mathbf{a}_G = \mathbf{a}_D + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{G/D} - \omega_{BD}^2 \mathbf{r}_{G/D}$$

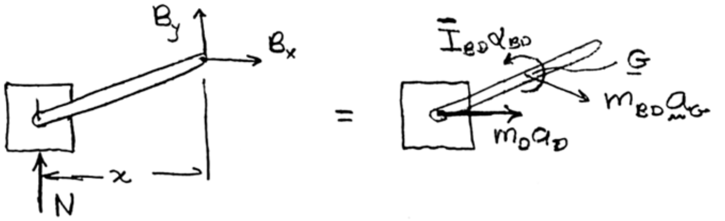
$$= -172.3\mathbf{i} + (-1723\mathbf{k}) \times (0.11455\mathbf{i} + 0.05\mathbf{j}) - 0$$

$$= -172.3\mathbf{i} - 197.34\mathbf{j} + 86.15\mathbf{i}$$

$$= -(86.15\ \text{m/s}^2)\mathbf{i} - (197.34\ \text{m/s}^2)\mathbf{j}$$

Force on bar *BD* at *P* in *B*.

Use piston + bar *BD* as a free body



$$m_D\mathbf{a}_D = (1.8)(-172.3\mathbf{i})$$

$$= -(310.14\ \text{N})\mathbf{i}$$

$$m_{BD}\mathbf{a}_G = (1.2)(-86.15\mathbf{i} - 197.34\mathbf{j})$$

$$= -(103.38\ \text{N})\mathbf{i} - (236.88\ \text{N})\mathbf{j}$$

$$\bar{I}_{BD}\alpha_{BD} = \frac{1}{12}(1.2)(0.25)^2(-1723) = -10.769\ \text{N}\cdot\text{m}$$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad B_x = m_D a_D + (m_{BD}a_G)_x$$

$$B_x = -310.14 - 103.38 \quad B_x = -413.52\ \text{N}$$

$$+\Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad -xN\mathbf{k} = \bar{I}_{BD}\alpha_{BD}\mathbf{k} + \mathbf{r}_{D/B} \times (m_D\mathbf{a}_D) + \mathbf{r}_{G/B} \times (m_{BD}\mathbf{a}_G)$$

$$-0.2291\,N\mathbf{k} = -10.769\mathbf{k} + (0.1)(-310.14)\mathbf{k} + (-0.11455\mathbf{i} - 0.05\mathbf{j})$$

$$\times(-103.38\mathbf{i} - 236.38\mathbf{j})$$

$$= -10.769\mathbf{k} - 31.014\mathbf{k} + 27.038\mathbf{k} - 5.169\mathbf{k}$$

$$N = 86.923\ \text{N}$$

$$+\Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad N + B_y = (m_{BD}a_G)_y$$

$$86.923 + B_y = -236.88 \quad B_y = -323.80\ \text{N}$$

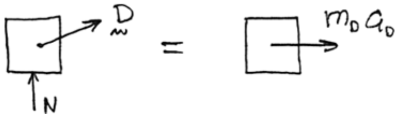
$$B = \sqrt{413.52^2 + 323.80^2} = 525.2\ \text{N}$$

$$\tan\beta = \frac{323.80}{413.52} \quad \beta = 38.1^\circ$$

B = 525 N ↙ 38.1° ◀

PROBLEM 16.138 (Continued)

Force exerted by bar *BD* as piston *D*.



Use piston *D* as a free body $\Sigma \mathbf{F} = \Sigma \mathbf{F}_{\text{eff}}$

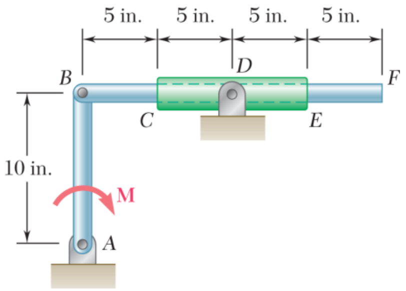
$$\mathbf{D} + N\mathbf{j} = m_D a_D \mathbf{i}$$

$$\mathbf{D} = m_D a_D \mathbf{i} - N\mathbf{j}$$

$$= -(310.14 \text{ N})\mathbf{i} + (86.923 \text{ N})\mathbf{j}$$

Force exerted by the piston on bar *BD*. By Newton's third law,

$$\mathbf{D}' = -\mathbf{D} = (310.14 \text{ N})\mathbf{i} - (86.923 \text{ N})\mathbf{j} \qquad \mathbf{D}' = 322 \text{ N} \searrow 15.7° \blacktriangleleft$$



PROBLEM 16.139

The 4-lb uniform slender rod *AB*, the 8-lb uniform slender rod *BF*, and the 4-lb uniform thin sleeve *CE* are connected as shown and move without friction in a vertical plane. The motion of the linkage is controlled by the couple **M** applied to rod *AB*. Knowing that at the instant shown the angular velocity of rod *AB* is 15 rad/s and the magnitude of the couple **M** is 5 ft·lb, determine (*a*) the angular acceleration of rod *AB*, (*b*) the reaction at point *D*.

SOLUTION

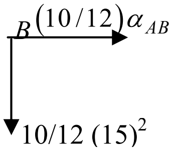
Kinematics: $\omega_{BF} = 0$ Radial acceleration of *B*/*A* is equal to the tangential acceleration of *B*/*D*,

Therefore $l_{AB}\omega_{AB}^2 = l_{BD}\alpha_{BE}, \quad l_{AB} = l_{BD}$

$$\alpha_{BE} = \omega_{AB}^2 = 225 \text{ rad/s}^2 \circlearrowright$$

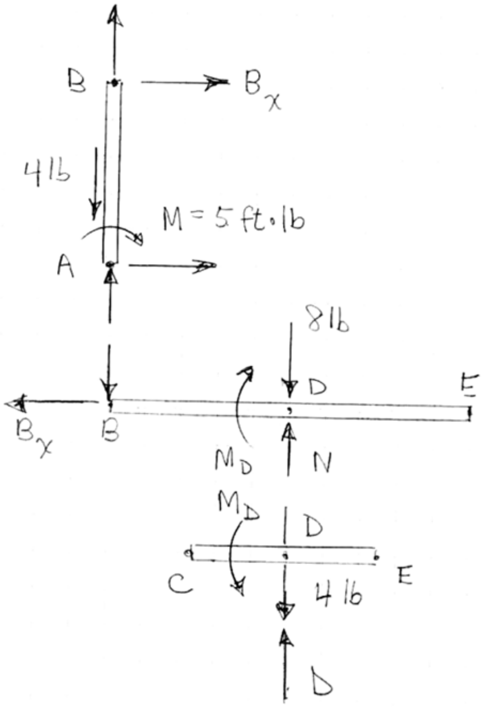
Bar BE

$a_G =$



$$+\uparrow\ (10/12)(15)^2 = \overrightarrow{(10/12)\ \alpha_{AB}}$$

Kinetics:



$$AB: +\circlearrowright\ \Sigma M_A = 5 \text{ ft}\cdot\text{lb} + B_x\left(\frac{10}{12}\text{ ft}\right)$$

$$= \frac{1}{3}\left(\frac{4 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{10}{12}\text{ ft}\right)^2 \alpha_{AB}$$

$$BE: \xrightarrow{+}\ \Sigma F_y = -B_y = \left(\frac{8 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{10}{12}\text{ ft}\right)\alpha_{AB}$$

(*a*) $$\alpha_{AB} = \frac{5}{0.1725 + 0.02875} = 24.84 \text{ rad/s}^2$$

$$\boldsymbol{\alpha}_{AB} = 24.8 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

$$M_D = \frac{1}{12}\left(\frac{4 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{10}{12}\text{ ft}\right)^2 (225 \text{ rad/s}^2)$$

$$D = N + 4 \text{ lb}$$

$$BE: +\circlearrowright\ \Sigma M_B = N\left(\frac{10}{12}\text{ ft}\right) - (8 \text{ lb})\left(\frac{10}{12}\text{ ft}\right)$$

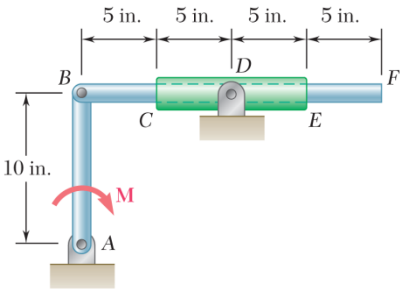
$$-M_D = \frac{1}{12}\left(\frac{8 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{20}{12}\right)^2 (225 \text{ rad/s}^2)$$

$$N = 25.469 \text{ N}$$

(*b*) $$D = 25.469 + (4 \text{ lb})$$

$$= 29.46 \text{ lb}\uparrow$$

$$\mathbf{D} = 29.5 \text{ lb}\uparrow \blacktriangleleft$$

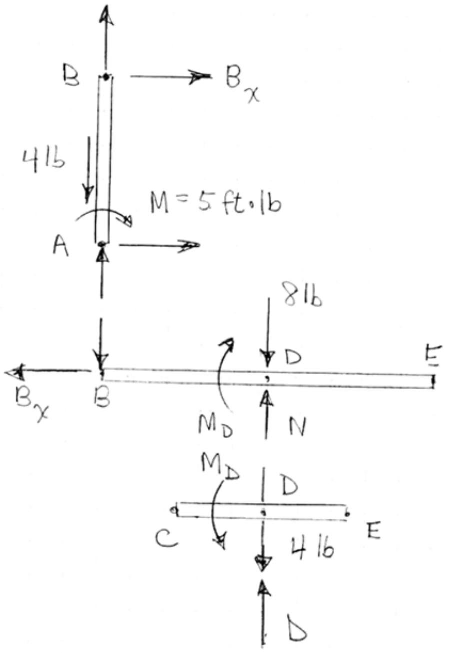


PROBLEM 16.140

The 4-lb uniform slender rod *AB*, the 8-lb uniform slender rod *BF*, and the 4-lb uniform thin sleeve *CE* are connected as shown and move without friction in a vertical plane. The motion of the linkage is controlled by the couple **M** applied to rod *AB*. Knowing that at the instant shown the angular velocity of rod *AB* is 30 rad/s and the angular acceleration of rod *AB* is 96 rad/s^2 clockwise, determine (*a*) the magnitude of the couple **M**, (*b*) the reaction at point *D*.

SOLUTION

From Prob. 16.139



$$\omega_{BE} = 0, \qquad \alpha_{BE} = \omega_{AB}^2 = 900 \text{ rad/s}^2 \curvearrowleft$$

$$a_G = \left(\frac{10}{12}\text{ ft}\right)(\alpha_{AB}) = \left(\frac{10}{12}\text{ ft}\right)(96 \text{ rad/s}^2)$$

$$= 80 \text{ ft/s}^2 \longrightarrow$$

$$AB: +\curvearrowright \Sigma M_A = M + B_x\left(\frac{10}{12}\text{ ft}\right)$$

$$= \frac{1}{3}\left(\frac{4 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{10}{12}\text{ ft}\right)^2 (96 \text{ rad/s}^2)$$

$$BE: \overset{+}{\longrightarrow} \Sigma F_x = -B_x = \left(\frac{8 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(80 \text{ ft/s}^2)$$

$$B_x = -19.876 \text{ lb}$$

(*a*) $M = 2.761 + 16.563 = 19.324$ ft·lb

$M = 19.3$ ft·lb $\curvearrowright$ ◀

Sleeve: $M_D = \frac{1}{12}\left(\frac{4 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{10}{12}\text{ ft}\right)^2 (900 \text{ rad/s}^2)$

$$M_D = 6.470 \text{ ft}\cdot\text{lb}, \quad D = N + 4 \text{ lb}$$

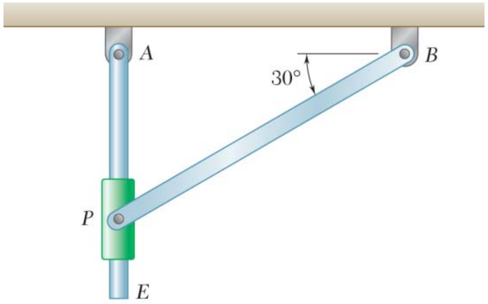
$$BE: +\curvearrowright \Sigma M_B = N\left(\frac{10}{12}\text{ ft}\right) - (8 \text{ lb})\left(\frac{10}{12}\text{ ft}\right)$$

$$-M_D = \frac{1}{12}\left(\frac{8 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{20}{12}\text{ ft}\right)^2 (900 \text{ rad/s}^2)$$

$$N = 77.876 \text{ lb}$$

(*b*) $D = (77.876 + 4)\text{ lb} = 81.876$ lb

or **D** $= 81.9$ lb$\uparrow$ ◀



PROBLEM 16.141

Two rotating rods in the vertical plane are connected by a slider block *P* of negligible mass. The rod attached at *A* has a weight of 1.6 lb and a length of 8 in. Rod *BP* weighs 2 lb and is 10 in. long and the friction between block *P* and *AE* is negligible. The motion of the system is controlled by a couple **M** applied to rod *BP*. Knowing that rod *BP* has a constant angular velocity of 20 rad/s clockwise, determine (*a*) the couple **M**, (*b*) the components of the force exerted on *AE* by block *P*.

SOLUTION

Unit vectors: $\mathbf{i} = 1 \longrightarrow,\ \mathbf{j} = 1\uparrow,\ \mathbf{k} = 1 \circlearrowleft.$

Geometry:

$$\mathbf{r}_{P/A} = -\left(\frac{10}{12}\sin 30°\ \text{ft}\right)\mathbf{j}$$

$$\mathbf{r}_{P/B} = -\left(\frac{10}{12}\cos 30°\ \text{ft}\right)\mathbf{i} - \left(\frac{10}{12}\sin 30°\ \text{ft}\right)\mathbf{j}$$

$$\mathbf{r}_{P/A} = -\left(\frac{8}{12}\ \text{ft}\right)\mathbf{j}$$

Kinematics: $\boldsymbol{\omega}_{BP} = -(20\ \text{rad/s})\mathbf{k}, \qquad \boldsymbol{\alpha}_{BP} = 0$

Velocity analysis.

Rod *BP*:

$$\mathbf{v}_P = \boldsymbol{\omega}_{BP} \times \mathbf{r}_{P/B}$$

$$= -20\mathbf{k} \times \left(-\frac{10}{12}\cos 30°\mathbf{i} - \frac{10}{12}\sin 30°\mathbf{j}\right)$$

$$= -(8.3333\ \text{ft/s})\mathbf{i} + (14.4338\ \text{ft/s})\mathbf{j}$$

Rod *AE*: Use a frame of reference rotating with angular velocity $\boldsymbol{\omega}_{AE} = \omega_{AE}\mathbf{k}$. The collar *P* slides on the rod with relative velocity $\mathbf{v}_{P/A} = u\mathbf{j}$

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/AE} = \boldsymbol{\omega}_{AE} \times \mathbf{r}_{P/A} + u\,\mathbf{j}$$

$$= \omega_{AE}\mathbf{k} \times \left(-\frac{10}{12}\sin 30°\mathbf{j}\right) + u\,\mathbf{j} = 0.41667\,\omega_{AE}\mathbf{i} + u\,\mathbf{j}$$

Equate the two expressions for $\mathbf{v}_P$ and resolve into components.

$\mathbf{i}$: $-8.3333 = 0.41667\,\omega_{AE}$ $\qquad \omega_{AE} = -20\ \text{rad/s}$

$\mathbf{j}$: $14.4388 = u$ $\qquad u = 14.4338\ \text{ft/s}$

Acceleration analysis.

Rod *BP*:

$$\mathbf{a}_P = \boldsymbol{\alpha}_{BP} \times \mathbf{r}_{P/B} - \omega_{BP}^2\mathbf{r}_{P/B}$$

$$= 0 - (20)^2\left(-\frac{10}{12}\cos 30°\mathbf{i} - \frac{10}{12}\sin 30°\mathbf{j}\right)$$

$$= (288.68\ \text{ft/s}^2)\mathbf{i} + (166.67\ \text{ft/s}^2)\mathbf{j}$$

Rod *AE*: $\boldsymbol{\alpha}_{AE} = \boldsymbol{\alpha}_{AE}\mathbf{k}, \qquad \mathbf{a}_{P/AE} = \dot{u}\,\mathbf{j}$

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/AE} + 2\boldsymbol{\omega}\times\mathbf{v}_{P/AE}$$

where
$$\begin{aligned}\mathbf{a}_{P'} &= \boldsymbol{\alpha}_{AB}\times r_{P/A} - \omega_{AE}^2\mathbf{r}_{P/A}\\ &= \alpha_{AE}\mathbf{k}\times\left(-\frac{10}{12}\sin 30°\mathbf{j}\right) - (20)^2\left(-\frac{10}{12}\sin 30°\mathbf{j}\right)\\ &= 0.41667\alpha_{AE}\mathbf{i} + (166.67\text{ ft/s}^2)\mathbf{j}\end{aligned}$$

and
$$\begin{aligned}2\boldsymbol{\omega}_{AE}\times\mathbf{v}_{P/AE} &= (2)(-20\mathbf{k})\times(14.4338\,\mathbf{j})\\ &= 577.35\text{ ft/s}^2\mathbf{i}\end{aligned}$$

Equate the two expressions for $\mathbf{a}_P$ and resolve into components.

$$\mathbf{i}:\quad 288.68 = 0.41667\alpha_{AE} + 577.35$$

$$\alpha_{AE} = -692.8\text{ rad/s}^2$$

Summary:
$$\boldsymbol{\omega}_{BP} = -(20\text{ rad/s})\mathbf{k}, \quad \boldsymbol{\omega}_{AE} = -(20\text{ rad/s})\mathbf{k}$$
$$\boldsymbol{\alpha}_{BP} = 0, \qquad \boldsymbol{\alpha}_{AE} = -(692.8\text{ rad/s}^2)\mathbf{k}$$

Masses and moments of inertia.

$$m_{AE} = \frac{W_{AE}}{g} = \frac{1.6\text{ lb}}{32.2\text{ ft/s}^2} = 0.049689\text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m_{BP} = \frac{W_{BP}}{g} = \frac{2\text{ lb}}{32.2\text{ ft/s}^2} = 0.062112\text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\bar{I}_{AE} = \frac{1}{12}m_{AE}l_{AE}^2 = \frac{1}{12}(0.049689\text{ lb}\cdot\text{s}^2/\text{ft})\left(\frac{8}{12}\text{ ft}\right)^2 = 1.8403\times10^{-3}\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{BP} = \frac{1}{12}m_{BP}l_{BP}^2 = \frac{1}{12}(0.062112\text{ lb}\cdot\text{s}^2/\text{ft})\left(\frac{10}{12}\text{ ft}\right)^2 = 3.5944\times10^{-3}\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Mass centers: Let Point *G* be the mass center of rod *AE* and Point *H* be that of rod *BP*.

$$\mathbf{r}_{G/A} = -\left(\frac{4}{12}\text{ ft}\right)\mathbf{j}$$

$$\mathbf{r}_{H/B} = -\left(\frac{10}{12}\cos 30°\text{ ft}\right)\mathbf{i} - \left(\frac{10}{12}\sin 30°\text{ ft}\right)\mathbf{j}$$

Acceleration of mass centers.

$$\begin{aligned}\mathbf{a}_G &= \boldsymbol{\alpha}_{AE}\times\mathbf{r}_{G/A} - \omega_{AE}^2\mathbf{r}_{G/A}\\ &= (-692.8\mathbf{k})\times(-0.33333\mathbf{j}) - (20)^2(-0.33333\mathbf{j}) = -(230.93\text{ ft/s}^2)\mathbf{i} + (133.33\text{ ft/s}^2)\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_H &= \boldsymbol{\alpha}_{BP}\times\mathbf{r}_{H/B} - \omega_{BP}^2\mathbf{r}_{H/B}\\ &= 0 - (20)^2(-0.72169\mathbf{i} - 0.41667\mathbf{j}) = (288.68\text{ ft/s}^2)\mathbf{i} + (166.67\text{ ft/s}^2)\mathbf{j}\end{aligned}$$

Inertial terms at mass centers.

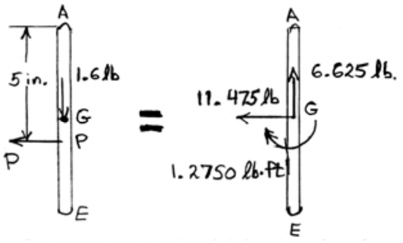
Rod *AE*: $m_{AE}\mathbf{a}_G = (0.049689)(-230.93\mathbf{i} + 133.33\mathbf{j}) = -(11.475 \text{ lb})\mathbf{i} + (6.625 \text{ lb})\,\mathbf{j}$

Rod *BP*: $m_{BP}\mathbf{a}_H = (0.062112)(288.68\mathbf{i} + 166.67\mathbf{j}) = (17.930 \text{ lb})\mathbf{i} + (10.352 \text{ lb})\,\mathbf{j}$

Effective couples at mass centers.

Rod *AE*: $\bar{I}_{AE}\boldsymbol{\alpha}_{AE} = (1.8403 \times 10^{-3})(-692.8\mathbf{k}) = -(1.2750 \text{ lb}\cdot\text{ft})\,\mathbf{k}$

Rod *BP*: $\bar{I}_{BP}\boldsymbol{\alpha}_{BP} = 0$



Kinetics.

Rod *AE*: $\Sigma\mathbf{M}_A = \Sigma(\mathbf{M}_A)_{\text{eff}}: \quad \mathbf{r}_{P/A} \times (-P\mathbf{i}) = \mathbf{r}_{G/A} \times (m_{AE}\mathbf{a}_G) + \bar{I}_{AE}\boldsymbol{\alpha}_{AE}$

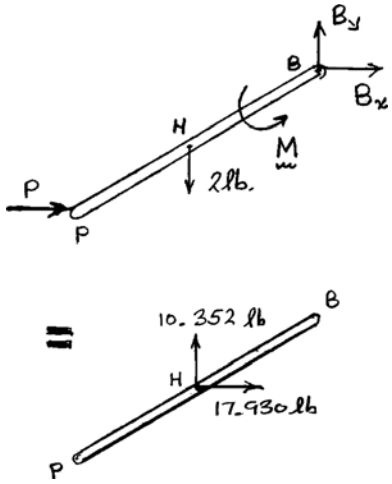
$$\left(-\frac{5}{12}\mathbf{j}\right) \times (-P\mathbf{i}) = \left(-\frac{4}{12}\mathbf{j}\right) \times (-11.475\mathbf{i} + 16.625\mathbf{j}) - 1.2750\mathbf{k}$$

$$-\frac{5}{12}P\mathbf{k} = -3.8249\mathbf{k} - 1.2750\mathbf{k} \qquad P = 12.240 \text{ lb}$$

Rod *BP*: $\Sigma\mathbf{M}_B = \Sigma(\mathbf{M}_B)_{\text{eff:}} \quad \mathbf{r}_{P/B} \times P\mathbf{i} + \mathbf{r}_{H/B} \times (-W_{BP}\mathbf{j}) + M\mathbf{k} = \mathbf{r}_{H/B} \times (m_{BP}\mathbf{a}_H) + \bar{I}_{BP}\boldsymbol{\alpha}_{BP}$

$$\frac{5}{12}P\mathbf{k} + \left(\frac{5\cos 30^\circ}{12} \text{ ft}\right)(2 \text{ lb})\mathbf{k} + M\mathbf{k}$$

$$= (-0.36084\mathbf{i} - 0.20833\mathbf{j}) \times (17.930\mathbf{i} + 10.352\mathbf{j}) + 0$$

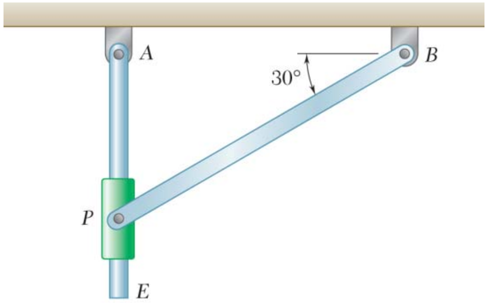


$$5.1\mathbf{k} + 0.72169\mathbf{k} + M\mathbf{k} = -3.7354\mathbf{k} + 3.7354\mathbf{k}$$

$$M = -5.82 \text{ lb}\cdot\text{ft}$$

(*a*) *Couple* **M**. $\mathbf{M} = 5.82 \text{ lb}\cdot\text{ft} \circlearrowright$ ◀

(*b*) *Force exerted on AE by block P*. $\mathbf{P} = 12.24 \text{ lb} \longleftarrow$ ◀



PROBLEM 16.142

Two rotating rods in the vertical plane are connected by a slider block *P* of negligible mass. The rod attached at *A* has a mass of 0.8 kg and a length of 160 mm. Rod *BP* has a mass of 1 kg and is 200 mm long and the friction between block *P* and *AE* is negligible. The motion of the system is controlled by a couple **M** applied to bar *AE*. Knowing that at the instant shown rod *BP* has an angular velocity of 20 rad/s clockwise and an angular acceleration of 80 rad/s^2 clockwise, determine (*a*) the couple **M**, (*b*) the components of the force exerted on *AE* by block *P*.

SOLUTION

Geometry: $\mathbf{r}_{P/A} = (0.200\text{ m})\sin 30^\circ \downarrow = 0.100\text{ m}\downarrow$

$\mathbf{r}_{P/B} = 0.200\text{ m} \swarrow 30^\circ$

$\mathbf{r}_{E/A} = 0.160\text{ m}\downarrow$

Kinematics: $\boldsymbol{\omega}_{BP} = 20\text{ rad/s} \circlearrowright$

$\boldsymbol{\alpha}_{BP} = 80\text{ rad/s}^2 \circlearrowright$

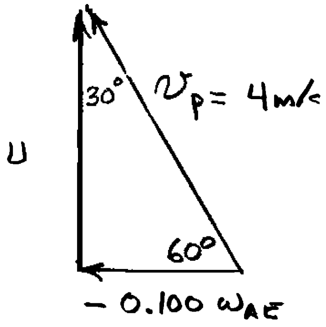
Velocity analysis.

Rod *BP*: $\mathbf{v}_P = \boldsymbol{\omega}_{BP} \times \mathbf{r}_{P/B} = (20\text{ rad/s})\cdot(0.200\text{ m}) \nwarrow 60^\circ = 4\text{ m/s} \nwarrow 60^\circ$

Rod *AE*: Use a frame of reference rotating with angular velocity $\boldsymbol{\omega}_{AE} = \omega_{AE} \circlearrowleft$. The collar slides on the rod with relative velocity $\mathbf{v}_{P/AE} = u\uparrow$.

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/AE} = \omega_{AE} \times \mathbf{r}_{P/A} + u\uparrow$$

$$= 0.100\,\omega_{AE} \longrightarrow + u\uparrow$$



Equate the two expressions for $\mathbf{v}_P$ using a triangle construction for vector addition.

$$-0.100\omega_{AE} = 4\cos 60^\circ$$

$$\omega_{AE} = -20\text{ rad/s} \quad \boldsymbol{\omega}_{AE} = 20\text{ rad/s} \circlearrowright$$

$$u = 4\sin 60^\circ = 3.461\text{ m/s} \qquad \mathbf{v}_{P/AE} = 3.4641\text{ m/s}\uparrow$$

Acceleration analysis. $\boldsymbol{\alpha}_{BP} = \alpha_{BP} \circlearrowleft$

Rod *BP*: $\mathbf{a}_P = \boldsymbol{\alpha}_{BP} \times \mathbf{r}_{P/B} - \omega_{BP}^2 \mathbf{r}_{P/B}$

$$= [(80\text{ rad/s}^2)(0.200\text{ m}) \nwarrow 60^\circ] + [(20\text{ rad/s})^2(0.200\text{ m}) \nearrow 30^\circ]$$

$$= [16\text{ m/s}^2 \nwarrow 60^\circ] + [80\text{ m/s}^2 \nearrow 30^\circ]$$

PROBLEM 16.142 (Continued)

Rod *AE*: $\boldsymbol{\alpha}_{AE} = \alpha_{AE} \circlearrowleft, \quad \mathbf{a}_{P/AE} = \dot{u} \uparrow$

$$\boldsymbol{\alpha}_{P} = \mathbf{a}_{P'} + \mathbf{a}_{P/AE} + 2\boldsymbol{\omega}_{AE} \times \mathbf{v}_{P/AE}$$

where $\mathbf{a}_{P'} = \boldsymbol{\alpha}_{AE} \times \mathbf{r}_{P/A} - \omega_{AE}^2 \mathbf{r}_{P/A}$

$$= [(0.100\text{ m})\alpha_{AE} \longrightarrow] + [(20\text{ rad/s}^2)(0.100\text{ m}) \uparrow]$$

$$= [0.100\alpha_{AE} \longrightarrow] + [40\text{ m/s}^2 \uparrow]$$

and $2\boldsymbol{\omega}_{AE} \times \mathbf{v}_{P/AE} = [(2)(20\text{ rad/s})(3.4641\text{ m/s}) \longrightarrow] = 138.564\text{ m/s}^2 \longrightarrow$

Equate the two expressions for $\mathbf{a}_P$ and resolve into components.

$$\longrightarrow: -(16\text{ m/s}^2)\cos 60° + (80\text{ m/s}^2)\cos 30°$$
$$= 0.100\alpha_{AE} + 138.564\text{ m/s}^2$$

$$\alpha_{AE} = -772.82\text{ rad/s}^2 \qquad \boldsymbol{\alpha}_{AE} = 772.82\text{ rad/s}^2 \circlearrowright$$

Masses, weights, and moments of inertia.

$$m_{AE} = 0.8\text{ kg} \quad W_{AE} = m_{AE}g = (0.8\text{ kg})(9.81\text{ m/s}^2) = 7.848\text{ N}$$
$$m_{BP} = 1.0\text{ kg} \quad W_{BP} = m_{BP}g = (1.0\text{ kg})(9.81\text{ m/s}^2) = 9.81\text{ N}$$
$$\bar{I}_{AE} = \frac{1}{12}m_{AE}l_{AE}^2 = \frac{1}{12}(0.8\text{ kg})(0.16\text{ m})^2 = 1.70667 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$\bar{I}_{BP} = \frac{1}{12}m_{BP}l_{BP}^2 = \frac{1}{12}(1.0\text{ kg})(0.20\text{ m})^2 = 3.3333 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

Mass centers: Let Point *G* be the mass center of rod *AE* and Point *H* be that of rod *BP*.

$$\mathbf{r}_{G/A} = 0.08\text{ m} \downarrow \qquad \mathbf{r}_{H/B} = 0.10\text{ m} \swarrow 30°$$

Accelerations of mass centers.

$$\mathbf{a}_G = \boldsymbol{\alpha}_{AE} \times \mathbf{r}_{G/A} - \omega_{AE}^2 \mathbf{r}_{G/A}$$
$$= (772.82\text{ rad/s}^2)(0.08\text{ m}) \longleftarrow + (20\text{ rad/s}^2)(0.08\text{ m}) \uparrow$$
$$= [61.826\text{ m/s}^2 \longleftarrow] + [32\text{ m/s}^2 \uparrow]$$

$$\mathbf{a}_H = \boldsymbol{\alpha}_{BP} \times \mathbf{r}_{H/A} - \omega_{BP}^2 \mathbf{r}_{H/B}$$
$$= [(80\text{ rad/s}^2)(0.10\text{ m}) \nwarrow 60°] + [(20\text{ rad/s})^2(0.10\text{ m}) \nearrow 30°]$$
$$= [8\text{ m/s}^2 \nwarrow 60°] + [40\text{ m/s}^2 \nearrow 30°]$$

PROBLEM 16.142 (Continued)

Inertial terms at mass centers.

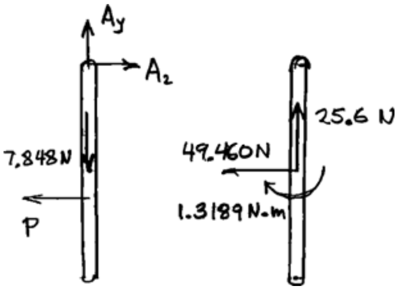
Rod *AE*: $m_{AE}\mathbf{a}_G = [49.460\text{ N}\longleftarrow] + [25.6\text{ N}\uparrow]$

Rod *BP*: $m_{BP}\mathbf{a}_H = [8\text{ N}\searrow 60^\circ] + [40\text{ N}\measuredangle 30^\circ]$

Effective couples at mass centers.

Rod *AE*: $\bar{I}_{AE}\boldsymbol{\alpha}_{AE} = 1.3189\text{ N}\cdot\text{m}\circlearrowright$

Rod *BP*: $\bar{I}_{BP}\boldsymbol{\alpha}_{BP} = 0.2667\text{ N}\cdot\text{m}\circlearrowright$



Kinetics:

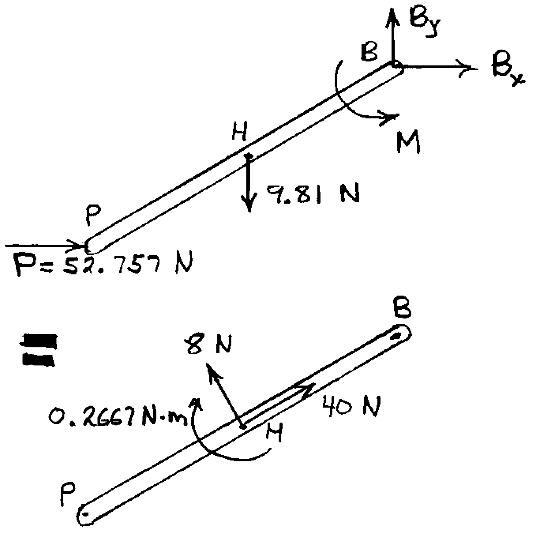
Rod *AE*: $+\circlearrowleft\Sigma M_A = \Sigma(M_A)_{\text{eff}}$: $-0.10P = -(0.08)(49.460) - 1.3189$

$$P = 52.757\text{ N}$$

Rod *BP*: $+\circlearrowleft\Sigma M_B = \Sigma(M_B)_{\text{eff}}$: $(52.757\text{ N})(0.1\text{ m}) + (9.81\text{ N})(0.086603\text{ m}) + M$

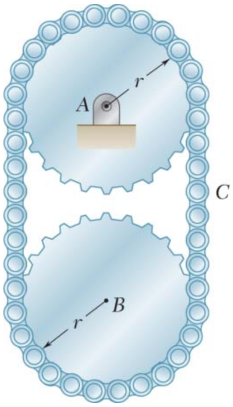
$$= -(8\text{ N})(0.1\text{ m}) - 0.2667\text{ N}\cdot\text{m}$$

$$M = 7.1919\text{ N}\cdot\text{m}$$



(*a*) *Couple* **M**. $M = 7.19\text{ N}\cdot\text{m}\circlearrowright$ ◀

(*b*) *Force exerted on AE by force by block P*. $P = 52.8\text{ N}\longleftarrow$ ◀

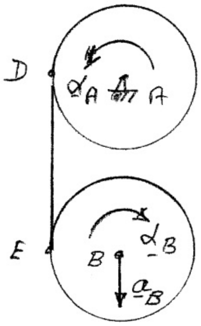


PROBLEM 16.143

Two disks, each of mass *m* and radius *r* are connected as shown by a continuous chain belt of negligible mass. If a pin at point *C* of the chain belt is suddenly removed, determine (*a*) the angular acceleration of each disk, (*b*) the tension in the left-hand portion of the belt, (*c*) the acceleration of the center of disk *B*.

SOLUTION

Kinematics: Assume α_A ↺ and α_B ↻



$$\omega_A = \omega_B = 0$$

$$\mathbf{a}_D = r\alpha_A \downarrow$$

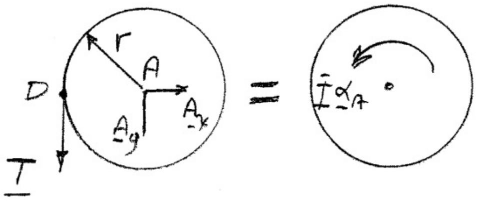
$$\mathbf{a}_E = a_D = r\alpha_A \downarrow$$

$$\overline{\mathbf{a}}_B = a_E + a_{B/E}$$

$$= (r\alpha_A + r\alpha_B)$$

$$\overline{\mathbf{a}}_B = r(\alpha_A + \alpha_B) \downarrow$$

Kinetics: *Disk A*:



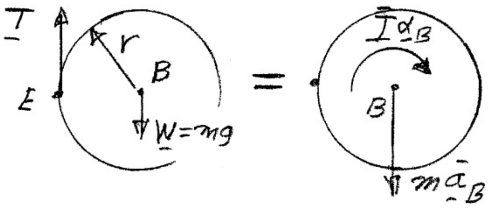
$$+↺\ \Sigma M_A = \Sigma(M_A)_{\text{eff}}:$$

$$Tr = \bar{I}\alpha_A$$

$$Tr = \frac{1}{2}mr^2\alpha_A$$

$$\alpha_A = \frac{2T}{mr} ↺ \qquad (1)$$

Disk B:



$$+↻\ \Sigma M_B = \Sigma(M_B)_{\text{eff}}:$$

$$Tr = \bar{I}\alpha_B$$

$$Tr = \frac{1}{2}mr^2\alpha_B$$

$$\alpha_B = \frac{2T}{mr} ↻ \qquad (2)$$

PROBLEM 16.143 (Continued)

From (1) and (2) we note that $\alpha_A = \alpha_B$

$$\overset{+}{\circlearrowright}\ \Sigma M_E = \Sigma (M_E)_{\text{eff}}: \quad Wr = \bar{I}\alpha_B + (m\bar{a}_B)r$$

$$Wr = \frac{1}{2}mr^2\alpha_B + mr(\alpha_A + \alpha_B)r$$

$\alpha_A = \alpha_B$:

$$Wr = \frac{5}{2}mr^2\alpha_A$$

$$\boldsymbol{\alpha}_A = \frac{2}{5}\frac{g}{r}\ \circlearrowleft$$ ◀

$$\boldsymbol{\alpha}_B = \frac{2}{5}\frac{g}{r}\ \circlearrowright$$ ◀

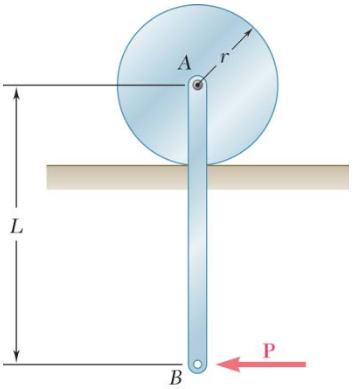
Substitute for α_A into (1):

$$\frac{2}{5}\frac{g}{r} = \frac{2T}{mr}$$

$$T = \frac{1}{5}mg$$ ◀

$$a_B = r(\alpha_A + \alpha_B) = r(2\alpha_A) = 2r\left(\frac{2}{5}\frac{g}{r}\right)$$

$$\mathbf{a}_B = \frac{4}{5}g \downarrow$$ ◀



PROBLEM 16.144*

A uniform slender bar *AB* of mass *m* is suspended as shown from a uniform disk of the same mass *m*. Neglecting the effect of friction, determine the accelerations of Points *A* and *B* immediately after a horizontal force **P** has been applied at *B*.

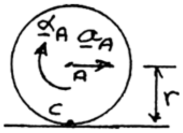
SOLUTION

Kinematics:

$$\omega = 0$$

Cylinder:

Rolling without sliding $(a_C)_x = 0$

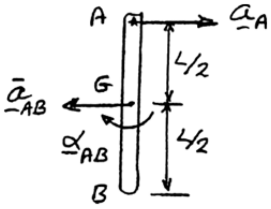

α_A
a_A
A
C
r

$$\xrightarrow{+} \mathbf{a}_A = (\mathbf{a}_C)_x + \mathbf{a}_{A/C} = 0 + r\alpha_A$$

$$\mathbf{a}_A = r\alpha_A \longrightarrow$$

$$\boldsymbol{\alpha}_A = \frac{a_A}{r} \circlearrowright$$

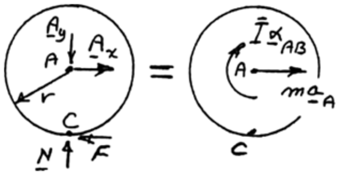
Rod *AB*:


A
a_A
L/2
G
$\bar{a}_{AB}$
α_{AB}
L/2
B

$$\xrightarrow{+} a_A = \frac{L}{2}\alpha_{AB} - \bar{a}_{AB}$$

Kinetics:

Cylinder:


A_y
A
A_x
r
C
N
F
=
$\bar{I}\alpha_{AB}$
A
ma_A
C

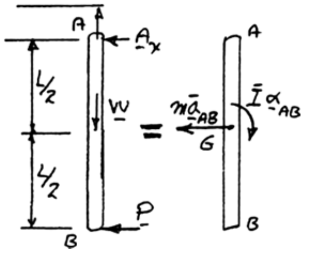
$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad A_x r = ma_A r + \bar{I}\alpha_A$$

$$A_x r = ma_A r + \frac{1}{2}mr^2\left(\frac{a_A}{r}\right)$$

$$A_x = \frac{3}{2}ma_A$$

$$A_x = \frac{3}{2}m\left(\frac{L}{2}\alpha_{AB} - \bar{a}_{AB}\right) \quad (1)$$

Rod *AB*:


A
A_x
L/2
W
L/2
P
B
=
A
$m\bar{a}_{AB}$
G
$\bar{I}\alpha_{AB}$
B

$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad PL = m\bar{a}_{AB}\frac{L}{2} + \bar{I}\alpha_{AB}$$

$$PL = m\bar{a}_{AB}\frac{L}{2} + \frac{m}{12}L^2\alpha_{AB} \quad (2)$$

$$\xleftarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad P + A_x = m\bar{a}_{AB}$$

PROBLEM 16.144* (Continued)

Substitute from (1):

$$P+\frac{3}{2}m\left(\frac{L}{2}\alpha_{AB}-\bar{a}_{AB}\right)=m\bar{a}_{AB}$$

$$P=\frac{5}{2}m\bar{a}_{AB}-\frac{3}{4}mL\alpha_{AB} \qquad (3)$$

Multiply by $\frac{L}{9}$:

$$\frac{1}{9}PL=\frac{5L}{18}m\bar{a}_{AB}-\frac{1}{12}mL^2\alpha_{AB} \qquad (4)$$

(4) + (2):

$$\frac{10}{9}PL=\left(\frac{1}{2}+\frac{5}{18}\right)mL\bar{a}_{AB}=\frac{7}{9}mL\bar{a}_{AB}$$

$$\bar{\mathbf{a}}_{AB}=\frac{10}{7}\frac{P}{m}\longleftarrow \qquad (5)$$

(5) ⟶ (3)

$$P=\frac{5}{2}m\left(\frac{10}{7}\frac{P}{m}\right)-\frac{3}{4}mL\alpha_{AB}$$

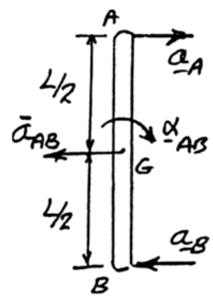
$$P=\frac{25}{7}P-\frac{3}{4}mL\alpha_{AB}$$

$$-\frac{18}{7}P=\frac{3}{4}mL\alpha_{AB}$$

$$\boldsymbol{\alpha}_{AB}=\frac{24}{7}\frac{P}{mL}\curvearrowright$$

$\xrightarrow{+}$: $a_A=\frac{L}{2}\alpha_{AB}-\bar{a}_{AB}$

$$=\frac{L}{2}\left(\frac{24}{7}\frac{P}{mL}\right)-\frac{10}{7}\frac{P}{m}$$



$$a_A=\left(\frac{12}{7}-\frac{10}{7}\right)\frac{P}{m}$$

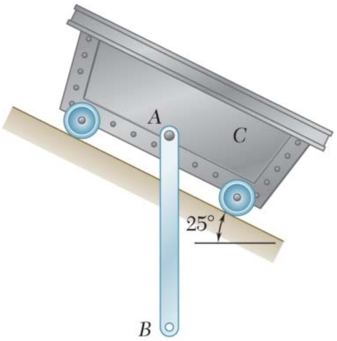
$$\mathbf{a}_A=\frac{2}{7}\frac{P}{m}\longrightarrow \blacktriangleleft$$

$\xleftarrow{+}$: $a_B=\frac{L}{2}\alpha_{AB}+\bar{a}_{AB}$

$$=\frac{L}{2}\left(\frac{24}{7}\frac{P}{mL}\right)+\frac{10}{7}\frac{P}{m}$$

$$a_B=\left(\frac{12}{7}+\frac{10}{7}\right)\frac{P}{m}$$

$$\mathbf{a}_B=\frac{22}{7}\frac{P}{m}\longleftarrow \blacktriangleleft$$



PROBLEM 16.145

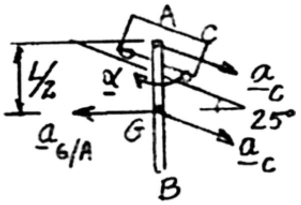
A uniform rod *AB*, of mass 15 kg and length 1 m, is attached to the 20-kg cart *C*. Neglecting friction, determine immediately after the system has been released from rest, (*a*) the acceleration of the cart, (*b*) the angular acceleration of the rod.

SOLUTION

Kinematics: We resolve the acceleration of *G* into the acceleration of the cart and the acceleration of *G* relative to *A*:

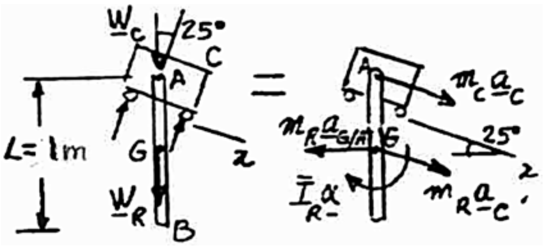
$$\bar{\mathbf{a}}_R = \mathbf{a}_G = \mathbf{a}_A + \mathbf{a}_{G/A}$$

$$\bar{\mathbf{a}}_R = \mathbf{a}_C + \mathbf{a}_{G/A}$$


A
C
L/2
α
$\underline{a}_C$
25°
$\underline{a}_{G/A}$
G
$\underline{a}_C$
B

where $$a_{G/A} = \frac{1}{2}L\alpha$$

Kinetics: Cart and rod


$\underline{W}_C$
25°
C
A
L = 1m
G
z
$\underline{W}_R$
B
=
A
$m_C\underline{a}_C$
$m_R\underline{a}_{G/A}$
G
25°
$\bar{\underline{I}}_R\underline{\alpha}$
$m_R\underline{a}_C$
z

$$m_R = 15 \text{ kg}$$

$$m_C = 20 \text{ kg}$$

$$L = 1 \text{ m}$$

$$\bar{I}_R = \frac{1}{12}m_R L^2$$

$$a_{G/A} = \frac{1}{2}(1)\alpha = 0.5\alpha$$

$$\overset{+}{\searrow} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad (m_C + m_R)g\sin 25° = (m_C + m_R)\mathbf{a}_C - m_R a_{G/A}\cos 25°$$

$$g\sin 25° = a_C - \left(\frac{m_g}{m_C + m_R}\right)\left(\frac{L}{2}\alpha\right)\cos 25°$$

$$a_C = (9.81)\sin 25° + \left(\frac{15}{20+15}\right)(0.5\cos 25°)\alpha$$

$$a_C = (9.81)\sin 25° + 0.19421\alpha \qquad (1)$$

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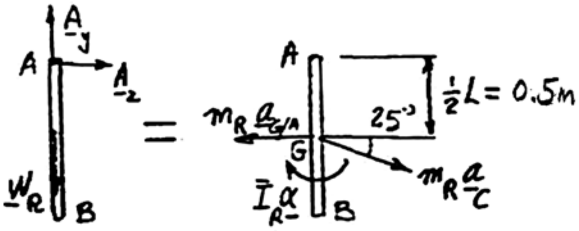
PROBLEM 16.145 (Continued)

Rod



$$+\circlearrowright \Sigma M_A = \Sigma (M_A)_{\text{eff}}: \quad 0 = \bar{I}\alpha + (m_R a_{G/R})\frac{L}{2} - (m_R a_C \cos 25°)\frac{L}{2}$$

$$\frac{1}{12}15(1)^2\alpha + 15(0.5\alpha)(0.5) - (15a_C \cos 25°)(0.5) = 0$$

$$1.25\alpha + 3.75\alpha - (7.5\cos 25°)a_C = 0$$

$$\alpha = (1.50\cos 25°)a_C \qquad (2)$$

(*a*) Acceleration of the cart.

Substitute for α from (2) into (1):

$$a_C = (9.81)\sin 25° + 0.19421(1.5\cos 25°)a_C$$

$$a_C = \frac{9.81\sin 25°}{1 - 0.19421(1.5\cos 25°)}$$

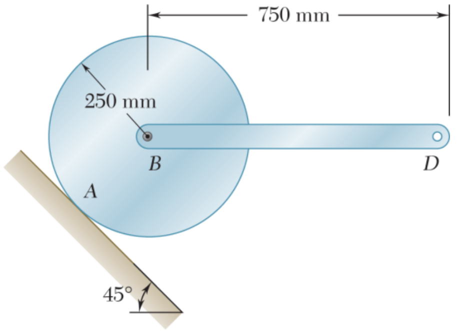
$$= 5.6331 \text{ m/s}^2 \qquad \mathbf{a}_C = 5.63 \text{ m/s}^2 \searrow 25° \blacktriangleleft$$

(*b*) Angular acceleration.

From (2):

$$\alpha = (1.50\cos 25°)(5.6331)$$

$$= 7.6580 \text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 7.66 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$


750 mm
250 mm
B
D
A
45°

PROBLEM 16.146

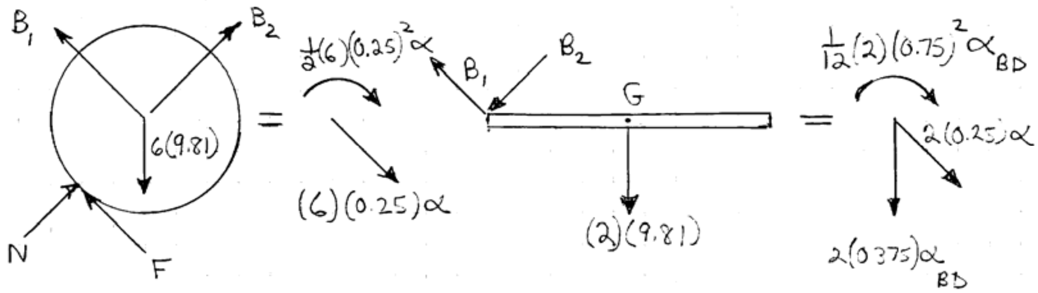
The uniform slender 2-kg bar *BD* is attached to the uniform 6-kg uniform disk by a pin at *B* and released from rest in the position shown. Assuming that the disk rolls without slipping, determine (*a*) the initial reaction at the contact point *A*, (*b*) the corresponding smallest allowable value of the coefficient of static friction.

SOLUTION

Kinematics:

$$a_B = \searrow 0.25\alpha \qquad a_G = \searrow 0.25\alpha + \downarrow 0.375\alpha_{BD}$$

Kinetics:


B1
B2
6(9.81)
N
F
=
½(6)(0.25)² α
(6)(0.25)α
B1
B2
G
(2)(9.81)
=
1/12(2)(0.75)² α_BD
2(0.25)α
2(0.375)α_BD

Disk: $\searrow\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}: B - F + (58.86\text{ N})(\cos 45°) - (6\text{ kg})(0.25\text{ m})\alpha$

$\nearrow\ \Sigma F_y = \Sigma(F_y)_{\text{eff}}: N + B_2 - (58.86\text{ N})(\sin 45°) = 0$

$\circlearrowleft\ \Sigma M_G = \Sigma(M_G)_{\text{eff}}:\ F(0.25\text{ m}) = \frac{1}{2}(6\text{ kg})(0.25\text{ m})^2\alpha$

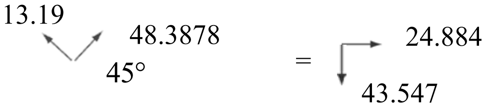
Bar: x-Dir $\searrow$: $-B_1 + (19.62\text{ N})\cos 45° = (2\text{ kg})(0.25\text{ m})\alpha + (\cos 45°)(2\text{ kg})(0.375\text{ m})\alpha_{BD}$

y-Dir $\swarrow$: $B_2 + (19.62\text{ N})\sin 45° = \sin 45°(2\text{ kg})(0.375\text{ m})\alpha_{BD}$

Six equations and six unknowns:

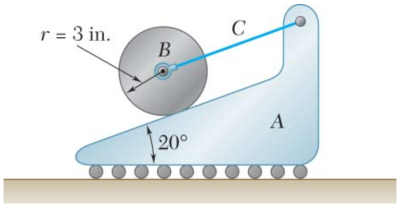
$F = 13.1967\text{ N}, \qquad N = 48.3878\text{ N}$

(*a*)


13.19
48.3878
45°
=
24.884
43.547

$= \mathbf{N} + \mathbf{F} = 50.2\text{ N} \measuredangle 60.3°$ ◀

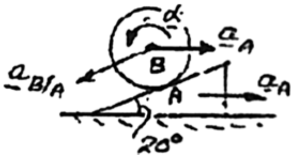
(*b*) $\dfrac{F}{N} = \dfrac{13.1967}{48.3878} = 0.273$ ◀



PROBLEM 16.147*

The 6-lb cylinder *B* and the 4-lb wedge *A* are held at rest in the position shown by cord *C*. Assuming that the cylinder rolls without sliding on the wedge and neglecting friction between the wedge and the ground, determine, immediately after cord *C* has been cut, (*a*) the acceleration of the wedge, (*b*) the angular acceleration of the cylinder.

SOLUTION


α
$\underline{a}_A$
$\underline{a}_{B/A}$
B
A
$\underline{a}_A$
20°

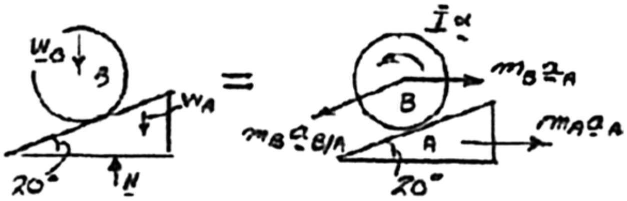
Kinematics: We resolve $\mathbf{a}_B$ into $\mathbf{a}_A$ and a component parallel to the incline

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

Where $a_{B/A} = r\alpha$, since the cylinder rolls on wedge *A*.

$$a_{B/A} = (0.25 \text{ ft})\alpha$$

Kinetics: Cylinder and wedge

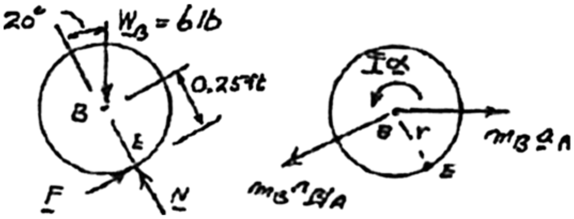

$\underline{W}_B$
B
$\underline{W}_A$
20°
$\underline{N}$
=
$\bar{I}\alpha$
$m_B\underline{a}_A$
B
$m_B\underline{a}_{B/A}$
A
$m_A\underline{a}_A$
20°

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad 0 = m_A a_A + m_B a_A - m_B a_{B/A}\cos 20°$$

$$0 = \frac{(4+6)\text{lb}}{g}a_A - \frac{6 \text{ lb}}{g}\left(\frac{3}{12}\text{ft}\right)\alpha\cos 20°$$

$$a_n = (0.15\cos 20°)\alpha \qquad (1)$$

Cylinder


20°
$\underline{W}_B = 6 lb$
0.25 ft
B
E
$\underline{F}$
$\underline{N}$
$\bar{I}\alpha$
B
r
E
$m_B\underline{a}_A$
$m_B a_{B/A}$

$$\bar{I} = \frac{1}{2}\frac{W}{g}r^2 = \frac{1}{2}\frac{6 \text{ lb}}{g}(0.25 \text{ ft})^2$$

$$\bar{I} = \frac{3}{16g}$$

PROBLEM 16.147* (Continued)

(*b*) Angular acceleration of the cylinder.

$$+\circlearrowleft \Sigma M_E = \Sigma(M_E)_{\text{eff}}: \quad (6\text{ lb})\sin 20°(0.25\text{ ft}) = \bar{I}\alpha + m_B a_{B/A}(0.25\text{ ft}) - m_B a_A \cos 20°(0.25\text{ ft})$$

$$1.5\sin 20° = \frac{3}{16(32.2)}\alpha + \frac{6\text{ lb}}{32.2}(0.25\alpha)(0.25)$$

$$-\frac{6\text{ lb}}{32.2}a_A \cos 20°(0.25)$$

$$0.51303 = 0.00582\alpha + 0.01165\alpha - 0.04378a_A$$

Substitute from (1):

$$0.51303 = 0.01747\alpha - 0.04378(0.15\cos 20°)\alpha$$

$$0.51303 = (0.01747 - 0.00617)\alpha$$

$$\alpha = 45.41\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 45.4\text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$

(*a*) Acceleration of the wedge.

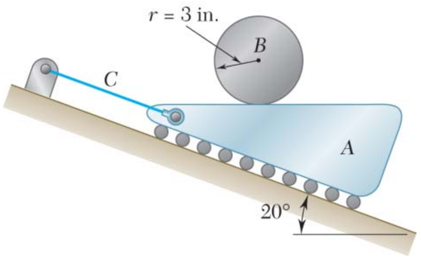
Eq. (1):

$$a_A = (0.15\cos 20°)\alpha$$

$$= (0.15\cos 20°)(45.41)$$

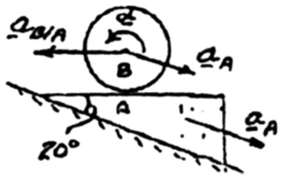
$$a_A = 6.401\text{ ft/s}^2 \qquad \mathbf{a}_A = 6.40\text{ ft/s}^2 \longrightarrow \blacktriangleleft$$

PROBLEM 16.148*



The 6-lb cylinder B and the 4-lb wedge A are held at rest in the position shown by cord C. Assuming that the cylinder rolls without sliding on the wedge and neglecting friction between the wedge and the ground, determine, immediately after cord C has been cut, (a) the acceleration of the wedge, (b) the angular acceleration of the cylinder.

SOLUTION



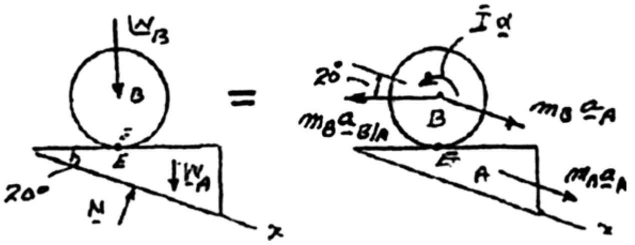
Kinematics: We resolve $\mathbf{a}_B$ into $\mathbf{a}_A$ and a horizontal component $\mathbf{a}_{B/A}$

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

Where $a_{B/A} = r\alpha,$ since the cylinder B rolls on wedge A.

$$a_{B/A} = (0.25\text{ ft})\alpha$$

Kinetics: Cylinder and wedge:



$$\overset{+}{\searrow} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad (W_A + W_B)\sin 20° = (m_A + m_B)a_A - m_B a_{B/A}\cos 20°$$

$$(10\text{ lb})\sin 20° = \left(\frac{10}{g}\right)a_A - \left(\frac{6}{g}\right)(0.25\alpha)\cos 20°$$

$$a_A = g\sin 20° + \frac{6}{10}(0.25)\cos 20°\alpha$$

$$a_A = g\sin 20° + 0.15\cos 20°\alpha \qquad (1)$$

Cylinder: $$+\circlearrowleft \Sigma M_E = \Sigma (M_E)_{\text{eff}}: \quad 0 = \bar{I}\alpha + (m_B a_{B/A})(0.25\text{ ft}) - (m_B a_A \cos 20°)(0.25\text{ ft})$$

$$0 = \frac{1}{2}\frac{6\text{ lb}}{g}(0.25\text{ ft})^2\alpha + \frac{6\text{ lb}}{g}(0.25\alpha)(0.25)$$

$$-\frac{6\text{ lb}}{g}a_A\cos 20°(0.25)$$

$$0 = \frac{1}{g}[0.1875\alpha + 0.325\alpha - 1.4095a_A]$$

$$0 = 0.5625\alpha - 1.4095a_A$$

$$\alpha = 2.506a_A \qquad (2)$$

PROBLEM 16.148* (Continued)

(*a*) Acceleration of the wedge.

Substitute for α from (2) into (1):

$$a_A = g\sin 20° + 0.15\cos 20°(2.506a_A)$$
$$a_A = 11.013 + 0.3532a_A$$
$$(1-0.3532)a_A = 11.013$$
$$a_A = 17.027 \text{ ft/s}^2$$

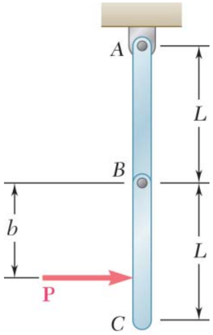
$\mathbf{a}_A = 17.03 \text{ ft/s}^2 \searrow 20°$ ◀

(*b*) Angular acceleration of the cylinder.

Eq. (2):

$$\alpha = 2.506a_A$$
$$= 2.506(17.027)$$
$$\alpha = 42.7 \text{ rad/s}^2$$

$\boldsymbol{\alpha} = 42.7 \text{ rad/s}^2$ ↻ ◀

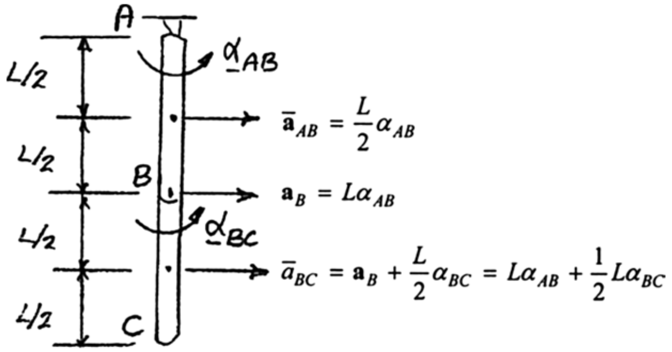


PROBLEM 16.149*

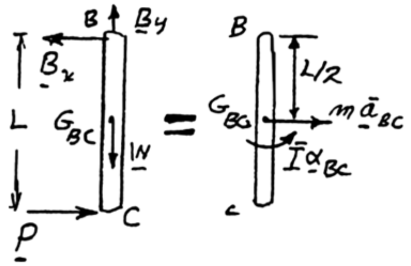
Each of the 3-kg bars *AB* and *BC* is of length L = 500 mm. A horizontal force **P** of magnitude 20 N is applied to bar *BC* as shown. Knowing that $b = L$ (**P** is applied at C), determine the angular acceleration each bar.

SOLUTION

Kinematics: Assume $\alpha_{AB}\circlearrowleft$ $\alpha_{BC}\circlearrowleft$ and $\omega_{AB} = \omega_{BC} = 0$


A
L/2
L/2
L/2
L/2
B
C
α_{AB}
$\bar{\mathbf{a}}_{AB} = \frac{L}{2}\alpha_{AB}$
$\mathbf{a}_B = L\alpha_{AB}$
α_{BC}
$\bar{a}_{BC} = \mathbf{a}_B + \frac{L}{2}\alpha_{BC} = L\alpha_{AB} + \frac{1}{2}L\alpha_{BC}$

Kinetics: Bar *BC*


B
B_y
B_x
L
G_{BC}
W
P
C
=
B
L/2
G_{BC}
$m\bar{a}_{BC}$
$\bar{I}\alpha_{BC}$
C

$$+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad PL = \bar{I}\alpha_{BC} + (m\bar{a}_{BC})\frac{L}{2}$$

$$= \frac{m}{12}L^2\alpha_{BC} + m\left(L\alpha_{AB} + \frac{L}{2}\alpha_{BC}\right)\frac{L}{2}$$

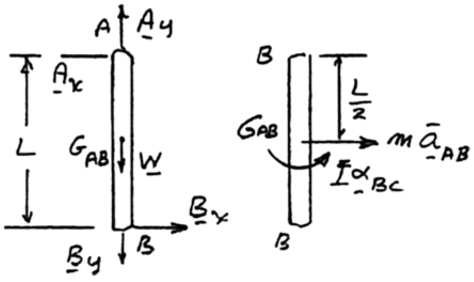
$$P = \frac{1}{2}mL\alpha_{AB} + \frac{1}{3}mL\alpha_{BC} \tag{1}$$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad P - B_x = m\bar{a}_{BC}$$

$$P - B_x = m\left(L\alpha_{AB} + \frac{1}{2}L\alpha_{BC}\right) \tag{2}$$

PROBLEM 16.149* (Continued)

Bar *AB*:



$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad B_x L = \bar{I}\alpha_{AB} + (m\bar{a}_{AB})\frac{L}{2}$$

$$= \frac{m}{12}L^2\alpha_{AB} + m\left(\frac{L}{2}\alpha_{AB}\right)\frac{L}{2}$$

$$B_x = \frac{1}{3}mL\alpha_{AB} \qquad (3)$$

Add (2) and (3):
$$P = \frac{4}{3}mL\alpha_{AB} + \frac{1}{2}mL\alpha_{BC} \qquad (4)$$

Subtract (1) from (4)
$$0 = \frac{5}{6}mL\alpha_{AB} + \frac{1}{6}mL\alpha_{BC}$$

$$\alpha_{BC} = -5\alpha_{AB} \qquad (5)$$

Substitute for α_{BC} in (1):
$$P = \frac{1}{2}mL\alpha_{AB} + \frac{1}{3}mL(-5\alpha_{AB}) = -\frac{7}{6}mL\alpha_{AB}$$

$$\alpha_{AB} = -\frac{6}{7}\frac{P}{mL} \qquad (6)$$

Eq. (5)
$$\alpha_{BC} = -5\left(-\frac{6}{7}\frac{P}{mL}\right) \qquad \alpha_{BC} = \frac{30}{7}\frac{P}{mL} \qquad (7)$$

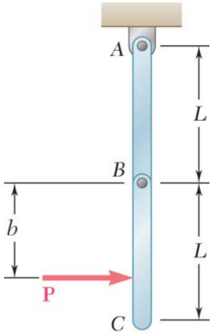
Data:
$$L = 500 \text{ mm} = 0.5 \text{ m}, \quad m = 3 \text{ kg}, \quad P = 20 \text{ N}$$

$$\alpha_{AB} = -\frac{6}{7}\frac{P}{mL} = -\frac{6}{7}\frac{20 \text{ N}}{(3 \text{ kg})(0.5 \text{ m})}$$

$$= -11.249 \text{ rad/s}^2 \qquad \boldsymbol{\alpha}_{AB} = 11.43 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

$$\alpha_{BC} = \frac{30}{7}\frac{P}{mL} = \frac{30}{7}\frac{20 \text{ N}}{(3 \text{ kg})(0.5 \text{ m})}$$

$$= 57.143 \text{ rad/s}^2 \qquad \boldsymbol{\alpha}_{BC} = 57.1 \text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$

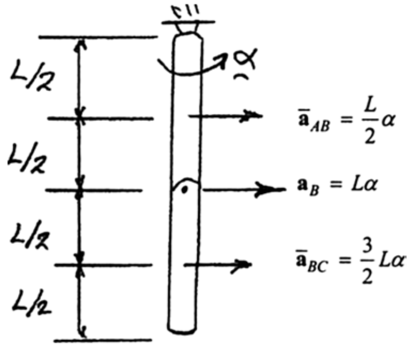


PROBLEM 16.150*

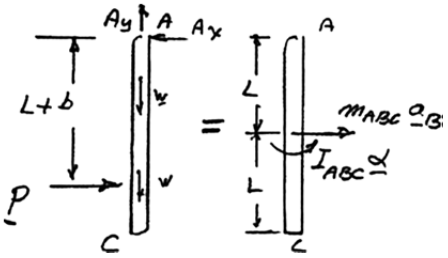
Each of the 3-kg bars *AB* and *BC* is of length $L = 500$ mm. A horizontal force **P** of magnitude 20 N is applied to bar *BC*. For the position shown, determine (*a*) the distance *b* for which the bars move as if they formed a single rigid body, (*b*) the corresponding angular acceleration of the bars.

SOLUTION

Kinematics: We choose $\alpha = \alpha_{AB} = \alpha_{BC}$ ↻


L/2
L/2
L/2
L/2
α
$\bar{\mathbf{a}}_{AB} = \frac{L}{2}\alpha$
$\mathbf{a}_B = L\alpha$
$\bar{\mathbf{a}}_{BC} = \frac{3}{2}L\alpha$

Kinetics: Bars *AB* and *BC* (acting as rigid body)


Ay
A
Ax
L+b
w
w
P
C
=
A
L
L
$m_{ABC}\,\bar{a}_B$
$\bar{I}_{ABC}\,\alpha$
C

$$m_{ABC} = 2m$$

$$\bar{I} = \frac{1}{12}(2m)(2L)^2$$

$$\bar{I} = \frac{2}{3}mL^2$$

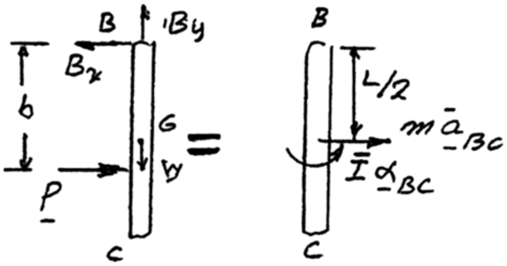
$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad P(L+b) = \bar{I}_{ABC}\alpha + m_{ABC}a_B L$$

$$P(L+b) = \frac{2}{3}mL^2\alpha + (2m)(L\alpha)L$$

$$P(L+b) = \frac{8}{3}mL^2\alpha \qquad (1)$$

PROBLEM 16.150* (Continued)

Bar *BC*:



$$+\circlearrowright \Sigma M_B = \Sigma (M_B)_{\text{eff}}: \quad Pb = \bar{I}_{BC}\alpha + (m\bar{a}_{BC})\frac{L}{2}$$

$$= \frac{m}{12}L^2\alpha + m\left(\frac{3}{2}L\alpha\right)\frac{L}{2}$$

$$Pb = \frac{5}{6}mL^2\alpha$$

$$\alpha = \frac{6}{5}\frac{Pb}{mL^2} \qquad (2)$$

Substitute for α into (1):

$$P(L+b) = \frac{8}{3}mL^2\left(\frac{6}{5}\frac{Pb}{mL^2}\right)$$

$$PL + Pb = \frac{16}{5}Pb$$

$$L = \left(\frac{16}{5} - 1\right)b = \frac{11}{5}b$$

$$b = \frac{5}{11}L$$

Eq. (2)

$$\alpha = \frac{6}{5}\frac{P}{mL^2}\left(\frac{5}{11}L\right) \qquad \alpha = \frac{6}{11}\frac{P}{mL}$$

Data:

$L = 500 \text{ mm} = 0.5 \text{ m}$

$m = 3 \text{ kg}$

$P = 20 \text{ N}$

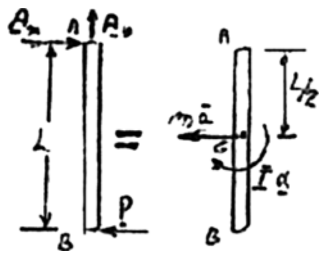
(*a*) $b = \frac{5}{11}L = \frac{5}{11}(0.5) = 0.22727 \text{ m}$ $\qquad b = 227 \text{ mm}$ ◀

(*b*) $\alpha = \frac{6}{11}\frac{P}{mL} = \frac{6}{11}\frac{(20 \text{ N})}{(3 \text{ kg})(0.5 \text{ m})} = 7.2727 \text{ rad/s}^2$ $\qquad \boldsymbol{\alpha} = 7.27 \text{ rad/s}^2 \circlearrowright$ ◀

PROBLEM 16.151*

(*a*) Determine the magnitude and the location of the maximum bending moment in the rod of Problem 16.78.
(*b*) Show that the answer to Part *a* is independent of the weight of the rod.

SOLUTION



Rod *AB*: $$\bar{a} = \frac{L}{2}\alpha$$

$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad PL = (m\bar{a})\frac{L}{2} + \bar{I}\alpha$$

$$= \left(m\frac{1}{2}\alpha\right)\frac{L}{2} + \frac{1}{12}mL^2\alpha$$

$$\alpha = \frac{3P}{mL} \circlearrowright \qquad (1)$$

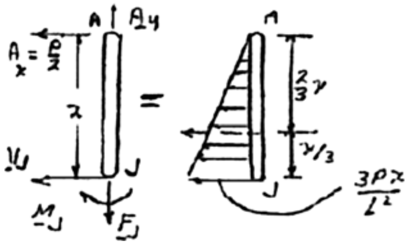
$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad A_x - P = -m\bar{a}$$

$$A_x = P - m\frac{L}{2}\alpha = P - m\frac{L}{2}\left(\frac{3P}{mL}\right) = -\frac{P}{2} \qquad \mathbf{A}_x = \frac{1}{2}P \longleftarrow$$

Portion *AJ* of Rod:

External forces: A_x, W_{AJ}, axial force F_J, shear V_J, and bending moment M_J.

Inertial terms: Since acceleration at any point is proportional to distance from *A*, inertial terms are linearly distributed. Since mass per unit length is *m*/*L*, at Point *J* we find



$$\left(\frac{m}{L}\right)a_J = \frac{m}{L}(x\alpha)$$

Using (1): $$\frac{m}{L}a_J = \frac{m}{L}\left(\frac{3P}{mL}\right)$$

$$\frac{m}{L}a_J = \frac{3Px}{L^2}$$

$$+\circlearrowright \Sigma M_J = \Sigma(M_J)_{\text{eff}}: \quad M_J - A_x x = -\frac{1}{2}\left(\frac{3Px}{L^2}\right)\left(\frac{2x}{3}\right)$$

$$M_J = \frac{1}{2}Px - \frac{1}{2}\frac{P}{L^2}x^3 \qquad (2)$$

For $M_{\max}$: $$\frac{dM_J}{dx} = \frac{P}{2} - \frac{3}{2}\frac{P}{L^2}x^2 = 0$$

$$x = \frac{L}{\sqrt{3}} \qquad (3)$$

PROBLEM 16.151* (Continued)

Substituting into (2)

$$(M_J)_{\max} = \frac{1}{2}\frac{PL}{\sqrt{3}} - \frac{1}{2}\frac{P}{L^2}\left(\frac{L}{\sqrt{3}}\right)^3 = \frac{1}{2}\frac{PL}{\sqrt{3}}\left(\frac{2}{3}\right)$$

$$(M_J)_{\max} = \frac{PL}{3\sqrt{3}} \qquad (4)$$

Note: Eqs. (3) and (4) are independent of W.

Data: $L = 36 \text{ in.}, \quad P = 1.5 \text{ lb}$

Eq. (3): $x = \frac{L}{\sqrt{3}} = \frac{36 \text{ in.}}{\sqrt{3}} = 20.78 \text{ in.}$

Eq. (4):

$$(M_J)_{\max} = \frac{(1.5 \text{ lb})(36 \text{ in.})}{3\sqrt{3}}$$

$$= 10.392 \text{ lb} \cdot \text{in.}$$

$M_{\max} = 10.39 \text{ lb} \cdot \text{in.}$ located 20.8 in. below A ◀

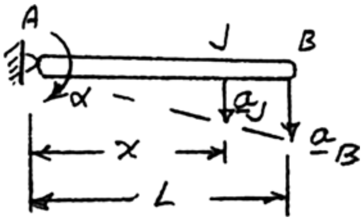
PROBLEM 16.152*

Draw the shear and bending-moment diagrams for the beam of Problem 16.84 immediately after the cable at *B* breaks.

SOLUTION

From answers to Problem 16.84:

$$\mathbf{a}_B = \frac{3}{2}g\downarrow \qquad \mathbf{A} = \frac{1}{4}mg\uparrow$$



We now find

$$\boldsymbol{\alpha} = \frac{a_B}{L}\circlearrowright = \frac{3g}{2L}\circlearrowright$$

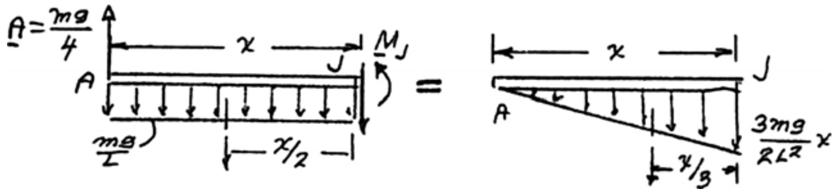
$$\mathbf{a}_J = x\alpha = \frac{3g}{2L}x\downarrow$$

Portion *AJ* of rod:

External forces: Reaction *A*, distributed load per unit length *mg/L*, shear V_J, bending moment M_J.

Inertial terms: Since *a* is proportional to *x*, the inertial terms are linearly distributed. The effective force per unit length at *J* is:

$$\frac{m}{L}a_J = \frac{m}{L}\cdot\frac{3g}{2L}x = \frac{3mg}{2L^2}x$$



$$+\downarrow\Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad \frac{mg}{L}x - \frac{mg}{4} + V_J = \frac{1}{2}\left(\frac{3mg}{2L^2}x\right)x$$

$$V_J = \frac{mg}{4} - \frac{mg}{L}x + \frac{3}{4}\frac{mg}{L^2}x^2$$

$$+\circlearrowleft\Sigma M_J = \Sigma(M_J)_{\text{eff}}: \quad \left(\frac{mg}{L}x\right)\frac{x}{2} - \frac{mg}{4}x + M_J = \frac{1}{2}\left(\frac{3mg}{2L^2}x\right)x\left(\frac{x}{3}\right)$$

$$M_J = \frac{mg}{4}x - \frac{1}{2}\frac{mg}{L}x^2 + \frac{1}{4}\frac{mg}{L^2}x^3$$

Find $V_{\min}$:

$$\frac{dV_J}{dx} = -\frac{mg}{L} + \frac{3}{2}\frac{mg}{L^2}x = 0; \qquad x = \frac{2}{3}L$$

$$V_{\min} = \frac{mg}{4} - \frac{mg}{L}\left(\frac{2}{3}L\right) + \frac{3}{4}\frac{mg}{L^2}\left(\frac{2}{3}L\right)^2; \qquad V_{\min} = -\frac{mg}{12}$$

PROBLEM 16.152* (Continued)

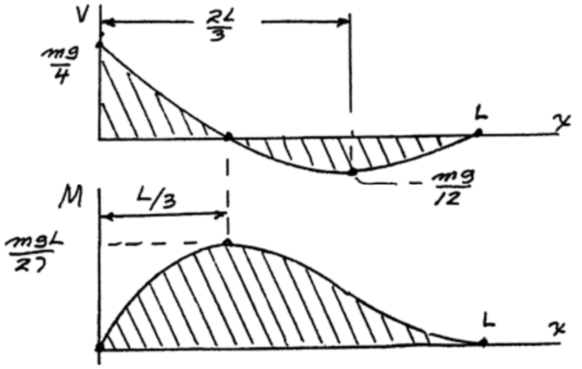
Find $M_{\max}$ where $V_J = 0$: $V_J = \frac{mg}{4} - \frac{mg}{L}x + \frac{3}{4}\frac{mg}{L}x^2 = 0$

$$3x^2 - 4Lx + L^2 = 0$$

$$(3x - L)(x - L) = 0 \qquad x = \frac{L}{3} \text{ and } x = L$$

$$M_{\max} = \frac{mg}{4}\left(\frac{L}{3}\right) - \frac{1}{2}\frac{mg}{L}\left(\frac{L}{3}\right)^2 + \frac{1}{4}\left(\frac{mg}{L}\right)\left(\frac{L}{3}\right)^3 = \frac{mgL}{27}$$

$$M_{\min} = \frac{mg}{4}L - \frac{1}{2}\frac{mg}{L}L^2 + \frac{1}{4}\frac{mg}{L}L^3 = 0$$

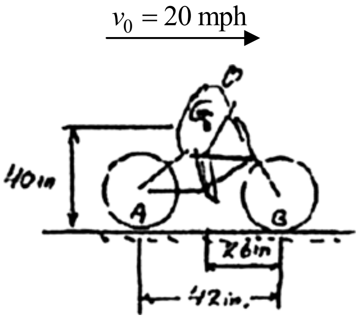


$M_{\max} = \frac{mgL}{27}$ at $\frac{L}{3}$ from A ◀

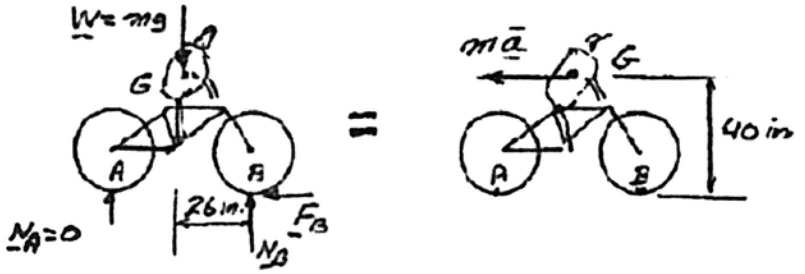
PROBLEM 16.153

A cyclist is riding a bicycle at a speed of 20 mph on a horizontal road. The distance between the axles is 42 in., and the mass center of the cyclist and the bicycle is located 26 in. behind the front axle and 40 in. above the ground. If the cyclist applies the brakes only on the front wheel, determine the shortest distance in which he can stop without being thrown over the front wheel.

SOLUTION



When cyclist is about to be thrown over the front wheel, $N_A = 0$



$$+\circlearrowleft \Sigma M_B = \Sigma (M_B)_{\text{eff}}: \quad mg(26\text{ in.}) = m\bar{a}(40\text{ in.})$$

$$a = \frac{26}{40}g = \frac{26}{40}(32.2\text{ ft/s}^2) = 20.93\text{ ft/s}^2$$

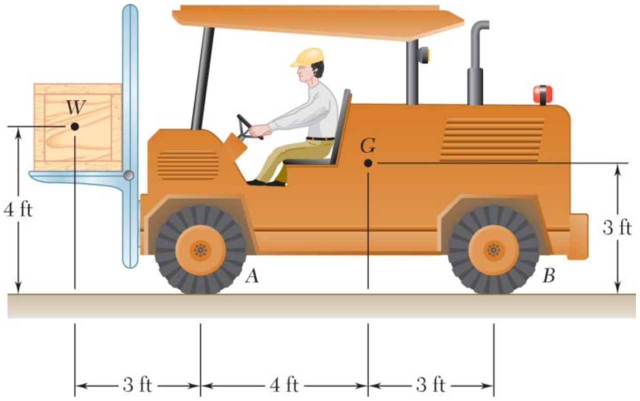
Uniformly accelerated motion:

$$v_0 = 20\text{ mph} = 29.333\text{ ft/s}$$

$$v^2 - v_0^2 = 2as: \quad 0 - (29.333\text{ ft/s})^2 = 2(-20.93\text{ ft/s}^2)s$$

$$s = 20.555\text{ ft}$$

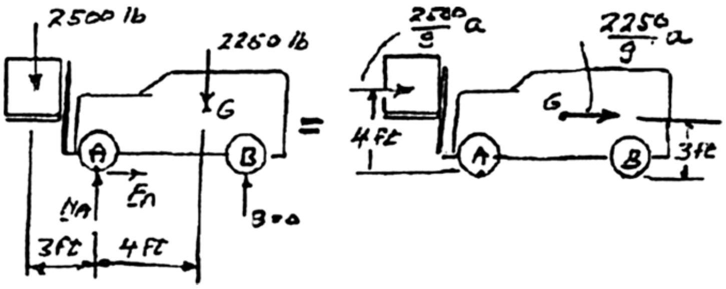
$s = 20.6$ ft ◀



PROBLEM 16.154

The forklift truck shown weighs 2250 lb and is used to lift a crate of weight $W = 2500$ lb. The truck is moving to the left at a speed of 10 ft/s when the brakes are applied on all four wheels. Knowing that the coefficient of static friction between the crate and the fork lift is 0.30, determine the smallest distance in which the truck can be brought to a stop if the crate is not to slide and if the truck is not to tip forward.

SOLUTION



Assume crate does not slide and that tipping impends about A. $(B = 0)$

$$+\circlearrowright \Sigma M_A = \Sigma (M_A)_{\text{eff}}:$$

$$(2500 \text{ lb})(3 \text{ ft}) - (2250 \text{ lb})(4 \text{ ft}) = -\left(2500\frac{a}{g}\right)(4 \text{ ft}) - \left(2250\frac{a}{g}\right)(3 \text{ ft})$$

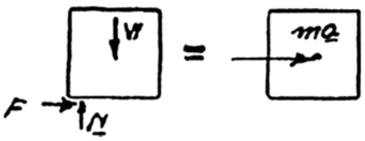
$$7500 - 9000 = -(10{,}000 + 6750)\frac{a}{g}$$

$$\frac{a}{g} = 0.09; \quad a = 0.09(32.2 \text{ ft/s}^2) \qquad \mathbf{a} = 2.884 \text{ ft/s}^2 \longrightarrow$$

Uniformly accelerated motion

$$v^2 = v_0^2 + 2ax; \quad 0 = (10 \text{ ft/s})^2 - 2(2.884 \text{ ft/s}^2)x \qquad x = 17.34 \text{ ft} \blacktriangleleft$$

Check whether crate slides

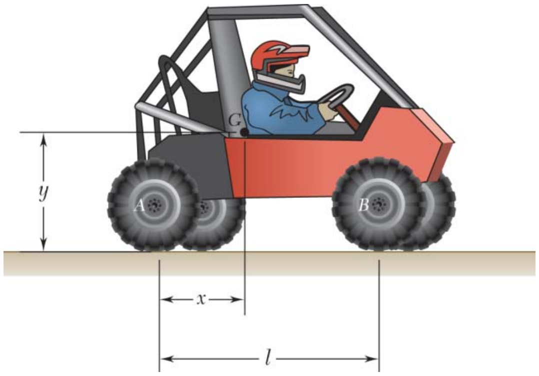


$$N = W$$

$$F = ma = \frac{W}{g}a$$

$$\mu_{\text{req}} = \frac{F}{N} = \frac{a}{g} = \frac{2.884 \text{ ft/s}^2}{32.2 \text{ ft/s}^2}$$

$\mu_{\text{req}} = 0.09 < 0.30.$ The crate does not slide. ◀



PROBLEM 16.155

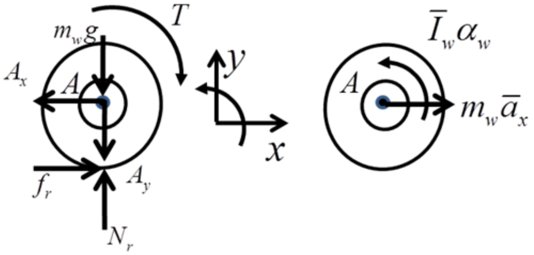
The total mass of the baja car and driver, including the wheels, is 250 kg. Each pair of 58-cm radius wheels and axle has a total mass of 20 kg and a mass moment of inertia of 2.9 kg-m^2. The center of gravity of the driver and baja body (not including the wheels) is located x= 0.70 m from the rear axle A and y = 0.55 m from the ground, and the wheelbase is L= 1.60 m. If the engine exerts a torque of 500 N-m on the rear axle, what is the car's acceleration?

SOLUTION

Given:

$$m_c = 210 \text{ kg},\ \text{m}_w = 20 \text{ kg}$$
$$r_w = 0.58 \text{ m},\ \overline{I}_w = 2.9 \text{ kg}\bullet m^2$$
$$x = 0.70 \text{ m},\ \text{y} = 0.55 \text{ m},\ \text{L} = 1.60 \text{ m}$$
$$\alpha_c = 0,\ T = 500\text{N}\bullet\text{m}$$
$$\overline{a}_{r,x} = \overline{a}_{f,x} = \overline{a}_{c,x} = a_x$$
$$\alpha_r = \alpha_f = \alpha_w$$

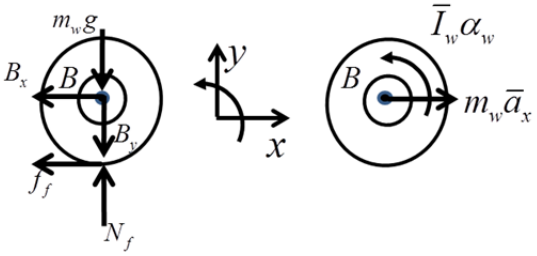
Free Body Diagram of Rear Wheels:



Kinetics:

$$\sum F_x = m_w\overline{a}_x$$
$$f_r - A_x = 20\overline{a}_x \quad\Rightarrow\quad f_r - A_x - 20\overline{a}_x = 0 \qquad (1)$$
$$\sum F_y = m_w \cancel{\overline{a}_y}$$
$$N_r - m_w g - A_y = 0 \quad\Rightarrow\quad N_r - A_y = 196.2 \qquad (2)$$
$$\sum M_A = \overline{I}_w\alpha_w$$
$$r_w f_r - T = \overline{I}_w\alpha_w \quad\Rightarrow\quad 0.58 f_r - 2.9\alpha_w = 500 \qquad (3)$$

Free Body Diagram of Front Wheels:



Kinetics:

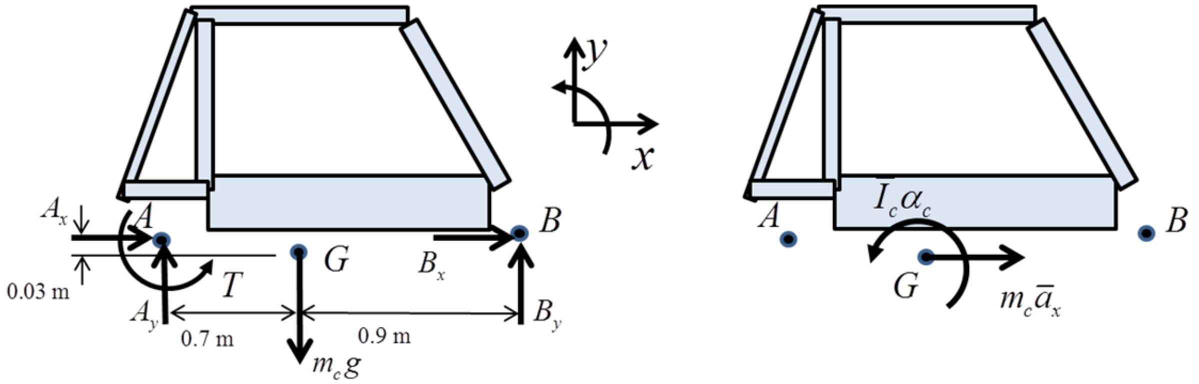
$$\sum F_x = m_w\overline{a}_x$$
$$-f_f - B_x = 20\overline{a}_x \quad\Rightarrow\quad -f_f - B_x - 20\overline{a}_x = 0 \qquad (4)$$
$$\sum F_y = m_w \cancel{\overline{a}_y}$$
$$N_f - m_w g - B_y = 0 \quad\Rightarrow\quad N_f - B_y = 196.2 \qquad (5)$$
$$\sum M_B = \overline{I}_w\alpha_w$$
$$-r_w f_f = \overline{I}_w\alpha_w \quad\Rightarrow\quad -0.58 f_f - 2.9\alpha_w = 0 \qquad (6)$$

Rolling Kinematics:

$$\overline{a}_x = -r_w\alpha_w \quad\Rightarrow\quad \overline{a}_x + 0.58\alpha_w = 0 \qquad (7)$$

PROBLEM 16.155 (Continued)

Free Body Diagram of Car Body:



Kinetics:

$$\sum F_x = m_c \bar{a}_x$$

$$A_x + B_x = m_c \bar{a}_x \quad \Rightarrow \quad A_x + B_x - 210\bar{a}_x = 0 \tag{8}$$

$$\sum F_y = m_c \cancel{\bar{a}_y}$$

$$A_y + B_y - m_c g = 0 \quad \Rightarrow \quad A_y + B_y = 2060.1 \tag{9}$$

$$\sum M_B = \bar{I}_c \cancel{\alpha_c}$$

$$-0.03A_x - 0.7A_y - 0.03B_x + 0.9B_y + T = 0$$

$$-0.03A_x - 0.7A_y - 0.03B_x + 0.9B_y = -500 \tag{10}$$

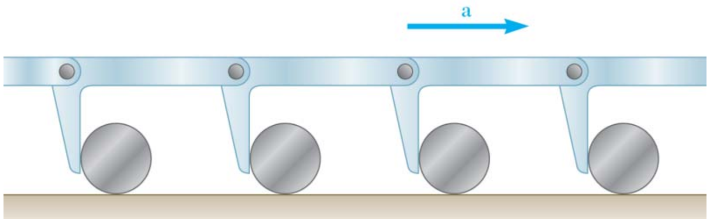
Set up System of Linear Equations (1)→(10):

$$\begin{vmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -20 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.58 & 0 & 0 & 0 & 0 & -2.9 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & -20 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.58 & 0 & 0 & -2.9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.58 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -210 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.03 & -0.7 & -0.03 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \begin{Bmatrix} A_x \\ A_y \\ B_x \\ B_y \\ f_r \\ N_r \\ f_f \\ N_f \\ \bar{a}_x \\ \alpha_w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 196.2 \\ 500 \\ 0 \\ 196.2 \\ 0 \\ 0 \\ 0 \\ 2060.1 \\ -500 \end{Bmatrix}$$

Solving: $\bar{a}_x = 3.23 \text{ m/s}^2$ ◀

PROBLEM 16.156

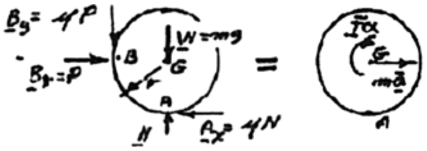
Identical cylinders of mass *m* and radius *r* are pushed by a series of moving arms. Assuming the coefficient of friction between all surfaces to be $\mu < 1$ and denoting by *a* the magnitude of the acceleration of the arms, derive an expression for (*a*) the maximum allowable value of *a* if each cylinder is to roll without sliding, (*b*) the minimum allowable value of *a* if each cylinder is to move to the right without rotating.



SOLUTION

(*a*) Cylinder rolls without sliding $a = r\alpha$ or $\alpha = \dfrac{a}{r}$

P is horizontal component of force that the arm exerts on cylinder.



$$+\circlearrowleft \Sigma M_A = \Sigma (M_A)_{\text{eff}}: \quad Pr - (\mu_k P)r = I\alpha + (m\bar{a})r$$

$$P(1-\mu)r = \frac{1}{2}mr^2\left(\frac{\bar{a}}{r}\right) + (m\bar{a})r$$

$$P = \frac{3}{2}\frac{m\bar{a}}{(1-\mu)} \tag{1}$$

$$+\uparrow \Sigma F_y = 0: \qquad N - \mu P - mg = 0 \tag{2}$$

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \qquad P - \mu N = m\bar{a} \tag{3}$$

Solve (2) for *N* and substitute for *N* into (3).

$$P - \mu^2 P - \mu mg = m\bar{a}$$

Substitute *P* from (1): $$(1-\mu^2)\frac{3}{2}\frac{m\bar{a}}{(1-\mu)} - \mu mg = m\bar{a}$$

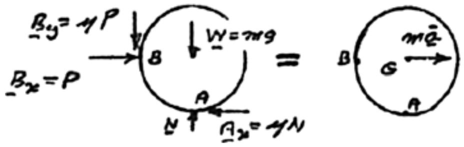
$$3(1+\mu)\bar{a} - 2\mu g = 2\bar{a}$$

$$\bar{a}(1+3\mu) - 2\mu g = 0$$

$$\bar{a} = \frac{2\mu}{1+3\mu}g \blacktriangleleft$$

PROBLEM 16.156 (Continued)

(*b*) Cylinder translates: $\alpha = 0$



Sliding occurs at *A*: $A_x = \mu N$

Assume sliding impends at *B*: $B_y = \mu P$

$$+\circlearrowright \Sigma M_A = \Sigma (M_A)_{\text{eff}}: \quad Pr - \mu Pr = (m\bar{a})r$$

$$P(1-\mu)r = m\bar{a}r$$

$$P = \frac{m\bar{a}}{1-\mu} \tag{4}$$

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad P - \mu N = m\bar{a} \tag{5}$$

$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad N - \mu P - mg = 0 \tag{6}$$

Solve (5) for *N* and substitute for *N* into (6).

$$P - P\mu^2 - \mu mg = m\bar{a}$$

Substitute for *P* from (4):

$$\frac{m\bar{a}}{1-\mu}(1-\mu^2) - \mu mg = m\bar{a}$$

$$\bar{a}(1+\mu) - \mu g = \bar{a}$$

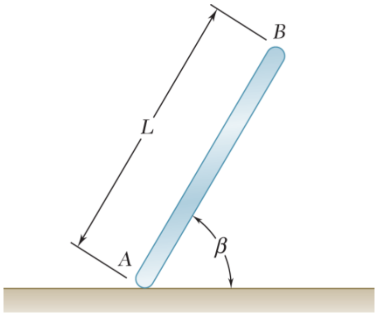
$$\bar{a}\mu - \mu g = 0$$

$\bar{a} = g$ ◀

Summary: $a < \dfrac{2\mu}{1+3\mu} g$: Rolling

$\dfrac{3\mu}{1+3\mu} g < a < g$: Rotating and sliding

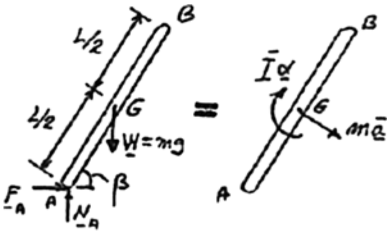
$a > g$: Translation



PROBLEM 16.157

The uniform rod *AB* of weight *W* is released from rest when $\beta = 70°$. Assuming that the friction force between end *A* and the surface is large enough to prevent sliding, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the normal reaction at *A*, (*c*) the friction force at *A*.

SOLUTION



We note that rod rotates about *A*. $\omega = 0$

$$\bar{I} = \frac{1}{12} mL^2$$

$$\bar{a} = \frac{L}{2}\alpha$$

$$+\circlearrowleft \Sigma M_A = \Sigma (M_A)_{\text{eff}}: \quad mg\left(\frac{L}{2}\cos\beta\right) = \bar{I}\alpha + (m\bar{a})\frac{L}{2}$$

$$\frac{1}{2} mgL\cos\beta = \frac{1}{12} mL^2\alpha + \left(m\frac{L}{2}\alpha\right)\frac{L}{2}$$

$$= \frac{1}{3} mL^2\alpha$$

$$\alpha = \frac{3}{2}\frac{g\cos\beta}{L} \tag{1}$$

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad F_A = m\bar{a}\sin\beta$$

$$F_A = m\frac{L}{2}\alpha\sin\beta = m\frac{L}{2}\left(\frac{3}{2}\frac{g\cos\beta}{L}\right)\sin\beta$$

$$F_A = \frac{3}{4} mg\sin\beta\cos\beta \tag{2}$$

$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad N_A - mg = -m\bar{a}\cos\beta = -m\left(\frac{L}{2}\alpha\right)\cos\beta$$

$$N_A - mg = -m\frac{L}{2}\left(\frac{3}{2}\frac{g\cos\beta}{L}\right)\cos\beta$$

$$N_A = mg\left(1 - \frac{3}{4}\cos^2\beta\right) \tag{3}$$

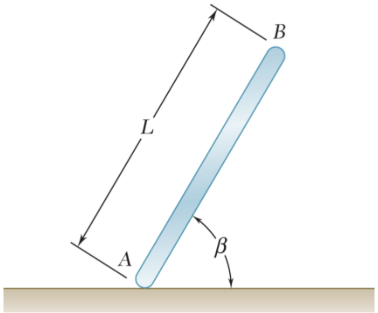
PROBLEM 16.157 (Continued)

For $\beta = 70°$:

(*a*) Eq. (1): $\alpha = \frac{3}{2}\frac{g\cos 70°}{L}$ $\boldsymbol{\alpha} = 0.513\frac{g}{L} \circlearrowright$ ◀

(*b*) Eq. (3): $N_A = mg\left(1 - \frac{3}{4}\cos^2 70°\right)$ $\mathbf{N}_A = 0.912mg \uparrow$ ◀

(*c*) Eq. (2) $F_A = \frac{3}{4}mg\sin 70°\cos 70°$ $\mathbf{F}_A = 0.241mg \longrightarrow$ ◀

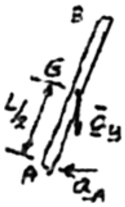


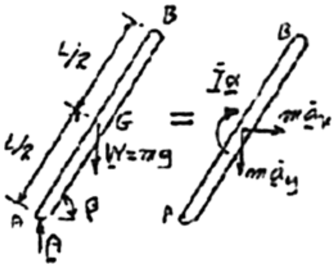
PROBLEM 16.158

The uniform rod *AB* of weight *W* is released from rest when $\beta = 70°$. Assuming that the friction force is zero between end *A* and the surface, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the acceleration of the mass center of the rod, (*c*) the reaction at *A*.

SOLUTION

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \qquad a = m\bar{\mathbf{a}}_x \quad \bar{\mathbf{a}}_x = 0$$





$$\mathbf{a}_A = \bar{\mathbf{a}}_y + \frac{L}{2}\alpha \searrow \beta$$

$$+\!\downarrow 0 = a_y - \frac{L}{2}\alpha\cos\beta$$

$$\mathbf{a}_y = \frac{L}{2}\alpha\cos\beta \downarrow$$

$$+\!\downarrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \qquad mg - A = m\bar{a}_y = m\left(\frac{L}{2}\alpha\cos\beta\right) \qquad (1)$$

$$+\circlearrowright \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad A\left(\frac{L}{2}\cos\beta\right) = \bar{I}\alpha = \frac{1}{12}mL^2\alpha$$

$$A = \frac{mL}{6}\frac{\alpha}{\cos\beta} \qquad (2)$$

Substitute (2) into (1):

$$mg - \frac{mL}{6}\frac{\alpha}{\cos\beta} = m\frac{L}{2}\alpha\cos\beta$$

$$g = \left(\frac{L}{2}\cos\beta + \frac{L}{6\cos\beta}\right)\alpha$$

$$g = \frac{L}{6}\left(3\cos\beta + \frac{1}{\cos\beta}\right)\alpha$$

$$g = \frac{L}{6}\left(\frac{3\cos^2\beta + 1}{\cos\beta}\right)\alpha$$

$$\alpha = \frac{6g}{L}\left(\frac{\cos\beta}{1 + 3\cos^2\beta}\right)\circlearrowright$$

PROBLEM 16.158 (Continued)

$$\bar{\mathbf{a}} = \frac{L}{2}\alpha\cos\beta = \frac{L}{2}\left(\frac{6g}{L}\cdot\frac{\cos\beta}{1+3\cos^2\beta}\right)\cos\beta = 3g\left(\frac{\cos^2\beta}{1+3\cos^2\beta}\right)\leftarrow$$

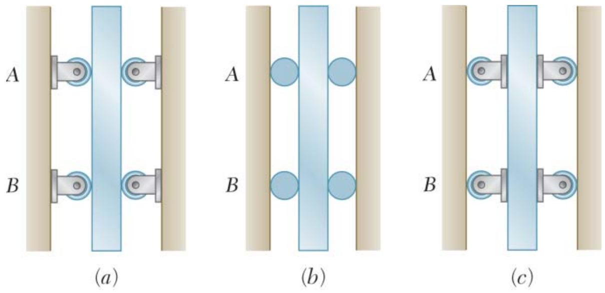
$$\mathbf{A} = \frac{mL}{6}\cdot\frac{\alpha}{\cos\beta} = \frac{mL}{6}\cdot\left(\frac{6g}{L}\cdot\frac{\cos\beta}{1+3\cos^2\beta}\right)\frac{1}{\cos\beta} = mg\frac{1}{1+3\cos^2\beta}\uparrow$$

For $\beta = 70°$:

(*a*) $$\alpha = \frac{6g}{L}\frac{\cos 70°}{1+3\cos^2 70°}$$ $\boldsymbol{\alpha} = 1.519\dfrac{g}{L}\circlearrowright$ ◀

(*b*) $$\bar{a} = 3g\frac{\cos^2 70}{1+3\cos^2 70°}$$ $\bar{\mathbf{a}} = 0.260g\downarrow$ ◀

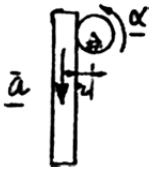
(*c*) $$A = mg\frac{1}{1+3\cos^2 70°}$$ $\mathbf{A} = 0.740\,mg\uparrow$ ◀



PROBLEM 16.159

A bar of mass $m = 5$ kg is held as shown between four disks, each of mass $m' = 2$ kg and radius $r = 75$ mm. Knowing that the normal forces on the disks are sufficient to prevent any slipping, for each of the cases shown determine the acceleration of the bar immediately after it has been released from rest.

SOLUTION

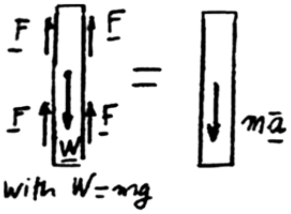


(*a*) Configuration (*a*)

Kinematics:

$$\bar{a} = r\alpha \qquad \alpha = \frac{\bar{a}}{r}$$

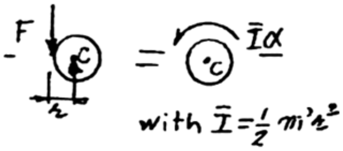
Kinetics of bar



$$+\!\downarrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}:$$

$$W - 4F = m\bar{a} \qquad (1)$$

Kinetics of one disk



$$+\circlearrowright \Sigma M_C = \Sigma (M_C)_{\text{eff}}: \quad Fr = \bar{I}\alpha$$

$$Fr = \frac{1}{2}m'r^2\left(\frac{\bar{a}}{r}\right)$$

$$F = \frac{1}{2}m'\bar{a} \qquad (2)$$

Substitute for F from (2) into (1).

$$mg - 4\left(\frac{1}{2}m'\bar{a}\right) = m\bar{a}$$

$$mg = (m + 2m')\bar{a} \qquad \bar{a} = \frac{mg}{m + 2m'}$$

$$m = 5 \text{ kg}, \quad m' = 2 \text{ kg}: \quad \bar{a} = \frac{5}{5 + 2(2)}g \qquad \bar{\mathbf{a}} = \frac{5}{9}g\downarrow \blacktriangleleft$$

(*b*) Configuration (*b*)

Kinematics:

Disk is rolling on vertical wall

$$\bar{a}' = a_C = r\alpha$$

$$\bar{a} = a_r = 2r\alpha$$

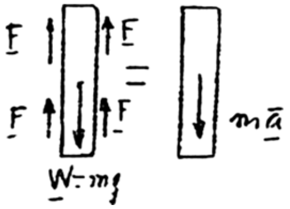
Therefore:

$$\alpha = \frac{\bar{a}}{2r}$$

$$\bar{a}' = r\alpha = \frac{1}{2}\bar{a}$$

PROBLEM 16.159 (Continued)

Kinetics of bar



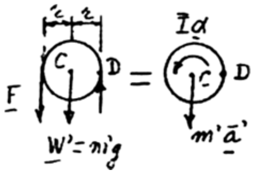
$$+\downarrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}:$$

$$W - 4F = m\bar{a}$$

$$mg - 2(2F) = m\bar{a} \qquad (1)$$

Kinetics of one disk

$$\bar{I} = \frac{1}{2} m' r^2$$



$$+\circlearrowright \Sigma M_D = \Sigma (M_D)_{\text{eff}}:$$

$$F(2r) + m'gr = \bar{I}\alpha + m'\bar{a}'r$$

$$2Fr = m'gr = \frac{1}{2} m' r^2 \left(\frac{\bar{a}}{2r} \right) + m' \frac{\bar{a}}{2} r$$

$$2F = m' \left(\frac{1}{4} \bar{a} + \frac{1}{2} \bar{a} \right) - m'g \qquad (2)$$

Substitute for $2F$ from (2) into (1):

$$mg - 2m' \left(\frac{3}{4} \bar{a} \right) + 2m'g = m\bar{a}$$

$$\left(m + \frac{3}{2} m' \right) \bar{a} = (m + 2m')g \qquad \bar{a} = \frac{m + 2m'}{m + \frac{3}{2} m'} g$$

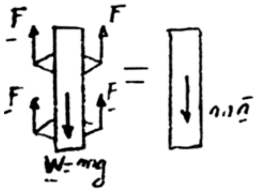
$$m = 5\text{ kg}, \quad m' = 2\text{ kg}: \quad \bar{a} = \frac{5+4}{5+3} g \qquad \bar{\mathbf{a}} = \frac{9}{8} g \downarrow \blacktriangleleft$$

(*c*) Configuration (*c*)

Kinematics:

Disk is rolling on vertical wall $\quad \bar{a} = \bar{a}' = r\alpha$

Kinetics of bar



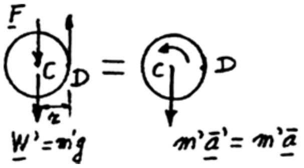
$$+\downarrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}:$$

$$W - 4F = m\bar{a}$$

$$mg - 4F = m\bar{a} \qquad (1)$$

Kinetics of one disk

$$\bar{I}\alpha = \frac{1}{2} m' r^2 \left(\frac{\bar{a}}{r} \right)$$



$$+\circlearrowright \Sigma M_D = \Sigma (M_D)_{\text{eff}}:$$

$$(F + w')r = \bar{I}\alpha + m'\bar{a}'r$$

$$(F + m'g)r = \frac{1}{2} m' r' \left(\frac{\bar{a}}{r} \right) + m'\bar{a}t$$

$$F = \frac{3}{2} m'\bar{a} - m'g \qquad (2)$$

Substitute for F from (2) into (1):

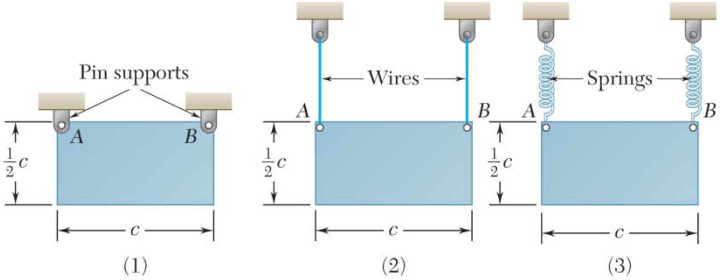
$$mg - 6m'\overline{a} + 4m'g = m\overline{a}$$

$$(m + 6m')\overline{a} = (m + 4m')g \quad \overline{a} = \frac{m + 4m'}{m + 6m'}g$$

$$m = 5\text{ kg}, \quad m' = 2\text{ kg}: \quad \overline{a} = \frac{5 + 8}{5 + 12} \qquad \overline{\mathbf{a}} = \frac{13}{17}g \downarrow ◀$$

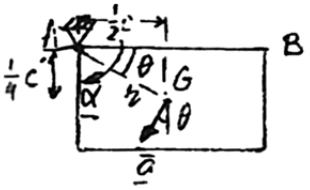
PROBLEM 16.160

A uniform plate of mass *m* is suspended in each of the ways shown. For each case determine immediately after the connection at *B* has been released (*a*) the angular acceleration of the plate, (*b*) the acceleration of its mass center.


Pin supports
A
B
$\frac{1}{2}c$
c
(1)
Wires
A
B
$\frac{1}{2}c$
c
(2)
Springs
A
B
$\frac{1}{2}c$
c
(3)

SOLUTION

(1) Plate attached to pins


A
$\frac{1}{2}c$
B
$\frac{1}{4}c$
θ
G
r
α
θ
$\bar{a}$

Kinematics: Assume $\alpha \circlearrowright$ $(\omega = 0)$

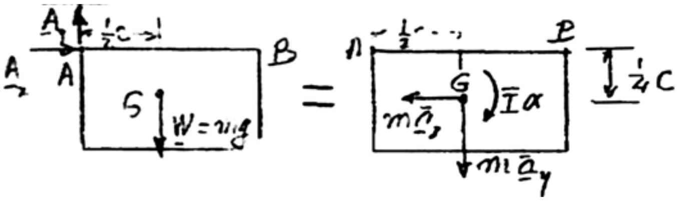
$$\bar{\mathbf{a}} = r\alpha \measuredangle \theta$$

$\xleftarrow{+}$ *x* comp: $\bar{a}_x = r\alpha \sin\theta = (r\sin\theta)\alpha$

$+\!\downarrow$ *y* comp: $\bar{a}_y = r\alpha\cos\theta = (r\cos\theta)\alpha$

Thus: $\bar{\mathbf{a}}_x = \frac{1}{4}c\alpha \longleftarrow; \quad \bar{\mathbf{a}}_y = \frac{1}{2}c\alpha \downarrow$ (1)

Kinetics: $\bar{I} = \frac{1}{12}m\left[c^2 + \left(\frac{c}{2}\right)^2\right] = \frac{5}{48}mc^2$


A_y
$\frac{1}{2}c$
A_x
A
B
G
$W = mg$
=
A
$\frac{1}{2}c$
B
G
$\bar{I}\alpha$
$m\bar{a}_x$
$m\bar{a}_y$
$\frac{1}{4}c$

(*a*) $+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad W\left(\frac{c}{2}\right) = \bar{I}\alpha + (m\bar{a}_x)\left(\frac{c}{4}\right) + (m\bar{a}_y)\left(\frac{c}{2}\right)$

$$\frac{1}{2}mgc = \frac{5}{48}mc^2\alpha + m\left(\frac{1}{4}c\alpha\right)\left(\frac{c}{4}\right) + m\left(\frac{1}{2}c\alpha\right)\left(\frac{c}{2}\right)$$

$$\frac{1}{2}mgc = \frac{20}{48}mc^2\alpha \qquad \alpha = 1.2\frac{g}{c}$$

$\alpha = 1.2\frac{g}{c} \circlearrowright$ ◀

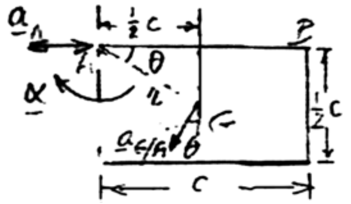
PROBLEM 16.160 (Continued)

(*b*) From (1): $\bar{a}_x = \frac{1}{4}c\alpha = \frac{1}{4}(1.2g)$ $\quad \bar{\mathbf{a}}_x = 0.3g \leftarrow$

$\bar{a}_y = \frac{1}{2}c\alpha = \frac{1}{2}(1.2g)$ $\quad \bar{\mathbf{a}}_y = 0.6g \downarrow$ $\qquad \bar{\mathbf{a}} = 0.671g \swarrow 63.4°$ ◀

(2) Plate suspended from wires.

Kinematics: Assume $\alpha\circlearrowright$ $(\boldsymbol{\omega} = 0)$



$\underline{a}_A$
$\frac{1}{2}c$
P
θ
$\underline{\alpha}$
r
$\frac{1}{2}c$
G
$\underline{a}_{G/A}$
θ
c

$$\bar{\mathbf{a}} = \mathbf{a}_C = \mathbf{a}_A + \mathbf{a}_{G/A}$$
$$= a_A \leftrightarrow + r\alpha \swarrow \theta$$

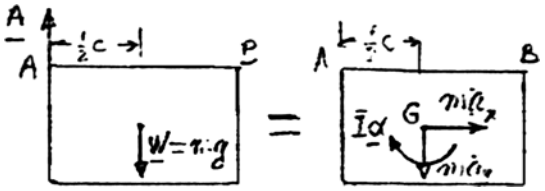
$+\uparrow$ *y* comp.

$$\bar{a}_y = 0 - r\alpha\cos\theta$$
$$= -(r\cos\theta)\alpha$$

$$r\cos\theta = \frac{1}{2}c$$

Thus: $\bar{a}_y = -\frac{1}{2}c\alpha \qquad \bar{\mathbf{a}}_y = \frac{1}{2}c\alpha\downarrow$ (2)

Kinetics: $\bar{I} = \frac{1}{12}m\left[c^2 + \left(\frac{c}{2}\right)^2\right] = \frac{I}{48}mc^2$



$\underline{A}$
A
$\frac{1}{2}c$
P
$\underline{W} = mg$
=
A
$\frac{1}{2}c$
B
$\bar{I}\underline{\alpha}$
G
$m\bar{a}_x$
$m\bar{a}_y$

$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\qquad 0 = m\bar{a}_2$

$$\bar{a}_x = 0$$

$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}$: $\quad W\left(\frac{1}{2}c\right) = \bar{I}\alpha + (m\bar{a}_y)\left(\frac{1}{2}c\right)$

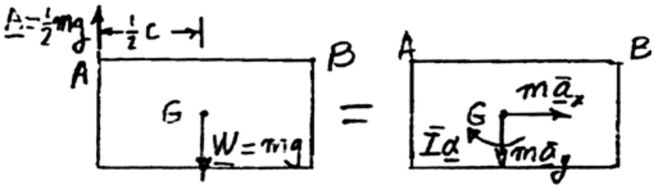
Recalling (1): $\frac{1}{2}mgc = \frac{5}{48}mc^2\alpha + \left(\frac{1}{2}mc\alpha\right)\left(\frac{1}{2}c\right)$

$\frac{1}{2}mgc = \frac{17}{48}mc^2\alpha \quad \alpha = \frac{24}{17}\frac{g}{c}$ $\qquad \alpha = \frac{24}{17}\frac{g}{c}\circlearrowright$ ◀

$\bar{a}_y = \frac{1}{2}c\alpha = \frac{1}{2}c\left(\frac{24}{17}\frac{g}{c}\right) = \frac{12}{17}g$ $\qquad \bar{\mathbf{a}} = \frac{12}{17}g\downarrow$ ◀

PROBLEM 16.160 (Continued)

(3) Plate suspended from springs. Immediately after spring B is released, the tension in spring A is still $\frac{1}{2}mg$ since its elongation is unchanged.



(*a*) Angular acceleration.

$$+\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \quad \left(\frac{1}{2}mg\right)\left(\frac{1}{2}c\right) = \bar{I}\alpha$$

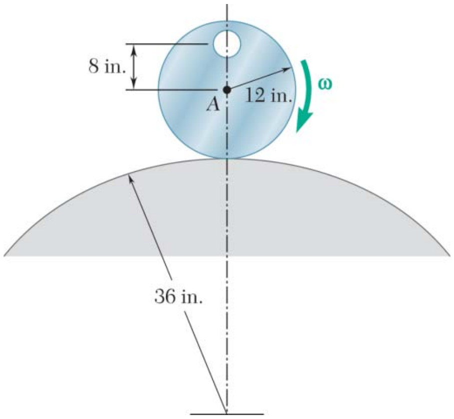
$$\frac{1}{4}mgc = \frac{5}{48}mc^2\alpha \qquad \boldsymbol{\alpha} = 2.4\frac{g}{c} \circlearrowright \blacktriangleleft$$

(*b*) Acceleration at mass center.

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad 0 = m\bar{a}_x \quad \bar{a}_x = 0$$

$$+\downarrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad mg - \frac{1}{2}mg = m\bar{a}_y$$

$$\bar{a}_y = \frac{1}{2}g \qquad \bar{\mathbf{a}} = 0.5g\downarrow \blacktriangleleft$$



PROBLEM 16.161

A cylinder with a circular hole is rolling without slipping on a fixed curved surface as shown. The cylinder would have a weight of 16 lb without the hole, but with the hole it has a weight of 15 lb. Knowing that at the instant shown the disk has an angular velocity of 5 rad/s clockwise, determine (*a*) the angular acceleration of the disk, (*b*) the components of the reaction force between the cylinder and the ground at this instant.

SOLUTION

Geometry: Let the mass center G of the cylinder lie a distance b below the geometric center for the position shown. Let C, the contact point between the cylinder and the fixed curved surface, be the origin of a coordinate system, as shown. The position vector of a point is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

Let r be the radius of the cylinder and R that of the fixed curved surface

Kinematics:

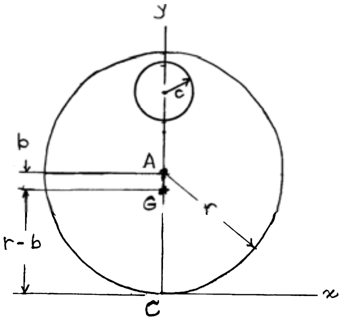
The acceleration $\mathbf{a}_P$ at a point is given by

$$\mathbf{a}_P = \mathbf{a}_C + \boldsymbol{\alpha} \times \mathbf{r}_{P/C} - \omega^2 \mathbf{r}_{P/C}$$

Let

$$\boldsymbol{\alpha} = \alpha \curvearrowright$$

Then, using the coordinate system



$$\begin{aligned}\mathbf{a}_P &= [(a_C)_x \longrightarrow] + [(a_C)_y \uparrow] \\ &+ [y\alpha \longrightarrow] + [x\alpha \downarrow] \\ &+ [y\omega^2 \downarrow] + [x\omega^2 \longleftarrow]\end{aligned}$$

Since the cylinder rolls without slipping on a fixed surface,

$$(a_C)_x = 0$$

For Points G and A,

$$\mathbf{a}_G = [(a_C)_y \uparrow] + [(r-b)\alpha \longrightarrow] + [(r-b)\omega^2 \downarrow] \tag{1}$$

$$\mathbf{a}_A = [(a_C)_y \uparrow] + [r\alpha \longrightarrow] + [r\omega^2 \downarrow] \tag{2}$$

Subtract Eq. (2) from Eq. (1) to eliminate $(a_C)_y$

$$\mathbf{a}_G = \mathbf{a}_A = [b\alpha \longleftarrow] + [b\omega^2 \uparrow]$$

$$\begin{aligned}\mathbf{a}_G &= \mathbf{a}_A + [b\alpha \longleftarrow] + [b\omega^2 \uparrow] \\ &= [(a_A)_x \longrightarrow] + [(\mathbf{a}_A)_y \uparrow] + [b\alpha \longleftarrow] + [b\omega^2 \uparrow] \\ &= [(r-b)\alpha \longrightarrow] + [(\mathbf{a}_A)_y \uparrow] + [b\omega^2 \uparrow]\end{aligned} \tag{3}$$

PROBLEM 16.161 (Continued)

Point C is the instantaneous center, so that

$$\mathbf{v}_A = r\omega \longrightarrow$$

Point A is constrained to move on a circle of radius

$$\rho = R + r$$

so its vertical component of acceleration is

$$(\mathbf{a}_A)_y = \frac{v_A^2}{\rho}\downarrow = \frac{r^2\omega^2}{\rho}\downarrow$$

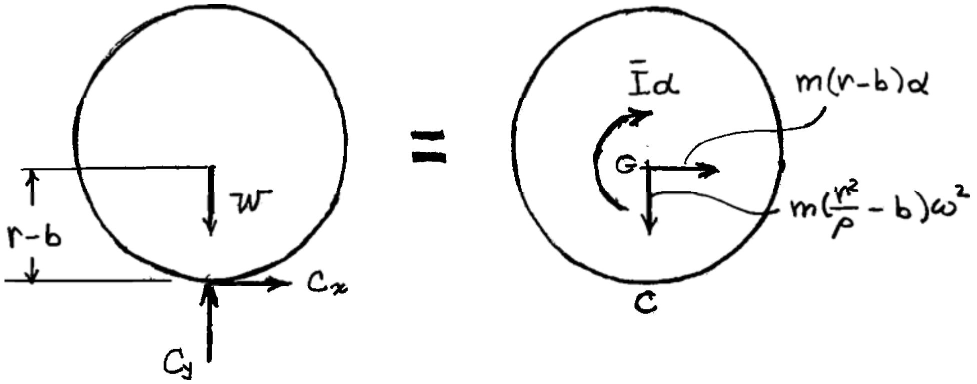
Using Eq. 3,
$$\mathbf{a}_G = [(r-b)\alpha \longrightarrow] + \left(\frac{r^2}{\rho} - b\right)\downarrow$$

The effective force at the mass center is

$$m\mathbf{a}_G = [m(r-b)\alpha \longrightarrow] + \left(\frac{r^2}{\rho} - b\right)\omega^2\downarrow$$

Kinetics:

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad 0 + \bar{I}\alpha + (r-b)m(r-b)\alpha$$
$$= [\bar{I} + m(r-b)^2]\alpha$$



(*a*) *Angular acceleration.* $\boldsymbol{\alpha} = 0$ ◀

(*b*) *Force components at C.*

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad C_x = m(r-b)\alpha = 0$$

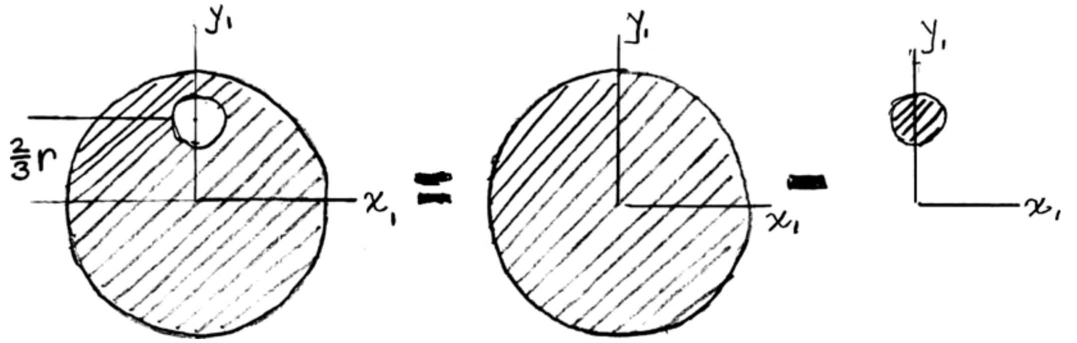
$\mathbf{C}_x = 0$ ◀

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad C_y - W = -m\left(\frac{r^2}{\rho} - b\right)\omega^2$$

$$C_y = W - \frac{W}{g}\left(\frac{r^2}{\rho} - b\right)\omega^2$$

It remains to determine the distance *b* from the mass distribution of the cylinder.

The mass center *G* coincides with the centroid of a circular cylinder of area $A_1 = \pi r^2$ with a circular cut out of area $A_2 = \frac{1}{16}A,$ with its center located $\frac{2}{3}r$ above the center of A_1.



	A_1	$\overline{y}_1$	$A_1\overline{y}_1$
(1)	πr^2	0	0
(2)	$-\frac{1}{16}\pi r^2$	$\frac{2}{3}r$	$-\frac{1}{24}\pi r^3$
Σ	$\frac{15}{16}\pi r^2$		$-\frac{1}{24}\pi r^3$

$$\overline{Y}\Sigma A = \Sigma A\overline{y}_1$$

$$\frac{15}{16}\pi r^2\overline{Y} = -\frac{1}{24}\pi r^3$$

$$\overline{Y} = -\frac{2}{45}r$$

$$b = \frac{2}{45}r$$

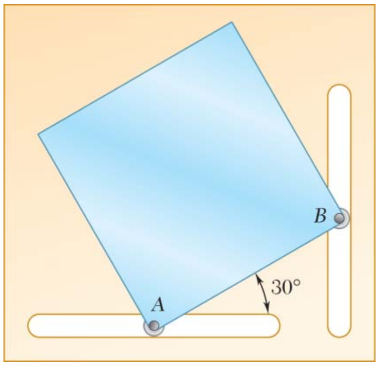
Data: $r = 12 \text{ in.} = 1 \text{ ft}, \quad R = 36 \text{ in.}, \quad \rho = 48 \text{ in.}$

$$b = \frac{2}{45}(12) = 0.53333 \text{ in.}$$

$$\left(\frac{r^2}{\rho} - b\right) = \frac{144}{48} - 0.53333 = 2.4667 \text{ in.} = 0.20556 \text{ ft}$$

$$C_y = 15 \text{ lb} - \frac{15 \text{ lb}}{32.2 \text{ ft/s}^2}(0.20556 \text{ ft})(5 \text{ rad/s})^2$$

$$\mathbf{C} = 12.61 \text{ lb}\uparrow \blacktriangleleft$$



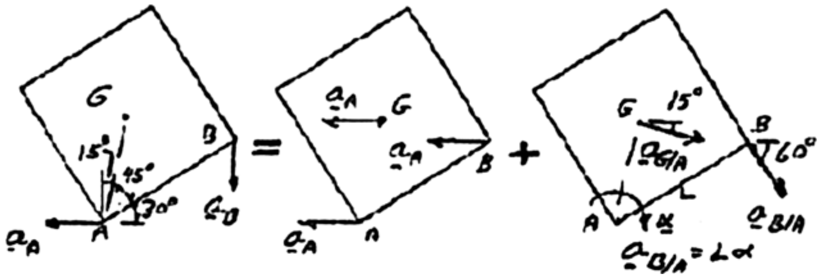
PROBLEM 16.162

The motion of a square plate of side 150 mm and mass 2.5 kg is guided by pins at corners *A* and *B* that slide in slots cut in a vertical wall. Immediately after the plate is released from rest in the position shown, determine (*a*) the angular acceleration of the plate, (*b*) the reaction at corner *A*.

SOLUTION

Kinematics:

$$AG = \frac{L}{2}\sqrt{2} = \frac{L}{\sqrt{2}} \qquad a_{G/A} = (AB)\alpha = \frac{L\alpha}{\sqrt{2}}$$

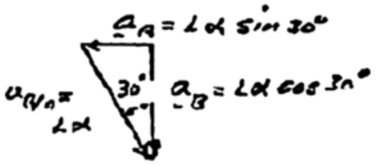

G
15°
45°
30°
B
a_B
a_A
A
=
a_A
G
a_A
B
a_A
A
+
G
15°
B
60°
$a_{G/A}$
L
A
α
$a_{B/A}$
$a_{B/A} = L\alpha$

Plane motion = Translation + Rotation

$$a_B \downarrow = a_A \leftarrow + \, a_{B/A} \searrow 60°$$

$$a_B \downarrow = a_A \leftarrow + \, L\alpha \searrow 60°$$

$$\bar{a} = a_A \leftarrow + \, a_{G/A} \searrow 15°$$


$a_A = L\alpha \sin 30°$
30°
$a_B = L\alpha \cos 30°$
$a_{B/A} = L\alpha$

$$\bar{\mathbf{a}} = [L\alpha \sin 30° \leftarrow] + \left[\frac{L\alpha}{\sqrt{2}} \searrow 15°\right] = [0.5L\alpha \leftarrow] + [0.707L\alpha \searrow 15°]$$

Law of cosines

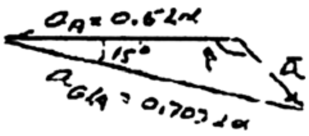
$$\bar{a}^2 = a_A^2 \cdot a_{G/A}^2 - 2a_A a_{G/A}\cos 15°$$

$$\bar{a}^2 = (0.5L\alpha)^2 + (0.707L\alpha)^2 - 2(0.5L\alpha)(0.707L\alpha)\cos 15°$$

$$\bar{a}^2 = L^2\alpha^2(0.25 + 0.5 - 0.68301)$$

$$a^2 = L^2\alpha^2(0.06699)$$

$$\bar{a} = 0.25882L\alpha$$


$a_A = 0.5L\alpha$
15°
β
$\bar{a}$
$a_{G/A} = 0.707L\alpha$

PROBLEM 16.162 (Continued)

Law of sines.

$$\frac{\bar{a}}{\sin 15^\circ} = \frac{a_{G/A}}{\sin\beta}; \quad \sin\beta = \frac{a_{G/A}}{\bar{a}}\sin 15^\circ = \frac{0.707L\alpha}{0.25950L\alpha}\sin 15^\circ$$

$$\sin\beta = 0.707; \quad \beta = 135^\circ$$

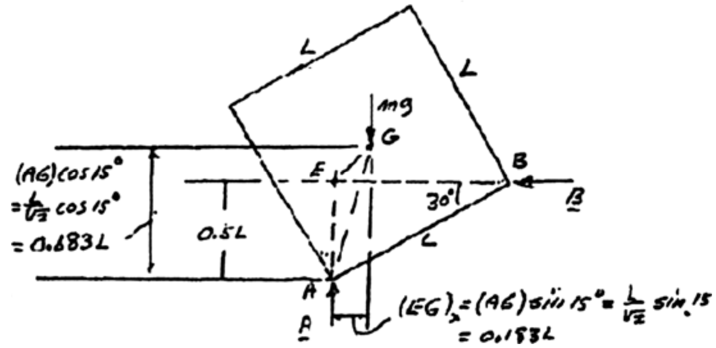
$$\bar{\mathbf{a}} = 0.2583L\alpha \searrow 45^\circ$$

Kinetics: $(\omega = 0)$

We find the location of Point E where lines of action of A and B intersect.

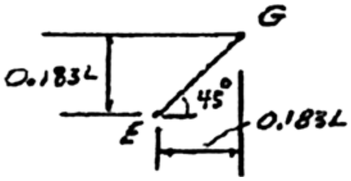
$$MG = \frac{L}{\sqrt{2}}$$

$$\measuredangle EAG = 15^\circ$$

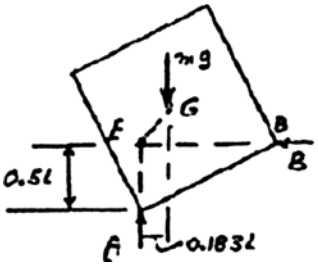

L
L
mg
G
E
B
B
30°
L
A
A
(AG) cos 15°
= L/√2 cos 15°
= 0.683L
0.5L
(EG)y = (AG) sin 15° = L/√2 sin 15
= 0.183L

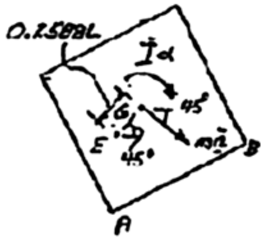
$$(EG)_y = 0.6829L - 0.5L = 0.183L$$

$$(EG) = (0.183L)\sqrt{2} = 0.2588L$$


G
0.183L
45°
E
0.183L

$$\bar{I} = \frac{1}{6}mL^2$$


mg
G
E
B
B
0.5L
A
0.183L


0.2588L
Īα
45°
G
E
45°
mā
B
A

PROBLEM 16.162 (Continued)

(*a*) Angular acceleration.

$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad mg(0.183L) = \bar{I}\alpha + (m\bar{a})(0.2588L)$$

$$0.183mgL = \frac{1}{6}mL^2\alpha + m(0.2588L\alpha)(0.2588L)$$

$$0.183gL = L^2\alpha\left(\frac{1}{6} + 0.06698\right)$$

$$0.183\frac{g}{L} = 0.2336\alpha; \quad \alpha = 0.7834\frac{g}{L}$$

$$\alpha = 0.7834\frac{9.81 \text{ m/s}^2}{0.15 \text{ m}}$$

$\boldsymbol{\alpha} = 51.2 \text{ rad/s}^2 \circlearrowright$ ◀

(*b*) Reaction at corner *A*.

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \quad A - mg = -m\bar{a}\sin 45^\circ$$

$$= -m(0.2588L\alpha)\sin 45^\circ$$

$$= -m(0.2588L)\left(0.7834\frac{g}{L}\right)\sin 45^\circ$$

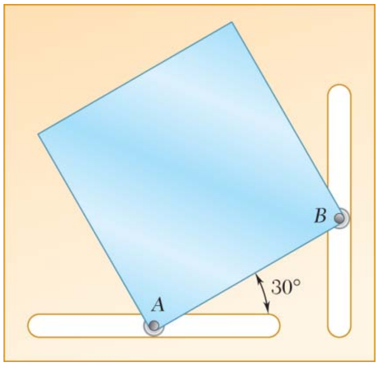
$$A - mg = 0.1434mg$$

$$A = 0.8566mg$$

$$= 0.8566(2.5 \text{ kg})(9.81 \text{ m/s}^2)$$

$$= 21.01 \text{ N}$$

$\mathbf{A} = 21.0 \text{ N}\uparrow$ ◀

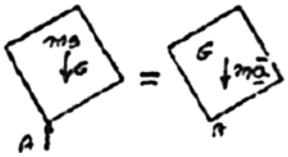


PROBLEM 16.163

Solve Problem 16.162, assuming that the plate is fitted with a single pin at corner *A*.

Problem 16.162 The motion of a square plate of side 150 mm and mass 2.5 kg is guided by pins at corners *A* and *B* that slide in slots cut in a vertical wall. Immediately after the plate is released from rest in the position shown, determine (*a*) the angular acceleration of the plate, (*b*) the reaction of corner *A*.

SOLUTION

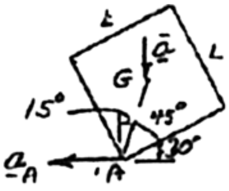


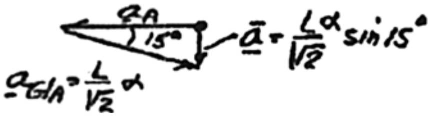
Since both *A* and *mg* are vertical, $\bar{a}_x = 0$ and $\bar{\mathbf{a}}$ is $\downarrow$

Kinematics:

$$AG = \frac{L}{\sqrt{2}} \searrow 15^\circ \quad \mathbf{a}_{G/A} = (AG)\alpha \searrow 15^\circ$$

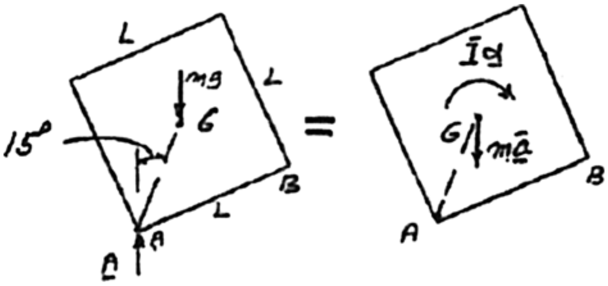
$$\bar{\mathbf{a}}\downarrow = \mathbf{a}_A \longleftarrow + \mathbf{a}_{G/A} \searrow 15^\circ$$





$$\bar{\mathbf{a}} = 0.183L\alpha \searrow 15^\circ$$

Kinetics: $$\bar{I} = \frac{1}{6}mL^2$$



PROBLEM 16.163 (Continued)

(*a*) Angular acceleration.

$$\circlearrowright\!\!+\ \Sigma M_A = \Sigma(M_A)_{\text{eff}}:\quad mg(AG)\sin 15° = \bar{I}\alpha + m\bar{a}(AG)\sin 15°$$

$$mg\left(\frac{L}{\sqrt{2}}\right)\sin 15° = \frac{1}{6}mL^2\alpha + m(0.183L\alpha)\left(\frac{L}{\sqrt{2}}\right)\sin 15°$$

$$0.183\frac{g}{L} = \left(\frac{1}{6} + 0.033494\right)\alpha$$

$$0.183\frac{g}{L} = 0.2002\alpha$$

$$\alpha = 0.9143\frac{g}{L}$$

$$= 0.9143\frac{9.81\text{ m/s}^2}{0.15\text{ m}}$$

$\boldsymbol{\alpha} = 59.8\text{ rad/s}^2 \circlearrowright$ ◀

(*b*) Reaction at corner *A*.

$$+\!\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}:\quad A - mg = -m\bar{a}$$

$$A - mg = -m(0.183L\alpha)$$

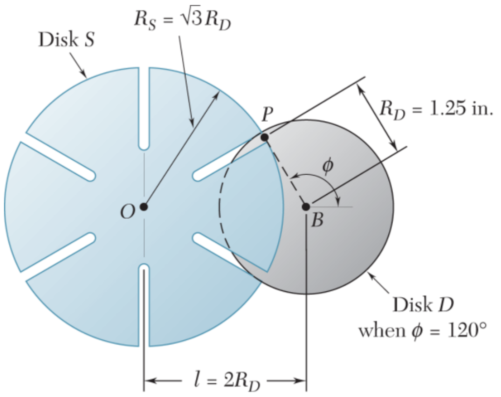
$$= -m(0.183L)\left(0.9143\frac{g}{L}\right)$$

$$A - mg = -0.1673mg$$

$$A = 0.8326mg$$

$$A = 0.8326(2.5\text{ kg})(9.81\text{ m/s}^2)$$

$\mathbf{A} = 20.4\text{ N}\uparrow$ ◀

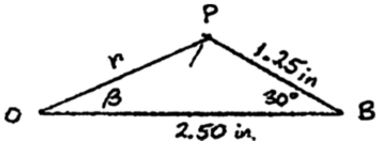


PROBLEM 16.164

The Geneva mechanism shown is used to provide an intermittent rotary motion of disk S. Disk D weighs 2 lb and has a radius of gyration of 0.9 in. and disk S weighs 6 lb and has a radius of gyration of 1.5 in. The motion of the system is controlled by a couple **M** applied to disk D. A pin P is attached to disk D and can slide in one of the six equally spaced slots cut in disk S. It is desirable that the angular velocity of disk S be zero as the pin enters and leaves each of the six slots; this will occur if the distance between the centers of the disks and the radii of the disks are related as shown. Knowing disk D rotates with a constant counterclockwise angular velocity of 8 rad/s and the friction between the slot and pin P is negligible, determine when $\phi = 150°$ (*a*) the couple **M**, (*b*) the magnitude of the force pin P applies to disk S.

SOLUTION

Geometry:



Law of cosines. $r^2 = 1.25^2 + 2.50^2 - (2)(1.25)(2.50)\cos 30°$

$$r = 1.54914 \text{ in.}$$

Law of sines. $\dfrac{\sin\beta}{1.25} = \dfrac{\sin 30°}{r}$

$$\beta = 23.794°$$

Let disk S be a rotating frame of reference.

$$\mathbf{\Omega} = \omega_S \circlearrowright, \quad \dot{\mathbf{\Omega}} = \alpha_S \circlearrowright$$

Motion of coinciding Point P′ on the disk.

$$\mathbf{v}_{P'} = r\omega_S = 1.54914\omega_S \searrow \beta$$

$$\mathbf{a}_{P'} = -\alpha_S \mathrm{k} \times \mathrm{r}_{P/O} - \omega_S^2 \mathrm{r}_{P/O} = [1.54914\alpha_S \searrow \beta] + [1.54914\omega_S^2 \swarrow \beta]$$

Motion relative to the frame.

$$\mathbf{v}_{P/S} = u \swarrow \beta \qquad \mathrm{a}_{P/S} = \dot{u} \swarrow \beta$$

Coriolis acceleration. $2\omega_S u \nwarrow \beta$

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/S} = [1.54914\omega_S \searrow \beta] + [u \swarrow \beta]$$

$$\mathbf{a}_P = \mathbf{a}_P + \mathbf{a}_{P/S} + 2\omega_S u \nwarrow$$

$$= [1.54914\alpha_S \searrow \beta] + [1.54914\omega_S^2 \swarrow \beta] + [\dot{u} \swarrow \beta] + [2\omega_S u \nwarrow \beta]$$

Motion of disk D. (rotation about *B*)

$$\mathbf{v}_P = (BP)\omega_D = (1.25)(8) = 10 \text{ in./s} \swarrow 30°$$

$$\mathbf{a}_P = [(BP)\alpha_D \swarrow 60°] + [(BP)\omega_S^2 \searrow 30°] = 0 + [(1.25)(8)^2 \searrow 30°]$$

$$= 80 \text{ in./s}^2 \searrow 30°$$

Equate the two expressions for $\mathbf{v}_P$ and resolve into components.

$$\searrow \beta: \quad 1.54914\omega_S = 10\cos(30° + \beta)$$

$$\omega_S = \frac{10\cos 53.794°}{1.54914}$$
$$= 3.8130 \text{ rad/s}$$

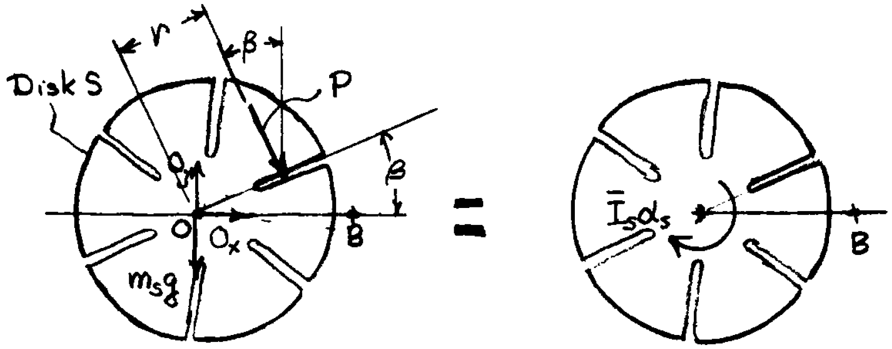
$$\swarrow \beta: \quad u = 10\sin(30° + \beta) = 10\sin 53.794° = 8.0690 \text{ in./s}$$

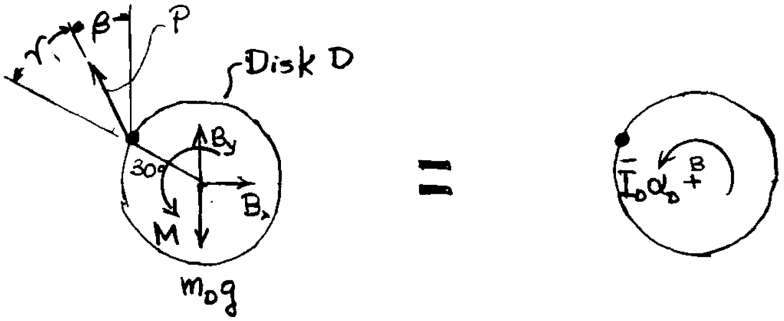
Equate the two expressions for $\mathbf{a}_P$ and resolve into components.

$$\searrow \beta: \quad 1.54914\alpha_S - 2\omega_S u = 80\sin\,(30° + \beta)$$

$$\alpha_S = \frac{80\sin 53.794° + (2)(3.8130)(8.0690)}{1.54914}$$
$$= 81.391 \text{ rad/s}^2$$

Kinetics:





$$\bar{I}_s\alpha_s = \left(\frac{6\text{ lb}}{32.2\text{ ft/s}^2}\right)\left(\frac{1.5}{12}\text{ ft}\right)^2 (81.391\text{ rad/s}^2) = 0.23697\text{ lb}\cdot\text{ft}$$

$$\bar{I}\alpha_D = 0 \qquad \text{since} \quad \boldsymbol{\alpha}_D = 0$$

Disk S:

$$\circlearrowright \Sigma M_0 = \Sigma M_{\text{eff}}: \quad Pr = \bar{I}_S\alpha_S$$

$$P\left(\frac{1.54914}{12}\text{ ft}\right) = 0.23697\text{ lb}\cdot\text{ft} \qquad P = 1.8356\text{ lb}$$

Disk D:

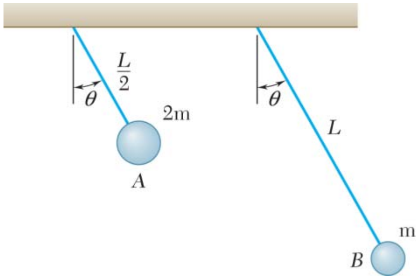
$$r = 90° - 30° - \beta = 36.206°$$

$$+\circlearrowleft \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad M - (P\sin r)\left(\frac{1.25}{12}\right)\text{ ft}$$

$$M = \frac{(1.25)(1.8356\sin 36.206°)}{12} = 0.11294\text{ lb}\cdot\text{ft}$$

(*a*) *Couple* **M**. $\mathbf{M} = 1.355\text{ lb}\cdot\text{in}\ \circlearrowleft$ ◀

(*b*) *Magnitude of contact force*. $P = 1.836\text{ lb}$ ◀



PROBLEM 16.CQ1

Two pendulums, *A* and *B*, with the masses and lengths shown are released from rest. Which system has a larger mass moment of inertia about its pivot point?

(*a*) *A*

(*b*) *B*

(*c*) They are the same.

SOLUTION

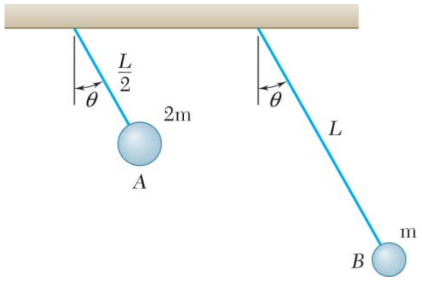
The mass moment of inertia is defined as: $\int r^2 dm$. Since each mass is a point mass, making *r* a constant, we can find the mass moment of inertia about the pivot point *O* as: $I_O = mr^2$.

For system *A*: $I_O = 2m\frac{L^2}{2^2} = \frac{mL^2}{2}$

For system *B*: $I_O = mL^2$

Therefore system *B* has twice the mass moment of inertia about its pivot than system *A*.

Answer: *(b)* ◀



PROBLEM 16.CQ2

Two pendulums, *A* and *B*, with the masses and lengths shown are released from rest. Which system has a larger angular acceleration immediately after release?

(*a*) *A*

(*b*) *B*

(*c*) They are the same.

SOLUTION

The mass moment of inertia is defined as: $\int r^2 dm$. Since each mass is a point mass, making *r* a constant, we can find the mass moment of inertia about the pivot point *O* as: $I_O = mr^2$.

For system *A*: $$I_O = 2m\frac{L^2}{2^2} = \frac{mL^2}{2}$$

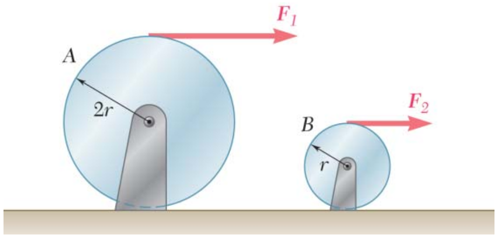
For system *B*: $$I_O = mL^2$$

Therefore system *B* has twice the mass moment of inertia about its pivot than system *A*.

The angular acceleration of each system can be found using: $\sum M_O = I_O\alpha$. The moment created about point O for both systems is the same. Since the inertia of system B is higher, its angular acceleration will be lower.

Answer: *(a)* ◀

PROBLEM 16.CQ3



Two solid cylinders, A and B, have the same mass m and the radii $2r$ and r respectively. Each is accelerated from rest with a force applied as shown. In order to impart identical angular accelerations to both cylinders, what is the relationship between F_1 and F_2?

(*a*) $F_1 = 0.5F_2$

(*b*) $F_1 = F_2$

(*c*) $F_1 = 2F_2$

(*d*) $F_1 = 4F_2$

(*e*) $F_1 = 8F_2$

SOLUTION

$$\sum M_G = I\alpha$$

$$Fr = I\alpha$$

$$\alpha = \frac{FR}{\frac{1}{2}mR^2} = \frac{F_1(2r)}{\frac{1}{2}m(2r)^2} = \frac{F_2 r}{\frac{1}{2}mr^2}$$

$$\frac{F_1}{mr} = \frac{2F_2}{mr}$$

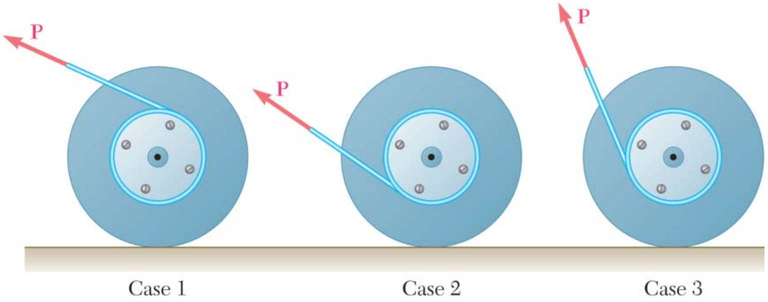
$$F_1 = 2F_2$$

Answer: (*c*) ◀

PROBLEM 16.CQ4

A cord is attached to a spool when a force **P** is applied to the cord as shown. Assuming the spool rolls without slipping, what direction does the spool move for each case?

Case 1:	(*a*) left	(*b*) right	(*c*) It would not move.
Case 2:	(*a*) left	(*b*) right	(*c*) It would not move.
Case 3:	(*a*) left	(*b*) right	(*c*) It would not move.



SOLUTION

For each case consider that the force P creates a moment about the point where the spool contacts the ground. This moment must have the same sense (CW or CCW) as the resultant of the couple $\bar{I}\alpha$ plus the moment of the $m\bar{\mathbf{a}}$ vector about the contact point.

Answer:

Case 1: The force P creates a CCW moment about the contact point. If the spool accelerates to the left, the angular acceleration will have a CCW sense and the moment of the $m\bar{\mathbf{a}}$ vector about the contact point will also be CCW. This means the spool will move to the left.. Answer: (*a*)

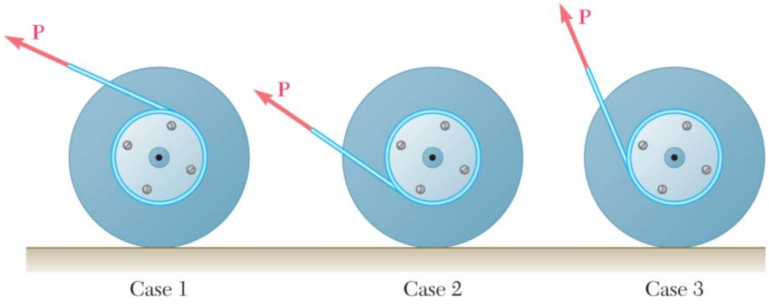
Case 2: The force P creates a CCW moment about the contact point. If the spool accelerates to the left, the angular acceleration will have a CCW sense and the moment of the $m\bar{\mathbf{a}}$ vector about the contact point will also be CCW. This means the spool will move to the left.. Answer: (*a*)

Case 3: The force P creates a CW moment about the contact point. If the spool accelerates to the right, the angular acceleration will have a CW sense and the moment of the $m\bar{\mathbf{a}}$ vector about the contact point will also be CW. This means the spool will move to the right. Answer: (*b*)

PROBLEM 16.CQ5

A cord is attached to a spool when a force **P** is applied to the cord as shown. Assuming the spool rolls without slipping, in what direction does the friction force act for each case?

Case 2:	(*a*) left	(*b*) right	(*c*) The friction force would be zero.
Case 3:	(*a*) left	(*b*) right	(*c*) The friction force would be zero.



SOLUTION

The direction of the friction force on a rolling object is not determined exclusively by the direction of rolling. To determine the direction of the friction force, consider all forces acting on the body and note that the sum of the forces must be in the same direction as the acceleration of the mass center and that the sum of the moments about the mass center must have the same sense as the angular acceleration.

Case 2: From Problem 16.CQ4, the spool must accelerate to the left. Since P causes a CW moment about the mass center and the net moment must be CCW, the friction force must be to the right to create the necessary net CCW moment about the mass center. Answer: (*b*)

Case 3: From Problem 16.CQ4, the spool must accelerate to the right. Since P causes a CW moment about the mass center and the net moment must be CCW, the friction force must be to the right to create the necessary net CCW moment about the mass center. Answer: (*b*)

PROBLEM 16.CQ6

A front wheel drive car starts from rest and accelerates to the right. Knowing that the tires do not slip on the road, what is the direction of the friction force the road applies to the front tires?

(*a*) left

(*b*) right

(*c*) The friction force is zero.

SOLUTION

Since the engine/transmission is applying a moment to the front wheels and the friction force from the ground on the wheels makes the care go forward, the friction force must be in the direction of acceleration.

Answer: (*b*)

PROBLEM 16.CQ7

A front wheel drive car starts from rest and accelerates to the right. Knowing that the tires do not slip on the road, what is the direction of the friction force the road applies to the rear tires?

(*a*) left

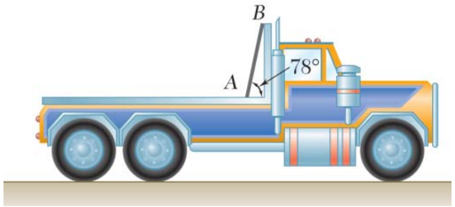
(*b*) right

(*c*) The friction force is zero.

SOLUTION

Since the car is front wheel drive, there is no moment applied to the read wheels by the engine/transmission. Since the rear wheel has angular acceleration, the moment applied about the mass center can only come from the friction force. This force must be to the left in order for the moment to be CCW which cause acceleration to the right.

Answer: (*a*)

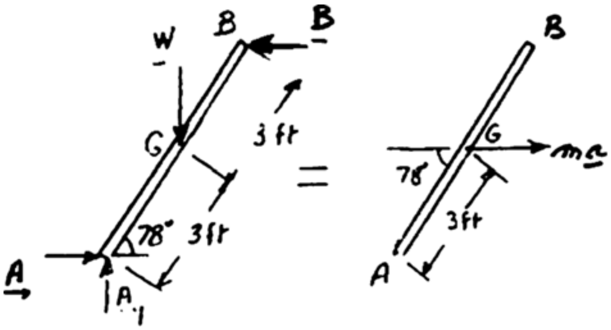


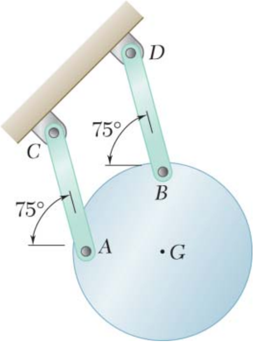
PROBLEM 16.F1

A 6-ft board is placed in a truck with one end resting against a block secured to the floor and the other leaning against a vertical partition. Draw the FBD and KD necessary to determine the maximum allowable acceleration of the truck if the board is to remain in the position shown.

SOLUTION

Answer:



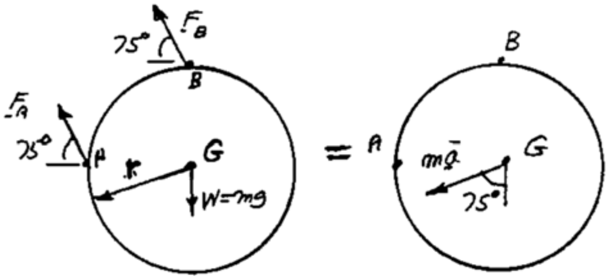


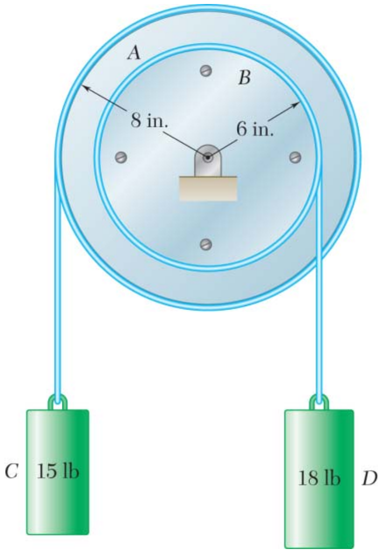
PROBLEM 16.F2

A uniform circular plate of mass 3 kg is attached to two links *AC* and *BD* of the same length. Knowing that the plate is released from rest in the position shown, in which lines joining *G* to *A* and *B* are, respectively, horizontal and vertical, draw the FBD and KD for the plate.

SOLUTION

Answer:


F_B
75°
B
F_A
75°
A
r
G
W = mg
=
A
B
$m\bar{a}$
G
75°

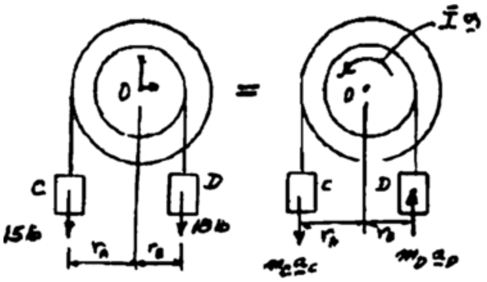

A
B
8 in.
6 in.
C
15 lb
18 lb
D

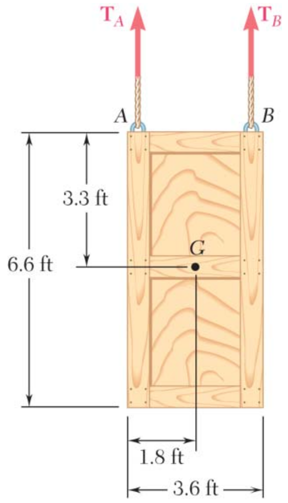
PROBLEM 16.F3

Two uniform disks and two cylinders are assembled as indicated. Disk *A* weighs 20 lb and disk *B* weighs 12 lb. Knowing that the system is released from rest, draw the FBD and KD for the whole system.

SOLUTION

Answer:


O
=
O
$\bar{I}\alpha$
C
D
c
D
15 lb
18 lb
r_A
r_B
r_A
r_B
$m_C\underline{a}_C$
$m_D\underline{a}_D$

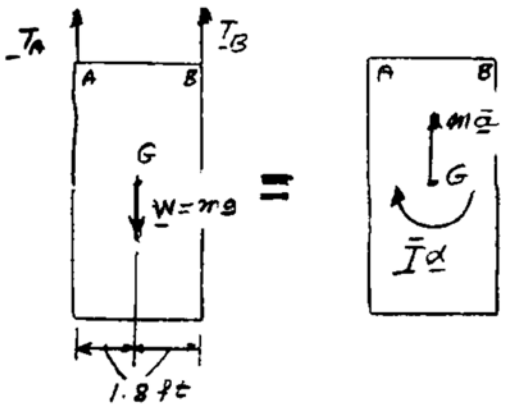

T_A
T_B
A
B
3.3 ft
6.6 ft
G
1.8 ft
3.6 ft

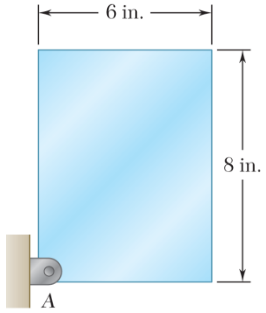
PROBLEM 16.F4

The 400-lb crate shown is lowered by means of two overhead cranes. Knowing the tension in each cable, draw the FBD and KD that can be used to determine the angular acceleration of the crate and the acceleration of the center of gravity.

SOLUTION

Answer:


T_A
T_B
A
B
G
$W = mg$
=
A
B
$m\bar{a}$
G
$\bar{I}\alpha$
1.8 ft

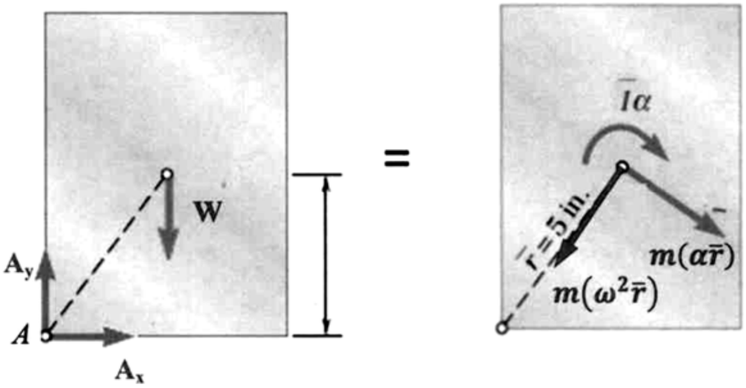


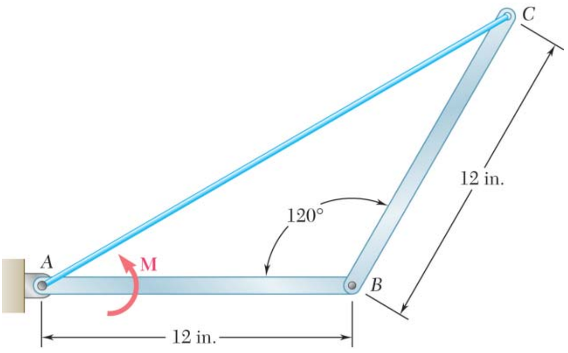
PROBLEM 16.F5

A uniform 6 × 8-in. rectangular plate of mass *m* is pinned at *A*. Knowing the angular velocity of the plate at the instant shown is *ω*, draw the FBD and KD.

SOLUTION

Answer:


W
A_y
A
A_x
=
$\bar{I}\alpha$
$\bar{r} = 5$ in.
$m(\alpha\bar{r})$
$m(\omega^2\bar{r})$

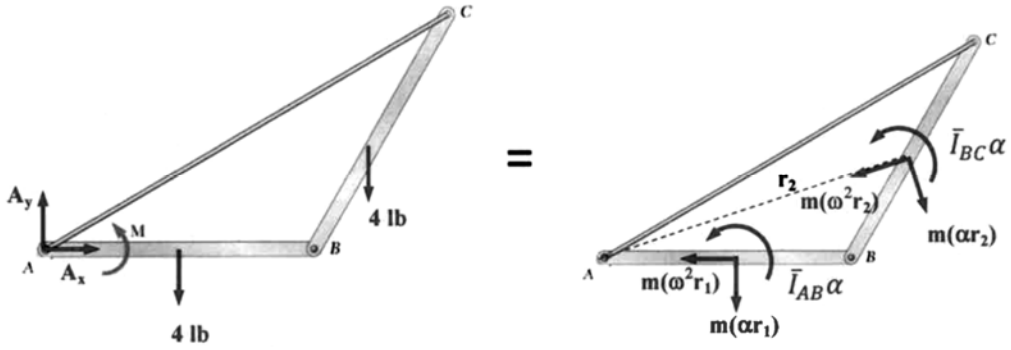

C
12 in.
120°
A
M
B
12 in.

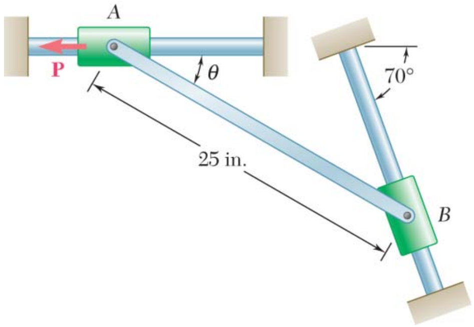
PROBLEM 16.F6

Two identical 4-lb slender rods *AB* and *BC* are connected by a pin at *B* and by the cord *AC*. The assembly rotates in a vertical plane under the combined effect of gravity and a couple **M** applied to rod *AB*. Knowing that in the position shown the angular velocity of the assembly is ω, draw the FBD and KD that can be used to determine the angular acceleration of the assembly and the tension in cord *AC*.

SOLUTION

Answer:


C
A_y
M
4 lb
B
A
A_x
4 lb
=
C
$\bar{I}_{BC}\alpha$
r_2
$m(\omega^2 r_2)$
$m(\alpha r_2)$
A
B
$m(\omega^2 r_1)$
$\bar{I}_{AB}\alpha$
$m(\alpha r_1)$

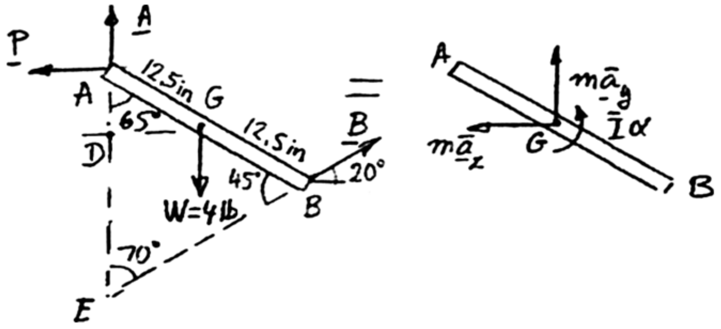

A
P
θ
70°
25 in.
B

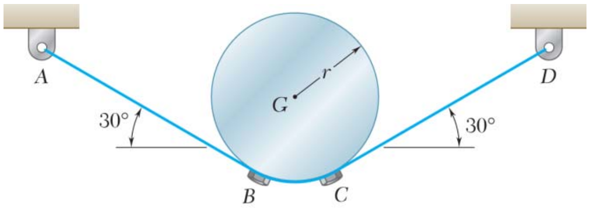
PROBLEM 16.F7

The 4-lb uniform rod *AB* is attached to collars of negligible mass which may slide without friction along the fixed rods shown. Rod *AB* is at rest in the position $\theta = 25°$ when a horizontal force **P** is applied to collar *A* causing it to start moving to the left. Draw the FBD and KD for the rod.

SOLUTION

Answer:


P
A
A
12.5in G
12.5in
65°
D
45°
W=4lb
B
B
20°
70°
E
=
A
$m\bar{a}_y$
$\bar{I}\alpha$
$m\bar{a}_x$
G
B

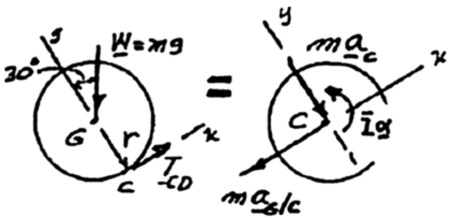

A
D
r
G
30°
30°
B
C

PROBLEM 16.F8

A uniform disk of mass $m = 4$ kg and radius $r = 150$ mm is supported by a belt *ABCD* that is bolted to the disk at *B* and *C*. If the belt suddenly breaks at a point located between *A* and *B*, draw the FBD and KD for the disk immediately after the break.

SOLUTION

Answer:


y
30°
W = mg
G
r
C
x
T_CD
=
y
ma_C
x
C
Īα
ma_G/C

CHAPTER 17

PROBLEM 17.1

A 200-kg flywheel is at rest when a constant 300 $\text{N}\cdot\text{m}$ couple is applied. After executing 560 revolutions, the flywheel reaches its rated speed of 2400 rpm. Knowing that the radius of gyration of the flywheel is 400 mm, determine the average magnitude of the couple due to kinetic friction in the bearing.

SOLUTION

Moment of inertia. $\bar{I} = mk^2 = (200)(0.400)^2 = 32\ \text{kg}\cdot\text{m}^2$

Initial state. $\omega_1 = 0, \quad T_1 = 0$

Rotation angle. $\theta_2 - \theta_1 = 560\ \text{rev} = 1120\pi\ \text{radians}$

Let M_f be the couple due to friction.

Work. $U_{1\to 2} = M(\theta_2 - \theta_1) = (300 - M_f)(1120\pi)$

Final state. $\omega_2 = 2400\ \text{rpm} = 80\pi\ \text{rad/s}$

$$T_2 = \frac{1}{2}\bar{I}\omega_2^2 = \frac{1}{2}(32)(80\pi)^2 = 1.01065 \times 10^6\ \text{J}$$

Principle of work and energy: $T_1 + U_{1\to 2} = T_2$

$$0 + (300 - M_f)(1120\pi) = 1.01065 \times 10^6$$

$$300 - M_f = 287.23 \qquad M_f = 12.77\ \text{N}\cdot\text{m} \blacktriangleleft$$

PROBLEM 17.2

The rotor of an electric motor has an angular velocity of 3600 rpm when the load and power are cut off. The 110-lb rotor, which has a centroidal radius of gyration of 9 in., then coasts to rest. Knowing that the kinetic friction of the rotor produces a couple of magnitude $2.5\ \text{lb}\cdot\text{ft}$, determine the number of revolutions that the rotor executes before coming to rest.

SOLUTION

Kinetic energy. *Position 1.* $\omega_1 = 3600 \text{ rpm} = 120\pi \text{ rad/s}$

$$m = \frac{W}{g} \qquad I = \frac{W}{g}k^2 = \frac{110}{32.2}\left(\frac{9}{12}\right)^2 = 1.92158 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$T_1 = \frac{1}{2}I\omega_1^2 = \frac{1}{2}(1.92158)(120\pi)^2 = 136.55\times10^3 \text{ ft}\cdot\text{lb}$$

Position 2. $\omega_2 = 0 \qquad T_2 = 0$

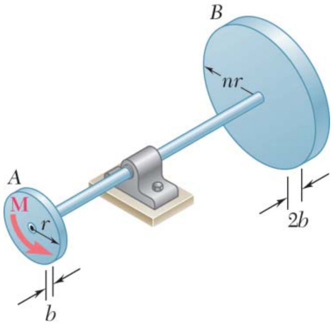
Work. $M = 2.5 \text{ lb}\cdot\text{ft} \qquad U_{1\to2} = -M\theta = -2.5\theta$

Principle of work and energy.

$$T_1 + U_{1\to2} = T_2:$$

$$136.55\times10^3 - 2.5\theta = 0 \qquad \theta = 54620 \text{ radians}$$

$\theta = 8690$ rev ◀



PROBLEM 17.3

Two uniform disks of the same material are attached to a shaft as shown. Disk *A* has a weight of 10 lb and a radius $r = 6$ in. Disk *B* is twice as thick as disk *A*. Knowing that a couple **M** of magnitude $22\ \text{lb}\cdot\text{ft}$ is applied to disk *A* when the system is at rest, determine the radius *nr* of disk *B* if the angular velocity of the system is to be 480 rpm after 5 revolutions.

SOLUTION

Moments of inertia.

Disk *A*: $$I_A = \frac{1}{2}m_A r_A^2 = \frac{1}{2}\left(\frac{10}{32.2}\right)\left(\frac{6}{12}\right)^2 = 0.03882\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

Disk *B*: $$m_B = m_A\left(\frac{t_B}{t_A}\right)\left(\frac{r_B}{r_A}\right)^2 = \left(\frac{10}{32.2}\right)(2)(n)^2 = 0.62112n^2\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$I_B = \frac{1}{2}m_B r_B^2 = \frac{1}{2}\left(0.62112n^2\right)\left(\frac{6}{12}n\right)^2 = 0.07764n^4\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

Total: $$I = I_A + I_B = \left(0.03882 + 0.07764n^4\right)\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

Work. $$\theta_2 - \theta_1 = 5\ \text{rev} = 10\pi\ \text{radians}$$

$$U_{1\to 2} = M(\theta_2 - \theta_1) = (22)(10\pi) = 220\pi\ \text{ft}\cdot\text{lb}$$

Kinetic energy. $$\omega_1 = 0 \qquad T_1 = 0$$

$$\omega_2 = 480\ \text{rpm} = 16\pi\ \text{rad/s} \qquad T_2 = \frac{1}{2}I\omega_2^2$$

Principle of work and energy. $$T_1 + U_{1\to 2} = T_2$$

$$0 + U_{1\to 2} = \frac{1}{2}I\omega_2^2$$

Solving for *I*, $$I = \frac{2U_{1\to 2}}{\omega_2^2} = \frac{(2)(220\pi)}{(16\pi)^2} = 0.547095\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

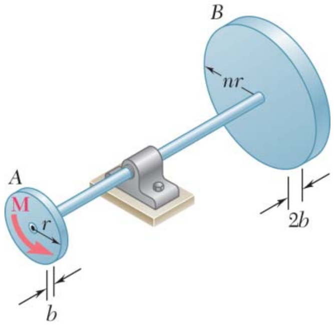
Radius of Disk *B*.

Equating the two expressions for *I*, $$0.03882 + 0.07764n^4 = 0.547095$$

$$n^4 = 6.5466 \qquad n = 1.5996$$

$$r_B = nr_A = (1.5996)(6\ \text{in.}) = 9.597\ \text{in.}$$

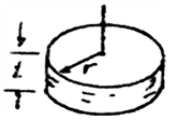
$r_B = 9.60$ in. ◀



PROBLEM 17.4

Two disks of the same material are attached to a shaft as shown. Disk *A* has radius *r* and has a thickness *b*, while disk *B* has radius *nr* and thickness 2*b*. A couple **M** of constant magnitude is applied when the system is at rest and is removed after the system has executed 2 revolutions. Determine the value of *n* which results in the largest final speed for a point on the rim of disk *B*.

SOLUTION



For any disk:

$$m = \rho(\pi r^2 t)$$

$$\bar{I} = \frac{1}{2}mr^2$$

$$= \frac{1}{2}\pi\rho t r^4$$

Moment of inertia.

Disk *A*:

$$I_A = \frac{1}{2}\pi\,\rho\,b\,r^4$$

Disk *B*:

$$I_B = \frac{1}{2}\pi\,\rho(2b)(nr)^4$$

$$= 2n^4\left[\frac{1}{2}\pi\rho b r^4\right]$$

$$= 2n^4 I_A$$

$$I_{\text{total}} = I_A + I_B$$

$$= (1+2n^4)I_A$$

Work-energy.

$$T_1 = 0 \qquad U_{1\to2} = M\theta = M(4\pi\text{ rad})$$

$$T_2 = \frac{1}{2}I_{\text{total}}\,\omega_2^2$$

$$T_1 + U_{1\to2} = T_2: \quad 0 + M(4\pi) = \frac{1}{2}(1+2n^4)I_A\omega_2^2$$

$$\omega_2^2 = \frac{8\pi M}{(1+2n^4)I_A}$$

For Point *D* on rim of disk *B*

$$v_D = (nr)\omega_2 \quad \text{or} \quad v_D^2 = n^2r^2\omega_2^2 = \frac{8\pi M r^2}{I_A}\cdot\frac{n^2}{1+2n^4}$$

PROBLEM 17.4 (Continued)

Value of n for maximum final speed.

For maximum v_D: $\frac{d}{dn}\left(\frac{n^2}{1+2n^4}\right)=0$

$$\frac{1}{(1+2n^4)^2}[n^2(8n^3)-(1+2n^4)(2n)]=0$$

$$8n^5-2n-4n^5=0$$

$$2n(2n^4-1)=0$$

$n=0$ and $n=\left(\frac{1}{2}\right)^{0.25}=0.8409$ $n=0.841$ ◀

PROBLEM 17.5

The flywheel of a small punch rotates at 300 rpm. It is known that 1800 ft · lb of work must be done each time a hole is punched. It is desired that the speed of the flywheel after one punching be not less that 90 percent of the original speed of 300 rpm. (*a*) Determine the required moment of inertia of the flywheel. (*b*) If a constant 25-lb · ft couple is applied to the shaft of the flywheel, determine the number of revolutions which must occur between each punching, knowing that the initial velocity is to be 300 rpm at the start of each punching.

SOLUTION

Angular velocities: $\omega_1 = 300 \text{ rpm} = 10\pi \text{ rad/s}$

$$\omega_2 = 0.90\omega_1 = 9\pi \text{ rad/s}$$

Principle of work and energy: $T_1 + U_{1\to 2} = T_2$

$$T_1 = \frac{1}{2}\bar{I}\omega_1^2 = \frac{1}{2}\bar{I}(10\pi)^2$$

$$T_2 = \frac{1}{2}\bar{I}\omega_2^2 = \frac{1}{2}\bar{I}(9\pi)$$

$$U_{1\to 2} = -1800 \text{ ft} \cdot \text{lb}$$

$$\frac{1}{2}\bar{I}(10\pi)^2 - 1800 = \frac{1}{2}\bar{I}(9\pi)^2$$

(*a*) *Required moment of inertia.*

$$\bar{I} = \frac{2(1800)}{\pi^2(100-81)} = 19.198 \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \qquad \bar{I} = 19.20 \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \blacktriangleleft$$

(*b*) *Number of revolution between each punching.*

Definition of work: $U_{2\to 1} = M\theta$:

$$1800 \text{ ft} \cdot \text{lb} = (25 \text{ lb} \cdot \text{ft})\theta$$

$$\theta = 72 \text{ rad} = 11.459 \text{ rev} \qquad \theta = 11.46 \text{ rev} \blacktriangleleft$$

PROBLEM 17.6

The flywheel of a punching machine has a mass of 300 kg and a radius of gyration of 600 mm. Each punching operation requires 2500 J of work. (*a*) Knowing that the speed of the flywheel is 300 rpm just before a punching, determine the speed immediately after the punching. (*b*) If a constant 25-N · m couple is applied to the shaft of the flywheel, determine the number of revolutions executed before the speed is again 300 rpm.

SOLUTION

Moment of inertia.

$$I = mk^2 = (300\text{ kg})(0.6\text{ m})^2 = 108\text{ kg}\cdot\text{m}^2$$

Kinetic energy. *Position 1*.

$$\omega_1 = 300\text{ rpm} = 10\pi\text{ rad/s}$$

$$T_1 = \frac{1}{2}I\omega_1^2 = \frac{1}{2}(108)(10\pi)^2 = 53.296\times10^3\text{ J}$$

Position 2.

$$T_2 = \frac{1}{2}I\omega_2^2 = 54\omega_2^2$$

Work.

$$U_{1\to2} = -2500\text{ J}$$

Principle of work and energy for punching.

$$T_1 + U_{1\to2} = T_2:\quad 53.296\times10^3 - 2500 = 54\omega_2^2$$

(*a*)

$$\omega_2^2 = 940.66$$

$$\omega_2 = 30.67\text{ rad/s}\qquad \omega_2 = 293\text{ rpm} ◀$$

Principle of work and energy for speed recovery.

$$T_2 + U_{2\to1} = T_1$$

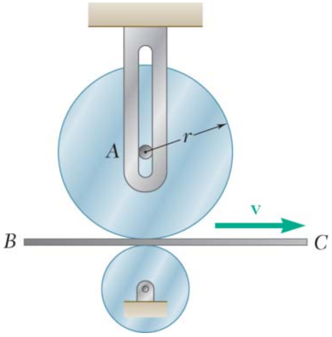
$$U_{2\to1} = 2500\text{ J}$$

$$M = 25\text{ N}\cdot\text{m}$$

$$U_{2\to1} = M\theta\quad 2500 = 25\theta\qquad \theta = 100\text{ rad}$$

(*b*)

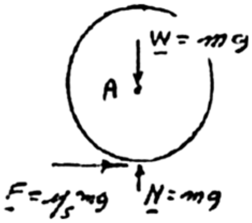
$$\theta = 15.92\text{ rev} ◀$$



PROBLEM 17.7

Disk *A*, of weight 10 lb and radius $r = 6$ in., is at rest when it is placed in contact with belt *BC,* which moves to the right with a constant speed $v = 40$ ft/s. Knowing that $\mu_k = 0.20$ between the disk and the belt, determine the number of revolutions executed by the disk before it attains a constant angular velocity.

SOLUTION



Work of external friction force on disk *A*.

Only force doing work is *F*. Since its moment about *A* is $M = rF$, we have

$$\begin{aligned} U_{1\to 2} &= M\theta \\ &= rF\theta \\ &= r(\mu_k mg)\theta \end{aligned}$$

Kinetic energy of disk *A*.

Angular velocity becomes constant when $\omega_2 = \dfrac{v}{r}$

$$\begin{aligned} T_1 &= 0 \\ T_2 &= \frac{1}{2}\bar{I}\omega_2^2 \\ &= \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2 \\ &= \frac{mv^2}{4} \end{aligned}$$

Principle of work and energy for disk *A*.

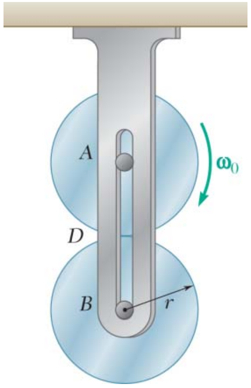
$$T_1 + U_{1-2} = T_2: \quad 0 + r\mu_k mg\theta = \frac{mv^2}{4}$$

Angle change $\theta = \dfrac{v^2}{4r\,\mu_k g}$ rad $\qquad \theta = \dfrac{v^2}{8\pi r\,\mu_k g}$ rev

Data:

$$\begin{aligned} r &= 0.5 \text{ ft} \\ \mu_k &= 0.20 \\ v &= 40 \text{ ft/s} \end{aligned}$$

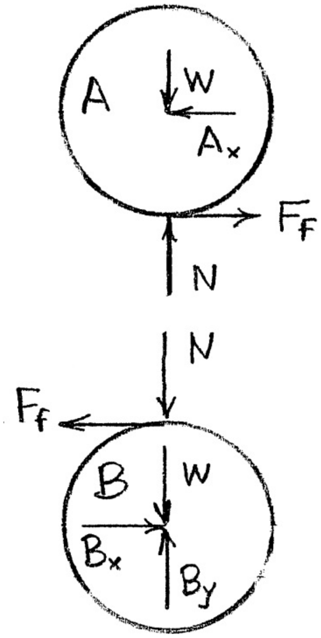
$$\theta = \frac{(40 \text{ ft/s})^2}{8\pi(0.5 \text{ ft})(0.20)(32.2 \text{ ft/s}^2)} \qquad \theta = 19.77 \text{ rev} \blacktriangleleft$$



PROBLEM 17.8

The uniform 4-kg cylinder *A*, of radius $r = 150$ mm, has an angular velocity $\omega_0 =$ 50 rad/s when it is brought into contact with an identical cylinder *B* which is at rest. The coefficient of kinetic friction at the contact point *D* is μ_k. After a period of slipping, the cylinders attain constant angular velocities of equal magnitude and opposite direction at the same time. Knowing that cylinder *A* executes three revolutions before it attains a constant angular velocity and cylinder *B* executes one revolution before it attains a constant angular velocity, determine (*a*) the final angular velocity of each cylinder, (*b*) the coefficient of kinetic friction μ_k.

SOLUTION



Moment of inertia.

$$I = \frac{1}{2}mr^2 = \frac{1}{2}(4)(0.15)^2 = 0.045 \text{ kg}\cdot\text{m}^2$$

Cylinder A: $+\uparrow \Sigma F_y = 0$: $N - W = 0 \quad N = W = (4)(9.81) = 39.24$ N

Kinetic friction force. $F_f = \mu_k N = 39.24\mu_k$ N

Kinetic energy. $T_1 = \frac{1}{2}I\omega_0^2 = \frac{1}{2}(0.045)(50)^2$

$$= 56.25 \text{ N}\cdot\text{m}$$

$$T_2 = \frac{1}{2}I\omega_f^2 = (0.0225)\omega_A^2$$

$$+\circlearrowright \Sigma M_A = F_f r = 39.24\mu_k(0.15) = 5.886\mu_k \text{ N}\cdot\text{m}$$

$$U_{1\to 2} = -M_A\theta_A = -(5.886\mu_k)(6\pi) = -110.949\mu_k \text{ N}\cdot\text{m}$$

Principle of work and energy for cylinder *A*.

$$T_1 + U_{1\to 2} = T_2: \quad 56.25 - 110.949\mu_k = 0.0225\omega_A^2$$

$$0.0225\omega_A^2 + 110.949\mu_k = 56.25 \qquad (1)$$

Kinematics. At the time of no slipping,

$$r_A\omega_A = r_B\omega_B$$

$$\omega_A^2 - \omega_B^2 = 0 \qquad (2)$$

Cylinder B: Kinetic energy. $T_1 = 0$

$$T_2 = \frac{1}{2}I\omega_B^2 = 0.0225\omega_B^2$$

$$\Sigma M_B = F_f r = 5.886\mu_k \text{ N}\cdot\text{m}$$

PROBLEM 17.8 (Continued)

$$U_{1\to 2} = M_B\theta_B = (5.886\mu_k)(2\pi) = 36.983\mu_k \text{ N}\cdot\text{m}$$

Principle of work and energy for cylinder *B*.

$$T_1 + U_{1\to 2} = T_2: \quad 0 + 36.983\mu_k = 0.0225\omega_B^2$$

$$0.0225\omega_B^2 - 36.983\mu_k = 0 \qquad (3)$$

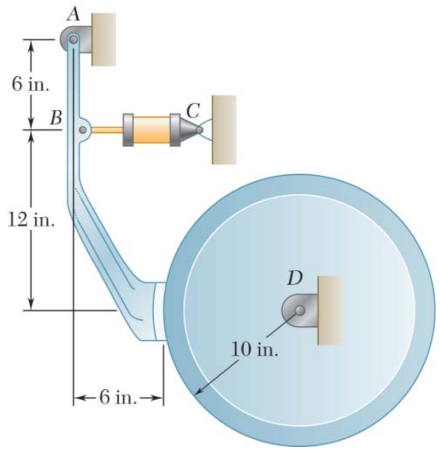
Solving (1), (2), and (3) simultaneously,

$$\omega_A^2 = 625(\text{rad/s})^2, \;\; \omega_B^2 = 625(\text{rad/s})^2, \;\; \mu_k = 0.380$$

(*a*) $\boldsymbol{\omega}_A = 25 \text{ rad/s}$ ↻ ◀

$\boldsymbol{\omega}_B = 25 \text{ rad/s}$ ↺ ◀

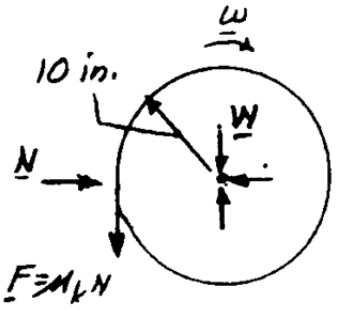
(*b*) $\mu_k = 0.380$ ◀



PROBLEM 17.9

The 10-in.-radius brake drum is attached to a larger flywheel which is not shown. The total mass moment of inertia of the flywheel and drum is 16 lb · ft · s^2 and the coefficient of kinetic friction between the drum and the brake shoe is 0.40. Knowing that the initial angular velocity is 240 rpm clockwise, determine the force which must be exerted by the hydraulic cylinder if the system is to stop in 75 revolutions.

SOLUTION



Kinetic energies.

$$\omega_1 = 240 \text{ rpm} = 8\pi \text{ rad/s}$$

$$\bar{I} = 16 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$T_1 = \frac{1}{2}\bar{I}\omega_1^2 = \frac{1}{2}(16)(8\pi)^2 = 5053 \text{ ft} \cdot \text{lb}$$

$$\omega_2 = 0 \qquad T_2 = 0$$

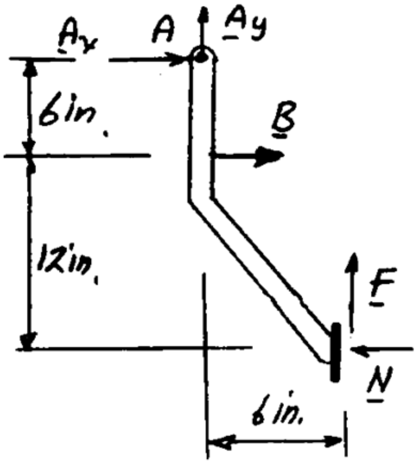
Angular displacement. $\theta = 75 \text{ rev} = 75(2\pi) = 150\pi \text{ rad}$

Work. $U_{1-2} = -M\theta = -\left[F\left(\frac{10}{12}\text{ ft}\right)\right](150\pi \text{ rad}) = -392.7F$

Principle of work and energy. $T_1 + U_{1-2} = T_2$:

$$5053 - 392.7F = 0 \qquad F = 12.868 \text{ lb}$$

$$F = \mu_k N: \quad 12.868 = (0.40)N \qquad N = 32.17 \text{ lb}$$



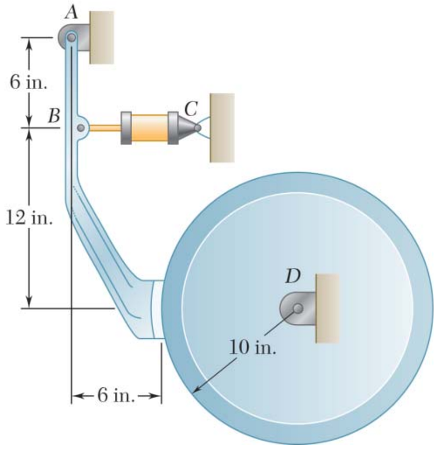
Free body brake arm:

$$+\circlearrowleft \Sigma M_A = 0: \quad B(6 \text{ in.}) + F(6 \text{ in.}) - N(18 \text{ in.}) = 0$$

$$B(6 \text{ in.}) + (12.868 \text{ lb})(6 \text{ in.}) - (32.17 \text{ lb})(18 \text{ in.}) = 0$$

$$B = 83.64 \text{ lb}$$

$B = 83.6$ lb ◀

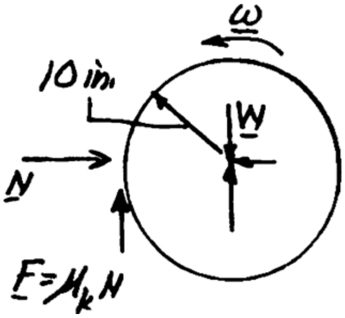


PROBLEM 17.10

Solve Problem 17.9, assuming that the initial angular velocity of the flywheel is 240 rpm counterclockwise.

PROBLEM 17.9 The 10-in.-radius brake drum is attached to a larger flywheel which is not shown. The total mass moment of inertia of the flywheel and drum is $16\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ and the coefficient of kinetic friction between the drum and the brake shoe is 0.40. Knowing that the initial angular velocity is 240 rpm clockwise, determine the force which must be exerted by the hydraulic cylinder if the system is to stop in 75 revolutions.

SOLUTION



Kinetic energies.

$$\omega_1 = 240\text{ rpm} = 8\pi\text{ rad/s}$$

$$\overline{I} = 16\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$T_1 = \frac{1}{2}\overline{I}\,\omega_1^2 = \frac{1}{2}(16)(8\pi)^2 = 5053\text{ ft}\cdot\text{lb}$$

$$\omega_2 = 0 \qquad T_2 = 0$$

Angular displacement. $\theta = 75\text{ rev} = 75(2\pi) = 150\pi\text{ rad}$

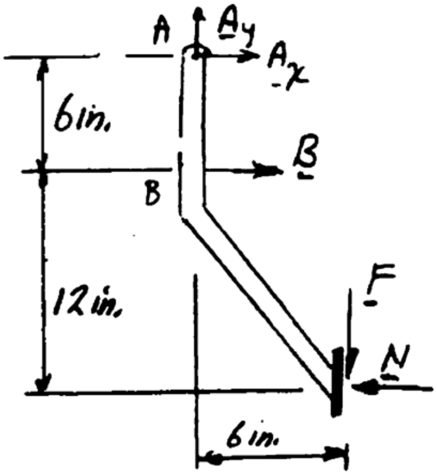
Work. $U_{1-2} = -M\theta = -\left[F\left(\frac{10}{12}\text{ft}\right)\right](150\pi\text{ rad}) = -392.7F$

Principle of work and energy. $T_1 + U_{1-2} = T_2:$

$$5053 - 392.7F = 0 \qquad F = 12.868\text{ lb}$$

$$F = \mu_k N:\quad 12.868 = (0.40)N \qquad N = 32.17\text{ lb}$$

Free body brake arm:

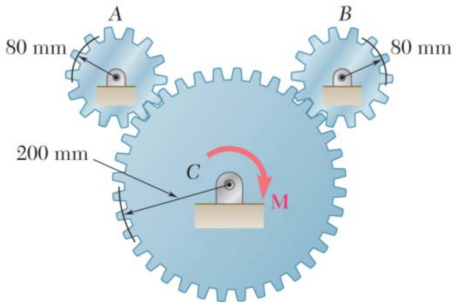


$$+\circlearrowleft \Sigma M_A = 0:\quad B(6\text{ in.}) - F(6\text{ in.}) - N(18\text{ in.}) = 0$$

$$B(6\text{ in.}) - (12.868\text{ lb})(6\text{ in.}) - (32.17\text{ lb})(18\text{ in.}) = 0$$

$$B = 109.37\text{ lb}$$

$B = 109.4\text{ lb}$ ◀



PROBLEM 17.11

Each of the gears A and B has a mass of 2.4 kg and a radius of gyration of 60 mm, while gear C has a mass of 12 kg and a radius of gyration of 150 mm. A couple **M** of constant magnitude 10 N · m is applied to gear C. Determine (a) the number of revolutions of gear C required for its angular velocity to increase from 100 to 450 rpm, (b) the corresponding tangential force acting on gear A.

SOLUTION

Moments of inertia.

Gears A and B: $I_A = I_B = mk^2 = (2.4)(0.06)^2 = 8.64\times 10^{-3}\,\text{kg}\cdot\text{m}^2$

Gear C: $I_C = (12)(0.15)^2 = 270\times 10^{-3}\,\text{kg}\cdot\text{m}^2$

Kinematics. $r_A\omega_A = r_B\omega_B = r_C\omega_C$

$$\omega_A = \omega_B = \frac{200}{80}\omega_C = 2.5\omega_C$$

$$\theta_A = \theta_B = 2.5\theta_C$$

Kinetic energy. $T = \dfrac{1}{2}I\omega^2$:

Position 1. $\omega_C = 100\text{ rpm} = \dfrac{10}{3}\pi\text{ rad/s}$

$$\omega_A = \omega_B = 250\text{ rpm} = \frac{25}{3}\pi\text{ rad/s}$$

Gear A: $(T_1)_A = \dfrac{1}{2}(8.64\times 10^{-3})\left(\dfrac{25\pi}{3}\right)^2 = 2.9609\text{ J}$

Gear B: $(T_1)_B = \dfrac{1}{2}(8.64\times 10^{-3})\left(\dfrac{25\pi}{3}\right)^2 = 2.9609\text{ J}$

Gear C: $(T_1)_C = \dfrac{1}{2}(270\times 10^{-3})\left(\dfrac{10\pi}{3}\right)^2 = 14.8044\text{ J}$

System: $T_1 = (T_1)_A + (T_1)_B + (T_1)_C = 20.726\text{ J}$

Position 2. $\omega_C = 450\text{ rpm} = 15\pi\text{ rad/s}$

$$\omega_A = \omega_B = 37.5\pi\text{ rad/s}$$

PROBLEM 17.11 (Continued)

Gear *A*: $(T_2)_A = \frac{1}{2}(8.64\times 10^{-3})(37.5\pi)^2 = 59.957 \text{ J}$

Gear *B*: $(T_2)_B = \frac{1}{2}(8.64\times 10^{-3})(37.5\pi)^2 = 59.957 \text{ J}$

Gear *C*: $(T_2)_C = \frac{1}{2}(270\times 10^{-3})(15\pi)^2 = 299.789 \text{ J}$

System: $T_2 = (T_2)_A + (T_2)_B + (T_2)_C = 419.7 \text{ J}$

Work of couple. $U_{1\to 2} = M\theta_C = 10\theta_C$

Principle of work and energy for system.

$$T_1 + U_{1\to 2} = T_2: \quad 20.726 + 10\theta_C = 419.7$$

$$\theta_C = 39.898 \text{ radians}$$

(*a*) Rotation of gear *C*. $\theta_C = 6.35 \text{ rev}$ ◀

Rotation of gear *A*.

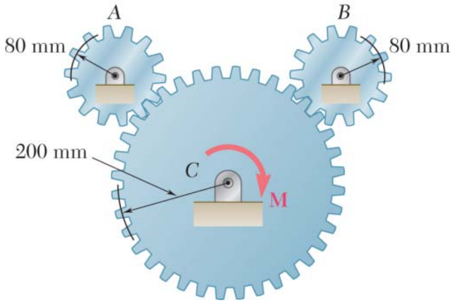
$$\theta_A = (2.5)(39.898)$$
$$= 99.744 \text{ radians}$$

Principle of work and energy for gear *A*.

$$(T_1)_A + M_A\theta_A = (T_2)_A: \quad 2.9609 + M_A(99.744) = 59.957$$

$$M_A = 0.57142 \text{ N}\cdot\text{m}$$

(*b*) Tangential force on gear *A*. $F_t = \frac{M_A}{r_A} = \frac{0.57142}{0.08}$ $F_t = 7.14 \text{ N}$ ◀



PROBLEM 17.12

Solve Problem 17.11, assuming that the 10-N · m couple is applied to gear *B*.

PROBLEM 17.11 Each of the gears *A* and *B* has a mass of 2.4 kg and a radius of gyration of 60 mm, while gear *C* has a mass of 12 kg and a radius of gyration of 150 mm. A couple **M** of constant magnitude 10 N · m is applied to gear *C*. Determine (*a*) the number of revolutions of gear *C* required for its angular velocity to increase from 100 to 450 rpm, (*b*) the corresponding tangential force acting on gear *A*.

SOLUTION

Moments of inertia.

Gears *A* and *B*: $I_A = I_B = mk^2 = (2.4)(0.06)^2 = 8.64\times 10^{-3}\,\text{kg}\cdot\text{m}^2$

Gear *C*: $I_C = (12)(0.15)^2 = 270\times 10^{-3}\,\text{kg}\cdot\text{m}^2$

Kinematics. $r_A\omega_A = r_B\omega_B = r_C\omega_C$

$$\omega_A = \omega_B = \frac{200}{80}\omega_C = 2.5\omega_C$$

$$\theta_A = \theta_B = 2.5\theta_C$$

Kinetic energy. $T = \frac{1}{2}I\omega^2$:

Position 1. $\omega_C = 100 \text{ rpm} = \frac{10}{3}\pi \text{ rad/s}$

$$\omega_A = \omega_B = 250 \text{ rpm} = \frac{25}{3}\pi \text{ rad/s}$$

Gear *A*: $(T_1)_A = \frac{1}{2}(8.64\times 10^{-3})\left(\frac{25\pi}{3}\right)^2 = 2.9609 \text{ J}$

Gear *B*: $(T_1)_B = \frac{1}{2}(8.64\times 10^{-3})\left(\frac{25\pi}{3}\right)^2 = 2.9609 \text{ J}$

Gear *C*: $(T_1)_C = \frac{1}{2}(270\times 10^{-3})\left(\frac{10\pi}{3}\right)^2 = 14.8044 \text{ J}$

System: $T_1 = (T_1)_A + (T_1)_B + (T_1)_C = 20.726 \text{ J}$

PROBLEM 17.12 (Continued)

Position 2. $\omega_C = 450 \text{ rpm} = 15\pi \text{ rad/s}$

$\omega_A = \omega_B = 37.5\pi \text{ rad/s}$

Gear *A*: $(T_2)_A = \frac{1}{2}(8.64\times 10^{-3})(37.5\pi)^2 = 59.957 \text{ J}$

Gear *B*: $(T_2)_B = \frac{1}{2}(8.64\times 10^{-3})(37.5\pi)^2 = 59.957 \text{ J}$

Gear *C*: $(T_2)_C = \frac{1}{2}(270\times 10^{-3})(15\pi)^2 = 299.789 \text{ J}$

System: $T_2 = (T_2)_A + (T_2)_B + (T_2)_C = 419.7 \text{ J}$

Work of couple. $U_{1\to 2} = M\theta_B = 10\theta_B$

Principle of work and energy for system.

$$T_1 + U_{1\to 2} = T_2: \quad 20.726 + 10\theta_B = 419.7$$

$$\theta_B = 39.898 \text{ radians}$$

(*a*) Rotation of gear *C*. $\theta_C = \dfrac{39.898}{2.5} = 15.959 \text{ radians}$ $\theta_C = 2.54 \text{ rev}$ ◀

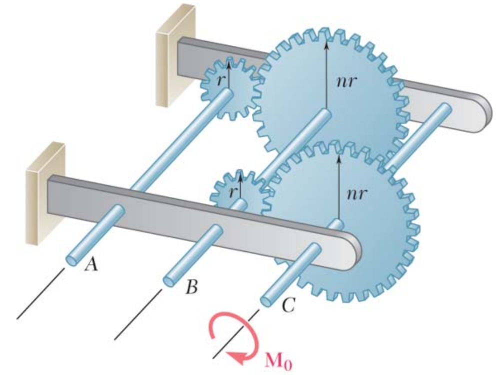
Rotation of gear *A*. $\theta_A = \theta_B = 39.898 \text{ radians}$

Principle of work and energy for gear *A*.

$$(T_1)_A + M_A\theta_A = (T_2)_A: \quad 2.9609 + M_A(39.898) = 59.957$$

$$M_A = 1.4285 \text{ N}\cdot\text{m}$$

(*b*) Tangential force on gear *A*. $F_t = \dfrac{M_A}{r_A} = \dfrac{1.4285}{0.08}$ $F_t = 17.86 \text{ N}$ ◀



PROBLEM 17.13

The gear train shown consists of four gears of the same thickness and of the same material; two gears are of radius *r*, and the other two are of radius *nr*. The system is at rest when the couple $\mathbf{M}_0$ is applied to shaft *C*. Denoting by I_0 the moment of inertia of a gear of radius *r*, determine the angular velocity of shaft *A* if the couple $\mathbf{M}_0$ is applied for one revolution of shaft *C*.

SOLUTION

Mass and moment of inertia:

For a disk of radius *r* and thickness *t*: $m = \rho(\pi r^2)t = \rho\pi t r^2$

$$\bar{I}_0 = \frac{1}{2}mr^2 = \frac{1}{2}(\rho\pi t r^2)r^2 = \frac{1}{2}\rho\pi t r^4$$

For a disk of radius *nr* and thickness *t*, $\bar{I} = \frac{1}{2}\rho\pi t(nr)^4 \qquad \bar{I} = n^4\bar{I}_0$

Kinematics: If for shaft *A* we have ω_A ↻

Then, for shaft *B* we have $\omega_B = \omega_A/n$ ↺

And, for shaft *C* we have $\omega_C = \omega_A/n^2$ ↻

Principle of work-energy:

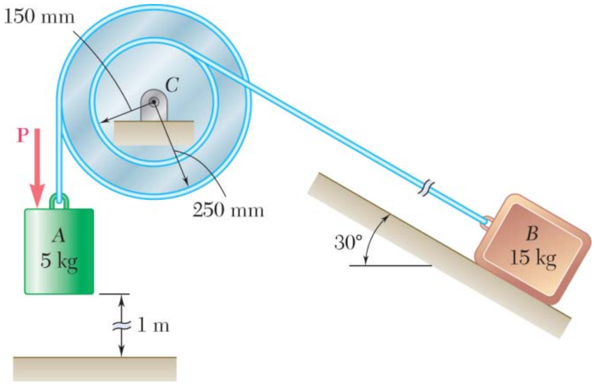
Couple M_0 applied to shaft *C* for one revolution. $\theta = 2\pi$ radians, $T_1 = 0$,

$$U_{1-2} = M_0\theta = M_0(2\pi \text{ radians}) = 2\pi M_0$$

$$\begin{aligned}
T_2 &= \frac{1}{2}(\bar{I}_{\text{shaft }A})w_A^2 + \frac{1}{2}(\bar{I}_{\text{shaft }B})\omega_B^2 + \frac{1}{2}(\bar{I}_{\text{shaft }C})\omega_C^2 \\
&= \frac{1}{2}\bar{I}_0 w_A^2 + \frac{1}{2}(\bar{I}_0 + n^4\bar{I}_0)\left(\frac{\omega_A}{n}\right)^2 + \frac{1}{2}(n^4\bar{I}_0)\left(\frac{\omega_A}{n^2}\right)^2 \\
&= \frac{1}{2}\bar{I}_0\omega_A^2\left(n^2 + 2 + \frac{1}{n^2}\right) \\
&= \frac{1}{2}\bar{I}_0\omega_A^2\left(n + \frac{1}{n}\right)^2
\end{aligned}$$

$$T_1 + U_{1-2} = T_2: \quad 0 + 2\pi M_0 = \frac{1}{2}\bar{I}_0\omega_A^2\left(n + \frac{1}{n}\right)^2$$

Angular velocity. $\omega_A^2 = \dfrac{4\pi M_0}{\bar{I}_0}\dfrac{1}{\left(n + \frac{1}{n}\right)^2}$ $\qquad \omega_A = \dfrac{2n}{n^2+1}\sqrt{\dfrac{\pi M_0}{\bar{I}_0}}$ ◀



PROBLEM 17.14

The double pulley shown has a mass of 15 kg and a centroidal radius of gyration of 160 mm. Cylinder *A* and block *B* are attached to cords that are wrapped on the pulleys as shown. The coefficient of kinetic friction between block *B* and the surface is 0.2. Knowing that the system is at rest in the position shown when a constant force $P = 200$ N is applied to cylinder *A*, determine (*a*) the velocity of cylinder *A* as it strikes the ground, (*b*) the total distance that block *B* moves before coming to rest.

SOLUTION

Kinematics. Let r_A be the radius of the outer pulley and r_B that of the inner pulley.

$$v_A = r_A\omega_C \qquad v_B = r_B\omega_C = \frac{r_B}{r_A}v_A$$

$$s_A = r_A\theta_C \qquad s_B = \frac{r_B}{r_A}s_A$$

Use the principle of work and energy with position 1 being the initial rest position and position 2 being when cylinder *A* strikes the ground.

$$T_1 + U_{1\to2} = T_2:$$

where $$T_1 = 0$$

and $$T_2 = \frac{1}{2}m_Av_A^2 + \frac{1}{2}m_Bv_B^2 + \frac{1}{2}\bar{I}_C\omega_C^2$$

with $m_A = 5\text{ kg}, \quad m_B = 15\text{ kg}, \quad \bar{I}_C = m_C\bar{k}_C^2 = (15\text{ kg})(0.160\text{ m})^2 = 0.384\text{ kg}\cdot\text{m}^2$

$$T_2 = \frac{1}{2}\left[m_A + \frac{m_Br_B^2}{r_A^2} + \frac{\bar{I}_C}{r_A^2}\right]v_A^2$$

$$= \frac{1}{2}\left[5\text{ kg} + \frac{(15\text{ kg})(0.150\text{ m})^2}{(0.250\text{ m})^2} + \frac{0.384\text{ kg}\cdot\text{m}^2}{(0.250\text{ m})^2}\right]v_A^2$$

$$= (8.272\text{ kg})v_A^2$$

Principle of work and energy applied to the system consisting of blocks *A* and *B* and the double pulley *C*.

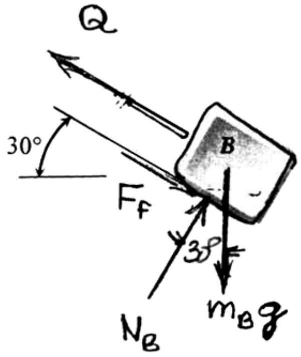
Work. $$U_{1\to2} = Ps_A + m_Ag\,s_A - F_Fs_B - m_Bg\,s_B\sin30°$$

where $$s_A = 1\text{ m}$$

PROBLEM 17.14 (Continued)

and $$s_B = \frac{r_B}{r_A} s_A = \frac{0.150\text{ m}}{0.250\text{ m}}(1\text{ m}) = 0.6\text{ m}$$

To find F_f use the free body diagram of block B.



$$\measuredangle 60^\circ\ \Sigma F = 0: \quad N_B - m_B g\cos 30^\circ = 0$$

$$N_B = m_B g\cos 30^\circ = (15\text{ kg})(9.81\text{ m/s})\cos 30^\circ = 127.44\text{ N}$$

$$F_f = \mu_k N_B = (0.2)(127.44\text{ N}) = 25.487\text{ N}$$

$$U_{1\to 2} = (200\text{ N})(1\text{ m}) + (5\text{ kg})(9.81\text{ m/s}^2)(1\text{ m})$$
$$-(25.487\text{ N})(0.6\text{ m}) - (15\text{ kg})(9.81\text{ m/s}^2)(0.6\text{ m})\sin 30^\circ$$
$$= 189.613\text{ J}$$

Work-energy: $0 + 189.613\text{ J} = (8.272\text{ kg})v_A^2$

(*a*) *Velocity of A*. $v_A = 4.7877\text{ m/s}$ $\mathbf{v}_A = 4.79\text{ m/s}\uparrow$ ◀

when the cylinder strikes the ground,

$$v_B = \frac{r_B}{r_A} v_A = \frac{0.150\text{ m}}{0.250\text{ m}}(4.7877\text{ m/s}) = 2.8726\text{ m/s}$$

$$\omega_C = \frac{v_A}{r_A} = \frac{4.7877\text{ m/s}}{0.250\text{ m}} = 19.1508\text{ rad/s}$$

After the cylinder strikes the ground use the principle of work and energy applied to a system consisting of block B and double pulley C.

Let T_3 be its kinetic energy when A strikes the ground.

$$T_3 = \frac{1}{2}m_B v_B^2 + \frac{1}{2}\bar{I}_C\omega_C^2$$
$$= \frac{1}{2}(15\text{ kg})(2.8726\text{ m/s})^2 + \frac{1}{2}(0.384\text{ kg}\cdot\text{m}^2)(19.1508\text{ rad/s})^2$$
$$= 132.305\text{ J}$$

When the system comes to rest, $T_4 = 0$

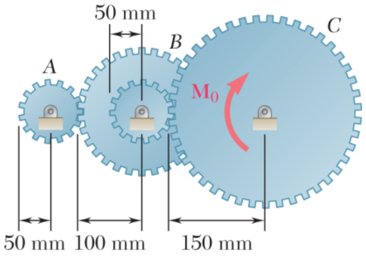
$$U_{3\to 4} = -(25.487\text{ N})s'_B - (15\text{ kg})(9.81\text{ m/s}^2)(s'_B\sin 30^\circ)$$
$$= -(99.062\text{ N})s'_B$$

where s'_B is the additional travel of block B.

$$T_3 + U_{3\to 4} = T_4: \quad 132.305\text{ J} - (99.062\text{ N})s'_B = 0$$

$$s'_B = 1.3356\text{ m}$$

(*b*) *Total distance:* $s_B + s'_B = 1.936\text{ m}$ ◀



PROBLEM 17.15

Gear *A* has a mass of 1 kg and a radius of gyration of 30 mm; gear *B* has a mass of 4 kg and a radius of gyration of 75 mm; gear *C* has a mass of 9 kg and a radius of gyration of 100 mm. The system is at rest when a couple $\mathbf{M}_0$ of constant magnitude 4 N · m is applied to gear *C*. Assuming that no slipping occurs between the gears, determine the number of revolutions required for disk *A* to reach an angular velocity of 300 rpm.

SOLUTION

Moments of inertia: $\bar{I} = mk^2$

Gear *A*: $I_A = (1\text{ kg})(0.030\text{ m})^2 = 0.9\times 10^{-3}\text{ kg}\cdot\text{m}^2$

Gear *B*: $I_B = (4\text{ kg})(0.075\text{ m})^2 = 22.5\times 10^{-3}\text{ kg}\cdot\text{m}^2$

Gear *C*: $I_C = (9\text{ kg})(0.100\text{ m})^2 = 90\times 10^{-3}\text{ kg}\cdot\text{m}^2$

Let r_A be the radius of gear *A*, r_1 the outer radius of gear *B*, r_2 the inner radius of gear *B*, and r_C the radius of gear *C*.

$$r_A = 50\text{ mm},\quad r_1 = 100\text{ mm},\quad r_2 = 50\text{ mm},\quad r_C = 150\text{ mm}$$

At the contact point between gears *A* and *B*,

$$r_1\omega_B = r_A\omega_A:\quad \omega_B = \frac{r_A}{r_1}\omega_A = 0.5\omega_A$$

At the contact point between gear *B* and *C*.

$$r_C\omega_C = r_2\omega_B:\quad \omega_C = \frac{r_2}{r_C}\omega_B = 0.33333\omega_B$$

$$\omega_C = 0.16667\omega_A$$

Kinetic energy: $T = \frac{1}{2}I_A\omega_A^2 + \frac{1}{2}I_B\omega_B^2 + \frac{1}{2}I_C\omega_C^2$

$$T = \frac{1}{2}[0.9\times 10^{-3}\omega_A^2 + (22.5\times 10^{-3})(0.5\omega_A)^2 + (90\times 10^{-3})(0.16667\omega_A)^2]$$

$$= (4.5125\times 10^{-3}\text{ kg}\cdot\text{m}^2)\omega_A^2$$

PROBLEM 17.15 (Continued)

Use the principle of work and energy applied to the system of all three gears with position 1 being the initial rest position and position 2 being when $\omega_A = 300$ rpm.

$$\omega_A = \frac{2\pi \text{ rad}}{\text{rev}} \cdot \frac{1\text{min}}{60 \text{ s}} \cdot 300 \frac{\text{rev}}{\text{min}} = 31.416 \text{ rad/s}$$

$$T_1 = 0$$

$$T_2 = (4.5125 \times 10^{-3} \text{ kg} \cdot \text{m}^2)(31.416 \text{ rad/s})^2 = 4.4565 \text{ J}$$

$$U_{1\to 2} = M\theta_C = (4 \text{ N} \cdot \text{m})\theta_C$$

Principle of work and energy.

$$T_1 + U_{1\to 2} = T_2: \quad 0 + 4.4565 \text{ J} = 4(\text{N} \cdot \text{m})\theta_C$$

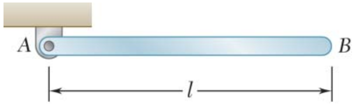
$$\theta_C = 1.11413 \text{ rad}$$

$$\theta_A = \frac{\theta_C}{0.16667} = 6\theta_C = 6.6848 \text{ rad}$$

$$\theta_A = \frac{6.6848 \text{ rad}}{2\pi \text{ rad/rev}}$$

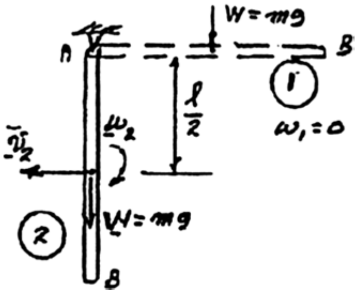
$\theta_A = 1.063$ rev ◀

PROBLEM 17.16



A slender rod of length l and weight W is pivoted at one end as shown. It is released from rest in a horizontal position and swings freely. (*a*) Determine the angular velocity of the rod as it passes through a vertical position and determine the corresponding reaction at the pivot, (*b*) Solve part *a* for $W = 1.8$ lb and $l = 3$ ft.

SOLUTION



Position 1:

$$v_1 = 0$$
$$\omega_1 = 0$$
$$T_1 = 0$$
$$\bar{v}_2 = \frac{l}{2}\omega_2$$

Position 2:

$$T_2 = \frac{1}{2}m\bar{r}_2^2 + \frac{1}{2}\bar{I}\omega_2^2$$
$$= \frac{1}{2}m\left(\frac{l}{2}\omega_2\right)^2 + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\omega_2^2$$
$$T_2 = \frac{1}{6}ml^2\omega_2^2$$

Work:

$$U_{1\to 2} = mg\frac{l}{2}$$

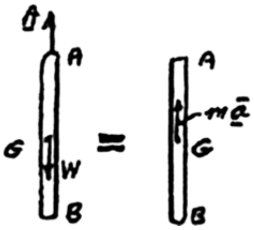
Principle of work and energy:

$$T_1 + U_{1\to 2} = T_2$$
$$0 + mg\frac{l}{2} = \frac{1}{6}ml^2\omega_2^2$$

(*a*) Expressions for angular velocity and reactions.

$$\omega_2^2 = \frac{3g}{l} \qquad \boldsymbol{\omega}_2 = \sqrt{\frac{3g}{l}} \circlearrowright \blacktriangleleft$$

$$\bar{a} = \frac{l}{2}\omega_2^2 = \frac{l}{2}\cdot\frac{3g}{l} = \frac{3}{2}g$$



$$+\uparrow \Sigma F = \Sigma(F)_{\text{eff}}: \quad A - W = m\bar{a}$$

$$A - mg = m\frac{3}{2}g$$

$$A = \frac{5}{2}mg \qquad \mathbf{A} = \frac{5}{2}W\uparrow \blacktriangleleft$$

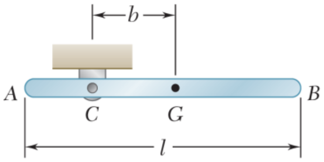
PROBLEM 17.16 (Continued)

(*b*) Application of data:

$$W = 1.8 \text{ lb}, \quad l = 3 \text{ ft}$$

$$\omega_2^2 = \frac{3g}{l} = \frac{3g}{3} = 32.2 \text{ rad}^2/\text{s}^2 \qquad \boldsymbol{\omega}_2 = 5.67 \text{ rad/s} \circlearrowright \blacktriangleleft$$

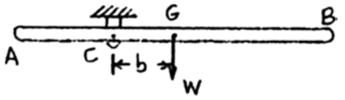
$$A = \frac{5}{2}W = \frac{5}{2}(1.8 \text{ lb}) \qquad \mathbf{A} = 4.5 \text{ lb} \uparrow \blacktriangleleft$$



PROBLEM 17.17

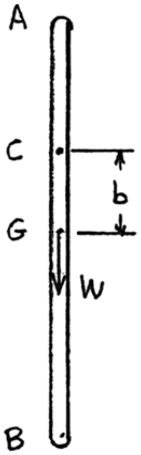
A slender rod of length l is pivoted about a Point C located at a distance b from its center G. It is released from rest in a horizontal position and swings freely. Determine (*a*) the distance b for which the angular velocity of the rod as it passes through a vertical position is maximum, (*b*) the corresponding values of its angular velocity and of the reaction at C.

SOLUTION



Position 1. $\bar{v} = 0, \quad \omega = 0 \quad T_1 = 0$

Elevation: $h = 0 \quad V_1 = mgh = 0$



Position 2. $\bar{v}_2 = b\omega_2$

$$I = \frac{1}{12}ml^2$$

$$T_2 = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2$$

$$= \frac{1}{2}m\left(b^2 + \frac{1}{12}l^2\right)\omega_2^2$$

Elevation: $h = -b \quad V_2 = -mgb$

Principle of conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad 0 + 0 = \frac{1}{2}m\left(b^2 + \frac{1}{12}l^2\right)\omega_2^2 - mgb$$

$$\omega_2^2 = \frac{2gb}{b^2 + \frac{1}{12}l^2}$$

(*a*) Value of b for maximum ω_2.

$$\frac{d}{db}\left(\frac{b}{b^2 + \frac{1}{12}l^2}\right) = \frac{\left(b^2 + \frac{1}{12}l^2\right) - b(2b)}{\left(b^2 + \frac{1}{12}l^2\right)^2} = 0 \qquad b^2 = \frac{1}{12}l^2 \qquad b = \frac{l}{\sqrt{12}} \blacktriangleleft$$

(*b*) Angular velocity.

$$\omega_2^2 = \frac{2g\frac{l}{\sqrt{12}}}{\frac{l^2}{12} + \frac{l^2}{12}}$$

$$= \sqrt{12}\frac{g}{l}$$

$$\omega_2 = 12^{1/4}\sqrt{\frac{g}{l}} \qquad \omega_2 = 1.861\sqrt{\frac{g}{l}} \blacktriangleleft$$

PROBLEM 17.17 (Continued)

Reaction at *C*.

$$a_n = b\omega_2^2$$

$$= \frac{l}{\sqrt{12}}\sqrt{12}\,\frac{g}{l}$$

$$= g$$

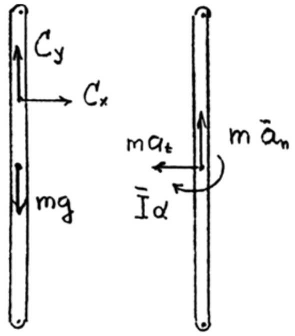
$$+\uparrow \Sigma F_y = ma_n: \quad C_y - mg = mg$$

$$C_y = 2mg$$

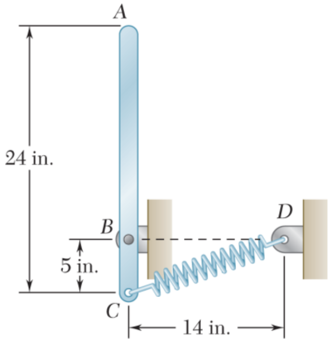
$$+\circlearrowright \Sigma M_C = mba_t + \bar{I}\alpha: \quad 0 = (mb^2 + \bar{I})\alpha$$

$$\alpha = 0, \quad a_t = 0$$

$$\xrightarrow{+} \Sigma F_x = ma_t: \qquad C_x = -ma_t = 0$$



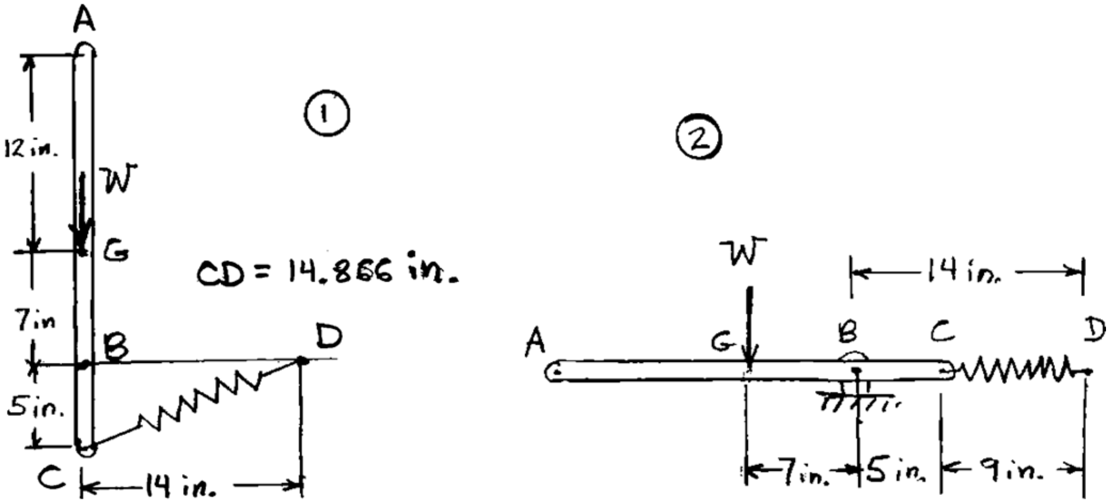
$$\mathbf{C} = 2mg \uparrow \blacktriangleleft$$



PROBLEM 17.18

A slender 9 lb rod can rotate in a vertical plane about a pivot at *B*. A spring of constant k = 30 lb/ft and of unstretched length 6 in. is attached to the rod as shown. Knowing that the rod is released from rest in the position shown, determine its angular velocity after it has rotated through 90°.

SOLUTION



Position 1:

Spring: $$x_1 = CD - (\overset{\text{Unstretched Length}}{\overbrace{6\text{ in.}}}) = 14.866 - 6 = 8.8661\text{ in.} = 0.73884\text{ ft}$$

$$V_e = \frac{1}{2}kx_1^2 = \frac{1}{2}(30\text{ lb/ft})(0.73884)^2 = 8.1882\text{ lb}\cdot\text{ft}$$

Gravity: $$V_g = Wh = (9\text{ lb})\left(\frac{7}{12}\text{ft}\right) = 5.25\text{ lb}\cdot\text{ft}$$

$$V_1 = V_e + V_g = 8.1882\text{ lb}\cdot\text{ft} + 5.25\text{ lb}\cdot\text{ft} = 13.438\text{ lb}\cdot\text{ft}$$

Kinetic energy: $$T_1 = 0$$

Position 2:

Spring: $$x_2 = 9\text{ in.} - 6\text{ in.} = 3\text{ in.} = 0.25\text{ ft}$$

$$V_e = \frac{1}{2}kx_2^2 = \frac{1}{2}(30\text{ lb/ft})(0.25\text{ ft})^2 = 0.9375\text{ lb}\cdot\text{ft}$$

PROBLEM 17.18 (Continued)

Gravity:

$$V_g = Wh = 0$$

$$V_2 = V_e + V_g$$

$$= 0.9375 \text{ lb} \cdot \text{ft}$$

Kinetic energy:

$$\bar{v}_2 = r\omega_2 = \left(\frac{7}{12}\text{ ft}\right)\omega_2$$

$$\bar{I} = \frac{1}{12}mL^2 = \frac{1}{12}\left(\frac{9 \text{ lb}}{32.2}\right)(2 \text{ ft})^2 = 0.093168 \text{ slug} \cdot \text{ft}^2$$

$$T_2 = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2$$

$$= \frac{1}{2}\left(\frac{9 \text{ lb}}{32.2}\right)\left(\left(\frac{7}{12}\text{ ft}\right)\omega_2\right)^2 + \frac{1}{2}(0.093168)\omega_2^2$$

$$T_2 = 0.094138\omega_2^2$$

Conservation of energy:

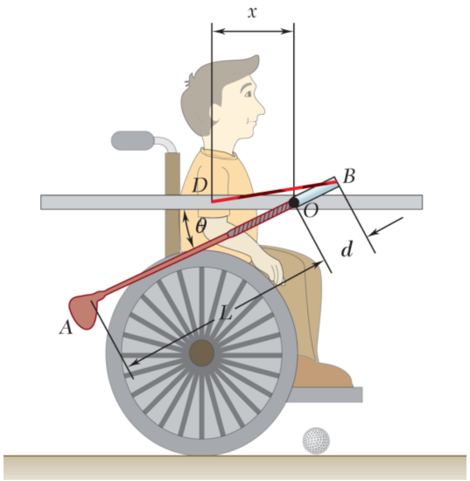
$$T_1 + V_1 = T_2 + V_2$$

$$0 + 13.438 = 0.094138\,\omega_2^2 + 0.9375$$

$$\omega_2^2 = 132.79$$

$$\omega_2 = 11.524 \text{ rad/s}$$

$\boldsymbol{\omega}_2 = 11.52$ rad/s ↺ ◀



PROBLEM 17.19

An adapted golf device attaches to a wheelchair to help people with mobility impairments play putt-putt. The stationary frame *OD* is attached to the wheelchair, and a club holder *OB* is attached to the pin at *O*. Holder *OB* is 6 in. long and weighs 8 oz, and the distance between *O* and *D* is *x*= 1 ft. The putter shaft has a length *L*= 36 in. and weighs 10 oz, while the putter head at *A* weighs 12 oz. Knowing that the 1- lb/in. spring between *D* and *B* is unstretched when θ = 90° and that the putter is released from rest at θ = 0°, determine the putter head speed when it hits the golf ball.

SOLUTION

Given:

$$m_A = \frac{12 \text{ oz}}{16\left(32.2 \text{ ft/s}^2\right)} = \frac{3}{4g} \text{ slugs}$$

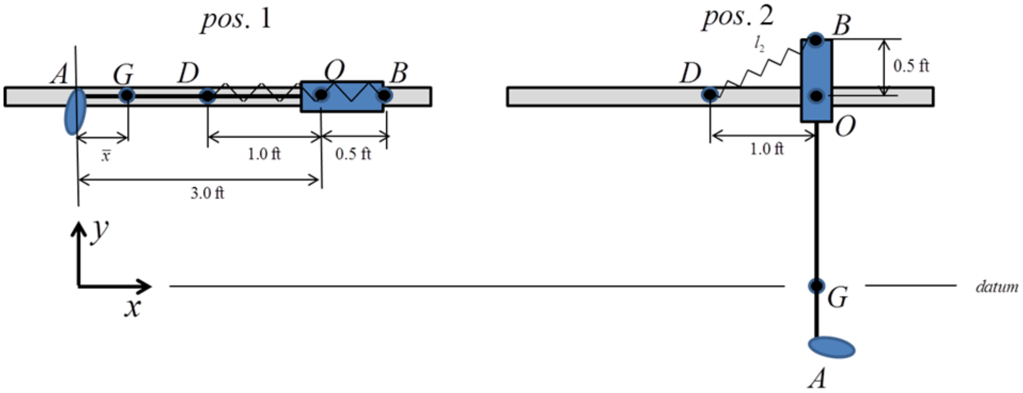
$$m_{OA} = \frac{10 \text{ oz}}{16\left(32.2 \text{ ft/s}^2\right)} = \frac{5}{8g} \text{ slugs}$$

$$m_{OB} = \frac{8 \text{ oz}}{16\left(32.2 \text{ ft/s}^2\right)} = \frac{1}{2g} \text{ slugs}$$

$$l_{OB} = 0.5 \text{ ft},\ l_{OD} = 1.0 \text{ ft},\ l_{OA} = 3.0 \text{ ft},\ l_{BD} = 1.5 \text{ ft}$$

$$k = 12 \text{ lb/ft},\quad \theta_1 = 0°,\ \theta_2 = 90°$$

Diagram of two positions:



Location of Mass Center:

$$\bar{x} = \frac{\sum m_i \bar{x}_i}{\sum m_i} = \frac{m_{OA}(0) + m_{OA}(1.5) + m_{OB}(3.25)}{m_A + m_{OA} + m_{OB}}$$

$$= \frac{41}{30} \text{ m}$$

PROBLEM 17.19 (Continued)

Stretch of Spring:

$$l_2 = \sqrt{l_{OD}^2 + l_{OB}^2}$$
$$= \sqrt{1.25}\ \text{m}$$
$$s_1 = l_{OD} + l_{OB} - l_2$$
$$= 1.5 - \sqrt{1.25}\ \text{m}$$
$$s_2 = 0$$

Mass Moment of Inertia:

$$I_O = \sum I_{O,i}$$
$$= I_{O,A} + I_{O,O} + I_{O,OB}$$
$$= m_A l_{OA}^2 + \frac{1}{3} m_{OA} \left(l_{OA}\right)^2 + \frac{1}{3} m_{OB} l_{OB}{}^2$$
$$= \frac{26}{3g}\ \text{slug}{\bullet}\text{ft}^2$$

Work Energy Equation:

$$\cancel{T_1} + V_{g1} + V_{e,1} + \cancel{U'_{1-2}} = T_2 + \cancel{V_{g2}} + \cancel{V_{e,2}}$$

$$V_{g,1} = m_T g h_1$$
$$= \left(m_A + m_{OA} + m_{OB}\right) g \left(l_{OA} - \overline{x}\right)$$
$$= \frac{49}{16}\ \text{ft}{\bullet}\text{lb}$$

$$V_{e,1} = \frac{1}{2} k s_1^2$$
$$= \frac{1}{2} 12 \left(1.5 - \sqrt{1.25}\right)^2$$
$$= 6\left(1.5 - \sqrt{1.25}\right)^2\ \text{ft}{\bullet}\text{lb}$$

$$T_2 = \frac{1}{2} I_O \omega_2^2$$
$$= \frac{13}{3g} \omega_2^2$$

$$\frac{49}{16}\ \text{ft}{\bullet}\text{lb} + 6\left(1.5 - \sqrt{1.25}\right)^2\ \text{ft}{\bullet}\text{lb} = \frac{13}{3g}\omega_2^2$$

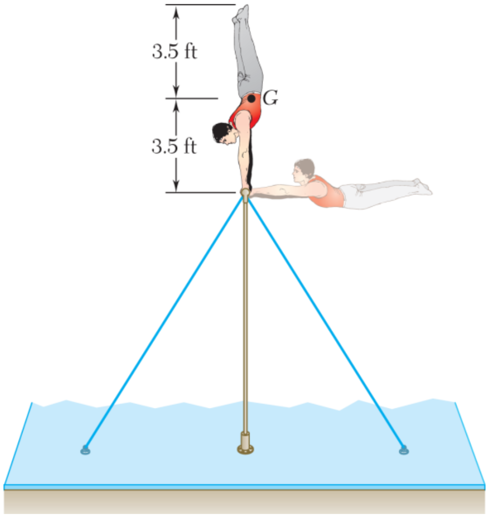
Solving for angular Velocity at positon 2:

$$\omega_2 = 5.409\ \text{rad/s}$$

Putter Head Speed at Position 2:

$$v_{A,2} = l_{OA}\omega_2$$

$v_{A,2} = 16.23$ ft/s ◀



PROBLEM 17.20

A 160-lb gymnast is executing a series of full-circle swings on the horizontal bar. In the position shown he has a small and negligible clockwise angular velocity and will maintain his body straight and rigid as he swings downward. Assuming that during the swing the centroidal radius of gyration of his body is 1.5 ft, determine his angular velocity and the force exerted on his hands after he has rotated through (*a*) 90°, (*b*) 180°.

SOLUTION

Position 1. (*Directly above the bar*).

Elevation: $h_1 = 3.5 \text{ ft}$

Potential energy: $V_1 = Wh_1 = (160 \text{ lb})(3.5 \text{ ft}) = 560 \text{ ft} \cdot \text{lb}$

Speeds: $\omega_1 = 0, \quad \bar{v}_1 = 0$

Kinetic energy: $T_1 = 0$

(*a*) *Position 2*. (*Body at level of bar after rotating* 90°).

Elevation: $h_2 = 0.$

Potential energy: $V_2 = 0$

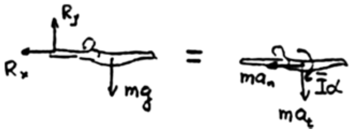
Speeds: $\bar{v}_2 = 3.5\omega_2.$

Kinetic energy:

$$T_2 = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}mk^2\omega_2^2$$

$$T_2 = \frac{1}{2}\left(\frac{160}{32.2}\right)(3.5\omega_2)^2 + \frac{1}{2}\left(\frac{160}{32.2}\right)(1.5)^2\omega_2^2$$

$$= 36.025\omega_2^2$$



Principle of conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad 0 + 560 = 36.025\omega_2^2$$

$$\omega_2^2 = 15.545$$

$\boldsymbol{\omega}_2 = 3.94 \text{ rad/s} \circlearrowright$ ◀

PROBLEM 17.20 (Continued)

Kinematics: $\bar{a}_t = 3.5\alpha$

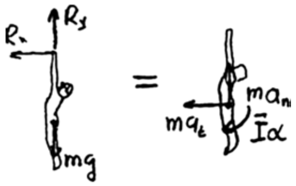
$$\bar{a}_n = 3.5\omega_2^2 = (3.5)(15.545) = 54.407 \text{ ft/s}^2 \longleftarrow$$

$$+\circlearrowright \Sigma M_0 = \Sigma\left(M_0\right)_{\text{eff}}: \quad (3.5)(160) = \left(\frac{160}{32.2}\right)(3.5)(3.5\alpha) + \left(\frac{160}{32.2}\right)(1.5)^2\alpha$$

$$\alpha = 7.7724 \text{ rad/s}^2 \qquad \bar{a}_t = 27.203 \text{ ft/s}^2 \downarrow$$

$$\overset{+}{\longleftarrow} \Sigma F_x = ma_n: \qquad R_x = \left(\frac{160}{32.2}\right)(54.407) = 270.35 \text{ lb} \longleftarrow$$

$$+\uparrow \Sigma F_y = -ma_t: \qquad R_y - 160 = -\left(\frac{160}{32.2}\right)(27.203) \uparrow$$



$$R_y = 24.83 \text{ lb} \uparrow$$

R = 271 lb ⦣ 5.25° ◀

(*b*) *Position 3*. (*Directly below bar after rotating* 180°).

Elevation: $h_3 = -3.5$ ft.

Potential energy: $V_3 = Wh_3 = (160)(-3.5) = -560 \text{ ft} \cdot \text{lb}$

Speeds: $\bar{v}_3 = 3.5\omega_3$.

Kinetic energy: $T_3 = 36.025\omega_3^2$

Principle of conservation of energy.

$$T_1 + V_1 = T_3 + V_3: \quad 0 + 560 = 36.025\omega_3^2 - 560$$

$$\omega_3^2 = 31.09$$

$\boldsymbol{\omega}_3 = 5.58$ rad/s $\circlearrowright$ ◀

Kinematics: $a_n = (3.5)(31.09) = 108.81 \text{ ft/s}^2 \uparrow$

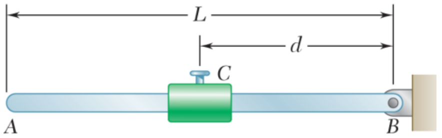
From $\Sigma M_0 = \Sigma(M_0)_{\text{eff}}$ and $\Sigma F_x = 0$,

$$\alpha = 0, \qquad a_t = 0 \qquad R_x = 0$$

$$+\uparrow \Sigma F_y = ma_n: \quad R_y - 160 = \left(\frac{160}{32.2}\right)(108.81)$$

$$R_y = 700.62 \text{ lb}$$

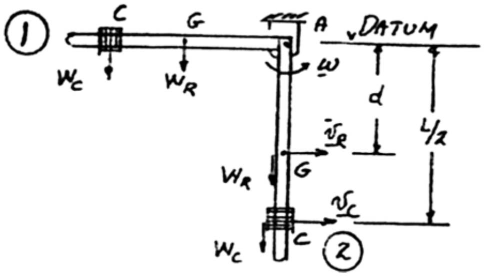
R = 701 lb ↑ ◀



PROBLEM 17.21

A collar with a mass of 1 kg is rigidly attached at a distance $d = 300$ mm from the end of a uniform slender rod AB. The rod has a mass of 3 kg and is of length $L = 600$ mm. Knowing that the rod is released from rest in the position shown, determine the angular velocity of the rod after it has rotated through 90°.

SOLUTION



Kinematics.

Rod $$v_R = \frac{L}{2}\omega$$

Collar $$v_C = d\omega$$

Position 1. $$\omega = 0$$
$$T_1 = 0 \quad V_1 = 0$$

Position 2. $$T_2 = \frac{1}{2}m_R\bar{v}_R^2 + \frac{1}{2}\bar{I}_R\omega^2 + \frac{1}{2}m_C v_C^2$$
$$= \frac{1}{2}m_R\left(\frac{L}{2}\omega\right)^2 + \frac{1}{2}\left(\frac{1}{12}m_R L^2\right)\omega^2 + \frac{1}{2}m_C d^2\omega^2$$
$$= \frac{1}{6}m_R L^2\omega^2 + \frac{1}{2}m_C d^2\omega^2$$
$$V_2 = -W_C d - W_R\frac{L}{2}$$

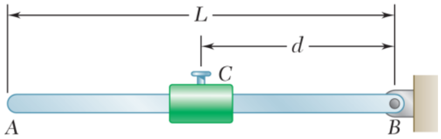
$$T_1 + V_1 = T_2 + V_2: \quad 0 + 0 = \left(\frac{1}{6}m_R L^2 + \frac{1}{2}m_C d^2\right)\omega^2 - W_C d - W_R\frac{L}{2}$$

$$\omega^2 = \frac{3(2W_C d + W_C L)}{3m_C d^2 + m_R L^2} = \frac{3g(2m_C d + m_R L)}{3m_C d^2 + m_R L^2} \qquad (1)$$

Data: $m_C = 1$ kg, $d = 0.3$ m, $m_R = 3$ kg, $L = 0.6$ m

From Eq. (1), $$\omega^2 = 3(9.81)\left[\frac{(2)(1)(0.3) + 3(0.6)}{3(1)(0.3)^2 + 3(0.6)^2}\right]$$
$$= 52.32$$

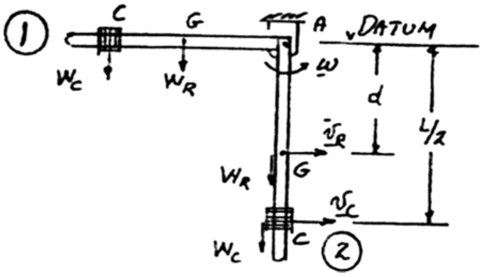
$\bar{\omega} = 7.23$ rad/s ↺ ◀



PROBLEM 17.22

A collar with a mass of 1 kg is rigidly attached to a slender rod *AB* of mass 3 kg and length $L = 600$ mm. The rod is released from rest in the position shown. Determine the distance *d* for which the angular velocity of the rod is maximum after it has rotated 90°.

SOLUTION



Kinematics.

Rod $$v_R = \frac{L}{2}\omega$$

Collar $$v_C = d\omega$$

Position 1. $$\omega = 0$$
$$T_1 = 0 \quad V_1 = 0$$

Position 2.
$$T_2 = \frac{1}{2}m_R\bar{v}_R^2 + \frac{1}{2}\bar{I}_R\omega^2 + \frac{1}{2}m_C v_C^2$$
$$= \frac{1}{2}m_R\left(\frac{L}{2}\omega\right)^2 + \frac{1}{2}\left(\frac{1}{12}m_R L^2\right)\omega^2 + \frac{1}{2}m_C d^2\omega^2$$
$$= \frac{1}{6}m_R L^2\omega^2 + \frac{1}{2}m_C d^2\omega^2$$
$$V_2 = -W_C d - W_R\frac{L}{2}$$

$$T_1 + V_1 = T_2 + V_2: \quad 0 + 0 = \left(\frac{1}{6}m_R L^2 + \frac{1}{2}m_C d^2\right)\omega^2 - W_C d - W_R\frac{L}{2}$$

$$\omega^2 = \frac{3(2W_C d + W_C L)}{3m_C d^2 + m_R L^2} = \frac{3g(2m_C d + m_R L)}{3m_C d^2 + m_R L^2} \qquad (1)$$

Let $$x = \frac{d}{L}.$$

$$\omega^2 = \frac{3g}{L}\cdot\frac{2x + \frac{m_R}{m_C}}{3x^2 + \frac{m_R}{m_C}}$$

Data: $$m_C = 1\text{ kg}, \quad m_R = 3\text{ kg}$$
$$\frac{L\omega^2}{3g} = \frac{2x + 3}{3x^2 + 3}$$

PROBLEM 17.22 (Continued)

$L\omega^2/3g$ is maximum. Set its derivative with respect to x equal to zero.

$$\frac{d}{dx}\left(\frac{L\omega^2}{3g}\right) = \frac{(3x^2+3)(2)-(2x+3)(6x)}{(3x^2+3)^2} = 0$$

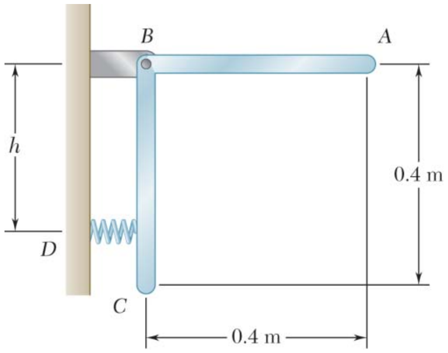
$$-6x^2 - 18x + 6 = 0$$

Solving the quadratic equation

$$x = -3.30 \quad \text{and} \quad x = 0.30278$$

$$\begin{aligned} d &= 0.30278L \\ &= (0.30278)(0.6) \\ &= 0.1817 \text{ m} \end{aligned}$$

$d = 181.7$ mm ◀



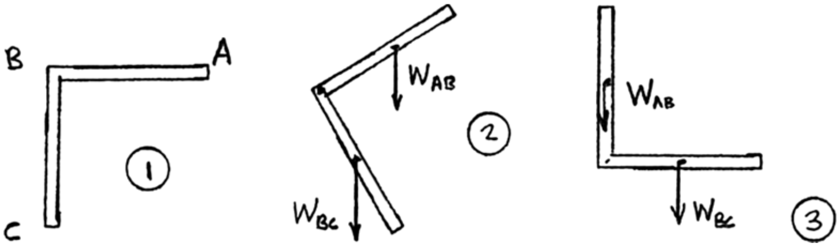
PROBLEM 17.23

Two identical slender rods *AB* and *BC* are welded together to form an L-shaped assembly. The assembly is pressed against a spring at *D* and released from the position shown. Knowing that the maximum angle of rotation of the assembly in its subsequent motion is 90° counterclockwise, determine the magnitude of the angular velocity of the assembly as it passes through the position where rod *AB* forms an angle of 30° with the horizontal.

SOLUTION

Moment of inertia about *B*.

$$I_B = \frac{1}{3}m_{AB}l^2 + \frac{1}{3}m_{BC}l^2$$



Position 2.

$$\theta = 30°$$

$$V_2 = W_{AB}(h_{AB})_2 + W_{BC}(h_{BC})_2$$

$$= W_{AB}\frac{l}{2}\sin 30° + W_{BC}\left(-\frac{l}{2}\cos 30°\right)$$

$$T_2 = \frac{1}{2}I_B\omega_2^2 = \frac{1}{6}(m_{AB}+m_{BC})l^2\omega_2^2$$

Position 3.

$$\theta = 90°$$

$$V_3 = W_{AB}\frac{l}{2} \quad T_3 = 0$$

Conservation of energy.

$$T_2 + V_2 = T_3 + V_3:$$

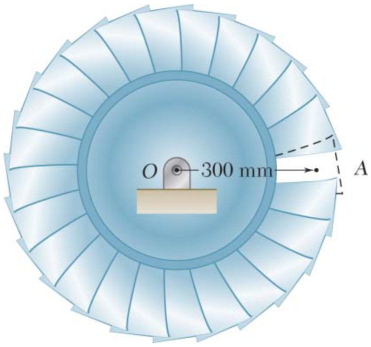
$$\frac{1}{6}(m_{AB}+m_{BC})l^2\omega_2^2 + W_{AB}\frac{l}{2}\sin 30° - W_{BC}\frac{l}{2}\cos 30° = 0 + W_{AB}\frac{l}{2}$$

$$\omega_2^2 = \frac{3}{l}\cdot\frac{W_{AB}(1-\sin 30°) + W_{BC}\cos 30°}{m_{AB}+m_{BC}}$$

$$= \frac{3}{2}\frac{g}{l}[1-\sin 30° + \cos 30°]$$

$$= 2.049\frac{g}{l} = 2.049\frac{9.81}{0.4} = 50.25$$

$\omega_2 = 7.09$ rad/s ◀



PROBLEM 17.24

The 30-kg turbine disk has a centroidal radius of gyration of 175 mm and is rotating clockwise at a constant rate of 60 rpm when a small blade of weight 0.5 N at Point *A* becomes loose and is thrown off. Neglecting friction, determine the change in the angular velocity of the turbine disk after it has rotated through (*a*) 90°, (*b*) 270°.

SOLUTION

Mass of blade. $m_A = 51\,\text{grams} = 0.051\,\text{kg}$

Weight of blade. $m_A g = (0.051)(9.81) = 0.5\ \text{N}$

Moment of inertia about *O*. $I_O = mk^2 - m_A r_2^2 = 30(0.175)^2 - 51\times 10^{-3}(0.3)^2 = 0.91416\ \text{kg}\cdot\text{m}^2$

Location of mass center for the position shown.

$$(m - m_A)\overline{x} = -m_A r_A \qquad \overline{x} = -\frac{m_A r_A}{m - m_A}$$

Position 1. $\theta = 0°, \quad \omega_1 = 60\ \text{rpm} = 2\pi\ \text{rad/s}$

Kinetic energy: $T_1 = \frac{1}{2} I_O \omega_1^2$

Center of gravity lies at the level of Point *O*. $h_1 = 0$

Potential energy: $V_1 = (mg - m_A g)h_1 = 0$

(*a*) *Position 2.* $\theta = 90°$

Kinetic energy: $T_2 = \frac{1}{2} I_O \omega_2^2$

Center of gravity lies a distance $\frac{m_A r_A}{m - m_A}$ above Point *O*.

$$h_2 = \frac{m_A r_A}{m - m_A}$$

Potential energy: $V_2 = (mg - m_A g)h_2 = m_A g r_A = (0.5)(0.3) = 0.150\ \text{N}\cdot\text{m}$

PROBLEM 17.24 (Continued)

Conservation of energy. $T_1 + V_1 = T_2 + V_2:$

$$\frac{1}{2}I_O\omega_1^2 + 0 = \frac{1}{2}I_O\omega_2^2 + V_2$$

$$\omega_2^2 = \omega_1^2 - \frac{2V_2}{I_O} - (2\pi)^2 - \frac{(2)(0.15)}{0.91416} \qquad \omega_2 = 6.257016 \text{ rad/s}$$

$$\Delta\omega = \omega_2 - \omega_1 = 6.257016 - 2\pi = -0.02617 \text{ rad/s}$$

$\Delta\omega = -0.250$ rpm ◀

(*b*) *Position 3.* $\theta = 270°$

Kinetic energy: $T_3 = \frac{1}{2}I_O\omega_3^2$

Center of gravity lies a distance $\frac{m_A r_A}{m - m_A}$ below Point *O*.

$$h_3 = -\frac{m_A r_A}{m - m_A}$$

Potential energy: $V_3 = (mg - m_A g)h_3 = -m_A g r_A = -(0.5)(0.3) = -0.15 \text{ N}\cdot\text{m}$

Conservation of energy. $T_1 + V_1 = T_3 + V_3:$

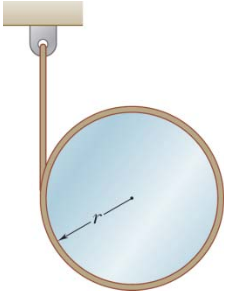
$$\frac{1}{2}I_O\omega_1^2 + 0 = \frac{1}{2}I_O\omega_3^2 + V_3$$

$$\omega_3^2 = \omega_1^2 - \frac{2V_3}{I_O} = (2\pi)^2 - \frac{(2)(-0.15)}{0.91416}$$

$$\omega_3 = 6.309246 \text{ rad/s}$$

$$\Delta\omega = \omega_3 - \omega_1 = 6.309246 - 2\pi = 0.026061 \text{ rad/s}$$

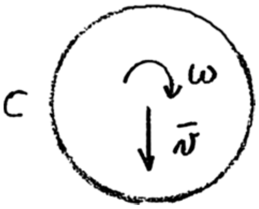
$\Delta\omega = 0.249$ rpm ◀



PROBLEM 17.25

A rope is wrapped around a cylinder of radius r and mass m as shown. Knowing that the cylinder is released from rest, determine the velocity of the center of the cylinder after it has moved downward a distance s.

SOLUTION



Point C is the instantaneous center.

$$\bar{v} = r\omega \qquad \omega = \frac{\bar{v}}{r}$$

Position 1. At rest. $T_1 = 0$

Position 2. Cylinder has fallen through distance s.

$$T_2 = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$

$$= \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{\bar{v}}{r}\right)^2$$

$$= \frac{3}{4}m\bar{v}^2$$

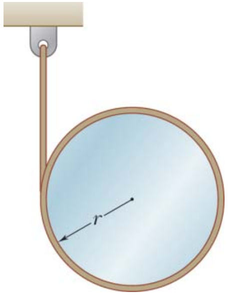
Work. $U_{1\to 2} = mgs$

Principle of work and energy.

$$T_1 + U_{1\to 2} = T_2: \quad 0 + mgs = \frac{3}{4}m\bar{v}^2$$

$$\bar{v}^2 = \frac{4gs}{3}$$

$$\bar{\mathbf{v}} = \sqrt{\frac{4gs}{3}}\downarrow \blacktriangleleft$$

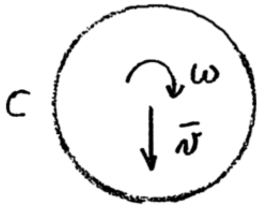


PROBLEM 17.26

Solve Problem 17.25, assuming that the cylinder is replaced by a thin-walled pipe of radius r and mass m.

PROBLEM 17.25 A rope is wrapped around a cylinder of radius r and mass m as shown. Knowing that the cylinder is released from rest, determine the velocity of the center of the cylinder after it has moved downward a distance s.

SOLUTION



Point C is the instantaneous center.

$$\bar{v} = r\omega \qquad \omega = \frac{\bar{v}}{r}$$

Position 1. At rest. $\quad T_1 = 0$

Position 2. Cylinder has fallen through distance s.

$$T_2 = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$

$$= \frac{1}{2}m\bar{v}^2 + \frac{1}{2}(mr^2)\left(\frac{\bar{v}}{r}\right)^2$$

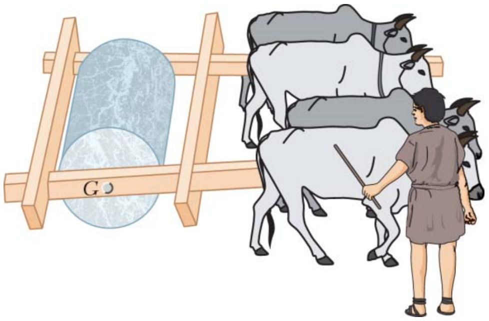
$$= m\bar{v}^2$$

Work. $\quad U_{1\to 2} = mgs$

Principle of work and energy.

$$T_1 + U_{1\to 2} = T_2: \quad 0 + mgs = m\bar{v}^2$$

$$\bar{v}^2 = gs \qquad \bar{\mathbf{v}} = \sqrt{gs}\downarrow \blacktriangleleft$$



PROBLEM 17.27

Greek engineers had the unenviable task of moving large columns from the quarries to the city. One engineer, Chersiphron, tried several different techniques to do this. One method was to cut pivot holes into the ends of the stone and then use oxen to pull the column. The 4-ft diameter column weighs 12,000 lbs, and the team of oxen generates a constant pull force of 1500 lbs on the center of the cylinder *G*. Knowing that the column starts from rest and rolls without slipping, determine *(a)* the velocity of its center *G* after it has moved 5 ft, *(b)* the minimum static coefficient of friction that will keep it from slipping.

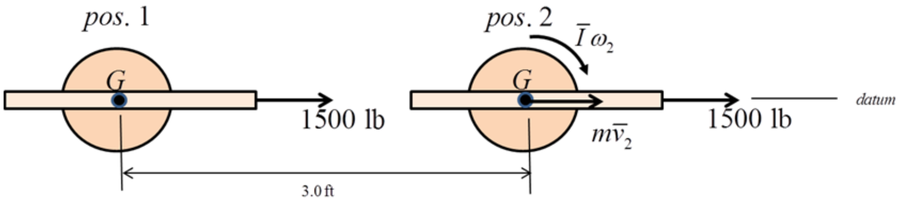
SOLUTION

Given:

$$m = \frac{12000 \text{ lb}}{32.2 \text{ ft/s}^2} = 372.67 \text{ slugs}$$

$$\bar{I} = \frac{1}{2}mr^2, \; r = 2 \text{ ft}, \; F = 1500 \text{ lbs}, \; \Delta x = 5 \text{ ft}$$

Diagram of two positions:



Work Energy Equation:

$$\cancel{T_1} + \cancel{V_{g1}} + \cancel{V_{e,1}} + U'_{1-2} = T_2 + \cancel{V_{g2}} + \cancel{V_{e,2}}$$

$$U'_{1-2} = \int_0^5 F dx$$

$$= 7500 \text{ ft}\bullet\text{lb}$$

$$T_2 = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2$$

Kinematics:

$$\omega_2 = \frac{\bar{v}}{r}$$

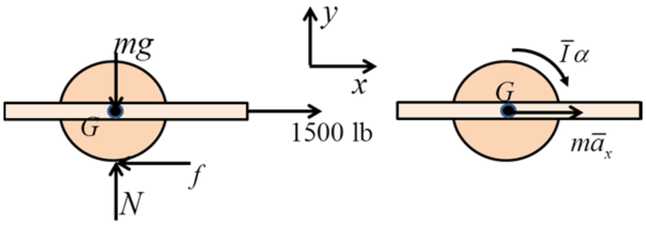
(a) Solving for Velocity at positon 2:

$$7500 = \frac{12000}{2g}\bar{v}_2^2 + \frac{12000}{4g}\cancel{r^2}\left(\frac{\bar{v}}{\cancel{r}}\right)^2$$

$\bar{v}_2 = 5.18$ ft/s ◀

PROBLEM 17.27 (Continued)

Free Body Diagram:



Kinetics:

$$\sum F_x = m\bar{a}_x \qquad \sum F_y = m\cancel{\bar{a}_y} \qquad \sum M_G = \bar{I}\alpha$$

$$1500 - f = m\bar{a}_x \qquad N - mg = 0 \qquad fr = \bar{I}\alpha$$

$$f = 1500 - m\bar{a}_x \qquad N = 12000 \text{ lb}$$

Kinematics:

$$\alpha = \frac{\bar{a}_x}{r}$$

Substitute into moment equation:

$$fr = \frac{1}{2}mr^2\frac{\bar{a}_x}{r}$$

$$\bar{a}_x = \frac{2f}{m}$$

Substitute into x-direction equation:

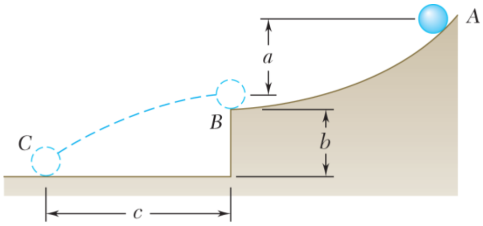
$$f = 1500 - \cancel{m}\frac{2f}{\cancel{m}}$$

$$f = 500 \text{ lbs}$$

(b) Minimum Necessary Coefficient of Friction:

$$\mu_s = \frac{f}{N}$$

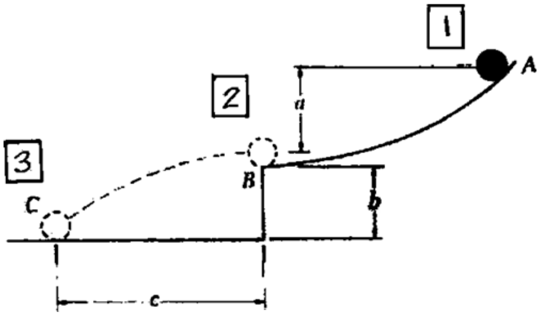
$\mu_s = 0.042$ ◀



PROBLEM 17.28

A small sphere of mass m and radius r is released from rest at A and rolls without sliding on the curved surface to Point B where it leaves the surface with a horizontal velocity. Knowing that $a = 1.5$ m and $b = 1.2$ m, determine (a) the speed of the sphere as it strikes the ground at C, (b) the corresponding distance c.

SOLUTION



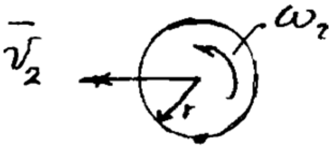
Work: $U_{1\to 2} = mga$

Kinetic energy: $T_1 = 0$

Rolling motion at position 2. $\bar{v}_2 = r\omega$ or $60\omega = \dfrac{v}{r}$

$$T_2 = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{7}{10}m\bar{v}^2$$



Principle of work and energy.

$$T_1 + U_{1\to 2} = T_2: \quad 0 + mga = \frac{7}{10}m\bar{v}^2$$

$$\bar{v}^2 = \frac{10ga}{7} = \frac{(10)(9.81\text{ m/s}^2)(1.5\text{ m})}{7} = 21.021\text{ m/s}^2$$

$$\bar{v} = 4.5849\text{ m/s}$$

For path B to C the motion is projectile motion. Let $t = 0$ at Point B. Let $y = 0$ at Point C.

Vertical motion: $v_y = (v_y)_0 - gt = -gt$

$$y = y_0 + (v_y)_0 t - \frac{1}{2}gt^2$$

At Point C, $0 = b + 0 - \dfrac{1}{2}gt_C^2$

$$t_C = \sqrt{\frac{2b}{g}} = \sqrt{\frac{(2)(1.2\text{ m})}{9.81\text{ m/s}^2}} = 0.49462\text{ s}$$

$$(v_y)_C = -gt_C = -(9.81\text{ m/s}^2)(0.49462\text{ s}) = -4.8522\text{ m/s}$$

PROBLEM 17.28 (Continued)

Horizontal motion: Let the x coordinate point to the left with origin below B.

$$v_x = (v_x)_B = \overline{v} = 4.5849 \text{ m/s}$$

(*a*) *Speed at C.*

$$v_C = \sqrt{(v_x)_C^2 + (v_y)_C^2}$$

$$v_C = \sqrt{(4.5849)^2 + (4.8522)^2}$$

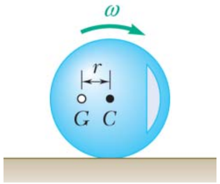
$v_C = 6.68$ m/s ◀

(*b*) *Distance c.*

$$c = v_x t_C$$

$$c = (4.5849 \text{ m/s})(0.49462 \text{ s})$$

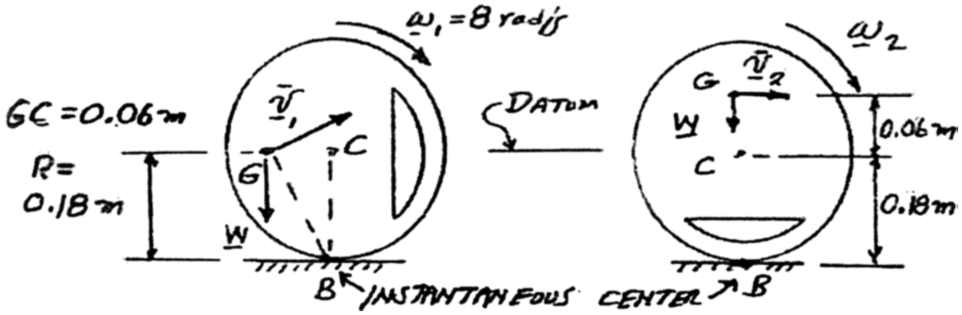
$c = 2.27$ m ◀



PROBLEM 17.29

The mass center G of a 3-kg wheel of radius $R = 180\text{ mm}$ is located at a distance $r = 60$ mm from its geometric center C. The centroidal radius of gyration of the wheel is $\bar{k} = 90\text{ mm}.$ As the wheel rolls without sliding, its angular velocity is observed to vary. Knowing that $\omega = 8\text{ rad/s}$ in the position shown, determine (a) the angular velocity of the wheel when the mass center G is directly above the geometric center C, (b) the reaction at the horizontal surface at the same instant.

SOLUTION


$\omega_1 = 8$ rad/s
GC = 0.06 m
R = 0.18 m
$\bar{v}_1$
C
G
W
DATUM
ω_2
G
$\bar{v}_2$
W
C
0.06 m
0.18 m
B
INSTANTANEOUS CENTER
B

$$\bar{v}_1 = (BG)\omega_1$$
$$= \sqrt{(0.18)^2 + (0.06)^2}\,(8)$$
$$= 8\sqrt{0.036}\ \text{m/s}$$
$$\bar{v}_2 = 0.24\omega_2$$
$$m = 3\text{ kg}$$
$$\bar{k} = 0.09\text{ m}$$

Position 1.

$$V_1 = 0$$
$$T_1 = \frac{1}{2}m\bar{v}_1^2 + \frac{1}{2}\bar{I}\omega_1^2$$
$$= \frac{1}{2}(3)(8\sqrt{0.036})^2 + \frac{1}{2}(3)(0.09)^2(8)^2$$
$$= 4.2336\text{ J}$$

Position 2.

$$V_2 = Wh$$
$$= mgh$$
$$= (3)(9.81)(0.06)$$
$$= 1.7658\text{ J}$$
$$T_2 = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2$$
$$= \frac{1}{2}(3)(0.24\omega_2)^2 + \frac{1}{2}(3)(0.09)^2\omega_2^2$$
$$= 0.09855\omega_2^2$$

PROBLEM 17.29 (Continued)

(*a*) Conservation of energy.

$$T_1 + V_1 = T_2 + V_2$$

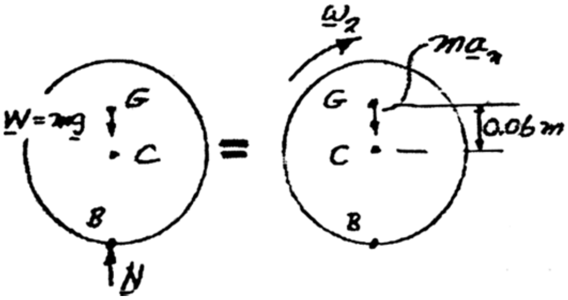
$$4.2336 \text{ J} + 0 = 0.09855\omega_2^2 + 1.7658 \text{ J}$$

$$\omega_2^2 = 25.041$$

$$\omega_2 = 5.004 \text{ rad/s}$$

$\boldsymbol{\omega}_2 = 5.00$ rad/s ◀

(*b*) Reaction at *B*.



$$ma_n = m(CG)\omega_2^2$$

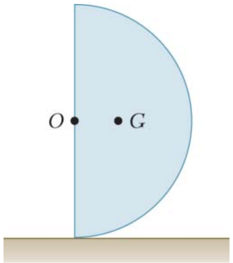
$$= (3 \text{ kg})(0.06 \text{ m})(5.00 \text{ rad/s})^2$$

$$= 4.5 \text{ N}\downarrow$$

$$+\uparrow \Sigma F_y = ma_y: \quad N - mg = -ma_n$$

$$N - (3)(9.81) = -4.5$$

N = 24.9 N↑ ◀

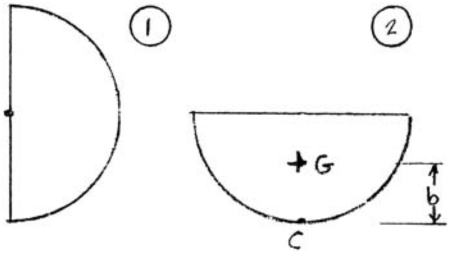


PROBLEM 17.30

A half-cylinder of mass m and radius r is released from rest in the position shown. Knowing that the half-cylinder rolls without sliding, determine (a) its angular velocity after it has rolled through $90°$, (b) the reaction at the horizontal surface at the same instant. [*Hint*: Note that $GO = 4r/3\pi$ and that, by the parallel-axis theorem,

$$\bar{I} = \frac{1}{2}mr^2 - m(GO)^2.$$
]

SOLUTION



Position 1. $T_1 = 0 \qquad V_1 = 0$

Position 2. $$V_2 = -mg(OG) = -\frac{4}{3\pi}mgr$$

Moments of inertia: $$\bar{I} = I_0 - m(OG)^2$$

$$\bar{I} = \frac{1}{2}mr^2 - m\left(\frac{4r}{3\pi}\right)^2 = 0.319873mr^2$$

Kinematics: Point C is the instantaneous center.

$$\bar{v} = v_G = b\omega_2 = \left(r - \frac{4r}{3\pi}\right)\omega_2 = 0.57559r\omega_2$$

Kinetic energy: $$T_2 = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 = \frac{1}{2}m\left(0.57559r\omega_2\right)^2 + \frac{1}{2}\left(0.319873\right)mr^2\omega_2^2$$

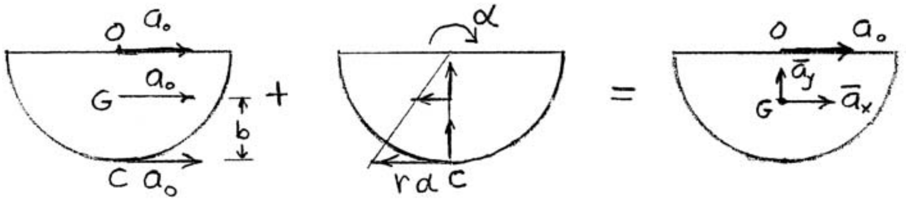
$$= 0.32559mr^2\omega_2^2$$

(a) Conservation of energy. $T_1 + V_1 = T_2 + V_2$:

$$0 + 0 = 0.32559mr^2\omega_2^2 - \frac{4}{3\pi}mgr \qquad \omega_2^2 = 1.3035\frac{g}{r}$$

$$\omega_2 = 1.142\sqrt{\frac{g}{r}} \circlearrowright \blacktriangleleft$$

Kinematics:

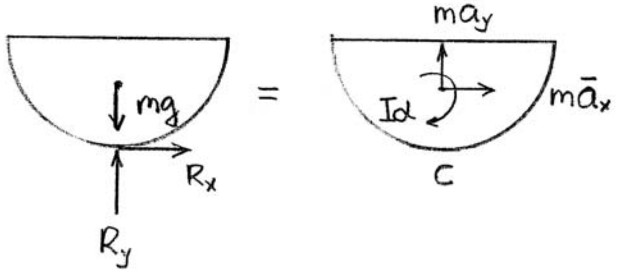


Translation + Rotation about O = Rolling Motion

(*b*) Acceleration of *C*. *x* component. $a_0 - r\alpha = 0 \qquad a_0 = r\alpha$

Acceleration of *G*. *x* component. $\bar{a}_x = a_0 \quad (OG)\alpha = b\alpha \longrightarrow$

y component. $\bar{a}_y = (OG)\omega_2^2 = \dfrac{4}{3\pi}r\omega_2^2 \uparrow$



$$+\circlearrowright\ \Sigma M_C = \Sigma(M_C)_{\text{eff}}:$$

$$0 = bm\bar{a}_x + \bar{I}\alpha$$

$$= mb^2\alpha + \bar{I}\alpha$$

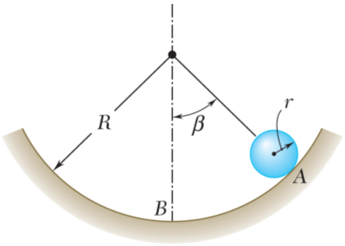
$$\alpha = 0, \qquad \bar{a}_x = 0$$

$$\xrightarrow{+}\ \Sigma F_x = m\bar{a}_x: \quad R_x = m\bar{a}_x \qquad R_x = 0$$

$$+\uparrow\ \Sigma F_y = m\bar{a}_y: \quad R_y - mg = m\left(\frac{4r}{3\pi}\right)\omega^2$$

$$R_y = mg + m\left(\frac{4r}{3\pi}\right)1.3035\frac{g}{r} \qquad R_y = 1.553\ mg$$

$\mathbf{R} = 1.553\ mg \uparrow$ ◀

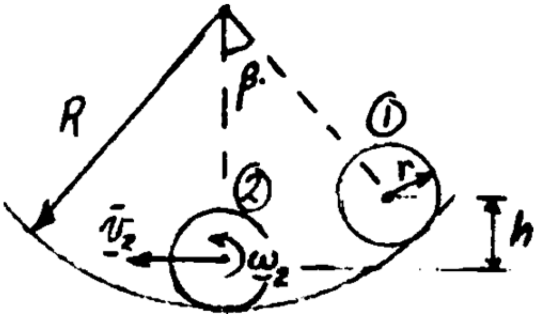


PROBLEM 17.31

A sphere of mass m and radius r rolls without slipping inside a curved surface of radius R. Knowing that the sphere is released from rest in the position shown, derive an expression (a) for the linear velocity of the sphere as it passes through B, (b) for the magnitude of the vertical reaction at that instant.

SOLUTION

Kinematics: The sphere rolls without slipping.



$$\bar{v} = r\omega \quad \omega = \frac{\bar{v}}{r}$$

Kinetic energy.

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$

$$= \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{\bar{v}}{r}\right)^2$$

$$T = \frac{7}{10}m\bar{v}^2$$

$$T_1 = 0 \quad T_2 = \frac{7}{10}m\bar{v}_2^2$$

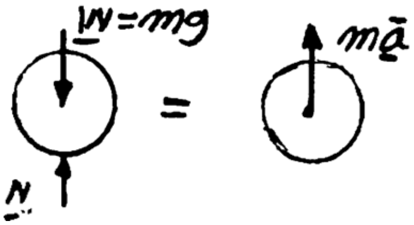
Work. $U_{1-2} = mgh = mg(R-r)(1-\cos\beta)$

Principle of work and energy. $T_1 + U_{1-2} = T_2$:

$$0 + mg(R-r)(1-\cos\beta) = \frac{7}{10}m\bar{v}_2^2$$

(a) *Linear velocity at B.* $\bar{v}_2 = \sqrt{\frac{10}{7}g(R-r)(1-\cos\beta)}$ ◀

Free body diagram when $\beta = 0$.



$\xrightarrow{+} \Sigma F = ma_t$: $\quad a_t = 0$

$+\circlearrowleft \Sigma M_G = \bar{I}\alpha$: $\quad \alpha = 0$

PROBLEM 17.31 (Continued)

The sphere rolls so that its mass center moves on a circle of radius $\rho = R - r$.

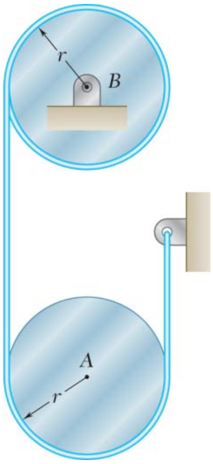
$$\bar{a} = a_n = \frac{\bar{v}_2^2}{R - r}\uparrow$$

$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad N - mg = m\bar{a}$$

$$N - mg = m\left(\frac{1}{R - r}\right)\left[\frac{10}{7} g(R - r)(1 - \cos\beta)\right]$$

$$N = mg\left[1 + \frac{10}{7}(1 - \cos\beta)\right]$$

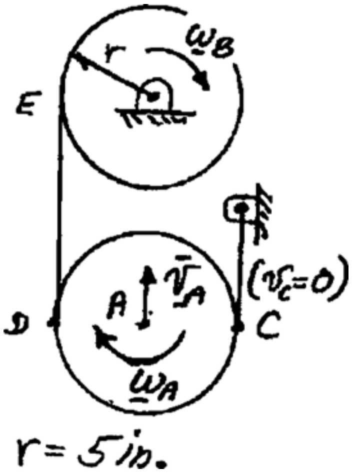
(*b*) *Vertical reaction.* $N = \frac{1}{7} mg[17 - 10\cos\beta]\uparrow$ ◀



PROBLEM 17.32

Two uniform cylinders, each of weight $W = 14$ lb and radius $r = 5$ in., are connected by a belt as shown. Knowing that at the instant shown the angular velocity of cylinder B is 30 rad/s clockwise, determine (a) the distance through which cylinder A will rise before the angular velocity of cylinder B is reduced to 5 rad/s, (b) the tension in the portion of belt connecting the two cylinders.

SOLUTION



Kinematics.

$$v_D = v_E = r\omega_B$$

Point C is the instantaneous center of cylinder A.

$$\omega_A = \frac{v_D}{cd} = \frac{r\omega_B}{2r} = \frac{1}{2}\omega_B$$

$$\bar{v}_A = r\omega_A = \frac{1}{2}r\omega_B$$

$$v_D = 2\bar{v}_A$$

Kinetic energy of the system.

$$T = \frac{1}{2}m\bar{v}_A^2 + \frac{1}{2}\bar{I}\omega_A^2 + \frac{1}{2}\bar{I}\omega_B^2$$

$$T = \frac{1}{2}m\left(\frac{r}{2}\omega_B\right)^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{1}{2}\omega_B\right)^2 + \left(\frac{1}{2}mr^2\right)\omega_B^2$$

$$T = \frac{7}{16}mr^2\omega_B^2 \qquad (1)$$

Position 1: $(\omega_B)_1 = 30$ rad/s ↻

Position 2: $(\omega_B)_2 = 5$ rad/s ↻

Work. For the system considered, the only force which does work is the weight of disk A.

$$U_{1-2} = -Wh = -mgh$$

where h is the rise of cylinder A.

PROBLEM 17.32 (Continued)

Principle of work and energy.

$$T_1 + U_{1-2} = T_2: \quad \frac{7}{16} mr^2 (\omega_B)_1^2 - mgh = \frac{7}{16} mr^2 (\omega_B)_2^2$$

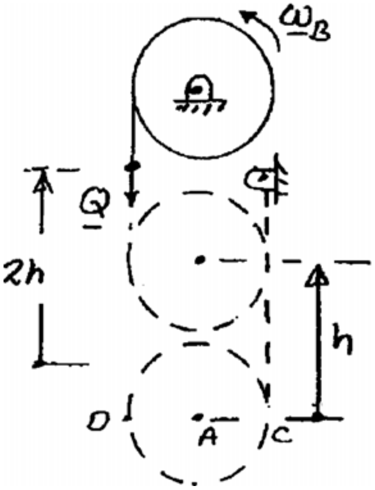
$$h = \frac{7}{16}\frac{r^2}{g}[(\omega_B)_1^2 - (\omega_B)_2^2] \qquad (2)$$

$$h = \frac{7}{16}\left(\frac{5}{12}\text{ ft}\right)^2 \frac{1}{32.2\text{ ft/s}^2}[(30\text{ rad/s})^2 - (5\text{ rad/s})^2] = 2.064\text{ ft}$$

(*a*) *Rise of cylinder A.* $h = 2.06$ ft ◀

(*b*) *Tension in cord DE.* Let Q be its value.

Recall that $v_D = 2v_A$ thus D moves twice the distance that A moves, i.e $2h\uparrow$



$$T_1 = \frac{1}{2}\bar{I}(\omega_B)_1^2$$

$$T_2 = \frac{1}{2}\bar{I}(\omega_B)_2^2$$

$$U_{1-2} = -Q(2h)$$

$$T_1 + U_{1-2} = T_2$$

$$\frac{1}{2}\bar{I}(\omega_B)_1^2 - 2Qh = \frac{1}{2}\bar{I}(\omega_B)_2^2$$

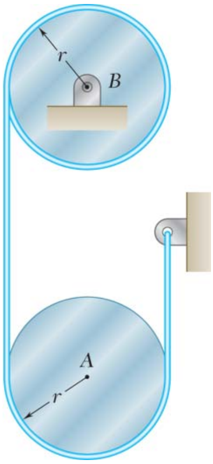
$$Qh = \frac{1}{4}\bar{I}[(\omega_B)_1^2 - (\omega_B)_2^2] \qquad (3)$$

Divide Equation (3) by Equation (2):

$$Q = \frac{1}{4}\bar{I}\frac{16\text{ g}}{7r^2} = \frac{1}{4}\left(\frac{1}{2}mr^2\right)\frac{16\text{ g}}{7r^2} = \frac{2}{7}mg = \frac{2}{7}W \qquad (4)$$

$$Q = \frac{2}{7}(14\text{ lb})$$

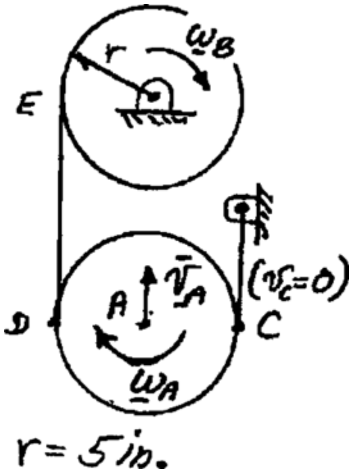
Tension $= Q = 4.00$ lb ◀



PROBLEM 17.33

Two uniform cylinders, each of weight $W = 14$ lb and radius $r = 5$ in., are connected by a belt as shown. If the system is released from rest, determine (*a*) the velocity of the center of cylinder *A* after it has moved through 3 ft, (*b*) the tension in the portion of belt connecting the two cylinders.

SOLUTION



Kinematics.

$$v_D = v_E = r\omega_B$$

Point *C* is the instantaneous center of cylinder *A*.

$$\omega_A = \frac{v_D}{CD} = \frac{r\omega_B}{2r} = \frac{1}{2}\omega_B$$

$$\bar{v}_A = r\omega_A = \frac{1}{2}r\omega_B$$

$$v_D = 2\bar{v}_A$$

Kinetic energy of the system.

$$T = \frac{1}{2}m\bar{v}_A^2 + \frac{1}{2}\bar{I}\omega_A^2 + \frac{1}{2}\bar{I}\omega_B^2$$

$$T = \frac{1}{2}m\left(\frac{r}{2}\omega_B\right)^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{1}{2}\omega_B\right)^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega_B^2$$

$$T = \frac{7}{16}mr^2\omega_B^2 \qquad (1)$$

Position 1: At rest $T_1 = 0$

Position 2: Center of cylinder *C* has moved 3 ft↓.

Work. For the system considered, the only force which does work is the weight of disk *A*.

$$U_{1-2} = Wh = (14 \text{ lb})(3 \text{ ft}) = 42 \text{ ft} \cdot \text{lb}$$

where *h* is the distance that cylinder *A* falls.

PROBLEM 17.33 (Continued)

Principle of work and energy:

$$T_1 + U_{1\to 2} = T_2: \quad 0 + 42\text{ ft}\cdot\text{lb} = \frac{7}{16}\frac{14\text{ lb}}{32.2\text{ ft/s}^2}\left(\frac{5}{12}\text{ft}\right)^2(\omega_B)_2^2$$

$$(\omega_B)_2 = 35.662\text{ rad/s}$$

(*a*) *Velocity of A.* $\bar{v}_A = \frac{1}{2}r\omega_B = \frac{1}{2}\left(\frac{5}{12}\text{ft}\right)(35.66\text{ rad/s})$ $\qquad \bar{\mathbf{v}}_A = 7.43\text{ ft/s}\downarrow$ ◀

(*b*) *Tension in cord DE.* Let Q be its value.

Recall that $v_D = 2v_A$ thus D moves twice the distance that A moves, i.e $2h\downarrow$

$$T_1 = 0$$

$$T_2 = \frac{1}{2}\bar{I}(\omega_B)_2^2$$

$$U_{1-2} = Q(2h)$$

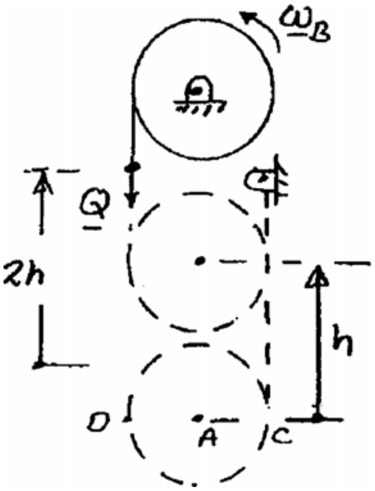
$$T_1 + U_{1-2} = T_2$$

$$0 + Qh = \frac{1}{2}\left(\frac{1}{2}\frac{W}{g}r^2\right)\omega_B^2$$

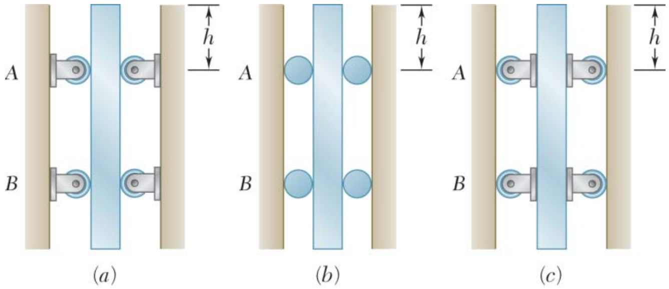
$$Q = \frac{1}{4}\frac{W}{g}r^2\frac{\omega_B^2}{h}$$

$$= \frac{1}{4}\frac{14\text{ lb}}{32.2\text{ ft/s}^2}\left(\frac{5}{12}\text{ft}\right)^2\frac{(35.662\text{ rad/s})^2}{6\text{ ft}}$$

$$= 4.00\text{ lb}$$



$Q = 4.00$ lb. ◀



(a) (b) (c)

PROBLEM 17.34

A bar of mass $m = 5$ kg is held as shown between four disks each of mass $m' = 2$ kg and radius $r = 75$ mm. Knowing that the forces exerted on the disks are sufficient to prevent slipping and that the bar is released from rest, for each of the cases shown determine the velocity of the bar after it has moved through the distance h.

SOLUTION

Let **v** be the velocity of the bar $(\mathbf{v} = v\downarrow)$, $\overline{\mathbf{v}}'$ be the velocity of the mass center G of the upper left disk, $(\overline{\mathbf{v}}' = \overline{v}'\downarrow)$ and $\boldsymbol{\omega}$ be its angular velocity.

For all three arrangements, the magnitudes of mass center velocities are the same for all disks. Likewise, the angular speeds are the same for all disks.

Moment of inertia of one disk. $\quad \overline{I} = \frac{1}{2}m'r^2$

Kinetic energy.

$$T = \frac{1}{2}mv^2 + 4\left[\frac{1}{2}m'(\overline{v}')^2 + \frac{1}{2}\overline{I}\,\omega^2\right]$$

$$T = \frac{1}{2}(5)v^2 + 4\left[\frac{1}{2}(2)(\overline{v}')^2 + \frac{1}{2}\left(\frac{1}{2}\right)(2)r^2\omega^2\right]$$

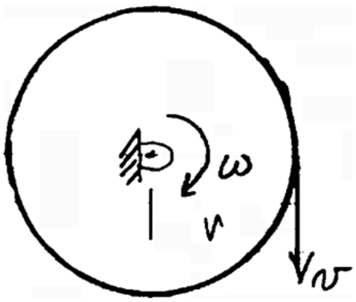
$$= 2.5v^2 + 4(\overline{v}')^2 + 2r^2\omega^2$$

Position 1. Initial at rest position. $\quad T_1 = 0$

Position 2. Bar has moved down a distance h. All the disks move down a distance h'.

Work $\quad U_{1\to 2} = mgh + 4m'gh' = 5gh + 8gh'$

Kinematics and kinetic energy for case (*a*).



The mass center of each disk is not moving.

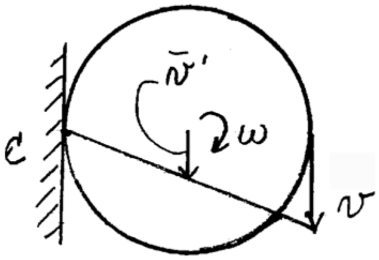
$$\overline{v}' = 0, \qquad h' = 0$$

$$\omega = \frac{v}{r} \qquad r\omega = v$$

$$T_{2a} = 2.5v^2 + 0 + 2v^2 = 4.5v^2$$

PROBLEM 17.34 (Continued)

Kinematics and kinetic energy for case (*b*).



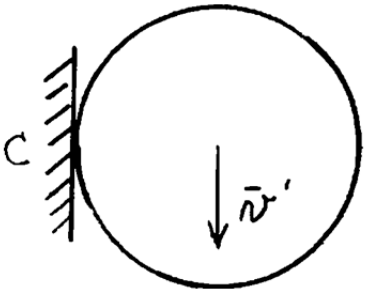
The instantaneous center C of a typical disk lies at its point of contact with the fixed wall.

$$\omega = \frac{v}{2r}$$

$$\vec{v}' = r\omega = \frac{1}{2}v, \qquad h' = \frac{1}{2}h$$

$$T_{2b} = 2.5v^2 + (4)\left(\frac{1}{2}v\right)^2 + 2\left(\frac{1}{2}v\right)^2 = 4.0v^2$$

Kinematics and kinetic energy for case (*c*).



The mass center of each disk moves with the bar.

$$\vec{v}' = v, \qquad h'' = h$$

The instantaneous center C of a typical disk lies at its point of contact with the fixed wall.

$$\vec{v}' = r\omega = v,$$

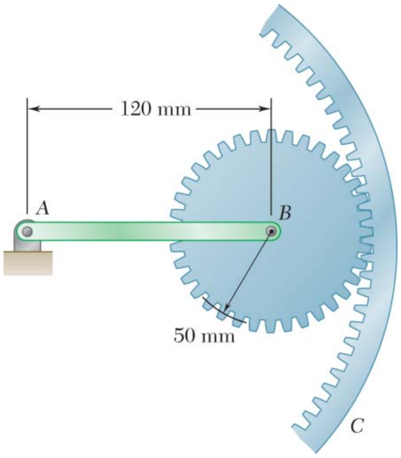
$$T_{2c} = 2.5v^2 + (4)v^2 + 2v^2 = 8.5v^2$$

Principle of Work and Energy. $T_1 + U_{1\to 2} = T_2$

(*a*) $0 + 5gh + 0 = 4.5v^2$ $\quad v = 1.054\sqrt{gh}$ ◀

(*b*) $0 + 5gh + 4gh = 4.0v^2$ $\quad v = 1.500\sqrt{gh}$ ◀

(*c*) $0 + 5gh + 8gh = 8.5v^2$ $\quad v = 1.237\sqrt{gh}$ ◀



PROBLEM 17.35

The 1.5-kg uniform slender bar *AB* is connected to the 3-kg gear *B* which meshes with the stationary outer gear *C*. The centroidal radius of gyration of gear *B* is 30 m. Knowing that the system is released from rest in the position shown, determine (*a*) the angular velocity of the bar as it passes through the vertical position, (*b*) the corresponding angular velocity of gear *B*.

SOLUTION

Kinematics:
$$v_B = l_{AB}\omega_{AB} = r_B\omega_B \qquad \omega_B = \frac{l_{AB}}{r_B}\omega_{AB}$$

$$\bar{v}_{AB} = \frac{1}{2}v_B = \frac{1}{2}l_{AB}\omega_{AB}$$

Moments of inertia.
$$\bar{I}_{AB} = \frac{1}{12}m_{AB}l_{AB}^2, \quad \bar{I}_B = m_Bk^2$$

Kinetic energy.
$$T = \frac{1}{2}m_{AB}\bar{v}_{AB}^2 + \frac{1}{2}\bar{I}_{AB}\omega_{AB}^2 + \frac{1}{2}m_Bv_B^2 + \frac{1}{2}\bar{I}_B\omega_B^2:$$

$$T = \frac{1}{2}\left(\frac{1}{4}m_{AB} + \frac{1}{12}m_{AB} + m_B + m_B\frac{k^2}{r_B^2}\right)l_{AB}^2\omega_{AB}^2$$

$$= \frac{1}{2}\left[\frac{1}{4}(1.5) + \frac{1}{12}(1.5) + 3 + 3\frac{(0.030)^2}{(0.050)^2}\right](0.12)^2\,\omega_{AB}^2 = 0.032976\omega_{AB}^2$$

Position 1. As shown in the drawing above. $V_1 = 0$

Position 2. Point *B* is directly below *A*.

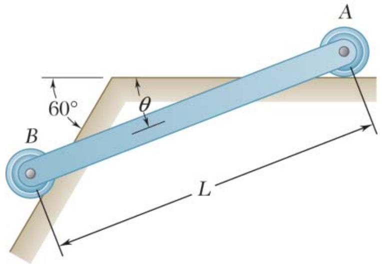
$$V_2 = -W_{AB}\frac{l_{AB}}{2} - W_Bl_{AB} = -(1.5)(9.81)(0.060) - (3)(9.81)(0.120) = -4.4145 \text{ J}$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2:$

$$0 + 0 = 0.032976(\omega_{AB})_2^2 - 4.4145 \qquad (\omega_{AB})_2^2 = 133.87$$

(*a*) Angular velocity of the bar. $\boldsymbol{\omega}_{AB} = 11.57$ rad/s ↻ ◀

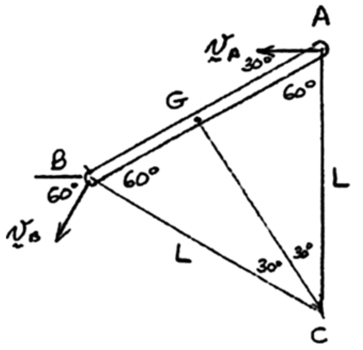
(*b*) $$\omega_B = \frac{0.120}{0.050}(11.57)$$ $\boldsymbol{\omega}_B = 27.8$ rad/s ↺ ◀



PROBLEM 17.36

The motion of the uniform rod *AB* is guided by small wheels of negligible mass that roll on the surface shown. If the rod is released from rest when $\theta = 0$, determine the velocities of *A* and *B* when $\theta = 30°$.

SOLUTION



Position 1.

$$\theta = 0$$

$$v_A = v_B = 0$$

$$\omega = 0$$

$$T_1 = 0$$

$$V_1 = 0$$

Position 2. $\theta = 30°$

Kinematics. Locate the instantaneous center *C*. Triangle *ABC* is equilateral.

$$v_A = v_B = L\omega$$

$$v_G = L\omega \cos 30°$$

Moment of inertia. $\bar{I} = \frac{1}{12}ml^2$

Kinetic energy. $T_2 = \frac{1}{2}mv_G^2 + \frac{1}{2}\bar{I}\omega^2$: $T_2 = \frac{1}{2}m(L\omega\cos 30°)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega^2$

$$= \frac{5}{12}ml^2\omega^2$$

Potential energy. $V_2 = -mg\frac{L}{2}\sin 30° = -\frac{1}{4}mgL$

Conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad 0 + 0 = \frac{5}{12}mL^2\omega^2 - \frac{1}{4}mgL$$

$$\omega^2 = 0.6\frac{g}{L}$$

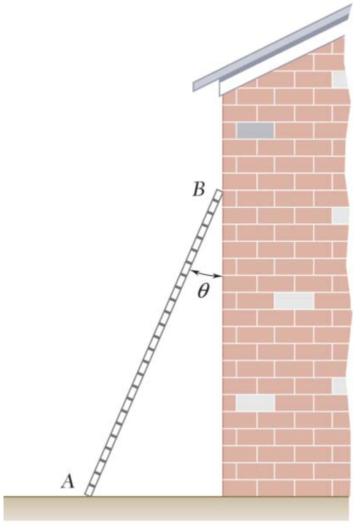
$$\omega = 0.775\sqrt{\frac{g}{L}}$$

$$v_A = 0.775\sqrt{gL}$$

$$v_B = 0.775\sqrt{gL}$$

$\mathbf{v}_A = 0.775\sqrt{gL}$ ← ◀

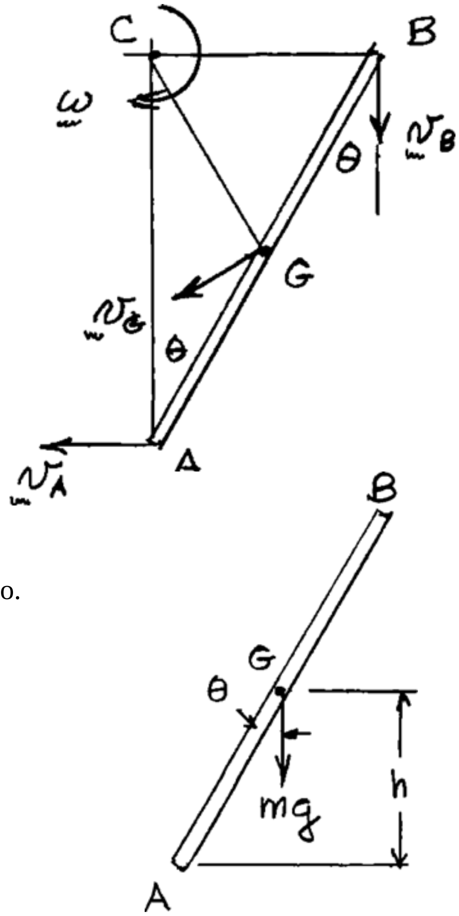
$\mathbf{v}_B = 0.775\sqrt{gL}$ ↙ 60° ◀



PROBLEM 17.37

A 5-m long ladder has a mass of 15 kg and is placed against a house at an angle $\theta = 20°$. Knowing that the ladder is released from rest, determine the angular velocity of the ladder and the velocity of A when $\theta = 45°$. Assume the ladder can slide freely on the horizontal ground and on the vertical wall.

SOLUTION



Kinematics:

Let $\mathbf{v}_A = v_A \longleftarrow$, $\mathbf{v}_B = v_B \downarrow$, and $\boldsymbol{\omega} = \omega \circlearrowright$. Locate the instantaneous center C by drawing AC perpendicular to $\mathbf{v}_A$ and BC perpendicular to $\mathbf{v}_B$. Triangle GCB is isosceles.

$GA = GB = GC = L/2$. The velocity of the mass center G is

$$\bar{v} = v_G = L\omega/2$$

Kinetic energy:

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$

$$= \frac{1}{2}\left(\bar{I} + \frac{1}{4}mL^2\right)\omega^2$$

Since the ladder can slide freely, the friction forces at A and B are zero.

Use the principle of conservation of energy.

$$T_1 + V_1 = T_2 + V_2:$$

Potential energy: Use the ground as the datum.

$$V = mgh$$

where

$$h = \frac{L}{2}\cos\theta$$

PROBLEM 17.37 (Continued)

Position 1. $\theta = 20°$; rest $(T_1 = 0)$

Position 2. $\theta = 45°$; $\omega = ?$

$$0 + mg\frac{L}{2}\cos 20° = \frac{1}{2}\left(\bar{I} + \frac{1}{4}mL^2\right)\omega^2 + mg\frac{L}{2}\cos 45°$$

Data: $m = 15\ \text{kg}.\quad L = 5\ \text{m}\quad g = 9.81\ \text{m/s}^2$

Assume $\bar{I} = \frac{1}{12}mL^2 = \frac{1}{12}(15\ \text{kg})(5\ \text{m})^2 = 31.25\ \text{kg}\cdot\text{m}^2$

$$\bar{I} + \frac{1}{4}mL^2 = 31.25\ \text{kg}\cdot\text{m}^2 + \frac{1}{4}(15\ \text{kg})(5\ \text{m})^2 = 125\ \text{kg}\cdot\text{m}^2$$

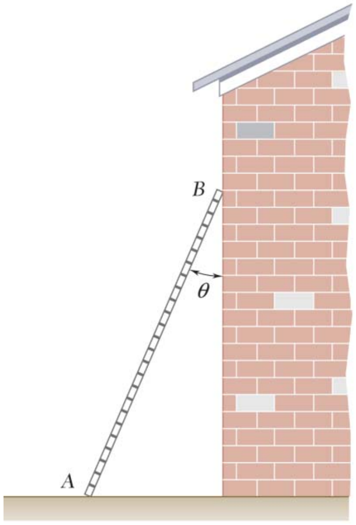
$$(15\ \text{kg})(9.81\ \text{m/s}^2)(2.5\ \text{m})(\cos 20° - \cos 45°) = \frac{1}{2}(125\ \text{kg}\cdot\text{m}^2)\omega^2$$

$$\omega^2 = 1.3690\ \text{rad}^2/\text{s}^2 \qquad \omega = 1.17004\ \text{rad/s}$$

Angular velocity. $\boldsymbol{\omega} = 1.170$ rad/s ↻ ◀

Velocity of end A.

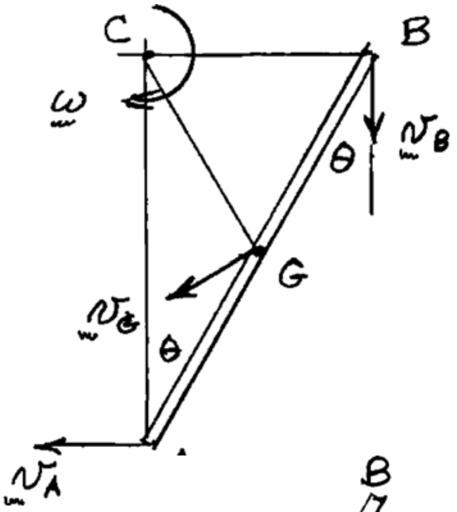
$v_A = \omega L\cos\theta = (1.17004\ \text{rad/s})(5\ \text{m})\cos 30°$ $\mathbf{v}_A = 5.07$ m/s ← ◀



PROBLEM 17.38

A long ladder of length l, mass m, and centroidal mass moment of inertia $\bar{I}$ is placed against a house at an angle $\theta = \theta_0$. Knowing that the ladder is released from rest, determine the angular velocity of the ladder when $\theta = \theta_2$. Assume the ladder can slide freely on the horizontal ground and on the vertical wall.

SOLUTION



Kinematics:

Let $\mathbf{v}_A = v_A \longleftarrow$, $\mathbf{v}_B = v_B \downarrow$, and $\boldsymbol{\omega} = \omega \circlearrowleft$. Locate the instantaneous center C by drawing AC perpendicular to $\mathbf{v}_A$ and BC perpendicular to $\mathbf{v}_B$. Triangle GCB is isosceles.

$GA = GB = GC = L/2.$ The velocity of the mass center G is

$$\bar{v} = v_G = L\omega/2$$

Kinetic energy:

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$

$$= \frac{1}{2}\left(\bar{I} + \frac{1}{4}mL^2\right)\omega^2$$

Since the ladder can slide freely, the friction forces at A and B are zero.

Use the principle of conservation of energy.

$$T_1 + V_1 = T_2 + V_2:$$

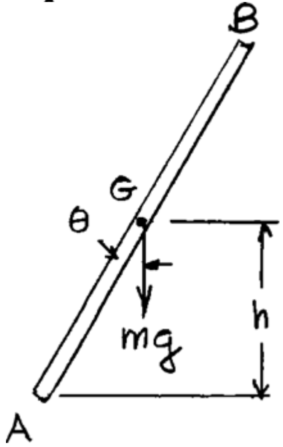
Potential energy: Use the ground as the datum.

$$V = mgh$$

where

$$h = \frac{L}{2}\cos\theta$$

Position 1. $\theta = \theta_0$; rest $(T_1 = 0)$



PROBLEM 17.38 (Continued)

Position 2. $\theta = \theta_2; \quad \omega = ?$

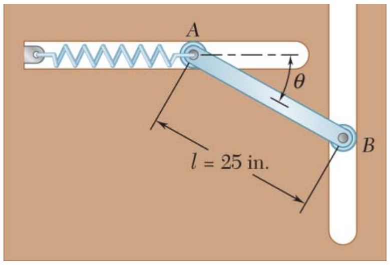
$$0 + mg\frac{L}{2}\cos\theta = \frac{1}{2}\left(\bar{I} + \frac{1}{4}mL^2\right)\omega^2 + mg\frac{L}{2}\cos\theta_2$$

Assume $\bar{I} = \frac{1}{12}mL^2$

$$\bar{I} + \frac{1}{4}mL^2 = \frac{1}{3}mL^2$$

$$\omega^2 = \frac{3g}{L}(\cos\theta_0 - \cos\theta_2)$$

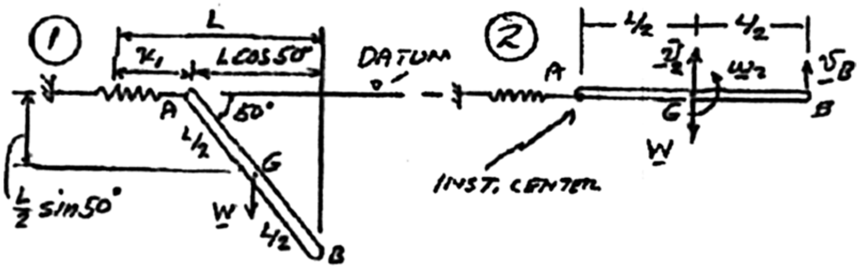
Angular velocity. $\boldsymbol{\omega} = \sqrt{3g(\cos\theta_0 - \cos\theta_2)/L} \circlearrowright$ ◀



PROBLEM 17.39

The ends of a 9-lb rod AB are constrained to move along slots cut in a vertical plate as shown. A spring of constant $k = 3$ lb/in. is attached to end A in such a way that its tension is zero when $\theta = 0$. If the rod is released from rest when $\theta = 50°$, determine the angular velocity of the rod and the velocity of end B when $\theta = 0$.

SOLUTION



$$\bar{v}_2 = \frac{L}{2}\omega_2$$

$$v_B = L\omega_2$$

$$\begin{aligned} x_1 &= L - L\cos 50° \\ &= (25\text{ in.})(1-\cos 50°) \\ &= 8.9303\text{ in.} \end{aligned}$$

Position 1.

$$V_1 = -W\frac{L}{2}\sin 50° + \frac{1}{2}kx_1^2$$

$$\begin{aligned} V_1 &= -(9\text{ lb})\left(\frac{25\text{ in.}}{2}\right)\sin 50° + \frac{1}{2}(3\text{ lb/in.})(8.9303\text{ in.})^2 \\ &= -86.18 + 119.63 \\ &= 33.45\text{ in.}\cdot\text{lb} \\ &= 2.787\text{ ft}\cdot\text{lb.} \end{aligned}$$

$$T_1 = 0$$

Position 2.

$$V_2 = (V_g)_2 + (V_e)_2 = 0$$

$$\begin{aligned} T_2 &= \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2 \\ &= \frac{1}{2}m\left(\frac{L}{2}\omega_2\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega_2^2 \\ &= \frac{1}{6}mL^2\omega_2^2 = \frac{1}{6}\left(\frac{9\text{ lb}}{32.2}\right)\left(\frac{25\text{ in.}}{12}\right)^2\omega_2^2 = 0.2022\omega_2^2 \end{aligned}$$

PROBLEM 17.39 (Continued)

Conservation of energy:

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 2.787\text{ ft}\cdot\text{lb} = 0.2022\omega_2^2$$

$$\omega_2^2 = 13.7849$$

$$\omega_2 = 3.713\text{ rad/s}$$

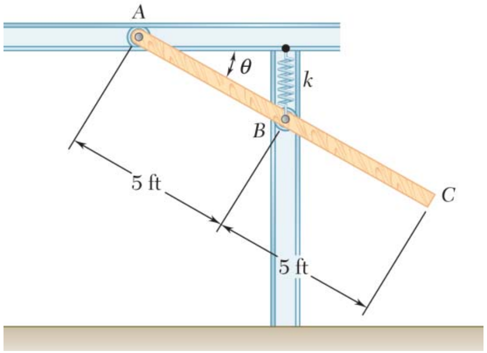
$\boldsymbol{\omega}_2 = 3.71\text{ rad/s}$ ↺ ◀

Velocity of *B*:

$$v_B = L\omega_2 = \left(\frac{25\text{ in.}}{12}\right)(3.713\text{ rad/s})$$

$$= 7.735\text{ ft/s}$$

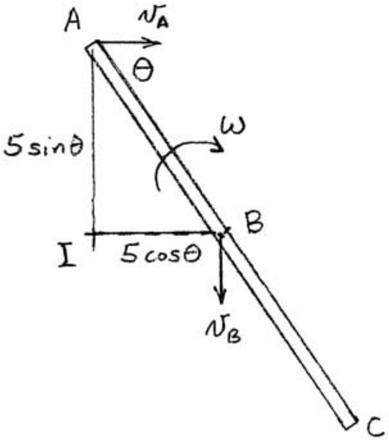
$\mathbf{v}_B = 7.74\text{ ft/s}$ ↑ ◀



PROBLEM 17.40

The mechanism shown is one of two identical mechanisms attached to the two sides of a 200-lb uniform rectangular door. Edge *ABC* of the door is guided by wheels of negligible mass that roll in horizontal and vertical tracks. A spring of constant $k = 40$ lb/ft is attached to wheel *B*. Knowing that the door is released from rest in the position $\theta = 30°$ with the spring unstretched, determine the velocity of wheel *A* just as the door reaches the vertical position.

SOLUTION



Kinematics. Locate the instantaneous center at point *I*.

$$v_A = (5\sin\theta)\omega \qquad \bar{v} = v_B = (5\cos\theta)\omega$$

Moment of inertia. $\bar{I} = \frac{1}{12}\frac{W}{g}l^2:$

$$\bar{I} = \frac{1}{12}\frac{200}{32.2}(10)^2 = 51.760 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Kinetic energy. $T = \frac{1}{2}\frac{W}{g}\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$

$$T = \frac{1}{2}\frac{200}{32.2}(5\cos\theta)^2\omega^2 + \frac{1}{2}(51.760)\omega^2$$

$$= (77.640\cos^2\theta + 25.880)\omega^2$$

Potential energy.

Gravity. Datum at level *A*. $V_g = Wh = (200)(-5\sin\theta)$

$$= -1000\sin\theta \text{ ft}\cdot\text{lb}$$

Two springs. $V_e = 2\left(\frac{1}{2}ke^2\right) = k(5\sin\theta - 5\sin 30°)^2$

$$= 25k(\sin\theta - 0.5)^2$$

Position 1. $\theta = 30° \qquad \omega = \omega_1 = 0 \qquad T_1 = 0$

$$V_1 = (V_e)_1 + (V_g)_1 = 0 - 1000\sin 30° = -500 \text{ ft}\cdot\text{lb}$$

Position 2. $\theta = 90° \qquad \omega = \omega_2$

$$T_2 = (77.640\cos^2 90° + 25.880)\omega^2 = 25.880\omega^2$$

PROBLEM 17.40 (Continued)

$$V_2 = (V_e)_2 + (V_g)_2 = (25)(40)(\sin 90° - 0.5)^2 - 500 \sin 90°$$

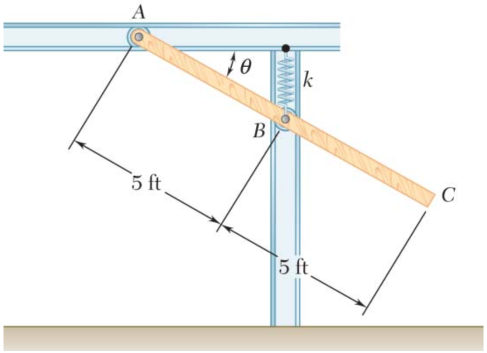
$$= -750 \text{ ft} \cdot \text{lb}$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2:$

$$0 - 500 = 25.880\omega^2 - 750 \qquad \omega = 3.1080 \text{ rad/s}$$

Kinematics. $v_A = (5 \sin 90°)(3.1080)$

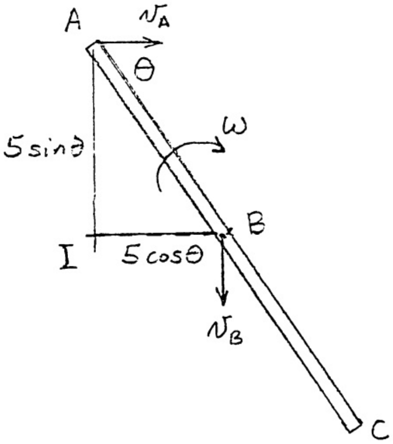
$$\mathbf{v}_A = 15.54 \text{ ft/s} \longrightarrow \blacktriangleleft$$



PROBLEM 17.41

The mechanism shown is one of two identical mechanisms attached to the two sides of a 200-lb uniform rectangular door. Edge *ABC* of the door is guided by wheels of negligible mass that roll in horizontal and vertical tracks. A spring of constant *k* is attached to wheel *B* in such a way that its tension is zero when $\theta = 30°$. Knowing that the door is released from rest in the position $\theta = 45°$ and reaches the vertical position with an angular velocity of 0.6 rad/s, determine the spring constant *k*.

SOLUTION



Kinematics. Locate the instantaneous center at point *I*.

$$v_A = (5\sin\theta)\omega \qquad \bar{v} = v_B = (5\cos\theta)\omega$$

Moment of inertia. $\bar{I} = \frac{1}{12}\frac{W}{g}l^2$

$$\bar{I} = \frac{1}{12}\frac{200}{32.2}(10)^2 = 51.760 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Kinetic energy. $T = \frac{1}{2}\frac{W}{g}\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$

$$T = \frac{1}{2}\frac{200}{32.2}(5\cos\theta)^2\omega^2 + \frac{1}{2}(51.760)\omega^2$$

$$= (77.640\cos^2\theta + 25.880)\omega^2$$

Potential energy.

Gravity. Datum at level *A*.

$$V_g = Wh = (200)(-5\sin\theta) = -1000\sin\theta \text{ ft}\cdot\text{lb}$$

Two springs. $V_e = 2\left(\frac{1}{2}ke^2\right) = k(5\sin\theta - 5\sin 30°)^2$

$$= 25k(\sin\theta - 0.5)^2$$

Position 1. $\theta = 45° \qquad \omega = \omega_1 = 0 \qquad T_1 = 0$

$$V_1 = (V_e)_1 + (V_g)_1 = 25k(\sin 45° - 0.5)^2 - 1000\sin 45°$$

$$= 1.07233k - 707.11$$

PROBLEM 17.41 (Continued)

Position 2. $\theta = 90°$ $\omega = \omega_2 = 0.6 \text{ rad/s}$

$$T_2 = \left(77.640\cos^2 90° + 25.880\right)(0.6)^2 = 9.3168 \text{ ft}\cdot\text{lb}$$

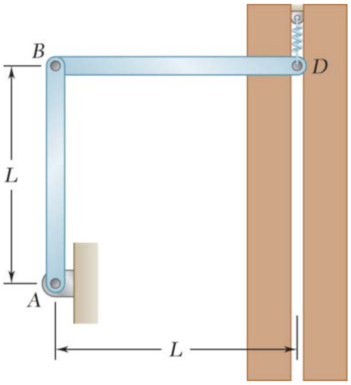
$$V_2 = (V_e)_2 + \left(V_g\right)_2 = 25k\left(\sin 90° - 0.5\right)^2 - 1000\sin 90°$$

$$= 6.25k - 1000$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$:

$$0 + 1.07233k - 707.11 = 9.3168 + 6.25k - 1000$$

$k = 54.8 \text{ lb/ft}$ ◀



PROBLEM 17.42

Each of the two rods shown is of length $L = 1$ m and has a mass of 5 kg. Point D is connected to a spring of constant $k = 20$ N/m and is constrained to move along a vertical slot. Knowing that the system is released from rest when rod BD is horizontal and the spring connected to Point D is initially unstretched, determine the velocity of Point D when it is directly to the right of Point A.

SOLUTION

Moments of inertia. $$\bar{I} = \frac{1}{12}mL^2, \quad I_A = \frac{1}{3}mL^2$$

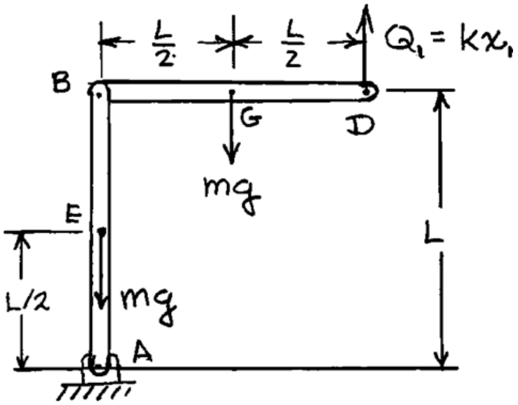
Use the principle of conservation of energy applied to the system consisting of both rods. Use the level at A as the datum for the potential energy of each rod.

Position 1. (no motion)

$$T_1 = 0$$

$$V_1 = mg\left(\frac{1}{2}L\right) + mgL + \frac{1}{2}kx_1^2$$

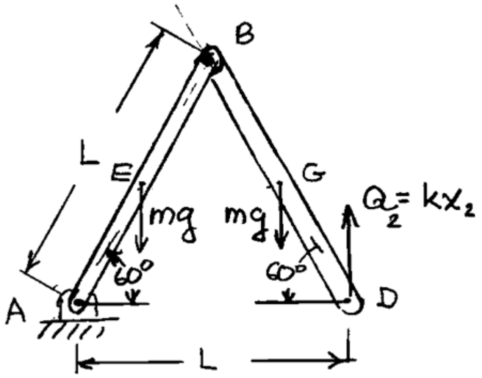
$$= \frac{3}{2}mgL + \frac{1}{2}kx_1^2$$



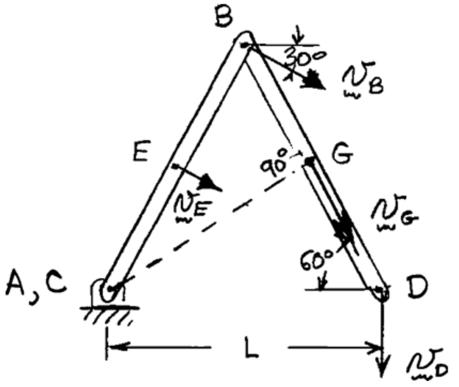
Position 2.

$$V_2 = mg\frac{L}{2}\sin 60° + mg\frac{L}{2}\sin 60°$$

$$= \frac{\sqrt{3}}{2}mgL + \frac{1}{2}kx_2^2$$



PROBLEM 17.42 (Continued)



Kinematics. $\boldsymbol{\omega}_{AB} = \omega_{AB} \circlearrowright$

$$v_B = L\omega_{AB} \qquad \mathbf{v}_B = L\omega_{AB} \searrow 30°$$

$$\mathbf{v}_D = v_D \downarrow$$

Locate the instantaneous center C of rod BD by drawing BC perpendicular to $\mathbf{v}_B$ and DC perpendicular to $\mathbf{v}_D$. Point C coincides with Point A in position 2.

Let $\boldsymbol{\omega}_{BD} = \omega_{BD} \circlearrowright$

$$\omega_{BD} = \frac{v_B}{L} = \omega_{AB}$$

$$v_E = \frac{L}{2}\omega_{AB}$$

$$v_G = (L\sin 60°)\omega_{BD} = \frac{\sqrt{3}}{2}L\omega_{AB}$$

$$v_D = L\omega_{BD} = L\omega_{AB} \tag{1}$$

$$T_2 = \frac{1}{2}I_A\omega_{AB}^2 + \frac{1}{2}\bar{I}\,\omega_{BD}^2 + \frac{1}{2}mv_G^2$$

$$= \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega_{AB}^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega_{AB}^2 + \frac{1}{2}m\left(\frac{\sqrt{3}}{2}\omega_{AB}\right)^2$$

$$= \left(\frac{1}{6} + \frac{1}{24} + \frac{3}{8}\right)mL^2\omega_{AB}^2 = \frac{7}{12}mL^2\omega_{AB}^2$$

Principle of conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad 0 + \frac{3}{2}mgL + \frac{1}{2}kx_1^2 = \frac{7}{12}mL^2\omega_{AB}^2 + \frac{\sqrt{3}}{2}mgL + \frac{1}{2}kx_2^2$$

$$\frac{7}{12}mL^2\omega_{AB}^2 = \left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)mgL - \frac{1}{2}k(x_2^2 - x_1^2) \tag{2}$$

Data: $m = 5\text{ kg}, \quad L = 1\text{ m}, \quad g = 9.81\text{ m/s}^2$

$k = 20\text{ N}\cdot\text{m}, \quad x_1 = 0, \quad x_2 = L = 1\text{ m}$

$$\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)mgL = (0.63397)(5\text{ kg})(9.81\text{ m/s}^2)(1\text{ m}) = 31.096\text{ J}$$

$$-\frac{1}{2}k(x_2^2 - x_1^2) = \frac{1}{2}(20\text{ N/m})(1\text{ m})^2 = -10\text{ J}$$

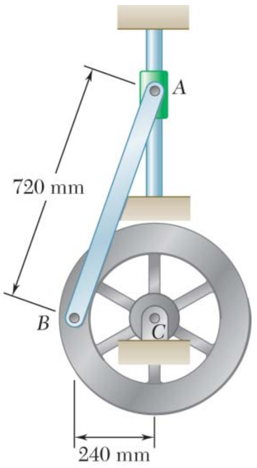
PROBLEM 17.42 (Continued)

By Eq. (2), $$\frac{7}{12}mL^2\omega_{AB}^2 = \left(\frac{35}{12}\,\text{kg}\cdot\text{m}^2\right)\omega_{AB}^2 = 21.096\ \text{J}$$

$$\omega_{AB}^2 = 7.2329\ \text{rad}^2/\text{s}^2 \qquad \omega_{AB} = 2.6894\ \text{rad/s}$$

By Eq. (1), $$v_D = (1\ \text{m})(2.6894\ \text{rad/s})$$

$$\mathbf{v}_D = 2.69\ \text{m/s}\downarrow \blacktriangleleft$$

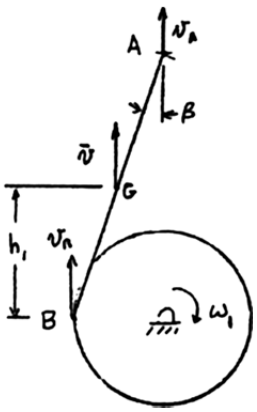


PROBLEM 17.43

The 4-kg rod *AB* is attached to a collar of negligible mass at *A* and to a flywheel at *B*. The flywheel has a mass of 16 kg and a radius of gyration of 180 mm. Knowing that in the position shown the angular velocity of the flywheel is 60 rpm clockwise, determine the velocity of the flywheel when Point *B* is directly below *C*.

SOLUTION

Moments of inertia.



Rod *AB*:

$$\bar{I}_{AB} = \frac{1}{12} m_{AB} L_{AB}^2$$

$$= \frac{1}{12}(4\text{ kg})(0.72\text{ m})^2$$

$$= 0.1728\text{ kg}\cdot\text{m}^2$$

Flywheel:

$$I_C = m\bar{k}^2$$

$$= (16\text{ kg})(0.18\text{ m})^2$$

$$= 0.5184\ \text{ kg}\cdot\text{m}^2$$

Position 1. As shown.

$$\omega = \omega_1 \circlearrowright$$

$$\sin\beta = \frac{0.24}{0.72} \qquad \beta = 19.471°$$

$$h_1 = \frac{1}{2}(0.72)\cos\beta = 0.33941\text{ m}$$

$$V_1 = W_{AB} h_1$$

$$= (4)(9.81)(0.33941)$$

$$= 13.3185\text{ J}$$

Kinematics.

$$v_B = r\omega_1 = 0.24\omega_1$$

Bar *AB* is in translation.

$$\omega_{AB} = 0, \quad \bar{v} = v_B$$

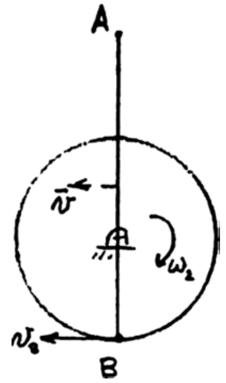
$$T_1 = \frac{1}{2} m_{AB}\bar{v}^2 + \frac{1}{2}\bar{I}_{AB}\omega_{AB}^2 + \frac{1}{2} I_C \omega_1^2$$

$$= \frac{1}{2}(4)(0.24\omega_1)^2 + 0 + \frac{1}{2}(0.5184)\omega_1^2$$

$$= 0.3744\omega_1^2$$

PROBLEM 17.43 (Continued)

Position 2. Point *B* is directly below *C*.



$$h_2 = \frac{1}{2}L_{AB} - r$$
$$= \frac{1}{2}(0.72) - 0.24$$
$$= 0.12 \text{ m}$$
$$V_2 = W_{AB}h_2$$
$$= (4)(9.81)(0.12)$$
$$= 4.7088 \text{ J}$$

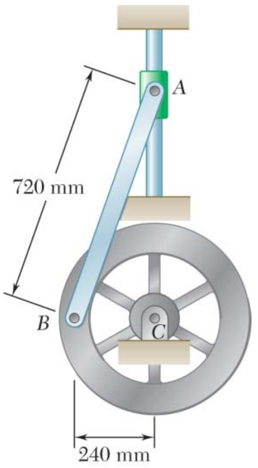
Kinematics. $v_B = r\omega_2 = 0.24\omega_2$

$$\omega_{AB} = \frac{v_B}{0.72} = 0.33333\omega_2$$
$$\bar{v} = \frac{1}{2}v_B = 0.12\omega_2$$
$$T_2 = \frac{1}{2}m_{AB}\bar{v}^2 + \frac{1}{2}\bar{I}_{AB}\omega_{AB}^2 + \frac{1}{2}I_C\omega_2^2$$
$$= \frac{1}{2}(4)(0.12\omega_2)^2 + \frac{1}{2}(0.1728)(0.33333\omega_2)^2 + \frac{1}{2}(0.5184)\omega_2^2$$
$$= 0.2976\omega_2^2$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$: $0.3744\omega_1^2 + 13.3185 = 0.2976\omega_2^2 + 4.7088$ (1)

Angular speed data: $\omega_1 = 60$ rpm $= 2\pi$ rad/s

Solving Equation (1) for ω_2, $\omega_2 = 8.8655$ rad/s $\omega_2 = 84.7$ rpm ↻ ◀



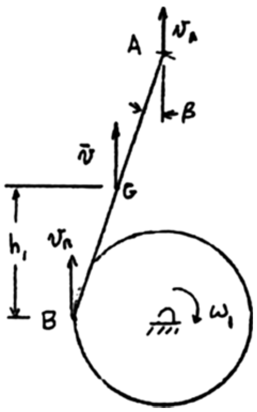
PROBLEM 17.44

If in Problem 17.43 the angular velocity of the flywheel is to be the same in the position shown and when Point *B* is directly above *C*, determine the required value of its angular velocity in the position shown.

PROBLEM 17.43 The 4-kg rod *AB* is attached to a collar of negligible mass at *A* and to a flywheel at *B*. The flywheel has a mass of 16 kg and a radius of gyration of 180 mm. Knowing that in the position shown the angular velocity of the flywheel is 60 rpm clockwise, determine the velocity of the flywheel when Point *B* is directly below *C*.

SOLUTION

Moments of inertia.



Rod *AB*:

$$\bar{I}_{AB} = \frac{1}{12} m_{AB} L_{AB}^2 = \frac{1}{12}(4\text{ kg})(0.72\text{ m})^2 = 0.1728\text{ kg}\cdot\text{m}^2$$

Flywheel:

$$I_C = m\bar{k}^2 = (16\text{ kg})(0.18\text{ m})^2 = 0.5184\text{ kg}\cdot\text{m}^2$$

Position 1. As shown.

$$\omega = \omega_1 \circlearrowright$$

$$\sin\beta = \frac{0.24}{0.72} \qquad \beta = 19.471°$$

$$h_1 = \frac{1}{2}(0.72)\cos\beta = 0.33941\text{ m}$$

$$V_1 = W_{AB}h_1 = (4)(9.81)(0.33941) = 13.3185\text{ J}$$

Kinematics.

$$v_B = r\omega_1 = 0.24\omega_1$$

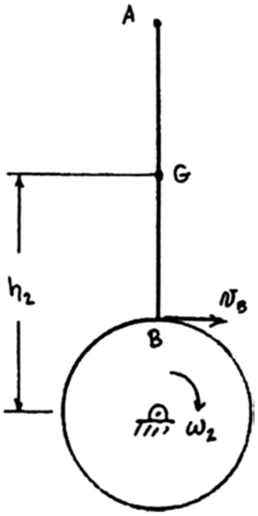
Bar *AB* is in translation.

$$\omega_{AB} = 0, \quad \bar{v} = v_B$$

$$T_1 = \frac{1}{2}m_{AB}\bar{v}^2 + \frac{1}{2}\bar{I}_{AB}\omega_{AB}^2 + \frac{1}{2}I_C\omega_1^2$$
$$= \frac{1}{2}(4)(0.24\omega_1)^2 + 0 + \frac{1}{2}(0.5184)\omega_1^2$$
$$= 0.3744\omega_1^2$$

PROBLEM 17.44 (Continued)

Position 2. Point B is directly above C.



$$\begin{aligned} h_2 &= \frac{1}{2}L_{AB} + r \\ &= \frac{1}{2}(0.72) + 0.24 \\ &= 0.6 \text{ m} \\ V_2 &= W_{AB}h_2 \\ &= (4)(9.81)(0.6) \\ &= 23.544 \text{ J} \end{aligned}$$

Kinematics. $v_B = r\omega_2 = 0.24\omega_2$

$$\omega_{AB} = \frac{v_B}{0.72} = 0.33333\omega_2$$

$$\bar{v} = \frac{1}{2}v_B = 0.12\omega_2$$

$$\begin{aligned} T_2 &= \frac{1}{2}m_{AB}\bar{v}^2 + \frac{1}{2}\bar{I}_{AB}\omega_{AB}^2 + \frac{1}{2}I_C\omega_2^2 \\ &= \frac{1}{2}(4)(0.12\omega_2)^2 + \frac{1}{2}(0.1728)(0.33333\omega_2)^2 + \frac{1}{2}(0.5184)\omega_2^2 \\ &= 0.2976\omega_2^2 \end{aligned}$$

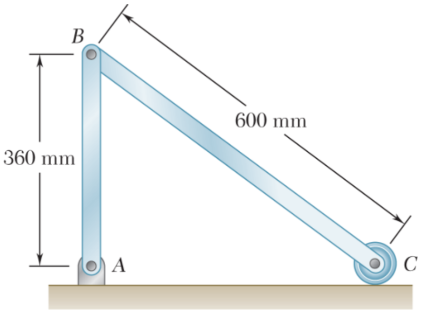
Conservation of energy. $T_1 + V_1 = T_2 + V_2$: $0.3744\omega_1^2 + 13.3135 = 0.2976\omega_2^2 + 23.544$

Angular speed data: $\omega_2 = \omega_1$

Then, $0.0760\omega_1^2 = +0.4105$

$\omega_1 = 11.602$ rad/s

$\omega_1 = 110.8$ rpm ↻ ◀



PROBLEM 17.45

The uniform rods *AB* and *BC* of masses 2.4 kg and 4 kg, respectively, and the small wheel at *C* is of negligible mass. If the wheel is moved slightly to the right and then released, determine the velocity of pin *B* after rod *AB* has rotated through 90°.

SOLUTION

Moments of inertia.

Rod *AB*: $$I_A = \frac{1}{3}m_{AB}L_{AB}^2 = \frac{1}{3}(2.4)(0.360)^2 = 0.10368\ \text{kg}\cdot\text{m}^2$$

Rod *BC*: $$\bar{I} = \frac{1}{12}m_{BC}L_{BC}^2 = \frac{1}{12}(4)(0.600)^2 = 0.1200\ \text{kg}\cdot\text{m}^2$$

Position 1. As shown with bar *AB* vertical. Point *G* is the midpoint of *BC*.

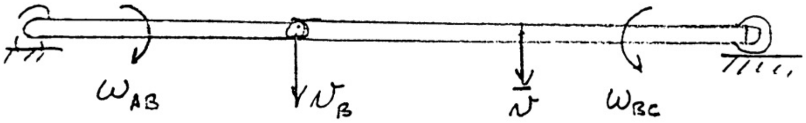
$$V_1 = m_{AB}gh_{AB} + m_{BC}gh_{BC} = (2.4)(9.81)(0.180) + (4)(9.81)(0.180) = 11.3011\ \text{J}$$

Rod *BC* is at rest. $$\omega_{BC} = 0$$

$$\bar{v} = v_G = v_B = v_C = 0 \qquad \omega_{AB} = \frac{v_B}{L_{AB}} = 0$$

$$T_1 = 0$$

Position 2. Rod *AB* is horizontal.



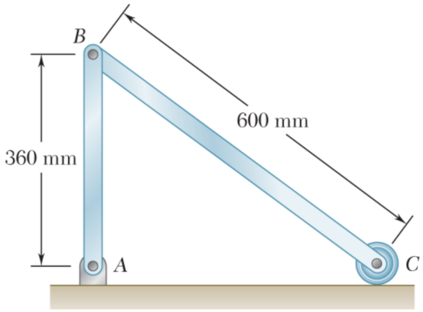
$$V_2 = 0$$

Kinematics. $$\omega_{AB} = \frac{v_B}{L_{AB}} = \frac{v_B}{0.360} \qquad \omega_{BC} = \frac{v_B}{L_{BC}} = \frac{v_B}{0.600} \qquad \bar{v} = \frac{1}{2}v_B$$

$$\begin{aligned} T_2 &= \frac{1}{2}I_A\omega_{AB}^2 + \frac{1}{2}m_{BC}\bar{v}^2 + \frac{1}{2}\bar{I}\,\omega_{BC}^2 \\ &= \frac{1}{2}(0.10368)\left(\frac{v_B}{0.360}\right)^2 + \frac{1}{2}(4)\left(\frac{1}{2}v_B\right)^2 + \frac{1}{2}(0.1200)\left(\frac{v_B}{0.600}\right)^2 \\ &= 1.06667v_B^2 \end{aligned}$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$: $0 + 11.3011 = 1.06667v_B^2$

$$v_B = 3.25\ \text{m/s} \qquad \mathbf{v}_B = 3.25\ \text{m/s}\ \downarrow \blacktriangleleft$$



PROBLEM 17.46

The uniform rods *AB* and *BC* of masses 2.4 kg and 4 kg, respectively, and the small wheel at *C* is of negligible mass. Knowing that in the position shown the velocity of wheel *C* is 2 m/s to the right, determine the velocity of pin *B* after rod *AB* has rotated through 90°.

SOLUTION

Moments of inertia.

Rod *AB*: $I_A = \frac{1}{3} m_{AB} L_{AB}^2 = \frac{1}{3}(2.4)(0.36)^2 = 0.10368 \text{ kg}\cdot\text{m}^2$

Rod *BC*: $\bar{I} = \frac{1}{12} m_{BC} L_{BC^2} = \frac{1}{12}(4)(0.600)^2 = 0.1200 \text{ kg}\cdot\text{m}^2$

Position 1. As shown with rod *AB* vertical. Point *G* is the midpoint of *BC*.

$$\begin{aligned} V_1 &= W_{AB} h_{AB} + W_{BC} h_{BC} \\ &= (2.4)(9.81)(0.180) + (4)(9.81)(0.180) \\ &= 11.301 \text{ J} \end{aligned}$$

Kinematics: At the instant shown in Position 1,

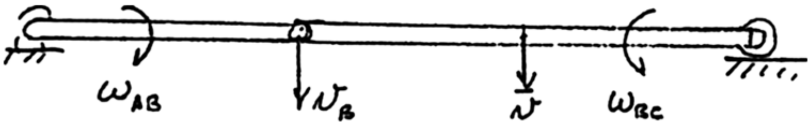
$$\omega_{BC} = 0$$

$$\bar{v} = v_G = v_B = v_C = 2 \text{ m/s}$$

$$\omega_{AB} = \frac{v_B}{L_{AB}} = \frac{2}{0.36} = 5.5556 \text{ rad/s}$$

$$\begin{aligned} T_1 &= \frac{1}{2} I_A \omega_{AB}^2 + \frac{1}{2} m_{BC} \bar{v}^2 + \frac{1}{2} I \omega_{BC}^2 \\ &= \frac{1}{2}(0.10368)(5.5556)^2 + \frac{1}{2}(4)(2)^2 + 0 \\ &= 9.6 \text{ J} \end{aligned}$$

Position 2. Rod *AB* is horizontal.



$$V_2 = 0$$

PROBLEM 17.46 (Continued)

Kinematics.

$$\omega_{AB} = \frac{v_B}{L_{AB}} = \frac{v_B}{0.36}$$

$$\omega_{BC} = \frac{v_B}{L_{BC}} = \frac{v_B}{0.60}$$

$$\bar{v} = \frac{1}{2} v_B$$

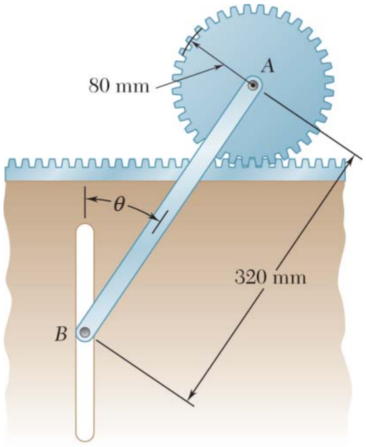
$$T_2 = \frac{1}{2} I_A \omega_{AB}^2 + \frac{1}{2} m_{BC} \bar{v}^2 + \frac{1}{2} \bar{I} \omega_{BC}^2$$

$$= \frac{1}{2}(0.10368)\left(\frac{v_B}{0.36}\right)^2 + \frac{1}{2}(4)\left(\frac{1}{2} v_B\right)^2 + \frac{1}{2}(0.12)\left(\frac{v_B}{0.60}\right)^2$$

$$= 1.0667 v_B^2$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$: $9.6 + 11.301 = 1.0667 v_B^2 + 0$

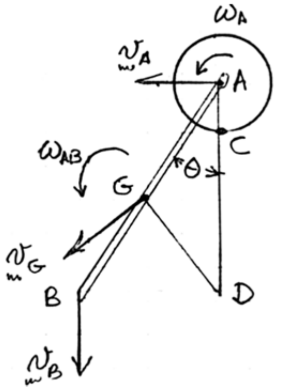
$v_B = 4.4266$ m/s $\quad$ $\mathbf{v}_B = 4.43$ m/s $\downarrow$ ◀



PROBLEM 17.47

The 80-mm-radius gear shown has a mass of 5 kg and a centroidal radius of gyration of 60 mm. The 4-kg rod *AB* is attached to the center of the gear and to a pin at *B* that slides freely in a vertical slot. Knowing that the system is released from rest when $\theta = 60°$, determine the velocity of the center of the gear when $\theta = 20°$.

SOLUTION



Kinematics.

$$\mathbf{v}_A = v_A \longleftarrow$$

$$\mathbf{v}_B = v_B \downarrow$$

Point *D* is the instantaneous center of rod *AB*.

$$\omega_{AB} = \frac{v_A}{L\cos\theta}$$

$$v_B = (L\sin\theta)\omega_{AB} = v_A \tan\theta$$

$$v_G = \frac{L}{2}\omega_{AB} = \frac{v_A}{2\cos\theta}$$

Gear *A* effectively rolls without slipping, with Point *C* being the contact point.

$$v_C = 0$$

Angular velocity of gear $\qquad \omega_A = \frac{v_A}{r}.$

Potential energy: Use the level of the center of gear *A* as the datum.

$$V = -W_{AB}\left(\frac{L}{2}\cos\theta\right) = -\frac{1}{2}m_{AB}gL\cos\theta$$

Kinetic energy: $\qquad T = \frac{1}{2}m_A v_A^2 + \frac{1}{2}I_A\omega_A^2 + \frac{1}{2}m_{AB}v_\theta^2 + \frac{1}{2}\bar{I}_{AB}\omega_{AB}^2$

Masses and moments of inertia: $\qquad m_A = 5 \text{ kg}, \quad m_{AB} = 4 \text{ kg}$

$$I_A = m_A k^2 = (5)(0.060)^2 = 0.018 \text{ kg}\cdot\text{m}^2$$

$$\bar{I}_{AB} = \frac{1}{12}m_{AB}L^2 = \frac{1}{12}(4)(0.320)^2 = 0.03413 \text{ kg}\cdot\text{m}^2$$

PROBLEM 17.47 (Continued)

Conservation of energy: $T_1 + V_1 = T_2 + V_2$

Position 1: $\theta = 60°$

$v_A = 0 \quad T_1 = 0$

$$V_1 = -\frac{1}{2}(4)(9.81)(0.320)\cos 60°$$

$$= -3.1392 \text{ J}$$

Position 2: $\theta = 20° \quad v_A = ?$

$$T_2 = \frac{1}{2}(5)v_A^2 + \frac{1}{2}(0.018)\left(\frac{v_A}{0.080}\right)^2 + \frac{1}{2}(4)\left(\frac{v_A}{2\cos 20°}\right)^2$$

$$+\frac{1}{2}(0.03413)\left(\frac{v_A}{0.320\cos 20°}\right)^2$$

$$= (2.5 + 1.40625 + 0.56624 + 0.18875)v_A^2$$

$$= 4.66124 v_A^2$$

$$V_2 = -\frac{1}{2}(4)(9.81)(0.320)\cos 20°$$

$$= -5.8998 \text{ J}$$

Conservation of energy: $T_1 + V_1 = T_2 + V_2$

$$0 - 3.1392 = 4.66124 v_A^2 - 5.8998$$

$$v_A^2 = 0.59225 \text{ m}^2/\text{s}^2$$

$$v_A = 0.770 \text{ m/s}$$

$\mathbf{v}_A = 0.770$ m/s ← ◀

PROBLEM 17.48

Knowing that the maximum allowable couple that can be applied to a shaft is 15.5 kip · in., determine the maximum horsepower that can be transmitted by the shaft at (*a*) 180 rpm, (*b*) 480 rpm.

SOLUTION

$$\begin{aligned} M &= 15.5 \text{ kip}\cdot\text{in.} \\ &= 1.2917 \text{ kip}\cdot\text{ft} \\ &= 1291.7 \text{ lb}\cdot\text{ft} \end{aligned}$$

(*a*)

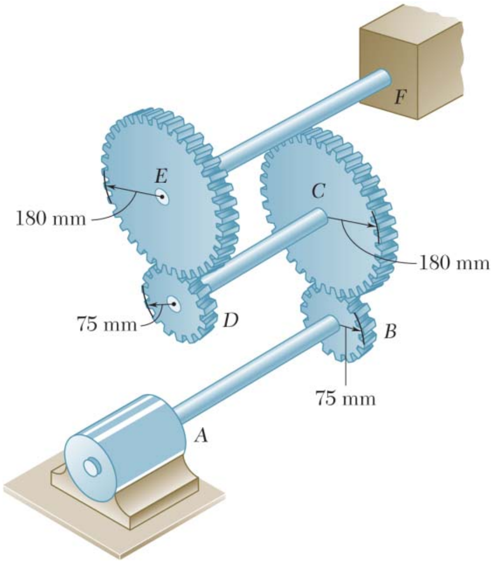
$$\begin{aligned} \omega &= 180 \text{ rpm} \\ &= 6\pi \text{ rad/s} \\ \text{Power} &= M\omega \\ &= (1291.7 \text{ lb}\cdot\text{ft})(6\pi \text{ rad/s}) \\ &= 24348 \text{ ft}\cdot\text{lb/s} \\ \text{Horsepower} &= \frac{24348}{550} \\ &= 44.3 \text{ hp} \end{aligned}$$

◀

(*b*)

$$\begin{aligned} \omega &= 480 \text{ rpm} \\ &= 16\pi \text{ rad/s} \\ \text{Power} &= M\omega \\ &= (1291.7 \text{ lb}\cdot\text{ft})(16\pi \text{ rad/s}) \\ &= 64930 \text{ ft}\cdot\text{lb/s} \\ \text{Horsepower} &= \frac{64930}{550} \\ &= 118.1 \text{ hp} \end{aligned}$$

◀



PROBLEM 17.49

Three shafts and four gears are used to form a gear train which will transmit 7.5 kW from the motor at *A* to a machine tool at *F*. (Bearings for the shafts are omitted from the sketch.) Knowing that the frequency of the motor is 30 Hz, determine the magnitude of the couple which is applied to shaft (*a*) *AB*, (*b*) *CD*, (*c*) *EF*.

SOLUTION

Kinematics.

$$\omega_{AB} = 30 \text{ Hz} = 30(2\pi)\text{rad/s} = 60\pi \text{ rad/s}$$

Gears *B* and *C*.

$$r_B = 75 \text{ mm}$$

$$r_C = 180 \text{ mm}$$

$$r_B\omega_{AB} = r_C\omega_{CD}: \quad (75 \text{ mm})(60\pi \text{ rad/s}) = (180 \text{ mm})(\omega_{CD})$$

Gears *D* and *E*.

$$\omega_{CD} = 25\pi \text{ rad/s}$$

$$r_D = 75 \text{ mm}$$

$$r_E = 180 \text{ mm}$$

$$r_D\omega_{CD} = r_E\omega_{EF}: \quad (75 \text{ mm})(25\pi \text{ rad/s}) = (180 \text{ mm})(\omega_{EF})$$

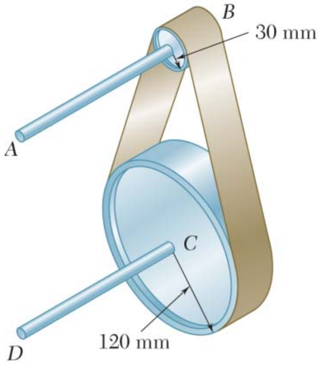
$$\omega_{EF} = 10.4167\pi \text{ rad/s}$$

$$\text{Power} = 7.5 \text{ kW}$$

(*a*) Shaft *AB*. Power $= M_{AB}\omega_{AB}$: $7500 \text{ W} = M_{AB}(60\pi \text{ rad/s})$ $M_{AB} = 39.8 \text{ N}\cdot\text{m}$ ◀

(*b*) Shaft *CD*. Power $= M_{CD}\omega_{CD}$: $7500 \text{ W} = M_{CD}(25\pi \text{ rad/s})$ $M_{CD} = 95.5 \text{ N}\cdot\text{m}$ ◀

(*c*) Shaft *EF*. Power $= M_{EF}\omega_{EF}$: $7500 \text{ W} = M_{EF}(10.4167\pi \text{ rad/s})$ $M_{EF} = 229 \text{ N}\cdot\text{m}$ ◀



PROBLEM 17.50

The shaft-disk-belt arrangement shown is used to transmit 2.4 kW from Point *A* to Point *D*. Knowing that the maximum allowable couples that can be applied to shafts *AB* and *CD* are $25\ \mathrm{N\cdot m}$ and $80\ \mathrm{N\cdot m}$, respectively, determine the required minimum speed of shaft *AB*.

SOLUTION

Power.
$$2.4\ \text{kW} = 2400\ \text{W}$$
$$M_{AB} < 25\ \mathrm{N\cdot m}$$
$$P = M_{AB}\omega_{AB}$$
$$\min \omega_{AB} = \frac{P}{\max M_{AB}} = \frac{2400}{25} = 96\ \text{rad/s}$$
$$M_{CD} < 80\ \mathrm{N\cdot m}$$
$$P = M_{CD}\omega_{CD}$$
$$\min \omega_{CD} = \frac{P}{\max M_{CD}} = \frac{2400}{80} = 30\ \text{rad/s}$$

Kinematics.
$$r_A\omega_{AB} = r_C\omega_{CD}$$
$$\min \omega_{AB} = \frac{r_C}{r_A}(\min \omega_{CD})$$
$$= \left(\frac{120}{30}\right)(30)$$
$$= 120\ \text{rad/s}$$

Choose the larger value for $\min \omega_{AB}$. $\min \omega_{AB} = 120\ \text{rad/s}$ $\min \omega_{AB} = 1146\ \text{rpm}$ ◀

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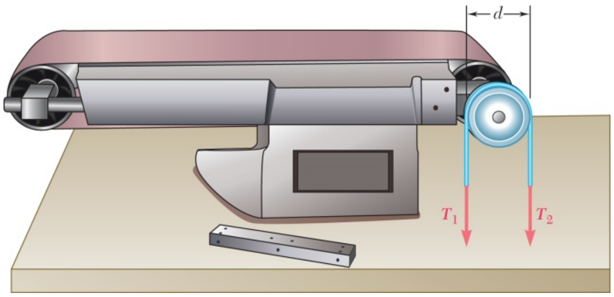


PROBLEM 17.51

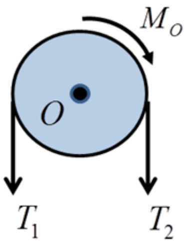
The drive belt on a vintage sander transmits ½ hp to a pulley that has a diameter of d = 4 in. Knowing that the pulley rotates at 1450 rpm, determine the tension difference $T1 - T2$ between the tight and slack sides of the belt.

SOLUTION

Given:

$$P = 0.5\ \text{HP} \times \frac{550\ \text{ft•lb/s}}{\text{HP}}$$
$$= 275\ \text{ft•lb/s}$$
$$\text{d}=\frac{1}{3}\ \text{ft}$$
$$\omega = 1450\ \frac{\text{rev}}{\text{min}} \times \frac{2\pi}{60}$$
$$= \frac{145\pi}{3}\ \text{rad/s}$$

Belt Tensions on Pulley:



Kinetics:

$$\sum M_O = \bar{I}\cancel{\alpha}$$
$$T_1\frac{d}{2} - T_2\frac{d}{2} - M_O = 0$$
$$M_O = \frac{d}{2}(T_1 - T_2)$$

Substitute into the equation for Power

$$P = M\omega$$
$$275 = \frac{1}{6}(T_1 - T_2)\frac{145\pi}{3}$$
$$(T_1 - T_2) = 10.87\ \text{lb}$$

$$(T_1 - T_2) = 10.87\ \text{lb} \blacktriangleleft$$

PROBLEM 17.52

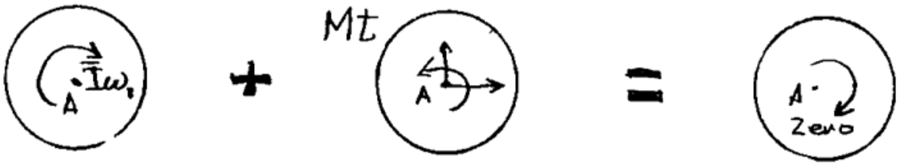
The rotor of an electric motor has a mass of 25 kg, and it is observed that 4.2 min is required for the rotor to coast to rest from an angular velocity of 3600 rpm. Knowing that kinetic friction produces a couple of magnitude 1.2 N·m, determine the centroidal radius of gyration for the rotor.

SOLUTION

Coasting time: $t = 4.2 \text{ min} = 252 \text{ s}$

Initial angular velocity: $\omega_1 = 3600 \text{ rpm} = 120\pi \text{ rad/s}$

Principle of impulse and momentum.



Syst. Momenta$_1$ + Σ**Syst. Impulses.**$_{1\to 2}$ = **Syst. Momenta**$_2$

↺ Moments about axle *A*: $\bar{I}\omega_1 - Mt = 0$

$$\bar{I} = \frac{Mt}{\omega_1}$$

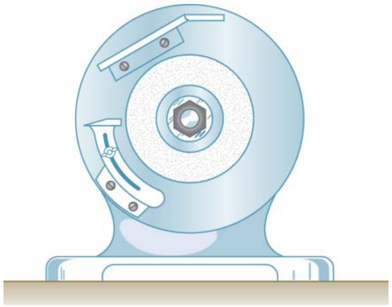
$$= \frac{(1.2 \text{ N}\cdot\text{m})(252 \text{ s})}{120\pi \text{ rad/s}}$$

$$= 0.80214 \text{ kg}\cdot\text{m}^2$$

$$\bar{I} = mk^2$$

Radius of gyration: $\bar{k} = \sqrt{\dfrac{\bar{I}}{m}} = \sqrt{\dfrac{0.80214 \text{ kg}\cdot\text{m}^2}{25 \text{ kg}}} = 0.1791 \text{ m}$

$\bar{k} = 179.1 \text{ mm}$ ◀



PROBLEM 17.53

A small grinding wheel is attached to the shaft of an electric motor which has a rated speed of 3600 rpm. When the power is turned off, the unit coasts to rest in 70 s. The grinding wheel and rotor have a combined weight of 6 lb and a combined radius of gyration of 2 in. Determine the average magnitude of the couple due to kinetic friction in the bearings of the motor.

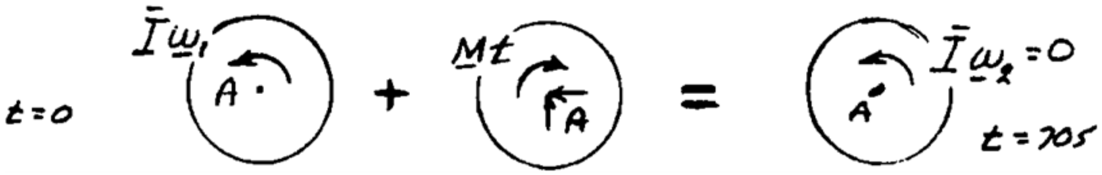
SOLUTION

Use the principle of impulse and momentum applied to the grinding wheel and rotor with

$$t_1 = 0 \qquad t_2 = 70 \text{ s}$$

$$\omega_1 = 3600 \text{ rpm} = 120\pi \text{ rad/s} \qquad \omega_2 = 0$$

Moment of inertia. $$\bar{I} = m\bar{k}^2 = \frac{6 \text{ lb}}{32.2 \text{ ft/s}^2}\left(\frac{2}{12}\text{ ft}\right)^2 = 0.00518 \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$



$$\textbf{Syst. Momenta}_1 \quad + \quad \textbf{Syst. Ext. Imp.}_{1\to 2} \quad = \quad \textbf{Syst. Momenta}_2$$

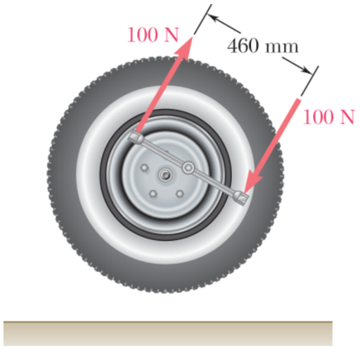
$+\circlearrowleft$ Moments about *A*: $$\bar{I}\omega_1 - Mt = 0$$

$$(0.00518)(120\pi) - M(70 \text{ s}) = 0$$

$$M = 0.02788 \text{ lb}\cdot\text{ft}$$

$$M = 0.33451 \text{ lb}\cdot\text{in.}$$

$$M = 0.335 \text{ lb}\cdot\text{in.} \blacktriangleleft$$



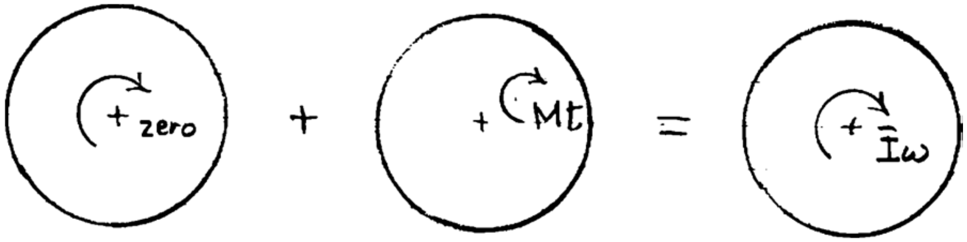
PROBLEM 17.54

A bolt located 50 mm from the center of an automobile wheel is tightened by applying the couple shown for 0.10 s. Assuming that the wheel is free to rotate and is initially at rest, determine the resulting angular velocity of the wheel. The wheel has a mass of 19 kg and has a radius of gyration of 250 mm.

SOLUTION

Moment of inertia. $\bar{I} = m\bar{k}^2 = (19\text{ kg})(0.25\text{ m})^2 = 1.1875\text{ kg-m}^2$

Applied couple. $M = (100\text{ N})(0.460\text{ m}) = 46\text{ N-m}$

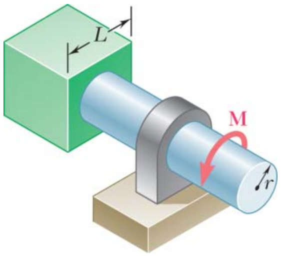


Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\rightarrow 2}$ = **Syst. Momenta**$_2$

Moments about axle: $0 + Mt = \bar{I}\omega$

$$0 + (46\text{ N-m})(0.10\text{ s}) = (1.1875\text{ kg-m}^2)\omega$$

$\omega = 3.87\text{ rad/s}$ ◀



PROBLEM 17.55

A uniform 144-lb cube is attached to a uniform 136-lb circular shaft as shown and a couple **M** of constant magnitude is applied to the shaft when the system is at rest. Knowing that $r = 4$ in., $L = 12$ in., and the angular velocity of the system is 960 rpm after 4 s, determine the magnitude of the couple **M**.

SOLUTION

Moments of inertia.

$$\text{Cube: } \frac{1}{12}m\left(L^2+L^2\right) = \frac{1}{12}\left(\frac{144}{32.2}\right)\left[(1)^2+(1)^2\right] = 0.74534 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\text{Cylinder: } \frac{1}{2}mr^2 = \frac{1}{2}\left(\frac{136}{32.2}\right)\left(\frac{4}{12}\right)^2 = 0.23464 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\text{Total: } \bar{I} = 0.97999 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Final angular velocity.

$$\omega_2 = 960 \text{ rpm} = 32\pi \text{ rad/s}$$

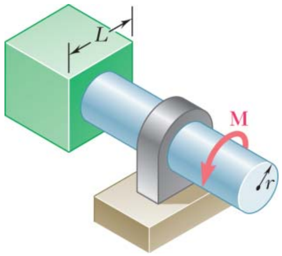
$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\to 2} = \textbf{Syst Momenta}_2$$

Moments about cylinder axis:

$$0 + Mt = \bar{I}\omega_2$$

$$M = \frac{\bar{I}\omega_2}{t} = \frac{(0.97999)(32\pi)}{4}$$

$$M = 24.6 \text{ ft}\cdot\text{lb} \blacktriangleleft$$



PROBLEM 17.56

A uniform 75-kg cube is attached to a uniform 70-kg circular shaft as shown and a couple **M** of constant magnitude $20\ \text{N}\cdot\text{m}$ is applied to the shaft. Knowing that $r = 100$ mm and $L = 300$ mm., determine the time required for the angular velocity of the system to increase from 1000 rpm to 2000 rpm.

SOLUTION

Moments of inertia.

$$\text{Cube:}\quad \frac{1}{12}m\left(L^2+L^2\right)=\frac{1}{12}75\left[(0.3)^2+(0.3)^2\right]=1.125\ \text{kg}\cdot\text{m}^2$$

$$\text{Cylinder:}\quad \frac{1}{2}mr^2=\frac{1}{2}70(0.1)^2=0.35\ \text{kg}\cdot\text{m}^2$$

$$\text{Total:}\quad \bar{I}=1.475\ \text{kg}\cdot\text{m}^2$$

Angular speeds.

$$\omega_1=1000\ \text{rpm}=\frac{100\pi}{3}\ \text{rad/s}$$

$$\omega_2=2000\ \text{rpm}=\frac{200\pi}{3}\ \text{rad/s}$$

Applied couple.

$$M=20\ \text{N}\cdot\text{m}$$

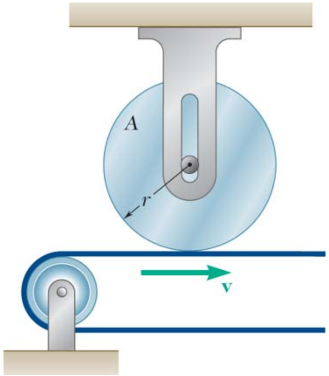
$$\textbf{Syst Momenta}_1+\textbf{Syst Ext Imp}_{1\to2}=\textbf{Syst Momenta}_2$$

Moments about cylinder axis:

$$\bar{I}\omega_1+Mt=\bar{I}\omega_2$$

$$t=\frac{\bar{I}(\omega_2-\omega_1)}{M}=\frac{(1.475)(200-100)\frac{\pi}{3}}{20}$$

$t = 7.72$ s ◀



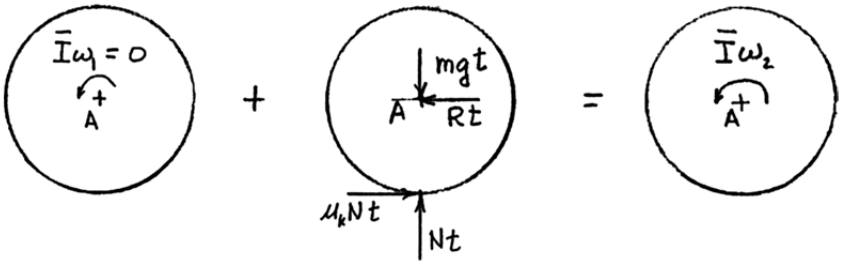
PROBLEM 17.57

A disk of constant thickness, initially at rest, is placed in contact with a belt that moves with a constant velocity **v**. Denoting by μ_k the coefficient of kinetic friction between the disk and the belt, derive an expression for the time required for the disk to reach a constant angular velocity.

SOLUTION

Moment of inertia. $\bar{I} = \frac{1}{2}mr^2$

Final state of constant angular velocity. $\omega_2 = \frac{v}{r}$



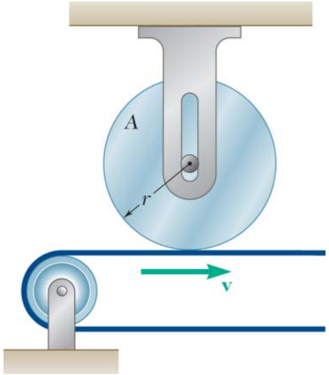
Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

$+\uparrow$ y components: $0 + Nt - mgt = 0 \quad N = mg$

$+\circlearrowleft$ Moments about A: $0 + \mu_k Ntr = \bar{I}\omega_2$

$$t = \frac{\bar{I}\omega_2}{\mu_k mgr} = \frac{\frac{1}{2}mr^2\frac{v}{r}}{\mu_k mgr} = \frac{v}{2\mu_k g}$$

$$t = \frac{v}{2\mu_k g} \blacktriangleleft$$



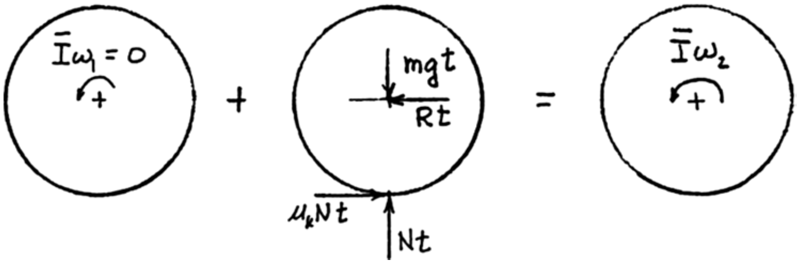
PROBLEM 17.58

Disk A, of weight 5 lb and radius $r = 3$ in., is at rest when it is placed in contact with a belt which moves at a constant speed $v = 50$ ft/s. Knowing that $\mu_k = 0.20$ between the disk and the belt, determine the time required for the disk to reach a constant angular velocity.

SOLUTION

Moment of inertia. $\bar{I} = \frac{1}{2}mr^2$

Final state of constant angular velocity. $\omega_2 = \frac{v}{r}$



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

$+\uparrow$ y components: $0 + Nt - mgt = 0 \qquad N = mg$

$+\circlearrowleft$ Moments about A: $0 + \mu_k Ntr = \bar{I}\omega_2$

$$t = \frac{\bar{I}\omega_2}{\mu_k mgr} = \frac{\frac{1}{2}mr^2 \frac{v}{r}}{\mu_k mgr} = \frac{v}{2\mu_k g}$$

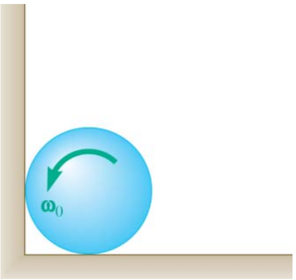
$$t = \frac{v}{2\mu_k g}$$

Data:

$$v = 50 \text{ ft/s}$$

$$\mu_k = 0.20$$

$$t = \frac{50}{(2)(0.20)(32.2)} \qquad t = 3.88 \text{ s} \blacktriangleleft$$



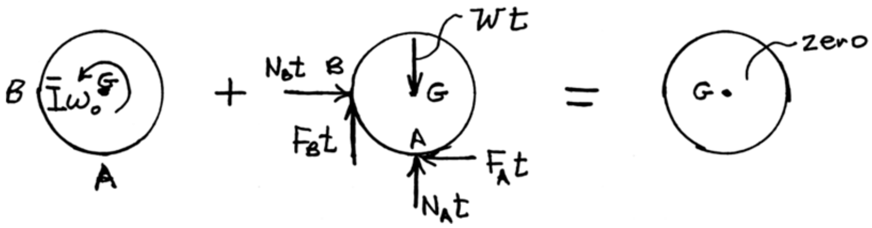
PROBLEM 17.59

A cylinder of radius r and weight W with an initial counterclockwise angular velocity ω_0 is placed in the corner formed by the floor and a vertical wall. Denoting by μ_k the coefficient of kinetic friction between the cylinder and the wall and the floor derive an expression for the time required for the cylinder to come to rest.

SOLUTION

For the cylinder $\bar{I} = \dfrac{1}{2}mr^2, \quad W = mg$

Principle of impulse and momentum.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

Linear momentum $\xrightarrow{+}$: $0 + N_Bt - F_At = 0$

$$N_B = F_A$$

Linear momentum $+\uparrow$: $0 + N_At + F_Bt - Wt = 0$

$$N_A + F_B = N_A + \mu_k N_B$$

$$= N_A + \mu_k F_A + N_A + \mu_k^2 N_A = W$$

$$N_A = \frac{W}{1+\mu_k^2}$$

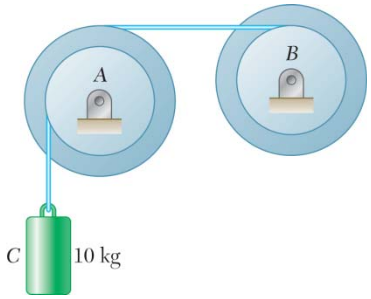
$$F_A = \mu_k N_A = \frac{\mu_k W}{1+\mu_k^2}$$

$$N_B = \frac{\mu_k W}{1+\mu_k^2}$$

$$F_B = \frac{\mu_k^2 W}{1+\mu_k^2}$$

$+\circlearrowleft$ Moments about G: $\bar{I}\omega_0 - F_Art - F_Brt = 0$

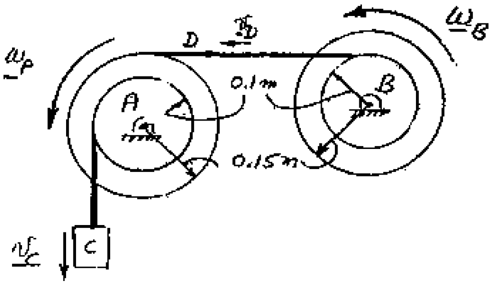
$$t = \frac{\bar{I}\omega_0}{(F_A + F_B)r} = \frac{(1+\mu_k^2)\bar{I}\omega_0}{\mu_k(1+\mu_k)Wr} \qquad t = \frac{1+\mu_k^2}{2\mu_k(1+\mu_k)}\frac{r\omega_0}{g} \blacktriangleleft$$



PROBLEM 17.60

Each of the double pulleys shown has a centroidal mass moment of inertia of 0.25 kg·m^2, an inner radius of 100 mm, and an outer radius of 150 mm. Neglecting bearing friction, determine (*a*) the velocity of the cylinder 3 s after the system is released from rest, (*b*) the tension in the cord connecting the pulleys.

SOLUTION


ω_B
ω_P
D
T_D
A
B
0.1 m
0.15 m
C
v_C

Kinematics.

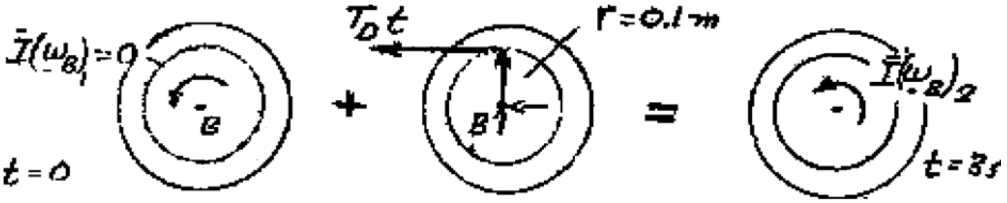
Cylinder *C*. $\mathbf{v}_C = v_C \downarrow$

Pulley *A*. $\omega_A = \dfrac{v_C}{0.100} = 10 v_C \circlearrowleft$

$\mathbf{v}_D = 0.150\ \omega_A = 1.5 v_C \longleftarrow$

Pulley *B*. $\omega_B = \dfrac{v_D}{0.100} = 15 v_C \circlearrowleft$

Principle of impulse and momentum for pulley B.

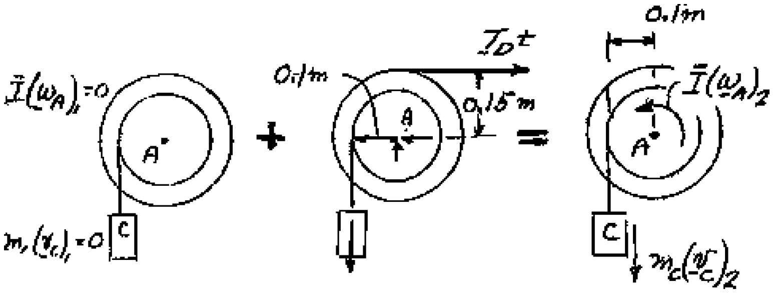

$\bar{I}(\omega_B)_1 = 0$
B
$t = 0$
+
$T_D t$
$r = 0.1$ m
B
=
$\bar{I}(\omega_B)_2$
$t = 3$ s

Syst momenta$_1$ + **Syst Ext Imp$_{1\to 2}$** = **Syst momenta$_2$**

Moments about *B*: $0 + (T_D t) r = \bar{I}(\omega_B)_2$

$(T_D t)(0.100) = (0.25)(15 v_C)$ $\qquad T_D t = 37.5 v_C \qquad (1)$

Principle of impulse and momentum for pulley A and cylinder.


$\bar{I}(\omega_A)_1 = 0$
A
C
$m_c(v_c)_1 = 0$
+
0.1 m
$T_D t$
0.15 m
A
=
0.1 m
$\bar{I}(\omega_A)_2$
A
C
$m_c(v_c)_2$

Syst momenta$_1$ + **Syst Ext Imp$_{1\to 2}$** = **Syst Momenta$_2$**

PROBLEM 17.60 (Continued)

Moments about A: $0 + (W_C t) r_i - (T_D t) r_o = \bar{I}\omega_A + (m_C v_C) r_i$

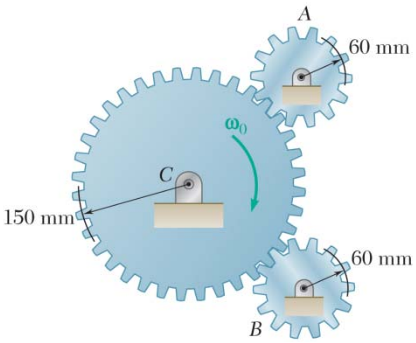
$$(10)(9.81)(3)(0.100) - (T_D t)(0.150) = (0.25)(10 v_C) + (10 v_C)(0.100)$$

$$29.43 - 0.15\, T_D t = 3.5 v_C$$
$$29.43 - (0.15)(37.5\, v_C) = 3.5 v_C$$

(*a*) $v_C = 3.2252 \text{ m/s}$ $\qquad \mathbf{v}_C = 3.23 \text{ m/s}\downarrow$ ◀

(*b*) From (1), $T_D t = (37.5)(3.2252) = 120.945 \text{ N}\cdot\text{s}$

$$T_D = \frac{T_D t}{t} = \frac{120.945}{3}$$ $T_D = 40.3 \text{ N}$ ◀



PROBLEM 17.61

Each of the gears A and B has a mass of 675 g and has a radius of gyration of 40 mm, while gear C has a mass of 3.6 kg and has a radius of gyration of 100 mm. Assume that kinetic friction in the bearings of gears A, B, and C produces couples of constant magnitude $0.15\ \text{N}\cdot\text{m}$, $0.15\ \text{N}\cdot\text{m}$, and $0.3\ \text{N}\cdot\text{m}$, respectively. Knowing that the initial angular velocity of gear C is 2000 rpm, determine the time required for the system to come to rest.

SOLUTION

Moments of inertia.

$$\bar{I}_A = \bar{I}_B = m_A k_A^2 = (0.675)(0.04)^2 = 1.08 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

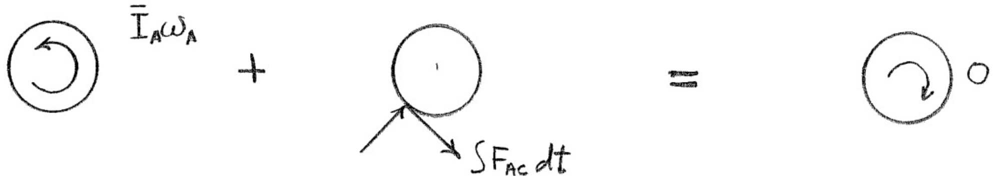
$$\bar{I}_C = m_C k_C^2 = (3.6)(0.1)^2 = 0.036\ \text{kg}\cdot\text{m}^2$$

Kinematics.

$$(\omega_C)_1 = 2000\ \text{rpm} = 209.44\ \text{rad/s}$$

$$r_A\omega_A = r_B\omega_B = r_C\omega_C \qquad (\omega_A)_1 = (\omega_B)_1 = \frac{r_C}{r_A}(\omega_C)_1 = \frac{0.15}{0.06}(209.44) = 523.60\ \text{rad/s}$$

Gear A:



Syst Momenta$_1$ + **Syst Ext Imp$_{1\to2}$** = **Syst Momenta$_2$**

↻ moments about A:

$$\bar{I}_A(\omega_A)_1 + r_A\int F_{AC}dt - (M_f)_A t = 0$$

$$(1.08\times10^{-3})(523.60) + 0.06\int F_{AC}dt - 0.15t$$

or

$$0.15t - 0.06\int F_{AC}dt = 0.56549 \qquad (1)$$

Gear B: By a similar analysis

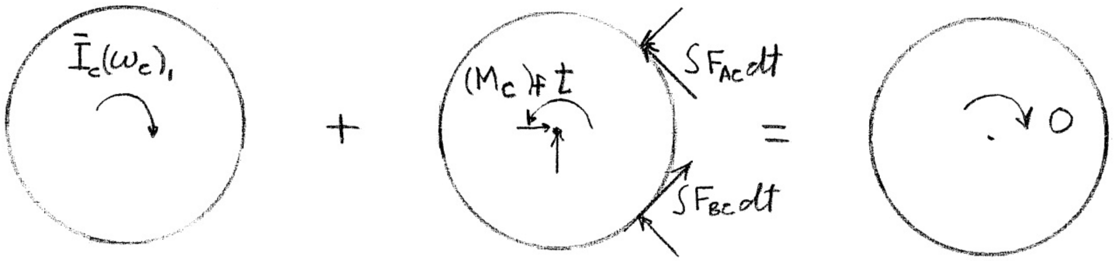
$$0.15t - 0.06\int F_{BC}dt = 0.56549 \qquad (2)$$

Let $x = \int F_{AC}dt + \int F_{BC}dt$ and add Equations (1) and (2).

$$0.3t - 0.06x = (0.56549)(2) = 1.13098 \qquad (3)$$

PROBLEM 17.61 (Continued)

Gear C:



Syst Momenta$_1$ + **Syst Ext Imp**$_{1\to 2}$ = **Syst Momenta**$_2$

↻ moments about *C*: $\bar{I}_C(\omega_C)_1 - (M_C)_f t - r_C\int F_{AC}dt - r_C\int F_{BC}dt = 0$

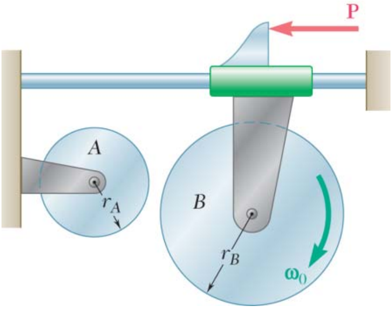
$$(0.036)(209.44) - 0.3t - 0.15\left[\int F_{AC}dt + \int F_{BC}dt\right]$$

$$0.3t + 0.15x = 7.5398 \qquad (4)$$

Solving Equations (3) and (4) simultaneously,

$$t = 9.87 \text{ s}, \quad x = 30.5 \text{ N}\cdot\text{s}$$

$t = 9.87$ s ◀



PROBLEM 17.62

Disk *B* has an initial angular velocity ω_0 when it is brought into contact with disk *A* which is at rest. Show that the final angular velocity of disk *B* depends only on ω_0 and the ratio of the masses m_A and m_B of the two disks.

SOLUTION

Let Points *A* and *B* be the centers of the two disks and Point *C* be the contact point between the two disks.
Let ω_A and ω_B be the final angular velocities of disks *A* and *B*, respectively, and let v_C be the final velocity at *C* common to both disks.

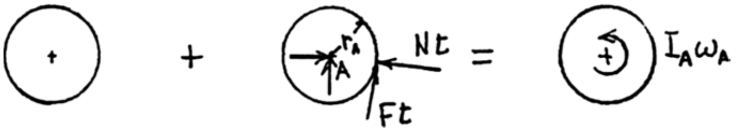
Kinematics: No slipping $\qquad v_C = r_A\omega_A = r_B\omega_B$

Moments of inertia. Assume that both disks are uniform cylinders.

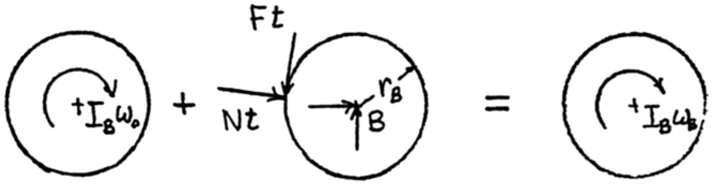
$$I_A = \frac{1}{2}m_A r_A^2 \qquad I_B = \frac{1}{2}m_B r_B^2$$

Principle of impulse and momentum.

Disk A



Disk B



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

Disk *A*: ↺ Moments about *A*: $\qquad 0 + r_A Ft = I_A\omega_A$

$$Ft = \frac{I_A\omega_A}{r_A} = \frac{1}{2}\frac{m_A r_A^2 v_C}{r_A^2}$$

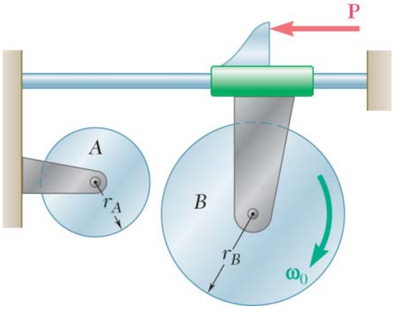
$$= \frac{1}{2}m_A v_C$$

$$= \frac{1}{2}m_A r_B\omega_B$$

Disk *B*: ↻ Moments about *B*: $\qquad I_B\omega_0 - r_B Ft = I_B\omega_B$

$$\frac{1}{2}m_B r_B^2\omega_0 - r_B\left(\frac{1}{2}m_A r_B\omega_B\right) = \frac{1}{2}mr_B^2\omega_B$$

$$\omega_B = \frac{\omega_0}{1+\frac{m_A}{m_B}} \blacktriangleleft$$



PROBLEM 17.63

The 7.5-lb disk A has a radius $r_A = 6$ in. and is initially at rest. The 10-lb disk B has a radius $r_B = 8$ in. and an angular velocity ω_0 of 900 rpm when it is brought into contact with disk A. Neglecting friction in the bearings, determine (*a*) the final angular velocity of each disk, (*b*) the total impulse of the friction force exerted on disk A.

SOLUTION

Let Points A and B be the centers of the two disks and Point C be the contact point between the two disks.

Let ω_A and ω_B be the final angular velocities of disks A and B, respectively, and let v_C be the final velocity at C common to both disks.

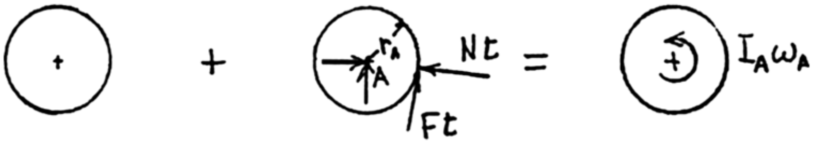
Kinematics: No slipping $\qquad v_C = r_A\omega_A = r_B\omega_B$

Moments of inertia. Assume that both disks are uniform cylinders.

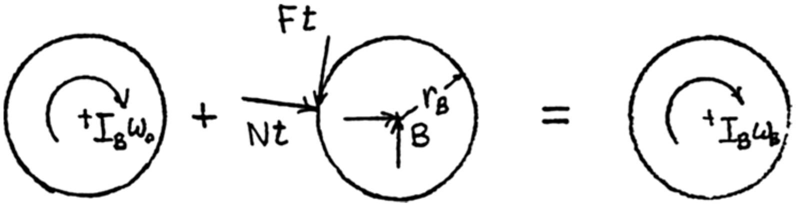
$$I_A = \frac{1}{2}m_A r_A^2 \qquad I_B = \frac{1}{2}m_B r_B^2$$

Principle of impulse and momentum.

Disk A



Disk B



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

Disk A: ↻ Moments about A:

$$0 + r_A Ft = I_A\omega_A$$

$$Ft = \frac{I_A\omega_A}{r_A}$$

$$= \frac{1}{2}\frac{m_A r_A^2 v_C}{r_A^2} = \frac{1}{2}m_A v_C$$

$$= \frac{1}{2}m_A r_B\omega_B$$

PROBLEM 17.63 (Continued)

Disk B: ↻ Moments about B:

$$I_B\omega_0 - r_B Ft = I_B\omega_B$$

$$\frac{1}{2}m_B r_B^2\omega_0 - r_B\left(\frac{1}{2}m_A r_B\omega_B\right) = \frac{1}{2}mr_B^2\omega_B$$

$$\omega_B = \frac{\omega_0}{1+\frac{m_A}{m_B}}$$

Data:

$$m_A = \frac{7.5}{32.2} = 0.23292\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$\frac{m_A}{m_B} = \frac{W_A}{W_B} = \frac{7.5}{10} = 0.75$$

$$r_B = \frac{8}{12}\ \text{ft}$$

$$\frac{r_B}{r_A} = \frac{8}{6}$$

$$\omega_0 = 900\ \text{rpm} = 30\pi\ \text{rad/s}$$

(*a*)

$$\omega_B = \frac{\omega_0}{1+0.75} = \frac{30\pi}{1.75} = 53.856\ \text{rad/s}$$

$$\omega_A = \frac{r_B}{r_A}\omega_B = \left(\frac{8}{6}\right)(53.856) = 71.808\ \text{rad/s}$$

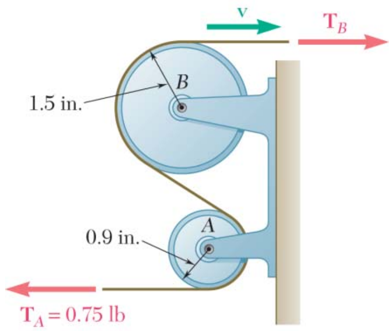
$\boldsymbol{\omega}_A = 686$ rpm ↺ ◀

$\boldsymbol{\omega}_B = 514$ rpm ↻ ◀

(*b*)

$$Ft = \frac{1}{2}\frac{(0.23292)\left(\frac{8}{12}\right)(30\pi)}{1+0.75}$$

$\mathbf{F}t = 4.18$ lb·s ↑ ◀



PROBLEM 17.64

A tape moves over the two drums shown. Drum A weighs 1.4 lb and has a radius of gyration of 0.75 in., while drum B weighs 3.5 lb and has a radius of gyration of 1.25 in. In the lower portion of the tape the tension is constant and equal to $T_A = 0.75$ lb. Knowing that the tape is initially at rest, determine (*a*) the required constant tension T_B if the velocity of the tape is to be $v = 10$ ft/s after 0.24 s, (*b*) the corresponding tension in the portion of tape between the drums.

SOLUTION

Kinematics. Drums A and B rotate about fixed axes. Let v be the tape velocity in ft/s.

$$v = r_A\omega_A = \frac{0.9}{12}\omega_A \qquad \omega_A = 13.3333v$$

$$v = r_B\omega_B = \frac{1.5}{12}\omega_B \qquad \omega_B = 8v$$

Moments of inertia.

$$\bar{I}_A = m_A\bar{k}_A^2 = \left(\frac{1.4}{32.2}\right)\left(\frac{0.75}{12}\right)^2 = 169.837\times 10^{-6}\,\text{lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_B = m_B\bar{k}_B^2 = \left(\frac{3.5}{32.2}\right)\left(\frac{1.25}{12}\right)^2 = 1.17942\times 10^{-3}\,\text{lb}\cdot\text{s}^2\cdot\text{ft}$$

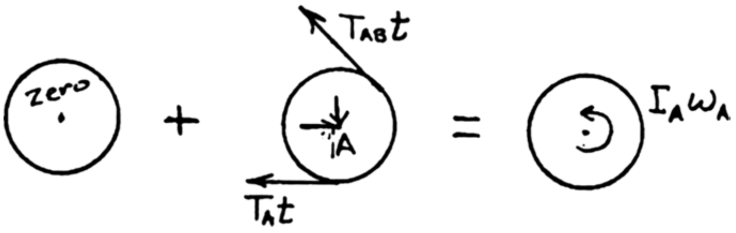
State 1. $t = 0 \qquad v = 0 \qquad \omega_A = \omega_B = 0$

State 2. $t = 0.24\text{ s}, \qquad v = 10\text{ ft/s}$

$$\omega_A = (13.3333)(10) = 133.333\text{ rad/s} \circlearrowleft$$

$$\omega_B = (8)(10) = 80\text{ rad/s} \circlearrowright$$

Drum A.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

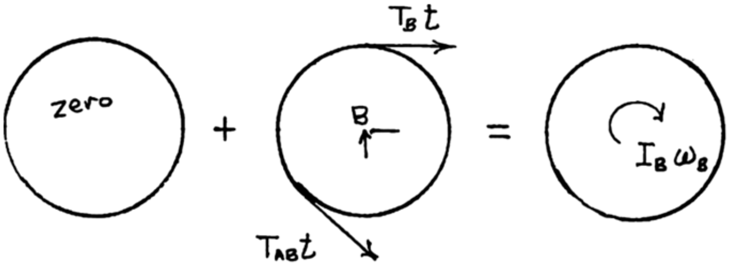
PROBLEM 17.64 (Continued)

$+\curvearrowright$ Moments about *A*: $0 + r_A T_{AB} t - r_A T_A t = I_A \omega_A$

$$0 + \left(\frac{0.9}{12}\right)(T_{AB}t) - \left(\frac{0.9}{12}\right)(0.75)(0.24) = (169.837 \times 10^{-6})(133.333)$$

$$T_{AB}t = 0.48193\ \text{lb} \cdot \text{s}$$

Drum *B*.



$$\textbf{Syst. Momenta}_1 \quad + \quad \textbf{Syst. Ext. Imp.}_{1\to 2} \quad = \quad \textbf{Syst. Momenta}_2$$

$+\curvearrowleft$ Moments about *B*: $0 + r_B T_B t - r_B T_{AB} t = I_B \omega_B$

$$0 + \frac{1.5}{12}(T_B t) - \left(\frac{1.5}{12}\right)(0.48193) = (1.17942 \times 10^{-3})(80)$$

$$T_B t = 1.23676\ \text{lb} \cdot \text{s}$$

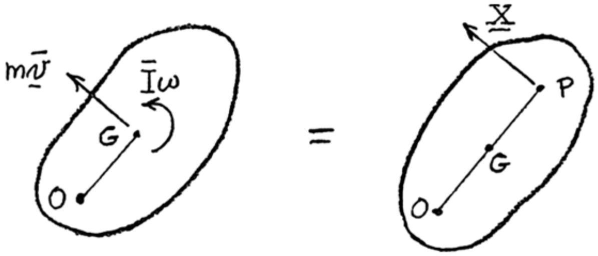
(*a*) $T_B = \dfrac{T_B t}{t} = \dfrac{1.23676}{0.24}$ $T_B = 5.15\ \text{lb}$ ◀

(*b*) $T_{AB} = \dfrac{T_{AB} t}{t} = \dfrac{0.48193}{0.24}$ $T_{AB} = 2.01\ \text{lb}$ ◀

PROBLEM 17.65

Show that the system of momenta for a rigid body in plane motion reduces to a single vector, and express the distance from the mass center *G* to the line of action of this vector in terms of the centroidal radius of gyration $\bar{k}$ of the body, the magnitude $\bar{v}$ of the velocity of *G*, and the angular velocity ω.

SOLUTION

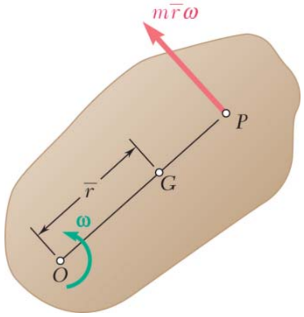


Syst. Momenta = **Single Vector**

↖ Components parallel to $m\bar{\mathbf{v}}$: $m\bar{\mathbf{v}} = \mathbf{X}$ $\qquad \mathbf{X} = m\bar{\mathbf{v}}$ ◀

↺ Moments about *G*: $\bar{I}\omega = (m\bar{v})d$

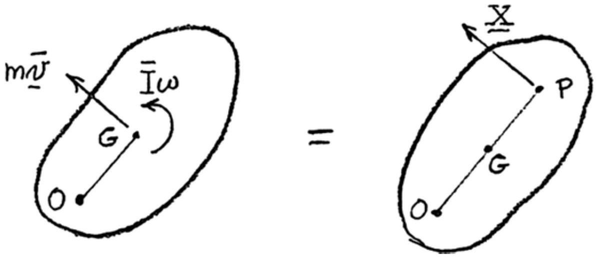
$$d = \frac{\bar{I}\omega}{m\bar{v}} = \frac{m\bar{k}^2\omega}{m\bar{v}} \qquad d = \frac{\bar{k}^2\omega}{\bar{v}} \blacktriangleleft$$



PROBLEM 17.66

Show that, when a rigid body rotates about a fixed axis through O perpendicular to the body, the system of the momenta of its particles is equivalent to a single vector of magnitude $m\bar{r}\omega$, perpendicular to the line OG, and applied to a Point P on this line, called the *center of percussion*, at a distance $GP = \bar{k}^2/\bar{r}$ from the mass center of the body.

SOLUTION



Kinematics. Point O is fixed. $\bar{v} = \bar{r}\omega$

System momenta.

Components parallel to $m\bar{\mathbf{v}}$: $X = m\bar{v} = m\bar{r}\omega$ $X = m\bar{r}\omega$ ◀

Moments about G: $(GP)X = \bar{I}\omega$

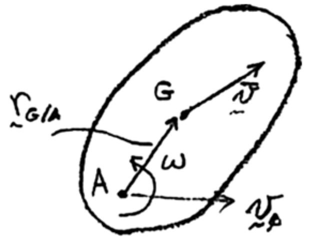
$$(GP)m\bar{r}\omega = m\bar{k}^2\omega$$ $$(GP) = \frac{\bar{k}^2}{\bar{r}}$$ ◀

PROBLEM 17.67

Show that the sum $\mathbf{H}_A$ of the moments about a Point A of the momenta of the particles of a rigid body in plane motion is equal to $I_A\omega$, where ω is the angular velocity of the body at the instant considered and I_A the moment of inertia of the body about A, if and only if one of the following conditions is satisfied: (*a*) A is the mass center of the body, (*b*) A is the instantaneous center of rotation, (*c*) the velocity of A is directed along a line joining Point A and the mass center G.

SOLUTION

Kinematics.

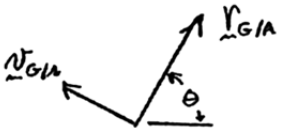


Let $\boldsymbol{\omega} = \omega\mathbf{k}$

and $\mathbf{r}_{G/A} = r_{G/A} \measuredangle\, \theta$

Then, $\mathbf{v}_{G/A} = \boldsymbol{\omega} \times \mathbf{r}_{G/A} = (\omega r_{G/A}) \measuredangle\, \beta$

where $\beta = \theta + 90°$

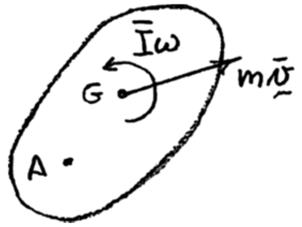


Also $\bar{\mathbf{v}} = \mathbf{v}_A + \mathbf{v}_{G/A}$

Define $\mathbf{h} = \mathbf{r}_{G/A} \times \mathbf{v}_{G/A}$

$$\mathbf{h} = (r_{G/A})(v_{G/A})\mathbf{k} = (r_{G/A})^2\,\omega\mathbf{k} = (r_{G/A})^2\,\boldsymbol{\omega}$$

System momenta. Moments about A:



$$\begin{aligned}
\mathbf{H}_A &= \mathbf{r}_{G/A} \times m\bar{\mathbf{v}} + \bar{I}\boldsymbol{\omega} \\
&= \mathbf{r}_{G/A} \times m(\mathbf{v}_A + \mathbf{v}_{G/A}) + \bar{I}\,\omega \\
&= \mathbf{r}_{G/A} \times m\mathbf{v}_A + m\mathbf{r}_{G/A} \times \mathbf{v}_{G/A} + \bar{I}\boldsymbol{\omega} \\
&= \mathbf{r}_{G/A} \times m\mathbf{v}_A + m\mathbf{h} + \bar{I}\boldsymbol{\omega} \\
&= \mathbf{r}_{G/A} \times m\mathbf{v}_A + mr_{G/A}^2\boldsymbol{\omega} + \bar{I}\boldsymbol{\omega} \\
&= \mathbf{r}_{G/A} \times m\mathbf{v}_A + (m\,r_{G/A}^2 + \bar{I})\,\boldsymbol{\omega}
\end{aligned}$$

The first term on the right hand side is equal to zero if

(*a*) $\mathbf{r}_{G/A} = 0$ (*A* is the mass center)

or (*b*) $\mathbf{v}_A = 0$ (*A* is the instantaneous center of rotation)

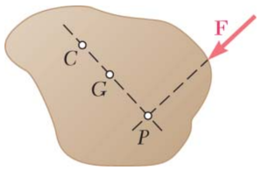
or (*c*) $\mathbf{r}_{G/A}$ is perpendicular to $\mathbf{v}_A$.

In the second term, $mr_{G/A}^2 + \bar{I} = I_A$ by the parallel axis theorem.

Thus, $\mathbf{H}_A = I_A\boldsymbol{\omega}$

when one or more of the conditions (*a*), (*b*) or (*c*) is satisfied.

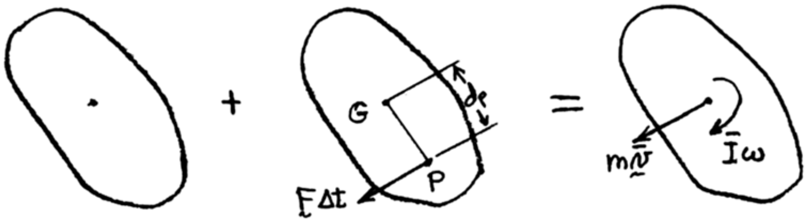
PROBLEM 17.68



Consider a rigid body initially at rest and subjected to an impulsive force **F** contained in the plane of the body. We define the *center of percussion P* as the point of intersection of the line of action of **F** with the perpendicular drawn from G. (a) Show that the instantaneous center of rotation C of the body is located on line GP at a distance $GC = \bar{k}^2/GP$ on the opposite side of G. (b) Show that if the center of percussion were located at C the instantaneous center of rotation would be located at P.

SOLUTION

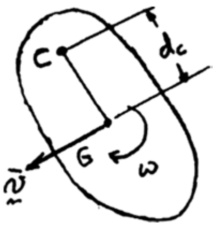
(a) Locate the instantaneous center C corresponding to center of percussion P. Let $d_P = GP$.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

Components parallel to $\mathbf{F}\Delta t$: $\quad 0 + F\Delta t = m\bar{v}$

Moments about G: $\quad 0 + d_P(F\Delta t) = \bar{I}\,\omega$



Eliminate $F\Delta t$ to obtain

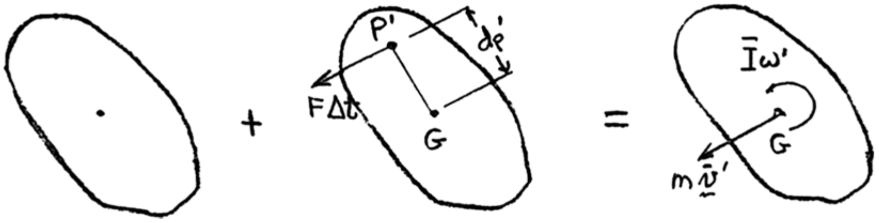
$$\frac{\bar{v}}{\omega} = \frac{\bar{I}}{md_P}$$

$$= \frac{\bar{k}^2}{d_P}$$

Kinematics. Locate Point C. $\quad GC = d_C = \dfrac{\bar{v}}{\omega} = \dfrac{\bar{k}^2}{d_P}$ $\qquad GC = \dfrac{\bar{k}^2}{GP}$ ◀

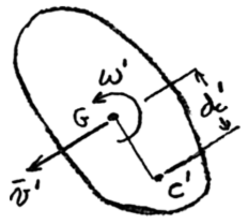
(b) Place the center of percussion at $P' = C$. Locate the corresponding instantaneous center C'. Let

$$d_{P'} = GP' = GC = d_C.$$



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

PROBLEM 17.68 (Continued)



Components parallel to $\mathbf{F}\Delta t$: $0 + F\Delta t = m\bar{v}'$

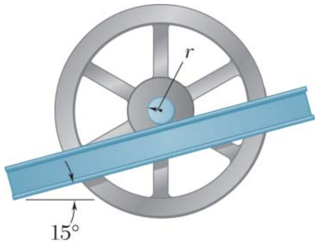
Moments about G: $0 + d_{P'}(F\Delta t) = \bar{I}\omega'$

Eliminate $F\Delta t$ to obtain $$\frac{\bar{v}'}{\omega'} = \frac{\bar{I}}{md_{P'}} = \frac{\bar{k}^2}{d_{P'}}$$

Kinematics. Locate Point C'. $$GC' = d_{C'} = \frac{\bar{v}'}{\omega'} = \frac{\bar{k}^2}{d_{P'}} = \frac{\bar{k}^2}{d_C}$$

Using $d_C = d_{P'} = \dfrac{\bar{k}^2}{d_P}$ gives $d_{C'} = d_P$ or $GC' = GP$ ◀

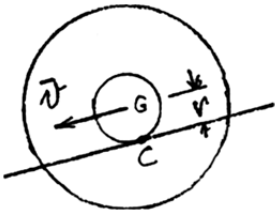
Thus Point C' coincides with Point P.



PROBLEM 17.69

A flywheel is rigidly attached to a 1.5-in.-radius shaft that rolls without sliding along parallel rails. Knowing that after being released from rest the system attains a speed of 6 in./s in 30 s, determine the centroidal radius of gyration of the system.

SOLUTION

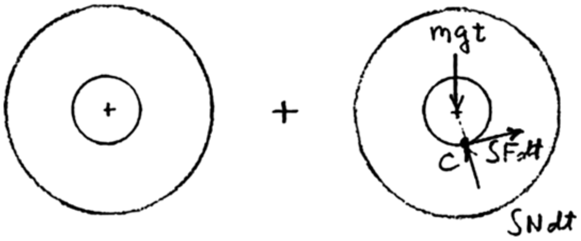


Kinematics. Rolling motion. Instantaneous center at *C*.

$$\bar{v} = v_G = r\omega$$

Moment of inertia. $\bar{I} = m\bar{k}^2$

Kinetics.



$$\textbf{Syst. Momenta}_1 \quad + \quad \textbf{Syst. Ext. Imp.}_{1\to 2} \quad = \quad \textbf{Syst. Momenta}_2$$

↺ Moments about *C*: $0 + (mgt)r\sin\beta = m\bar{v}r + \bar{I}\,\omega$

$$mgtr\sin\beta = m\left(r + \frac{\bar{k}^2}{r}\right)\bar{v}$$

Solving for $\bar{k}^2$, $\bar{k}^2 = r^2\left(\frac{gt\sin\beta}{\bar{v}} - 1\right)$

Data: $r = 1.5\text{ in.} = 0.125\text{ ft}$

$g = 32.2\text{ ft/s}$

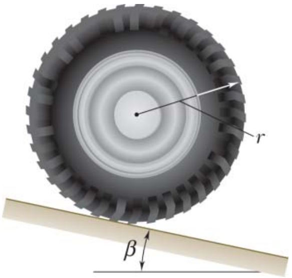
$t = 30\text{ s}$

$\bar{v} = 6\text{ in./s} = 0.5\text{ ft/s}$

$$\bar{k}^2 = (0.125)^2\left[\frac{(32.2)(30)\sin 15^\circ}{0.5} - 1\right]$$

$$= 7.7974\text{ ft}^2$$

$\bar{k} = 2.79\text{ ft}$ ◀

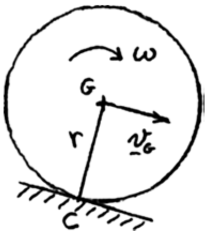


PROBLEM 17.70

A wheel of radius r and centroidal radius of gyration $\bar{k}$ is released from rest on the incline shown at time $t = 0$. Assuming that the wheel rolls without sliding, determine (*a*) the velocity of its center at time *t*, (*b*) the coefficient of static friction required to prevent slipping.

SOLUTION

Kinematics. Rolling motion. Instantaneous center at *C*.

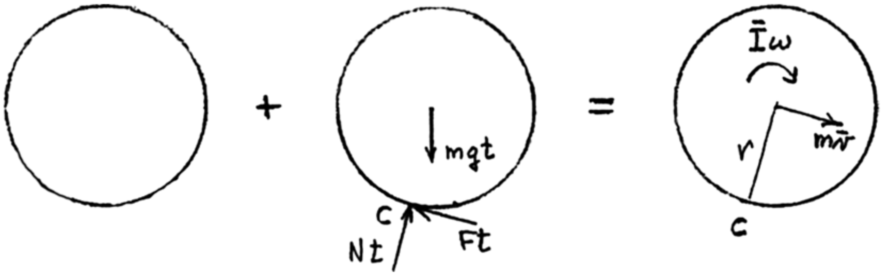


$$\bar{v} = v_G = r\omega$$

$$\omega = \frac{\bar{v}}{r}$$

Moment of inertia. $\bar{I} = m\bar{k}^2$

Kinetics.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

↷ moments about *C*: $0 + (mgt)r\sin\beta = m\bar{v}r + \bar{I}\omega$

$$(mgr\sin\beta)t = mr\bar{v} + \frac{m\bar{k}^2\bar{v}}{r}$$

(*a*) *Velocity of Point G.*
$$\bar{\mathbf{v}} = \frac{r^2 gt\sin\beta}{r^2 + \bar{k}^2} \searrow \beta \quad ◀$$

$+\searrow$ components parallel to incline:

$$0 + mgt\sin\beta - Ft = m\bar{v}$$

$$Ft = mgt\sin\beta - \frac{mr^2 gt\sin\beta}{r^2 + k^2}$$

$$= \frac{\bar{k}^2 mgt\sin\beta}{r^2 + k^2}$$

PROBLEM 17.70 (Continued)

+↗ components normal to incline:

$$0 + Nt - mgt\cos\beta = 0$$

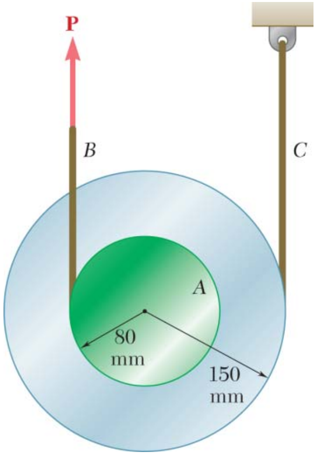
$$Nt = mgt\cos\beta$$

(*b*) *Required coefficient of static friction*.

$$\mu_s \geq \frac{F}{N}$$

$$= \frac{Ft}{Nt}$$

$$= \frac{\bar{k}^2 mgt\sin\beta}{(r^2 + \bar{k}^2)mgt\cos\beta} \qquad \mu_s \geq \frac{\bar{k}^2\tan\beta}{r^2 + \bar{k}^2} \blacktriangleleft$$



PROBLEM 17.71

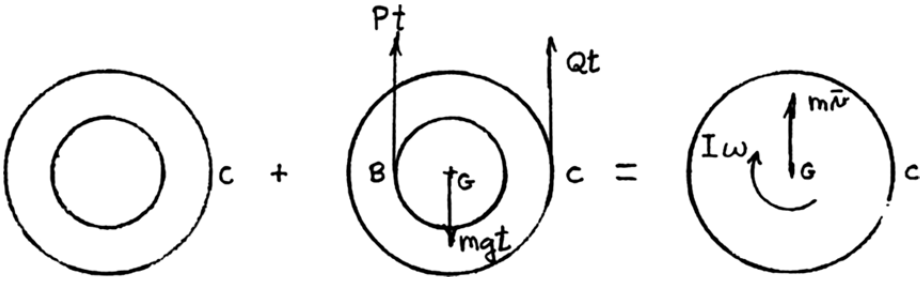
The double pulley shown has a mass of 3 kg and a radius of gyration of 100 mm. Knowing that when the pulley is at rest, a force **P** of magnitude 24 N is applied to cord *B*, determine (*a*) the velocity of the center of the pulley after 1.5 s, (*b*) the tension in cord *C*.

SOLUTION

For the double pulley,

$$r_C = 0.150 \text{ m}$$
$$r_B = 0.080 \text{ m}$$
$$k = 0.100 \text{ m}$$

Principle of impulse and momentum.



$$\textbf{Syst. Momenta}_1 \quad + \quad \textbf{Syst. Ext. Imp.}_{1\to2} \quad = \quad \textbf{Syst. Momenta}_2$$

Kinematics. Point *C* is the instantaneous center. $\bar{v} = r_C\omega$

↶ Moments about *C*: $0 + Pt(r_C + r_B) - mgtr_C = \bar{I}\omega + m\bar{v}r_C$

$$= mk^2\omega + m(r_C\omega)r_C$$

$$\omega = \frac{Pt(r_C + r_B) - mgtr_C}{m\left(k^2 + r_C^2\right)}$$

$$= \frac{(24)(1.5)(0.230) - (3)(9.81)(1.5)(0.150)}{3(0.100^2 + 0.150^2)}$$

$$= 17.0077 \text{ rad/s}$$

PROBLEM 17.71 (Continued)

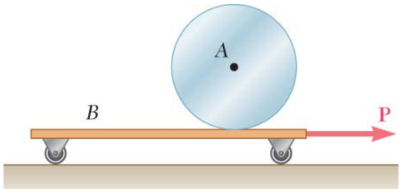
(*a*) $\bar{v} = (0.150)(17.0077) = 2.55115\ \text{m/s}$ $\qquad \bar{\mathbf{v}} = 2.55\ \text{m/s}\uparrow$ ◀

$+\uparrow$ Linear components: $0 + Pt - mgt + Qt = m\bar{v}$

$$Q = \frac{m\bar{v}}{t} + mg - P$$

$$= \frac{(3)(2.55115)}{1.5} + (3)(9.81) - 24$$

(*b*) *Tension in cord C.* $\qquad Q = 10.53\ \text{N}$ ◀



PROBLEM 17.72

A 9-in.-radius cylinder of weight 18 lb rests on a 6-lb carriage. The system is at rest when a force **P** of magnitude 2.5 lb is applied as shown for 1.2 s. Knowing that the cylinder rolls without sliding on the carriage and neglecting the mass of the wheels of the carriage, determine the resulting velocity of (*a*) the carriage, (*b*) the center of the cylinder.

SOLUTION

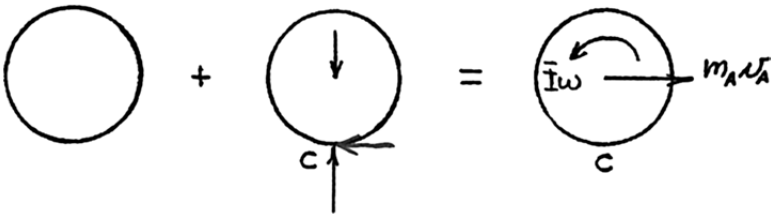
Moment of inertia.

$$\bar{I} = \frac{1}{2} m_A r^2$$

$$= \frac{1}{2}\left(\frac{18\text{ lb}}{32.2}\right)\left(\frac{9\text{ in.}}{12}\right)^2$$

$$= 0.15722\ \text{slug}\cdot\text{ft}^2$$

Cylinder alone:



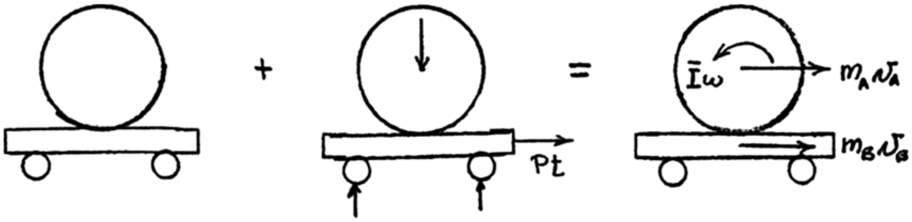
$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

↺ Moments about *C*: $0 + 0 = \bar{I}\omega - m_A v_A r$

or

$$0 = 0.15722\omega - \left(\frac{18}{32.2}\right)\left(\frac{9}{12}\right)v_A \qquad (1)$$

Cylinder and carriage:



$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

$\xrightarrow{+}$ Horizontal components: $0 + Pt = m_A v_A + m_B v_B$

or

$$0 + (2.5)(1.2) = \left(\frac{18}{32.2}\right)v_A + \left(\frac{6}{32.2}\right)v_B \qquad (2)$$

PROBLEM 17.72 (Continued)

Kinematics.

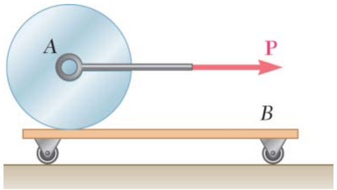
$$v_A = v_B - r\omega$$

$$v_A = v_B - \left(\frac{9}{12}\right)\omega \qquad (3)$$

Solving Equations (1), (2) and (3) simultaneously gives $\boldsymbol{\omega} = 7.16 \text{ rad/s}$ ↺

(*a*) Velocity of the carriage. $\mathbf{v}_B = 8.05 \text{ ft/s} \longrightarrow$ ◀

(*b*) Velocity of the center of the cylinder. $\mathbf{v}_A = 2.68 \text{ ft/s} \longrightarrow$ ◀



PROBLEM 17.73

A 9-in.-radius cylinder of weight 18 lb rests on a 6-lb carriage. The system is at rest when a force **P** of magnitude 2.5 lb is applied as shown for 1.2 s. Knowing that the cylinder rolls without sliding on the carriage and neglecting the mass of the wheels of the carriage, determine the resulting velocity of (*a*) the carriage, (*b*) the center of the cylinder.

SOLUTION

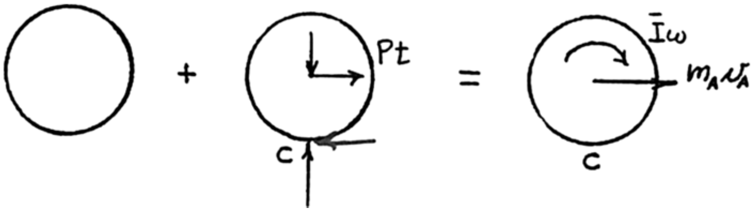
Moment of inertia.

$$\bar{I} = \frac{1}{2}m_A r^2$$

$$= \frac{1}{2}\left(\frac{18\text{ lb}}{32.2}\right)\left(\frac{9\text{ in.}}{12}\right)^2$$

$$= 0.15722\text{ slug}\cdot\text{ft}^2$$

Cylinder alone:



$$\textbf{Syst. Momenta}_1 \quad + \quad \textbf{Syst. Ext. Imp.}_{1\to 2} \quad = \quad \textbf{Syst. Momenta}_2$$

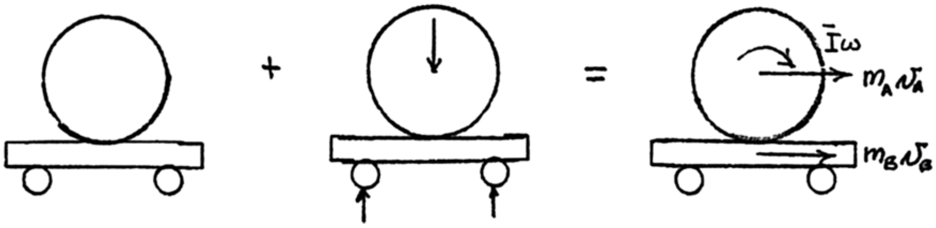
↻ Moments about *C*:

$$0 + Ptr = \bar{I}\omega + m_A v_A r$$

or

$$0 + (2.5)(1.2)\left(\frac{9}{12}\right) = 0.15722\omega + \left(\frac{18}{32.2}\right)\left(\frac{9}{12}\right)v_A \qquad (1)$$

Cylinder and carriage:



$$\textbf{Syst. Momenta}_1 \quad + \quad \textbf{Syst. Ext. Imp.}_{1\to 2} \quad = \quad \textbf{Syst. Momenta}_2$$

$\xrightarrow{+}$ Horizontal components:

$$0 + Pt = m_A v_A + m_B v_B$$

or

$$0 + (2.5)(1.2) = \left(\frac{18}{32.2}\right)v_A + \left(\frac{6}{32.2}\right)v_B \qquad (2)$$

PROBLEM 17.73 (Continued)

Kinematics.

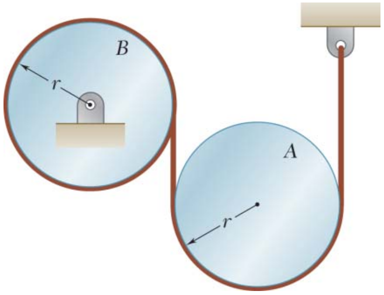
$$v_A = v_B + r\omega$$

$$v_A = v_B + \left(\frac{9}{12}\right)\omega \qquad (3)$$

Solving Equations (1), (2) and (3) simultaneously gives $\boldsymbol{\omega} = 2.39$ rad/s ↻

(*a*) Velocity of the carriage. $\mathbf{v}_B = 2.68$ ft/s → ◀

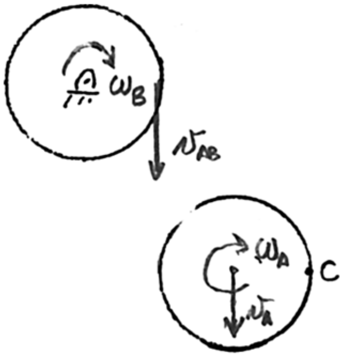
(*b*) Velocity of the center of the cylinder. $\mathbf{v}_A = 4.47$ ft/s → ◀



PROBLEM 17.74

Two uniform cylinders, each of mass $m = 6$ kg and radius $r = 125$ mm, are connected by a belt as shown. If the system is released from rest when $t = 0$, determine (*a*) the velocity of the center of cylinder *B* at $t = 3\text{ s}$, (*b*) the tension in the portion of belt connecting the two cylinders.

SOLUTION



Kinematics. $$v_{AB} = r\omega_B$$

Point *C* is the instantaneous center of cylinder *A*.

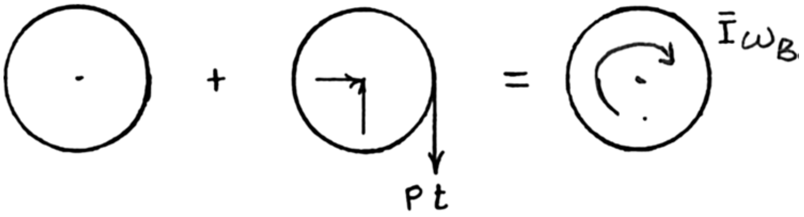
$$\omega_A = \frac{v_{AB}}{2r} = \frac{1}{2}\omega_B$$

$$\overline{v}_A = r\omega_A = \frac{1}{2}r\omega_B$$

Moment of inertia. $$\overline{I} = \frac{1}{2}mr^2$$

(*a*) Velocity of the center of *A*.

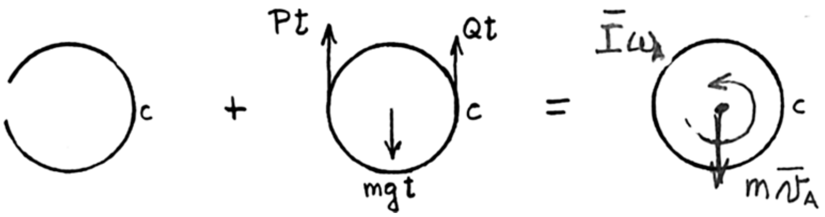
Cylinder B:



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

+↻ Moments about *B*: $$0 + Ptr = \overline{I}\omega_B \qquad (1)$$

Cylinder. A:



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

PROBLEM 17.74 (Continued)

↺ Moments about *C*: $$0-2Ptr+mgtr=m\bar{v}_A r+\bar{I}\omega_A$$

$$0-2\bar{I}\,\omega_B+mgtr=m\left(\frac{1}{2}r\omega_B\right)r+\bar{I}\left(\frac{1}{2}\omega_B\right)$$

$$\left(\frac{5}{2}\bar{I}+\frac{1}{2}mr^2\right)\omega_B=mgrt$$

$$\left(\frac{5}{2}\frac{mr^2}{2}+\frac{1}{2}mr^2\right)\omega_B=mgrt$$

$$\frac{7}{4}r\omega_B=gt$$

$$\omega_B=\frac{4}{7}\frac{gt}{r} \qquad (2)$$

$$\bar{v}_A=\frac{1}{2}r\omega_B=\frac{2}{7}gt=\frac{2}{7}(9.81)(3)$$

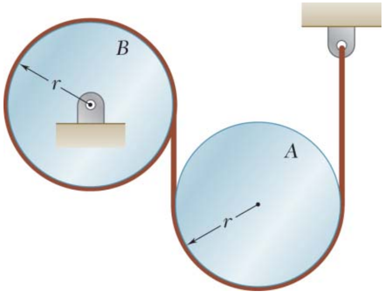
$\bar{\mathbf{v}}_A=8.41\text{ m/s}$ ↓ ◀

(*b*) Tension in the belt.

From Eqs. (1) and (2), $$Ptr=\bar{I}\left(\frac{4}{7}\frac{gt}{r}\right)$$

$$P=\frac{1}{tr}\left(\frac{1}{2}mr^2\right)\left(\frac{4}{7}\frac{gt}{r}\right)=\frac{2}{7}mg=\frac{2}{7}(6)(9.81)=16.817\text{ N}$$

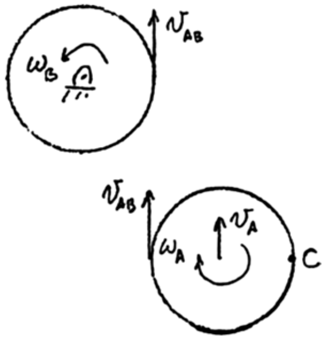
$P=16.82\text{ N}$ ◀



PROBLEM 17.75

Two uniform cylinders, each of mass $m = 6$ kg and radius $r = 125$ mm, are connected by a belt as shown. Knowing that at the instant shown the angular velocity of cylinder *A* is 30 rad/s counterclockwise, determine (*a*) the time required for the angular velocity of cylinder *A* to be reduced to 5 rad/s, (*b*) the tension in the portion of belt connecting the two cylinders.

SOLUTION


ω_B
v_{AB}
v_{AB}
ω_A
v_A
C

Kinematics. $v_{AB} = r\omega_B$

Point *C* is the instantaneous center of cylinder *A*.

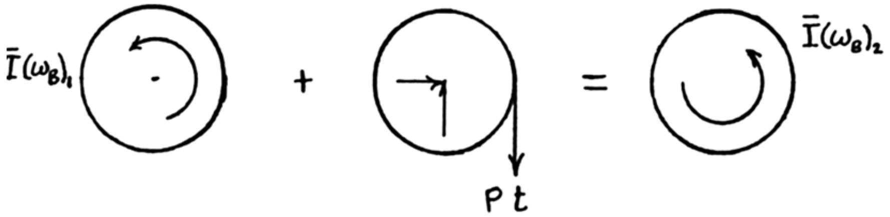
$$\omega_A = \frac{v_{AB}}{2r} = \frac{1}{2}\omega_B$$

$$\bar{v}_A = r\omega_A = \frac{1}{2}r\omega_B$$

Moment of inertia. $\bar{I} = \frac{1}{2}\frac{W}{g}r^2$

(*a*) *Required time.*

Cylinder B:


$\bar{I}(\omega_B)_1$
+
Pt
=
$\bar{I}(\omega_B)_2$

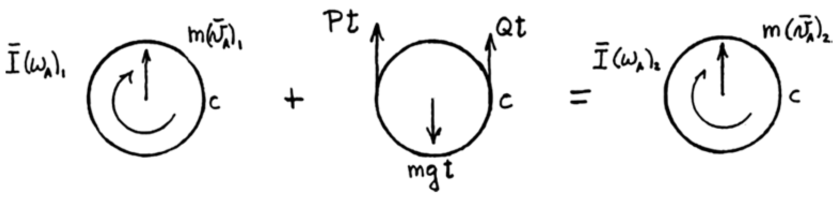
Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

↺ Moments about *B*: $\bar{I}(\omega_B)_1 - Ptr = \bar{I}(\omega_B)_2$

$$Ptr = \bar{I}[(\omega_B)_1 - (\omega_B)_2]$$

$$= \frac{1}{2}mr^2[(\omega_B)_1 - (\omega_B)_2] \qquad (1)$$

Cylinder A:


$\bar{I}(\omega_A)_1$
$m(\bar{v}_A)_1$
C
+
Pt
Qt
C
mg t
=
$\bar{I}(\omega_A)_2$
$m(\bar{v}_A)_2$
C

Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

PROBLEM 17.75 (Continued)

↺ Moments about C: $$\bar{I}(\omega_A)_1 + m(v_A)_1 r + 2Ptr - mgtr = \bar{I}(\omega_A)_2 + m(v_A)_2 r$$

$$\frac{1}{2}mr^2[(\omega_A)_1 - (\omega_A)_2 + mr[(\omega_A)_1 - (\omega_A)_2]r + 2Ptr - mgtr = 0$$

$$\frac{3}{2}mr^2\left[\left(\frac{1}{2}\omega_B\right)_1 - \frac{1}{2}(\omega_B)_2\right] + 2\left\{\frac{1}{2}mr^2[(\omega_B)_1 - (\omega_B)_2]\right\} - mgtr = 0$$

$$\frac{7}{4}mr^2[(\omega_B)_1 - (\omega_B)_2] - mgtr = 0$$

$$t = \frac{7r[(\omega_B)_1 - (\omega_B)_2]}{4g} \qquad (2)$$

Data: $m = 6\text{ kg}$

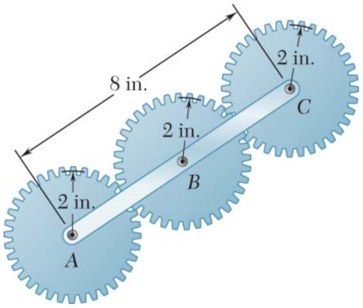
$r = 125\text{ mm} = 0.125\text{ m}$

From Equation (2), $$t = \frac{(7)(0.125)(30-5)}{(4)(9.81)} = 0.55747$$ $t = 0.557\text{ s}$ ◀

(*b*) *Tension in belt between cylinders.*

From Equation (1), $$Ptr = \frac{1}{2}(6)(0.125)^2(30-5)$$ $$= 1.172\text{ N}\cdot\text{m}\cdot\text{s}$$

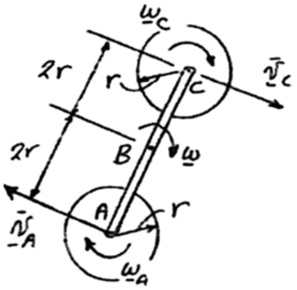
$$P = \frac{Ptr}{tr} = \frac{1.172}{(0.55747)(0.125)} = 16.817$$ $P = 16.82\text{ N}$ ◀


8 in.
2 in.
C
2 in.
B
2 in.
A

PROBLEM 17.76

In the gear arrangement shown, gears *A* and *C* are attached to rod *ABC*, which is free to rotate about *B*, while the inner gear *B* is fixed. Knowing that the system is at rest, determine the magnitude of the couple **M** which must be applied to rod *ABC*, if 2.5 s later the angular velocity of the rod is to be 240 rpm clockwise. Gears *A* and *C* weigh 2.5 lb each and may be considered as disks of radius 2 in.; rod *ABC* weighs 4 lb.

SOLUTION


ω_C
$\bar{v}_C$
2r
r
C
2r
B
ω
A
r
$\bar{v}_A$
ω_A

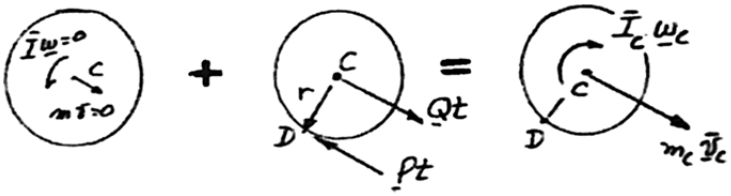
Kinematics of motion

Let $\omega_{ABC} = \omega$ $\qquad \bar{v}_A = \bar{v}_C = (BC)\omega = 2r\omega$

Since gears *A* and *C* roll on the fixed gear *B*,

$$\omega_A = \omega_C = \frac{v_C}{r} = \frac{2r\omega}{r} = 2\omega$$

Principle of impulse and momentum.

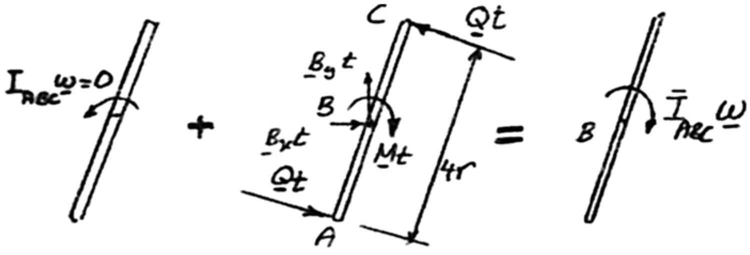

$\bar{I}\omega=0$
C
$m\bar{v}=0$
+
C
r
D
Qt
Pt
=
$\bar{I}_C\omega_C$
C
D
$m_C\bar{v}_C$

$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

+↻ Moments about *D*: $\qquad 0 + (Qt)r = m_C\bar{v}_C r + \bar{I}_C w_C$

$$(Qt)r = m_C(2r\omega)r + \frac{1}{2}m_C r^2(2\omega)$$

$$Qt = 3m_C r\omega \tag{1}$$


$I_{ABC}\omega=0$
+
C
Qt
$B_y t$
B
$B_x t$
Mt
4r
Qt
A
=
B
$\bar{I}_{ABC}\omega$

$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

PROBLEM 17.76 (Continued)

$+\circlearrowright$ Moments about *B*:

$$Mt - Qt(4r) = \overline{I}_{ABC}\omega$$

$$Mt - 4(Qt)r = \frac{1}{12}m_{ABC}(4r)^2\omega$$

$$Mt - 4(Qt)r = \frac{4}{3}m_{ABC}r^2\omega \qquad (2)$$

Substitute for (*Qt*) from (1) into (2):

$$Mt - 4(3m_C r\omega)r = \frac{4}{3}m_{ABC}r^2\omega$$

$$Mt = \frac{4}{3}r^2\omega(m_{ABC} + 9m_C) \qquad (3)$$

Couple *M*.

Data:

$$t = 2.5\text{ s}$$

$$r = \frac{2}{12}\text{ ft}$$

$$m_{ABC} = \frac{4\text{ lb}}{32.2\text{ ft/s}}$$

$$m_C = \frac{2.5\text{ lb}}{32.2\text{ ft/s}^2}$$

$$\omega = 240\text{ rpm}$$
$$= 8\pi\text{ rad/s}$$

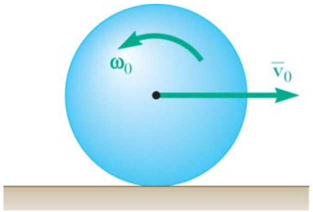
Eq. (3):

$$M(2.5\text{ s}) = \frac{4}{3}\left(\frac{2}{12}\text{ ft}\right)^2(8\pi\text{ rad/s})\left[\frac{4}{32.2} + 9\left(\frac{2.5}{32.2}\right)\right]$$

$$2.5\text{ M} = 0.76607$$

$$M = 0.3064\text{ lb}\cdot\text{ft}$$

$\mathbf{M} = 0.306\text{ lb}\cdot\text{ft}\circlearrowright$ ◀

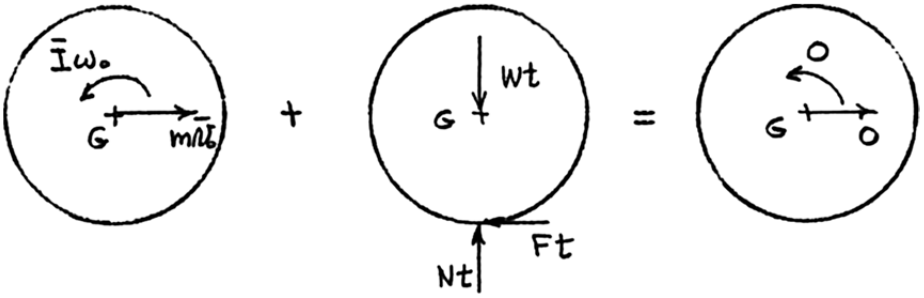


PROBLEM 17.77

A sphere of radius r and mass m is projected along a rough horizontal surface with the initial velocities shown. If the final velocity of the sphere is to be zero, express (a) the required magnitude of $\boldsymbol{\omega}_0$ in terms of v_0 and r, (b) the time required for the sphere to come to rest in terms of v_0 and coefficient of kinetic friction μ_k.

SOLUTION

Moment of inertia. Solid sphere. $\overline{I} = \frac{2}{5}mr^2$



$$\textbf{Syst. Momenta}_1 \quad + \quad \textbf{Syst. Ext. Imp.}_{1\to 2} \quad = \quad \textbf{Syst. Momenta}_2$$

$+\uparrow y$ components: $Nt - Wt = 0 \qquad N = W = mg$ (1)

$\xrightarrow{+}$ x components: $m\overline{v}_0 - Ft = 0 \qquad Ft = m\overline{v}_0$ (2)

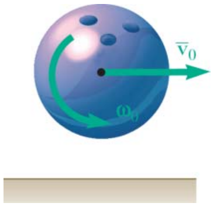
$+\circlearrowright$ Moments about G: $\overline{I}\omega_0 - Ftr = 0$ (3)

$$\frac{2}{5}mr^2\omega_0 - m\overline{v}_0 r = 0$$

(a) *Solving for* ω_0, $\omega_0 = \frac{5}{2}\frac{\overline{v}_0}{r}$ ◀

(b) *Time to come to rest.*

From Equation (2), $t = \frac{mv_0}{F} = \frac{m\overline{v}_0}{\mu_k mg}$ $\qquad t = \frac{\overline{v}_0}{\mu_k g}$ ◀



PROBLEM 17.78

A bowler projects an 8.5-in.-diameter ball weighing 16 lb along an alley with a forward velocity $\mathbf{v}_0$ of 25 ft/s and a backspin ω_0 of 9 rad/s. Knowing that the coefficient of kinetic friction between the ball and the alley is 0.10, determine (*a*) the time t_1 at which the ball will start rolling without sliding, (*b*) the speed of the ball at time t_1.

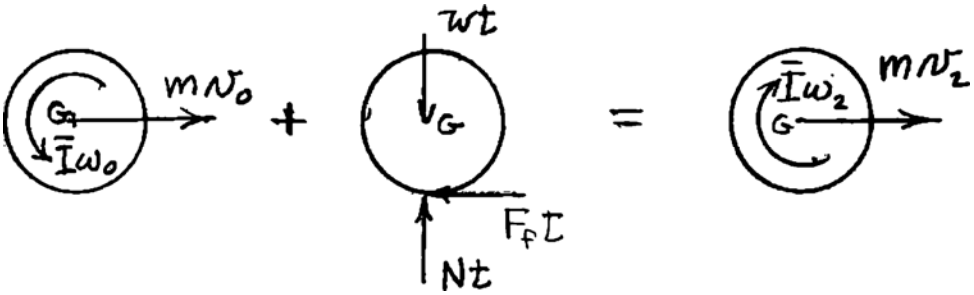
SOLUTION

Radius: $r = \frac{1}{2}d = \frac{1}{2}(8.5 \text{ in.}) = 4.25 \text{ in.} = 0.35417 \text{ ft}$

Mass: $m = \frac{W}{g} = \frac{16 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.49689 \text{ lb}\cdot\text{s}^2/\text{ft}$

Moment of inertia: $\bar{I} = \frac{2}{5}mr^2 = \left(\frac{2}{5}\right)(0.49689)(0.35417)^2 = 0.02493 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$

Use the principle of impulse and momentum.



$$\textbf{Syst. Momenta}_1 \quad + \quad \textbf{Syst. Ext. Imp.}_{1\to 2} \quad = \quad \textbf{Syst. Momenta}_2$$

$+\uparrow$: $Nt - Wt = 0 \qquad N = W = 16 \text{ lb}$

Friction force: $F_F = \mu_k N = (0.10)(16) = 1.6 \text{ lb.}$

$\xrightarrow{+}$: $mv_0 - F_F t = mv_2$

$$v_2 = v_0 - \frac{F_F t}{m} = 25 \text{ ft/s} - \frac{1.6t}{0.49689}$$
$$= 25 - 3.22t$$

$+\circlearrowright$ Moments about G: $\bar{I}\omega_0 - F_F t r = -\bar{I}\omega_2$

$$\omega_2 = \frac{F_F t r}{\bar{I}} - \omega_0 = \frac{(1.6t)(0.35417)}{0.02493} - 9$$
$$= 22.731t - 9$$

Slipping stops when $v_2 = r\omega_2$

PROBLEM 17.78 (Continued)

(*a*) Time t when slipping stops.

$$(25-3.22t)=(0.35417\text{ ft})(22.731t-9)$$

$$(25+3.1875)=(3.22+8.0506)t$$

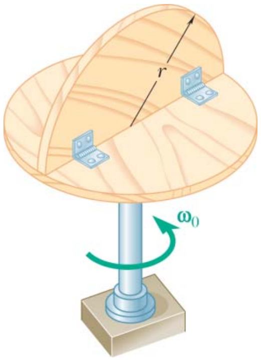
$$t=2.501\text{ s}$$

$t=2.50\text{ s}$ ◀

(*b*) Corresponding velocity.

$$v_2=25-3.22t$$

$v_2=16.95\text{ ft/s}$ ◀



PROBLEM 17.79

A semicircular panel of radius *r* is attached with hinges to a circular plate of radius *r* and initially held in the vertical position as shown. The plate and the panel are made of the same material and have the same thickness. Knowing that the entire assembly is rotating freely with an initial angular velocity $\boldsymbol{\omega}_0$, determine the angular velocity of the assembly after the panel has been released and comes to rest against the plate.

SOLUTION

Let *m* be the mass of the plate. Then, the mass of the panel is $\frac{1}{2}m$.

Moment of inertia of plate plus panel:

(1) Panel is vertical.

$$I_1 = \frac{1}{2}mr^2 + \frac{1}{4}\left(\frac{1}{2}m\right)r^2 = \frac{5}{8}mr^2$$

(2) Panel is horizontal.

$$I_2 = \frac{1}{2}mr^2 + \frac{1}{2}\left(\frac{1}{2}m\right)r^2 = \frac{3}{4}mr^2$$

Angular momentum about vertical axis.

(1) Panel is vertical.

$$H_1 = I_1\omega_0 = \frac{5}{8}mr^2\omega_0$$

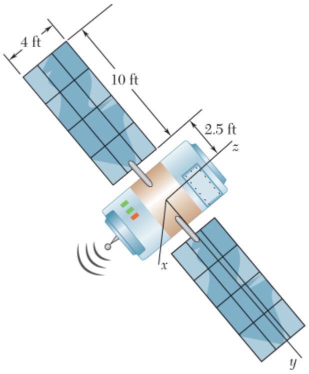
(2) Panel is horizontal.

$$H_2 = I_2\omega_2 = \frac{3}{4}mr^2\omega_2$$

Conservation of angular momentum. $H_1 = H_2:$

$$\frac{5}{8}mr^2\omega_0 = \frac{3}{4}mr^2\omega_2$$

$$\omega_2 = \frac{5}{6}\omega_0 \blacktriangleleft$$



PROBLEM 17.80

A satellite has a total weight (on Earth) of 250 lbs, and each of the solar panels weighs 15 lbs. The body of the satellite has mass moment of inertia about the *z*-axis of 6 slug-ft^2, and the panels can be modeled as flat plates. The satellite spins with a rate of 10 rpm about the *z*-axis when the solar panels are positioned in the *xy* plane. Determine the spin rate about *z* after a motor on the satellite has rotated both panels to be positioned in the *yz* plane (as shown in the figure).

SOLUTION

Given:

$$m_s = \frac{250}{32.2} \text{ slugs, } I_s = 6 \text{ slug•ft}^2$$

$$m_p = \frac{15}{32.2} \text{ slugs}$$

$$\omega_1 = 10 \ \frac{\text{rev}}{\text{min}} \times \frac{2\pi}{60}$$

$$= \frac{\pi}{3} \text{ rad/s}$$

Conservation of Momentum about the z-axis:

$$H_{O,1} = H_{O,2}$$

$$I_{O,1}\omega_1 = I_{O,2}\omega_2$$

$$\omega_2 = \frac{I_{O,1}}{I_{O,2}}\omega_1$$

Mass Moment of Inertia about the z-axis with panels in the first position:

$$I_{O,1} = I_s + 2I_{p,1}$$

$$= 6 + 2\left(\bar{I}_{p,1} + m_p d_p^2\right)$$

$$= 6 + 2\left(\frac{1}{12}m_p\left(4^2 + 10^2\right) + m_p\left(7.5\right)^2\right)$$

$$= 67.413 \text{ slug•ft}^2$$

Mass Moment of Inertia about the z-axis with panels in the second position:

$$I_{O,2} = I_s + 2I_{p,2}$$

$$= 6 + 2\left(\bar{I}_{p,2} + m_p d_p^2\right)$$

$$= 6 + 2\left(\frac{1}{12}m_p 10^2 + m_p\left(7.5\right)^2\right)$$

$$= 66.171 \text{ slug•ft}^2$$

Substitute into Momentum Equation:

$\omega_2 = 10.19$ rpm ◀

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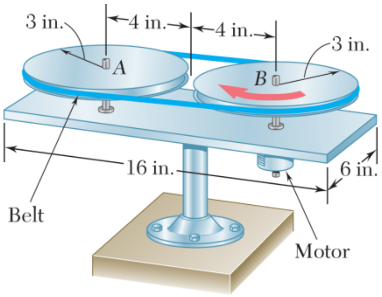


PROBLEM 17.81

Two 10-lb disks and a small motor are mounted on a 15-lb rectangular platform which is free to rotate about a central vertical spindle. The normal operating speed of the motor is 180 rpm. If the motor is started when the system is at rest, determine the angular velocity of all elements of the system after the motor has attained its normal operating speed. Neglect the mass of the motor and of the belt.

SOLUTION

Kinematics. Motor speed: $\omega_M = 180 \text{ rpm} = 6\pi \text{ rad/s}$

Let ω_A, ω_B, and ω_P be the angular velocities, respectfully, of disk *A*, disk *B* and the platform. Since the motor speed is the angular velocity of disk *B* relative to the platform,

$$\omega_B = \omega_P + \omega_M = \omega_P + 6\pi \qquad (1)$$

Since, the disks have the same outer radius, $\omega_B = \omega_A$ (2)

Velocity of the center of disk *A* $v_A = \dfrac{4}{12}\omega_P$ (3)

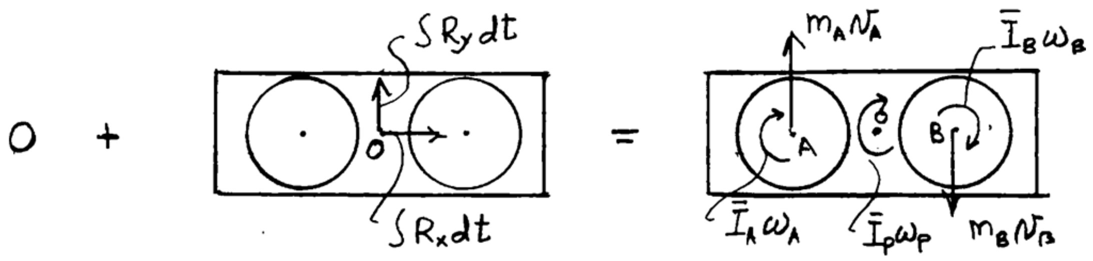
Velocity of the center of disk *B* $v_B = \dfrac{4}{12}\omega_P$ (4)

Moments of inertia.

Disks *A* and *B*: $\bar{I}_A = \bar{I}_B = \dfrac{1}{2}\dfrac{W}{g}r^2 = \dfrac{1}{2}\left(\dfrac{10}{32.2}\right)\left(\dfrac{3}{12}\right)^2 = 9.705 \times 10^{-3} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$

Platform: $\bar{I}_P = \dfrac{1}{12}\dfrac{W}{g}(a^2 + b^2) = \dfrac{1}{12}\left(\dfrac{15}{32.2}\right)\left[\left(\dfrac{16}{12}\right)^2 + \left(\dfrac{6}{12}\right)^2\right] = 78.718 \times 10^{-3} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$

Principle of impulse and momentum for system.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

PROBLEM 17.81 (Continued)

$+\circlearrowright$ Moments about O:

$$0 + 0 = \bar{I}_P\omega_P + m_A v_A l_{OA} + \bar{I}_A\omega_A + m_B v_B l_{OB} + \bar{I}_B\omega_B$$

$$= (78.718\times10^{-3})\omega_P + \left(\frac{10}{32.2}\right)\left(\frac{4}{12}\omega_P\right)\left(\frac{4}{12}\right) + (9.705\times10^{-3})(\omega_P + 6\pi) + \left(\frac{10}{32.2}\right)\left(\frac{4}{12}\omega_P\right)\left(\frac{4}{12}\right)$$

$$+ (9.705\times10^{-3})(\omega_P + 6\pi)$$

$$= 167.141\times10^{-3}\omega_P + 365.87\times10^{-3}$$

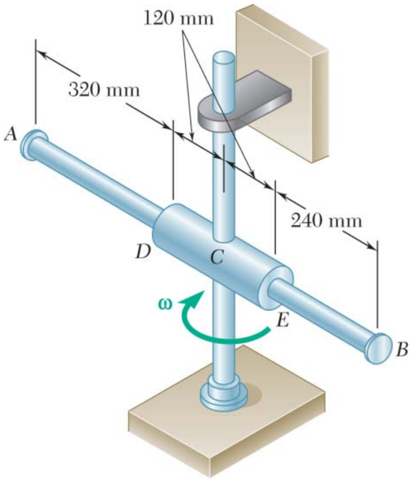
$$\omega_P = -2.189 \text{ rad/s}$$

$$\omega_A = \omega_B = -2.189 + 6\pi = 16.66 \text{ rad/s}$$

Angular velocities. $\omega_A = 159.1 \text{ rpm} \circlearrowright$ ◀

$\omega_B = 159.1 \text{ rpm} \circlearrowright$ ◀

$\omega_P = 20.9 \text{ rpm} \circlearrowleft$ ◀

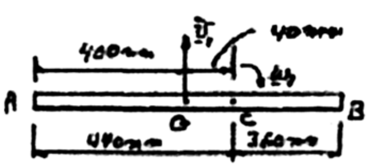

120 mm
320 mm
A
240 mm
D
C
ω
E
B

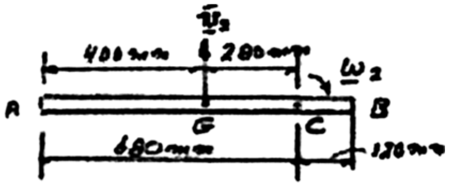
PROBLEM 17.82

A 3-kg rod of length 800 mm can slide freely in the 240-mm cylinder *DE*, which in turn can rotate freely in a horizontal plane. In the position shown the assembly is rotating with an angular velocity of magnitude $\omega = 40$ rad/s and end *B* of the rod is moving toward the cylinder at a speed of 75 mm/s relative to the cylinder. Knowing that the centroidal mass moment of inertia of the cylinder about a vertical axis is $0.025\ \text{kg}\cdot\text{m}^2$ and neglecting the effect of friction, determine the angular velocity of the assembly as end *B* of the rod strikes end *E* of the cylinder.

SOLUTION

Kinematics and geometry.


400 mm
$\bar{v}_1$
40 mm
ω_1
A
G
C
B
440 mm
360 mm


$\bar{v}_2$
400 mm
280 mm
ω_2
A
G
C
B
680 mm
120 mm

$$\bar{v}_1 = (0.04\ \text{m})\omega_1 = (0.4\ \text{m})(40\ \text{rad/s})$$

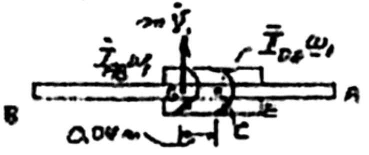
$$\bar{v}_1 = 1.6\ \text{m/s}$$

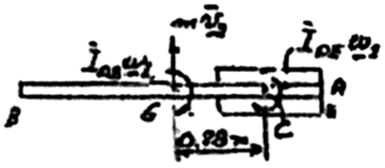
$$\bar{v}_2 = (0.28\ \text{m})\omega_2$$

Initial position

Final position

Conservation of angular momentum about *C*.


$m\bar{v}_1$
$\bar{I}_{AB}\omega_1$
$\bar{I}_{DE}\omega_1$
B
G
E
A
0.04 m
C


$m\bar{v}_2$
$\bar{I}_{AB}\omega_2$
$\bar{I}_{DE}\omega_2$
B
G
E
A
0.28 m
C

$+\circlearrowleft$ Moments about *C*:

$$\bar{I}_{AB} = \frac{1}{12}(3\ \text{kg})(0.8\ \text{m})^2 = 0.16\ \text{kg}\cdot\text{m}^2$$

$$\bar{I}_{AB}\omega_1 + m\bar{v}_1(0.04\ \text{m}) + \bar{I}_{DE}\omega_1 = \bar{I}_{AB}\omega_2 + m\bar{v}_2(0.028\ \text{m}) + \bar{I}_{DE}\omega_2$$

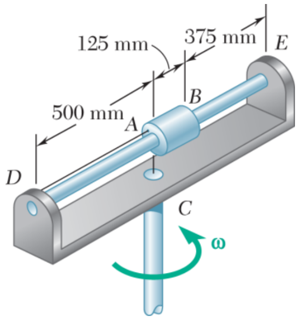
$$(0.16\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s}) + (3\ \text{kg})(1.6\ \text{m/s})(0.04\ \text{m}) + (0.025\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s})$$

$$= (0.16\ \text{kg}\cdot\text{m}^2)\omega_2 + (3\ \text{kg})(0.28\omega_2)(0.28) + (0.025\ \text{kg}\cdot\text{m}^2)\omega_2$$

$$(6.4 + 0.192 + 1.00) = (0.16 + 0.2352 + 0.025)\omega_2$$

$$7.592 = 0.4202\omega_2; \quad \omega_2 = 18.068\ \text{rad/s}$$

Angular velocity. $\omega_2 = 18.07\ \text{rad/s}$ ◀


125 mm
375 mm
E
500 mm
B
A
D
C
ω

PROBLEM 17.83

A 1.6-kg tube *AB* can slide freely on rod *DE*, which in turn can rotate freely in a horizontal plane. Initially the assembly is rotating with an angular velocity $\omega = 5$ rad/s and the tube is held in position by a cord. The moment of inertia of the rod and bracket about the vertical axis of rotation is $0.30 \text{ kg}\cdot\text{m}^2$ and the centroidal moment of inertia of the tube about a vertical axis is $0.0025 \text{ kg}\cdot\text{m}^2$. If the cord suddenly breaks, determine (*a*) the angular velocity of the assembly after the tube has moved to end *E*, (*b*) the energy lost during the plastic impact at *E*.

SOLUTION

Let Point *C* be the intersection of axle *C* and rod *DE*. Let Point *G* be the mass center of tube *AB*.

Masses and moments of inertia about vertical axes.

$$m_{AB} = 1.6 \text{ kg}$$
$$\bar{I}_{AB} = 0.0025 \text{ kg}\cdot\text{m}^2$$
$$\bar{I}_{DCE} = 0.30 \text{ kg}\cdot\text{m}^2$$

State 1.

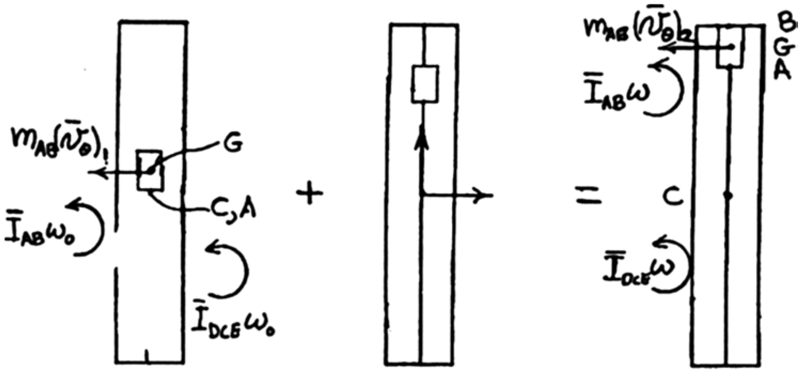
$$(r_{G/A})_1 = \frac{1}{2}(125) = 62.5 \text{ mm}$$
$$\omega_1 = 5 \text{ rad/s}$$

State 2.

$$(r_{G/A})_2 = 500 - 62.5 = 437.5 \text{ mm}$$
$$\omega = \omega_2$$

Kinematics.

$$(v_G)_\theta = \bar{v}_\theta = r_{G/C}\,\omega$$


$m_{AB}(\bar{v}_\theta)_1$
G
C, A
$\bar{I}_{AB}\omega_0$
$\bar{I}_{DCE}\omega_0$
+
=
$m_{AB}(\bar{v}_\theta)_2$
B
G
A
$\bar{I}_{AB}\omega$
C
$\bar{I}_{DCE}\omega$

Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

PROBLEM 17.83 (Continued)

Moments about C:

$$\bar{I}_{AB}\omega_1 + \bar{I}_{DCE}\omega_1 + m_{AB}(\bar{v}_\theta)_1(r_{G/C})_1 + 0 = \bar{I}_{AB}\omega_2 + \bar{I}_{DCE}\omega_2 + m_{AB}(\bar{v}_\theta)_2(r_{G/C})_2$$

$$\left[\bar{I}_{AB} + \bar{I}_{DCE} + m_{AB}(r_{G/G})_1^2\right]\omega_1 = \left[\bar{I}_{AB} + \bar{I}_{DCE} + m_{AB}(r_{G/C})_2^2\right]\omega_2$$

$$[0.0025 + 0.30 + (1.6)(0.0625)^2](5) = [0.0025 + 0.30 + (1.6)(0.4375)^2]\omega_2$$

$$(0.30875)(5) = 0.60875\omega_2$$

$$\omega_2 = 2.5359 \text{ rad/s}$$

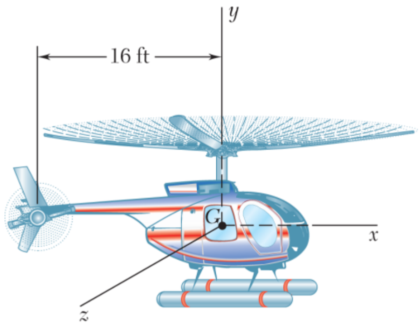
(a) *Angular velocity after the plastic impact.* 2.54 rad/s ◀

Kinetic energy. $T = \frac{1}{2}\bar{I}_{AB}\omega^2 + \frac{1}{2}\bar{I}_{DCE}\omega^2 + \frac{1}{2}m_{AB}\bar{v}^2$

$$T_1 = \frac{1}{2}(0.0025)(5)^2 + \frac{1}{2}(0.30)(5)^2 + \frac{1}{2}(1.6)(0.0625)^2(5)^2$$
$$= 3.859375 \text{ J}$$

$$T_2 = \frac{1}{2}(0.0025)(2.5359)^2 + \frac{1}{2}(0.30)(2.5359)^2 + \frac{1}{2}(1.6)(0.4375)^2(2.5359)^2$$
$$= 1.9573 \text{ J}$$

(b) *Energy lost.* $T_1 - T_2 = 1.902 \text{ J}$ ◀



PROBLEM 17.84

In the helicopter shown, a vertical tail propeller is used to prevent rotation of the cab as the speed of the main blades is changed. Assuming that the tail propeller is not operating, determine the final angular velocity of the cab after the speed of the main blades has been changed from 180 to 240 rpm. (The speed of the main blades is measured relative to the cab, and the cab has a centroidal moment of inertia of $650 \text{ lb}\cdot\text{ft}\cdot\text{s}^2$. Each of the four main blades is assumed to be a slender 14-ft rod weighting 55 lb.)

SOLUTION

Let $\mathbf{\Omega}$ be the angular velocity of the cab and ω be the angular velocity of the blades relative to the cab. The absolute angular velocity of the blades is $\Omega + \omega$.

$$\omega_1 = 180 \text{ rpm} = 6\pi \text{ rad/s}$$
$$\omega_2 = 240 \text{ rpm} = 8\pi \text{ rad/s}$$

Moments of inertia.

Cab: $$I_C = 650 \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

Blades: $$I_B = 4\left(\frac{1}{3}mL^2\right)$$
$$= (4)\left(\frac{1}{3}\right)\left(\frac{55}{32.2}\right)(14)^2$$
$$= 446.38 \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

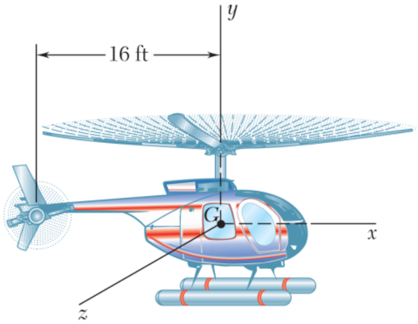
Assume $\Omega_1 = 0$.

Conservation of angular momentum about shaft.

$$I_B(\omega_1 + \Omega_1) + I_C\Omega_1 = I_B(\omega_2 + \Omega_2) + I_C\Omega_2$$

$$\Omega_2 = -\frac{I_B(\omega_2 - \omega_1)}{I_C + I_B}$$
$$= -\frac{(446.38)(8\pi - 6\pi)}{446.38 + 650}$$
$$= -2.5581 \text{ rad/s}$$

$\Omega_2 = -24.4$ rpm ◀



PROBLEM 17.85

Assuming that the tail propeller in Problem 17.84 is operating and that the angular velocity of the cab remains zero, determine the final horizontal velocity of the cab when the speed of the main blades is changed from 180 to 240 rpm. The cab weighs 1250 lb and is initially at rest. Also determine the force exerted by the tail propeller if the change in speed takes place uniformly in 12 s.

SOLUTION

Let $\mathbf{\Omega}$ be the angular velocity of the cab and ω be the angular velocity of the blades relative to the cab. The absolute angular velocity of the blades is $\Omega + \omega$.

$$\omega_1 = 180 \text{ rpm} = 6\pi \text{ rad/s}$$

$$\omega_2 = 240 \text{ rpm} = 8\pi \text{ rad/s}$$

Moments of inertia.

Cab: $I_C = 650 \text{ lb}\cdot\text{ft}\cdot\text{s}^2$

Blades: $I_B = 4\left(\frac{1}{3}mL^2\right) = (4)\left(\frac{1}{3}\right)\left(\frac{55}{32.2}\right)(14)^2$

$$= 446.38 \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

The cab does not rotate. $\Omega_1 = \Omega_2 = 0$

$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

Moments about shaft: $I_B(\omega_1 + \Omega_1) + I_C\Omega_1 + Frt = I_B(\omega_2 + \Omega_2) + I_C\Omega_2$

$$Frt = I_B(\omega_2 - \omega_1)$$

$$= (446.38)(8\pi - 6\pi)$$

$$= 2804.7 \text{ lb}\cdot\text{ft}\cdot\text{s}$$

$$Ft = \frac{Frt}{r} = \frac{2804.7}{16} = 175.29 \text{ lb}\cdot\text{s}$$

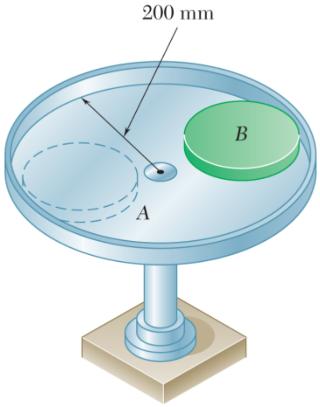
Linear components: $mv_1 + Ft = mv_2$

$$v_2 - v_1 = \frac{Ft}{m} = \frac{175.29}{\frac{1250}{32.2} + (4)\left(\frac{55}{32.2}\right)}$$

$$= 3.8398 \text{ ft/s}$$

(*a*) Assume $v_1 = 0$. $v_2 = 3.84 \text{ ft/s}$ ◀

(*b*) Force. $F = \frac{Ft}{t} = \frac{175.29}{12}$ $F = 14.61 \text{ lb}$ ◀



PROBLEM 17.86

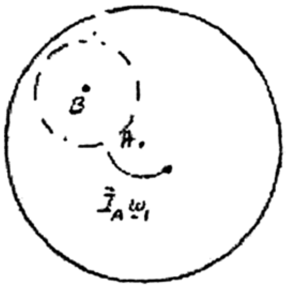
The circular platform A is fitted with a rim of 200-mm inner radius and can rotate freely about the vertical shaft. It is known that the platform-rim unit has a mass of 5 kg and a radius of gyration of 175 mm with respect to the shaft. At a time when the platform is rotating with an angular velocity of 50 rpm, a 3-kg disk B of radius 80 mm is placed on the platform with no velocity. Knowing that disk B then slides until it comes to rest relative to the platform against the rim, determine the final angular velocity of the platform.

SOLUTION

Moments of inertia.

$$\bar{I}_A = m_A k^2 = (5\text{ kg})(0.175\text{ m})^2 = 0.153125\text{ kg}\cdot\text{m}^2$$

$$\bar{I}_B = \frac{1}{2}m_B r_B^2 = \frac{1}{2}(3\text{ kg})(0.08\text{ m})^2 = 9.6\times10^{-3}\text{ kg}\cdot\text{m}^2$$



Syst. Momenta$_1$

State 1 Disk B is at rest.

State 2 Disk B moves with platform A.

Kinematics. In State 2, $\bar{v}_B = (0.12\text{ m})\omega_2$

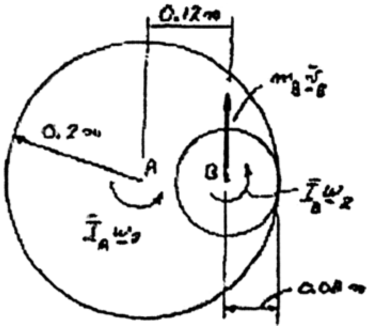
Principle of conservation of angular momentum.

+↻ Moments about D: $\bar{I}_A\omega_1 = \bar{I}_A\omega_2 + \bar{I}_B\omega_2 + m_B\bar{v}_B(0.12\text{ m})$

$$(0.153125\text{ kg}\cdot\text{m}^2)\omega_1 = (0.153125\text{ kg}\cdot\text{m}^2)\omega_2 + (9.6\times10^{-3}\text{ kg}\cdot\text{m}^2)\omega_2 + (3\text{ kg})(0.12\text{ m})^2\omega_2$$

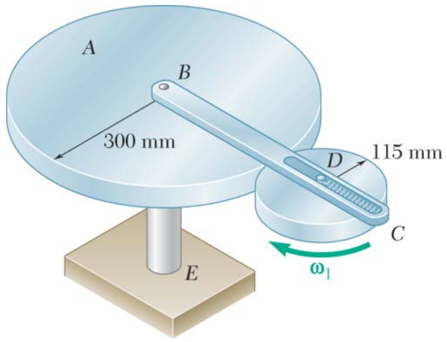
$$0.153125\omega_2 = 0.20593\omega_1$$

$$\omega_2 = 0.7436\omega_1 = 0.7436(50\text{ rpm})$$



Syst. Momenta$_2$

Final angular velocity $\omega_2 = 37.2\text{ rpm}$ ◀



PROBLEM 17.87

The 30-kg uniform disk *A* and the bar *BC* are at rest and the 5-kg uniform disk *D* has an initial angular velocity $\boldsymbol{\omega}_1$ of magnitude 440 rpm when the compressed spring is released and disk *D* contacts disk *A*. The system rotates freely about the vertical spindle *BE*. After a period of slippage, disk *D* rolls without slipping. Knowing that the magnitude of the final angular velocity of disk *D* is 176 rpm, determine the final angular velocities of bar *BC* and disk *A*. Neglect the mass of bar *BC*.

SOLUTION

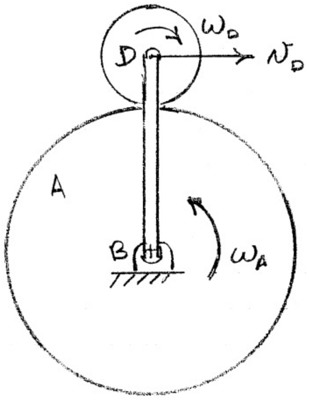
Data: $$r_A = 0.3 \text{ m}, \qquad r_D = 0.115 \text{ m}$$

$$\bar{I}_A = \frac{1}{2}m_A r_A^2 = \frac{1}{2}30(0.3)^2 = 1.35 \text{ kg}\cdot\text{m}^2$$

$$\bar{I}_D = \frac{1}{2}m_D r_D^2 = \frac{1}{2}5(0.115)^2 = 0.0330625 \text{ kg}\cdot\text{m}^2$$

$$(\omega_D)_1 = 440 \text{ rpm} = 46.077 \text{ rad/s}, \quad (\omega_D)_2 = 176 \text{ rpm} = 18.4307 \text{ rad/s}$$

Kinematics. Let *P* be the disk contact point on disk *D* and *Q* be the disk contact point on disk *A*.



$$\mathbf{v}_P = [r_D\omega_D \longleftarrow] + [v_D \longrightarrow] = [(r_D\omega_D - v_D) \longleftarrow]$$

$$\mathbf{v}_Q = [r_A\omega_A \longleftarrow]$$

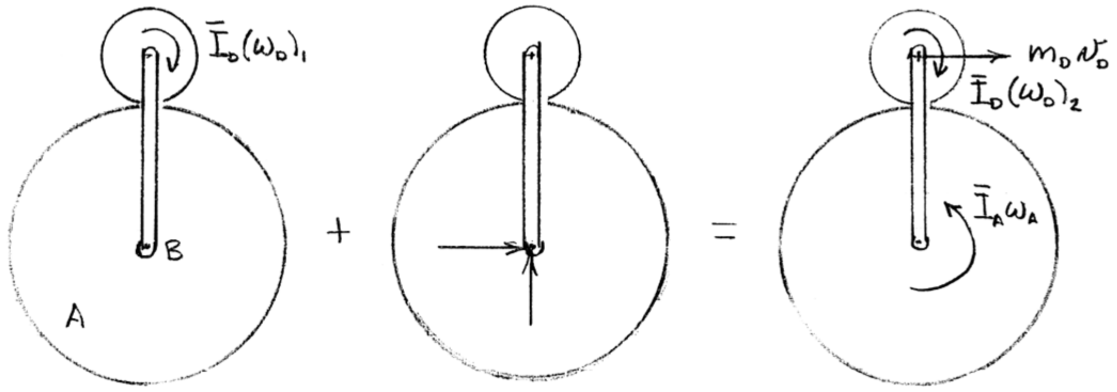
For rolling without slipping, $$v_Q = v_P$$

$$r_A\omega_A = r_D\omega_D - v_D$$

$$r_A\omega_A + v_D = r_D\omega_D \tag{1}$$

PROBLEM 17.87 (Continued)

Kinetics.



Syst Momenta$_1$ + **Syst Ext Imp**$_{1\to 2}$ = **Syst Momenta**$_2$

↻ Moments about *B*: $\bar{I}_D(\omega_D)_1 + 0 = \bar{I}_D(\omega_D)_2 + (r_A + r_D)m_D v_D - \bar{I}_A\omega_A$

$$\bar{I}_A\omega_A - (r_A + r_D)m_D v_D = -\bar{I}_D\left[(\omega_D)_1 - (\omega_D)_2\right] \tag{2}$$

Substituting numerical data into Equations (1) and (2),

$$0.3\omega_A + v_D = (0.115)(18.4307) \tag{1}$$

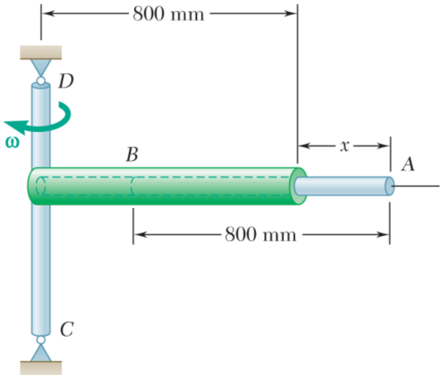
$$1.35\omega_A - (0.3 + 0.115)(5)v_D = -(0.0330625)(46.077 - 18.4307) \tag{2}$$

Solving simultaneously, $\omega_A = 1.7663 \text{ rad/s}, \quad v_D = 1.58965 \text{ m/s}$

$$\omega_{BC} = \frac{v_D}{r_A + r_D} = \frac{1.58965}{0.3 + 0.115} = 3.8305 \text{ rad/s}$$

$\omega_{BC} = 36.6$ rpm ↻ ◀

$\omega_A = 16.87$ rpm ↺ ◀



PROBLEM 17.88

The 4-kg rod *AB* can slide freely inside the 6-kg tube *CD*. The rod was entirely within the tube ($x = 0$) and released with no initial velocity relative to the tube when the angular velocity of the assembly was 5 rad/s. Neglecting the effect of friction, determine the speed of the rod relative to the tube when $x = 400$ mm.

SOLUTION

Let l be the length of the tube and the rod and Point O be the point of intersection of the tube and the axle.

Moments of inertia.

$$\bar{I}_T = \frac{1}{12}m_T l^2, \qquad \bar{I}_R = \frac{1}{12}m_R l^2$$

Kinematics.

$$(\bar{v}_\theta)_T = \bar{r}_T\omega = \frac{l}{2}\omega$$

$$(\bar{v}_\theta)_R = \bar{r}_R\omega = \left(\frac{l}{2} + x\right)\omega, \qquad (\bar{v}_r)_T = v_r$$

Angular momentum about Point O.

$$\begin{aligned} H_O &= m_T\bar{r}_T(\bar{v}_\theta) + \bar{I}_T\omega + m_R\bar{r}_R(\bar{v}_\theta)_R + \bar{I}_R\omega \\ &= m_T\frac{l}{2}\left(\frac{l}{2}\omega\right) + \frac{1}{12}m_T l^2\omega + m_R\left(\frac{l}{2} + x\right)\left(\frac{l}{2} + x\right)\omega + \frac{1}{12}m_R l^2\omega \\ &= \left[\frac{1}{3}m_T l^2 + m_R\left(\frac{1}{3}l^2 + lx + x^2\right)\right]\omega \end{aligned}$$

Kinetic energy.

$$\begin{aligned} T &= \frac{1}{2}m_T(\bar{v}_\theta)_T^2 + \frac{1}{2}\bar{I}_T\omega^2 + \frac{1}{2}m_R(\bar{v}_\theta^2)_R + \frac{1}{2}m_R v_r^2 + \frac{1}{2}\bar{I}_R\omega^2 \\ &= \frac{1}{2}\left[m_T\left(\frac{l}{2}\omega\right)^2 + \frac{1}{12}m_T l^2\omega^2 + \frac{1}{2}m_R\left(\frac{l}{2} + x\right)^2\omega^2 + \frac{1}{2}m_R v_r^2 + \frac{1}{12}m_R l^2\omega^2\right] \\ &= \frac{1}{2}\left[\frac{1}{3}m_T l^2 + m_R\left(\frac{1}{3}l^2 + lx + x^2\right)\right]\omega^2 + \frac{1}{2}m_R v_r^2 = \frac{1}{2}H_O\omega + \frac{1}{2}m_R v_r^2 \end{aligned}$$

Potential energy. All motion is horizontal. $V = 0$

PROBLEM 17.88 (Continued)

State 1. $x = 0, \quad \omega = \omega_1 = 5 \text{ rad/s}, \quad v_r = 0$

$$(H_O)_1 = \frac{1}{3}(m_T + m_R)l^2\omega = \frac{1}{3}(6 + 4)(0.800)^2(5) = 10.6667 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$T_1 = \frac{1}{2}\left[\frac{1}{3}(m_T + m_R)l^2\omega^2\right] + 0 = \frac{1}{2}(H_O)_1\omega = \frac{1}{2}(10.6667)(5) = 26.667 \text{ J}$$

$$V_1 = 0$$

State 2. $x = \frac{l}{2} = 0.400 \text{ m}, \quad \omega = \omega_2 = ?, \quad v_r = ?$

$$(H_O)_2 = \left[\frac{1}{3}(6)(0.800)^2 + (4)\left\{\frac{1}{3}(0.800)^2 + (0.800)(0.400) + (0.400)^2\right\}\right]\omega_2$$

$$= 4.05333\omega_2$$

$$T_2 = \frac{1}{2}(4.05333\omega_2)\omega_2 + \frac{1}{2}(4)v_r^2 = 2.026667\omega_2^2 + 2v_r^2$$

$$V_2 = 0$$

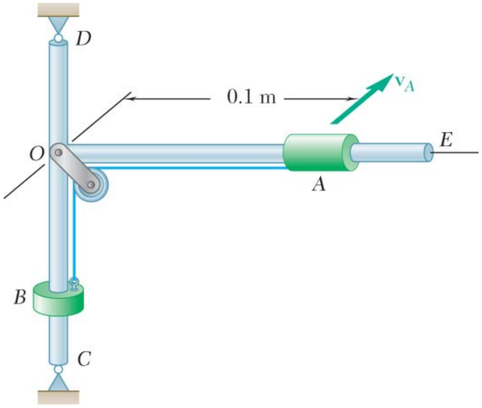
Conservation of angular momentum: $(H_O)_1 = (H_O)_2$

$$10.6667 = 4.05333\omega_2 \qquad \omega_2 = 2.6316 \text{ rad/s}$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$

$$26.667 + 0 = (2.02667)(2.6316)^2 + 2v_r^2 + 0$$

$$v_r^2 = 6.3158 \text{ m}^2/\text{s}^2 \qquad v_r = 2.51 \text{ m/s} \blacktriangleleft$$



PROBLEM 17.89

A 1.8-kg collar *A* and a 0.7-kg collar *B* can slide without friction on a frame, consisting of the horizontal rod *OE* and the vertical rod *CD*, which is free to rotate about its vertical axis of symmetry. The two collars are connected by a cord running over a pulley that is attached to the frame at *O*. At the instant shown, the velocity $\mathbf{v}_A$ of collar *A* has a magnitude of 2.1 m/s and a stop prevents collar *B* from moving. The stop is suddenly removed and collar *A* moves toward *E*. As it reaches a distance of 0.12 m from *O*, the magnitude of its velocity is observed to be 2.5 m/s. Determine at that instant the magnitude of the angular velocity of the frame and the moment of inertia of the frame and pulley system about *CD*.

SOLUTION

Components of velocity of collar *A*. $v_A^2 = (v_A)_r^2 + (v_A)_\theta^2$ (1)

Constraint of rod *OE*. $(v_A)_\theta = r_A\omega$ (2)

Constraint of cable *AB*. $\Delta r_A = \Delta y_B, \quad (v_A)_r = v_B$ (3)

Position 1. $(\Delta r_A) = 0.1\,\text{m}, \quad [(v_A)_r]_1 = 0, \quad (v_A)_1 = 2.1\,\text{m/s}$

From Equation (1), $(2.1)^2 = 0 + [(v_A)_\theta]_1^2 \qquad [(v_A)_\theta]_1 = 2.1\,\text{m/s}$

From Equation (2), $(2.1) = 0.1\omega_1 \qquad \omega_1 = 21\,\text{rad/s}$

From Equation (3), $v_B = 0$

Potential energy. Take position 1 as datum. $V_1 = 0$ (4)

Angular momentum. $(H_O)_1 = I\omega_1 + m_A[(v_A)_r]_1(r_A)_1$:

$$(H_O)_1 = I(21) + (1.8)(2.1)(0.1) \qquad (H_O)_1 = 21I + 0.378 \tag{5}$$

Kinetic energy. $T_1 = \frac{1}{2}I\omega_1^2 + \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$:

$$T_1 = \frac{1}{2}I(21)^2 + \frac{1}{2}(1.8)(2.1)^2 \qquad T_1 = 220.5I + 3.969 \tag{6}$$

Position 2. $(r_A)_2 = 0.12\,\text{m}, \quad (v_A)_2 = 2.5\,\text{m/s} \qquad \omega = \omega_2$

From Equation (2), $[(v_A)_\theta]_2 = 0.12\omega_2$

From Equation (1), $[(v_A)_r]_2^2 = (v_A)_2^2 - [(v_A)_\theta]_2^2 = (2.5)^2 - (0.12)^2\omega_2^2$

$$= 6.25 - 0.0144\omega_2^2$$

PROBLEM 17.89 (Continued)

From Equation (3), $v_B^2 = 6.25 - 0.0144\omega_2^2$

Change in radial position. $\Delta r_A = (r_A)_2 - (r_A)_1 = 0.02\text{ m}$

From Equation (3), $\Delta y_B = 0.02\text{ m}$

Potential energy. $V_2 = m_B g(\Delta y_B) = (0.7)(9.81)(0.02)$

$$V_2 = 0.13734\text{ J} \tag{7}$$

Angular momentum. $(H_O)_2 = I\omega_2 + m_A[(v_A)_\theta]_2(r_A)_2$:

$$(H_O)_2 = I\omega_2 + (1.8)(0.12\omega_2)(0.12) \qquad (H_O)_2 = (I + 0.02592)\omega_2 \tag{8}$$

Kinetic energy. $T_2 = \frac{1}{2}I\omega_2^2 + \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$:

$$T_2 = \frac{1}{2}I\omega_2^2 + \frac{1}{2}(1.8)(2.5)^2 + \frac{1}{2}(0.7)(6.25 - 0.0144\omega_2^2)$$

$$T_2 = (0.5I - 0.00504)\omega_2^2 + 7.8125 \tag{9}$$

Conservation of angular momentum. $(H_O)_1 = (H_O)_2$:

$$21I + 0.378 = (I + 0.02592)\omega_2$$

Solving for ω_2, $$\omega_2 = \frac{21I + 0.378}{I + 0.02592} = \frac{N}{D} \tag{10}$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$:

$$220.5I + 3.969 = (0.5I - 0.00504)\omega_2^2 + 7.8125 + 0.13734$$

$$220.5I - (0.5I - 0.00504)\frac{N^2}{D^2} - 3.98084 = 0$$

$$220.5ID^2 - 0.5IN^2 + 0.00504N^2 - 3.98084D^2 = 0$$

$$220.5I(I^2 + 0.05184I + 0.0006718464) - 0.5I(441I^2 + 15.876I + 0.142884)$$

$$+ 0.00504(441I^2 + 15.876I + 0.142884) - (3.98084)(I^2 + 0.05184I + 0.0006718464) = 0$$

$$0I^3 + 1.73452I^2 - 0.04965167I - 0.001954378 = 0$$

PROBLEM 17.89 (Continued)

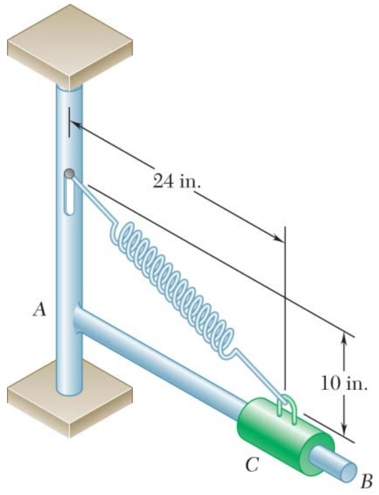
Solving the quadratic equation for I,

$$I = \frac{0.04965167 \pm 0.126590}{3.46904} = 0.050804 \qquad \text{and} \qquad -0.022179$$

Reject the negative root.

From Equation (10),
$$\omega_2 = \frac{(21)(0.050804) + 0.378}{0.050804 + 0.02592} \qquad \omega = 18.83 \text{ rad/s} \blacktriangleleft$$

$$I = 0.0508 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$



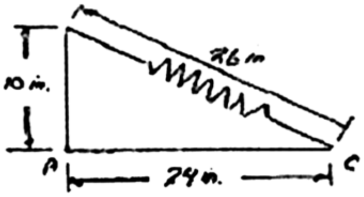
PROBLEM 17.90

A 6-lb collar *C* is attached to a spring and can slide on rod *AB*, which in turn can rotate in a horizontal plane. The mass moment of inertia of rod *AB* with respect to end *A* is 0.35 lb · ft · s^2. The spring has a constant $k = 15$ lb/in. and an undeformed length of 10 in. At the instant shown the velocity of the collar relative to the rod is zero, and the assembly is rotating with an angular velocity of 12 rad/s. Neglecting the effect of friction, determine (*a*) the angular velocity of the assembly as the collar passes through a point located 7.5 in. from end *A* of the rod, (*b*) the corresponding velocity of the collar relative to the rod.

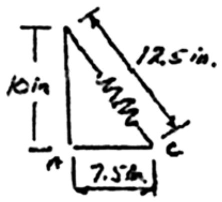
SOLUTION

Potential energy of spring: undeformed length = 10 in.

Position 1:



Position 2:



$\Delta = 26 \text{ in.} - 10 \text{ in.} = 16 \text{ in.}$

$$V_1 = \frac{1}{2}k\Delta^2 = \frac{1}{2}(15 \text{ lb/in.})(16 \text{ in.})^2$$

$$= 1920 \text{ in.} \cdot \text{lb}$$

$$V_1 = 160 \text{ ft} \cdot \text{lb}$$

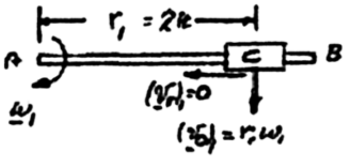
$\Delta = 12.5 \text{ in.} - 10 \text{ in.} = 2.5 \text{ in.}$

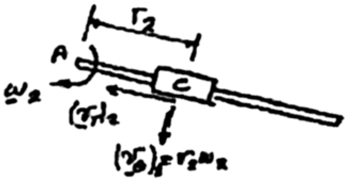
$$V_2 = \frac{1}{2}k\Delta^2 = \frac{1}{2}(15 \text{ lb/in.})(2.5 \text{ in.})^2$$

$$= 46.875 \text{ in.} \cdot \text{lb}$$

$$V_2 = 3.91 \text{ ft} \cdot \text{lb}$$

Kinematics:





Kinetics: Since moments of all forces about shaft at *A* are zero, $(\mathbf{H}_A)_1 = (\mathbf{H}_A)_2$

$$I_R\omega_1 + m_C(v_0)r_1 = I_R\omega_C + m_C(v_0)_2 r_2$$

$$\left(I_R + m_C r_1^2\right)\omega_1 = \left(I_R + m_C r_2^3\right)\omega_2$$

PROBLEM 17.90 (Continued)

Data:

$$I_R = 0.35\ \text{lb}\cdot\text{ft}\cdot\text{s}^2, \quad m_C = \frac{6\ \text{lb}}{32.2}$$

$$r_1 = 2\ \text{ft}, \quad r_2 = \frac{7.5}{12}\,\text{ft}, \quad \omega_1 = 12\ \text{rad/s}$$

$$\left[0.35\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 + \frac{6\ \text{lb}}{32.2}(2\ \text{ft})^2\right](12\ \text{rad/s}) = \left[0.35\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 + \frac{6\ \text{lb}}{32.2}\left(\frac{7.5}{12}\,\text{ft}\right)^2\right]\omega_2$$

$$13.1441 = 0.42279\omega_2; \quad \omega_2 = 31.089\ \text{rad/s}$$

(*a*) Angular velocity. $\omega_2 = 31.1\,\text{rad/s}$ ◀

Kinetic energy.

$$T_1 = \frac{1}{2}I_A\omega_1^2 + \frac{1}{2}m_C(v_D)_1^2 + \frac{1}{2}m_C(v_r)_1^2$$

$$= \frac{1}{2}(0.35\ \text{lb}\cdot\text{ft}\cdot\text{s}^2)(12\ \text{rad/s})^2 + \frac{1}{2}\left(\frac{6\ \text{lb}}{32.2}\right)(2\ \text{ft})^2(12\ \text{rad/s})^2 + 0$$

$$T_1 = 78.865\ \text{ft}\cdot\text{lb}$$

$$T_2 = \frac{1}{2}I_R\omega_2^2 + \frac{1}{2}m(v_B)_2^2 + \frac{1}{2}m_2(v_r)_2^2$$

$$= \frac{1}{2}(0.35\ \text{lb}\cdot\text{ft}\cdot\text{s}^2)(31.089\ \text{rad/s})^2$$

$$+\frac{1}{2}\left(\frac{6\ \text{lb}}{32.2}\right)\left(\frac{7.5}{12}\,\text{ft}\right)^2(31.089\ \text{rad/s})^2 + \frac{1}{2}\left(\frac{6\ \text{lb}}{32.2}\right)(v_r)_2^2$$

$$T_2 = 204.32 + 0.09317(v_r)_2^2$$

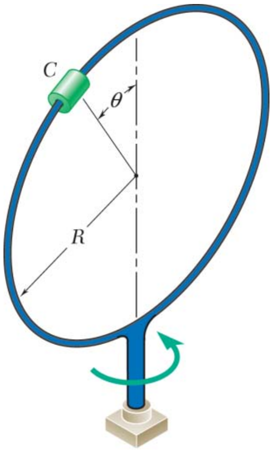
Principle of conservation of energy: $T_1 + V_1 = T_2 + V_2$

Recall: $V_1 = 160\ \text{ft}\cdot\text{lb}$ and $V_2 = 3.91\,\text{ft}\cdot\text{lb}$

$$78.865 + 160 = 204.32 + 0.09317(v_r)_2^2 + 3.91$$

$$30.638 = 0.09317(v_r)_2^2$$

(*b*) Velocity of collar relative to rod. $(v_r)_2 = 18.13\,\text{ft/s}$ ◀

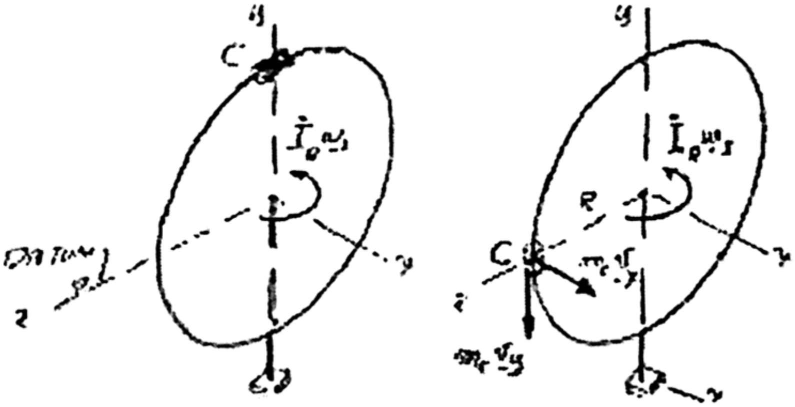


PROBLEM 17.91

A small 4-lb collar *C* can slide freely on a thin ring of weight 6 lb and radius 10 in. The ring is welded to a short vertical shaft, which can rotate freely in a fixed bearing. Initially the ring has an angular velocity of 35 rad/s and the collar is at the top of the ring $(\theta = 0)$ when it is given a slight nudge. Neglecting the effect of friction, determine (*a*) the angular velocity of the ring as the collar passes through the position $\theta = 90°$, (*b*) the corresponding velocity of the collar relative to the ring.

SOLUTION

Moment of inertia of ring. $\bar{I}_R = \frac{1}{2} m_R R^2$



Position 1 Position 2

Position 1. $\theta = 0$

$$v_C = 0$$

Position 2. $\theta = 90°$

$$(v_C)_y = v_y = R\omega_2$$

Conservation of angular momentum about *y* axis for system.

$$\bar{I}_R\omega_1 = \bar{I}_R\omega_2 + m_C v_y R$$

$$\frac{1}{2} m_R R^2 \omega_1 = \frac{1}{2} m_R R^2 \omega_2 + m_C R^2 \omega_2$$

$$m_R R^2 \omega_1 = (m_R + 2m_C) R^2 \omega_2$$

$$\omega_2 = \frac{m_R}{m_R + 2m_C}\omega_1 \qquad (1)$$

PROBLEM 17.91 (Continued)

Potential energy. Datum is the center of the ring.

$$V_1 = m_C g R \qquad V_2 = 0$$

Kinetic energy:

$$T_1 = \frac{1}{2}\bar{I}_R\omega_1^2 = \frac{1}{2}\left(\frac{1}{2}m_R R^2\right)\omega_1^2$$

$$= \frac{1}{4}m_R R^2\omega_1^2$$

$$T_2 = \frac{1}{2}\bar{I}_R\omega_2^2 + \frac{1}{2}m_C\left(v_x^2 + v_y^2\right)$$

$$= \frac{1}{4}m_R R^2\omega_2^2 + \frac{1}{2}m_C R^2\omega_2^2 + \frac{1}{2}m_C v_y^2$$

Principle of conservation of energy:

$$T_1 + V_1 = T_2 + V_2$$

$$\frac{1}{4}m_R R^2\omega_1^2 + m_C g R = \left(\frac{1}{4}m_R + \frac{1}{2}m_C\right)R^2\omega_2^2 + \frac{1}{2}m_C v_y^2 \tag{2}$$

Data:

$$W_C = 4 \text{ lb}$$

$$W_R = 6 \text{ lb}$$

$$R = 10 \text{ in.} = 0.83333 \text{ ft}$$

$$\omega_1 = 35 \text{ rad/s}$$

(*a*) Angular velocity.

From Eq. (1),

$$\omega_2 = \frac{\frac{6 \text{ lb}}{g}}{\frac{6 \text{ lb}}{g} + 2\left(\frac{4 \text{ lb}}{g}\right)}(35 \text{ rad/s}) \qquad \omega_2 = 15.00 \text{ rad/s} \blacktriangleleft$$

(*b*) Velocity of collar relative to ring.

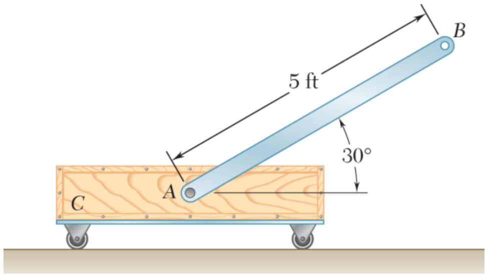
From Eq. (2),

$$\frac{1}{4}\left(\frac{6 \text{ lb}}{32.2}\right)\left(\frac{10}{12}\text{ft}\right)^2(35 \text{ rad/s})^2 + (4 \text{ lb})\left(\frac{10}{12}\text{ft}\right)$$

$$= \left[\frac{1}{4}\left(\frac{6 \text{ lb}}{32.2}\right) + \frac{1}{2}\left(\frac{4 \text{ lb}}{32.2}\right)\right]\left(\frac{10}{12}\text{ft}\right)^2(15 \text{ rad/s})^2 + \frac{1}{2}\left(\frac{4 \text{ lb}}{32.2}\right)v_y^2$$

$$39.629 + 3.3333 = 16.984 + 0.062112 v_y^2$$

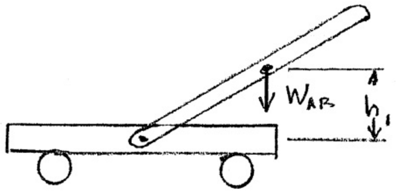
$$v_y^2 = 418.25 \qquad v_y = 20.5 \text{ ft/s} \blacktriangleleft$$



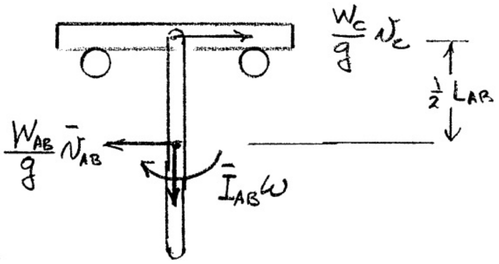
PROBLEM 17.92

Rod *AB* has a weight of 6 lb and is attached to a 10-lb cart *C*. Knowing that the system is released from rest in the position shown and neglecting friction, determine (*a*) the velocity of point *B* as rod *AB* passes through a vertical position, (*b*) the corresponding velocity of the cart *C*.

SOLUTION



Position 1



Position 2

Since the external forces are always vertical, the horizontal component of linear momentum is conserved.

$$\xrightarrow{+} L_1 = L_2:\quad 0 = \frac{W_C}{g}v_C - \frac{W_{AB}}{g}\bar{v}_{AB} \qquad \bar{v}_{AB} = \frac{W_C}{W_{AB}}v_C = \frac{10}{6}v_C = \frac{5}{3}v_C \tag{1}$$

Kinematics. In position 2,

$$v_C = -\bar{v}_{AB} + \frac{L_{AB}}{2}\omega \qquad L_{AB}\omega = 2\left(v_C + \bar{v}_{AB}\right)$$

$$L_{AB}\omega = 2\left(1 + \frac{10}{6}\right)v_C = \frac{16}{3}v_C \tag{2}$$

Position 1.

$$V_1 = W_{AB}h_1 = (6)\left(\frac{5}{2}\sin 30°\right) = 7.5\ \text{ft}\cdot\text{lb}$$

$$T_1 = 0$$

Position 2.

$$V_2 = -W_{AB}\frac{L_{AB}}{2} = -(6)\left(\frac{5}{2}\right) = -15.0\ \text{ft}\cdot\text{lb}$$

$$T_2 = \frac{1}{2}\frac{W_C}{g}v_C^2 + \frac{1}{2}\frac{W_{AB}}{g}\bar{v}_{AB}^2 + \frac{1}{2}\bar{I}_{AB}\omega^2$$

$$= \frac{1}{2}\left(\frac{10}{32.2}\right)v_C^2 + \frac{1}{2}\left(\frac{6}{32.2}\right)\left(\frac{5}{3}v_C\right)^2 + \frac{1}{2}\frac{1}{12}\cdot\frac{6}{32.2}L_{AB}^2\omega^2$$

$$= 0.41408v_C^2 + 0.0077640\left(\frac{16}{3}v_C\right)^2 = 0.63492v_C^2$$

PROBLEM 17.92 (Continued)

Conservation of energy. $$T_1 + V_1 = T_2 + V_2:$$

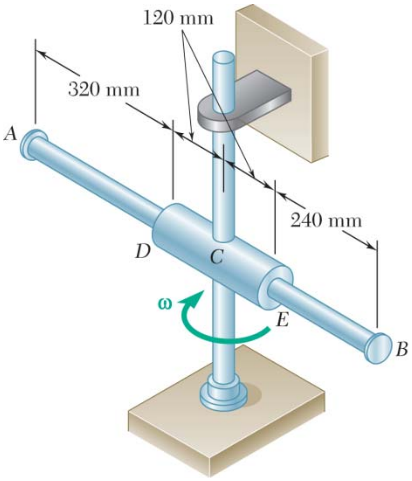
$$0 + 7.5 = 0.63492 v_C^2 - 15.0 \qquad v_C = 5.9529 \text{ ft/s}$$

From Equation (2), $$L_{AB}\omega = \left(\frac{16}{3}\right)(5.9529) = 31.749 \text{ ft/s}$$

(*a*) By kinematics, $$v_B = L_{AB}\omega - v_C = 31.749 - 5.9529$$

$$\mathbf{v}_B = 25.8 \text{ ft/s} \longleftarrow \blacktriangleleft$$

(*b*) Velocity of the cart. $$\mathbf{v}_C = 5.95 \text{ ft/s} \longrightarrow \blacktriangleleft$$


120 mm
320 mm
240 mm
A
B
C
D
E
ω

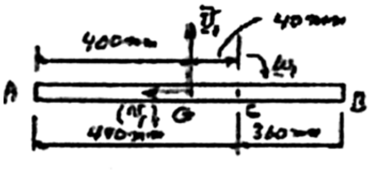
PROBLEM 17.93

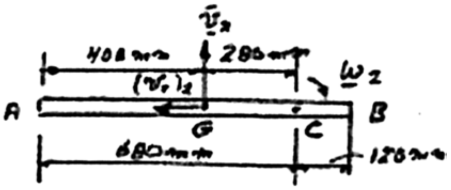
In Prob. 17.82, determine the velocity of rod *AB* relative to cylinder *DE* as end *B* of the rod strikes end *E* of the cylinder.

PROBLEM 17.82 A 3-kg rod of length 800 mm can slide freely in the 240-mm cylinder *DE*, which in turn can rotate freely in a horizontal plane. In the position shown the assembly is rotating with an angular velocity of magnitude $\omega = 40$ rad/s and end *B* of the rod is moving toward the cylinder at a speed of 75 mm/s relative to the cylinder. Knowing that the centroidal mass moment of inertia of the cylinder about a vertical axis is $0.025\ \text{kg}\cdot\text{m}^2$ and neglecting the effect of friction, determine the angular velocity of the assembly as end *B* of the rod strikes end *E* of the cylinder.

SOLUTION

Kinematics and geometry.


400 mm
$\bar{v}_1$
40 mm
ω_1
A
B
G
C
$(v_r)_1$
440 mm
360 mm


$\bar{v}_2$
400 mm
280 mm
ω_2
$(v_r)_2$
A
B
G
C
680 mm
120 mm

$$\bar{v}_1 = (0.04\ \text{m})\omega_1 = (0.4\ \text{m})(40\ \text{rad/s})$$

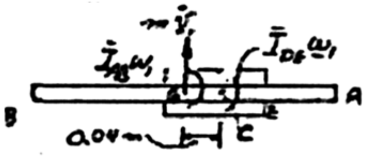
$$\bar{v}_1 = 1.6\ \text{m/s}$$

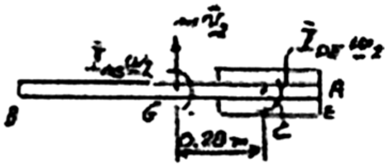
$$\bar{v}_2 = (0.28\ \text{m})\omega_2$$

Initial position

Final position

Conservation of angular momentum about *C*.


$m\bar{v}_1$
$\bar{I}_{AB}\omega_1$
$\bar{I}_{DE}\omega_1$
B
A
E
C
0.04 m


$m\bar{v}_2$
$\bar{I}_{AB}\omega_2$
$\bar{I}_{DE}\omega_2$
B
G
A
E
C
0.28 m

+↻ Moments about *C*: $\bar{I}_{AB} = \dfrac{1}{12}(3\ \text{kg})(0.8\ \text{m})^2 = 0.16\ \text{kg}\cdot\text{m}^2$

$$\bar{I}_{AB}\omega_1 + m\bar{v}_1(0.04\ \text{m}) + I_{DE}\omega_1 = I_{AB}\omega_2 + m\bar{v}_2(0.28\ \text{m}) + I_{DE}\omega_2$$

$$(0.16\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s}) + (3\ \text{kg})(1.6\ \text{m/s})(0.04\ \text{m}) + (0.025\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s})$$

$$= (0.16\ \text{kg}\cdot\text{m}^2)\omega_2 + (3\ \text{kg})(0.28\omega_2)(0.28) + (0.025\ \text{kg}\cdot\text{m}^2)\omega_2$$

$$(6.4 + 0.192 + 1.00) = (0.16 + 0.2352 + 0.025)\omega_2$$

$$7.592 = 0.4202\omega_2; \quad \omega_2 = 18.068\ \text{rad/s}: \quad \omega_2 = 18.07\ \text{rad/s}$$

PROBLEM 17.93 (Continued)

Conservation of energy $(v_r) = 0.075\text{ m/s}$

$$V_1 = V_2 = 0$$

$$T_1 = \frac{1}{2}\overline{I}_{DE}\omega_1^2 + \frac{1}{2}\overline{I}_{AB}\omega_1^2 + \frac{1}{2}m_{AB}\overline{v}_1^2 + \frac{1}{2}m_{AB}(v_r)_1^2$$

$$= \frac{1}{2}(0.025\text{ kg}\cdot\text{m}^2)(40\text{ rad/s})^2 + \frac{1}{2}(0.16\text{ kg}\cdot\text{m}^2)(40\text{ rad/s})^2$$

$$+\frac{1}{2}(3\text{ kg})(1.6\text{ m/s})^2 + \frac{1}{2}(3\text{ kg})(0.075\text{ m/s})^2$$

$$T_1 = 20\text{ J} + 128\text{ J} + 3.84\text{ J} + 0.008\text{ J} = 151.85\text{ J}$$

$$\overline{v}_2 = (0.28\text{ m})\omega_2 = (0.28\text{ m})(18.068\text{ rad/s}) = 5.059\text{ m/s}$$

$$T_2 = \frac{1}{2}\overline{I}_{DE}\omega_2^2 + \frac{1}{2}\overline{I}_{AB}\omega_2^2 + \frac{1}{2}m_{AB}v_2^2 + \frac{1}{2}m_{AB}(v_r)_2^2$$

$$= \frac{1}{2}(0.025\text{ kg}\cdot\text{m}^2)(18.068\text{ rad/s})^2$$

$$+\frac{1}{2}(0.16\text{ kg}\cdot\text{m}^2)(18.068\text{ rad/s})^2$$

$$= \frac{1}{2}(3\text{ kg})(5.059\text{ m/s})^2 + \frac{1}{2}(3\text{ kg})(v_r)_2^2$$

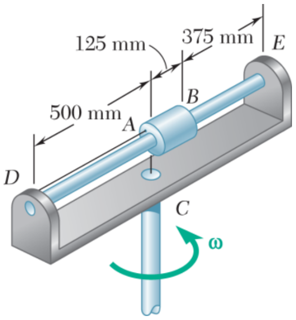
$$T_2 = 4.081\text{ J} + 26.116\text{ J} + 38.391\text{ J} + 1.5(v_r)_2^2$$

$$T_2 = 68.587\text{ J} + 1.5(v_r)_2^2$$

$$T_1 + V_1 = T_2 + V_2: \quad 151.85\text{ J} + 0 = 68.587\text{ J} + 1.5(v_r)_2^2$$

$$83.263 = 1.5(v_r)_2^2$$

Velocity of rod relative to cylinder. $(v_r)_2 = 7.45\text{ m/s}$ ◀



PROBLEM 17.94

In Problem 17.83 determine the velocity of the tube relative to the rod as the tube strikes end *E* of the assembly.

PROBLEM 17.83 A 1.6-kg tube *AB* can slide freely on rod *DE* which in turn can rotate freely in a horizontal plane. Initially the assembly is rotating with an angular velocity $\omega = 5\text{ rad/s}$ and the tube is held in position by a cord. The moment of inertia of the rod and bracket about the vertical axis of rotation is $0.30\text{ kg}\cdot\text{m}^2$ and the centroidal moment of inertia of the tube about a vertical axis is $0.0025\text{ kg}\cdot\text{m}^2$. If the cord suddenly breaks, determine (*a*) the angular velocity of the assembly after the tube has moved to end *E*, (*b*) the energy lost during the plastic impact at *E*.

SOLUTION

Let Point *C* be the intersection of axle *C* and rod *DE*. Let Point *G* be the mass center of tube *AB*.

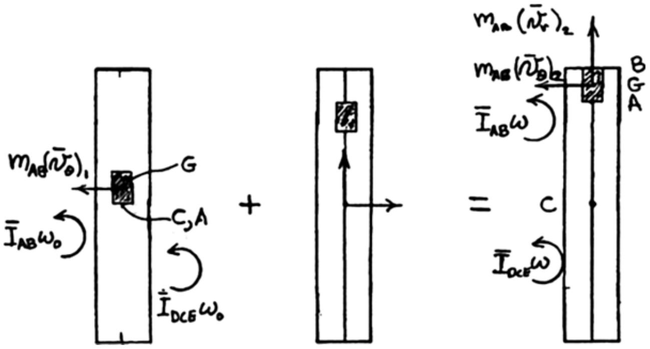
Masses and moments of inertia about vertical axes.

$$m_{AB} = 1.6\text{ kg},\qquad \bar{I}_{AB} = 0.0025\text{ kg}\cdot\text{m}^2,\qquad \bar{I}_{DCE} = 0.30\text{ kg}\cdot\text{m}^2$$

State 1. $\quad (r_{G/A})_1 = \frac{1}{2}(125) = 62.5\text{ mm},\qquad \omega_1 = 5\text{ rad/s},\quad (v_r)_1 = 0$

State 2. $\quad (r_{G/A})_2 = 500 - 62.5 = 437.5\text{ mm},\qquad \omega = \omega_2,\qquad v_r = (v_r)_2 = 0$

Kinematics. $\quad (v_G)_\theta = \bar{v}_\theta = r_{G/C}\,\omega$



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

Moments about *C*:

$$\bar{I}_{AB}\omega_1 + \bar{I}_{DCE}\omega_1 + m_{AB}(\bar{v}_\theta)_1(r_{G/C})_1 + 0 = \bar{I}_{AB}\omega_2 + \bar{I}_{DCE}\omega_2 + m_{AB}(\bar{v}_\theta)_2(r_{G/C})_2$$

$$\left[\bar{I}_{AB} + \bar{I}_{DCE} + m_{AB}(r_{G/C})_1^2\right]\omega_1 = \left[\bar{I}_{AB} + \bar{I}_{DCE} + m_{AB}(r_{G/C})_2^2\right]\omega_2$$

$$[0.0025 + 0.30 + (1.6)(0.0625)^2](5) = [0.0025 + 0.30 + (1.6)(0.4375)^2]\omega_2$$

$$(0.30875)(5) = 0.60875\omega_2 \qquad \omega_2 = 2.5359\text{ rad/s}$$

PROBLEM 17.94 (Continued)

Kinetic energy.

$$T = \frac{1}{2}\bar{I}_{AB}\omega^2 + \frac{1}{2}\bar{I}_{DCE}\omega^2 + \frac{1}{2}m_{AB}\bar{v}^2$$

$$= \frac{1}{2}\bar{I}_{AB}\omega^2 + \frac{1}{2}\bar{I}_{DCE}\omega^2 + \frac{1}{2}m_{AB}\left(r_{G/C}^2\omega^2 + \bar{v}_r^2\right)$$

$$T_1 = \frac{1}{2}(0.0025)(5)^2 + \frac{1}{2}(0.3)(5)^2 + \frac{1}{2}(1.6)(0.0625)^2 + 0 = 3.859375 \text{ J}$$

$$T_2 = \frac{1}{2}(0.0025)(2.5359)^2 + \frac{1}{2}(0.30)(2.5359)^2$$

$$+\frac{1}{2}(1.6)(0.4375)^2(2.5359)^2 + \frac{1}{2}(1.6)(\bar{v}_r)_2^2$$

$$= 1.95737 + 0.8(\bar{v}_r)_2^2$$

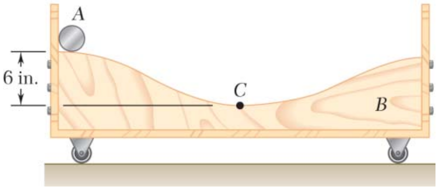
Work. The work of the bearing reactions at C is zero. Since the sliding contact between the rod and the tube is frictionless, the work of the contact force is zero.

$$U_{1\to 2} = 0$$

Principle of work and energy. $T_1 + U_{1\to 2} = T_2$

$$3.859375 + 0 = 1.95737 + 0.8(\bar{v}_r)_2^2$$

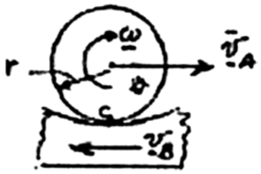
Velocity of the tube relative to the rod. $(\bar{v}_r)_2 = 1.542 \text{ m/s}$ ◀



PROBLEM 17.95

The 6-lb steel cylinder *A* and the 10-lb wooden cart *B* are at rest in the position shown when the cylinder is given a slight nudge, causing it to roll without sliding along the top surface of the cart. Neglecting friction between the cart and the ground, determine the velocity of the cart as the cylinder passes through the lowest point of the surface at *C*.

SOLUTION

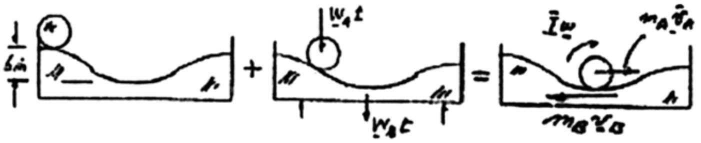


Kinematics (when cylinder is passing *C*)

$$\xleftarrow{+}\ v_B = v_C = r\omega - \bar{v}_A$$

$$\omega = \frac{\bar{v}_A + v_B}{r}$$

Principle of impulse and momentum.



Syst. of Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

$\xrightarrow{+}$ *x* components: $m_A\bar{v}_A - m_B v_B = 0$

$$\frac{6\text{ lb}}{\text{g}}\bar{v}_A = \frac{10\text{ lb}}{\text{g}}v_B; \quad v_B = 0.6\,\bar{v}_A$$

Work: $U_{1\to 2} = W_A(6\text{ in.}) = (6\text{ lb})\left(\frac{6}{12}\text{ ft}\right) = 3\text{ ft}\cdot\text{lb}; \quad T_1 = 0$

Kinetic energy: $T_2 = \frac{1}{2}m_A\bar{v}_A^2 + \frac{1}{2}\bar{I}\omega^2 + \frac{1}{2}m_B v_3^2$

$$v_B = 0.6\bar{v}_A; \quad \omega = \frac{\bar{v}_A + v_0}{r} = \frac{v_A + 0.6v_A}{r} = \frac{1.6v_A}{r}$$

$$T_2 = \frac{1}{2}\left(\frac{6\text{ lb}}{g}\right)\bar{v}_A^2 + \frac{1}{2}\left[\frac{1}{2}\frac{6\text{ lb}}{g}r^2\right]\left(\frac{1.6v_A}{r}\right)^2 + \frac{1}{2}\frac{10\text{ lb}}{g}(0.6v_A)^2$$

$$= \frac{3}{g}\bar{v}_A^2 + \frac{3.84}{g}\bar{v}_A^2 + \frac{1.8}{g}\bar{v}_A^2 = \frac{8.64}{g}\bar{v}_A^2$$

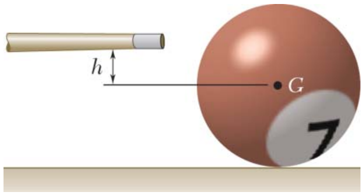
Principle of work and energy: $T_1 + U_{1\to 2} = T_2$

$$0 + 3\text{ ft}\cdot\text{lb} = \frac{8.64}{32.2}v_A^2$$

$$\bar{v}_A^2 = 11.181 \quad \bar{\mathbf{v}}_A = 3.344\text{ ft/s} \longrightarrow$$

$$v_B = 0.6\bar{v}_A = 0.6(3.344)$$

$$\mathbf{v}_B = 2.01\text{ ft/s} \longleftarrow ◀$$



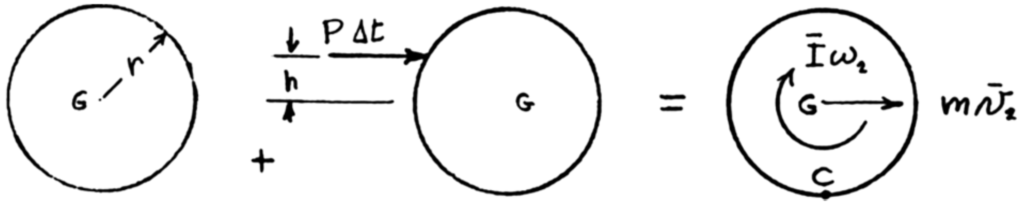
PROBLEM 17.96

At what height h above its center G should a billiard ball of radius r be struck horizontally by a cue if the ball is to start rolling without sliding?

SOLUTION

Moment of inertia. $$\bar{I} = \frac{2}{5}mr^2$$

Principle of impulse and momentum.



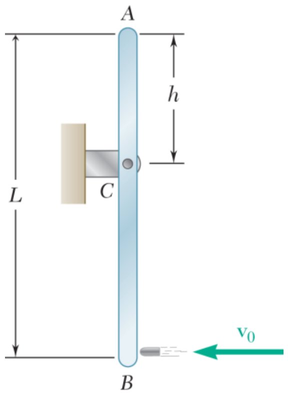
Syst. Momenta$_1$ + **Syst. Ext. Imp.** $_{1\to 2}$ = **Syst. Momenta**$_2$

Kinematics. Rolling without sliding. Point C is the instantaneous center of rotation.

$\xrightarrow{+}$ Linear components: $$0 + P\Delta t = m\bar{v}_2$$
$$= mr\omega_2$$

↺ Moments about G: $$0 + hP\Delta t = \bar{I}\omega_2$$
$$0 + h(mr\omega_2) = \left(\frac{2}{5}mr^2\right)\omega_2$$

$$h = \frac{2}{5}r \blacktriangleleft$$



PROBLEM 17.97

A bullet weighing 0.08 lb is fired with a horizontal velocity of 1800 ft/s into the lower end of a slender 15-lb bar of length $L = 30$ in. Knowing that $h = 12$ in. and that the bar is initially at rest, determine (*a*) the angular velocity of the bar immediately after the bullet becomes embedded, (*b*) the impulsive reaction at *C*, assuming that the bullet becomes embedded in 0.001 s.

SOLUTION

Bar:

$$L = 30 \text{ in.} = 2.5 \text{ ft} \quad m = \frac{15}{32.2} = 0.46584 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\bar{I} = \frac{1}{12} mL^2 = \frac{1}{12}(0.46584)(2.5)^2 = 0.24262 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Bullet:

$$m_0 = \frac{0.08}{32.2} = 0.0024845 \text{ lb}\cdot\text{s}^2/\text{ft}$$

Support location:

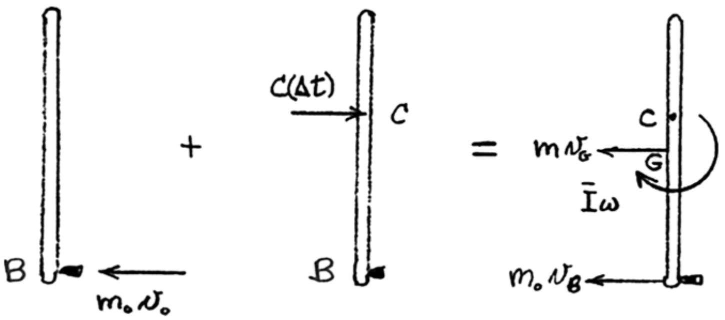
$$h = 12 \text{ in.} = 1.0 \text{ ft}$$

Kinematics.

$$v_B = (L-h)\omega = (2.5-1.0)\omega = 1.5\omega$$

$$v_G = \left(\frac{L}{2} - h\right)\omega = (1.25-1.0)\omega = 0.25\omega$$

Kinetics.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

↻ Moments about *C*: $m_0 v_0 (L-h) = m_0 v_B (L-h) + m v_0 \left(\frac{L}{2} - h\right) + \bar{I}\,\omega$

$$(0.0024845)(1800)(1.5) = (0.0024845)(1.5\omega) + (0.46584)(0.25\omega)(0.25) + (0.24262\omega)$$

PROBLEM 17.97 (Continued)

(*a*) $$6.7082 = 0.27546\omega \quad \text{or} \quad \omega = 24.353 \qquad \boldsymbol{\omega} = 24.4 \text{ rad/s} \circlearrowright \blacktriangleleft$$

$$v_B = (1.5)(24.353) = 36.53 \text{ ft/s}$$

$$v_G = (0.25)(24.353) = 6.0881 \text{ ft/s}$$

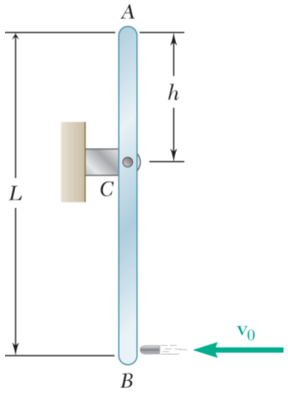
$\xrightarrow{+}$ Horizontal components:

$$-m_0 v_0 + C(\Delta t) = -m_0 v_B - m v_G: \quad C(\Delta t) = m_0(v_0 - v_B) - m v_0$$

$$C(\Delta t) = (0.0024845)(1800 - 36.53) - (0.46584)(6.0881)$$

$$= 1.545 \text{ lb} \cdot \text{s}$$

(*b*) $$C = \frac{C\Delta t}{\Delta t} = \frac{1.545}{0.001} \qquad \mathbf{C} = 1545 \text{ lb} \longrightarrow \blacktriangleleft$$



PROBLEM 17.98

In Problem 17.97, determine (*a*) the required distance *h* if the impulsive reaction at *C* is to be zero, (*b*) the corresponding angular velocity of the bar immediately after the bullet becomes embedded.

PROBLEM 17.97 A bullet weighing 0.08 lb is fired with a horizontal velocity of 1800 ft/s into the lower end of a slender 15-lb bar of length $L = 30$ in. Knowing that $h = 12$ in. and that the bar is initially at rest, determine (*a*) the angular velocity of the bar immediately after the bullet becomes embedded, (*b*) the impulsive reaction at *C*, assuming that the bullet becomes embedded in 0.001 s.

SOLUTION

Bar:

$$L = 30 \text{ in.} = 2.5 \text{ ft} \quad m = \frac{15}{32.2} = 0.46584 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$\bar{I} = \frac{1}{12}mL^2 = \frac{1}{12}(0.46584)(2.5)^2 = 2.24262 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Bullet:

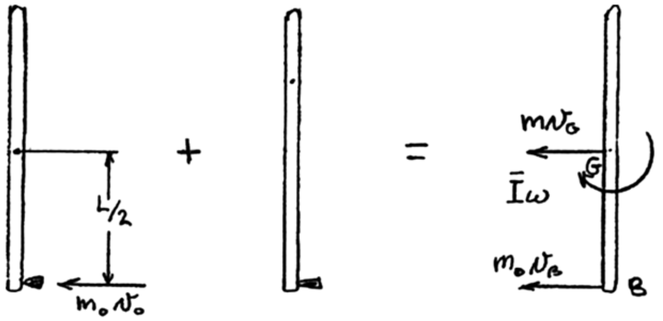
$$m_0 = \frac{0.08}{32.2} = 0.0024845 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Kinematics.

$$v_B = (L - h)\omega = (2.5 - h)\omega$$

$$v_G = \left(\frac{L}{2} - h\right)\omega = (1.25 - h)\omega$$

Kinetics.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

↻ moment about *B*:

$$0 + 0 = \bar{I}\omega - mv_G\left(\frac{L}{2}\right)$$

$$0 + 0 = 0.24262\omega - (0.46584)(1.25 - h)\omega(1.25)$$

Divide by ω

$$0 = 0.24262 - 0.5823(1.25 - h)$$

PROBLEM 17.98 (Continued)

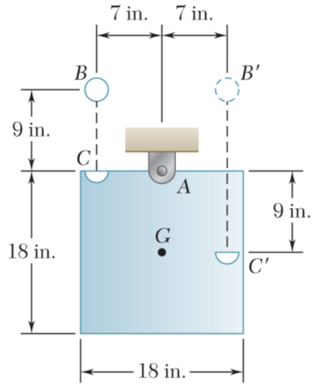
(*a*) $h = 0.8333\text{ ft}$ $h = 10.00\text{ in.}$ ◀

$$v_B = (2.5 - 0.8333)\omega = 1.6667\omega$$
$$v_G = (1.25 - 0.8333)\omega = 0.4167\omega$$

$\xleftarrow{+}$ Horizontal components: $m_0 v_0 + 0 = m v_G + m_0 v_B$

$$(0.0024845)(1800) + 0 = (0.46584)(0.4167\omega) + (0.0024845)(1.6667\omega)$$

(*b*) $\omega = 22.56$ $\boldsymbol{\omega} = 22.6\text{ rad/s}$ ↻ ◀


7 in.
7 in.
B
B′
9 in.
C
A
9 in.
G
18 in.
C′
18 in.

PROBLEM 17.99

A 16-lb wooden panel is suspended from a pin support at *A* and is initially at rest. A 4-lb metal sphere is released from rest at *B* and falls into a hemispherical cup *C* attached to the panel at a point located on its top edge. Assuming that the impact is perfectly plastic, determine the velocity of the mass center *G* of the panel immediately after the impact.

SOLUTION

Mass and moment of inertia

$$W_s = 4\text{ lb} \quad W_P = 16\text{ lb}$$

$$\bar{I} = \frac{1}{6}m_P(L)^2 = \frac{1}{6}\left(\frac{16}{32.2}\right)\left(\frac{18}{12}\right)^2 = 0.18634\text{ slug}\cdot\text{ft}^2$$

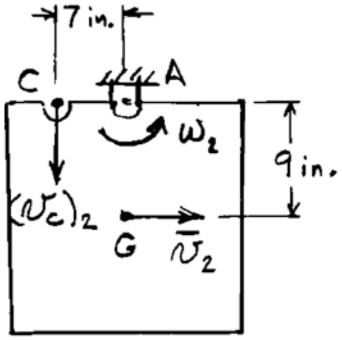
Velocity of sphere at *C*. $(v_C)_1 = \sqrt{2gy} = \sqrt{2(32.2\text{ ft/s}^2)\left(\tfrac{9}{12}\text{ ft}\right)} = 6.9498\text{ ft/s}$

Impact analysis.

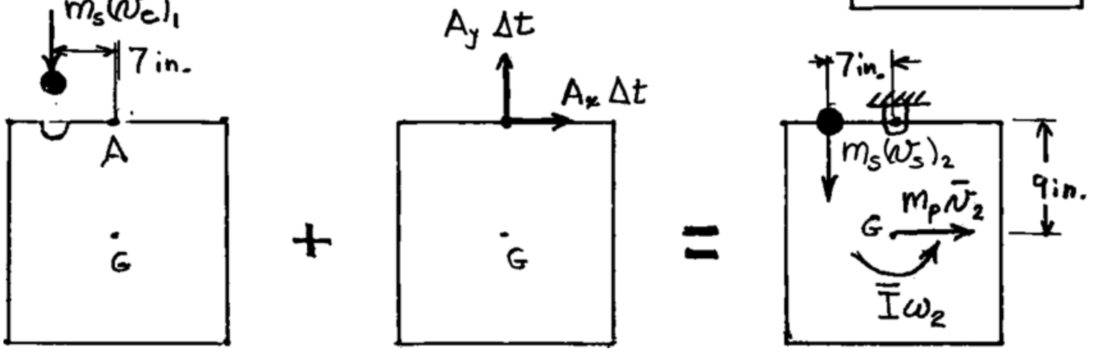
Kinematics: Immediately after impact in terms of ω_2

$$\bar{v}_2 = \frac{9}{12}\omega_2$$

$$(v_C)_2 = \frac{7}{12}\omega_2$$


7 in.
C
A
ω_2
9 in.
$(v_C)_2$
G
$\bar{v}_2$

Principle of impulse and momentum.


$m_s(v_C)_1$
7 in.
A
G
+
$A_y\,\Delta t$
$A_x\,\Delta t$
G
=
7in.
$m_s(v_s)_2$
$m_P\bar{v}_2$
G
$\bar{I}\omega_2$
9in.

Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

PROBLEM 17.99 (Continued)

$+\circlearrowleft$ Moments about A:

$$m_s(v_C)_1\left(\frac{7}{12}\text{ ft}\right)+0=m_s(v_C)_2\left(\frac{7}{12}\text{ ft}\right)+\bar{I}\omega_2+m_P\bar{v}_2\left(\frac{9}{12}\text{ ft}\right)$$

$$\left(\frac{4\text{ lb}}{32.2}\right)(6.9498\text{ ft/s})\left(\frac{7}{12}\text{ ft}\right)=\left(\frac{4\text{ lb}}{32.2}\right)\left(\frac{7}{12}\omega_2\right)\left(\frac{7}{12}\text{ ft}\right)+0.18634\omega_2+\left(\frac{16\text{ lb}}{32.2}\right)\left(\frac{9}{12}\omega_2\right)\left(\frac{9}{12}\text{ ft}\right)$$

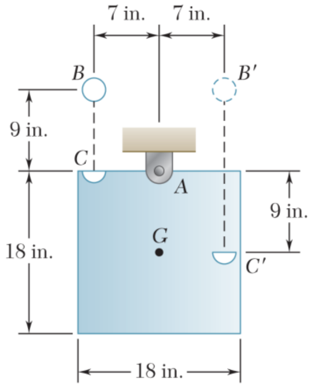
$$0.50361=(0.042271+0.18634+0.2795)\omega_2$$

$$\omega_2=0.99115\text{ rad/s}\quad \boldsymbol{\omega}_2=0.99115\text{ rad/s}\circlearrowleft$$

Velocity of the mass center

$$\bar{v}_2=\left(\frac{9}{12}\text{ ft}\right)\omega_2=\left(\frac{9}{12}\text{ ft}\right)(0.99115\text{ rad/s})$$

$$\bar{v}_2=0.74336\text{ ft/s}\qquad \bar{\mathbf{v}}_2=8.92\text{ in./s}\longrightarrow \blacktriangleleft$$



PROBLEM 17.100

An 16-lb wooden panel is suspended from a pin support at *A* and is initially at rest. A 4-lb metal sphere is released from rest at *B′* and falls into a hemispherical cup *C′* attached to the panel at the same level as the mass center *G*. Assuming that the impact is perfectly plastic, determine the velocity of the mass center *G* of the panel immediately after the impact.

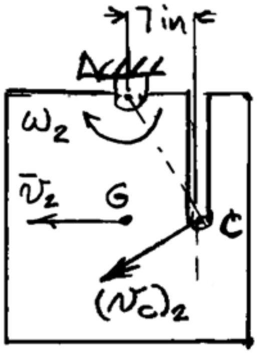
SOLUTION

Mass and moment of inertia.

$$W_s = 4 \text{ lb}$$
$$W_P = 16 \text{ lb}$$
$$\bar{I} = \frac{1}{6} m_P (0.5 \text{ m})^2$$
$$= \frac{1}{6}\left(\frac{16}{32.2}\right)\left(\frac{18}{12}\right)^2$$
$$= 0.18634 \text{ slug} \cdot \text{ft}^2$$

Velocity of sphere at *C′*.

$$(v_{C'})_1 = \sqrt{2gy}$$
$$= \sqrt{2(32.2 \text{ ft/s}^2)\left(\tfrac{18}{12} \text{ ft}\right)}$$
$$= 9.8285 \text{ ft/s}$$



Impact analysis.

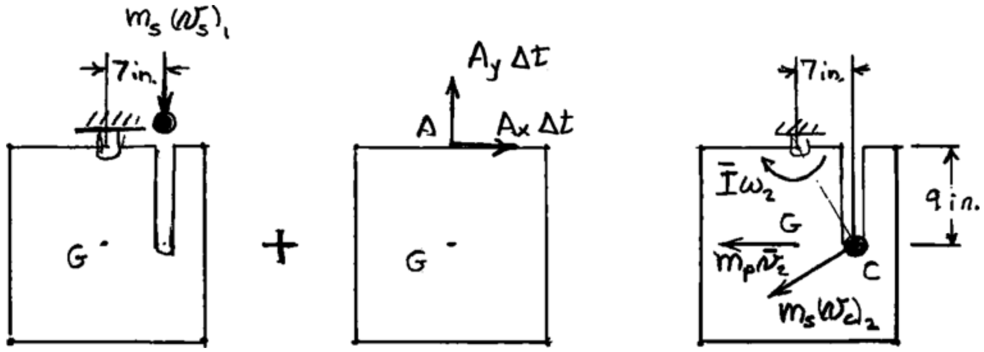
Kinematics: Immediately after impact in terms of ω_2.

$$AC' = \sqrt{\left(\frac{7}{12}\right)^2 + \left(\frac{9}{12}\right)^2} = 0.95015 \text{ ft}$$

$$(\mathbf{v}_{C'})_2 = AC'\omega_2$$
$$= 0.95015\omega_2 \swarrow \theta \quad \text{(perpendicular to } AC.)$$
$$\bar{v}_2 = \frac{9}{12}\omega_2$$

PROBLEM 17.100 (Continued)

Principle of impulse and momentum.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1-2}$ = **Syst. Momenta**$_2$

$\overset{+}{\circlearrowright}$ Moments about A:

$$m_s(v_C)_1\left(\frac{7}{12}\text{ ft}\right)+0=m_s(v_C)_2(0.95015\text{ ft})+\bar{I}\omega_2+m_P\bar{v}_2\left(\frac{9}{12}\text{ ft}\right)$$

$$\left(\frac{4\text{ lb}}{32.2}\right)(9.8285\text{ ft/s})\left(\frac{7}{12}\text{ ft}\right)=\left(\frac{4\text{ lb}}{32.2}\right)(0.95015\omega_2)(0.95015\text{ ft})+0.18634\omega_2$$

$$+\left(\frac{16\text{ lb}}{32.2}\right)\left(\frac{9}{12}\omega_2\right)\left(\frac{9}{12}\text{ ft}\right)$$

$$0.71221=(0.11215+0.18634+0.2795)\omega_2$$

$$\omega_2=1.2322$$

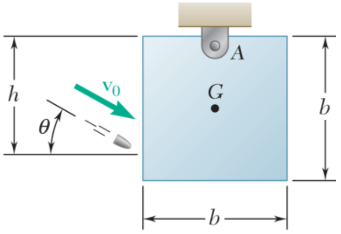
Velocity of the mass center.

$$\bar{v}_2=\left(\frac{9\text{ ft}}{12}\right)\omega_2$$

$$=\left(\frac{9\text{ ft}}{12}\right)(1.2322\text{ rad/s})$$

$$=0.92418\text{ ft/s}$$

$\bar{\mathbf{v}}_2=11.09\text{ in./s}\longleftarrow$ ◀



PROBLEM 17.101

A 45-g bullet is fired with a velocity of 400 m/s at $\theta = 30°$ into a 9-kg square panel of side $b = 200$ mm. Knowing that $h = 150$ mm and that the panel is initially at rest, determine (*a*) the velocity of the center of the panel immediately after the bullet becomes embedded, (*b*) the impulsive reaction at *A*, assuming that the bullet becomes embedded in 2 ms.

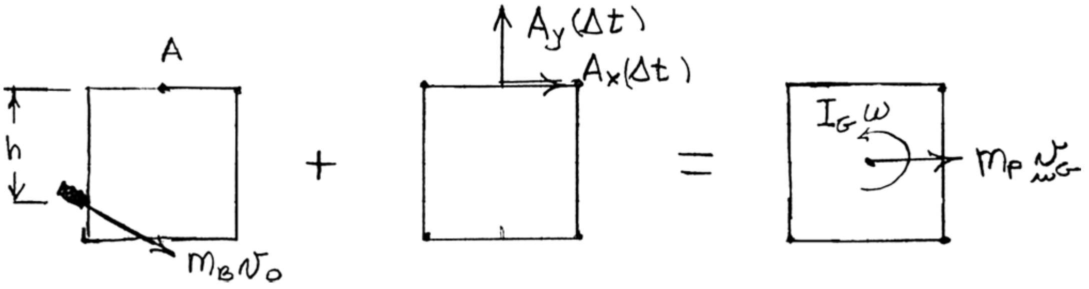
SOLUTION

$$m_B = 0.045\text{ kg} \quad m_P = 9\text{ kg} \quad I_G = \frac{1}{6}m_P b^2 = \frac{1}{6}(9)(0.200)^2 = 0.06\text{ kg}\cdot\text{m}^2$$

Kinematics. After impact, the plate is rotating about the fixed Point *A* with angular velocity $\boldsymbol{\omega} = \omega \circlearrowleft$.

$$\mathbf{v}_G = \frac{b}{2}\omega \longrightarrow$$

Principle of impulse and momentum. To simplify the analysis, neglect the mass of the bullet after impact.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

(*a*) $+\circlearrowleft$ Moments about *A*:

$$(m_B v_0\cos 30°)h + m_B v_0 \sin 30°\left(\frac{b}{2}\right) + 0 = I_G\omega + m_P v_G\frac{b}{2}$$

$$m_B v_0\left(h\cos 30° + \frac{b}{2}\sin 30°\right) = \left(I_G + \frac{1}{4}m_P b^2\right)\omega$$

$$(0.045)(400)(0.150\cos 30° + 0.100\sin 30°)$$

$$= \left[0.06 + \frac{1}{4}(9)(0.2)^2\right]\omega = 0.15\omega$$

$$\omega = 21.588\text{ rad/s}$$

$$v_B = (0.100)(21.588) = 2.1588\text{ m/s} \qquad \mathbf{v}_G = 2.16\text{ m/s} \longrightarrow ◀$$

PROBLEM 17.101 (Continued)

(*b*) $\xrightarrow{+}$ Linear momentum:

$$m_B v_0 \cos 30^\circ + A_x(\Delta t) = m_P v_G$$

$$(0.045)(400 \cos 30^\circ) + A_x(0.002) = (9)(2.1588)$$

$$A_x = 1920\text{ N} \qquad \mathbf{A}_x = 1920\text{ N} \longrightarrow$$

$+\uparrow$ Linear momentum:

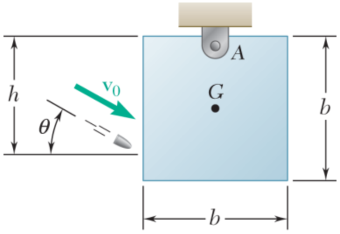
$$-m_B v_0 \sin 30^\circ + A_y(\Delta t) = 0$$

$$-(0.045)(400) \sin 30^\circ + A_y(0.002) = 0$$

$$A_y = 4500\text{ N} \qquad \mathbf{A}_y = 4500\text{ N} \uparrow$$

$$A = 4892\text{ N} = 4.892\text{ kN} \qquad \tan \beta = \frac{4500}{1920} \quad \beta = 66.9^\circ$$

$$\mathbf{A} = 4.87\text{ kN} \measuredangle 66.9^\circ$$ ◀



PROBLEM 17.102

A 45-g bullet is fired with a velocity of 400 m/s at $\theta = 5°$ into a 9-kg square panel of side $b = 200$ mm. Knowing that the panel is initially at rest, determine (*a*) the required distance *h* if the horizontal component of the impulsive reaction at *A* is to be zero, (*b*) the corresponding velocity of the center of the panel immediately after the bullet becomes embedded.

SOLUTION

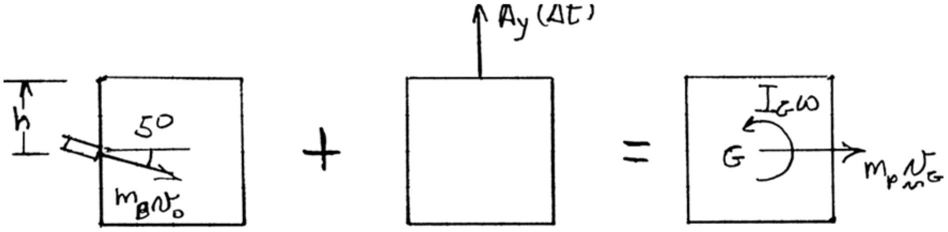
$$m_B = 0.045 \text{ kg} \quad m_P = 9 \text{ kg} \quad I_G = \frac{1}{6}m_P b^2 = \frac{1}{6}(9)(0.200)^2 = 0.06 \text{ kg}\cdot\text{m}^2$$

Kinematics. After impact, the plate is rotating about the fixed Point *A* with angular velocity $\boldsymbol{\omega} = \omega \circlearrowright$.

$$\mathbf{v}_G = \frac{b}{2}\omega \longrightarrow.$$

Principle of impulse and momentum. To simplify the analysis, neglect the mass of the bullet after impact.

Also $$A_X(\Delta t) = 0.$$



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

$\xrightarrow{+}$ Linear momentum: $$m_B v_0 \cos 5° + 0 = m_P v_G = m_P\left(\frac{b}{2}\omega\right)$$

$$(0.045)(400\cos 5°) = (9)(0.100)\omega \quad \omega = 19.9239 \text{ rad/s}$$

$$v_G = (0.100)(19.9239) = 1.99239 \text{ m/s} \qquad (1)$$

$+\circlearrowright$ Moments about *A*: $$(m_B v_0 \cos 5°)h + (m_B v_0 \sin 5°)\frac{b}{2} = I_G\omega + m_P v_G \frac{b}{2}$$

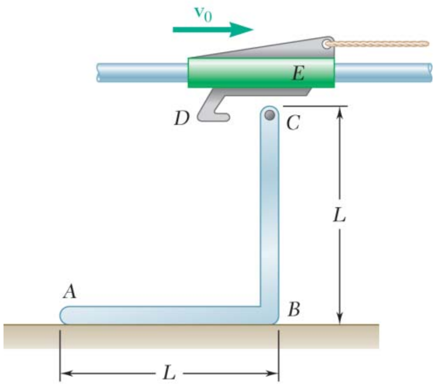
$$m_B v_0\left(h\cos 5° + \frac{b}{2}\sin 5°\right) = \left(I_G + \frac{1}{4}m_P b^2\right)\omega$$

$$(0.045)(400)(h\cos 5° + 0.100\sin 5°) = \left[0.06 + \frac{1}{4}(9)(0.200)^2\right](19.9239)$$

$$17.9315h + 0.1569 = 2.9886$$

(*a*) $h = 0.15792$ m $\qquad h = 158$ mm ◀

(*b*) From Eq. (1), $\qquad \mathbf{v}_G = 1.992$ m/s $\longrightarrow$ ◀



PROBLEM 17.103

The uniform rods, each of mass m, form the L-shaped rigid body ABC which is initially at rest on the frictionless horizontal surface when hook D of the carriage E engages a small pin at C. Knowing that the carriage is pulled to the right with a constant velocity $\mathbf{v}_0$, determine immediately after the impact (*a*) the angular velocity of the body, (*b*) the velocity of corner B. Assume that the velocity of the carriage is unchanged and that the impact is perfectly plastic.

SOLUTION

Kinematics: $\mathbf{v}_B = v_B \longrightarrow, \quad \boldsymbol{\omega} = \omega \circlearrowright$

$$\bar{\mathbf{v}}_{BC} = [v_0 \longrightarrow] + \left[\frac{L}{2}\omega \longleftarrow\right]$$

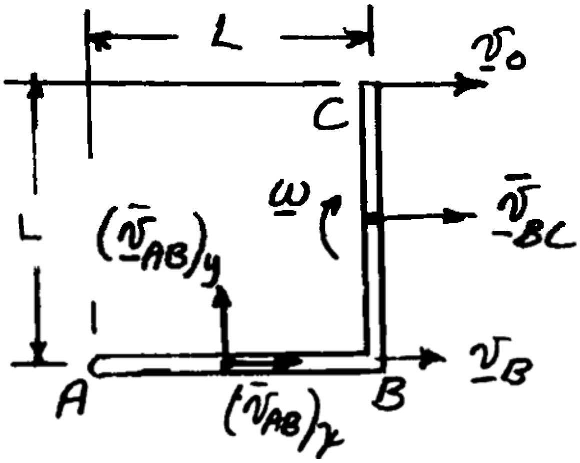
$$\bar{\mathbf{v}}_{BC} = \left[v_0 - \frac{L}{2}\omega\right] \longrightarrow$$

$$\mathbf{v}_B = [v_0 \longrightarrow] + [L\omega \longleftarrow]$$

$$\mathbf{v}_B = [v_0 - L\omega] \longrightarrow$$

$$(\bar{v}_{AB})_x = v_B = [v_0 - L\omega] \longrightarrow$$

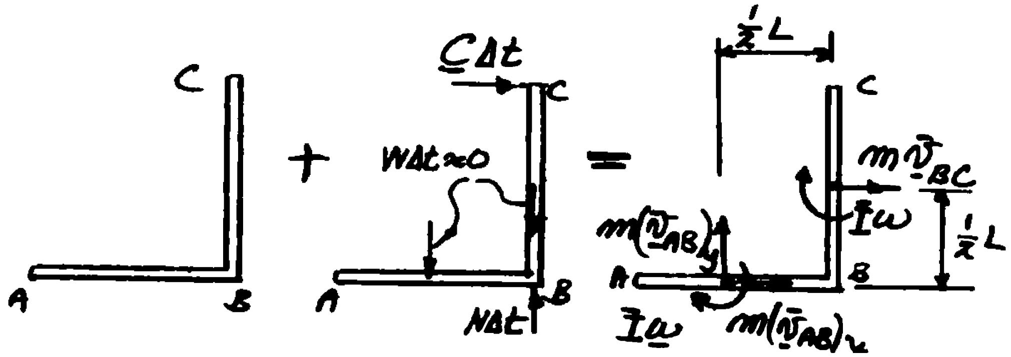
$$(\bar{v}_{AB})_y = (v_B)_y + \frac{L}{2}\omega \uparrow = \frac{L}{2}\omega \uparrow$$


L
v_0
C
ω
$\bar{v}_{BC}$
L
$(\bar{v}_{AB})_y$
A
B
v_B
$(\bar{v}_{AB})_x$

Let m be the mass of each rod.

Moment of inertia of each rod. $\bar{I} = \frac{1}{12}mL^2$

(a) *Principle of impulse and momentum.*


C
A
B
+
$C\Delta t$
C
$W\Delta t \approx 0$
A
B
$N\Delta t$
=
$\frac{1}{2}L$
C
$m\bar{v}_{BC}$
$\bar{I}\omega$
$\frac{1}{2}L$
$m(\bar{v}_{AB})_y$
A
B
$\bar{I}\omega$
$m(\bar{v}_{AB})_x$

Syst. Momenta$_0$ + **Syst. Ext. Imp.**$_{0-1}$ = **Syst. Momenta**$_1$

PROBLEM 17.103 (Continued)

$+\circlearrowright$ Moments about C:

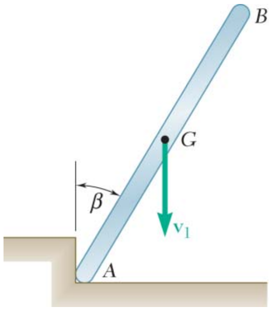
$$0+0=m(v_{AB})_y\left(\frac{1}{2}L\right)-m(v_{AB})_x L+\bar{I}\,\omega-m(\bar{v}_{BC})\left(\frac{1}{2}L\right)+\bar{I}\,\omega$$

$$0=m\left(\frac{L}{2}\omega\right)\left(\frac{1}{2}L\right)-m(v_0-L\omega)L+\frac{1}{12}mL^2\omega-m\left(v_0-\frac{L}{2}\omega\right)\left(\frac{1}{2}L\right)+\frac{1}{12}mL^2\omega$$

$$0=-\frac{3}{2}mLv_0+\frac{5}{3}mL^2\omega \qquad \omega=\frac{9}{10}\frac{v_0}{L}$$

(*a*) *Angular velocity* $\boldsymbol{\omega}=0.900\ v_0/L \circlearrowright$ ◀

(*b*) *Velocity of B.* $v_B=v_0-L\omega=\frac{1}{10}v_0$ $\mathbf{v}_B=0.100\ v_0 \longrightarrow$ ◀

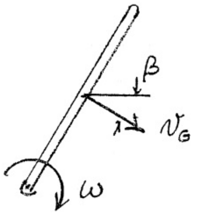


PROBLEM 17.104

The uniform slender rod *AB* of weight 5 lb and length 30 in. forms an angle $\beta = 30°$ with the vertical as it strikes the smooth corner shown with a vertical velocity $\mathbf{v}_1$ of magnitude 8 ft/s and no angular velocity. Assuming that the impact is perfectly plastic, determine the angular velocity of the rod immediately after the impact.

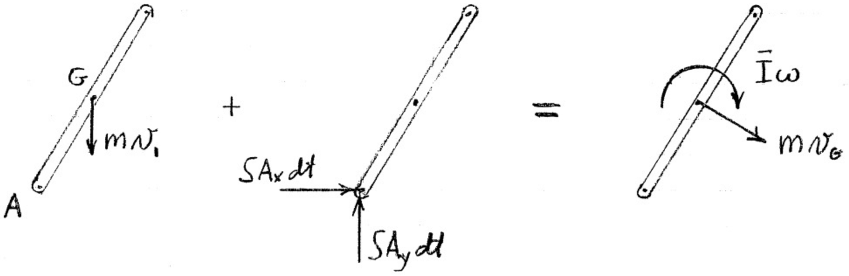
SOLUTION

Moment of inertia.
$$\bar{I} = \frac{1}{12}mL^2 = \frac{1}{12}\left(\frac{5}{32.2}\right)\left(\frac{30}{12}\right)^2 = 80.875\times 10^{-3}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$



Kinematics. (Rotation about *A*)
$$\beta = 30°$$
$$v_G = \frac{L}{2}\omega = \frac{15}{12}\omega$$

Kinetics.

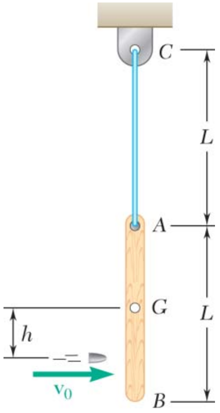


Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

↻ moments about *A*:
$$mv_1\frac{L}{2}\sin\beta + 0 = \bar{I}\omega + mv_G\frac{L}{2}$$
$$\left(\frac{5}{32.2}\right)(8)\left(\frac{15}{12}\right)\sin 30° + 0 = 80.875\times 10^{-3}\omega + \left(\frac{5}{32.2}\right)\left(\frac{15}{12}\omega\right)\left(\frac{15}{12}\right)$$
$$\omega = 2.4\ \text{rad/s}$$

$\omega = 2.40$ rad/s ↻ ◀

PROBLEM 17.105



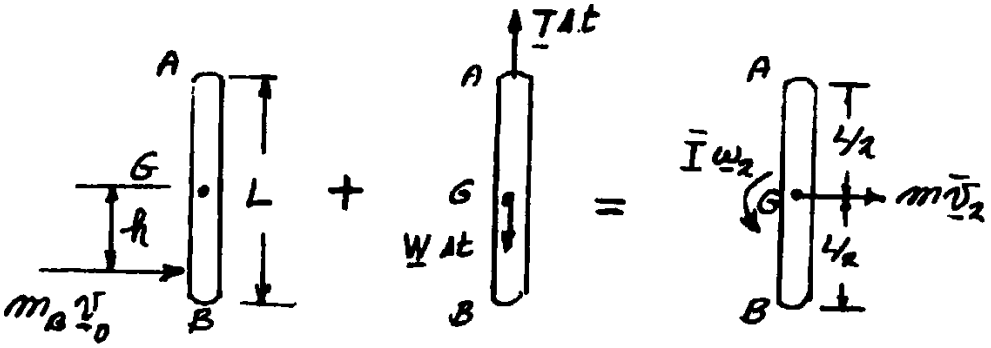
A bullet weighing 0.08 lb is fired with a horizontal velocity of 1800 ft/s into the 15-lb wooden rod *AB* of length *L* = 30 in. The rod, which is initially at rest, is suspended by a cord of length *L* = 30 in. Determine the distance *h* for which, immediately after the bullet becomes embedded, the instantaneous center of rotation of the rod is Point *C*.

SOLUTION

Let m_B be the mass of the bullet and *m* the mass of the rod. The moment of inertia *I* of the rod is

$$\bar{I} = \frac{1}{12} m L^2$$

Principle of impulse and momentum.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

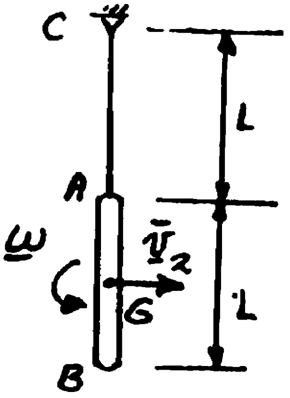
$+\circlearrowright$ Moments about *G*: $\quad m_B v_0 h = \bar{I}\omega_2 \qquad (1)$

$\xrightarrow{+}$ *x* components: $\quad m_B v_0 = m\bar{v}_2 \qquad (2)$

From Eq. (2). $\quad \bar{v}_2 = \frac{m_B}{m} v_0 = \frac{W_B}{W} v_0 \qquad (3)$

From Eq. (3). $\quad w_2 = \frac{m_B v_0 h}{\bar{I}} = \frac{\frac{W_B}{g} v_0 h}{\frac{1}{12}\frac{W}{g} L^2} = 12 \frac{W_B}{W} \frac{v_0 h}{L^2} \qquad (4)$

For the instantaneous center to lie at Point *C*, $\quad \bar{v}_2 = \frac{3}{2} L\omega_2$

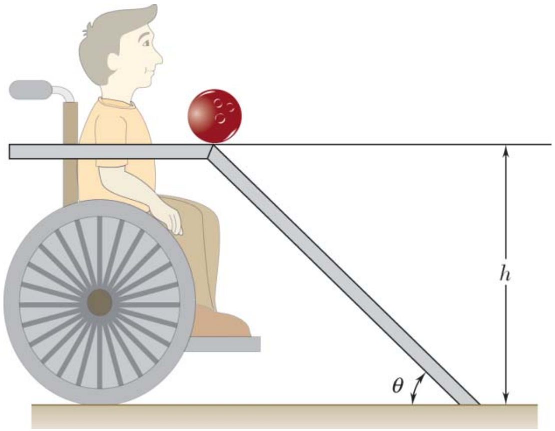


PROBLEM 17.105 (Continued)

Substitute for $\bar{v}_2$ and ω_2 from Equations (3) and (4).

$$\frac{W_B}{W}v_0 = \frac{3}{2}L\left[12\frac{W_B}{W}\cdot\frac{v_0 h}{L^2}\right]$$

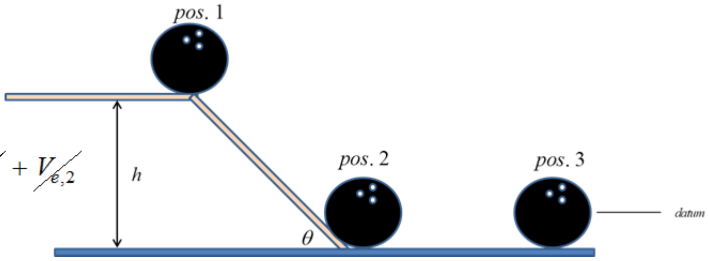
$$h = \frac{1}{18}L = \frac{30\text{ in.}}{18} \qquad h = 1.667\text{ in.} \blacktriangleleft$$



PROBLEM 17.106

A prototype of an adapted bowling device is a simple ramp that attaches to a wheelchair. The bowling ball has a mass moment of inertia about its center of gravity of cmr^2, where c is a unitless constant, r is the radius, and m is its mass. The athlete nudges the ball slightly from a height of h, and the ball rolls down the ramp without sliding. It hits the bowling lane, and after slipping for a short distance, it begins to roll again. Assuming that the ball does not bounce as it hits the lane, determine the angular velocity and velocity of the mass center of the ball after it has resumed rolling.

SOLUTION



Given: $\bar{I} = cmr^2$

Work Energy Equation Pos. 1 and 2:

$$\cancel{T_1} + V_{g1} + \cancel{V_{e,1}} + \cancel{U'_{1-2}} = T_2 + \cancel{V_{g2}} + \cancel{V_{e,2}}$$

$$V_{g1} = T_2$$

$$mgh = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2$$

Rolling Kinematics: $\omega_2 = \dfrac{\bar{v}_2}{r}$

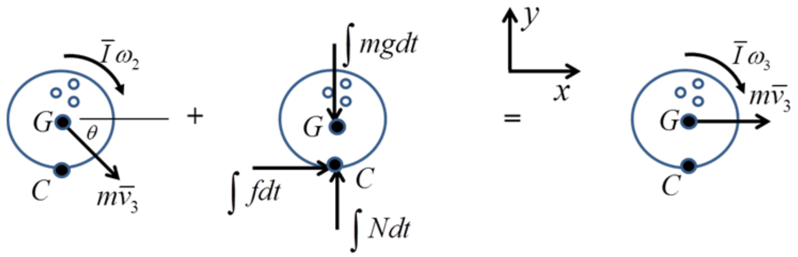
Substitute into the Work Energy Equation:

$$\cancel{m}gh = \frac{1}{2}\cancel{m}\bar{v}_2^2 + \frac{1}{2}c\cancel{m}\cancel{r^2}\frac{\bar{v}_2^2}{\cancel{r^2}}$$

$$2gh = \bar{v}_2^2(1+c)$$

$$\bar{v}_2 = \sqrt{\frac{2gh}{1+c}}$$

Impulse Momentum Diagram between time 2 and time 3:



$$H_{C,2} + \int \cancel{M_c}\,dt = H_{O,3}$$

$$\bar{I}\omega_2 + m\bar{v}_2\cos\theta = \bar{I}\omega_3 + mv_3r$$

PROBLEM 17.106 (Continued)

Substitute rolling Kinematics and Mass Moment Equations into Momentum Equation:

$$c\cancel{m}r^2\frac{\overline{v}_2}{r}+\cancel{m}\overline{v}_2 r\cos\theta = c\cancel{m}r^2\omega_3+\cancel{m}r\omega_3 r$$
$$(c+\cos\theta)\overline{v}_2=(c+1)r\omega_3$$

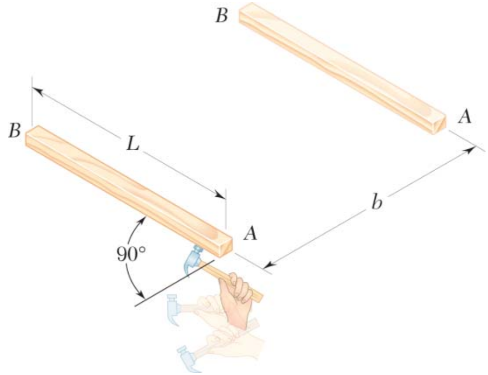
Substitute in the velocity at position 2 and solve for the angular velocity at position 3:

$$(c+\cos\theta)\overline{v}_2=(c+1)r\omega_3$$
$$\omega_3=\frac{(c+\cos\theta)}{r(c+1)}\sqrt{\frac{2gh}{1+c}}$$

$$\omega_3=\frac{(c+\cos\theta)}{r(c+1)^{\frac{3}{2}}}\sqrt{2gh} \blacktriangleleft$$

Velocity at position 3: $\overline{v}_3=r\omega_3$

$$\overline{v}_3=\frac{(c+\cos\theta)}{(c+1)^{\frac{3}{2}}}\sqrt{2gh} \blacktriangleleft$$



PROBLEM 17.107

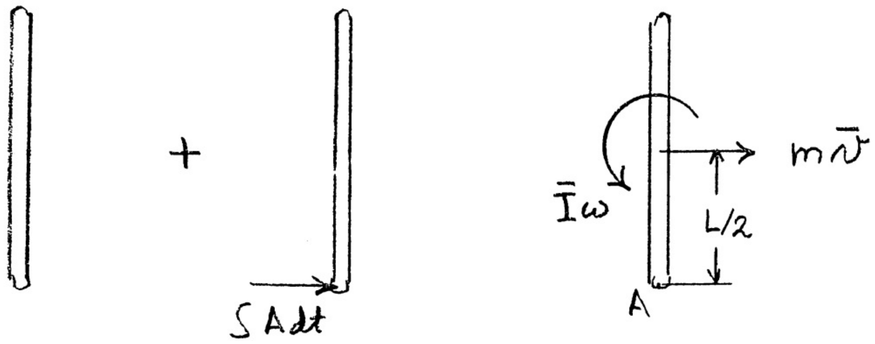
A uniform slender rod *AB* is at rest on a frictionless horizontal table when end *A* of the rod is struck by a hammer which delivers an impulse that is perpendicular to the rod. In the subsequent motion, determine the distance *b* through which the rod will move each time it completes a full revolution.

SOLUTION

Moment of inertia.

$$\bar{I} = \frac{1}{12}mL^2$$

Kinetics.



Syst Momenta$_1$ + **Syst Ext Imp**$_{1\to2}$ = **Syst Momenta**$_2$

$\curvearrowleft$ moments about *A*:

$$0 + 0 + \bar{I}\omega + m\bar{v}\frac{L}{2}$$

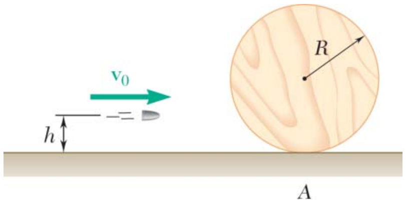
$$\bar{v} = \frac{2\bar{I}\omega}{mL} = \frac{2\frac{1}{12}mL^2\omega}{mL} = \frac{1}{6}\omega L$$

Motion after impact.

$$\theta = \omega t \qquad t = \frac{\theta}{\omega} = \frac{2\pi}{\omega}$$

$$b = \bar{v}t = \left(\frac{1}{6}\omega L\right)\frac{2\pi}{\omega}$$

$$b = \frac{\pi}{3}L \blacktriangleleft$$



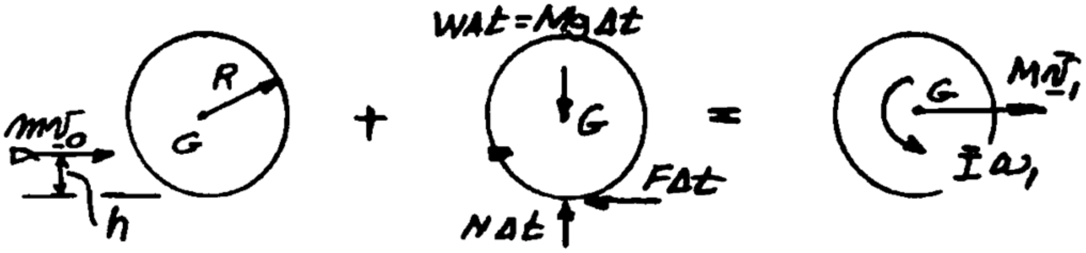
PROBLEM 17.108

A bullet of mass m is fired with a horizontal velocity v_0 and at a height $h = \frac{1}{2}R$ into a wooden disk of much larger mass M and radius R. The disk rests on a horizontal plane and the coefficient of friction between the disk and the plane is finite. (*a*) Determine the linear velocity $\bar{\mathbf{v}}_1$ and the angular velocity $\boldsymbol{\omega}_1$ of the disk immediately after the bullet has penetrated the disk. (*b*) Describe the ensuing motion of the disk and determine its linear velocity after the motion has become uniform.

SOLUTION

(*a*) Conditions immediately after the bullet has penetrated the disk.

Principle of impulse and momentum:



Syst. Momenta$_0$ + **Syst. Ext. Imp.**$_{0\to1}$ = **Syst. Momenta**$_1$

$+\uparrow$ y components: $0 + N\Delta t - W\Delta t = 0 \quad N = W$

$\xrightarrow{+}$ x components:

$$mv_0 - F\Delta t = M\bar{v}_1$$

$$mv_0 - \mu W\Delta t = M\bar{v}_1$$

Since $\Delta t \approx 0$, $\quad mv_0 = m\bar{v}_1$

$$\bar{\mathbf{v}}_1 = \frac{mv_0}{M} \longrightarrow \quad (1) ◀$$

$+\circlearrowright$ Moments about G: $\quad mv_0 = (R-h) - R(\mu W\Delta t) = \bar{I}\omega_1$

Since $\Delta t \approx 0$, $\quad mv_0 = (R-h) = \frac{1}{2}MR^2\omega_1$

$$\omega_1 = 2\frac{m}{M}\frac{R-h}{R^2}v_0 \circlearrowright \quad (2)$$

But $\quad h = \frac{1}{2}R$

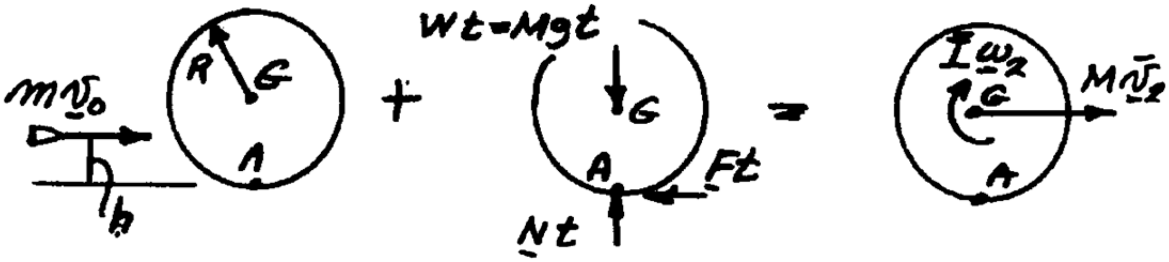
$$\omega_1 = 2\frac{m}{M}\frac{R - \frac{1}{2}R}{R^2}$$

$$\omega_1 = \frac{mv_0}{MR} \circlearrowright ◀$$

PROBLEM 17.108 (Continued)

(*b*) After the motion becomes uniform, the disk rolls without slipping.

Kinematics. $\bar{v}_2 = R\omega_2$



Syst. Momenta$_0$ + **Syst. Ext. Imp.**$_{0\to 2}$ = **Syst. Momenta**$_2$

$+\circlearrowright$ Moments about A: $mv_0 h + 0 = M\bar{v}_2 R + \bar{I}\omega_2$

Since $h = \frac{1}{2}R$ is given:

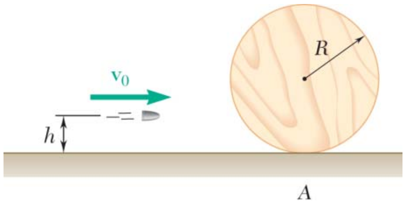
$$mv_0\left(\frac{1}{2}R\right) = (MR\omega_2)R + \frac{1}{2}MR^2\omega_2$$

$$\frac{1}{2}mv_0 = \frac{3}{2}MR\omega_2$$

$$\omega_2 = \frac{mv_0}{3MR} \circlearrowright$$

$$\bar{v}_2 = R\omega_2 \qquad \bar{\mathbf{v}}_2 = \frac{mv_0}{3M} \longrightarrow \blacktriangleleft$$

At first the disk slides $\longrightarrow$ and rotates $\circlearrowright$, it latter rolls with a constant linear velocity $\bar{\mathbf{v}}_2$ and a constant angular velocity $\boldsymbol{\omega}_2$.

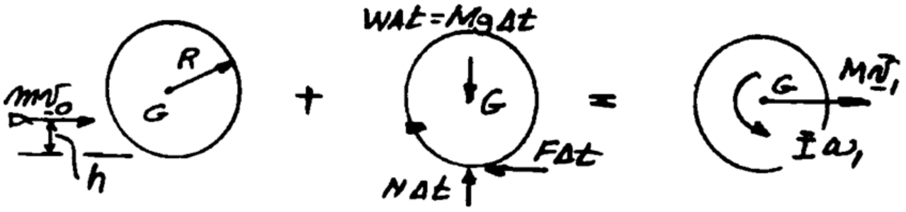


PROBLEM 17.109

Determine the height *h* at which the bullet of Problem 17.108 should be fired (*a*) if the disk is to roll without sliding immediately after impact, (*b*) if the disk is to slide without rolling immediately after impact.

SOLUTION

Principle of impulse and momentum:



Syst. Momenta$_0$ + **Syst. Ext. Imp.** = **Syst. Momenta$_1$**

$+\uparrow$ *y* components: $0 + N\Delta t - W\Delta t = 0 \qquad N = W$

$\xrightarrow{+}$ *x* components: $mv_0 - F\Delta t = M\bar{v}_1$

$$mv_0 - \mu W\Delta t = M\bar{v}_1$$

Since $\Delta t \approx 0$; $\quad mv_0 = m\bar{v}_1 \qquad \bar{\mathbf{v}}_1 = \dfrac{mv_0}{M} \longrightarrow \quad (1)$

$+\circlearrowright$ Moments about *G*: $mv_0 = (R-h) - R(\mu W\Delta t) = \bar{I}\,\omega_1$

Since $\Delta t \approx 0$; $\quad mv_0 = (R-h) = \dfrac{1}{2}MR^2\omega_1$

$$\omega_1 = 2\frac{m}{M}\frac{R-h}{R^2}v_0 \circlearrowright \qquad (2)$$

(*a*) If disk is to roll without sliding immediately after impact, we must have

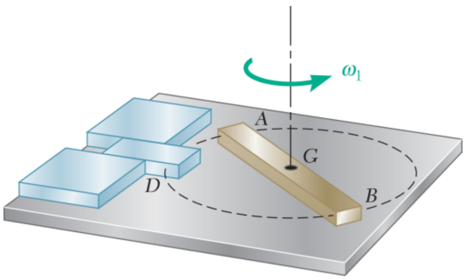
$\omega_1 \circlearrowright$ and $\bar{v}_1 = R\omega_1 \longrightarrow$

$$\frac{mv_0}{M} = -R\left[\frac{2m}{M}\cdot\frac{R-h}{R^2}v_0\right]$$

$$1 = -2\frac{R-h}{R} \qquad h = \frac{3}{2}R \blacktriangleleft$$

(*b*) If disk is to slide without rotating,

$$\omega_1 = \frac{2m}{M}\cdot\frac{R-h}{R^2}v_0 = 0 \qquad h = R \blacktriangleleft$$



PROBLEM 17.110

A uniform slender bar of length $L = 200$ mm and mass $m = 0.5$ kg is supported by a frictionless horizontal table. Initially the bar is spinning about its mass center G with a constant angular velocity $\boldsymbol{\omega}_1 = 6$ rad/s. Suddenly latch D is moved to the right and is struck by end A of the bar. Knowing that the coefficient of restitution between A and D is $e = 0.6$, determine the angular velocity of the bar and the velocity of its mass center immediately after the impact.

SOLUTION

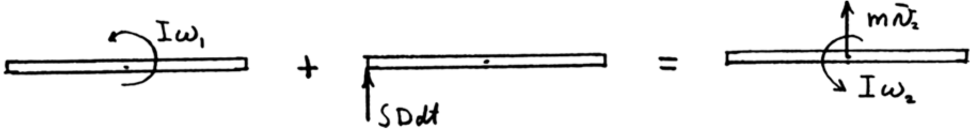
Moment of inertia. $$\bar{I} = \frac{1}{12}mL^2$$

Before impact. $$(\mathbf{v}_A)_1 = \frac{L}{2}\omega_1 \downarrow$$

Impact condition. $$(\mathbf{v}_A)_2 = -e(\mathbf{v}_A)_1 = \frac{1}{2}eL\omega_1 \uparrow$$

Kinematics after impact. $$\bar{v}_2 = (v_A)_2 + \frac{L}{2}\omega_2 = \frac{1}{2}eL\omega_1 + \frac{1}{2}L\omega_2$$

Principle of impulse-momentum at impact.



Syst. Momenta$_1$ + **Syst. Ext. Imp.$_{1\to 2}$** = **Syst. Momenta$_2$**

$\circlearrowright$ Moments about D: $$I\omega_1 + 0 = I\omega_2 + m\bar{v}_2\frac{L}{2}$$

$$I\omega_1 = I\omega_2 + m\left(\frac{1}{2}eL\omega_1 + \frac{1}{2}L\omega_2\right)\frac{L}{2}$$

$$\frac{1}{12}mL^2\omega_1 = \frac{1}{12}mL^2\omega_2 + \frac{1}{4}mL^2e\omega_1 + \frac{1}{4}mL^2\omega_2$$

$$\omega_2 = \frac{1}{4}(1-3e)\omega_1$$

$$\bar{v}_2 = \frac{1}{2}Le\omega_1 + \frac{1}{2}L\left(\frac{1}{4}\right)(1-3e)\omega_1 = \frac{1}{8}(1+e)\omega_1 L$$

Coefficient of restitution, $$e = 0.6$$

$$\omega_2 = \frac{1}{4}(1-(3)(0.6))(6\text{ rad/s}) = -1.200 \qquad \boldsymbol{\omega}_2 = 1.200\text{ rad/s} \circlearrowright$$ ◀

$$v_2 = \frac{1}{8}(1+0.6)(6\text{ rad/s})(0.2\text{ m}) = 0.240\text{ m/s} \qquad \mathbf{v}_2 = 0.240\text{ m/s}\uparrow$$ ◀

PROBLEM 17.110 (Continued)

$$v_A = \frac{2\bar{I}\omega}{ma} = \frac{2\left(\frac{1}{6}ma^2\right)\omega}{ma} = \frac{1}{3}a\omega$$

$\searrow$ 45° linear components: $$\frac{\sqrt{2}}{2}mv_0 + 0 = mv_A + \frac{\sqrt{2}}{2}mb\omega$$

$$\frac{\sqrt{2}}{2}mv_0 = m\left(\frac{1}{3}a\omega\right) + \frac{\sqrt{2}}{2}m\left(\frac{\sqrt{2}}{2}a\right)\omega = \frac{5}{6}ma\omega$$

$$\omega = \frac{3\sqrt{2}}{5}\frac{v_0}{a}$$

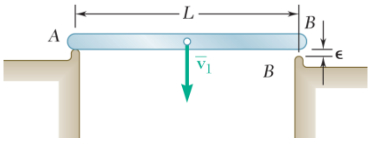
$\boldsymbol{\omega} = 0.849\dfrac{v_0}{a} \circlearrowleft$ ◀

$$\bar{\mathbf{v}} = \left[\frac{1}{3}a\omega \searrow 45°\right] + \left[b\omega \downarrow\right]$$

$$= \left[\frac{1}{3}b\omega \longrightarrow\right] + \left[\frac{1}{3}b\omega \downarrow\right] + \left[b\omega \downarrow\right]$$

$$= \left[\frac{1}{5}v_0 \longrightarrow\right] + \left[\frac{4}{5}v_0 \downarrow\right]$$

$\bar{\mathbf{v}} = 0.825v_0 \searrow 76.0°$ ◀



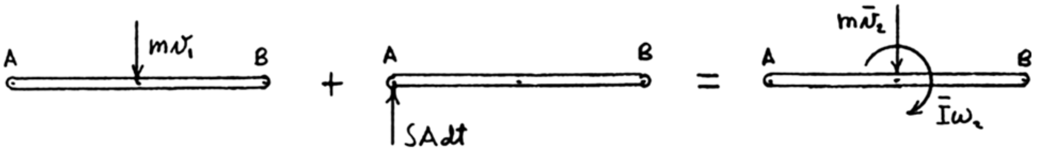
PROBLEM 17.111

A uniform slender rod of length L is dropped onto rigid supports at A and B. Since support B is slightly lower than support A, the rod strikes A with a velocity $\bar{\mathbf{v}}_1$ before it strikes B. Assuming perfectly elastic impact at both A and B, determine the angular velocity of the rod and the velocity of its mass center immediately after the rod (*a*) strikes support A, (*b*) strikes support B, (*c*) again strikes support A.

SOLUTION

Moment of inertia. $\bar{I} = \frac{1}{12} mL^2$

(*a*) *First impact at A.*



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

Condition of impact: $e = 1: \quad (\mathbf{v}_A)_2 = v_1 \uparrow$

Kinematics: $\bar{v}_2 = \frac{L}{2}\omega - (v_A)_2 = \frac{L}{2}\omega - v_1$

↻ Moments about A: $mv_1 \frac{L}{2} + 0 = m\bar{v}_2 \frac{L}{2} + \bar{I}\,\omega_2$

$$= m\left(\frac{L}{2}\omega - v_1\right)\frac{L}{2} + \left(\frac{1}{12}mL^2\right)\omega_2$$

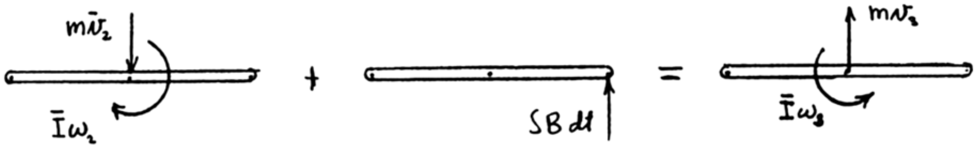
$$\boldsymbol{\omega}_2 = \frac{3v_1}{L} \circlearrowright \blacktriangleleft$$

$$\bar{v}_2 = \frac{L}{2}\left(\frac{3v_1}{L}\right) - v_1 = \frac{1}{2}v_1$$

$$\bar{\mathbf{v}}_2 = \frac{1}{2}v_1 \downarrow \blacktriangleleft$$

$$(v_B)_2 = L\omega - (v_A)_2 = 3v_1 - v_1 = 2v_1 \downarrow$$

(*b*) *Impact at B.*



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_3$

Condition of impact. $e = 1: \quad (\mathbf{v}_B)_3 = 2v_1 \uparrow$

Kinematics: $\bar{v}_3 = (v_B)_2 - \frac{L}{2}\omega = 2v_1 - \frac{L}{2}\omega$

PROBLEM 17.111 (Continued)

$\circlearrowleft$ Moments about B: $$-m\bar{v}_2\frac{L}{2}+\bar{I}\omega_2+0=m\bar{v}_3\frac{L}{2}-\bar{I}\omega_3$$

$$-m\left(\frac{1}{2}v_1\right)\frac{L}{2}+\left(\frac{1}{12}mL^2\right)\left(\frac{3v_1}{L}\right)+0=m\left(2v_1-\frac{L}{2}\omega_3\right)\frac{L}{2}-\left(\frac{1}{12}mL^2\right)\omega_3$$

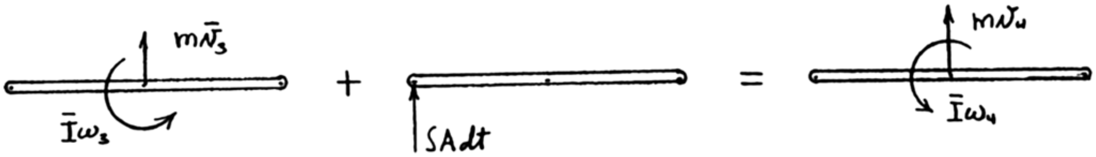
$\boldsymbol{\omega}_3=\frac{3v_1}{L}$ $\circlearrowright$ ◀

$$\bar{v}_3=2v_1-\frac{L}{2}\left(\frac{3v_1}{L}\right)=\frac{1}{2}v_1$$

$\bar{\mathbf{v}}_3=\frac{1}{2}v_1$ ↑ ◀

$$(v_A)_3=L\omega-(v_B)_3=3v_1-2v_1=v_1\downarrow$$

(*c*) *Second impact at A.*



Syst. Momenta$_3$ + **Syst. Ext. Imp.**$_{3\to4}$ = **Syst. Momenta**$_4$

Condition of impact. $e=1$: $(\mathbf{v}_A)_4=v_1\uparrow$

Kinematics: $$\bar{v}_4=(v_A)_4+\frac{L}{2}\omega_4=v_1+\frac{L}{2}\omega_4$$

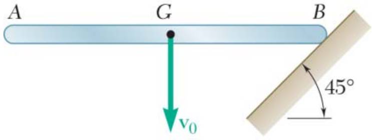
$\circlearrowright$ Moments about A: $$m\bar{v}_3\frac{L}{2}+\bar{I}\omega_3+0=m\bar{v}_4\frac{L}{2}+\bar{I}\omega_4$$

$$m\left(\frac{1}{2}v_1\right)\frac{L}{2}+\left(\frac{1}{12}mL^2\right)\left(\frac{3v_1}{L}\right)+0=m\left(v_1+\frac{L}{2}\omega_4\right)\frac{L}{2}+\left(\frac{1}{12}mL^2\right)\omega_4$$

$\boldsymbol{\omega}_4=0$ ◀

$$\bar{v}_4=v_1+0$$

$\bar{\mathbf{v}}_4=v_1$ ↑ ◀



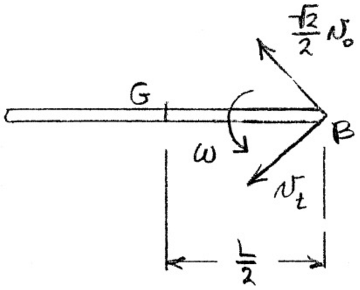
PROBLEM 17.112

A uniform slender rod *AB* of mass *m* and length *L* is falling freely with a velocity $\mathbf{v}_0$ when end *B* strikes a smooth inclined surface as shown. Assuming that the impact is perfectly elastic, determine the angular velocity of the rod and the velocity of its mass center immediately after the impact.

SOLUTION

Moment of inertia.
$$\bar{I} = \frac{1}{12}mL^2$$

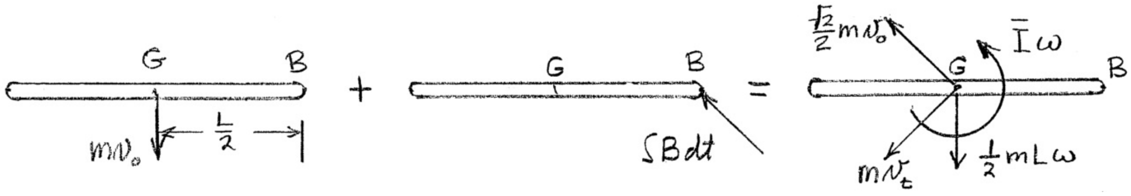
Perfectly elastic impact.
$$e = 1 \qquad \mathbf{v}_B = \left[\frac{\sqrt{2}}{2}v_0 \nwarrow 45°\right] + \left[v_t \swarrow 45°\right]$$


$\frac{\sqrt{2}}{2}v_0$
G
B
ω
v_t
$\frac{L}{2}$

Kinematics.
$$\bar{\mathbf{v}} = \mathbf{v}_B + \mathbf{v}_{G/B}$$
$$= \left[\frac{\sqrt{2}}{2}v_0 \nwarrow 45°\right] + \left[v_t \swarrow 45°\right] + \left[\frac{L}{2}\omega \downarrow\right]$$

Linear momentum.
$$m\bar{v} = \left[\frac{\sqrt{2}}{2}mv_0 \nwarrow 45°\right] + \left[mv_t \swarrow 45°\right] + \left[\frac{1}{2}mL\omega \downarrow\right]$$

Kinetics.


G
B
mv_0
$\frac{L}{2}$
+
G
B
$\int B\,dt$
=
$\frac{\sqrt{2}}{2}mv_0$
$\bar{I}\omega$
G
B
mv_t
$\frac{1}{2}mL\omega$

Syst Momenta$_1$ + **Syst Ext Imp**$_{1\to 2}$ = **Syst Momenta**$_2$

$\swarrow$ 45° linear components:
$$\frac{\sqrt{2}}{2}mv_0 = mv_t + \frac{\sqrt{2}}{2}\left(\frac{1}{2}mL\omega\right)$$
$$v_t = \frac{\sqrt{2}}{2}v_0 - \frac{\sqrt{2}}{4}L\omega$$

PROBLEM 17.112 (Continued)

$\circlearrowleft$ moments about B:

$$mv_0\frac{L}{2}+0=-\left(\frac{\sqrt{2}}{2}mv_0\right)\left(\frac{\sqrt{2}}{4}L\right)+mv_t\left(\frac{\sqrt{2}}{4}L\right)+\left(\frac{1}{2}mL\omega\right)\left(\frac{1}{2}L\right)+\bar{I}\,\omega$$

$$=-\frac{1}{4}mv_0L+m\left(\frac{\sqrt{2}}{2}v_0-\frac{\sqrt{2}}{4}L\omega\right)\left(\frac{\sqrt{2}}{4}L\right)+\frac{1}{4}mL^2\omega+\frac{1}{12}mL^2\omega$$

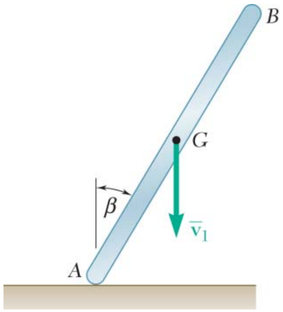
$$=\frac{5}{24}mL^2\omega$$

$$\omega=\frac{12}{5}\frac{v_0}{L}\qquad\qquad \boldsymbol{\omega}=2.4\frac{v_0}{L}\circlearrowleft \blacktriangleleft$$

$$v_t=\frac{\sqrt{2}}{2}v_0-\frac{\sqrt{2}}{4}\cdot\frac{12}{5}v_0=-\frac{\sqrt{2}}{10}v_0$$

$$\bar{\mathbf{v}}=\left[\frac{\sqrt{2}}{2}v_0\ \nwarrow 45^\circ\right]+\left[\frac{\sqrt{2}}{10}v_0\ \nearrow 45^\circ\right]+\left[\frac{L}{2}\left(\frac{12}{5}\frac{v_0}{L}\right)\downarrow\ \right]=\left[\frac{2}{5}v_0\ \leftarrow\ \right]+\left[\frac{3}{5}v_0\downarrow\ \right]$$

$$\bar{\mathbf{v}}=0.721v_0\ \swarrow 56.3^\circ \blacktriangleleft$$



PROBLEM 17.113

The slender rod *AB* of length $L = 1$ m forms an angle $\beta = 30°$ with the vertical as it strikes the frictionless surface shown with a vertical velocity $\bar{\mathbf{v}}_1 = 2$ m/s and no angular velocity. Knowing that the coefficient of restitution between the rod and the ground is $e = 0.8$, determine the angular velocity of the rod immediately after the impact.

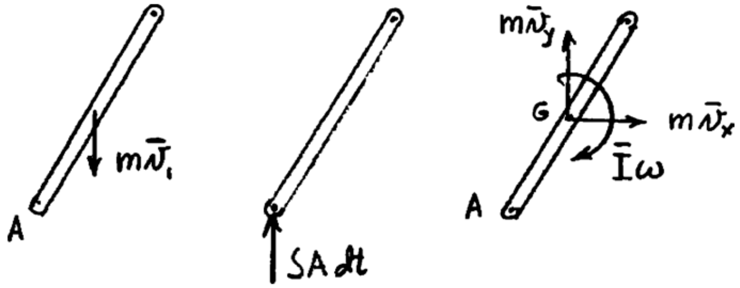
SOLUTION

Moment of inertia. $$\bar{I} = \frac{1}{12}mL^2$$

Apply coefficient of restitution. $$[(v_A)_y]_2 = -e[(v_A)_y]_1 = e\bar{v}_1 \uparrow$$

$$\mathbf{v}_A = (v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} = (v_A)_x\mathbf{i} + e\bar{v}_1\mathbf{j}$$

Kinetics.



$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

$\xrightarrow{+}$ horizontal components: $$0 + 0 = m\bar{v}_x \qquad m\bar{v}_x = 0$$

Kinematics. $$\mathbf{v}_G = \mathbf{v}_A + \mathbf{v}_{G/A} \qquad [\bar{v}_y \uparrow] = [e\bar{v}_1 \uparrow] + [(v_A)_x \longrightarrow] + \left[\frac{L}{2}\omega \nwarrow \beta\right]$$

Velocity components $\uparrow$: $$v_y = e\bar{v}_1 - \frac{L}{2}\omega\sin\beta$$

$\circlearrowright$ Moments about *A*: $$mv_1\frac{L}{2}\sin\beta + 0 = -m\bar{v}_y\frac{L}{2}\sin\beta + \bar{I}\omega$$

$$mv_1\frac{L}{2}\sin\beta = m\left(\frac{L}{2}\omega\sin\beta - e\bar{v}_1\right)\frac{L}{2}\sin\beta + \frac{1}{12}mL^2\omega$$

$$\left(\frac{1}{12}mL^2 + \frac{1}{4}mL^2\sin^2\beta\right)\omega = \frac{1+e}{2}mv_1L\sin\beta$$

$$\omega = \frac{6(1+e)\sin\beta}{1+3\sin^2\beta}\frac{\bar{v}_1}{1L}$$

PROBLEM 17.113 (Continued)

Data: $L = 1\text{ m},\quad \beta = 30°,\quad \bar{v}_1 = 2\text{ m/s},\quad e = 0.8$

$$\omega = \frac{(6)(1.8)\sin 30°}{1 + 3\sin^2 30°} \cdot \frac{2\text{ m/s}}{1\text{ m}}$$

$\omega = 6.17$ rad/s ↻ ◀

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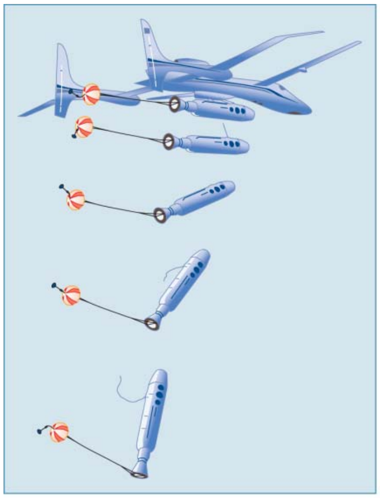
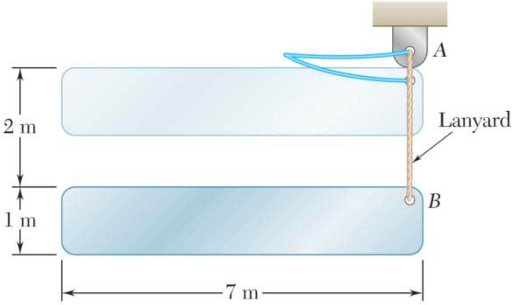
PROBLEM 17.114

The trapeze/lanyard air drop (t/LAD) launch is a proposed innovative method for airborne launch of a payload-carrying rocket. The release sequence involves several steps as shown in (1) where the payload rocket is shown at various instances during the launch. To investigate the first step of this process where the rocket body drops freely from the carrier aircraft until the 2-m lanyard stops the vertical motion of *B*, a trial rocket is tested as shown in (2). The rocket can be considered a uniform 1-m by 7-m rectangle with a mass of 4000 kg. Knowing that the rocket is released from rest and falls vertically 2 m before the lanyard becomes taut, determine the angular velocity of the rocket immediately after the lanyard is taut.



(1)



(2)

SOLUTION

While the lanyard is slack, the rocket falls freely without rotation. Considering its motion relative to the airplane (a Newtonian frame of reference), its vertical velocity is

$$v_1^2 = v_0^2 + 2gy = 2gy$$

$$v_1 = \sqrt{2gy} = \sqrt{(2)(9.81\text{ m/s})(2\text{ m})} \qquad \mathbf{v}_1 = 6.2642\text{ m/s}\downarrow \qquad (1)$$

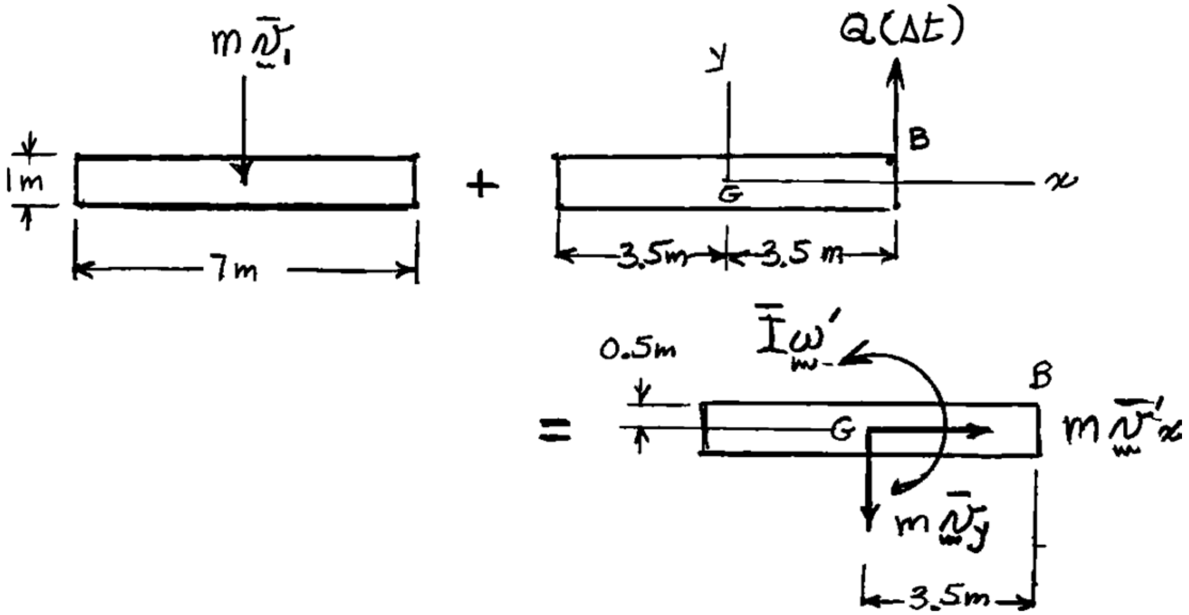
Moment of inertia:

$$\bar{I} = \frac{1}{12}m(a^2+b^2)$$

$$= \frac{1}{12}(4000\text{ kg})[(7\text{ m})^2 + (1\text{ m})^2] = 16.667\text{ kg}\cdot\text{m}^2$$

PROBLEM 17.114 (Continued)

For impact use the principle of impulse and momentum.



$$\textbf{Syst. Momenta}_1 \quad + \quad \textbf{Syst. Ext. Imp.}_{1\to 2} \quad = \quad \textbf{Syst. Momenta}_2$$

x-components $\longrightarrow$: $\quad 0 + 0 = m\bar{v}'_x \qquad \bar{v}'_x = 0$

$+\circlearrowleft$ Moments about B: $\quad mv_1(3.5) + 0 = \bar{I}\,\omega' + m\bar{v}'_y(3.5) \qquad (2)$

Kinematics. $\quad \mathbf{v}_B = v_B \longrightarrow$

$$\bar{v} = [\bar{v}_x \longrightarrow] + [\bar{v}_y \downarrow] = [v_B \longrightarrow] + [3.5\omega' \downarrow] + [0.5\omega \longrightarrow]$$

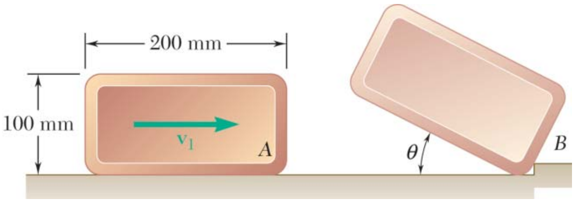
y-components $\downarrow$: $\quad \bar{v}_y = 3.5\omega' \qquad (3)$

Substitute from Eqs. (1) and (3) into Eq. (2).

$$mv_1(3.5) = [\bar{I} + m(3.5)^2]\omega'$$

$$(4000\text{ kg})(6.2642\text{ m/s})(3.5\text{ m}) = [16667\text{ kg}\cdot\text{m}^2 + (4000\text{ kg})(3.5\text{ m})^2]\omega'$$

Angular velocity. $\quad \boldsymbol{\omega}' = 1.336\text{ rad/s} \circlearrowright$ ◀



PROBLEM 17.115

The uniform rectangular block shown is moving along a frictionless surface with a velocity $\bar{\mathbf{v}}_1$ when it strikes a small obstruction at *B*. Assuming that the impact between corner *A* and obstruction *B* is perfectly plastic, determine the magnitude of the velocity $\bar{\mathbf{v}}_1$ for which the maximum angle θ through which the block will rotate is 30°.

SOLUTION

Let *m* be the mass of the block.

Dimensions: $a = 0.200 \text{ m}$

$b = 0.100 \text{ m}$

Moment of inertia about the mass center.

$$\bar{I} = \frac{1}{12}m(a^2 + b^2)$$

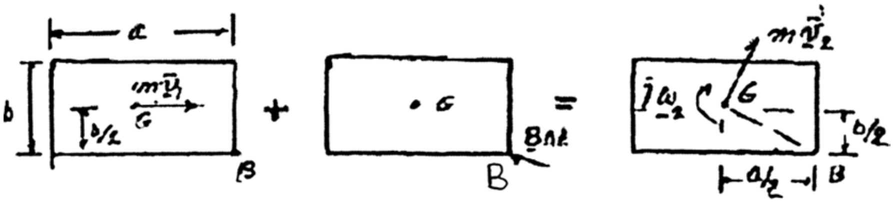
Let *d* be one half the diagonal. $d = \frac{1}{2}\sqrt{a^2 + b^2} = 0.1118 \text{ m}$

Kinematics. Before impact $\bar{\mathbf{v}}_1 = v_1 \longrightarrow, \quad \boldsymbol{\omega}_1 = 0$

After impact, the block is rotating about corner at *B*.

$$\boldsymbol{\omega}_2 = \omega_2 \circlearrowright \qquad \mathbf{v}_2 = d\omega_2 \nearrow$$

Principle of impulse and momentum.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

↺+ Moments about *B*:

$$\frac{mv_1 b}{2} + 0 = \bar{I}\omega_2 + mdv_2$$

$$\frac{1}{2}mv_1 b = \frac{1}{12}m(a^2 + b^2)\omega_2 + md^2\omega_2$$

$$= \frac{1}{3}m(a^2 + b^2)\omega_2$$

PROBLEM 17.115 (Continued)

Angular velocity after impact $\qquad \boldsymbol{\omega}_2 = \dfrac{3v_1 b}{2(a^2+b^2)} \circlearrowright \qquad (1)$

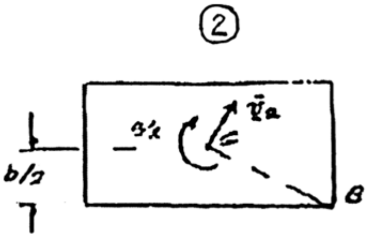
The motion after impact is a rotation about corner *B*.

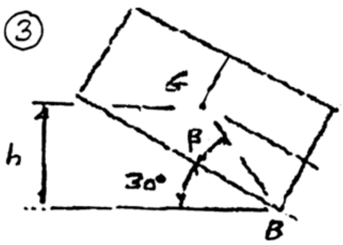
Position 2 (immediately after impact). $\qquad \overline{v}_2 = d\omega_2$

Position 3 $\quad (\theta = 30°)$. $\qquad \beta = \tan^{-1}\dfrac{b}{a} = \tan^{-1} 0.5 = 26.565°$

$$h = d\sin(\beta + 30°) = 0.11180 \sin 56.565° = 0.093301 \text{ m}$$

$$\omega_3 = 0 \qquad\qquad \overline{v}_3 = 0$$





Potential energy: $\qquad V_2 = \dfrac{mgb}{2} \qquad V_3 = mgh$

Kinetic energy: $\qquad T_2 = \dfrac{1}{2}\overline{I}\omega_2^2 + \dfrac{1}{2}m\overline{v}_2^2 = \dfrac{1}{2}(\overline{I} + md^2)\omega_2^2$

$$= \frac{1}{6}m(a^2+b^2)\omega_2^2 \qquad T_3 = 0$$

Principle of conservation of energy:

$$T_2 + V_3 = T_3 + V_3$$

$$\frac{1}{6}m(a^2+b^2)\omega_2^2 + \frac{mgb}{2} = 0 + mgh$$

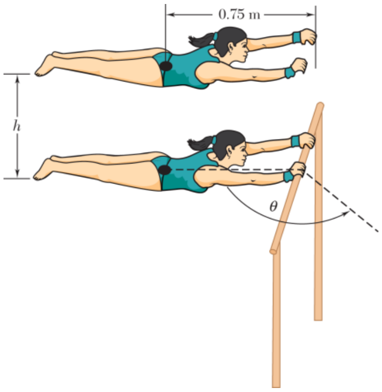
$$\omega_2^2 = \frac{3g(2h-b)}{(a^2+b^2)} = \frac{(3)(9.81)(0.18660 - 0.100)}{(0.200)^2 + (0.100)^2}$$

$$= 50.974 \text{ (rad/s)}^2 \qquad\qquad \boldsymbol{\omega}_2 = 7.1396 \text{ rad/s} \circlearrowright$$

Magnitude of initial velocity.

Solving Eq. (1) for v_1 $\qquad v_1 = \dfrac{2(a^2+b^2)\omega_2}{3b}$

$$v_1 = \frac{(2)[(0.200)^2 + (0.100)^2](7.1396)}{(3)(0.100)} \qquad\qquad v_1 = 2.38 \text{ m/s} ◀$$



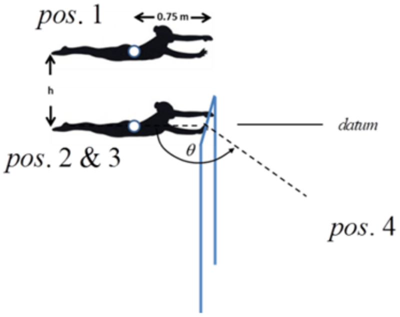
PROBLEM 17.116

The 40-kg gymnast drops from her maximum height of h = 0.5 m straight down to the bar as shown. Her hands hit the bar and clasp onto it, and her body remains straight in the position shown. Her center of mass is 0.75 meters away from her hands, and her mass moment of inertia about her center of mass is 7.5 kg·m^2. Assuming that friction between the bar and her hands is negligible and that she remains in the same position throughout the swing, determine her angular velocity when she swings around to θ = 135°.

SOLUTION

Given: $m = 40$ kg, $h = 0.5$ m, $\bar{I} = 7.5$ kg•m^2, $\bar{r} = 0.75$ m

$\theta_1 = 0,\ \theta_2 = 135°$



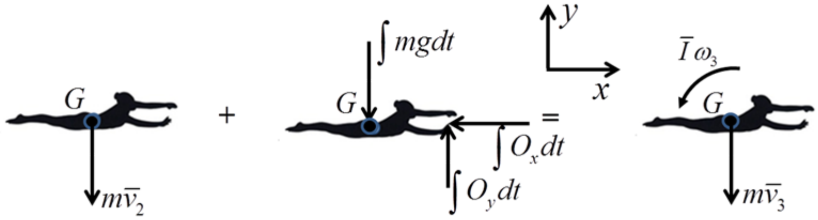
Work Energy Equation Pos. 1 and 2:

$$\cancel{T_1} + V_{g1} + \cancel{V_{e,1}} + \cancel{U'_{1-2}} = T_2 + \cancel{V_{g2}} + \cancel{V_{e,2}}$$

$$V_{g1} = T_2$$

$$\cancel{m} gh = \frac{1}{2}\cancel{m}\bar{v}_2^2 \quad \Rightarrow \quad \bar{v}_2 = \sqrt{2gh}$$

Impulse-Momentum During Impact (position 2&3):



$$H_{O,2} = H_{O,3}$$

$$m\bar{v}_2\bar{r} = I_O\omega_3$$

$$m\sqrt{2gh}\bar{r} = \left(\bar{I} + m\bar{r}^2\right)\omega_3$$

$$\omega_3 = \frac{40\sqrt{2g(0.5)}(0.75)}{7.5 + 40(0.75)^2} \quad \Rightarrow \quad \omega_3 = \sqrt{g}$$

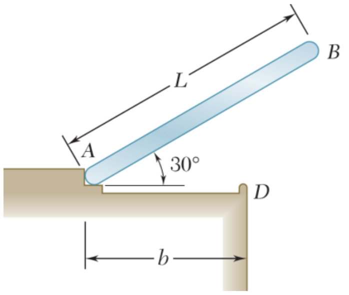
Work Energy Equation Pos. 3 and 4:

$$T_3 + \cancel{V_{g3}} + \cancel{V_{e,3}} + \cancel{U'_{1-2}} = T_4 + V_{g4} + \cancel{V_{e,4}}$$

$$\frac{1}{2}I_O\omega_3^2 = \frac{1}{2}I_O\omega_4^2 - mg\bar{r}\cos 45°$$

$$\omega_4 = \sqrt{g + \frac{2(40)g(0.75)\cos 45°}{7.5 + 40(0.75)^2}}$$

$\omega_4 = 4.867$ rad/s ◀

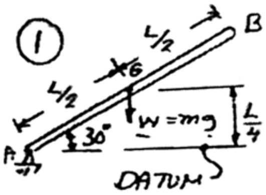


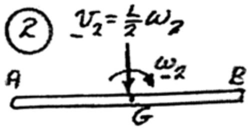
PROBLEM 17.117

A slender rod of mass *m* and length *L* is released from rest in the position shown and hits edge *D*. Assuming perfectly plastic impact at *D*, determine for $b = 0.6L$, (*a*) the angular velocity of the rod immediately after the impact, (*b*) the maximum angle through which the rod will rotate after the impact.

SOLUTION

For analysis of the falling motion before impact use the principle of conservation of energy.


①
B
L/2
G
L/2
W = mg
L/4
30°
A
DATUM


②
$v_2 = \frac{L}{2}\omega_2$
ω_2
A
B
G

Position 1: $T_1 = 0, \quad V_1 = mg\dfrac{L}{4}$

Position 2: $V_2 = 0$

$$T_2 = \frac{1}{2}m\left(\frac{L}{2}\omega_2\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega_2^2$$

$$T_2 = \frac{1}{6}mL^2\omega_2^2$$

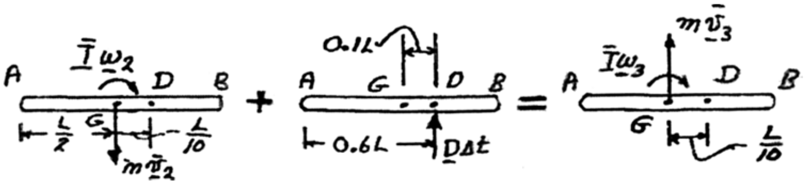
$$T_1 + V_1 = T_2 + V_2: \quad 0 + mg\frac{L}{4} = \frac{1}{6}mL^2\omega_2^2 \qquad \omega_2 = \sqrt{\frac{3g}{2L}}$$

Analysis of impact. Kinematics

Before impact, rotation is about Point *A*. $\qquad \bar{v}_2 = \dfrac{L}{2}\omega_2$

After impact, rotation is about Point *D*. $\qquad \bar{v}_3 = \dfrac{L}{10}\omega_3$

Principle of impulse-momentum.


A
$\bar{I}\omega_2$
D
B
L/2
G
L/10
$m\bar{v}_2$
+
0.1L
A
G
D
B
0.6L
$D\Delta t$
=
A
$\bar{I}\omega_3$
$m\bar{v}_3$
D
B
G
L/10

Syst. Momenta$_2$ + **Syst. Ext. Imp.** $_{2\to 3}$ = **Syst. Momenta**$_3$

+↻ Moments about *D*: $\qquad \bar{I}\omega_2 - m\bar{v}_2\left(\dfrac{L}{10}\right) = \bar{I}\omega_3 + m\bar{v}_3\left(\dfrac{L}{10}\right) \qquad (1)$

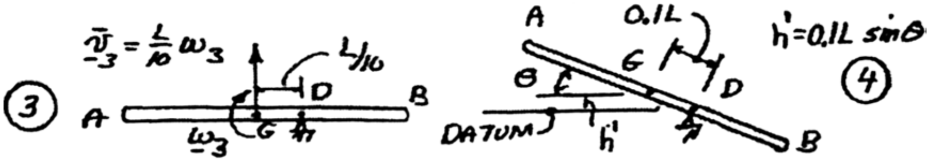
$$\frac{1}{12}mL^2\omega_2 - m\left(\frac{L}{2}\omega_2\right)\frac{L}{10} = \frac{1}{12}mL^2\omega_3 + m\left(\frac{L}{10}\omega_3\right)\frac{L}{10}$$

$$\left(\frac{1}{12} - \frac{1}{20}\right)\omega_2 = \left(\frac{1}{12} + \frac{1}{100}\right)\omega_3$$

PROBLEM 17.117 (Continued)

(*a*) Angular velocity.
$$\omega_3 = \frac{5}{14}\omega_2 = \frac{5}{14}\sqrt{\frac{3g}{2L}} \qquad \boldsymbol{\omega}_3 = 0.437\sqrt{\frac{g}{L}} \circlearrowright \blacktriangleleft$$

For analysis of the rotation about Point *D* after the impact use the principle of conservation of energy.



Position 3 (just after impact)

$$\bar{v}_3 = \frac{L}{10}\omega_3 \quad V_3 = 0$$

$$T_3 = \frac{1}{2}m\left(\frac{L}{10}\omega_3\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega_3^2 = \frac{14}{300}mL^2\omega_3^2$$

$$= \frac{14}{300}mL^2\left(\frac{5}{14}\sqrt{\frac{3g}{2L}}\right)^2 = \frac{mgL}{112}$$

Position 4. $\theta =$ maximum rotation angle.

$$h' = \frac{L}{10}\sin\theta$$

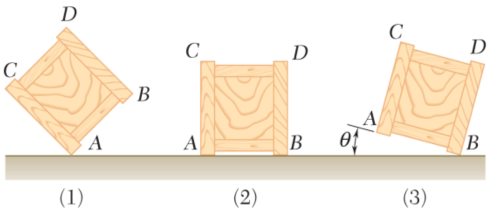
$$V_4 = mgh' = \frac{mgL}{10}\sin\theta$$

$$\bar{v}_4 = 0, \quad \omega_4 = 0, \quad T_4 = 0$$

$$T_3 + V_3 = T_4 + V_4; \quad \frac{mgL}{112} + 0 = 0 + \frac{mgL}{10}\sin\theta$$

(*b*) Maximum rotation angle.
$$\sin\theta = \frac{10}{112} \qquad \theta = 5.12° \blacktriangleleft$$

PROBLEM 17.118

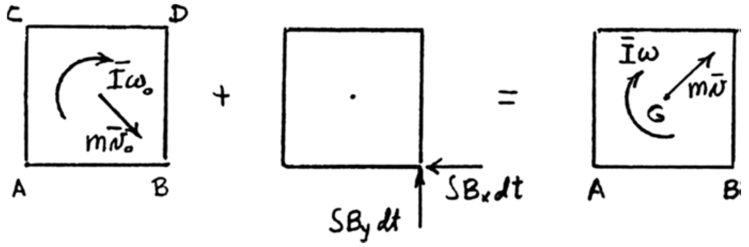


A uniformly loaded square crate is released from rest with its corner *D* directly above *A*; it rotates about *A* until its corner *B* strikes the floor, and then rotates about *B*. The floor is sufficiently rough to prevent slipping and the impact at *B* is perfectly plastic. Denoting by $\boldsymbol{\omega}_0$ the angular velocity of the crate immediately before *B* strikes the floor, determine (*a*) the angular velocity of the crate immediately after *B* strikes the floor, (*b*) the fraction of the kinetic energy of the crate lost during the impact, (*c*) the angle θ through which the crate will rotate after *B* strikes the floor.

SOLUTION

Let *m* be the mass of the crate and *c* be the length of an edge.

Moment of inertia
$$\bar{I} = \frac{1}{12}m(c^2 + c^2) = \frac{1}{6}mc^2$$



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

Kinematics:
$$\bar{v}_0 = r_{G/A}\omega_0 = \frac{1}{2}\sqrt{2}c\omega_0$$
$$\bar{v} = r_{G/B}\omega = \frac{1}{2}\sqrt{2}c\omega$$

↺ Moments about *B*:
$$\bar{I}\omega_0 + 0 = \bar{I}\omega + r_{G/B}m\bar{v}$$
$$\frac{1}{6}mc^2\omega_0 + 0 = \frac{1}{6}mc^2\omega + \left(\frac{1}{2}\sqrt{2}c\right)m\left(\frac{1}{2}\sqrt{2}c\omega\right) = \frac{2}{3}mc^2\omega$$

(*a*) Solving for ω,
$$\boldsymbol{\omega} = \frac{1}{4}\omega_0 \circlearrowright \blacktriangleleft$$

Kinetic Energy.

Before impact:
$$T_1 = \frac{1}{2}\bar{I}\omega_0^2 + \frac{1}{2}m\bar{v}_0^2$$
$$= \frac{1}{2}\left(\frac{1}{6}mc^2\right)\omega_0^2 + \frac{1}{2}m\left(\frac{1}{2}\sqrt{2}c\omega_0\right)^2$$
$$= \frac{1}{3}mc^2\omega_0^2$$

PROBLEM 17.118 (Continued)

After impact: $T_2 = \frac{1}{2}\bar{I}\omega^2 + \frac{1}{2}m\bar{v}^2 = \frac{1}{2}\left(\frac{1}{6}mc^2\right)\omega^2 + \frac{1}{2}m\left(\frac{1}{2}\sqrt{2}c\omega\right)^2$

$$= \frac{1}{3}mc^2\omega^2 = \frac{1}{3}mc^2\left(\frac{1}{4}\omega_0\right)^2 = \frac{1}{48}mc^2\omega_0$$

(*b*) Fraction of energy lost: $\frac{T_1 - T_2}{T_1} = \frac{\frac{1}{3} - \frac{1}{48}}{\frac{1}{3}} = 1 - \frac{1}{16}$ $\qquad \frac{15}{16}$ ◀

Conservation of energy during falling. $T_0 + V_0 = T_1 + V_1$ (1)

Conservation of energy during rising. $T_3 + V_3 = T_2 + V_2$ (2)

Conditions: $T_0 = 0, \quad T_3 = 0 \quad T_2 = \frac{1}{16}T_1$

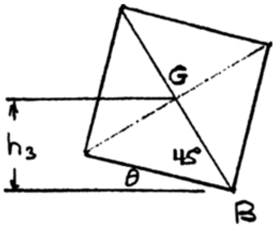
$$V_0 = mg\left(\frac{1}{2}\sqrt{2}c\right) \quad V_1 = V_2 = mg\left(\frac{1}{2}c\right) \quad V_3 = mgh_3$$

From Equation (1), $T_1 = V_0 - V_1 = \frac{1}{2}(\sqrt{2} - 1)mgc$

From Equation (2), $T_2 = V_3 - V_2 = mgh_3 - \frac{1}{2}mgc$

$$\frac{h_3 - \frac{1}{2}c}{\frac{1}{2}\sqrt{2} - 1} = \frac{1}{16} \qquad h_3 = \left[\frac{1}{2} + \frac{1}{16}(\sqrt{2} - 1)\right]c$$

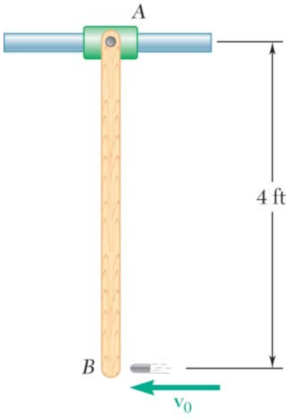
(*c*) From geometry, $h_3 = \frac{1}{2}\sqrt{2}c\sin(\theta + 45°)$



Equating the two expressions for h_3,

$$\sin(45° + \theta) = \frac{\frac{1}{2} + \frac{1}{16}(\sqrt{2} - 1)}{\frac{1}{2}\sqrt{2}}$$

$45° + \theta = 46.503°$ $\qquad \theta = 1.50°$ ◀



PROBLEM 17.119

A 1-oz bullet is fired with a horizontal velocity of 750 mi/h into the 18-lb wooden beam AB. The beam is suspended from a collar of negligible mass that can slide along a horizontal rod. Neglecting friction between the collar and the rod, determine the maximum angle of rotation of the beam during its subsequent motion.

SOLUTION

Mass of bullet. $W' = 1\text{ ounce} = 0.0625\text{ lb}$

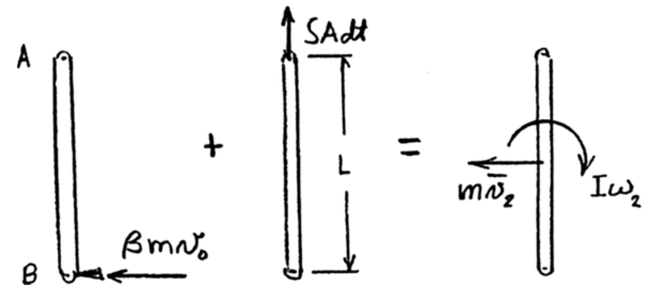
Mass of beam AB. $W = 18\text{ lb}$

Mass ratio. $\beta = \dfrac{W'}{W} = 0.0034722 \qquad W' = \beta W \text{ and } m' = \beta m$

Since β is so small, the mass of the bullet will be neglected in comparison with that of the beam in determining the motion after the impact.

Moment of inertia. $\bar{I} = \dfrac{1}{12} mL^2$

Impact kinetics.



$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

$\xleftarrow{+}$ linear components: $-\beta m v_0 + 0 = m\bar{v}_2 \qquad \bar{v}_2 = \beta v_0$

↻ Moments about B: $0 + 0 = \bar{I}\,\omega - m\bar{v}_2\dfrac{L}{2}$

$$\omega = \frac{m\bar{v}_2 L}{2\bar{I}} = \frac{12 m\beta v_0 L}{2mL^2}$$

$$\omega = \frac{6\beta v_0}{L}$$

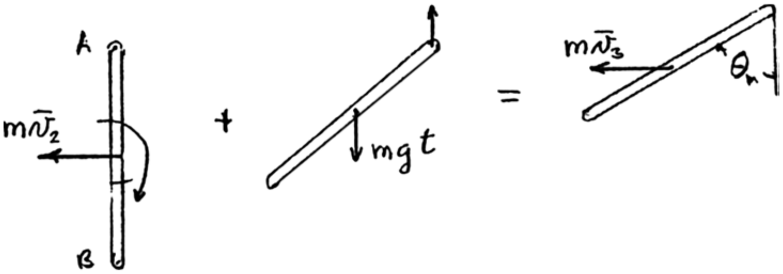
PROBLEM 17.119 (Continued)

Motion during rising. *Position 2*. Just after the impact.

$$V_2 = -mg\frac{L}{2} \quad \text{(datum at level } A\text{)}$$

$$\begin{aligned} T_2 &= \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2 \\ &= \frac{1}{2}m(\beta v_0)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\frac{6\beta v_0}{L}\right)^2 \\ &= 2\beta^2 m v_0^2 \end{aligned}$$

Position 3. $\omega = 0, \quad \theta = \theta_m.$



Syst. Momenta$_2$ + **Syst. Ext. Imp.** $_{2\to3}$ = **Syst. Momenta**$_3$

$$V_3 = -mg\frac{L}{2}\cos\theta_m$$

$$T_3 = \frac{1}{2}m\bar{v}_3^2$$

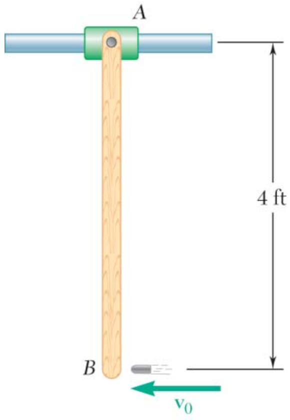
$\xleftarrow{+}$ linear components: $m\bar{v}_2 + 0 = m\bar{v}_3 \qquad \bar{v}_3 = \bar{v}_2 = \beta v_0$ where $v_0 = 750\text{ mi/h} = 1100\text{ ft/s}$

Conservation of energy. $T_2 + V_2 = T_3 + V_3: \quad 2\beta^2 m v_0^2 - mg\frac{L}{2} = \frac{1}{2}m(\beta v_0)^2 - mg\frac{L}{2}\cos\theta_m$

$$\frac{3\beta^2 v_0^2}{gL} = 1 - \cos\theta_m$$

$$\begin{aligned} \cos\theta_m &= 1 - \frac{3\beta^2 v_0^2}{gL} \\ &= 1 - \frac{(3)(0.0034722)^2(1100)^2}{(32.2)(4)} \\ &= 0.66021 \end{aligned}$$

$\theta_m = 48.7°$ ◀



PROBLEM 17.120

For the beam of Problem 17.119, determine the velocity of the 1-oz bullet for which the maximum angle of rotation of the beam will be 90°.

PROBLEM 17.119 A 1-oz bullet is fired with a horizontal velocity of 350 m/s into the 18-lb wooden beam *AB*. The beam is suspended from a collar of negligible weight that can slide along a horizontal rod. Neglecting friction between the collar and the rod, determine the maximum angle of rotation of the beam during its subsequent motion.

SOLUTION

Mass of bullet. $W' = 1 \text{ ounce} = 0.0625 \text{ lb}$

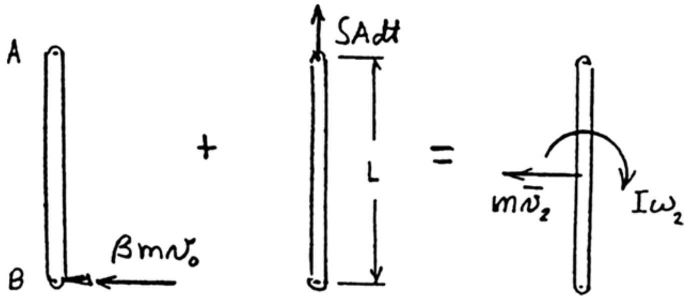
Mass of beam *AB*. $W = 18 \text{ lb}$

Mass ratio. $\beta = \dfrac{W'}{W} = 0.0034722 \qquad W' = \beta W \text{ and } m' = \beta m$

Since β is so small, the mass of the bullet will be neglected in comparison with that of the beam in determining the motion after the impact.

Moment of inertia. $\bar{I} = \dfrac{1}{12} m L^2$

Impact Kinetics.



$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to2} = \textbf{Syst. Momenta}_2$$

$\xleftarrow{+}$ linear components: $-\beta m v_0 + 0 = m\bar{v}_2 \qquad \bar{v}_2 = \beta v_0$

↻ Moments about *B*: $0 + 0 = \bar{I}\omega - m\bar{v}_2 \dfrac{L}{2}$

$$\omega = \frac{m\bar{v}_2 L}{2\bar{I}} = \frac{12 m \beta v_0 L}{2 m L^2}$$

$$\omega = \frac{6\beta v_0}{L}$$

PROBLEM 17.120 (Continued)

Motion during rising. *Position 2*. Just after the impact.

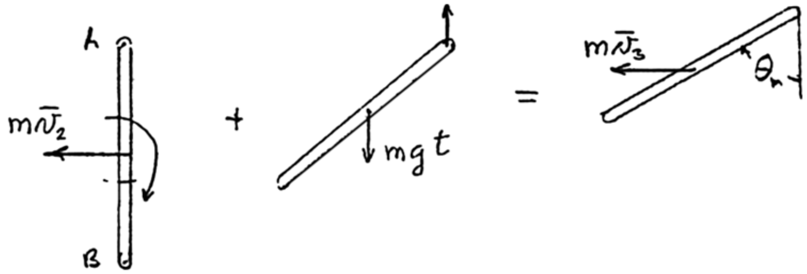
$$V_2 = -mg\frac{L}{2} \quad \text{(datum at level } A\text{)}$$

$$T_2 = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\,\omega_2^2$$

$$= \frac{1}{2}m(\beta v_0)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\frac{6\beta v_0}{L}\right)^2$$

$$= 2\beta^2 m v_0^2$$

Position 3. $\omega = 0, \quad \theta = \theta_m.$



$$\textbf{Syst. Momenta}_2 \quad + \quad \textbf{Syst. Ext. Imp.}_{2\to 3} \quad = \quad \textbf{Syst. Momenta}_3$$

$$V_3 = -mg\frac{L}{2}\cos\theta_m$$

$$T_3 = \frac{1}{2}m\bar{v}_3^2$$

$\xleftarrow{+}$ linear components: $m\bar{v}_2 + 0 = m\bar{v}_3 \quad \bar{v}_3 = \bar{v}_2 = \beta v_0$

Conservation of energy.

$$T_2 + V_2 = T_3 + V_3: \quad 2\beta^2 m v_0^2 - mg\frac{L}{2} = \frac{1}{2}m(\beta v_0)^2 - mg\frac{L}{2}\cos\theta_m$$

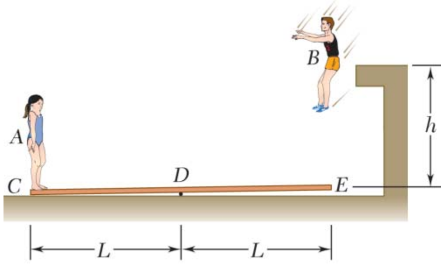
$$\beta v_0 = \sqrt{\frac{1}{3}gL(1-\cos\theta_m)}$$

$$= \sqrt{\left(\frac{1}{3}\right)(32.2)(4)(1-\cos 90°)}$$

$$= 6.5524 \text{ ft/s}$$

$$v_0 = \frac{6.5524}{0.0034722}$$

$v_0 = 1887$ ft/s ◀



PROBLEM 17.121

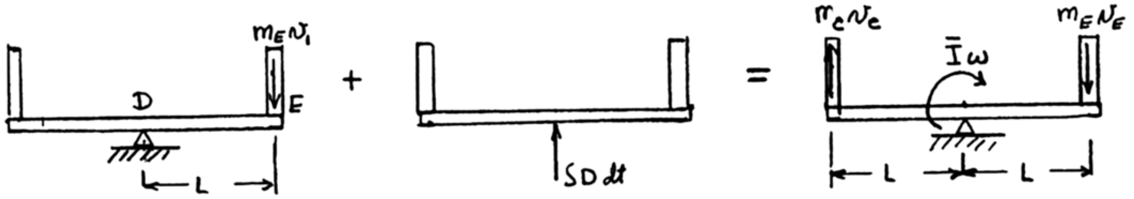
The plank *CDE* has a mass of 15 kg and rests on a small pivot at *D*. The 55-kg gymnast *A* is standing on the plank at *C* when the 70-kg gymnast *B* jumps from a height of 2.5 m and strikes the plank at *E*. Assuming perfectly plastic impact and that gymnast *A* is standing absolutely straight, determine the height to which gymnast *A* will rise.

SOLUTION

Moment of inertia. $\bar{I} = \frac{1}{12} m_P (2L)^2 = \frac{1}{3} m_P L^2$

Velocity of jumper at *E*. $(v)_1 = \sqrt{2gh_1}$ (1)

Principle of impulse-momentum.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

Kinematics: $v_C = L\omega \quad v_D = L\omega$

↻ Moments about *D*:

$$m_E v_1 L + 0 = m_E v_E L + m_C v_C L + \bar{I}\omega$$

$$= m_E L^2 \omega + m_C L^2 \omega + \frac{1}{3} m_P L^2 \omega$$

$$\omega = \frac{m_E}{m_E + m_C + \frac{1}{3} m_P} \frac{v_1}{L}$$

$$v_C = L\omega = \frac{m_E v_1}{m_E + m_C + \frac{1}{3} m_P} \quad (2)$$

Gymnast (flier) rising. $h_C = \frac{v_C^2}{2g}$ (3)

Data:

$m_E = m_B = 70 \text{ kg}$

$m_C = m_A = 55 \text{ kg}$

$m_P = 15 \text{ kg}$

$h_1 = 2.5 \text{ m}$

PROBLEM 17.121 (Continued)

From Equation (1)

$$v_1 = \sqrt{(2)(9.81)(2.5)}$$
$$= 7.0036 \text{ m/s}$$

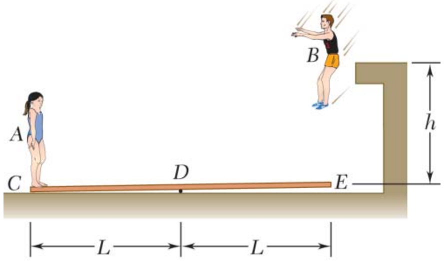
From Equation (2)

$$v_C = \frac{(70)(7.0036)}{70+55+5}$$
$$= 3.7712 \text{ m/s}$$

From Equation (3)

$$h_2 = \frac{(3.7712)^2}{(2)(9.81)}$$
$$= 0.725 \text{ m}$$

$h_2 = 725 \text{ mm}$ ◀



PROBLEM 17.122

Solve Problem 17.121, assuming that the gymnasts change places so that gymnast *A* jumps onto the plank while gymnast *B* stands at *C*.

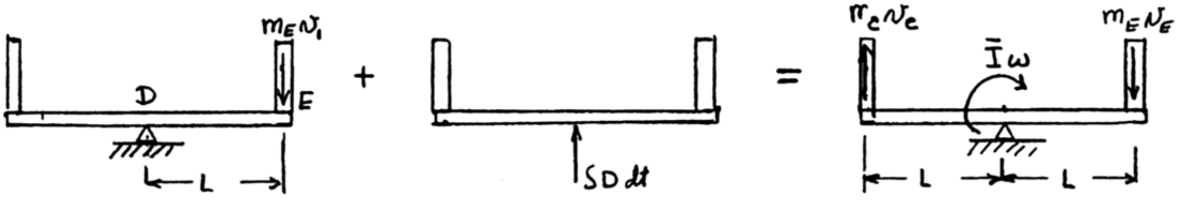
PROBLEM 17.121 The plank *CDE* has a mass of 15 kg and rests on a small pivot at *D*. The 55-kg gymnast *A* is standing on the plank at *C* when the 70-kg gymnast *B* jumps from a height of 2.5 m and strikes the plank at *E*. Assuming perfectly plastic impact and that gymnast *A* is standing absolutely straight, determine the height to which gymnast *A* will rise.

SOLUTION

Moment of inertia.
$$\bar{I} = \frac{1}{12} m_P (2L)^2 = \frac{1}{3} m_P L^2$$

Velocity of jumper at *E*.
$$(v)_1 = \sqrt{2gh_1} \qquad (1)$$

Principle of impulse-momentum.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

Kinematics:
$$v_C = L\omega \quad v_D = L\omega$$

↻ Moments about *D*:
$$m_E v_1 L + 0 = m_E v_E L + m_C v_C L + \bar{I}\omega$$
$$= m_E L^2 \omega + m_C L^2 \omega + \frac{1}{3} m_P L^2 \omega$$
$$\omega = \frac{m_E}{m_E + m_C + \frac{1}{3} m_P} \frac{v_1}{L}$$
$$v_C = L\omega = \frac{m_E v_1}{m_E + m_C + \frac{1}{3} m_P} \qquad (2)$$

Gymnast (flier) rising.
$$h_C = \frac{v_C^2}{2g} \qquad (3)$$

PROBLEM 17.122 (Continued)

Data:

$$m_E = m_A = 55 \text{ kg}$$
$$m_C = m_B = 70 \text{ kg}$$
$$m_P = 15 \text{ kg}$$
$$h_1 = 2.5 \text{ m}$$

From Equation (1)

$$v_1 = \sqrt{(2)(9.81)(2.5)}$$
$$= 7.0036 \text{ m/s}$$

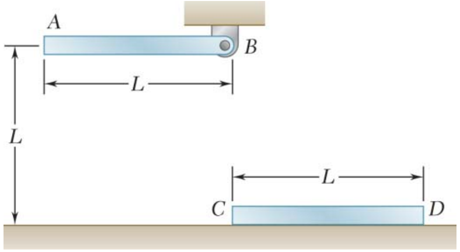
From Equation (2)

$$v_C = \frac{(55)(7.0036)}{55 + 70 + 5}$$
$$= 2.9631 \text{ m/s}$$

From Equation (3)

$$h_2 = \frac{(2.9631)^2}{(2)(9.81)}$$
$$= 0.447 \text{ m}$$

$h_2 = 447 \text{ mm}$ ◀



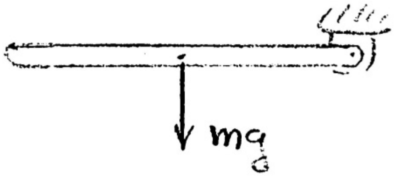
PROBLEM 17.123

A slender rod *AB* is released from rest in the position shown. It swings down to a vertical position and strikes a second and identical rod *CD* which is resting on a frictionless surface. Assuming that the coefficient of restitution between the rods is 0.4, determine the velocity of rod *CD* immediately after the impact.

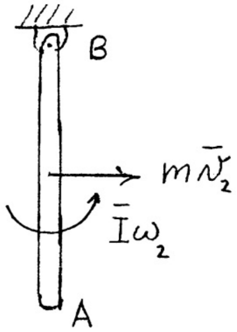
SOLUTION

Moment of inertia. $\bar{I} = \frac{1}{12}mL^2$ for each rod.

Rod *AB* swings to vertical position.



Position 1



Position 2

Position 1 $V_1 = 0 \qquad T_1 = 0$

Position 2 $V_2 = -mg\frac{L}{2}$

$$T_2 = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2$$

$$= \frac{1}{2}m\left(\frac{L}{2}\omega_2\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega_2^2$$

$$= \frac{1}{6}mL^2\omega_2^2$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$:

$$0 + 0 = \frac{1}{6}mL^2\omega_2^2 - mg\frac{L}{2} \qquad \boldsymbol{\omega}_2 = \sqrt{\frac{3g}{L}} \circlearrowright$$

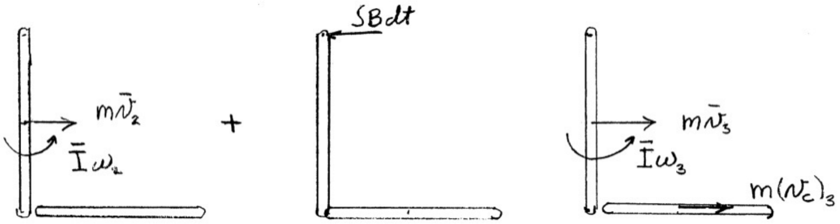
$$\bar{\mathbf{v}}_2 = \frac{L}{2}\sqrt{\frac{3g}{L}} \longrightarrow \qquad (v_B)_2 = L\omega = \sqrt{3gL}$$

Impact condition: $(v_C)_3 - (v_B)_3 = e(v_B)_2$

$$(v_C)_3 - L\omega_3 = e\sqrt{3gL} \qquad (v_C)_3 = L\omega_3 + e\sqrt{3gL}$$

PROBLEM 17.123 (Continued)

Principle of impulse-momentum at impact.



Syst Momenta$_2$ + **Syst Ext Imp**$_{2\to3}$ = **Syst Momenta**$_3$

↺ moments about B: $$m\bar{v}_2\frac{L}{2}+\bar{I}\omega_2+0=m\bar{v}_3\frac{L}{2}+\bar{I}\omega_3+m(v_C)_3L$$

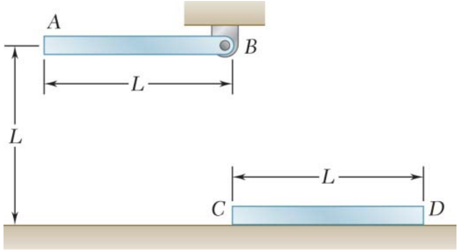
$$m\left(\frac{L}{2}\sqrt{\frac{3g}{L}}\right)\frac{L}{2}+\frac{1}{12}mL^2\sqrt{\frac{3g}{L}}+0=m\left(\frac{L}{2}\omega_3\right)\frac{L}{2}+\frac{1}{12}mL^2\omega_3+m\left(L\omega_3+e\sqrt{3gL}\right)L$$

$$\omega_3=\left(\frac{1}{4}-\frac{3}{4}e\right)\sqrt{\frac{3g}{L}}$$

$$(v_C)_3=\left(\frac{1}{4}-\frac{3}{4}e\right)\sqrt{3gL}+e\sqrt{3gL}\qquad (v_C)_3=\frac{1}{4}(1+e)\sqrt{3gL}$$

For $e=0.4$, $$(v_C)_3=\frac{1}{4}(1+0.4)\sqrt{3gL}$$

$$(\mathbf{v}_C)_3=0.606\sqrt{gL}\longrightarrow \blacktriangleleft$$



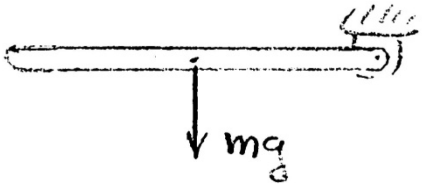
PROBLEM 17.124

A slender rod AB is released from rest in the position shown. It swings down to a vertical position and strikes a second and identical rod CD which is resting on a frictionless surface. Assuming that the impact between the rods is perfectly elastic, determine the velocity of rod CD immediately after the impact.

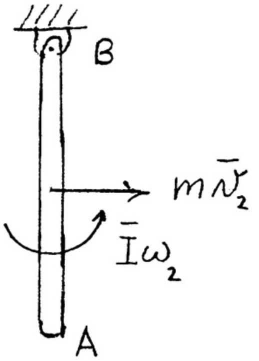
SOLUTION

Moment of inertia. $\bar{I} = \frac{1}{12}mL^2$ for each rod.

Rod AB swings to vertical position.



Position 1



Position 2

Position 1 $V_1 = 0 \qquad T_1 = 0$

Position 2 $V_2 = -mg\frac{L}{2}$

$$T_2 = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2$$

$$= \frac{1}{2}m\left(\frac{L}{2}\omega_2\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega_2^2$$

$$= \frac{1}{6}mL^2\omega_2^2$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$:

$$0 + 0 = \frac{1}{6}mL^2\omega_2^2 - mg\frac{L}{2} \qquad \boldsymbol{\omega}_2 = \sqrt{\frac{3g}{L}} \circlearrowright$$

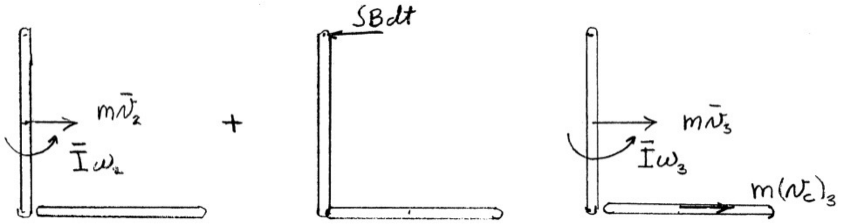
$$\bar{\mathbf{v}}_2 = \frac{L}{2}\sqrt{\frac{3g}{L}} \longrightarrow \qquad (v_B)_2 = L\omega = \sqrt{3gL}$$

PROBLEM 17.124 (Continued)

Impact condition:
$$-(v_C)_3 - (v_B)_3 = e(v_B)_2$$

$$(v_C)_3 - L\omega_3 = e\sqrt{3gL} \qquad (v_C)_3 = L\omega_3 + e\sqrt{3gL}$$

Principle of impulse-momentum at impact.



Syst Momenta$_2$ + **Syst Ext Imp$_{2\to3}$** = **Syst Momenta$_3$**

↶ moments about B:
$$m\bar{v}_2\frac{L}{2} + \bar{I}\omega_2 + 0 = m\bar{v}_3\frac{L}{2} + \bar{I}\omega_3 + m(v_C)_3 L$$

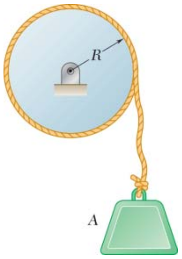
$$m\left(\frac{L}{2}\sqrt{\frac{3g}{L}}\right)\frac{L}{2} + \frac{1}{12}mL^2\sqrt{\frac{3g}{L}} + 0 = m\left(\frac{L}{2}\omega_3\right)\frac{L}{2} + \frac{1}{12}mL^2\omega_3 + m\left(L\omega_3 + e\sqrt{3gL}\right)L$$

$$\omega_3 = \left(\frac{1}{4} - \frac{3}{4}e\right)\sqrt{\frac{3g}{L}}$$

$$(v_C)_3 = \left(\frac{1}{4} - \frac{3}{4}e\right)\sqrt{3gL} + e\sqrt{3gL} \qquad (v_C)_3 = \frac{1}{4}(1+e)\sqrt{3gL}$$

For perfectly elastic impact, $e = 1$.
$$(v_C)_3 = \frac{1}{4}(1+1)\sqrt{3gL}$$

$$(\mathbf{v}_C)_3 = 0.866\sqrt{gL} \longrightarrow \blacktriangleleft$$



PROBLEM 17.125

Block A of mass m is attached to a cord which is wrapped around a uniform disk of mass M. The block is released from rest and falls through a distance h before the cord becomes taut. Derive expressions for the velocity of the block and the angular velocity of the disk immediately after the impact. Assume that the impact is (*a*) perfectly plastic, (*b*) perfectly elastic.

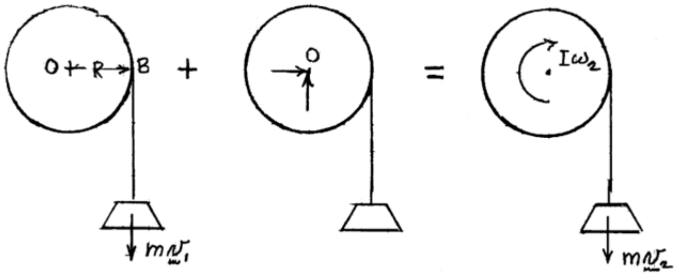
SOLUTION

After freely falling a distance h, block A has a velocity

$$\mathbf{v}_1 = \sqrt{2gh}\downarrow \qquad (1)$$

This result can be demonstrated by *conservation of energy*.

Principle of impulse and momentum.



Syst Momenta$_1$ + **Syst Ext Imp**$_{1\to 2}$ = **Syst Momenta**$_2$

↻ moments about O: $mv_1R + 0 = mv_2R + \bar{I}\omega_2$

Setting $\bar{I} = \frac{1}{2}MR^2$ for the disk and dividing by R,

$$mv_2 + \frac{1}{2}MR\omega_2 = mv_1$$

(2)

(*a*) *Plastic impact*. $e = 0$ and $v_2 = R\omega_2$

(3)

Substituting (3) into (2) and using (1) for v_1,

$$\left(m + \frac{1}{2}M\right)v_2 = mv_1$$

$$v_2 = \frac{2m}{2m+M}\sqrt{2gh}\downarrow \blacktriangleleft$$

$$\omega_2 = \frac{2m}{2m+M}\frac{\sqrt{2gh}}{R}\circlearrowright \blacktriangleleft$$

(*b*) *Elastic impact*. $e = 1$

$$(v_B)_2 - (v_A)_2 = -e\left[(v_B)_1 - (v_A)_1\right]$$

$$R\omega_2 - v_2 = -[0 - v_1] = v_1$$

$$R\omega_2 = v_1 + v_2$$

(4)

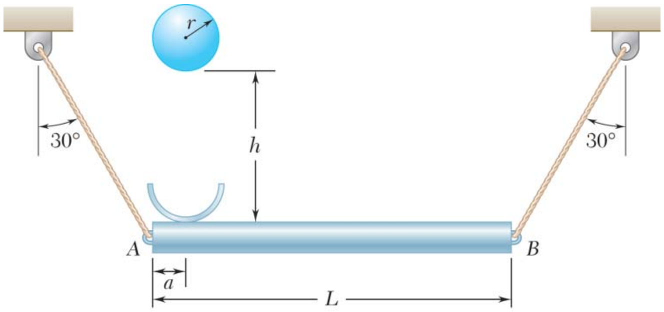
PROBLEM 17.125 (Continued)

Substituting (4) into (2) and using (1) for v_1

$$mv_2 + \frac{1}{2}M\left(v_1 + v_2\right) = mv_1$$

$$v_2 = \frac{2m - M}{2m + M}v_1 \qquad\qquad v_2 = \frac{2m - M}{2m + M}\sqrt{2gh}\downarrow \blacktriangleleft$$

$$R\omega_2 = \left(1 + \frac{2m - M}{2m + M}\right)v_1 \qquad\qquad \omega_2 = \frac{4m}{2m + M}\frac{\sqrt{2gh}}{R} \circlearrowright \blacktriangleleft$$



PROBLEM 17.126

A 2-kg solid sphere of radius r = 40 mm is dropped from a height h = 200 mm and lands on a uniform slender plank AB of mass 4 kg and length L = 500 mm which is held by two inextensible cords. Knowing that the impact is perfectly plastic and that the sphere remains attached to the plank at a distance a = 40 mm from the left end, determine the velocity of the sphere immediately after impact. Neglect the thickness of the plank.

SOLUTION

Masses and moments of inertia.

Sphere:

$$m_S = 2\text{ kg}, \quad r = 40\text{ mm} = 0.040\text{ m}$$

$$\bar{I}_S = \frac{2}{5}m_S r^2 = \left(\frac{2}{5}\right)(2\text{ kg})(0.04\text{ m})^2 = 1.28\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

Plank AB:

$$m_{AB} = 4\text{ kg}, \quad L = 500\text{ mm} = 0.5\text{ m}$$

$$\bar{I}_{AB} = \frac{1}{12}m_{AB}L^2 = \left(\frac{1}{12}\right)(4\text{ kg})(0.5\text{ m})^2 = 83.333\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

Velocity of sphere at impact.

$$v_S = \sqrt{2gh} = \sqrt{(2)(9.81\text{ m/s})(0.200\text{ m})} = 1.9809\text{ m/s}$$

Before impact.

Linear momentum: $m_S\mathbf{v}_S = (4\text{ kg})(1.9809\text{ m/s})\downarrow = 7.9236\text{ kg}\cdot\text{m/s}\downarrow$

with its line of action lying at distance $\frac{L}{2} - a$ from the midpoint of the plank.

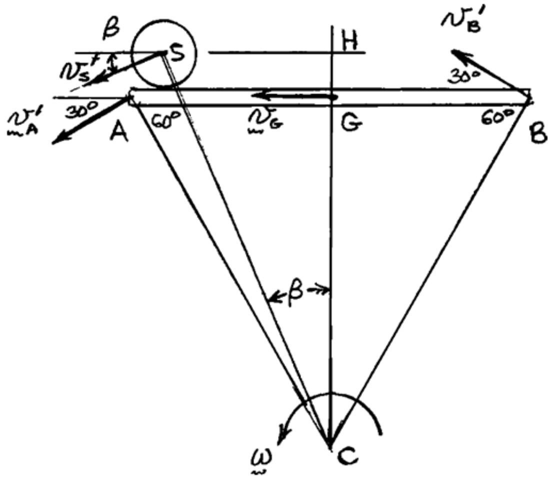
$$\frac{L}{2} - a = 0.25\text{ m} - 0.04\text{ m} = 0.21\text{ m}.$$

After impact. Assume that both cables are taut so $\mathbf{v}_A$ is perpendicular to the cable at A and $\mathbf{v}_B$ is perpendicular to the cable at B.

PROBLEM 17.126 (Continued)

Kinematics.

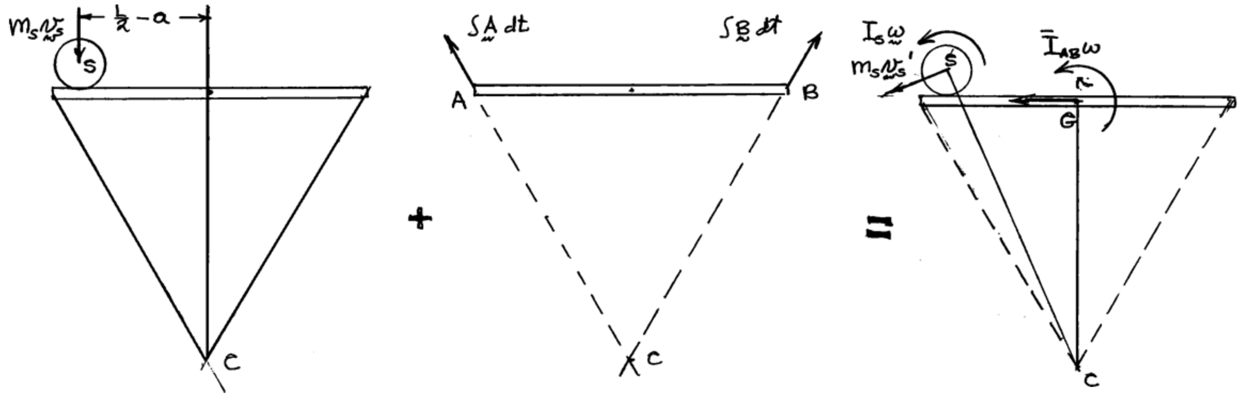
To locate the instantaneous center C draw line $\overline{AC}$ perpendicular to $\mathbf{v}_A$ and line $\overline{BC}$ perpendicular to $\mathbf{v}_B$. Let point G be the mass center of the plank AB and Point S be that of the sphere.


β
S
H
v_B'
30°
v_S'
v_A'
30°
A
60°
v_G
G
60°
B
β
ω
C

$$\overline{CH} = L\cos 30° + r$$
$$= (0.500\text{ m})\cos 30° + 0.040\text{ m}$$
$$= 0.47301\text{ m}$$

$$\overline{HS} = \frac{L}{2} - a = 0.21\text{ m}$$
$$\overline{CS} = \sqrt{\overline{CH}^2 + \overline{HS}^2} = 0.51753\text{ m}$$
$$\tan\beta = \frac{\overline{HS}}{\overline{CH}} = 0.44397 \qquad \beta = 23.94°$$
$$v_S = (\overline{CS})\omega = 0.51753\omega$$
$$v_G = (L\cos 30°)\omega = 0.43301\omega$$

Principle of impulse and momentum.


$m_S v_S$
$\frac{L}{2} - a$
S
C
$\int A\,dt$
$\int B\,dt$
A
B
C
+
$I_S\omega$
$\bar{I}_{AB}\omega$
$m_S v_S'$
S
G
C
=

Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

$+\circlearrowright$ Moments about C:

$$m_S v_S\left(\frac{L}{2} - a\right) + 0 = m_S v_S^1(\overline{CS}) + m_{AB} v_G^1(CH) + \bar{I}_S\omega + \bar{I}_{AB}\omega$$
$$m_S v_S\left(\frac{L}{2} - a\right) = [m_S(\overline{CS})^2 + m_{AB}\overline{CG}^2 + \bar{I}_S + \bar{I}_{AB}]\omega = I_C\omega$$

PROBLEM 17.126 (Continued)

where $$m_S v_S \left(\frac{L}{2} - a\right) = (2\text{ kg})(1.9809\text{ m/s})(0.21\text{ m}) = 0.83198\text{ kg}\cdot\text{m}^2/\text{s}$$

and $$\begin{aligned} I_C &= (2\text{ kg})(0.51753\text{ m})^2 + (4\text{ kg})(0.43301\text{ m})^2 + 1.28\times 10^{-3}\text{ kg}\cdot\text{m}^2 + 83.333\times 10^{-3}\text{ kg}\cdot\text{m}^2 \\ &= (0.53567 + 0.75 + 0.00128 + 0.08333)\text{ kg}\cdot\text{m}^2 = 1.37028\text{ kg}\cdot\text{m}^2 \end{aligned}$$

$$0.83198\text{ kg}\cdot\text{m}^2/\text{s} = (1.37028\text{ kg}\cdot\text{m}^2)\omega \qquad \boldsymbol{\omega} = 0.60716\text{ rad/s} ↻ ◀$$

$$v_S^1 = (0.51753\text{ m})(0.60716\text{ rad/s}) = 0.31422\text{ m/s}$$
$$v_G^1 = (0.43301\text{ m})(0.60716\text{ rad/s}) = 0.26291\text{ m/s}$$

To check that neither cable becomes slack during the impact, we show that $\int A dt$ and $\int B\, dt$ are positive quantities.

+↑ components: $$-m_S v_S + \left(\int A dt + \int B dt\right)\cos 30° = -m v_S^1 \sin\beta$$
$$\begin{aligned} \frac{\sqrt{3}}{2}\left(\int A dt + \int B dt\right) &= [m_S v_S - m_S v_1^1 \sin\beta]/\cos 30° \\ &= 7.9236 - (2)(0.31422)\sin 23.94° \\ &= 7.6686\text{ N}\cdot\text{s} \end{aligned}$$

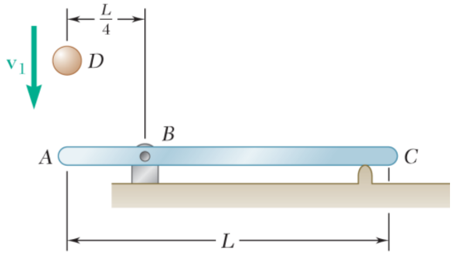
←+ components: $$0 + \left(\int A dt - \int B dt\right)\sin 30° = m_{AB} v_G^1 + m_S v_S \cos\beta$$
$$\begin{aligned} \frac{1}{2}\left(\int A dt - \int B dt\right) &= (4)(0.26291) + (2)(0.31422)\cos 23.94° \\ &= 1.6260\text{ N}\cdot\text{s} \end{aligned}$$

Solving the simultaneous equation gives

$$\int A dt = 6.05\text{ N}\cdot\text{s} \qquad \int B dt = 2.80\text{ N}\cdot\text{s}$$

The cables remain taut as assumed.

Velocity of sphere: $\mathbf{v}_S^1 = 0.314\text{ m/s}$ ↙ 23.9° ◀



PROBLEM 17.127

Member *ABC* has a mass of 2.4 kg and is attached to a pin support at *B*. An 800-g sphere *D* strikes the end of member *ABC* with a vertical velocity $\mathbf{v}_1$ of 3 m/s. Knowing that $L = 750$ mm and that the coefficient of restitution between the sphere and member *ABC* is 0.5, determine immediately after the impact (*a*) the angular velocity of member *ABC*, (*b*) the velocity of the sphere.

SOLUTION

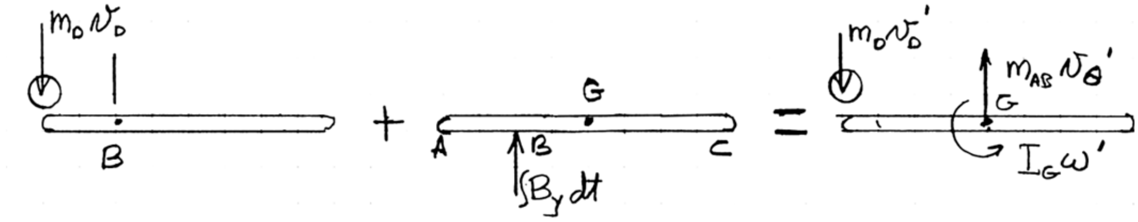
$$m_D = 0.800 \text{ kg}$$
$$L = 0.750 \text{ m}$$
$$\frac{1}{4}L = 0.1875 \text{ m}$$
$$m_{AC} = 2.4 \text{ kg}$$

Let Point *G* be the mass center of member *ABC*.

$$I_G = \frac{1}{12} m_{AC} L^2$$
$$= \frac{1}{12}(2.4)(0.750)^2$$
$$= 0.1125 \text{ kg} \cdot \text{m}^2$$

Kinematics after impact. $\boldsymbol{\omega}' = \omega' \circlearrowright$, $\mathbf{v}'_G = \frac{L}{4}\omega' \uparrow$, $\mathbf{v}'_A = \frac{L}{4}\omega' \downarrow$

Conservation of momentum.



$+\circlearrowright$ Moments about *B*:

$$m_D v_D \frac{L}{2} + 0 = m_D v'_D \frac{L}{2} + I_G \omega' + m_{AC} v'_G \frac{L}{4}$$

$$m_D v_D \frac{L}{4} = m_D v'_D \frac{L}{4} + \left[I_G + m_{AD} \left(\frac{L}{4}\right)^2 \right] \omega'$$

$$(0.800)(3)(0.1875) = (0.800)(0.1875)v'_D + [0.1125 + (2.4)(0.1875)^2]\omega'$$

$$0.45 = 0.15 v'_D + 0.196875\omega' \qquad (1)$$

PROBLEM 17.127 (Continued)

Coefficient of restitution.
$$v'_D - v'_A = v'_D - \frac{L}{4}\omega'$$
$$= -e(v_D - v_A)$$

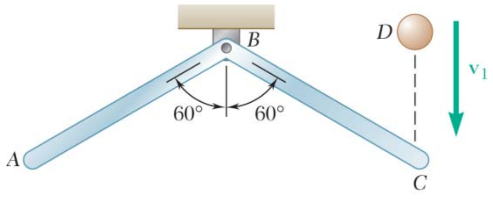
$$v'_D - 0.1875\omega' = -(0.5)(3-0) \tag{2}$$

Solving Eqs. (1) and (2) simultaneously.

(*a*) Angular velocity. $\omega' = 3$ $\boldsymbol{\omega}' = 3.00 \text{ rad/s} \circlearrowright$ ◀

(*b*) Velocity of *D*. $v'_D = -0.9375$ $v'_D = 0.938 \text{ m/s}\uparrow$ ◀

PROBLEM 17.128


B
D
v_1
60°
60°
A
C

Member *ABC* has a mass of 2.4 kg and is attached to a pin support at *B*. An 800-g sphere *D* strikes the end of member *ABC* with a vertical velocity $\mathbf{v}_1$ of 3 m/s. Knowing that $L = 750$ mm and that the coefficient of restitution between the sphere and member *ABC* is 0.5, determine immediately after the impact (*a*) the angular velocity of member *ABC*, (*b*) the velocity of the sphere.

SOLUTION

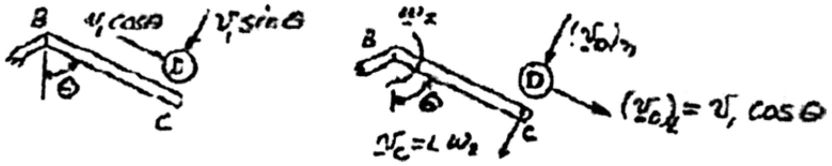
Let *M* be the mass of member *ABC* and $\bar{I}$ its moment of inertia about *B*.

$$M = 2.4 \text{ kg} \qquad \bar{I} = \frac{1}{12}M(2L)^2$$

where $L = 750 \text{ mm} = 0.75 \text{ m}$

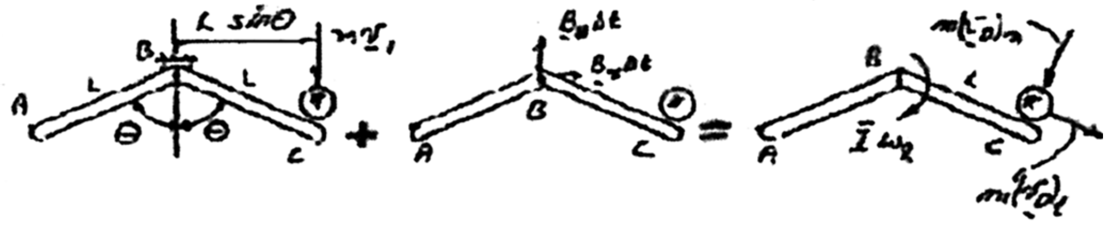
Let *m* be the mass of sphere *D*. $m = 800 \text{ g} = 0.8 \text{ kg}$

Impact kinematics and coefficient of restitution.


B
v_1 cos θ
v_1 sin θ
D
θ
C
ω_2
B
(v_D)_n
D
(v_D)_t = v_1 cos θ
θ
C
v_C = Lω_2

$$(v_1 \sin\theta)e = L\omega_2 - (v_D)_n: \quad (v_D)_n = L\omega_2 - (v_1 \sin\theta)e \tag{1}$$

Principle of impulse and momentum.


L sin θ
mv_1
B
L
L
A
θ
θ
C
+
B_y Δt
B_x Δt
B
A
C
=
m(v_D)_n
B
L
A
Ī ω_2
C
m(v_D)_t

Syst. Momenta$_1$ + **Syst. Ext. Imp.$_{1\to2}$** = **Syst. Momenta$_2$**

$+\circlearrowright$ Moments about *B*: $mv_1 L\sin\theta = \bar{I}\omega_2 + m(v_D)_n L$

$$mv_1 L\sin\theta = \frac{1}{12}M(2L)^2\omega_2 + m[L\omega_2 - (v_1\sin\theta)e]L$$

$$mv_1 \sin\theta = \frac{1}{3}ML\omega_2 - mL\omega_2 - m(v_1\sin\theta)e$$

$$m(1+e)\frac{v_1}{L}\sin\theta = \left(\frac{1}{3}M + m\right)\omega_2$$

PROBLEM 17.128 (Continued)

(*a*) Angular velocity.

$$\omega_2 = \frac{(3)(1+e)mv_1 \sin\theta}{(M+3m)L}$$

$$\omega_2 = \frac{(3)(1.5)(0.8)(3)\sin 60^\circ}{(2.4+2.4)(0.75)}$$

$$= 2.5981$$

$\boldsymbol{\omega}_2 = 2.60 \text{ rad/s}$ ↻ ◀

(*b*) Velocity of *D*.

From Eq. (1),

$$(v_D)_n = (0.75)(2.5981) - (3\sin 60^\circ)(0.5)$$

$$= 0.64976 \text{ m/s}$$

$$(v_D)_t = v_1 \cos 60^\circ$$

$$= 3\cos 60^\circ$$

$$= 1.5 \text{ m/s}$$

$(\mathbf{v}_D)_n = 0.64976$ m/s ↙ 30°

$(\mathbf{v}_D)_t = 1.5$ m/s ↘ 30°

$$v_D = \sqrt{(0.64976)^2 + (1.5)^2}$$

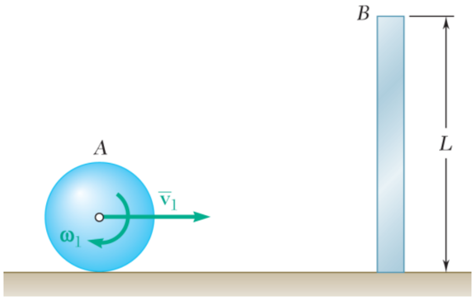
$$= 1.63468 \text{ m/s}$$

$$\tan\theta = \frac{0.64976}{1.5}$$

$$\theta = 23.4^\circ$$

$$\theta + 30^\circ = 53.4^\circ$$

$\mathbf{v}_D = 1.635$ m/s ↘ 53.4° ◀



PROBLEM 17.129

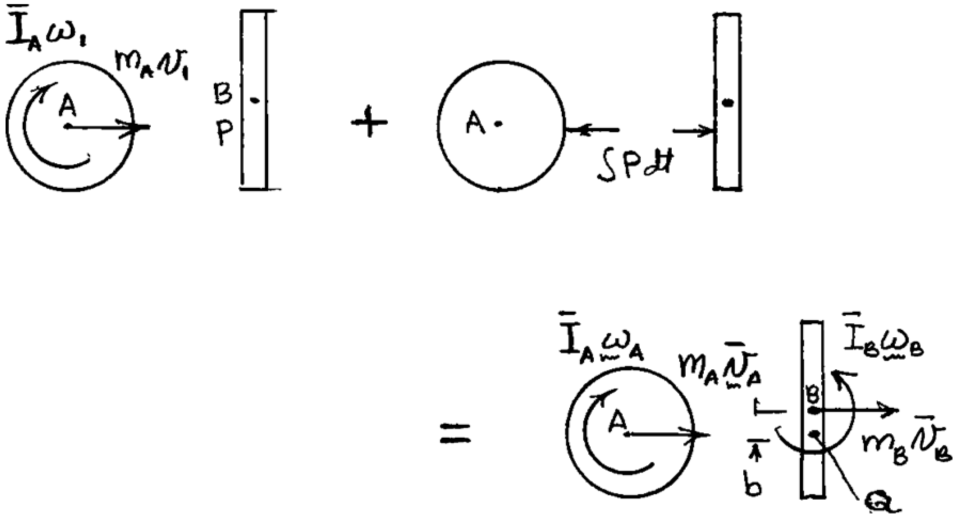
Sphere A of mass $m_A = 2$ kg and radius $r = 40$ mm rolls without slipping with a velocity $\bar{\mathbf{v}}_1 = 2$ m/s on a horizontal surface when it hits squarely a uniform slender bar B of mass $m_B = 0.5$ kg and length $L = 100$ mm that is standing on end and at rest. Denoting by μ_k the coefficient of kinetic friction between the sphere and the horizontal surface, neglecting friction between the sphere and the bar, and knowing the coefficient of restitution between A and B is 0.1, determine the angular velocities of the sphere and the bar immediately after the impact.

SOLUTION

Before impact sphere A rolls without slipping so that its instantaneous center of rotation is its contact point with the floor.

$$\omega_1 = \frac{v_1}{r} = \frac{2 \text{ m/s}}{0.040 \text{ m}} = 50 \text{ rad/s} \qquad \boldsymbol{\omega}_1 = 50 \text{ rad/s} \circlearrowright \blacktriangleleft$$

Analysis of impact. Use the principle of impulse and momentum. Let point A be the center of sphere A, point B be the mass center of bar B, and Points P and Q the contact point between the sphere and the bar, Point P being on sphere A and Point Q being on the bar B.



Syst. Momenta$_1$ + **Syst. Ext. Imp.$_{1\to 2}$** = **Syst. Momenta$_2$**

$\boldsymbol{\omega}_A = \omega_A \circlearrowright$ $\qquad$ $\boldsymbol{\omega}_B = \omega_B \circlearrowleft$

$\bar{\mathbf{v}}_A = \bar{v}_A \longrightarrow$ $\qquad$ $\bar{\mathbf{v}}_B = \bar{v}_B \longrightarrow,$

Sphere *A* alone.

$\circlearrowleft +$ Moments about *A*: $\bar{I}_A\omega_1 + 0 = \bar{I}_A\omega_A \qquad \omega_A = \omega_1$

$$\boldsymbol{\omega}_A = 50.0 \text{ rad/s} \circlearrowright \blacktriangleleft$$

Kinematics: $b = \dfrac{L}{2} - r = 50 \text{ mm} - 40 \text{ mm} = 10 \text{ mm} = 0.010 \text{ m}$

$$\mathbf{v}_Q = (\bar{v}_B + b\omega_B) \longrightarrow$$

Condition of impact. $\bar{\mathbf{v}}_A - \mathbf{v}_Q = -e\bar{\mathbf{v}}_1$

$$v_A - v_B - b\omega_B = -e\bar{v}_1 \tag{1}$$

Bar *B* alone:

$+\circlearrowright$ Moments about *Q*: $0 + 0 = \bar{I}_B\omega_B - bm_Bv_B$

$$\frac{1}{12}m_BL^2\omega_B - bm_Bv_B = 0$$

$$bv_B - \frac{1}{12}L^2\omega_B = 0 \tag{2}$$

Sphere *A* and bar *B* together.

$\xrightarrow{+}$ components:

$$m_A\bar{v}_1 + 0 = m_Av_A + m_Bv_B$$

$$m_A\bar{v}_A + m_B\bar{v}_B = m_A\bar{v}_1 \tag{3}$$

Data: $m_A = 2 \text{ kg}, \quad m_B = 0.5 \text{ kg}, \quad e = 0.1$

$\bar{v}_1 = 2 \text{ m/s}, \quad L = 0.100 \text{ m}, \quad b = 0.010 \text{ m}$

$$v_A - v_B - (0.010 \text{ m})\omega_B = -(0.1)(2 \text{ m/s}) \tag{1$'$}$$

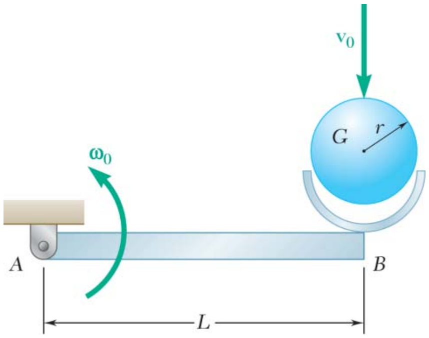
$$(0.010 \text{ m})v_B - \frac{1}{12}(0.100 \text{ m})^2\omega_B = 0 \tag{2$'$}$$

$$(2 \text{ kg})v_A + (0.5 \text{ kg})v_B = (2 \text{ kg})(2 \text{ m/s}) \tag{3$'$}$$

Solving Eqs. (1)′, (2)′, and (3)′ simultaneously,

$$v_A = 1.599 \text{ m/s} \qquad v_B = 1.606 \text{ m/s} \qquad \omega_B = 19.27 \text{ rad/s}$$

$$\boldsymbol{\omega}_B = 19.27 \text{ rad/s} \circlearrowright \blacktriangleleft$$



PROBLEM 17.130

A large 3-lb sphere with a radius r = 3 in. is thrown into a light basket at the end of a thin, uniform rod weighing 2 lb and length L = 10 in. as shown. Immediately before the impact the angular velocity of the rod is 3 rad/s counterclockwise and the velocity of the sphere is 2 ft/s down. Assume the sphere sticks in the basket. Determine after the impact (*a*) the angular velocity of the bar and sphere, (*b*) the components of the reactions at *A*.

SOLUTION

Let Point *G* be the mass center of the sphere and Point *C* be that of the rod *AB*.

Rod *AB*:

$$W_{AB} = 2\text{ lb.}\quad m_{AB} = \frac{2}{32.2} = 0.06211\text{ lb}\cdot\text{s}^2/\text{ft}$$

$$I_{AB} = \frac{1}{12}m_P L^2 = \frac{1}{12}(0.06211)\left(\frac{10}{12}\right)^2 = 0.003594\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Sphere:

$$W_S = 3\text{ lb}\quad m_S = \frac{3}{32.2} = 0.09317\text{ lb}\cdot\text{s}^2/\text{ft}$$

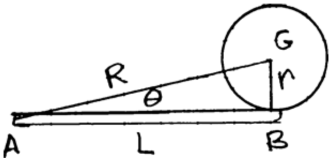
$$I_G = \frac{2}{5}m_S r^2 = \frac{2}{5}(0.09317)\left(\frac{3}{12}\right)^2 = 0.002329\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Impact. Before impact, bar *AB* is rotating about *A* with angular velocity $\boldsymbol{\omega}_0 = \omega_0 \circlearrowleft$ $(\omega_0 = 3\text{ rad/s})$ and the sphere is falling with velocity $\mathbf{v}_0 = v_0 \downarrow (v_0 = 2\text{ ft/s}).$ After impact, the rod and the sphere move together, rotating about *A* with angular velocity $\boldsymbol{\omega} = \omega \circlearrowright$.

Geometry.

$$R = \sqrt{L^2 + r^2} = \sqrt{10^2 + 3^2} = 10.44\text{ in.} = 0.8700\text{ ft}$$

$$\tan\theta = \frac{r}{L} = \frac{3}{10}\quad \theta = 16.7^\circ$$



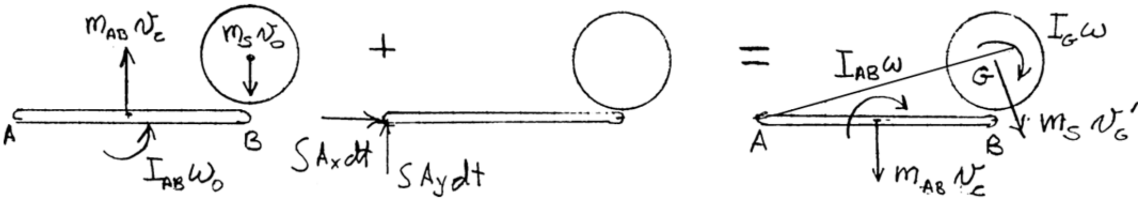
Kinematics: Before impact,

$$v_C = \frac{L}{2}\omega_0 = \left(\frac{5}{12}\right)(3) = 1.25\text{ ft/s}\uparrow$$

After impact,

$$\mathbf{v}_C = \frac{L}{2}\omega'\downarrow,\quad \mathbf{v}_G = R\omega' \searrow \theta$$

Principle of impulse and momentum. Neglect weights of the rod and sphere over the duration of the impact.



PROBLEM 17.130 (Continued)

(*a*) $+\circlearrowright$ Moments about *A*:

$$m_S v_0 L - I_{AB}\omega_0 - m_{AB}v_C\frac{L}{2} + 0 = I_G\omega' + m_S v'_G R + I_{AB}\omega' + m_{AB}v_C\frac{L}{2}$$

or
$$m_S v_0 L - I_{AB}\omega_0^2 - m_{AB}v_C\frac{L}{2} = \left(I_G + m_S R^2 + I_{AB} + \frac{1}{4}m_{AB}L^2\right)\omega' \qquad (1)$$

$$(0.09317)(2)\left(\frac{10}{12}\right) - (0.003594)(3) - (0.06211)(1.25)\left(\frac{5}{12}\right)$$

$$= \left[0.002329 + (0.09317)(0.87)^2 + 0.003594 + \frac{1}{4}(0.06211)\left(\frac{10}{12}\right)^2\right]\omega'$$

$$0.112152 = 0.087226\omega' \quad \omega' = 1.2858 \qquad \boldsymbol{\omega}' = 1.286 \text{ rad/s} \circlearrowright \blacktriangleleft$$

Normal accelerations at *C* and *G*.

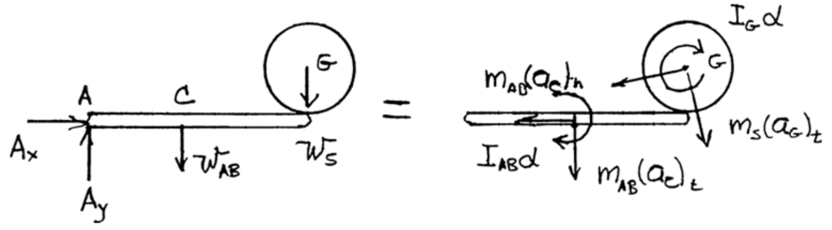
$$(\mathbf{a}_C)_n = \frac{L}{2}(\omega')^2 = \left(\frac{5}{12}\right)(1.2858)^2 = 0.6889 \text{ ft/s}^2 \longleftarrow$$

$$(a_G)_n = R(\omega')^2 = (0.87)(1.2858)^2 = 1.4384 \text{ ft/s}^2 \swarrow 16.7°$$

Tangential accelerations at *C* and *G*. $\boldsymbol{\alpha} = \alpha \circlearrowright$

$$(a_C)_t = \frac{L}{2}\alpha = \frac{5}{12}\alpha \downarrow \qquad (a_G)_t = R\alpha = 0.87\alpha \searrow 16.7°$$

(*b*) Kinetics. Use bar *AB* and the sphere as a free body.



$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}:$$

$$W_{AB}\frac{L}{2} + W_S L = I_{AB}\alpha + \frac{L}{2}m_{AB}(a_C)_t + I_G\alpha + m_S(a_G)_t R$$

$$= \left(I_{AB} + \frac{1}{4}m_{AB}L^2 + I_G + m_S R^2\right)\alpha$$

$$(2)\left(\frac{5}{12}\right) + (3)\left(\frac{10}{12}\right) = \left[0.003594 + \frac{1}{4}(0.06211)\left(\frac{10}{12}\right)^2 + 0.002329 + (0.09317)(0.87)^2\right]\alpha$$

$$3.3333 = 0.087226\alpha \qquad \boldsymbol{\alpha} = 38.214 \text{ rad/s}^2 \circlearrowright$$

$$(\mathbf{a}_C)_t = \left(\frac{5}{12}\right)(38.214) = 15.923 \text{ ft/s}^2 \downarrow, \quad (\mathbf{a}_G)_t = (0.87)(38.214) = 33.247 \text{ ft/s}^2 \searrow 16.7°$$

PROBLEM 17.130 (Continued)

$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}:$

$$A_x = -m_{AB}(a_C)_n - m_S(a_G)_n \cos 16.7° + m_S(a_G)_t \sin 16.70$$

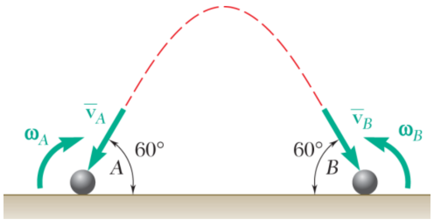
$$A_x = -(0.06211)(0.6889) - (0.09317)(1.4384)\cos 16.7° + (0.09317)(33.247)\sin 16.7°$$

$$A_x = 0.719 \text{ lb} \qquad \mathbf{A}_x = 0.719 \text{ lb} \longrightarrow \blacktriangleleft$$

$+\!\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad A_y - W_{AB} - W_S = -m_{AB}(a_C)_t - m_S(a_G)_t \cos 16.7° - m_S(a_G)_n \sin 16.7°$

$$A_y - 2 - 3 = -(0.06211)(15.923) - (0.09317)(33.247)\cos 16.7° - (0.09317)(1.4384)\sin 16.7°$$

$$A_y = 1.006 \text{ lb} \qquad \mathbf{A}_y = 1.006 \text{ lb} \uparrow \blacktriangleleft$$


ω_A
$\overline{\mathbf{v}}_A$
60°
A
60°
B
$\overline{\mathbf{v}}_B$
ω_B

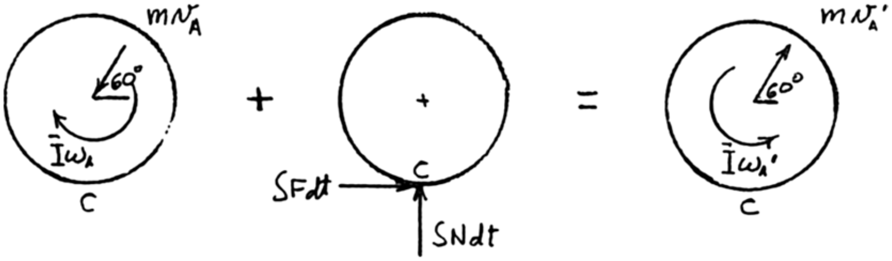
PROBLEM 17.131

A small rubber ball of radius r is thrown against a rough floor with a velocity $\overline{\mathbf{v}}_A$ of magnitude v_0 and a backspin $\boldsymbol{\omega}_A$ of magnitude ω_0. It is observed that the ball bounces from A to B, then from B to A, then from A to B, etc. Assuming perfectly elastic impact, determine the required magnitude ω_0 of the backspin in terms of $\overline{v}_0$ and r.

SOLUTION

Moment of inertia. $\quad \bar{I} = \frac{2}{5}mr^2$ Ball is assumed to be a solid sphere.

Impact at A.


$m\bar{v}_A$
60°
$\bar{I}\omega_A$
C
+
$\int F dt$
C
$\int N dt$
=
$m\bar{v}_A'$
60°
$\bar{I}\omega_A'$
C

$$\textbf{Syst. Momenta}_1 \quad + \quad \textbf{Syst. Ext. Imp.}_{1\rightarrow 2} \quad = \quad \textbf{Syst. Momenta}_2$$

For the velocity of the ball to be reversed on each impact,

$$v_A' = v_A = v_0$$
$$\omega_A' = \omega_A = \omega_0$$

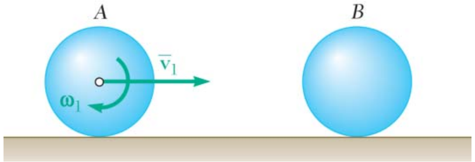
This is consistent with the assumption of perfectly elastic impact.

↺ Moments about C: $\quad mv_A r\cos 60° - \bar{I}\omega_A + 0 = \bar{I}\omega_A' - mv_A' r\cos 60°$

$$mv_0 r\cos 60° - \frac{2}{5}mr^2\omega_0 + 0 = \frac{2}{5}mr^2\omega_0 - mv_0 r\cos 60°$$

$$\frac{2}{5}r\omega_0 = v_0\cos 60°$$

$$\omega_0 = \frac{5}{4}\frac{v_0}{r} \quad ◀$$



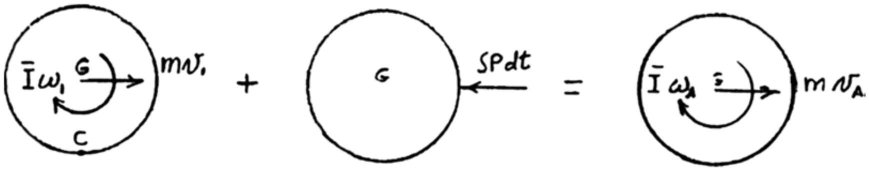
PROBLEM 17.132

Sphere *A* of mass *m* and radius *r* rolls without slipping with a velocity $\bar{\mathbf{v}}_1$ on a horizontal surface when it hits squarely an identical sphere *B* that is at rest. Denoting by μ_k the coefficient of kinetic friction between the spheres and the surface, neglecting friction between the spheres, and assuming perfectly elastic impact, determine (*a*) the linear and angular velocities of each sphere immediately after the impact, (*b*) the velocity of each sphere after it has started rolling uniformly.

SOLUTION

Moment of inertia. $\bar{I} = \frac{2}{5}mr^2$

Analysis of impact. Sphere *A*.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

Kinematics: Rolling without slipping in Position 1.

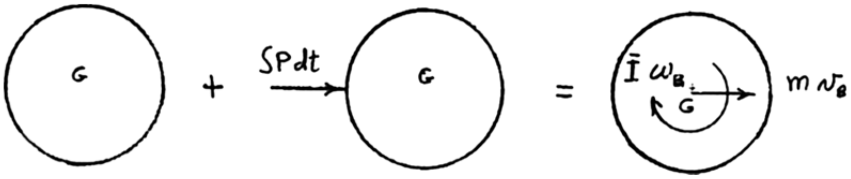
$$\omega_A = \frac{v_1}{r}$$

↻ Moments about *G*: $\bar{I}\omega_1 + 0 = \bar{I}\omega_A$

$$\omega_A = \omega_1 = \frac{v_1}{r}$$

$\xrightarrow{+}$ Linear components: $mv_1 - \int Pdt = mv_A$ (1)

Analysis of impact. *Sphere B*.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

$\xrightarrow{+}$ Linear components: $0 + \int Pdt = mv_B$ (2)

PROBLEM 17.132 (Continued)

Add Equations (1) and (2) to eliminate $\int P dt$.

$$mv_1 = mv_A + mv_B \quad \text{or} \quad v_B + v_A = v_1 \tag{3}$$

Condition of impact. $e = 1$. $\quad v_B - v_A = ev_1 = v_1 \quad (4)$

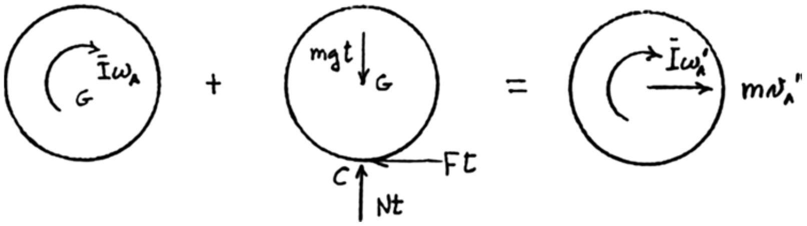
Solving Equations (3) and (4) simultaneously,

$$v_A = 0, \qquad v_B = v_1$$

↺ Moments about G: $\quad 0 + 0 = \bar{I}\omega_B \quad \omega_B = 0$

(*a*) *Velocities after impact.* $\quad \mathbf{v}_A = 0; \quad \boldsymbol{\omega}_A = \dfrac{v_1}{r}$ ↻; $\quad \mathbf{v}_B = v_1 \longrightarrow; \quad \boldsymbol{\omega}_B = 0$ ◀

Motion after Impact. *Sphere A*.



$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

Condition of rolling without slipping: $\quad v'_A = \omega'_A r$

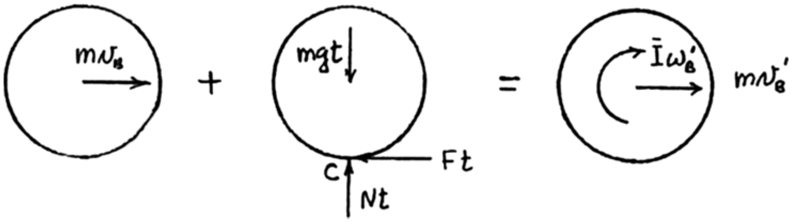
↺ Moments about C: $\quad \bar{I}\omega_A + 0 + \bar{I}\omega'_A + mv'_A r$

$$\left(\frac{2}{3}mr^2\right)\left(\frac{v_1}{r}\right) + 0 = \left(\frac{2}{5}mr^2\right)\omega'_A + m(r\omega'_A)r$$

$$\omega'_A = \frac{2}{7}\frac{v_1}{r}$$

$$v'_A = \frac{2}{7}v_1$$

Motion after impact. *Sphere B.*



$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

PROBLEM 17.132 (Continued)

Condition of rolling without slipping: $v'_B = r\omega'_B$

↻ Moments about *C*:

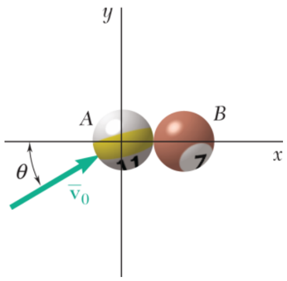
$$mv_B r + 0 = \bar{I}\,\omega'_B + mv'_B r$$

$$mv_1 r + 0 = \left(\frac{2}{5}mr^2\right)\omega'_B + m(r\omega'_B)r$$

$$\omega'_B = \frac{5}{7}\frac{v_1}{r}$$

$$v'_B = \frac{5}{7}v_1$$

(*b*) *Final Rolling Velocities.* $\mathbf{v}'_A = \frac{2}{7}v_1 \longrightarrow; \quad \mathbf{v}'_B = \frac{5}{7}v_1 \longrightarrow$ ◀



PROBLEM 17.133

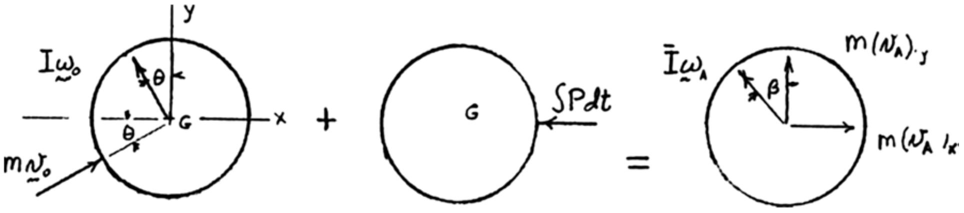
In a game of pool, ball A is rolling without slipping with a velocity $\bar{\mathbf{v}}_0$ as it hits obliquely ball B, which is at rest. Denoting by r the radius of each ball and by μ_k the coefficient of kinetic friction between a ball and the table and assuming perfectly elastic impact, determine (a) the linear and angular velocity of each ball immediately after the impact, (b) the velocity of ball B after it has started rolling uniformly.

SOLUTION

Moment of inertia. $\bar{I} = \frac{2}{5}mr^2$

(a) Impact analysis.

Ball A:



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

Kinematics of rolling: $\omega_0 = \frac{v_0}{r}$

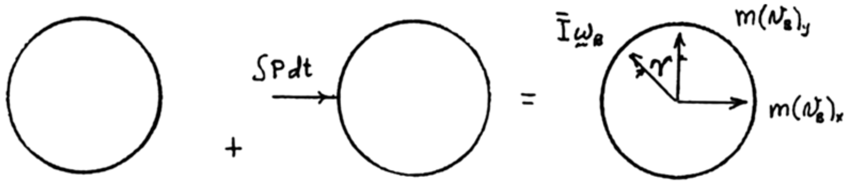
$\xrightarrow{+}$ Linear components: $mv_0 \cos\theta - \int P dt = m(v_A)_x$ (1)

$+\uparrow$ Linear components: $mv_0 \sin\theta + 0 = m(v_A)_y$ (2)

Moments about y axis: $\bar{I}\omega_0 \cos\theta + 0 = \bar{I}\omega_A \cos\beta$ (3)

Moments about x axis: $-\bar{I}\omega_0 \sin\theta + 0 = -\bar{I}\omega_A \sin\beta$ (4)

Ball B:



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

$\xrightarrow{+}$ Linear components: $0 + \int P dt = m(v_B)_x$ (5)

$+\uparrow$ Linear components: $0 + 0 = m(v_B)_y$ (6)

PROBLEM 17.133 (Continued)

Moments about y axis: $$0+0=\bar{I}\,\omega_B\cos\gamma \qquad (7)$$

Moments about x axis: $$0+0=\bar{I}\,\omega_B\sin\gamma \qquad (8)$$

Adding Equations (1) and (5) to eliminate $\int P\,dt$,

$$mv_0\cos\theta+0=m(v_A)_x+m(v_B)_x$$

or $$(v_B)_x+(v_A)_x=v_0\cos\theta \qquad (9)$$

Condition of impact. $e=1$: $$(v_B)_x-(v_A)_x=ev_0\cos\theta=v_0\cos\theta \qquad (10)$$

Solving Equations (9) and (10) simultaneously,

$$(v_A)_x=0,\quad (v_B)_x=v_0\cos\theta$$

From Equations (2) and (6), $$(v_A)_y=v_0\sin\theta,\quad (v_B)_y=0$$

$v_A=(v_0\sin\theta)\mathbf{j}$ ◀

$v_B=(v_0\cos\theta)\mathbf{i}$ ◀

From Equations (3) and (4) simultaneously,

$$\beta=\theta,\quad \omega_A=\omega_0=\frac{v_0}{r}$$

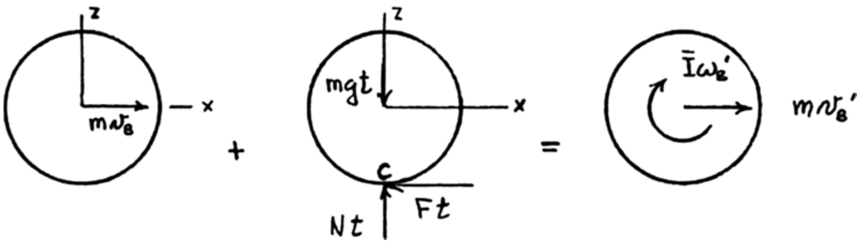
$\boldsymbol{\omega}_A=\frac{v_0}{r}(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})$ ◀

From Equations (7) and (8) simultaneously,

$$\omega_B=0$$

$\boldsymbol{\omega}_B=0$ ◀

(*b*) *Subsequent motion of ball B.*



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

Kinematics of rolling without slipping. $v_B'=r\omega_B'$

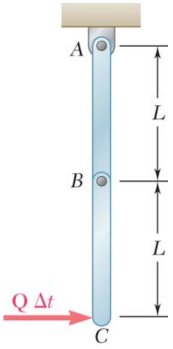
↻ Moments about C: $$mv_Br+0=\bar{I}\,\omega_B'+mv_B'r$$

$$=\frac{2}{5}mr^2\omega_B'+m(r\omega_B')r$$

$$\omega_B'=\frac{5}{7}\frac{v_B'}{r}=\frac{5}{7}\frac{v_1\cos\theta}{r}$$

$$v_B'=\frac{5}{7}v_1\cos\theta$$

$\mathbf{v}_B'=\frac{5}{7}(v_0\cos\theta)\mathbf{i}$ ◀



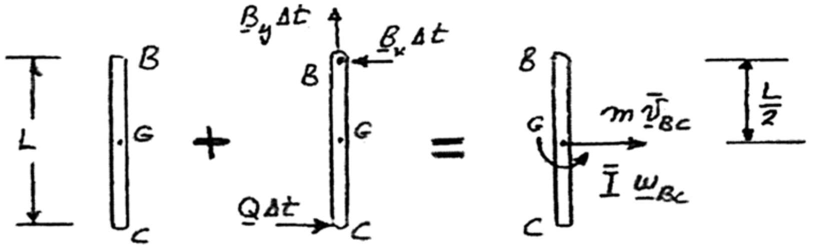
PROBLEM 17.134

Each of the bars AB and BC is of length $L = 400$ mm and mass $m = 1.2$ kg. Determine the angular velocity of each bar immediately after the impulse $Q\Delta t = (1.5\ \text{N}\cdot\text{s})\,\mathbf{i}$ is applied at C.

SOLUTION

Principle of impulse and momentum.

Bar BC:


L
B
G
C
+
$B_y \Delta t$
$B_x \Delta t$
B
G
$Q\Delta t$
C
=
B
G
$m\bar{v}_{BC}$
$\bar{I}\,\omega_{BC}$
C
$\frac{L}{2}$

$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

Kinematics $$\bar{v}_{BC} = v_B + \frac{L}{2}\omega_{BC} = L\omega_{AB} + \frac{L}{2}\omega_{BC}$$

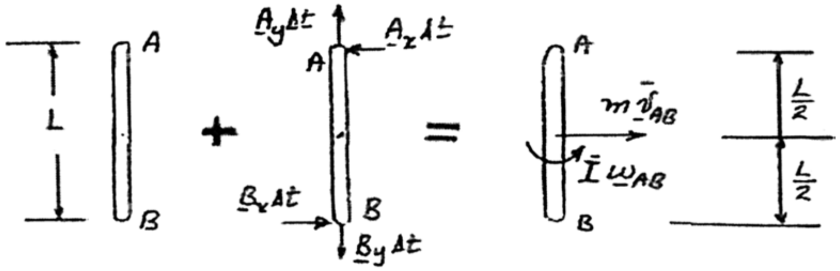
+↻ Moments about B: $$0 + (Q\Delta t)L = \bar{I}\,\omega_{BC} + m\bar{v}_{BC}\frac{L}{2}$$

$$(Q\Delta t)L = \frac{1}{12}mL^2\omega_{BC} + m\left(L\omega_{AB} + \frac{L}{2}\omega_{BC}\right)\frac{L}{2}$$

$$Q\Delta t = \frac{1}{2}mL\omega_{AB} + \frac{1}{3}mL\omega_{BC} \qquad (1)$$

$\xrightarrow{+}$ x components: $$Q\Delta t - B_x\Delta t = m\left(L\omega_{AB} + \frac{L}{2}\omega_{BC}\right) \qquad (2)$$

Bar AB:


L
A
B
+
$A_y \Delta t$
$A_x \Delta t$
A
$B_x \Delta t$
B
$B_y \Delta t$
=
A
$m\bar{v}_{AB}$
$\bar{I}\,\omega_{AB}$
B
$\frac{L}{2}$
$\frac{L}{2}$

$$\textbf{Syst. Momenta}_1 + \textbf{Syst. Ext. Imp.}_{1\to 2} = \textbf{Syst. Momenta}_2$$

PROBLEM 17.134 (Continued)

+↺ Moments about *A*:

$$0+(B_x\Delta t)L=\bar{I}\,\omega_{AB}+m\bar{v}_{AB}\frac{L}{2}$$

$$(B_x\Delta t)L=\frac{1}{12}mL^2\omega_{AB}+m\left(\frac{L}{2}\omega_{AB}\right)\frac{L}{2}$$

$$B_x\Delta t=\frac{1}{3}mL\omega_{AB} \qquad (3)$$

We now have 3 unknowns ($B_x\,\Delta\,t_1$, ω_{AB}, and ω_{BC}) and 3 equations.

Add Eqs. (2) and (3):

$$Q\Delta t=\frac{4}{3}mL\omega_{AB}+\frac{1}{2}mL\omega_{BC} \qquad (4)$$

Subtract Eq. (1) from Eq. (4):

$$0=\frac{5}{6}mL\omega_{AB}+\frac{1}{6}mL\omega_{BC}$$

$$\omega_{BC}=-5\omega_{AB} \qquad (5)$$

Substitute for ω_{BC} in Eq. (1):

$$Q\Delta t=\frac{1}{2}mL\omega_{AB}+\frac{1}{3}mL(-5\omega_{AB})$$

$$=-\frac{7}{6}mL\omega_{AB}$$

$$\omega_{AB}=-\frac{6}{7}\frac{Q\Delta t}{mL} \qquad (6)$$

Substituting into Eq. (5):

$$\omega_{BC}=-5\left(-\frac{6}{7}\frac{Q\Delta t}{mL}\right)$$

$$\omega_{BC}=\frac{30}{7}\frac{Q\Delta t}{mL} \qquad (7)$$

Given data:

$$L=0.400\text{ m}$$

$$Q\Delta t=1.5\text{ N}\cdot\text{s}$$

$$m=1.2\text{ kg}$$

Angular velocity of bar *AB*.

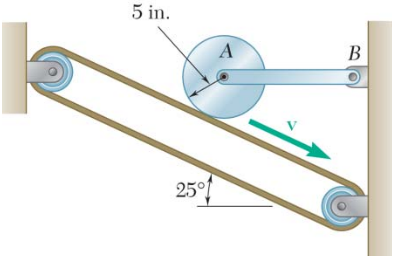
$$\omega_{AB}=-\frac{6}{7}\frac{Q\Delta t}{mL}=-\frac{(6)(1.5)}{(7)(1.2)(0.4)}$$

$\boldsymbol{\omega}_{AB}=2.68$ rad/s ↻ ◀

Angular velocity of bar *BC*.

$$\omega_{BC}=\frac{30}{7}\frac{Q\Delta t}{mL}=\frac{(30)(1.5)}{(7)(1.2)(0.4)}$$

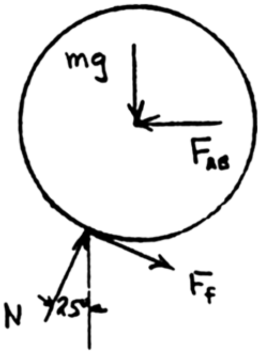
$\boldsymbol{\omega}_{BC}=13.39$ rad/s ↺ ◀



PROBLEM 17.135

A uniform disk of constant thickness and initially at rest is placed in contact with the belt shown, which moves at a constant speed $v = 80$ ft/s. Knowing that the coefficient of kinetic friction between the disk and the belt is 0.15, determine (*a*) the number of revolutions executed by the disk before it reaches a constant angular velocity, (*b*) the time required for the disk to reach that constant angular velocity.

SOLUTION



Kinetic friction. $F_f = \mu_k N = 0.15\text{ N}$

$$+\uparrow \Sigma F_y = N\cos 25° - F_f \sin 25° - mg = 0$$

$$(\cos 25° - \mu_k \sin 25°)N = mg$$

$$N = \frac{mg}{\cos 25° - 0.15\sin 25°}$$

$$= 1.18636\, mg$$

$$F_f = (0.15)(1.18636)mg$$

$$= 0.177954\, mg$$

Final angular velocity. $\omega_2 = \dfrac{v}{r}$

Moment of inertia. $\bar{I} = \dfrac{1}{2}mr^2$

(*a*) *Principle of work and energy.*

$$T_1 + W_{1\to 2} = T_2: \quad T_1 = 0$$

$$W_{1\to 2} = F_f r\theta = 0.177954 mgr\theta$$

$$T_2 = \frac{1}{2}\bar{I}\omega_2^2 = \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{1}{4}mv^2$$

$$0 + 0.177954 mgr\theta = \frac{1}{4}mv^2$$

$$\theta = 1.40486\frac{v^2}{gr}$$

$$= \frac{(1.40486)(80)^2}{(32.2)\left(\frac{5}{12}\right)}$$

$$= 670.14 \text{ radians}$$

$\theta = 106.7$ rev ◀

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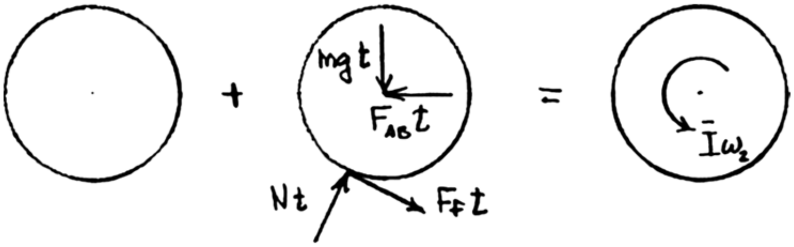
PROBLEM 17.135 (Continued)

(*b*) *Principle of impulse-momentum*.



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

$\circlearrowright$ Moments about *A*: $0 + F_f tr = \bar{I}\omega_2$

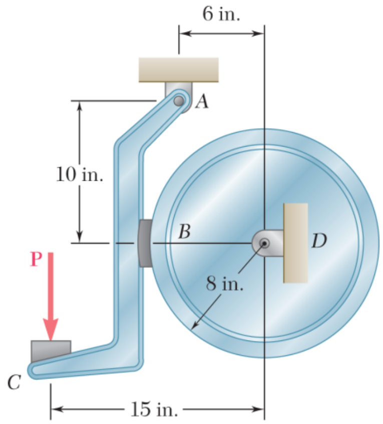
$$t = \frac{\bar{I}\omega_2}{F_f r}$$

$$= \frac{\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)}{0.177954mgr}$$

$$= 2.8097\frac{v}{g}$$

$$= \frac{(2.8097)(80)}{32.2}$$

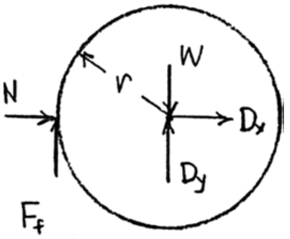
$t = 6.98$ s ◀



PROBLEM 17.136

The 8-in.-radius brake drum is attached to a larger flywheel that is not shown. The total mass moment of inertia of the flywheel and drum is $14\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ and the coefficient of kinetic friction between the drum and the brake shoe is 0.35. Knowing that the initial angular velocity of the flywheel is 360 rpm counterclockwise, determine the vertical force **P** that must be applied to the pedal *C* if the system is to stop in 100 revolutions.

SOLUTION



Kinetic energy.

$$\omega_1 = 360\text{ rpm} = 12\pi\text{ rad/s}$$

$$\omega_2 = 0$$

$$T_1 = \frac{1}{2}I\omega_1^2$$

$$= \frac{1}{2}(14)(12\pi)^2$$

$$= 9.9486\times 10^3\ \text{ft}\cdot\text{lb}$$

$$T_2 = 0$$

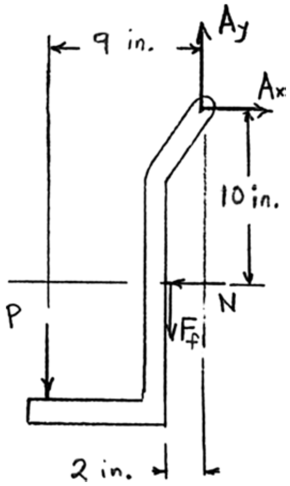
Work.

$$\theta = (100)(2\pi) = 628.32\text{ rad}$$

$$M_D = F_f r = F_f\left(\frac{8}{12}\right)$$

$$U_{1\to 2} = -M_D\theta = -F_f\left(\frac{8}{12}\right)(628.32)$$

$$= -418.88F_f$$



Principle of work and energy.

$$T_1 + U_{1\to 2} = T_2:\quad 9.9486\times 10^3 - 418.88F_f = 0$$

$$F_f = 23.75\text{ lb}$$

Kinetic friction force.

$$F_f = \mu_k N$$

$$N = \frac{F_f}{\mu_k} = \frac{23.75}{0.35} = 67.859\text{ lb}$$

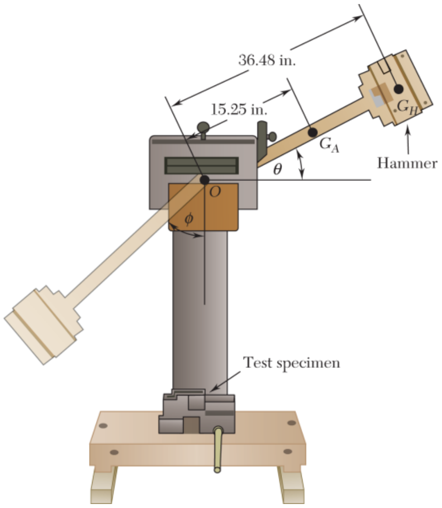
Statics.

$$+\circlearrowright\Sigma M_A = 0:\quad (9\text{ in.})P + (2\text{ in.})F_f - (10\text{ in.})N = 0$$

$$9P + (2)(23.75) - (10)(67.859) = 0$$

$$P = 70.12$$

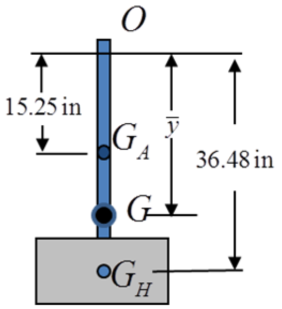
P = 70.1 lb ↓ ◀



PROBLEM 17.137

Charpy impact test pendulums are used to determine the amount of energy a test specimen absorbs during an impact (see ASTM Standard E23). The hammer weighs 71.2 lbs and has a mass moment of inertia about its center of gravity G_H of 20.9 slug·in^2. The arm weighs 19.5 lbs and has a mass moment of inertia about its own center of gravity G_A of 47.1 slug·in^2. The pendulum is released from rest from an initial position of θ = 39°. Knowing that the friction at pin O is negligible, determine *(a)* the impact speed when the hammer hits the test specimen, *(b)* the force on the pin O just before the hammer hits the test specimen, *(c)* the amount of energy that the test specimen absorbs if the hammer swings up to a maximum of ϕ = 70° after the impact.

SOLUTION



Given:

$$m_H = \frac{71.2}{g} = 2.211 \text{ slugs}, \ \bar{I}_H = 20.8 \text{ slug}\bullet\text{in}^2, \ \bar{r}_H = 36.48 \text{ in}$$

$$m_A = \frac{19.5}{g} = 0.6056 \text{ slugs}, \ \bar{I}_A = 47.1 \text{ slug}\bullet\text{in}^2, \ \bar{r}_A = 15.25 \text{ in}$$

$$\theta_1 = 39°, \ \theta_1 = 129°, \ \theta_1 = 199°$$

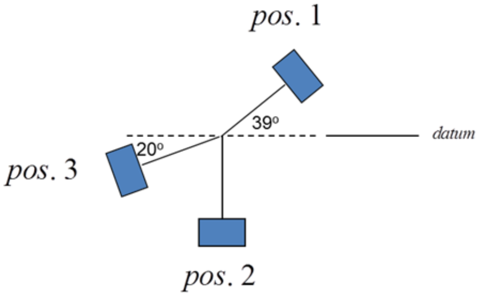
Mass properties of the system:

$$\bar{y} = \frac{\sum m_i y_i}{\sum m_i} = \frac{2.211(36.48) + 0.6056(15.25)}{2.211 + 0.6056} = 31.916 \text{ in}$$

$$m_T = m_A + m_H = 2.8168 \text{ slugs}$$

$$I_{O,\mathrm{T}} = I_{O,A} + I_{O,\mathrm{H}} = \bar{I}_A + m_A (15.25)^2 + \bar{I}_H + m_H (36.48)^2 = 3151.36 \text{ slug}\bullet\text{in}^2$$

Work Energy Equation Pos. 1 &2:



$$\cancel{T_1} + V_{g1} + \cancel{V_{e,1}} + \cancel{U'_{1-2}} = T_2 + V_{g2} + \cancel{V_{e,2}}$$

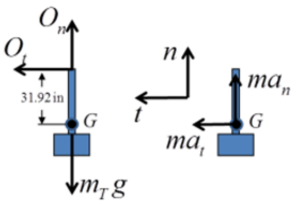
$$m_T g \frac{\bar{y}}{12} \sin 39° = \frac{1}{2} \frac{I_O}{144} \omega_2^2 - m_T g \frac{\bar{y}}{12}$$

$$\omega_2 = 5.993 \text{ rad/s}$$

Kinematics: $v_{H2} = r_H \omega_2$

(a) $v_{H2} = \dfrac{36.48}{12} 5.993$ $\qquad v_{H2} = 18.22$ ft/s ◄

Free Body Diagram at Position 2:



PROBLEM 17.137 (Continued)

Kinetics:

$$\sum M_O = I_O \alpha$$
$$0 = \alpha$$

$$\sum F_t = m_T a_t$$
$$O_t = m_T \bar{y}\, \cancel{\alpha}$$
$$O_t = 0$$

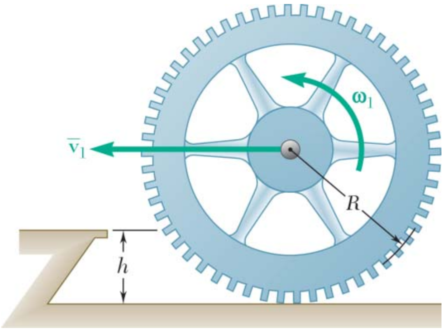
$$\sum F_n = m_T a_n$$
$$O_n - m_T g = m_T \frac{\bar{y}}{12} \omega^2$$
$$O_n = 90.7 + 2.8168 \frac{31.91}{12} 5.993^2$$
$$O_n = 359.7 \text{ lbs}$$

(b) $O_n = 359.7$ lbs ◀

(c) Work Energy Equation Pos. 1 and 3:

$$\cancel{T_1} + V_{g1} + \cancel{V_{e,1}} + U'_{1-3} = \cancel{T_3} + V_{g3} + \cancel{V_{e,3}}$$
$$U'_{1-3} = V_{g3} - V_{g1}$$
$$U'_{1-3} = -m_T g \frac{\bar{y}}{12} \sin 20° - m_T g \frac{\bar{y}}{12} \sin 39°$$
$$U'_{1-3} = -234.2 \text{ ft} \bullet \text{lbs}$$

234.2 ft•lbs ◀



PROBLEM 17.138

The gear shown has a radius R = 150 mm and a radius of gyration $\bar{k} = 125$ mm. The gear is rolling without sliding with a velocity $\bar{\mathbf{v}}_1$ of magnitude 3 m/s when it strikes a step of height h = 75 mm. Because the edge of the step engages the gear teeth, no slipping occurs between the gear and the step. Assuming perfectly plastic impact, determine (a) the angular velocity of the gear immediately after the impact, (*b*) the angular velocity of the gear after it has rotated to the top of the step.

SOLUTION

Part (*a*) Conditions just after impact.

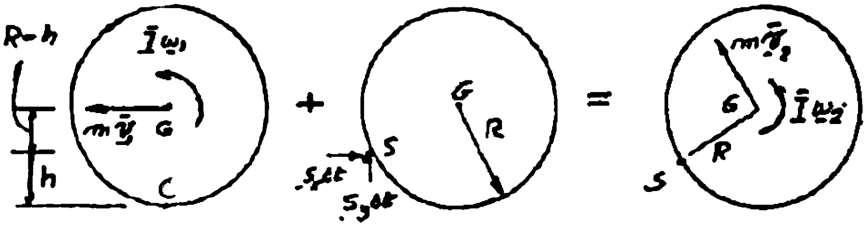
Kinematics. Just before impact, the contact Point C with the floor the instantaneous center of rotation of the gear.

$$\bar{\mathbf{v}}_1 = R\omega_1 \longleftarrow$$

Just after impact, Point S is the instantaneous center of rotation.

$$\mathbf{v}_2 = R\omega_2 \searrow \theta \quad \text{(perpendicular to } GS\text{)}$$

Principle of impulse and momentum.



+↺ Moments about S: $\quad m\bar{v}_2(R-h) + \bar{I}\omega_1 = m\bar{v}_2 R + \bar{I}\omega_2$

$$m(R\omega_1)(R-h) + m\bar{k}^2\omega_1 = m(R\omega_2)R + m\bar{k}^2\omega_2$$

$$[R(R-h) + \bar{k}^2]\omega_1 = (R^2 + \bar{k}^2)\omega_2$$

$$\omega_2 = \frac{R^2 + \bar{k}^2 - Rh}{R^2 + \bar{k}^2}\omega_1 \qquad \omega_2 = \left[1 - \frac{Rh}{R^2 + \bar{k}^2}\right]\omega_1 \qquad (1)$$

Data: $\quad R = 150\text{ mm}, \quad \bar{k} = 125\text{ mm}, \quad v_1 = 3\text{ m/s}, \quad h = 75\text{ mm}$

$$\omega_1 = \frac{v_1}{R} = \frac{3\text{ m/s}}{0.150\text{ m}} = 20\text{ rad/s}$$

Angular velocity.

From (1), $\quad \omega_2 = \left[1 - \frac{(150)(75)}{(150^2 + 125^2)}\right](20\text{ rad/s}) = 0.7049(20) \qquad \boldsymbol{\omega}_2 = 14.10\text{ rad/s}$ ↺ ◀

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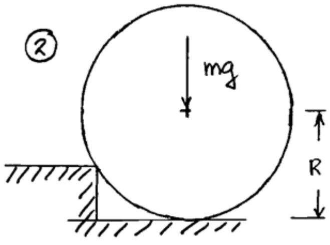
PROBLEM 17.138 (Continued)

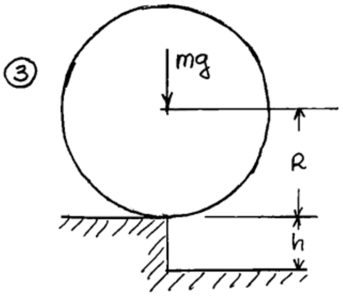
Part (*b*) Conditions at the top of the step.

The gear pivots about the edge of the step. Use the principle of conservation of energy.

Position (2): The gear has just broken contact with the floor.

Position (3): The center of the gear is above the edge of the step.





Kinematics: (Rotation about *S*) $\bar{v} = \omega R$

Kinetic energy:

$$T = \frac{1}{2}\bar{I}\omega^2 + \frac{1}{2}m\bar{v}^2$$

$$= \frac{1}{2}m\bar{k}^2\omega^2 + \frac{1}{2}mR^2\omega^2$$

$$= \frac{1}{2}m(\bar{k}^2 + R^2)\omega^2$$

Position (2):

$$T_2 = \frac{1}{2}m(\bar{k}^2 + R^2)\omega_2^2$$

$$V_2 = mgR$$

Position (3):

$$T_3 = \frac{1}{2}m(\bar{k}^2 + R^2)\omega_3^2$$

$$V_3 = mg(R + h)$$

Principle of conservation of energy: $T_2 + V_2 = T_3 + V_3$

$$\frac{1}{2}m(\bar{k}^2 + R^2)\omega_2^2 + mgR = \frac{1}{2}m(\bar{k}^2 + R^2)\omega_3^2 + mg(R + h)$$

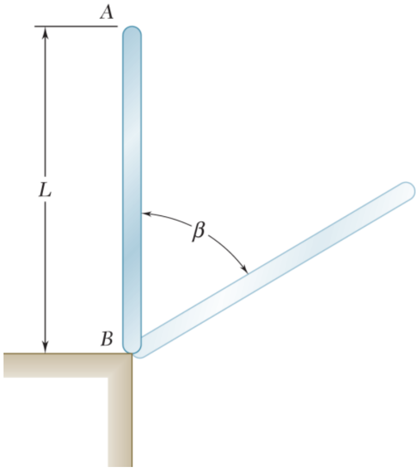
Angular velocity:

$$\omega_3^2 = \omega_3^2 - \frac{2gh}{\bar{k}^2 + R^2}$$

$$= (14.10\ \text{rad/s})^2 - \frac{(2)(9.81\ \text{m/s}^2)(0.075\ \text{m})}{(0.125\ \text{m})^2 + (0.150\ \text{m})^2}$$

$$= 160.21\ \text{rad}^2/\text{s}^2$$

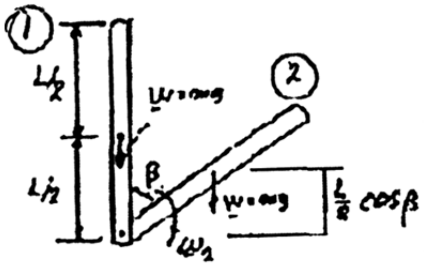
$\omega_3 = 12.66\ \text{rad/s}$ ↻ ◀



PROBLEM 17.139

A uniform slender rod is placed at corner B and is given a slight clockwise motion. Assuming that the corner is sharp and becomes slightly embedded in the end of the rod, so that the coefficient of static friction at B is very large, determine (*a*) the angle β through which the rod will have rotated when it loses contact with the corner, (*b*) the corresponding velocity of end A.

SOLUTION



Position 1

$$T_1 = 0$$

$$V_1 = mgh_1 = \frac{mgL}{2}$$

Position 2

$$V_2 = mgh_2 = \frac{mgL\cos\beta}{2}$$

$$T_2 = \frac{1}{2}I\omega_2^2 = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega_2^2$$

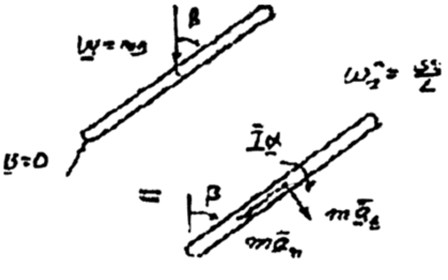
Principle of conservation of energy.

$$T_1 + V_1 = T_2 + V_2$$

$$0 + \frac{mgL}{2} = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega_2^2 + \frac{mgL\cos\beta}{2}$$

$$\omega_2^2 = \frac{3g}{L}(1-\cos\beta) \qquad (1)$$

Normal acceleration of mass center.



$$a_n = \frac{L}{2}\omega_2^2 = \frac{3}{2}g(1-\cos\beta)$$

$$+\swarrow \Sigma F = +\Sigma F_{\text{eff}} = ma_n$$

$$mg\cos\beta = \frac{3}{2}mg(1-\cos\beta)$$

(*a*) Angle β.

$$\frac{5}{2}\cos\beta = \frac{3}{2} \qquad \cos\beta = 0.6$$

$\beta = 53.1°$ ◀

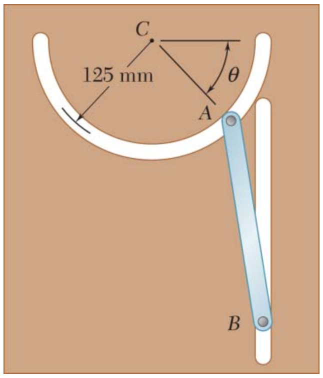
From (1)

$$\omega_2^2 = \frac{3g}{L}(1-0.6) = 1.2\frac{g}{L} \qquad \omega_2 = 1.09545\sqrt{\frac{g}{L}}$$

(*b*) Velocity of end A

$$v_A = L\omega_2$$

$\mathbf{v}_A = 1.095\sqrt{gL}$ ↖ 53.1° ◀



PROBLEM 17.140

The motion of the slender 250-mm rod AB is guided by pins at A and B that slide freely in slots cut in a vertical plate as shown. Knowing that the rod has a mass of 2 kg and is released from rest when $\theta = 0$, determine the reactions at A and B when $\theta = 90°$.

SOLUTION

Let Point G be the mass center of rod AB.

$$m = 2 \text{ kg}$$

$$L = 0.25 \text{ m}$$

$$I_G = \frac{1}{12}mL^2 = 0.0104667 \text{ kg}\cdot\text{m}^2$$

Kinematics. $\theta = 90°$

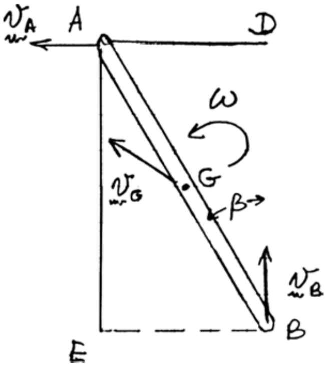
$$\overline{AD} = R = 0.125 \text{ m}$$

$$AB = L = 0.25 \text{ m}$$

$$\sin\beta = \frac{R}{L} = \frac{1}{2} \qquad \beta = 30°$$

$$\overline{AG} = \frac{L}{2} = 0.125 \text{ m}$$

$$\overline{BG} = 0.125 \text{ m}$$



Point E is the instantaneous center of rotation of bar AB.

$$v_G = \frac{L}{2}\omega = 0.125\ \omega$$

$$v_A = (L\cos 30°)\omega = 0.21651\omega$$

$$v_B = (L\sin 30°)\omega = 0.125\omega$$

Use principle of conservation of energy to obtain the velocities when $\theta = 90°$:

Use level A as the datum for potential energy.

Position 1 $\theta = 0$

$$T_1 = 0$$

$$V_1 = -mg\frac{L}{2} = -(2)(9.81)(0.125) = -2.4525 \text{ J}$$

PROBLEM 17.140 (Continued)

Position 2 $\theta = 90°$

$$\begin{aligned}
T_2 &= \frac{1}{2}I_G\omega^2 + \frac{1}{2}mv_G^2 \\
&= \frac{1}{2}(0.0104667)\omega^2 + \frac{1}{2}(2)(0.125\omega)^2 \\
&= 0.0208583\omega^2 \\
V_2 &= -mg\left(R + \frac{L}{2}\cos\beta\right) \\
&= -(2)(9.81)(0.125 + 0.125\cos 30°) \\
&= -4.5764 \text{ J}
\end{aligned}$$

$$T_1 + V_1 = T_2 + V_2: \quad 0 - 2.4525 = 0.0208583\omega^2 - 4.5764$$

$$\begin{aligned}
\omega^2 &= 101.826 \text{ rad}^2/\text{s}^2 \\
\omega &= 10.091 \text{ rad/s} \\
v_A &= (0.21651)(10.091) \\
&= 2.1848 \text{ m/s} \\
v_G &= (0.125)(10.091) \\
&= 1.2614 \text{ m/s}
\end{aligned}$$

More kinematics: For Point *A* moving in the curved slot,

$$\begin{aligned}
\mathbf{a}_A &= (a_C)_x\mathbf{i} + \frac{v_A^2}{R}\mathbf{j} \\
&= (a_C)_x\mathbf{i} + \frac{(2.1847)^2}{0.125}\mathbf{j} \\
&= (a_C)_x\mathbf{i} + 38.1833\mathbf{j}
\end{aligned}$$

For the rod *AB*, $\boldsymbol{\alpha} = \alpha\mathbf{k}, \quad \mathbf{v}_B = v_B\mathbf{j}$

$$\begin{aligned}
\mathbf{r}_{A/B} &= -L\sin 30°\mathbf{i} + L\cos 30°\mathbf{j} \\
&= -0.125\mathbf{i} + 0.21651\mathbf{j} \\
\mathbf{r}_{G/B} &= \frac{1}{2}\mathbf{r}_{A/B} \\
&= -0.0625\mathbf{i} + 0.108253\mathbf{j} \\
\mathbf{a}_A &= \mathbf{a}_B + \boldsymbol{\alpha} + \mathbf{r}_{A/B} - \omega^2\mathbf{r}_{A/B} \\
&= a_B\mathbf{j} + \alpha\mathbf{k} \times (-0.125\mathbf{i} + 0.21651\mathbf{j}) \\
&\quad -(10.091)^2(-0.125\mathbf{i} + 0.21651\mathbf{j}) \\
&= a_B\mathbf{j} - 0.125\alpha\mathbf{j} - 0.21651\alpha\mathbf{i} + 12.7285\mathbf{i} - 22.0468\mathbf{j}
\end{aligned}$$

PROBLEM 17.140 (Continued)

Matching vertical components of $\mathbf{a}_A$

$$38.1833 = a_B - 0.125\alpha - 22.0468$$

$$a_B = 0.125\alpha + 60.2301$$

$$\mathbf{a}_G = \mathbf{a}_B + \mathbf{a}_{G/B}$$

$$= \mathbf{a}_B + \alpha\mathbf{k}\times\mathbf{r}_{G/B} - \omega^2 r_{G/B}$$

$$= (0.125\alpha + 60.2324)\mathbf{j} + \alpha\mathbf{k}\times(-0.0625\mathbf{i} + 0.108253\mathbf{j})$$

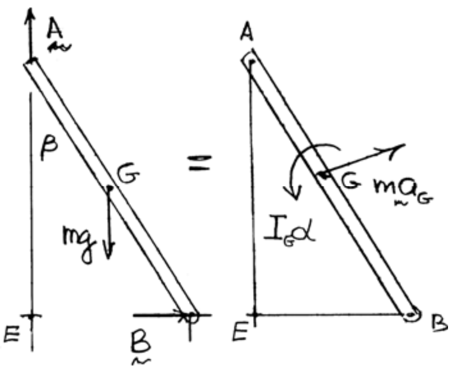
$$-(10.091)^2(-0.0625\mathbf{i} + 0.108253\mathbf{j})$$

$$= 0.125\alpha\mathbf{j} + 60.2301\mathbf{j} - 0.0625\alpha\mathbf{j} - 0.108253\alpha\mathbf{i}$$

$$+6.3643\mathbf{i} - 11.0232\mathbf{j}$$

$$\mathbf{a}_G = (-0.108253\alpha + 6.3643)\mathbf{i} + (0.0625\alpha + 49.2069)\mathbf{j}$$

Kinetics: Use rod *AB* as a free body.



$$+\circlearrowright \Sigma\mathbf{M}_E = \Sigma(\mathbf{M}_E)_{\text{eff}}:$$

$$-mg\frac{L}{2}\sin\beta\mathbf{k} = I_G\boldsymbol{\alpha} + \mathbf{r}_{G/E}\times(m\mathbf{a}_G)$$

$$-(2)(9.81)(0.125)\sin 30°\mathbf{k}$$

$$= 0.0104667\boldsymbol{\alpha} + (0.0625\mathbf{i} + 0.10825\mathbf{j})\times(m\mathbf{a}_G)$$

$$-1.22625 = 0.0104667\alpha + 0.03125\alpha + 4.7730$$

$$0.0417167\alpha = -5.99925$$

$$\alpha = -143.808 \text{ rad/s}^2$$

$$\mathbf{a}_G = (-21.933 \text{ m/s}^2)\mathbf{i} + (40.2189 \text{ m/s}^2)\mathbf{j}$$

$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}} = m(a_G)_x: \quad -B = (2)(-21.932) = 43.864 \text{ N}$ $\qquad$ $\mathbf{B} = 43.9 \text{ N} \longrightarrow$ ◀

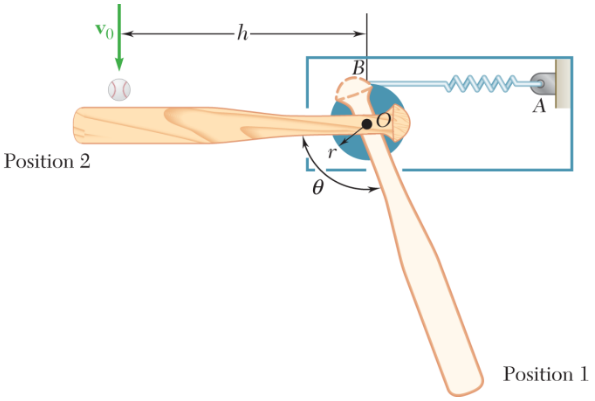
$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}} = m(a_G)_y: \quad A - mg = (2)(40.4289) = 80.4378$

$A = (2)(9.81) + 80.4378 = 100.058$ $\qquad$ $\mathbf{A} = 100.1 \text{ N} \uparrow$ ◀

Check by considering

$$\circlearrowleft + \Sigma M_G = \Sigma M_{G\,\text{eff}}: \quad \circlearrowleft \Sigma M_G = (0.0625)A - 0.108253B = 1.5052 \text{ N}\cdot\text{m} \circlearrowleft$$

$$\circlearrowleft \Sigma(M_G)_{\text{eff}} = I_G(-\alpha) = (0.0104667)(143.808) = 1.5052 \text{ N}\cdot\text{m} \circlearrowleft$$



PROBLEM 17.141

A baseball attachment that helps people with mobility impairments play T-ball and baseball is powered by a spring that is unstretched at position 2. The spring is attached to a cord, which is fastened to point B on the 75-mm radius pulley. As the pulley, which is fixed at point *O*, rotates backwards to the cocked position at *θ*, the rope wraps around the pulley and stretches the spring of stiffness *k* = 2000 N/m. The combined mass moment of inertia of all rotating components about point *O* is 0.40 kg·m^2. The swing is timed perfectly to strike a 145 gram baseball travelling with a speed of v_0= 10 m/s at a distance of h = 0.7 m away from point *O*. Knowing that the coefficient of restitution between the bat and ball is 0.59, determine the velocity of the baseball immediately after the impact. Assume that the ball is travelling primarily in the horizontal plane and that its spin is negligible.

SOLUTION

Given:

$$r = 0.075 \text{ m},\ k = 2000 \text{ N/m},\ I_O = 0.40 \text{ kg}\bullet\text{m}^2,\ m_B = 0.145 \text{ kg}$$

$$v_b = 10 \text{ m/s},\ h = 0.7 \text{ m},\ e = 0.59,\ \theta = 120°$$

$$s_1 = 120\frac{\pi}{180}0.075 \ \Rightarrow\ s_1 = 0.05\pi,\ s_2 = 0$$

Find the speed of the Bat at impact using Work Energy:

$$\cancel{T_1} + \cancel{V_{g1}} + V_{e,1} + \cancel{U'_{1-2}} = T_2 + \cancel{V_{g2}} + \cancel{V_{e,2}}$$

$$\frac{1}{2}ks_1^2 = \frac{1}{2}I_O\omega_2^2$$

$$\omega_2 = \sqrt{\frac{ks_1^2}{I_O}}$$

$$\omega_2 = 11.107 \text{ rad/s}$$

Kinematics:

$$v_{Bat} = h\omega_2$$
$$= 7.775 \text{ m/s}$$

Coefficient of Restitution:

$$e = \frac{v'_{Bat} - v'_b}{v_b - v_{Bat}}$$

$$0.59 = \frac{v'_{Bat} - v'_b}{10 - 7.775}$$

$$v'_{Bat} = v'_b - 10.487$$

PROBLEM 17.141 (Continued)

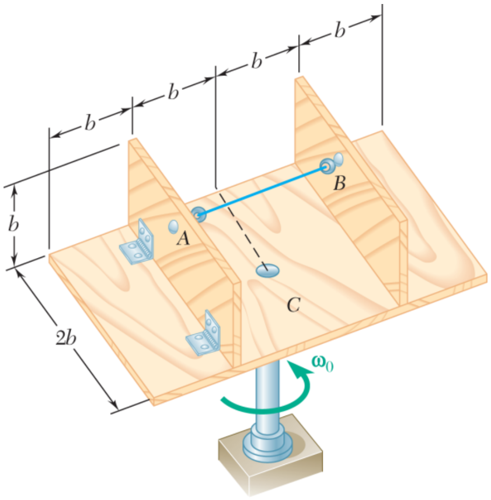
Conservation of Momentum about point O during impact:

$$H_{O,2} = H_{O,3}$$

$$m_b v_O h - I_O \omega_2 = -m_b v_b' h - I_O \frac{v_{Bat}'}{h}$$

$$v_B' = -\frac{m_b v_O h - I_O \omega_2 + \frac{I_O}{h}\left(v_b' - 10.487\right)}{h m_B}$$

$v_B' = 14.0$ m/s ◀

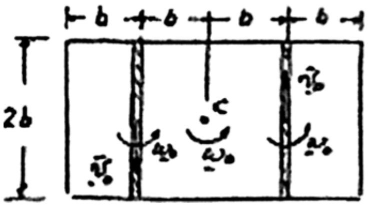


PROBLEM 17.142

Two panels A and B are attached with hinges to a rectangular plate and held by a wire as shown. The plate and the panels are made of the same material and have the same thickness. The entire assembly is rotating with an angular velocity ω_0 when the wire breaks. Determine the angular velocity of the assembly after the panels have come to rest against the plate.

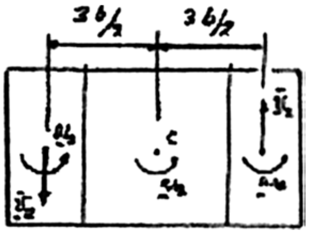
SOLUTION

Geometry and kinematics:



Panels in up position

$$\bar{v}_0 = b\omega_0$$



Panels in down position

$$\bar{v}_2 = \frac{3}{2}b\,\omega_0$$

Let ρ = mass density, t = thickness

Plate: $$m_{\text{plate}} = \rho t(2b)(4b) = 8\rho tb^2$$

$$\bar{I}_{\text{plate}} = \frac{1}{12}(8\rho tb^2)[(2b)^2 + (4b)^2]$$

$$= \frac{160}{12}\rho tb^4$$

$$= \frac{40}{3}\rho tb^4$$

Each panel: $$m_{\text{panel}} = \rho t(b)(2b) = 2\rho tb^2$$

Panel in up position $$(\bar{I}_{\text{panel}})_0 = \frac{1}{12}(2\rho t b^2)(2b)^2$$

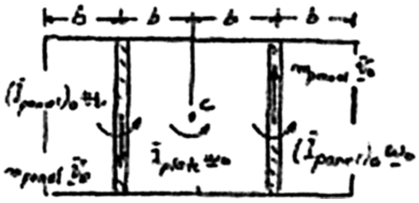
$$= \frac{8}{12}\rho t b^4 = \frac{2}{3}\rho tb^4$$

PROBLEM 17.142 (Continued)

Panel in down position

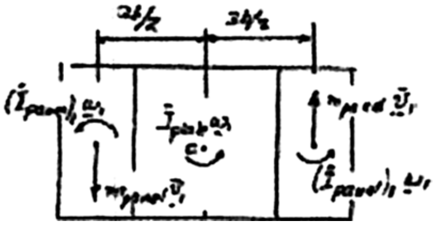
$$(\bar{I}_{\text{panel}})_1 = \frac{1}{12}(2\rho t b^2)[b^2 + (2b)^2]$$
$$= \frac{10}{12}\rho t b^4$$
$$= \frac{5}{6}\rho t b^4$$

Conservation of angular momentum about the vertical spindle.



b
b
b
b
$(\bar{I}_{panel})_0\omega_0$
$m_{panel}\bar{v}_0$
C
$\bar{I}_{plate}\omega_0$
$(\bar{I}_{panel})_0\omega_0$
$m_{panel}\bar{v}_0$

Initial momenta



3b/2
3b/2
$(\bar{I}_{panel})_1\omega_1$
$\bar{I}_{plate}\omega_1$
C
$m_{panel}\bar{v}_1$
$m_{panel}\bar{v}_1$
$(\bar{I}_{panel})_1\omega_1$

Final momenta

+↻ Moments about *C*:

$$\bar{I}_{\text{plate}}\omega_0 + 2[(\bar{I}_{\text{panel}})_0\,\omega_0 + m_{\text{panel}}v_0(b)] = \bar{I}_{\text{plate}}\omega_1 + 2\left[(\bar{I}_{\text{panel}})_1\omega_1 + m_{\text{panel}}v_1\left(\frac{3b}{2}\right)\right]$$

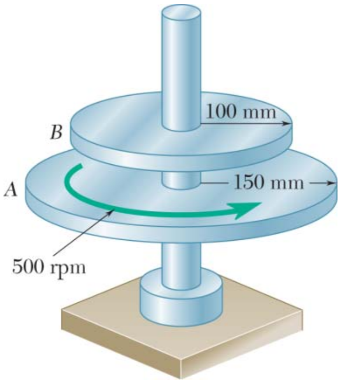
$$\frac{40}{3}\rho t b^4\omega_0 + 2\left[\frac{2}{3}\rho t b^4\omega_0 + (2\rho t b^2)(b\omega_0)b\right] = \frac{40}{3}\rho t b^4\omega_0 + 2\left[\frac{5}{6}\rho t b^4\omega_1 + 2\rho t b^2\left(\frac{3}{2}b\omega_0\right)\left(\frac{3}{2}b\right)\right]$$

$$\left[\frac{40}{3} + \frac{4}{3} + 4\right]\rho t b^4\omega_0 = \left[\frac{40}{3} + \frac{10}{6} + 9\right]\rho t b^4\omega_1$$

$$\frac{56}{3}\omega_0 = 24\omega_1$$

$$\omega_1 = \frac{56}{(3)(24)}\omega_0$$

$\omega_1 = \frac{7}{9}\omega_0$ ◀



PROBLEM 17.143

Disks *A* and *B* are made of the same material and are of the same thickness; they can rotate freely about the vertical shaft. Disk *B* is at rest when it is dropped onto disk *A*, which is rotating with an angular velocity of 500 rpm. Knowing that disk *A* has a mass of 8 kg, determine (*a*) the final angular velocity of the disks, (*b*) the change is kinetic energy of the system.

SOLUTION

Disk *A*: $m_A = 8\text{ kg} \quad r_A = 0.15\text{ m}$

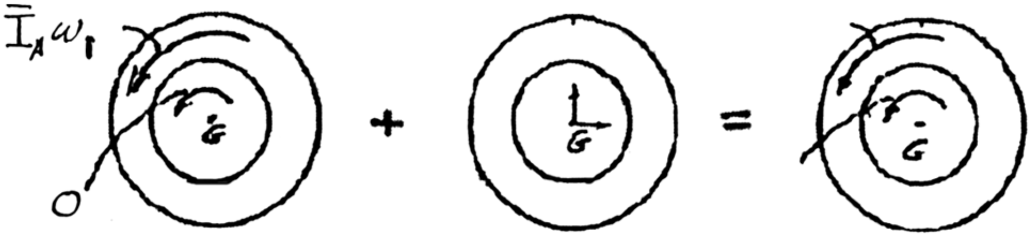
$$\bar{I}_A = \frac{1}{2}m_A r_A^2 = \frac{1}{2}(8)(0.15)^2 = 0.0900\text{ kg}\cdot\text{m}^2$$

Disk *B*: $r_B = 0.100\text{ m}$

$$m_B = m_A\left(\frac{r_B}{r_A}\right)^2 = (8)\left(\frac{0.10}{0.15}\right)^2 = 3.5556\text{ kg}$$

$$\bar{I}_B = \frac{1}{2}m_B r_B^2 = \frac{1}{2}(3.5556)(0.10)^2 = 0.017778\text{ kg}\cdot\text{m}^2$$

Principle of impulse and momentum.



Syst. Momenta$_1$ + **Syst. Ext. Imp.** $_{1\to2}$ = **Syst. Momenta**$_2$

+↻ Moments about *B*: $\bar{I}_A\omega_0 + 0 + 0 = \bar{I}_A\omega_2 + \bar{I}_B\omega_2$

$$\omega_2 = \frac{\bar{I}_A}{\bar{I}_A + \bar{I}_B}\omega_1 = \frac{0.09}{0.10778}\omega_1 = 0.83505\omega_1$$

Initial angular velocity of disk *A*: $\omega_1 = 500\text{ rpm} = 52.36\text{ rad/s}$

PROBLEM 17.143 (Continued)

(*a*) Final angular velocity of system: $\omega_2 = (0.83505)(52.36)$

$\omega_2 = 43.723 \text{ rad/s}$ $\qquad \omega_2 = 418 \text{ rpm}$ ◀

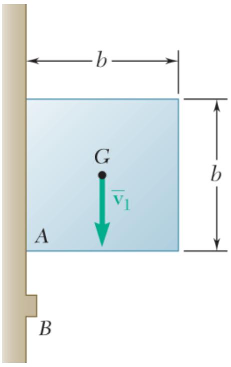
Initial kinetic energy: $T_1 = \frac{1}{2}\bar{I}_A\omega_1^2$

$$T_1 = \frac{1}{2}(0.09)(52.36)^2 = 123.37 \text{ J}$$

Final kinetic energy: $T_2 = \frac{1}{2}(\bar{I}_A + \bar{I}_B)\omega_2^2$

$$T_2 = \frac{1}{2}(0.10778)(43.723)^2 = 103.02 \text{ J}$$

(*b*) Change in energy: $T_2 - T_1 = -20.35 \text{ J}$ $\qquad \Delta T = -20.4 \text{ J}$ ◀

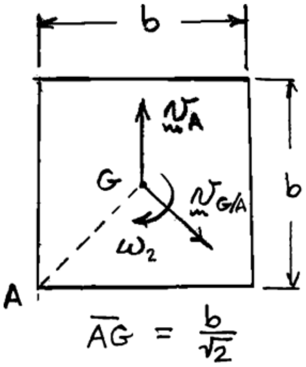

b
b
G
$\overline{\mathbf{v}}_1$
A
B

PROBLEM 17.144

A square block of mass m is falling with a velocity $\overline{\mathbf{v}}_1$ when it strikes a small obstruction at B. Knowing that the coefficient of restitution for the impact between corner A and the obstruction B is e = 0.5, determine immediately after the impact (a) the angular velocity of the block, (b) the velocity of its mass center G.

SOLUTION

Moments of inertia. $\overline{I} = \frac{1}{6}mb^2$


b
v_A
G
$v_{G/A}$
ω_2
b
A
$\overline{AG} = \frac{b}{\sqrt{2}}$

Kinematics. Before impact, block is translating.

$$\mathbf{v}_1 = v_1\downarrow \qquad \omega_1 = 0$$

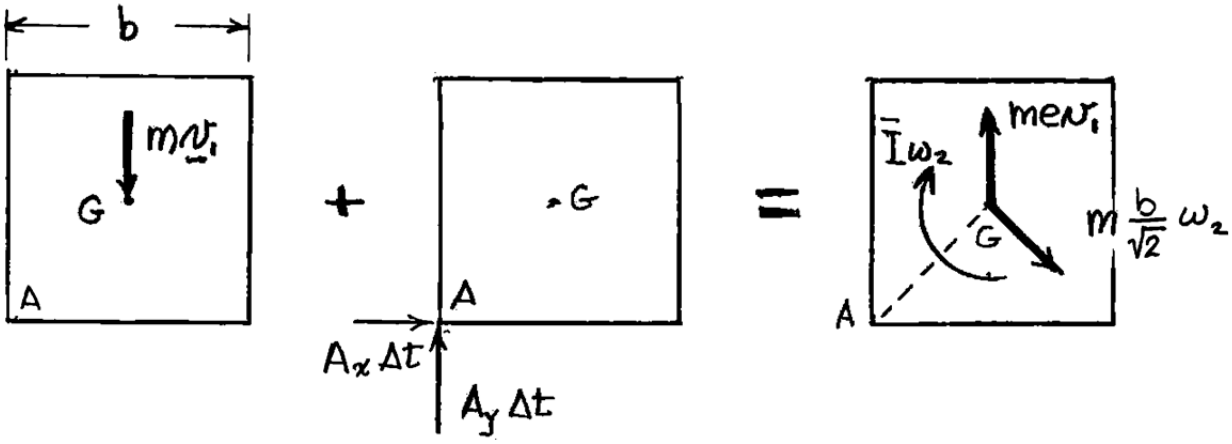
After impact,

$$\mathbf{v}_A = ev_1\uparrow$$

$$\overline{\mathbf{v}}_2 = \mathbf{v}_A + \mathbf{v}_{G/A}$$

$$= [ev_1\uparrow] + \left[\frac{b}{\sqrt{2}}\omega_2 \searrow 45^\circ\right] \qquad (1)$$

Principle of impulse and momentum.


b
mv_1
G
A
+
G
A
$A_x\Delta t$
$A_y\Delta t$
=
$\overline{I}\omega_2$
mev_1
G
$m\frac{b}{\sqrt{2}}\omega_2$
A

Syst. Momenta$_1$ + **Syst. Ext. Imp.** $_{1\to 2}$ = **Syst. Momenta**$_2$

PROBLEM 17.144 (Continued)

$+\circlearrowright$ Moments about A:

$$m\bar{v}_1\frac{b}{2}=\bar{I}\omega_1-me\bar{v}_1\frac{b}{2}+m\frac{b}{\sqrt{2}}\omega_2\frac{b}{\sqrt{2}}$$

$$\bar{I}=\frac{1}{6}mb^2$$

$$m\bar{v}_1\frac{b}{2}=\frac{1}{6}mb^2\omega_2-me\bar{v}_1\frac{b}{2}+m\left(\frac{b}{\sqrt{2}}\right)^2\omega_2$$

$$\frac{(1+e)\bar{v}_1b}{2}=\frac{2}{3}b^2\omega_2$$

(*a*) Angular velocity. $\omega_2=\dfrac{3}{4b}(1+e)v_1$

$$\boldsymbol{\omega}_2=1.125\frac{v_1}{b}\circlearrowright \blacktriangleleft$$

(*b*) Velocity of the mass center.

From Eq. (1),

$$\bar{\mathbf{v}}_2=e\bar{v}_1\uparrow+\left[\frac{b}{\sqrt{2}}\frac{3}{4b}(1+e)v_1\searrow 45^\circ\right]$$

$$=e\bar{v}_1\uparrow+\left[\frac{3}{4\sqrt{2}}(1+e)v_1\sin 45^\circ\downarrow\right]+\left[\frac{3}{4\sqrt{2}}(1+e)v_1\cos 45^\circ\longrightarrow\right]$$

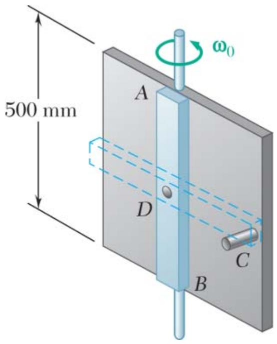
$$=e\bar{v}_1\uparrow+\left[\frac{3}{8}(1+e)v_1\downarrow\right]+\left[\frac{3}{8}(1+e)v_1\longrightarrow\right]$$

$$=\left[\left(\frac{5}{8}ev_1-\frac{3}{8}v_1\right)\uparrow\right]+\left[\frac{3}{8}(1+e)v_1\longrightarrow\right]$$

$$=\left[\left(\frac{5}{8}(0.5)-\frac{3}{8}\right)v_1\uparrow\right]+\frac{3}{8}(1+0.5)v_1\longrightarrow\Big]$$

$$=[-0.0625\uparrow]+[0.5625\longrightarrow]$$

$$\bar{\mathbf{v}}_2=0.566\text{ m/s}\searrow 6.34^\circ\blacktriangleleft$$



PROBLEM 17.145

A 3-kg bar *AB* is attached by a pin at *D* to a 4-kg square plate, which can rotate freely about a vertical axis. Knowing that the angular velocity of the plate is 120 rpm when the bar is vertical, determine (*a*) the angular velocity of the plate after the bar has swung into a horizontal position and has come to rest against pin *C*, (*b*) the energy lost during the plastic impact at *C*.

SOLUTION

Moments of inertia about the vertical centroidal axis.

Square plate. $\bar{I} = \frac{1}{12}mL^2 = \frac{1}{12}(4)(0.500)^2 = 0.083333 \text{ kg}\cdot\text{m}^2$

Bar *AB* vertical. $\bar{I} = \text{approximately zero}$

Bar *AB* horizontal. $\bar{I} = \frac{1}{12}mL^2 = \frac{1}{12}(3)(0.500)^2 = 0.0625 \text{ kg}\cdot\text{m}^2$

Position 1. Bar *AB* is vertical. $I_1 = 0.083333 \text{ kg}\cdot\text{m}^2$

Angular velocity. $\omega_1 = 120 \text{ rpm} = 4\pi \text{ rad/s}$

Angular momentum about the vertical axis.

$$(H_O)_1 = I_1\omega_1 = (0.083333)(4\pi) = 1.04720 \text{ kg}\cdot\text{m}^2/\text{s}$$

Kinetic energy. $T_1 = \frac{1}{2}I_1\omega_1^2 = \frac{1}{2}(0.083333)(4\pi)^2 = 6.5797 \text{ J}$

Position 2. Bar *AB* is horizontal. $I_2 = 0.145833 \text{ kg}\cdot\text{m}^2$

$$(H_O)_2 = I_2\omega_2 = 0.145833\omega_2$$

Conservation of angular momentum. $(H_O)_1 = (H_O)_2$:

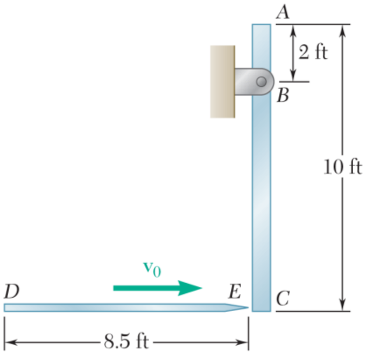
$$1.04720 = 0.145833\omega_2 \quad \omega_2 = 7.1808 \text{ rad/s}$$

(*a*) Final angular velocity. $\omega_2 = 68.6 \text{ rpm}$ ◀

(*b*) Loss of energy.

$$T_1 - T_2 = T_1 - \frac{1}{2}I_2\omega_2^2 = 6.5797 - \frac{1}{2}(0.145833)(7.1808)^2$$

$T_1 - T_2 = 2.82 \text{ J}$ ◀



PROBLEM 17.146

A 1.8-lb javelin *DE* impacts a 10-lb slender rod *ABC* with a horizontal velocity of $\mathbf{v}_0$ = 30 ft/s as shown. Knowing that the javelin becomes embedded into the end of the rod at Point *C* and does not penetrate very far into it, determine immediately after the impact (*a*) the angular velocity of the rod *ABC* after the impact, (*b*) the components of the reaction at *B*. Assume that the javelin and the rod move as a single body after the impact.

SOLUTION

Masses and moments of inertia.

$$m_{AC} = \frac{W_{AC}}{g} = \frac{10 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.31056 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_{DE} = \frac{W_{DE}}{g} = \frac{1.8 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.05590 \text{ lb} \cdot \text{s}^2/\text{ft}$$

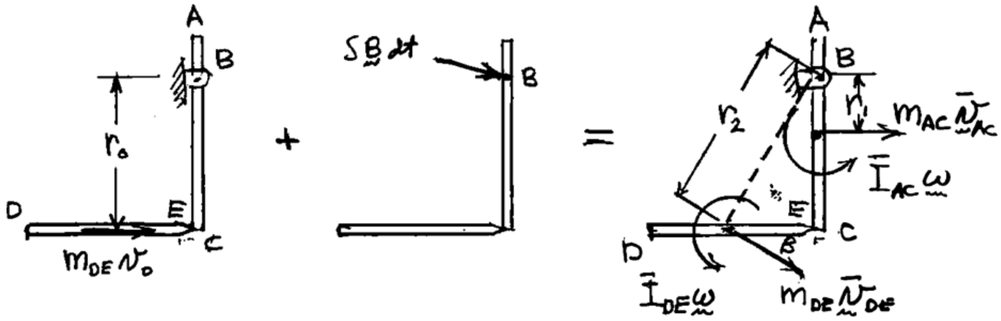
$$\overline{I}_{AC} = \frac{1}{12} m_{AC} L_{AC}^2 = \frac{1}{12}(0.31056)(10)^2 = 2.5880 \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

$$\overline{I}_{DE} = \frac{1}{12} m_{DE} L_{DE}^2 = \frac{1}{12}(0.05590)(8.5)^2 = 0.3365 \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

(*a*) *Angular velocity immediately after the impact.*

Principle of impulse and momentum.

$$\boldsymbol{\omega}_{DE} = \boldsymbol{\omega}_{AB} = \boldsymbol{\omega} = \omega \circlearrowleft$$


A
B
r_0
D
E
C
$m_{DE} v_0$
$\int B\,dt$
B
A
B
r_2
r_1
$m_{AC} \overline{v}_{AC}$
$\overline{I}_{AC} \omega$
E
C
D
B
$\overline{I}_{DE} \omega$
$m_{DE} \overline{v}_{DE}$

Syst. Momenta$_1$ + **Syst. Ext. Imp.** $_{1\to2}$ = **Syst. Momenta**$_2$

$+\circlearrowleft$ Moments about *B*: $\quad m_{DE} v_0 r_0 + 0 = \overline{I}_{AC}\omega + m_{AC}\overline{v}_{AC} r_1 + \overline{I}_{DE}\omega + m_{DE}\overline{v}_{DE} r_2$

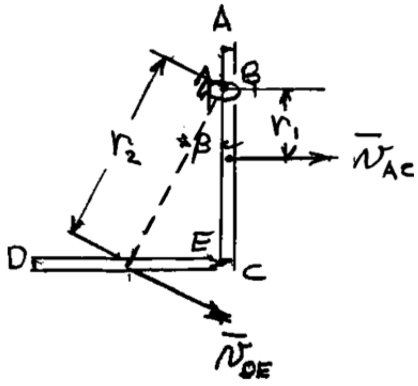
where

$$v_0 = 30 \text{ ft/s} \quad r_0 = 10 \text{ ft} - 2 \text{ ft} = 8 \text{ ft}$$

$$r_1 = 5 \text{ ft} - 2 \text{ ft} = 3 \text{ ft}$$

$$r_2 = \sqrt{(4.25 \text{ ft})^2 + (8 \text{ ft})^2} = 9.0588 \text{ ft}$$

PROBLEM 17.146 (Continued)


A
B
r_2
β
r_1
$\bar{v}_{AC}$
D
E
C
$\bar{v}_{DE}$

Kinematics: (Rotation about *B*)

$$\bar{v}_{AC} = r_1\omega$$

$$\bar{v}_{DE} = r_2\omega$$

$$\tan\beta = \frac{4.25\text{ ft}}{8\text{ ft}} \qquad \beta = 27.98°$$

$$m_{DE}v_0r_0 = \bar{I}_{AC}\omega + m_{AC}r_1^2\omega + \bar{I}_{DE}\omega + m_{DE}r_2^2\omega$$

$$= I_B\omega$$

where $I_B = \bar{I}_{AC} + m_{AC}r_1^2 + \bar{I}_{DE} + m_{DE}r_2^2$

$$= 2.5880 + (0.31056)(3)^2 + 0.3365 + (0.05590)(9.0588)^2$$

$$= 10.3068\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\omega = \frac{m_{DE}v_0r_0}{I_B} = \frac{(0.05590\text{ lb}\cdot\text{s}^2/\text{ft})(30\text{ ft/s})(8\text{ ft})}{10.3068\text{ lb}\cdot\text{s}^2\cdot\text{ft}}$$

$$= 1.30167\text{ rad/s}$$

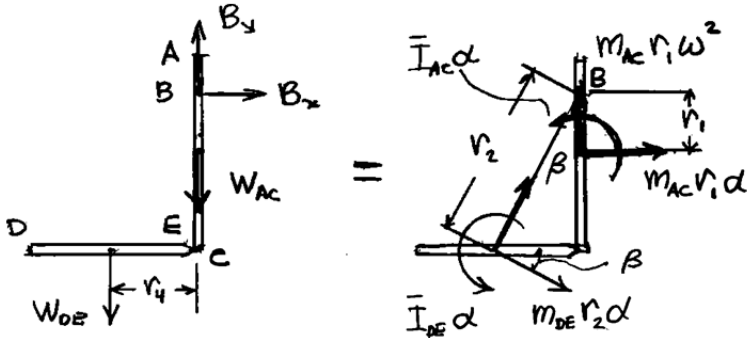
$\boldsymbol{\omega} = 1.302\text{ rad/s} \circlearrowright$ ◀

Accelerations: $\boldsymbol{\alpha} = \alpha \circlearrowright$

$$\bar{a}_{AD} = [r_1\alpha \longrightarrow] + [r_1\omega^2 \uparrow]$$

$$\bar{a}_{DE} = [r_2\alpha \searrow \beta] + [r_2\omega^2 \nearrow \beta]$$

Free body and kinetic diagrams.


B_y
A
B
B_x
W_{AC}
D
E
C
r_4
W_{DE}
=
$\bar{I}_{AC}\alpha$
$m_{AC}r_1\omega^2$
B
r_1
r_2
β
$m_{AC}r_1\alpha$
β
$\bar{I}_{DE}\alpha$
$m_{DE}r_2\alpha$

$+\circlearrowright$ Moments about *B*:

$$W_{DE}r_4 = \bar{I}_{AC}\alpha + m_{AC}(\bar{a}_{AC})_t r_1 + \bar{I}_{DE}\alpha + m_{DE}(\bar{a}_{DE})_t r_2$$

$$= \bar{I}_{AC}\alpha + m_{AC}r_1^2\alpha + \bar{I}_{DE}\alpha + m_{DE}r_2^2\alpha$$

$$= I_B\alpha$$

$$\alpha = \frac{W_{DE}r_4}{I_B} = \frac{(1.8\text{ lb})(4.25)}{10.3068\text{ lb}\cdot\text{s}^2\cdot\text{ft}}$$

$$= 0.74223\text{ rad/s}^2$$

$\boldsymbol{\alpha} = 0.742\text{ rad/s}^2 \circlearrowright$

PROBLEM 17.146 (Continued)

(*b*) *Components of reaction at B.*

$\Sigma F = \Sigma m\bar{a}$:

$$\mathbf{B} + [W_{AC}\downarrow] + [W_{DE}\downarrow] = [m_{AC} r_1 \alpha \longrightarrow] + [m_{AC} r_1 \omega^2 \uparrow] + [m_{DE} r_2 \alpha \searrow \beta] + [m_{DE} r_2 \omega^2 \nearrow \beta]$$

Component $\xrightarrow{+}$:

$$\begin{aligned} B_x &= m_{AC} r_1 \alpha + m_{DE}(r_2 \cos\beta)\alpha + m_{DE}(r_2 \sin\beta)\omega^2 \\ &= (0.31056)(3)(0.74223) + (0.05590)(8)(0.74223) + (0.05590)(4.25)(1.30167)^2 \\ &= 0.6915 + 0.3319 + 0.4025 \end{aligned}$$

$\mathbf{B}_x = 1.426 \text{ lb} \longrightarrow$ ◀

Component $+\uparrow$:

$$B_y - W_{AC} - W_{DC} = m_{AC} r_1 \omega^2 - m_{DE}(r_1 \sin\beta)\alpha + m_{DE}(r_2 \cos\beta)\omega^2$$

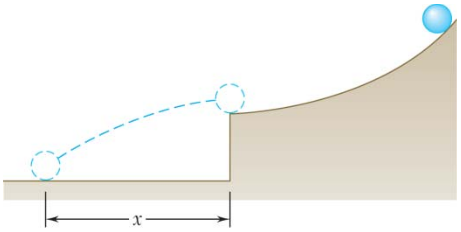
$$B_y - 10 - 1.8 = (0.31056)(3)(1.30167)^2 - (0.05590)(4.25)(0.74223) + (0.05590)(8)(1.30167)^2$$

$$B_y = 11.8 + 1.5785 - 0.1763 + 0.7577$$

$\mathbf{B}_y = 13.96 \text{ lb} \uparrow$ ◀

PROBLEM 17.CQ1

A round object of mass *m* and radius *r* is released from rest at the top of a curved surface and rolls without slipping until it leaves the surface with a horizontal velocity as shown. Will a solid sphere, a solid cylinder or a hoop travel the greatest distance x?



(*a*) A solid sphere
(*b*) A solid cylinder
(*c*) A hoop
(*d*) They will all travel the same distance.

SOLUTION

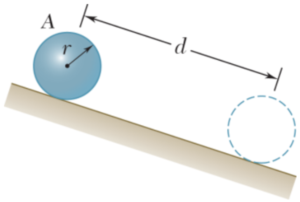
All objects have the same potential energy at the top of the ramp and therefore the same kinetic energy at the bottom of the ramp since they have the same m. The object with the smallest mass moment of inertia will have the greatest linear kinetic energy and the smallest rotation kinetic energy at the bottom of the ramp. Therefore its linear speed at the bottom of the curved ramp will be the greatest.

Answer: *(a)*

PROBLEM 17.CQ2

A solid steel sphere *A* of radius r and mass m is released from rest and rolls without slipping down an incline as shown. After traveling a distance d the sphere has a speed v. If a solid steel sphere of radius $2r$ is released from rest on the same incline, what will its speed be after rolling a distance d?

- (*a*) 0.25 v
- (*b*) 0.5 v
- (*c*) v
- (*d*) $2v$
- (*e*) $4v$



SOLUTION

Using conservation of energy you can show that the speed after traveling a distance d will be independent of the mass and the radius; therefore, any solid sphere will have the same speed.

Answer: *(c)*

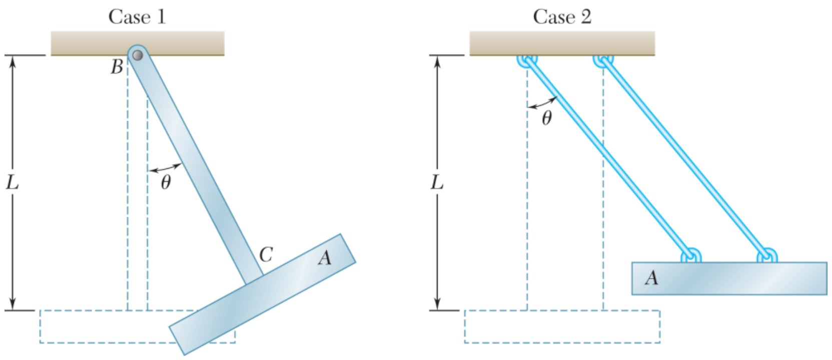
PROBLEM 17.CQ3

Slender bar *A* is rigidly connected to a massless rod *BC* in Case 1 and two massless cords in Case 2 as shown. The vertical thickness of bar *A* is negligible compared to *L*. In both cases *A* is released from rest at an angle $\theta = \theta_0$. When $\theta = 0°$ which system will have the larger kinetic energy?

(*a*) Case 1

(*b*) Case 2

(*c*) The kinetic energy will be the same.



SOLUTION

In the first position, the mass center of both Cases is at the same height. Since both objects have the same potential energy before release, by conservation of energy, the will have the same kinetic energy when they reach the second position.

Answer: *(c)*

PROBLEM 17.CQ4

In Problem 17.CQ3, how will the speeds of the centers of gravity compare for the two cases when $\theta = 0°$?

(*a*) Case 1 will be larger.

(*b*) Case 2 will be larger.

(*c*) The speeds will be the same.

SOLUTION

Kinetic energy of rotational bodies consists of both linear motion and rotational motion. Case 1 will have rotational kinetic energy while Case 2 does not; therefore, the speed of the mass center of Case 2 will be larger.

Answer: *(b)*

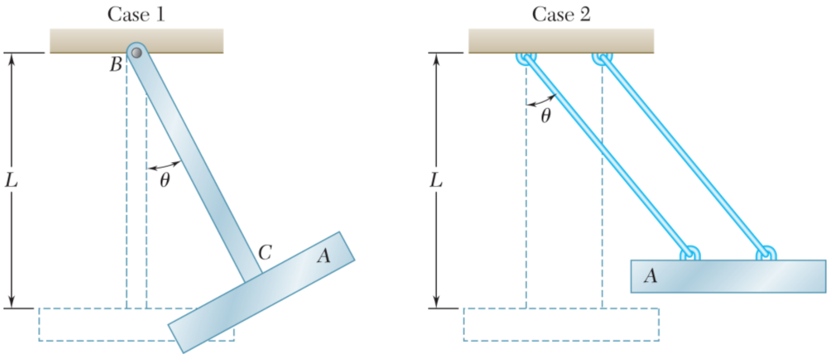
PROBLEM 17.CQ5

Slender bar *A* is rigidly connected to a massless rod *BC* in Case 1 and two massless cords in Case 2 as shown. The vertical thickness of bar *A* is not negligible compared to *L*. In both cases *A* is released from rest at an angle $\theta = \theta_0$. When $\theta = \theta°$ which system will have the largest kinetic energy?

(*a*) Case 1

(*b*) Case 2

(*c*) The kinetic energy will be the same.



SOLUTION

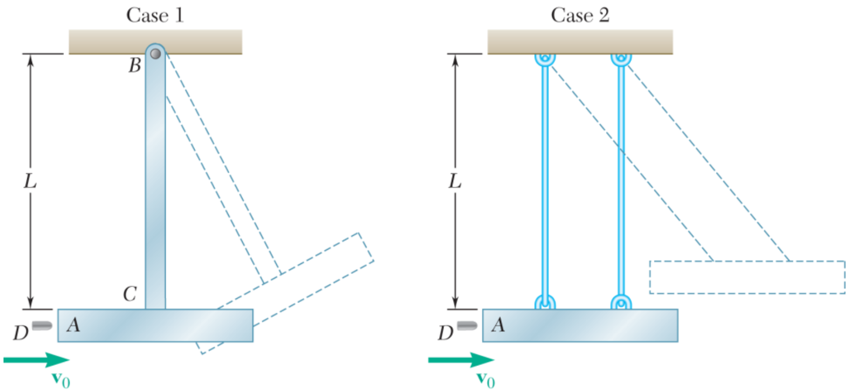
Case 1 will have a greater change in gravitational potential energy, so the kinetic energy will be larger.

Answer: *(a)*

PROBLEM 17.CQ6

Slender bar *A* is rigidly connected to a massless rod *BC* in Case 1 and two massless cords in Case 2 as shown. The vertical thickness of bar *A* is negligible compared to *L*. If bullet *D* strikes *A* with a speed v_0 and becomes embedded in it, how will the speeds of the center of gravity of *A* immediately after the impact compare for the two cases?

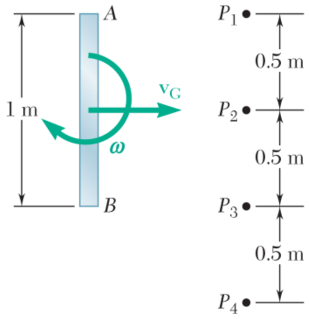
- (*a*) Case 1 will be larger.
- (*b*) Case 2 will be larger.
- (*c*) The speeds will be the same.



SOLUTION

The angular momentum about point B will be conserved before and after impact. In Case 2, the angular momentum is due only to velocity of the mass center where in Case 1, the angular moment after impact is the sum of the rotational momentum plus the momentum of the mass center about point B. Therefore the velocity of the center of gravity of Case 2 will be larger.

Answer: *(b)*



PROBLEM 17.CQ7

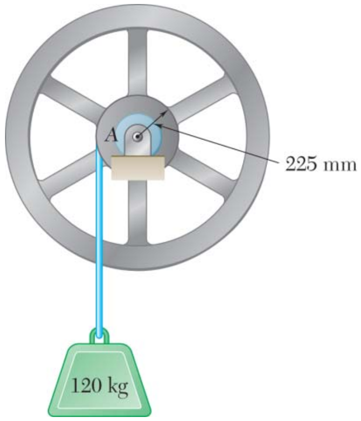
A 1-m long uniform slender bar *AB* has an angular velocity of 12 rad/s and its center of gravity has a velocity of 2 m/s as shown. About which point is the angular momentum of *A* smallest at this instant?

(*a*) P_1
(*b*) P_2
(*c*) P_3
(*d*) P_4
(*e*) It is the same about all the points.

SOLUTION

The angular momentum of a planar rigid body about a point is equal to $\bar{I}\omega + r \times m\bar{v}$. Since the first term is the same for all points in the figure. The second term is in the opposite sense for point P_1 and therefore the angular momentum of the body about that point is the smallest.

Answer: *(a)*

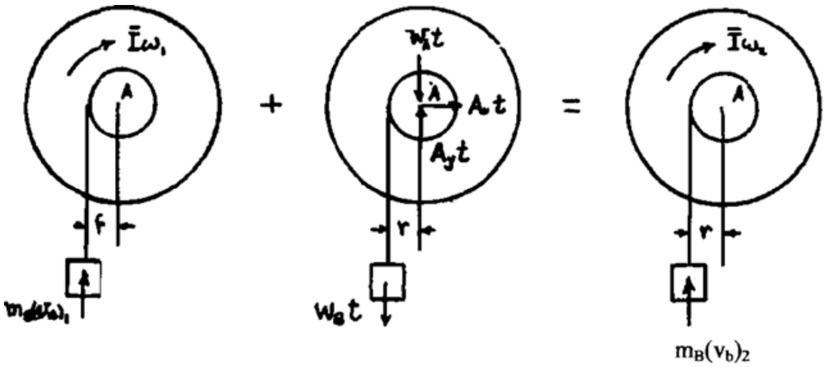

A
225 mm
120 kg

PROBLEM 17.F1

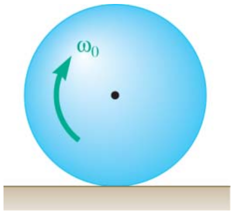
The 350-kg flywheel of a small hoisting engine has a radius of gyration of 600 mm. If the power is cut off when the angular velocity of the flywheel is 100 rpm clockwise, draw an impulse-momentum diagram that can be used to determine the time required for the system to come to rest.

SOLUTION

Answer:


$\bar{I}\omega_1$
A
r
$m_B(v_B)_1$
+
$W_A t$
A
$A_x t$
$A_y t$
r
$W_B t$
=
$\bar{I}\omega_2$
A
r
$m_B(v_b)_2$

Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\rightarrow 2}$ = **Syst. Momenta**$_2$

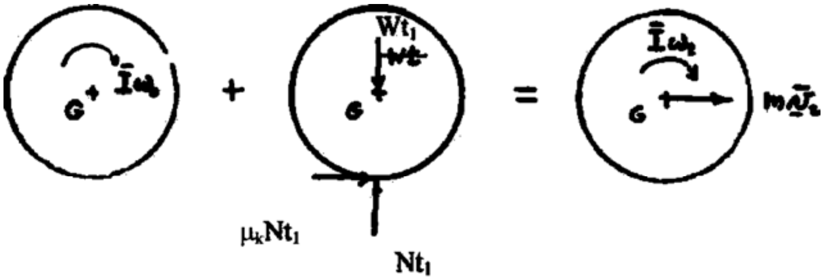


PROBLEM 17.F2

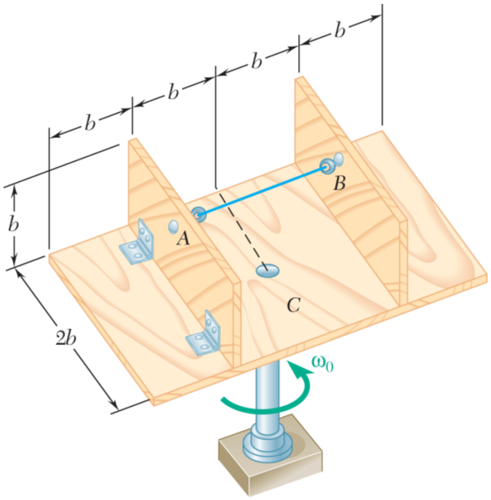
A sphere of radius *r* and mass *m* is placed on a horizontal floor with no linear velocity but with a clockwise angular velocity $\boldsymbol{\omega}_0$. Denoting by μ_k the coefficient of kinetic friction between the sphere and the floor, draw the impulse-momentum diagram that can be used to determine the time t_1 at which the sphere will start rolling without sliding.

SOLUTION

Answer:


$\bar{I}\omega_0$
G
+
Wt_1
Wt
G
$\mu_k N t_1$
$N t_1$
=
$\bar{I}\omega_2$
G
$m\bar{v}_2$

Syst. Momenta$_1$ + Syst. Ext. Imp.$_{1\to2}$ = Syst. Momenta$_2$

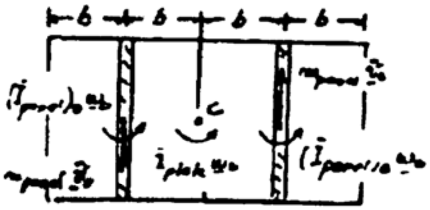

b
b
b
b
b
b
2b
A
B
C
ω_0

PROBLEM 17.F3

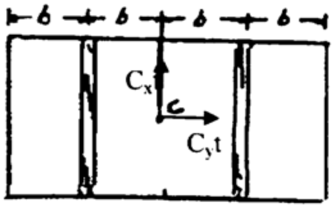
Two panels *A* and *B* are attached with hinges to a rectangular plate and held by a wire as shown. The plate and the panels are made of the same material and have the same thickness. The entire assembly is rotating with an angular velocity ω_0 when the wire breaks. Draw the impulse-momentum diagram that is needed to determine the angular velocity of the assembly after the panels have come to rest against the plate.

SOLUTION

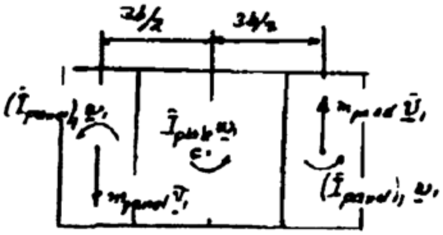
Answer:

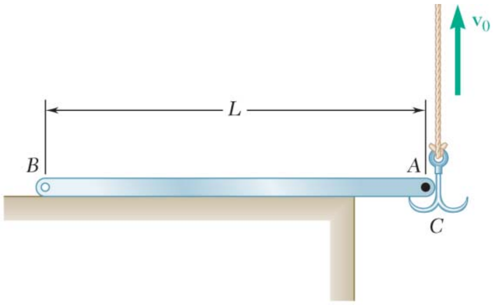

b
b
b
b
$(\bar{I}_{panel})_0\omega_0$
$m_{panel}\bar{v}_0$
c
$\bar{I}_{plate}\omega_0$
$(\bar{I}_{panel})_0\omega_0$
$m_{panel}\bar{v}_0$

+


b
b
b
b
$C_x t$
c
$C_y t$

=


3b/2
3b/2
$(\bar{I}_{panel})_1\omega_1$
$\bar{I}_{plate}\omega_1$
c
$m_{panel}\bar{v}_1$
$m_{panel}\bar{v}_1$
$(\bar{I}_{panel})_1\omega_1$

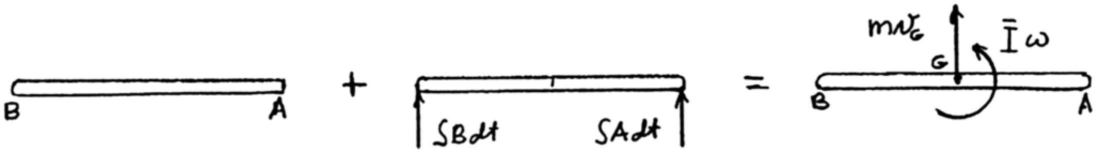


PROBLEM 17.F4

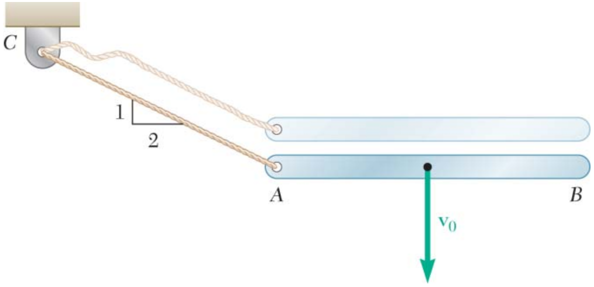
A uniform slender rod *AB* of mass *m* is at rest on a frictionless horizontal surface when hook *C* engages a small pin at *A*. Knowing that the hook is pulled upward with a constant velocity $\mathbf{v}_0$, draw the impulse-momentum diagram that is needed to determine the impulse exerted on the rod at *A* and *B*. Assume that the velocity of the hook is unchanged and that the impact is perfectly plastic.

SOLUTION

Answer:



Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$



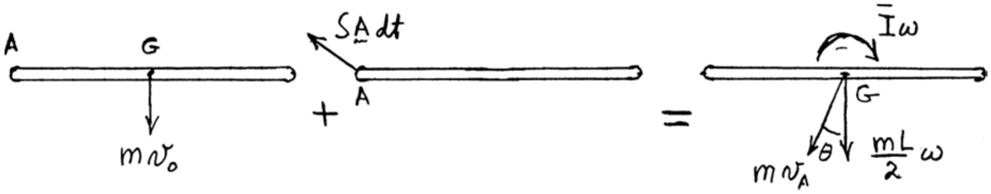
PROBLEM 17.F5

A uniform slender rod *AB* of length *L* is falling freely with a velocity $\mathbf{v_0}$ when cord *AC* suddenly becomes taut. Assuming that the impact is perfectly plastic, draw the impulse-momentum diagram that is needed to determine the angular velocity of the rod and the velocity of its mass center immediately after the cord becomes taut.

SOLUTION

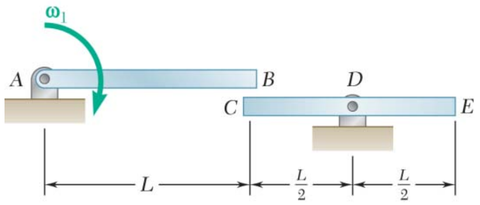
Answer:

Principle of impulse and momentum.


A
G
mv_o
$\int \underset{\sim}{A}\, dt$
A
+
=
$\bar{I}\omega$
G
θ
mv_A
$\frac{mL}{2}\omega$

Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to2}$ = **Syst. Momenta**$_2$

Note: For the momentum after the impact a general a_{Gx} and a_{Gy} can be used. These can be related to ω and $\mathbf{v}_A$ using kinematics.


ω_1
A
B
D
C
E
L
$\frac{L}{2}$
$\frac{L}{2}$

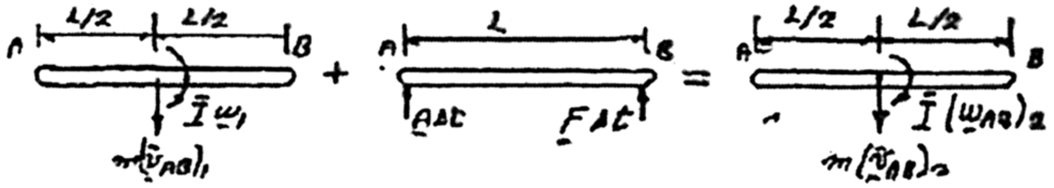
PROBLEM 17.F6

A slender rod *CDE* of length *L* and mass *m* is attached to a pin support at its midpoint *D*. A second and identical rod *AB* is rotating about a pin support at *A* with an angular velocity ω_1 when its end *B* strikes end *C* of rod *CDE*. The coefficient of restitution between the rods is *e*. Draw the impulse-momentum diagrams that are needed to determine the angular velocity of each rod immediately after the impact.

SOLUTION

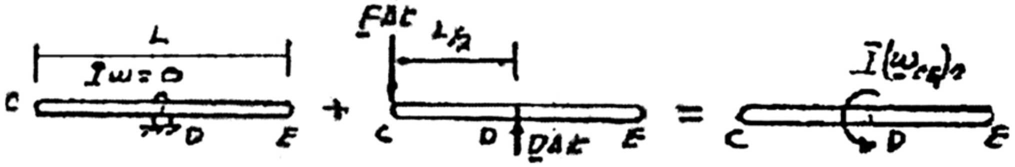
Answer:

Rod *AB*.


L/2
L/2
A
B
$\bar{I}\omega_1$
$m(\bar{v}_{AB})_1$
+
A
L
B
$A\Delta t$
$F\Delta t$
=
A
L/2
L/2
B
$\bar{I}(\omega_{AB})_2$
$m(\bar{v}_{AB})_2$

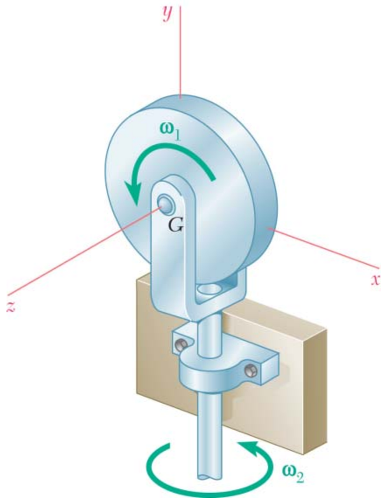
Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

Rod *CE*.


L
$I\omega=0$
C
$m\bar{v}$
D
E
+
$F\Delta t$
L/2
C
D
$D\Delta t$
E
=
$\bar{I}(\omega_{CE})_2$
C
D
E

Syst. Momenta$_1$ + **Syst. Ext. Imp.**$_{1\to 2}$ = **Syst. Momenta**$_2$

CHAPTER 18



PROBLEM 18.1

A thin, homogeneous disk of mass m and radius r spins at the constant rate ω_1 about an axle held by a fork-ended vertical rod, which rotates at the constant rate ω_2. Determine the angular momentum $\mathbf{H}_G$ of the disk about its mass center G.

SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega_2\mathbf{j} + \omega_1\mathbf{k}$

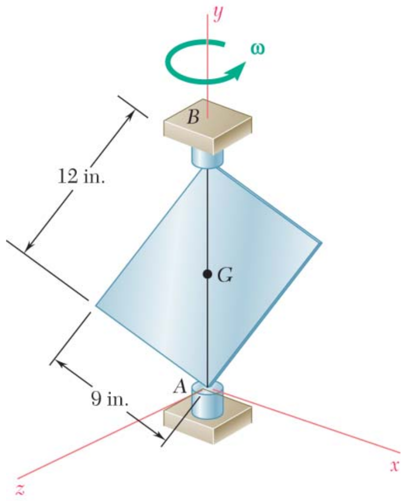
Moments of inertia: $\bar{I}_x = \frac{1}{4}mr^2, \quad \bar{I}_y = \frac{1}{4}mr^2, \quad \bar{I}_z = \frac{1}{2}mr^2$

Products of inertia: by symmetry, $\bar{I}_{xy} = \bar{I}_{yz} = \bar{I}_{zx} = 0$

Angular momentum: $\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$

$$\mathbf{H}_G = 0 + \frac{1}{4}mr^2\omega_2\mathbf{j} + \frac{1}{2}mr^2\omega_1\mathbf{k}$$

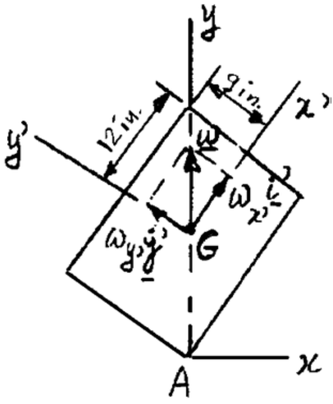
$$\mathbf{H}_G = \frac{1}{4}mr^2\omega_2\mathbf{j} + \frac{1}{2}mr^2\omega_1\mathbf{k} \blacktriangleleft$$



PROBLEM 18.2

A thin rectangular plate of weight 15 lb rotates about its vertical diagonal *AB* with an angular velocity ω. Knowing that the *z* axis is perpendicular to the plate and that ω is constant and equal to 5 rad/s, determine the angular momentum of the plate about its mass center *G*.

SOLUTION



$$h = \sqrt{(9\text{ in.})^2 + (12\text{ in.})^2} = 15\text{ in.}$$

Resolving ω along the principal axes x', y', z:

$$\omega_{x'} = \frac{12}{15}\omega = 0.8(5\text{ rad/s}) = 4\text{ rad/s}$$

$$\omega_{y'} = \frac{9}{15}\omega = 0.6(5\text{ rad/s}) = 3\text{ rad/s}$$

$$\omega_z = 0$$

Moments of inertia:

$$I_{x'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{9}{12}\text{ ft}\right)^2 = 0.021836\text{ slug}\cdot\text{ft}^2$$

$$I_{y'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{12}{12}\text{ ft}\right)^2 = 0.038820\text{ slug}\cdot\text{ft}^2$$

From Eqs. (18.10):

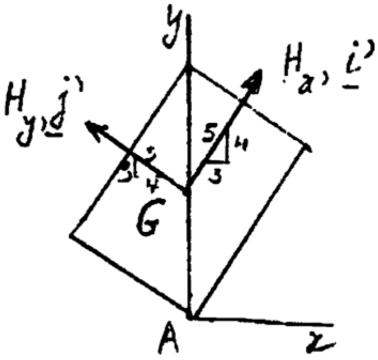
$$H_{x'} = I_{x'}\omega_{x'} = (0.021836)(4) = 0.087345\text{ slug}\cdot\text{ft}^2/\text{s}$$

$$H_{y'} = I_{y'}\omega_{y'} = (0.038820)(3) = 0.11646\text{ slug}\cdot\text{ft}^2/\text{s}$$

$$H_{z'} = I_z\omega_z = 0$$

$$\mathbf{H}_G = (0.087345\text{ slug}\cdot\text{ft}^2/\text{s})\mathbf{i}' + (0.11646\text{ slug}\cdot\text{ft}^2/\text{s})\mathbf{j}'$$

PROBLEM 18.2 (Continued)



Components along *x* and *y* axes:

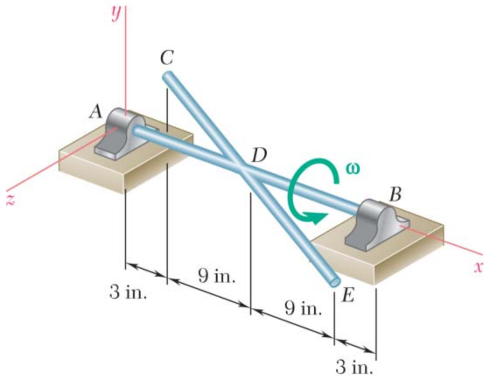
$$H_x = \frac{3}{5}H_{x'} - \frac{4}{5}H_{y'} = \left(\frac{3}{5}\right)(0.087345) - \left(\frac{4}{5}\right)(0.11646)$$

$$= -0.040761$$

$$H_y = \frac{4}{5}H_{x'} + \frac{3}{5}H_{y'}$$

$$= \left(\frac{4}{5}\right)(0.087345) + \left(\frac{3}{5}\right)(0.11646) = 0.13975$$

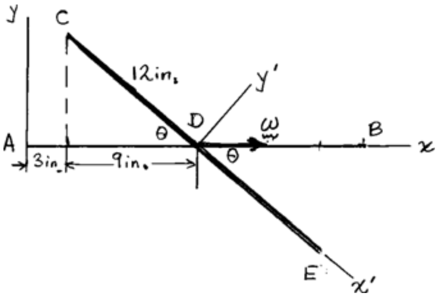
$$\mathbf{H}_G = -(0.0408\,\text{slug}\cdot\text{ft}^2/\text{s})\mathbf{i} + (0.1398\,\text{slug}\cdot\text{ft}^2/\text{s})\mathbf{j} \blacktriangleleft$$



PROBLEM 18.3

Two uniform rods *AB* and *CE*, each of weight 3 lb and length 2 ft, are welded to each other at their midpoints. Knowing that this assembly has an angular velocity of constant magnitude $\omega = 12$ rad/s, determine the magnitude and direction of the angular momentum $\mathbf{H}_D$ of the assembly about *D*.

SOLUTION



$$m = \frac{W}{g} = \frac{3}{32.2} = 0.093168\ \text{lb}\cdot\text{s}^2/\text{ft}, \qquad l = 2\ \text{ft},$$

$$\boldsymbol{\omega} = (12\ \text{rad/s})\mathbf{i}$$

For rod *ADB*, $\mathbf{H}_D = \bar{I}_x\omega\mathbf{i} \approx 0$, since $\bar{I}_x \approx 0.$

For rod *CDE*, use principal axes x', y' as shown.

$$\cos\theta = \frac{9}{12}, \qquad \theta = 41.410°$$

$$\omega_{x'} = \omega\cos\theta = 9\ \text{rad/s}^2$$

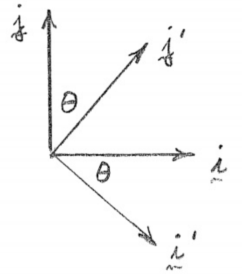
$$\omega_{y'} = \omega\sin\theta = 7.93725\ \text{rad/s}^2$$

$$\omega_{z'} = 0$$

$$\bar{I}_{x'} \approx 0$$

$$\bar{I}_{y'} = \frac{1}{12}ml^2 = \frac{1}{12}(0.093168)(2)^2$$

$$= 0.0310559\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$



$$\mathbf{H}_D = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_{z'}\omega_{z'}\mathbf{k}'$$

$$= 0 + (0.0310559)(7.93725)\mathbf{j}' + 0$$

$$= 0.246498\mathbf{j}'$$

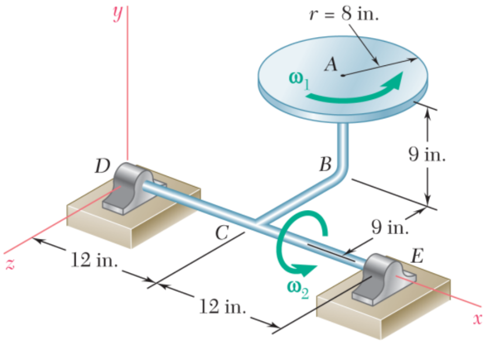
$$H_D = 0.246\ \text{lb}\cdot\text{s}\cdot\text{ft} \blacktriangleleft$$

$$\mathbf{H}_D = 0.246498(\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = 0.163045\mathbf{i} + 0.184874\mathbf{j}$$

$$\cos\theta_x = \frac{0.163045}{0.246498} \qquad \theta_x = 48.6° \blacktriangleleft$$

$$\cos\theta_y = \frac{0.184874}{0.246498} \qquad \theta_y = 41.4° \blacktriangleleft$$

$$\cos\theta_z = 0 \qquad \theta_z = 90° \blacktriangleleft$$



PROBLEM 18.4

A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm *ABC*, which is welded to a shaft *DCE* rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the angular momentum $\mathbf{H}_A$ of the disk about its center *A*.

SOLUTION

$$\boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{i} = (8\text{ rad/s})\mathbf{i} + (16\text{ rad/s})\mathbf{j}$$

For axes x', y', z' parallel to x, y, z with origin at A,

$$m = \frac{W}{g} = \frac{6}{32.2} = 0.186335\text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\bar{I}_{x'} = \frac{1}{4}mr^2 = \frac{1}{4}(0.186335)\left(\frac{8}{12}\right)^2$$

$$= 0.020704\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{z'} = \bar{I}_{x'} = 0.020704\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

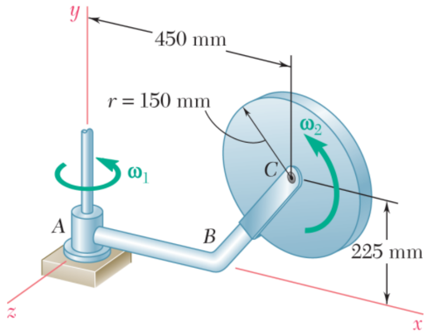
$$\bar{I}_{y'} = \bar{I}_{x'} + \bar{I}_{z'} = 0.041408\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\mathbf{H}_A = \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + \bar{I}_{z'}\omega_{z'}\mathbf{k}$$

$$= (0.020704)(8)\mathbf{i} + (0.041408)(16)\mathbf{j}$$

$$= 0.1656\mathbf{i} + 0.6625\mathbf{j}$$

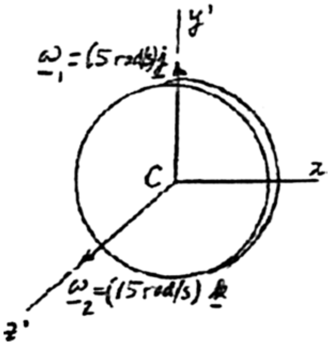
$$\mathbf{H}_A = (0.1656\text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{i} + (0.663\text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{j} \blacktriangleleft$$



PROBLEM 18.5

A thin disk of mass $m = 4$ kg rotates at the constant rate $\omega_2 = 15$ rad/s with respect to arm *ABC*, which itself rotates at the constant rate $\omega_1 = 5$ rad/s about the *y* axis. Determine the angular momentum of the disk about its center *C*.

SOLUTION



$$r = 150 \text{ mm}$$

Angular velocity of disk:

$$\boldsymbol{\omega} = \omega_1\,\mathbf{j} + \omega_2\mathbf{k}$$
$$= (5 \text{ rad/s})\mathbf{j} + (15 \text{ rad/s})\mathbf{k}$$

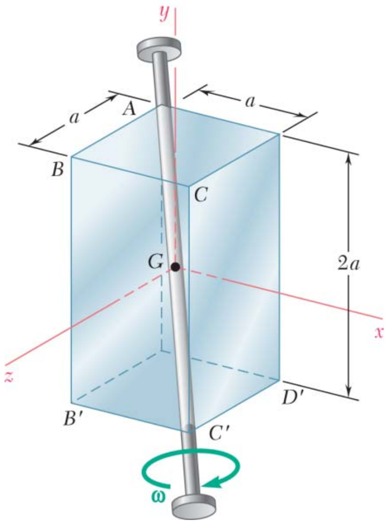
Centroidal moments of inertia:

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4}mr^2$$
$$= \frac{1}{4}(4)(0.150 \text{ m})^2 = 0.0225 \text{ kg}\cdot\text{m}^2$$
$$\bar{I}_{z'} = \frac{1}{2}mr^2 = 0.045 \text{ kg}\cdot\text{m}^2$$

Angular momentum about Point *C*.

$$\mathbf{H}_C = \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + I_{z'}\omega_{z'}\mathbf{k}$$
$$= 0 + (0.0225)(5)\mathbf{j} + (0.045)(15)\mathbf{k}$$
$$= (0.1125 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (0.6750 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

$$\mathbf{H}_C = (0.1125 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (0.675 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k} \blacktriangleleft$$



PROBLEM 18.6

A solid rectangular parallelepiped of mass m has a square base of side a and a length $2a$. Knowing that it rotates at the constant rate ω about its diagonal AC' and that its rotation is observed from A as counterclockwise, determine (*a*) the magnitude of the angular momentum $\mathbf{H}_G$ of the parallelepiped about its mass center G, (*b*) the angle that $\mathbf{H}_G$ forms with the diagonal AC'.

SOLUTION

Body diagonal:

$$d = \sqrt{a^2 + (2a)^2 + a^2} = \sqrt{6}a$$

$$\boldsymbol{\omega} = \frac{\omega}{d}(-a\mathbf{i} + 2a\mathbf{j} - a\mathbf{k}) = -\frac{\omega}{\sqrt{6}}\mathbf{i} + \frac{2\omega}{\sqrt{6}}\mathbf{j} - \frac{\omega}{\sqrt{6}}\mathbf{k}$$

$$I_x = \frac{1}{12}m[(2a)^2 + a^2] = \frac{5}{12}ma^2$$

$$I_y = \frac{1}{12}m[a^2 + a^2] = \frac{1}{6}ma^2$$

$$I_z = \frac{1}{12}m[a^2 + (2a)^2] = \frac{5}{12}ma^2$$

(*a*)

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= \left(\frac{5}{12}ma^2\right)\left(-\frac{\omega}{\sqrt{6}}\right)\mathbf{i} + \left(\frac{1}{6}ma^2\right)\left(\frac{2\omega}{\sqrt{6}}\right)\mathbf{j} + \left(\frac{5}{12}ma^2\right)\left(-\frac{\omega}{\sqrt{6}}\right)\mathbf{k}$$

$$= \frac{ma^2\omega}{12\sqrt{6}}(-5\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$$

$$H_G = \frac{ma^2\omega}{12\sqrt{6}}\sqrt{5^2 + 4^2 + 5^2} = \frac{\sqrt{11}ma^2\omega}{12}$$

$\mathbf{H}_G = 0.276\, ma^2\omega$ ◀

(*b*)

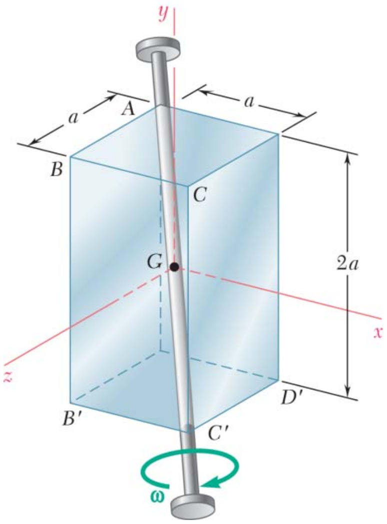
$$\mathbf{H}_G \cdot \boldsymbol{\omega} = \frac{ma^2\omega^2}{(12)(6)}(-5\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

$$= \frac{18ma^2\omega^2}{(12)(6)} = \frac{1}{4}ma^2\omega$$

$$H_G\omega = \frac{\sqrt{11}}{12}ma^2\omega^2$$

$$\cos\theta = \frac{\mathbf{H}_G \cdot \boldsymbol{\omega}}{H_G\omega} = \frac{12}{4\sqrt{11}} = 0.90453$$

$\theta = 25.2°$ ◀



PROBLEM 18.7

Solve Problem 18.6, assuming that the solid rectangular parallelepiped has been replaced by a hollow one consisting of six thin metal plates welded together.

PROBLEM 18.6 A solid rectangular parallelepiped of mass m has a square base of side a and a length $2a$. Knowing that it rotates at the constant rate ω about its diagonal AC' and that its rotation is observed from A as counterclockwise, determine (a) the magnitude of the angular momentum $\mathbf{H}_G$ of the parallelepiped about its mass center G, (b) the angle that $\mathbf{H}_G$ forms with the diagonal AC'.

SOLUTION

Body diagonal:

$$d = \sqrt{a^2 + (2a)^2 + a^2} = \sqrt{6}a$$

$$\boldsymbol{\omega} = \frac{\omega}{d}(-a\mathbf{i} + 2a\mathbf{j} - a\mathbf{k}) = -\frac{\omega}{\sqrt{6}}\mathbf{i} + \frac{2\omega}{\sqrt{6}}\mathbf{j} - \frac{\omega}{\sqrt{6}}\mathbf{k}$$

$$\text{Total area} = 2(a^2 + 2a^2 + 2a^2) = 10a^2$$

For each square plate:

$$m' = \frac{1}{10}m$$

$$I_x = \frac{1}{12}m'a^2 + m'a^2 = \frac{13}{12}m'a^2 = \frac{13}{120}ma^2$$

$$I_y = \frac{1}{6}m'a^2 = \frac{1}{60}ma^2$$

$$I_z = I_x = \frac{13}{120}ma^2$$

For each plate parallel to the yz plane:

$$m' = \frac{1}{5}m$$

$$I_x = \frac{1}{12}m'[a^2 + (2a)^2] = \frac{5}{12}m'a^2 = \frac{1}{12}ma^2$$

$$I_y = \frac{1}{12}m'a^2 + m'\left(\frac{a}{2}\right)^2 = \frac{1}{3}m'a^2 = \frac{1}{15}ma^2$$

$$I_z = \frac{1}{12}m(2a)^2 + m'\left(\frac{a}{2}\right)^2 = \frac{7}{12}m'a^2 = \frac{7}{60}ma^2$$

PROBLEM 18.7 (Continued)

For each plate parallel to the *xy* plane: $m' = \frac{1}{5}m$

$$I_x = \frac{1}{12}m'(2a)^2 + m'\left(\frac{a}{2}\right)^2 = \frac{7}{12}m'a^2 = \frac{7}{60}ma^2$$

$$I_y = \frac{1}{12}m'a^2 + m'\left(\frac{a}{2}\right)^2 = \frac{1}{3}m'a^2 = \frac{1}{15}ma^2$$

$$I_z = \frac{1}{12}m'[a^2 + (2a)^2] = \frac{5}{12}m'a^2 = \frac{1}{12}ma^2$$

Total moments of inertia:

$$I_x = 2\left(\frac{13}{120} + \frac{1}{12} + \frac{7}{60}\right)ma^2 = \frac{37}{60}ma^2$$

$$I_y = 2\left(\frac{1}{60} + \frac{1}{15} + \frac{1}{15}\right)ma^2 = \frac{3}{10}ma^2$$

$$I_z = 2\left(\frac{13}{120} + \frac{7}{60} + \frac{1}{12}\right)ma^2 = \frac{37}{60}ma^2$$

(*a*)

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k} = \frac{ma^2\omega}{60\sqrt{6}}(-37\mathbf{i} + 36\mathbf{j} - 37\mathbf{k})$$

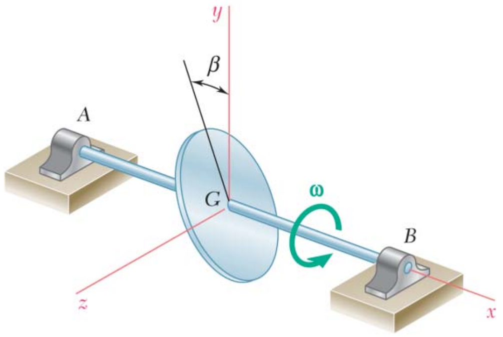
$$H_G = \frac{ma^2\omega}{60\sqrt{6}}\sqrt{(37)^2 + (36)^2 + (37)^2} = 0.43216\ ma^2\omega$$

$$H_G = 0.432\ ma^2\omega \quad ◀$$

(*b*)

$$\mathbf{H}_G \cdot \boldsymbol{\omega} = \frac{ma^2\omega}{(60)(6)}(-37\mathbf{i} + 36\mathbf{j} - 37\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0.40556\ ma^2\omega^2$$

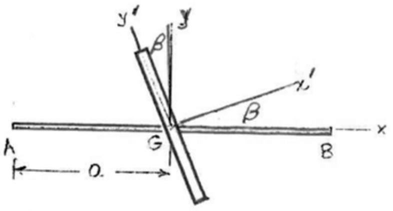
$$\cos\theta = \frac{\mathbf{H}_G \cdot \boldsymbol{\omega}}{H_G\omega} = \frac{0.40556}{0.43216} = 0.93844 \qquad \theta = 20.2° \quad ◀$$



PROBLEM 18.8

A thin homogeneous disk of mass m and radius r is mounted on the horizontal axle AB. The plane of the disk forms an angle $\beta = 20°$ with the vertical. Knowing that the axle rotates with an angular velocity $\boldsymbol{\omega}$, determine the angle θ formed by the axle and the angular momentum of the disk about G.

SOLUTION



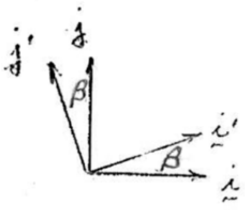
Use principal axes x', y', z' as shown.

$$\mathbf{i}' = \cos\beta\mathbf{i} + \sin\beta\mathbf{j}$$

$$\mathbf{j}' = -\sin\beta\mathbf{i} + \cos\beta\mathbf{j}$$

$$\mathbf{i} = \cos\beta\,\mathbf{i}' - \sin\beta\mathbf{j}'$$

$$\mathbf{j} = \sin\beta\,\mathbf{i}' + \cos\beta\,\mathbf{j}'$$



$$\boldsymbol{\omega} = \omega\mathbf{i} = \omega\cos\beta\mathbf{i}' - \omega\sin\beta\mathbf{j}'$$

Principal moments of inertia:

$$I_{x'} = \frac{1}{2}mr^2, \qquad I_{y'} = I_{z'} = \frac{1}{4}mr^2$$

Angular momentum:

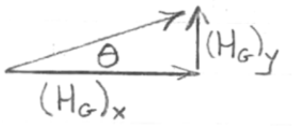
$$\mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + I_{z'}\omega_{z'}\mathbf{k}'$$

$$= \frac{1}{2}mr^2\omega\cos\beta\mathbf{i}' - \frac{1}{4}mr^2\omega\sin\beta\mathbf{j}'$$

$$\mathbf{H}_G = \left(\frac{1}{2}mr^2\omega\cos\beta\right)(\cos\beta\mathbf{i} + \sin\beta\mathbf{j}) - \left(\frac{1}{4}mr^2\omega\sin\beta\right)(-\sin\beta\mathbf{i} + \cos\beta\mathbf{j})$$

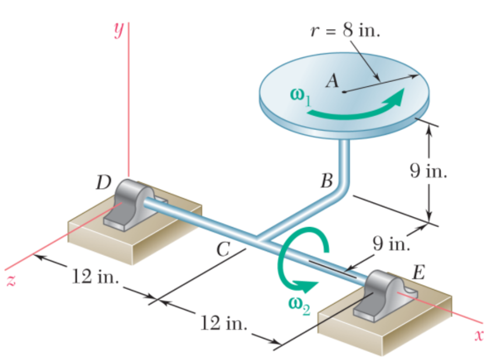
$$= mr^2\omega\left(\frac{1}{2}\cos^2\beta + \frac{1}{4}\sin^2\beta\right)\mathbf{i} + \frac{1}{4}mr^2\omega\sin\beta\cos\beta\mathbf{j}$$

For $\beta = 20°$, $\quad \mathbf{H}_G = \left(0.47076\,mr^2\omega\right)\mathbf{i} + \left(0.080348\,mr^2\omega\mathbf{j}\right)$



$$\tan\theta = \frac{(H_G)_y}{(H_G)_x} = \frac{0.080348}{0.47076}$$

$\theta = 9.7°$ ◀



PROBLEM 18.9

Determine the angular momentum $\mathbf{H}_D$ of the disk of Problem 18.4 about Point *D*.

PROBLEM 18.4 A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm *ABC*, which is welded to a shaft *DCE* rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the angular momentum $\mathbf{H}_A$ of the disk about its center *A*.

SOLUTION

$$\boldsymbol{\omega} = \omega_2\mathbf{i} + \omega_1\mathbf{j} = (8 \text{ rad/s})\mathbf{i} + (16 \text{ rad/s})\mathbf{j}$$

For axes x', y', z' parallel to x, y, z with origin at A,

$$m = \frac{W}{g} = \frac{6}{32.2} = 0.186335 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\bar{I}_{x'} = \frac{1}{4}mr^2 = \frac{1}{4}(0.186335)\left(\frac{8}{12}\right)^2$$

$$= 0.020704 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{z'} = \bar{I}_{x'} = 0.020704 \text{ lb}\cdot\text{s}^2\cdot\text{ft},$$

$$\bar{I}_{y'} = \bar{I}_{x'} + \bar{I}_{z'} = 0.041408 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\mathbf{H}_A = \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + \bar{I}_{z'}\omega_{z'}\mathbf{k}$$

$$= (0.020704)(8)\mathbf{i} + (0.041408)(16)\mathbf{j}$$

$$= (0.1656 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{i} + (0.6625 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{j}$$

Point *A* is the mass center of the disk.

$$\mathbf{r}_{A/D} = (1.0 \text{ ft})\mathbf{i} + (0.75 \text{ ft})\mathbf{j} - (0.75 \text{ ft})\mathbf{k}$$

$$\bar{\mathbf{v}} = \mathbf{v}_A = \omega_2\mathbf{i} \times \mathbf{r}_{A/D}$$

$$= 8\mathbf{i} \times (1.0\mathbf{i} + 0.75\mathbf{j} - 0.75\mathbf{k})$$

$$= (6 \text{ ft/s})\mathbf{j} + (6 \text{ ft/s})\mathbf{k}$$

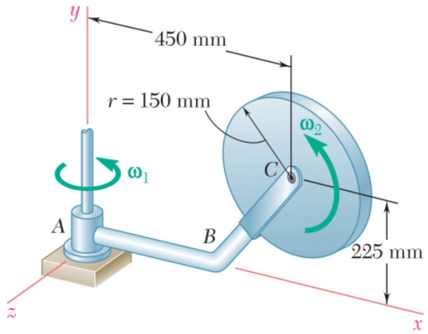
$$m\bar{\mathbf{v}} = (1.118 \text{ lb}\cdot\text{s})\mathbf{j} + (1.118 \text{ lb}\cdot\text{s})\mathbf{k}$$

$$\mathbf{r}_{A/D} \times m\bar{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.0 & 0.75 & -0.75 \\ 0 & 1.118 & 1.118 \end{vmatrix}$$

$$= (1.677 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{i} - (1.118 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{j} + (1.118 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{k}$$

$$\mathbf{H}_D = \mathbf{H}_A + \mathbf{r}_{A/D} \times m\bar{\mathbf{v}}$$

$$\mathbf{H}_D = (1.843 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{i} - (0.455 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{j} + (1.118 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{k} \quad \blacktriangleleft$$

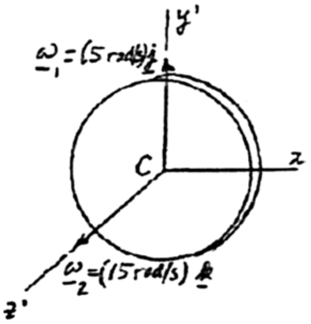


PROBLEM 18.10

Determine the angular momentum of the disk of Problem 18.5 about Point *A*.

PROBLEM 18.5 A thin disk of mass $m = 4\text{ kg}$ rotates at the constant rate $\omega_2 = 15\text{ rad/s}$ with respect to arm *ABC*, which itself rotates at the constant rate $\omega_1 = 5\text{ rad/s}$ about the *y* axis. Determine the angular momentum of the disk about its center *C*.

SOLUTION



$$r = 150\text{ in.}$$

Angular velocity of disk:

$$\boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k} = (5\text{ rad/s})\mathbf{j} + (15\text{ rad/s})\mathbf{k}$$

Centroidal moments of inertia:

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4}mr^2$$

$$= \frac{1}{4}(4)(0.150\text{ m})^2 = 0.0225\text{ kg}\cdot\text{m}^2$$

$$I_{z'} = \frac{1}{2}mr^2 = 0.045\text{ kg}\cdot\text{m}^2$$

Angular momentum about Point *C*.

$$\begin{aligned}\mathbf{H}_C &= \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + I_{z'}\omega_{z'}\mathbf{k} \\ &= 0 + (0.0225)(5)\mathbf{j} + (0.045)(15)\mathbf{k} \\ &= (0.1125\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (0.6750\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}\end{aligned}$$

Location of mass center. $\mathbf{r}_{C/A} = (0.450\text{ m})\mathbf{i} + (0.225\text{ m})\mathbf{j}$

Velocity of mass center.

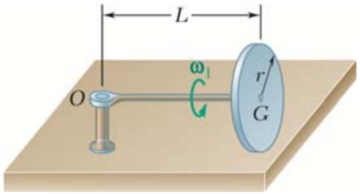
$$\begin{aligned}\bar{\mathbf{v}} &= \boldsymbol{\omega}_1 \times \mathbf{r}_{C/A} = 5\mathbf{j} \times (0.45\mathbf{i} + 0.225\mathbf{j}) \\ &= -(2.25\text{ m/s})\mathbf{k}\end{aligned}$$

Angular momentum about Point *A*.

$$\mathbf{H}_A = \mathbf{H}_C + \mathbf{r}_{C/A} \times (m\bar{\mathbf{v}})$$

$$\begin{aligned}\mathbf{H}_A &= 0.1125\mathbf{j} + 0.675\mathbf{k} + (0.45\mathbf{i} + 0.225\mathbf{j}) \times [-(4)(2.25)\mathbf{k}] \\ &= 0.1125\mathbf{j} + 0.675\mathbf{k} + 4.05\mathbf{j} - 2.025\mathbf{i}\end{aligned}$$

$$\mathbf{H}_A = -(2.03\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (4.16\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (0.675\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k} \blacktriangleleft$$



PROBLEM 18.11

Determine the angular momentum $\mathbf{H}_O$ of the disk of Sample Problem 18.2 from the expressions obtained for its linear momentum $m\overline{\mathbf{v}}$ and its angular momentum $\mathbf{H}_G$ using Eq. (18.11). Verify that the result obtained is the same as that obtained by direct computation.

PROBLEM 18.2 A homogeneous disk of radius *r* and mass *m* is mounted on an axle *OG* of length *L* and negligible mass. The axle is pivoted at the fixed Point *O*, and the disk is constrained to roll on a horizontal floor. Knowing that the disk rotates counterclockwise at the rate ω_1 about the axle *OG*, determine (*a*) the angular velocity of the disk, (*b*) its angular momentum about *O*, (*c*) its kinetic energy, (*d*) the vector and couple at *G* equivalent to the momenta of the particles of the disk.

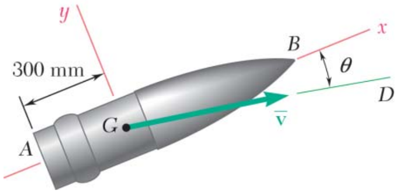
SOLUTION

Using Equation (18.11),

$$\begin{aligned}\mathbf{H}_O &= \overline{\mathbf{r}} \times m\overline{\mathbf{v}} + \overline{\mathbf{H}}_G \\ &= (L\mathbf{i}) \times (mr\omega_1\mathbf{k}) + \frac{1}{2}mr^2\omega_1\left(\mathbf{i} - \frac{r}{2L}\mathbf{j}\right) \\ &= -mrL\omega_1\mathbf{j} + \frac{1}{2}mr^2\omega_1\mathbf{i} - \frac{1}{4}m\frac{r^3}{L}\omega_1\mathbf{j}\end{aligned}$$

$$\mathbf{H}_O = \frac{1}{2}mr^2\omega_1\mathbf{i} - m\left(L^2 + \frac{1}{4}r^2\right)\left(\frac{r\omega_1}{L}\right)\mathbf{j}$$

which is the answer obtained in Part *b* of Sample Problem 18.2.



PROBLEM 18.12

The 100-kg projectile shown has a radius of gyration of 100 mm about its axis of symmetry Gx and a radius of gyration of 250 mm about the transverse axis Gy. Its angular velocity $\boldsymbol{\omega}$ can be resolved into two components; one component, directed along Gx, measures the *rate of spin* of the projectile, while the other component, directed along GD, measures its *rate of precession.* Knowing that $\theta = 6°$ and that the angular momentum of the projectile about its mass center G is $\mathbf{H}_G = (500\ \text{g}\cdot\text{m}^2/\text{s})\mathbf{i} - (10\ \text{g}\cdot\text{m}^2/\text{s})\mathbf{j}$, determine ($a$) the rate of spin, ($b$) the rate of precession.

SOLUTION

$$m = 100\ \text{kg},\quad k_x = 100\ \text{mm} = 0.1\ \text{m},\quad k_y = 250\ \text{mm} = 0.25\ \text{m}$$

$$\bar{I}_x = mk_x^2 = (100)(0.1)^2 = 1\ \text{kg}\cdot\text{m}^2$$

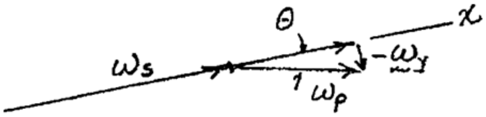
$$\bar{I}_y = \bar{I}_z = mk_y^2 = (100)(0.25)^2 = 6.25\ \text{kg}\cdot\text{m}^2$$

$$\mathbf{H}_G = (H_G)_x\mathbf{i} + (H_G)_y\mathbf{j} + (H_G)_z\mathbf{k} = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$\omega_x = \frac{(H_G)_x}{\bar{I}_x} = \frac{0.500\ \text{kg}\cdot\text{m}^2/\text{s}}{1\ \text{kg}\cdot\text{m}^2} = 0.5\ \text{rad/s}$$

$$\omega_y = \frac{(H_G)_y}{\bar{I}_y} = \frac{-0.01\ \text{kg}\cdot\text{m}^2/\text{s}}{6.25\ \text{kg}\cdot\text{m}^2} = -0.0016\ \text{rad/s}$$

$$\omega_z = 0$$



$$\omega_P \sin\theta = -\omega_y$$

$$\omega_P = \frac{-\omega_y}{\sin\theta} = \frac{0.0016}{\sin 6°} = 0.015307\ \text{rad/s}$$

(a) *Rate of spin.*

$$\omega_x = \omega_s + \omega_P\cos\theta$$

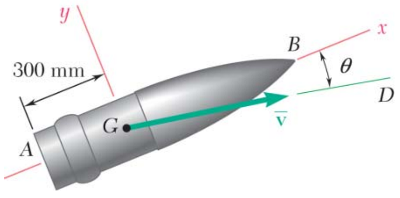
$$\omega_s = \omega_x - \omega_P\cos\theta$$

$$= 0.5 - 0.15307\cos 6°$$

$$= 0.4847$$

$\omega_s = 0.485\ \text{rad/s}$ ◀

(b) *Rate of precession.* $\omega_P = 0.01531\ \text{rad/s}$ ◀



PROBLEM 18.13

Determine the angular momentum $\mathbf{H}_A$ of the projectile of Problem 18.12 about the center A of its base, knowing that its mass center G has a velocity $\overline{\mathbf{v}}$ of 750 m/s. Give your answer in terms of components respectively parallel to the x and y axes shown and to a third axis z pointing toward you.

PROBLEM 18.12 The 100-kg projectile shown has a radius of gyration of 100 mm about its axis of symmetry Gx and a radius of gyration of 250 mm about the transverse axis Gy. Its angular velocity $\boldsymbol{\omega}$ can be resolved into two components; one component, directed along Gx, measures the *rate of spin* of the projectile, while the other component, directed along GD, measures its *rate of precession.* Knowing that $\theta = 6°$ and that the angular momentum of the projectile about its mass center G is $\mathbf{H}_G = (500\ \text{g}\cdot\text{m}^2/\text{s})\mathbf{i} - (10\ \text{g}\cdot\text{m}^2/\text{s})\mathbf{j}$, determine ($a$) the rate of spin, ($b$) the rate of precession.

SOLUTION

$$m = 100\ \text{kg}, \quad \mathbf{r}_{G/A} = (0.300\ \text{m})\mathbf{i}$$

$$\overline{\mathbf{v}} = \bar{v}\cos\theta\mathbf{i} - \bar{v}\sin\theta\mathbf{j}$$

$$\mathbf{H}_G = (0.50\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (0.10\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{j}$$

$$m\overline{\mathbf{v}} = (100)(750)(\cos 6°\mathbf{i} - \sin 6°\mathbf{j})$$

$$= (74589\ \text{kg}\cdot\text{m/s})\mathbf{i} - (7839\ \text{kg}\cdot\text{m/s})\mathbf{j}$$

$$\mathbf{r}_{G/A} \times m\overline{\mathbf{v}} = 0.3\mathbf{i} \times (74589\mathbf{i} - 7839.6\mathbf{j})$$

$$= -(2351.9\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

$$\mathbf{H}_A = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\overline{\mathbf{v}}$$

$$= 0.5\mathbf{i} - 0.1\mathbf{j} + (-2351.9\mathbf{k})$$

$$\mathbf{H}_A = (0.500\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (0.100\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{j} - (2350\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{k} \blacktriangleleft$$

PROBLEM 18.14

(*a*) Show that the angular momentum $\mathbf{H}_B$ of a rigid body about Point *B* can be obtained by adding to the angular momentum $\mathbf{H}_A$ of that body about Point *A* the vector product of the vector $\mathbf{r}_{A/B}$ drawn from *B* to *A* and the linear momentum $m\overline{\mathbf{v}}$ of the body:

$$\mathbf{H}_B = \mathbf{H}_A + \mathbf{r}_{A/B} \times m\overline{\mathbf{v}}$$

(*b*) Further show that when a rigid body rotates about a fixed axis, its angular momentum is the same about any two Points *A* and *B* located on the fixed axis $(\mathbf{H}_A = \mathbf{H}_B)$ if, and only if, the mass center *G* of the body is located on the fixed axis.

SOLUTION

(*a*) Angular momenta $\mathbf{H}_A$ and $\mathbf{H}_B$ are related to $\mathbf{H}_G$ and $m\overline{\mathbf{v}}$ by

$$\mathbf{H}_A = \mathbf{r}_{G/A} \times m\overline{\mathbf{v}} + \mathbf{H}_G \quad \text{and} \quad \mathbf{H}_B = \mathbf{r}_{G/B} \times m\overline{\mathbf{v}} + \mathbf{H}_G$$

Subtracting,

$$\begin{aligned} \mathbf{H}_B - \mathbf{H}_A &= \mathbf{r}_{G/B} \times m\overline{\mathbf{v}} - \mathbf{r}_{G/A} \times m\overline{\mathbf{v}} \\ \mathbf{H}_B &= \mathbf{H}_A + (\mathbf{r}_{G/B} - \mathbf{r}_{G/A}) \times m\overline{\mathbf{v}} \\ &= \mathbf{H}_A + (\mathbf{r}_{G/B} + \mathbf{r}_{A/G}) \times m\overline{\mathbf{v}} \\ \mathbf{H}_B &= \mathbf{H}_A + \mathbf{r}_{A/B} \times m\overline{\mathbf{v}} \end{aligned}$$

(*b*) It follows that $\mathbf{H}_A = \mathbf{H}_B$ if, and only if

$$\mathbf{r}_{A/B} \times m\overline{\mathbf{v}} = 0$$

With Points *A* and *B* located on the fixed axis,

$$\boldsymbol{\omega} = \omega\boldsymbol{\lambda}$$

where $\boldsymbol{\lambda}$ is a unit vector along the fixed axis, and

$$\overline{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{r}_{G/A} = \omega\boldsymbol{\lambda} \times \mathbf{r}_{G/A}$$

Then

$$\mathbf{r}_{A/B} \times (m\omega\boldsymbol{\lambda} \times \mathbf{r}_{G/A}) = 0$$

but $\mathbf{r}_{A/B}$ is parallel to $\boldsymbol{\lambda}$, hence,

$$\boldsymbol{\lambda} \times (\boldsymbol{\lambda} \times \mathbf{r}_{G/A}) = 0$$

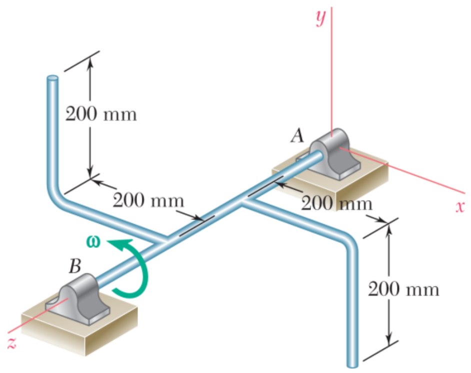
Let $\mathbf{u} = \boldsymbol{\lambda} \times \mathbf{r}_{G/A}$, so that $\boldsymbol{\lambda} \times \mathbf{u} = 0$.

Note that **u** must be either perpendicular to $\boldsymbol{\lambda}$ or equal to zero. But if **u** is perpendicular to $\boldsymbol{\lambda}$, $\boldsymbol{\lambda} \times \mathbf{u}$ cannot be equal to zero.

Hence,

$$\mathbf{u} = \boldsymbol{\lambda} \times \mathbf{r}_{G/A} = 0$$

$\mathbf{r}_{G/A}$ is parallel to $\boldsymbol{\lambda}$ and Point *G* lies on the fixed axis.



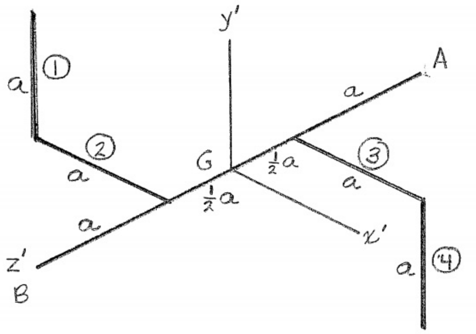
PROBLEM 18.15

Two L-shaped arms, each of mass 5 kg, are welded at the one-third points of the 600 mm shaft AB to form the assembly shown. Knowing that the assembly rotates at the constant rate of 360 rpm, determine (a) the angular momentum $\mathbf{H}_A$ of the assembly about point A, (b) the angle formed by $\mathbf{H}_A$ and AB.

SOLUTION

$$m = 5 \text{ kg}, \qquad a = 200 \text{ mm} = 0.2 \text{ m}$$

$$\omega = \frac{(2\pi)(360)}{60} = 12\pi \text{ rad/s.} \qquad \omega_x = 0, \qquad \omega_y = 0, \qquad \omega_z = 12\pi \text{ rad/s}$$



Use parallel axes x', y', z' with origin at the mass center as shown.

$$(H_G)_x = -\bar{I}_{xz}\omega$$

$$(H_G)_y = -\bar{I}_{yz}\omega$$

$$(H_G)_z = \bar{I}_z\omega$$

Segments 1, 2, 3, and 4, each of mass $m' = 2.5$ kg, contribute to $\bar{I}_{xz}$, $\bar{I}_{yz}$ and $\bar{I}_z$.

Part	I_{xz}	I_{yz}	I_z
①	$-\frac{1}{2}m'a^2$	$\frac{1}{4}m'a^2$	$\left(\frac{1}{12}+\frac{1}{4}+1\right)m'a^2$
②	$-\frac{1}{4}m'a^2$	0	$\frac{1}{3}m'a^2$
③	$-\frac{1}{4}m'a^2$	0	$\frac{1}{3}m'a^2$
④	$-\frac{1}{2}m'a^2$	$\frac{1}{4}m'a^2$	$\left(\frac{1}{12}+\frac{1}{4}+1\right)m'a^2$
Σ	$-\frac{3}{2}m'a^2$	$\frac{1}{2}m'a^2$	$\frac{10}{3}m'a^2$

$$(H_G)_x = -\left(-\frac{3}{2}m'a^2\right)\omega$$

$$= \left(\frac{3}{2}\right)(2.5)(0.2)^2(12\pi)$$

$$= 5.6549 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_G)_y = -\left(\frac{1}{2}m'a^2\right)\omega$$

$$= -\left(\frac{1}{2}\right)(2.5)(0.2)^2(12\pi)$$

$$= -1.8850 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_G)_z = \left(\frac{10}{3}m'a^2\right)\omega$$

$$= \left(\frac{10}{3}\right)(2.5)(0.2)^2(12\pi)$$

$$= 12.5664 \text{ kg}\cdot\text{m}^2/\text{s}$$

Velocity of the mass center: $\bar{\mathbf{v}} = 0$

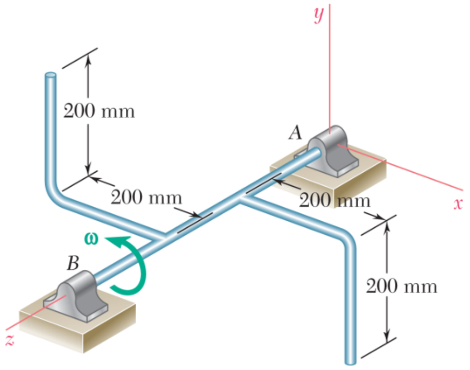
(*a*) $\mathbf{H}_A = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}} = \mathbf{H}_G$

$$\mathbf{H}_A = \left(5.65 \text{ kg}\cdot\text{m}^2/\text{s}\right)\mathbf{i} - \left(1.885 \text{ kg}\cdot\text{m}^2/\text{s}\right)\mathbf{j} + \left(12.57 \text{ kg}\cdot\text{m}^2/\text{s}\right)\mathbf{k} \blacktriangleleft$$

(*b*) $\boldsymbol{\lambda}_{AB} = \mathbf{k}, \qquad \boldsymbol{\lambda}_{AB}\cdot\mathbf{H}_A = \mathbf{k}\cdot\mathbf{H}_A = (H_A)_z$

$$H_A = \sqrt{(5.6549)^2 + (1.8850)^2 + (12.5664)^2} = 13.9085 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$\cos\theta = \frac{(H_A)_z}{H_A} = \frac{12.5664}{13.9085} \qquad \theta = 25.4° \blacktriangleleft$$



PROBLEM 18.16

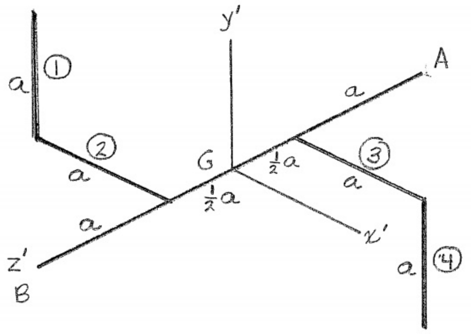
For the assembly of Prob. 18.15, determine (*a*) the angular momentum $\mathbf{H}_B$ of the assembly about point *B*, (*b*) the angle formed by $\mathbf{H}_B$ and *BA*.

PROBLEM 18.15: Two L-shaped arms, each of mass 5 kg, are welded at the one-third points of the 600 mm shaft *AB* to form the assembly shown. Knowing that the assembly rotates at the constant rate of 360 rpm, determine (*a*) the angular momentum $\mathbf{H}_A$ of the assembly about point *A*, (*b*) the angle formed by $\mathbf{H}_A$ and *AB*.

SOLUTION

$$m = 5 \text{ kg}, \qquad a = 200 \text{ mm} = 0.2 \text{ m}$$

$$\omega = \frac{(2\pi)(360)}{60} = 12\pi \text{ rad/s.} \qquad \omega_x = 0, \qquad \omega_y = 0, \qquad \omega_z = 12\pi \text{ rad/s}$$



Use parallel axes *x*′, *y*′, *z*′ with origin at the mass center as shown.

$$(H_G)_x = -\bar{I}_{xz}\omega$$

$$(H_G)_y = -\bar{I}_{yz}\omega$$

$$(H_G)_z = \bar{I}_z\omega$$

Segments 1, 2, 3, and 4, each of mass $m' = 2.5$ kg, contribute to $\bar{I}_{xz}$, $\bar{I}_{yz}$ and $\bar{I}_z$.

Part	I_{xz}	I_{yz}	I_z
①	$-\frac{1}{2}m'a^2$	$\frac{1}{4}m'a^2$	$\left(\frac{1}{12}+\frac{1}{4}+1\right)m'a^2$
②	$-\frac{1}{4}m'a^2$	0	$\frac{1}{3}m'a^2$
③	$-\frac{1}{4}m'a^2$	0	$\frac{1}{3}m'a^2$
④	$-\frac{1}{2}m'a^2$	$\frac{1}{4}m'a^2$	$\left(\frac{1}{12}+\frac{1}{4}+1\right)m'a^2$
Σ	$-\frac{3}{2}m'a^2$	$\frac{1}{2}m'a^2$	$\frac{10}{3}m'a^2$

PROBLEM 18.16 (Continued)

$$(H_G)_x = -\left(-\frac{3}{2}m'a^2\right)\omega$$

$$= \left(\frac{3}{2}\right)(2.5)(0.2)^2(12\pi)$$

$$= 5.6549 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_G)_y = -\left(\frac{1}{2}m'a^2\right)\omega$$

$$= -\left(\frac{1}{2}\right)(2.5)(0.2)^2(12\pi)$$

$$= -1.8850 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_G)_z = \left(\frac{10}{3}m'a^2\right)\omega$$

$$= \left(\frac{10}{3}\right)(2.5)(0.2)^2(12\pi)$$

$$= 12.5664 \text{ kg}\cdot\text{m}^2/\text{s}$$

Velocity of the mass center: $\bar{\mathbf{v}} = 0$

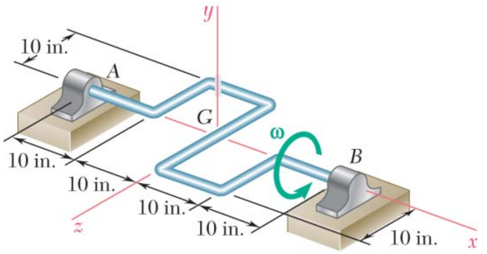
(*a*) $$\mathbf{H}_B = \mathbf{H}_G + \mathbf{r}_{G/B} \times m\bar{\mathbf{v}} = \mathbf{H}_G$$

$$\mathbf{H}_B = \left(5.65 \text{ kg}\cdot\text{m}^2/\text{s}\right)\mathbf{i} - \left(1.885 \text{ kg}\cdot\text{m}^2/\text{s}\right)\mathbf{j} + \left(12.57 \text{ kg}\cdot\text{m}^2/\text{s}\right)\mathbf{k} \blacktriangleleft$$

(*b*) $$\boldsymbol{\lambda}_{BA} = -\mathbf{k}, \qquad \boldsymbol{\lambda}_{BA}\cdot\mathbf{H}_A = -\mathbf{k}\cdot\mathbf{H}_A = -(H_A)_z$$

$$H_A = \sqrt{(5.6549)^2 + (1.8850)^2 + (12.5664)^2} = 13.9085 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$\cos\theta = -\frac{(H_A)_z}{H_A} = -\frac{12.5664}{13.9085} \qquad \theta = 154.6° \blacktriangleleft$$



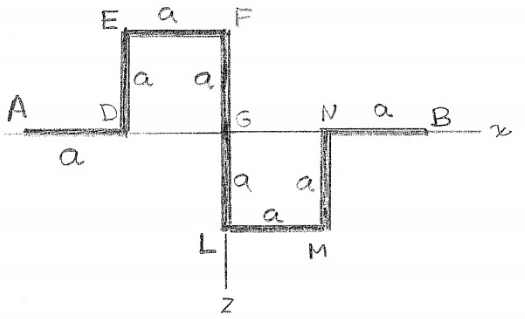
PROBLEM 18.17

A 10-lb rod of uniform cross section is used to form the shaft shown. Knowing that the shaft rotates with a constant angular velocity $\boldsymbol{\omega}$ of magnitude 12 rad/s, determine (*a*) the angular momentum $\mathbf{H}_G$ of the shaft about its mass center G, (*b*) the angle formed by $\mathbf{H}_G$ and the axis AB.

SOLUTION

$$m = \frac{W}{g} = \frac{10}{32.2} = 0.31056 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\omega = (12 \text{ rad/s})\mathbf{i}, \qquad \omega_y = \omega_z = 0$$



$$(H_G)_x = \bar{I}_x\omega$$

$$(H_G)_y = -\bar{I}_{xy}\omega$$

$$(H_G)_z = -\bar{I}_{xz}\omega$$

The shaft is comprised of 8 sections, each of length $a = \frac{10}{12}$ ft and of mass $m' = \frac{m}{8} = 0.03882 \text{ lb}\cdot\text{s}^2/\text{ft}.$

$$\bar{I}_x = (4)\left(\frac{1}{3}m'a^2\right) + (2)\left(m'a^2\right) = \frac{10}{3}m'a^2 = \frac{10}{3}(0.03882)\left(\frac{10}{12}\right)^2 = 0.089861 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{xy} = 0$$

$$\bar{I}_{xz} = (4)\left(m'a\frac{a}{2}\right) = 2m'a^2 = (2)(0.03882)\left(\frac{10}{12}\right)^2 = 0.053916 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$(H_G)_x = (0.089861)(12) = 1.0783 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

$$(H_G)_y = 0$$

$$(H_G)_z = -(0.053916)(12) = -0.6470 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

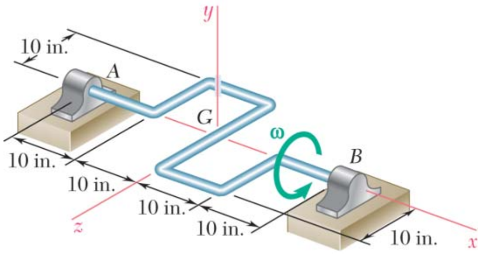
(*a*) $\mathbf{H}_G = (1.078 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{i} - (0.647 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{k}$ ◀

$$H_G = \sqrt{(1.0783)^2 + (0.6470)^2} = 1.2575 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

(*b*) $\mathbf{H}_G \cdot \boldsymbol{\omega} = (1.0783\mathbf{i} - 0.6470\mathbf{k}) \cdot 12\mathbf{i} = 12.940 \text{ lb}\cdot\text{ft}$

$$\cos\theta = \frac{\mathbf{H}_G \cdot \boldsymbol{\omega}}{H_G\omega} = \frac{12.940}{(1.2575)(12)} = 0.85752$$

$\theta = 31.0°$ ◀



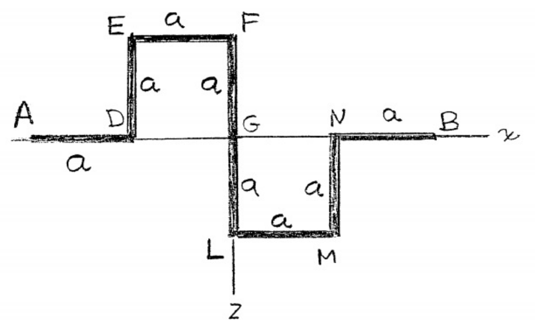
PROBLEM 18.18

Determine the angular momentum of the shaft of Prob. 18.17 about (*a*) point *A*, (*b*) point *B*.

SOLUTION

$$m = \frac{W}{g} = \frac{10}{32.2} = 0.31056\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$\omega = (12\ \text{rad/s})\mathbf{i}, \qquad \omega_y = \omega_z = 0$$



$$(H_G)_x = \bar{I}_x\omega$$

$$(H_G)_y = -\bar{I}_{xy}\omega$$

$$(H_G)_z = -\bar{I}_{xz}\omega$$

The shaft is comprised of 8 sections, each of length $a = \frac{10}{12}$ ft and of mass $m' = \frac{m}{8} = 0.03882\ \text{lb}\cdot\text{s}^2/\text{ft}$.

$$\bar{I}_x = (4)\left(\frac{1}{3}m'a^2\right) + (2)\left(m'a^2\right) = \frac{10}{3}m'a^2 = \frac{10}{3}(0.03882)\left(\frac{10}{12}\right)^2 = 0.089861\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{xy} = 0$$

$$\bar{I}_{xz} = (4)\left(m'a\frac{a}{2}\right) = 2m'a^2 = (2)(0.03882)\left(\frac{10}{12}\right)^2 = 0.053916\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

$$(H_G)_x = (0.089861)(12) = 1.0783\ \text{lb}\cdot\text{s}\cdot\text{ft}$$

$$(H_G)_y = 0$$

$$(H_G)_z = -(0.053916)(12) = -0.6470\ \text{lb}\cdot\text{s}\cdot\text{ft}$$

$$\mathbf{H}_G = (1.078\ \text{lb}\cdot\text{s}\cdot\text{ft})\mathbf{i} - (0.647\ \text{lb}\cdot\text{s}\cdot\text{ft})\mathbf{k}$$

Since point *G* lies on the axis of rotation, its velocity is zero.

$$\bar{\mathbf{v}} = \mathbf{v}_G = 0$$

(*a*)

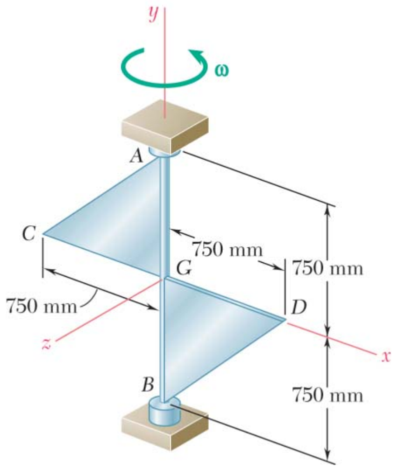
$$\mathbf{H}_A = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}} = \mathbf{H}_G$$

$$\mathbf{H}_A = (1.078\ \text{lb}\cdot\text{s}\cdot\text{ft})\mathbf{i} - (0.647\ \text{lb}\cdot\text{s}\cdot\text{ft})\mathbf{k} \blacktriangleleft$$

(*b*)

$$\mathbf{H}_B = \mathbf{H}_G + \mathbf{r}_{G/B} \times m\bar{\mathbf{v}} = \mathbf{H}_G$$

$$\mathbf{H}_B = (1.078\ \text{lb}\cdot\text{s}\cdot\text{ft})\mathbf{i} - (0.647\ \text{lb}\cdot\text{s}\cdot\text{ft})\mathbf{k} \blacktriangleleft$$



PROBLEM 18.19

Two triangular plates, each of mass 8 kg, are welded to a vertical shaft *AB*. Knowing that the system rotates at the constant rate $\omega = 6$ rad/s, determine its angular momentum about *G*.

SOLUTION

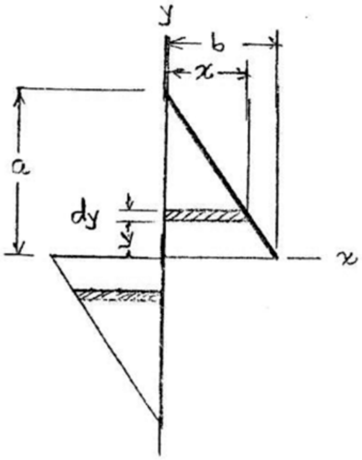
Angular velocity: $\boldsymbol{\omega} = \omega\mathbf{j}$

Angular momentum about the mass center: $\mathbf{H}_G = -\bar{I}_{xy}\omega\mathbf{i} + \bar{I}_y\omega\mathbf{j} - \bar{I}_{yz}\omega\mathbf{k}$ (1)

Since the body lies in the *xy* plane, $\bar{I}_{yz} = 0.$

Calculation of $\bar{I}_{xy}$ and $\bar{I}_y$: Let $\rho = \dfrac{m}{A} = \dfrac{2m}{ab}$ = mass per unit area.

$$(I)_{\text{mass}} = \rho(I)_{\text{area}}.$$



For the pairs of area elements shown,

$$dI_{xy} = 2\bar{x}_{\text{el}}\bar{y}_{\text{el}}dA = 2\frac{x}{2}yxdy = x^2ydy$$

$$I_{xy} = \int_0^a x^2ydy = \int_0^a\left(b - \frac{by}{a}\right)^2 ydy$$

$$= \int_0^a b^2ydy - 2\frac{b^2}{a}\int_0^a y^2dy + \frac{b^2}{a^2}\int_0^a y^3dy$$

$$= \frac{1}{2}b^2a^2 - \frac{2}{3}b^2a^3 + \frac{1}{4}b^2a^2 = \frac{1}{12}b^2a^2$$

$$I_y = 2\left(\frac{1}{12}ab^3\right) = \frac{1}{6}ab^3$$

For mass, $\bar{I}_{xy} = \dfrac{1}{6}mab, \qquad \bar{I}_y = \dfrac{1}{3}mb^2$

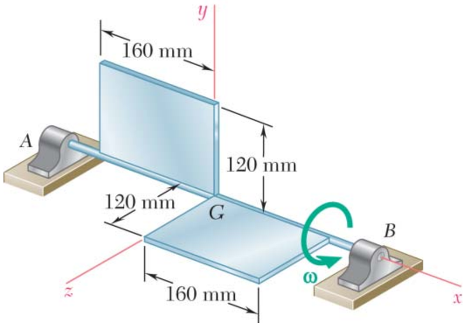
Data: $m = 8$ kg, $a = b = 750$ mm $= 0.75$ m, $\omega = 6$ rad/s

$$\bar{I}_{xy} = \frac{1}{6}(8)(0.75)(0.75) = 0.75\ \text{kg}\cdot\text{m}^2$$

$$\bar{I}_y = \frac{1}{3}(8)(0.75)^2 = 1.50\ \text{kg}\cdot\text{m}^2$$

From (1), $\mathbf{H}_G = -\left(4.50\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{i} + \left(9.00\ \text{kg}\cdot\text{m}^2/\text{s}\right)\mathbf{j}$ ◀

PROBLEM 18.20



The assembly shown consists of two pieces of sheet aluminum of uniform thickness and total mass 1.6 kg welded to a light axle supported by bearings *A* and *B*. Knowing that the assembly rotates with an angular velocity of constant magnitude $\omega = 20$ rad/s, determine the angular momentum $\mathbf{H}_G$ of the assembly about point *G*.

SOLUTION

Mass of each sheet $m = \frac{1.6}{2} = 0.8 \text{ kg}$

Required moment and products of inertia.

$$I_x = 2\left[\frac{1}{3}mb^2\right] = (2)\left(\frac{1}{3}\right)(0.8)(0.120)^2 = 7.68 \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

$$I_{xy} = m\left(-\frac{a}{2}\right)\left(\frac{b}{2}\right) = (0.8)(-0.080)(0.060) = -3.84 \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

$$I_{xz} = m\left(\frac{a}{2}\right)\left(\frac{b}{2}\right) = (0.8)(0.080)(0.060) = 3.84 \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

Angular velocity. $\omega_x = 20$ rad/s, $\omega_y = \omega_z = 0$

Angular momentum about *G*.

$$(H_G)_x = I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$

$$= (7.68 \times 10^{-3})(20) - 0 - 0 = 0.1536 \text{ kg}\cdot\text{m}^2/\text{s}$$

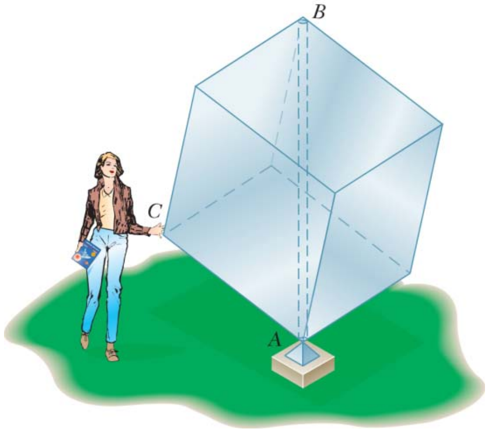
$$(H_G)_y = -I_{xy}\omega_x + I_y\omega_y - I_{yz}\omega_z$$

$$= -(-3.84 \times 10^{-3})(20) + 0 - 0 = 0.0768 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_G)_z = -I_{xz}\omega_x - I_{yz}\omega_y + I_z\omega_z$$

$$= -(3.84 \times 10^{-3})(20) - 0 + 0 = -0.0768 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$\mathbf{H}_G = (0.1536 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (0.0768 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} - (0.0768 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k} \blacktriangleleft$$



PROBLEM 18.21

One of the sculptures displayed on a university campus consists of a hollow cube made of six aluminum sheets, each 1.5×1.5 m, welded together and reinforced with internal braces of negligible weight. The cube is mounted on a fixed base at A and can rotate freely about its vertical diagonal AB. As she passes by this display on the way to a class in mechanics, an engineering student grabs corner C of the cube and pushes it for 1.2 s in a direction perpendicular to the plane ABC with an average force of 50 N. Having observed that it takes 5 s for the cube to complete one full revolution, she flips out her calculator and proceeds to determine the mass of the cube. What is the result of her calculation? (*Hint:* The perpendicular distance from the diagonal joining two vertices of a cube to any of its other six vertices can be obtained by multiplying the side of the cube by $\sqrt{2/3}$.)

SOLUTION

Let $m' = \frac{1}{6}m$ be the mass of one side of the cube. Choose x, y, and z axes perpendicular to the face of the cube. Let a be the side of the cube.

For a side perpendicular to the x axis, $(I_x)_1 = \frac{1}{6}m'a^2$.

For a side perpendicular to the y or z axis, $$(I_x)_2 = \left(\frac{1}{12} + \frac{1}{4}\right)m'a^2 = \frac{1}{3}m'a^2$$

Total moment of inertia: $$I_x = 2(I_x)_1 + 4(I_x)_2 = \frac{5}{3}m'a^2 = \frac{5}{18}ma^2$$

By symmetry, $I_y = I_x$ and $I_z = I_x$.

Since all three moments of inertia are equal, the ellipsoid of inertia is a sphere. All centroidal axes are principal axes.

Moment of inertia about the vertical axis: $$I_v = \frac{5}{18}ma^2$$

Let $b = \sqrt{\frac{2}{3}}a$ be the moment arm of the impulse applied to the corner.

Using the impulse-momentum principle and taking moments about the vertical axis,

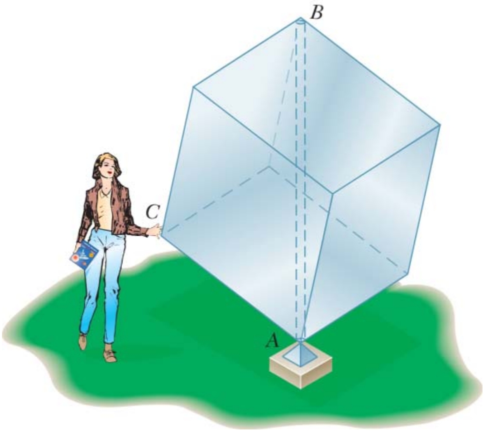
$$bF(\Delta t) = H_v = I_v\omega = \frac{5}{18}ma^2\omega \qquad (1)$$

Data: $$a = 1.5 \text{ m}, \qquad b = \sqrt{\frac{2}{3}}(1.5) = 1.22474 \text{ m}$$

$$\omega = \frac{2\pi}{5} = 1.25664 \text{ rad/s}, \qquad F = 50 \text{ N}, \qquad \Delta t = 1.2 \text{ s.}$$

Solving Equation (1) for m, $$m = \frac{18}{5}\frac{bF(\Delta t)}{a^2\omega} = \frac{18}{5}\frac{(1.22474)(50)(1.2)}{(1.5)^2(1.25664)} = 93.563 \text{ kg}$$

$m = 93.6$ kg ◀



PROBLEM 18.22

If the aluminum cube of Problem 18.21 were replaced by a cube of the same size, made of six plywood sheets with mass 8 kg each, how long would it take for that cube to complete one full revolution if the student pushed its corner *C* in the same way that she pushed the corner of the aluminum cube?

SOLUTION

Let $m' = \frac{1}{6}m$ be the mass of one side of the cube. Choose *x*, *y*, and *z* axes perpendicular to the face of the cube. Let *a* be the side of the cube.

For a side perpendicular to the *x* axis, $(I_x)_1 = \frac{1}{6}m'a^2$.

For a side perpendicular to the *y* or *z* axis, $$(I_x)_2 = \left(\frac{1}{12} + \frac{1}{4}\right)m'a^2 = \frac{1}{3}m'a^2$$

Total moment of inertia: $$I_x = 2(I_x)_1 + 4(I_x)_2 = \frac{5}{3}m'a^2 = \frac{5}{18}ma^2$$

By symmetry, $I_y = I_x$ and $I_z = I_x$.

Since all three moments of inertia are equal, the ellipsoid of inertia is a sphere. All centroidal axes are principal axes.

Moment of inertia about the vertical axis: $$I_v = \frac{5}{18}ma^2$$

Let $b = \sqrt{\frac{2}{3}}a$ be the moment arm of the impulse applied to the corner.

Using the impulse-momentum principle and taking moments about the vertical axis,

$$bF(\Delta t) = H_v = I_v\omega = \frac{5}{18}ma^2\omega \qquad (1)$$

Data: $$m' = 8 \text{ kg}, \qquad m = 6m' = 48 \text{ kg}, \qquad a = 1.5 \text{ m},$$

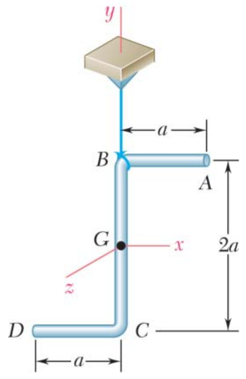
$$b = \sqrt{\frac{2}{3}}a = 1.22474 \text{ m}, \qquad F = 50 \text{ N}, \qquad \Delta t = 1.2 \text{ s}$$

Solving (1) for ω,

$$\omega = \frac{18}{5}\frac{bF(\Delta t)}{ma^2} = \frac{18}{5}\frac{(1.22474)(50)(1.2)}{(48)(1.5)^2} = 2.4495 \text{ rad/s}$$

$$t = \frac{2\pi}{\omega} = \frac{2\pi}{2.4495}$$

$t = 2.57$ s ◀

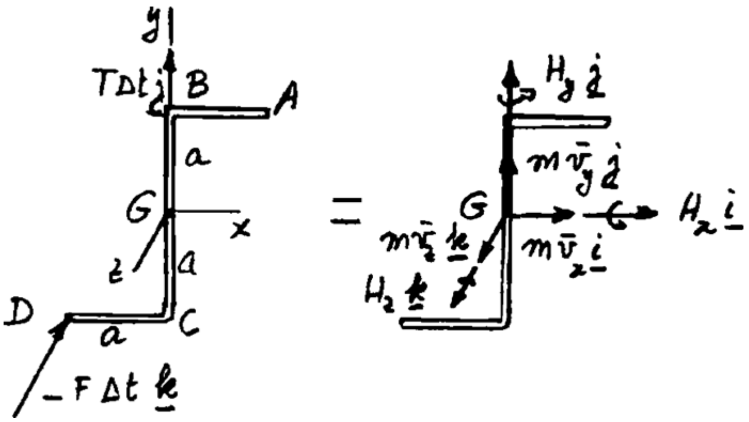


PROBLEM 18.23

A uniform rod of total mass m is bent into the shape shown and is suspended by a wire attached at B. The bent rod is hit at D in a direction perpendicular to the plane containing the rod (in the negative z direction). Denoting the corresponding impulse by $\mathbf{F}\Delta t$, determine (a) the velocity of the mass center of the rod, (b) the angular velocity of the rod.

SOLUTION

We apply the principle of impulse and momentum, considering only the impulsive forces.



(a) *Velocity of mass center*

From constraints: $\bar{v}_y = 0$

$\xrightarrow{+}$ x components: $0 = m\bar{v}_x$ $\qquad \bar{v}_x = 0$

$+\!\uparrow$ y components: $T\Delta t = m\bar{v}_y = 0$ $\qquad T\Delta t = 0$

$+\!\swarrow$ z components: $-F\Delta t = m\bar{v}_t$ $\qquad \bar{v}_z = -\dfrac{F\Delta t}{m}$

$$\bar{\mathbf{v}} = -\frac{F\Delta t}{m}\mathbf{k} \blacktriangleleft$$

(b) *Angular velocity*

Equating moments about G:

$$(-a\mathbf{i} - a\,\mathbf{j}) \times (-F\Delta t\,\mathbf{k}) = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

$$aF\Delta t\,\mathbf{i} - aF\Delta t\,\mathbf{j} = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

PROBLEM 18.23 (Continued)

Thus: $H_x = aF\Delta t, \quad H_y = -aF\Delta t, \quad H_z = 0$ (1)

To determine angular velocity, we shall use Eqs. (18.7).

First, we determine the moments & products of inertia:

$$\bar{I}_x = \frac{1}{12}\left(\frac{m}{2}\right)(2a)^2 + \frac{m}{4}a^2 + \frac{m}{4}a^2 = \frac{2}{3}ma^2 \tag{2}$$

$$\bar{I}_y = 2\left[\frac{1}{3}\left(\frac{m}{4}\right)a^2\right] = \frac{1}{6}ma^2 \tag{3}$$

$$\bar{I}_{xy} = \frac{m}{4}\left(\frac{a}{2}\right)(a) + \frac{m}{4}\left(-\frac{a}{2}\right)(-a) = +\frac{1}{4}ma^2 \tag{4}$$

$$\bar{I}_{xz} = 0 \qquad \bar{I}_{yz} = 0 \tag{5}$$

We substitute the expressions (1) through (5) into Eqs. (18.7):

$$aF\Delta t = \frac{2}{3}ma^2\omega_x - \frac{1}{4}ma^2\omega_y + 0 \tag{6}$$

$$-aF\Delta t = -\frac{1}{4}ma^2\omega_x + \frac{1}{6}ma^2\omega_y + 0 \tag{7}$$

$$0 = 0 + 0 + \bar{I}_z\omega_z \tag{8}$$

Multiplying Eq. (7) by 3/2 and adding to Eq. (6):

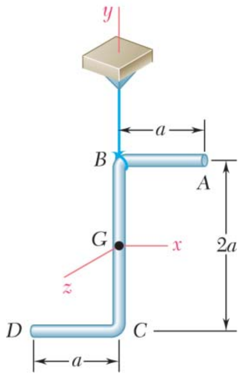
$$-\frac{1}{2}aF\Delta t = \frac{7}{24}ma^2\omega_x \qquad \omega_x = -\frac{12}{7}\frac{F\Delta t}{ma}$$

Substituting for ω into (7):

$$-aF\Delta t + \frac{1}{4}\left(-\frac{12^2}{7}\right)aF\Delta t = \frac{1}{6}ma^2\omega_y, \quad \omega_y = -\frac{60}{7}\frac{F\Delta t}{ma}$$

From Eq. (8): $\bar{I}_z\omega_z = 0 \quad \omega_z = 0$

Thus: $$\boldsymbol{\omega} = \frac{12}{7}\frac{F\Delta t}{ma}(-\mathbf{i} - 5\mathbf{j}) \blacktriangleleft$$


y
a
B
A
G
x
2a
z
D
C
a

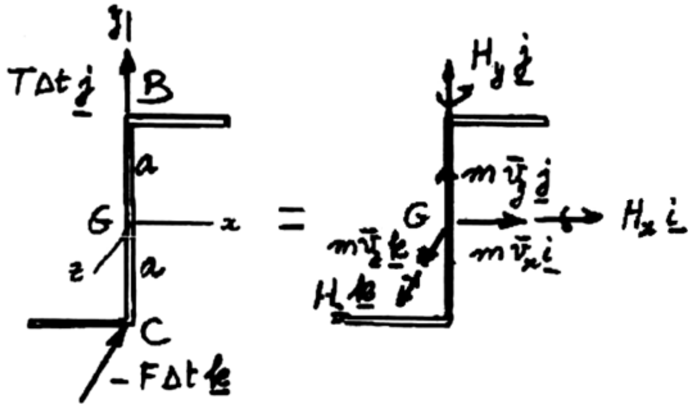
PROBLEM 18.24

Solve Problem 18.23, assuming that the bent rod is hit at *C*.

PROBLEM 18.23 A uniform rod of total mass *m* is bent into the shape shown and is suspended by a wire attached at *B*. The bent rod is hit at *D* in a direction perpendicular to the plane containing the rod (in the negative *z* direction). Denoting the corresponding impulse by $\mathbf{F}\Delta t$, determine (*a*) the velocity of the mass center of the rod, (*b*) the angular velocity of the rod.

SOLUTION

We apply the principle of impulse and momentum, consider only impulsive forces.


$\mathbf{j}$
$T\Delta t\,\mathbf{j}$
B
a
G
x
z
a
C
$-F\Delta t\,\mathbf{k}$
=
$H_y\mathbf{j}$
$m\bar{v}_y\mathbf{j}$
G
$H_x\mathbf{i}$
$m\bar{v}_z\mathbf{k}$
$m\bar{v}_x\mathbf{i}$
$H_z\mathbf{k}$

(*a*) *Velocity of mass center*

From constraints: $\bar{v}_y = 0$

$\xrightarrow{+}$ *x* components: $0 = m\bar{v}_x \qquad \bar{v}_x = 0$

$+\uparrow$ *y* components: $T\Delta t = m\bar{v}_y = 0 \qquad T\Delta t = 0$

$+\swarrow$ *z* components: $-F\Delta t = m\bar{v}_z \qquad \bar{v}_z = -\dfrac{F\Delta t}{m}$

$$\bar{\mathbf{v}} = \frac{F\Delta t}{m}\mathbf{k} \blacktriangleleft$$

(*b*) *Angular velocity*

Equating moments about *G*:

$$-a\,\mathbf{j}\times(-F\Delta t\,\mathbf{k}) = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

$$aF\Delta t\,\mathbf{i} = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

Thus: $H_x = aF\Delta t, \quad H_y = 0, \quad H_z = 0 \qquad (1)$

PROBLEM 18.24 (Continued)

To determine angular velocity, we shall use Eqs. (18.7) first, we determine the moments & products of inertia:

$$\bar{I}_x = \frac{1}{12}\left(\frac{m}{2}\right)(2a)^2 + \frac{m}{4}a^2 + \frac{m}{4}a^2 = \frac{2}{3}ma^2 \tag{2}$$

$$\bar{I}_y = 2\left[\frac{1}{3}\left(\frac{m}{4}\right)a^2\right] = \frac{1}{6}ma^2 \tag{3}$$

$$\bar{I}_{xy} = \frac{m}{4}\left(\frac{a}{2}\right)(a) + \frac{m}{4}\left(-\frac{a}{2}\right)(-a) = +\frac{1}{4}ma^2 \tag{4}$$

$$\bar{I}_{xz} = 0 \qquad \bar{I}_{yz} = 0 \tag{5}$$

$$aF\Delta t = \frac{2}{3}ma^2\omega_x - \frac{1}{4}ma^2\omega_y + 0 \tag{6}$$

$$0 = -\frac{1}{4}ma^2\omega_x + \frac{1}{6}ma^2\omega_y + 0 \tag{7}$$

$$0 = 0 + 0 + \bar{I}_z\omega_z \tag{8}$$

Multiplying Eq. (7) by 3/2 and adding to Eq. (6):

$$aF\Delta t = \left(\frac{2}{3} - \frac{3}{8}\right)ma^2\omega_x, \qquad \omega_x = \frac{24}{7}\frac{F\Delta t}{ma}$$

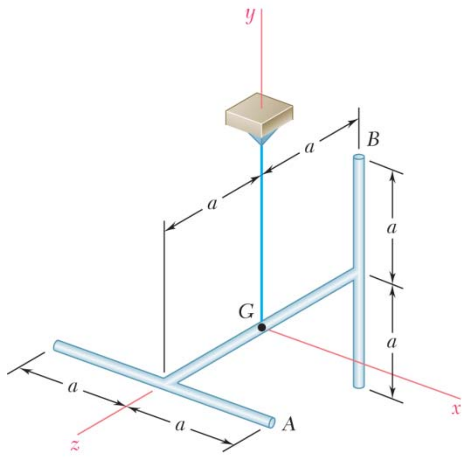
Substituting for ω_x into (7):

$$\omega_y = \frac{3}{2}\omega_x = \frac{3}{2}\frac{24}{7}aF\Delta t, \quad \omega_y = \frac{36}{7}\frac{F\Delta t}{ma}$$

From Eq. (8): $\bar{I}_z\omega_z = 0, \quad \omega_z = 0$

Thus: $\boldsymbol{\omega} = \frac{12}{7}\frac{F\Delta t}{ma}(2\mathbf{i} + 3\mathbf{j})$ ◀

Note that $\omega_y \neq 0$, even though Point *C* where impulse is applied is on the *y* axis.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PROBLEM 18.25

Three slender rods, each of mass *m* and length 2*a*, are welded together to form the assembly shown. The assembly is hit at *A* in a vertical downward direction. Denoting the corresponding impulse by $\mathbf{F}\,\Delta t,$ determine immediately after the impact (*a*) the velocity of the mass center *G*, (*b*) the angular velocity of the rod.

SOLUTION

Computation of moments and products of inertia.

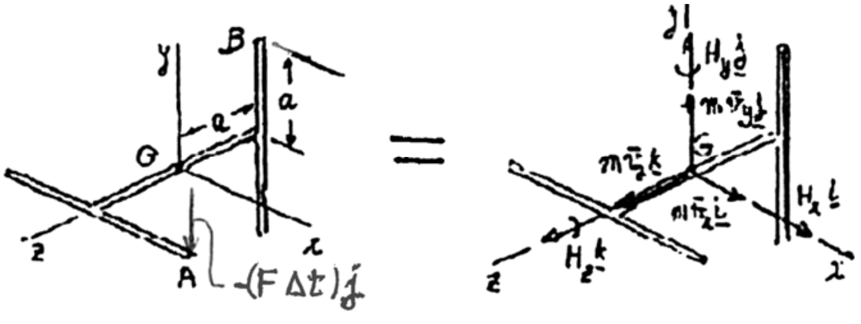
$$\bar{I}_x = (I_x)_1 + (I_x)_2 + (I_x)_3 = ma^2 + \frac{1}{3}ma^2 + m\left(a^2 + \frac{a^2}{3}\right) = \frac{8}{3}ma^2$$

$$\bar{I}_y = (I_y)_1 + (I_y)_2 + (I_y)_3 = m\left(a^2 + \frac{a^2}{3}\right) + \frac{1}{3}ma^2 + ma^2 = \frac{8}{3}ma^2 \qquad (1)$$

$$\bar{I}_z = (I_z)_1 + (I_z)_2 + (I_z)_3 = \frac{1}{3}ma^2 + 0 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$$

$$\bar{I}_{xy} = 0, \quad \bar{I}_{yz} = 0, \quad I_{zx} = 0$$

Impulse-momentum principle.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$-(F\Delta t)\mathbf{j}$
=
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$H_y\mathbf{j}$
$m\bar{v}_y\mathbf{j}$
$m\bar{v}_z\mathbf{k}$
G
$m\bar{v}_x\mathbf{i}$
$H_x\mathbf{i}$
$H_z\mathbf{k}$
z
x

The impulses consist of $-(F\Delta t)\mathbf{j}$ applied at *A* and $(T\Delta t)\mathbf{j}$ applied at *G*. Because of constraints, $\bar{v}_y = 0.$

(*a*) Velocity of mass center.

Equate sums of vectors: $(T\Delta t)\mathbf{j} - (F\Delta t)\mathbf{j} = m\bar{v}_x\mathbf{i} + m\bar{v}_z\mathbf{k}$

Thus, $T\Delta t = F\Delta t, \quad v_x = 0, \quad v_z = 0.$ Since $v_y = 0$ from above, $\bar{\mathbf{v}} = 0$ ◀

PROBLEM 18.25 (Continued)

(*b*) Angular velocity.

Equate moments about G:

$$(a\mathbf{i}+a\mathbf{k})\times(-F\Delta t)\mathbf{j}=H_x\mathbf{i}+H_y\mathbf{j}+H_z\mathbf{k}$$

$$-(aF\Delta t)\mathbf{k}+(aF\Delta t)\mathbf{i}=H_x\mathbf{i}+H_y\mathbf{j}+H_z\mathbf{k}$$

Thus, $$H_x=aF\Delta t,\quad H_y=0,\quad H_z=-aF\Delta t \qquad (2)$$

Since the three products of inertia are zero, the x, y, and z axes are principal centroidal axes and we can use Eqs. (18.10). Substituting from Eqs. (1) and (2) into these equations, we have

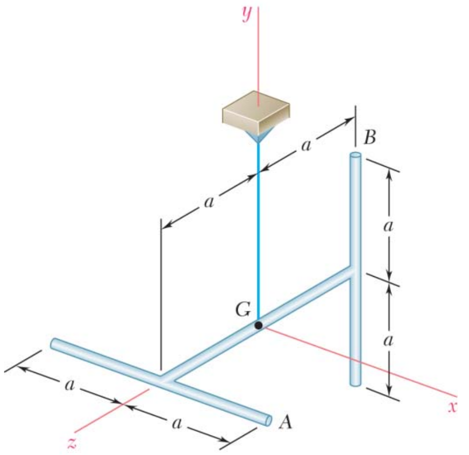
$$H_x=\bar{I}_x\omega_x:\quad aF\Delta t=\frac{8}{3}ma^2\omega_x\quad \omega_x=3F\Delta t/8ma \qquad (3)$$

$$H_y=\bar{I}_y\omega_y:\quad 0=\frac{8}{3}ma^2\omega_y\quad \omega_y=0 \qquad (4)$$

$$H_z=\bar{I}_z\omega_z:\quad -aF\Delta t=\frac{2}{3}ma^2\quad \omega_z=-3F\Delta t/2ma \qquad (5)$$

Therefore,

$$\boldsymbol{\omega}=(3F\Delta t/8ma)(\mathbf{i}-4\mathbf{k})\ ◀$$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PROBLEM 18.26

Solve Problem 18.25, assuming that the assembly is hit at *B* in the negative x direction.

PROBLEM 18.25 Three slender rods, each of mass *m* and length 2*a*, are welded together to form the assembly shown. The assembly is hit at *A* in a vertical downward direction. Denoting the corresponding impulse by $\mathbf{F}\Delta t$, determine immediately after the impact (*a*) the velocity of the mass center *G*, (*b*) the angular velocity of the rod.

SOLUTION

Computation of moments and products of inertia.

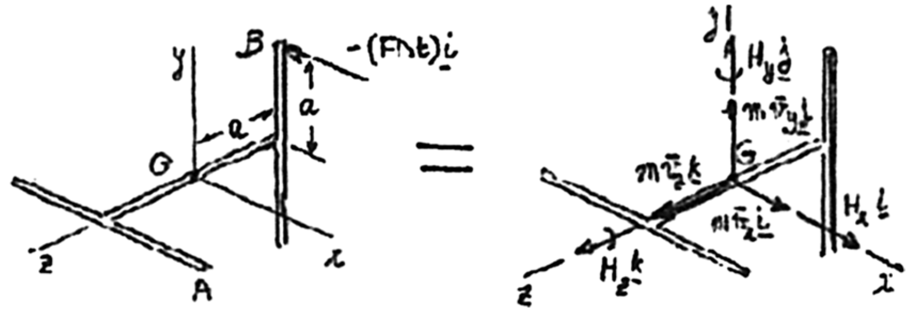
$$\bar{I}_x = (I_x)_1 + (I_x)_2 + (I_x)_3 = ma^2 + \frac{1}{3}ma^2 + m\left(a^2 + \frac{a^2}{3}\right) = \frac{8}{3}ma^2$$

$$\bar{I}_y = (I_y)_1 + (I_y)_2 + (I_y)_3 = m\left(a^2 + \frac{a^2}{3}\right) + \frac{1}{3}ma^2 + ma^2 = \frac{8}{3}ma^2 \qquad (1)$$

$$\bar{I}_z = (I_z)_1 + (I_z)_2 + (I_z)_3 = \frac{1}{3}ma^2 + 0 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$$

$$\bar{I}_{xy} = \bar{I}_{yz} = \bar{I}_{zx} = 0$$

Impulse-momentum principle.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=
y
$H_y\mathbf{j}$
$m\bar{v}_y\mathbf{j}$
G
$m\bar{v}_z\mathbf{k}$
$m\bar{v}_x\mathbf{i}$
$H_x\mathbf{i}$
z
$H_z\mathbf{k}$
x

The only impulse is $F\Delta t = -(F\Delta t)\mathbf{i}$.

(*a*) Velocity of mass center.

Equate sums of vectors: $-(F\Delta t)\mathbf{i} = m\bar{v}_x\mathbf{i} + mv_y\mathbf{j} + mv_z\mathbf{k}$

Thus, $v_x = -F\Delta t/m, \quad v_y = 0, \quad v_z = 0$

$\bar{\mathbf{v}} = -(F\Delta t/m)\mathbf{i}$ ◀

PROBLEM 18.26 (Continued)

(*b*) Angular velocity.

Equate moments about G:

$$(a\mathbf{j} - a\mathbf{k}) \times (-F\Delta t)\mathbf{i} = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

$$(aF\Delta t)\mathbf{k} + (aF\Delta t)\mathbf{j} = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

Thus, $$H_x = 0, \quad H_y = aF\Delta t, \quad H_z = aF\Delta t \tag{2}$$

Since the three products of inertia are zero, the x, y, and z axes are principal centroidal axes and we can use Eqs. (18.10). Substituting from Eqs. (1) and (2) into these equations, we have

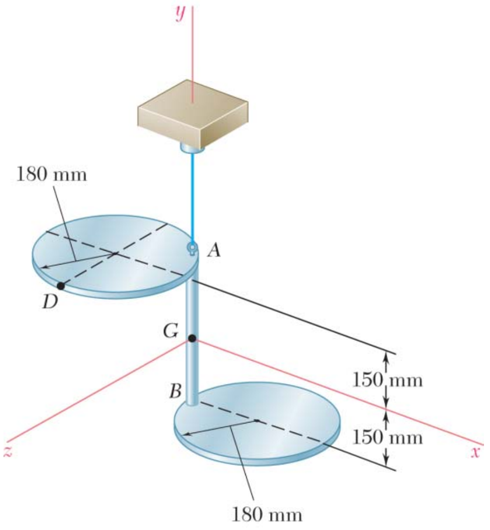
$$H_x = \bar{I}_x\omega_x: \qquad 0 = \frac{8}{3}ma^2\omega_x \quad \omega_x = 0 \tag{3}$$

$$H_y = \bar{I}_y\omega_y: \qquad aF\Delta t = \frac{8}{3}ma^2\omega_y \quad \omega_y = 3F\Delta t/8ma \tag{4}$$

$$H_z = \bar{I}_z\omega_z: \qquad aF\Delta t = \frac{2}{3}ma^2\omega_z \quad \omega_z = 3F\Delta t/2ma \tag{5}$$

Therefore,

$$\boldsymbol{\omega} = (3F\Delta t/8ma)(\mathbf{j} + 4\mathbf{k}) \blacktriangleleft$$



PROBLEM 18.27

Two circular plates, each of mass 4 kg, are rigidly connected by a rod *AB* of negligible mass and are suspended from Point *A* as shown. Knowing that an impulse $\mathbf{F}\,\Delta t = -(2.4\ \text{N}\cdot\text{s})\mathbf{k}$ is applied at Point *D*, determine (*a*) the velocity of the mass center *G* of the assembly, (*b*) the angular velocity of the assembly.

SOLUTION

Moments and products of inertia:

$$I_x = 2\left(\frac{1}{4}mr^2 + mb^2\right) = 2\left[\frac{1}{4}(4)(0.18)^2 + (4)(0.15)^2\right] = 0.2448\ \text{kg}\cdot\text{m}^2$$

$$I_y = 2\left(\frac{1}{2}mr^2 + mr^2\right) = 3mr^2 = (3)(4)(0.18)^2 = 0.3888\ \text{kg}\cdot\text{m}^2$$

$$I_z = 2\left[\frac{1}{4}mr^2 + m(b^2 + r^2)\right]$$

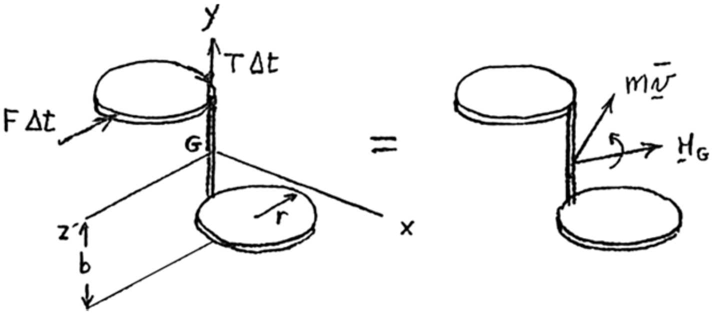
$$= 2\left[\frac{1}{4}(4)(0.18)^2 + (4)(0.15^2 + 0.18^2)\right] = 0.504\ \text{kg}\cdot\text{m}^2$$

$$I_{xy} = mb(-r) + m(-b)(r) = -2mbr = -(2)(4)(0.15)(0.18) = -0.216\ \text{kg}\cdot\text{m}^2$$

$$I_{xz} = 0, \quad I_{yz} = 0, \qquad \text{total mass} = 2m$$

Constraint of cable: $(v_G)_y = 0 \quad \bar{\mathbf{v}} = \mathbf{v}_G$

Principle of impulse-momentum: $2m\bar{\mathbf{v}} = 0, \quad \mathbf{H}_G = 0$ initially.



PROBLEM 18.27 (Continued)

(*a*) Direct components:

$$\mathbf{F}\Delta t = 2m\overline{\mathbf{v}}$$

$$0 = 2m\overline{v}_x \qquad \overline{v}_x = 0$$

$$T\Delta t = 2m\overline{v}_y = 0$$

$$-F\Delta t = 2m\overline{v}_z$$

$$\overline{v}_z = -\frac{F\Delta t}{2m} = -\frac{2.4}{(2)(4)} = -0.3 \text{ m/s} \qquad \overline{\mathbf{v}} = -(0.300 \text{ m/s})\mathbf{k} \blacktriangleleft$$

(*b*) Moments about *G*. $(b\mathbf{j} - r\mathbf{i}) \times (-F\Delta t\mathbf{k}) = \mathbf{H}_G$

$$-bF(\Delta t)\mathbf{i} - rF\Delta t\mathbf{j} = (I_x\omega_x - I_{xy}\omega_y)\mathbf{i} + (-I_{xy}\omega_x + I_y\omega_y)\mathbf{j} + I_z\omega_z\mathbf{k}$$

Resolve into components and apply the numerical data.

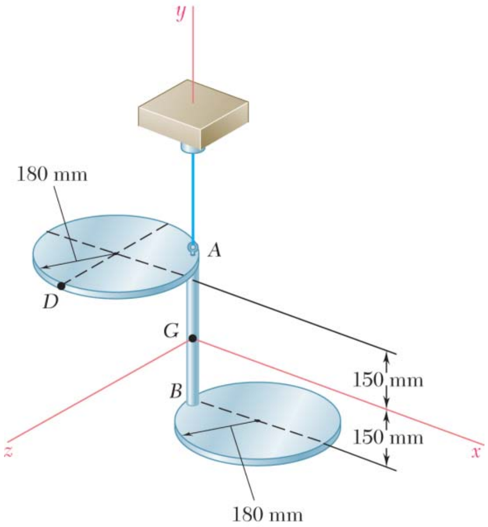
$$\mathbf{i}: \quad -(0.15)(2.4) = 0.2448\omega_x - (-0.216)\omega_y \tag{1}$$

$$\mathbf{j}: \quad -(0.18)(2.4) = -(-0.216)\omega_x + 0.3888\omega_y \tag{2}$$

$$\mathbf{k}: \quad 0 = 0.504\omega_z \qquad \omega_z = 0$$

Solving Eqs. (1) and (2), $\omega_x = -0.962 \text{ rad/s}, \quad \omega_y = -0.577 \text{ rad/s}$

$$\boldsymbol{\omega} = -(0.962 \text{ rad/s})\mathbf{i} - (0.577 \text{ rad/s})\mathbf{j} \blacktriangleleft$$



PROBLEM 18.28

Two circular plates, each of mass 4 kg, are rigidly connected by a rod *AB* of negligible mass and are suspended from Point *A* as shown. Knowing that an impulse $\mathbf{F}\Delta t = (2.4\ \text{N}\cdot\text{s})\mathbf{j}$ is applied at Point *D*, determine (*a*) the velocity of the mass center *G* of the assembly, (*b*) the angular velocity of the assembly.

SOLUTION

Moments and products of inertia:

$$I_x = 2\left(\frac{1}{4}mr^2 + mb^2\right) = 2\left[\frac{1}{4}(4)(0.18)^2 + (4)(0.15)^2\right] = 0.2448\ \text{kg}\cdot\text{m}^2$$

$$I_y = 2\left(\frac{1}{2}mr^2 + mr^2\right) = 3mr^2 = (3)(4)(0.18)^2 = 0.3888\ \text{kg}\cdot\text{m}^2$$

$$I_z = 2\left[\frac{1}{4}mr^2 + m(b^2 + r^2)\right]$$

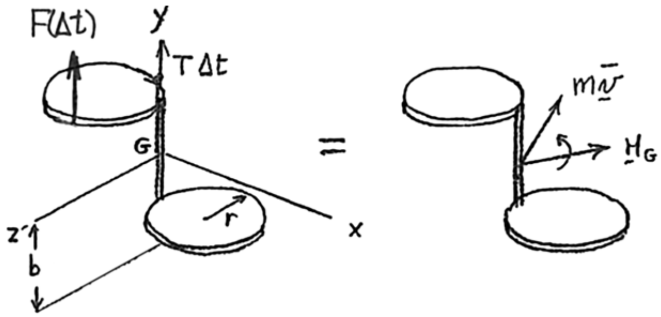
$$= 2\left[\frac{1}{4}(4)(0.18)^2 + (4)(0.15^2 + 0.18^2)\right] = 0.504\ \text{kg}\cdot\text{m}^2$$

$$I_{xy} = mb(-r) + m(-b)(r) = -2mbr = -(2)(4)(0.15)(0.18) = -0.216\ \text{kg}\cdot\text{m}^2$$

$$I_{xz} = 0, \quad I_{yz} = 0, \qquad \text{total mass} = 2m$$

Constraint of cable: $(v_G)_y = 0 \quad \overline{\mathbf{v}} = \mathbf{v}_G$

Principle of impulse-momentum: $2m\overline{\mathbf{v}} = 0, \quad \mathbf{H}_G = 0$ initially.



PROBLEM 18.28 (Continued)

(*a*) Direct components:

$$\mathbf{F}\Delta t = 2m\overline{\mathbf{v}}$$

$$0 = 2m\overline{v}_x \qquad \overline{v}_x = 0$$

$$F(\Delta t) + T\Delta t = 2m\overline{v}_y = 0 \qquad \overline{v}_y = \frac{F(\Delta t) + T(\Delta t)}{2m}$$

$$0 = 2m\overline{v}_z \quad v_z = 0$$

Point *A* moves upward. The cord becomes slack. $T(\Delta t) = 0$

$$v_y = \frac{2.4}{(2)(4)} = 0.300 \text{ m/s} \qquad \overline{\mathbf{v}} = (0.300 \text{ m/s})\mathbf{j} \blacktriangleleft$$

(*b*) Moments about *G*. $(b\mathbf{j} - r\mathbf{i}) \times (-F\Delta t\mathbf{j}) = \mathbf{H}_G$

$$-rF(\Delta t)\mathbf{i} - rF(\Delta t)\mathbf{k} = (I_x\omega_x - I_{xy}\omega_y)\mathbf{i} + (-I_{xy}\omega_x + I_y\omega_y)\mathbf{j} + I_z\omega_z\mathbf{k}$$

Resolve into components and apply the numerical data.

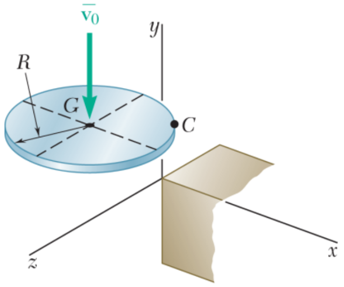
$$\mathbf{i}: \quad -(0.18)(2.4) = 0.2448\omega_x - (-0.216)\omega_y \tag{1}$$

$$\mathbf{j}: \quad 0 = -(-0.216)\omega_x + 0.3888\omega_y \tag{2}$$

$$\mathbf{k}: \quad -(0.18)(2.4) = 0.504\omega_z \quad \omega_z = -0.857 \text{ rad/s}$$

Solving Eqs. (1) and (2), $\omega_x = -3.46 \text{ rad/s}, \quad \omega_y = -1.923 \text{ rad/s}$

$$\boldsymbol{\omega} = -(3.46 \text{ rad/s})\mathbf{i} + (1.923 \text{ rad/s})\mathbf{j} - (0.857 \text{ rad/s})\mathbf{k} \blacktriangleleft$$



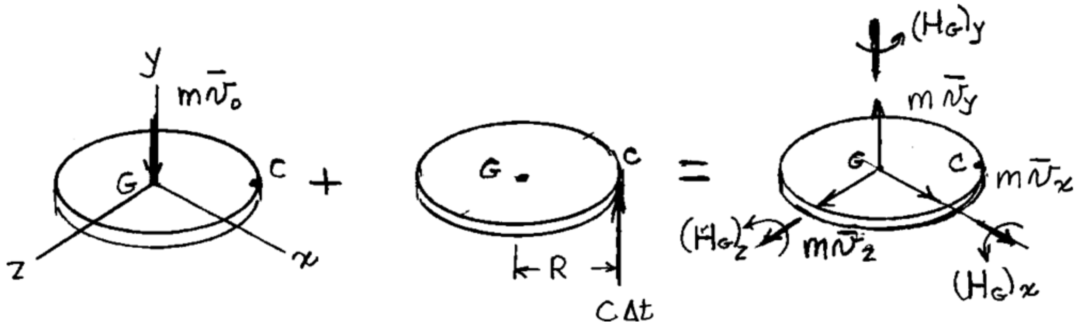
PROBLEM 18.29

A circular plate of mass m is falling with a velocity $\overline{\mathbf{v}}_0$ and no angular velocity when its edge C strikes an obstruction. Assuming the impact to be perfectly plastic $(e = 0)$, determine the angular velocity of the plate immediately after the impact.

SOLUTION

Principal moments of inertia. $\quad \overline{I}_y = \frac{1}{2}mR^2, \quad \overline{I}_x = \overline{I}_z = \frac{1}{4}mR^2$

Principle of impulse and momentum.



Syst. Momenta$_1$ + **Syst. Ext. Imp.$_{1\to2}$** = **Syst. Momenta$_2$**

Linear momentum: $\quad -mv_0\mathbf{j} + C\Delta t\,\mathbf{j} = m\bar{v}_x\mathbf{i} + m\bar{v}_y\mathbf{j} + m\bar{v}_z\mathbf{k}$

i: $\quad 0 = m\bar{v}_x \qquad \bar{v}_x = 0$

j: $\quad -m\overline{\mathbf{v}}_0 + C\Delta t = m\bar{v}_y \qquad C\Delta t = m(\bar{v}_0 + \bar{v}_y) \qquad (1)$

k: $\quad 0 = m\bar{v}_z \qquad \bar{v}_z = 0$

Geometry: $\quad \mathbf{r}_{C/G} = \frac{1}{\sqrt{2}}R(\mathbf{i} - \mathbf{k})$

Condition of impact: $\quad (e = 0) \qquad (v_C)_y = 0$

Kinematics: $\quad \overline{\mathbf{v}}_C = \overline{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{r}_{C/G}$

$$(v_C)_x\mathbf{i} + (v_C)_z\mathbf{k} = \bar{v}_y\mathbf{j} + (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times \frac{R}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$$

$$= \bar{v}_y\mathbf{j} + \frac{R}{\sqrt{2}}\omega_x\mathbf{j} - \frac{R}{\sqrt{2}}\omega_y(\mathbf{k} + \mathbf{i}) + \frac{R}{\sqrt{2}}\omega_z\mathbf{j}$$

j: $\quad 0 = \bar{v}_y + \frac{R}{\sqrt{2}}(\omega_x + \omega_z) \qquad \bar{v}_y = -\frac{R}{\sqrt{2}}(\omega_x + \omega_z)$

PROBLEM 18.29 (Continued)

Moments about G:

$$0+\mathbf{r}_{C/G}\times C\Delta t\,\mathbf{j}=(H_G)_x\,\mathbf{i}+(H_G)_y\,\mathbf{j}+(H_G)_z\,\mathbf{k}$$

$$\frac{R}{\sqrt{2}}(\mathbf{i}-\mathbf{k})\times(C\Delta t\,\mathbf{j})=\bar{I}_x\omega_x\mathbf{i}+\bar{I}_y\omega_y\mathbf{j}+I_z\omega_z\mathbf{k}$$

$$\frac{R}{\sqrt{2}}(C\Delta t)(\mathbf{k}+\mathbf{i})=\frac{1}{4}mR^2\omega_x\mathbf{i}+\frac{1}{2}mR^2\omega_y\mathbf{j}+\frac{1}{4}mR^2\omega_z\mathbf{k} \qquad (2)$$

i: $$\frac{R}{\sqrt{2}}C\Delta t=\frac{1}{4}mR^2\omega_x \qquad (3)$$

j: $$0=\frac{1}{2}mR^2\omega_y \qquad \omega_y=0$$

k: $$\frac{R}{\sqrt{2}}C\Delta t=\frac{1}{4}mR^2\omega_z \qquad (4)$$

From Eqs. (1) and (2), $$C\Delta t=m\left[\bar{v}_0-\frac{R}{\sqrt{2}}(\omega_x+\omega_z)\right]$$

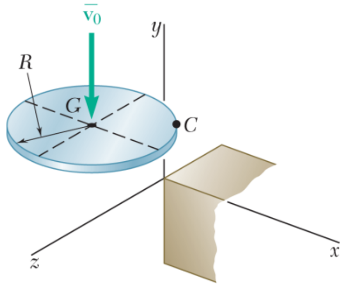
$$\omega_x=2\sqrt{2}\,\frac{\bar{v}_0}{R}-2(\omega_x+\omega_z) \qquad 3\omega_x+2\omega_z=2\sqrt{2}\,\frac{\bar{v}_0}{R} \qquad (5)$$

$$\omega_z=2\sqrt{2}\,\frac{\bar{v}_0}{R}-2(\omega_x+\omega_z) \qquad 2\omega_x+3\omega_z=2\sqrt{2}\,\frac{\bar{v}_0}{R} \qquad (6)$$

Solving Eqs. (5) and (6) simultaneously,

$$\omega_x=\omega_z=\frac{2\sqrt{2}}{5}\frac{\bar{v}_0}{R}$$

Angular velocity. $$\boldsymbol{\omega}=\omega_x\mathbf{i}+\omega_y\mathbf{j}+\omega_z\mathbf{k}$$ $$\boldsymbol{\omega}=\frac{2\sqrt{2}}{5}\frac{v_0}{R}(\mathbf{i}+\mathbf{k})$$ ◀



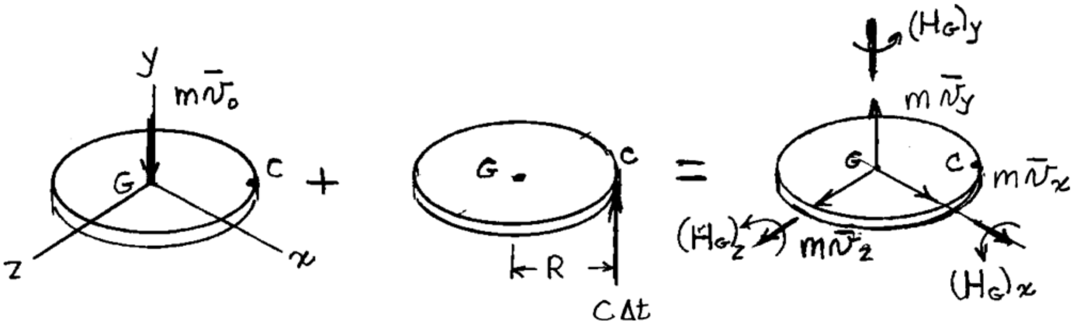
PROBLEM 18.30

For the plate of Problem 18.29, determine (*a*) the velocity of its mass center *G* immediately after the impact, (*b*) the impulse exerted on the plate by the obstruction during the impact.

SOLUTION

Principal moments of inertia. $\overline{I}_y = \frac{1}{2}mR^2, \quad \overline{I}_x = \overline{I}_z = \frac{1}{4}mR^2$

Principle of impulse and momentum.



Syst. Momenta$_1$ + **Syst. Ext. Imp.$_{1\to2}$** = **Syst. Momenta$_2$**

Linear momentum: $-mv_0\mathbf{j} + C\Delta t\,\mathbf{j} = m\overline{v}_x\mathbf{i} + m\overline{v}_y\mathbf{j} + m\overline{v}_z\mathbf{k}$

i: $0 = m\overline{v}_x \qquad \overline{v}_x = 0$

j: $-m\overline{v}_0 + C\Delta t = m\overline{v}_y \qquad C\Delta t = m(\overline{v}_0 + \overline{v}_y)$ (1)

k: $0 = m\overline{v}_z \qquad \overline{v}_z = 0$

Geometry: $\mathbf{r}_{C/G} = \frac{1}{\sqrt{2}}R(\mathbf{i} - \mathbf{k})$

Condition of impact: $(e = 0) \qquad (v_C)_y = 0$

Kinematics: $\overline{\mathbf{v}}_C = \overline{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{r}_{C/G}$

$$(v_C)_x\mathbf{i} + (v_C)_z\mathbf{k} = \overline{v}_y\mathbf{j} + (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times \frac{R}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$$

$$= \overline{v}_y\mathbf{j} + \frac{R}{\sqrt{2}}\omega_x\mathbf{j} - \frac{R}{\sqrt{2}}\omega_y(\mathbf{k} + \mathbf{i}) + \frac{R}{\sqrt{2}}\omega_z\mathbf{j}$$

j: $0 = \overline{v}_y + \frac{R}{\sqrt{2}}(\omega_x + \omega_z) \qquad \overline{v}_y = -\frac{R}{\sqrt{2}}(\omega_x + \omega_z)$

PROBLEM 18.30 (Continued)

Moments about G:

$$0 + \mathbf{r}_{C/G} \times C\Delta t\,\mathbf{j} = (H_G)_x\mathbf{i} + (H_G)_y\mathbf{j} + (H_G)_z\mathbf{k}$$

$$\frac{R}{\sqrt{2}}(\mathbf{i}-\mathbf{k}) \times (C\Delta t\,\mathbf{j}) = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$\frac{R}{\sqrt{2}}(C\Delta t)(\mathbf{k}+\mathbf{i}) = \frac{1}{4}mR^2\omega_x\mathbf{i} + \frac{1}{2}mR^2\omega_y\mathbf{j} + \frac{1}{4}mR^2\omega_z\mathbf{k} \qquad (2)$$

i: $$\frac{R}{\sqrt{2}}C\Delta t = \frac{1}{4}mR^2\omega_x \qquad (3)$$

j: $$0 = \frac{1}{2}mR^2\omega_y \qquad \omega_y = 0$$

k: $$\frac{R}{\sqrt{2}}C\Delta t = \frac{1}{4}mR^2\omega_z \qquad (4)$$

From Eqs. (1) and (2), $$C\Delta t = m\left[\bar{v}_0 - \frac{R}{\sqrt{2}}(\omega_x + \omega_z)\right]$$

$$\omega_x = 2\sqrt{2}\frac{\bar{v}_0}{R} - 2(\omega_x + \omega_z) \qquad 3\omega_x + 2\omega_z = 2\sqrt{2}\frac{\bar{v}_0}{R} \qquad (5)$$

$$\omega_z = 2\sqrt{2}\frac{\bar{v}_0}{R} - 2(\omega_x + \omega_z) \qquad 2\omega_x + 3\omega_z = 2\sqrt{2}\frac{\bar{v}_0}{R} \qquad (6)$$

Solving Eqs. (5) and (6) simultaneously,

$$\omega_x = \omega_z = \frac{2\sqrt{2}}{5}\frac{\bar{v}_0}{R}$$

(*a*) *Velocity of the mass center*.

From Eq. (2), $$\bar{v}_y = -\frac{R}{\sqrt{2}}(\omega_x + \omega_z) = -\frac{4}{5}\bar{v}_0$$

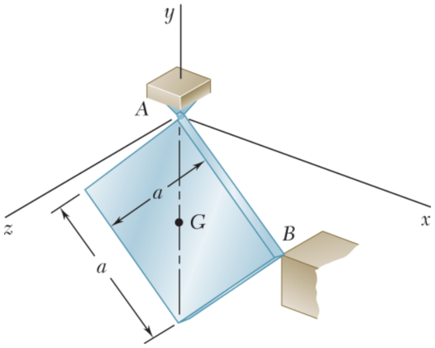
$$\bar{\mathbf{v}} = \bar{v}_x\mathbf{i} + \bar{v}_y\mathbf{j} + \bar{v}_z\mathbf{k}$$

$$\bar{\mathbf{v}} = -\frac{4}{5}\bar{v}_0\mathbf{j}$$ ◀

(*b*) *Impulse at C*.

From Eq. (1), $$C\Delta t = m\left(\bar{v}_0 - \frac{4}{5}\bar{v}_0\right) = \frac{1}{5}m\bar{v}_0$$

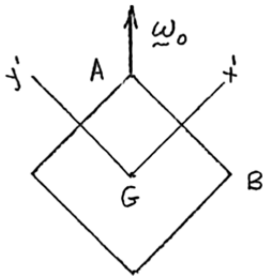
$$\mathbf{C}\Delta t = \frac{1}{5}m\bar{v}_0\mathbf{j}$$ ◀



PROBLEM 18.31

A square plate of side a and mass m supported by a ball-and-socket joint at A is rotating about the y axis with a constant angular velocity $\boldsymbol{\omega} = \omega_0 \mathbf{j}$ when an obstruction is suddenly introduced at B in the xy plane. Assuming the impact at B to be perfectly plastic $(e = 0)$, determine immediately after impact (a) the angular velocity of the plate, (b) the velocity of its mass center G.

SOLUTION



For the x' and y' axes shown, the initial angular velocity $\omega_0 \mathbf{j}$ has components

$$\omega_{x'} = \frac{\sqrt{2}}{2}\omega_0, \qquad \omega_{y'} = \frac{\sqrt{2}}{2}\omega_0,$$

Initial angular momentum about the mass center:

$$(\mathbf{H}_G)_0 = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' = \frac{1}{12}ma^2\frac{\sqrt{2}}{2}\omega_0(\mathbf{i}' + \mathbf{j}')$$

Initial velocity of the mass center: $\bar{\mathbf{v}}_0 = 0$

Let $\boldsymbol{\omega}$ be the angular velocity and $\bar{\mathbf{v}}$ be the velocity of the mass center immediately after impact.
Let $(F\Delta t)\mathbf{k}$ be the impulse at B.

Kinematics: $\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = (\omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times (-a\mathbf{j}')$

$$\mathbf{v}_B = a(\omega_{z'}\mathbf{i}' + \omega_{x'}\mathbf{k}')$$

Since the corner B does not rebound, $(v_B)_{z'} = 0$ or $\omega_{x'} = 0$

$$\bar{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{r}_{G/A} = (\omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times \left(\frac{1}{2}a\right)(-\mathbf{i}' - \mathbf{j}')$$

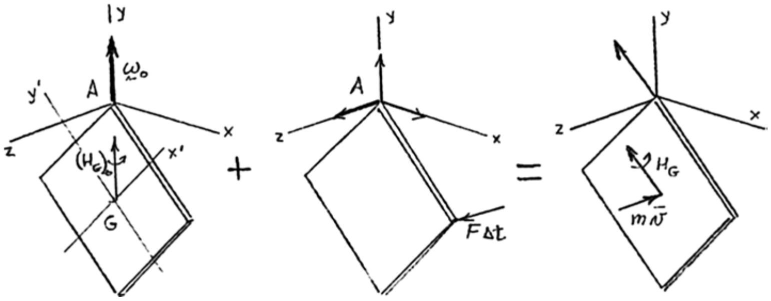
$$= \frac{1}{2}a(\omega_{z'}\mathbf{i}' - \omega_{z'}\mathbf{j} + \omega_{y}\mathbf{k}')$$

Also, $\mathbf{r}_{G/A} \times m\bar{\mathbf{v}} = \frac{1}{4}ma^2(-\omega_{y'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + 2\omega_{z'}\mathbf{k}')$

and $\mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_{z'}\omega_{z'}\mathbf{k}' = \frac{1}{12}ma^2\omega_{y'}\mathbf{j}' + \frac{1}{6}ma^2\omega_{z'}\mathbf{k}'$

PROBLEM 18.31 (Continued)

Principle of impulse-momentum.



Moments about A: $$(\mathbf{H}_A)_0 + (-a\mathbf{j}) \times (F\Delta t)\mathbf{k} = \mathbf{H}_A$$

$$(\mathbf{H}_G)_0 + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}}_0 - (aF\Delta t)\mathbf{i} = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}}$$

Resolve into components.

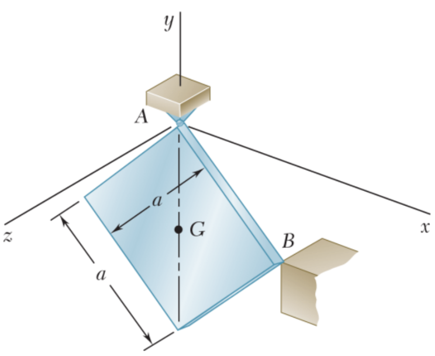
$$\mathbf{i}': \quad \frac{1}{24}\sqrt{2} m a^2 \omega_0 - aF(\Delta t) = -\frac{1}{4} m a^2 \omega_{y'}$$

$$\mathbf{j}': \quad \frac{1}{24}\sqrt{2} m a^2 \omega_0 = \frac{1}{12} m a^2 \omega_{y'} + \frac{1}{4} m a^2 \omega_{y'} \qquad \omega_{y'} = \frac{\sqrt{2}}{8}\omega_0$$

$$\mathbf{k}': \quad 0 = \frac{1}{6} m a^2 \omega_{z'} + \frac{1}{2} m a^2 \omega_{z'} \qquad \omega_{z'} = 0$$

(*a*) $$\boldsymbol{\omega} = \frac{\sqrt{2}}{8}\omega_0 \mathbf{j}' = \frac{1}{8}\sqrt{2}\omega_0 \frac{\sqrt{2}}{2}(\mathbf{j} - \mathbf{i})$$ $$\boldsymbol{\omega} = \frac{1}{8}\omega_0(-\mathbf{i} + \mathbf{j}) \quad ◀$$

(*b*) $$\bar{\mathbf{v}} = \frac{1}{2} a\omega_y \mathbf{k}' = \frac{\sqrt{2}}{16} a\omega_0 \mathbf{k}$$ $$\bar{\mathbf{v}} = 0.0884 a\omega_0 \mathbf{k} \quad ◀$$

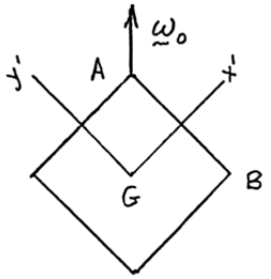


PROBLEM 18.32

Determine the impulse exerted on the plate of Problem 18.31 during the impact by (*a*) the obstruction at *B*, (*b*) the support at *A*.

PROBLEM 18.31 A square plate of side *a* and mass *m* supported by a ball-and-socket joint at *A* is rotating about the *y* axis with a constant angular velocity $\boldsymbol{\omega} = \omega_0 \mathbf{j}$ when an obstruction is suddenly introduced at *B* in the *xy* plane. Assuming the impact at *B* to be perfectly plastic $(e = 0)$, determine immediately after impact (*a*) the angular velocity of the plate, (*b*) the velocity of its mass center *G*.

SOLUTION



For the simpler x' and y' axes, the initial angular velocity $\omega_0 \mathbf{j}$ has components

$$\omega_{x'} = \frac{\sqrt{2}}{2}\omega_0, \qquad \omega_{y'} = \frac{\sqrt{2}}{2}\omega_0,$$

Initial angular momentum about the mass center:

$$(\mathbf{H}_G)_0 = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' = \frac{1}{12}ma^2\frac{\sqrt{2}}{2}\omega_0(\mathbf{i}' + \mathbf{j}')$$

Initial velocity of the mass center: $\bar{\mathbf{v}}_0 = 0$

Let $\boldsymbol{\omega}$ be the angular velocity and $\bar{\mathbf{v}}$ be the velocity of the mass center immediately after impact.
Let $(F\Delta t)\mathbf{k}$ be the impulse at *B*.

Kinematics:

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = (\omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times (-a\mathbf{j}')$$

$$\mathbf{v}_B = a(\omega_{z'}\mathbf{i}' + \omega_{x'}\mathbf{k}')$$

Since the corner *B* does not rebound, $(v_B)_{z'} = 0 \quad \text{or} \quad \omega_{x'} = 0$

$$\bar{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{r}_{G/A} = (\omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times \left(\frac{1}{2}a\right)(-\mathbf{i}' - \mathbf{j}')$$

$$= \frac{1}{2}a(\omega_{z'}\mathbf{i}' - \omega_{z'}\mathbf{j} + \omega_y\mathbf{k}')$$

Also,

$$\mathbf{r}_{G/A} \times m\bar{\mathbf{v}} = \frac{1}{4}ma^2(-\omega_{y'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + 2\omega_{z'}\mathbf{k}')$$

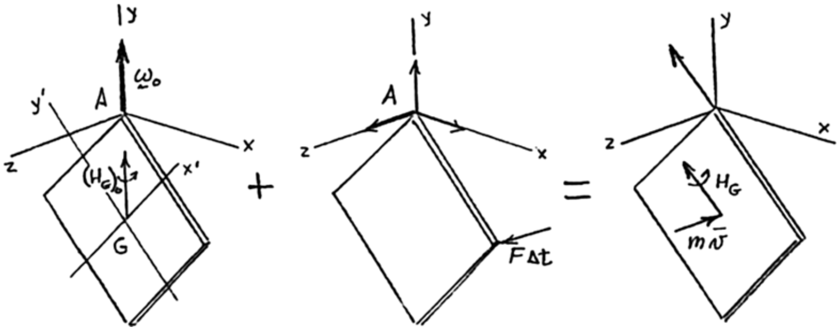
and

$$\mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_{z'}\omega_{z'}\mathbf{k}'$$

$$= \frac{1}{12}ma^2\omega_{y'}\mathbf{j}' + \frac{1}{6}ma^2\omega_{z'}\mathbf{k}'$$

PROBLEM 18.32 (Continued)

Principle of impulse-momentum.



Moments about A: $$(\mathbf{H}_A)_0 + (-a\mathbf{j}') \times (F\Delta t)\mathbf{k} = \mathbf{H}_A$$

$$(\mathbf{H}_G)_0 + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}}_0 + aF\Delta t\mathbf{i}' = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}}$$

Resolve into components.

$$\mathbf{i}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 - aF(\Delta t) = -\frac{1}{4}ma^2\omega_{y'} \qquad (1)$$

$$\mathbf{j}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 = \frac{1}{12}ma^2\omega_{y'} + \frac{1}{4}ma^2\omega_{y'} \quad \omega_{y'} = \frac{1}{8}\sqrt{2}\omega_0$$

$$\mathbf{k}': \quad 0 = \frac{1}{6}ma^2\omega_{z'} + \frac{1}{2}ma^2\omega_{z'} \quad \omega_{z'} = 0$$

(*a*) From Eq. (1), $$F\Delta t = \frac{1}{24}\sqrt{2}ma\omega_0 + \frac{1}{32}\sqrt{2}ma\omega_0 = \frac{7}{96}\sqrt{2}ma\omega_0$$

$$(F\Delta t)\mathbf{k} = 0.1031ma\omega_0\mathbf{k} \blacktriangleleft$$

$$\bar{\mathbf{v}} = \frac{1}{2}a\omega_y\mathbf{k}' = \frac{1}{16}\sqrt{2}a\omega_0\mathbf{k}'$$

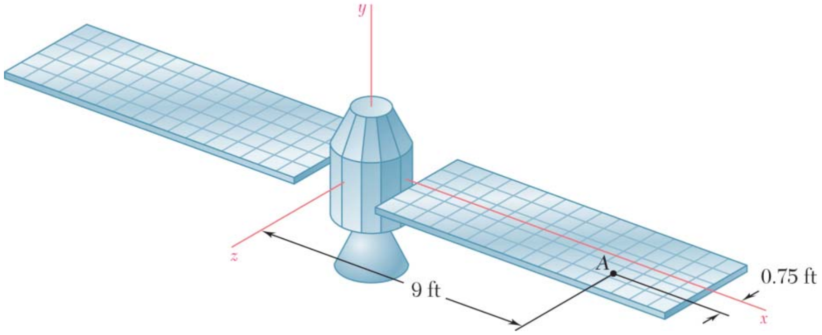
Linear momentum: $$m\bar{\mathbf{v}}_0 + \mathbf{A}\Delta t + F\Delta t\mathbf{k} = m\bar{\mathbf{v}}$$

$$0 + \mathbf{A}\Delta t + \frac{7}{96}\sqrt{2}ma\omega_0\mathbf{k}' = \frac{1}{16}\sqrt{2}ma\omega_0\mathbf{k}'$$

(*b*) $$\mathbf{A}\Delta t = -\frac{1}{96}\sqrt{2}ma\omega_0 \qquad \mathbf{A}\Delta t = -0.01473ma\omega_0\mathbf{k} \blacktriangleleft$$

PROBLEM 18.33

The coordinate axes shown represent the principal centroidal axes of inertia of a 3000-lb space probe whose radii of gyration are $k_x = 1.375\text{ ft},\ k_y = 1.425\text{ ft},$ and $k_z = 1.250\text{ ft}.$ The probe has no angular velocity when a 5-oz meteorite strikes one of its solar panels at Point A with a velocity $\mathbf{v}_0 = (2400\text{ ft/s})\mathbf{i} - (3000\text{ ft/s})\mathbf{j} + (3200\text{ ft/s})\mathbf{k}$ relative to the probe. Knowing that the meteorite emerges on the other side of the panel with no change in the direction of its velocity, but with a speed reduced by 20 percent, determine the final angular velocity of the probe.



SOLUTION

Masses: Space probe: $m' = \dfrac{3000}{32.2} = 93.17\text{ lb}\cdot\text{s}^2/\text{ft}$

Meteorite: $m = \dfrac{5}{(16)(32.2)} = 0.009705\text{ lb}\cdot\text{s}^2/\text{ft}$

Point of impact: $\mathrm{r}_A = (9\text{ ft})\mathbf{i} + (0.75\text{ ft})\mathbf{k}$

Initial linear momentum of the meteorite, $(\text{lb}\cdot\text{s})$:

$$m\mathbf{v}_0 = (0.009705)(2400\mathbf{i} + 3000\mathbf{j} + 3200\mathbf{k}) = 23.292\mathbf{i} - 29.115\mathbf{j} + 31.056\mathbf{k}$$

Its moment about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$\mathbf{r}_A \times m\mathbf{v}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & 0 & 0.75 \\ 23.292 & -29.115 & 31.056 \end{vmatrix} = 21.836\mathbf{i} - 262.04\mathbf{j} - 262.04\mathbf{k}$$

Final linear momentum of meteorite and its moment about the origin, $(\text{lb}\cdot\text{s})$ and $(\text{lb}\cdot\text{s}\cdot\text{ft})$:

$$0.8m\mathbf{v}_0 = 18.634\mathbf{i} - 23.292\mathbf{j} + 24.845\mathbf{k}$$

$$\mathbf{r}_A \times (0.8m\mathbf{v}_0) = 17.469\mathbf{i} - 209.63\mathbf{j} - 209.63\mathbf{k}$$

PROBLEM 18.33 (Continued)

Let $\mathbf{H}_A$ be the angular momentum of the probe and m' be its mass. Conservation of angular momentum about the origin for a system of particles consisting of the probe plus the meteorite:

$$\mathbf{r}_A \times m\mathbf{v}_0 = \mathbf{H}_A + \mathbf{r}_A \times (0.8m\mathbf{v}_0)$$

$$\mathbf{H}_A = (4.367 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{i} - (52.41 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{j} - (52.41 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{k}$$

$$I_x\omega_x = (H_A)_x \qquad \omega_x = \frac{(H_A)_x}{m'k_x^2} = \frac{4.367}{(93.17)(1.375)^2} = 0.02479 \text{ rad/s}$$

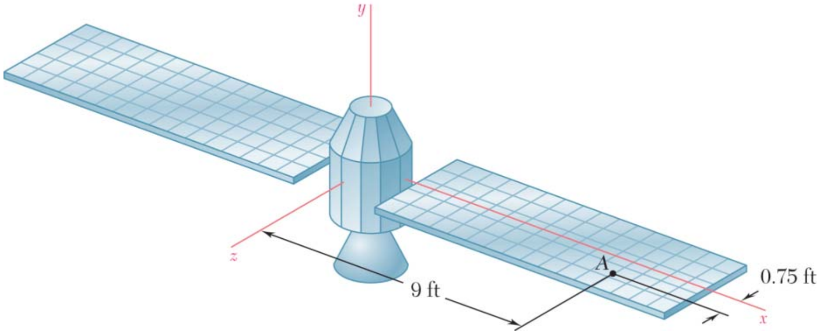
$$I_y\omega_y = (H_A)_y \qquad \omega_y = \frac{(H_A)_y}{m'k_y^2} = \frac{-52.41}{(93.17)(1.425)^2} = -0.2770 \text{ rad/s}$$

$$I_z\omega_z = (H_A)_z \qquad \omega_z = \frac{(H_A)_z}{m'k_z^2} = \frac{-52.41}{(93.17)(1.250)^2} = -0.3600 \text{ rad/s}$$

$$\boldsymbol{\omega} = (0.0248 \text{ rad/s})\mathbf{i} - (0.277 \text{ rad/s})\mathbf{j} - (0.360 \text{ rad/s})\mathbf{k} \blacktriangleleft$$

PROBLEM 18.34

The coordinate axes shown represent the principal centroidal axes of inertia of a 3000-lb space probe whose radii of gyration are $k_x = 1.375$ ft, $k_y = 1.425$ ft, and $k_z = 1.250$ ft. The probe has no angular velocity when a 5-oz meteorite strikes one of its solar panels at Point A and emerges on the other side of the panel with no change in the direction of its velocity, but with a speed reduced by 25 percent. Knowing that the final angular velocity of the probe is $\boldsymbol{\omega} = (0.05 \text{ rad/s})\mathbf{i} - (0.12 \text{ rad/s})\mathbf{j} + \omega_z\mathbf{k}$ and that the x component of the resulting change in the velocity of the mass center of the probe is -0.675 in./s, determine (*a*) the component ω_z of the final angular velocity of the probe, (*b*) the relative velocity $\mathbf{v}_0$ with which the meteorite strikes the panel.



SOLUTION

Masses: Space probe: $m' = \dfrac{3000}{32.2} = 93.17 \text{ lb}\cdot\text{s}^2/\text{ft}$

Meteorite: $m = \dfrac{5}{(16)(32.2)} = 0.009705 \text{ lb}\cdot\text{s}^2/\text{ft}$

Point of impact: $\mathbf{r}_A = (9 \text{ ft})\mathbf{i} + (0.75 \text{ ft})\mathbf{k}$

Initial linear momentum of the meteorite, $(\text{lb}\cdot\text{s})$:

$$m\mathbf{v}_0 = (0.009705)(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})$$

Its moment about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$(\mathbf{H}_A)_0 = \mathbf{r}_A \times m\mathbf{v}_0 = 0.009705\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & 0 & 0.75 \\ v_x & v_y & v_z \end{vmatrix}$$

$$= 0.009705[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}]$$

Final linear momentum of the meteorite, $(\text{lb}\cdot\text{s})$:

$$0.75m\mathbf{v}_0 = 0.007279(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})$$

Its moment about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$\mathbf{r}_A \times (0.75m\mathbf{v}_0) = 0.007279[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}]$$

PROBLEM 18.34 (Continued)

Initial linear momentum of the space probe, $(\mathrm{lb}\cdot\mathrm{s})$: $m'\mathbf{v}'_0 = 0$

Final linear momentum of the space probe, $(\mathrm{lb}\cdot\mathrm{s})$:

$$m'(v'_x\mathbf{i} + v'_y\mathbf{j} + v'_z\mathbf{k}) = 93.17\left(-\frac{0.675}{12}\mathbf{i} + v'_y\mathbf{j} + v'_z\mathbf{k}\right)$$

Final angular momentum of space probe, $(\mathrm{lb}\cdot\mathrm{ft}\cdot\mathrm{s})$:

$$\mathbf{H}_A = m'\left(k_x^2\omega_x\mathbf{i} + k_y^2\omega_y\mathbf{j} + k_z^2\omega_z\mathbf{k}\right)$$

$$= 93.17[(1.375)^2(0.05)\mathbf{i} + (1.425)^2(-0.12)\mathbf{j} + (1.250)^2\omega_z\mathbf{k}]$$

$$= 8.8075\mathbf{i} - 22.703\mathbf{j} + 145.58\omega_z\mathbf{k}$$

Conservation of linear momentum of the probe plus the meteorite, $(\mathrm{lb}\cdot\mathrm{s})$:

$$0.009705(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) = 0.007279(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) + 93.17(-0.05625\mathbf{i} + v'_y\mathbf{j} + v'_z\mathbf{k})$$

i: $0.002426v_x = -5.2408$ $\qquad v_x = -2160$ ft/s

j: $0.002426v_y = 93.17v'_y$

k: $0.002426v_z = 93.17v'_z$

Conservation of angular momentum about the origin, $(\mathrm{lb}\cdot\mathrm{ft}\cdot\mathrm{s})$:

$$(0.009705)[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}]$$

$$= (0.007279)[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}] + 8.8075\mathbf{i} - 22.703\mathbf{j} + 145.58\omega_z\mathbf{k}$$

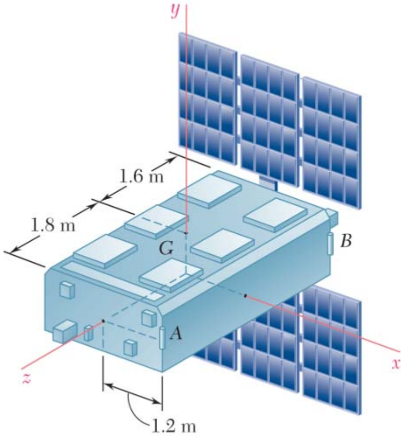
i: $-0.0018195v_y = 8.8075$ $\qquad v_y = 4840.5$ ft/s

j: $-0.021834v_z + 0.0018195v_x = -22.703$ $\qquad v_z = 0.08333v_x + 1039.8 = 859.8$ ft/s

k: $-0.021834v_y = 145.58\omega_z$

(*a*) $\omega_z = -149.98\times 10^{-6}v_y$ $\qquad \omega_z = -0.726$ rad/s ◀

(*b*) $\mathbf{v}_0 = -(2160\text{ ft/s})\mathbf{i} - (4840\text{ ft/s})\mathbf{j} + (860\text{ ft/s})\mathbf{k}$ ◀



PROBLEM 18.35

A 1200-kg satellite designed to study the sun has an angular velocity ω_0 = (0.050 rad/s)**i** + (0.075 rad/s)**k** when two small jets are activated at A and B in a direction parallel to the y axis. Knowing that the coordinate axes are principal centroidal axes, that the radii of gyration of the satellite are $\bar{k}_x = 1.120$ m, $\bar{k}_y = 1.200$ m, $\bar{k}_z = 0.900$ m, and that each jet produces a 50-N thrust, determine (a) the required operating time of each jet if the angular velocity of the satellite is to be reduced to zero, (b) the resulting change in the velocity of the mass center G.

SOLUTION

Principal moments of inertia.

$$\bar{I}_x = m\bar{k}_x^2 = (1200)(1.120)^2 = 1505.28\ \text{kg}\cdot\text{m}^2$$

$$\bar{I}_y = m\bar{k}_y^2 = (1200)(1.200)^2 = 1728\ \text{kg}\cdot\text{m}^2$$

$$\bar{I}_z = m\bar{k}_z^2 = (1200)(0.900)^2 = 972\ \text{kg}\cdot\text{m}^2$$

Initial angular velocity.

$$(\omega_x)_1 = 0.050\ \text{rad/s}, \qquad (\omega_y)_1 = 0, \qquad (\omega_z)_1 = 0.075\ \text{rad/s}$$

Initial angular momentum.

$$(\mathbf{H}_G)_1 = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$= (75.264\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (72.9\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

Final angular momentum if $(\boldsymbol{\omega})_2 = 0.$ $\qquad (\mathbf{H}_G)_z = 0$

Positions of jets A and B relative to mass center.

$$\mathbf{r}_{A/G} = (1.2\ \text{m})\mathbf{i} + (1.8\ \text{m})\mathbf{k} \qquad \mathbf{r}_{B/G} = (1.2\ \text{m})\mathbf{i} - (1.6\ \text{m})\mathbf{k}$$

*Impulses from jets A and B***.** $\qquad (F\Delta t)_A\,\mathbf{j}$ and $(F\Delta t)_B\,\mathbf{j}$

Principle of impulse and momentum. $\qquad$ Moments about **G**:

$$(\mathbf{H}_G)_1 + \mathbf{r}_{A/G} \times (F\Delta t)_A\,\mathbf{j} + \mathbf{r}_{B/G} \times (F\Delta t)_B\,\mathbf{j} = (\mathbf{H}_{\text{G}})_z$$

$$75.264\mathbf{i} + 72.9\mathbf{k} + (1.2\mathbf{i} + 1.8\mathbf{k}) \times (F\Delta t)_A\,\mathbf{j} + (1.2\mathbf{i} - 1.6\mathbf{k}) \times (F\Delta t)_B\,\mathbf{j} = 0$$

$$75.264\mathbf{i} + 72.9\mathbf{k} + 1.2(F\Delta t)_A\,\mathbf{k} - 1.8(F\Delta t)_A\,\mathbf{i} + 1.2(F\Delta t)_B\,\mathbf{k} + 1.6(F\Delta t)_B\,\mathbf{i} = 0$$

Components: $\qquad$ **i**: $\quad 75.264 - 1.8(F\Delta t)_A + 1.6(F\Delta t)_B = 0 \qquad (1)$

k: $\quad 72.9 + 1.2(F\Delta t)_A + 1.2(F\Delta t)_B = 0 \qquad (2)$

PROBLEM 18.35 (Continued)

Solving (1) and (2) simultaneously,

$$(F\Delta t)_A = -6.4518\ \text{N}\cdot\text{s}, \qquad (F\Delta t)_B = -54.298\ \text{N}\cdot\text{s}$$

Since both impulse components are negative, the 50-N thrusts of jets A and B act in the negative y direction.

$$F = -50\ \text{N}$$

(a) *Operating times of jets.*

$$\Delta t_A = \frac{(F\Delta t)_A}{F} = \frac{-6.4518}{-50} \qquad \Delta t_A = 0.1290\ \text{s} \blacktriangleleft$$

$$\Delta t_B = \frac{(F\Delta t)_B}{F} = \frac{-54.298}{-50} \qquad \Delta t_B = 1.086\ \text{s} \blacktriangleleft$$

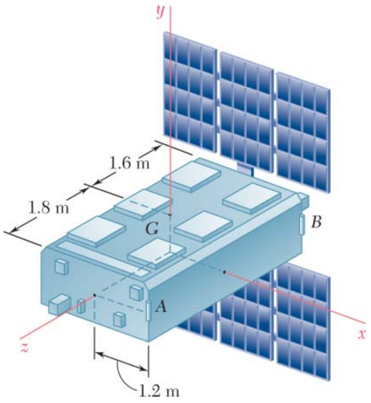
Principle of impulse and momentum. Linear momentum:

$$(F\Delta t)_A\,\mathbf{j} + (F\Delta t)_B\,\mathbf{j} = m\Delta\bar{\mathbf{v}}$$

$$-6.4518\mathbf{j} - 54.298\mathbf{j} = 1200\ \Delta\bar{\mathbf{v}}$$

$$\Delta\bar{\mathbf{v}} = -\left(50.6 \times 10^{-3}\text{m/s}\right)\mathbf{j}$$

(b) *Change in velocity of mass center.* $\Delta\bar{\mathbf{v}} = -\left(50.6\ \text{mm/s}\right)\mathbf{j}$ ◀



PROBLEM 18.36

If jet A in Prob. 18.35 is inoperative, determine (*a*) the required operating time of jet B to reduce to zero the x component of the angular velocity of the satellite, (*b*) the resulting final angular velocity, (*c*) the resulting change in the velocity of the mass center G.

SOLUTION

Data from Problem 18.33.

Mass of satellite: $m = 1200$ kg.

Initial angular velocity: $\boldsymbol{\omega}_0 = (0.050 \text{ rad/s})\mathbf{i} + (0.075 \text{ rad/s})\mathbf{j}$

Principal radii of gyration: $\bar{k}_x = 1.120$ m, $\bar{k}_y = 1.200$ m, $\bar{k}_z = 0.900$ m

Jet thrust $= 50$ N parallel to the y axis.

Principal moments of inertia.

$$\bar{I}_x = m\bar{k}_x^2 = (1200)(1.120)^2 = 1505.28 \text{ kg}\cdot\text{m}^2$$

$$\bar{I}_y = m\bar{k}_y^2 = (1200)(1.200)^2 = 1728 \text{ kg}\cdot\text{m}^2$$

$$\bar{I}_z = m\bar{k}_z^2 = (1200)(0.900)^2 = 972 \text{ kg}\cdot\text{m}^2$$

Initial angular velocity.

$$(\omega_x)_1 = 0.050 \text{ rad/s}, \qquad (\omega_y)_1 = 0, \qquad (\omega_z)_1 = 0.075 \text{ rad/s}$$

Initial angular momentum.

$$(\mathbf{H}_G)_1 = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$= \left(75.264 \text{ kg}\cdot\text{m}^2/\text{s}\right)\mathbf{i} + \left(72.9 \text{ kg}\cdot\text{m}^2/\text{s}\right)\mathbf{k}$$

Final angular momentum if $(\boldsymbol{\omega}_x)_2 = 0.$

$$(\mathbf{H}_G) = \bar{I}_x(\omega_x)_2\,\mathbf{i} + \bar{I}_y(\omega_y)_2\,\mathbf{j} + \bar{I}_z(\omega_z)_2\,\mathbf{k}$$

$$= 0 + 1728(\omega_y)_2\,\mathbf{j} + 972(\omega_z)_2\,\mathbf{k}$$

Positions of jet B relative to mass center. $\quad \mathbf{r}_{B/G} = (1.2 \text{ m})\mathbf{i} - (1.6 \text{ m})\mathbf{j}$

Impulse from jet B. $\quad (F\Delta t)_B\,\mathbf{j}$

PROBLEM 18.36 (Continued)

Principle of impulse and momentum. Moments about G:

$$(\mathbf{H}_G)_1 + \mathbf{r}_{B/G} \times (F\Delta t)_B\,\mathbf{j} = (\mathbf{H}_\text{G})_2$$

$$75.264\mathbf{i} + 72.9\mathbf{k} + (1.2\mathbf{i} - 1.6\mathbf{k}) \times (F\Delta t)_B\,\mathbf{j} = 1728(\omega_y)_2\,\mathbf{j} + 972(\omega_z)_2\,\mathbf{k}$$

$$75.264\mathbf{i} + 72.9\mathbf{k} + 1.2(F\Delta t)_B\,\mathbf{k} + 1.6(F\Delta t)_B\,\mathbf{i} = 1728(\omega_y)_2\,\mathbf{j} + 972(\omega_z)_2\,\mathbf{k}$$

Components: **i**: $75.264 + 1.6(F\Delta t)_B = 0$

$$(F\Delta t)_B = -47.04 \text{ N}\cdot\text{s}$$

j: $0 = 1728(\omega_y)_2 \qquad (\omega_y)_2 = 0$

k: $72.9 + 1.2(F\Delta t)_B = 972(\omega_z)_2$

$$(\omega_z)_2 = \frac{72.9 + (1.2)(-47.04)}{972} = 0.01693 \text{ rad/s}$$

Since $(F\Delta t)_B$ is negative, the 50-N thrust of jet B acts in the negative y direction.

(*a*) *Operating time of jet B.*

$$\Delta t_B = \frac{(F\Delta t)_B}{F} = \frac{-47.04}{-50} \qquad \Delta t_B = 0.941 \text{ s} \blacktriangleleft$$

(*b*) *Final angular velocity.* $\boldsymbol{\omega} = (\omega_x)_2\,\mathbf{i} + (\omega_y)_2\,\mathbf{j} + (\omega_z)_2\,\mathbf{k}$

$$\boldsymbol{\omega} = (0.01693 \text{ rad/s})\mathbf{k} \blacktriangleleft$$

Principle of impulse and momentum. Linear components:

$$(F\Delta t)_B\,\mathbf{j} = m(\Delta\overline{\mathbf{v}})$$

$$-47.04\mathbf{j} = 1200\,\Delta\overline{\mathbf{v}} \qquad \Delta\overline{\mathbf{v}} = -(0.0392 \text{ m/s})\,\mathbf{j}$$

(*c*) *Change in velocity of mass center.* $\Delta\overline{\mathbf{v}} = -(39.2 \text{ mm/s})\,\mathbf{j} \blacktriangleleft$

PROBLEM 18.37

Denoting, respectively, by $\boldsymbol{\omega}$, $\mathbf{H}_O$, and T the angular velocity, the angular momentum, and the kinetic energy of a rigid body with a fixed Point O, (*a*) prove that $\mathbf{H}_O \cdot \boldsymbol{\omega} = 2T$; (*b*) show that the angle θ between ω and $\mathbf{H}_O$ will always be acute.

SOLUTION

(*a*)

$$\mathbf{H}_0 = (I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\mathbf{i} + (-I_{xy}\omega_x + I_y\omega_y - I_{yz}\omega_z)\mathbf{j} + (-I_{xz}\omega_x - I_{yz}\omega_y + I_z\omega_z)\mathbf{k}$$

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

$$\mathbf{H}_0 \cdot \boldsymbol{\omega} = I_x\omega_x^2 - I_{xy}\omega_y\omega_x - I_{xz}\omega_z\omega_x - I_{xy}\omega_x\omega_y + I_y\omega_y^2 - I_{yz}\omega_z\omega_y$$

$$-I_{xz}\omega_x\omega_z - I_{yz}\omega_y\omega_z + I_z\omega_z^2$$

$$= (2)\left(\frac{1}{2}\right)\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{yz}\omega_y\omega_z - 2I_{xz}\omega_x\omega_z\right)$$

$$= 2T$$

(*b*)

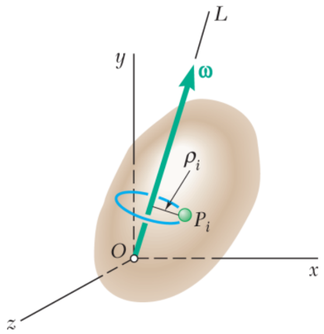
$$\mathbf{H}_0 \cdot \boldsymbol{\omega} = H_0\omega\cos\theta$$

$$2T = H_0\omega\cos\theta$$

$$\cos\theta = \frac{2T}{H_0\omega}$$

But $T > 0,\ H_0 > 0,\ \omega > 0$

$$\cos\theta > 0 \qquad \theta < 90°$$



PROBLEM 18.38

Show that the kinetic energy of a rigid body with a fixed Point O can be expressed as $T = \frac{1}{2} I_{OL}\omega^2$, where ω is the instantaneous angular velocity of the body and I_{OL} is its moment of inertia about the line of action OL of ω. Derive this expression (*a*) from Eqs. (9.46) (or Eq. B.19 in the Appendix) and (18.19), (*b*) by considering T as the sum of the kinetic energies of particles P_i describing circles of radius ρ_i about line OL.

SOLUTION

(*a*)

$$T = \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{yz}\omega_y\omega_z - 2I_{xz}\omega_x\omega_z\right)$$

Let

$$\omega_x = \omega\cos\theta_x = \omega\lambda_x$$

$$\omega_y = \omega\cos\theta_y = \omega\lambda_y$$

$$\omega_z = \omega\cos\theta_z = \omega\lambda_z$$

$$T = \frac{1}{2}\left(I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{xz}\lambda_x\lambda_z\right)\omega^2$$

$$= \frac{1}{2} I_{OL}\omega^2$$

$$T = \frac{1}{2} I_{OL}\omega^2 \blacktriangleleft$$

(*b*) Each particle of mass $(\Delta m)_i$ describes a circle of radius ρ_i.

The speed of the particle is $v_i = \rho_i\omega$.

Its kinetic energy is

$$(\Delta T)_i = \frac{1}{2}(\Delta m_i)v_i^2 = \frac{1}{2}(\Delta m_i)\rho_i^2\omega^2$$

The kinetic energy of the entire body is

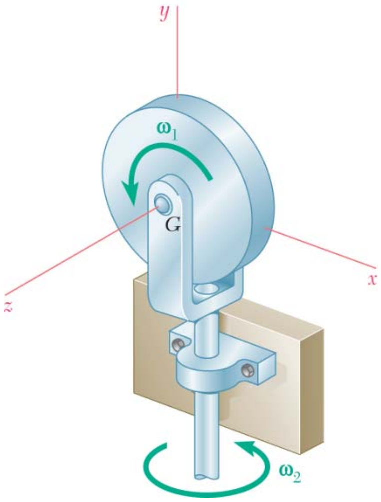
$$T = \Sigma(\Delta T)_i = \frac{1}{2}\Sigma(\Delta m_i)\rho_i^2\omega^2$$

but

$$I_{OL} = \Sigma(\Delta m_i)\rho_i^2$$

Hence,

$$T = \frac{1}{2} I_{OL}\omega^2 \blacktriangleleft$$



PROBLEM 18.39

Determine the kinetic energy of the disk of Problem 18.1.

PROBLEM 18.1 A thin, homogeneous disk of mass m and radius r spins at the constant rate ω_1 about an axle held by a fork-ended vertical rod, which rotates at the constant rate ω_2. Determine the angular momentum $\mathbf{H}_G$ of the disk about its mass center G.

SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega_2 \mathbf{j} + \omega_1 \mathbf{k}$

Moments of inertia: $\bar{I}_x = \frac{1}{4}mr^2, \quad \bar{I}_y = \frac{1}{4}mr^2, \quad \bar{I}_z = \frac{1}{2}mr^2$

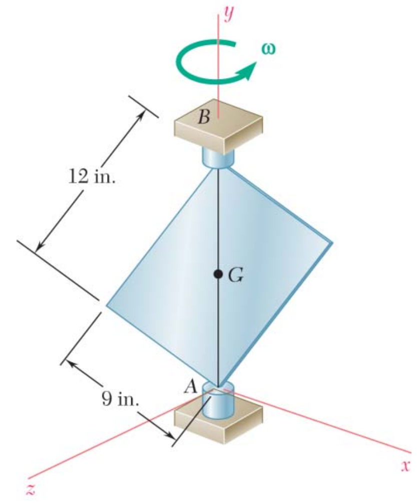
Products of inertia: by symmetry, $\bar{I}_{xy} = \bar{I}_{yz} = \bar{I}_{zx} = 0$

Kinetic energy:

$$T = \frac{1}{2}\left(\bar{I}_x \omega_x^2 + \bar{I}_y \omega_y^2 + \bar{I}_z \omega_z\right)$$

$$T = \frac{1}{2}\left[0 + \left(\frac{1}{4}mr^2\right)\omega_2^2 + \left(\frac{1}{2}mr^2\right)\omega_1^2\right]$$

$$T = \frac{1}{8}mr^2\left(\omega_2^2 + 2\omega_1^2\right) \blacktriangleleft$$



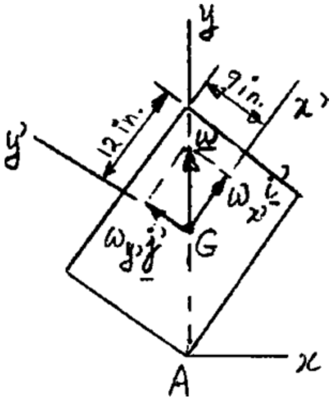
PROBLEM 18.40

Determine the kinetic energy of the plate of Problem 18.2.

PROBLEM 18.2 A thin rectangular plate of weight 15 lb rotates about its vertical diagonal *AB* with an angular velocity *ω*. Knowing that the *z* axis is perpendicular to the plate and that *ω* is constant and equal to 5 rad/s, determine the angular momentum of the plate about its mass center *G*.

SOLUTION

$$h = \sqrt{(9\text{ in.})^2 + (12\text{ in.})^2} = 15\text{ in.}$$



Resolving ω along the principal axes x', y', z:

$$\omega_{x'} = \frac{12}{15}\omega = 0.8(5\text{ rad/s}) = 4\text{ rad/s}$$

$$\omega_{y'} = \frac{9}{15}\omega = 0.6(5\text{ rad/s}) = 3\text{ rad/s}$$

$$\omega_z = 0$$

Moments of inertia:

$$I_{x'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{9}{12}\text{ ft}\right) = 0.021836\text{ slug}\cdot\text{ft}^2$$

$$I_{y'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{12}{12}\text{ ft}\right)^2 = 0.038820\text{ slug}\cdot\text{ft}^2$$

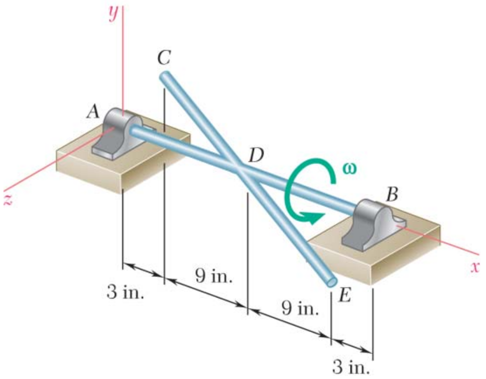
From Eqs. (18.20):

$$T = \frac{1}{2}(\bar{I}_{x'}\omega_{x'}^2 + \bar{I}_{y'}\omega_{y'}^2 + \bar{I}_{z'}\omega_{z'}^2)$$

$$= \frac{1}{2}[(0.021836)(4)^2 + (0.038820)(3)^2 + 0]$$

$$T = 0.34938\text{ ft}\cdot\text{lb}$$

$T = 0.349\text{ ft}\cdot\text{lb}$ ◀

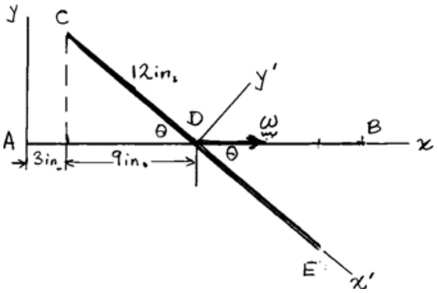


PROBLEM 18.41

Determine the kinetic energy of the assembly of Problem 18.3.

PROBLEM 18.3 Two uniform rods *AB* and *CE*, each of weight 3 lb and length 2 ft, are welded to each other at their midpoints. Knowing that this assembly has an angular velocity of constant magnitude $\omega = 12$ rad/s, determine the magnitude and direction of the angular momentum $\mathbf{H}_D$ of the assembly about *D*.

SOLUTION



$$m = \frac{W}{g} = \frac{3}{32.2} = 0.093168 \text{ lb}\cdot\text{s}^2/\text{ft}, \qquad l = 24 \text{ in.} = 2 \text{ ft},$$

$$\boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{i}$$

For rod *ADB*, $T = \frac{1}{2} I_x \omega^2 \approx 0,$ since $I_x \approx 0.$

For rod *CDE*, use principal axes x', y' as shown.

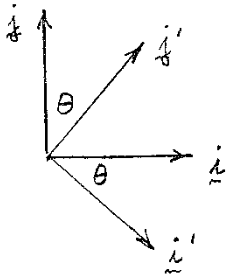
$$\cos\theta = \frac{9}{12}, \qquad \theta = 41.410°$$

$$\omega_{x'} = \omega\cos\theta = 9 \text{ rad/s}^2$$

$$\omega_{y'} = \omega\sin\theta = 7.93725 \text{ rad/s}^2$$

$$\omega_{z'} = 0$$

$$\bar{I}_{x'} \approx 0$$



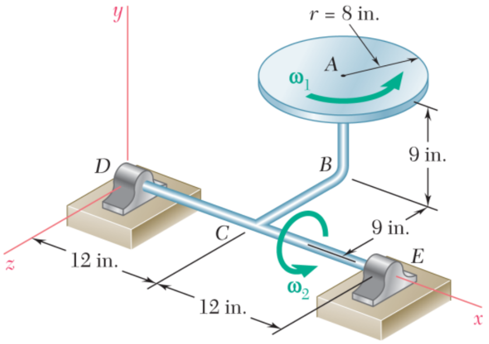
$$\bar{I}_{y'} = \frac{1}{12} m l^2 = \frac{1}{12}(0.093168)(2)^2$$

$$= 0.0310559 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$T = \frac{1}{2} m \bar{v}^2 + \frac{1}{2}\bar{I}_{x'}\omega_{x'}^2 + \frac{1}{2}\bar{I}_{y'}\omega_{y'}^2 + \frac{1}{2}\bar{I}_{z'}\omega_{z'}^2$$

$$= 0 + 0 + \frac{1}{2}(0.0310559)(7.93725)^2 + 0$$

$$= 0.97826 \text{ ft}\cdot\text{lb} \qquad T = 0.978 \text{ ft}\cdot\text{lb} \blacktriangleleft$$



PROBLEM 18.42

Determine the kinetic energy of the disk of Problem 18.4.

PROBLEM 18.4 A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm *ABC*, which is welded to a shaft *DCE* rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the angular momentum $\mathbf{H}_A$ of the disk about its center *A*.

SOLUTION

$$m = \frac{W}{g} = \frac{6}{32.2} = 0.1863 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\boldsymbol{\omega} = \omega_2\mathbf{i} + \omega_1\mathbf{j} = (8 \text{ rad/s})\mathbf{i} + (16 \text{ rad/s})\mathbf{j}$$

For axes x', y', z' parallel to x, y, z with origin at A,

$$\bar{I}_{x'} = \frac{1}{4}mr^2 = \frac{1}{4}(0.1863)\left(\frac{8}{12}\right)^2 = 0.0207 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{z'} = \bar{I}_{x'} = 0.0207 \text{ lb}\cdot\text{s}^2\cdot\text{ft}, \quad \bar{I}_{y'} = \bar{I}_{x'} + \bar{I}_{z'} = 0.0414 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Point A is the mass center of the disk.

$$\mathbf{r}_{A/C} = (9 \text{ in.})\mathbf{i} - (9 \text{ in.})\mathbf{k} = (0.75 \text{ ft})\mathbf{i} - (0.75 \text{ ft})\mathbf{k}$$

$$\bar{\mathbf{v}} = \mathbf{v}_A = \omega_2\mathbf{i} \times \mathbf{r}_{A/C} = 8\mathbf{i} \times (0.75\mathbf{j} - 0.75\mathbf{k})$$

$$= (6 \text{ ft/s})\mathbf{j} + (6 \text{ ft/s})\mathbf{k}$$

$$\bar{v} = \sqrt{(6)^2 + (6)^2} = 8.4853 \text{ ft/s}$$

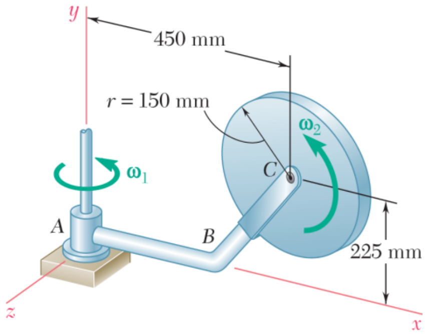
Kinetic energy:

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}_{x'}\omega_x^2 + \frac{1}{2}\bar{I}_{y'}\omega_y^2 + \frac{1}{2}\bar{I}_{z'}\omega_z^2$$

$$T = \frac{1}{2}(0.1863)(8.4853)^2 + \frac{1}{2}(0.0207)(8)^2 + \frac{1}{2}(0.0414)(16)^2 + 0$$

$$= 6.7068 + 0.6624 + 5.2992 = 12.6684 \text{ ft}\cdot\text{lb}$$

$T = 12.67$ ft · lb ◀

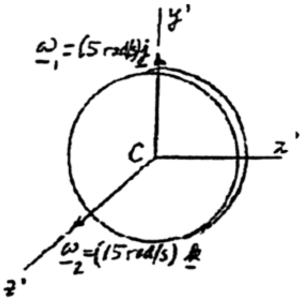


PROBLEM 18.43

Determine the kinetic energy of the disk of Problem 18.5.

PROBLEM 18.5 A thin disk of mass $m = 4\text{ kg}$ rotates at the constant rate $\omega_2 = 15\text{ rad/s}$ with respect to arm *ABC*, which itself rotates at the constant rate $\omega_1 = 5\text{ rad/s}$ about the y axis. Determine the angular momentum of the disk about its center *C*.

SOLUTION



$$r = 150\text{ mm}$$

Angular velocity of disk:

$$\boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k}$$
$$= (5\text{ rad/s})\mathbf{j} + (15\text{ rad/s})\mathbf{k}$$

Centroidal moments of inertia:

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4}mr^2$$
$$= \frac{1}{4}(4)(0.150\text{ m})^2 = 0.0225\text{ kg}\cdot\text{m}^2$$
$$\bar{I}_{z'} = \frac{1}{2}mr^2 = 0.045\text{ kg}\cdot\text{m}^2$$

Location of mass center. $\mathbf{r}_{C/A} = (0.450\text{ m})\mathbf{i} + (0.225\text{ m})\mathbf{j}$

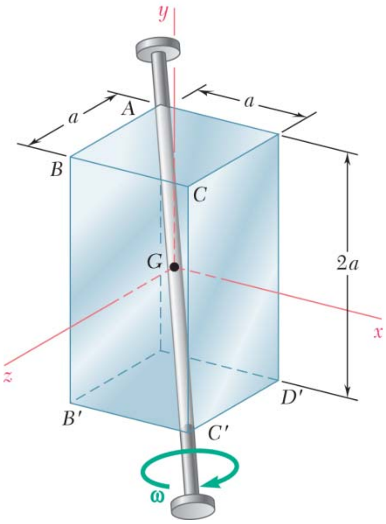
Velocity of mass center.

$$\bar{\mathbf{v}} = \boldsymbol{\omega}_1 \times \mathbf{r}_{C/A} = 5\mathbf{j} \times (0.45\mathbf{i} + 0.225\mathbf{j})$$
$$= -(2.25\text{ m/s})\mathbf{k}$$

Kinetic energy:

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}_x\omega_x^2 + \frac{1}{2}\bar{I}_y\omega_y^2 + \frac{1}{2}\bar{I}_z\omega_z^2$$
$$= \frac{1}{2}(4)(2.25)^2 + 0 + \frac{1}{2}(0.0225)(5)^2 + \frac{1}{2}(0.0450)(15)^2$$
$$= 10.125 + 0 + 0.28125 + 5.0625$$

$T = 15.47\text{ J}$ ◀



PROBLEM 18.44

Determine the kinetic energy of the solid parallelepiped of Problem 18.6.

PROBLEM 18.6 A solid rectangular parallelepiped of mass m has a square base of side a and a length $2a$. Knowing that it rotates at the constant rate ω about its diagonal AC' and that its rotation is observed from A as counterclockwise, determine (a) the magnitude of the angular momentum $\mathbf{H}_G$ of the parallelepiped about its mass center G, (b) the angle that $\mathbf{H}_G$ forms with the diagonal AC'.

SOLUTION

Body diagonal:

$$d = \sqrt{a^2 + (2a)^2 + a^2} = \sqrt{6}a$$

$$\boldsymbol{\omega} = \frac{\omega}{d}(-a\mathbf{i} + 2a\mathbf{j} - a\mathbf{k}) = -\frac{\omega}{\sqrt{6}}\mathbf{i} + \frac{2\omega}{\sqrt{6}}\mathbf{j} - \frac{\omega}{\sqrt{6}}\mathbf{k}$$

$$I_x = \frac{1}{12}m[(2a)^2 + a^2] = \frac{5}{12}ma^2$$

$$I_y = \frac{1}{12}m[a^2 + a^2] = \frac{1}{6}ma^2$$

$$I_z = \frac{1}{12}m[a^2 + (2a)^2] = \frac{5}{12}ma^2$$

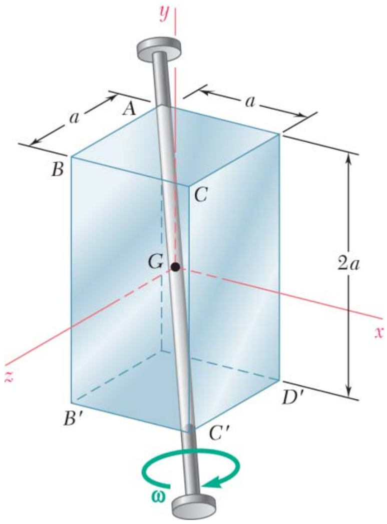
Axis of rotation passes through the mass center, hence $\overline{\mathbf{v}} = 0$.

Kinetic energy:

$$T = \frac{1}{2}m\overline{v}^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$T = 0 + \frac{1}{2}\left(\frac{5}{12}ma^2\right)\left(\frac{\omega}{\sqrt{6}}\right)^2 + \frac{1}{2}\left(\frac{1}{6}ma^2\right)\left(\frac{2\omega}{\sqrt{6}}\right)^2 + \frac{1}{2}\left(\frac{5}{12}ma^2\right)\left(\frac{\omega}{\sqrt{6}}\right)^2 = \frac{1}{8}ma^2\omega^2$$

$$T = 0.1250\ ma^2\omega^2 \blacktriangleleft$$



PROBLEM 18.45

Determine the kinetic energy of the hollow parallelepiped of Problem 18.7.

PROBLEM 18.7 Solve Problem 18.6, assuming that the solid rectangular parallelepiped has been replaced by a hollow one consisting of six thin metal plates welded together.

SOLUTION

Body diagonal:

$$d = \sqrt{a^2 + (2a)^2 + a^2} = \sqrt{6}a$$

$$\boldsymbol{\omega} = \frac{\omega}{d}(-a\mathbf{i} + 2a\mathbf{j} - a\mathbf{k}) = -\frac{\omega}{\sqrt{6}}\mathbf{i} + \frac{2\omega}{\sqrt{6}}\mathbf{j} - \frac{\omega}{\sqrt{6}}\mathbf{k}$$

$$\text{Total area} = 2(a^2 + 2a^2 + 2a^2) = 10a^2$$

For each square plate,

$$m' = \frac{1}{10}m$$

$$I_x = \frac{1}{12}m'a^2 + m'a^2 = \frac{13}{12}m'a^2 = \frac{13}{120}ma^2$$

$$I_y = \frac{1}{6}m'a^2 = \frac{1}{60}ma^2$$

$$I_z = I_x = \frac{13}{120}ma^2$$

For each plate parallel to the *yz* plane,

$$m' = \frac{1}{5}m$$

$$I_x = \frac{1}{12}m'[a^2 + (2a)^2] = \frac{5}{12}m'a^2 = \frac{1}{12}ma^2$$

$$I_y = \frac{1}{12}m'a^2 + m'\left(\frac{a}{2}\right)^2 = \frac{1}{3}m'a^2 = \frac{1}{15}ma^2$$

$$I_z = \frac{1}{12}m'(2a)^2 + m'\left(\frac{a}{2}\right)^2 = \frac{7}{12}m'a^2 = \frac{7}{60}ma^2$$

For each plate parallel to the *xy* plane, $m' = \frac{1}{5}m$

$$I_x = \frac{1}{12}m'(2a)^2 + m'\left(\frac{a}{2}\right)^2 = \frac{7}{12}m'a^2 = \frac{7}{60}ma^2$$

$$I_y = \frac{1}{12}m'a^2 + m'\left(\frac{a}{2}\right)^2 = \frac{1}{3}m'a^2 = \frac{1}{15}ma^2$$

$$I_z = \frac{1}{12}m'[a^2 + (2a)^2] = \frac{5}{12}m'a^2 = \frac{1}{12}ma^2$$

Total moments of inertia:

$$I_x = 2\left(\frac{13}{120} + \frac{1}{12} + \frac{7}{60}\right)ma^2 = \frac{37}{60}ma^2$$

$$I_y = 2\left(\frac{1}{60} + \frac{1}{15} + \frac{1}{15}\right)ma^2 = \frac{3}{10}ma^2$$

$$I_z = 2\left(\frac{13}{120} + \frac{7}{60} + \frac{1}{12}\right)ma^2 = \frac{37}{60}ma^2$$

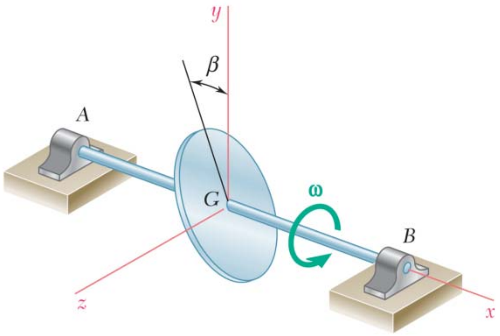
Axis of rotation passes through the mass center, hence $\overline{\mathbf{v}} = 0$.

Kinetic energy:

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$T = 0 + \frac{1}{2}\left(\frac{37}{60}ma^2\right)\left(\frac{\omega}{\sqrt{6}}\right)^2 + \frac{1}{2}\left(\frac{3}{10}ma^2\right)\left(\frac{2\omega}{\sqrt{6}}\right)^2 + \frac{1}{2}\left(\frac{37}{60}ma^2\right)\left(\frac{\omega}{\sqrt{6}}\right)^2 = \frac{73}{360}\ ma^2\omega^2$$

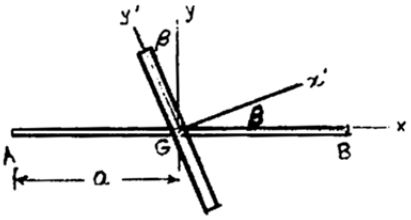
$T = 0.203\ ma^2\omega^2$ ◀

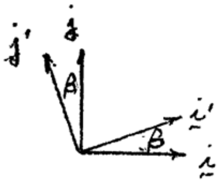


PROBLEM 18.46

Determine the kinetic energy of the disk of Prob. 18.8.

SOLUTION





Use principal axes x', y', z' as shown.

$$\mathbf{i}' = \cos\beta\mathbf{i} + \sin\beta\mathbf{j}$$

$$\mathbf{j}' = -\sin\beta\,\mathbf{i} + \cos\beta\,\mathbf{j}$$

$$\mathbf{i} = \cos\beta\ \mathbf{i}' - \sin\beta\,\mathbf{j}'$$

$$\mathbf{j} = \sin\beta\,\mathbf{i}' + \cos\beta\,\mathbf{j}'$$

$$\boldsymbol{\omega} = \omega\,\mathbf{i} = \omega\ \cos\beta\,\mathbf{i}' - \omega\ \sin\ \beta\mathbf{j}'$$

$$\omega_{x'} = \omega\cos\beta \qquad \omega_{y'} = -\omega\sin\beta$$

Principal moments of inertia:

$$I_{x'} = \frac{1}{2}mr^2, \qquad I_{y'} = I_{z'} = \frac{1}{4}mr^2$$

Kinetic energy:

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\,I_{x'}\omega_{x'}^2 + \frac{1}{2}I_{y'}\omega_{y'}^2 + \frac{1}{2}I_{z'}\omega_{z'}^2$$

$$= 0 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)(\omega\cos\beta)^2 + \frac{1}{2}\left(\frac{1}{4}mr^2\right)(-\omega\sin\beta)^2 + 0$$

$$= \frac{1}{8}mr^2\omega^2\left(2\cos^2\beta + \sin^2\beta\right)$$

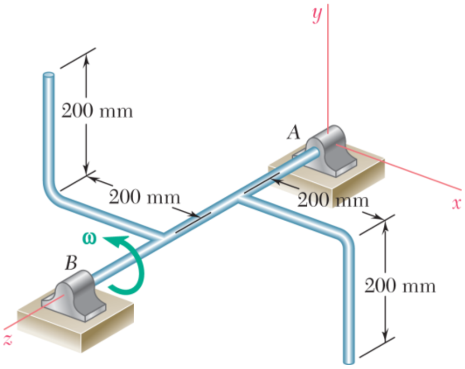
For $\beta = 20°$,

$$T = \frac{1}{8}mr^2\omega\left(2\cos^2 20° + \sin^2 20°\right)$$

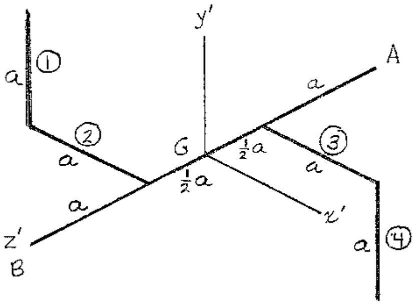
$$T = 0.235mr^2\omega^2 \blacktriangleleft$$

PROBLEM 18.47

Determine the kinetic energy of the assembly of Prob. 18.15.



SOLUTION



$$m = 5 \text{ kg}, \qquad a = 200 \text{ mm} = 0.2 \text{ m}$$

$$\omega = \frac{(2\pi)(360)}{60} = 12\pi \text{ rad/s.}$$

$$\omega_x = 0, \qquad \omega_y = 0, \qquad \omega_z = 12\pi \text{ rad/s}$$

Use parallel axes x', y', z' with origin at the mass center as shown.

$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 - I_{xy}\omega_x\omega_y - I_{yz}\omega_y\omega_z - I_{xz}\omega_x\omega_z$$

$$= \frac{1}{2}I_x\omega^2$$

Part	I_z
①	$\left(\frac{1}{12} + \frac{1}{4} + 1\right)m'a^2$
②	$\frac{1}{3}m'a^2$
③	$\frac{1}{3}m'a^2$
④	$\left(\frac{1}{12} + \frac{1}{4} + 1\right)m'a^2$
Σ	$\frac{10}{3}m'a^2$

Segments 1, 2, 3, and 4, each of mass $m' = 2.5$ kg, contribute to $\bar{I}_z$.

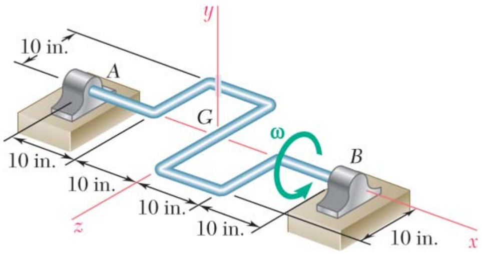
$$I_z = \frac{10}{3}(2.5)(0.2)^2$$

$$= 0.33333 \text{ kg}\cdot\text{m}^2$$

$$T = \frac{1}{2}(0.33333)(12\pi)^2$$

$$= 237 \text{ kg}\cdot\text{m}^2/\text{s}^2 = 237 \text{ N}\cdot\text{m}$$

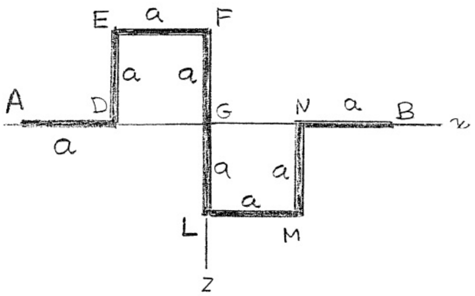
$T = 237$ J ◀



PROBLEM 18.48

Determine the kinetic energy of the shaft of Prob. 18.17.

SOLUTION



$$m = \frac{W}{g} = \frac{10}{32.2} = 0.31056 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\omega = (12 \text{ rad/s})\mathbf{i}, \qquad \omega_y = \omega_z = 0$$

$$\bar{}_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 - I_{xy}\omega_x\omega_y - I_{yz}\omega_y\omega_z - I_{xz}\omega_x\omega_z$$

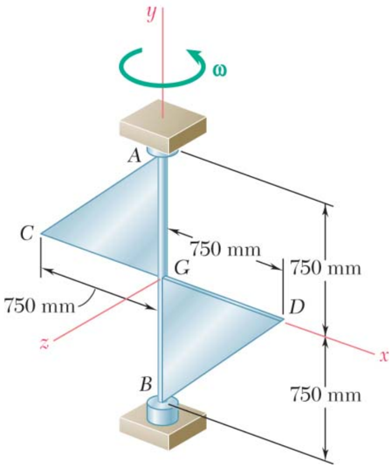
$$= \frac{1}{2}I_x\omega^2$$

The shaft is comprised of eight sections, each of length $a = \frac{10}{12}$ ft and of mass $m' = \frac{m}{8} = 0.03882 \text{ lb}\cdot\text{s}^2/\text{ft}$.

$$\bar{I}_x = (4)\left(\frac{1}{3}m'a^2\right) + (2)\left(m'a^2\right) = \frac{10}{3}m'a^2 = \frac{10}{3}(0.3882)\left(\frac{10}{12}\right)^2 = 0.089861 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$T = \frac{1}{2}(0.089861)(12)^2 = 6.47 \text{ ft}\cdot\text{lb}$$

$T = 6.47$ ft·lb ◀



PROBLEM 18.49

Determine the kinetic energy of the assembly of Prob. 18.19.

SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega\mathbf{j} = (6 \text{ rad/s})\,\mathbf{j}$

$\omega_x = 0, \quad \omega_y = 6 \text{ rad/s}, \quad \omega_z = 0$

Velocity of mass center *G*. $\bar{\mathbf{v}} = 0.$

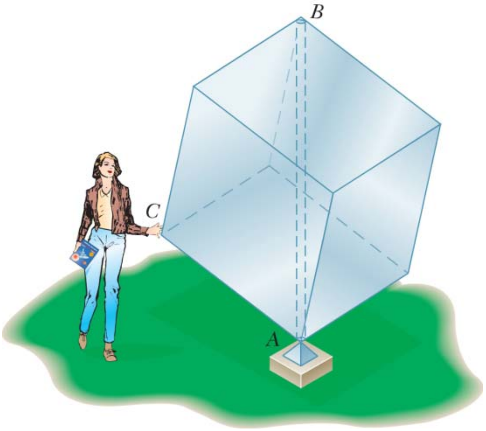
From the solution to Prob. 18.19, $\bar{I}_y = 1.50 \text{ kg}\cdot\text{m}^2$

Kinetic energy.

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}_x\omega_x^2 + \frac{1}{2}\bar{I}_y\omega_y^2 + \frac{1}{2}\bar{I}_z\omega_z^2 - \bar{I}_{xy}\omega_x\omega_y - \bar{I}_{yz}\omega_y\omega_z - \bar{I}_{xz}\omega_x\omega_z$$

$$T = 0 + \frac{1}{2}(1.50)(6)^2 + 0 + 0 + 0 + 0 + 0$$

$T = 27.0 \text{ J}$ ◀



PROBLEM 18.50

Determine the kinetic energy imparted to the cube of Problem 18.21.

PROBLEM 18.21 One of the sculptures displayed on a university campus consists of a hollow cube made of six aluminum sheets, each 1.5×1.5 m, welded together and reinforced with internal braces of negligible weight. The cube is mounted on a fixed base at *A* and can rotate freely about its vertical diagonal *AB*. As she passes by this display on the way to a class in mechanics, an engineering student grabs corner *C* of the cube and pushes it for 1.2 s in a direction perpendicular to the plane *ABC* with an average force of 50 N. Having observed that it takes 5 s for the cube to complete one full revolution, she flips out her calculator and proceeds to determine the mass of the cube. What is the result of her calculation? (*Hint:* The perpendicular distance from the diagonal joining two vertices of a cube to any of its other six vertices can be obtained by multiplying the side of the cube by $\sqrt{2/3}$.)

SOLUTION

Let $m' = \frac{1}{6}m$ be the mass of one side of the cube. Choose *x*, *y*, and *z* axes perpendicular to the face of the cube. Let *a* be the side of the cube. For a side perpendicular to the *x* axis, $(I_x)_1 = \frac{1}{6}m'a^2$.

For a side perpendicular to the *y* or *z* axis, $(I_x)_2 = \left(\frac{1}{12} + \frac{1}{4}\right)m'a^2 = \frac{1}{3}m'a^2$

Total moment of inertia: $I_x = 2(I_x)_1 + 4(I_x)_2 = \frac{5}{3}m'a^2 = \frac{5}{18}ma^2$

By symmetry, $I_y = I_x$ and $I_z = I_x$.

Since all three moments of inertia are equal, the ellipsoid of inertia is a sphere. All centroidal axes are principal axes.

Moment of inertia about the vertical axis: $I_v = \frac{5}{18}ma^2$

Let $b = \sqrt{\frac{2}{3}}a$ be the moment arm of the impulse applied to the corner.

Using the impulse-momentum principle and taking moments about the vertical axis,

$$bF(\Delta t) = H_v = I_v\omega = \frac{5}{18}ma^2\omega \qquad (1)$$

Data: $a = 1.5 \text{ m}, \quad b = \sqrt{\frac{2}{3}}(1.5) = 1.22474 \text{ m}$

$$\omega = \frac{2\pi}{5} = 1.25664 \text{ rad/s}, \quad F = 50 \text{ N}, \quad \Delta t = 1.2 \text{ s}.$$

PROBLEM 18.50 (Continued)

Solving Equation (1) for *m*,

$$m = \frac{18}{5}\frac{bF(\Delta t)}{a^2\omega} = \frac{18}{5}\frac{(1.22474)(50)(1.2)}{(1.5)^2(1.25664)} = 93.563\ \text{kg}$$

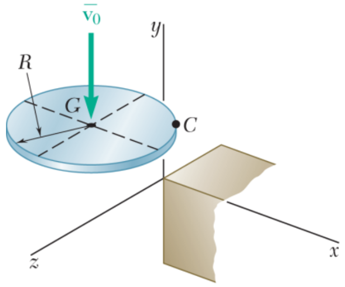
$$m = 93.6\ \text{kg}$$

For principal axes,

$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$T = \frac{1}{2}I_v\omega^2 = \frac{1}{2}\left(\frac{5}{18}ma^2\right)\omega^2$$

$$= \frac{1}{2}\frac{5}{18}(93.563)(1.5)^2(1.2566)^2 \qquad T = 46.2\ \text{J} \blacktriangleleft$$



PROBLEM 18.51

Determine the kinetic energy lost when edge *C* of the plate of Problem 18.29 hits the obstruction.

SOLUTION

For principal moments of inertia, the kinetic energy is

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}_x\omega_x^2 + \frac{1}{2}\bar{I}_y\omega_y^2 + \frac{1}{2}\bar{I}_z\omega_z^2$$

Before impact:

$$\bar{v} = \bar{v}_0$$

$$\omega_x = \omega_y = \omega_z = 0$$

$$T_0 = \frac{1}{2}m\bar{v}_0^2$$

After impact:

From Problem 18.30,

$$\bar{v}_x = \bar{v}_z = 0$$

$$\bar{v}_y = -\frac{4}{5}\bar{v}_0$$

$$\bar{v} = \frac{4}{5}\bar{v}_0$$

From Problem 18.29,

$$\omega_x = \omega_z = \frac{2\sqrt{2}}{5}\frac{v_0}{R}$$

$$\omega_y = 0$$

$$\bar{I}_x = \bar{I}_z = \frac{1}{4}mR^2$$

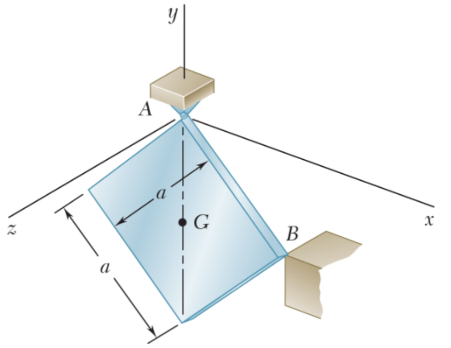
$$T = \frac{1}{2}m\left(\frac{4}{5}v_0\right)^2 + \left(\frac{1}{2}\right)\left(\frac{1}{4}mR^2\right)\left(\frac{2\sqrt{2}}{5}\frac{v_0}{R}\right)^2 + 0 + \left(\frac{1}{2}\right)\left(\frac{1}{4}mR^2\right)\left(\frac{2\sqrt{2}}{5}\frac{\bar{v}_0}{R}\right)^2$$

$$= \frac{1}{2}\left[\frac{16}{25} + \frac{2}{25} + 0 + \frac{2}{25}\right]m\bar{v}_0^2 = \frac{2}{5}m\bar{v}_0^2$$

Energy loss:

$$T_0 - T = \frac{1}{2}m\bar{v}_0^2 - \frac{2}{5}m\bar{v}_0^2$$

$$T_0 - T = \frac{1}{10}m\bar{v}_0^2 \blacktriangleleft$$



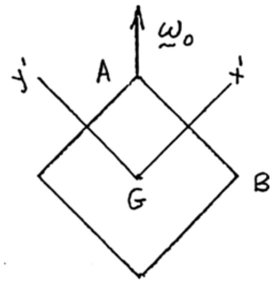
PROBLEM 18.52

Determine the kinetic energy lost when the plate of Problem 18.31 hits the obstruction at *B*.

SOLUTION

For the x' and y' axes shown, the initial angular velocity $\omega_0 \mathbf{j}$ has components

$$\omega_{x'} = \frac{\sqrt{2}}{2}\omega_0, \quad \omega_{y'} = \frac{\sqrt{2}}{2}\omega_0$$



Initial angular momentum about the mass center:

$$(\mathbf{H}_G)_0 = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}'$$

$$= \frac{1}{12}ma^2\frac{\sqrt{2}}{2}\omega_0(\mathbf{i}' + \mathbf{j}')$$

Initial velocity of the mass center: $\quad \bar{\mathbf{v}}_0 = 0$

Let $\boldsymbol{\omega}$ be the angular velocity and $\bar{\mathbf{v}}$ be the velocity of the mass center immediately after impact.
Let $(F\Delta t)\mathbf{k}$ be the impulse at *B*.

Kinematics:

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = (\omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times (-a\mathbf{j}')$$

$$\mathbf{v}_B = a(\omega_{z'}\mathbf{i}' + \omega_{x'}\mathbf{k}')$$

Since the corner *B* does not rebound, $\quad (v_B)_{z'} = 0 \quad \text{or} \quad \omega_{x'} = 0$

$$\bar{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{r}_{G/A} = (\omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times \left(\frac{1}{2}a\right)(-\mathbf{i}' - \mathbf{j}')$$

$$= \frac{1}{2}a(\omega_{z'}\mathbf{i}' - \omega_{z'}\mathbf{j} + \omega_y\mathbf{k}')$$

Also,

$$\mathbf{r}_{G/A} \times m\bar{\mathbf{v}} = \frac{1}{4}ma^2(-\omega_{y'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + 2\omega_{z'}\mathbf{k}')$$

and

$$\mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_{z'}\omega_{z'}\mathbf{k}'$$

$$= \frac{1}{12}ma^2\omega_{y'}\mathbf{j}' + \frac{1}{6}ma^2\omega_{z'}\mathbf{k}'$$

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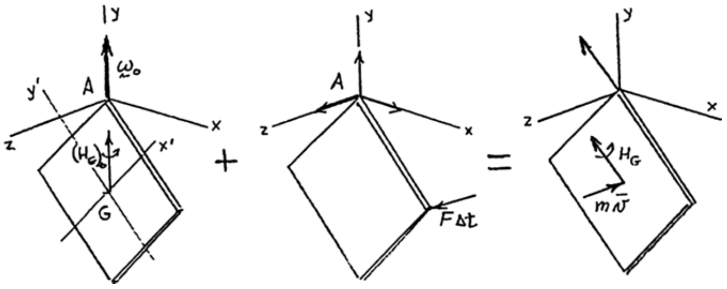
PROBLEM 18.52 (Continued)

Principle of impulse-momentum.



Moments about A:

$$(\mathbf{H}_A)_0 + (-a\mathbf{j}) \times (F\Delta t)\mathbf{k} = \mathbf{H}_A$$

$$(\mathbf{H}_G)_0 + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}}_0 - (aF\Delta t)\mathbf{i} = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}}$$

Resolve into components.

$$\mathbf{i}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 - aF(\Delta t) = -\frac{1}{4}ma^2\omega_{y'}$$

$$\mathbf{j}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 = \frac{1}{12}ma^2\omega_{y'} + \frac{1}{4}ma^2\omega_{y'} \qquad \omega_{y'} = \frac{\sqrt{2}}{8}\omega_0$$

$$\mathbf{k}': \quad 0 = \frac{1}{6}ma^2\omega_{z'} + \frac{1}{2}ma^2\omega_{z'} \qquad \omega_{z'} = 0$$

$$\boldsymbol{\omega} = \frac{\sqrt{2}}{8}\omega_0\mathbf{j}' = \frac{1}{8}\sqrt{2}\omega_0\frac{\sqrt{2}}{2}(\mathbf{j} - \mathbf{i})$$

$$\bar{\mathbf{v}} = \frac{1}{2}a\omega_y\mathbf{k}' = \frac{\sqrt{2}}{16}a\omega_0\mathbf{k}$$

Kinetic energy: $$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}_x\omega_{x'}^2 + \frac{1}{2}\bar{I}_y\omega_{y'}^2 + \frac{1}{2}\bar{I}_{z'}\omega_{z'}^2$$

Before impact: $$\bar{v}_0 = 0, \quad \omega_{x'} = \frac{\sqrt{2}}{2}\omega_0, \quad \omega_{y'} = \frac{\sqrt{2}}{2}\omega_0, \quad \omega_{z'} = 0$$

$$T_0 = 0 + \frac{1}{2}\left(\frac{1}{12}ma^2\right)\left(\frac{\sqrt{2}}{2}\omega_0\right)^2 + \frac{1}{2}\left(\frac{1}{12}ma^2\right)\left(-\frac{\sqrt{2}}{2}\omega_0\right)^2 + 0 = \frac{1}{24}ma^2\omega_0^2$$

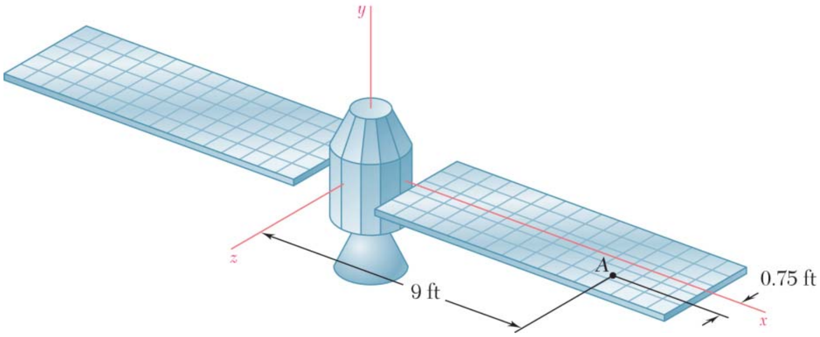
After impact: $$\bar{v} = \frac{\sqrt{2}}{16}a\omega_0, \quad \omega_{x'} = 0, \quad \omega_{y'} = \frac{\sqrt{2}}{8}\omega_0, \quad \omega_{z'} = 0$$

$$T_1 = \frac{1}{2}m\left(\frac{\sqrt{2}}{16}a\omega_0\right)^2 + 0 + \frac{1}{2}\left(\frac{1}{12}ma^2\right)\left(\frac{\sqrt{2}}{8}\omega_0\right)^2 + 0 = \frac{1}{192}ma^2\omega_0^2$$

Kinetic energy lost. $$T_0 - T_1 = \frac{7}{192}ma^2\omega_0^2 \blacktriangleleft$$

PROBLEM 18.53

Determine the kinetic energy of the space probe of Problem 18.33 in its motion about its mass center after its collision with the meteorite.



SOLUTION

Masses: Space probe: $m' = \dfrac{3000}{32.2} = 93.17 \text{ lb}\cdot\text{s}^2/\text{ft}$

Meteorite: $m = \dfrac{5}{(16)(32.2)} = 0.009705 \text{ lb}\cdot\text{s}^2/\text{ft}$

Point of impact: $\mathbf{r}_A = (9 \text{ ft})\mathbf{i} + (0.75 \text{ ft})\mathbf{k}$

Initial linear momentum of the meteorite, $(\text{lb}\cdot\text{s})$:

$$m\mathbf{v}_0 = (0.009705)(2400\mathbf{i} + 3000\mathbf{j} + 3200\mathbf{k}) = 23.292\mathbf{i} - 29.115\mathbf{j} + 31.056\mathbf{k}$$

Its moment about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$\mathbf{r}_A \times m\mathbf{v}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & 0 & 0.75 \\ 23.292 & -29.115 & 31.056 \end{vmatrix} = 21.836\mathbf{i} - 262.04\mathbf{j} - 262.04\mathbf{k}$$

Final linear momentum of meteorite and its moment about the origin, $(\text{lb}\cdot\text{s})$ and $(\text{lb}\cdot\text{s}\cdot\text{ft})$:

$$0.8m\mathbf{v}_0 = 18.634\mathbf{i} - 23.292\mathbf{j} + 24.845\mathbf{k}$$
$$\mathbf{r}_A \times (0.8m\mathbf{v}_0) = 17.469\mathbf{i} - 209.63\mathbf{j} - 209.63\mathbf{k}$$

Let $\mathbf{H}_A$ be the angular momentum of the probe and m' be its mass. Conservation of angular momentum about the origin for a system of particles consisting of the probe plus the meteorite:

$$\mathbf{r}_A \times m\mathbf{v}_0 = \mathbf{H}_A + \mathbf{r}_A \times (0.8m\mathbf{v}_0)$$

$$\mathbf{H}_A = (4.367 \text{ lb} \cdot \text{s} \cdot \text{ft})\mathbf{i} - (52.41 \text{ lb} \cdot \text{s} \cdot \text{ft})\mathbf{j} - (52.41 \text{ lb} \cdot \text{s} \cdot \text{ft})\mathbf{k}$$

$$I_x\omega_x = (H_A)_x \qquad \omega_x = \frac{(H_A)_x}{m'k_x^2} = \frac{4.367}{(93.17)(1.375)^2} = 0.02479 \text{ rad/s}$$

$$I_y\omega_y = (H_A)_y \qquad \omega_y = \frac{(H_A)_y}{m'k_y^2} = \frac{-52.41}{(93.17)(1.425)^2} = -0.2770 \text{ rad/s}$$

$$I_z\omega_z = (H_A)_z \qquad \omega_z = \frac{(H_A)_z}{m'k_z^2} = \frac{-52.41}{(93.17)(1.250)^2} = -0.3600 \text{ rad/s}$$

Kinetic energy of motion of the probe about its mass center:

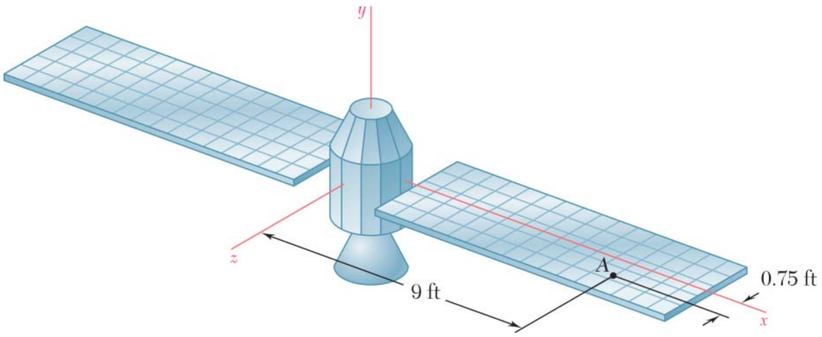
$$T' = \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right) = \frac{m}{2}\left(k_x^2\omega_x^2 + k_y^2\omega_y^2 + k_z^2\omega_z^2\right)$$

$$= \frac{93.17}{2}[(1.375)^2(0.02479)^2 + (1.425)^2(-0.2770)^2 + (1.250)^2(-0.3600)^2]$$

$$= 16.75 \text{ ft} \cdot \text{lb} \qquad T' = 16.75 \text{ ft} \cdot \text{lb} \blacktriangleleft$$

PROBLEM 18.54

Determine the kinetic energy of the space probe of Problem 18.34 in its motion about its mass center after its collision with the meteorite.



SOLUTION

Masses: Space probe: $m' = \dfrac{3000}{32.2} = 93.17\ \text{lb}\cdot\text{s}^2/\text{ft}$

Meteorite: $m = \dfrac{5}{(16)(32.2)} = 0.009705\ \text{lb}\cdot\text{s}^2/\text{ft}$

Point of impact: $\mathbf{r}_A = (9\ \text{ft})\mathbf{i} + (0.75\ \text{ft})\mathbf{k}$

Initial linear momentum of the meteorite, $(\text{lb}\cdot\text{s})$:

$$m\mathbf{v}_0 = (0.009705)(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})$$

Its moment about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$(\mathbf{H}_A)_0 = \mathbf{r}_A \times m\mathbf{v}_0 = 0.009705\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & 0 & 0.75 \\ v_x & v_y & v_z \end{vmatrix}$$

$$= 0.009705[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}]$$

Final linear momentum of the meteorite, $(\text{lb}\cdot\text{s})$:

$$0.75m\mathbf{v}_0 = 0.007279(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})$$

Its moment about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$\mathbf{r}_A \times (0.75m\mathbf{v}_0) = 0.007279[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}]$$

Initial linear momentum of the space probe, $(\text{lb}\cdot\text{s})$: $m'\mathbf{v}'_0 = 0$

Final linear momentum of the space probe, $(\mathrm{lb}\cdot\mathrm{s})$:

$$m'(v'_x\mathbf{i}+v'_y\mathbf{j}+v'_z\mathbf{k}) = 93.17\left(-\frac{0.675}{12}\mathbf{i}+v'_y\mathbf{j}+v'_z\mathbf{k}\right)$$

Final angular momentum of space probe, $(\mathrm{lb}\cdot\mathrm{ft}\cdot\mathrm{s})$:

$$\begin{aligned}\mathbf{H}_A &= m'(k_x^2\omega_x\mathbf{i}+k_y^2\omega_y\mathbf{j}+k_z^2\omega_z\mathbf{k})\\ &= 93.17[(1.375)^2(0.05)\mathbf{i}+(1.425)^2(-0.12)\mathbf{j}+(1.250)^2\omega_z\mathbf{k}]\\ &= 8.8075\mathbf{i}-22.703\mathbf{j}+145.58\omega_z\mathbf{k}\end{aligned}$$

Conservation of linear momentum of the probe plus the meteorite $(\mathrm{lb}\cdot\mathrm{s})$:

$$0.009705(v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{k}) = 0.007279(v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{k})+93.17(-0.05625\mathbf{i}+v'_y\mathbf{j}+v'_z\mathbf{k})$$

$$\mathbf{i}:\quad 0.002426v_x = -5.2408 \qquad v_x = -2160\text{ ft/s}$$

$$\mathbf{j}:\quad 0.002426v_y = 93.17v'_y$$

$$\mathbf{k}:\quad 0.002426v_z = 93.17v'_z$$

Conservation of angular momentum about the origin $(\mathrm{lb}\cdot\mathrm{ft}\cdot\mathrm{s})$:

$$\begin{aligned}(0.009705)[-0.75v_y\mathbf{i}+(0.75v_x-9v_z)\mathbf{j}+9v_y\mathbf{k}] &= (0.007279)[-0.75v_y\mathbf{i}+(0.75v_x-9v_z)\mathbf{j}+9v_y\mathbf{k}]\\ &\quad +8.8075\mathbf{i}-22.703\mathbf{j}+145.58\omega_z\mathbf{k}\end{aligned}$$

$$\mathbf{i}:\quad -0.0018195v_y = 8.8075 \qquad v_y = -4840.5\text{ ft/s}$$

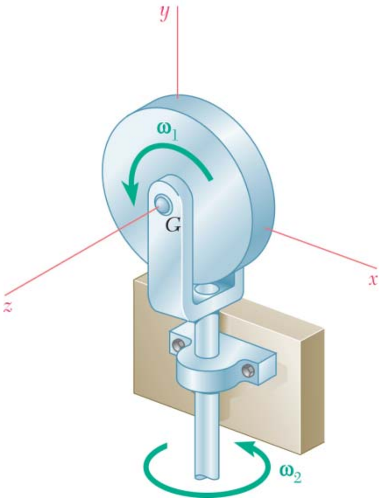
$$\mathbf{k}:\quad -0.021834v_y = 145.58\omega_z$$

$$\omega_z = -149.98\times10^{-6}v_y \qquad \omega_z = -0.726\text{ rad/s}$$

Kinetic energy of motion of probe relative to its mass center:

$$\begin{aligned}T' &= \frac{1}{2}\left(I_x\omega_x^2+I_y\omega_y^2+I_z\omega_z^2\right) = \frac{1}{2}m\left(k_x^2\omega_x^2+k_y^2\omega_z^2\omega_z^2\right)\\ &= \frac{1}{2}(93.17)[(1.375)^2(0.05)^2+(1.425)^2(-0.12)^2+(1.250)^2(-0.726)^2]\\ &= 39.9\text{ ft}\cdot\text{lb}\end{aligned}$$

$T' = 39.9\text{ ft}\cdot\text{lb}$ ◀



PROBLEM 18.55

Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Problem 18.1.

PROBLEM 18.1 A thin, homogeneous disk of mass *m* and radius *r* spins at the constant rate ω_1 about an axle held by a fork-ended vertical rod, which rotates at the constant rate ω_2. Determine the angular momentum $\mathbf{H}_G$ of the disk about its mass center *G*.

SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega_2 \mathbf{j} + \omega_1 \mathbf{k}$

Moments of inertia: $\bar{I}_x = \frac{1}{4} mr^2, \qquad \bar{I}_y = \frac{1}{4} mr^2, \qquad \bar{I}_z = \frac{1}{2} mr^2$

Products of inertia: by symmetry, $\bar{I}_{xy} = \bar{I}_{yz} = \bar{I}_{zx} = 0$

Angular momentum:

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k}$$

$$\mathbf{H}_G = 0 + \left(\frac{1}{4} mr^2\right)\omega_2 \mathbf{j} + \frac{1}{2} mr^2 \omega_1 \mathbf{k}$$

$$\mathbf{H}_G = \frac{1}{4} mr^2 \omega_2 \mathbf{j} + \frac{1}{2} mr^2 \omega_1 \mathbf{k}$$

Rate of change of angular momentum. Let the frame of reference *Gxyz* be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega_1 \mathbf{j}$$

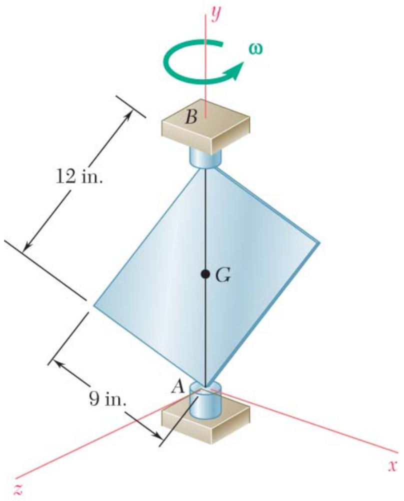
Then

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

$$= 0 + \omega_2 \mathbf{j} \times \left(\frac{1}{4} mr^2 \omega_2 \mathbf{j} + \frac{1}{2} mr^2 \omega_1 \mathbf{k}\right)$$

$$= \frac{1}{2} mr^2 \omega_1 \omega_2 \mathbf{i}$$

$$\dot{\mathbf{H}}_G = \frac{1}{2} mr^2 \omega_1 \omega_2 \mathbf{i} \blacktriangleleft$$

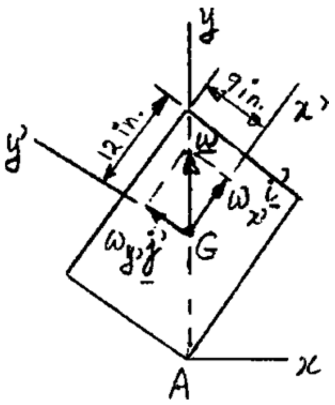


PROBLEM 18.56

Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the plate of Problem 18.2.

PROBLEM 18.2 A thin rectangular plate of weight 15 lb rotates about its vertical diagonal *AB* with an angular velocity *ω*. Knowing that the *z* axis is perpendicular to the plate and that *ω* is constant and equal to 5 rad/s, determine the angular momentum of the plate about its mass center *G*.

SOLUTION



$$h = \sqrt{(9\text{ in.})^2 + (12\text{ in.})^2} = 15\text{ in.}$$

Resolving *ω* along the principal axes x', y', z:

$$\omega_{x'} = \frac{12}{15}\omega = 0.8(5\text{ rad/s}) = 4\text{ rad/s}$$

$$\omega_{y'} = \frac{9}{15}\omega = 0.6(5\text{ rad/s}) = 3\text{ rad/s}$$

$$\omega_z = 0$$

Moments of inertia:

$$I_{x'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{9}{12}\text{ft}\right)^2 = 0.021836\text{ slug}\cdot\text{ft}^2$$

$$I_{y'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{12}{12}\text{ft}\right)^2 = 0.038820\text{ slug}\cdot\text{ft}^2$$

From Eqs. (18.10):

$$H_{x'} = I_{x'}\omega_{x'} = (0.021836)(4) = 0.087345\text{ slug ft}^2\text{/s}$$

$$H_{y'} = I_{y'}\omega_{y'} = (0.038820)(3) = 0.11646\text{ slug ft}^2\text{/s}$$

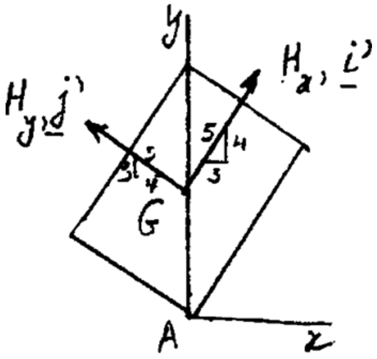
$$H_z = I_z\omega_z = 0$$

$$\mathbf{H}_G = (0.087345\text{ slug ft}^2\text{/s})\mathbf{i}' + (0.11646\text{ slug ft}^2\text{/s})\mathbf{j}'$$

PROBLEM 18.56 (Continued)

Components along x and y axes:

$$H_x = \frac{3}{5}H_{x'} - \frac{4}{5}H_{y'}$$
$$= \frac{3}{5}(0.087345) - \frac{4}{5}(0.11646)$$
$$= -0.040761$$
$$H_y = \frac{4}{5}H_{x'} + \frac{3}{5}H_{y'}$$
$$= \frac{4}{5}(0.087345) + \frac{3}{5}(0.11646)$$
$$= 0.13975$$



$$\mathbf{H}_G = (-0.040761\,\text{slug}\cdot\text{ft}^2/\text{s})\mathbf{i} + (0.13975\,\text{slug}\cdot\text{ft}^2/\text{s})\mathbf{j}$$

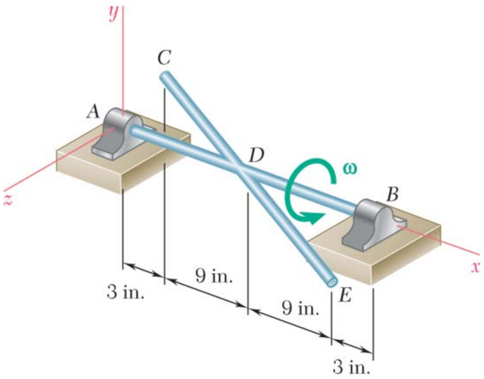
where the frame *Axyz* rotates with the plate with the angular velocity.

$$\boldsymbol{\Omega} = \boldsymbol{\omega} = (5\text{ rod/s})\mathbf{j}$$

We have $(\dot{\mathbf{H}}_G)_{Axyz} = 0.$ Substituting into Eq. (18.22):

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_G = 0 + 5\mathbf{j}\times(-0.040761\mathbf{i} + 0.13975\mathbf{j})$$
$$= (0.20380\text{ ft/lb})\mathbf{k}$$

$$\dot{\mathbf{H}}_G = (0.204\text{ ft/lb})\mathbf{k}$$ ◀

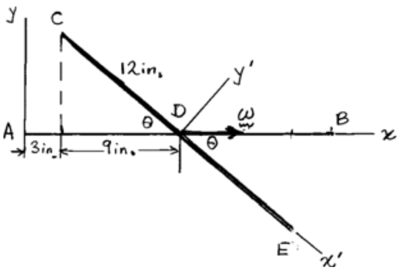


PROBLEM 18.57

Determine the rate of change $\dot{\mathbf{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Problem 18.3.

PROBLEM 18.3 Two uniform rods *AB* and *CE*, each of weight 3 lb and length 2 ft, are welded to each other at their midpoints. Knowing that this assembly has an angular velocity of constant magnitude $\omega = 12$ rad/s, determine the magnitude and direction of the angular momentum $\mathbf{H}_D$ of the assembly about *D*.

SOLUTION



$$m = \frac{W}{g} = \frac{3}{32.2} = 0.093168 \text{ lb}\cdot\text{s}^2/\text{ft}, \qquad l = 24 \text{ in.} = 2 \text{ ft},$$

$$\boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{i}$$

For rod *ADB*, $\mathbf{H}_D = I_x \omega \mathbf{i} \approx 0$, since $I_x \approx 0$.

For rod *CDE*, use principal axes x', y' as shown.

$$\cos\theta = \frac{9}{12}, \qquad \theta = 41.410°$$

$$\omega_{x'} = \omega\cos\theta = 9 \text{ rad/s}^2$$

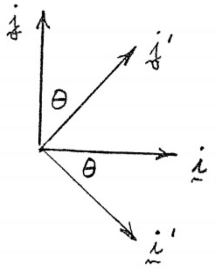
$$\omega_{y'} = \omega\sin\theta = 7.93725 \text{ rad/s}^2$$

$$\omega_{z'} = 0$$

$$I_{x'} \approx 0$$

$$I_{y'} = \frac{1}{12}ml^2 = \frac{1}{12}(0.093168)(2)^2$$

$$= 0.0310559 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$



$$\mathbf{H}_D = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + I_{z'}\omega_{z'}\mathbf{k}'$$

$$= 0 + (0.0310559)(7.93725)\mathbf{j}' + 0$$

$$= 0.246498\mathbf{j}'$$

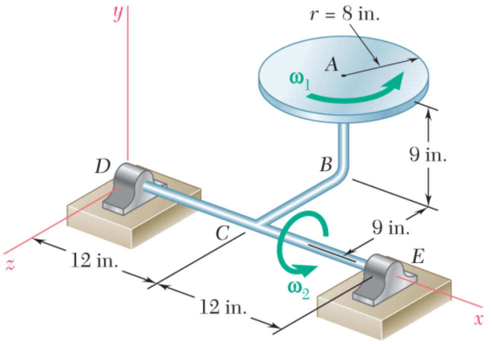
$$\mathbf{H}_D = 0.246498(\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = 0.163045\mathbf{i} + 0.184874\mathbf{j}$$

Let the frame of reference *Dxyz* be rotating with angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{i}$$

Then, $\dot{\mathbf{H}}_D = (\dot{\mathbf{H}}_D)_{Dxyz} + \boldsymbol{\Omega} \times \mathbf{H}_D = 0 + \boldsymbol{\omega} \times \mathbf{H}_D$

$$\dot{\mathbf{H}}_D = 12\mathbf{i} \times (0.163045\mathbf{i} + 0.184874\mathbf{j}) \qquad \dot{\mathbf{H}}_D = (2.22 \text{ lb}\cdot\text{ft})\mathbf{k} \blacktriangleleft$$



PROBLEM 18.58

Determine the rate of change $\dot{\mathbf{H}}_A$ of the angular momentum $\mathbf{H}_A$ of the disk of Problem 18.4.

PROBLEM 18.4 A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm *ABC*, which is welded to a shaft *DCE* rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the angular momentum $\mathbf{H}_A$ of the disk about its center *A*.

SOLUTION

$$\boldsymbol{\omega} = \omega_2\mathbf{i} + \omega_1\mathbf{j} = (8 \text{ rad/s})\mathbf{i} + (16 \text{ rad/s})\mathbf{j}$$

For axes x', y', z' parallel to x, y, z with origin at A,

$$m = \frac{W}{g} = \frac{6}{32.2} = 0.186335 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$\bar{I}_{x'} = \frac{1}{4}mr^2 = \frac{1}{4}(0.186335)\left(\frac{8}{12}\right)^2 = 0.020704 \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

$$\bar{I}_{z'} = \bar{I}_{x'} = 0.020704 \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

$$\bar{I}_{y'} = \bar{I}_{x'} + \bar{I}_{z'} = 0.041408 \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

$$\begin{aligned} \mathbf{H}_A &= \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\,\omega_{y'}\,\mathbf{j} + I_{z'}\,\omega_{z'}\mathbf{k} \\ &= (0.020704)(8)\mathbf{i} + (0.041408)(16)\mathbf{j} \\ &= (0.1656 \text{ lb} \cdot \text{s} \cdot \text{ft})\mathbf{i} + (0.6625 \text{ lb} \cdot \text{s} \cdot \text{ft})\mathbf{j} \end{aligned}$$

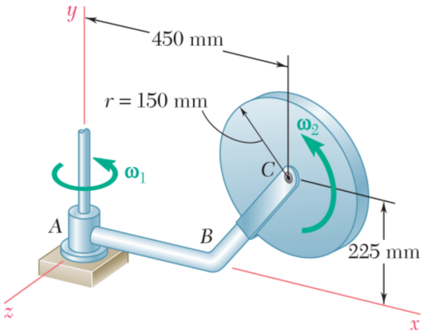
Let the frame of reference *Axyz* be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega_2\mathbf{i} = (8 \text{ rad/s})\mathbf{i}$$

Then

$$\dot{\mathbf{H}}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega} \times \mathbf{H}_A = 0 + \omega_2\mathbf{i} \times \mathbf{H}_A$$

$$\dot{\mathbf{H}}_A = 8\mathbf{i} \times (0.1656\mathbf{i} + 0.6625\mathbf{j}) = (5.30 \text{ lb} \cdot \text{ft})\mathbf{k} \qquad \dot{\mathbf{H}}_A = (5.30 \text{ lb} \cdot \text{ft})\mathbf{k} \blacktriangleleft$$


y
450 mm
r = 150 mm
ω_2
ω_1
C
A
B
225 mm
z
x

PROBLEM 18.59

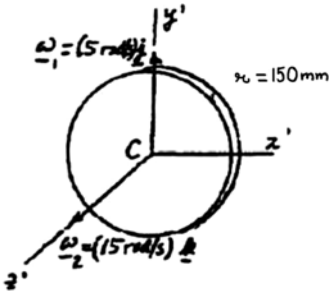
Determine the rate of change $\dot{\mathbf{H}}_C$ of the angular momentum $\mathbf{H}_C$ of the disk of Problem 18.5.

PROBLEM 18.5 A thin disk of mass $m = 4$ kg rotates at the constant rate $\omega_2 = 15$ rad/s with respect to arm *ABC*, which itself rotates at the constant rate $\omega_1 = 5$ rad/s about the *y* axis. Determine the angular momentum of the disk about its center *C*.

SOLUTION

$$r = 150 \text{ mm}$$

Angular velocity of disk:


y'
$\underline{\omega}_1 = (5 \text{ rad/s})\underline{j}$
r = 150 mm
C
z'
$\underline{\omega}_2 = (15 \text{ rad/s})\underline{k}$
z'

$$\boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k}$$
$$= (5 \text{ rad/s})\mathbf{j} + (15 \text{ rad/s})\mathbf{k}$$

Centroidal moments of inertia:

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4} m r^2$$
$$= \frac{1}{4}(4)(0.150 \text{ m})^2 = 0.0225 \text{ kg} \cdot \text{m}^2$$
$$\bar{I}_{z'} = \frac{1}{2} m r^2 = 0.045 \text{ kg} \cdot \text{m}^2$$

Angular momentum about *C*.

$$\mathbf{H}_C = \bar{I}_{x'}\, \omega_{x'} \mathbf{i} + \bar{I}_{y'}\, \omega_{y'} \mathbf{j} + I_{z'}\, \omega_{z'} \mathbf{k}$$
$$= 0 + (0.0225)(5)\mathbf{j} + (0.045)(15)\mathbf{k}$$
$$= (0.1125 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{j} + (0.6750 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{k}$$

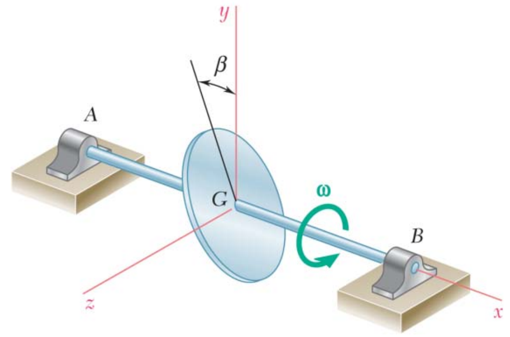
Rate of change of angular momentum. Let the reference frame *Oxyz* be rotating with angular velocity.

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_1 = (5 \text{ rad/s})\mathbf{j}$$

Then

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{H}_C$$
$$= 0 + 5\mathbf{j} \times (0.1125\mathbf{j} + 0.6750\mathbf{k})$$
$$= (3.3750 \text{ N} \cdot \text{m})\mathbf{i}$$

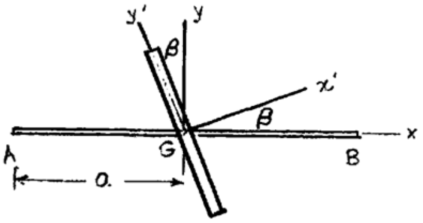
$$\dot{\mathbf{H}}_C = (3.38 \text{ N} \cdot \text{m})\mathbf{i} \blacktriangleleft$$



PROBLEM 18.60

Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.8 for an arbitrary value of β, knowing that its angular velocity $\boldsymbol{\omega}$ remains constant.

SOLUTION



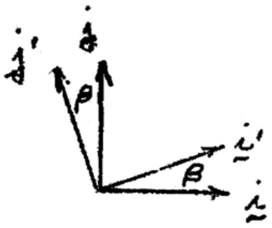
Use principal axes x', y', z' as shown.

$$\mathbf{i}' = \cos\beta\mathbf{i} + \sin\beta\mathbf{j}$$

$$\mathbf{j}' = -\sin\beta\mathbf{i} + \cos\beta\mathbf{j}$$

$$\mathbf{i} = \cos\beta\mathbf{i}' - \sin\beta\mathbf{j}'$$

$$\mathbf{j} = \sin\beta\mathbf{i}' + \cos\beta\mathbf{j}'$$



$$\boldsymbol{\omega} = \omega\mathbf{i} = (\omega\cos\beta)\mathbf{i}' - (\omega\sin\beta)\mathbf{j}'$$

$$\boldsymbol{\alpha} = \dot{\omega}\mathbf{i} = (\dot{\omega}\cos\beta)\mathbf{i}' - (\dot{\omega}\sin\beta)\mathbf{j}' = 0$$

Principal moments of inertia:

$$I_{x'} = \frac{1}{2}mr^2, \qquad I_{y'} = I_{z'} = \frac{1}{4}mr^2$$

Angular momentum:

$$\mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + I_{z'}\omega_{z'}\mathbf{k}'$$

$$= \frac{1}{2}mr^2\omega\cos\beta\mathbf{i}' - \frac{1}{4}mr^2\omega\sin\beta\mathbf{j}'$$

Let the frame of reference $Gx'y'z'$ be rotating with angular velocity

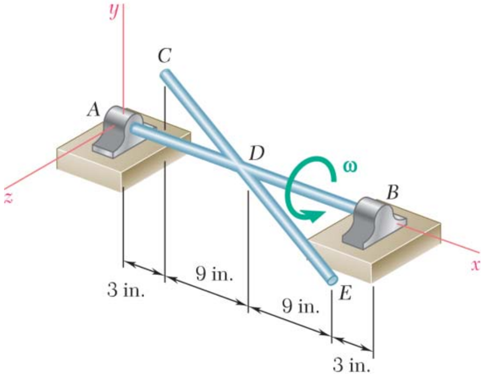
$$\boldsymbol{\Omega} = \omega$$

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gx'y'z'} + \boldsymbol{\Omega} \times \mathbf{H}_G = 0 + \boldsymbol{\Omega} \times \mathbf{H}_G$$

$$= (\omega\cos\beta\mathbf{i}' - \omega\sin\beta\mathbf{j}') \times \left(\frac{1}{2}mr^2\omega\cos\beta\mathbf{i}' - \frac{1}{4}mr^2\sin\beta\mathbf{j}'\right)$$

$$= \frac{1}{4}mr^2\omega^2\sin\beta\cos\beta\mathbf{k}$$

$$\dot{\mathbf{H}}_G = \frac{1}{4}mr^2\omega^2\sin\beta\cos\beta\mathbf{k} \blacktriangleleft$$



PROBLEM 18.61

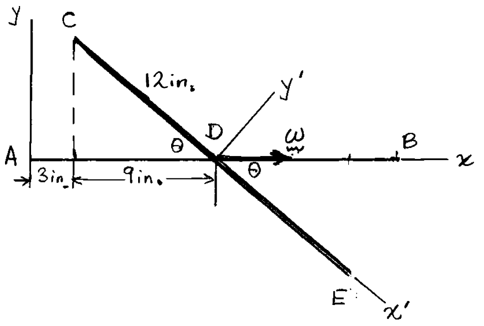
Determine the rate of change $\dot{\mathbf{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Problem 18.3, assuming that at the instant considered the assembly has an angular velocity $\boldsymbol{\omega} = (12\,\text{rad/s})\mathbf{i}$ and an angular acceleration $\alpha = -(96\text{ rad/s}^2)\mathbf{i}$.

SOLUTION

$$m = \frac{W}{g} = \frac{3}{32.2} = 0.093168\text{ lb}\cdot\text{s}^2/\text{ft}^2, \quad l = 24\text{ in.} = 2\text{ ft}, \quad \boldsymbol{\omega} = (12\text{ rad/s})\mathbf{i}$$

For rod *ADB*, $\mathbf{H}_D = I_x\omega\mathbf{i} \approx 0,$ since $I_x \approx 0.$

For rod *CDE*, use principal axes x', y' as shown.



$$\cos\theta = \frac{9}{12}, \quad \theta = 41.410°$$

$$\omega_{x'} = \omega\cos\theta = 9\text{ rad/s}^2$$

$$\omega_{y'} = \omega\sin\theta = 7.93725\text{ rad/s}^2$$

$$\omega_{z'} = 0$$

$$\alpha = -(96\text{ rad/s}^2)\mathbf{i}$$

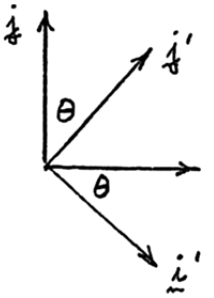
$$\alpha_{x'} = \alpha\cos\theta = -72\text{ rad/s}$$

$$\alpha_{y'} = \alpha\sin\theta = -63.498\text{ rad/s}^2$$

$$I_{x'} \approx 0$$

$$I_{y'} = \frac{1}{12}ml^2 = \frac{1}{12}(0.93168)(2)^2 = 0.0310559\text{ kg}\cdot\text{m}^2$$

$$I_{z'} = I_{y'}$$



$$\mathbf{H}_D = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + I_{z'}\omega_{z'}\mathbf{k}'$$

$$= 0 + (0.0310559)(7.93725)\mathbf{j}' + 0 = (0.24698\text{ lb}\cdot\text{s/ft})\mathbf{j}'$$

Let the reference frame $Dx'y'z'$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega\mathbf{i} = \omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' = (9\text{ rad/s})\mathbf{i}' + (7.93725\text{ rad/s})\mathbf{j}'$$

$$\dot{\mathbf{H}}_D = (\dot{\mathbf{H}}_D)_{Dx'y'z'} + \boldsymbol{\Omega}\times\mathbf{H}_D$$

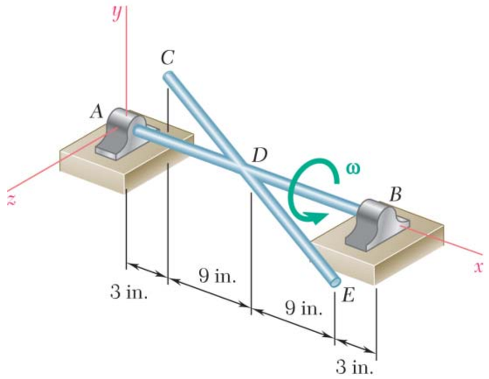
$$= I_{x'}\alpha_{x'}\mathbf{i}' + I_{y'}\alpha_{y'}\mathbf{j}' + I_{z'}\alpha_{z'}\mathbf{k}' + \boldsymbol{\Omega}\times\mathbf{H}_D$$

$$= 0 + (0.0310559)(-63.498)\mathbf{j} + 0 + (9\mathbf{i}' + 7.93725\mathbf{j}')\times(0.246498\mathbf{j}')$$

$$= -1.97199\mathbf{j}' + 2.21848\mathbf{k}$$

$$= -(1.97199\text{ kg}\cdot\text{m}^2/\text{s}^2)(\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + (2.21848\text{ kg}\cdot\text{m}^2/\text{s}^2)\mathbf{k}$$

$$\dot{\mathbf{H}}_D = -(1.304\text{ N}\cdot\text{m})\mathbf{i} - (1.479\text{ N}\cdot\text{m})\mathbf{j} + (2.22\text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$$



PROBLEM 18.62

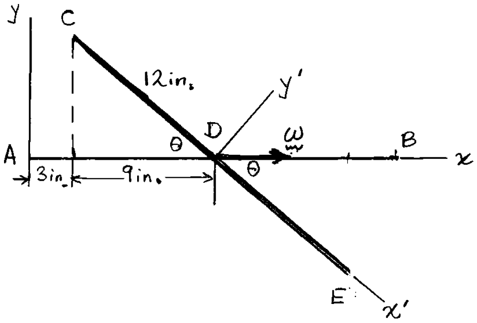
Determine the rate of change $\dot{\mathbf{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Problem 18.3, assuming that at the instant considered the assembly has an angular velocity $\boldsymbol{\omega} = (12\text{ rad/s})\mathbf{i}$ and an angular acceleration $\boldsymbol{\alpha} = (96\text{ rad/s}^2)\mathbf{i}$.

SOLUTION

$$m = \frac{W}{g} = \frac{3}{32.2} = 0.093168\text{ lb}\cdot\text{s}^2/\text{ft}, \quad l = 24\text{ in.} = 2\text{ ft}, \quad \boldsymbol{\omega} = (12\text{ rad/s})\mathbf{i}$$

For rod *ADB*, $\mathbf{H}_D = I_x\omega\mathbf{i} \approx 0$, since $I_x \approx 0$.

For rod *CDE*, use principal axes x', y' as shown.



$$\cos\theta = \frac{9}{12}, \quad \theta = 41.410°$$

$$\omega_{x'} = \omega\cos\theta = 9\text{ rad/s}^2$$

$$\omega_{y'} = \omega\sin\theta = 7.93725\text{ rad/s}^2$$

$$\omega_{z'} = 0$$

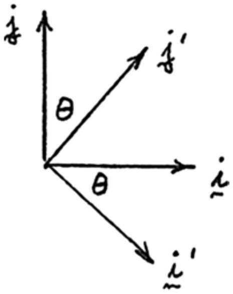
$$\alpha = (96\text{ rad/s}^2)\mathbf{i}$$

$$\alpha_{x'} = \alpha\cos\theta = 72\text{ rad/s}$$

$$\alpha_{y'} = \alpha\sin\theta = 63.498\text{ rad/s}^2$$

$$I_{x'} \approx 0$$

$$I_{y'} = \frac{1}{12}ml^2 = \frac{1}{12}(0.93168)(2)^2 = 0.0310559\text{ kg}\cdot\text{m}^2$$



$$I_{z'} = I_{y'}$$

$$\mathbf{H}_D = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + I_{z'}\omega_{z'}\mathbf{k}'$$

$$= 0 + (0.0310559)(7.93725)\mathbf{j} + 0 = (0.246498\text{ lb}\cdot\text{s/ft})\mathbf{j}'$$

Let the reference frame $Dx'y'z'$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega\mathbf{i} = \omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' = (9\text{ rad/s})\mathbf{i}' + (7.93725\text{ rad/s})\mathbf{j}'$$

$$\dot{\mathbf{H}}_D = (\dot{\mathbf{H}}_D)_{Dx'y'z'} + \boldsymbol{\Omega}\times\mathbf{H}_D$$

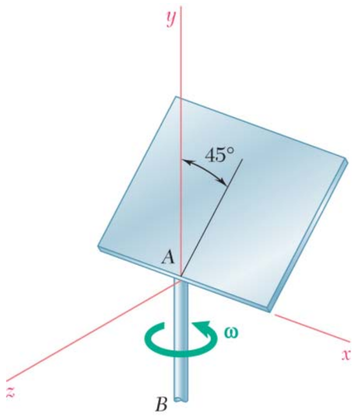
$$= I_{x'}\alpha_{x'}\mathbf{i}' + I_{y'}\alpha_{y'}\mathbf{j}' + I_{z'}\alpha_{z'}\mathbf{k}' + \boldsymbol{\Omega}\times\mathbf{H}_D$$

$$= 0 + (0.310559)(63.498)\mathbf{j} + 0 + (9\mathbf{i}' + 7.93725\mathbf{j}')\times(0.246498\mathbf{j}')$$

$$= 1.97199\mathbf{j}' + 2.21848\mathbf{k}$$

$$= (1.97199\text{ kg}\cdot\text{m}^2/\text{s}^2)(\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + (2.21848\text{ kg}\cdot\text{m}^2/\text{s}^2)\mathbf{k}$$

$$\dot{\mathbf{H}}_D = (1.304\text{ N}\cdot\text{m})\mathbf{i} + (1.479\text{ N}\cdot\text{m})\mathbf{j} + (2.22\text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$$

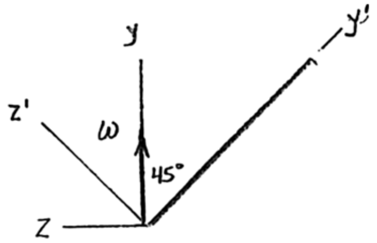


PROBLEM 18.63

A thin homogeneous square of mass m and side a is welded to a vertical shaft AB with which it forms an angle of 45°. Knowing that the shaft rotates with an angular velocity $\boldsymbol{\omega} = \omega \mathbf{j}$ and an angular acceleration $\boldsymbol{\alpha} = \alpha \mathbf{j}$, determine the rate of change $\dot{\mathbf{H}}_A$ of the angular momentum $\mathbf{H}_A$ of the plate assembly.

SOLUTION

Use principal axes y', z' as shown.



$$\omega_{y'} = \omega \cos 45°, \quad \omega_{z'} = \omega \sin 45°$$

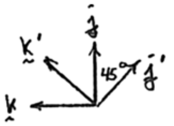
$$\omega_{x'} = 0$$

$$I_{x'} = \frac{1}{3} ma^2, \quad I_{y'} = \frac{1}{12} ma^2$$

$$I_{z'} = I_{x'} + I_{y'} = \frac{5}{12} ma^2$$

$$\mathbf{H}_A = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + I_{z'}\omega_{z'}\mathbf{k}'$$

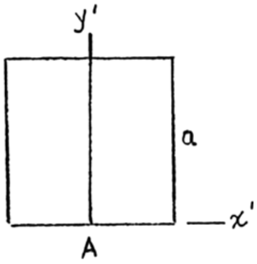
$$= 0 + \left(\frac{1}{12} ma^2\right)(\omega \cos 45°)\mathbf{j}' + \left(\frac{5}{12} ma^2\right)(\omega \sin 45°)\mathbf{k}'$$



$$\mathbf{H}_A = \left(\frac{1}{12} ma^2\right)(\omega \cos 45°)(\cos 45° \mathbf{j} - \sin 45° \mathbf{k})$$

$$+ \left(\frac{5}{12} ma^2\right)(\omega \sin 45°)(\sin 45° \mathbf{j} + \cos 45° \mathbf{k})$$

$$\mathbf{H}_A = \frac{ma^2\omega}{12}(3\mathbf{j} + 2\mathbf{k})$$



Angular acceleration. $\boldsymbol{\alpha} = \alpha \mathbf{j}; \quad \dot{\omega}_{y'} = \alpha \cos 45°, \quad \dot{\omega}_{z'} = \alpha \sin 45°$

Let the reference frame $Axyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega \mathbf{j}$$

With respect to this frame,

$$(\dot{\mathbf{H}}_A)_{Axyz} = I_{x'}\dot{\omega}_{x'}\mathbf{i}' + I_{y'}\dot{\omega}_{y'}\mathbf{j}' + I_{z'}\dot{\omega}_{z}\mathbf{k}'$$

$$= 0 + \left(\frac{1}{12} ma^2\right)(\alpha \cos 45°)\mathbf{i}' + \left(\frac{5}{12} ma^2\right)(\alpha \sin 45°)\mathbf{k}'$$

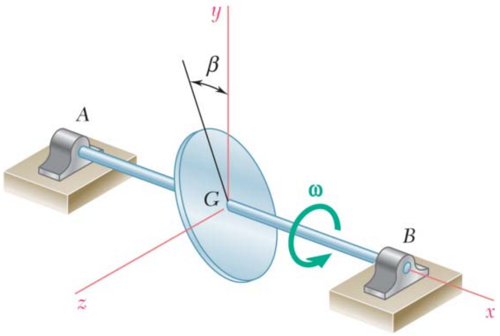
$$= \frac{ma^2\alpha}{12}(3\mathbf{j} + 2\mathbf{k})$$

PROBLEM 18.63 (Continued)

With respect to the fixed reference frame,

$$\begin{aligned}\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega} \times \mathbf{H}_A \\ &= \frac{ma^2\alpha}{12}(3\mathbf{j} + 2\mathbf{k}) + \omega\mathbf{j} \times \left[\frac{ma^2\omega}{12}(3\mathbf{j} + 2\mathbf{k})\right]\end{aligned}$$

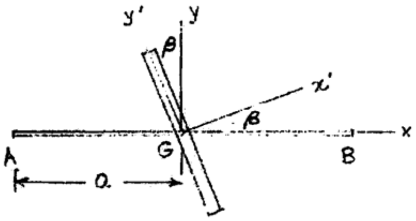
$$\dot{\mathbf{H}}_A = \frac{ma^2}{12}(2\omega^2\mathbf{i} + 3\alpha\mathbf{j} + 2\alpha\mathbf{k}) \blacktriangleleft$$

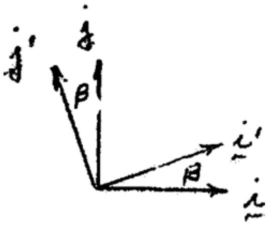


PROBLEM 18.64

Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.8 for an arbitrary value of β, knowing that the disk has an angular velocity $\boldsymbol{\omega} = \omega\mathbf{i}$ and an angular acceleration $\boldsymbol{\alpha} = \alpha\mathbf{i}$.

SOLUTION





Use principal axes x', y', z' as shown.

$$\mathbf{i}' = \cos\beta\mathbf{i} + \sin\beta\mathbf{j}$$

$$\mathbf{j}' = -\sin\beta\mathbf{i} + \cos\beta\mathbf{j}$$

$$\mathbf{i} = \cos\beta\mathbf{i}' - \sin\beta\mathbf{j}'$$

$$\mathbf{j} = \sin\beta\mathbf{i}' + \cos\beta\mathbf{j}'$$

$$\boldsymbol{\omega} = \omega\mathbf{i} = (\omega\cos\beta)\mathbf{i}' - (\omega\sin\beta)\mathbf{j}'$$

$$\boldsymbol{\alpha} = \alpha\mathbf{i} = (\alpha\cos\beta)\mathbf{i}' - (\alpha\sin\beta)\mathbf{j}'$$

Principal moments of inertia:

$$I_{x'} = \frac{1}{2}mr^2, \qquad I_{y'} = I_{z'} = \frac{1}{4}mr^2$$

Angular momentum:

$$\mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j} + I_{z'}\omega_{z'}\mathbf{k}$$

$$= \frac{1}{2}mr^2\omega\cos\beta\mathbf{i} - \frac{1}{4}mr^2\omega\sin\beta\mathbf{j}$$

Let the frame of reference $Gx'y'z'$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega$$

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gx'y'z'} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

$$= I_{x'}\alpha_{x'}\mathbf{i}' + I_{y'}\alpha_{y'}\mathbf{j}' + I_{z'}\alpha_{z'}\mathbf{k}$$

$$+ (\omega\cos\beta\mathbf{i}' - \omega\sin\beta\mathbf{j}') \times \left(\frac{1}{2}mr^2\omega\cos\beta\mathbf{i}' - \frac{1}{4}mr^2\sin\beta\mathbf{j}'\right)$$

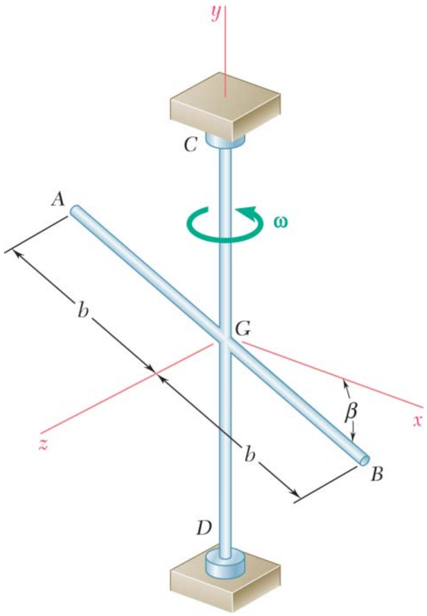
PROBLEM 18.64 (Continued)

$$= \left(\frac{1}{2}mr^2\right)(\alpha\cos\beta)(\cos\beta\mathbf{i} + \sin\beta\mathbf{j})$$

$$+ \left(\frac{1}{4}mr^2\right)(-\alpha\sin\beta)(-\sin\beta\mathbf{i} + \cos\beta\mathbf{j}) + 0$$

$$+ \frac{1}{4}mr^2\omega^2\sin\beta\cos\beta\mathbf{k}$$

$$\dot{\mathbf{H}}_G = \frac{1}{4}mr^2\alpha\left(2\cos^2\beta + \sin^2\beta\right)\mathbf{i} \qquad \frac{1}{4}mr^2\alpha\sin\beta\cos\beta\mathbf{j} + \frac{1}{4}mr^2\omega^2\sin\beta\cos\beta\mathbf{k} \blacktriangleleft$$



PROBLEM 18.65

A slender, uniform rod *AB* of mass *m* and a vertical shaft *CD*, each of length 2*b*, are welded together at their midpoints *G*. Knowing that the shaft rotates at the constant rate *ω*, determine the dynamic reactions at *C* and *D*.

SOLUTION

Using the principal axes $Gx'y'z$:

$$\bar{I}_{x'} = 0, \quad \bar{I}_{y'} = \bar{I}_z = \frac{1}{3}mb^2$$

$$\omega_{x'} = -\omega \sin\beta, \quad \omega_{y'} = \omega\cos\beta, \quad \omega_z = 0$$

Angular momentum about *G*.

$$\mathbf{H}_G = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_z\omega_z\mathbf{k}$$

$$\mathbf{H}_G = \frac{1}{3}mb^2\omega\cos\beta\,\mathbf{j}'$$

or, since $\mathbf{j}' = \mathbf{i}\sin\beta + \mathbf{j}\cos\beta$:

$$\mathbf{H}_G = \frac{1}{3}mb^2\omega\cos\beta(\sin\beta\,\mathbf{i} + \cos\beta\,\mathbf{j}) \qquad (1)$$

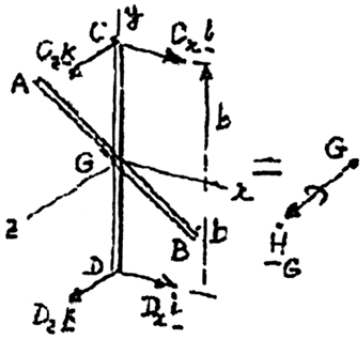
Rate of change of angular momentum.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega}\times\mathbf{H}_G = 0 + \boldsymbol{\omega}\times\mathbf{H}_B$$

$$= \omega\mathbf{j}\times\left[\frac{1}{3}mb^2\omega\cos\beta(\sin\beta\,\mathbf{i} + \cos\beta\,\mathbf{j})\right]$$

$$= -\frac{1}{3}mb^2\omega^2\sin\beta\cos\beta\,\mathbf{k}$$

PROBLEM 18.65 (Continued)


C
y
$C_z\mathbf{k}$
$C_x\mathbf{i}$
A
b
G
x
=
G
z
B
b
$\dot{\mathbf{H}}_G$
D
$D_z\mathbf{k}$
$D_x\mathbf{i}$

Equations of motion.

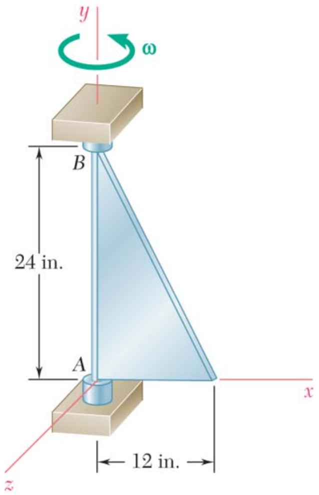
We equate the systems of external and effective forces

$$\Sigma\mathbf{M}_D = \Sigma(\mathbf{M}_D)_{\text{eff}}: \quad 2b\mathbf{j}\times(C_x\mathbf{i}+C_z\mathbf{k}) = \dot{\mathbf{H}}_G$$

$$-2bC_x\mathbf{k} + 2bC_z\mathbf{i} = -\frac{1}{3}mb^2\omega^2\sin\beta\cos\beta\,\mathbf{k}$$

Thus, $\quad C_x = \frac{1}{6}mb\omega^2\sin\beta\cos\beta, \quad C_z = 0 \qquad \mathbf{C} = \frac{1}{6}mb\omega^2\sin\beta\cos\beta\mathbf{i}$ ◀

$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}: \quad \mathbf{C}+\mathbf{D} = 0 \qquad \mathbf{D} = -\frac{1}{6}mb\omega^2\sin\beta\cos\beta\mathbf{i}$ ◀



PROBLEM 18.66

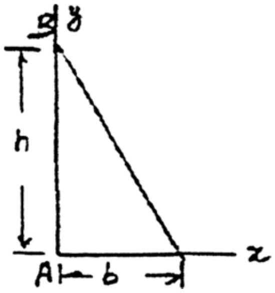
A thin homogeneous triangular plate of weight 10 pounds is welded to a light vertical axle supported by bearings at *A* and *B*. Knowing that the plate rotates at the constant rate $\omega = 8$ rad/s, determine the dynamic reactions at *A* and *B*.

SOLUTION

We shall use Eqs. (18.1) and (18.28):

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \quad \Sigma \mathbf{M}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega} \times \mathbf{H}_A$$

Computation of $\mathbf{H}_A$:



$$\boldsymbol{\omega} = \omega \mathbf{j}$$

$$\mathbf{H}_A = -I_{xy}\omega\mathbf{i} + I_y\omega\mathbf{j} - I_{yz}\omega\mathbf{k}$$

The moment of inertia of the triangle is

$$(I_y)_{\text{area}} = \frac{1}{12}b^3h, \quad A = \frac{1}{2}bh$$

$$(I_y)_{\text{mass}} = \frac{m}{A}(I_y)_{\text{area}} = \frac{1}{6}mb^2$$

The product of inertia of the triangle is

$$(I_{xy})_{\text{area}} = \frac{1}{24}b^2h^2$$

$$(I_{xy})_{\text{mass}} = \frac{m}{A}(I_{xy})_{\text{area}} = \frac{1}{12}mbh$$

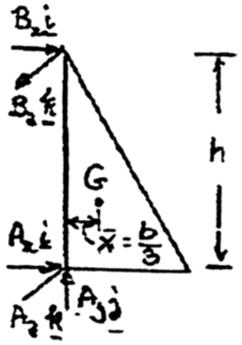
Since the *z*-coordinate is negligible,

$$(I_{yz})_{\text{mass}} = 0$$

Thus, $$\mathbf{H}_A = -\frac{1}{12}mbh\omega\mathbf{i} + \frac{1}{6}mb^2\omega\mathbf{j}. \qquad (1)$$

where the frame of reference *Axyz* rotates with the plate with the angular velocity

$$\mathbf{\Omega} = \boldsymbol{\omega} = \omega\mathbf{j}$$



Equations of motion. (Weight is omitted for dynamic reactions.)

Eq. (18.28),

$$\Sigma\mathbf{M}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega}\times\mathbf{H}_A.$$

Since $\omega = \text{constant}$, it follows from Eq. (1) that

$$(\dot{\mathbf{H}}_A)_{Axyz} = 0$$

Thus,

$$h\mathbf{j}\times(B_x\mathbf{i} + B_z\mathbf{k}) = 0 + \omega\mathbf{j}\times\left(-\frac{mbh}{12}\omega\mathbf{i} + \frac{mb^2}{6}\omega\mathbf{j}\right) - hB_x\mathbf{k} + h\mathbf{B}_z\mathbf{i}$$

$$= +\frac{1}{12}mbh\omega^2\mathbf{k}$$

Equating coefficients of unit vectors: $B_x = -\frac{1}{12}mb\omega^2, \quad B_z = 0$

$$\mathbf{B} = -\frac{1}{12}mb\omega^2\mathbf{i}$$

The dynamic reactions must also satisfy Eq. (18.1):

$$\Sigma\mathbf{F} = m\bar{\mathbf{a}}: \quad \mathbf{A} + \mathbf{B} = -m\bar{x}\omega^2\mathbf{i} = -m\left(\frac{b}{3}\right)\omega^2\mathbf{i}$$

$$\mathbf{A} = -\frac{1}{3}mb\omega^2\mathbf{i} - \left(-\frac{1}{12}mb\omega^2\mathbf{i}\right), \quad \mathbf{A} = -\frac{1}{4}mb\omega^2\mathbf{i}$$

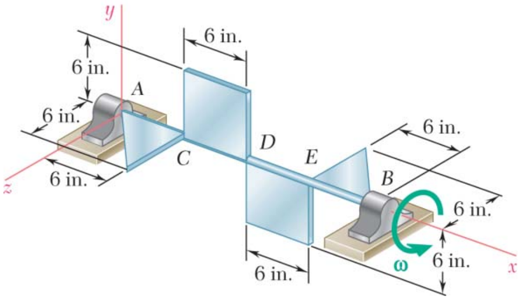
Given data: $W = 10$ lb, $b = 12$ in. $= 1$ ft, $h = 24$ in. $= 2$ ft, $\omega = 8$ rad/s

$$\mathbf{A} = -\frac{1}{4}\left(\frac{10\text{ lbs}}{32.2}\right)(1\text{ ft})(8\text{ rad/s})^2\mathbf{i}$$

A = −(4.97 lb)**i** ◀

$$\mathbf{B} = -\frac{1}{12}\left(\frac{10\text{ lbs}}{32.2}\right)(1\text{ ft})(8\text{ rad/s})^2\mathbf{i}$$

B = −(1.656 lb)**i** ◀



PROBLEM 18.67

The assembly shown consists of pieces of sheet aluminum of uniform thickness and of total weight 2.7 lb welded to a light axle supported by bearings at *A* and *B*. Knowing that the assembly rotates at the constant rate $\omega = 240\,\text{rpm}$, determine the dynamic reactions at *A* and *B*.

SOLUTION

Mass of sheet metal: $m = \dfrac{2.7}{32.2} = 0.08385\ \text{lb}\cdot\text{s}^2/\text{ft}$

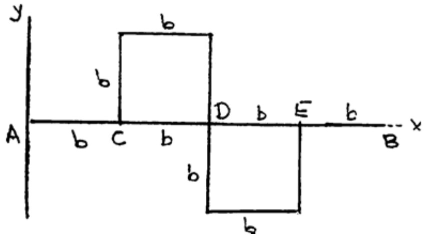
Sheet metal dimension: $b = 6\ \text{in.} = 0.5\ \text{ft}$

Area of sheet metal: $A = \dfrac{1}{2}b^2 + b^2 + b^2 + \dfrac{1}{2}b^2 = 3b^2 = 0.75\ \text{ft}^2$

Let $\rho = \dfrac{m}{A} = \dfrac{0.08385}{0.75} = 0.1118\ \text{lb}\cdot\text{s}^2/\text{ft}^3 =$ mass per unit area.

Moments and products of inertia: $I_\text{mass} = \rho I_\text{area}$

xy plane (rectangles)



$$I_x = \frac{1}{3}b^4 + \frac{1}{3}b^4 = \frac{2}{3}b^4$$

$$I_x = \frac{2}{3}\rho b^4$$

$$= \frac{2}{3}(0.1118)(0.5)^4$$

$$= 4.658\times 10^{-3}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

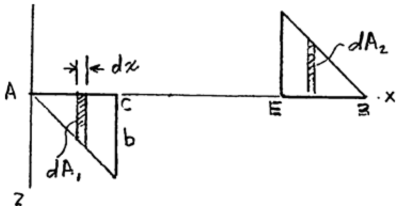
$$I_{xy} = (b^2)\left(\frac{3}{2}b\right)\left(\frac{1}{2}b\right) + (b^2)\left(\frac{5}{2}b\right)\left(-\frac{1}{2}b\right)$$

$$= -\frac{1}{2}b^4$$

$$I_{xy} = -\frac{1}{2}\rho b^4 = -\frac{1}{2}(0.1118)(0.5)^4$$

$$= -3.4938\times 10^{-3}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

xz plane (triangles)



$$I_x = \frac{1}{12}b^4 + \frac{1}{12}b^4 = \frac{1}{6}b^4$$

$$I_x = \frac{1}{6}\rho b^4 = \frac{1}{6}(0.1118)(0.5)^4$$

$$= 1.1646\times 10^{-3}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

For calculation of I_{xz}, use pairs of elements dA_1 and dA_2:

$$dA_2 = dA_1.$$

$$I_{xz} = \int x\frac{z}{2}dA_1 + \int(4b-x)\left(-\frac{z}{2}\right)dA_2 = -\int(2b-x)z\,dA_1 = -\int_0^b(2b-x)z^2dx$$

but $z = x.$

Hence, $I_{xz} = -\int_0^a(2bx^2 - x^3)dx = -\left(\frac{2}{3}b^4 - \frac{1}{4}b^4\right) = -\frac{5}{12}b^4$

$$I_{xz} = -\frac{5}{12}\rho b^4 = -\frac{5}{12}(0.1118)(0.5)^4 = -2.9115\times 10^{-3}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

Total for I_x: $I_x = 4.658\times 10^{-3} + 1.1646\times 10^{-3} = 5.823\times 10^{-3}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$

The mass center lies on the rotation axis, therefore

$$\bar{\mathbf{a}} = 0$$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = m\bar{\mathbf{a}} = 0 \quad \mathbf{A} = -\mathbf{B}$$

$$\mathbf{H}_A = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k} \quad \boldsymbol{\omega} = \omega\mathbf{i}, \quad \boldsymbol{\alpha} = \alpha\mathbf{i}$$

Let the frame of reference *Axyz* be rotating with angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega\mathbf{i}$$

$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_A$$

$$M_0\mathbf{i} + 4b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k}) = I_x\alpha\mathbf{i} - I_{xy}\alpha\mathbf{j} - I_{xz}\alpha\mathbf{k} + \omega\mathbf{i}\times(I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k})$$

$$M_0\mathbf{i} - 4bB_z\mathbf{j} + 4bB_y\mathbf{k} = I_x\alpha\mathbf{i} - (I_{xy}\alpha - I_{xz}\omega^2)\mathbf{j} - (I_{xz}\alpha + I_{xy}\omega^2)\mathbf{k}$$

Resolve into components and solve for B_y and B_z.

i: $M_0 = I_x\alpha$

j: $B_z = \dfrac{(I_{xy}\alpha - I_{xz}\omega^2)}{4b}$

k: $B_y = -\dfrac{(I_{xz}\alpha + I_{xy}\omega^2)}{4b}$

Data: $\alpha = 0, \quad \omega = \dfrac{2\pi(240)}{60} = 25.133\ \text{rad/s}, \quad b = 0.5\ \text{ft} \quad M_0 = 0$

$$B_z = \frac{0 - (-2.9115\times 10^{-3})(25.133)^2}{(4)(0.5)} = 0.91955\ \text{lb.}$$

PROBLEM 18.67 (Continued)

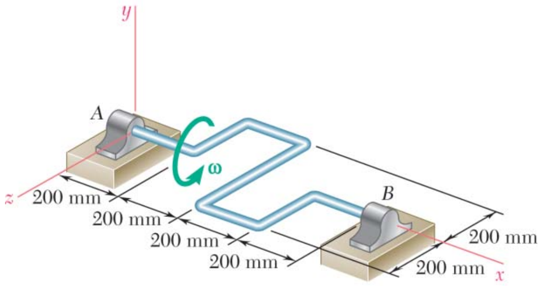
$$B_y = \frac{0-(-3.4938\times 10^{-3})(25.133)^2}{(4)(0.5)} = 1.10346 \text{ lb.}$$

$$A_y = -B_y = -1.10346 \text{ lb.}$$

$$A_z = -B_z = -0.91955 \text{ lb.}$$

$\mathbf{A} = -(1.103 \text{ lb})\mathbf{j} - (0.920 \text{ lb})\mathbf{k}$ ◀

$\mathbf{B} = (1.103 \text{ lb})\mathbf{j} + (0.920 \text{ lb})\mathbf{k}$ ◀



PROBLEM 18.68

The 8-kg shaft shown has a uniform cross section. Knowing that the shaft rotates at the constant rate $\omega = 12$ rad/s, determine the dynamic reactions at *A* and *B*.

SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega \mathbf{i}$

Angular momentum about the mass center *G*:

$$\mathbf{H}_G = \bar{I}_x \omega \mathbf{i} - \bar{I}_{xy} \omega \mathbf{j} - \bar{I}_{xz} \omega \mathbf{k}$$

Let the reference frame *Axyz* be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega \mathbf{i}.$$

$$\begin{aligned} \dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Axyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \\ &= 0 + \omega \mathbf{i} \times (\bar{I}_x \omega \mathbf{i} - \bar{I}_{xy} \omega \mathbf{j} - \bar{I}_{xz} \omega \mathbf{k}) \\ &= \bar{I}_{xz} \omega^2 \mathbf{j} - \bar{I}_{xy} \omega^2 \mathbf{k} \end{aligned}$$

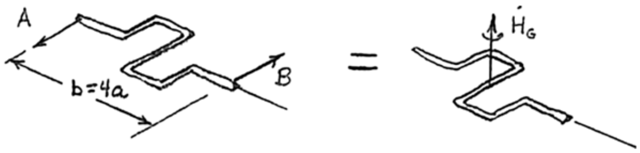
Since the shaft lies in the *xz* plane, $\bar{I}_{xy} = 0$.

By symmetry, the mass center lies on line *AB*.

$$m\bar{\mathbf{a}} = 0$$

$$\Sigma \mathbf{F} = \Sigma F_{\text{eff}}: \quad \mathbf{A} + \mathbf{B} = m\bar{\mathbf{a}} = 0 \quad \mathbf{A} \text{ and } \mathbf{B} \text{ form a couple.}$$

$$\mathbf{A} = -\mathbf{B}$$



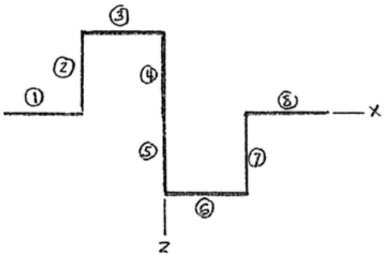
$$b\mathbf{i} \times (B_y \mathbf{j} + B_z \mathbf{k}) = \dot{\mathbf{H}}_G$$

$$-bB_z \mathbf{j} + bB_y \mathbf{k} = \bar{I}_{xz} \omega \mathbf{j} \quad B_z = -\frac{\bar{I}_{xz} \omega^2}{b}, \quad B_y = 0$$

Calculation of $\bar{I}_{xz}$.

Divide the shaft into eight segments, each of length

$$a = 200 \text{ mm} = 0.2 \text{ m}$$



Let m' be the mass of one segment.

$$m' = \frac{8 \text{ kg}}{8} = 1 \text{ kg}$$

For ①, ④, ⑤, and ⑧, $I_{xz} = 0$

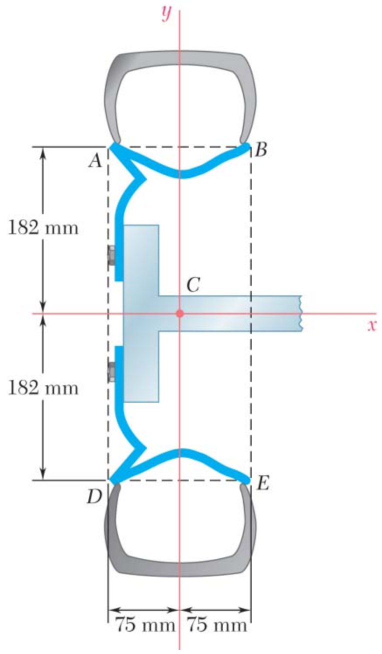
Then $$I_{xz} = m'(-a)\left(-\frac{a}{2}\right) + m'\left(-\frac{a}{2}\right)(-a) + m'\left(\frac{a}{2}\right)(a) + m'(a)\left(\frac{a}{2}\right) = 2m'a^2$$

$$B_z = -\frac{2m'a^2\omega^2}{4a} = -\frac{(2)(1)(0.2)^2(12)^2}{(4)(0.2)} = -14.4 \text{ N}$$

$$A_y = 0, \qquad A_z = -B_z = 14.4 \text{ N}$$

A = (14.4 N)**k** ◀

B = −(14.4 N)**k** ◀



PROBLEM 18.69

After attaching the 18-kg wheel shown to a balancing machine and making it spin at the rate of 15 rev/s, a mechanic has found that to balance the wheel both statically and dynamically, he should use two corrective masses, a 170-g mass placed at *B* and a 56-g mass placed at *D*. Using a right-handed frame of reference rotating with the wheel (with the *z* axis perpendicular to the plane of the figure), determine before the corrective masses have been attached (*a*) the distance from the axis of rotation to the mass center of the wheel and the products of inertia I_{xy} and I_{zx}, (*b*) the force-couple system at *C* equivalent to the forces exerted by the wheel on the machine.

SOLUTION

$$m = 18 \text{ kg}, \quad \boldsymbol{\omega} = 2\pi(15)\mathbf{i} = (94.248 \text{ rad/s})\mathbf{i}$$

$$m_B = 170 \text{ g} = 0.17 \text{ kg}$$

$$m_D = 56 \text{ g} = 0.056 \text{ kg}$$

$$x_B = 75 \text{ mm} = 0.075 \text{ m}, \quad y_B = 182 \text{ mm} = 0.182 \text{ m}$$

$$x_D = -0.075 \text{ m}, \quad y_D = -0.182 \text{ m}$$

(*a*) Balance masses are added to move the mass center to Point *C* and to reduce the products of inertia to zero.

$$m_B y_B + m_D y_D + m\bar{y} = 0$$

$$(0.17)(0.182) + (0.056)(-0.182) + 18\bar{y} = 0$$

$$\bar{y} = -1.15267 \times 10^{-3} \text{ m} \qquad \bar{y} = -1.153 \text{ mm} \blacktriangleleft$$

$$\bar{z} = 0 \qquad \bar{z} = 0 \blacktriangleleft$$

$$m_B x_B y_B + m_D x_D y_D + I_{xy} = 0$$

$$(0.17)(0.075)(0.182) + (0.056)(-0.075)(-0.182) + I_{xy} = 0$$

$$I_{xy} = -3.0848 \times 10^{-3} \qquad I_{xy} = -3.08 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{zx} = 0 \blacktriangleleft$$

PROBLEM 18.69 (Continued)

(*b*) Force-couple system exerted on the wheel:

$$\mathbf{F}' = m\bar{\mathbf{a}}_n = -m\bar{\mathbf{r}}\omega^2 = -(18)(-1.15267\times 10^{-3}\,\mathbf{j})(94.248)^2$$
$$= (184.3\text{ N})\mathbf{j}$$

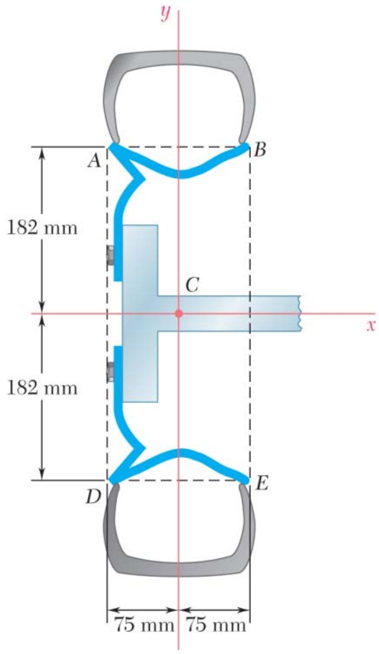
$$\mathbf{H}_C = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k}$$

$$\mathbf{M}'_C = \dot{\mathbf{H}}_C = \boldsymbol{\omega}\times\mathbf{H}_C = I_{xz}\omega^2\mathbf{j} - I_{xy}\omega^2\mathbf{k}$$
$$= 0 - (-3.0848\times 10^{-3})(94.248)^2\mathbf{k} = (27.4\text{ N}\cdot\text{m})\mathbf{k}$$

Force-couple system exerted by the wheel on the machine:

$$\mathbf{F} = -\mathbf{F}' \qquad \mathbf{F} = -(184.3\text{ N})\mathbf{j} \blacktriangleleft$$

$$\mathbf{M}_C = -\mathbf{M}'_C \qquad \mathbf{M}_C = -(27.4\text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$$



PROBLEM 18.70

When the 18-kg wheel shown is attached to a balancing machine and made to spin at a rate of 12.5 rev/s, it is found that the forces exerted by the wheel on the machine are equivalent to a force-couple system consisting of a force $\mathbf{F} = (160\text{ N})\mathbf{j}$ applied at C and a couple $\mathbf{M}_C = (14.7\text{ N}\cdot\text{m})\mathbf{k}$, where the unit vectors form a triad which rotates with the wheel. (*a*) Determine the distance from the axis of rotation to the mass center of the wheel and the products of inertia I_{xy} and I_{zx}. (*b*) If only two corrective masses are to be used to balance the wheel statically and dynamically, what should these masses be and at which of the Points A, B, D, or E should they be placed?

SOLUTION

$$m = 18\text{ kg}, \quad \omega = 2\pi(12.5)\mathbf{i} = (78.54\text{ rad/s})\mathbf{i}$$

(*a*) The force-couple system acting on the wheel is

$$\mathbf{F}' = -(160\text{ N})\mathbf{j}, \quad \mathbf{M}'_C = -(14.7\text{ N}\cdot\text{m})\mathbf{k}$$

$$\mathbf{F}' = m\bar{\mathbf{a}}_n = -m\bar{\mathbf{r}}\omega^2 \qquad \bar{\mathbf{r}} = \bar{y}\mathbf{j} + \bar{z}\mathbf{k} = \frac{-\mathbf{F}'}{m\omega^2}$$

$$\bar{y} = \frac{-F'_y}{m\omega^2} = \frac{160}{(18)(78.54)^2} = 1.441\times 10^{-3}\text{ m} \qquad \bar{y} = 1.441\text{ mm}$$

$$\bar{z} = \frac{-F'_z}{m\omega^2} = 0 \qquad \bar{r} = 1.441\text{ mm} \blacktriangleleft$$

$$\mathbf{H}_C = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k}$$

$$\mathbf{M}'_C = \dot{\mathbf{H}}_C = \boldsymbol{\omega}\times\mathbf{H}_C = \omega\mathbf{i}\times(I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k})$$

$$(M'_C)_y\mathbf{j} + (M'_C)_z\mathbf{k} = I_{xz}\omega^2\mathbf{j} - I_{xy}\omega^2\mathbf{k}$$

$$I_{xy} = -\frac{(M'_C)_z}{\omega^2} = \frac{-(-14.7)}{(78.54)^2} = 2.3831\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$I_{xy} = 2.38\times 10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

$$I_{xz} = \frac{(M'_C)_y}{\omega^2} = 0 \qquad I_{xz} = 0 \blacktriangleleft$$

PROBLEM 18.70 (Continued)

(*b*) Positions for the balance masses.

$$y_A = y_B = -y_E = -y_D = 182\text{ mm} = 0.182\text{ m}$$

$$-x_A = x_B = x_E = -x_D = 75\text{ mm} = 0.075\text{ m}$$

Balance masses must be added to move the mass center to Point *C* and to reduce the product of inertia to zero.

$$m_A y_A + m_B y_B + m_E y_E + m_D y_D + m\bar{y} = 0$$

$$(0.182)(m_A + m_B - m_E - m_D) + (18)(1.441\times 10^{-3})$$

$$m_A + m_B - m_E - m_D = -0.1425\text{ kg} \tag{1}$$

$$m_A x_A y_A + m_B x_B y_B + m_E x_E y_E + m_D x_D y_D + I_{xy} = 0$$

$$(0.075)(0.182)(-m_A + m_B - m_E + m_D) + 2.3831\times 10^{-3} = 0$$

$$-m_A + m_B - m_E + m_D = -0.17459\text{ kg} \tag{2}$$

To solve Eqs. (1) and (2), set $m_B = 0$ and let $m_{AD} = m_A - m_D$.

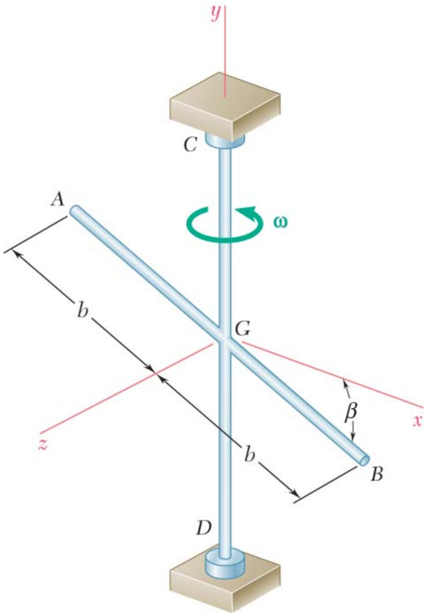
Then $\quad m_{AD} - m_E = -0.1425$

and $\quad -m_{AD} - m_E = -0.17459$

Solving, $\quad m_{AD} = 0.01605\text{ kg}, \quad m_E = 0.1585\text{ kg}$

Set $m_D = 0,$ so that $m_A = 0.01605\text{ kg}$ $\qquad m_E = 158.5\text{ g}$ ◀

$m_A = 16.05\text{ g}$ ◀



PROBLEM 18.71

Knowing that the assembly of Problem 18.65 is initially at rest $(\omega = 0)$ when a couple of moment $\mathbf{M}_0 = M_0\mathbf{j}$ is applied to shaft *CD*, determine (*a*) the resulting angular acceleration of the assembly, (*b*) the dynamic reactions at *C* and *D* immediately after the couple is applied.

SOLUTION

Using the principal axes $Gx'y'z$:

$$\bar{I}_{x'} = 0, \quad \bar{I}_{y'} = \bar{I}_z = \frac{1}{3}mb^2$$

$$\omega_{x'} = -\omega\sin\beta, \quad \omega_{y'} = \omega\cos\beta, \quad \omega_z = 0$$

Angular momentum about *G*.

$$\mathbf{H}_G = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' + I_z\omega_z\mathbf{k}$$

$$\mathbf{H}_G = \frac{1}{3}mb^2\omega\cos\beta\,\mathbf{j}'$$

or, since $\mathbf{j}' = \mathbf{i}\sin\beta + \mathbf{j}\cos\beta$: $\quad \mathbf{H}_G = \frac{1}{3}mb^2\omega\cos\beta(\sin\beta\mathbf{i} + \cos\beta\mathbf{j}) \qquad (1)$

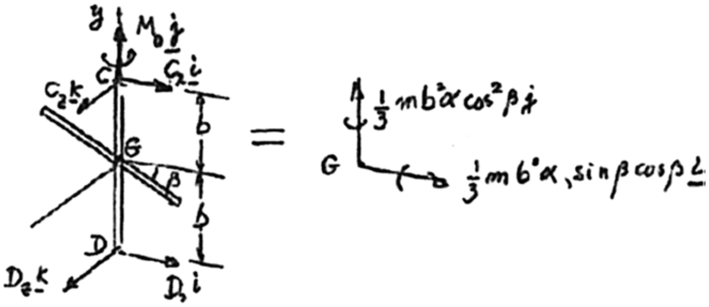
Rate of change of angular momentum.

Eq. (18.22): $\quad \dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega}\times\mathbf{H}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + 0$

Since $\mathbf{\Omega} = \boldsymbol{\omega} = 0$ when couple is applied, thus,

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} = \frac{1}{3}mb^2\alpha\cos\beta(\sin\beta\mathbf{i} + \cos\beta\mathbf{j}) \qquad (2)$$

Equations of motion: Equivalence of applied and effective forces.


y
$M_0\mathbf{j}$
C
$C_x\mathbf{i}$
$C_z\mathbf{k}$
G
β
b
b
D
$D_z\mathbf{k}$
$D_x\mathbf{i}$
=
$\frac{1}{3}mb^2\alpha\cos^2\beta\,\mathbf{j}$
G
$\frac{1}{3}mb^2\alpha\sin\beta\cos\beta\,\mathbf{i}$

$$\Sigma\mathbf{M}_D = \Sigma(\mathbf{M}_D)_{\text{eff}}: \quad 2b\mathbf{j}\times(C_x\mathbf{i}+C_z\mathbf{k})+M_0\mathbf{j}=\frac{1}{3}mb^2\alpha\sin\beta\cos\beta\mathbf{i}+\frac{1}{3}mb^2\alpha\cos^2\beta\mathbf{j}$$

$$-2bC_x\mathbf{k}+2bC_z\mathbf{i}+M_0\mathbf{j}=\frac{1}{3}mb^2\alpha\sin\beta\cos\beta\mathbf{i}+\frac{1}{3}mb^2\alpha\cos^2\beta\mathbf{j}$$

Equating the coefficients of **i**, **j**, **k**:

$$\mathbf{i}: \qquad 2bC_z = \frac{1}{3}mb^2\alpha\sin\beta\cos\beta \tag{3}$$

$$\mathbf{j}: \qquad M_0 = \frac{1}{3}mb^2\alpha\cos^2\beta \tag{4}$$

$$\mathbf{k}: \qquad C_x = 0 \tag{5}$$

(*a*) Angular acceleration.

From Eq. (4), $$\alpha = \frac{3M_0}{mb^2\cos^2\beta} \blacktriangleleft$$

(*b*) Initial dynamic reactions.

From Eq. (3),

$$C_z = \frac{1}{6}mb\alpha\sin\beta\cos\beta = \frac{1}{6}mb\sin\beta\cos\beta\left(\frac{3M_0}{mb^2\cos^2\beta}\right)$$

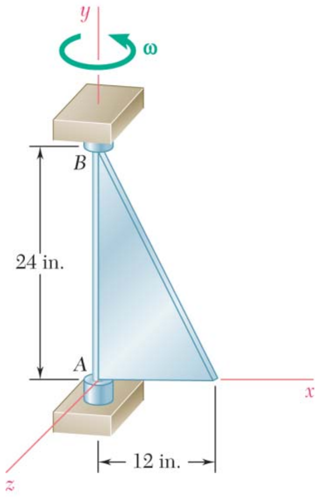
$$C_z = \left(\frac{M_0}{2b}\right)\tan\beta$$

Recalling Eq. (5), $C_x = 0$

$$\mathbf{C} = \left(\frac{M_0}{2b}\right)\tan\beta\mathbf{k} \blacktriangleleft$$

$\Sigma\mathbf{F} = \Sigma(\mathbf{F})_{\text{eff}}:$ $\mathbf{C}+\mathbf{D}=0, \quad \mathbf{D}=-\mathbf{C}$

$$\mathbf{D} = -\left(\frac{M_0}{2b}\right)\tan\beta\mathbf{k} \blacktriangleleft$$



PROBLEM 18.72

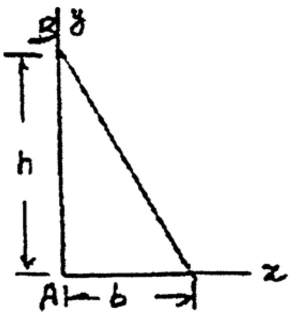
Knowing that the plate of Problem 18.66 is initially at rest $(\omega = 0)$ when a couple of moment $\mathbf{M}_0 = (0.75\ \text{ft} \cdot \text{lb})\mathbf{j}$ is applied to it, determine (*a*) the resulting angular acceleration of the plate, (*b*) the dynamic reactions *A* and *B* immediately after the couple has been applied.

PROBLEM 18.66 A thin homogeneous triangular plate of weight 10 pounds is welded to a light vertical axle supported by bearings at *A* and *B*. Knowing that the plate rotates at the constant rate $\omega = 8\ \text{rad/s}$, determine the dynamic reactions at *A* and *B*.

SOLUTION

We shall use Eqs. (18.1) and (18.28):

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \quad \Sigma \mathbf{M}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega} \times \mathbf{H}_A$$



Computation of $\mathbf{H}_A$:

$$\boldsymbol{\omega} = \omega \mathbf{j}$$

$$\mathbf{H}_A = -I_{xy}\omega \mathbf{i} + I_y \omega \mathbf{j} - I_{yz}\omega \mathbf{k}$$

The moment of inertia of the triangle is

$$(I_y)_{\text{area}} = \frac{1}{12}b^3 h, \quad A = \frac{1}{2}bh$$

$$(I_y)_{\text{mass}} = \frac{m}{A}(I_y)_{\text{area}} = \frac{1}{6}mb^2$$

The product of inertia of the triangle is

$$(I_{xy})_{\text{area}} = \frac{1}{24}b^2 h^2$$

$$(I_{xy})_{\text{mass}} = \frac{m}{A}(I_{xy})_{\text{area}} = \frac{1}{12}mbh$$

Since the *z*-coordinate is negligible,

$$(I_{yz})_{\text{mass}} = 0$$

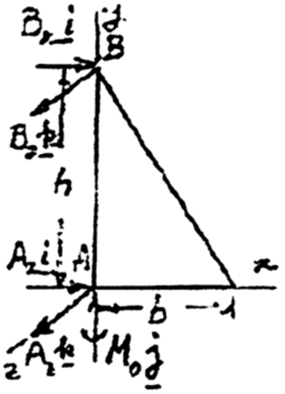
Thus,

$$\mathbf{H}_A = -\frac{1}{12}mbh\omega \mathbf{i} + \frac{1}{6}mb^2 \omega \mathbf{j} \qquad (1)$$

where the frame of reference *Axyz* rotates with the plate with the angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega \mathbf{j}$$

PROBLEM 18.72 (Continued)



Equations of motion. We first use Eq. (18.28):

$$\Sigma \mathbf{M}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega} \times \mathbf{H}_A$$

Where $\mathbf{H}_A$ was obtained in Eq. (1) of Problem 18.66:

$$\mathbf{H}_A = -\frac{1}{12}mbh\omega\mathbf{i} + \frac{1}{6}mb^2\omega\mathbf{j}$$

Differentiating with respect to the rotating frame:

$$(\dot{\mathbf{H}}_A)_{Axyz} = -\frac{1}{12}mbh\alpha\mathbf{i} + \frac{1}{6}mb^2\alpha\mathbf{j}$$

Substituting for $\mathbf{H}_A$ and $(\dot{\mathbf{H}}_A)_{Axyz}$ into Eq. (18.28), noting that $\omega = 0$, and computing $\Sigma\mathbf{M}_A$ from diagram:

$$M_0\mathbf{j} + h\mathbf{j} \times (B_x\mathbf{i} + B_z\mathbf{k}) = -\frac{1}{12}mbh\alpha\mathbf{i} + \frac{1}{6}mb^2\alpha\mathbf{j} + 0$$

$$M_0\mathbf{j} - hB_x\mathbf{k} + hB_z\mathbf{i} = -\frac{1}{12}mbh\alpha\mathbf{i} + \frac{1}{6}mb^2\alpha\mathbf{j}$$

Equating the coefficients of the unit vectors:

$$\mathbf{j}: \qquad M_0 = \frac{1}{6}mb^2\alpha \qquad\qquad \alpha = \frac{6M_0}{mb^2}$$

(*a*) $$\alpha = \frac{6(0.75\text{ lb}\cdot\text{ft})}{(\frac{10\text{ lb}}{32.2})(1\text{ ft})^2} = 14.49\text{ rad/s}^2$$ $\boldsymbol{\alpha} = (14.49\text{ rad/s}^2)\mathbf{j}$ ◀

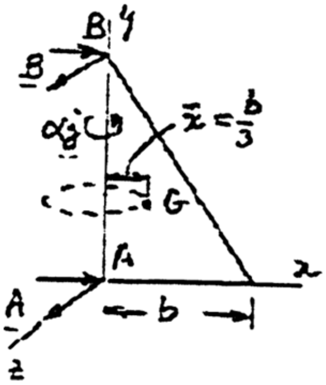
$$\mathbf{k}: \qquad -hB_x = 0 \qquad\qquad B_x = 0$$

$$\mathbf{i}: \qquad hB_z = -\frac{1}{12}mbh\alpha, \quad B_z = -\frac{1}{12}mb\alpha = -\frac{1}{12}\left(\frac{10\text{ lb}}{32.2}\right)(1)(14.49)$$

(*b*) $$B_z = -0.375\text{ lb}$$ $\mathbf{B} = -(0.375\text{ lb})\mathbf{k}$ ◀

We shall now apply Eq. (18.1): $\Sigma\mathbf{F} = m\bar{\mathbf{a}}$:

Since $\omega = 0$: $$\bar{\mathbf{a}} = \bar{\mathbf{a}}_t = \alpha\mathbf{j} \times \frac{b}{3}\mathbf{i} = -\frac{1}{3}b\alpha\mathbf{k}$$



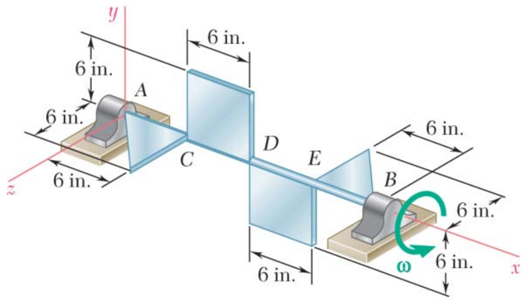
Substituting in Eq. (12.1):

$$\mathbf{A} + \mathbf{B} = -\frac{1}{3}mb\alpha\mathbf{k}$$

$$\mathbf{A} = -\frac{1}{3}mb\alpha\mathbf{k} - B$$

$$= -\frac{1}{3}mb\alpha\mathbf{k} + \frac{1}{12}mb\alpha\mathbf{k}$$

$$\mathbf{A} = -\frac{1}{4}mb\alpha\mathbf{k} = -\frac{1}{4}\left(\frac{10\text{ lb}}{32.2}\right)(1\text{ ft})(14.49\text{ rad/s}^2)\mathbf{k}$$ $\mathbf{A} = -(1.125\text{ lb})\mathbf{k}$ ◀



PROBLEM 18.73

The assembly of Problem 18.67 is initially at rest $(\omega = 0)$ when a couple $\mathbf{M}_0$ is applied to axle AB. Knowing that the resulting angular acceleration of the assembly is $\boldsymbol{\alpha} = (150 \text{ rad/s}^2)\mathbf{i}$, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at A and B immediately after the couple is applied.

SOLUTION

Mass of sheet metal: $m = \dfrac{2.7}{32.2} = 0.08385 \text{ lb} \cdot \text{s}^2/\text{ft}$

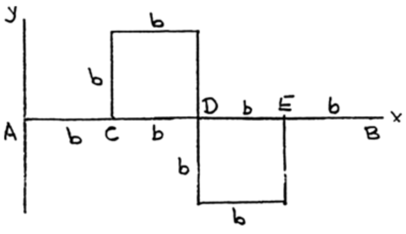
Sheet metal dimension: $b = 6 \text{ in.} = 0.5 \text{ ft}$

Area of sheet metal: $A = \dfrac{1}{2}b^2 + b^2 + b^2 + \dfrac{1}{2}b^2 = 3b^2 = 0.75 \text{ ft}^2$

Let $\rho = \dfrac{m}{A} = \dfrac{0.08385}{0.75} = 0.1118 \text{ lb} \cdot \text{s}^2/\text{ft}^3 = \text{mass per unit area}$

Moments and products of inertia: $(I)_{\text{mass}} = \rho I_{(\text{area})}$

xy plane (rectangles)



$$I_x = \frac{1}{3}b^4 + \frac{1}{3}b^4 = \frac{2}{3}b^4$$

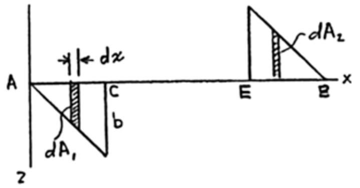
$$I_x = \frac{2}{3}\rho b^4 = \frac{2}{3}(0.1118)(0.5)^4$$

$$= 4.658 \times 10^{-3} \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

$$I_{xy} = (b^2)\left(\frac{3}{2}b\right)\left(\frac{1}{2}b\right) + (b^2)\left(\frac{5}{2}b\right)\left(-\frac{1}{2}b\right) = -\frac{1}{2}b^4$$

$$I_{xy} = -\frac{1}{4}\rho b^4 = -\frac{1}{4}(0.1118)(0.5)^4$$

$$= -3.4938 \times 10^{-3} \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$



xz plane (triangles)

$$I_x = \frac{1}{12}b^4 + \frac{1}{12}b^4 = \frac{1}{6}b^4$$

$$I_x = \frac{1}{6}\rho b^4 = \frac{1}{6}(0.1118)(0.5)^4$$

$$= 1.1646 \times 10^{-3} \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

PROBLEM 18.73 (Continued)

For calculation of I_{xz}, use pairs of elements dA_1 and dA_2: $\quad dA_2 = dA_1$

$$I_{xz} = \int x\frac{z}{2}dA_1 + \int (4b - x)\left(-\frac{z}{2}\right)dA_2 = -\int (2b - x)z\,dA_1 = -\int_0^b (2b - x)z^2 dx$$

But $z = x$.

Hence, $$I_{xz} = -\int_0^a (2bx^2 - x^3)dx = -\left(\frac{2}{3}b^4 - \frac{1}{4}b^4\right) = -\frac{5}{12}b^4$$

$$I_{xz} = -\frac{5}{12}\rho b^4 = -\frac{5}{12}(0.1118)(0.5)^4 = -2.9115\times 10^{-3}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

Total for I_x: $\quad I_x = 4.658\times 10^{-3} + 1.1646\times 10^{-3} = 5.823\times 10^{-3}\ \text{kg}\cdot\text{m}^2$

The mass center lies on the rotation axis, therefore, $\bar{\mathbf{a}} = 0$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = m\bar{\mathbf{a}} = 0 \qquad \mathbf{A} = -\mathbf{B}$$

$$\mathbf{H}_A = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k} \qquad \boldsymbol{\omega} = \omega\mathbf{i}, \qquad \boldsymbol{\alpha} = \alpha\mathbf{i}$$

Let the frame of reference *Axyz* be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega\mathbf{i}$.

$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_A$$

$$M_0\mathbf{i} + 4b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k}) = I_x\alpha\mathbf{i} - I_{xy}\alpha\mathbf{j} - I_{xz}\alpha\mathbf{k} + \omega\mathbf{i}\times(I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k})$$

$$M_0\mathbf{i} - 4bB_z\mathbf{j} + 4bB_y\mathbf{k} = I_x\alpha\mathbf{i} - (I_{xy}\alpha - I_{xz}\omega^2)\mathbf{j} - (I_{xz}\alpha + I_{xy}\omega^2)\mathbf{k}$$

Resolve into components and solve for B_y and B_z.

i: $\quad M_0 = I_x\alpha$

j: $\quad B_z = \dfrac{(I_{xy}\alpha - I_{xz}\omega^2)}{4b}$

k: $\quad B_y = -\dfrac{(I_{xz}\alpha + I_{xy}\omega^2)}{4b}$

Data: $\quad \alpha = 150\ \text{rad/s}^2, \quad \omega = 0, \quad b = 0.5\ \text{ft}$

(*a*) $\quad M_0 = (5.823\times 10^{-3})(150) = 0.87345\ \text{lb}\cdot\text{ft} \qquad M_0 = (0.873\ \text{lb}\cdot\text{ft})\mathbf{i}$ ◀

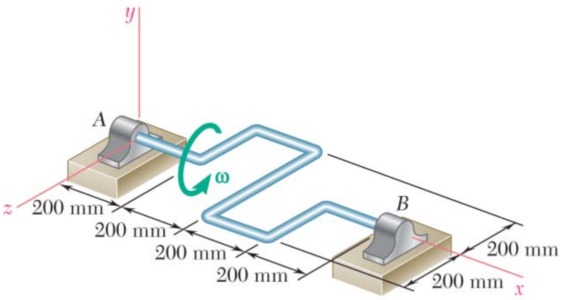
(*b*) $$B_z = \frac{(-3.4938\times 10^{-3})(150) - 0}{(4)(0.5)} = -0.262\ \text{lb}$$

$$B_y = -\frac{(-2.9115\times 10^{-3})(150) + 0}{(4)(0.5)} = 0.218\ \text{lb}$$

$$A_y = -B_y = -0.218\ \text{lb}$$

$A_z = -B_z = 0.262\ \text{lb} \qquad \mathbf{A} = -(0.218\ \text{lb})\mathbf{j} + (0.262\ \text{lb})\mathbf{k}$ ◀

$\mathbf{B} = (0.218\ \text{lb})\mathbf{j} - (0.262\ \text{lb})\mathbf{k}$ ◀



PROBLEM 18.74

The shaft of Problem 18.68 is initially at rest $(\omega = 0)$ when a couple $\mathbf{M}_0$ is applied to it. Knowing that the resulting angular acceleration of the shaft is $\boldsymbol{\alpha} = (20\text{ rad/s}^2)\mathbf{i}$, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at *A* and *B* immediately after the couple is applied.

PROBLEM 18.68 The 8-kg shaft shown has a uniform cross section. Knowing that the shaft rotates at the constant rate $\omega = 12\text{ rad/s}$, determine the dynamic reactions at *A* and *B*.

SOLUTION

Angular velocity and angular acceleration: $\dot{\boldsymbol{\omega}} = \dot{\omega}\mathbf{i} = \alpha\,\mathbf{i}$

Angular momentum about the mass center *G*: $\mathbf{H}_G = \bar{I}_x\omega\mathbf{i} - \bar{I}_{xy}\omega\mathbf{j} - \bar{I}_{xz}\omega\mathbf{k}$

Let the reference frame *Axyz* be rotating with angular velocity $\Omega = \omega\mathbf{i} = 0$

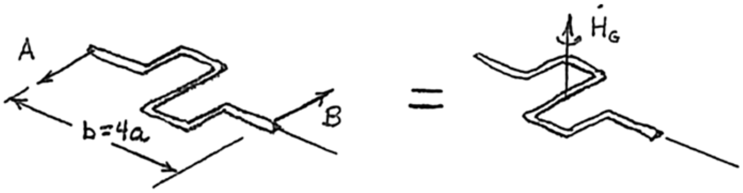
$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_G\\ &= \bar{I}_x\alpha\mathbf{i} - \bar{I}_{xy}\alpha\mathbf{j} - \bar{I}_{xz}\alpha\mathbf{k} + 0\\ &= \bar{I}_x\alpha\mathbf{i} - \bar{I}_{xy}\alpha\mathbf{j} - \bar{I}_{xz}\alpha\mathbf{k}\end{aligned}$$

Since the shaft lies in the *xz* plane, $\bar{I}_{xy} = 0$.

By symmetry, the mass center lies on line *AB*. $m\bar{\mathbf{a}} = 0$

$\Sigma\mathbf{F} = \Sigma F_{\text{eff}}$: $\mathbf{A} + \mathbf{B} = m\bar{\mathbf{a}} = 0$ **A** and **B** form a couple.

$\mathbf{A} = -\mathbf{B}$



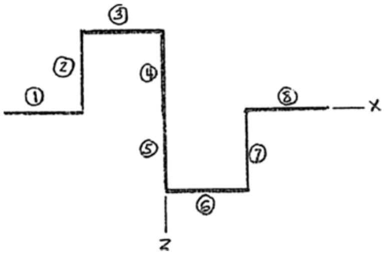
$$M_0\mathbf{i} + b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k}) = \dot{\mathbf{H}}_G$$

$$M_0\mathbf{i} + bB_y\mathbf{k} - bB_z\mathbf{j} = \bar{I}_x\alpha\mathbf{i} - \bar{I}_{xz}\alpha\mathbf{k}$$

$$M_0 = \bar{I}_x\alpha,\quad B_y = -\frac{I_{xz}\alpha}{b},\quad B_z = 0$$

PROBLEM 18.74 (Continued)

Calculation of $\bar{I}_x$ and I_{xz}.



Divide the shaft into eight segments, each of length

$$a = 200\text{ mm} = 0.2\text{ m}$$

Let m' be the mass of one segment.

$$m' = \frac{8\text{ kg}}{8} = 1\text{ kg}$$

For ① and ⑧, $I_x = 0$

For ②, ④, ⑤, and ⑦, $I_x = \frac{1}{3}ma^2$

For ③ and ⑥, $I_x = m'a^2$

Total: $\bar{I}_x = \frac{10}{3}m'a^2$

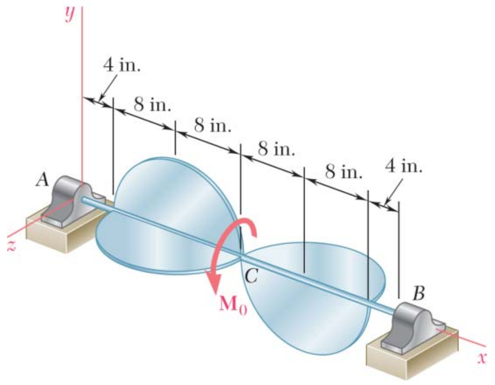
For ①, ④, ⑤, and ⑧, $I_{xz} = 0$

For ②, ③, ⑥, and ⑦, $I_{xz} = m'(-a)\left(-\frac{a}{2}\right) + m'\left(-\frac{a}{2}\right)(-a) + m'\left(\frac{a}{2}\right)(a) + m'(a)\left(\frac{a}{2}\right) = 2m'a^2$

(*a*) $M_0 = \frac{10}{3}m'a^2\alpha = \frac{10}{3}(1)(0.2)^2(20)$ $\quad\mathbf{M}_0 = (2.67\text{ N}\cdot\text{m})\mathbf{i}$ ◀

(*b*) $B_y = -\frac{2m'a^2\alpha}{b} = -\frac{(2)(1)(0.2)^2(20)}{(4)(0.2)}$ $\quad\mathbf{B} = -(2.00\text{ N})\mathbf{j}$ ◀

$A_z = 0, \quad A_y = -B_y = 2\text{ N}$ $\quad\mathbf{A} = (2.00\text{ N})\mathbf{j}$ ◀



PROBLEM 18.75

The assembly shown weighs 12 lb and consists of 4 thin 16-in.-diameter semicircular aluminum plates welded to a light 40-in.-long shaft *AB*. The assembly is at rest $(\omega = 0)$ at time $t = 0$ when a couple $\mathbf{M}_0$ is applied to it as shown, causing the assembly to complete one full revolution in 2 s. Determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at *A* and *B* at $t = 0$.

SOLUTION

Fixed axis rotation with constant angular acceleration. $\boldsymbol{\alpha} = \alpha\mathbf{i}, \quad \boldsymbol{\omega} = \omega\mathbf{i}$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2 \qquad \alpha = \frac{2\theta}{t^2} = \frac{2(2\pi)}{2^2} = 3.1416 \text{ rad/s}^2$$

$$\omega = 0$$

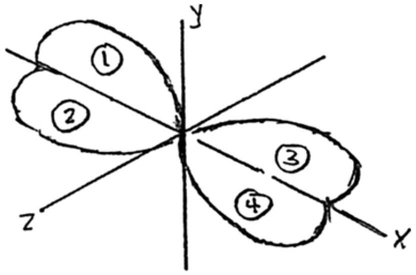
Use centroidal axes *x*, *y*, *z* with origin at *C*. $\mathbf{H}_C = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k}$

Let the reference frame *Cxyz* rotate with angular velocity $\mathbf{\Omega} = \boldsymbol{\omega} = \omega\mathbf{i}$

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Cxyz} + \mathbf{\Omega}\times\mathbf{H}_C = I_x\alpha\mathbf{i} - I_{xy}\alpha\mathbf{j} - I_{xz}\alpha\mathbf{k} + I_{xz}\omega^2\mathbf{j} - I_{xy}\omega^2\mathbf{k}$$

Required moments and products of inertia. Let $\rho = \dfrac{m}{A}$ = mass per unit area.

$$I_{\text{mass}} = \rho I_{\text{area}} \qquad r = 8 \text{ in.} = 0.66667 \text{ ft} \qquad m = \frac{12}{32.2} = 0.37267 \text{ lb}\cdot\text{s}^2/\text{ft}$$



Part	A	I_x	I_{xy}	I_{xz}
①	$\frac{1}{2}\pi r^2$	$\frac{1}{8}\pi r^4$	$-\frac{2}{3}r^4$	0
②	$\frac{1}{2}\pi r^2$	$\frac{1}{8}\pi r^4$	0	$-\frac{2}{3}r^4$
③	$\frac{1}{2}\pi r^2$	$\frac{1}{8}\pi r^4$	0	$-\frac{2}{3}r^4$
④	$\frac{1}{2}\pi r^2$	$\frac{1}{8}\pi r^4$	$-\frac{2}{3}r^4$	0
Σ	$2\pi r^2$	$\frac{1}{2}\pi r^4$	$-\frac{4}{3}r^4$	$-\frac{4}{3}r^4$

$$\rho = \frac{m}{2\pi r^2} = \frac{0.37267}{2\pi(0.66667)^2} = 0.13345\ \text{lb}\cdot\text{s}^2/\text{ft}^3$$

$$I_x = (0.13345)\left(\frac{1}{2}\pi\right)(0.66667)^4 = 0.041407\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

$$I_{xy} = (0.13345)\left(-\frac{4}{3}\right)(0.66667)^4 = -0.035147\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

$$I_{xz} = (0.13345)\left(-\frac{4}{3}\right)(0.66667)^4 = -0.035147\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

Since the mass center lies on the rotation axis, $\bar{\mathbf{a}} = 0$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = \Sigma\mathbf{F}_{\text{eff}} = m\bar{\mathbf{a}} = 0 \qquad \mathbf{A} = -\mathbf{B}$$

$$\Sigma\mathbf{M}_C = M_0\mathbf{i} + (-b\mathbf{i})\times\mathbf{A} + b\mathbf{i}\times\mathbf{B} = M_0\mathbf{i} + 2b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k})$$

$$= M_0\mathbf{i} - 2bB_z\mathbf{j} + 2bB_y\mathbf{k} \qquad \text{where} \qquad 2b = 40\ \text{in.} = 3.3333\ \text{ft}$$

$$\Sigma M_C = \dot{\mathbf{H}}_C$$ Resolve into components.

(*a*) **i**: $M_0 = I_x\alpha = (0.041407)(3.1416)$ $\qquad \mathbf{M}_0 = (0.1301\ \text{lb}\cdot\text{ft})\mathbf{i}$ ◀

(*b*) **j**: $-2b\,B_z = -I_{xy}\alpha + I_{xz}\omega^2$

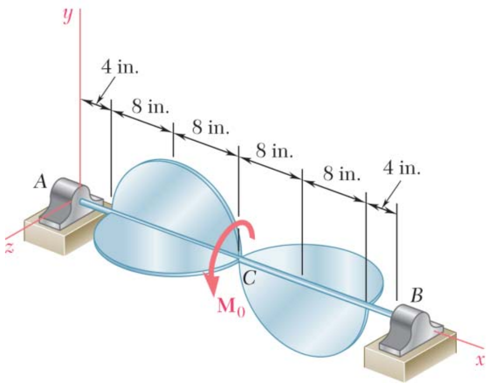
$$B_z = -\frac{-(-0.035147)(3.1416) + 0}{3.3333} = -0.0331\ \text{lb} \qquad A_z = 0.0331\ \text{lb}$$

k: $2bB_y = -I_{xz}\alpha - I_{xy}\omega^2$

$$B_y = -\frac{(-0.035147)(3.1416) + 0}{3.3333} = 0.0331\ \text{lb} \qquad A_y = -0.0331\ \text{lb}$$

$\mathbf{A} = -(0.0331\ \text{lb})\mathbf{j} + (0.0331\ \text{lb})\mathbf{k}$ ◀

$\mathbf{B} = (0.0331\ \text{lb})\mathbf{j} - (0.0331\ \text{lb})\mathbf{k}$ ◀



PROBLEM 18.76

For the assembly of Problem 18.75, determine the dynamic reactions at *A* and *B* at $t = 2$ s.

PROBLEM 18.75 The assembly shown weighs 12 lb and consists of four thin 16-in.-diameter semicircular aluminum plates welded to a light 40-in.-long shaft *AB*. The assembly is at rest $(\omega = 0)$ at time $t = 0$ when a couple $\mathbf{M}_0$ is applied to it as shown, causing the assembly to complete one full revolution in 2 s. Determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at *A* and *B* at $t = 0$.

SOLUTION

Fixed axis rotation with constant angular acceleration. $\quad \boldsymbol{\alpha} = \alpha\mathbf{i}, \quad \boldsymbol{\omega} = \omega\mathbf{i}$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2 \qquad \alpha = \frac{2\theta}{t^2} = \frac{2(2\pi)}{2^2} = 3.1416 \text{ rad/s}^2$$

$$\omega = \omega_0 + \alpha t = \alpha t = (3.1416)(2) = 6.2832 \text{ rad/s}$$

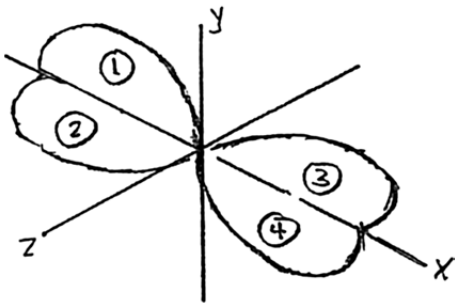
Use centroidal axes *x*, *y*, *z* with origin at *C*. $\quad \mathbf{H}_C = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k}$

Let the reference frame *Cxyz* rotate with angular velocity $\quad \boldsymbol{\Omega} = \boldsymbol{\omega} = \omega\mathbf{i}$

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Cxyz} + \boldsymbol{\Omega} \times \mathbf{H}_C = I_x\alpha\mathbf{i} - I_{xy}\alpha\mathbf{j} - I_{xz}\alpha\mathbf{k} + I_{xz}\omega^2\mathbf{j} - I_{xy}\omega^2\mathbf{k}$$

Required moments and products of inertia. Let $\rho = \frac{m}{A} =$ mass per unit area.

$$I_{\text{mass}} = \rho I_{\text{area}} \qquad r = 8 \text{ in.} = 0.66667 \text{ ft} \qquad m = \frac{12}{32.2} = 0.37267 \text{ lb}\cdot\text{s}^2/\text{ft}$$



Part	A	I_x	I_{xy}	I_{xz}
①	$\frac{1}{2}\pi r^2$	$\frac{1}{8}\pi r^4$	$-\frac{2}{3}r^4$	0
②	$\frac{1}{2}\pi r^2$	$\frac{1}{8}\pi r^4$	0	$-\frac{2}{3}r^4$
③	$\frac{1}{2}\pi r^2$	$\frac{1}{8}\pi r^4$	0	$-\frac{2}{3}r^4$
④	$\frac{1}{2}\pi r^2$	$\frac{1}{8}\pi r^4$	$-\frac{2}{3}r^4$	0
Σ	$2\pi r^2$	$\frac{1}{2}\pi r^4$	$-\frac{4}{3}r^4$	$-\frac{4}{3}r^4$

PROBLEM 18.76 (Continued)

$$\rho = \frac{m}{2\pi r^2} = \frac{0.37267}{2\pi(0.66667)^2} = 0.13345 \text{ lb}\cdot\text{s}^2/\text{ft}^3$$

$$I_x = (0.13345)\left(\frac{1}{2}\pi\right)(0.66667)^4 = 0.041407 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$I_{xy} = (0.13345)\left(-\frac{4}{3}\right)(0.66667)^4 = -0.035147 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$I_{xz} = (0.13345)\left(-\frac{4}{3}\right)(0.66667)^4 = -0.035147 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Since the mass center lies on the rotation axis, $\bar{\mathbf{a}} = 0$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = \Sigma\mathbf{F}_{\text{eff}} = m\bar{\mathbf{a}} = 0 \qquad \mathbf{A} = -\mathbf{B}$$

$$\Sigma\mathbf{M}_C = M_0\mathbf{i} + (-b\mathbf{i})\times\mathbf{A} + b\mathbf{i}\times\mathbf{B} = M_0\mathbf{i} + 2b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k})$$

$$= M_0\mathbf{i} - 2bB_z\mathbf{j} + 2bB_y\mathbf{k} \qquad \text{where} \quad 2b = 40 \text{ in.} = 3.3333 \text{ ft}$$

$$\Sigma M_C = \dot{\mathbf{H}}_C \qquad \text{Resolve into components.}$$

i: $M_0 = I_x\alpha = (0.041407)(3.1416)$ $\qquad M_0 = 0.1301 \text{ lb}\cdot\text{ft}$

j: $-2b\,B_z = -I_{xy}\alpha + I_{xz}\omega^2$

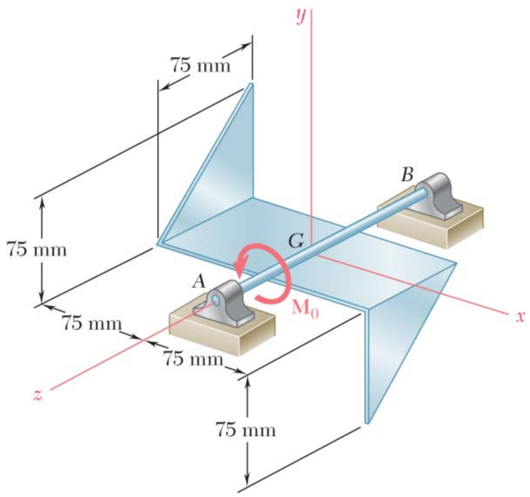
$$B_z = -\frac{-(-0.035147)(3.1416) + (-0.035147)(6.2832)^2}{3.3333} = 0.383 \text{ lb}, \qquad A_z = -0.383 \text{ lb}$$

k: $2bB_y = -I_{xz}\alpha - I_{xy}\omega^2$

$$B_y = -\frac{(-0.035147)(3.1416) + (-0.035147)(6.2832)^2}{3.3333} = 0.449 \text{ lb}, \qquad A_y = -0.449 \text{ lb}$$

$\mathbf{A} = -(0.449 \text{ lb})\mathbf{j} - (0.383 \text{ lb})\mathbf{k}$ ◀

$\mathbf{B} = (0.449 \text{ lb})\mathbf{j} + (0.383 \text{ lb})\mathbf{k}$ ◀



PROBLEM 18.77

The sheet-metal component shown is of uniform thickness and has a mass of 600 g. It is attached to a light axle supported by bearings at *A* and *B* located 150 mm apart. The component is at rest when it is subjected to a couple $\mathbf{M}_0$ as shown. If the resulting angular acceleration is $\boldsymbol{\alpha} = (12\text{ rad/s}^2)\mathbf{k}$, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at *A* and *B* immediately after the couple has been applied.

SOLUTION

The sheet metal component rotates about the fixed *z* axis, so that Equations (18.29) of the textbook apply. These are relisted below as equations (1), (2), and (3).

$$\Sigma M_x = -I_{xz}\alpha + I_{yz}\omega^2 \tag{1}$$

$$\Sigma M_y = -I_{yz}\alpha - I_{xz}\omega^2 \tag{2}$$

$$\Sigma M_z = I_z\alpha \tag{3}$$

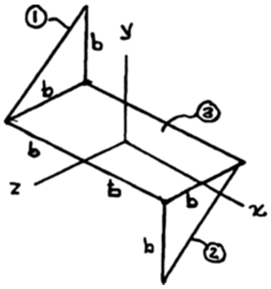
Calculation of the required moment and products of inertia.

Total mass: $m = 600\text{ g} = 0.6\text{ kg}$

Total area: $A = \frac{1}{2}(0.075)^2 + (0.150)(0.075) + \frac{1}{2}(0.075)^2 = 16.875\times 10^{-3}\text{ m}^2$

Let ρ be the mass per unit area. $\rho = \frac{m}{A} = 35.556\text{ kg/m}^2$

The component is comprised of 3 parts: triangle ①, triangle ②, and rectangle ③ as shown. Let the lengths of 75 mm be labeled *b*.



Triangle ①. The equation of the upper edge is

$$y = \frac{b}{2} - z$$

Use elements of width *dz* and height *y*.

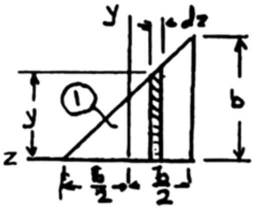
$$dm = \rho y dz \qquad (d\bar{I}_z)_{\text{el}} = \frac{1}{12}y^2 dm$$

$$(d\bar{I}_{xz})_{\text{el}} = 0 \qquad (d\bar{I}_{yz})_{\text{el}} = 0$$

PROBLEM 18.77 (Continued)

Coordinates of the element mass center:

$$\overline{x}_{\text{el}} = -b, \qquad \overline{y}_{\text{el}} = \frac{1}{2}y, \qquad \overline{z}_{\text{el}} = z$$



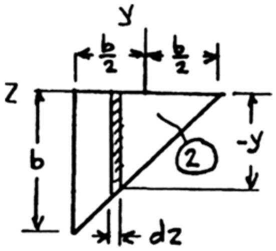
$$dI_{xz} = (d\overline{I}_{xz})_{\text{el}} + \overline{x}_{\text{el}}\overline{z}_{\text{el}}dm$$

$$= 0 + (-b)z(\rho y d_z) = -\rho bz\left(\frac{b}{2} - z\right)dz$$

$$I_{xz} = -\rho b\int_{-b/2}^{b/2} z\left(\frac{b}{2} - z\right)dz = \frac{1}{12}\rho b^4$$

$$dI_{yz} = (d\overline{I}_{yz})_{\text{el}} + \overline{y}_{\text{el}}\overline{z}_{\text{el}}dm$$

$$= 0 + \left(\frac{1}{2}y\right)z\left(\rho y dz\right) = \frac{1}{2}\rho\left(\frac{b}{2} - z\right)^2 zdz$$



$$I_{yz} = \frac{1}{2}\rho\int_{-b/2}^{b/2}\left(\frac{b}{2} - z\right)^2 zdz = -\frac{1}{24}\rho b^4$$

$$dI_z = (d\overline{I}_z)_{\text{el}} + (\overline{x}_{\text{el}}^2 + \overline{y}_{\text{el}}^2)dm$$

$$= \left(\frac{1}{12}y^2 + b^2 + \frac{1}{4}y^2\right)\rho y dz = \rho\left(b^2 y + \frac{1}{3}y^3\right)dz$$

$$= \rho\left(\frac{13}{24}b^3 - \frac{5}{4}b^2 z + \frac{1}{2}bz^2 - \frac{1}{3}z^3\right)dz$$

$$I_z = \rho\int_{-b/2}^{b/2}\left(\frac{13}{24}b^3 - \frac{5}{4}b^2 z + \frac{1}{2}bz^2 - \frac{1}{3}z^3\right)dz = \frac{7}{12}\rho b^4$$

Applying the data,

$$I_{xz} = \frac{1}{12}(35.556)(0.075)^4 = 93.75\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$I_{yz} = -\frac{1}{24}(35.556)(0.075)^4 = -46.875\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$I_z = \frac{7}{12}(35.556)(0.075)^4 = 656.25\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

Triangle ②. The equation of the lower edge is $y = -\left(\frac{b}{2} - z\right)$.

Use elements of width dz and height $-y$.

$$dm = -\rho y dz, \qquad (d\bar{I}_z)_{\text{el}} = \frac{1}{12} y^2 dm, \qquad (d\bar{I}_{zx})_{\text{el}} = 0, \qquad (d\bar{I}_{yz})_{\text{el}} = 0$$

Coordinates of the element mass center:

$$\bar{x}_{\text{el}} = b, \qquad \bar{y}_{\text{el}} = \frac{1}{2} y, \qquad \bar{z}_{\text{el}} = z$$

The integrals for $I_{xz}, I_{yz},$ and I_z turn out to be the same as those of triangle ①.

$$I_{xz} = 93.75 \times 10^{-6}\ \text{kg} \cdot \text{m}^2, \quad I_{yz} = -46.875 \times 10^{-6}\ \text{kg} \cdot \text{m}^2, \quad I_z = 656.25 \times 10^{-6}\ \text{kg} \cdot \text{m}^2$$

Rectangle ③. Area: $A = (0.150)(0.075) = 11.25 \times 10^{-3}\ \text{m}^2$

Mass: $m = \rho A = 400 \times 10^{-3}\ \text{kg}$

$$I_{xz} = 0, \qquad I_{yz} = 0$$

$$I_z = \frac{1}{12} m (2b)^2 = \frac{1}{12}(400 \times 10^{-3})(0.150)^2 = 750 \times 10^{-6}\ \text{kg} \cdot \text{m}^2$$

Totals.

$$I_{xz} = \Sigma I_{xz} = 187.5 \times 10^{-6}\ \text{kg} \cdot \text{m}^2$$

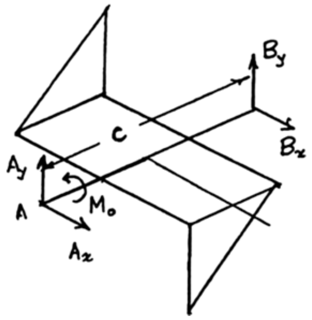
$$I_{yz} = \Sigma I_{yz} = -93.75 \times 10^{-6}\ \text{kg} \cdot \text{m}^2$$

$$I_z = \Sigma I_z = 2062.5 \times 10^{-6}\ \text{kg} \cdot \text{m}^2$$

Since the mass center lies on the fixed axis, the acceleration $\bar{\mathbf{a}}$ of the mass center is zero.

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} = 0$$

The reactions at A and B form a couple.



$$\mathbf{B} = -\mathbf{A}$$

Let $\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j}$

Resultant couple acting on the body:

$$\mathbf{M} = c\mathbf{k} \times (A_x \mathbf{i} + A_y \mathbf{j}) + M_0 \mathbf{k}$$

$$= -cA_y \mathbf{i} + cA_x \mathbf{j} + M_0 \mathbf{k}$$

(*a*) Moment M_0. Using Equation (3),

$$\Sigma M_z = M_0 = I_z \alpha = (2062.5 \times 10^{-6})(12) \qquad M_0 = 24.8 \times 10^{-3}\ \text{N} \cdot \text{m} \ ◀$$

PROBLEM 18.77 (Continued)

(*b*) Reactions at A and B for the case $\omega = 0$.

$$\Sigma M_x = -cA_y = -I_{xz}\alpha - I_{yz}\omega^2 = -I_{xz}\alpha$$

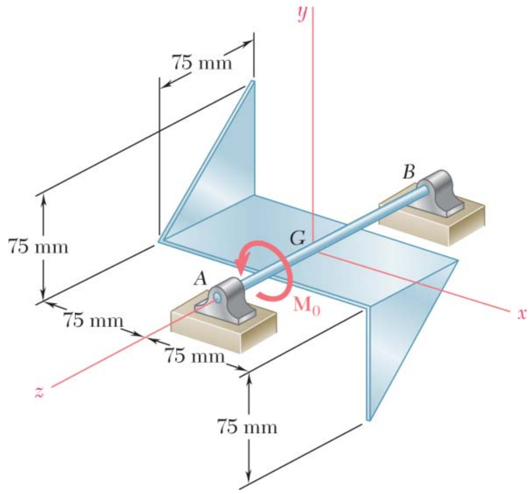
$$A_y = \frac{I_{xz}\alpha}{c} = \frac{(187.5\times10^{-6})(12)}{0.150} = 15\times10^{-3}\ \text{N}$$

$$\Sigma M_y = cA_x = -I_{yz}\alpha - I_{xz}\omega^2 = -I_{yz}\alpha$$

$$A_x = -\frac{I_{yz}\alpha}{c} = -\frac{(-93.75\times10^{-6})(12)}{0.150} = 7.5\times10^{-3}\ \text{N}$$

$$\mathbf{A} = (7.50\times10^{-3}\ \text{N})\mathbf{i} + (15.00\times10^{-3}\ \text{N})\mathbf{j}\ \blacktriangleleft$$

$$\mathbf{B} = -(7.50\times10^{-3}\ \text{N})\mathbf{i} - (15.00\times10^{-3}\ \text{N})\mathbf{j}\ \blacktriangleleft$$



PROBLEM 18.78

For the sheet-metal component of Problem 18.77, determine (*a*) the angular velocity of the component 0.6 s after the couple $\mathbf{M}_0$ has been applied to it, (*b*) the magnitude of the dynamic reactions at *A* and *B* at that time.

SOLUTION

The sheet metal component rotates about the fixed *z* axis with angular acceleration $\boldsymbol{\alpha} = (12 \text{ rad/s}^2)\mathbf{k}$.

(*a*) *Angular velocity at* $t = 0.6$ s.

$$\omega = \omega_0 + \alpha t = 0 + (12)(0.6) = 7.2 \text{ rad/s} \qquad \boldsymbol{\omega} = (7.20 \text{ rad/s})\mathbf{k} \blacktriangleleft$$

(*b*) *Dynamic reactions.*

Equations (18.29) of the textbook apply. These are relisted below as equations (1), (2), and (3).

$$\Sigma M_x = -I_{xz}\alpha + I_{yz}\omega^2 \tag{1}$$

$$\Sigma M_y = -I_{yz}\alpha - I_{xz}\omega^2 \tag{2}$$

$$\Sigma M_z = I_z\alpha \tag{3}$$

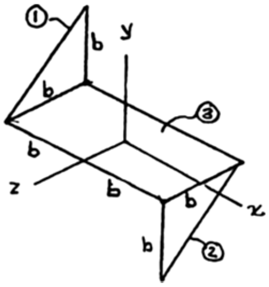
Calculation of the required moment and products of inertia.

Total mass: $m = 600 \text{ g} = 0.6 \text{ kg}$

Total area: $A = \frac{1}{2}(0.075)^2 + (0.150)(0.075) + \frac{1}{2}(0.075)^2 = 16.875 \times 10^{-3} \text{ m}^2$

Let ρ be the mass per unit area. $\rho = \frac{m}{A} = 35.556 \text{ kg/m}^2$

The component is comprised of 3 parts: triangle ①, triangle ②, and rectangle ③ as shown. Let the lengths of 75 mm be labeled *b*.



Triangle ①. The equation of the upper edge is

$$y = \frac{b}{2} - z$$

Use elements of width *dz* and height *y*.

$$dm = \rho y dz \qquad (d\bar{I}_z)_{\text{el}} = \frac{1}{12} y^2 dm$$

$$(d\bar{I}_{xz})_{\text{el}} = 0 \qquad (d\bar{I}_{yz})_{\text{el}} = 0$$

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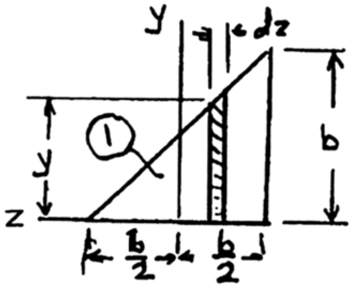
PROBLEM 18.78 (Continued)

Coordinates of the element mass center:

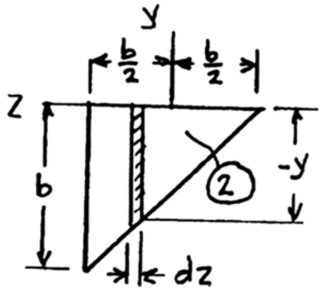


$$\bar{x}_{\text{el}} = -b,\ \bar{y}_{\text{el}} = \frac{1}{2}y, \quad \bar{z}_{\text{el}} = z$$

$$dI_{xz} = (d\bar{I}_{xz})_{\text{el}} + \bar{x}_{\text{el}}\bar{z}_{\text{el}}dm$$

$$= 0 + (-b)z(\rho y dz) = -\rho bz\left(\frac{b}{2} - z\right)dz$$

$$I_{xz} = -\rho b\int_{-b/2}^{b/2} z\left(\frac{b}{2} - z\right)dz = \frac{1}{12}\rho b^4$$



$$dI_{yz} = (d\bar{I}_{yz})_{\text{el}} + \bar{y}_{\text{el}}\bar{z}_{\text{el}}dm$$

$$= 0 + \left(\frac{1}{2}y\right)z(\rho y dz) = \frac{1}{2}\rho\left(\frac{b}{2} - z\right)^2 zdz$$

$$I_{yz} = \frac{1}{2}\rho\int_{-b/2}^{b/2}\left(\frac{b}{2} - z\right)^2 zdz = -\frac{1}{24}\rho b^4$$

$$dI_z = (d\bar{I}_z)_{\text{el}} + \left(\bar{x}_{\text{el}}^2 + \bar{y}_{\text{el}}^2\right)dm$$

$$= \left(\frac{1}{12}y^2 + b^2 + \frac{1}{4}y^2\right)\rho y_i dz = \rho\left(b^2 y + \frac{1}{3}y^3\right)dz$$

$$= \rho\left(\frac{13}{24}b^3 - \frac{5}{4}b^2 z + \frac{1}{2}bz^2 - \frac{1}{3}z^3\right)dz$$

$$I_z = \rho\int_{-b/2}^{b/2}\left(\frac{13}{24}b^3 - \frac{5}{4}b^2 z + \frac{1}{2}bz^2 - \frac{1}{3}z^3\right)dz = \frac{7}{12}\rho b^4$$

Applying the data,

$$I_{xz} = \frac{1}{12}(35.556)(0.075)^4 = 93.75\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$I_{yz} = -\frac{1}{24}(35.556)(0.075)^4 = -46.875\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$I_z = \frac{7}{12}(35.556)(0.075)^4 = 656.25\times 10^{6}\ \text{kg}\cdot\text{m}^2$$

Triangle ②. The equation of the lower edge is $y = -\left(\frac{b}{2} - z\right).$

Use elements of width dz and height $-y$.

$$dm = -\rho y dz, \quad (d\bar{I}_z)_{\text{el}} = \frac{1}{12}y^2 dm, \quad (d\bar{I}_{zx})_{\text{el}} = 0, \quad (d\bar{I}_{yz})_{\text{el}} = 0$$

PROBLEM 18.78 (Continued)

Coordinates of the element mass center:

$$\bar{x}_{\text{el}} = b, \qquad \bar{y}_{\text{el}} = \frac{1}{2}y, \qquad \bar{z}_{\text{el}} = z$$

The integrals for $I_{xz}, I_{yz},$ and I_z turn out to be the same as those of triangle ①.

$$I_{xz} = 93.75\times10^{-3}\ \text{kg}\cdot\text{m}^2, \quad I_{yz} = -46.875\times10^{-6}\ \text{kg}\cdot\text{m}^2, \quad I_z = 656.25\times10^{-6}\ \text{kg}\cdot\text{m}^2$$

Rectangle ③. Area: $A = (0.150)(0.075) = 11.25\times10^{-3}\ \text{m}^2$

Mass: $m = \rho A = 400\times10^{-3}\ \text{kg}$

$$I_{xz} = 0, \qquad I_{yz} = 0$$

$$I_z = \frac{1}{12}m(2b)^2 = \frac{1}{12}(400\times10^{-3})(0.150)^2 = 750\times10^{-6}\ \text{kg}\cdot\text{m}^2$$

Totals.

$$I_{xz} = \Sigma I_{xz} = 187.5\times10^{-6}\ \text{kg}\cdot\text{m}^2$$

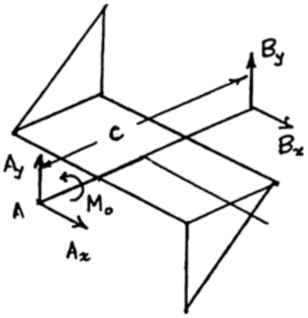
$$I_{yz} = \Sigma I_{yz} = -93.75\times10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$I_z = \Sigma I_z = 2062.5\times10^{-6}\ \text{kg}\cdot\text{m}^2$$

Since the mass center lies on the fixed axis, the acceleration $\bar{\mathbf{a}}$ of the mass center is zero.

$$\Sigma\mathbf{F} = m\bar{\mathbf{a}} = 0$$

The reactions at *A* and *B* form a couple.



$$\mathbf{B} = -\mathbf{A}$$

Let $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$

Resultant couple acting on the body:

$$\mathbf{M} = c\mathbf{k}\times(A_x\mathbf{i} + A_y\mathbf{j}) + M_0\mathbf{k}$$

$$= -cA_y\mathbf{i} + cA_x\mathbf{j} + M_0\mathbf{k}$$

From Eq. (1), $-cA_y = -I_{xz}\alpha + I_{yz}\omega^2$

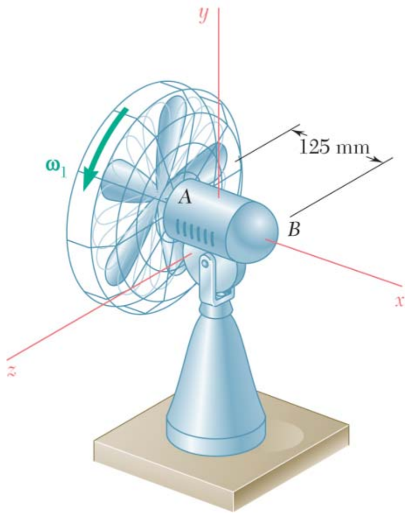
$$A_y = \frac{I_{xz}\alpha}{c} - \frac{I_{yz}\omega^2}{c} = \frac{(187.5\times10^{-6})(12)}{0.150} - \frac{(-93.75\times10^{-6})(7.2)^2}{0.150} = 47.4\times10^{-3}\ \text{N}$$

From Eq. (2), $cA_x = -I_{yz}\alpha - I_{xz}\omega^2$

$$A_x = -\frac{I_{yz}\alpha}{c} - \frac{I_{xz}\omega^2}{c} = -\frac{(-93.75\times10^{-6})(12)}{0.150} - \frac{(187.5\times10^{-6})(7.2)^2}{0.150} = -57.3\times10^{-3}\ \text{N}$$

$\mathbf{A} = -(57.3\times10^{-3}\ \text{N})\mathbf{i} + (47.3\times10^{-3}\ \text{N})\mathbf{j}$ ◀

$\mathbf{B} = (57.3\times10^{-3}\ \text{N})\mathbf{i} - (47.3\times10^{-3}\ \text{N})\mathbf{j}$ ◀



PROBLEM 18.79

The blade of an oscillating fan and the rotor of its motor have a total weight of 300 g and a combined radius of gyration of 75 mm. They are supported by bearings at *A* and *B*, 125 mm apart, and rotate at the rate $\omega_1 = 1800$ rpm. Determine the dynamic reactions at *A* and *B* when the motor casing has an angular velocity $\omega_2 = (0.6 \text{ rad/s})\mathbf{j}$.

SOLUTION

$$\boldsymbol{\omega}_1 = \omega_1 \mathbf{i}$$

$$\omega_1 = \frac{2\pi(1800)}{60} = 188.5 \text{ rad/s}$$

Angular velocity: $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j}$

Angular momentum: $\mathbf{H}_G = I_x \omega_1 \mathbf{i} + I_y \omega_2 \mathbf{j}$

Let the reference frame be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2 \mathbf{j}$.

$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \\ &= 0 + \omega_2 \mathbf{j} \times (I_x \omega_1 \mathbf{i} + I_y \omega_2 \mathbf{j}) \\ &= -I_x \omega_1 \omega_2 \mathbf{k}\end{aligned}$$

Assume that the acceleration of the mass center is negligible. Then the dynamic reactions at *A* and *B* reduce to a couple.

$$\mathbf{A} = -\mathbf{B}$$

$$\begin{aligned}\mathbf{M} &= b\mathbf{i} \times \mathbf{B} \\ &= b\mathbf{i} \times (B_y \mathbf{j} + B_z \mathbf{k}) \\ &= -b B_z \mathbf{j} + b B_y \mathbf{k}\end{aligned}$$

$$\mathbf{M} = \dot{\mathbf{H}}_G \qquad \text{Resolve into components.}$$

$\mathbf{j}$: $\quad -b B_z = 0 \qquad B_z = 0, \qquad A_z = 0$

$\mathbf{k}$: $\quad b B_y = -I_x \omega_1 \omega_2 \qquad B_y = -\dfrac{I_x \omega_1 \omega_2}{b} = -\dfrac{m k_x^2 \omega_1 \omega_2}{b}$

PROBLEM 18.79 (Continued)

Data:

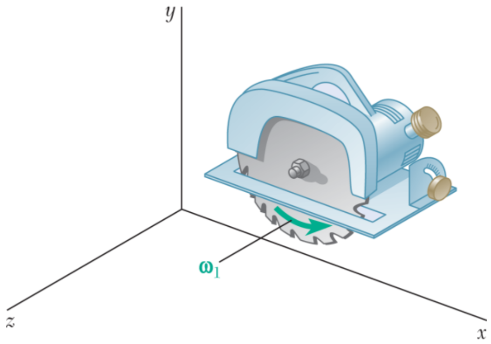
$$m = 0.300 \text{ kg}$$
$$k_x = 0.075 \text{ m}$$
$$b = 0.125 \text{ m}$$

$$B_y = -\frac{(0.300)(75\times10^{-3})^2(188.5)(0.6)}{0.125} = -1.527 \text{ N}$$

$$A_y = 1.527 \text{ N}$$

$\mathbf{A} = (1.527 \text{ N})\mathbf{j}$ ◀

$\mathbf{B} = -(1.527 \text{ N})\mathbf{j}$ ◀



PROBLEM 18.80

The blade of a portable saw and the rotor of its motor have a total weight of 2.5 lb and a combined radius of gyration of 1.5 in. Knowing that the blade rotates as shown at the rate $\omega_1 = 1500$ rpm, determine the magnitude and direction of the couple **M** that a worker must exert on the handle of the saw to rotate it with a constant angular velocity $\boldsymbol{\omega}_2 = -(2.4 \text{ rad/s})\mathbf{j}$.

SOLUTION

$$\boldsymbol{\omega}_1 = \frac{(2\pi)(1500)}{60}$$

$$= 157.08 \text{ rad/s}$$

$$\boldsymbol{\omega}_1 = \omega_1 \mathbf{k}$$

Angular velocity: $\boldsymbol{\omega} = \omega_2 \mathbf{j} + \omega_1 \mathbf{k}$

Angular momentum of rotor: $\mathbf{H}_G = I_y \omega_2 \mathbf{j} + I_z \omega_1 \mathbf{k}$

Let the reference frame *Gxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2 \mathbf{j}$.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

$$= 0 + \omega_2 \mathbf{j} \times (I_y \omega_2 \mathbf{j} + I_z \omega_1 \mathbf{k})$$

$$= I_z \omega_1 \omega_2 \mathbf{i}$$

Couple exerted on the saw: $\mathbf{M} = \dot{\mathbf{H}}_G$

$$= I_z \omega_1 \omega_2 \mathbf{i}$$

$$= m k_z^2 \omega_1 \omega_2 \mathbf{i}$$

Data: $W = 2.5 \text{ lb}$

$$m = \frac{2.5}{32.2}$$

$$= 0.07764 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$k_z = 1.5 \text{ in.}$$

$$= 0.125 \text{ ft}$$

$$\mathbf{M} = (0.07764)(0.125)^2(157.08)(-2.4)\,\mathbf{i}$$

$\mathbf{M} = -(0.457 \text{ lb} \cdot \text{ft})\,\mathbf{i}$ ◀

PROBLEM 18.81

The flywheel of an automobile engine, which is rigidly attached to the crankshaft, is equivalent to a 400-mm-diameter, 15-mm-thick steel plate. Determine the magnitude of the couple exerted by the flywheel on the horizontal crankshaft as the automobile travels around an unbanked curve of 200-m radius at a speed of 90 km/h, with the flywheel rotating at 2700 rpm. Assume the automobile to have (*a*) a rear-wheel drive with the engine mounted longitudinally, (*b*) a front-wheel drive with the engine mounted transversely. (Density of steel $= 7860\ \text{kg/m}^3$.)

SOLUTION

Let the *x* axis be a horizontal axis directed along the engine mounting, i.e., longitudinally for rear-wheel drive and transversely for front-wheel drive.

Let the *y* axis be vertical.

The angular velocity of the automobile, $\boldsymbol{\omega}_2$, is equal to $(v/\rho)\mathbf{j}$, where

$$v = 90\ \text{km/h} = 25\ \text{m/s} \ \text{ and } \ \rho = 200\ \text{m}.$$

$$\boldsymbol{\omega}_2 = \frac{25}{200}\mathbf{j} = (0.125\ \text{rad/s})\mathbf{j}$$

Angular velocity of the fly wheel relative to the automobile: $\boldsymbol{\omega}_1 = \omega_1\mathbf{i}$

where $$\omega_1 = \frac{2\pi(2700)}{60} = 282.74\ \text{rad/s}$$

Angular momentum of fly wheel: $\mathbf{H}_G = I_x\omega_1\mathbf{i} + I_y\omega_y\mathbf{j}$

Let the reference frame *Gxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2\mathbf{j}$

$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega}\times\mathbf{H}_G\\ &= 0 + \omega_2\mathbf{j}\times(I_x\omega_1\mathbf{i} + I_y\omega_2\mathbf{j})\\ &= -I_x\omega_1\omega_2\mathbf{k}\end{aligned}$$

Couple exerted by the shaft on the fly wheel: $\mathbf{M} = -I_x\omega_1\omega_2\mathbf{k}$

Couple exerted by the fly wheel on the shaft: $\mathbf{M}' = -\mathbf{M} = I_x\omega_1\omega_2\mathbf{k}$ (1)

Data for fly wheel: $$m = \rho\left(\frac{\pi}{4}d^2\right)t = (7860)\frac{\pi}{4}(0.4)^2(0.015) = 14.816\ \text{kg}$$

For a circular plate, $$I_x = \frac{1}{2}mr^2 = \frac{1}{2}(14.816)(0.2)^2 = 0.29632\ \text{kg}\cdot\text{m}^2$$

Using Equation (1), $$\mathbf{M}' = (0.29632)(282.74)(0.125)\mathbf{k} = (10.47\ \text{N}\cdot\text{m})\mathbf{k}$$

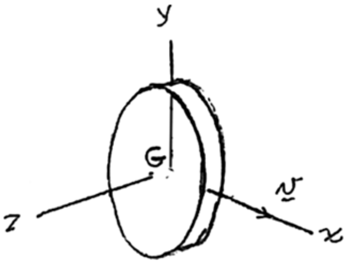
(*a*) Magnitude of couple for rear-wheel drive: $M' = 10.47\ \text{N}\cdot\text{m}$ ◀

(*b*) For front-wheel drive: $M' = 10.47\ \text{N}\cdot\text{m}$ ◀

PROBLEM 18.82

Each wheel of an automobile has a mass of 22 kg, a diameter of 575 mm, and a radius of gyration of 225 mm. The automobile travels around an unbanked curve of radius 150 m at a speed of 95 km/h. Knowing that the transverse distance between the wheels is 1.5 m, determine the additional normal force exerted by the ground on each outside wheel due to the motion of the car.

SOLUTION



For each wheel,

$$v = 95 \text{ km/h} = 26.389 \text{ m/s}$$

$$\omega_x = 0$$

$$\omega_y = \frac{v}{\rho} = \frac{26.389}{150} = 0.17593 \text{ rad/s}$$

$$r = \frac{d}{2} = \frac{575}{2} = 287.5 \text{ mm} = 0.2875 \text{ m}$$

$$\bar{I}_z = mk^2 = (22)(0.225)^2 = 1.11375 \text{ kg}\cdot\text{m}^2$$

$$\omega_z = -\frac{v}{r} = -\frac{26.389}{0.2875} = -91.787 \text{ rad/s}$$

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$= \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

Let reference frame *Gxyz* be rotating with angular velocity $\mathbf{\Omega} = \omega_y\mathbf{j}$.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega}\times\mathbf{H}_G$$

$$= 0 + \omega_y\mathbf{j}\times(\bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k})$$

$$= \bar{I}_z\omega_z\omega_y\mathbf{i}$$

$$\dot{\mathbf{H}}_G = (1.11375)(-91.787)(0.17593)\mathbf{i}$$

$$= -(17.985 \text{ N}\cdot\text{m})\mathbf{i}$$

Let O be the point at the center of the axle. For the two wheels plus the axle,

$$\dot{\mathbf{H}}_O = \dot{\mathbf{H}}_G + \dot{\mathbf{H}}_G = -(35.97 \text{ N}\cdot\text{m})\mathbf{i}$$

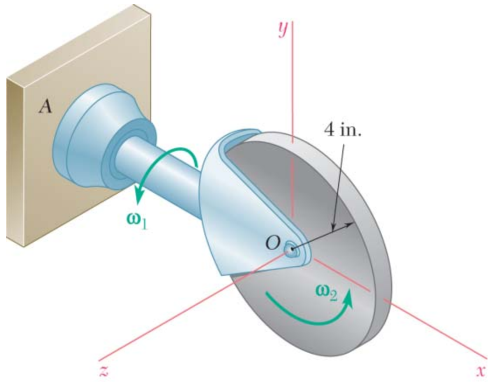
The distance between the wheels is 1.5 m.

$$\mathbf{M}_O = 1.5\mathbf{k}\times F_y\mathbf{j} = -1.5F_y\mathbf{i}$$

Set $\mathbf{M}_O = \dot{\mathbf{H}}_O$ and solve for F_y.

$$F_y = \frac{-35.97}{-1.5} = 23.98 \text{ N}$$

$\mathbf{F}_y = 24.0 \text{ N}\uparrow$ ◀



PROBLEM 18.83

The uniform thin 5-lb disk spins at a constant rate $\omega_2 = 6$ rad/s about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $\omega_1 = 3$ rad/s. Determine the couple which represents the dynamic reaction at the support *A*.

SOLUTION

Angular velocity: $\omega_x = \omega_1, \qquad \omega_y = 0, \qquad \omega_z = \omega_2.$

$$\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{k}$$

Angular momentum:

$$\mathbf{H}_O = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$= \bar{I}_x\omega_1\mathbf{i} + \bar{I}_z\omega_2\mathbf{k}$$

Let frame *Oxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \omega_1\,\mathbf{i}$.

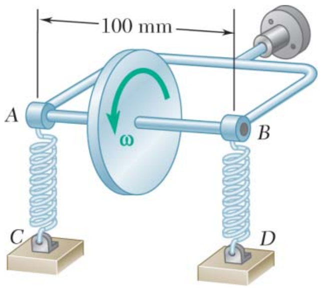
Rate of change of angular momentum.

$$\begin{aligned}\dot{\mathbf{H}}_O &= (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\Omega}\times\mathrm{H}_O\\ &= \bar{I}_x\dot{\omega}_1\mathbf{i} + \bar{I}_z\dot{\omega}_2\mathbf{k} + \omega_1\mathbf{i}\times(\bar{I}_x\omega_1\mathbf{i} + \bar{I}_z\omega_2\mathbf{k})\\ &= 0 + 0 + 0 - \bar{I}_z\omega_1\omega_2\mathbf{j} = -\frac{1}{2}mr^2\omega_1\omega_2\mathbf{j}\end{aligned}$$

Dynamic reaction couple: $\mathbf{M} = \dot{\mathbf{H}}_O$

$$\mathbf{M} = -\frac{1}{2}mr^2\omega_1\omega_2\mathbf{j} = -\frac{1}{2}\left(\frac{5}{32.2}\right)\left(\frac{4}{12}\right)^2(3)(6)\mathbf{j}$$

$\mathbf{M} = -(0.1553\ \mathrm{lb\cdot ft})\mathbf{j}$ ◀

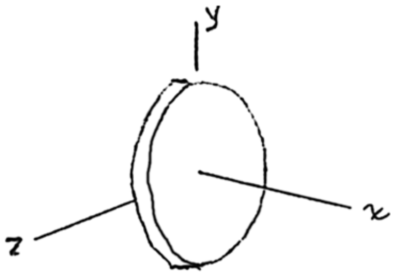


PROBLEM 18.84

The essential structure of a certain type of aircraft turn indicator is shown. Each spring has a constant of 500 N/m, and the 200-g uniform disk of 40-mm radius spins at the rate of 10,000 rpm. The springs are stretched and exert equal vertical forces on yoke *AB* when the airplane is traveling in a straight path. Determine the angle through which the yoke will rotate when the pilot executes a horizontal turn of 750-m radius to the right at a speed of 800 km/h. Indicate whether Point *A* will move up or down.

SOLUTION

Let the *x* axis lie along the axle *AB* and the *y* axis be vertical.



$$\omega_x = \frac{2\pi(10{,}000)}{60} = 1047.2 \text{ rad/s}$$

$$v = 800 \text{ km/h} = 222.22 \text{ m/s}$$

$$\rho = 750 \text{ m}$$

$$\omega_y = -\frac{v}{\rho} = -\frac{22.222}{750} = -0.2963 \text{ rad/s}$$

$$\omega_z = 0$$

Angular momentum:
$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$
$$= I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j}$$

Let the reference frame *Gxyz* be turning about the *y* axis with angular velocity $\mathbf{\Omega} = \omega_y\mathbf{j}$.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega}\times\mathbf{H}_G$$
$$= \omega_y\mathbf{j}\times(\bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j})$$
$$= -I_x\omega_x\omega_y\mathbf{k}$$

Data for the disk:
$$m = 200 \text{ g} = 0.2 \text{ kg}$$

$$I_x = \frac{1}{2}mr^2$$
$$= \frac{1}{2}(0.2)(0.040)^2$$
$$= 160\times10^{-6} \text{ kg}\cdot\text{m}^2$$

$$\mathbf{M}_G = \dot{\mathbf{H}}_G$$
$$= -(160\times10^{-6})(1047.2)(-0.2963)\mathbf{k}$$
$$= (0.049646 \text{ N}\cdot\text{m})\mathbf{k}$$

PROBLEM 18.84 (Continued)

The spring forces $\mathbf{F}_A$ and $\mathbf{F}_B$ exerted on the yoke provide the couple $\mathbf{M}_G$. The force exerted by spring *B* is upward.

Let $$\mathbf{F}_B = F\mathbf{j}$$

Then $$\mathbf{F}_A = -F\mathbf{j}$$

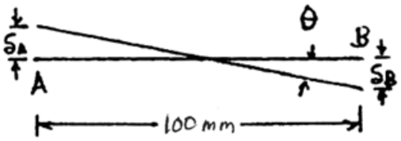
$$\begin{aligned}\mathbf{M}_G &= \mathbf{r}_{B/A}\mathbf{i} \times F\mathbf{j}\\ &= 0.100\mathbf{i} \times F\mathbf{j}\\ &= 0.1F\mathbf{k}\end{aligned}$$

From $$\mathbf{M}_G = \dot{\mathbf{H}}_G,$$

$$\begin{aligned}0.1F &= 0.049646\\ F &= 0.49646 \text{ N.}\end{aligned}$$

Compression of spring *B*: $$\begin{aligned}\delta_B &= \frac{F}{k}\\ &= \frac{0.49646}{500}\\ &= 0.99291 \times 10^{-3} \text{ m}\\ &= 0.9929 \text{ mm}\end{aligned}$$

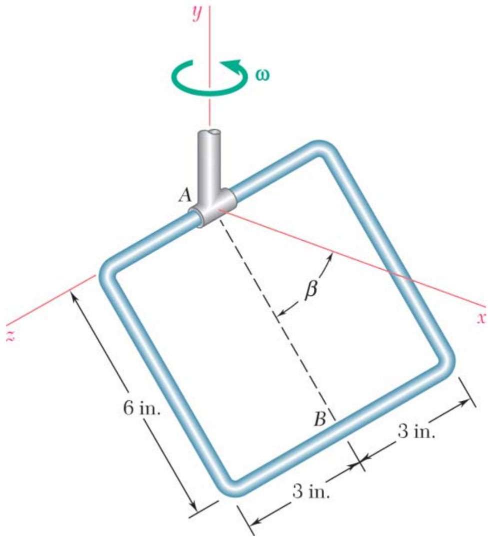
Point *B* moves 0.9929 mm *down*. Point *A* moves 0.9929 mm *up*.



Turning angle for yoke: $$\begin{aligned}\theta &= \frac{0.9929 + 0.9929}{100}\\ &= 0.019858 \text{ rad}\end{aligned}$$

$\boldsymbol{\theta} = 1.138^\circ \curvearrowright$ ◀

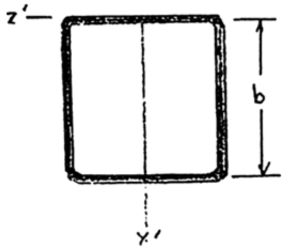
Point *A* moves *up*. ◀



PROBLEM 18.85

A slender rod is bent to form a square frame of side 6 in. The frame is attached by a collar at A to a vertical shaft which rotates with a constant angular velocity $\boldsymbol{\omega}$. Determine the value of ω for which line AB forms an angle $\beta = 48°$ with the horizontal x axis.

SOLUTION

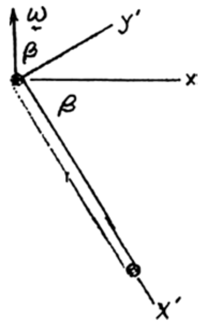


Choose principal axes x', y', z' with origin at the fixed Point A.

$$I_{x'} = 2\left(\frac{m}{4}\right)\left(\frac{1}{12}a^2\right) + 2\left(\frac{m}{4}\right)\left(\frac{a}{2}\right)^2 = \frac{1}{6}ma^2$$

$$I_{z'} = 2\left(\frac{m}{4}\right)\left(\frac{1}{3}a^2\right) + \left(\frac{m}{4}\right)(a)^2 = \frac{5}{12}ma^2$$

$$I_{y'} = I_{x'} + I_{z'} = \frac{7}{12}ma^2$$

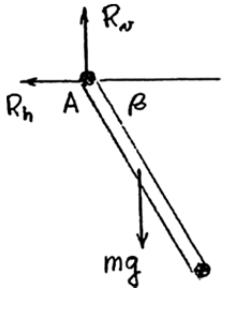


Angular velocity: $\boldsymbol{\omega} = -\omega\sin\beta\,\mathbf{i}' + \omega\cos\beta\,\mathbf{j}'$

Angular momentum about A: $\mathbf{H}_A = -I_{x'}\omega\sin\beta\,\mathbf{i}' + I_{y'}\omega\cos\beta\,\mathbf{j}'$

Let the reference from $Axyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

$$\begin{aligned}\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_A = 0 + \boldsymbol{\omega}\times\mathbf{H}_A\\ &= (-\omega\sin\beta\,\mathbf{i}' + \omega\cos\beta\,\mathbf{j}')\\ &\quad\times(-I_{x'}\,\omega\sin\beta\,\mathbf{i}' + I_{y'}\,\omega\cos\beta\,\mathbf{j}')\\ &= -(I_y - I_z)\,\omega^2\sin\beta\cos\beta\,\mathbf{k}\\ &= -\frac{5}{12}ma^2\omega^2\sin\beta\cos\beta\,\mathbf{k}\end{aligned}$$

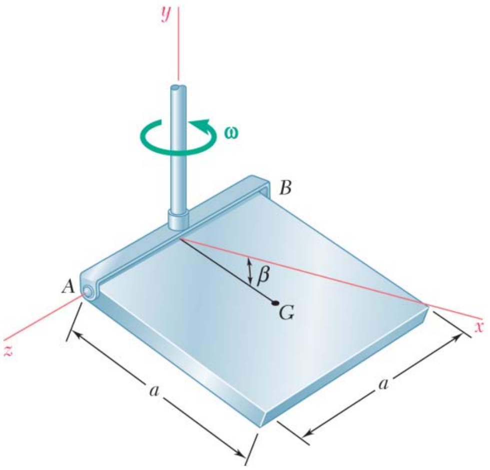


$$\Sigma\mathbf{M}_A = -mg\frac{a}{2}\cos\beta\,\mathbf{k} = \dot{\mathbf{H}}_A$$

$$-mg\frac{a}{2}\cos\beta = -\frac{5}{12}ma^2\omega^2\sin\beta\cos\beta$$

$$\omega^2 = \frac{6}{5}\frac{g}{a\sin\beta} = \frac{(6)(32.2)}{(5)(0.5)\sin 48°} = 103.99\,(\text{rad/s})^3$$

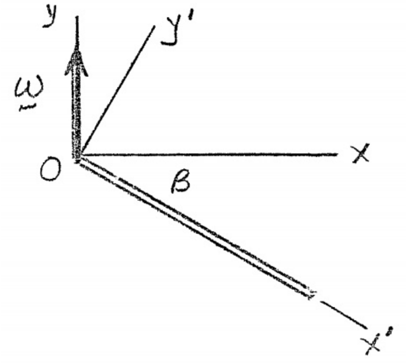
$\omega = 10.20$ rad/s ◀



PROBLEM 18.86

A uniform square plate of side $a = 225$ mm is hinged at points A and B to a clevis which rotates with a constant angular velocity $\boldsymbol{\omega}$ about a vertical axis. Determine (a) the constant angle β that the plate forms with the horizontal x axis when $\omega = 12$ rad/s, (b) the largest value of ω for which the plate remains vertical $(\beta = 90°)$.

SOLUTION



Let point O be the midpoint of axle AB. Choose principal axes x', y', z' with origin at point O.

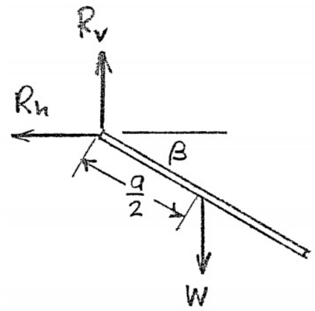
$$I_{x'} = \frac{1}{12}ma^2, \qquad I_{z'} = \frac{1}{3}ma^2,$$

$$I_{y'} = I_{x'} + I_{z'} = \frac{5}{12}ma^2$$

$$\boldsymbol{\omega} = -\omega \sin\beta \mathbf{i}' + \omega\cos\beta\mathbf{j}'$$

$$\mathbf{H}_O = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j} + I_{z'}\omega_z\mathbf{k}'$$

$$= -I_x\omega\sin\beta\mathbf{i}$$



Let the reference frame $Ox'y'z'$ be rotating about the fixed point O with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

$$\dot{\mathbf{H}}_O = \left(\dot{\mathbf{H}}_O\right)_{Ox'y'z'} + \boldsymbol{\Omega}\times\mathbf{H}_O = 0 + \boldsymbol{\omega}\times\mathbf{H}_O$$

$$= \left(-\omega\sin\beta\mathbf{i}' + \omega\cos\beta\mathbf{j}'\right)\times\left(-I_{x'}\omega\sin\beta\mathbf{i}' + I_{y'}\omega\cos\beta\mathbf{j}'\right)$$

$$= -\left(I_{y'} - I_{x'}\right)\omega^2\sin\beta\cos\beta\mathbf{k}$$

$$= -\frac{1}{3}ma^2\sin\beta\cos\beta\mathbf{k}$$

$$\mathbf{M}_O = -W\frac{a}{2}\cos\beta\mathbf{k} = \dot{\mathbf{H}}_O$$

$$W = \frac{2}{3}ma\omega^2\sin\beta = mg$$

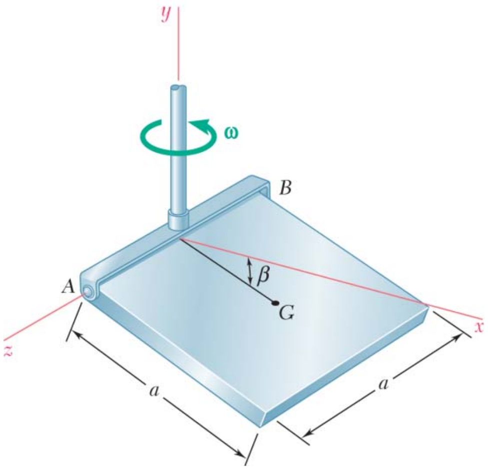
$$g = \frac{2}{3}a\omega^2\sin\beta$$

PROBLEM 18.86 (Continued)

(*a*) $$\sin\beta = \frac{3g}{2a\omega^2} = \frac{(3)(9.81)}{(2)(0.225)(12)^2} = 0.45417 \qquad \beta = 27.0° \blacktriangleleft$$

(*b*) Set $$\beta = 90°, \qquad \sin\beta = 1$$

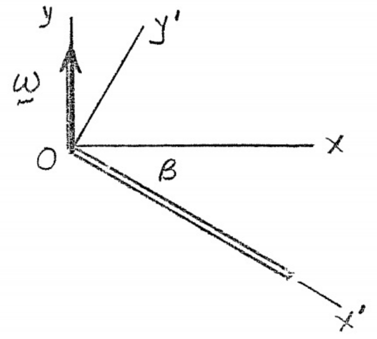
$$\omega^2 = \frac{3g}{2a} = \frac{(3)(9.81)}{(2)(0.225)} = 65.4 \qquad \omega = 8.09 \text{ rad/s} \blacktriangleleft$$



PROBLEM 18.87

A uniform square plate of side $a = 300$ mm is hinged at points A and B to a clevis which rotates with a constant angular velocity $\boldsymbol{\omega}$ about a vertical axis. Determine (*a*) the value of ω for which the plate forms a constant angle $\beta = 60°$ with the horizontal x axis, (*b*) the largest value of ω for which the plate remains vertical $(\beta = 90°)$.

SOLUTION



Let point O be the midpoint of axle AB. Choose principal axes x', y', z' with origin at point O.

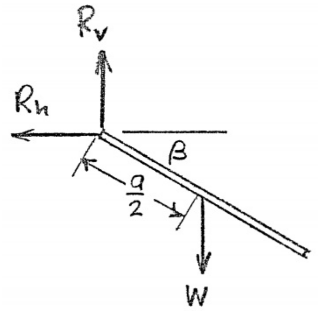
$$I_{x'} = \frac{1}{12}ma^2, \qquad I_{z'} = \frac{1}{3}ma^2,$$

$$I_{y'} = I_{x'} + I_{z'} = \frac{5}{12}ma^2$$

$$\boldsymbol{\omega} = -\omega \sin\beta\,\mathbf{i}' + \omega\cos\beta\,\mathbf{j}'$$

$$\mathbf{H}_O = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j} + I_{z'}\omega_{z'}\mathbf{k}'$$

$$= -I_x\omega\sin\beta\,\mathbf{i}$$



Let the reference frame $Ox'y'z'$ be rotating about the fixed point O with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

$$\dot{\mathbf{H}}_O = \left(\dot{\mathbf{H}}_O\right)_{Ox'y'z'} + \boldsymbol{\Omega}\times\mathbf{H}_O = 0 + \boldsymbol{\omega}\times\mathbf{H}_O$$

$$= \left(-\omega\sin\beta\,\mathbf{i}' + \omega\cos\beta\,\mathbf{j}'\right)\times\left(-I_{x'}\omega\sin\beta\,\mathbf{i}' + I_{y'}\omega\cos\beta\,\mathbf{j}'\right)$$

$$= -\left(I_{y'} - I_{x'}\right)\omega^2\sin\beta\cos\beta\,\mathbf{k}$$

$$= -\frac{1}{3}ma^2\sin\beta\cos\beta\,\mathbf{k}$$

$$\mathbf{M}_O = -W\frac{a}{2}\cos\beta\,\mathbf{k} = \dot{\mathbf{H}}_O$$

$$W = \frac{2}{3}ma\omega^2\sin\beta = mg$$

$$g = \frac{2}{3}a\omega^2\sin\beta$$

PROBLEM 18.87 (Continued)

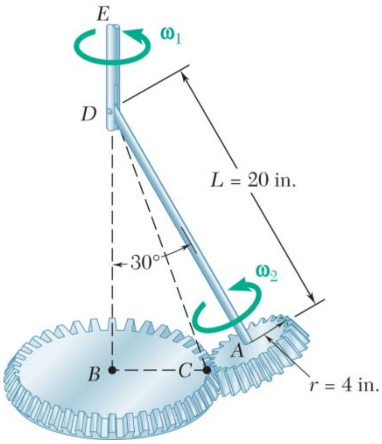
(*a*)
$$\omega^2 = \frac{3g}{2a\sin\beta} = \frac{(3)(9.81)}{(2)(0.3)\sin 60^\circ} = 56.638(\text{rad/s})^2$$

$\omega = 7.53 \text{ rad/s}$ ◀

(*b*) Set $\beta = 90^\circ, \qquad \sin\beta = 1$

$$\omega^2 = \frac{3g}{2a} = \frac{(3)(9.81)}{(2)(0.3)} = 49.05(\text{rad/s})^2$$

$\omega = 7.00 \text{ rad/s}$ ◀



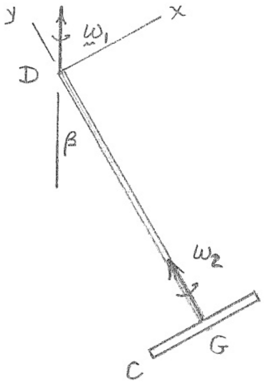
PROBLEM 18.88

The 2-lb gear *A* is constrained to roll on the fixed gear *B*, but is free to rotate about axle *AD*. Axle *AD*, of length 20 in. and negligible weight, is connected by a clevis to the vertical shaft *DE* which rotates as shown with a constant angular velocity $\boldsymbol{\omega}_1$. Assuming that gear *A* can be approximated by a thin disk of radius 4 in., determine the largest allowable value of ω_1 if gear *A* is not to lose contact with gear *B*.

SOLUTION

$$\beta = 30°, \qquad L = 20 \text{ in.} = \frac{20}{12} \text{ ft}, \qquad r = 4 \text{ in.} = \frac{4}{12} \text{ ft}$$

Choose principal axes *x*, *y*, *z* as shown.



Kinematics:

$$\boldsymbol{\omega}_1 = \omega_1 \sin\beta \mathbf{i} + \omega_1 \cos\beta \mathbf{j} \qquad \boldsymbol{\omega}_2 = \omega_2 \mathbf{j}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \omega_1 \sin\beta \mathbf{i} + \left(\omega_1 \cos\beta + \omega_2\right)\mathbf{j}$$

$$\omega_x = \omega_1 \sin\beta \qquad \omega_y = \omega_1 \cos\beta + \omega_2$$

$$\mathbf{v}_G = \boldsymbol{\omega}_1 \times \mathbf{r}_{G/D} = \boldsymbol{\omega}_1 \mathbf{j} \times \mathbf{r}_{G/D} = -\omega_1 L \sin\beta \mathbf{k}$$

$$\mathbf{a}_G = \boldsymbol{\omega}_1 \times \mathbf{v}_G = \omega_1^2 L \sin\beta \longleftarrow$$

$$\mathbf{v}_C = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{C/G}$$

$$0 = -\omega_1 L \sin\beta \mathbf{k} + \left(\omega_x \mathbf{i} + \omega_y \mathbf{j}\right) \times \left(-r\mathbf{i}\right)$$

$$= -\omega_1 L \sin\beta \mathbf{k} + \omega_y r \mathbf{k}$$

$$\omega_y = \omega_1 \frac{L}{r} \sin\beta$$

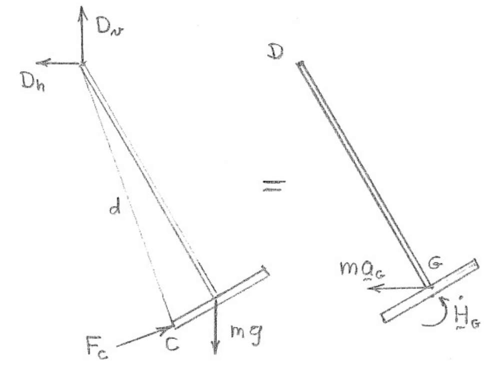
Angular momentum:

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j}$$

Let the reference frame $Dxyz$ be rotating with angular velocity $\mathbf{\Omega} = \boldsymbol{\omega}_1$.

$$\dot{\mathbf{H}}_G = \left(\dot{\mathbf{H}}_G\right)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G = 0 + \left(\omega_1 \sin\beta \mathbf{i} + \omega_1 \cos\beta \mathbf{j}\right) \times \left(\bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j}\right)$$

$$= \left(\bar{I}_y \omega_y \omega_1 \sin\beta - \bar{I}_x \omega_x \omega_1 \cos\beta\right)\mathbf{k} = \left(\frac{1}{2} mr^2 \omega_y \sin\beta - \frac{1}{4} mr^2 \omega_x \cos\beta\right)\omega_1 \mathbf{k}$$



Moments about D:

$$\mathbf{M}_D = Fd\mathbf{k} - mgL \sin\beta \mathbf{k},$$

where $$d = \sqrt{L^2 + r^2}$$

$$\left(\mathbf{M}_D\right)_{\text{eff}} = \dot{\mathbf{H}}_G + \mathbf{r}_{G/D} \times m\mathbf{a}_G$$

$$= \dot{\mathbf{H}}_G + m\omega_1^2 L^2 \sin\beta \cos\beta \mathbf{k}$$

Equating $\mathbf{M}_D = \left(\mathbf{M}_D\right)_{\text{eff}}$ and taking the z component,

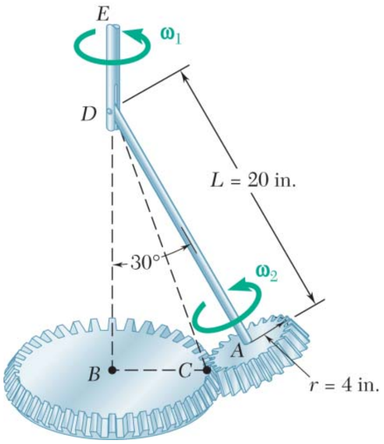
$$F_C d - mgL \sin\beta = \left(\frac{1}{2} mr^2 \frac{L}{r} \sin^2\beta - \frac{1}{4} mr^2 \sin\beta \cos\beta\right)\omega_1^2 + m\omega_1^2 L^2 \sin\beta \cos\beta$$

$$= m\omega_1^2 \sin\beta \left(\frac{1}{2} rL \sin\beta - \frac{1}{4} r^2 \cos\beta - L^2 \cos\beta\right)$$

Set $F = 0$ and solve for ω_1^2:

$$\omega_1^2 = \frac{gL}{L^2 \cos\beta + \frac{1}{4} r^2 \cos\beta - \frac{1}{2} rL \sin\beta} = \frac{(32.2)\left(\frac{20}{12}\right)}{\left(\frac{20}{12}\right)^2 \cos 30° + \frac{1}{4}\left(\frac{4}{12}\right)^2 \cos 30° - \frac{1}{2}\left(\frac{4}{12}\right)\left(\frac{20}{12}\right) \sin 30°}$$

$$= 23.427 \qquad \omega_1 = 4.84 \text{ rad/s} \blacktriangleleft$$



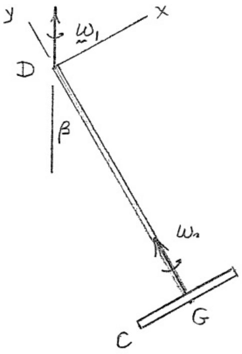
PROBLEM 18.89

Determine the force **F** exerted by gear *B* on gear *A* of Prob. 18.88 when shaft *DE* rotates with the constant angular velocity $\boldsymbol{\omega}_1 = 4$ rad/s. (*Hint*: The force **F** must be perpendicular to the line drawn from *D* to *C*.)

SOLUTION

$$\beta = 30^\circ, \qquad L = 20\text{ in.} = \frac{20}{12}\text{ ft}, \qquad r = 4\text{ in.} = \frac{4}{12}\text{ ft}$$

Choose principal axes *x*, *y*, *z* as shown.



Kinematics:

$$\boldsymbol{\omega}_1 = \omega_1 \sin\beta\,\mathbf{i} + \omega_1\cos\beta\,\mathbf{j} \qquad \boldsymbol{\omega}_2 = \omega_2\mathbf{j}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \omega_1\sin\beta\,\mathbf{i} + \left(\omega_1\cos\beta + \omega_2\right)\mathbf{j}$$

$$\omega_x = \omega_1\sin\beta \qquad \omega_y = \omega_1\cos\beta + \omega_2$$

$$\mathbf{v}_G = \boldsymbol{\omega}_1 \times \mathbf{r}_{G/D} = \boldsymbol{\omega}_1\mathbf{j} \times \mathbf{r}_{G/D} = -\omega_1 L\sin\beta\,\mathbf{k}$$

$$\mathbf{a}_G = \boldsymbol{\omega}_1 \times \mathbf{v}_G = \omega_1^2 L\sin\beta \longleftarrow$$

$$\mathbf{v}_C = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{C/G}$$

$$0 = -\omega_1 L\sin\beta\,\mathbf{k} + \left(\omega_x\mathbf{i} + \omega_y\mathbf{j}\right) \times \left(-r\mathbf{i}\right)$$

$$= -\omega_1 L\sin\beta\,\mathbf{k} + \omega_y r\mathbf{k}$$

$$\omega_y = \omega_1\frac{L}{r}\sin\beta$$

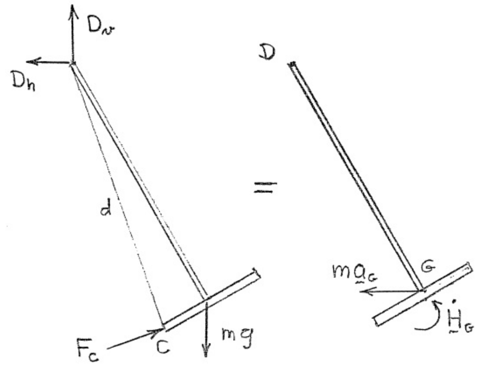
Angular momentum:

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j}$$

Let the reference frame $Dxyz$ be rotating with angular velocity $\mathbf{\Omega} = \boldsymbol{\omega}_1$.

$$\dot{\mathbf{H}}_G = \left(\dot{\mathbf{H}}_G\right)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G = 0 + \left(\omega_1 \sin\beta \mathbf{i} + \omega_1 \cos\beta \mathbf{j}\right) \times \left(\bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j}\right)$$

$$= \left(\bar{I}_y \omega_y \omega_1 \sin\beta - \bar{I}_x \omega_x \omega_1 \cos\beta\right)\mathbf{k} = \left(\frac{1}{2} m r^2 \omega_y \sin\beta - \frac{1}{4} m r^2 \omega_x \cos\beta\right)\omega_1 \mathbf{k}$$



Moments about D:

$$\mathbf{M}_D = Fd\mathbf{k} - mgL\sin\beta\mathbf{k},$$

where $$d = \sqrt{L^2 + r^2}$$

$$\left(\mathbf{M}_D\right)_{\text{eff}} = \dot{\mathbf{H}}_G + \mathbf{r}_{G/D} \times m\mathbf{a}_G$$

$$= \dot{\mathbf{H}}_G + m\omega_1^2 L^2 \sin\beta\cos\beta\mathbf{k}$$

Equating $\mathbf{M}_D = \left(\mathbf{M}_D\right)_{\text{eff}}$ and taking the z component,

$$F_C d - mgL\sin\beta = \left(\frac{1}{2} m r^2 \frac{L}{r} \sin^2\beta - \frac{1}{4} m r^2 \sin\beta\cos\beta\right)\omega_1^2 + m\omega_1^2 L^2 \sin\beta\cos\beta$$

$$= m\omega_1^2 \sin\beta\left(\frac{1}{2} rL\sin\beta - \frac{1}{4} r^2 \cos\beta - L^2\cos\beta\right)$$

Solving for F_C,

$$F_C = \frac{m\sin\beta}{d}\left[gL - \omega_1^2\left(L^2\cos\beta + \frac{1}{4} r^2 \cos\beta - \frac{1}{2} rL\sin\beta\right)\right]$$

Additional data: $$m = \frac{2}{32.2} = 0.062112 \text{ lb}\cdot\text{s}^2/\text{ft}, \qquad \omega_1 = 4 \text{ rad/s}$$

$$d = \sqrt{\left(\frac{20}{12}\right)^2 + \left(\frac{4}{12}\right)^2} = 1.69967 \text{ ft}$$

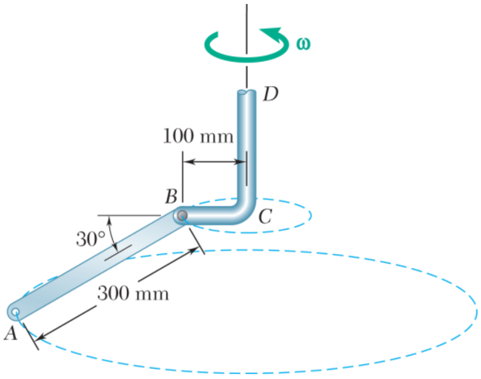
PROBLEM 18.89 (Continued)

$$F_C = \frac{(0.061112)\sin 30^\circ}{1.69967}\left\{(32.2)\left(\frac{20}{12}\right) - (4)^2\left[\left(\frac{20}{12}\right)^2 \cos 30^\circ + \frac{1}{4}\left(\frac{4}{20}\right)^2 \cos 30^\circ - \frac{1}{2}\left(\frac{20}{12}\right)\left(\frac{4}{12}\right)\sin 30^\circ\right]\right\}$$

$$= 0.311\text{ lb}$$

$$\alpha = \beta - \tan^{-1}\frac{r}{L} = 30^\circ - \tan^{-1}\left(\frac{4}{20}\right) \qquad \alpha = 18.7^\circ$$

$0.311\text{ lb} \measuredangle 18.7^\circ$ ◀



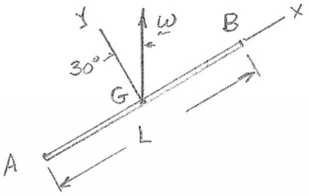
PROBLEM 18.90

The slender rod *AB* is attached by a clevis to arm *BCD* which rotates with a constant angular velocity $\boldsymbol{\omega}$ about the centerline of its vertical portion *CD*. Determine the magnitude of the angular velocity $\boldsymbol{\omega}$.

SOLUTION

Let $AB = L = 0.3\text{ m}$, and $BC = b = 0.1\text{ m}$.

Choose x, y, z axes as shown. $\bar{I}_x \approx 0, \quad \bar{I}_y = \bar{I}_z = \frac{1}{12}mL^2$



Angular velocity: $\boldsymbol{\omega} = \omega \sin 30° \mathbf{i} + \omega \cos 30° \mathbf{j}$

Angular momentum of rod *AB* about its mass center *G*:

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = \bar{I}_y \omega \cos 30° \mathbf{j}$$

Let the reference frame *Gxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

$$\dot{\mathbf{H}}_G = \left(\dot{\mathbf{H}}_G\right)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

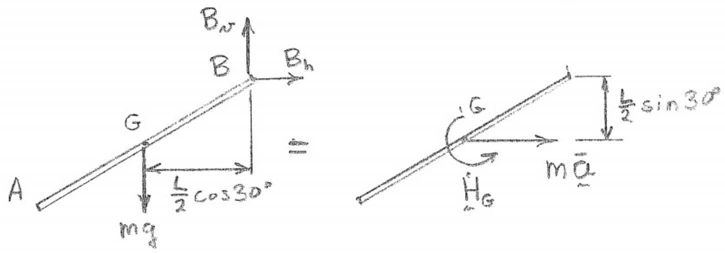
$$= 0 + \left(\omega \sin 30° \mathbf{i} + \omega \cos 30° \mathbf{j}\right) \times \bar{I}_y \omega \cos 30° \mathbf{j}$$

$$= \bar{I}_y \omega^2 \sin 30° \cos 30° \mathbf{k} = \frac{\sqrt{3}}{48} mL^2 \omega^2 \mathbf{k}$$

Radius of circular path of point *G*: $r = \frac{L}{2}\cos 30° + b = 0.22990\text{ m}$

Acceleration of the mass center: $\bar{\mathbf{a}} = r\omega^2 \longrightarrow$

Equations of motion:



PROBLEM 18.90 (Continued)

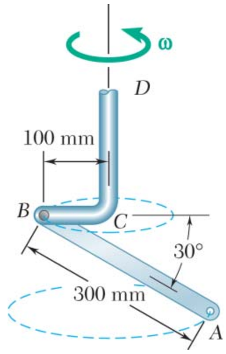
$$+\circlearrowleft\ \Sigma\mathbf{M}_B = mg\frac{L}{2}\cos 30°\mathbf{k} = \left(\frac{L}{2}\sin 30°\right)mr\omega^2\mathbf{k} + \dot{\mathbf{H}}_G$$

$$\frac{\sqrt{3}}{4}mgL\mathbf{k} = \left(\frac{1}{4}mLr + \frac{\sqrt{3}}{48}mL^2\right)\omega^2\mathbf{k}$$

$$\frac{\sqrt{3}}{4}g = \left(\frac{1}{4}r + \frac{\sqrt{3}}{48}L\right)\omega^2$$

$$\frac{\sqrt{3}}{4}(9.81) = \left[\frac{1}{4}(0.22990) + \frac{\sqrt{3}}{48}(0.3)\right]\omega^2$$

$\omega^2 = 62.194$ $\qquad$ $\omega = 7.89$ rad/s ◀



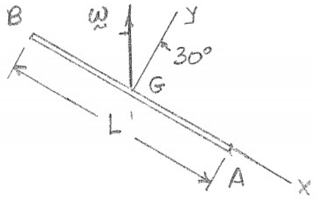
PROBLEM 18.91

The slender rod *AB* is attached by a clevis to arm *BCD* which rotates with a constant angular velocity ω about the centerline of its vertical portion *CD*. Determine the magnitude of the angular velocity ω.

SOLUTION

Let $AB = L = 0.3\text{ m},$ and $BC = b = 0.1\text{ m}.$

Choose x, y, z axes as shown. $\bar{I}_x \approx 0, \qquad \bar{I}_y = \bar{I}_z = \frac{1}{12}mL^2$



Angular velocity: $\boldsymbol{\omega} = -\omega\sin 30°\mathbf{i} + \omega\cos 30°\mathbf{j}$

Angular momentum of rod *AB* about its mass center *G*:

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} = \bar{I}_y\omega\cos 30°\mathbf{j}$$

Let the reference frame *Gxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

$$\dot{\mathbf{H}}_G = \left(\dot{\mathbf{H}}_G\right)_{Gxyz} + \boldsymbol{\Omega}\times\mathbf{H}_G$$

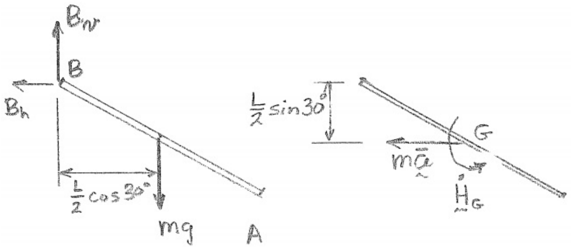
$$= 0 + \left(-\omega\sin 30°\mathbf{i} + \omega\cos 30°\mathbf{j}\right)\times\bar{I}_y\omega\cos 30°\mathbf{j}$$

$$= -\bar{I}_y\omega^2\sin 30°\cos 30°\mathbf{k} = -\frac{\sqrt{3}}{48}mL^2\omega^2\mathbf{k}$$

Radius of circular path of point *G*: $r = \frac{L}{2}\cos 30° - b = 0.029904\text{ m}$

Acceleration of the mass center: $\bar{\mathbf{a}} = r\omega^2 \longleftarrow$

Equations of motion:



PROBLEM 18.91 (Continued)

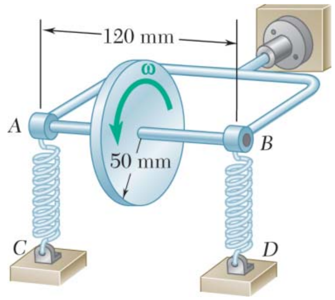
$$+\circlearrowleft\ \Sigma\mathbf{M}_B = -mg\frac{L}{2}\cos 30°\mathbf{k} = -\left(\frac{L}{2}\sin 30°\right)mr\omega^2\mathbf{k} + \dot{\mathbf{H}}_G$$

$$-\frac{\sqrt{3}}{4}mgL\mathbf{k} = -\left(\frac{1}{4}mLr + \frac{\sqrt{3}}{48}mL^2\right)\omega^2\mathbf{k}$$

$$\frac{\sqrt{3}}{4}g = \left(\frac{1}{4}r + \frac{\sqrt{3}}{48}L\right)\omega^2$$

$$\frac{\sqrt{3}}{4}(9.81) = \left[\frac{1}{4}(0.029904) + \frac{\sqrt{3}}{48}(0.3)\right]\omega^2$$

$$\omega^2 = 232.11 \qquad\qquad \omega = 15.24 \text{ rad/s} \blacktriangleleft$$



PROBLEM 18.92

The essential structure of a certain type of aircraft turn indicator is shown. Springs *AC* and *BD* are initially stretched and exert equal vertical forces at *A* and *B* when the airplane is traveling in a straight path. Each spring has a constant of 600 N/m and the uniform disk has a mass of 250 g and spins at the rate of 12,000 rpm. Determine the angle through which the yoke will rotate when the pilot executes a horizontal turn of 800-m radius to the right at a speed of 720 km/h. Indicate whether point *A* will move up or down.

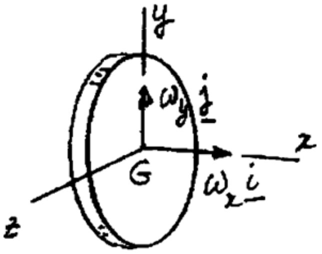
SOLUTION

Aircraft speed: $v = 720 \text{ km/h} = 200 \text{ m/s}$

Radius of turn: $\rho = 800 \text{ m}$

Angular velocity: $\omega_x = 12000 \text{ rpm} = 1256.6 \text{ rad/s}$

ω_y is negative since the aircraft is turning to the right.



$$\omega_y = -\frac{v}{\rho} = -\frac{200 \text{ m/s}}{800 \text{ m}} = -0.25 \text{ rad/s}$$

$$\omega_z = 0$$

Angular momentum: $\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j}$

where the reference frame *Gxyz* is turning with the aircraft with an angular velocity

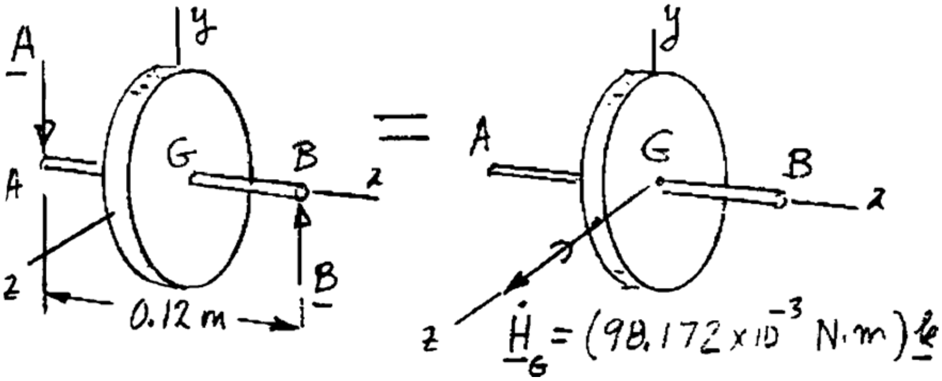
$$\mathbf{\Omega} = \omega_y \mathbf{j}$$

Rate of change of angular momentum:

Since ω_x and ω_y are constant, $(\dot{\mathbf{H}}_G)_{Gxyz} = 0$ and Eq. (18.22) yields

$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G = 0 + \omega_y \mathbf{j} \times (\bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j}) \\ &= -\bar{I}_x \omega_x \omega_y \mathbf{k} = -\left(\frac{1}{2} m r^2\right) \omega_x \omega_y \mathbf{k} \\ &= -\frac{1}{2}(0.25 \text{ kg})(0.05 \text{ m})^2 (1256.6 \text{ rad/s})(-0.250 \text{ rad/s})\mathbf{k} \\ \dot{\mathbf{H}}_G &= +(98.172 \times 10^{-3} \text{ N} \cdot \text{m})\mathbf{k}\end{aligned}$$

Free Body and kinetic diagrams


A
y
A
G
B
z
z
B
0.12 m
=
A
y
G
B
z
z
$\dot{\mathbf{H}}_G = (98.172\times10^{-3}\ \mathrm{N\cdot m})\mathbf{k}$

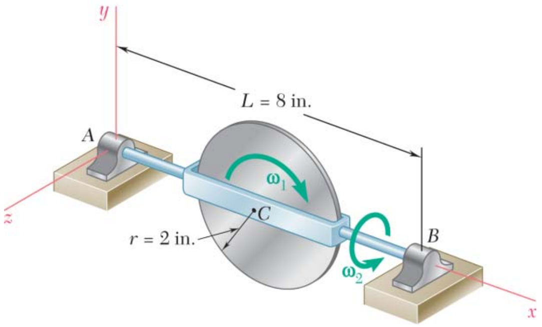
The forces A and B exerted by the springs must be equivalent to the couple $\dot{\mathbf{H}}_G$. They must therefore be directed as shown, which means that the spring at A will stretch and A will move up. ◀

We have $A = B$, $(0.12\ \mathrm{m})A = 98.172\times10^{-3}\ \mathrm{N\cdot m}$

$$F = A = 0.81810\ \mathrm{N}$$

$$\text{Deflection of spring } = x = \frac{F}{k} = \frac{0.81810\ \mathrm{N}}{600\ \mathrm{N/m}} = 1.3635\times10^{-3}\ \mathrm{m}$$

$$\text{Angle of rotation } = \frac{x}{GA} = \frac{1.3635\times10^{-3}\ \mathrm{m}}{0.06\ \mathrm{m}} = 0.022725\ \mathrm{rad} = 1.30^\circ$$ ◀



PROBLEM 18.93

The 10-oz disk shown spins at the rate $\omega_1 = 750$ rpm, while axle *AB* rotates as shown with an angular velocity ω_2 of 6 rad/s. Determine the dynamic reactions at *A* and *B*.

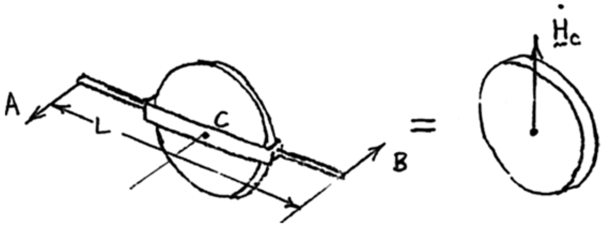
SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega_2\mathbf{i} - \omega_1\mathbf{k}$

Angular momentum: $\mathbf{H}_C = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} = \bar{I}_x\omega_2\mathbf{i} - \bar{I}_z\omega_1\mathbf{k}$

Let the reference frame *Cxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2\mathbf{i}$.

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Cxyz} + \boldsymbol{\Omega} \times \mathbf{H}_C = 0 + \omega_2\mathbf{i} \times (I_x\omega_2\mathrm{i} - \bar{I}_z\omega_1\mathbf{k}) = \bar{I}_z\omega_2\omega_1\mathbf{j}$$



Acceleration of mass center:

$$\bar{\mathbf{a}} = 0$$

$$\Sigma\mathbf{F} = m\bar{\mathbf{a}}$$

$$A\mathbf{k} - B\mathbf{k} = 0$$

$$A = B$$

$$\mathbf{M}_C = \dot{\mathbf{H}}_C$$

$$LB\mathbf{j} = \bar{I}_z\omega_2\omega_1\mathbf{j} \quad B = \frac{\bar{I}_z\omega_2\omega_1}{L}$$

Data: $m = \frac{W}{g} = \frac{10}{(16)(32.2)} = 0.01941 \text{ lb}\cdot\text{s}^2/\text{ft} \quad r = 2 \text{ in.} = 0.16667 \text{ ft}$

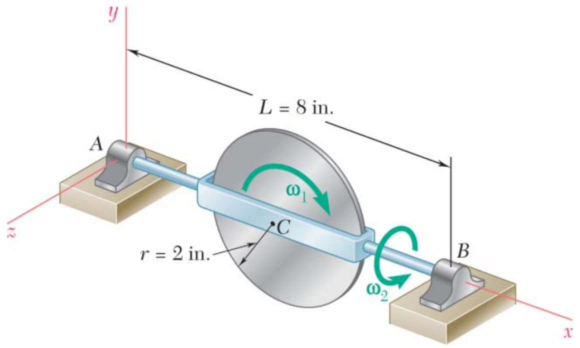
$$\bar{I}_Z = \frac{1}{2}mr^2 = \frac{1}{2}(0.01941)(0.16667)^2 = 269.6\times10^{-6} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\omega_1 = \frac{2\pi(750)}{60} = 25\pi \text{ rad/s}, \quad \omega_2 = 6 \text{ rad/s}, \quad L = 8 \text{ in.} = 0.66667 \text{ ft}$$

$$A = B = \frac{(269.6\times10^{-6})(6)(25\pi)}{0.66667} = 0.1906 \text{ lb}$$

$\mathbf{A} = (0.1906 \text{ lb})\mathbf{k}$ ◀

$\mathbf{B} = -(0.1906 \text{ lb})\mathbf{k}$ ◀



PROBLEM 18.94

The 10-oz disk shown spins at the rate $\omega_1 = 750$ rpm, while axle *AB* rotates as shown with an angular velocity ω_2. Determine the maximum allowable magnitude of ω_2 if the dynamic reactions at *A* and *B* are not to exceed 0.25 lb each.

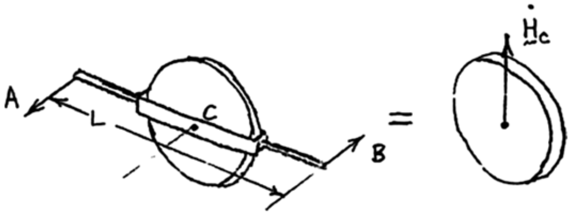
SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega_2\mathbf{i} - \omega_1\mathbf{k}$

Angular momentum: $\mathbf{H}_C = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} = \bar{I}_x\omega_2\mathbf{i} - \bar{I}_z\omega_1\mathbf{k}$

Let the reference frame *Cxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2\mathbf{i}$.

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Cxyz} + \boldsymbol{\Omega}\times\mathbf{H}_C = 0 + \omega_2\mathbf{i}\times(I_x\omega_2\mathbf{i} - \bar{I}_z\omega_1\mathbf{k}) = \bar{I}_z\omega_2\omega_1\mathbf{j}$$



Acceleration of mass center:

$$\bar{\mathbf{a}} = 0$$

$$\Sigma\mathbf{F} = m\bar{\mathbf{a}}$$

$$A\mathbf{k} - B\mathbf{k} = 0$$

$$A = B$$

$$\mathbf{M}_C = \dot{\mathbf{H}}_C$$

$$LB\mathbf{j} = \bar{I}_z\omega_2\omega_1\mathbf{j} \quad B = \frac{\bar{I}_z\omega_2\omega_1}{L}$$

Data: $m = \dfrac{W}{g} = \dfrac{10}{(16)(32.2)} = 0.01941 \text{ lb}\cdot\text{s}^2/\text{ft} \quad r = 2 \text{ in.} = 0.16667 \text{ ft}$

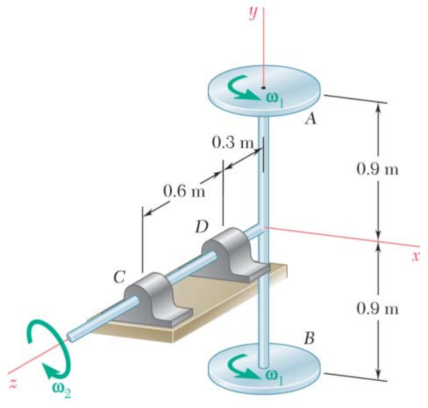
$$\bar{I}_z = \frac{1}{2}mr^2 = \frac{1}{2}(0.01941)(0.16667)^2 = 269.6\times10^{-6} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\omega_1 = \frac{2\pi(750)}{60} = 25\pi \text{ rad/s}, \quad L = 8 \text{ in.} = 0.66667 \text{ ft}$$

$$A = B = 0.25 \text{ lb}$$

$$\omega_2 = \frac{LB}{\bar{I}_z\omega_1} = \frac{(0.66667)(0.25)}{(269.6\times10^{-6})(25\pi)}$$

$\omega_2 = 7.87$ rad/s ◀



PROBLEM 18.95

Two disks, each of mass 5 kg and radius 300 mm, spin as shown at the rate $\omega_1 = 1200$ rpm about a rod *AB* of negligible mass which rotates about the horizontal *z* axis at the rate $\omega_2 = 60$ rpm. (*a*) Determine the dynamic reactions at points *C* and *D*. (*b*) Solve part *a* assuming that the direction of spin of disk *A* is reversed.

SOLUTION

For disk *A*: $\boldsymbol{\omega}_A = \omega_1\mathbf{j} - \omega_2\mathbf{k}$

$$\mathbf{H}_A = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} = 0 + \bar{I}_y\omega_1\mathbf{j} - \bar{I}_z\omega_2\mathbf{j}$$

$$= \frac{1}{2}mr^2\omega_1\mathbf{j} - \frac{1}{4}mr^2\omega_2\mathbf{k}$$

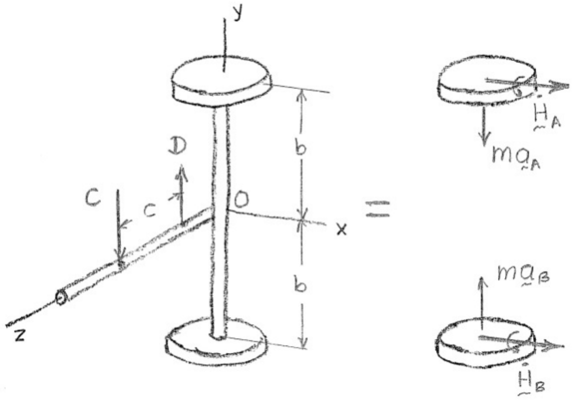
Let the reference frame *Oxyz* be rotating with angular velocity $\boldsymbol{\Omega} = -\omega_2\mathbf{k}$.

$$\dot{\mathbf{H}}_A = \left(\dot{\mathbf{H}}_A\right)_{Oxyz} + \boldsymbol{\Omega}\times\mathbf{H}_A = 0 + \left(-\omega_2\mathbf{k}\right)\times\left(\frac{1}{2}mr^2\omega_1\mathbf{j} - \frac{1}{4}mr^2\omega_2\mathbf{k}\right)$$

$$= \frac{1}{2}mr^2\omega_2\omega_1\mathbf{i}$$

Acceleration of its mass center: $\mathbf{a}_A = -b\omega_2^2\mathbf{j}$

For disk *B*: $\boldsymbol{\omega}_B = \boldsymbol{\omega}_A, \quad \mathbf{H}_B = \mathbf{H}_A, \quad \dot{\mathbf{H}}_B = \dot{\mathbf{H}}_A, \quad \mathbf{a}_B = b\omega_2^2\mathbf{j}$



Let the system of particles be disks *A* and *B* together with axle *AB* and shaft *CD*.
Neglect the mass of the axle and the shaft.

$$\Sigma\mathbf{F} = m\mathbf{a}_A + m\mathbf{a}_B$$

$$-C\mathbf{j} + D\mathbf{j} = -b\omega_2^2\mathbf{j} + b\omega_2^2\mathbf{j}$$

$$D = C$$

PROBLEM 18.95 (Continued)

$$\Sigma\mathbf{M}_0 = cC\mathbf{i} = \dot{\mathbf{H}}_A + \dot{\mathbf{H}}_B = mr^2\omega_2\omega_1\mathbf{i} \qquad C = \frac{mr^2\omega_2\omega_1}{c}$$

(*a*) Data: $m = 5\text{ kg}, \quad r = 0.3\text{ m}, \quad c = 0.6\text{ m},$

$$\omega_1 = \frac{2\pi(1200)}{60} = 40\pi\text{ rad/s} \qquad \omega_2 = \frac{2\pi(60)}{60} = 2\pi\text{ rad/s}$$

$$C = \frac{(5)(0.3)^2(40\pi)(2\pi)}{0.6} = 592\text{ N}$$

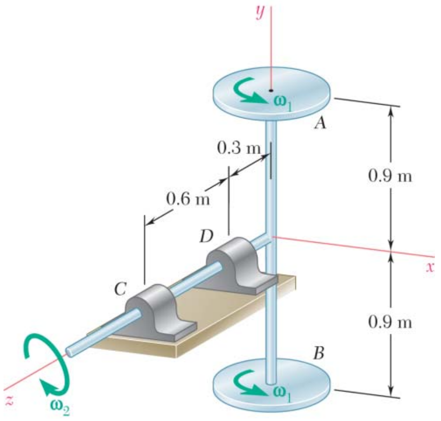
$\mathbf{C} = -(592\text{ N})\mathbf{j}$ ◀

$\mathbf{D} = (592\text{ N})\mathbf{j}$ ◀

(*b*) For disk **A**: $\boldsymbol{\omega}_A = -\omega_1\mathbf{j} - \omega_2\mathbf{k}, \qquad \mathbf{H}_A = -\bar{I}_y\omega_1\mathbf{j} - \bar{I}_z\omega_2\mathbf{k}$

$$\dot{\mathbf{H}}_A = -\frac{1}{2}mr^2\omega_2\omega_1\mathbf{i}, \qquad \dot{\mathbf{H}}_A + \dot{\mathbf{H}}_B = 0$$

$\mathbf{C} = \mathbf{D} = 0$ ◀



PROBLEM 18.96

Two disks, each of mass 5 kg and radius 300 mm, spin as shown at the rate $\omega_1 = 1200 \text{ rpm}$ about a rod *AB* of negligible mass which rotates about the horizontal *z* axis at the rate ω_2. Determine the maximum allowable value of ω_2 if the magnitudes of the dynamic reactions at points *C* and *D* are not to exceed 350 N each.

SOLUTION

For disk *A*:

$$\boldsymbol{\omega}_A = \omega_1\mathbf{j} - \omega_2\mathbf{k}$$

$$\mathbf{H}_A = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} = 0 + \bar{I}_y\omega_1\mathbf{j} - \bar{I}_z\omega_2\mathbf{j}$$

$$= \frac{1}{2}mr^2\omega_1\mathbf{j} - \frac{1}{4}mr^2\omega_2\mathbf{k}$$

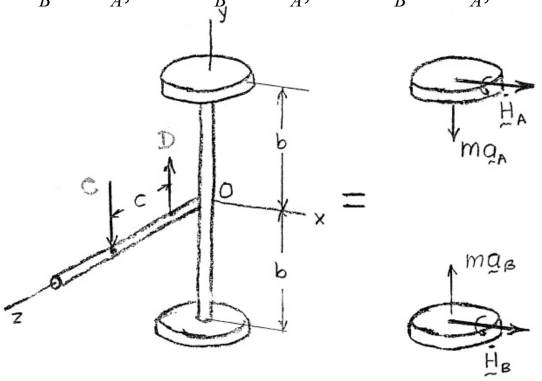
Let the reference frame *Oxyz* be rotating with angular velocity $\boldsymbol{\Omega} = -\omega_2\mathbf{k}$.

$$\dot{\mathbf{H}}_A = \left(\dot{\mathbf{H}}_A\right)_{Oxyz} + \boldsymbol{\Omega}\times\mathbf{H}_A = 0 + \left(-\omega_2\mathbf{k}\right)\times\left(\frac{1}{2}mr^2\omega_1\mathbf{j} - \frac{1}{4}mr^2\omega_2\mathbf{k}\right)$$

$$= \frac{1}{2}mr^2\omega_2\omega_1\mathbf{i}$$

Acceleration of its mass center: $\mathbf{a}_A = -b\omega_2^2\mathbf{j}$

For disk *B*: $\boldsymbol{\omega}_B = \boldsymbol{\omega}_A, \quad \mathbf{H}_B = \mathbf{H}_A, \quad \dot{\mathbf{H}}_B = \dot{\mathbf{H}}_A, \quad \mathbf{a}_B = b\omega_2^2\mathbf{j}$



Let the system of particles be disks *A* and *B* together with axle *AB* and shaft *CD*.

Neglect the mass of the axle and the shaft.

$$\Sigma\mathbf{F} = m\mathbf{a}_A + m\mathbf{a}_B$$

$$-C\mathbf{j} + D\mathbf{j} = -b\omega_2^2\mathbf{j} + b\omega_2^2\mathbf{j}$$

$$D = C$$

PROBLEM 18.96 (Continued)

$$\Sigma\mathbf{M}_0 = cC\mathbf{i} = \dot{\mathbf{H}}_A + \dot{\mathbf{H}}_B = mr^2\omega_2\omega_1\mathbf{i} \qquad C = \frac{mr^2\omega_2\omega_1}{c}$$

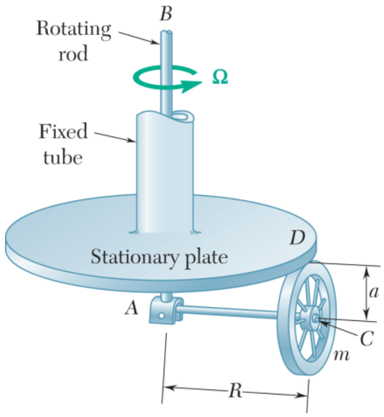
Solving for ω_2, $$\omega_2 = \frac{cC}{mr^2\omega_1}$$

Data: $m = 5\text{ kg},\quad r = 0.3\text{ m},\quad c = 0.6\text{ m}$

$$\omega_1 = \frac{2\pi(1200)}{60} = 40\pi\text{ rad/s}, \qquad C = D = 350\text{ N}$$

$$\omega_2 = \frac{(0.6)(350)}{(5)(0.3)^2(40\pi)} = 3.7136\text{ rad/s}$$

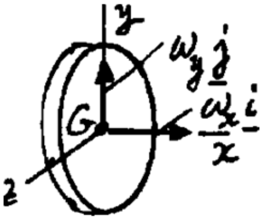
$\omega_2 = 35.5\text{ rpm}$ ◀


B
Rotating rod
Ω
Fixed tube
Stationary plate
D
a
A
C
m
R

PROBLEM 18.97

A stationary horizontal plate is attached to the ceiling by means of a fixed vertical tube. A wheel of radius *a* and mass *m* is mounted on a light axle *AC* which is attached by means of a clevis at *A* to a rod *AB* fitted inside the vertical tube. The rod *AB* is made to rotate with a constant angular velocity **Ω** causing the wheel to roll on the lower face of the stationary plate. Determine the minimum angular velocity **Ω** for which contact is maintained between the wheel and the plate. Consider the particular cases (*a*) when the mass of the wheel is concentrated in the rim, (*b*) when the wheel is equivalent to a thin disk of radius *a*.

SOLUTION


y
$\omega_y \mathbf{j}$
G
$\omega_x \mathbf{i}$
x
z

Angular momentum $\mathbf{H}_G$ of wheel: $\omega_y = \Omega, \quad \omega_x = \frac{R}{a}\Omega, \quad \omega_z = 0$

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k}$$

$$\mathbf{H}_G = \bar{I}_x \frac{R}{a}\Omega \mathbf{i} + \bar{I}_y \Omega \mathbf{j}$$

Rate of change of angular momentum:

Since ω_x and ω_y are constant, and observing that the frame *Gxyz* rotates with the angular velocity $\boldsymbol{\Omega} = \Omega\mathbf{i}$:

$$(\dot{\mathbf{H}}_G)_{Gxyz} = 0$$

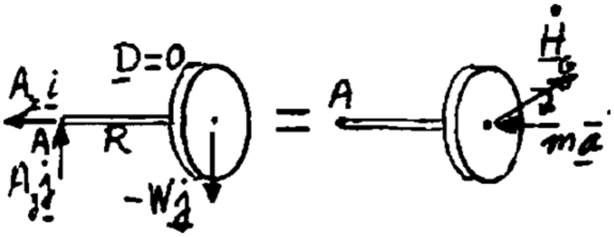
Eq. (18.22):

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

$$\dot{\mathbf{H}}_G = 0 + \Omega\mathbf{j} \times \left(\bar{I}_x \frac{R}{a} a\mathbf{i} + \bar{I}_y \Omega \mathbf{j} \right)$$

$$\dot{\mathbf{H}}_G = -\bar{I}_z \frac{R}{a} \Omega^2 \mathbf{k} \qquad (1)$$

The free body and kinetic diagrams


D=0
$A_x \mathbf{i}$
A
R
$A_y \mathbf{j}$
$-W\mathbf{j}$
=
A
$\dot{\mathbf{H}}_G$
$m\bar{\mathbf{a}}$

PROBLEM 18.97 (Continued)

Equating moments about A:

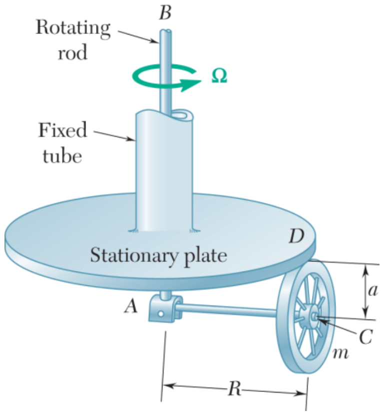
$$R\,\mathbf{i}\times(-W\,\mathbf{j})=\dot{\mathbf{H}}_G$$

$$-Rmg\,\mathbf{k}=-\bar{I}_z\,\frac{R}{a}\Omega^2\mathbf{k}$$

$$\Omega=\sqrt{\frac{mga}{\bar{I}_x}}$$

(a) mass in rim: $\bar{I}_x=ma^2$ $\Omega=\sqrt{g/a}$ ◀

(b) thin disk: $\bar{I}_x=\frac{1}{2}ma^2$ $\Omega=\sqrt{2g/a}$ ◀



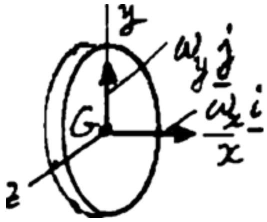
PROBLEM 18.98

Assuming that the wheel of Problem 18.97 weights 8 lb, has a radius $a = 4$ in. and a radius of gyration of 3 in., and that $R = 20$ in., determine the force exerted by the plate on the wheel when $\Omega = 25$ rad/s.

PROBLEM 18.97 A stationary horizontal plate is attached to the ceiling by means of a fixed vertical tube. A wheel of radius a and mass m is mounted on a light axle AC which is attached by means of a clevis at A to a rod AB fitted inside the vertical tube. The rod AB is made to rotate with a constant angular velocity $\mathbf{\Omega}$ causing the wheel to roll on the lower face of the stationary plate. Determine the minimum angular velocity $\mathbf{\Omega}$ for which contact is maintained between the wheel and the plate. Consider the particular cases (*a*) when the mass of the wheel is concentrated in the rim, (*b*) when the wheel is equivalent to a thin disk of radius a.

SOLUTION

Angular momentum $\mathbf{H}_G$ of wheel: $\quad \omega_y = \Omega, \quad \omega_x = \frac{R}{a}\Omega, \quad \omega_z = 0$



$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k}$$

$$\mathbf{H}_G = \bar{I}_x \frac{R}{a}\Omega \mathbf{i} + \bar{I}_y \Omega \mathbf{j}$$

Rate of change of angular momentum:

Since ω_x and ω_y are constant, and observing that the frame $Gxyz$ rotates with the angular velocity $\mathbf{\Omega} = \Omega \mathbf{j}$:

$$(\dot{\mathbf{H}}_G)_{Gxyz} = 0$$

Eq. (18.22):

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G$$

$$\dot{\mathbf{H}}_G = 0 + \Omega \mathbf{j} \times \left(\bar{I}_x \frac{R}{a} a \mathbf{i} + \bar{I}_y \Omega \mathbf{j} \right)$$

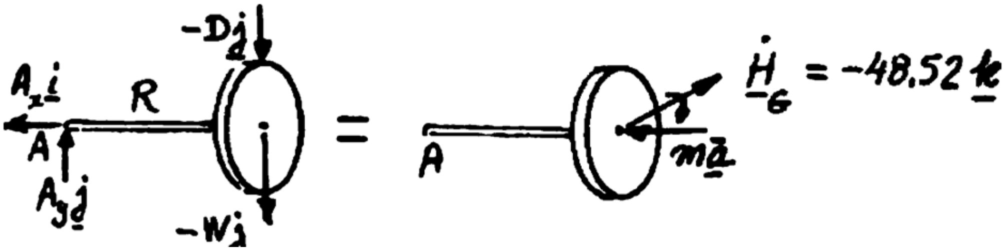
$$\dot{\mathbf{H}}_G = -\bar{I}_z \frac{R}{a} \Omega^2 \mathbf{k} \qquad (1)$$

With $\quad R = 20 \text{ in.}, \quad a = 4 \text{ in.}, \quad \Omega = 25 \text{ rad/s}, \quad m = \frac{8}{g}$

and $\quad \bar{I}_x = m\bar{k}_x^2 = \frac{8}{g}\left(\frac{3}{12}\text{ ft}\right)^2 = 15.53 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$

$$\dot{\mathbf{H}}_G = -15.53 \times 10^{-3} \left(\frac{20}{4}\right)(25)^2 \mathbf{k} = 48.52\mathbf{k}$$

PROBLEM 18.98 (Continued)


$-D\mathbf{j}$
$A_x\mathbf{i}$
R
A
$A_y\mathbf{j}$
$-W\mathbf{j}$
=
A
$\dot{\mathbf{H}}_G = -48.52\mathbf{k}$
$m\bar{\mathbf{a}}$

Taking moments about *A*:

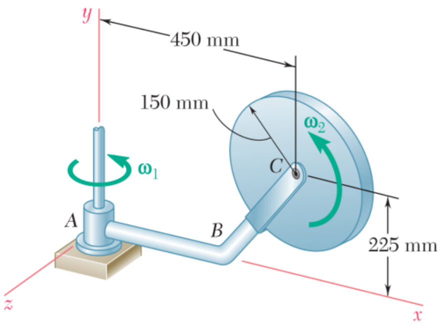
$$R\mathbf{i}\times(-D\mathbf{j}-W\mathbf{j}) = \dot{\mathbf{H}}_G$$

$$\frac{20}{12}\mathbf{i}\times(-D\mathbf{j}-8\mathbf{j}) = -48.52\mathbf{k}$$

$$-\frac{5}{3}(D+8)\mathbf{k} = -48.52\mathbf{k}$$

$$D = \frac{3}{5}(48.52)-8 = 21.1\ \text{lb}$$

D = 21.1 lb ↓ ◀



PROBLEM 18.99

A thin disk of mass $m = 4$ kg rotates with an angular velocity ω_2 with respect to arm *ABC*, which itself rotates with an angular velocity ω_1 about the *y* axis. Knowing that $\omega_1 = 5$ rad/s and $\omega_2 = 15$ rad/s and that both are constant, determine the force-couple system representing the dynamic reaction at the support at *A*.

SOLUTION

Angular velocity of the disk. $\boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k} = (5\text{ rad/s})\mathbf{j} + (15\text{ rad/s})\mathbf{k}$

Moments of inertia about principal axes passing through the mass center.

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4}mr^2$$
$$= \frac{1}{4}(4)(0.150\text{ m})^2 = 0.0225\text{ kg}\cdot\text{m}^2$$
$$\bar{I}_{z'} = \frac{1}{2}mr^2 = 0.045\text{ kg}\cdot\text{m}^2$$

Angular momentum about mass center C.

$$\mathbf{H}_C = \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + \bar{I}_{z'}\omega_{z'}\mathbf{k}$$
$$= 0 + (0.0225)5\mathbf{j} + (0.045)15\mathbf{k}$$
$$\mathbf{H}_C = (0.1125\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (0.6750\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

Rate of change of $\mathbf{H}_C$. Let the frame *Axyz* be turning with angular velocity $\boldsymbol{\Omega} = \omega_1\mathbf{j}$.

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_C = 0 + \boldsymbol{\Omega}\times\mathbf{H}_C$$
$$= 5\mathbf{j}\times(0.1125\mathbf{j} + 0.675\mathbf{k}) = (3.375\text{ N}\cdot\text{m})\mathbf{i}$$

Position vector of Point C.

$$\mathbf{r}_{C/A} = (0.450\text{ m})\mathbf{i} + (0.225\text{ m})\mathbf{j}$$

Velocity of Point C, the mass center of the disk.

$$\mathbf{v}_C = \omega_1\times\mathbf{r}_{C/A} = 5\mathbf{j}\times(0.45\mathbf{i} + 0.225\mathbf{j})$$
$$= -(2.25\text{ m/s})\mathbf{k}$$

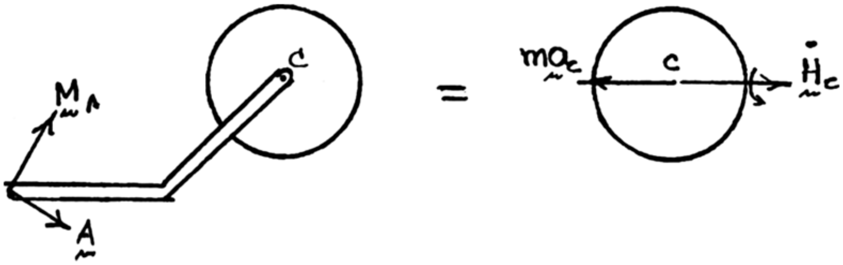
Acceleration of Point C.

$$\mathbf{a}_C = \alpha_1\mathbf{j}\times\mathbf{r}_{C/A} + \omega_1\mathbf{j}\times\mathbf{v}_C = 0 + 5\mathbf{j}\times(-2.25\mathbf{k}) = -(11.25\text{ m/s}^2)\mathbf{i}$$

PROBLEM 18.99 (Continued)

$$m\mathbf{a}_C = (4)(-11.25\mathbf{i}) = -(45\text{ N})\mathbf{i}$$

Free body and kinetic diagrams


C
$\mathbf{M}_A$
$\mathbf{A}$
=
$m\mathbf{a}_c$
C
$\dot{\mathbf{H}}_c$

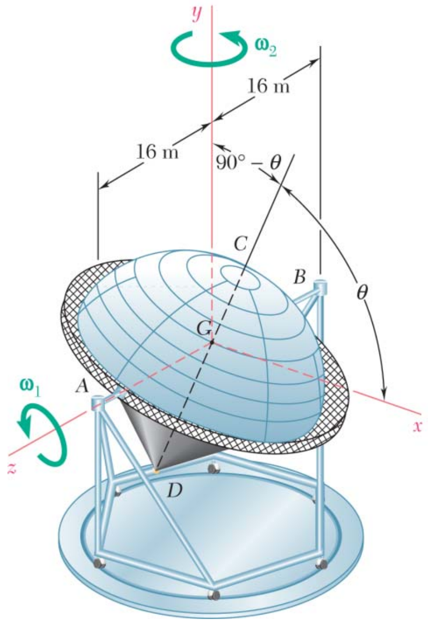
Linear components: $\mathbf{A} = m\mathbf{a}_C$ $\mathbf{A} = -(45\text{ N})\mathbf{i}$ ◀

Moments about *A*.

$$\mathbf{M}_A = \mathbf{r}_{C/A} \times m\mathbf{a}_C + \dot{\mathbf{H}}_C$$

$$\mathbf{M}_A = (0.450\mathbf{i} + 0.225\mathbf{j}) \times (-45\mathbf{i}) + 3.375\mathbf{i}$$

$$\mathbf{M}_A = (3.38\text{ N}\cdot\text{m})\mathbf{i} + (10.13\text{ N}\cdot\text{m})\mathbf{k}$$ ◀



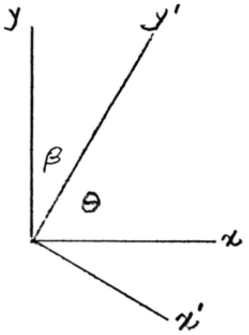
PROBLEM 18.100

An experimental Fresnel-lens solar-energy concentrator can rotate about the horizontal axis *AB*, which passes through its mass center *G*. It is supported at *A* and *B* by a steel framework, which can rotate about the vertical *y* axis. The concentrator has a mass of 30 Mg, a radius of gyration of 12 m about its axis of symmetry *CD*, and a radius of gyration of 10 m about any transverse axis through *G*. Knowing that the angular velocities ω_1 and ω_2 have constant magnitudes equal to 0.20 rad/s and 0.25 rad/s, respectively, determine for the position $\theta = 60^\circ$ (*a*) the forces exerted on the concentrator at *A* and *B*, (*b*) the couple $M_2\mathbf{k}$ applied to the concentrator at that instant.

SOLUTION

Let the y axis be vertical and the y' axis be the symmetry axis.

Let the z axis be directed along axle BA as shown and the x' axis be the transverse axis perpendicular to BA.



Unit vectors.

$$\beta = 90^\circ - \theta$$

$$\mathbf{i}' = \mathbf{i}\cos\beta - \mathbf{j}\sin\beta$$

$$\mathbf{j}' = \mathbf{i}\sin\beta + \mathbf{j}\cos\beta$$

$$\mathbf{i} = \mathbf{i}'\cos\beta + \mathbf{j}'\sin\beta$$

$$\mathbf{j} = -\mathbf{i}'\sin\beta + \mathbf{j}\cos\beta$$

Angular velocity.

$$\boldsymbol{\omega} = \omega_2\mathbf{j} + \omega_1\mathbf{k}$$

$$\omega = -(\omega_2\sin\beta)\mathbf{i}' + (\omega_2\cos\beta)\mathbf{j}' + \omega_1\mathbf{k}$$

$$= -\omega_2\cos\theta\mathbf{i} + \omega_2\sin\theta\mathbf{j} + \omega_1\mathbf{k}$$

$$\omega_{x'} = \omega_2\cos\theta \qquad \omega_{y'} = \omega_2\sin\theta \qquad \omega_z = \omega_1$$

$\dot{\theta} = \omega_1$ $\quad \omega_1$ and ω_2 are constant.

$$\dot{\omega}_{x'} = \omega_2\omega_1\sin\theta$$

$$\dot{\omega}_{y'} = \omega_2\omega_1\cos\theta, \quad \dot{\omega}_2 = 0$$

Radii of gyration:

$$k_{x'} = 10 \text{ m}$$

$$k_{y'} = 12 \text{ m}$$

$$k_z = 10 \text{ m}$$

Moments of inertia:

$$I = mk^2$$

PROBLEM 18.100 (Continued)

Angular momentum about Point *G*. $\mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_y\omega_{y'}\mathbf{j} + I_z\omega_z\mathbf{k}$

$$\mathbf{H}_G = m[(10)^2\omega_{x'}\mathbf{i}' + (12)^2\omega_{y'}\mathbf{j}' + (10)^2\omega_z\mathbf{k}]$$
$$= m(100\omega_z\cos\theta\mathbf{i}' + 144\omega_2\sin\theta\mathbf{j}' + 100\omega_1\mathbf{k})$$

where *m* is the mass. $m = 30,000\text{ kg}$

Acceleration of the mass center.

Since the mass center lies at the center of the axle *BA*, $\bar{\mathbf{a}} = 0$

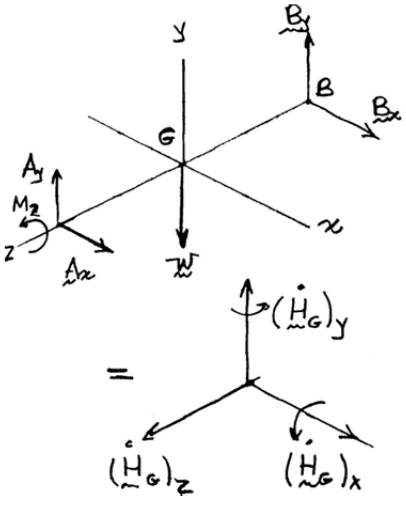
Rate of change of angular momentum.

Let the reference frame $G_{x'y'z}$ be turning with angular velocity $\mathbf{\Omega} = \boldsymbol{\omega}$

$$\dot{\mathbf{H}}_G = \dot{\mathbf{H}}_{Gx'y'z} + \mathbf{\Omega}\times\mathbf{H}_G$$

where
$$\dot{\mathbf{H}}_{Gx'y'z} = I_{x'}\dot{\omega}_{x'}\mathbf{i}' + I_{y'}\dot{\omega}_y\mathbf{j}' + I_z\dot{\omega}_z\mathbf{k}$$
$$= m[(10)^2\omega_2\omega_1\sin\theta\mathbf{i}' + (12)^2\omega_2\omega_1\cos\theta\mathbf{j}' + 0]$$
$$= m[(100)(0.25)(0.20)\sin 60°\mathbf{i}' + (144)(0.25)(0.20)\cos 60°\mathbf{j}']$$
$$= m(4.3301\mathbf{i}' + 3.6\mathbf{j}')$$

and
$$\mathbf{\Omega}\times\mathbf{H}_G = m\begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k} \\ -\omega_2\cos\theta & \omega_2\sin\theta & \omega_1 \\ -100\omega_2\cos\theta & 144\omega_2\sin\theta & 100\omega_1 \end{vmatrix}$$
$$= m[-44\omega_1\omega_2\sin\theta\mathbf{i}' - 44\omega_2^2\sin\theta\cos\theta\mathbf{k}]$$
$$= m[-(44)(0.20)(0.25)\sin 60°\mathbf{i}'$$
$$+(44)(0.25)^2\sin 60°\cos 60°\mathbf{k}]$$
$$= m(-1.9053\mathbf{i}' + 1.1908\mathbf{k})$$
$$\dot{\mathbf{H}}_G = m[2.4248\mathbf{i}' + 3.6\mathbf{j}' - 1.1908\mathbf{k}]$$
$$= (30,000)[2.4248(\mathbf{i}\cos 30° - \mathbf{j}\sin 30°)$$
$$+3.6(\mathbf{i}\sin 30° + \mathbf{j}\cos 30°) - 1.1908\mathbf{k}]$$
$$= (117\times 10^3\text{ N}\cdot\text{m})\mathbf{i} + (57.158\times 10^3\text{ N}\cdot\text{m})\mathbf{j}$$
$$-(35.724\times 10^3\text{ N}\cdot\text{m})\mathbf{k}$$



Weight: $\mathbf{W} = -mg\mathbf{j} = -(30\times 10^3)(9.81)\mathbf{j}$
$$= -(294.3\times 10^3\text{ N})\mathbf{j}$$

Equations of motion.

$$\Sigma\mathbf{M}_B = \Sigma(\mathbf{M}_B)_{\text{eff}}: \quad \mathbf{r}_{A/B}\times(A_x\mathbf{i} + B_y\mathbf{j}) + \mathbf{r}_{G/B}\times\mathbf{W}) + M_2\mathbf{k} = \dot{\mathbf{H}}_G$$
$$32\mathbf{k}\times(A_x\mathbf{i} + A_y\mathbf{j}) + 16\mathbf{k}\times(-294.3\times 10^3)\mathbf{j} + M_2\mathbf{k} = \dot{\mathbf{H}}_G$$
$$(-32A_y + 4.7088\times 10^6)\mathbf{i} + 32A_x\mathbf{j} + M_2\mathbf{k} = \dot{\mathbf{H}}_G$$

PROBLEM 18.100 (Continued)

Equate like components:

$$\mathbf{i}: \quad -32A_y + 4.7088\times 10^6 = 117\times 10^3 \qquad A_y = 143.5\times 10^3 \text{ N}$$

$$\mathbf{j}: \quad 32A_x = 57.158\times 10^3 \qquad A_x = 1.786\times 10^3 \text{ N}$$

$$\mathbf{k}: \quad M_2 = -35.724\times 10^3$$

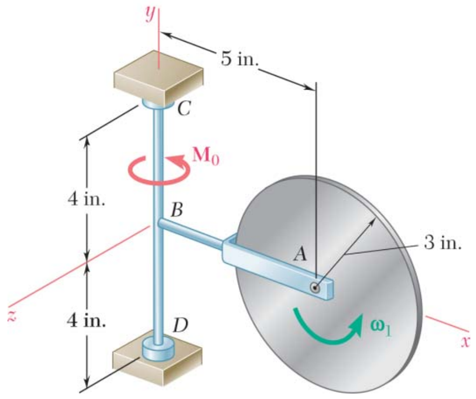
(*a*) Reading: Reaction at *A*. $\mathbf{A} = (1.786 \text{ kN})\mathbf{i} + (143.5 \text{ kN})\mathbf{j}$ ◀

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}: \quad \mathbf{B} + \mathbf{A} + \mathbf{W} = 0$$

$$\mathbf{B} = -\mathbf{W} - \mathbf{A} = 294.3\times 10^3\mathbf{j} - \mathbf{A}$$

Reaction at *B*. $\mathbf{B} = -(1.786 \text{ kN})\mathbf{i} + (150.8 \text{ kN})\mathbf{j}$ ◀

(*b*) Couple $M_2\mathbf{k}$: $M_2\mathbf{k} = -(35.7 \text{ kN}\cdot\text{m})\mathbf{k}$ ◀



PROBLEM 18.101

A 6-lb homogeneous disk of radius 3 in. spins as shown at the constant rate $\omega_1 = 60$ rad/s. The disk is supported by the fork-ended rod *AB*, which is welded to the vertical shaft *CBD*. The system is at rest when a couple $\mathbf{M}_0 = (0.25\ \text{ft}\cdot\text{lb})\mathbf{j}$ is applied to the shaft for 2 s and then removed. Determine the dynamic reactions at *C* and *D* after the couple has been removed.

SOLUTION

Angular velocity of shaft *CBD* and arm *AB*: $\mathbf{\Omega} = \omega_2\mathbf{j}$

Angular velocity of disk *A*: $\boldsymbol{\omega} = \omega_2\mathbf{j} + \omega_1\mathbf{k}$

Its angular momentum about *A*: $\mathbf{H}_A = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} = \bar{I}_y\omega_2\mathbf{j} + \bar{I}_z\omega_1\mathbf{k}$

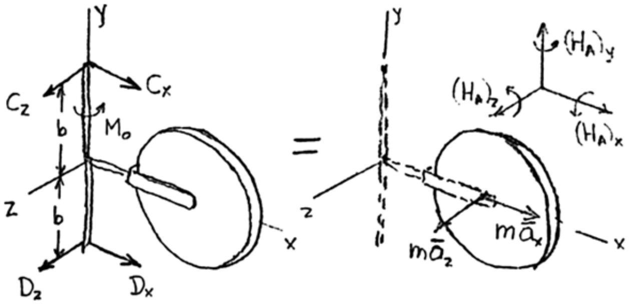
Let the reference frame *Bxyz* be rotating with angular velocity $\mathbf{\Omega}$.

$$\begin{aligned}
\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega}\times\mathbf{H}_A\\
&= \bar{I}_y\dot{\omega}_2\mathbf{j} + \bar{I}_z\dot{\omega}_1\mathbf{k} + \omega_2\mathbf{j}\times(\bar{I}_y\omega_2\mathbf{j} + \bar{I}_z\omega_1\mathbf{k})\\
&= \bar{I}_z\omega_1\omega_2\mathbf{i} + \bar{I}_y\dot{\omega}_2\mathbf{j} + \bar{I}_z\dot{\omega}_1\mathbf{k}\\
&= \frac{1}{2}mr^2\omega_1\omega_2\mathbf{i} + \frac{1}{4}mr^2\dot{\omega}_2\mathbf{j} + \frac{1}{2}mr^2\dot{\omega}_1\mathbf{k}
\end{aligned}$$

Velocity and acceleration of the mass center *A* of the disk:

$$\bar{\mathbf{v}} = \omega_2\mathbf{j}\times c\mathbf{i} = -c\omega_2\mathbf{k},$$

$$\bar{\mathbf{a}} = \dot{\omega}_2\mathbf{j}\times c\mathbf{i} + \omega_2\mathbf{j}\times\bar{\mathbf{v}} = -c\dot{\omega}_2\mathbf{k} - c\omega_2^2\mathbf{i}$$



$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}} = m\bar{\mathbf{a}}$$

$$C_x\mathbf{i} + C_y\mathbf{k} + D_x\mathbf{i} + D_z\mathbf{k} = m\bar{\mathbf{a}}$$

Resolve into components.

$$C_x + D_x = -mc\omega_2^2$$

$$C_z + D_z = -mc\dot{\omega}_2$$

PROBLEM 18.101 (Continued)

$$\Sigma\mathbf{M}_D = \dot{\mathbf{H}}_D = \dot{\mathbf{H}}_A + \mathbf{r}_{A/D} \times m\overline{\mathbf{a}}$$

$$= \dot{\mathbf{H}}_A + (c\mathbf{i} + b\mathbf{j}) \times (-c\dot{\omega}_2\mathbf{k} - c\omega_2^2\mathbf{i})$$

$$M_0\mathbf{j} + 2b\mathbf{j} \times (C_x\mathbf{i} + C_z\mathbf{k}) = \dot{\mathbf{H}}_A - mbc\dot{\omega}_2\mathbf{i} + mc^2\dot{\omega}_2\mathbf{j} + mbc\omega_2^2\mathbf{k}$$

$$2bC_z\mathbf{i} + M_0\mathbf{j} - 2bC_x\mathbf{k} = m\left(\frac{1}{2}r^2\omega_1\omega_2 - bc\dot{\omega}_2\right)\mathbf{i} + m\left(\frac{1}{4}r^2 + c^2\right)\dot{\omega}_2\mathbf{j} + m\left(\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right)\mathbf{k}$$

$$\mathbf{j}: \quad M_0 = m\left(\frac{1}{4}r^2 + c^2\right)\dot{\omega}_2 \qquad (1)$$

$$\mathbf{k}: \quad C_x = -\frac{m}{2b}\left(\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right) \quad D_x = -\frac{m}{2b}\left(-\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right) \qquad (2)$$

$$\mathbf{i}: \quad C_z = \frac{m}{2b}\left(\frac{1}{2}r^2\omega_1\omega_2 - bc\dot{\omega}_2\right) \quad D_z = -\frac{m}{2b}\left(\frac{1}{2}r^2\omega_1\omega_2 + bc\dot{\omega}_2\right) \qquad (3)$$

Data: $W = 6\text{ lb}, \quad r = 3\text{ in.} = 0.25\text{ ft}, \quad b = 4\text{ in.} = 0.33333\text{ ft},$

$c = 5\text{ in.} = 0.41667\text{ ft}, \quad \omega_1 = 60\text{ rad/s}, \quad \dot{\omega}_1 = 0$

While the couple is applied, $M_0 = 0.25\text{ ft}\cdot\text{lb}$

Rearranging Equation (1)

$$\dot{\omega}_2 = \frac{M_0}{m\left(\frac{1}{4}r^2 + c^2\right)} = \frac{0.25}{\left(\frac{6}{32.2}\right)\left[\frac{1}{4}(0.25)^2 + (0.41667)^2\right]} = 7.0899\text{ rad/s}^2$$

At $t = 2\text{s}$, $\omega_2 = (\omega_2)_0 + \dot{\omega}_2 t = 0 + (7.0899)(2) = 14.18\text{ rad/s}$

For $t > 2\text{s}$, $M_0 = 0, \quad \dot{\omega}_2 = 0$

From Equations (2), (3)

$$C_x = \frac{\frac{6}{32.2}}{(2)(0.33333)}[0 + (0.33333)(0.41667)(14.1798)^2] = -7.8054\text{ lb}$$

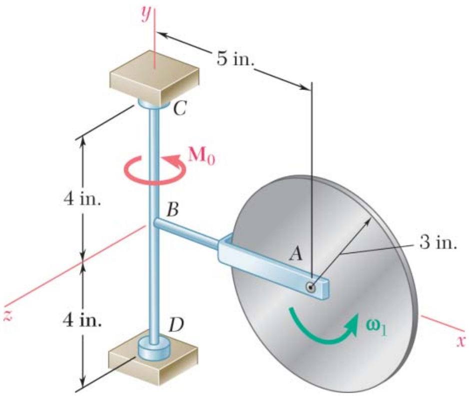
$$D_x = \frac{\frac{6}{32.2}}{(2)(0.33333)}[0 + (0.33333)(0.41667)(14.1798)^2] = -7.8054\text{ lb}$$

$$C_z = \frac{\frac{6}{32.2}}{(2)(0.33333)}\left[\frac{1}{2}(0.25)^2(60)(14.1798) - 0\right] = 7.4312\text{ lb}$$

$$D_z = \frac{\frac{6}{32.2}}{(2)(0.33333)}\left[-\frac{1}{2}(0.25)^2(60)(14.1798) - 0\right] = -7.4312\text{ lb}$$

$\mathbf{C} = -(7.81\text{ lb})\mathbf{i} + (7.43\text{ lb})\mathbf{k}$ ◀

$\mathbf{D} = -(7.81\text{ lb})\mathbf{i} - (7.43\text{ lb})\mathbf{k}$ ◀



PROBLEM 18.102

A 6-lb homogeneous disk of radius 3 in. spins as shown at the constant rate $\omega_1 = 60$ rad/s. The disk is supported by the fork-ended rod *AB*, which is welded to the vertical shaft *CBD*. The system is at rest when a couple $\mathbf{M}_0$ is applied as shown to the shaft for 3 s and then removed. Knowing that the maximum angular velocity reached by the shaft is 18 rad/s, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at *C* and *D* after the couple has been removed.

SOLUTION

Angular velocity of shaft *CBD* and arm *AB*: $\mathbf{\Omega} = \omega_2 \mathbf{j}$

Angular velocity of disk *A*: $\boldsymbol{\omega} = \omega_2 \mathbf{j} + \omega_1 \mathbf{k}$

Its angular momentum about *A*: $\mathbf{H}_A = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = \bar{I}_y \omega_2 \mathbf{j} + \bar{I}_z \omega_1 \mathbf{k}$

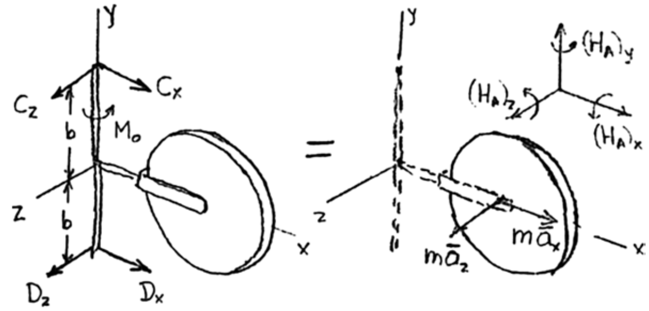
Let the reference frame *Bxyz* be rotating with angular velocity $\mathbf{\Omega}$.

$$\begin{aligned}\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega} \times \mathbf{H}_A \\ &= \bar{I}_y \dot{\omega}_2 \mathbf{j} + \bar{I}_z \dot{\omega}_1 \mathbf{k} + \omega_2 \mathbf{j} \times (\bar{I}_y \omega_2 \mathbf{j} + \bar{I}_z \omega_1 \mathbf{k}) \\ &= \bar{I}_z \omega_1 \omega_2 \mathbf{i} + \bar{I}_y \dot{\omega}_2 \mathbf{j} + \bar{I}_z \dot{\omega}_1 \mathbf{k} \\ &= \frac{1}{2} m r^2 \omega_1 \omega_2 \mathbf{i} + \frac{1}{4} m r^2 \dot{\omega}_2 \mathbf{j} + \frac{1}{2} m r^2 \dot{\omega}_1 \mathbf{k}\end{aligned}$$

Velocity and acceleration of the mass center *A* of the disk:

$$\bar{\mathbf{v}} = \omega_2 \mathbf{j} \times c\mathbf{i} = -c\omega_2 \mathbf{k},$$

$$\bar{\mathbf{a}} = \dot{\omega}_2 \mathbf{j} \times c\mathbf{i} + \omega_2 \mathbf{j} \times \bar{\mathbf{v}} = -c\dot{\omega}_2 \mathbf{k} - c\omega_2^2 \mathbf{i}$$



$$\Sigma \mathbf{F} = \Sigma \mathbf{F}_{\text{eff}} = m\bar{\mathbf{a}}$$

$$C_x \mathbf{i} + C_y \mathbf{k} + D_x \mathbf{i} + D_z \mathbf{k} = m\bar{\mathbf{a}}$$

Resolve into components.

$$C_x + D_x = -mc\omega_2^2$$

$$C_z + D_z = -mc\dot{\omega}_2$$

PROBLEM 18.102 (Continued)

$$\Sigma\mathbf{M}_D = \dot{\mathbf{H}}_D = \dot{\mathbf{H}}_A + \mathbf{r}_{A/D} \times m\bar{\mathbf{a}}$$

$$= \dot{\mathbf{H}}_A + (c\mathbf{i} + b\mathbf{j}) \times (-c\dot{\omega}_2\mathbf{k} - c\omega_2^2\mathbf{i})$$

$$M_0\mathbf{j} + 2b\mathbf{j} \times (C_x\mathbf{i} + C_z\mathbf{k}) = \dot{\mathbf{H}}_A - mbc\dot{\omega}_2\mathbf{i} + mc^2\dot{\omega}_2\mathbf{j} + mbc\omega_2^2\mathbf{k}$$

$$2bC_z\mathbf{i} + M_0\mathbf{j} - 2bC_x\mathbf{k} = m\left(\frac{1}{2}r^2\omega_1\omega_2 - bc\dot{\omega}_2\right)\mathbf{i} + m\left(\frac{1}{4}r^2 + c^2\right)\dot{\omega}_2\mathbf{j} + m\left(\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right)\mathbf{k}$$

$$\mathbf{j}: \quad M_0 = m\left(\frac{1}{4}r^2 + c^2\right)\dot{\omega}_2 \qquad (1)$$

$$\mathbf{k}: \quad C_x = -\frac{m}{2b}\left(\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right) \quad D_x = -\frac{m}{2b}\left(-\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right) \qquad (2)$$

$$\mathbf{i}: \quad C_z = \frac{m}{2b}\left(\frac{1}{2}r^2\omega_1\omega_2 - bc\dot{\omega}_2\right) \quad D_z = -\frac{m}{2b}\left(\frac{1}{2}r^2\omega_1\omega_2 + bc\dot{\omega}_2\right) \qquad (3)$$

Data: $W = 6 \text{ lb}, \quad r = 3 \text{ in.} = 0.25 \text{ ft}, \quad b = 4 \text{ in.} = 0.33333 \text{ ft},$

$c = 5 \text{ in.} = 0.41667 \text{ ft}, \quad \omega_1 = 60 \text{ rad/s}, \quad \dot{\omega}_1 = 0$

(*a*) While the couple is applied $\dot{\omega}_2 = \frac{\omega_2}{t} = \frac{18}{3} = 6 \text{ rad/s}^2$

From Equation (1) $M_0 = m\left(\frac{1}{4}r^2 + c^2\right)\dot{\omega}_2$

$$= \left(\frac{6}{32.2}\right)\left[\frac{1}{4}(0.25)^2 + (0.41667)^2\right](6)$$

$M_0 = (0.212 \text{ ft}\cdot\text{lb})\mathbf{j}$ ◀

(*b*) For $t > 3$ s, $M_0 = 0, \quad \dot{\omega}_2 = 0$

From Equations (2), (3)

$$C_x = \frac{\frac{6}{32.2}}{(2)(0.33333)}[0 + (0.33333)(0.41667)(18)^2] = -12.578 \text{ lb}$$

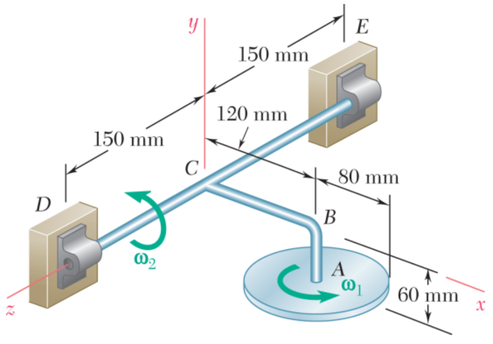
$$D_x = \frac{\frac{6}{32.2}}{(2)(0.33333)}[0 + (0.33333)(0.41667)(18)^2] = -12.578 \text{ lb}$$

$$C_z = \frac{\frac{6}{32.2}}{(2)(0.33333)}\left[\frac{1}{2}(0.25)^2(60)(18) - 0\right] = 9.4332 \text{ lb}$$

$$D_z = \frac{\frac{6}{32.2}}{(2)(0.33333)}\left[-\frac{1}{2}(0.25)^2(60)(18) - 0\right] = -9.4332 \text{ lb}$$

$\mathbf{C} = -(12.58 \text{ lb})\mathbf{i} + (9.43 \text{ lb})\mathbf{k}$ ◀

$\mathbf{D} = -(12.58 \text{ lb})\mathbf{i} - (9.43 \text{ lb})\mathbf{k}$ ◀



PROBLEM 18.103

A 2.5 kg homogeneous disk of radius 80 mm rotates with an angular velocity $\boldsymbol{\omega}_1$ with respect to arm *ABC*, which is welded to a shaft *DCE* rotating as shown at the constant rate $\omega_2 = 12$ rad/s. Friction in the bearing at *A* causes ω_1 to decrease at the rate of $15\ \text{rad/s}^2$. Determine the dynamic reactions at *D* and *E* at a time when ω_1 has decreased to 50 rad/s.

SOLUTION

Angular velocity of shaft *DCE* and arm *CBA*: $\boldsymbol{\Omega} = \omega_2 \mathbf{k}$

Angular velocity of disk *A*: $\boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k}$

Its angular momentum about *A*: $\mathbf{H}_A = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = \bar{I}_y \omega_1 \mathbf{j} + \bar{I}_z \omega_2 \mathbf{k}$

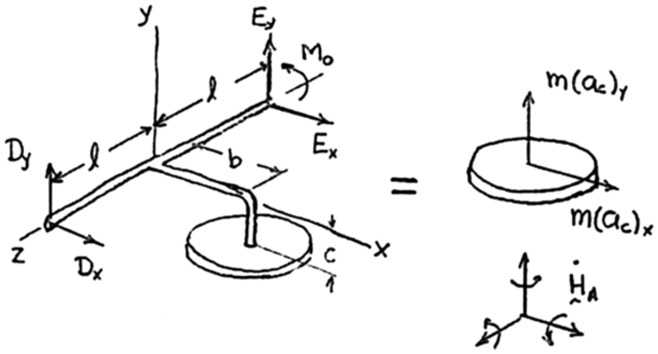
Let the reference frame *Cxyz* be rotating with angular velocity $\boldsymbol{\Omega}$.

$$\begin{aligned}\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega} \times \mathbf{H}_A \\ &= \bar{I}_y \dot{\omega}_1 \mathbf{j} + \bar{I}_z \dot{\omega}_2 \mathbf{k} + \omega_2 \mathbf{k} \times (\bar{I}_y \omega_1 \mathbf{j} + \bar{I}_z \omega_2 \mathbf{k}) \\ &= -\bar{I}_y \omega_2 \omega_1 \mathbf{i} + \bar{I}_y \dot{\omega}_1 \mathbf{j} + \bar{I}_z \dot{\omega}_2 \mathbf{k} \\ &= -\frac{1}{2} m r^2 \omega_2 \omega_1 \mathbf{i} + \frac{1}{2} m r^2 \dot{\omega}_1 \mathbf{j} + \frac{1}{4} m r^2 \dot{\omega}_2 \mathbf{k}\end{aligned}$$

Velocity and acceleration of the mass center *A* of the disk:

$$\mathbf{v}_A = \boldsymbol{\omega}_2 \mathbf{k} \times \mathbf{r}_{A/C} = \omega_2 \mathbf{k} \times (b\mathbf{i} - c\mathbf{j}) = c\omega_2 \mathbf{i} + b\omega_2 \mathbf{j}$$

$$\mathbf{a}_A = \dot{\omega}_2 \mathbf{k} \times \mathbf{r}_{A/C} + \omega_2 \mathbf{k} \times \mathbf{v}_A = (c\dot{\omega}_2 - b\omega_2^2)\mathbf{i} + (b\dot{\omega}_2 + c\omega_2^2)\mathbf{j}$$



$$\Sigma \mathbf{F} = \Sigma \mathbf{F}_{\text{eff}}$$

$$D_x \mathbf{i} + D_y \mathbf{j} + E_x \mathbf{i} + E_y \mathbf{j} = m\mathbf{a}_A$$

PROBLEM 18.103 (Continued)

Resolve into components.

$$D_x + E_x = m(c\dot{\omega}_2 - b\omega_2^2)$$
$$D_y + E_y = m(b\dot{\omega}_2 + c\omega_2^2)$$
$$\Sigma\mathbf{M}_E = \dot{\mathbf{H}}_E = \dot{\mathbf{H}}_A + \mathbf{r}_{A/E} \times m\mathbf{a}_A = \dot{\mathbf{H}}_A + (b\mathbf{i} - c\mathbf{j} + l\mathbf{k}) \times m\mathbf{a}_A$$
$$2l\mathbf{k} \times (D_x\mathbf{i} + D_y\mathbf{j}) + M_0\mathbf{k} = \dot{\mathbf{H}}_A + m(bl\dot{\omega}_2 - cl\omega_2^2)\mathbf{i} + m(cl\dot{\omega}_2 + bl\omega_2^2)\mathbf{j} + m(b^2 + c^2)\dot{\omega}_2\mathbf{k}$$
$$-2lD_y\mathbf{i} + 2lD_x\mathbf{j} + M_0\mathbf{k} = m\left(-\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 - cl\omega_2^2\right)\mathbf{i}$$
$$+ m\left(\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right)\mathbf{j} + m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2\mathbf{k}$$

$$\mathbf{k}: \quad M_0 = m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2$$
$$\mathbf{j}: \quad D_x = \frac{m}{2l}\left(\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right) \quad E_x = \frac{m}{2l}\left(-\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right)$$
$$\mathbf{i}: \quad D_y = \frac{m}{2l}\left(\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 + cl\omega_2^2\right) \quad E_y = \frac{m}{2l}\left(-\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 + cl\omega_2^2\right)$$

Data:

$m = 2.5$ kg, $r = 80$ mm $= 0.08$ m

$b = 120$ mm $= 0.12$ m, $c = 60$ mm $= 0.06$ m, $l = 150$ mm $= 0.15$ m

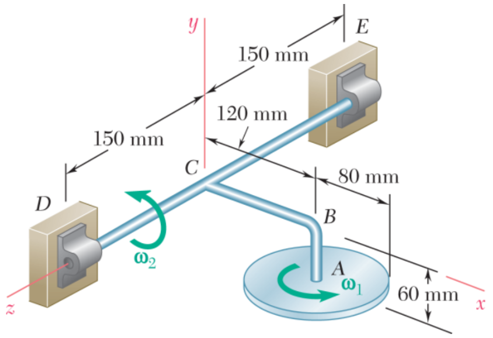
$\omega_1 = 50$ rad/s, $\dot{\omega}_1 = -15$ rad/s^2, $\omega_2 = 12$ rad/s, $\dot{\omega}_2 = 0$

$$D_x = \frac{2.5}{(2)(0.15)}\left[\frac{1}{2}(0.08)^2(-15) + 0 - (0.12)(0.15)(12)^2\right] = -22.0 \text{ N}$$
$$D_y = \frac{2.5}{(2)(0.15)}\left[\frac{1}{2}(0.08)^2(12)(50) + 0 + (0.06)(0.15)(12)^2\right] = 26.8 \text{ N}$$

$$\mathbf{D} = -(22.0 \text{ N})\mathbf{i} + (26.8 \text{ N})\mathbf{j} \ ◀$$

$$E_x = \frac{2.5}{(2)(0.15)}\left[-\frac{1}{2}(0.08)^2(-15) + 0 - (0.12)(0.15)(12)^2\right] = -21.2 \text{ N}$$
$$E_y = \frac{2.5}{(2)(0.15)}\left[-\frac{1}{2}(0.08)^2(12)(50) + 0 + (0.06)(0.15)(12)^2\right] = -5.20 \text{ N}$$

$$\mathbf{E} = -(21.2 \text{ N})\mathbf{i} - (5.20 \text{ N})\mathbf{j} \ ◀$$



PROBLEM 18.104

A 2.5-kg homogeneous disk of radius 80 mm rotates at the constant rate $\omega_1 = 50$ rad/s with respect to arm *ABC*, which is welded to a shaft *DCE*. Knowing that at the instant shown shaft *DCE* has an angular velocity $\omega_2 = (12 \text{ rad/s})\mathbf{k}$ and an angular acceleration $\alpha_2 = (8 \text{ rad/s}^2)\mathbf{k}$, determine (*a*) the couple which must be applied to shaft *DCE* to produce that acceleration, (*b*) the corresponding dynamic reactions at *D* and *E*.

SOLUTION

Angular velocity of shaft *DCE* and arm *CBA*: $\boldsymbol{\Omega} = \omega_2\mathbf{k}$

Angular velocity of disk *A*: $\boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k}$

Its angular momentum about *A*: $\mathbf{H}_A = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} = \bar{I}_y\omega_1\mathbf{j} + \bar{I}_z\omega_2\mathbf{k}$

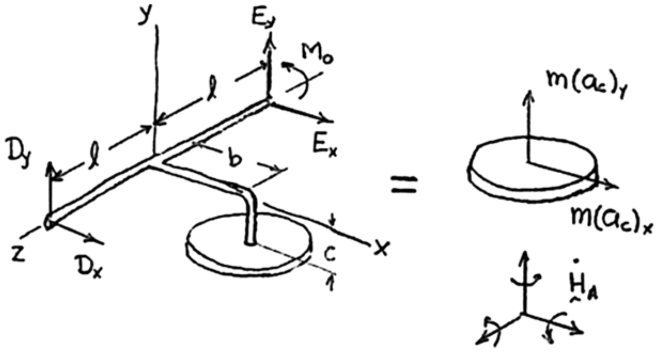
Let the reference frame *Cxyz* be rotating with angular velocity $\boldsymbol{\Omega}$.

$$\begin{aligned}\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_A\\ &= \bar{I}_y\dot{\omega}_1\mathbf{j} + \bar{I}_z\dot{\omega}_2\mathbf{k} + \omega_2\mathbf{k}\times(\bar{I}_y\omega_1\mathbf{j} + \bar{I}_z\omega_2\mathbf{k})\\ &= -\bar{I}_y\omega_2\omega_1\mathbf{i} + \bar{I}_y\dot{\omega}_1\mathbf{j} + \bar{I}_z\dot{\omega}_2\mathbf{k}\\ &= -\frac{1}{2}mr^2\omega_2\omega_1\mathbf{i} + \frac{1}{2}mr^2\dot{\omega}_1\mathbf{j} + \frac{1}{4}mr^2\dot{\omega}_2\mathbf{k}\end{aligned}$$

Velocity and acceleration of the mass center *A* of the disk:

$$\mathbf{v}_A = \boldsymbol{\omega}_2\mathbf{k}\times\mathbf{r}_{A/C} = \omega_2\mathbf{k}\times(b\mathbf{i} - c\mathbf{j}) = c\omega_2\mathbf{i} + b\omega_2\mathbf{j}$$

$$\mathbf{a}_A = \dot{\omega}_2\mathbf{k}\times\mathbf{r}_{A/C} + \omega_2\mathbf{k}\times\mathbf{v}_A = (c\dot{\omega}_2 - b\omega_2^2)\mathbf{i} + (b\dot{\omega}_2 + c\omega_2^2)\mathbf{j}$$



$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}$$

$$D_x\mathbf{i} + D_y\mathbf{j} + E_x\mathbf{i} + E_y\mathbf{j} = m\mathbf{a}_A$$

PROBLEM 18.104 (Continued)

Resolve into components.

$$D_x + E_x = m(c\dot{\omega}_2 - b\omega_2^2)$$

$$D_y + E_y = m(b\dot{\omega}_2 + c\omega_2^2)$$

$$\Sigma \mathbf{M}_E = \dot{\mathbf{H}}_E = \dot{\mathbf{H}}_A + \mathbf{r}_{A/E} \times m\mathbf{a}_A = \dot{\mathbf{H}}_A + (b\mathbf{i} - c\mathbf{j} + l\mathbf{k}) \times m\mathbf{a}_A$$

$$2l\mathbf{k} \times (D_x\mathbf{i} + D_y\mathbf{j}) + M_0\mathbf{k} = \dot{\mathbf{H}}_A + m(bl\dot{\omega}_2 - cl\omega_2^2)\mathbf{i} + m(cl\dot{\omega}_2 + bl\omega_2^2)\mathbf{j} + m(b^2 + c^2)\dot{\omega}_2\mathbf{k}$$

$$-2lD_y\mathbf{i} + 2lD_x\mathbf{j} + M_0\mathbf{k} = m\left(-\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 - cl\omega_2^2\right)\mathbf{i}$$

$$+ m\left(\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right)\mathbf{j} + m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2\mathbf{k}$$

$$\mathbf{k}: \quad M_0 = m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2$$

$$\mathbf{j}: \quad D_x = \frac{m}{2l}\left(\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right) \quad E_x = \frac{m}{2l}\left(-\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right)$$

$$\mathbf{i}: \quad D_y = \frac{m}{2l}\left(\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 + cl\omega_2^2\right) \quad E_y = \frac{m}{2l}\left(-\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 + cl\omega_2^2\right)$$

Data:

$m = 2.5$ kg, $r = 80$ mm $= 0.08$ m

$b = 120$ mm $= 0.12$ m, $c = 60$ mm $= 0.06$ m, $l = 150$ mm $= 0.15$ m

$\omega_1 = 50$ rad/s, $\dot{\omega}_1 = 0$, $\omega_2 = 12$ rad/s, $\dot{\omega}_2 = \alpha_2 = 8$ rad/s^2

(*a*)

$$M_0 = (2.5)\left[\frac{1}{4}(0.08)^2 + (0.12)^2 + (0.06)^2\right](8)$$

$\mathbf{M}_0 = (0.392\ \text{N}\cdot\text{m})\mathbf{k}$ ◀

(*b*)

$$D_x = \frac{2.5}{(2)(0.15)}[0 + (0.06)(0.15)(8) - (0.12)(0.15)(12)^2] = -21.0\ \text{N}$$

$$D_y = \frac{2.5}{(2)(0.15)}\left[\frac{1}{2}(0.08)^2(12)(50) + (0.12)(0.15)(8) + (0.06)(0.15)(12)^2\right] = 28.0\ \text{N}$$

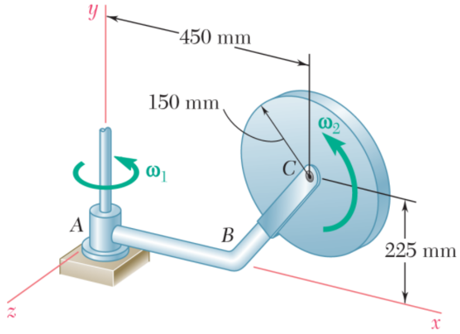
$\mathbf{D} = -(21.0\ \text{N})\mathbf{i} + (28.0\ \text{N})\mathbf{j}$ ◀

$$E_x = \frac{2.5}{(2)(0.15)}[-0 + (0.06)(0.15)(8) - (0.12)(0.15)(12)^2] = -21.0\ \text{N}$$

$$E_y = \frac{2.5}{(2)(0.15)}\left[-\frac{1}{2}(0.08)^2(12)(50) + (0.12)(0.15)(8) + (0.06)(0.15)(12)^2\right]$$

$$= -4.00\ \text{N}$$

$\mathbf{E} = -(21.0\ \text{N})\mathbf{i} - (4.00\ \text{N})\mathbf{j}$ ◀



PROBLEM 18.105

For the disk of Problem 18.99, determine (*a*) the couple $M_1\mathbf{j}$ which should be applied to arm *ABC* to give it an angular acceleration $\alpha_1 = -(7.5 \text{ rad/s}^2)\mathbf{j}$ when $\omega_1 = 5 \text{ rad/s}$, knowing that the disk rotates at the constant rate $\omega_2 = 15 \text{ rad/s}$, (*b*) the force-couple system representing the dynamic reaction at *A* at that instant. Assume that *ABC* has a negligible mass.

PROBLEM 18.99 A thin disk of mass $m = 4 \text{ kg}$ rotates with an angular velocity ω_2 with respect to arm *ABC*, which itself rotates with an angular velocity ω_1 about the *y* axis. Knowing that $\omega_1 = 5 \text{ rad/s}$ and $\omega_2 = 15 \text{ rad/s}$ and that both are constant, determine the force-couple system representing the dynamic reaction at the support at *A*.

SOLUTION

Angular velocity of the disk. $\boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k} = (5 \text{ rad/s})\mathbf{j} + (15 \text{ rad/s})\mathbf{k}$

Moments of inertia about principal axes passing through the mass center.

$$\bar{I}_x = \bar{I}_y = \frac{1}{4}mr^2$$

$$= \frac{1}{4}(4)(0.150 \text{ m})^2 = 0.0225 \text{ kg}\cdot\text{m}^2$$

$$\bar{I}_z = \frac{1}{2}mr^2 = 0.045 \text{ kg}\cdot\text{m}^2$$

Angular momentum about mass center C.

$$\mathbf{H}_C = \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + I_{z'}\omega_{z'}\mathbf{k}$$

$$= 0 + (0.0225)5\mathbf{j} + (0.045)15\mathbf{k}$$

$$= (0.1125 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (0.6750 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

Rate of change of $\mathbf{H}_C$. Let the frame *Axyz* be turning with angular velocity $\boldsymbol{\Omega} = \omega_1\mathbf{j}$.

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_C = \bar{I}_x\dot{\omega}_x\mathbf{i} + \bar{I}_y\dot{\omega}_y\mathbf{j} + \bar{I}_z\dot{\omega}_z\mathbf{k} + \boldsymbol{\Omega}\times\mathbf{H}_C$$

$$= 0 + (0.0225)(-7.5)\mathbf{j} + 0 + 5\mathbf{j}\times(0.1125\mathbf{j} + 0.675\mathbf{k})$$

$$= -(0.16875 \text{ N}\cdot\text{m})\mathbf{j} + (3.375 \text{ N}\cdot\text{m})\mathbf{i}$$

Position vector of Point C. $\mathbf{r}_{C/A} = (0.450 \text{ m})\mathbf{i} + (0.225 \text{ m})\mathbf{j}$

Velocity of Point C, the mass center of the disk.

$$\mathbf{v}_C = \omega_1\times\mathbf{r}_{C/A} = 5\mathbf{j}\times(0.45\mathbf{i} + 0.225\mathbf{j}) = -(2.25 \text{ m/s})\mathbf{k}$$

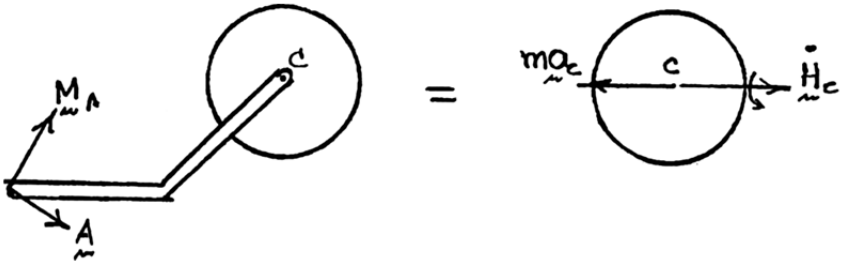
PROBLEM 18.105 (Continued)

Acceleration of Point C.

$$\mathbf{a}_C = \alpha_1\mathbf{j}\times\mathbf{r}_{C/A} + \omega_1\mathbf{j}\times\mathbf{v}_C = (-7.5\mathbf{j})\times(0.45\mathbf{i}+0.225\mathbf{j}) + 5\mathbf{j}\times(-2.25\mathbf{k})$$
$$= (3.3750\text{ m/s}^2)\mathbf{k} - (11.25\text{ m/s}^2)\mathbf{i}$$

$$m\mathbf{a}_C = (4)(-11.25\mathbf{i} + 3.3750\mathbf{k})$$
$$= -(45\text{ N})\mathbf{i} + (13.5\text{ N})\mathbf{k}$$

Free body and kinetic diagrams



Linear components: $\mathbf{A} = m\mathbf{a}_C$ $\mathbf{A} = -(45\text{ N})\mathbf{i} + (13.5\text{ N})\mathbf{k}$

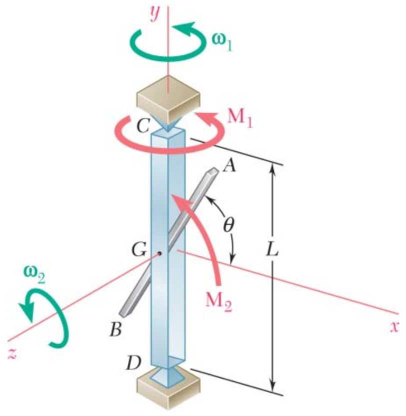
Moments about **A**.

$$\mathbf{M}_A = \mathbf{r}_{C/A}\times m\mathbf{a}_C + \dot{\mathbf{H}}_C$$
$$\mathbf{M}_A = (0.450\mathbf{i}+0.225\mathbf{j})\times(-45\mathbf{i}+13.5\mathbf{k}) - 0.16875\mathbf{j} + 3.375\mathbf{i}$$
$$= -6.0750\mathbf{j} + 10.125\mathbf{k} + 3.0375\mathbf{i} - 0.16875\mathbf{j} + 3.375\mathbf{i}$$
$$= 6.4125\mathbf{i} - 6.2438\mathbf{j} + 10.125\mathbf{k}$$

(*a*) Required couple. $\mathbf{M}_1\mathbf{j} = -(6.24\text{ N}\cdot\text{m})\mathbf{j}$ ◀

(*b*) Dynamic reaction. $\mathbf{A} = -(45.0\text{ N})\mathbf{i} + (13.50\text{ N})\mathbf{k}$ ◀

$\mathbf{M}_A = (6.41\text{ N}\cdot\text{m})\mathbf{i} + (10.13\text{ N}\cdot\text{m})\mathbf{k}$ ◀



PROBLEM 18.106*

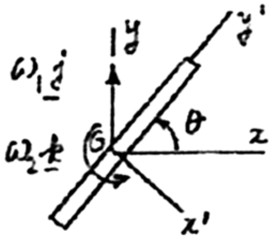
A slender homogeneous rod AB of mass m and length L is made to rotate at the constant rate ω_2 about the horizontal z axis, while frame CD is made to rotate at the constant rate ω_1 about the y axis. Express as a function of the angle θ (*a*) the couple $\mathbf{M}_1$ required to maintain the rotation of the frame, (*b*) the couple $\mathbf{M}_2$ required to maintain the rotation of the rod, (*c*) the dynamic reactions at the supports C and D.

SOLUTION

Angular momentum $\mathbf{H}_G$:

We resolve the angular velocity $\boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k}$ into components along the principal axes $Gx'y'z$:

$$\omega_{x'} = -\omega_1 \cos\theta$$
$$\omega_{y'} = \omega_1 \sin\theta$$
$$\omega_z = \omega_2$$



Moments of inertia:

$$\bar{I}_{x'} = \bar{I}_z = \frac{1}{12}mL^2 \qquad \bar{I}_{y'} = 0$$

We have
$$H_{x'} = \bar{I}_{x'}\,\omega_{x'} = -\frac{1}{12}mL^2\,\omega_1\cos\theta$$
$$H_{y'} = \bar{I}_{y'}\,\omega_{y'} = 0$$
$$H_z = \bar{I}_z\,\omega_z = \frac{1}{12}mL^2\,\omega_2$$

Computing the components of $\mathbf{H}_G$ along the x, y, z axes:

$$H_x = H_{x'}\sin\theta = -\frac{1}{12}mL^2\,\omega_1\cos\theta\sin\theta = -\frac{1}{24}mL^2\,\omega_1\sin 2\theta$$
$$H_y = -H_{x'}\cos\theta = +\frac{1}{12}mL^2\,\omega_1\cos^2\theta$$
$$H_z = \frac{1}{12}mL^2\,\omega_2$$

The angular momentum is therefore

$$\mathbf{H}_G = -\frac{1}{24}mL^2\omega_1\sin 2\theta\,\mathbf{i} + \frac{1}{12}mL^2\omega_1\cos^2\theta\,\mathbf{j} + \frac{1}{12}mL^2\omega_2\mathbf{k}$$

where the reference frame $Gxyz$ rotates with the angular velocity

$$\boldsymbol{\Omega} = \omega_1\mathbf{j}$$

PROBLEM 18.106* (Continued)

Rate of change of $\mathbf{H}_G$. We note that ω_1 and ω_2 are constant, while θ varies with *t*, with $\dot{\theta} = \omega_2$.

Eq. (18.22) yields

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G = -\frac{1}{24} mL^2 \omega_1 (2\cos 2\theta \dot{\theta})\mathbf{i} + \frac{1}{12} mL^2 \omega_1 (-2\cos\theta \sin\theta \dot{\theta})\mathbf{j}$$

$$+ \omega_1 \mathbf{j} \times \left(-\frac{1}{24} mL^2 \omega_1 \sin 2\theta \mathbf{i} + \frac{1}{12} mL^2 \omega_2 \mathbf{k} \right)$$

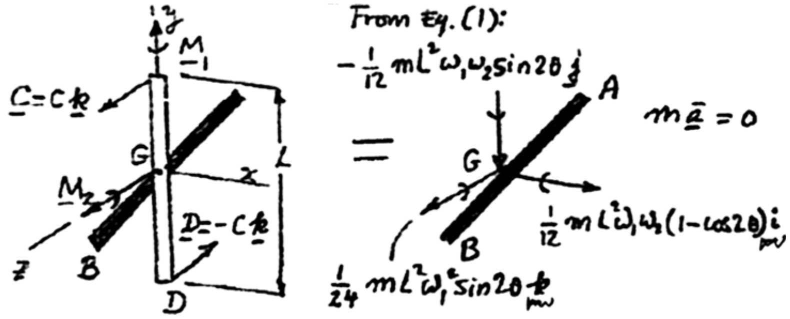
$$= -\frac{1}{12} mL^2 \omega_1 \omega_2 \cos 2\theta \mathbf{i} - \frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta \mathbf{j}$$

$$+ \frac{1}{24} mL^2 \omega_1^2 \sin 2\theta \mathbf{k} + \frac{1}{12} mL^2 \omega_1 \omega_2 \mathbf{i}$$

$$\dot{\mathbf{H}}_G = \frac{1}{12} mL^2 \omega_1 \omega_2 (1 - \cos 2\theta)\mathbf{i} - \frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta \mathbf{j}$$

$$+ \frac{1}{24} mL^2 \omega_1^2 \sin 2\theta \mathbf{k} \qquad (1)$$

Equivalence of external and effective forces.


y
$\mathbf{M}_1$
$\mathbf{C} = C\mathbf{k}$
G
L
x
$\mathbf{M}_2$
$\mathbf{D} = -C\mathbf{k}$
z
B
D
=
From Eq. (1):
$-\frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta \mathbf{j}$
A
$m\bar{\mathbf{a}} = 0$
G
B
$\frac{1}{12} mL^2 \omega_1 \omega_2 (1 - \cos 2\theta)\mathbf{i}$
$\frac{1}{24} mL^2 \omega_1^2 \sin 2\theta \mathbf{k}$

Equating the moments of the variable couples:

$$L\mathbf{j} \times C\mathbf{k} + M_1 \mathbf{j} + M_2 \mathbf{k} = \dot{\mathbf{H}}_G$$

$$LC\mathbf{i} + M_1 \mathbf{j} + M_2 \mathbf{k} = \frac{1}{12} mL^2 \omega_1 \omega_2 (1 - \cos 2\theta)\mathbf{i} - \frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta \mathbf{j}$$

$$+ \frac{1}{24} mL^2 \omega_1^2 \sin 2\theta \mathbf{k}$$

Equating the coefficients of the unit vectors:

$$\mathbf{j}: \quad M_1 = -\frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta$$

(*a*) Couple M_1: $\mathbf{M}_1 = -\frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta \mathbf{j}$ ◀

$$\mathbf{k}: \quad M_2 = \frac{1}{24} mL^2 \omega_1^2 \sin 2\theta$$

PROBLEM 18.106* (Continued)

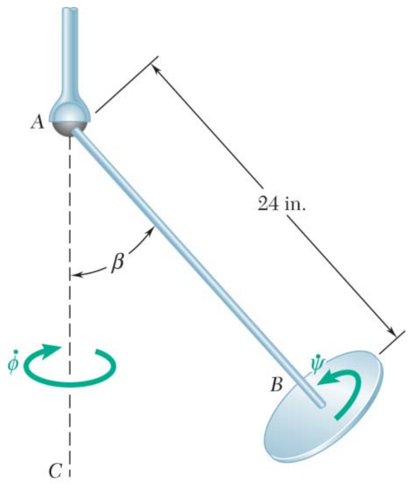
(*b*) Couple M_2: $\mathbf{M}_2 = \dfrac{1}{24} mL^2\omega_1^2 \sin 2\theta\mathbf{k}$ ◀

$$\mathbf{i}:\quad LC = \frac{1}{12} mL^2\omega_1\omega_2(1-\cos 2\theta)$$

$$C = \frac{1}{12} mL\omega_1\omega_2(1-\cos 2\theta) = \frac{1}{6} mL\omega_1\omega_2 \sin^2\theta$$

Dynamic reactions.

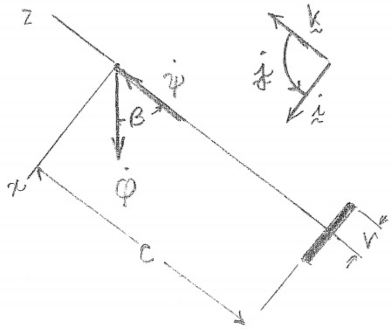
$\mathbf{C} = \dfrac{1}{6} mL\omega_1\omega_2 \sin^2\theta\mathbf{k};\ \mathbf{D} = -\dfrac{1}{6} mL\omega_1\omega_2 \sin^2\theta\mathbf{k}$ ◀



PROBLEM 18.107

A uniform thin disk of 6-in. diameter is attached to the end of a rod *AB* of negligible mass which is supported by a ball-and-socket joint at point *A*. Knowing that the disk is observed to precess about the vertical axis *AC* at the constant rate of 36 rpm in the sense indicated and that its axis of symmetry *AB* forms an angle $\beta = 60°$ with *AC*, determine the rate at which the disk spins about rod *AB*.

SOLUTION



Use principal axes *x*, *y*, *z* with origin at *A*.

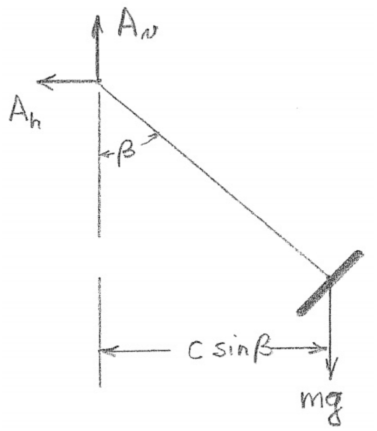
Angular velocity: $$\boldsymbol{\omega} = \dot{\phi}\sin\beta\mathbf{i} + \left(\dot{\psi} - \dot{\phi}\cos\beta\right)\mathbf{k}$$

$$\omega_x = \dot{\phi}\sin\beta, \qquad \omega_y = 0, \qquad \omega_z = \dot{\psi} - \dot{\phi}\cos\beta$$

Angular momentum about *A*:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= I'\dot{\phi}\sin\beta\mathbf{i} + I\left(\dot{\psi} - \dot{\phi}\cos\beta\right)\mathbf{k}$$



Let the reference frame *Axyz* be rotating with angular velocity $\boldsymbol{\Omega} = \dot{\phi}\sin\beta\mathbf{i} - \dot{\phi}\cos\beta\mathbf{k}$.

$$\dot{\mathbf{H}}_A = \boldsymbol{\Omega}\times\mathbf{H}_A$$

$$= -\dot{\phi}\left[I\dot{\psi}\sin\beta + \left(I' - I\right)\dot{\phi}\sin\beta\cos\beta\right]\mathbf{j}$$

$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A$$

$$-mgc\sin\beta\mathbf{j} = -\dot{\phi}\left[I\dot{\psi}\sin\beta + \left(I' - I\right)\dot{\phi}\sin\beta\cos\beta\right]\mathbf{j}$$

$$mgc = I\dot{\phi}\dot{\psi} + \left(I' - I\right)\dot{\phi}^2\cos\beta$$

$$\frac{I\dot{\phi}\dot{\psi}}{mgc} + \frac{\left(I' - I\right)\dot{\phi}^2\cos\beta}{mgc} - 1 = 0 \qquad (1)$$

Data: $r = 3\text{ in.} = 0.25\text{ ft}, \qquad c = 24\text{ in.} = 2\text{ ft},$

$$g = 32.2\text{ ft/s}^2$$

$$\frac{I}{m} = \frac{1}{2}r^2 = \frac{1}{2}(0.25)^2 = 0.03125 \text{ ft}^2$$

$$\frac{I'}{m} = \frac{1}{4}r^2 + c^2 = \frac{1}{4}(0.25)^2 + (2)^2 = 4.015625 \text{ ft}^2$$

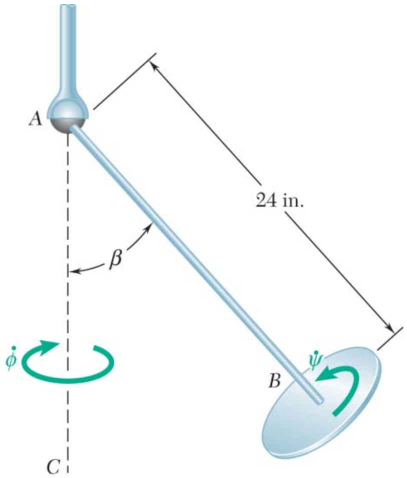
$$\beta = 60°, \qquad \dot{\phi} = 36 \text{ rpm} = 3.7699 \text{ rad/s}$$

Substituting into (1),

$$\frac{(0.03125)(3.7699)\dot{\psi}}{(32.2)(2)} + \frac{(4.015625 - 0.03125)(3.7699)^2 \cos 60°}{(32.2)(2)} - 1 = 0$$

$$1.82934 \times 10^{-3}\dot{\psi} + 0.43965 - 1 = 0$$

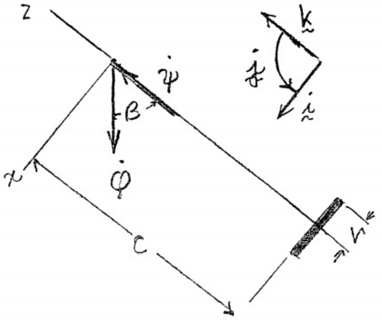
$$\dot{\psi} = 306.3 \text{ rad/s}, \qquad \dot{\psi} = 2930 \text{ rpm} \blacktriangleleft$$



PROBLEM 18.108

A uniform thin disk of 6-in. diameter is attached to the end of a rod *AB* of negligible mass which is supported by a ball-and-socket joint at point *A*. Knowing that the disk is spinning about its axis of symmetry *AB* at the rate of 2100 rpm in the sense indicated and that *AB* forms an angle $\beta = 45°$ with the vertical axis *AC*, determine the two possible rates of steady precession of the disk about the axis *AC*.

SOLUTION



Use principal axes *x*, *y*, *z* with origin at *A*.

Angular velocity: $$\boldsymbol{\omega} = \dot{\phi}\sin\beta\mathbf{i} + \left(\dot{\psi} - \dot{\phi}\cos\beta\right)\mathbf{k}$$

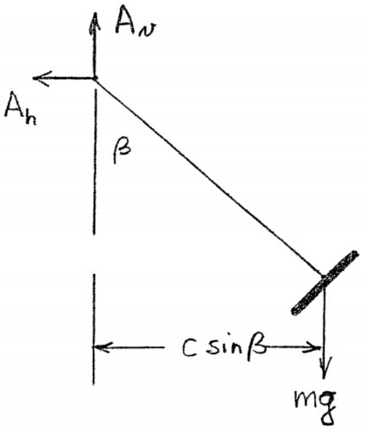
$$\omega_x = \dot{\phi}\sin\beta, \qquad \omega_y = 0, \qquad \omega_z = \dot{\psi} - \dot{\phi}\cos\beta$$

Angular momentum about *A*:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= I'\dot{\phi}\sin\beta\mathbf{i} + I\left(\dot{\psi} - \dot{\phi}\cos\beta\right)\mathbf{k}$$

Let the reference frame *Axyz* be rotating with angular velocity $\boldsymbol{\Omega} = \dot{\phi}\sin\beta\mathbf{i} - \dot{\phi}\cos\beta\mathbf{k}$.



$$\dot{\mathbf{H}}_A = \boldsymbol{\Omega} \times \mathbf{H}_A$$

$$= -\dot{\phi}\left[I\dot{\psi}\sin\beta + (I' - I)\dot{\phi}\sin\beta\cos\beta\right]\mathbf{j}$$

$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A$$

$$-mgc\sin\beta\mathbf{j} = -\dot{\phi}\left[I\dot{\psi}\sin\beta + (I' - I)\dot{\phi}\sin\beta\cos\beta\right]\mathbf{j}$$

$$mgc = I\dot{\phi}\dot{\psi} + (I' - I)\dot{\phi}^2\cos\beta$$

$$\frac{I\dot{\phi}\dot{\psi}}{mgc} + \frac{(I' - I)\dot{\phi}^2\cos\beta}{mgc} - 1 = 0 \tag{1}$$

Data: $r = 3\text{ in.} = 0.25\text{ ft}, \qquad c = 24\text{ in.} = 2\text{ ft},$

$$g = 32.2\text{ ft/s}^2$$

$$\frac{I}{m} = \frac{1}{2}r^2 = \frac{1}{2}(0.25)^2 = 0.03125\ \text{ft}^2$$

$$\frac{I'}{m} = \frac{1}{4}r^2 + c^2 = \frac{1}{4}(0.25)^2 + (2)^2 = 4.015625\ \text{ft}^2$$

$$\beta = 45°, \qquad \dot{\psi} = 2100\ \text{rpm} = 219.91\ \text{rad/s}$$

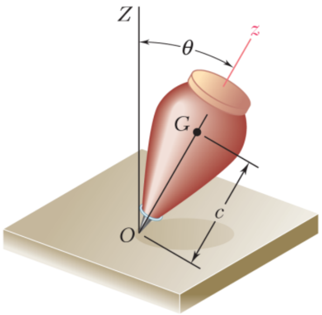
Substituting into (1),

$$\frac{(0.03125)(219.91)\dot{\phi}}{(32.2)(2)} + \frac{(4.015625 - 0.03125)\dot{\phi}^2 \cos 45°}{(32.2)(2)} - 1 = 0$$

$$(0.106711)\dot{\phi} + 0.043748\dot{\phi}^2 - 1 = 0$$

$$\dot{\phi} = -6.1537\ \text{rad/s} \quad \text{or} \quad \dot{\phi} = 3.7145\ \text{rad/s}$$

$$\dot{\phi} = -58.8\ \text{rpm},\ 35.5\ \text{rpm}\ ◀$$

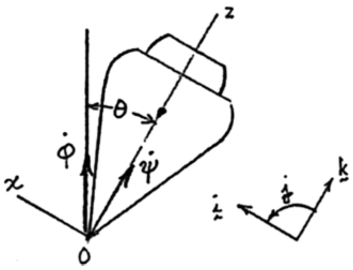


PROBLEM 18.109

The 85-g top shown is supported at the fixed Point O. The radii of gyration of the top with respect to its axis of symmetry and with respect to a transverse axis through O are 21 mm and 45 mm, respectively. Knowing that $c = 37.5\text{ mm}$ and that the rate of spin of the top about its axis of symmetry is 1800 rpm, determine the two possible rates of steady precession corresponding to $\theta = 30°$.

SOLUTION

Use principal axes x, y, z with origin at O.



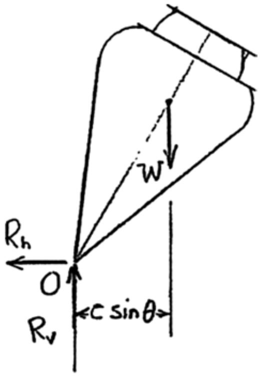
Angular velocity: $\boldsymbol{\omega} = \dot{\varphi}\sin\theta\mathbf{i} + (\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$

$$\omega_x = \dot{\varphi}\sin\theta, \quad \omega_y = 0, \quad \omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$$

Angular momentum about O:

$$\begin{aligned}\mathbf{H}_O &= I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k} \\ &= I'\dot{\varphi}\sin\theta\mathbf{i} + I\omega_z\mathbf{k}\end{aligned}$$

Let the reference frame $Oxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$.



$$\begin{aligned}\dot{\mathbf{H}}_O &= \boldsymbol{\Omega}\times\mathbf{H}_O \\ &= (\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k})\times(I'\dot{\varphi}\sin\theta + I\omega_z\mathbf{k}) \\ &= (I'\dot{\varphi}\sin\theta\cos\theta - I\omega_z\sin\theta)\dot{\varphi}\mathbf{j}\end{aligned}$$

$$\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O$$

$$-Wc\sin\theta\mathbf{j} = (I'\dot{\varphi}\sin\theta\cos\theta - I\omega_z\sin\theta)\mathbf{j}$$

$$Wc = (I\omega_z - I'\dot{\varphi}\cos\theta)\dot{\varphi}$$

$$Wc = [I\dot{\psi} - (I' - I)\dot{\varphi}\cos\theta]\dot{\varphi} \qquad (1)$$

Data:

$$m = 85\text{ g} = 0.085\text{ kg}$$
$$W = mg = (0.085)(9.81) = 0.83385\text{ N}$$

$$I = mk_z^2 = (0.085)(0.021)^2 = 37.485\times10^{-6}\text{ kg}\cdot\text{m}^2$$
$$I' = mk_x^2 = (0.085)(0.045)^2 = 172.125\times10^{-6}\text{ kg}\cdot\text{m}^2$$

$$c = 37.5\text{ mm} = 0.0375\text{ m}, \quad \dot{\psi} = 1800\text{ rpm} = 188.496\text{ rad/s}$$

$$\theta = 30°$$

PROBLEM 18.109 (Continued)

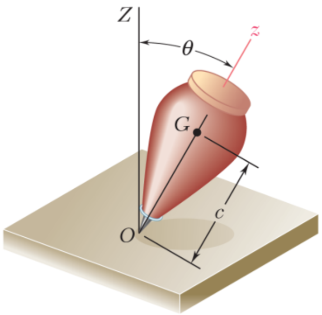
Substituting into Eq. (1),

$$(0.83385)(0.0375) = [(37.485\times 10^{-6})(188.496) - (134.64\times 10^{-6})\dot{\varphi}\cos 30°]\dot{\varphi}$$

$$116.602\times 10^{-6}\dot{\varphi}^2 - 7.0658\times 10^{-3}\dot{\varphi} + 31.269\times 10^{-3} = 0$$

$$\dot{\varphi} = 30.299 \pm 25.492 \qquad \dot{\varphi} = 4.807 \text{ rad/s}, \quad 55.791 \text{ rad/s}$$

$$\dot{\varphi} = 45.9 \text{ rpm}, \quad 533 \text{ rpm} \blacktriangleleft$$



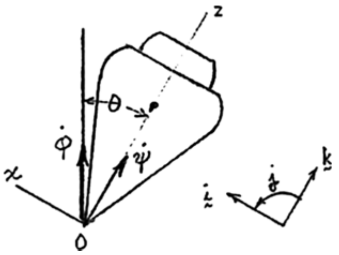
PROBLEM 18.110

The top shown is supported at the fixed Point O and its moments of inertia about its axis of symmetry and about a transverse axis through O are denoted, respectively, by I and I'. (a) Show that the condition for steady precession of the top is

$$(I\omega_z - I'\dot{\phi}\cos\theta)\dot{\phi} = Wc$$

where $\dot{\phi}$ is the rate of precession and ω_z is the rectangular component of the angular velocity along the axis of symmetry of the top. (b) Show that if the rate of spin $\dot{\psi}$ of the top is very large compared with its rate of precession $\dot{\phi}$, the condition for steady precession is $I\dot{\psi}\dot{\phi} \approx Wc$. ($c$) Determine the percentage error introduced when this last relation is used to approximate the slower of the two rates of precession obtained for the top of Problem 18.109.

SOLUTION



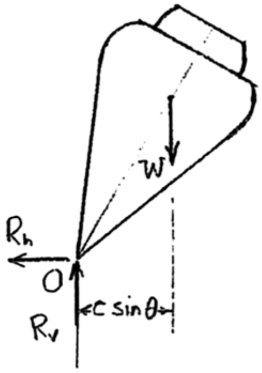
Use principal axes x, y, z with origin at O.

Angular velocity: $\boldsymbol{\omega} = \dot{\varphi}\sin\theta\mathbf{i} + (\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$

$$\omega_x = \dot{\varphi}\sin\theta, \quad \omega_y = 0, \quad \omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$$

Angular momentum about O:

$$\mathbf{H}_O = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k} = I'\dot{\varphi}\sin\theta\mathbf{i} + I\omega_z\mathbf{k}$$



Let the reference frame $Oxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$.

$$\begin{aligned}\dot{\mathbf{H}}_O &= \boldsymbol{\Omega}\times\mathbf{H}_O \\ &= (\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k})\times(I'\dot{\varphi}\sin\theta + I\omega_z\mathbf{k}) \\ &= (I'\dot{\varphi}\sin\theta\cos\theta - I\omega_z\sin\theta)\dot{\varphi}\mathbf{j}\end{aligned}$$

$$\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O$$

$$-Wc\sin\theta\mathbf{j} = (I'\dot{\varphi}\sin\theta\cos\theta - I\omega_z\sin\theta)\mathbf{j}$$

(a) $Wc = (I\omega_z - I'\dot{\varphi}\cos\theta)\dot{\varphi}$ ◀

$$Wc = [I\dot{\psi} - (I' - I)\dot{\varphi}\cos\theta]\dot{\varphi} \qquad (1)$$

(b) For $|\dot{\varphi}| << \dot{\psi}$, $Wc \approx I\dot{\psi}\dot{\varphi}$ (2) ◀

PROBLEM 18.110 (Continued)

(*c*) Data:

$$m = 85 \text{ g} = 0.085 \text{ kg}$$

$$W = mg = (0.085)(9.81) = 0.83385 \text{ N}$$

$$I = mk_z^2 = (0.085)(0.021)^2 = 37.485 \times 10^{-6} \text{ kg} \cdot \text{m}^2$$

$$I = mk_x^2 = (0.085)(0.045)^2 = 172.125 \times 10^{-6} \text{ kg} \cdot \text{m}^2$$

$$c = 37.5 \text{ mm} = 0.0375 \text{ m}, \quad \dot{\psi} = 1800 \text{ rpm} = 188.496 \text{ rad/s}$$

$$\theta = 30°$$

Substituting into Eq. (1),

$$(0.83385)(0.0375) = [(37.485 \times 10^{-6})(188.496) - (134.64 \times 10^{-6})\dot{\varphi} \cos 30°]\dot{\varphi}$$

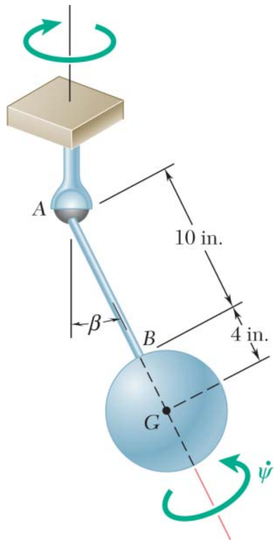
$$116.602 \times 10^{-6} \dot{\varphi}^2 - 7.0658 \times 10^{-3} \dot{\varphi} + 31.269 \times 10^{-3} = 0$$

$$\dot{\varphi} = 30.299 \pm 25.492 \qquad \dot{\varphi} = 4.807 \text{ rad/s, } 55.791 \text{ rad/s}$$

From Eq. (2),

$$\dot{\varphi} = \frac{Wc}{I\dot{\psi}} = \frac{(0.83385)(0.0375)}{(37.485 \times 10^{-6})(188.496)} = 4.425 \text{ rad/s}$$

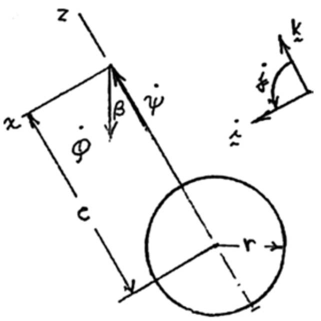
$$\% \text{ error} = \frac{4.425 - 4.807}{4.807} \times 100\% \qquad \% \text{ error} = -7.95\% \blacktriangleleft$$



PROBLEM 18.111

A solid aluminum sphere of radius 4 in. is welded to the end of a10-in.-long rod *AB* of negligible mass which is supported by a ball-and-socket joint at *A*. Knowing that the sphere is observed to precess about a vertical axis at the constant rate of 60 rpm in the sense indicated and that rod *AB* forms an angle $\beta = 20°$ with the vertical, determine the rate of spin of the sphere about line *AB*.

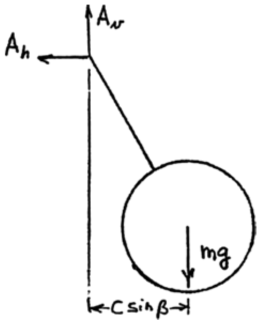
SOLUTION



Use principal axes *x*, *y*, *z* with origin at *A*.

Angular velocity:
$$\boldsymbol{\omega} = \dot{\varphi}\sin\beta\mathbf{i} + (\dot{\psi} - \dot{\varphi}\cos\beta)\mathbf{k}$$
$$\omega_x = \dot{\varphi}\sin\beta, \quad \omega_y = 0, \quad \omega_z = \dot{\psi} - \dot{\varphi}\cos\beta$$

Angular momentum about *A*:
$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$
$$= I'\dot{\varphi}\sin\beta\mathbf{i} + I(\dot{\psi} - \dot{\varphi}\cos\beta)\mathbf{k}$$



Let the reference frame *Axyz* be rotating with angular velocity $\boldsymbol{\Omega} = \dot{\varphi}\sin\beta\mathbf{i} - \dot{\varphi}\cos\beta\mathbf{k}$.

$$\dot{\mathbf{H}}_A = \boldsymbol{\Omega}\times\mathbf{H}_A$$
$$= -\dot{\varphi}[I\dot{\psi}\sin\beta + (I' - I)\dot{\varphi}\sin\beta\cos\beta]\mathbf{j}$$
$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A$$
$$-mgc\sin\beta\mathbf{j} = -\dot{\varphi}[I\dot{\psi}\sin\beta + (I' - I)\dot{\varphi}\sin\beta\cos\beta]\mathbf{j}$$
$$mgc = I\dot{\varphi}\dot{\psi} - (I' - I)\dot{\varphi}^2\cos\beta$$
$$\frac{I\dot{\varphi}\dot{\psi}}{mgc} - \frac{(I' - I)\dot{\varphi}^2\cos\beta}{mgc} + 1 = 0 \qquad (1)$$

Data:
$$r = 4\text{ in.} = 0.333\text{ ft}$$
$$c = 10\text{ in.} + 4\text{ in.}$$
$$= 14\text{ in.} = 1.1667\text{ ft},$$
$$g = 32.2\text{ ft/s}^2$$

PROBLEM 18.111 (Continued)

$$\frac{I}{m} = \frac{2}{5}r^2 = \frac{2}{5}(0.3333)^2 = 0.04444\text{ ft}^2$$

$$\frac{I'}{m} = \frac{2}{5}r^2 + c^2 = \frac{2}{5}(0.3333)^2 + (1.16667)^2 = 1.4056\text{ ft}^2$$

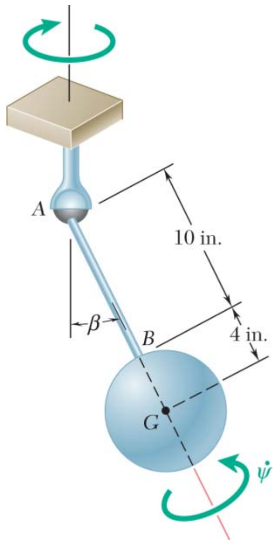
$$\beta = 20°, \quad \dot{\varphi} = 60\text{ rpm} = 6.2832\text{ rad/s}$$

Substituting into Eq. (1),

$$\frac{(0.04444)(6.2832)\dot{\psi}}{(32.2)(1.1667)} - \frac{(1.4056 - 0.04444)(6.2832)^2\cos 20°}{(32.2)(1.1667)} + 1 = 0$$

$$7.4335\times 10^{-3}\dot{\psi} - 1.34412 + 1 = 0$$

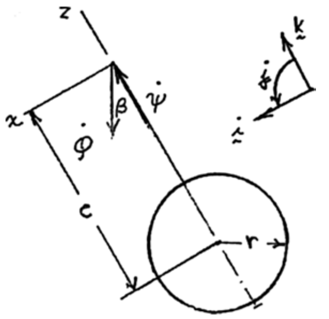
$$\dot{\psi} = 46.293\text{ rad/s} = 442\text{ rpm} \qquad \dot{\psi} = 442\text{ rpm} \blacktriangleleft$$



PROBLEM 18.112

A solid aluminum sphere of radius 4 in. is welded to the end of a 10-in.-long rod *AB* of negligible mass which is supported by a ball-and-socket joint at *A*. Knowing that the sphere spins as shown about line *AB* at the rate of 600 rpm, determine the angle β for which the sphere will precess about a vertical axis at the constant rate of 60 rpm in the sense indicated.

SOLUTION



Use principal axes *x*, *y*, *z* with origin at *A*.

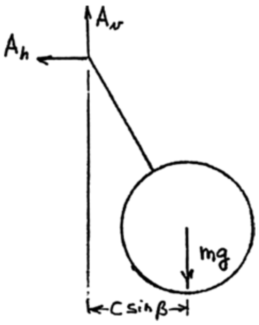
Angular velocity:

$$\boldsymbol{\omega} = \dot{\varphi}\sin\beta\mathbf{i} + (\dot{\psi} - \dot{\varphi}\cos\beta)\mathbf{k}$$

$$\omega_x = \dot{\varphi}\sin\beta, \quad \omega_y = 0, \quad \omega_z = \dot{\psi} - \dot{\varphi}\cos\beta$$

Angular momentum about *A*:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$
$$= I'\dot{\varphi}\sin\beta\mathbf{i} + I(\dot{\psi} - \dot{\varphi}\cos\beta)\mathbf{k}$$



Let the reference frame *Axyz* be rotating with angular velocity $\boldsymbol{\Omega} = \dot{\varphi}\sin\beta\mathbf{i} - \dot{\varphi}\cos\beta\mathbf{k}$.

$$\dot{\mathbf{H}}_A = \boldsymbol{\Omega} \times \mathbf{H}_A$$
$$= -\dot{\varphi}[I\dot{\psi}\sin\beta + (I' - I)\dot{\varphi}\sin\beta\cos\beta]\mathbf{j}$$
$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A$$
$$-mgc\sin\beta\mathbf{j} = -\dot{\varphi}[I\dot{\psi}\sin\beta + (I' - I)\dot{\varphi}\sin\beta\cos\beta]\mathbf{j}$$
$$mgc = I\dot{\varphi}\dot{\psi} - (I' - I)\dot{\varphi}^2\cos\beta$$
$$\frac{I\dot{\varphi}\dot{\psi}}{mgc} - \frac{(I' - I)\dot{\varphi}^2\cos\beta}{mgc} + 1 = 0 \qquad (1)$$

Data:

$$r = 4\text{ in.} = \frac{1}{3}\text{ ft}$$
$$c = 10 + 4 = 14\text{ in.}$$
$$= \frac{14}{12} = \frac{7}{6}\text{ ft},$$
$$g = 32.2\text{ ft/s}^2$$

PROBLEM 18.112 (Continued)

$$\frac{I}{m} = \frac{2}{5}r^2 = \frac{2}{5}\left(\frac{1}{3}\right)^2 = 0.04444 \text{ ft}^2$$

$$\frac{I'}{m} = \frac{2}{5}r^2 + c^2 = \frac{2}{5}\left(\frac{1}{3}\right)^2 + \left(\frac{7}{6}\right)^2 = 1.4056 \text{ ft}^2$$

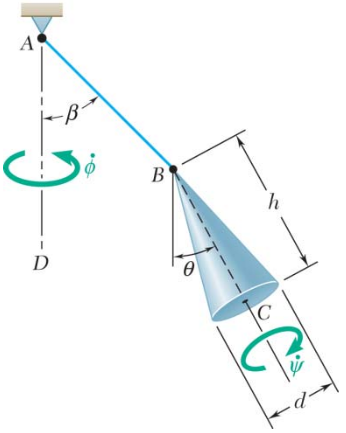
$$\dot{\psi} = 600 \text{ rpm} = 62.832 \text{ rad/s} \qquad \dot{\varphi} = -60 \text{ rpm} = -6.2832 \text{ rad/s}$$

Substituting into Eq. (1),

$$\frac{(0.044444)(-6.2832)(62.832)}{(32.2)(1.16667)} - \frac{(1.40556 - 0.04444)(-6.2832)^2}{(32.2)(1.16667)}\cos\beta + 1 = 0$$

$$-0.46706 - 1.43038\cos\beta + 1 = 0$$

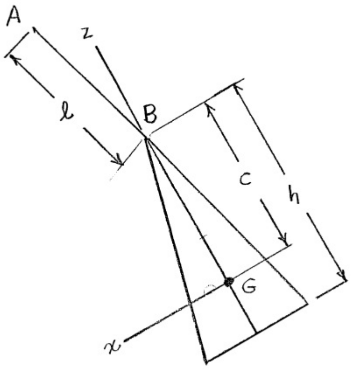
$$\cos\beta = 0.37259 \qquad \beta = 68.1° \blacktriangleleft$$



PROBLEM 18.113

A homogeneous cone of height h and with a base of diameter $d < h$ is attached as shown to a cord AB. The cone spins about its axis BC at the constant rate $\dot\psi$ and precesses about the vertical through A at the constant rate $\dot\phi$. Determine the angle β for which the axis BC of the cone is aligned with the cord AB ($\theta = \beta$).

SOLUTION



Use principal centroidal axes x, y, z as shown.

Angular velocity: $$\boldsymbol{\omega} = -\dot\phi\sin\theta\mathbf{i} + \left(\dot\psi + \dot\phi\cos\theta\right)\mathbf{k}$$

Angular momentum about the mass center G:

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

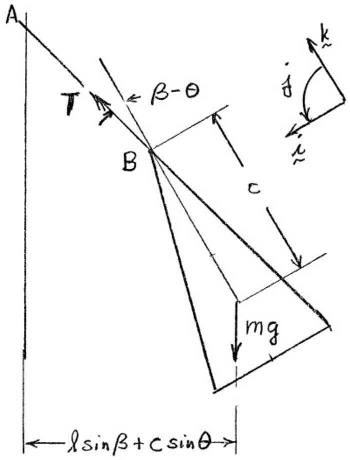
$$= -I'\dot\phi\sin\theta\mathbf{i} + I\left(\dot\psi + \dot\phi\cos\theta\right)\mathbf{k}$$

Let the reference frame $Gxyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = -\dot\phi\sin\theta\mathbf{i} + \dot\phi\cos\theta\mathbf{k}$$

$$\dot{\mathbf{H}}_G = \boldsymbol{\Omega}\times\mathbf{H}_G$$

$$= \left[I\dot\phi\dot\psi\sin\theta - \left(I' - I\right)\dot\phi^2\sin\theta\cos\theta\right]\mathbf{j}$$



Acceleration of the mass center:

$$\overline{\mathbf{a}} = \left(l\sin\beta + c\sin\theta\right)\dot\phi^2 \longleftarrow$$

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}:$$

$$+\uparrow: \quad T\cos\beta - mg = 0, \qquad T = \frac{mg}{\cos\beta} \qquad (1)$$

$$+\longleftarrow: \quad T\sin\beta = m\overline{a}$$

$$g\tan\beta = \left(l\sin\beta + c\sin\theta\right)\dot\phi^2 \qquad (2)$$

PROBLEM 18.113 (Continued)

$+\circlearrowleft$: $\Sigma M_G = Tc\sin(\beta - \theta) = \dot{H}_G$

(3)

For alignment of axis BC with cord AB, $\theta = \beta$.

Substituting into (3),

$$0 = \frac{I}{m}\dot{\psi} - \frac{I' - I}{m}\dot{\phi}\cos\beta \qquad\qquad \cos\beta = \frac{I\dot{\psi}}{(I' - I)\dot{\phi}}$$

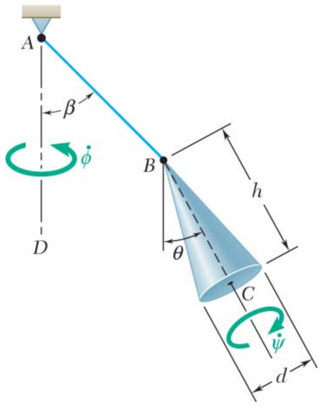
For a cone, $c = \frac{3}{4}h \qquad I = \frac{3}{10}mr^2 = \frac{3}{40}md^2$

$$I' = \frac{3}{20}mr^2 + \frac{3}{5}mh^2 - mc^2 = \frac{3}{80}md^2 - \frac{3}{80}mh^2$$

$$I' - I = \frac{3}{80}md^2 + \frac{3}{80}mh^2$$

$$\frac{I' - I}{I} = \frac{2d^2}{d^2 + h^2}$$

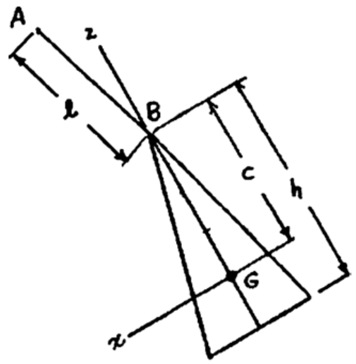
$$\cos\beta = \frac{2d^2\dot{\psi}}{(d^2 + h^2)\dot{\phi}} \qquad\qquad \beta = \cos^{-1}\left[\frac{2d^2\dot{\psi}}{(d^2 + h^2)\dot{\phi}}\right] \blacktriangleleft$$



PROBLEM 18.114

A homogeneous cone of height $h = 12$ in. and with a base diameter $d = 6$ in. is attached as shown to a cord AB. Knowing that the angles that cord AB and the axis BC of the cone form with the vertical are, respectively, $\beta = 45°$ and $\theta = 30°$, and that the cone precesses at the constant rate $\dot{\phi} = 8$ rad/s in the sense indicated, determine (*a*) the rate of spin $\dot{\psi}$ of the cone about its axis BC, (*b*) the length of cord AB.

SOLUTION



Use principal centroidal axes *x*, *y*, *z* as shown.

Angular velocity: $\boldsymbol{\omega} = -\dot{\phi}\sin\theta\mathbf{i} + \left(\dot{\psi} + \dot{\phi}\cos\theta\right)\mathbf{k}$

Angular momentum about the mass center *G*:

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

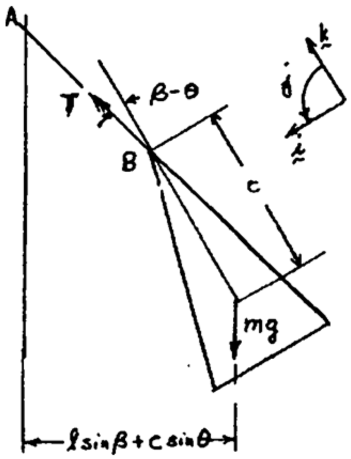
$$= -I'\dot{\phi}\sin\theta\mathbf{i} + I\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\mathbf{k}$$

Let the reference frame $Gxyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = -\dot{\phi}\sin\theta\mathbf{i} + \dot{\phi}\cos\theta\mathbf{k}$$

$$\dot{\mathbf{H}}_G = \boldsymbol{\Omega}\times\mathbf{H}_G$$

$$= \left[I\dot{\phi}\dot{\psi}\sin\theta - \left(I' - I\right)\dot{\phi}^2\sin\theta\cos\theta\right]\mathbf{j}$$



Acceleration of the mass center:

$$\bar{\mathbf{a}} = \left(l\sin\beta + c\sin\theta\right)\dot{\phi}^2 \longleftarrow$$

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}:$$

$+\uparrow$: $T\cos\beta - mg = 0, \qquad T = \dfrac{mg}{\cos\beta}$ (1)

$+\longleftarrow$: $T\sin\beta = m\bar{a}$

$$g\tan\beta = \left(l\sin\beta + c\sin\theta\right)\dot{\phi}^2 \qquad (2)$$

$+\circlearrowleft$: $\Sigma M_G = Tc\sin\left(\beta - \theta\right) = \dot{H}_G$

$$\frac{mgc\sin\left(\beta - \theta\right)}{\cos\beta} = I\dot{\psi}\dot{\phi}\sin\theta - \left(I' - I\right)\dot{\phi}^2\sin\theta\cos\theta \qquad (3)$$

PROBLEM 18.114 (Continued)

Data: $r = 3 \text{ in.} = 0.25 \text{ ft}, \qquad h = 12 \text{ in.} = 1 \text{ ft},$

$$c = \frac{3}{4}h = 0.75 \text{ ft}, \qquad g = 32.2 \text{ ft/s}^2$$

$$\frac{I}{m} = \frac{3}{10}r^2 = \frac{3}{10}(0.25)^2 = 0.01875 \text{ ft}^2$$

$$\frac{I'}{m} = \frac{3}{20}r^2 + \frac{3}{5}h^2 - c^2 = \frac{3}{20}(0.25)^2 + \frac{3}{5}(1)^2 - (0.75)^2$$

$$= 0.046875 \text{ ft}^2$$

$$\frac{I' - I}{m} = 0.028125 \text{ ft}^2$$

$$\beta = 45°, \qquad \theta = 30° \qquad \beta - \theta = 15°$$

(*a*) Dividing Eq. (3) by *m* and substituting numerical data,

$$\frac{(32.2)(0.75)\sin 15°}{\cos 45°} = \left(0.01875\,\dot{\psi}(8)\sin 30°\right) - (0.028125)(8)^2 \sin 30° \cos 30°$$

$$8.8395 = 0.075\,\dot{\psi} - 0.77942$$

$$\dot{\psi} = 128.252 \text{ rad/s}$$

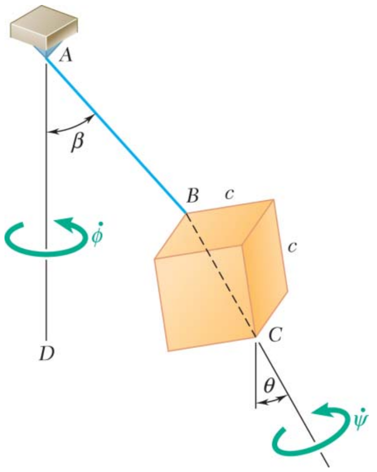
$\dot{\psi} = 128.3 \text{ rad/s}$ ◀

(*b*) Substituting data into Eq. (2),

$$32.2 \tan 45° = \left(l \sin 45° + 0.75 \sin 30°\right)(8)^2$$

$$32.2 = 45.255l + 24, \qquad l = 0.1812 \text{ ft}$$

$l = 2.17 \text{ in.}$ ◀



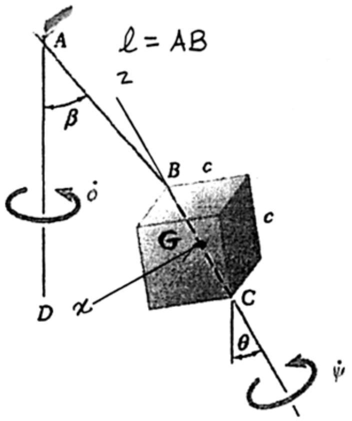
PROBLEM 18.115

A solid cube of side $c = 80$ mm is attached as shown to cord AB. It is observed to spin at the rate $\dot{\psi} = 40$ rad/s about its diagonal BC and to precess at the constant rate $\dot{\phi} = 5$ rad/s about the vertical axis AD. Knowing that $\beta = 30°$, determine the angle ϕ that the diagonal BC forms with the vertical. (*Hint:* The moment of inertia of a cube about an axis through its center is independent of the orientation of that axis.)

SOLUTION

Use centroidal axes x, y, z such that the z axis lies along the body diagonal BC and the x axis lies in the plane of A, B, C, and D. Let e be the length GB.

$$e = \frac{\sqrt{3}}{2}c$$



Moment of inertia: $$I_x = I_y = I_z = \frac{1}{6}mc^2$$

Angular velocity: $$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\mathbf{i} + (\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$$

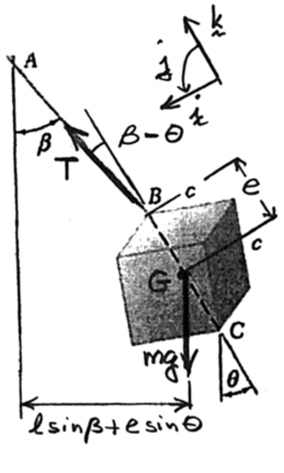
Angular momentum about the mass center G:

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k} = -I'\dot{\varphi}\sin\theta\mathbf{i} + I(\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$$

Let the reference frame $Gxyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$$

$$\begin{aligned}\dot{\mathbf{H}}_G &= \boldsymbol{\Omega}\times\mathbf{H}_G\\ &= [I\dot{\varphi}\dot{\psi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta]\mathbf{j}\end{aligned}$$



Acceleration of the mass center:

$$\bar{\mathbf{a}} = (l\sin\beta + e\sin\theta)\dot{\varphi}^2 \longleftarrow$$

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}:$$

$$+\uparrow: \quad T\cos\beta - mg = 0, \quad T = \frac{mg}{\cos\beta} \qquad (1)$$

$$+\longleftarrow: \quad T\sin\beta = m\bar{a}$$

$$g\tan\beta = (l\sin\beta + e\sin\theta)\dot{\varphi}^2 \qquad (2)$$

$$+\circlearrowleft \Sigma M_{G.} = Te\sin(\beta - \theta) = \dot{H}_G$$

$$\frac{mge\sin(\beta - \theta)}{\cos\beta} = I\dot{\psi}\dot{\varphi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta \qquad (3)$$

PROBLEM 18.115 (Continued)

Data:

$$c = 80 \text{ mm} = 0.08 \text{ m} \quad e = \frac{\sqrt{3}}{2}(0.08) = 0.069282 \text{ m}$$

$$\frac{I}{m} = \frac{I'}{m} = \frac{1}{6}c^2 = 1.06667 \times 10^{-3} \text{ m}^2 \quad I' - I = 0$$

$$\beta = 30° \quad \dot{\psi} = 40 \text{ rad/s} \quad \dot{\varphi} = 5 \text{ rad/s}$$

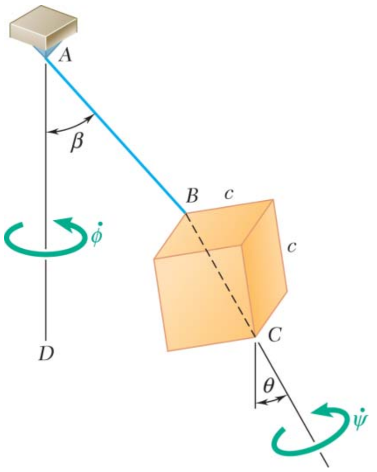
Substituting into Eq. (3),

$$\frac{(9.81)(0.069282)\sin(\beta - \theta)}{\cos\beta} = (1.06667 \times 10^{-3})(40)(5)\sin\theta$$

$$0.67966(\sin\beta\cos\theta - \cos\beta\sin\theta) = 0.21333\sin\theta\cos\beta$$

$$0.89299\sin\theta\cos\beta = 0.67966\sin\beta\cos\theta$$

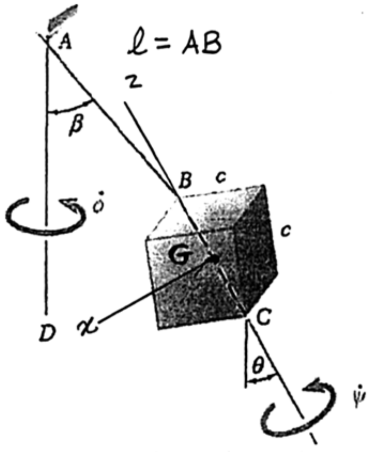
$$\tan\theta = 0.76111\tan\beta = 0.76111\tan 30° = 0.43942 \qquad \theta = 23.7° \blacktriangleleft$$

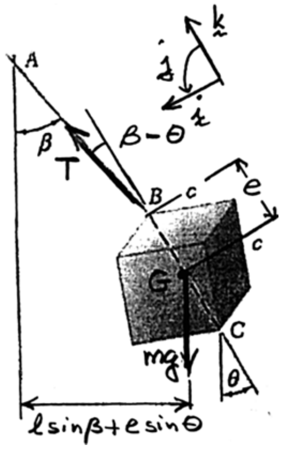


PROBLEM 18.116

A solid cube of side $c = 120$ mm is attached as shown to a cord *AB* of length 240 mm. The cube spins about its diagonal *BC* and precesses about the vertical axis *AD*. Knowing that $\theta = 25°$ and $\beta = 40°$, determine (*a*) the rate of spin of the cube, (*b*) its rate of precession. (See hint of Problem 18.115.)

SOLUTION





Use centroidal axes *x*, *y*, *z* such that the *z* axis lies along the body diagonal *BC* and the *x* axis lies in the plane of *A*, *B*, *C*, and *D*. Let *e* be the length *GB*.

$$e = \frac{\sqrt{3}}{2}c$$

Moment of inertia: $$I_x = I_y = I_z = \frac{1}{6}mc^2$$

Angular velocity: $$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\mathbf{i} + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k}$$

Angular momentum about the mass center *G*:

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k} = -I'\dot{\varphi}\sin\theta\mathbf{i} + I(\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$$

Let the reference frame *Gxyz* be rotating with angular velocity

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$$

$$\dot{\mathbf{H}}_G = \boldsymbol{\Omega}\times\mathbf{H}_G$$

$$= [I\dot{\varphi}\dot{\psi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta]\mathbf{j}$$

Acceleration of the mass center:

$$\bar{\mathbf{a}} = (l\sin\beta + e\sin\theta)\dot{\varphi}^2 \longleftarrow$$

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}:$$

$$+\uparrow: \quad T\cos\beta - mg = 0, \quad T = \frac{mg}{\cos\beta} \qquad (1)$$

$$+\longleftarrow: \quad T\sin\beta = m\bar{a}$$

$$g\tan\beta = (l\sin\beta + e\sin\theta)\dot{\varphi}^2 \qquad (2)$$

$$+\circlearrowleft \Sigma M_G = Te\sin(\beta - \theta) = \dot{H}_G$$

$$\frac{mge\sin(\beta - \theta)}{\cos\beta} = I\dot{\psi}\dot{\varphi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta \qquad (3)$$

PROBLEM 18.116 (Continued)

Data:

$$c = 120\text{ mm} = 0.12\text{ m}$$

$$e = \frac{\sqrt{3}}{2}(0.12) = 0.103923\text{ m}$$

$$l = 240\text{ mm} = 0.24\text{ m}$$

$$\frac{I}{m} = \frac{I'}{m} = \frac{1}{6}c^2 = \frac{1}{6}(0.12)^2 = 2.4\times 10^3\,\text{m}^2 \quad I' - I = 0$$

$$\theta = 25° \quad \beta = 40° \quad \theta - \beta = 15°$$

From Eq. (2),

$$(9.81)\tan 40° = (0.24\sin 40° + 0.103923\sin 25°)\dot{\varphi}^2$$

$$\dot{\varphi}^2 = 41.534 \qquad \dot{\varphi} = 6.4447\text{ rad/s}$$

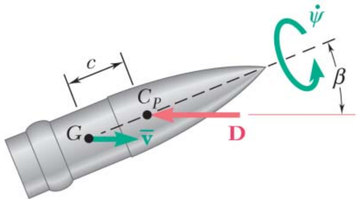
From Eq. (3),

$$\frac{(9.81)(0.103923)\sin 15°}{\cos 40°} = (2.4\times 10^{-3})\dot{\psi}(6.4447)\sin 25°$$

(*a*) $\dot{\psi} = 52.694\text{ rad/s}$ $\qquad \dot{\psi} = 52.7\text{ rad/s}$ ◀

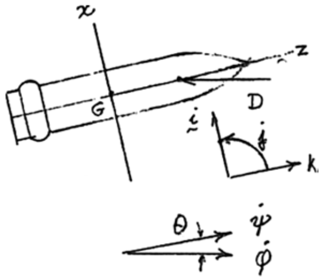
(*b*) $\dot{\varphi} = 6.44\text{ rad/s}$ ◀

PROBLEM 18.117



A high-speed photographic record shows that a certain projectile was fired with a horizontal velocity $\bar{\mathbf{v}}$ of 2000 ft/s and with its axis of symmetry forming an angle $\beta = 3°$ with the horizontal. The rate of spin $\dot{\psi}$ of the projectile was 6000 rpm, and the atmospheric drag was equivalent to a force **D** of 25 lb acting at the center of pressure C_P located at a distance $c = 6$ in. from G. (*a*) Knowing that the projectile has a weight of 45 lb and a radius of gyration of 2 in. with respect to its axis of symmetry, determine its approximate rate of steady precession. (*b*) If it is further known that the radius of gyration of the projectile with respect to a transverse axis through G is 8 in., determine the exact values of the two possible rates of precession.

SOLUTION



Choose principal centroidal axes x, y, z as shown.

The symmetry (spin) axis is the z axis.

Reduce the drag force to a force-couple system at the mass center.

$$\mathbf{F} = (D\sin\beta - W\cos\beta)\mathbf{i} - (D\cos\beta - W\sin\beta)\mathbf{k}$$

$$\mathbf{M}_G = Dc\sin\beta\,\mathbf{j}$$

For the occurrence of steady precession, the precession axis must be parallel to the drag force.

Thus, $\theta = \beta = 3°.$

(*a*) Using Equation (18.44), $\Sigma M_G = \dot{\mathbf{H}}_G$

$$Dc\sin\beta = (I\omega_z - I'\dot{\varphi}\cos\theta)\dot{\varphi}\sin\theta$$

Using $\beta = \theta$ and $\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$ gives

$$I\dot{\psi}\dot{\varphi} - (I' - I)\dot{\varphi}^2\cos\theta = Dc \qquad (1)$$

Neglecting the quadratic term in $\dot{\varphi}$,

$$I\dot{\psi}\dot{\varphi} \approx Dc \qquad \dot{\varphi} \approx \frac{Dc}{I\dot{\psi}}$$

Data: $W = 45\text{ lb}, \quad m = \dfrac{W}{g} = \dfrac{45}{32.2} = 1.3975\text{ slug}$

$$I = mk_z^2 = (1.3975)\left(\frac{2}{12}\right)^2 = 0.0388205\text{ slug}\cdot\text{ft}^2$$

$$\dot{\psi} = 6000\text{ rpm} = 628.32\text{ rad/s}, \quad c = \frac{6}{12}\text{ in.} = 0.5\text{ ft}$$

$$\dot{\varphi} \approx \frac{(25)(0.5)}{(0.0388205)(628.32)} = 0.51248\text{ rad/s}$$

$\dot{\varphi} \approx 4.89$ rpm ◀

PROBLEM 18.117 (Continued)

(*b*) $$I' = mk_x^2 = (1.3975)\left(\frac{8}{12}\right)^2 = 0.62112 \text{ slug}\cdot\text{ft}^2$$

$$I' - I = 0.58230 \text{ slug}\cdot\text{ft}^2$$

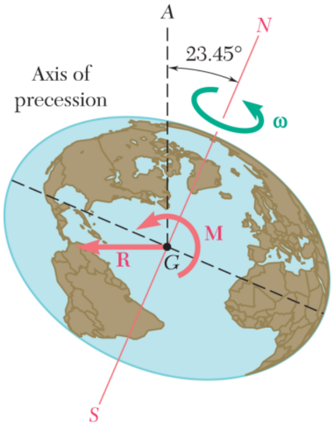
Substituting into Equation (1),

$$(0.0388205)(628.32)\dot{\varphi} - (0.58230)\dot{\varphi}^2\cos 3° = (25)(0.5)$$

$$0.58150\dot{\varphi}^2 - 24.3912\dot{\varphi} + 12.5 = 0$$

$$\dot{\varphi} = 20.9727 \pm 20.4538$$

$$= 0.51890 \text{ rad/s}, \quad 41.427 \text{ rad/s} \qquad \dot{\varphi} = 4.96 \text{ rpm}, 396 \text{ rpm} \blacktriangleleft$$



PROBLEM 18.118

If the earth were a sphere, the gravitational attraction of the sun, moon, and planets would at all times be equivalent to a single force **R** acting at the mass center of the earth. However, the earth is actually an oblate spheroid and the gravitational system acting on the earth is equivalent to a force **R** and a couple **M**. Knowing that the effect of the couple **M** is to cause the axis of the earth to precess about the axis *GA* at the rate of one revolution in 25,800 years, determine the average magnitude of the couple **M** applied to the earth. Assume that the average density of the earth is $5.51\ \text{g/cm}^3$, that the average radius of the earth is 6370 km, and that $\bar{I} = \frac{2}{5}mR^2$. (*Note:* This forced precession is known as the precession of the equinoxes and is not to be confused with the free precession discussed in Problem 18.123.)

SOLUTION

$$25{,}800 \text{ years} = (25{,}800 \text{ yr})(365.24 \text{ day/yr})(24 \text{ h/day})(3600 \text{ s/h})$$

$$= 814.16\times 10^9 \text{ s}$$

$$\dot{\varphi} = \frac{2\pi}{814.16\times 10^9} = 7.7173\times 10^{-12} \text{ rad/s}$$

$$\dot{\psi} = \frac{2\pi}{(23.93 \text{ h})(3600 \text{ s/h})} = 72.935\times 10^{-6} \text{ rad/s}$$

Mass density of Earth: $\rho = 5.51 \text{ g/cm}^3 = 5510 \text{ kg/m}^3$

Radius of Earth: $R = 6370 \text{ km} = 6.370\times 10^6 \text{ m}$

Mass of Earth: $\frac{4}{3}\pi R^3\rho = \frac{4}{3}\pi(6.370\times 10^6)^3(5510) = 5.9657\times 10^{24} \text{ kg}$

$$\bar{I} = \frac{2}{5}mR^2 = \frac{2}{5}(5.9657\times 10^{24})(6.370\times 10^6)^2$$

$$= 96.827\times 10^{36} \text{ kg}\cdot\text{m}^2$$

Using Equation (18.44),

$$M = (I\omega_z - I'\dot{\varphi}\cos\theta)\dot{\varphi}\sin\theta$$

$$= [I\dot{\psi} + (I - I')\dot{\varphi}\cos\theta]\dot{\varphi}\sin\theta$$

$$= I\dot{\psi}\dot{\varphi}\sin\theta$$

$$= (96.827\times 10^{36})(72.935\times 10^{-6})(7.7173\times 10^{-12})\sin 23.45°$$

$$= 21.688\times 10^{21} \text{ N}\cdot\text{m}$$

$$M = 21.7\times 10^{21} \text{ N}\cdot\text{m} \blacktriangleleft$$

PROBLEM 18.119

Show that for an axisymmetrical body under no force, the rates of precession and spin can be expressed, respectively, as

$$\dot{\varphi} = \frac{H_G}{I'}$$

and

$$\dot{\psi} = \frac{H_G \cos\theta (I' - I)}{II'}$$

where H_G is the constant value of the angular momentum of the body.

SOLUTION

By Equations (18.48), $\omega_x = -\frac{H_G \sin\theta}{I'}, \qquad \omega_y = 0, \qquad \omega_z = \frac{H_G \cos\theta}{I}$

By Equation (18.35) with $\dot{\theta} = 0,$

$$\omega_x = -\dot{\varphi}\sin\theta, \qquad \omega_y = 0, \qquad \omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$$

Eliminating ω_x and ω_z,

$$\dot{\varphi} = \frac{H_G}{I'}$$

$$\dot{\psi} + \dot{\varphi}\cos\theta = \frac{H_G \cos\theta}{I}$$

$$\dot{\psi} = \frac{H_G \cos\theta}{I} - \dot{\varphi}\cos\theta = \frac{H_G \cos\theta}{I} - \frac{H_G \cos\theta}{I'} = \frac{H_G (I' - I)\cos\theta}{II'}$$

PROBLEM 18.120

(*a*) Show that for an axisymmetrical body under no force, the rate of precession can be expressed as

$$\dot{\phi} = \frac{I\omega_z}{I'\cos\theta}$$

where ω_z is the rectangular component of $\boldsymbol{\omega}$ along the axis of symmetry of the body. (*b*) Use this result to check that the condition (18.44) for steady precession is satisfied by an axisymmetrical body under no force.

SOLUTION

(*a*) Angular velocity of the body: $\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \omega_2\mathbf{k}$

Its angular momentum about *G*: $\mathbf{H}_G = -I'\dot{\varphi}\sin\theta\mathbf{i} + I\omega_2\mathbf{k}$

Let the reference frame $Gxyz$ be rotating with angular velocity $\boldsymbol{\Omega}$, where

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$$

$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega}\times\mathbf{H}_G \\ &= 0 + (-\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k})\times(-I'\dot{\varphi}\sin\theta\mathbf{i} + I\omega_z\mathbf{k}) \\ &= I\dot{\varphi}\omega_z\sin\theta - I'\dot{\varphi}^2\sin\theta\cos\theta\end{aligned}$$

For no force, $\dot{\mathbf{H}}_G = 0$

Hence, $(I\omega_z - I'\dot{\varphi}\cos\theta)\dot{\varphi}\sin\theta = 0 \quad \text{or} \quad I\omega_z - I'\dot{\varphi}\cos\theta = 0 \qquad (1)$

Solving for $\dot{\varphi}$,

$$\dot{\varphi} = \frac{I\omega_z}{I'\cos\theta}$$

(*b*) Comparing Equation (1) with Equation (18.44) yields $\Sigma\mathbf{M}_0 = 0$, which is the condition for no force.

PROBLEM 18.121

Show that the angular velocity vector $\boldsymbol{\omega}$ of an axisymmetrical body under no force is observed from the body itself to rotate about the axis of symmetry at the constant rate

$$n = \frac{I' - I}{I'}\omega_z$$

where ω_z is the rectangular component of $\boldsymbol{\omega}$ along the axis of symmetry of the body.

SOLUTION

Angular velocity of the body: $\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + (\dot{\varphi}\cos\theta + \dot{\psi})\mathbf{k}$

Let the reference frame $Gxyz$ be rotating with angular velocity $\boldsymbol{\Omega}$, where

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{j}$$

Angular acceleration of the body: $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}_{Gxyz} + \boldsymbol{\Omega}\times\boldsymbol{\omega}$

$$\boldsymbol{\alpha} = 0 + \dot{\varphi}\dot{\psi}\sin\theta\mathbf{j}$$

The rate of change of angular velocity as observed from the body is $-\boldsymbol{\alpha}$.

Assume that $-\boldsymbol{\alpha}$ may be represented as the angular velocity vector rotating with angular velocity $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$.

$$\begin{aligned} -\boldsymbol{\alpha} &= (l\mathbf{i} + m\mathbf{j} + n\mathbf{k})\times(\omega_x\mathbf{i} + \omega_z\mathbf{k}) \\ &= m\omega_z\mathbf{i} + (n\omega_x - l\omega_z)\mathbf{j} - m\omega_x\mathbf{k} \end{aligned}$$

Matching components: $\mathbf{i}$: $0 = m\omega_z \quad m = 0$

$\mathbf{j}$: $-\dot{\varphi}\dot{\psi}\sin\theta = n\omega_x - l\omega_z$ (1)

$\mathbf{k}$: $0 = -m\omega_x \quad m = 0$

From Equation (1), $-\dot{\varphi}\dot{\psi}\sin\theta = -n\dot{\varphi}\sin\theta - l(\dot{\varphi}\cos\theta + \dot{\psi})$

from which $l = 0$ and $n = \dot{\psi}$.

Using Equation (18.44) with $\Sigma\mathbf{M}_0 = 0$ yields $I\omega_z - I'\dot{\varphi}\cos\theta = 0$

or $\dot{\varphi}\cos\theta = \dfrac{I\omega_z}{I'}$

But $\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta = \dot{\psi} + \dfrac{I\omega_z}{I'}$

Solving for $\dot{\psi}$, $\dot{\psi} = \dfrac{I' - I}{I}\omega_z$

Using $\omega_z = \omega_2$ and $n = \dot{\psi}$ yields $n = \dfrac{I' - I}{I'}\omega_z$. ◀

PROBLEM 18.122

For an axisymmetrical body under no force, prove (*a*) that the rate of retrograde precession can never be less than twice the rate of spin of the body about its axis of symmetry, (*b*) that in Figure 18.24 the axis of symmetry of the body can never lie within the space cone.

SOLUTION

For no force, $\mathbf{M}_G = 0$

(*a*) Using Equation (18.44), $0 = (\bar{I}\,\omega_z - \bar{I}'\dot{\varphi}\cos\theta)\dot{\varphi}\sin\theta$

$$\bar{I}\,\omega_z - \bar{I}'\dot{\varphi}\cos\theta = 0$$

Using $\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$ yields $\bar{I}\,\dot{\psi} - (\bar{I}' - I)\dot{\varphi}\cos\theta = 0$

$$\dot{\varphi} = \frac{\bar{I}\,\dot{\psi}}{(\bar{I}' - \bar{I})\cos\theta}$$

$$|\dot{\varphi}| = \frac{\bar{I}\,\dot{\psi}}{(\bar{I} - \bar{I}')}\sec\theta$$

For a flat disk, $\bar{I}' = \frac{1}{2}\bar{I}$

For any other shape, $\bar{I}' > \frac{1}{2}\bar{I}$

Hence, $\bar{I} - \bar{I}' < \frac{1}{2}I$ and $\frac{\bar{I}}{\bar{I} - \bar{I}'} > 2$

Also, $\sec\theta > 1$: $|\dot{\varphi}| > 2\dot{\psi}$

(*b*) $\tan\gamma = \frac{I}{I'}\tan\theta$ or $\tan\theta = \frac{I'}{I}\tan\gamma > \frac{1}{2}\tan\gamma$

Angle of surface of space cone is $\gamma - \theta$.

Angle of axis is θ.

$$\tan\theta > \frac{1}{2}\tan[\theta + (\gamma - \theta)] = \frac{1}{2}\,\frac{\tan\theta + \tan(\gamma - \theta)}{1 + \tan\theta\tan(\gamma - \theta)}$$

$$[1 + \tan\theta\tan(\gamma - \theta)]\tan\theta > \frac{1}{2}\tan\theta + \frac{1}{2}\tan(\gamma - \theta)$$

PROBLEM 18.122 (Continued)

$$\tan\theta + \tan^2\theta \tan(\gamma-\theta) > \frac{1}{2}\tan\theta + \frac{1}{2}\tan(\gamma-\theta)$$

$$\frac{1}{2}\tan\theta > \frac{1}{2}\tan(\gamma-\theta) - \tan^2\theta\tan(\gamma-\theta)$$

$$\frac{1}{2}\tan\theta > \frac{1}{2}\tan(\gamma-\theta)$$

$$\tan\theta > \tan(\gamma-\theta)$$

$$\theta > (\gamma-\theta)$$

The axis lies outside the space cone. ◀

PROBLEM 18.123

Using the relation given in Problem 18.121, determine the period of precession of the north pole of the earth about the axis of symmetry of the earth. The earth may be approximated by an oblate spheroid of axial moment of inertia I and of transverse moment of inertia $I' = 0.9967I$. (*Note:* Actual observations show a period of precession of the north pole of about 432.5 mean solar days; the difference between the observed and computed periods is due to the fact that the earth is not a perfectly rigid body. The free precession considered here should not be confused with the much slower precession of the equinoxes, which is a forced precession. See Problem 18.118.)

SOLUTION

Angular velocity of the body: $\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + (\dot{\varphi}\cos\theta + \dot{\psi})\mathbf{k}$

Let the reference frame $Gxyz$ be rotating with angular velocity $\boldsymbol{\Omega}$, where

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{j}$$

Angular acceleration of the body: $\boldsymbol{\alpha} = \dot{\omega}_{Gxyz} + \boldsymbol{\Omega}\times\boldsymbol{\omega}$

The rate of change of angular velocity as observed from the body is $-\boldsymbol{\alpha}$.

Assume that $-\alpha$ may be represented as the angular velocity vector rotating with angular velocity

$$l\mathbf{i} + m\mathbf{j} + n\mathbf{k}.$$

$$\begin{aligned} -\boldsymbol{\alpha} &= (l\mathbf{i} + m\mathbf{j} + n\mathbf{k})\times(\omega_x\mathbf{i} + \omega_z\mathbf{k}) \\ &= -m\omega_z\mathbf{i} + (n\omega_x - l\omega_z\mathbf{j}) - m\omega_x\mathbf{k} \end{aligned}$$

Matching components: $\mathbf{i}$: $0 = m\omega_z \qquad m = 0$

$\mathbf{j}$: $-\dot{\varphi}\dot{\psi}\sin\theta = n\omega_x - l\omega_z$ (1)

$\mathbf{k}$: $0 = -m\omega_x \qquad m = 0$

From Equation (1), $-\dot{\varphi}\dot{\psi}\sin\theta = -n\dot{\varphi}\sin\theta - l(\dot{\varphi}\cos\theta + \psi)$

From which $l = 0$ and $n = \dot{\psi}$.

Using Equation (18.44) with $\Sigma\mathbf{M}_0 = 0$ yields $I\omega_z - I'\dot{\varphi}\cos\theta = 0$

or $$\dot{\varphi}\cos\theta = \frac{I\omega_z}{I'}$$

But $$\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta = \dot{\psi} + \frac{I\omega_z}{I'}$$

Solving for $\dot{\psi}$, $$\dot{\psi} = \frac{I' - I}{I}\omega_z$$

PROBLEM 18.123 (Continued)

Using $\omega_z = \omega_2$ and $n = \dot{\psi}$ yields $n = \dfrac{I' - I}{I'}\omega_2$

Data for Earth: $I' = 0.9967I$

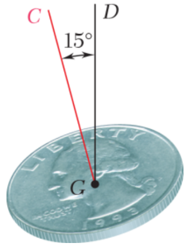
$$I' - I = -0.0033I,$$

$$n = -\frac{0.0033}{0.9967}\omega_2 = -0.003311\omega_2$$

$$\text{period} = \frac{2\pi}{|n|} = \frac{1}{0.003311}\frac{2\pi}{\omega_2} = 302.03\frac{2\pi}{\omega_2}$$

$$= (302.03)(24\text{ h}) = 7248\text{ h}$$

period = 302 days ◀



PROBLEM 18.124

A coin is tossed into the air. It is observed to spin at the rate of 600 rpm about an axis *GC* perpendicular to the coin and to precess about the vertical direction *GD*. Knowing that *GC* forms an angle of 15° with *GD*, determine (*a*) the angle that the angular velocity $\boldsymbol{\omega}$ of the coin forms with *GD*, (*b*) the rate of precession of the coin about *GD*.

SOLUTION

Moments of inertia:
$$I = \frac{1}{2}mr^2$$
$$I' = \frac{1}{4}mr^2$$

Euler angle θ for steady precession: $\theta = 15°$

For axisymmetric body under no force, Equation (18.49) gives for the body cone angle:

$$\tan\gamma = \frac{I}{I'}\tan\theta = 2\tan 15° \qquad \gamma = 28.187°$$

(*a*) Angular velocity: $\boldsymbol{\omega} = \omega(-i\sin\gamma + \mathbf{k}\cos\gamma)$

Its projection onto the vertical direction is

$$\omega\cos\beta = \boldsymbol{\omega}\cdot\mathbf{k} = \omega(-i\sin\gamma + \mathbf{k}\cos\gamma)\cdot(-\sin\theta\mathbf{i} + \cos\theta\mathbf{k})$$
$$= \omega(\sin\gamma\sin\theta + \cos\gamma\cos\theta) = \omega\cos(\gamma - \theta)$$

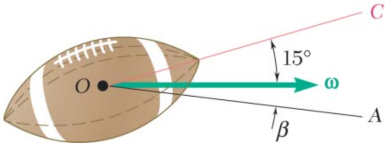
$$\cos\beta = \cos(\gamma - \theta) \qquad \beta = |\gamma - \theta| = 28.187° - 15° = 13.187°$$

Angle between $\boldsymbol{\omega}$ and vertical direction *GD*: $\beta = 13.19°$ ◀

(*b*)
$$\tan\gamma = -\frac{\omega_x}{\omega_z} = \frac{\dot{\varphi}\sin\theta}{\dot{\varphi}\cos\theta + \dot{\psi}}$$

from which
$$\dot{\varphi} = \frac{\dot{\psi}\sin\gamma}{\sin(\theta - \gamma)} = \frac{\dot{\psi}\sin 28.187°}{-\sin 13.187°} = -2.0705\dot{\psi}$$

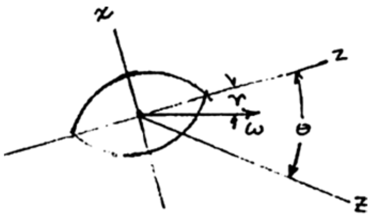
With $\dot{\psi} = 600$ rpm, $\dot{\varphi} = (-2.0705)(600)$ $|\dot{\varphi}| = 1242$ rpm (retrograde) ◀



PROBLEM 18.125

The angular velocity vector of a football which has just been kicked is horizontal, and its axis of symmetry *OC* is oriented as shown. Knowing that the magnitude of the angular velocity is 200 rpm and that the ratio of the axis and transverse moments of inertia is $I/I' = \frac{1}{3}$, determine (*a*) the orientation of the axis of precession *OA*, (*b*) the rates of precession and spin.

SOLUTION



$$\tan\gamma = -\frac{\omega_x}{\omega_z}$$

$$\gamma = 15°$$

For steady precession with no force,

$$\tan\theta = \frac{I'}{I}\tan\gamma$$

$$= 3\tan 15°$$

$$\theta = 38.794°$$

(*a*) $\beta = \theta - \gamma = 38.794 - 15°$ $\beta = 23.8°$ ◀

(*b*) $\omega_x = -\dot{\varphi}\sin\theta = -\omega\sin\gamma$

$$\dot{\varphi} = \frac{\omega\sin\gamma}{\sin\theta} = \frac{(200\text{ rpm})\sin 15°}{\sin(38.794°)}$$

$$= 82.621\text{ rpm}$$

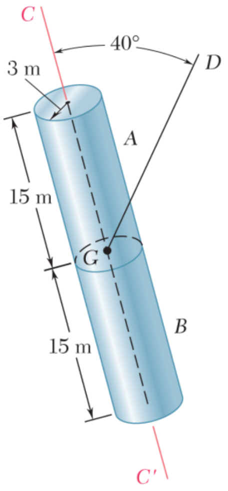
precession: $\dot{\varphi} = 82.6$ rpm ◀

$$\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta = \omega\cos\gamma$$

$$\dot{\psi} = \omega\cos\gamma - \dot{\varphi}\cos\theta$$

$$= 200\cos 15° - 82.621\cos 38.794°$$

spin: $\dot{\psi} = 128.8$ rpm ◀



PROBLEM 18.126

A space station consists of two sections *A* and *B* of equal masses, which are rigidly connected. Each section is dynamically equivalent to a homogeneous cylinder of length 15 m and radius 3 m. Knowing that the station is precessing about the fixed direction *GD* at the constant rate of 2 rev/h, determine the rate of spin of the station about its axis of symmetry *CC*′.

SOLUTION

For entire station:

$$I' = \frac{1}{12}m\left(3a^2 + L^2\right), \quad I = \frac{1}{2}ma^2$$

$$\frac{I}{I'} = \frac{6a^2}{3a^2 + L^2} = \frac{(6)(3)^2}{(3)(3)^2 + (30)^2} = 0.0582524$$

From Equation (18.49),

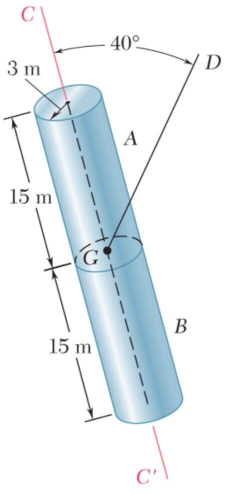
$$\tan\gamma = \frac{I}{I'}\tan\theta = 0.0582524\tan 40° \qquad \gamma = 2.7984°$$

$$\omega_x = -\dot{\varphi}\sin\theta = -(2\text{ rev/h})\sin 40° = -1.28557\text{ rev/h}$$

$$\omega_z = -\frac{\omega_x}{\tan\gamma} = \frac{1.28557}{\tan\gamma} = 26.3005\text{ rev/h}$$

$$\dot{\psi} = \omega_z - \dot{\varphi}\cos\theta = 26.3005 - 2\cos 40°$$

$$\dot{\psi} = 24.8\text{ rev/h} \blacktriangleleft$$



PROBLEM 18.127

If the connection between sections *A* and *B* of the space station of Prob. 18.126 is severed when the station is oriented as shown and if the two sections are gently pushed apart along their common axis of symmetry, determine (*a*) the angle between the spin axis and the new precession axis of section *A*, (*b*) the rate of precession of section *A*, (*c*) its rate of spin.

SOLUTION

For entire station:

$$I' = \frac{1}{12}m\left(3a^2 + L^2\right), \qquad I = \frac{1}{2}ma^2$$

$$\frac{I}{I'} = \frac{6a^2}{3a^2 + L^2} = \frac{(6)(3)^2}{(3)(3)^3 + (30)^2} = 0.0582524$$

From Equation (18.49),

$$\tan\gamma = \frac{I}{I'}\tan\theta = 0.0582524\tan 40° \qquad \gamma = 2.7984°$$

$$\omega_x = -\dot{\varphi}\sin\theta = -(2\text{ rev/h})\sin 40° = -1.28557\text{ rev/h}$$

$$\omega_z = -\frac{\omega_x}{\tan\gamma} = \frac{1.28557}{\tan\gamma} = 26.3005\text{ rev/h}$$

$$\dot{\psi} = \omega_z - \dot{\varphi}\cos\theta = 26.3005 - 2\cos 40° \qquad \dot{\psi} = 24.8\text{ rev/h}$$

During severing, the angular momentum about *G* remains constant, thus $\boldsymbol{\omega}$ and γ do not change.

For each section:

$$I' = \frac{1}{12}m\left(3a^2 + l^2\right), \qquad I = \frac{1}{2}ma^2$$

$$\frac{I}{I'} = \frac{6a^2}{3a^2 + l^2} = \frac{(6)(3)^2}{(3)(3)^2 + (15)^2} = 0.214286$$

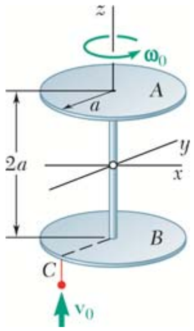
From Equation (18.49),

$$\tan\gamma = \frac{I}{I'}\tan\theta$$

(*a*) $\tan\theta = \frac{I'}{I}\tan\gamma = \frac{\tan 2.7984°}{0.214286}, \qquad \theta = 12.8497°$ $\qquad \theta = 12.85°$ ◀

(*b*) $\omega_x = -\dot{\varphi}\sin\theta \qquad \dot{\varphi} = -\frac{\omega_x}{\sin\theta} = \frac{1.28557}{\sin 12.8497°}$ $\qquad \dot{\varphi} = 5.78\text{ rev/h}$ ◀

(*c*) $\dot{\psi} = \omega_z - \dot{\varphi}\cos\theta = 26.3005 - 5.7806\cos 12.8497°$ $\qquad \dot{\psi} = 20.7\text{ rev/h}$ ◀



PROBLEM 18.128

Solve Sample Problem 18.6, assuming that the meteorite strikes the satellite at C with a velocity $\mathbf{v}_0 = (2000\text{ m/s})\mathbf{i}$.

PROBLEM 18.6 A space satellite of mass m is known to be dynamically equivalent to two thin disks of equal mass. The disks are of radius a = 800 mm and are rigidly connected by a light rod of length $2a$. Initially the satellite is spinning freely about its axis of symmetry at the rate $\omega_0 = 60$ rpm. A meteorite, of mass $m_0 = m/1000$ and traveling with a velocity $\mathbf{v}_0$ of 2000 m/s relative to the satellite, strikes the satellite and becomes embedded at C. Determine (*a*) the angular velocity of the satellite immediately after impact, (*b*) the precession axis of the ensuing motion, (*c*) the rates of precession and spin of the ensuing motion.

SOLUTION

(*a*) *Angular velocity after impact.*

From Sample Problem 18.6:

$$I = I_z = \frac{1}{2}ma^2 \qquad I' = I_x = I_y = \frac{5}{4}ma^2$$

Conservation of angular momentum: Angular momentum after impact:

$$\begin{aligned}\mathbf{H}_G &= \mathbf{r}_C \times m_0\mathbf{v}_0 + I\omega_0\mathbf{k}\\ &= (-a\mathbf{j} - a\mathbf{k}) \times m_0 v_0 \mathbf{i} + I\omega_0\mathbf{k}\\ &= -am_0v_0\mathbf{j} + (I\omega_0 + am_0v_0)\mathbf{k}\end{aligned}$$

Data:

$$a = 800\text{ mm} = 0.8\text{ m}$$

$$\omega_0 = 60\text{ rpm} = 2\pi\text{ rad/s}$$

$$v_0 = 2000\text{ m/s}, \quad m_0 = \frac{m}{1000}$$

$$\omega_x = \frac{(H_G)_x}{I_x} = 0$$

$$\omega_y = \frac{(H_G)_y}{I_y} = -\frac{am_0v_0}{\frac{5}{4}ma^2} = -\frac{4}{5}\frac{m_0}{m}\frac{v_0}{a} = -\frac{4}{5}\left(\frac{1}{1000}\right)\frac{2000}{0.8} = -2\text{ rad/s}$$

$$\omega_z = \frac{(H_G)_z}{I_z} = \omega_0 + \frac{am_0v_0}{\frac{1}{2}ma^2} = \omega_0 + 2\frac{m_0}{m}\frac{v_0}{a}$$

$$= 2\pi + 2\left(\frac{1}{1000}\right)\frac{2000}{0.8} = 11.2832\text{ rad/s}$$

$$\boldsymbol{\omega} = -(2.00\text{ rad/s})\mathbf{j} + (11.28\text{ rad/s})\mathbf{k}$$

$$\omega = \sqrt{(2)^2 + (11.2832)^2} = 11.4591\text{ rad/s} = 109.426\text{ rpm}$$

$\omega = 109.4$ rpm ◀

PROBLEM 18.128 (Continued)

$$\tan\gamma = \left|\frac{\omega_y}{\omega_z}\right| = \frac{2}{11.2832} \qquad \gamma = 10.0515^\circ \qquad \gamma_x = 90^\circ,\quad \gamma_y = 100.05^\circ,\quad \gamma_z = 10.05^\circ \ ◀$$

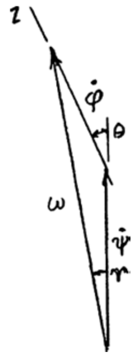
(*b*) *Precession axis*.

$$\tan\theta = \frac{I'}{I}\tan\gamma$$

$$= \left(\frac{5}{4}\right)(2)\frac{2}{11.2832}$$

$$\theta = 23.900^\circ \qquad \theta_x = 90^\circ,\quad \theta_y = 113.9^\circ,\quad \theta_z = 23.9^\circ \ ◀$$

(*c*) *Rates of precession and spin*.



$$\theta - \gamma = 13.8484^\circ$$

Law of sines.

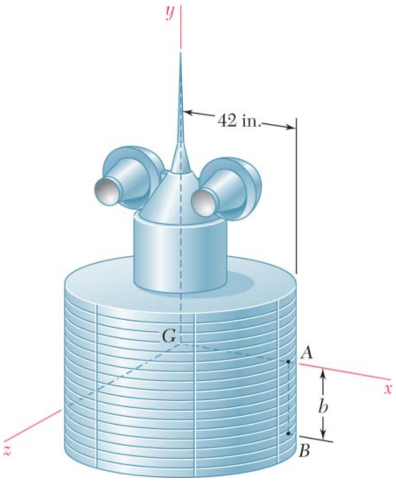
$$\frac{\dot{\varphi}}{\sin\gamma} = \frac{\dot{\psi}}{\sin(\theta-\gamma)} = \frac{\omega}{\sin\theta}$$

$$\dot{\varphi} = \frac{\omega\sin\gamma}{\sin\theta} = \frac{109.4\sin 10.05^\circ}{\sin 23.9^\circ}$$

precession: $\dot{\varphi} = 47.1$ rpm ◀

$$\dot{\psi} = \frac{\omega\sin(\theta-\gamma)}{\sin\theta} = \frac{109.4\sin 13.85^\circ}{\sin 23.9^\circ}$$

spin: $\dot{\psi} = 64.6$ rpm ◀



PROBLEM 18.129

An 800-lb geostationary satellite is spinning with an angular velocity $\omega_0 = (1.5 \text{ rad/s})\mathbf{j}$ when it is hit at B by a 6-oz meteorite traveling with a velocity $\mathbf{v}_0 = -(1600 \text{ ft/s})\mathbf{i} + (1300 \text{ ft/s})\mathbf{j} + (4000 \text{ ft/s})\mathbf{k}$ relative to the satellite. Knowing that $b = 20$ in. and that the radii of gyration of the satellite are $\bar{k}_x = \bar{k}_z = 28.8$ in. and $\bar{k}_y = 32.4$ in., determine the precession axis and the rates of precession and spin of the satellite after the impact.

SOLUTION

Mass of satellite: $m = \frac{W}{g} = \frac{800}{32.2} = 24.845 \text{ lb}\cdot\text{s}^2/\text{ft}$

Principal moments of inertia:

$$\bar{I}_x = m k_x^2 = (24.845)\left(\frac{28.8}{12}\right)^2$$

$$= 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_y = m k_y^2 = (24.845)\left(\frac{32.4}{12}\right)^2$$

$$= 181.118 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_z = \bar{I}_x = 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Mass of meteorite: $m' = \frac{6}{(16)(32.2)} = 0.011649 \text{ lb}\cdot\text{s}^2/\text{ft}$

Initial momentum of meteorite:

$$m'\mathbf{v}_0 = (0.011649)(-1600\mathbf{i} + 1300\mathbf{j} + 4000\mathbf{k})$$

$$= -(18.633 \text{ lb}\cdot\text{s})\mathbf{i} + (15.140 \text{ lb}\cdot\text{s})\mathbf{j} + (46.584 \text{ lb}\cdot\text{s})\mathbf{k}$$

Assume that the position of the mass center of the satellite plus the meteorite is essentially that of the satellite alone.

Position of Point B relative to the mass center:

$$\mathbf{r}_{B/G} = \left(\frac{42}{12}\mathbf{i} - \frac{20}{12}\mathbf{j}\right)$$

$$= (3.5 \text{ ft})\mathbf{i} - (1.66667 \text{ ft})\mathbf{j}$$

Angular velocity of satellite before impact:

$$\boldsymbol{\omega}_0 = (1.5 \text{ rad/s})\mathbf{j} \qquad (\omega_0)_x = (\omega_0)_z = 0, \quad (\omega_0)_y = 1.5 \text{ rad/s}$$

Angular momentum of satellite-meteorite system before impact:

$$(\mathbf{H}_G)_0 = \bar{I}_y\omega_0\mathbf{j} + \mathbf{r}_{B/G} \times (m'\mathbf{v}_0)$$

$$= (181.118)(1.5)\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3.5 & -1.66667 & 0 \\ -18.633 & 15.140 & 46.584 \end{vmatrix}$$

$$= -(77.64 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{i} + (108.637 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{j} + (21.935 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{k}$$

Principle of impulse and momentum for satellite-meteorite system. Moments about G:

$$(\mathbf{H}_G)_1 = (\mathbf{H}_G)_0 = \mathbf{H}_G$$

Angular velocity immediately after impact.

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

Neglect the mass of the meteorite.

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$\omega_x = \frac{(H_G)_x}{\bar{I}_x} = \frac{-77.64}{143.106} = -0.54253 \text{ rad/s}$$

$$\omega_y = \frac{(H_G)_y}{\bar{I}_y} = \frac{108.637}{181.118} = 0.59981 \text{ rad/s}$$

$$\omega_z = \frac{(H_G)_z}{\bar{I}_z} = \frac{21.934}{143.106} = 0.15327 \text{ rad/s}$$

$$\boldsymbol{\omega} = -(0.54253 \text{ rad/s})\mathbf{i} + (0.59981 \text{ rad/s})\mathbf{j} + (0.15327 \text{ rad/s})\mathbf{k}$$

$$\omega = \sqrt{(0.54253)^2 + (0.59981)^2 + (0.15327)^2} = 0.82317 \text{ rad/s}$$

$$H_G = \sqrt{(77.64)^2 + (108.637)^2 + (21.935)^2} = 135.319 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

Motion after impact. Since the moments of inertia $\bar{I}_x$ and $\bar{I}_z$ are equal, the body moves as an axisymmetrical body with the y axis as the symmetry axis.

Moment of inertia about the symmetry axis: $I = \bar{I}_y = 181.118 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$

Moment of inertia about a transverse axis through G: $I' = \bar{I}_x = \bar{I}_z = 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$

The motion is a steady precession $\dot{\phi}$ about the precession axis together with a steady spin $\dot{\psi}$ about the spin or symmetry axis. Since $I > I'$, the precession is retrograde.

Precession axis. The precession axis is directed along the angular momentum vector $\mathbf{H}_G$, which remains fixed. Immediately after impact, its direction cosines relative to the body axes x, y, z are:

$$\cos\theta_x = \frac{(H_G)_x}{H_G} = \frac{-77.64}{135.319} = -0.57376 \qquad \theta_x = 125.0° ◀$$

$$\cos\theta_y = \frac{(H_G)_y}{H_G} = \frac{108.637}{135.319} = 0.80282 \qquad \theta_y = 36.6° ◀$$

$$\cos\theta_z = \frac{(H_G)_z}{H_G} = \frac{21.935}{135.319} = 0.16210 \qquad \theta_z = 80.7° ◀$$

The angle θ between the spin axis (y axis) and the precession axis remains constant.

$$\theta = \theta_y = 36.600°$$

The angle γ between the angular velocity vector and the spin axis (y axis) is

$$\cos\gamma = \frac{\omega_y}{\omega} = \frac{0.59981}{0.82317} \qquad \gamma = 43.226°$$

The angle γ could also have been calculated from

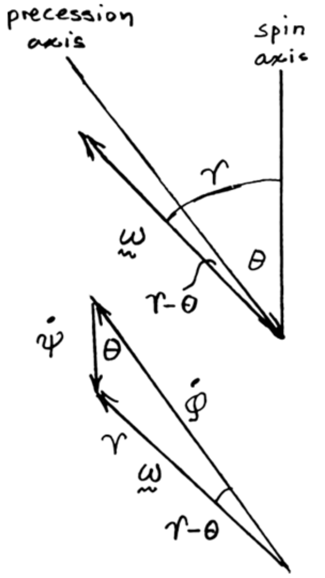
$$\tan\gamma = \frac{I}{I'}\tan\beta = \frac{181.118}{143.106}\tan 36.600°$$

The angle between the precession axis and the angular velocity vector is

$$\gamma - \theta = 6.626°$$

Rates of precession and spin.

Set up the triangle of vector addition for the components of angular velocity. Apply the law of sines.



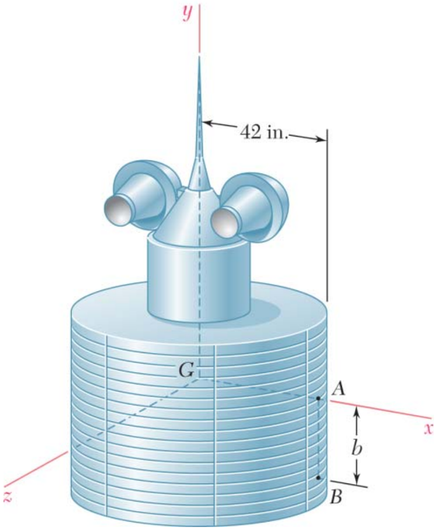
$$\frac{\omega}{\sin\theta} = \frac{\dot\psi}{\sin(\gamma-\theta)} = \frac{\dot\varphi}{\sin\gamma}$$

$$\frac{0.82317}{\sin 36.600°} = \frac{\dot\psi}{\sin 6.626°} = \frac{\dot\varphi}{\sin 43.226°}$$

Rate of precession: $\dot\varphi = 0.946$ rad/s ◀

Rate of spin: $\dot\psi = 0.1593$ rad/s ◀

Since $\gamma > \theta$, precession is retrograde. ◀



PROBLEM 18.130

Solve Problem 18.129, assuming that the meteorite hits the satellite at *A* instead of *B*.

PROBLEM 18.129 An 800-lb geostationary satellite is spinning with an angular velocity $\omega_0 = (1.5 \text{ rad/s})\mathbf{j}$ when it is hit at *B* by a 6-oz meteorite traveling with a velocity $\mathbf{v}_0 = -(1600 \text{ ft/s})\mathbf{i} + (1300 \text{ ft/s})\mathbf{j} + (4000 \text{ ft/s})\mathbf{k}$ relative to the satellite. Knowing that $b = 20$ in. and that the radii of gyration of the satellite are $\bar{k}_x = \bar{k}_z = 28.8$ in. and $\bar{k}_y = 32.4$ in., determine the precession axis and the rates of precession and spin of the satellite after the impact.

SOLUTION

Mass of satellite: $m = \dfrac{W}{g} = \dfrac{800}{32.2} = 24.845 \text{ lb}\cdot\text{s}^2/\text{ft}$

Principal moments of inertia: $\bar{I}_x = m k_x^2 = (24.845)\left(\dfrac{28.8}{12}\right)^2 = 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$

$$\bar{I}_y = m k_y^2 = (24.845)\left(\frac{32.4}{12}\right)^2 = 181.118 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_z = \bar{I}_x = 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Mass of meteorite: $m' = \dfrac{6}{(16)(32.2)} = 0.011649 \text{ lb}\cdot\text{s}^2/\text{ft}$

Initial momentum of meteorite:

$$\begin{aligned} m'\mathbf{v}_0 &= (0.011649)(-1600\mathbf{i} + 1300\mathbf{j} + 4000\mathbf{k}) \\ &= -(18.633 \text{ lb}\cdot\text{s})\mathbf{i} + (15.140 \text{ lb}\cdot\text{s})\mathbf{j} + (46.584 \text{ lb}\cdot\text{s})\mathbf{k} \end{aligned}$$

Assume that the position of the mass center of the satellite plus the meteorite is essentially that of the satellite alone.

Position of Point *A* relative to the mass center:

$$\mathbf{r}_{A/G} = \frac{42}{12}\mathbf{i} = (3.5 \text{ ft})\mathbf{i}$$

Angular velocity of satellite before impact:

$$\boldsymbol{\omega}_0 = (1.5 \text{ rad/s})\mathbf{j}, \qquad (\omega_0)_x = (\omega_0)_z = 0, \quad (\omega_0)_y = 1.5 \text{ rad/s}$$

PROBLEM 18.130 (Continued)

Angular momentum of satellite-meteorite system before impact:

$$(\mathbf{H}_G)_0 = \bar{I}_y\omega_0\mathbf{j} + \mathbf{r}_{A/G} \times (m'\mathbf{v}_0)$$

$$= (181.118)(1.5)\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3.5 & 0 & 0 \\ -18.633 & 15.140 & 46.584 \end{vmatrix}$$

$$= (108.637 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{j} + (52.99 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{k}$$

Principle of impulse and momentum for satellite-meteorite system. Moments about G:

$$(\mathbf{H}_G)_1 = (\mathbf{H}_G)_0 = \mathbf{H}_G$$

Angular velocity immediately after impact.

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

Neglect the mass of the meteorite.

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$\omega_x = \frac{(H_G)_x}{\bar{I}_x} = 0$$

$$\omega_y = \frac{(H_G)_y}{\bar{I}_y} = \frac{108.637}{181.118} = 0.59981 \text{ rad/s}$$

$$\omega_z = \frac{(H_G)_z}{\bar{I}_z} = \frac{52.99}{143.106} = 0.37028 \text{ rad/s}$$

$$\boldsymbol{\omega} = (0.59981 \text{ rad/s})\mathbf{j} + (0.37028 \text{ rad/s})\mathbf{k}$$

$$\omega = \sqrt{(0.59981)^2 + (0.37028)^2} = 0.70490 \text{ rad/s}$$

$$H_G = \sqrt{(108.637)^2 + (52.99)^2} = 120.872 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

Motion after impact. Since the moments of inertia $\bar{I}_x$ and $\bar{I}_z$ are equal, the body moves as an axisymmetrical body with the y axis as the symmetry axis.

Moment of inertia about the symmetry axis:

$$I = \bar{I}_y = 181.118 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Moment of inertia about a transverse axis through G:

$$I' = \bar{I}_x = \bar{I}_z = 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

The motion is a steady precession $\dot{\phi}$ about the precession axis together with a steady spin $\dot{\psi}$ about the spin or symmetry axis. Since $I > I'$, the precession is retrograde.

PROBLEM 18.130 (Continued)

Precession axis. The precession axis is directed along the angular momentum vector $\mathbf{H}_G$, which remains fixed. Immediately after impact, its direction cosines relative to the body axes x, y, z are:

$$\cos\theta_x = \frac{(H_G)_x}{H_G} = 0 \qquad \theta_x = 90.0° ◀$$

$$\cos\theta_y = \frac{(H_G)_y}{H_G} = \frac{108.637}{120.872} = 0.89878 \qquad \theta_y = 26.0° ◀$$

$$\cos\theta_z = \frac{(H_G)_z}{H_G} = \frac{52.99}{120.872} = 0.43840 \qquad \theta_z = 64.0° ◀$$

The angle θ between the spin axis (y axis) and the precession axis remains constant.

$$\theta = \theta_y = 26.002°$$

The angle γ between the angular velocity vector and the spin axis (y axis) is

$$\cos\gamma = \frac{\omega_y}{\omega} = \frac{0.59981}{0.70490} \qquad \gamma = 31.689°$$

The angle γ could also have been calculated from

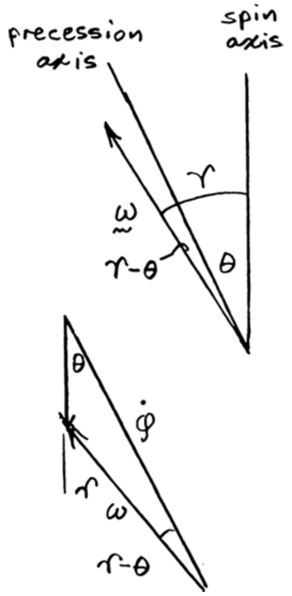
$$\tan\gamma = \frac{I}{I'}\tan\theta = \frac{181.118}{143.106}\tan 26.002°$$

The angle between the precession axis and the angular velocity vector is

$$\gamma - \theta = 5.687°$$

Rates of precession and spin.

Set up the triangle of vector addition for the components of angular velocity. Apply the law of sines.



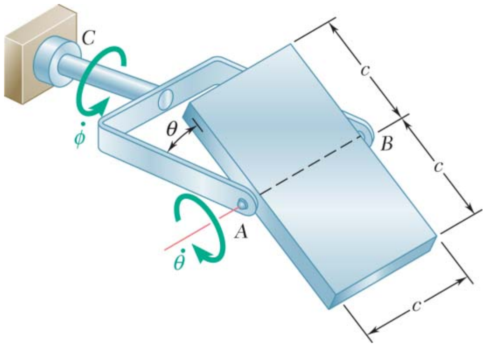
$$\frac{\omega}{\sin\theta} = \frac{\dot{\psi}}{\sin(\gamma-\theta)} = \frac{\dot{\varphi}}{\sin\gamma}$$

$$\frac{0.70490}{\sin 26.002°} = \frac{\dot{\psi}}{\sin 5.687°} = \frac{\dot{\varphi}}{\sin 31.689°}$$

Rate of precession: $\dot{\varphi} = 0.844$ rad/s ◀

Rate of spin: $\dot{\psi} = 0.1593$ rad/s ◀

Since $\gamma > \theta$, the precession is retrograde. ◀



PROBLEM 18.131

A homogeneous rectangular plate of mass m and sides c and $2c$ is held at A and B by a fork-ended shaft of negligible mass, which is supported by a bearing at C. The plate is free to rotate about AB, and the shaft is free to rotate about a horizontal axis through C. Knowing that, initially, $\theta_0 = 40°$, $\dot\theta_0 = 0$, and $\dot\phi_0 = 10$ rad/s, determine for the ensuring motion (a) the range of values of θ, (b) the minimum value of $\dot\phi$, (c) the maximum value of $\dot\theta$.

SOLUTION

Let the fixed Z axis lie along the axle of the fork-ended shaft. Let the axes $Gxyz$ be attached at the mass center with x perpendicular to the plate, y along the axle AB and z parallel to the long edges of the plate.

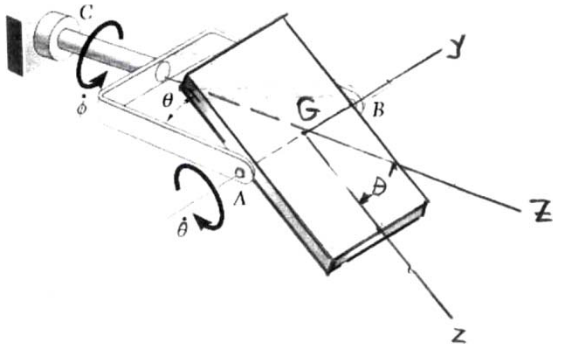
Angular velocity vector: $\dot{\boldsymbol{\omega}} = \dot\varphi\mathbf{k} + \dot\theta\mathbf{j}$

$$= \dot\varphi\cos\theta\,\mathbf{i} + \dot\theta\,\mathbf{j} + \dot\varphi\sin\theta\,\mathbf{k}$$

Conservation of angular momentum.

Since plate is free to rotate about Z axis,

$$H_Z = \text{constant} \qquad (1)$$



But
$$H_Z = H_x\cos\theta + H_z\sin\theta$$
$$H_Z = I_x\omega_x\cos\theta + I_z\omega_z\sin\theta$$
$$= \frac{1}{12}mc^2\dot\phi\cos^2\theta + \frac{5}{12}mc^2\dot\phi\sin^2\theta$$
$$= \frac{1}{12}mc^2\dot\phi(\cos^2\theta + 5\sin^2\theta)$$
$$= \frac{1}{12}mc^2\dot\phi(1 + 4\sin^2\theta)$$

Using the initial conditions, Eq. (1) yields

$$\dot\phi(1+4\sin^2\theta) = \dot\phi_0(1+4\sin^2\theta_0) \qquad (2)$$

Conservation of energy.

Since no work is done, we have $T = \text{constant}$ (3)

where
$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$
$$T = \frac{1}{2}\left(\frac{1}{12}mc^2\dot\phi^2\cos^2\theta + \frac{1}{3}mc^2\dot\theta^2 + \frac{5}{12}mc^2\dot\phi^2\sin^2\theta\right)$$
$$= \frac{1}{24}mc^2[4\dot\theta^2 + \dot\phi^2(\cos^2\theta + 5\sin^2\theta)]$$
$$= \frac{1}{24}mc^2[4\dot\theta^2 + \dot\phi^2(1+4\sin^2\theta)]$$

PROBLEM 18.131 (Continued)

Using the initial conditions, including $\dot{\theta}_0 = 0$, Eq. (3) yields

$$4\dot{\theta}^2 + \dot{\phi}^2(1+4\sin^2\theta) = \dot{\phi}_0^2(1+4\sin^2\theta_0) \qquad (4)$$

(*a*) With $\theta_0 = 40°$ and $\dot{\phi}_0 = 10$ rad/s in. Eqs. (2) and (4),

$$\dot{\phi}(1+4\sin^2\theta) = 26.527$$

$$\dot{\phi} = \frac{26.527}{1+4\sin^2\theta} \qquad (2')$$

$$4\dot{\theta}^2 + \dot{\phi}^2(1+4\sin^2\theta) = 265.27$$

Eliminate $\dot{\phi}$ and solve for $\dot{\theta}^2$:

$$4\dot{\theta}^2 = 265.27 - \frac{(26.527)^2}{1+4\sin^2\theta} \qquad (5)$$

For $\dot{\theta}^2 = 0, \quad 1+4\sin^2\theta = \frac{(26.527)^2}{265.27} = 2.6527$

$$\sin^2\theta = 0.4132 \quad \sin\theta = 0.6428$$

From which $\theta = 40°$ and $140°$ $\qquad 40° < \theta < 140°$ ◀

(*b*) From Eq. (2′), $\dot{\phi}$ is minimum for $\theta = 90°$.

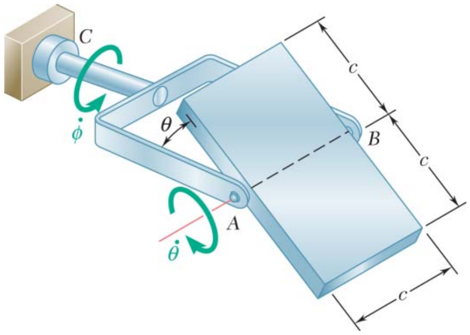
$\dot{\phi}_{\text{min}} = 5.3054$ $\qquad \dot{\phi}_{\text{min}} = 5.31$ rad/s ◀

(*c*) From Eq. (5),

$$\dot{\theta}^2 = 66.318 - \frac{175.92}{1+4\sin^2\theta}$$

$\dot{\theta}^2$ is maximum for $\theta = 90°$.

$\dot{\theta}^2_{\text{max}} = 31.134\ \text{rad}^2/\text{s}^2$ $\qquad \dot{\theta}_{\text{max}} = 5.58$ rad/s ◀



PROBLEM 18.132

A homogeneous rectangular plate of mass m and sides c and $2c$ is held at A and B by a fork-ended shaft of negligible mass which is supported by a bearing at C. The plate is free to rotate about AB, and the shaft is free to rotate about a horizontal axis through C. Initially the plate lies in the plane of the fork $(\theta_0 = 0)$ and the shaft has an angular velocity $\dot{\phi}_0 = 10\,\text{rad/s}$. If the plate is slightly disturbed, determine for the ensuring motion (*a*) the minimum value of $\dot{\phi}$, (*b*) the maximum value of $\dot{\theta}$.

SOLUTION

Let the fixed Z axis lie along the axle of the fork-ended shaft. Let the axes $Gxyz$ be attached at the mass center with x perpendicular to the plate, y along the axle AB and z parallel to the long edges of the plate.

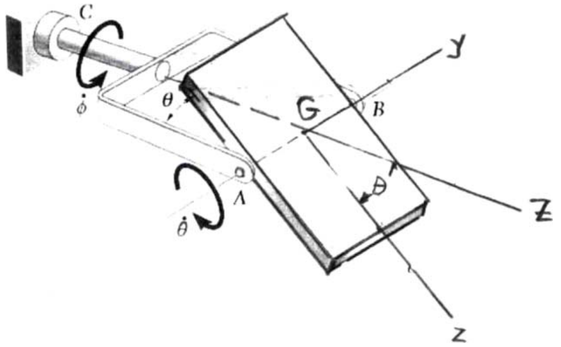
Angular velocity vector: $\dot{\boldsymbol{\omega}} = \dot{\varphi}\mathbf{k} + \dot{\theta}\mathbf{j}$

$$= \dot{\varphi}\cos\theta\,\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\varphi}\sin\theta\,\mathbf{k}$$

Conservation of angular momentum.

Since plate is free to rotate about Z axis,

$$H_Z = \text{constant} \qquad (1)$$



But $H_Z = H_x\cos\theta + H_z\sin\theta$

$$H_Z = I_x\omega_x\cos\theta + I_z\omega_z\sin\theta$$

$$= \frac{1}{12}mc^2\dot{\phi}\cos^2\theta + \frac{5}{12}mc^2\dot{\phi}\sin^2\theta$$

$$= \frac{1}{12}mc^2\dot{\phi}(\cos^2\theta + 5\sin^2\theta)$$

$$= \frac{1}{12}mc^2\dot{\phi}(1 + 4\sin^2\theta)$$

Using the initial conditions, Eq. (1) yields

$$\dot{\phi}(1 + 4\sin^2\theta) = \dot{\phi}_0(1 + 4\sin^2\theta_0) \qquad (2)$$

Conservation of energy.

Since no work is done, we have $T = \text{constant}$ (3)

where $T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$

$$T = \frac{1}{2}\left(\frac{1}{12}mc^2\dot{\phi}^2\cos^2\theta + \frac{1}{3}mc^2\dot{\theta}^2 + \frac{5}{12}mc^2\dot{\phi}^2\sin^2\theta\right)$$

$$= \frac{1}{24}mc^2[4\dot{\theta}^2 + \dot{\phi}^2(\cos^2\theta + 5\sin^2\theta)]$$

$$= \frac{1}{24}mc^2[4\dot{\theta}^2 + \dot{\phi}^2(1 + 4\sin^2\theta)]$$

PROBLEM 18.132 (Continued)

Using the initial conditions, including $\dot{\theta}_0 = 0$, Eq. (3) yields

$$4\dot{\theta}^2 + \dot{\phi}^2(1+4\sin^2\theta) = \dot{\phi}_0^2(1+4\sin^2\theta_0) \tag{4}$$

(*a*) With $\theta_0 = 0, \quad \dot{\phi}_0 = 10 \text{ rad/s}$

Eq. (2) yields $\dot{\phi} = \dfrac{10}{1+4\sin^2\theta}$

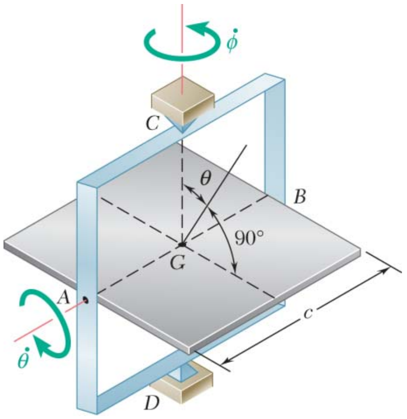
$\dot{\phi}$ is minimum for $\theta = 90°$: $\dot{\phi}_{\text{min}} = 2.00 \text{ rad/s}$ ◀

(*b*) Eq. (4) yields $4\dot{\theta}^2 = 100 - \dot{\phi}^2(1+4\sin^2\theta) = 100\left(1 - \dfrac{1}{1+4\sin^2\theta}\right)$

$\dot{\theta}^2$ is largest for $\theta = 90°$:

$$4\dot{\theta}_{\text{max}}^2 = 100\left(1-\frac{1}{5}\right)$$

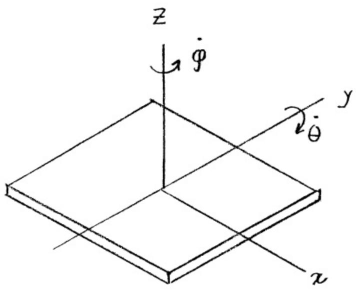
$\dot{\theta}_{\text{max}}^2 = 20$ $\dot{\theta}_{\text{max}} = 4.47 \text{ rad/s}$ ◀

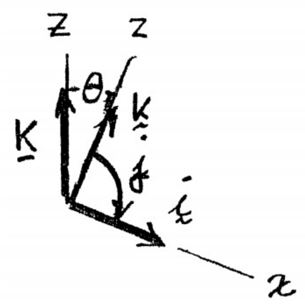


PROBLEM 18.133

A homogeneous square plate of mass m and side c is held at points A and B by a frame of negligible mass which is supported by bearings at points C and D. The plate is free to rotate about AB, and the frame is free to rotate about the vertical CD. Knowing that, initially, $\theta_0 = 45°$, $\dot\theta_0 = 0$, and $\dot\phi_0 = 8$ rad/s, determine for the ensuing motion (*a*) the range of values of θ, (*b*) the minimum value of $\dot\phi$, (*c*) the maximum value of $\dot\theta$.

SOLUTION





Place the origin at the center of mass and let $Oxyz$ be a principal axis frame of reference with the y axis directed along the moving axle AB. Let the Z axis lie along the fixed axle. Useful unit vectors are **i**, **j**, and **k** along the x, y, z axes and **K** along the Z axis.

$$\mathbf{K} = -\mathbf{i}\sin\theta + \mathbf{k}\cos\theta$$

Angular velocity: $\boldsymbol{\omega} = \dot\phi\mathbf{K} + \dot\theta\mathbf{j}$

$$\omega = -\dot\phi\sin\theta\mathbf{i} + \dot\theta\mathbf{j} + \dot\phi\cos\theta\mathbf{k}$$

Moments of inertia: $I_x = e_1 I_y, \quad I_z = e_2 I_y$

Angular momentum about O:

$$\mathbf{H}_0 = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= I_y\left(-e_1\dot\phi\sin\theta\mathbf{i} + \dot\theta\mathbf{j} + e_2\dot\phi\cos\theta\mathbf{k}\right)$$

The moment about the fixed Z axis is zero, hence $\mathbf{H}_O\cdot\mathbf{K} = \text{constant}$.

$$\mathbf{H}_O\cdot\mathbf{K} = I_y\left(e_1\sin^2\theta + e_2\cos^2\theta\right)\dot\phi = I_y C_1$$

$$\dot\phi = \frac{C_1}{e_1\sin^2\theta + e_2\cos^2\theta} \qquad C_1 = \left(e_1\sin^2\theta_0 + e_2\cos^2\theta_0\right)\dot\phi_0$$

Twice the kinetic energy: $2T = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = \text{constant}$

$$2T = I_y\left[\left(e_1\sin^2\theta + e_2\cos^2\theta\right)\dot\phi^2 + \dot\theta^2\right]$$

$$= I_y\left(C_1\dot\phi^2 + \dot\theta^2\right) = I_y C_2$$

PROBLEM 18.133 (Continued)

$$\dot{\theta}^2 = C_2 - C_1\dot{\phi} \qquad C_2 = \dot{\theta}_0^2 + C_1\dot{\phi}_0$$

Data: $e_1 = 1, \qquad e_2 = 2 \qquad e_1 \sin^2\theta + e_2\cos^2\theta = 1 + \cos^2\theta$

$$\theta_0 = 45°, \qquad \dot{\theta}_0 = 0, \qquad \dot{\phi}_0 = 8 \text{ rad/s}$$

$$C_1 = \left(1 + \cos^2 45°\right)\dot{\phi}_0 = \left(1 + \frac{1}{2}\right)(8) = 12 \text{ rad/s} \qquad \dot{\phi} = \frac{12}{1 + \cos^2\theta}$$

$$C_2 = 0 + (12)(96) = 96 \ (\text{rad/s})^2$$

(*a*) $$\dot{\theta}^2 = 96 - 12\dot{\phi} = 96 - \frac{144}{1 + \cos^2\theta} \geq 0$$

$$\cos^2\theta > 0.5 \qquad \cos\theta > 0.70711$$

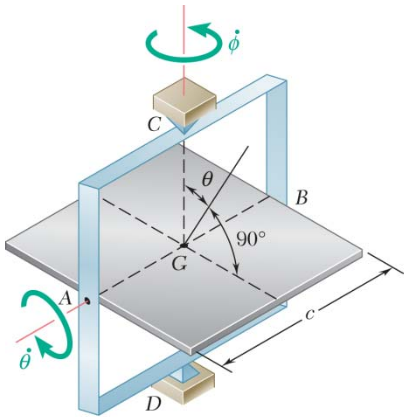
$-45° < \theta < 45°$ ◀

(*b*) $$\dot{\phi}_{\text{min}} = \frac{12}{1 + \cos 0} = 6 \text{ rad/s}$$

$\dot{\phi}_{\text{min}} = 6.00 \text{ rad/s}$ ◀

(*c*) $$\left(\dot{\theta}_{\text{max}}\right)^2 = C_2 - C_1\dot{\phi}_{\text{min}} = 96 - (12)(6) = 24 \ (\text{rad/s})^2$$

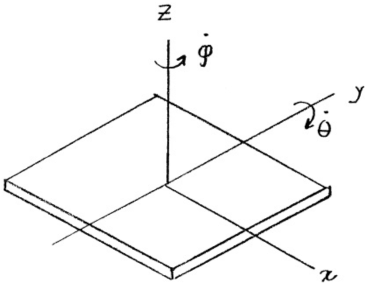
$\dot{\theta}_{\text{max}} = 4.90 \text{ rad/s}$ ◀



PROBLEM 18.134

A homogeneous square plate of mass m and side c is held at points A and B by a frame of negligible mass which is supported by bearings at points C and D. The plate is free to rotate about AB, and the frame is free to rotate about the vertical CD. Initially the plate lies in the plane of the frame $(\theta_0 = 90°)$ and the frame has an angular velocity $\dot{\phi}_0 = 8$ rad/s. If the plate is slightly disturbed, determine for the ensuing motion (*a*) the minimum value of $\dot{\phi}$, (*b*) the maximum value of $\dot{\theta}$.

SOLUTION



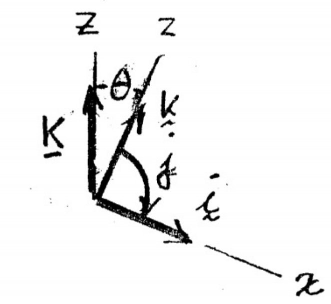
Place the origin at the center of mass and let $Oxyz$ be a principal axis frame of reference with the y axis directed along the moving axle AB. Let the Z axis lie along the fixed axle. Useful unit vectors are **i**, **j**, and **k** along the x, y, z axes and **K** along the Z axis.

$$\mathbf{K} = -\mathbf{i}\sin\theta + \mathbf{k}\cos\theta$$

Angular velocity: $\boldsymbol{\omega} = \dot{\phi}\mathbf{K} + \dot{\theta}\mathbf{j}$

$$\omega = -\dot{\phi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\phi}\cos\theta\mathbf{k}$$

Moments of inertia: $I_x = e_1 I_y, \quad I_z = e_2 I_y$



Angular momentum about O:

$$\mathbf{H}_0 = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= I_y\left(-e_1\dot{\phi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + e_2\dot{\phi}\cos\theta\mathbf{k}\right)$$

The moment about the fixed Z axis is zero, hence $\mathbf{H}_O\cdot\mathbf{K} = \text{constant}$.

$$\mathbf{H}_O\cdot\mathbf{K} = I_y\left(e_1\sin^2\theta + e_2\cos^2\theta\right)\dot{\phi} = I_y C_1$$

$$\dot{\phi} = \frac{C_1}{e_1\sin^2\theta + e_2\cos^2\theta} \qquad C_1 = \left(e_1\sin^2\theta_0 + e_2\cos^2\theta_0\right)\dot{\phi}_0$$

Twice the kinetic energy: $2T = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = \text{constant}$

$$2T = I_y\left[\left(e_1\sin^2\theta + e_2\cos^2\theta\right) + \dot{\phi}^2 + \dot{\theta}^2\right]$$

$$= I_y\left(C_1\dot{\phi}^2 + \dot{\theta}^2\right) = I_y C_2$$

PROBLEM 18.134 (Continued)

$$\dot{\theta}^2 = C_2 - C_1\dot{\phi} \qquad C_2 = \dot{\theta}_0^2 + C_1\dot{\phi}_0$$

Data: $e_1 = 1, \qquad e_2 = 2 \qquad e_1 \sin^2\theta + e_2\cos^2\theta = 1 + \cos^2\theta$

$$\theta_0 = 90°, \qquad \dot{\theta}_0 = 0, \qquad \dot{\phi}_0 = 8 \text{ rad/s}$$

$$C_1 = \left(1 + \cos^2 90°\right)(8) = 8 \text{ rad/s} \qquad \dot{\phi} = \frac{8}{1 + \cos^2\theta}$$

(*a*) $$\phi_{\min} = \frac{8}{1 + \cos^2\theta} = 4$$

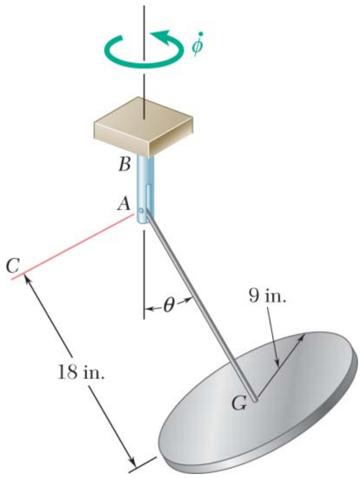
$$\dot{\phi}_{\min} = 4.00 \text{ rad/s} ◀$$

$$C_2 = 0 + (8)(8) = 64 \text{ rad/s}^2$$

$$\dot{\theta}^2 = C_2 - C_1\dot{\phi}$$

(*b*) $$\left(\dot{\theta}_{\max}\right)^2 = C_2 - C_1\dot{\phi}_{\min} = 64 - (8)(4) = 32 \ (\text{rad/s})^2$$

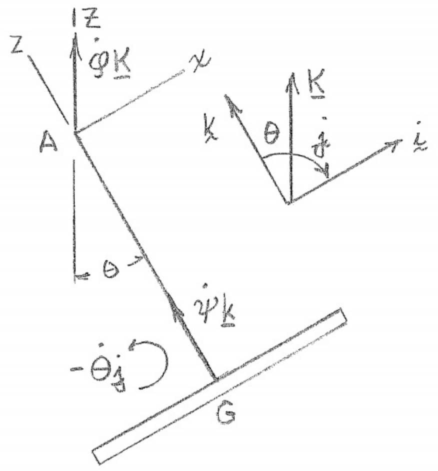
$$\dot{\theta}_{\max} = 5.66 \text{ rad/s} ◀$$



PROBLEM 18.135

A homogeneous disk of radius 9 in. is welded to a rod *AG* of length 18 in. and of negligible weight which is connected by a clevis to a vertical shaft *AB*. The rod and disk can rotate freely about a horizontal axis *AC*, and shaft *AB* can rotate freely about a vertical axis. Initially rod *AG* is horizontal $(\theta_0 = 90°)$ and has no angular velocity about *AC*. Knowing that the maximum value $\dot{\phi}_m$ of the angular velocity of shaft *AB* in the ensuing motion is twice its initial value $\dot{\phi}_0$, determine (*a*) the minimum value of θ, (*b*) the initial angular velocity $\dot{\phi}_0$ of shaft *AB*.

SOLUTION


Z
z
$\dot{\varphi}\underline{K}$
x
A
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$\underline{i}$
θ
$\dot{\psi}\underline{k}$
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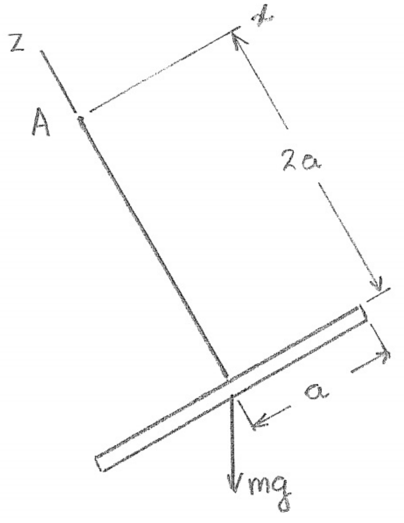


Let the *Z* axis be vertical.

For principal axes *xyz* with origin at *A*, the principal moments of inertia are

$$I' = I_x = I_y = m\left[\frac{1}{4}a^2 + (2a)^2\right] = \frac{17}{4}ma^2$$

$$I = I_z = \frac{1}{2}ma^2$$

Angular velocity components:

$$\omega_x = \dot{\phi}\sin\theta, \qquad \omega_y = -\dot{\theta}, \qquad \omega_z = \dot{\psi} + \dot{\phi}\cos\theta$$

Angular momentum about *A*:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= I'\dot{\phi}\sin\theta\mathbf{i} - I'\dot{\theta}\mathbf{j} + I\omega_z\mathbf{k}$$

Kinetic energy: $$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$T = \frac{1}{2}I'\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I\omega_z^2$$

Potential energy: $$V = -2mga\cos\theta$$

Conservation of angular momentum about fixed *Z* axis:

$$\mathbf{H}_A\cdot\mathbf{K} = \mathbf{H}_A\cdot(\mathbf{i}\sin\theta + \mathbf{k}\cos\theta)$$

$$= I'\dot{\phi}\sin^2\theta + I\omega_z\cos\theta$$

$$= \frac{17}{4}ma^2\dot{\phi}\sin^2\theta + \frac{1}{2}ma^2\omega_z\cos\theta = \alpha \qquad (1)$$

where α is a constant.

Conservation of energy: $T + V = E,$ where E is a constant.

$$\frac{17}{8}ma^2\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{4}ma^2\omega_z^2 - 2mga\cos\theta = E \tag{2}$$

Constraint of clevis: $\dot{\psi} = 0 \qquad \omega_z = \dot{\phi}\cos\theta$

(*a*) From (1),

$$\frac{17}{4}ma^2\dot{\phi}_m\sin^2\theta_m + \frac{1}{2}ma^2\dot{\phi}_m\cos^2\theta_m = \frac{17}{4}ma^2\dot{\phi}_0\sin^2 90° + \frac{1}{2}ma^2\dot{\phi}_0\cos^2 90°$$

$$\frac{17}{4}\sin^2\theta_m + \frac{1}{2}\cos^2\theta_m = \frac{17}{4}\frac{\dot{\phi}_0}{\dot{\phi}_m} = \left(\frac{17}{4}\right)\left(\frac{1}{2}\right) = \frac{17}{8}$$

$$\frac{17}{4}\sin^2\theta_m + \frac{1}{2}\left(1 - \sin^2\theta_m\right) = \frac{17}{8}$$

$$\sin^2\theta_m = \frac{13}{30}, \qquad \sin\theta_m = 0.65828 \qquad \theta_m = 41.169°$$

$$\theta_m = 41.2° \blacktriangleleft$$

(*b*) At the minimum value of θ, $\dot{\theta} = 0$

From (2),

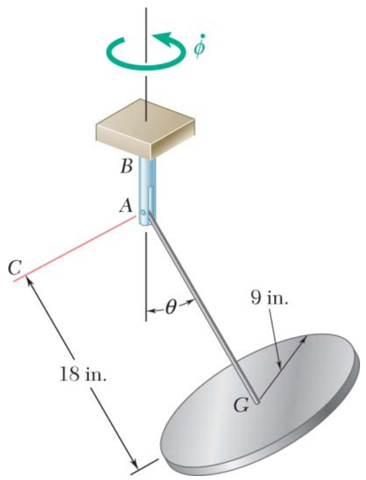
$$\frac{17}{8}ma^2\left(\dot{\phi}_m^2\sin^2\theta_m + 0\right) + \frac{1}{4}ma^2\dot{\phi}_m^2\cos^2\theta_m - 2mga\cos\theta_m$$

$$= \frac{17}{8}ma^2\left(\dot{\phi}_0^2\sin^2\theta_0 + 0\right) + \frac{1}{4}ma^2\dot{\phi}_0^2\cos^2 90° - 2mga\cos 90°$$

$$\left[\left(\frac{17}{8}\right)(2)^2\sin^2\theta_m + \left(\frac{1}{4}\right)(2)^2\cos^2\theta_m - \frac{17}{8}\right]ma^2\dot{\phi}_0^2 = 2mga\cos\theta_m$$

$$2.1250\, ma^2\dot{\phi}_0^2 = 1.5055mga$$

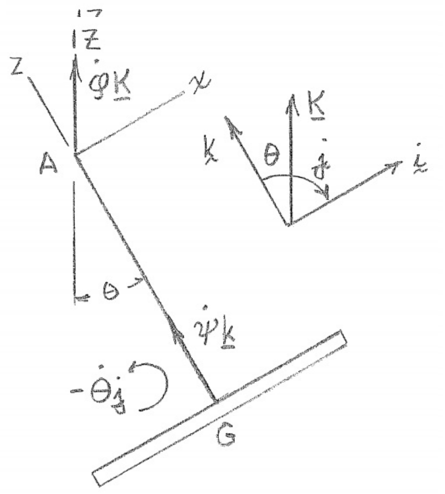
$$\dot{\phi}_0^2 = 0.70849\frac{g}{a} = 0.70849\frac{32.2}{0.75} = 30.418 \qquad \dot{\phi}_0 = 5.52 \text{ rad/s} \blacktriangleleft$$

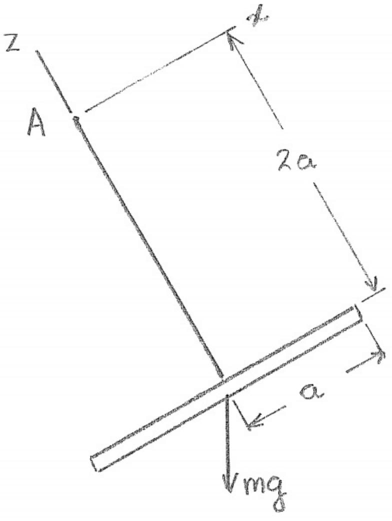


PROBLEM 18.136

A homogeneous disk of radius 9 in. is welded to a rod *AG* of length 18 in. and of negligible weight which is connected by a clevis to a vertical shaft *AB*. The rod and disk can rotate freely about a horizontal axis *AC*, and shaft *AB* can rotate freely about a vertical axis. Initially rod *AG* is horizontal $\left(\theta_0 = 90°\right)$ and has no angular velocity about *AC*. Knowing that the smallest value of θ in the ensuing motion is 30°, determine (*a*) the initial angular velocity of shaft *AB*, (*b*) its maximum angular velocity.

SOLUTION


Z
z
$\dot{\varphi}\underline{K}$
x
A
$\underline{K}$
k
θ
j
i
θ
$\dot{\psi}\underline{k}$
$-\dot{\theta}\underline{j}$
G



Let the *Z* axis be vertical.

For principal axes *xyz* with origin at *A*, the principal moments of inertia are

$$I' = I_x = I_y = m\left[\frac{1}{4}a^2 + (2a)^2\right] = \frac{17}{4}ma^2$$

$$I = I_z = \frac{1}{2}ma^2$$

Angular velocity components:

$$\omega_x = \dot{\phi}\sin\theta, \qquad \omega_y = -\dot{\theta}, \qquad \omega_z = \dot{\psi} + \dot{\phi}\cos\theta$$

Angular momentum about *A*:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= I'\dot{\phi}\sin\theta\mathbf{i} - I'\dot{\theta}\mathbf{j} + I\omega_z\mathbf{k}$$

Kinetic energy: $$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$T = \frac{1}{2}I'\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I\omega_z^2$$

Potential energy: $$V = -2mga\cos\theta$$

Conservation of angular momentum about fixed *Z* axis:

$$\mathbf{H}_A\cdot\mathbf{K} = \mathbf{H}_A\cdot\left(\mathbf{i}\sin\theta + \mathbf{k}\cos\theta\right)$$

$$= I'\dot{\phi}\sin^2\theta + I\omega_z\cos\theta$$

$$= \frac{17}{4}ma^2\dot{\phi}\sin^2\theta + \frac{1}{2}ma^2\omega_z\cos\theta = \alpha \qquad (1)$$

where α is a constant.

PROBLEM 18.136 (Continued)

Conservation of energy: $T + V = E,$ where E is a constant.

$$\frac{17}{8}ma^2\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{4}ma^2\omega_z^2 - 2mga\cos\theta = E \qquad (2)$$

Constraint of clevis: $\dot{\psi} = 0 \qquad \omega_z = \dot{\phi}\cos\theta$

(a) At $\theta = \theta_m = 30°$ and at $\theta = \theta_o = 90°, \quad \dot{\theta} = 0.$

From (1),

$$\frac{17}{4}ma^2\dot{\phi}_m\sin^2\theta_m + \frac{1}{2}ma^2\dot{\phi}_m\cos^2\theta_m = \frac{17}{4}ma^2\dot{\phi}_0\sin^2\theta_0 + 0$$

$$\left[\left(\frac{17}{4}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2\right]ma^2\dot{\phi}_m = \frac{17}{4}ma^2\dot{\phi}_0 \qquad \dot{\phi}_m = \frac{68}{23}\dot{\phi}_0$$

From (2),

$$\frac{17}{8}ma^2\left(\frac{68}{23}\dot{\phi}_0\right)^2\sin^2 30° + \frac{1}{4}ma^2\left(\frac{68}{23}\dot{\phi}_0\right)^2\cos^2 30° - 2mga\cos 30°$$

$$= \frac{17}{8}ma^2\dot{\phi}_0^2\sin^2 90° + \frac{1}{4}ma^2\dot{\phi}_0^2\cos^2 90° - 2mga\cos 90°$$

$$\left[\left(\frac{17}{32} + \frac{3}{16}\right)\left(\frac{68}{23}\right)^2 - \frac{17}{8}\right]ma^2\dot{\phi}_0^2 = 2mga\cos 30°$$

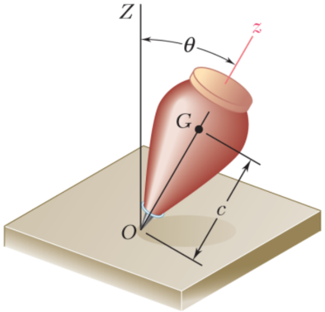
$$\dot{\phi}_0^2 = 0.41660\frac{g}{a} = 0.41660\ \frac{32.2}{0.75} = 17.886$$

$\dot{\phi}_0 = 4.2292$ rad/s $\qquad \dot{\phi}_0 = 4.23$ rad/s ◀

(b) $\dot{\phi}_m = \frac{68}{23}(4.2292) \qquad \dot{\varphi}_m = 12.50$ rad/s ◀

PROBLEM 18.137*

The top shown is supported at the fixed Point O. Denoting by ϕ, θ, and ψ the Eulerian angles defining the position of the top with respect to a fixed frame of reference, consider the general motion of the top in which all Eulerian angles vary.



(a) Observing that $\Sigma M_Z = 0$ and $\Sigma M_z = 0$, and denoting by I and I', respectively, the moments of inertia of the top about its axis of symmetry and about a transverse axis through O, derive the two first-order differential equations of motion

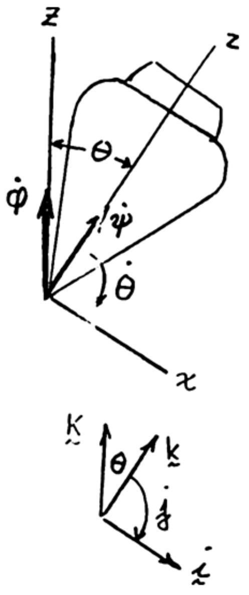
$$I'\dot{\phi}\sin^2\theta + I(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \alpha$$

$$I(\dot{\psi} + \dot{\phi}\cos\theta) = \beta$$

where α and β are constants depending upon the initial conditions. These equations express that the angular momentum of the top is conserved about both the Z and z axes, i.e., that the rectangular component of $\mathbf{H}_O$ along each of these axes is constant.

(b) Use Eqs. (1) and (2) to show that the rectangular component ω_z of the angular velocity of the top is constant and that the rate of precession $\dot{\phi}$ depends upon the value of the angle of nutation θ.

SOLUTION



Use a rotating frame of reference with the y axis pointing into the paper.

Angular velocity of the frame:

$$\mathbf{\Omega} = -\dot{\phi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\phi}\cos\theta\mathbf{k}$$

Angular velocity of the top:

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k}$$

Its angular momentum about O:

$$\mathbf{H}_O = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= -I'\dot{\phi}\sin\theta\mathbf{i} + I'\dot{\theta}\mathbf{j} + I(\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k}$$

where $I_x = I_y = I'$ and $I_z = I$.

The moment $\mathbf{M}_O$ about O is due to the weight mg.

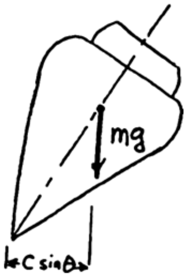
$$\mathbf{M}_0 = mgc\sin\theta\mathbf{j}$$

(a) Since the fixed Z axis refers to a Newtonian frame of reference and $(M_O)_Z = 0$, it follows that $(H_O)_Z$ is constant. Thus,

$$(H_O)_Z = \mathbf{H}_O \cdot \mathbf{K} = \mathbf{H}_O \cdot (-\mathbf{i}\sin\theta + \mathbf{k}\cos\theta)$$

$$= I'\dot{\phi}\sin^2\theta + I(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \alpha \qquad (1) \blacktriangleleft$$

where α is a constant.



$$\mathbf{M}_O = \dot{\mathbf{H}}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \mathbf{\Omega} \times \mathbf{H}_O$$

$$mgc\sin\theta\,\mathbf{j} = -I'\frac{d}{dt}(\dot{\phi}\sin\theta)\,\mathbf{i} + I'\ddot{\theta}\,\mathbf{j} + I\frac{d}{dt}(\dot{\psi}+\dot{\phi}\cos\theta)\,\mathbf{k}$$
$$+[I(\dot{\psi}+\dot{\phi}\cos\theta)\dot{\theta} - I'\dot{\phi}\dot{\theta}\cos\theta]\mathbf{i}$$
$$+[I(\dot{\psi}+\dot{\phi}\cos\theta)\dot{\phi}\sin\theta - I'\dot{\phi}^2\sin\theta\cos\theta]\mathbf{j}$$

z-components: $0 = \frac{d}{dt}(\dot{\psi}+\dot{\phi}\cos\theta)$

Integrating, $I(\dot{\psi}+\dot{\phi}\cos\theta) = \beta$ (2) ◀

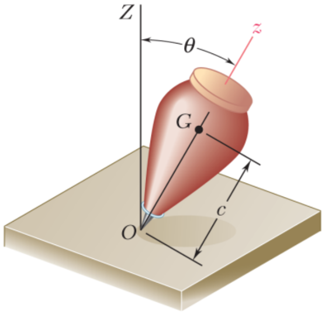
where β is a constant.

(*b*) Since $\omega_z = \dot{\psi}+\dot{\phi}\cos\theta$ $\omega_z = \frac{\beta}{I} =$ constant (3) ◀

From Eqs. (1) and (2), $I'\dot{\phi}\sin^2\theta + \beta\cos\theta = \alpha$ $\dot{\phi} = \frac{\alpha-\beta\cos\theta}{I'\sin^2\theta}$ (4) ◀

which is a function of θ.

PROBLEM 18.138*



(*a*) Applying the principle of conservation of energy, derive a third differential equation for the general motion of the top of Problem 18.137.

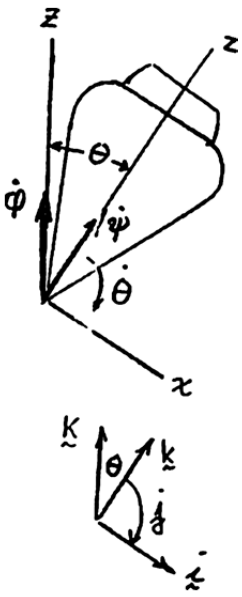
(*b*) Eliminating the derivatives $\dot{\phi}$ and $\dot{\psi}$ from the equation obtained and from the two equations of Problem 18.139, show that the rate of nutation $\dot{\theta}$ is defined by the differential equation $\dot{\theta}^2 = f(\theta)$, where

$$f(\theta) = \frac{1}{I'}\left(2E - \frac{\beta^2}{I} - 2mgc\cos\theta\right) - \left(\frac{\alpha - \beta\cos\theta}{I'\sin\theta}\right)^2$$

(*c*) Further show, by introducing the auxiliary variable $x = \cos\theta$, that the maximum and minimum values of θ can be obtained by solving for x the cubic equation

$$\left(2E - \frac{\beta^2}{I} - 2mgcx\right)(1 - x^2) - \frac{1}{I'}(\alpha - \beta x)^2 = 0$$

SOLUTION



(*a*) Angular velocity of the top:

$$\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + (\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$$

Kinetic energy:

$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$= \frac{1}{2}I'(\dot{\varphi}\sin\theta)^2 + \frac{1}{2}I'\dot{\theta}^2 + \frac{1}{2}I_z\omega_z^2$$

Potential energy: $V = mgc\cos\theta$

Principle of conservation of energy: $T + V = E$

$$\frac{1}{2}I'(\dot{\varphi}\sin\theta)^2 + \frac{1}{2}I'\dot{\theta}^2 + \frac{1}{2}I\omega_z^2 + mgc\cos\theta = E \blacktriangleleft$$

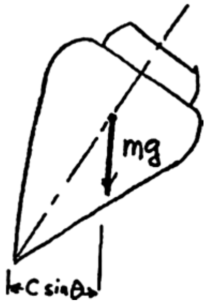
(*b*) Solving for $\dot{\theta}^2$,

$$\dot{\theta}^2 = \frac{1}{I'}(2E - I_z\omega_z^2 - 2mgc\cos\theta) - (\dot{\varphi}\sin\theta)^2 \qquad \text{(A)}$$

Equation (2) of Problem 18.137, with $\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$ gives

$$I_z\omega_z^2 = \frac{\beta^2}{I} \qquad \text{(B)}$$

PROBLEM 18.138* (Continued)



Equation (1) of Problem 18.137 gives

$$I'\dot{\varphi}\sin^2\theta + \beta\cos\theta = \alpha \qquad \text{(C)}$$

$$\dot{\varphi} = \frac{\alpha - \beta\cos\theta}{I'\sin^2\theta} \qquad \text{(D)}$$

Substituting Equations (D) and (B) into Equation (A),

$$\dot{\theta}^2 = f(\theta)$$

where $$f(\theta) = \frac{1}{I'}\left(2E - \frac{\beta^2}{I} - 2mgc\cos\theta\right) - \left(\frac{\alpha - \beta\cos\theta}{I'\sin\theta}\right)^2 \qquad (1) ◀$$

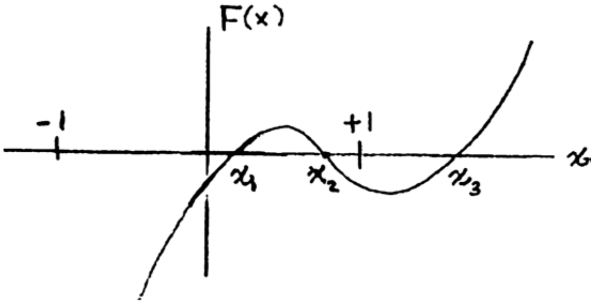
(*c*) Maximum and minimum values of θ occur when $f(\theta) = 0.$ Setting $\cos\theta = x$ and $\sin^2\theta = 1 - x^2$ in Equation (1), and letting $f(\theta) = 0$ gives

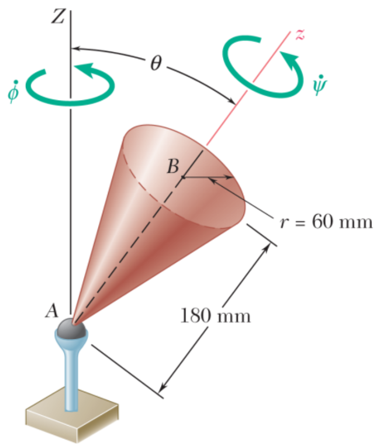
$$\frac{1}{I'}\left(2E - \frac{\beta^2}{I} + 2mgcx\right) - \frac{(\alpha - \beta x)^2}{(I')^2(1 - x^2)} = 0$$

Multiplying by $I'(1 - x^2)$ gives the cubic equation $F(x) = 0$:

$$\left(2E - \frac{\beta^2}{I} - 2mgcx\right)(1 - x^2) - \frac{1}{I'}(\alpha - \beta x)^2 = 0 \qquad (2) ◀$$

Solving this equation will yield three values of x. The two values lying between –1 and +1 correspond to the maximum and minimum values of θ.





PROBLEM 18.139*

A solid cone of height 180 mm with a circular base of radius 60 mm is supported by a ball and socket at *A*. The cone is released from the position $\theta_0 = 30°$ with a rate of spin $\dot{\psi}_0 = 300$ rad/s, a rate of precession $\dot{\phi}_0 = 20$ rad/s, and a zero rate of nutation. Determine (*a*) the maximum value of θ in the ensuing motion, (*b*) the corresponding values of the rates of spin and precession. [*Hint:* Use Eq. (2) of Prob. 18.138; you can either solve this equation numerically or reduce it to a quadratic equation, since one of its roots is known.]

SOLUTION

Data: $r = 60 \text{ mm} = 0.06 \text{ m}, \quad h = 180 \text{ mm} = 0.18 \text{ m}, \quad c = \frac{3}{4}h = 0.135 \text{ m},$

$$\theta_0 = 30°, \quad \dot{\psi}_0 = 300 \text{ rad/s}, \quad \dot{\phi}_0 = 20 \text{ rad/s}, \quad \dot{\theta}_0 = 0$$

Calculate the following:

$$\frac{I}{m} = \frac{3}{10}r^2 = \frac{3}{10}(0.06)^2 = 1.08 \times 10^{-3} \text{ m}^2$$

$$\frac{I'}{m} = \frac{3}{5}\left(\frac{1}{4}r^2 + h^2\right) = \frac{3}{5}\left[\frac{1}{4}(0.06)^2 + (0.18)^2\right] = 19.98 \times 10^{-3} \text{ m}^2$$

Initially, $\omega_x = -\dot{\phi}_0 \sin\theta_0 = -20\sin 30° = -10 \text{ rad/s}, \qquad \omega_y = \dot{\theta}_0 = 0$

$$\omega_z = \dot{\psi}_0 + \dot{\phi}_0 \cos\theta_0 = 300 + 20\cos 30° = 317.32 \text{ rad/s}$$

$$\frac{T}{m} = \frac{1}{2}\frac{I'}{m}(\omega_x^2 + \omega_y^2) + \frac{1}{2}\frac{I}{m}\omega_z^2$$

$$= \frac{1}{2}(19.98 \times 10^{-3})(10^2 + 0) + \frac{1}{2}(1.08 \times 10^{-3})(317.32)^2 = 55.3728 \text{ m}^2/\text{s}^2$$

$$\frac{V}{m} = gc\cos\theta_0 = (9.81)(0.135)\cos 30° = 1.1469 \text{ m}^2/\text{s}^2$$

$$\frac{2E}{m} = 2(55.3728 + 1.1469) = 113.0394 \text{ m}^2/\text{s}^2$$

$$\frac{\beta}{m} = \frac{I}{m}\omega_z = (1.08 \times 10^{-3})(317.32) = 0.342706 \text{ m}^2/\text{s}$$

$$\frac{\alpha}{m} = \frac{I'}{m}\dot{\phi}_0 \sin^2\theta_0 + \beta\cos\theta_0$$

$$= (19.98 \times 10^{-3})(20)\sin^2 30° + 0.342706\cos 30°$$

$$= 0.396692 \text{ m}^2/\text{s}$$

PROBLEM 18.139* (Continued)

After dividing by m, Equation (2) of Problem 18.138 becomes

$$F(x) = \left(\frac{2E}{m} - \frac{\frac{\beta^2}{m^2}}{\frac{I}{m}} - 2gcx\right)(1 - x^2) - \frac{m}{I'}\left(\frac{\alpha}{m} - \frac{\beta x}{m}\right)^2 = 0$$

$$\left[113.0394 - \frac{(0.342706)^2}{1.08 \times 10^{-3}} - (2)(9.81)(0.135)x\right](1 - x^2) - \frac{(0.396692 - 0.342706x)^2}{19.98 \times 10^{-3}} = 0$$

$$(4.29198 - 2.6487x)(1 - x^2) - 5.87825(1.157529 - x)^2 = 0$$

(a) Roots are: $x = \cos\theta = 0.68170, \quad 0.86603, \quad 2.2919$

$$\theta_{\max} = \cos^{-1}(0.68170) = 47.023°$$

$\theta_{\max} = 47.0°$ ◀

(b) By Equation (4) of Problem 18.137,

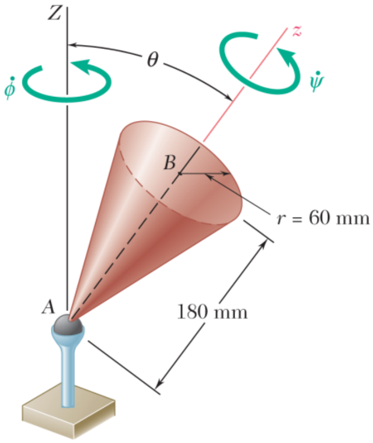
$$\dot{\phi} = \frac{\alpha - \beta\cos\theta}{I'\sin^2\theta} = \frac{0.396692 - (0.342706)(0.68170)}{(19.98 \times 10^{-3})\sin^2 47.023°} = 15.2474$$

$$\omega_z = 317.32 \text{ rad/s}$$

$$\dot{\psi} = \omega_z - \dot{\phi}\cos\theta = 317.32 - (15.2474)(0.68170)$$

spin: $\dot{\psi} = 307$ rad/s ◀

precession: $\dot{\phi} = 15.25$ rad/s ◀



PROBLEM 18.140*

A solid cone of height 180 mm with a circular base of radius 60 mm is supported by a ball and socket at A. The cone is released from the position $\theta_0 = 30°$ with a rate of spin $\dot{\psi}_0 = 300$ rad/s, a rate of precession $\dot{\phi}_0 = -4$ rad/s, and a zero rate of nutation. Determine (*a*) the maximum value of θ in the ensuing motion, (*b*) the corresponding values of the rates of spin and precession, (*c*) the value of θ for which the sense of the precession is reversed. (See hint of Problem 18.139.)

SOLUTION

Data: $r = 60 \text{ mm} = 0.06 \text{ m}, \quad h = 180 \text{ mm} = 0.18 \text{ m}, \quad c = \frac{3}{4}h = 0.135 \text{ m},$

$$\theta_0 = 30°, \quad \dot{\psi}_0 = 300 \text{ rad/s}, \quad \dot{\phi}_0 = -4 \text{ rad/s}, \quad \dot{\theta}_0 = 0$$

Calculate the following:

$$\frac{I}{m} = \frac{3}{10}r^2 = \frac{3}{10}(0.06)^2 = 1.08 \times 10^{-3} \text{ m}^2$$

$$\frac{I'}{m} = \frac{3}{5}\left(\frac{1}{4}r^2 + h^2\right) = \frac{3}{5}\left[\frac{1}{4}(0.06)^2 + (0.18)^2\right] = 19.98 \times 10^{-3} \text{ m}^2$$

Initially, $\omega_x = -\dot{\phi}_0 \sin\theta_0 = -(-4)\sin 30° = 2 \text{ rad/s}, \quad \omega_y = \dot{\theta}_0 = 0$

$$\omega_z = 300 + (-4)\cos 30° = 296.536 \text{ rad/s}$$

$$\frac{T}{m} = \frac{1}{2}\frac{I'}{m}(\omega_x^2 + \omega_y^2) + \frac{1}{2}\frac{I}{m}\omega_z^2$$

$$= \frac{1}{2}(19.98 \times 10^{-3})((2)^2 + 0) + \frac{1}{2}(1.08 \times 10^{-3})(296.536)^2 = 47.5241 \text{ m}^2/\text{s}^2$$

$$\frac{V}{m} = gc\cos\theta = (9.81)(0.135)\cos 30° = 1.1469 \text{ m}^2/\text{s}^2$$

$$\frac{2E}{m} = 2(47.5241 + 1.1469) = 97.3419 \text{ m}^2/\text{s}^2$$

$$\frac{\beta}{m} = \frac{I}{m}\omega_z = (1.08 \times 10^{-3})(296.536) = 0.320259 \text{ m}^2/\text{s}$$

$$\frac{\alpha}{m} = \frac{I'}{m}\dot{\phi}_0 \sin^2\theta_0 + \beta\cos\theta_0$$

$$= (19.98 \times 10^{-3})(-4)\sin^2 30° + 0.320259\cos 30°$$

$$= 0.257372 \text{ m}^2/\text{s}$$

After dividing by m, Equation (2) of Problem 18.138 becomes

$$F(x) = \left(\frac{2E}{m} - \frac{\frac{\beta^2}{m^2}}{\frac{I}{m}} - 2gcx\right)(1 - x^2) - \frac{m}{I'}\left(\frac{\alpha}{m} - \frac{\beta x}{m}\right)^2 = 0$$

$$\left[97.3419 - \frac{(0.320259)^2}{1.08 \times 10^{-3}} - (2)(9.81)(0.135)x\right](1 - x^2) - \frac{(0.257372 - 0.320259x)^2}{19.98 \times 10^{-3}} = 0$$

$$(2.37324 - 2.6487x)(1 - x^2) - 5.13342(0.80364 - x)^2 = 0$$

(*a*) Roots are: $x = \cos\theta = 0.23732, \quad 0.86603, \quad 1.73$

$$\theta_{\max} = \cos^{-1}(0.23732) = 76.272°$$

$\theta_{\max} = 76.3°$ ◀

(*b*) By Equation (4) of Problem 18.137,

$$\dot{\phi} = \frac{\alpha - \beta\cos\theta}{I'\sin^2\theta} = \frac{0.257372 - (0.320259)(0.23732)}{(19.98 \times 10^{-3})\sin^2 76.272°} = 9.6192 \text{ rad/s}$$

$$\omega_z = 296.536 \text{ rad/s}$$

$$\dot{\psi} = \omega_z - \dot{\phi}\cos\theta = 296.536 - (9.6192)(0.23732)$$

spin: $\dot{\psi} = 294$ rad/s ◀

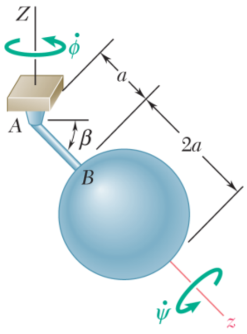
precession: $\dot{\phi} = 9.62$ rad/s ◀

(*c*)

$$\dot{\phi} = \frac{\alpha - \beta\cos\theta}{I'\sin^2\theta} = 0$$

$$\cos\theta = \frac{\alpha}{\beta} = \frac{0.257372}{0.320259}$$

$\theta = 36.5°$ ◀



PROBLEM 18.141*

A homogeneous sphere of mass m and radius a is welded to a rod AB of negligible mass, which is held by a ball-and-socket support at A. The sphere is released in the position $\beta = 0$ with a rate of precession $\dot{\phi}_0 = \sqrt{17g/11a}$ with no spin or nutation. Determine the largest value of β in the ensuing motion.

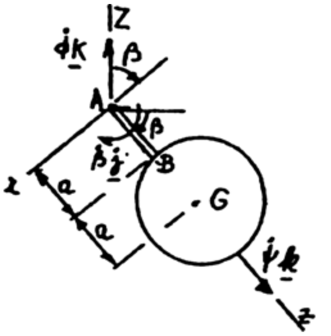
SOLUTION

Conservation of angular momentum about the Z and z axes.

Since the only external forces are the weight of the sphere and the reaction at A, a reasoning similar to that used in Problem 18.109 shows that the angular momentum is conserved about the Z and z axes.

Choosing the principal axes $Axyz$ shown (with y horizontal and pointing into the paper), we have

$$\boldsymbol{\omega} = -\dot{\phi}\cos\beta\mathbf{i} + \dot{\beta}\mathbf{j} + (\dot{\psi} - \dot{\phi}\sin\beta)\mathbf{k}$$



The moments of inertia are

$$I_z = \frac{2}{5}ma^2$$

$$I_x = I_y = \frac{2}{5}ma^2 + m(2a)^2 = \frac{22}{5}ma^2$$

Angular momentum about A:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$\mathbf{H}_A = -\frac{22}{5}ma^2\dot{\phi}\cos\beta\mathbf{i} + \frac{22}{5}ma^2\dot{\beta}\mathbf{j} + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)\mathbf{k}$$

We write $\mathbf{H}_z = \text{constant}$, or $\mathbf{H}_A \cdot \mathbf{K} = \text{constant}$

Since $\mathbf{i}\cdot\mathbf{K} = -\cos\beta$, $\mathbf{j}\cdot\mathbf{K} = 0$, $\mathbf{k}\cdot\mathbf{K} = -\sin\beta$

$$\mathbf{H}_0 \cdot \mathbf{K} = -\frac{22}{5}ma^2\dot{\phi}\cos\beta(-\cos\beta)$$

$$+\frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)(-\sin\beta) = \text{constant}$$

With the initial conditions $\dot{\phi} = \dot{\phi}_0$, $\dot{\psi} = \dot{\beta} = 0$, $\beta = 0$, we find that the constant is $\frac{22}{5}ma^2\dot{\phi}_0$. Thus,

$$11\dot{\phi}\cos^2\beta - (\dot{\psi} - \dot{\phi}\sin\beta)\sin\beta = 11\dot{\phi}_0 \qquad (1)$$

We now write $H_z = \text{constant}$:

$$H_z = \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta) = \text{constant}$$

and, from the initial conditions, we find that the constant is zero. Thus,

$$\dot{\psi} - \dot{\phi}\sin\beta = 0 \tag{2}$$

Conservation of energy.

We have

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

$$T = \frac{1}{2}\left[\frac{22}{5}ma^2\dot{\phi}^2\cos^2\beta + \frac{22}{5}ma^2\dot{\beta}^2 + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)^2\right]$$

and selecting the datum at $\beta = 0$:

$$V = -2mga\sin\beta$$

$$T + V = \text{constant:}\quad \frac{1}{2}\left[\frac{22}{5}ma^2\dot{\phi}^2\cos^2\beta + \frac{22}{5}ma^2\dot{\beta}^2 + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)^2\right] - 2mga\sin\beta = \text{constant}$$

From the initial conditions $\dot{\phi} = \dot{\phi}_0,\ \dot{\psi} = \dot{\beta} = 0,\ \beta = 0,$ we find that the constant is $\frac{11}{5}ma^2\dot{\phi}_0^2$. Thus,

$$11\dot{\phi}^2\cos^2\beta + 11\dot{\beta}^2 + (\dot{\psi} - \dot{\phi}\sin\beta)^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{3}$$

Substituting for $\dot{\psi} - \dot{\phi}\sin\beta$ from Eq. (2) into Eqs. (1) and (3):

Eq. (1): $$11\dot{\phi}\cos^2\beta = 11\dot{\phi}_0 \tag{1$'$}$$

Eq. (3): $$11\dot{\phi}^2\cos^2\beta + 11\dot{\beta}^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{3$'$}$$

Solving $(1')$ for $\dot{\phi}$,

$$\dot{\phi} = \dot{\phi}_0\sec^2\beta \tag{4}$$

Substituting for $\dot{\phi}$ from Eq. (4) into Eq. $(3')$:

$$11(\dot{\phi}_0\sec^2\theta)^2\cos^2\beta + 11\dot{\beta}^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{5}$$

For the maximum value of β, we have $\dot{\beta} = 0$ and Eq. (5) yields

$$\dot{\phi}_0^2\left(\frac{1}{\cos^2\theta} - 1\right) = \frac{10}{11}\frac{g}{a}\sin\beta$$

$$\dot{\phi}_0^2 = \frac{10}{11}\frac{g}{a}\frac{\cos^2\beta}{\sin\beta} \tag{6}$$

PROBLEM 18.141* (Continued)

Given data: $\dot{\phi}_0^2 = \frac{17}{11}\frac{g}{a}\cdot$ Substituting into Eq. (6),

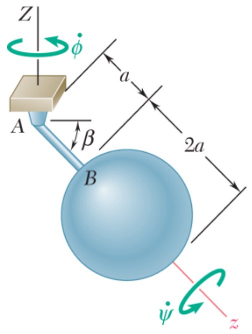
$$\frac{17}{11}\frac{g}{a} = \frac{10}{11}\frac{g}{a}\frac{\cos^2\beta}{\sin\beta} \qquad \cos^2\beta = 1.7\sin\beta$$

Letting $\cos^2\beta = 1 - \sin^2\beta$, we have

$$\sin^2\beta + 1.7\sin\beta - 1 = 0$$

Solving the quadratic,

$$\sin\beta = -2.162 \text{ (impossible)} \quad \text{and} \quad \sin\beta = 0.46244 \qquad \beta_{\text{max}} = 27.5° \blacktriangleleft$$



PROBLEM 18.142*

A homogeneous sphere of mass m and radius a is welded to a rod AB of negligible mass, which is held by a ball-and-socket support at A. The sphere is released in the position $\beta = 0$ with a rate of precession $\dot{\phi} = \dot{\phi}_0$ with no spin or nutation. Knowing that the largest value of β in the ensuing motion is 30°, determine (*a*) the rate of precession $\dot{\phi}_0$ of the sphere in its initial position, (*b*) the rates of precession and spin when $\beta = 30°$.

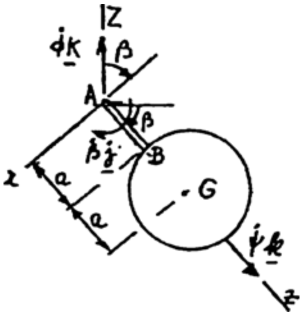
SOLUTION

Conservation of angular momentum about the *Z* and *z* axes.

Since the only external forces are the weight of the sphere and the reaction at *A*, a reasoning similar to that used in Problem 18.109 shows that the angular momentum is conserved about the *Z* and *z* axes.

Choosing the principal axes *Axyz* shown (with *y* horizontal and pointing into the paper), we have

$$\boldsymbol{\omega} = -\dot{\phi}\cos\beta\,\mathbf{i} + \dot{\beta}\,\mathbf{j} + (\dot{\psi} - \dot{\phi}\sin\beta)\mathbf{k}$$


Z
$\dot{\phi}\underline{K}$
β
A
β
$\dot{\beta}\underline{j}$
B
z
a
a
G
$\dot{\psi}\underline{k}$
z

The moments of inertia are

$$I_z = \frac{2}{5}ma^2$$

$$I_x = I_y = \frac{2}{5}ma^2 + m(2a)^2 = \frac{22}{5}ma^2$$

Angular momentum about *A*:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$\mathbf{H}_A = -\frac{22}{5}ma^2\dot{\phi}\cos\beta\,\mathbf{i} + \frac{22}{5}ma^2\dot{\beta}\,\mathbf{j} + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)\mathbf{k}$$

We write $\mathbf{H}_Z = \text{constant}$, or $\mathbf{H}_A \cdot \mathbf{K} = \text{constant}$

Since $\mathbf{i}\cdot\mathbf{K} = -\cos\beta$, $\mathbf{j}\cdot\mathbf{K} = 0$, $\mathbf{k}\cdot\mathbf{K} = -\sin\beta$

$$\mathbf{H}_0 \cdot \mathbf{K} = -\frac{22}{5}ma^2\dot{\phi}\cos\beta(-\cos\beta)$$
$$+\frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)(-\sin\beta) = \text{constant}$$

With the initial conditions $\dot{\phi} = \dot{\phi}_0$, $\dot{\psi} = \dot{\beta} = 0$, $\beta = 0$, we find that the constant is $\frac{22}{5}ma^2\dot{\phi}_0$. Thus,

$$11\dot{\phi}\cos^2\beta - (\dot{\psi} - \dot{\phi}\sin\beta)\sin\beta = 11\dot{\phi}_0 \qquad (1)$$

We now write $H_z = \text{constant}$:

$$H_z = \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta) = \text{constant}$$

and, from the initial conditions, we find that the constant is zero. Thus,

$$\dot{\psi} - \dot{\phi}\sin\beta = 0 \tag{2}$$

Conservation of energy.

We have

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

$$T = \frac{1}{2}\left[\frac{22}{5}ma^2\dot{\phi}^2\cos^2\beta + \frac{22}{5}ma^2\dot{\beta}^2 + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)^2\right]$$

and, selecting the datum at $\beta = 0$:

$$V = -2mga\sin\beta$$

$T + V = \text{constant}$: $\frac{1}{2}\left[\frac{22}{5}ma^2\dot{\phi}^2\cos^2\beta + \frac{22}{5}ma^2\dot{\beta}^2 + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)^2\right] - 2mga\sin\beta = \text{constant}$

From the initial conditions $\dot{\phi} = \dot{\phi}_0,\ \dot{\psi} = \dot{\beta} = 0,\ \beta = 0,$ we find that the constant is $\frac{11}{5}ma^2\dot{\phi}_0^2$. Thus,

$$11\dot{\phi}^2\cos^2\beta + 11\dot{\beta}^2 + (\dot{\psi} - \dot{\phi}\sin\beta)^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{3}$$

Substituting for $\dot{\psi} - \dot{\phi}\sin\beta$ from Eq. (2) into Eqs. (1) and (3),

Eq. (1): $$11\dot{\phi}\cos^2\beta = 11\dot{\phi}_0 \tag{1′}$$

Eq. (3): $$11\dot{\phi}^2\cos^2\beta + 11\dot{\beta}^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{3′}$$

Solving $(1')$ for $\dot{\phi}$,

$$\dot{\phi} = \dot{\phi}_0\sec^2\beta \tag{4}$$

Substituting for $\dot{\phi}$ from Eq. (4) into Eq. $(3')$,

$$11(\dot{\phi}_0\sec^2\theta)^2\cos^2\beta + 11\dot{\beta}^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{5}$$

For the maximum value of β, we have $\dot{\beta} = 0$ and Eq. (5) yields

$$\dot{\phi}_0^2\left(\frac{1}{\cos^2\theta} - 1\right) = \frac{10}{11}\frac{g}{a}\sin\beta$$

$$\dot{\phi}_0^2 = \frac{10}{11}\frac{g}{a}\frac{\cos^2\beta}{\sin\beta} \tag{6}$$

(*a*) Making $\beta = 30°$ in Eq. (6), we have

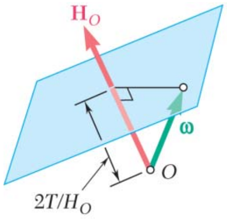
$$\dot{\phi}_0^2 = \frac{10}{11}\frac{g}{a}\frac{0.75}{0.5} = \frac{15}{11}\frac{g}{a} \qquad \dot{\phi}_0 = \sqrt{\frac{15}{11}\frac{g}{a}} \blacktriangleleft$$

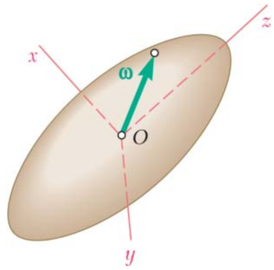
(*b*) Substituting for $\dot{\phi}_0$ in Eq. (4), and making $\beta = 30°$:

$$\dot{\phi} = \dot{\phi}_0 \sec^2 30° = \sqrt{\frac{15}{11}\frac{g}{a}}\left(\frac{4}{3}\right) = \sqrt{\frac{15(16)}{11(9)}\frac{g}{a}} \qquad \dot{\phi} = 2\sqrt{\frac{20}{33}\frac{g}{a}} \blacktriangleleft$$

Substituting for in Eq. (2), and making $\beta = 30°$:

$$\dot{\psi} = \dot{\phi}\sin 30° = \frac{1}{2}\dot{\phi} \qquad \dot{\psi} = \sqrt{\frac{20}{33}\frac{g}{a}} \blacktriangleleft$$





PROBLEM 18.143*

Consider a rigid body of arbitrary shape which is attached at its mass center O and subjected to no force other than its weight and the reaction of the support at O.

(*a*) Prove that the angular momentum $\mathbf{H}_O$ of the body about the fixed Point O is constant in magnitude and direction, that the kinetic energy T of the body is constant, and that the projection along $\mathbf{H}_O$ of the angular velocity $\boldsymbol{\omega}$ of the body is constant.

(*b*) Show that the tip of the vector $\boldsymbol{\omega}$ describes a curve on a fixed plane in space (called the *invariable plane*), which is perpendicular to $\mathbf{H}_O$ and at a distance $2T/H_O$ from O.

(*c*) Show that with respect to a frame of reference attached to the body and coinciding with its principal axes of inertia, the tip of the vector $\boldsymbol{\omega}$ appears to describe a curve on an ellipsoid of equation

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant}$$

The ellipsoid (called the *Poinsot ellipsoid*) is rigidly attached to the body and is of the same shape as the ellipsoid of inertia, but of a different size.

SOLUTION

(*a*) From Equation (18.27), $\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O$

Since $\Sigma\mathbf{M}_O = 0, \quad \dot{\mathbf{H}}_O = 0.$ $\qquad \mathbf{H}_O = \text{constant} \quad (1)$ ◀

Conservation of energy: $T + V = \text{constant}$

Since $V = 0,$ $\qquad T = \text{constant} \quad (2)$ ◀

For a rigid body rotating about Point O,

$$T = \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right)$$

$$\mathbf{H}_O = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

$$\mathbf{H}_O \cdot \omega = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T$$

Let β be the angle between the vectors $\mathbf{H}_O$ and $\boldsymbol{\omega}$.

$$\mathbf{H}_O \cdot \boldsymbol{\omega} = H_O\omega\cos\beta$$

The projection of $\boldsymbol{\omega}$ along $\mathbf{H}_O$ is $\omega\cos\beta$

$$\omega\cos\beta = \frac{2T}{H_O} = \text{constant}$$

$\omega\cos\beta = \text{constant} \quad (3)$ ◀

PROBLEM 18.143* (Continued)

(*b*) $\omega\cos\beta$ is the perpendicular distance from the invariable plane. This distance is equal to $\frac{2T}{H_O}$.

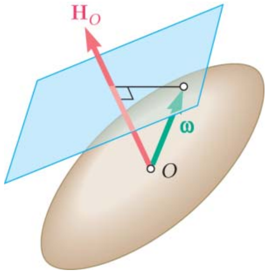
(*c*) For a frame of reference attached to the body, the moments of inertia with respect of orthogonal axes of the frame do not change.

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T \tag{4}$$

Let
$$a_1 = \sqrt{\frac{2T}{I_x}}, \quad b_1 = \sqrt{\frac{2T}{I_y}}, \quad c_1 = \sqrt{\frac{2T}{I_z}} \tag{5}$$

Then
$$\frac{\omega_x^2}{a_1^2} + \frac{\omega_y^2}{b_1^2} + \frac{\omega_z^2}{c_1^2} = 1 \tag{6}$$

which is the equation of an ellipsoid.



PROBLEM 18.144*

Referring to Problem 18.143, (*a*) prove that the Poinsot ellipsoid is tangent to the invariable plane, (*b*) show that the motion of the rigid body must be such that the Poinsot ellipsoid appears to roll on the invariable plane. [*Hint:* In part *a,* show that the normal to the Poinsot ellipsoid at the tip of $\boldsymbol{\omega}$ is parallel to $\mathbf{H}_O$. It is recalled that the direction of the normal to a surface of equation $F(x, y, z) = \text{constant}$ at a Point *P* is the same as that of **grad** *F* at Point *P*.]

SOLUTION

(*a*) From Problem 18.143, the equation of the Poinsot ellipsoid is

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant}$$

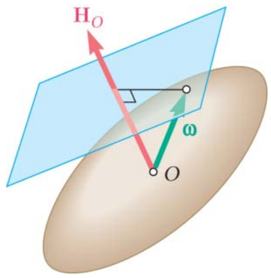
Let $$F(\omega_x, \omega_y, \omega_z) = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2$$

$$\mathbf{grad}\ F = \frac{\partial F}{\partial \omega_x}\mathbf{i} + \frac{\partial F}{\partial \omega_y}\mathbf{j} + \frac{\partial F}{\partial \omega_z}\mathbf{k}$$

$$= 2I_x\omega_x\mathbf{i} + 2I_y\omega_y\mathbf{j} + 2I_z\omega_z\mathbf{k} = 2\mathbf{H}_O$$

The direction of the normal at a point on the surface of the ellipsoid is parallel to **grad** *F*, which in turn is parallel to $\mathbf{H}_O$. Since $\mathbf{H}_O$ is normal also to the invariable plane, it follows that the Poinsot ellipsoid is tangent to the invariable plane at the point common to the plane and the ellipsoid.

(*b*) The Poinsot elipsoid moves with the body. Thus, its angular velocity is $\boldsymbol{\omega}$, the angular velocity of the body. Since Point *O* is regarded as fixed, the angular velocity vector lies along the axis of rotation, or the locus of points of zero velocity. Thus, the velocity of the point of contact of the Poinsot ellipsoid with the invariable plane is zero. The Poinsot ellipsoid rolls without slipping on the invariable plane.



PROBLEM 18.145*

Using the results obtained in Problems 18.143 and 18.144, show that for an axisymmetrical body attached at its mass center O and under no force other than its weight and the reaction at O, the Poinsot ellipsoid is an ellipsoid of revolution and the space and body cones are both circular and are tangent to each other. Further show that (*a*) the two cones are tangent externally, and the precession is direct, when $I < I'$, where I and I' denote, respectively, the axial and transverse moment of inertia of the body, (*b*) the space cone is inside the body cone, and the precession is retrograde, when $I > I'$.

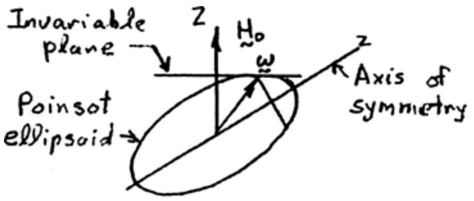
SOLUTION

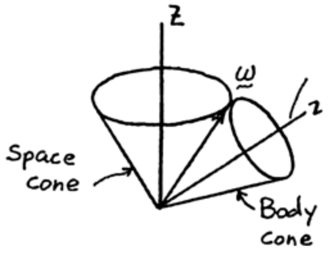
Let $I_x = I_y = I'$ and $I_z = I$ so that the z axis is the symmetry axis. Then, the equation of the Poinsot ellipsoid (Equation (4) of Problem 18.143) becomes

$$I'\left(\omega_x^2 + \omega_y^2\right) + I\omega_z^2 = 2T = \text{constant}$$

which is the equation of an *ellipsoid of revolution*. It follows that the tip of $\boldsymbol{\omega}$ describes circles on both the Poinsot ellipsoid and on the invariable plane, and that the vector $\boldsymbol{\omega}$ itself describes *circular body and space cones.* The Poinsot ellipsoid, the invariable plane, and the body and space cones are shown below for cases *a* and *b*.

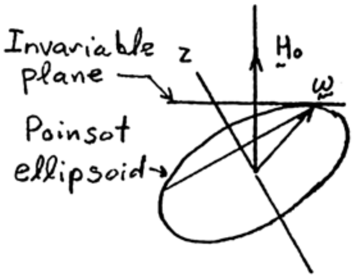
(*a*) $I < I'$

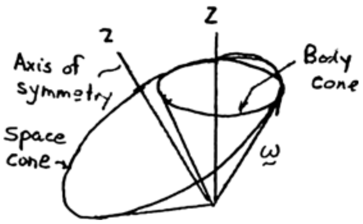

Invariable plane
Z
H_o
ω
z
Axis of symmetry
Poinsot ellipsoid


Z
ω
z
Space cone
Body cone

Direct Precession

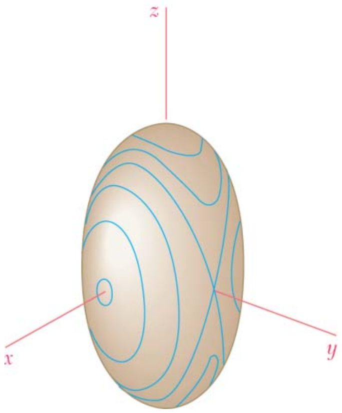
(*b*) $I > I'$


Invariable plane
z
H_o
ω
Poinsot ellipsoid


Z
z
Axis of symmetry
Body Cone
Space cone
ω

Retrograde Precession

PROBLEM 18.146*



Refer to Problems 18.143 and 18.144.

(*a*) Show that the curve (called *polhode*) described by the tip of the vector **ω** with respect to a frame of reference coinciding with the principal axes of inertia of the rigid body is defined by the equations

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant} \tag{1}$$

$$I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2 = H_O^2 = \text{constant} \tag{2}$$

and that this curve can, therefore, be obtained by intersecting the Poinsot ellipsoid with the ellipsoid defined by Eq. (2).

(*b*) Further show, assuming $I_x > I_y > I_z$, that the polhodes obtained for various values of H_O have the shapes indicated in the figure.

(*c*) Using the result obtained in part *b*, show that a rigid body under no force can rotate about a fixed centroidal axis if, and only if, that axis coincides with one of the principal axes of inertia of the body, and that the motion will be stable if the axis of rotation coincides with the major or minor axis of the Poinsot ellipsoid (*z* or *x* axis in the figure) and unstable if it coincides with the intermediate axis (*y* axis).

SOLUTION

(*a*) Equation (1) expresses conservation of energy as shown in the solution to Problem 18.143. It is the equation of the Poinsot ellipsoid.

Let $$a_1 = \sqrt{\frac{2T}{I_x}}, \quad b_1 = \sqrt{\frac{2T}{I_y}}, \quad c_1 = \sqrt{\frac{2T}{I_z}}$$

Then $$\frac{\omega_x^2}{a_1^2} + \frac{\omega_y^2}{b_1^2} + \frac{\omega_z^2}{c_1^2} = 1 \tag{3}$$

which is the equation of an ellipsoid.

Equation (2) expresses the constancy of $H_O^2 = (H_O)_x^2 + (H_O)_y^2 + (H_O)_z^2$, the square of the magnitude of the angular momentum vector.

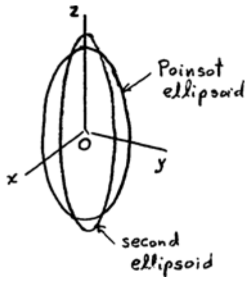
Let $$a_2 = \frac{H_O}{I_x}, \quad b_2 = \frac{H_O}{I_y}, \quad c_2 = \frac{H_O}{I_z}$$

Then $$\frac{\omega_x^2}{a_2^2} + \frac{\omega_y^2}{b_2^2} + \frac{\omega_z^2}{c_2^2} = 1 \tag{4}$$

which is the equation of a second ellipsoid.

Since the coordinates ω_x, ω_y, ω_z of the tip of the vector **ω** must satisfy both Equations (1) and (2), the curve described by the tip of **ω** is the intersection of the two ellipsoids.

PROBLEM 18.146* (Continued)

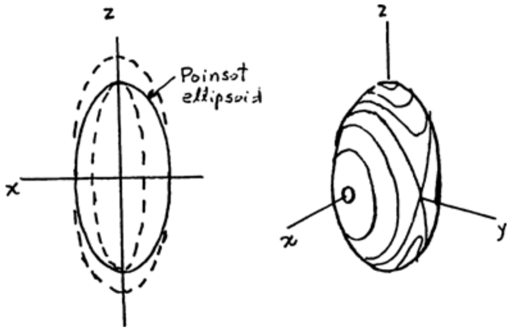


(*b*) Assume $I_x > I_y > I_z$.

Then $a_1 < b_1 < c_1$ and $a_2 < b_2 < c_2$.

Thus, for both ellipsoids, the minor axis is directed along the *x* axis, the intermediate axis along the *y* axis, and the major axis along the *z* axis. However, because the ratio of the major to minor semiaxis is $\sqrt{\frac{I_x}{I_z}}$ for the Poinsot ellipsoid and is $\frac{I_x}{I_z}$ for the second ellipsoid, the deviation from a spherical shape is more pronounced in the second ellipsoid.

The largest ellipsoid of the second type to be in contact with the Poinsot ellipsoid will lie outside that ellipsoid and touch it at its points of intersection with the *x* axis, and the smallest will lie inside the Poinsot ellipsoid and touch it at its points of intersection with the *z* axis (see left-hand sketch). All ellipsoids of the second type comprised between these two will intersect the Poinsot ellipsoid along the curves called polhodes, as shown in the right-hand figure.

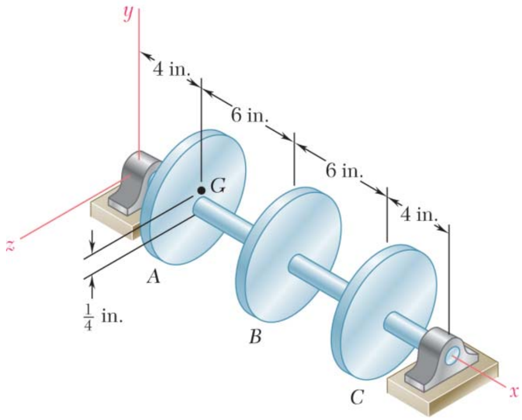


Note that the ellipsoid of the second type, which has the same intermediate axis as the Poinsot ellipsoid, intersects that ellipsoid along two ellipses whose planes contain the *y* axis. These curves are not polhodes, since the tip of ω will not describe them, but they separate the polhodes into four groups. Two groups loop around the minor axis (*x* axis) and the other two around the major axis (*z* axis).

(*c*) If the body is set to spin about one of the principal axes, the Poinsot ellipsoid will remain in contact with the invariable plane at the same point (on the *x*, *y*, or *z* axis); the rotation is steady. In any other case, the point of contact will be located on one of the polhodes, and the tip of ω will start describing that polhode, while the Poinsot ellipsoid rolls on the invariable plane.

A rotation about the *minor* or the *major* axis (*x* or *z* axis) is *stable*. If that motion is disturbed, the tip of ω will move to a very small polhode surrounding that axis and stay close to its original position.

On the other hand, a rotation about the *intermediate* axis (*y* axis) is *unstable.* If that motion is disturbed, the tip of ω will move to one of the polhodes located near that axis and start describing it, departing completely from its original position and causing the body to tumble.

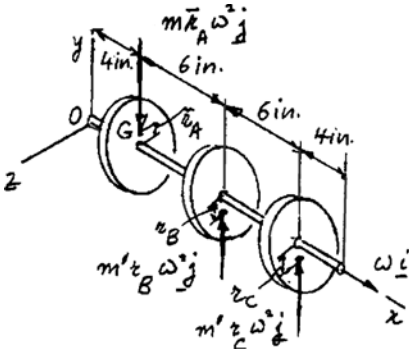


PROBLEM 18.147

Three 25-lb rotor disks are attached to a shaft which rotates at 720 rpm. Disk *A* is attached eccentrically so that its mass center is $\frac{1}{4}$ in. from the axis of rotation, while disks *B* and *C* are attached so that their mass centers coincide with the axis of rotation. Where should 2-lb weights be bolted to disks *B* and *C* to balance the system dynamically?

SOLUTION

The system is dynamically balanced if the effective forces are equivalent to zero. Let Points *A*, *B*, and *C* be the points where the disks are attached to the shaft and let Point *G* be the mass center of disk *A*. Let *m* be the mass of each rotor and *m*′ the magnitude of each added mass.



Since $\omega = \text{constant},$

$$\dot{H}_G = 0$$

for each disk. We treat the added masses as particles so that their moments of inertia about their mass centers are negligible. The effective force of disk *A* is.

$$(F_A)_{\text{eff}} = -m\bar{r}_A\omega^2\mathbf{j}$$

The effective forces of the added masses are

$$m' r_B \omega^2 \mathbf{j} \quad \text{and} \quad m' r_C \omega^2 \mathbf{j},$$

respectively for the masses added to disks *B* and *C*.

$$\Sigma F_{\text{eff}} = 0: \quad -m\bar{r}_A\omega^2\mathbf{j} + m' r_B\omega^2\mathbf{j} + m' r_C\omega^2\mathbf{j} = 0$$

$$r_B + r_C = \frac{m}{m'}\bar{r}_A \tag{1}$$

$$\Sigma(M_0)_{\text{eff}} = 0 \quad 4\mathbf{i}\times(-m\bar{r}_A\omega^2\mathbf{j}) + 10\mathbf{i}\times(m' r_B\omega^2\mathbf{j}) + 16\mathbf{i}\times(m' r_C\omega^2\mathbf{j})$$

$$10r_B + 16r_2 = 4\frac{m}{m'}\bar{r}_A \tag{2}$$

PROBLEM 18.147 (Continued)

Data: $\bar{r}_A = \frac{1}{4}\text{ in.} = 0.25\text{ in.} \qquad \frac{m}{m'} = \frac{25\text{ lb}}{2\text{ lb}} = 12.5$

$$r_A + r_B = 3.125\text{ in.} \tag{1$'$}$$

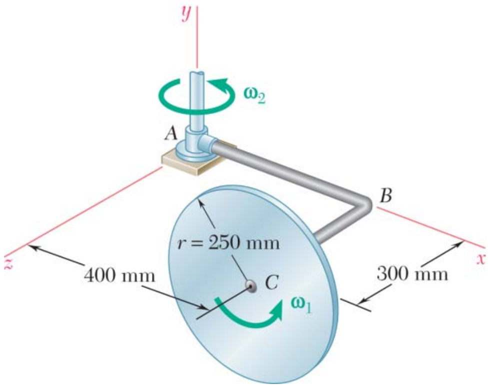
$$10r_A + 16r_B = 12.5\text{ in.} \tag{2$'$}$$

Solving the simultaneous equations.

$$r_B = 6.25\text{ in.} \qquad r_C = -3.125$$

Placement of added masses.

On B: $6\frac{1}{4}$ in. below shaft. On C: $3\frac{1}{8}$ in. above shaft ◀

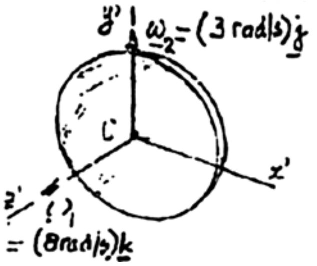


PROBLEM 18.148

A homogeneous disk of mass $m = 5$ kg rotates at the constant rate $\omega_1 = 8$ rad/s with respect to the bent axle *ABC*, which itself rotates at the constant rate $\omega_2 = 3$ rad/s about the *y* axis. Determine the angular momentum $\mathbf{H}_C$ of the disk about its center *C*.

SOLUTION

Using frame $Cx'y'z'$:



$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4}mr^2$$

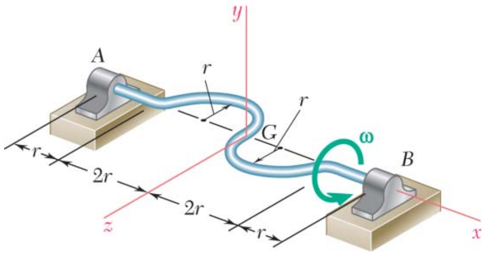
$$\bar{I}_{z'} = \frac{1}{2}mr^2$$

$$\mathbf{H}_C = \bar{I}_{y'}\omega_2\mathbf{j} + I_{z'}\omega_1\mathbf{k}$$

$$= \frac{1}{4}mr^2(\omega_2\mathbf{j} + 2\omega_1\mathbf{k})$$

$$\mathbf{H}_C = \frac{1}{4}(5\text{ kg})(0.25\text{ m})^2[(3\text{ rad/s})\mathbf{j} + 2(8\text{ rad/s})\mathbf{k}]$$

$$\mathbf{H}_C = (0.234\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (1.250\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k} \blacktriangleleft$$



PROBLEM 18.149

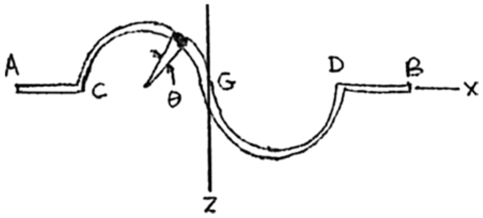
A rod of uniform cross section is used to form the shaft shown. Denoting by m the total mass of the shaft and knowing that the shaft rotates with a constant angular velocity $\boldsymbol{\omega}$, determine (a) the angular momentum $\mathbf{H}_G$ of the shaft about its mass center G, (b) the angle formed by $\mathbf{H}_G$ and the axis AB, (c) the angular momentum of the shaft about Point A.

SOLUTION

Length of rod: $L = 2r + 2\pi r = 8.2832r$

Angular velocity: $\boldsymbol{\omega} = \omega\mathbf{i}$

Angular momentum: $\mathbf{H}_G = \bar{I}_x\omega\mathbf{i} - \bar{I}_{xz}\omega\mathbf{k}$



Calculation of $\bar{I}_x$ and $\bar{I}_{xz}$:

Let $\rho =$ mass per unit length $= \dfrac{m}{L}.$

For portions AC and DB,

$$I_x = 0, \quad I_{xz} = 0$$

For portion CG, use polar coordinate θ.

$$x = -r(1-\cos\theta), \quad z = -r\sin\theta \quad dm = \rho r d\theta$$

$$I_x = \int z^2 dm = \int_0^\pi r^2 \sin^2\theta \rho r d\theta = \frac{\pi}{2}\rho r^3$$

$$I_{xz} = \int xz dm = \int_0^\pi r^2(1-\cos\theta)\sin\theta \rho r d\theta = 2\rho r^3$$

Likewise, for portion GD, $I_x = \dfrac{\pi}{2}\rho r^3, \quad I_{xz} = 2\rho r^3$

Total: $\bar{I}_x = \pi\rho r^3 = \dfrac{\pi m r^3}{L}, \quad \bar{I}_{xz} = 4\rho r^3 = \dfrac{4mr^3}{L}$

(a) Angular momentum $\mathbf{H}_G$. $\mathbf{H}_G = \dfrac{\pi m r^3}{L}\omega\mathbf{i} - \dfrac{4mr^3}{L}\omega\mathbf{k} = mr^2\omega(0.37927\mathbf{i} - 0.48291\mathbf{k})$

$$\mathbf{H}_G = mr^2\omega(0.379\mathbf{i} - 0.483\mathbf{k}) \blacktriangleleft$$

$$H_G = mr^2\omega(0.37927^2 + 0.48291^2)^{1/2} = 0.61404\, mr^2\omega$$

PROBLEM 18.149 (Continued)

(*b*) Angle formed by $\mathbf{H}_G$ and the axis AB.

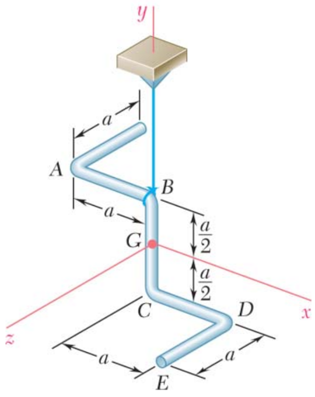
$$\mathbf{H}_G \cdot \mathbf{i} = 0.37927mr^2\omega$$

$$\cos\theta = \frac{\mathbf{H}_G \cdot \mathbf{i}}{H_G} = \frac{0.37927mr^2\omega}{0.61404mr^2\omega} = 0.61766 \qquad \theta = 51.9° \blacktriangleleft$$

(*c*) Angular momentum about Point A:

$$\mathbf{H}_A = \mathbf{H}_G + \mathbf{r}_{G/A} \times (m\bar{v})$$

$$\mathbf{H}_A = \mathbf{H}_G + 0 \qquad \mathbf{H}_A = mr^2\omega(0.379\mathbf{i} - 0.483\mathbf{k}) \blacktriangleleft$$



PROBLEM 18.150

A uniform rod of mass m and length $5a$ is bent into the shape shown and is suspended from a wire attached at Point B. Knowing that the rod is hit at Point A in the negative y direction and denoting the corresponding impulse by $-(F\Delta t)\mathbf{j}$, determine immediately after the impact (a) the velocity of the mass center G, (b) the angular velocity of the rod.

SOLUTION

Moments and products of inertia:

Part	m	I_x	I_y	I_z	I_{xy}
OA	$\frac{1}{5}m$	$\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+\frac{1}{4}a^2+\frac{1}{4}a^2\right)$	$\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+a^2+\frac{1}{4}a^2\right)$	$\left(\frac{1}{5}m\right)\left(a^2+\frac{1}{4}a^2\right)$	$\left(\frac{1}{5}m\right)\left(-\frac{1}{2}a^2\right)$
AB	$\frac{1}{5}m$	$\left(\frac{1}{5}m\right)\left(\frac{1}{4}a^2\right)$	$\left(\frac{1}{5}m\right)\left(\frac{1}{3}a^2\right)$	$\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+\frac{1}{4}a^2+\frac{1}{4}a^2\right)$	$\left(\frac{1}{5}m\right)\left(-\frac{1}{4}a^2\right)$
BC	$\frac{1}{5}m$	$\left(\frac{1}{5}m\right)\left(\frac{1}{12}a^2\right)$	0	$\left(\frac{1}{5}m\right)\left(\frac{1}{12}a^2\right)$	0
CD	$\frac{1}{5}m$	$\left(\frac{1}{5}m\right)\left(\frac{1}{4}a^2\right)$	$\left(\frac{1}{5}m\right)\left(\frac{1}{3}a^2\right)$	$\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+\frac{1}{4}a^2+\frac{1}{4}a^2\right)$	$\left(\frac{1}{5}m\right)\left(-\frac{1}{4}a^2\right)$
DE	$\frac{1}{5}m$	$\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+\frac{1}{4}a^2+\frac{1}{4}a^2\right)$	$\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+a^2+\frac{1}{4}a^2\right)$	$\left(\frac{1}{5}m\right)\left(a^2+\frac{1}{4}a^2\right)$	$\left(\frac{1}{5}m\right)\left(-\frac{1}{2}a^2\right)$
Σ	m	$0.35ma^2$	$0.66667ma^2$	$0.75ma^2$	$-0.3ma^2$

$$I_{xz}=\left(\frac{1}{5}m\right)\left(\frac{1}{2}a^2\right)+\left(\frac{1}{5}m\right)\left(\frac{1}{2}a^2\right)=0.2ma^2$$

$$I_{yz}=\left(\frac{1}{5}m\right)\left(-\frac{1}{4}a^2\right)+\left(\frac{1}{5}m\right)\left(-\frac{1}{4}a^2\right)=-0.1ma^2$$

Angular momentum about the mass center.

$$(H_G)_x=I_x\omega_x-I_{xy}\omega_y-I_{xz}\omega_z=0.35ma^2\omega_x+0.3ma^2\omega_y-0.2ma^2\omega_z$$

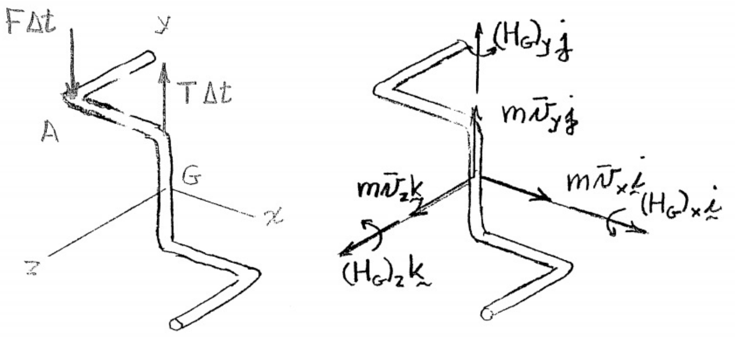
$$(H_G)_y=-I_{xy}\omega_x+I_y\omega_y-I_{yz}\omega_z=0.3ma^2\omega_x+0.66667ma^2\omega_y+0.1ma^2\omega_z$$

$$(H_G)_z=-I_{xz}\omega_x-I_{yz}\omega_z+I_z\omega_z=-0.2ma^2\omega_x+0.1ma^2\omega_y+0.75ma^2\omega_z$$

PROBLEM 18.150 (Continued)

Constraint of the supporting cable: $\bar{v}_y = 0$

Impulse-momentum principle: Before impact, $\bar{\mathbf{v}} = 0$, $\mathbf{H}_G = 0$.



(*a*) Linear momentum: $\mathbf{F}(\Delta t) + T\Delta t\mathbf{j} = m\bar{\mathbf{v}}$ Resolve into components.

$$0 = m\bar{v}_x, \qquad -F\Delta t + T\Delta t\mathbf{j} = 0, \qquad 0 = m\bar{v}_z$$

$$\bar{v}_x = 0, \qquad T\Delta t = F\Delta t, \qquad \bar{v}_z = 0 \qquad\qquad \bar{\mathbf{v}} = 0 \blacktriangleleft$$

(*b*) Angular momentum, moments about *G*: $\mathbf{r}_{A/G} \times \mathbf{F}\Delta t = \mathbf{H}_G$

$$\left(\frac{a}{2}\mathbf{j} - a\mathbf{i}\right) \times [-(F\Delta t)\mathbf{j}] = (aF\Delta t)\mathbf{k} = (H_G)_x\mathbf{i} + (H_G)_y\mathbf{j} + (H_G)_z\mathbf{k}$$

Using expressions for $(H_G)_x$, $(H_G)_y$, and $(H_G)_z$ and resolving into components,

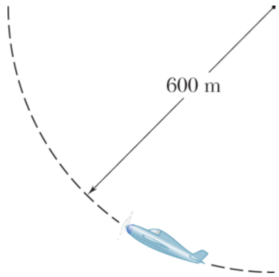
$$\mathbf{i}: \quad 0 = 0.35ma^2\omega_x + 0.3ma^2\omega_y - 0.2ma^2\omega_z$$

$$\mathbf{j}: \quad 0 = 0.3ma^2\omega_x + 0.66667ma^2\omega_y + 0.1ma^2\omega_z$$

$$\mathbf{k}: \quad aF\Delta t = -0.2ma^2\omega_x + 0.1ma^2\omega_y + 0.75ma^2\omega_z$$

Solving, $\omega_x = 2.5\dfrac{(F\Delta t)}{ma}$, $\omega_y = -1.454\dfrac{(F\Delta t)}{ma}$, $\omega_z = 2.19\dfrac{(F\Delta t)}{ma}$

$$\boldsymbol{\omega} = \left(\frac{F\Delta t}{ma}\right)(2.50\mathbf{i} - 1.454\mathbf{j} + 2.19\mathbf{k}) \blacktriangleleft$$

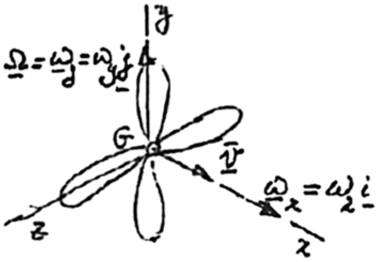


PROBLEM 18.151

A four-bladed airplane propeller has a mass of 160 kg and a radius of gyration of 800 mm. Knowing that the propeller rotates at 1600 rpm as the airplane is traveling in a circular path of 600-m radius at 540 km/h, determine the magnitude of the couple exerted by the propeller on its shaft due to the rotation of the airplane.

SOLUTION

We assume senses shown for $\boldsymbol{\omega}_{x,}$ $\boldsymbol{\omega}_y$, and $\mathbf{v}$.


y
$\boldsymbol{\Omega} = \boldsymbol{\omega}_y = \omega_y \mathbf{j}$
G
$\bar{\mathbf{v}}$
$\boldsymbol{\omega}_x = \omega_x \mathbf{i}$
z
x

$$\bar{v} = 540\text{ km/h} = 150\text{ m/s}$$

$$\omega_x = 1600\text{ rpm}\left(\frac{2\pi\text{ rad}}{60\text{ s}}\right)$$

$$= 167.55\text{ rad/s}$$

$$\omega_y = \frac{\bar{v}}{\rho} = \frac{150\text{ m/s}}{600\text{ m}} = 0.25\text{ rad/s}$$

$$\bar{I}_x = m\bar{k}^2 = (160\text{ kg})(0.8\text{ m})^2 = 102.4\text{ kg}\cdot\text{m}^2$$

Angular momentum about G:

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j}$$

Eq. (18.22),

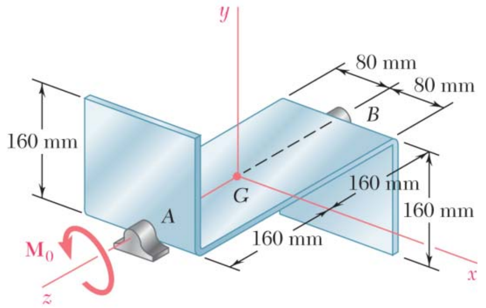
$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega}\times\mathbf{H}_G = 0 + \omega_y\mathbf{j}\times(\bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j})$$

$$\dot{\mathbf{H}}_G = -\bar{I}_x\omega_x\omega_y\mathbf{k} = -(102.4\text{ kg}\cdot\text{m}^2)(167.55\text{ rad/s})(0.25\text{ rad/s})\mathbf{k}$$

$$\dot{\mathbf{H}}_G = -(4289\text{ N}\cdot\text{m})\mathbf{k} = -(4.29\text{ kN}\cdot\text{m})\mathbf{k}$$

The couple exerted *on* the propeller, therefore, must be $\mathbf{M} = \dot{\mathbf{H}}_G = -(4.29\text{ kN}\cdot\text{m})\mathbf{k}$, and the couple exerted *by* the propeller on its shaft is $-\mathbf{M} = (4.29\text{ kN}\cdot\text{m})\mathbf{k}$.

Magnitude of couple. 4.29 kN · m ◀



PROBLEM 18.152

A 2.4-kg piece of sheet steel with dimensions 160×640 mm was bent to form the component shown. The component is at rest $(\omega = 0)$ when a couple $\mathbf{M}_0 = (0.8\ \text{N}\cdot\text{m})\mathbf{k}$ is applied to it. Determine (*a*) the angular acceleration of the component, (*b*) the dynamic reactions at *A* and *B* immediately after the couple is applied.

SOLUTION

$$m = 2.4\text{ kg}, \quad b = 160\text{ mm} = 0.16\text{ m}$$

Area of sheet metal:

$$A = b^2 + (2b)b + b^2 = 4b^2 = 0.1024\text{ m}^2$$

Let

$$\rho = \frac{m}{A} = \frac{2.4}{0.1024}$$
$$= 23.4375\text{ kg/m}^2$$
$$= \text{mass per unit area}$$

Angular velocity and angular acceleration:

$$\boldsymbol{\omega} = \omega\mathbf{k}$$
$$\boldsymbol{\alpha} = \alpha\mathbf{k}$$

Angular momentum about *G*:

$$\mathbf{H}_G = -I_{xz}\omega\mathbf{i} - I_{yz}\omega\mathbf{j} + I_z\omega\mathbf{k}$$
$$= -I_{yz}\omega\mathbf{j} + I_z\omega\mathbf{k}$$

Let the frame of reference *Gxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega\mathbf{k}$

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$
$$= -I_{yz}\alpha\mathbf{j} + I_z\alpha\mathbf{k} + I_{yz}\omega^2\mathbf{i}$$

Required moment and product of inertia: $I_{\text{mass}} = \rho I_{\text{area}}$

Part	A	I_z	I_{yz}
plate *A*	b^2	$\frac{1}{6}b^2 + b^2\left(\frac{1}{2}b\right)^2$	$b^2(b)\left(\frac{1}{2}b\right)$
plate *AB*	$2b^2$	$\frac{1}{12}(2b)b^3$	0
plate *B*	b^2	$\frac{1}{6}b^2 + b^2\left(\frac{1}{2}b\right)^2$	$b^2(-b)\left(-\frac{1}{2}b\right)$
Σ	$4b^2$	b^4	b^4

$$I_z = \rho b^4$$
$$= (23.4375)(0.16)^4$$
$$= 0.01536\text{ kg}\cdot\text{m}^2$$
$$I_{yz} = \rho b^4$$
$$= 0.01536\text{ kg}\cdot\text{m}^2$$

PROBLEM 18.152 (Continued)

Since the mass center lies on the rotation axis, $\overline{\mathbf{a}} = 0$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = m\overline{\mathbf{a}}$$

$$\mathbf{B} = -\mathbf{A}$$

$$\begin{aligned}\Sigma\mathbf{M}_G &= M_0\mathbf{k} + b\mathbf{k} \times \mathbf{A} + (-b\mathbf{k}) \times \mathbf{B} \\ &= M_0\mathbf{k} + 2b\mathbf{k} \times (A_x\mathbf{i} + A_y\mathbf{j}) \\ &= M_0\mathbf{k} + 2bA_x\mathbf{j} - 2bA_y\mathbf{i}\end{aligned}$$

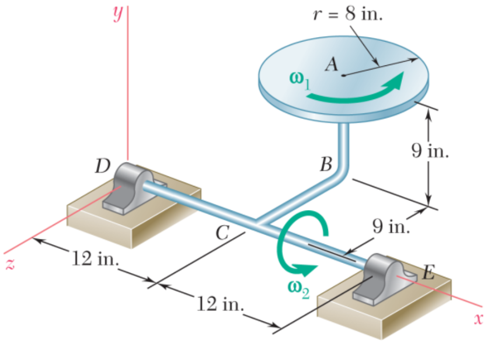
$\Sigma\mathbf{M}_G = \dot{\mathbf{H}}_G$ Resolve into components.

(*a*) $\mathbf{k}$: $M_0 = I_z\alpha \quad \alpha = \dfrac{M_0}{I_z} = \dfrac{0.8}{0.01536} = 52.083 \text{ rad/s}^2$ $\alpha = 52.1 \text{ rad/s}^2$ ◀

(*b*) $\mathbf{j}$: $2bA_x = -I_{yz}\alpha \quad A_x = -\dfrac{I_{yz}\alpha}{2b} = -\dfrac{(0.01536)(52.083)}{(2)(0.16)} = -2.50 \text{ N}$

$\mathbf{i}$: $-2bA_y = I_{yz}\omega^2 = 0 \qquad A_y = 0$ $\mathbf{A} = -(2.50 \text{ N})\mathbf{i}$ ◀

$\mathbf{B} = (2.50 \text{ N})\mathbf{i}$ ◀



PROBLEM 18.153

A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm *ABC*, which is welded to a shaft *DCE* rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the dynamic reactions at *D* and *E*.

SOLUTION

Angular velocity of shaft *DE* and arm *CBA*: $\boldsymbol{\Omega} = \omega_2\mathbf{i}$

Angular velocity of disk *A*: $\boldsymbol{\omega} = \omega_2\mathbf{i} + \omega_1\mathbf{j}$

Angular velocity about its mass center *A*:

$$\mathbf{H}_A = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$= \bar{I}_x\omega_2\mathbf{i} + \bar{I}_y\omega_1\mathbf{j}$$

$$= \frac{1}{4}mr^2\omega_2\mathbf{i} + \frac{1}{2}mr^2\omega_1\mathbf{j}$$

Let the reference frame *Oxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2\mathbf{i}$.

$$\dot{\mathbf{H}}_A = (\dot{\mathbf{H}}_A)_{Oxyz} + \boldsymbol{\Omega}\times\mathbf{H}_A$$

$$= \frac{1}{4}mr^2\dot{\omega}_2\mathbf{i} + \frac{1}{2}mr^2\dot{\omega}_1\mathbf{j} + \omega_2\mathbf{i}\times\mathbf{H}_A$$

$$= \frac{1}{4}mr^2\dot{\omega}_2\mathbf{i} + \frac{1}{2}mr^2\dot{\omega}_1\mathbf{j} + \frac{1}{2}mr^2\omega_1\omega_2\mathbf{k}$$

Velocity of Point *A*:

$$\mathbf{v}_A = \omega_2\mathbf{i}\times\mathbf{r}_{A/O}$$

$$= \omega_2\mathbf{i}\times(-b\mathbf{k} + c\mathbf{j})$$

$$= b\omega_2\mathbf{j} + c\omega_2\mathbf{k}$$

Acceleration of Point *A*:

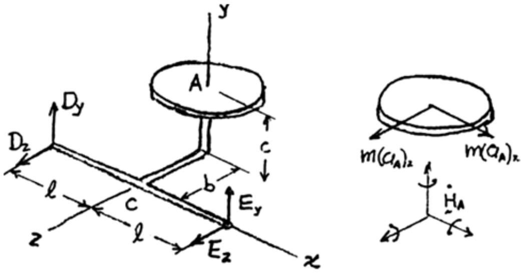
$$\mathbf{a}_A = \dot{\omega}_2\mathbf{j}\times\mathbf{r}_{A/O} + \omega_2\mathbf{j}\times\mathbf{v}_A$$

$$\mathbf{a}_A = (b\dot{\omega}_2 - c\omega_2^2)\mathbf{j} + (c\dot{\omega}_2 + b\omega_2)\mathbf{k}$$

Consider the system of particles consisting of the shaft, the arm, and the disk. Neglect the mass of the arm.

$$\Sigma\mathbf{F} = m\mathbf{a}_A$$

$$D_y\mathbf{j} + D_z\mathbf{k} + E_y\mathbf{j} + E_z\mathbf{k} = m\mathbf{a}_A$$



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PROBLEM 18.153 (Continued)

Resolve into components.

$$D_y + E_y = m(b\dot{\omega}_2 - c\omega_2^2)$$

$$D_z + E_z = m(c\dot{\omega}_2 + b\omega_2^2)$$

$$\Sigma \mathbf{M}_D = \dot{\mathbf{H}}_D = \dot{\mathbf{H}}_A + \mathbf{r}_{A/D} \times m\mathbf{a}_A$$

$$(M_0)\mathbf{i} + 2l\mathbf{i} \times (E_y\mathbf{j} + E_z\mathbf{k}) = \dot{\mathbf{H}}_A + (l\mathbf{i} + c\mathbf{j} - b\mathbf{k}) \times m\mathbf{a}_A$$

$$(M_0)\mathbf{i} - 2lE_z\mathbf{j} + 2lE_y\mathbf{k} = m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2\mathbf{i} + m\left(\frac{1}{2}r^2\dot{\omega}_1 - lc\dot{\omega}_2 - lb\omega_2^2\right)\mathbf{j}$$

$$+ m\left(\frac{1}{2}r^2\omega_1\omega_2 + lb\dot{\omega}_2 - lc\omega_2^2\right)\mathbf{k}$$

i: $$M_0 = m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2$$

k: $$E_y = \frac{m}{2l}\left(\frac{1}{2}r^2\omega_1\omega_2 + lb\dot{\omega}_2 - lc\omega_2^2\right)$$

$$D_y = \frac{m}{2l}\left(-\frac{1}{2}r^2\omega_1\omega_2 + lb\dot{\omega}_2 - lc\omega_2^2\right)$$

j: $$E_z = \frac{m}{2l}\left(lc\dot{\omega}_2 + lb\omega_2^2 - \frac{1}{2}r^2\dot{\omega}_1\right)$$

$$D_z = \frac{m}{2l}\left(lc\dot{\omega}_2 + lb\omega_2^2 + \frac{1}{2}r^2\dot{\omega}_1\right)$$

Data: $W = 6 \text{ lb.} \quad m = \frac{6}{32.2} = 0.186335 \quad r = 8 \text{ in.} = 0.66667 \text{ ft}$

$b = c = 9 \text{ in.} = 0.75 \text{ ft} \quad l = 12 \text{ in.} = 1.0 \text{ ft}$

$\omega_1 = 16 \text{ rad/s}, \quad \dot{\omega}_1 = 0, \quad \omega_2 = 8 \text{ rad/s}, \quad \dot{\omega}_2 = 0$

$$D_y = \frac{0.186335}{(2)(1.0)}\left[-\left(\frac{1}{2}\right)(0.66667)^2(16)(8) + 0 - (1.0)(0.75)(8)^2\right] = -7.12 \text{ lb}$$

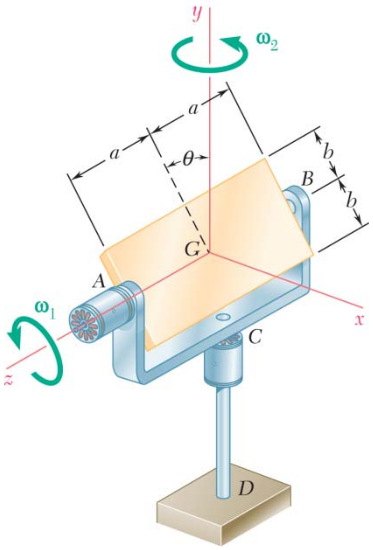
$$D_z = \frac{0.186335}{(2)(1.0)}[0 + (1.0)(0.75)(8)^2 + 0] = 4.47 \text{ lb}$$

$\mathbf{D} = -(7.12 \text{ lb})\mathbf{j} + (4.47 \text{ lb})\mathbf{k}$ ◀

$$E_y = \frac{0.186335}{(2)(1.0)}\left[\left(\frac{1}{2}\right)(0.66667)^2(16)(8) + 0 - (1.0)(0.75)(8)^2\right] = -1.822 \text{ lb}$$

$$E_z = \frac{0.186335}{(2)(1.0)}[0 + (1.0)(0.75)(8)^2 + 0] = 4.47 \text{ lb}$$

$\mathbf{E} = -(1.822 \text{ lb})\mathbf{j} + (4.47 \text{ lb})\mathbf{k}$ ◀

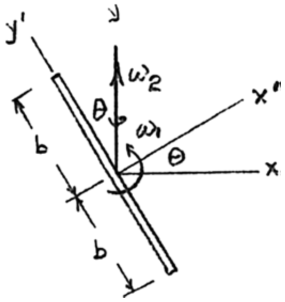


PROBLEM 18.154

A 48-kg advertising panel of length $2a = 2.4$ m and width $2b = 1.6$ m is kept rotating at a constant rate ω_1 about its horizontal axis by a small electric motor attached at *A* to frame *ACB*. This frame itself is kept rotating at a constant rate ω_2 about a vertical axis by a second motor attached at *C* to the column *CD*. Knowing that the panel and the frame complete a full revolution in 6 s and 12 s, respectively, express, as a function of the angle θ, the dynamic reaction exerted on column *CD* by its support at *D*.

SOLUTION

Use principal axes x', y', z' as shown.



Moments of inertia: $\bar{I}_{x'} = \frac{1}{3}m(a^2 + b^2)$

$$\bar{I}_{y'} = \frac{1}{3}ma^2, \quad \bar{I}_{z'} = \frac{1}{3}mb^2$$

Kinematics: $\dot{\theta} = \omega_1$

$$\omega_{x'} = \omega_2 \sin\theta, \quad \omega_{y'} = \omega_2 \cos\theta, \quad \omega_{z'} = \omega_1$$

$$\dot{\omega}_{x'} = \omega_1\omega_2 \cos\theta, \quad \dot{\omega}_{y'} = -\omega_1\omega_2 \sin\theta, \quad \dot{\omega}_{z'} = 0$$

Since the acceleration of the mass center is zero, the resultant force acting on the column *CD* is zero.

$\mathbf{R} = 0$ ◀

Euler's equations of motion for the plate:

$$\Sigma M_{x'} = \bar{I}_{x'}\dot{\omega}_{x'} - (\bar{I}_{y'} - \bar{I}_{z'})\omega_{y'}\omega_{z'}$$

$$= \bar{I}_{x'}\omega_1\omega_2 \cos\theta - (\bar{I}_{y'} - \bar{I}_{z'})\omega_1\omega_2 \cos\theta$$

$$= (\bar{I}_{x'} + \bar{I}_{z'} - \bar{I}_{y'})\omega_1\omega_2 \cos\theta = \frac{2}{3}mb^2\omega_1\omega_2 \cos\theta$$

$$\Sigma M_{y'} = \bar{I}_{y'}\dot{\omega}_{y'} - (\bar{I}_{z'} - \bar{I}_{x'})\omega_{x'}\omega_{z'}$$

$$= -\bar{I}_y\omega_1\omega_2 \sin\theta - (\bar{I}_z - \bar{I}_x)\omega_1\omega_2 \sin\theta$$

$$= (\bar{I}_{x'} - \bar{I}_{y'} - \bar{I}_{z'})\omega_1\omega_2 \sin\theta = 0$$

$$\Sigma M_{z'} = I_{z'}\dot{\omega}_{z'} - (\bar{I}_{x'} - \bar{I}_y)\omega_{x'}\omega_{y'}$$

$$= 0 - (\bar{I}_{x'} - \bar{I}_y)\omega_2^2 \sin\theta\cos\theta$$

$$= -\frac{1}{3}mb^2\omega_2^2 \sin\theta\cos\theta$$

$$\Sigma\mathbf{M} = \frac{2}{3}mb^2\omega_1\omega_2\cos\theta\mathbf{i}' - \frac{1}{3}mb^2\omega_2^2\sin\theta\cos\theta\mathbf{k}$$

$$= \frac{2}{3}mb^2\omega_1\omega_2\cos\theta(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) - \frac{1}{3}mb^2\omega_2^2\sin\theta\cos\theta\mathbf{k}$$

Resolve into components: $$\Sigma M_x = \frac{2}{3}mb^2\omega_1\omega_2\cos^2\theta$$

$$\Sigma M_y = \frac{2}{3}mb^2\omega_1\omega_2\sin\theta\cos\theta$$

$$\Sigma M_z = -\frac{1}{3}mb^2\omega_2^2\sin\theta\cos\theta$$

Data: $m = 48\text{ kg},\qquad b = 0.8\text{ m}$

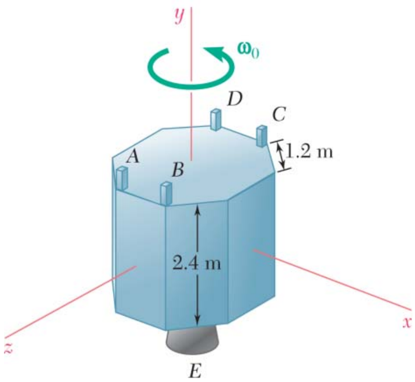
$$\omega_1 = \frac{2\pi}{6} = 1.0472\text{ rad/s},\quad \omega_2 = \frac{2\pi}{12} = 0.5236\text{ rad/s}$$

$$\Sigma M_x = \frac{2}{3}(48)(0.8)^2(1.0472)(0.5236)\cos^2\theta$$
$$= 11.23\cos^2\theta\text{ N}\cdot\text{m}$$

$$\Sigma M_y = \frac{2}{3}(48)(0.8)^2(1.0472)(0.5236)\sin\theta\cos\theta$$
$$= 11.23\sin\theta\cos\theta\text{ N}\cdot\text{m}$$

$$\Sigma M_z = -\frac{1}{3}(48)(0.8)^2(0.5236)^2\sin\theta\cos\theta$$
$$= -2.81\sin\theta\cos\theta\text{ N}\cdot\text{m}$$

$$\mathbf{M}_D = (11.23\text{ N}\cdot\text{m})\cos^2\theta\mathbf{i} + (11.23\text{ N}\cdot\text{m})\sin\theta\cos\theta\mathbf{j} - (2.81\text{ N}\cdot\text{m})\sin\theta\cos\theta\mathbf{k}$$ ◀



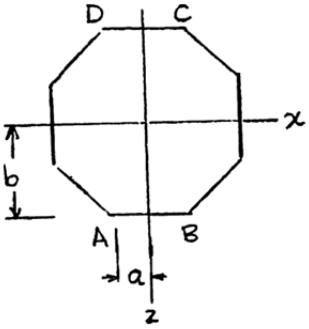
PROBLEM 18.155

A 2500-kg satellite is 2.4 m high and has octagonal bases of sides 1.2 m. The coordinate axes shown are the principal centroidal axes of inertia of the satellite, and its radii of gyration are $k_x = k_z = 0.90$ m and $k_y = 0.98$ m. The satellite is equipped with a main 500-N thruster *E* and four 20-N thrusters *A*, *B*, *C*, and *D*, which can expel fuel in the positive *y* direction. The satellite is spinning at the rate of 36 rev/h about its axis of symmetry G_y, which maintains a fixed direction in space, when thrusters *A* and *B* are activated for 2 s. Determine (*a*) the precession axis of the satellite, (*b*) its rate of precession, (*c*) its rate of spin.

SOLUTION

$$\omega_0 = (36 \text{ rev/h})\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right)$$
$$= 0.062832 \text{ rad/s}$$
$$m = 2500 \text{ kg}$$
$$I_x = mk_x^2 = (2500)(0.90)^2 = 2025 \text{ kg}\cdot\text{m}^2$$
$$I_z = I_x = 2025 \text{ kg}\cdot\text{m}^2$$
$$I_y = mk_y^2 = (2500)(0.98)^2 = 2401 \text{ kg}\cdot\text{m}^2$$
$$(\mathbf{H}_G)_0 = I_y\omega_0\mathbf{j} = (2401)(0.062832)\mathbf{j} = (150.86 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j}$$
$$a = 0.6 \text{ m}, \quad b = 0.6 + 1.2 \cos 45° = 1.4485$$

When thrusters *A* and *B* are activated,



$$\mathbf{M}_G = -b(F_A + F_B)\mathbf{i}$$
$$= -(1.4485)(40)\mathbf{i}$$
$$= -(57.941 \text{ N}\cdot\text{m})\mathbf{i}$$

Angular momentum after 2 s:

$$\mathbf{H}_G = (\mathbf{H}_G)_0 + \mathbf{M}_G(\Delta t)$$
$$= 150.86\mathbf{j} + (-57.941)(2)\mathbf{i}$$
$$= -(115.88 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (150.86 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j}$$
$$\omega_x = \frac{H_x}{I_x} = -\frac{115.88}{2025} = -0.057225 \text{ rad/s} = -32.788 \text{ rev/h}$$
$$\omega_y = \frac{H_y}{I_y} = 36 \text{ rev/h}$$
$$\omega_z = \frac{H_z}{I_z} = 0$$

(*a*) Precession axis:

$$\tan\theta = -\frac{H_x}{H_y} = \frac{115.88}{150.86} \quad \theta = 37.529°$$

$\theta_x = 52.5°, \quad \theta_y = 37.5°, \quad \theta_z = 90°$ ◀

$$\tan\gamma = -\frac{\omega_x}{\omega_y}$$

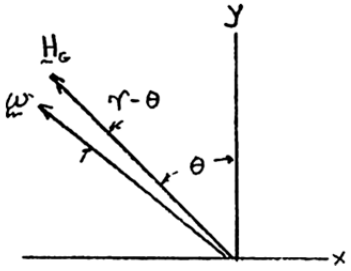
$$= \frac{32.788}{36}$$

$$\gamma = 42.327°$$

$$\gamma - \theta = 4.798°$$

$$\omega = \sqrt{\omega_x^2 + \omega_y^2}$$

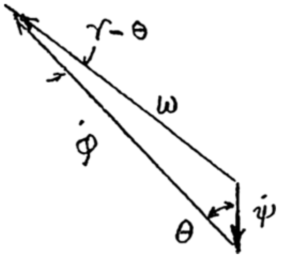
$$= 48.693 \text{ rev/h}$$



Y
$\underset{\sim}{H}_G$
$\underset{\sim}{\omega}$
γ - θ
θ
x

Law of sines.

$$\frac{\dot{\varphi}}{\sin\gamma} = \frac{\dot{\psi}}{\sin(\gamma - \theta)} = \frac{\omega}{\sin\theta}$$



γ - θ
ω
$\dot{\varphi}$
$\dot{\psi}$
θ

(*b*)

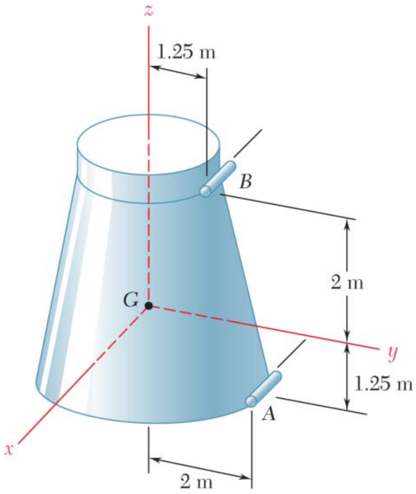
$$\dot{\varphi} = \frac{48.693 \sin 42.327°}{\sin 37.529°}$$

$\dot{\varphi} = 53.8$ rev/h ◀

(*c*)

$$\dot{\psi} = \frac{48.693 \sin 4.798°}{\sin 37.529°}$$

$\dot{\psi} = 6.68$ rev/h ◀



PROBLEM 18.156

The space capsule has no angular velocity when the jet at *A* is activated for 1 s in a direction parallel to the *x* axis. Knowing that the capsule has a mass of 1000 kg, that its radii of gyration are $\bar{k}_z = \bar{k}_y = 1.00\text{ m}$ and $\bar{k}_z = 1.25\text{ m}$, and that the jet at *A* produces a thrust of 50 N, determine the axis of precession and the rates of precession and spin after the jet has stopped.

SOLUTION

Initial angular momentum about the mass center:

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} = 0$$

Applied impulse at *A*: $\mathbf{A}\,\Delta t = (50\text{ N})(1\text{ s})\mathbf{i} = (50\text{ N}\cdot\text{s})\mathbf{i}$

Its moment about the mass center *G*:

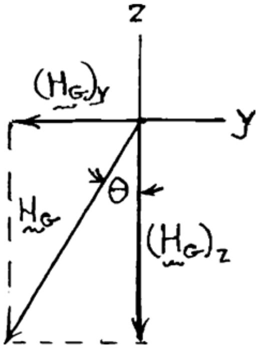
$$\begin{aligned}\mathbf{r}_{A/G} \times \mathbf{A}\,\Delta t &= [(2\text{ m})\mathbf{j} - (1.25\text{ m})\mathbf{k}] \times (50\text{ N}\cdot\text{s})\mathbf{i}\\ &= -(100\text{ N}\cdot\text{m}\cdot\text{s})\mathbf{k} - (62.5\text{ N}\cdot\text{m}\cdot\text{s})\mathbf{j}\end{aligned}$$

Principle of impulse and momentum. (Moments about *G*)

$$(\mathbf{H}_G)_0 = \mathbf{r}_{A/G} \times \mathbf{A}\,\Delta t = \mathbf{H}_G$$

where $\mathbf{H}_G$ is the final angular momentum about *G*.

$$0 + \mathbf{r}_{G/A} \times \mathbf{A}\,\Delta t = (H_G)_x\mathbf{i} + (H_G)_y\mathbf{j} + (H_G)_z\mathbf{k}$$



Angular momentum vector components:

i: $0 = (H_G)_x$

j: $-62.5\text{ N}\cdot\text{m}\cdot\text{s} = (H_G)_y = -62.5\text{ kg}\cdot\text{m}^2/\text{s}$

k: $-100\text{ N}\cdot\text{m}\cdot\text{s} = (H_G)_z = -100\text{ kg}\cdot\text{m}^2/\text{s}$

$$\tan\theta = \frac{(H_G)_y}{(H_G)_z} = 0.625$$

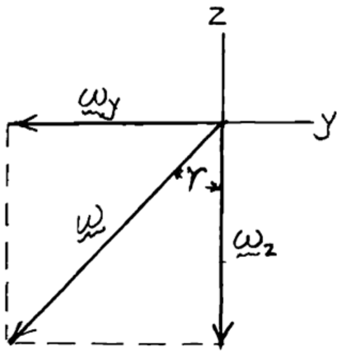
$\theta = 32.005°$ ◀

Moments of inertia:

$$\bar{I}_x = m\bar{k}_k^2 = (1000\text{ kg})(1\text{ m})^2 = 1000\text{ kg}\cdot\text{m}^2$$
$$\bar{I}_y = m\bar{k}_y^2 = (1000\text{ kg})(1\text{ m})^2 = 1000\text{ kg}\cdot\text{m}^2$$
$$\bar{I}_z = m\bar{k}_z^2 = (1000\text{ kg})(1.25\text{ m})^2 = 1562.5\text{ kg}\cdot\text{m}^2$$

Angular velocity vector components:

$$\omega_x = \frac{(H_G)_x}{\bar{I}_x} = 0$$
$$\omega_y = \frac{(H_G)_y}{\bar{I}_y} = \frac{-62.5}{1000} = -0.0625\text{ rad/s}$$
$$\omega_z = \frac{(H_G)_z}{\bar{I}_z} = \frac{-100}{1562.5} = -0.0640\text{ rad/s}$$
$$\tan\gamma = \frac{\omega_y}{\omega_z} = 0.97656 \qquad \gamma = 44.321°$$
$$\omega = \sqrt{\omega_y^2 + \omega_z^2} = 0.089455\text{ rad/s.}$$



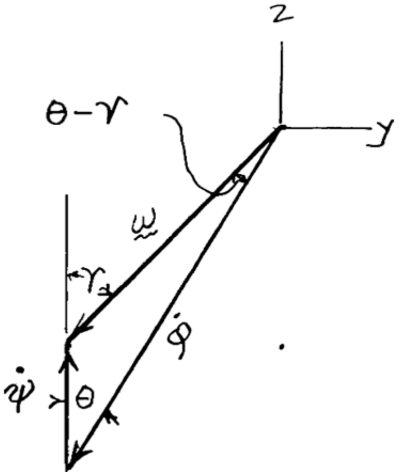
Rates of precession and spin.

The angular velocity is resolved into a component $\dot{\varphi}$ (rate of precession) parallel to $\mathbf{H}_G$ and $\dot{\psi}$ (rate of spin) parallel to the axis of symmetry.

$$\gamma - \theta = 12.316°$$

Law of sines:

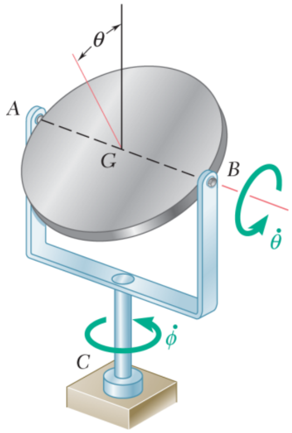
$$\frac{\dot{\varphi}}{\sin\gamma} = \frac{\dot{\psi}}{\sin(\gamma-\theta)} = \frac{\omega}{\sin\theta}$$
$$\dot{\varphi} = \frac{\omega\sin\gamma}{\sin\theta} = \frac{0.089455\sin 44.321°}{\sin 32.005°}$$
$$\dot{\psi} = \frac{\omega\sin(\gamma-\theta)}{\sin\theta} = \frac{0.089455\sin 12.316°}{\sin 32.005°}$$



Rate of precession: $\dot{\varphi} = 0.1179\text{ rad/s}$ $\dot{\varphi} = 1.126\text{ rpm}$ ◀

Rate of spin: $\dot{\psi} = 0.0360\text{ rad/s}$ $\dot{\psi} = 0.344\text{ rpm}$ ◀

Since $\gamma > \theta$, the precession is retrograde.

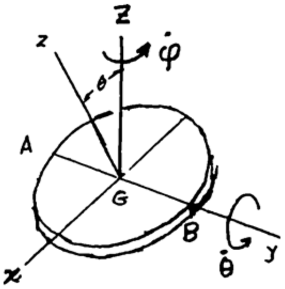


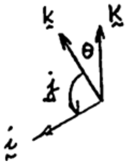
PROBLEM 18.157

A homogeneous disk of mass *m* connected at *A* and *B* to a fork-ended shaft of negligible mass which is supported by a bearing at *C*. The disk is free to rotate about its horizontal diameter *AB* and the shaft is free to rotate about a vertical axis through *C*. Initially, the disk lies in a vertical plane $(\theta_0 = 90°)$ and the shaft has an angular velocity $\dot{\phi}_0 = 8$ rad/s. If the disk is slightly disturbed, determine for the ensuring motion (*a*) the minimum value of $\dot{\phi}$, (*b*) the maximum value of $\dot{\theta}$.

SOLUTION

Place the origin at the center of mass and let *Oxyz* be a principal axis frame of reference with the *y* axis directed along the moving axle *AB*. Let the *Z* axis lie along the fixed axle. Useful unit vectors are **i**, **j** and **k** along the *x*, *y*, *z* axes and **K** along the *Z* axis.





$$\mathbf{K} = -\mathbf{i}\sin\theta + \mathbf{k}\cos\theta$$

Angular velocity:

$$\boldsymbol{\omega} = \dot{\varphi}\mathbf{k} + \dot{\theta}\mathbf{j}$$

$$\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\varphi}\cos\theta\mathbf{k}$$

Moments of inertia:

$$I_x = e_1 I_y, \qquad I_z = e_2 I_y$$

Angular momentum about *O*.

$$\mathbf{H}_O = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= I_y(-e_1\dot{\varphi}\sin\theta\,\mathbf{i} + \dot{\theta}\,\mathbf{j} + e_2\dot{\varphi}\cos\theta\,\mathbf{k})$$

The moment about the fixed *Z* axis is zero, hence, $\mathbf{H}_O \cdot \mathbf{K} = \text{constant}$.

$$\mathbf{H}_O \cdot \mathbf{K} = I_y(e_1\sin^2\theta + e_2\cos^2\theta)\dot{\varphi} = I_y C_1$$

$$\dot{\varphi} = \frac{C_1}{e_1\sin^2\theta + e_2\cos^2\theta}$$

$$C_1 = (e_1\sin^2\theta_0 + e_2\cos^2\theta_0)\dot{\varphi}_0$$

Twice the kinetic energy:

$$2T = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = \text{constant}$$

$$2T = I_y[(e_1\sin^2\theta + e_2\cos^2\theta)\dot{\varphi}^2 + \dot{\theta}^2]$$

$$= I_y(C_1\dot{\varphi} + \dot{\theta}^2)$$

$$= I_y C_2$$

$$\dot{\theta}^2 = C_2 - C_1\dot{\varphi}$$

$$C_2 = \dot{\theta}_0^2 + C_1\dot{\varphi}_0$$

PROBLEM 18.157 (Continued)

Data:

$$e_1 = 1$$

$$e_2 = 2$$

$$e_1 \sin^2\theta + e_2\cos^2\theta = 1 + \cos^2\theta$$

$$\theta_0 = 90°$$

$$\dot{\theta}_0 = 0$$

$$\dot{\varphi}_0 = 16 \text{ rad/s}$$

$$C_1 = (1 + \cos^2 90°)(8)$$

$$= 8 \text{ rad/s}$$

$$\dot{\varphi} = \frac{8}{1 + \cos\theta}$$

(*a*)

$$\dot{\varphi}_{\min} = \frac{8}{1 + \cos\theta} = 4$$

$\dot{\varphi}_{\min} = 4.00$ rad/s ◀

$$C_2 = 0 + (8)(8) = 64 \text{ (rad/s)}^2$$

$$\dot{\theta}^2 = C_2 - C_1\dot{\varphi}$$

(*b*)

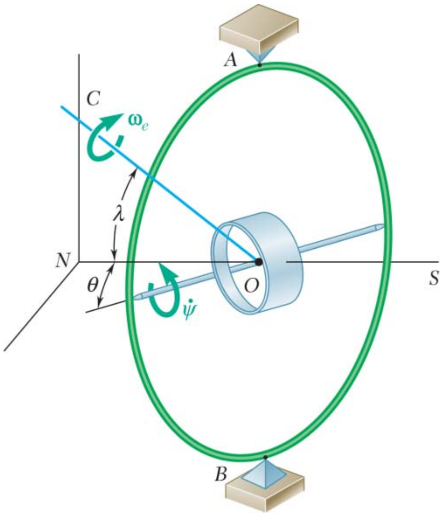
$$(\dot{\theta}^2)_{\max} = C_2 - C_1\dot{\varphi}_{\min}$$

$$= 64 - (8)(4)$$

$$= 32 \text{ (rad/s)}^2$$

$\dot{\theta}_{\max} = 5.66$ rad/s ◀

PROBLEM 18.158



The essential features of the gyrocompass are shown. The rotor spins at the rate $\dot{\psi}$ about an axis mounted in a single gimbal, which may rotate freely about the vertical axis AB. The angle formed by the axis of the rotor and the plane of the meridian is denoted by θ, and the latitude of the position on the earth is denoted by λ. We note that line OC is parallel to the axis of the earth, and we denote by $\boldsymbol{\omega}_e$ the angular velocity of the earth about its axis.

(*a*) Show that the equations of motion of the gyrocompass are

$$I'\ddot{\theta} + I\omega_z\omega_e\cos\lambda\sin\theta - I'\omega_e^2\cos^2\lambda\sin\theta\cos\theta = 0$$

$$I\dot{\omega}_z = 0$$

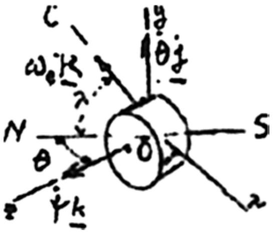
where ω_z is the rectangular component of the total angular velocity $\boldsymbol{\omega}$ along the axis of the rotor, and I and I' are the moments of inertia of the rotor with respect to its axis of symmetry and a transverse axis through O, respectively.

(*b*) Neglecting the term containing ω_e^2, show that for small values of θ, we have

$$\ddot{\theta} + \frac{I\omega_z\omega_e\cos\lambda}{I'}\theta = 0$$

and that the axis of the gyrocompass oscillates about the north-south direction.

SOLUTION



(*a*) Angular momentum about *O*.

We select a frame of reference *Oxyz* attached to the gimbal. The angular velocity of *Oxyz* with respect to a Newtonian frame is $\boldsymbol{\Omega} = \omega_e\mathbf{K} + \dot{\theta}\mathbf{j}$

where $\mathbf{K} = -\cos\lambda\sin\theta\mathbf{i} + \sin\lambda\mathbf{j} + \cos\lambda\cos\theta\mathbf{k}$

Thus, $$\boldsymbol{\Omega} = -\omega_e\cos\lambda\sin\theta\mathbf{i} + (\dot{\theta} + \omega_e\sin\lambda)\mathbf{j} + \omega_e\cos\lambda\cos\theta\mathbf{k} \quad (1)$$

The angular velocity $\boldsymbol{\omega}$ of the rotor is obtained by adding its spin $\dot{\psi}\mathbf{k}$ to $\boldsymbol{\Omega}$. Setting

$$\dot{\psi} + \omega_e\cos\lambda\cos\theta = \omega_z.$$

We have $$\boldsymbol{\omega} = -\omega_e\cos\lambda\sin\theta\mathbf{i} + (\dot{\theta} + \omega_e\sin\lambda)\mathbf{j} + \omega_z\mathbf{k} \quad (2)$$

PROBLEM 18.158 (Continued)

The angular momentum $\mathbf{H}_O$ of the rotor is

$$\mathbf{H}_O = I_x\omega_z\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

where $I_x = I_y = I'$ and $I_z = I.$ Recalling Eq. (2), we write

$$\mathbf{H}_O = -I'\omega_e\cos\lambda\sin\theta\mathbf{i} + I'(\dot{\theta} + \omega_e\sin\lambda)\mathbf{j} + I\omega_z\mathbf{k} \qquad (3)$$

Equations of motion.

Eq. (18.28): $\Sigma\mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \mathbf{\Omega}\times\mathbf{H}_O$ or, from Eqs. (1) and (3):

$$\Sigma\mathbf{M}_O = -I'\omega_e\cos\lambda\cos\theta\dot{\theta}\mathbf{i} + I'\ddot{\theta}\mathbf{j} + I\dot{\omega}_z\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\omega_e\cos\lambda\sin\theta & \dot{\theta} + \omega_e\sin\lambda & \omega_e\cos\lambda\cos\theta \\ -I'\omega_e\cos\lambda\sin\theta & I'(\dot{\theta} + \omega_e\sin\lambda) & I\omega_z \end{vmatrix} \qquad (4)$$

We observe that the rotor is free to spin about the z axis and free to rotate about the y axis. Therefore, the y and z components of $\Sigma\mathbf{M}_O$ must be zero. It follows that the coefficients of $\mathbf{j}$ and $\mathbf{k}$ at the right-hand member of Eq. (4) must also be zero.

Setting the coefficient of $\mathbf{j}$ in the right-hand member of Eq. (4) equal to zero,

$$I'\ddot{\theta} + (-I'\omega_e\cos\lambda\sin\theta)(\omega_e\cos\lambda\cos\theta) - (-\omega_e\cos\lambda\sin\theta)I\omega_z = 0$$

$$I'\ddot{\theta} + I\omega_z\omega_e\cos\lambda\sin\theta - I'\omega_e^2\cos^2\lambda\sin\theta\cos\theta = 0 \qquad (5) ◀$$

Q. E. D

Setting the coefficient of $\mathbf{k}$ equal to zero,

$$I\dot{\omega}_z + (-\omega_e\cos\lambda\sin\theta)I'(\theta + \omega_e\sin\lambda) - (-I'\omega_e\cos\lambda\sin\theta)(\dot{\theta} + \omega_e\sin\lambda) = 0$$

Observing that the last two terms cancel out, we have

$$I\dot{\omega}_z = 0 \qquad \text{Q. E. D.} \qquad (6) ◀$$

(*b*) It follows from Eq. (6) that $\omega_z = \text{constant}$ (7)

Rewrite Eq. (5) as follows: $I'\ddot{\theta} + (I\omega_z - I'\omega_e\cos\lambda\cos\theta)\omega_e\cos\lambda\sin\theta = 0$

It is evident that $\omega_z >>> \omega_e.$ We can therefore neglect the second term in the parenthesis and write

$$I'\ddot{\theta} + I\omega_z\omega_e\cos\lambda\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{I\omega_z\omega_e\cos\lambda}{I'}\sin\theta = 0 \qquad (8)$$

where the coefficient of sin θ is a constant. The rotor, therefore, oscillates about the line *NS* as a simple pendulum. For small oscillations, $\sin\theta \approx \theta$, and Eq. (8) yields

$$\ddot{\theta} + \frac{I\omega_z\omega_e \cos\lambda}{I'}\theta = 0 \qquad \text{Q. E. D.} \qquad (9) ◀$$

Eq. (9) is the equation of simple harmonic motion with period

$$\tau = 2\pi\sqrt{\frac{I'}{I\omega_z\omega_e \cos\lambda}} \qquad (10)$$

Since its rotor oscillates about the line *NS*, the gyrocompass can be used to determine the direction of that line. We should note, however, that for values of λ close to 90° or –90°, the period of oscillation becomes very large and the line about which the rotor oscillates cannot be determined. The gyrocompass, therefore, cannot be used in the polar regions.

CHAPTER 19

PROBLEM 19.1

A particle moves in simple harmonic motion. Knowing that the maximum velocity is 200 mm/s and the maximum acceleration is 4 m/s^2, determine the amplitude and frequency of the motion.

SOLUTION

Eq. 19.15: $v_m = x_m\omega_n$ $a_m = x_m\omega_n^2$

Given data $v_m = 0.2$ m/s $a_m = 4$ m/s^2

$v_m = x_m\omega_n$: $0.2 \text{ m/s} = x_m\omega_m$
(1)

$a_m = x_m\omega_n^2$: $4 \text{ m/s}^2 = x_m\omega_m^2$
(2)

Divide Equ. (2) by Equ. (1): $\omega_n = \dfrac{4 \text{ m/s}^2}{0.2 \text{ m/s}} = 20 \text{ rad/s}$

Eq. (1): $0.2 \text{ m/s} = x_m\left(20 \text{ rad/s}\right)$

$x_m = 0.01 \text{ m}$ $x_m = 10 \text{ mm}$ ◀

Frequency $f_n = \dfrac{\omega_n}{2\pi} = \dfrac{20 \text{ rad/s}}{2\pi}$ $f_n = 3.18 \text{ Hz}$ ◀

PROBLEM 19.2

A particle moves in simple harmonic motion. Knowing that the amplitude is 15 in. and the maximum acceleration is 15 ft/s^2, determine the maximum velocity of the particle and the frequency of its motion.

SOLUTION

Simple harmonic motion.

$$x = x_m \sin(\omega_n t + \phi)$$

$$x_m = 15 \text{ in.} = 1.25 \text{ ft}$$

$$\dot{x} = v = x_m \omega_n \cos(\omega_n t + \phi)$$

$$\ddot{x} = a = -x_m \omega_n^2 \sin(\omega_n t + \phi)$$

$$a_m = -x_m \omega_n^2$$

$$|a_m| = 15 \text{ ft/s}^2 = (1.25 \text{ ft})\omega_n^2$$

Natural frequency

$$\omega_n = 3.4641 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = 0.55133 \text{ Hz}$$

$$f_n = 0.551 \text{ Hz} \blacktriangleleft$$

Maximum velocity

$$v_m = x_m \omega_n = (1.25 \text{ ft})(3.4641 \text{ rad/s})$$

$$= 4.3301 \text{ ft/s}$$

$$v_m = 4.33 \text{ ft/s} \blacktriangleleft$$

PROBLEM 19.3

Determine the amplitude and maximum acceleration of a particle which moves in simple harmonic motion with a maximum velocity of 4 ft/s and a frequency of 6 Hz.

SOLUTION

$$x = x_m \sin \omega_n t, \quad \omega_n = 6\frac{\text{cycle}}{\text{s}} \times \frac{2\pi \text{ rad}}{\text{cycle}}$$

$$x = x_m \sin 12\pi t$$

$$\dot{x} = 12\pi x_m \cos 12\pi t$$

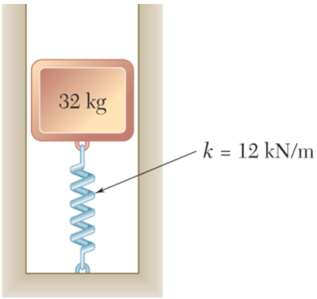
$$\ddot{x} = -144\pi^2 x_m \sin 12\pi t$$

$$12\pi x_m = 4 \text{ ft/s}$$

$$x_m = \frac{4}{12\pi} = 0.1061 \text{ ft}$$

$x_m = 1.273$ in. ◀

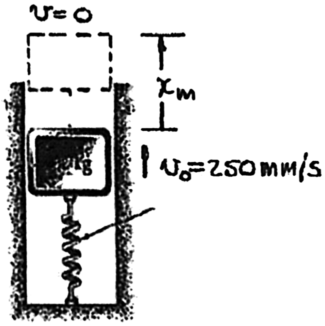
Max Acc. $= 144\pi^2 (0.1061) = 150.8 \text{ ft/s}^2$ ◀



PROBLEM 19.4

A 32-kg block is attached to a spring and can move without friction in a slot as shown. The block is in its equilibrium position when it is struck by a hammer, which imparts to the block an initial velocity of 250 mm/s. Determine (*a*) the period and frequency of the resulting motion, (*b*) the amplitude of the motion and the maximum acceleration of the block.

SOLUTION



(*a*)

$$x = x_m \sin(\omega_n t + \phi)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12\times 10^3 \text{ N/m}}{32 \text{ kg}}}$$

$$\omega_n = 19.365 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\tau_n = \frac{2\pi}{19.365}$$

$$\tau_n = 0.324 \text{ s} \blacktriangleleft$$

$$f_n = \frac{1}{\tau_n} = \frac{1}{0.324} = 3.08 \text{ Hz} \blacktriangleleft$$

(*b*) At $t = 0$, $x_0 = 0$, $\dot{x}_0 = v_0 = 250 \text{ mm/s}$

Thus, $x_0 = 0 = x_m \sin(\omega_n(0) + \phi)$

and $\phi = 0$

$$\dot{x}_0 = v_0 = x_m\omega_n \cos(\omega_n(0) + 0) = x_m\omega_n$$

$$v_0 = 0.250 \text{ m/s} = x_m(19.365 \text{ rad/s})$$

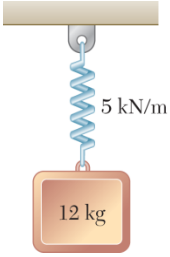
$$x_m = \frac{(0.250 \text{ m/s})}{(19.365 \text{ rad/s})}$$

$$x_m = 12.91\times 10^{-3} \text{ m}$$

$$x_m = 12.91 \text{ mm} \blacktriangleleft$$

$$a_m = x_m\omega_n^2 = (12.91\times 10^{-3} \text{ m})(19.365 \text{ rad/s})^2$$

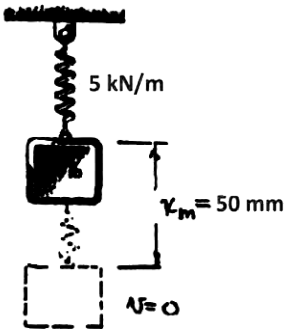
$$a_m = 4.84 \text{ m/s}^2 \blacktriangleleft$$



PROBLEM 19.5

A 12-kg block is supported by the spring shown. If the block is moved vertically downward from its equilibrium position and released, determine (*a*) the period and frequency of the resulting motion, (*b*) the maximum velocity and acceleration of the block if the amplitude of its motion is 50 mm.

SOLUTION



(*a*) Simple harmonic motion. $x = x_m \sin(\omega_n t + \phi)$

Natural frequency. $\omega_n = \sqrt{\dfrac{k}{m}}$ $\quad k = 5 \text{ kN/m} = 5000 \text{ N/m}$

$$\omega_n = \sqrt{\frac{(5000 \text{ N/m})}{12 \text{ kg}}}$$

$$\omega_n = 20.412 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\tau_n = \frac{2\pi}{20.412} = 0.30781 \text{ s} \qquad \tau_n = 0.308 \text{ s} \blacktriangleleft$$

$$f_n = \frac{1}{\tau_n} = \frac{1}{0.30781} = 3.25 \text{ Hz} \blacktriangleleft$$

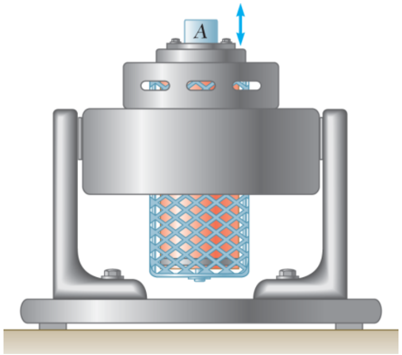
(*b*)

$$x_m = 50 \text{ mm} = 0.05 \text{ m}$$

$$x = 0.05 \sin(20.412t + \phi)$$

Maximum velocity. $v_m = x_m \omega_n = (0.05 \text{ m})(20.412 \text{ rad/s})$ $\qquad v_m = 1.021 \text{ m/s} \blacktriangleleft$

Maximum acceleration. $a_m = x_m \omega_n^2 = (0.05 \text{ m})(20.412 \text{ rad/s})^2$ $\qquad a_m = 20.8 \text{ m/s}^2 \blacktriangleleft$



PROBLEM 19.6

An instrument package *A* is bolted to a shaker table as shown. The table moves vertically in simple harmonic motion at the same frequency as the variable-speed motor which drives it. The package is to be tested at a peak acceleration of 150 ft/s^2. Knowing that the amplitude of the shaker table is 2.3 in., determine (*a*) the required speed of the motor in rpm, (*b*) the maximum velocity of the table.

SOLUTION

In simple harmonic motion,

$$a_{\max} = x_{\max}\omega_n^2$$

$$150 \text{ ft/s}^2 = \left(\frac{2.3}{12}\text{ ft}\right)\omega_n^2$$

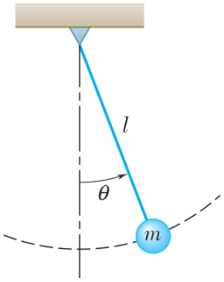
$$\omega_n^2 = (782.6 \text{ rad/s})^2$$

$$\omega_n = 27.98 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{27.98}{2\pi} = 4.452 \text{ Hz (cycles per second)}$$

(*a*) Motor speed. $(4.452 \text{ rev/s})(60 \text{ s/min})$ speed = 267 rpm ◀

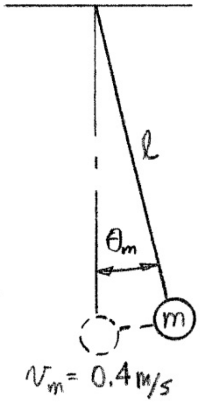
(*b*) Maximum velocity. $v_{\max} = x_{\max}\omega_n = \left(\frac{2.3}{12}\text{ ft}\right)(27.98 \text{ rad/s})$ $v_{\max} = 5.36$ ft/s ◀



PROBLEM 19.7

A simple pendulum consisting of a bob attached to a cord oscillates in a vertical plane with a period of 1.3 s. Assuming simple harmonic motion and knowing that the maximum velocity of the bob is 0.4 m/s, determine (*a*) the amplitude of the motion in degrees, (*b*) the maximum tangential acceleration of the bob.

SOLUTION



(*a*) Simple harmonic motion

$$\theta = \theta_m \sin(\omega_n t + \phi)$$

$$\omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{(1.3\text{ s})}$$

$$= 4.8332\text{ rad/s}$$

$$\dot{\theta} = \theta_m \omega_n \cos(\omega_n t + \phi)$$

$$\dot{\theta}_m = \theta_m \omega_n$$

$$v_m = l\dot{\theta}_m = l\theta_m \omega_n$$

Thus,

$$\theta_m = \frac{v_m}{l\omega_n} \qquad (1)$$

For a simple pendulum,

$$\omega_n = \sqrt{\frac{g}{l}}$$

Thus,

$$l = \frac{g}{\omega_n^2} = \frac{9.81\text{ m/s}^2}{(4.8332\text{ rad/s})^2}$$

$$= 0.41995\text{ m}$$

Amplitude from (1),

$$\theta_m = \frac{v_m}{l\omega_n} = \frac{0.4\text{ m/s}}{(0.42\text{ m})(4.833\text{ rad/s})}$$

$$= 0.19707\text{ rad}$$

$$= 11.291°$$

$\theta_m = 11.29°$ ◀

(*b*) Maximum tangential acceleration $a_t = l\ddot{\theta}$

The maximum tangential acceleration occurs when $\ddot{\theta}$ is maximum.

$$\ddot{\theta} = -\theta_m \omega_n^2 \sin(\omega_n t + \phi)$$

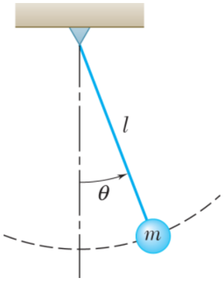
$$\ddot{\theta}_m = \theta_m \omega_n^2$$

$$(a_t)_m = l\theta_m \omega_n^2$$

$$(a_t)_m = (0.41995\text{ m})(0.19707\text{ rad})(4.8332\text{ rad/s})^2$$

$$= 1.933\text{ m/s}^2$$

$(a_t)_m = 1.933\text{ m/s}^2$ ◀



PROBLEM 19.8

A simple pendulum consisting of a bob attached to a cord of length $l = 800$ mm oscillates in a vertical plane. Assuming simple harmonic motion and knowing that the bob is released from rest when $\theta = 6°$, determine (*a*) the frequency of oscillation, (*b*) the maximum velocity of the bob.

SOLUTION

(*a*) Frequency.

$$\omega_n = \sqrt{\frac{g}{l}} = \sqrt{\frac{(9.81\text{ m/s}^2)}{(0.8\text{ m})}}$$

$$\omega_n = 3.502\text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{(3.502\text{ rad/s})}{2\pi}$$

$f_n = 0.557$ Hz ◀

(*b*) Simple harmonic motion.

$$\theta = \theta_m \sin(\omega_n t + \phi)$$

where

$$\theta_m = 6° = 0.10472\text{ rad}$$

Maximum velocity.

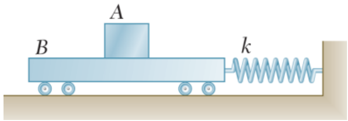
$$\dot{\theta} = \theta_m \omega_n \cos(\omega_n t + \phi)$$

$$\dot{\theta}_m = \theta_m \omega_n$$

$$v_m = l\dot{\theta}_m = l\theta_m\omega_n = (0.8\text{ m})(0.10472)(3.502)$$

$$v_m = 293.4 \times 10^{-3}\text{ m/s}$$

$v_m = 293$ mm/s ◀



PROBLEM 19.9

A 10-lb block A rests on a 40-lb plate B which is attached to an unstretched spring of constant $k = 60$ lb/ft. Plate B is slowly moved 2.4 in. to the left and released from rest. Assuming that block A does not slip on the plate, determine (*a*) the amplitude and frequency of the resulting motion, (*b*) the corresponding smallest allowable value of the coefficient of static friction.

SOLUTION

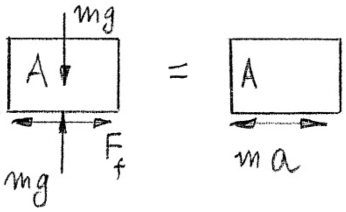
Simple Harmonic Motion:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{60 \text{ lb/ft}}{\frac{50 \text{ lb}}{32.2 \text{ ft/s}^2}}} = 6.2161 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{6.2161}{2\pi} \text{ Hz}$$

(*a*) $x_m = 2.4 \text{ in.} = 0.2 \text{ ft}$ $\qquad x_m = 0.2 \text{ ft}$ ◀

$f_n = 0.989 \text{ Hz}$ ◀



$$a_m = x_m\omega_n^2 = (0.2 \text{ ft})(6.2161 \text{ rad/s})^2$$

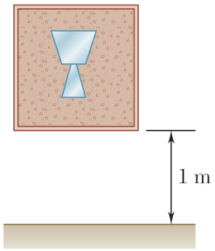
$$= 7.728 \text{ ft/s}^2$$

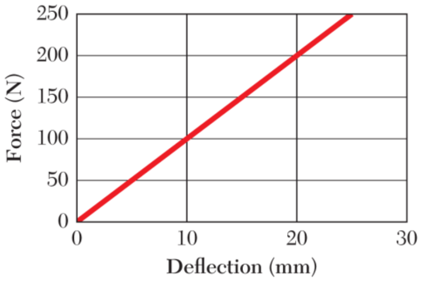
(*b*) $F_f = ma_m = \mu_s mg$

or $\qquad \mu_s = \frac{a_m}{g} = \frac{7.728 \text{ ft/s}^2}{32.2 \text{ ft/s}^2} = 0.240$ ◀

PROBLEM 19.10

A 5-kg fragile glass vase is surrounded by packing material in a cardboard box of negligible weight. The packing material has negligible damping and a force-deflection relationship as shown. Knowing that the box is dropped from a height of 1 m and the impact with the ground is perfectly plastic, determine (*a*) the amplitude of vibration for the vase, (*b*) the maximum acceleration the vase experiences in g's.



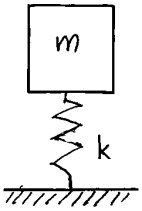


SOLUTION

Velocity at end of free fall: $v = \sqrt{2gh}$

$$v = \sqrt{(2)(9.81\text{ m/s}^2)(1\text{ m})} = 4.4294\text{ m/s}$$

Assume that the spring is unstretched during the free fall. Use a simple spring-mass model for the motion of the vase and the packing material.



$$m = 5\text{ kg}$$

$$k = \frac{100\text{ N}}{10\text{ mm}} \quad \text{(slope from graph)}$$

$$k = 10\text{ N/m} = 10000\text{ N/m}$$

Natural frequency: $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{10000\text{ N/m}}{5\text{ kg}}} = 44.721\text{ rad/s}$

Simple harmonic motion:

$$x = x_m \sin(\omega_n t + \phi)$$

$$v = \dot{x} = \omega_n x_m \cos(\omega_n t + \phi)$$

Let $t = 0$ at the instant when the box bottom hits the ground.

Then, at $t = 0$, $x = 0$ and $v = 4.4294\text{ m/s}$

from which $\phi = 0$

and $\omega_n x_m = 4.4294\text{ m/s}$

PROBLEM 19.10 (Continued)

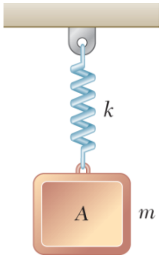
(*a*) Amplitude: $$x_m = \frac{4.4294 \text{ m/s}}{44.721 \text{ rad/s}} = 0.099045 \text{ m}$$

$x_m = 99.0 \text{ mm}$ ◀

(*b*) Maximum acceleration:

$$a_m = \omega_n^2 x_m = (44.721 \text{ rad/s})^2 (0.099045 \text{ m})$$
$$= 198.087 \text{ m/s}^2 = (20.192)(9.81 \text{ m/s}^2)$$

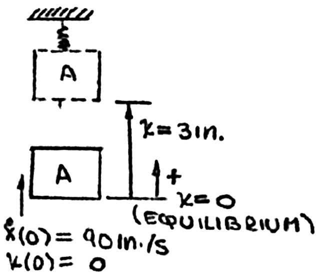
$a_m = 20.2\, g$ ◀



PROBLEM 19.11

A 3-lb block is supported as shown by a spring of constant $k = 2$ lb/in. which can act in tension or compression. The block is in its equilibrium position when it is struck from below by a hammer which imparts to the block an upward velocity of 90 in./s. Determine (*a*) the time required for the block to move 3 in. upward, (*b*) the corresponding velocity and acceleration of the block.

SOLUTION



Simple harmonic motion. $x = x_m \sin(\omega_n t + \phi)$

Natural frequency. $\omega_n = \sqrt{\dfrac{k}{m}}, \quad k = 2 \text{ lb/in.} = 24 \text{ lb/ft}$

$$\omega_n = \sqrt{\frac{24 \text{ lb/ft}}{\left(\frac{3 \text{ lb}}{32.2 \text{ ft/s}^2}\right)}}$$

$$\omega_n = 16.05 \text{ rad/s}$$

$$x(0) = 0 = x_m \sin(0 + \phi)$$

$$\phi = 0$$

$$\dot{x}(0) = x_m \omega_n \cos(0 + 0)$$

$$\dot{x}(0) = \frac{90}{12} = 7.5 \text{ ft/s}$$

$$7.5 = x_m(16.05) \qquad x_m = 0.4673 \text{ ft}$$

$$x = (0.4673)\sin(16.05t)(\text{ft/s}) \tag{1}$$

(*a*) Time at $x = 3$ in. $(x = 0.25 \text{ ft})$

$$0.25 = 0.4673\sin(16.05t)$$

$$t = \frac{\sin^{-1}\left(\frac{0.25}{0.4673}\right)}{16.05}$$

$t = 0.0352$ s ◀

(*b*) Velocity and acceleration.

$$\dot{x} = x_m \omega_n \cos(\omega_n t)$$

$$\ddot{x} = -x_m \omega_n^2 \sin \omega_n t$$

$$t = 0.0352$$

$$\dot{x} = (0.4673)(16.05)\cos[(16.05)(0.0352)]$$

$$\dot{x} = 6.34 \text{ ft/s}$$

$\mathbf{v} = 6.34$ ft/s ↑ ◀

$$\ddot{x} = -(0.4673)(16.05)^2 \sin[(16.05)(0.0352)]$$

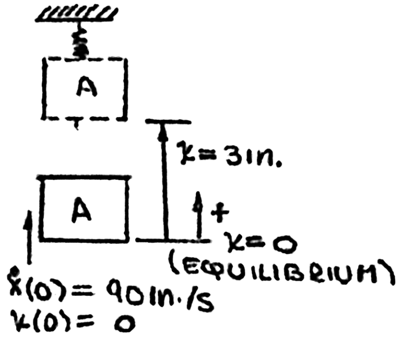
$$= -64.4 \text{ ft/s}^2$$

$\mathbf{a} = 64.4$ ft/s^2 ↓ ◀

PROBLEM 19.12

In Problem 19.11, determine the position, velocity, and acceleration of the block 0.90 s after it has been struck by the hammer.

SOLUTION



Simple harmonic motion. $x = x_m \sin(\omega_n t + \phi)$

Natural frequency. $\omega_n = \sqrt{\frac{k}{m}}, \quad k = 2 \text{ lb/in.} = 24 \text{ lb/ft}$

$$\omega_n = \sqrt{\frac{24 \text{ lb/ft}}{\left(\frac{3 \text{ lb}}{32.2 \text{ ft/s}^2}\right)}}$$

$$\omega_n = 16.05 \text{ rad/s}$$

$$x(0) = 0 = x_m \sin(0 + \phi)$$

$$\phi = 0$$

$$\dot{x}(0) = x_m \omega_n \cos(0 + 0) \qquad \dot{x}(0) = \frac{90}{12} = 7.5 \text{ ft/s}$$

$$7.5 = x_m (16.05) \qquad x_m = 0.4673 \text{ ft}$$

$$x = (0.4673)\sin(16.05t)(\text{ft/s})$$

Simple harmonic motion.

$$x = x_m \sin(\omega_n t + \phi)$$

$$\dot{x} = x_m \omega_n \cos(\omega_n t + \phi)$$

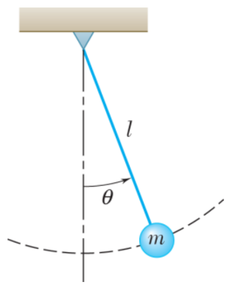
$$\ddot{x} = -x_m \omega_n^2 \sin(\omega_n t + \phi)$$

At 0.90 s:

$x = (0.4673)\sin[(16.05)(0.90)] = 0.445 \text{ ft}$ **x** $= 0.445$ ft ↑ ◀

$\dot{x} = (0.4673)(16.05)\cos[(16.05)(0.90)] = -2.27 \text{ ft/s}$ **v** $= 2.27$ ft/s ↓ ◀

$\ddot{x} = -(0.4673)(16.05)^2 \sin[(16.05)(0.90)] = -114.7 \text{ ft/s}^2$ **a** $= 114.7 \text{ ft/s}^2$ ↓ ◀



PROBLEM 19.13

The bob of a simple pendulum of length $l = 40$ in. is released from rest when $\theta = +5°$. Assuming simple harmonic motion, determine 1.6 s after release (*a*) the angle θ, (*b*) the magnitudes of the velocity and acceleration of the bob.

SOLUTION

For simple harmonic motion and $l = 40 \text{ in.} = 3.333 \text{ ft}$:

$$\omega_n = \sqrt{\frac{g}{l}} = \sqrt{\frac{32.2 \text{ ft/s}^2}{3.333 \text{ ft}}} = 3.1082 \text{ rad/s}$$

Angular displacement: $\theta = \theta_m \sin(\omega_n t + \phi)$

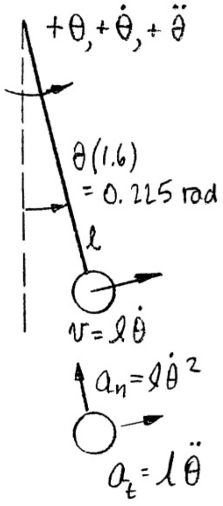
Initial conditions: $\theta(0) = 5° = 0.08727 \text{ rad}, \quad \text{and} \quad \dot{\theta}(0) = 0:$

$$\dot{\theta}(0) = 0 = \theta_m \omega_n \cos(0 + \phi) \qquad \phi = \frac{\pi}{2}$$

$$\theta_m = \theta(0) = \frac{5\pi}{180} = 0.08727 \text{ rad}$$

$$\theta = \frac{5\pi}{180} \sin\left[(3.1082 \text{ rad/s})t + \frac{\pi}{2}\right]$$

$$= (0.08727 \text{ rad}) \sin\left[(3.1082 \text{ rad/s})t + \frac{\pi}{2}\right]$$



(*a*) At $t = 1.6$ s.

$$\theta = \frac{5\pi}{180} \sin\left[(3.1082 \text{ rad/s})(1.6 \text{ s}) + \frac{\pi}{2}\right]$$

$$= 0.022496 \text{ rad} = 1.288°$$

$\theta = 1.288°$ ◀

(*b*) Velocity:

$$\dot{\theta} = \theta_m \omega_n \cos(\omega_n t + \phi)$$

$$= \frac{5\pi}{180}(3.1082 \text{ rad/s}) \cos\left[(3.1082 \text{ rad/s})(1.6 \text{ s}) + \frac{\pi}{2}\right]$$

$$= 0.262074 \text{ rad/s}$$

$$v = l\dot{\theta} = (3.3333 \text{ ft})(0.262074 \text{ rad/s}) = 0.874 \text{ ft/s}$$

$v = 0.874$ ft/s ◀

Angular acceleration:

$$\ddot{\theta} = -\theta_m \omega_n^2 \sin(\omega_n t + \phi) = -\frac{5\pi}{180}(3.1082 \text{ rad/s})^2 \cos\left[(3.1082 \text{ rad/s})(1.6 \text{ s}) + \frac{\pi}{2}\right]$$

$$= -0.21733 \text{ rad/s}^2$$

Acceleration:

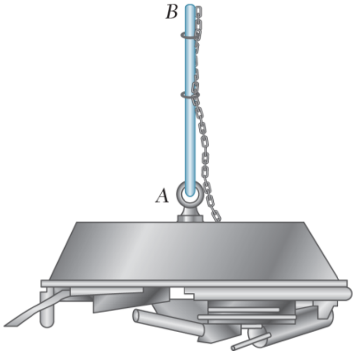
$$a = \sqrt{(a_n)^2 + (a_t)^2}$$

$$a_n = \frac{v^2}{l} = l\dot{\theta}^2 = (3.3333 \text{ ft})(0.26207 \text{ rad/s})^2 = 0.22894 \text{ ft/s}^2$$

$$a_t = l\ddot{\theta} = (3.333 \text{ ft})(-0.21733 \text{ rad/s}^2)^2 = -0.72443 \text{ m/s}^2$$

$$a = 0.75974 \text{ ft/s}^2$$

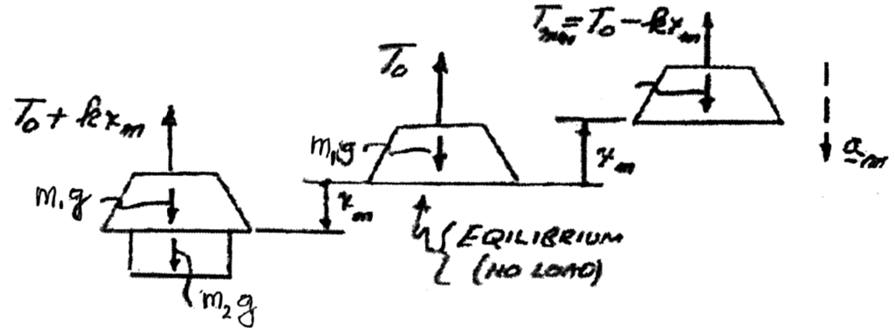
$a = 0.760 \text{ ft/s}^2$ ◀



PROBLEM 19.14

A 150-kg electromagnet is at rest and is holding 100 kg of scrap steel when the current is turned off and the steel is dropped. Knowing that the cable and the supporting crane have a total stiffness equivalent to a spring of constant 200 kN/m, determine (*a*) the frequency, the amplitude, and the maximum velocity of the resulting motion, (*b*) the minimum tension which will occur in the cable during the motion, (*c*) the velocity of the magnet 0.03 s after the current is turned off.

SOLUTION



Data: $m_1 = 150$ kg $\quad m_2 = 100$ kg $\quad k = 200 \times 10^3$ N/m

From the first two sketches, $\quad T_0 + kx_m = (m_1 + m_2)g \qquad (1)$

$$T_0 = m_1 g \qquad (2)$$

Subtracting Eq. (2) from Eq. (1), $\quad kx_m = m_2 g$

$$x_m = \frac{m_2 g}{k} = \frac{(100)(9.81)}{200 \times 10^3} = 4.905 \times 10^{-3} \text{ m} = 4.91 \text{ mm}$$

Natural circular frequency: $\quad \omega_n = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{200 \times 10^3}{150}} = 36.515$ rad/s

Natural frequency: $\quad f_n = \frac{\omega_n}{2\pi} = \frac{36.515}{2\pi} \qquad f_n = 5.81$ Hz

Maximum velocity: $\quad v_m = \omega_n x_m = (36.515)(4.905 \times 10^{-3}) = 0.1791$ m/s

(*a*) Resulting motion:

amplitude $x_m = 4.91$ mm ◀

frequency $f_n = 5.81$ Hz ◀

maximum velocity $v_m = 0.1791$ m/s ◀

PROBLEM 19.14 (Continued)

(*b*) Minimum value of tension occurs when $x = -x_m$.

$$\begin{aligned} T_{\min} &= T_0 - kx_m \\ &= m_1 g - m_2 g \\ &= (m_1 - m_2)g \\ &= (50)(9.81) \end{aligned}$$

$T_{\min} = 491 \text{ N}$ ◀

The motion is given by

$$x = x_m \sin(\omega_n t + \varphi)$$
$$\dot{x} = \omega_n x_m \cos(\omega_n t + \varphi)$$

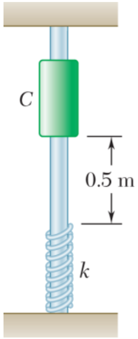
Initally,

$$x_0 = -x_m \quad \text{or} \quad \sin\varphi = -1$$
$$\dot{x}_0 = 0 \quad \text{or} \quad \cos\varphi = 0$$
$$\varphi = -\frac{\pi}{2}$$
$$\dot{x} = \omega_n x_m \cos\left(\omega_n t - \frac{\pi}{2}\right)$$

(*c*) Velocity at $t = 0.03$ s.

$$\omega_n t = (36.515)(0.03) = 1.09545 \text{ rad}$$
$$\omega_n t - \varphi = -0.47535 \text{ rad}$$
$$\cos(\omega_n t - \varphi) = 0.88913$$
$$\dot{x} = (36.515)(4.905 \times 10^{-3})(0.88913)$$

$\dot{\mathbf{x}} = 0.1592 \text{ m/s} \uparrow$ ◀



PROBLEM 19.15

A 5-kg collar C is released from rest in the position shown and slides without friction on a vertical rod until it hits a spring of constant $k = 720$ N/m which it compresses. The velocity of the collar is reduced to zero and the collar reverses the direction of its motion and returns to its initial position. The cycle is then repeated. Determine (*a*) the period of the motion of the collar, (*b*) the velocity of the collar 0.4 s after it was released. (*Note.* This is a periodic motion, but not simple harmonic motion.)

SOLUTION

With the given properties:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{720 \text{ N/m}}{5 \text{ kg}}} = 12 \text{ rad/s}$$

From free fall of the collar

$$v_0 = \sqrt{2gh} = \sqrt{2g(0.5 \text{ m})} = \sqrt{g} = 3.132 \text{ m/s}$$

The free-fall time is thus:

$$t_1 = \sqrt{\frac{2 \text{ m}}{g}} = \sqrt{\frac{2(0.5 \text{ m})}{g}} = \sqrt{\frac{1}{g}} = 0.3193 \text{ s}$$

Now, to simplify the analysis we measure the displacement from the position of static displacement of the spring, under the weight of the collar:

Note that the static deflection is:

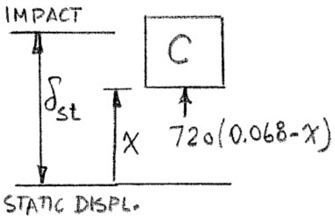
$$\delta_{ST} = \frac{W}{k} = \frac{(5 \text{ kg})(9.81 \text{ m/s}^2)}{720 \text{ N/m}} = 0.068125 \text{ m}$$

Then $5\ddot{x} + 720x = 0$, where x is measured positively up from the position of static deflection. The solution is:

$$x = x_m \sin(\omega_n t + \phi), \text{ with velocity}$$

$$\dot{x} = x_m \omega_n \cos(\omega_n t + \phi)$$

Now, to determine x_m and ϕ, impose the conditions at impact and count the time from there. Thus:



At impact:

$$t = 0, \quad x = \delta_{ST} = 0.068125 \text{ m and } v = -3.132 \text{ m/s} \quad \text{(downward)}$$

or

$$0.068125 \text{ m} = x_m \sin\phi$$

$$-3.132 \text{ m/s} = (12 \text{ rad/s}) x_m \cos\phi$$

PROBLEM 19.15 (Continued)

Solving for x_m and ϕ

$$x_m = -0.26975 \text{ m}$$

$$\phi = -0.25531 \text{ rad}$$

So, from time of impact, the 'time of flight' is the time necessary for the collar to come to rest on its downward motion. Thus, t_2 is the time such that

$$\dot{x}(t_2) = 0 \Rightarrow 12t_2 + \phi = \frac{\pi}{2}$$

or $$12t_2 - 0.25531 = \frac{\pi}{2}$$

$$\dot{x}(t_2) = 0 \Rightarrow 12t_2 + \phi = \frac{\pi}{2}$$

or $$12t_2 - 0.25531 = \frac{\pi}{2}$$

Hence, $t_2 = 0.15217$ s

Thus, the period of the motion is

$$\tau = 2(t_1 + t_2) = 2(0.3193 \text{ s} + 0.1522 \text{ s})$$

$$= 0.94274 \text{ s} \qquad \tau = 0.943 \text{ s} \blacktriangleleft$$

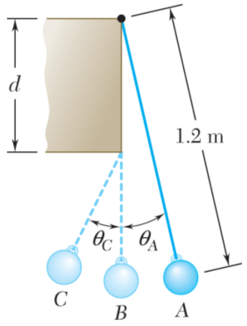
(*b*) After 0.4 seconds, the velocity is

$$\dot{x}(0.4) = x_m\omega_n \cos\left[\omega_n(0.4 - t_1) + \phi\right.$$

$$= (12 \text{ rad/s})(-0.26975 \text{ m})\cos\left[(12 \text{ rad/s})(0.4 - 0.3193 \text{ s})\right.$$

$$\left. - 0.25531 \text{ rad}\right]$$

$$= -2.4483 \text{ m/s} \qquad v(0.4) = 2.45 \text{ m/s} \downarrow \blacktriangleleft$$



PROBLEM 19.16

A small bob is attached to a cord of length 1.2 m and is released from rest when $\theta_A = 5°$. Knowing that $d = 0.6$ m, determine (*a*) the time required for the bob to return to Point *A*, (*b*) the amplitude θ_C.

SOLUTION

As the pendulum moves between Points *A* and *B*, the length of the pendulum is $l = l_{AB} = 1.2$ m.

$$\omega_n = \omega_{n1} = \sqrt{\frac{g}{l_{AB}}} = \sqrt{\frac{9.81\text{ m/s}^2}{1.2\text{ m}}} = 2.8592\text{ rad/s}$$

$$\tau_1 = \frac{2\pi}{\omega_{n1}} = \frac{2\pi}{2.8592\text{ rad/s}} = 2.1975\text{ s}$$

The falling from *A* to *B* is one quarter period.

$$\tau_{AB} = \frac{1}{4}\tau_1 = 0.54938\text{ s.}$$

As the pendulum moves between Points *B* and *C*, the length of the pendulum is
$l = l_{BC} = 1.2\text{ m} - 0.6\text{ m} = 0.6$ m.

$$\omega_n = \omega_{n2} = \sqrt{\frac{g}{l_{BC}}} = \sqrt{\frac{9.81\text{ m/s}^2}{0.6\text{ m}}} = 4.0435\text{ rad/s}$$

$$\tau_2 = \frac{2\pi}{\omega_{n2}} = \frac{2\pi}{4.0435\text{ rad/s}} = 1.55389\text{ s}$$

The motion from *B* to *C* and back to *B* is one half period

$$\tau_{BCB} = \frac{1}{2}\tau_2 = 0.77695\text{ s}$$

As the pendulum moves from *B* to *A*, the length is again 1.2 meters.

$$\tau_{BA} = \frac{1}{4}\tau_1 = 0.54938\text{ s}$$

(*a*) Time required to return to *A*.

$$\tau = \tau_{AB} + \tau_{BCB} + \tau_{BA}$$
$$\tau = 1.87571\text{ s}$$

$\tau = 1.876\text{ s}$ ◀

PROBLEM 19.16 (Continued)

For falling from *A* to *B*, $\theta_m = \theta_A$

At *B*, $\dot{\theta}_B = \dot{\theta}_m = \omega_{n1}\theta_A$

$$v_B = l_{AB}\dot{\theta}_B = l_{AB}\omega_{n1}\theta_A$$

For rising from *B* to *C*,

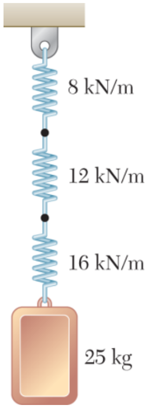
$$\dot{\theta}_B = \frac{v_B}{l_{BC}} = \frac{l_{AB}}{l_{BC}}\omega_{n1}\theta_A = \dot{\theta}_{\max}$$

$$\theta_C = \theta_{\max} = \frac{\dot{\theta}_{\max}}{\omega_{n2}} = \frac{l_{AB}\omega_{n1}}{l_{BC}\omega_{n2}}\theta_A$$

$$\theta_C = \frac{(1.2\text{ m})(2.8592\text{ rad/s})}{(0.6\text{ m})(4.0435\text{ rad/s})}\theta_A = 1.4142\,\theta_A$$

(*b*) Amplitude θ_C:

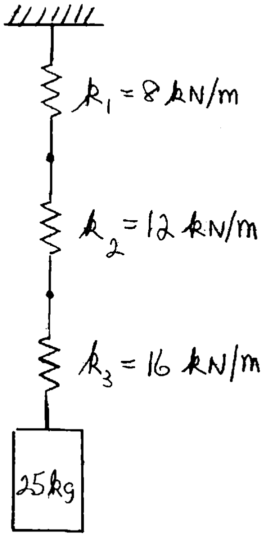
With $\theta_A = 5°$, $\theta_C = 7.07°$ ◀



PROBLEM 19.17

A 25-kg block is supported by the spring arrangement shown. If the block is moved vertically downward from its equilibrium position and released, determine (*a*) the period and frequency of the resulting motion, (*b*) the maximum velocity and acceleration of the block if the amplitude of the motion is 30 mm.

SOLUTION



For a static load P the total elongation of the spring is:

$$\delta = \frac{P}{k_1} + \frac{P}{k_2} + \frac{P}{k_3}$$

$$= P\left(\frac{1}{8\text{ kN/m}} + \frac{1}{12\text{ kN/m}} + \frac{1}{16\text{ kN/m}}\right)$$

$$\delta = P\left(\frac{6+4+3}{48}\right) = P\frac{13}{48}$$

$$k = \frac{P}{\delta} = \frac{48}{13} = 3.6923\text{ kN/m}$$

(*a*) $$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{3.6923 \times 10^3\text{ N/m}}{25\text{ kg}}} = 12.153\text{ rad/s}$$

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{12.153} = 0.517\text{ s}$$ $\tau = 0.517$ s ◀

$$f = \frac{\omega}{2\pi} = \frac{12.153}{2\pi} = 1.9342\text{ Hz}$$ $f = 1.934$ Hz ◀

(*b*) For $x_m = 30\text{ mm} = 0.03\text{ m}$

$$v_m = x_m\omega = (0.03\text{ m})(12.153\text{ rad/s})$$ $v_m = 0.365$ m/s ◀

$$a_m = x_m\omega^2 = (0.03\text{ m})(12.153\text{ rad/s})^2$$ $a_m = 4.43\text{ m/s}^2$ ◀

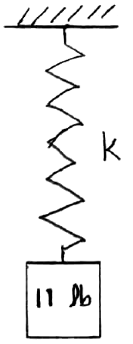
PROBLEM 19.18

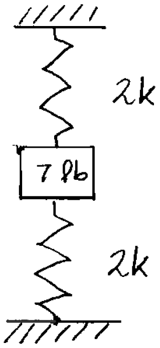
A 11-lb block, attached to the lower end of a spring whose upper end is fixed, vibrates with a period of 7.2 s. Knowing that the constant *k* of a spring is inversely proportional to its length (e.g., if you cut a 10 lb/in. spring in half, the remaining two springs each have a spring constant of 20 lb/in.), determine the period of a 7-lb block which is attached to the center of the same spring if the upper and lower ends of the spring are fixed.

SOLUTION

Equivalent spring constant.

$k' = 2k + 2k = 4k$ (Deflection of each spring is the same.)





For case ①, $\tau_{n1} = 7.2 \text{ s}$

$$\omega_{n1} = \frac{2\pi}{\tau_{n1}} = \frac{2\pi}{7.2} = 0.87266 \text{ rad/s}$$

$$\omega_n^2 = \frac{k}{m_1}$$

$$k = m_1\omega_{n1}^2 = \left(\frac{11}{32.2}\right)(0.87266)^2 = 0.26015 \text{ lb/ft}$$

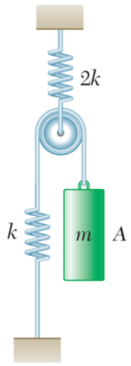
For case ②,

$$\omega_{n2}^2 = \frac{4k}{m_2} = \frac{(4)(0.26015)}{\frac{7}{32.2}} = 4.7868 \text{ (rad/s)}^2$$

$$\omega_{n2} = 2.1879 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_{n2}} = \frac{2\pi}{2.1879}$$

$\tau_{n2} = 2.87 \text{ s}$ ◀

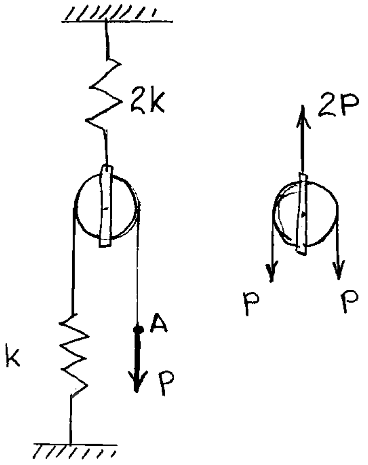


PROBLEM 19.19

Block A of mass m is supported by the spring arrangement as shown. Knowing that the mass of the pulley is negligible and that the block is moved vertically downward from its equilibrium position and released, determine the frequency of the motion.

SOLUTION

We first determine the constant k_{eq} of a single spring equivalent to the spring and pulley system supporting the block by finding the total displacement δ_A of the end of the cable under a given static load **P**. Owing to the force $2P$ in the upper spring the pulley moves down a distance



$$\delta_1 = \frac{2P}{2k}$$

Owing to the force P in the lower spring, Point A moves down an additional distance

$$\delta_2 = \frac{P}{k}$$

The total displacement is

$$\delta_A = \delta_1 + \delta_2 = \frac{2P}{2k} + \frac{P}{k} = \frac{2P}{k}$$

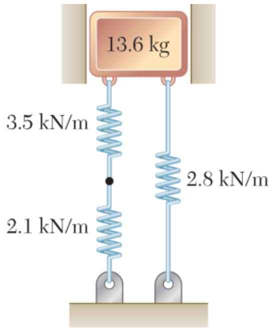
But $\delta_A = \dfrac{P}{k_{\text{eq}}}$ so that

$$k_{\text{eq}} = \frac{k}{2}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m}}$$

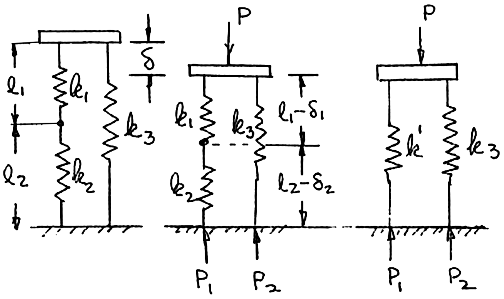
$$\omega_n = \sqrt{\frac{k}{2\,m}} \blacktriangleleft$$



PROBLEM 19.20

A 13.6-kg block is supported by the spring arrangement shown. If the block is moved from its equilibrium position 44 mm vertically downward and released, determine (*a*) the period and frequency of the resulting motion, (*b*) the maximum velocity and acceleration of the block.

SOLUTION



Determine the constant k of a single spring equivalent to the three springs shown.

Springs 1 and 2: $\delta = \delta_1 + \delta_2$, and $\dfrac{P_1}{k'} = \dfrac{P_1}{k_1} + \dfrac{P_1}{k_2}$

Hence, $k' = \dfrac{k_1 k_2}{k_1 + k_2}$

where k' is the spring constant of a single spring equivalent of springs 1 and 2.
Springs k' and 3: (Deflection in each spring is the same).

So $P = P_1 + P_2$, and $P = k\delta$, $P_1 = k'\delta$, $P_2 = k_3\delta$

Now $k\delta = k'\delta + k_3\delta$

$$k = k' + k_3 = \frac{k_1 k_2}{k_1 + k_2} + k_3$$

or

$$k = \frac{(3.5\text{ kN/m})(2.1\text{ kN/m})}{(3.5\text{ kN/m}) + (2.1\text{ kN/m})} + 2.8\text{ kN/m} = 4.11\text{ kN/m} = 4.11\times 10^3\text{ N/m}$$

(*a*) Period and frequency:

$$\tau_n = \frac{2\pi}{\sqrt{\frac{k}{m}}} = \frac{2\pi}{\sqrt{\frac{4.11\times 10^3\text{ N/m}}{13.6\text{ kg}}}}$$

$t_n = 0.361$ s ◀

$$f_n = \frac{1}{\tau_n} = \frac{1}{0.3614\text{ s}}$$

$f_n = 2.77$ Hz ◀

(*b*) Displacement:

$$x = x_m \sin(\omega_n t + \phi)$$

$$x_m = 44 \text{ mm} = 0.044 \text{ m}$$

$$\omega_n = 2\pi f_n = (2\pi)(2.77 \text{ Hz}) = 17.384 \text{ rad/s}$$

$$x = (0.044 \text{ m})\sin[(17.384 \text{ rad/s})t + \phi]$$
$$\dot{x} = (0.044 \text{ m})(17.384 \text{ rad/s})\cos[(17.384 \text{ rad/s})t + \phi]$$

Velocity: $v_{\max} = (0.044 \text{ m})(17.384 \text{ rad/s}) = 0.765 \text{ m/s}$

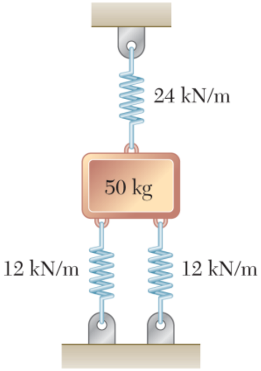
$$\ddot{x} = -(0.044 \text{ m})(17.384 \text{ rad/s})^2 \sin[(17.384 \text{ rad/s})t + \phi]$$

Acceleration: $a_{\max} = (0.044 \text{ m})(17.384 \text{ rad/s})^2 = 13.30 \text{ m/s}^2$

$v_{\max} = 0.765 \text{ m/s}$ ◀

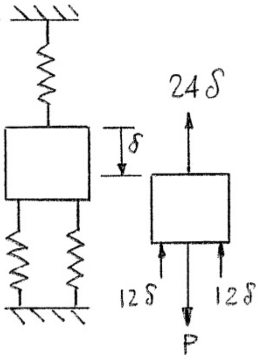
$a_{\max} = 13.30 \text{ m/s}^2$ ◀

PROBLEM 19.21



A 50-kg block is supported by the spring arrangement shown. The block is moved vertically downward from its equilibrium position and released. Knowing that the amplitude of the resulting motion is 60 mm, determine (*a*) the period and frequency of the motion, (*b*) the maximum velocity and maximum acceleration of the block.

SOLUTION



(*a*) First, calculate the spring constant

$$P = (24 \text{ kN/m})\delta + (12 \text{ kN/m})\delta + (12 \text{ kN/m})\delta = (48 \text{ kN/m})\delta$$

$$\therefore \; k = 48 \text{ kN/m}$$

Then

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40 \times 10^3 \text{ N/m}}{50 \text{ kg}}} = 30.984 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n} = 0.20279 \text{ s}$$

$$f_n = \frac{1}{\tau_n} = 4.9312 \text{ Hz}$$

$\tau_n = 0.203 \text{ s}$ ◀

$f_n = 4.93 \text{ Hz}$ ◀

(*b*) Now

$$x = x_m \sin(\omega_n t + \phi)$$

And, since $x_0 = 0.060 \text{ m}$

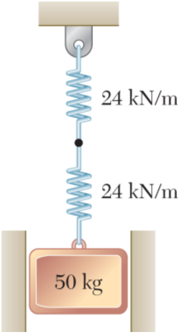
$$x = (0.060 \text{ m})\sin\left[(30.984 \text{ rad/s})t + \phi\right]$$

$$\dot{x} = (0.060 \text{ m})(30.984 \text{ rad/s})\cos\left[(30.984 \text{ rad/s})t + \phi\right]$$

$$\ddot{x} = -(0.060 \text{ m})(30.984 \text{ rad/s})^2 \sin\left[(30.984 \text{ rad/s})t + \phi\right]$$

Hence $v_{\text{max}} = 1.859 \text{ m/s}$ ◀

$a_{\text{max}} = 57.6 \text{ m/s}^2$ ◀



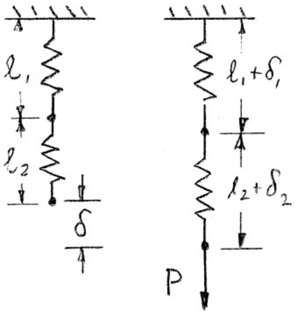
PROBLEM 19.22

A 50-kg block is supported by the spring arrangement shown. The block is moved vertically downward from its equilibrium position and released. Knowing that the amplitude of the resulting motion is 60 mm, determine (*a*) the period and frequency of the motion, (*b*) the maximum velocity and maximum acceleration of the block.

SOLUTION

(*a*) First, calculate the spring constant

$$\delta = \delta_1 + \delta_2 = \frac{P}{24} + \frac{P}{24} = \frac{P}{k}$$



Hence, $\frac{1}{k} = \frac{1}{24} + \frac{1}{24} \Rightarrow 12$ kN/m

Then $\tau_n = \frac{2\pi}{\sqrt{\frac{k}{m}}} = \frac{2\pi}{\sqrt{\frac{(12000 \text{ kN/m})}{50 \text{ kg}}}} = 0.40558 \text{ s}$

$\tau_n = 0.405$ s ◀

$f_n = \frac{1}{\tau_n} = 2.4656$ Hz $\qquad f_n = 2.47$ Hz ◀

(*b*)

$$\omega_n = 2\pi f_n = 15.492 \text{ rad/s}$$

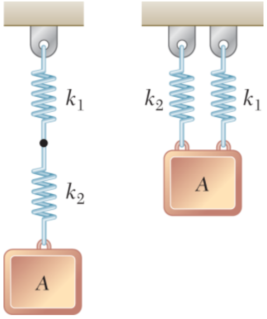
$$x = (0.060 \text{ m})\sin[(15.492 \text{ rad/s})t + \phi]$$

$$\dot{x} = (0.060 \text{ m})(15.492 \text{ rad/s})\cos[(15.492 \text{ rad/s})t + \phi]$$

$$\ddot{x} = -(0.060 \text{ m})(15.492 \text{ rad/s})^2 \sin[(15.492 \text{ rad/s})t + \phi]$$

Then $v_{\max} = 0.92951$ m/s ◀

$a_{\max} = 14.400 \text{ m/s}^2$ ◀



PROBLEM 19.23

Two springs of constants k_1 and k_2 are connected in series to a block *A* that vibrates in simple harmonic motion with a period of 5 s. When the same two springs are connected in parallel to the same block, the block vibrates with a period of 2 s. Determine the ratio k_1/k_2 of the two spring constants.

Solution

Equivalent Springs

Series:
$$k_s = \frac{k_1k_2}{k_1 + k_2}$$

Parallel:
$$k_p = k_1 + k_2$$

$$\tau_s = \frac{2\pi}{\omega_s} = \frac{2\pi}{\sqrt{\frac{k_s}{m}}}; \qquad \tau_p = \frac{2\pi}{\omega_p} = \frac{2\pi}{\sqrt{\frac{k_p}{m}}}$$

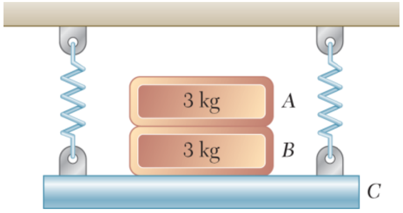
$$\left(\frac{\tau_s}{\tau_p}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{k_p}{k_s} = \frac{k_1 + k_2}{\frac{(k_1k_2)}{(k_1+k_2)}} = \frac{(k_1 + k_2)^2}{k_1k_2}$$

$$(6.25)(k_1k_2) = k_1^2 + 2k_1k_2 + k_2^2$$

$$k_1 = \frac{(4.25)k_2 \mp \sqrt{(4.25)^2 k_2^2 - 4k_2^2}}{2}$$

$$\frac{k_1}{k_2} = 2.125 \mp \sqrt{3.516}$$

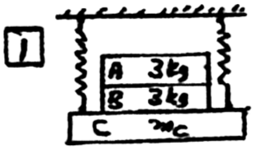
$$\frac{k_1}{k_2} = 4 \blacktriangleleft$$



PROBLEM 19.24

The period of vibration of the system shown is observed to be 0.8 s. If block *A* is removed, the period is observed to be 0.7 s. Determine (*a*) the mass of block *C*, (*b*) the period of vibration when both blocks *A* and *B* have been removed.

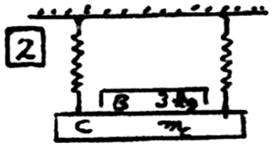
SOLUTION



$$m_1 = m_C + 6 \text{ kg} \qquad \tau_1 = 0.8 \text{ s}$$

$$\omega_1 = \frac{2\pi}{\tau_1} = \frac{2\pi}{0.8 \text{ s}} = \frac{2\pi}{0.8} \text{ rad/s}$$

$$\omega_1^2 = \frac{k}{m_1}; \quad k = m_1\omega_1^2 = (m_C + 6)\left(\frac{2\pi}{0.8}\right)^2 \qquad (1)$$



$$m_2 = m_C + 3 \text{ kg} \qquad \tau_2 = 0.7 \text{ s}$$

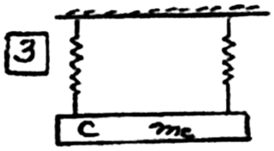
$$\omega_2 = \frac{2\pi}{\tau_2} = \frac{2\pi}{0.7 \text{ s}} = \frac{2\pi}{0.7} \text{ rad/s}$$

$$\omega_2^2 = \frac{k}{m_2}; \quad k = m_2\omega_2^2 = (m_C + 3)\left(\frac{2\pi}{0.7}\right)^2 \qquad (2)$$

Equating the expressions found for *k* in Eqs. (1) and (2):

$$(m_C + 6)\left(\frac{2\pi}{0.8}\right)^2 = (m_C + 3)\left(\frac{2\pi}{0.7}\right)^2$$

$$\frac{m_C + 6}{m_C + 3} = \left(\frac{0.8}{0.7}\right)^2; \quad \text{Solve for } m_C: \qquad m_C = 6.80 \text{ kg} \blacktriangleleft$$



$$\omega_3 = \frac{2\pi}{\tau_3}$$

$$\omega_3^2 = \frac{k}{m_C}; \quad k = m_C\omega_3^2 = m_C\left(\frac{2\pi}{\tau_3}\right)^2 \qquad (3)$$

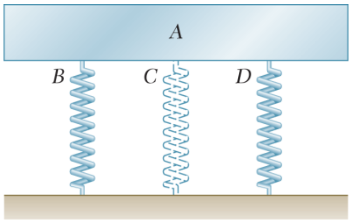
Equating expressions for *k* from Eqs. (2) and (3),

$$(m_C + 3)\left(\frac{2\pi}{0.7}\right)^2 = m_C\left(\frac{2\pi}{\tau_3}\right)^2$$

Recall $m_C = 6.8$ kg:

$$(6.8 + 3)\left(\frac{2\pi}{0.7}\right)^2 = 6.8\left(\frac{2\pi}{\tau_3}\right)^2$$

$$\left(\frac{\tau_3}{0.7}\right)^2 = \frac{6.8}{9.8}; \quad \frac{\tau_3}{0.7} = 0.833 \qquad \tau_3 = 0.583 \text{ s} \blacktriangleleft$$



PROBLEM 19.25

The 100-lb platform A is attached to springs B and D, each of which has a constant $k = 120$ lb/ft. Knowing that the frequency of vibration of the platform is to remain unchanged when an 80-lb block is placed on it and a third spring C is added between springs B and D, determine the required constant of spring C.

SOLUTION

Frequency of the original system.

Springs B and D are in parallel.

$$k_e = k_B + k_D = 2(120 \text{ lb/ft}) = 240 \text{ lb/ft}$$

$$\omega_n^2 = \frac{k_e}{m_A} = \frac{240 \text{ lb/ft}}{\left(\frac{100 \text{ lb}}{32.2 \text{ ft/s}^2}\right)}$$

$$\omega_n^2 = 77.28\,(\text{rad/s})^2$$

Frequency of new system.

Springs A, B, and C are in parallel.

$$k_e' = k_B + k_D + k_C = (2)(120) + k_e$$

$$(\omega_n')^2 = \frac{k_e'}{m_A + m_B} = \frac{(240 + k_C)(32.2 \text{ ft/s}^2)}{(100 \text{ lb} + 80 \text{ lb})}$$

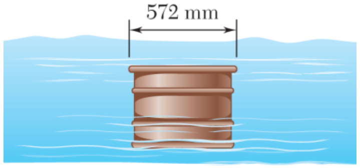
$$(\omega_n')^2 = (0.1789)(240 + k_C)$$

$$\omega_n^2 = (\omega_n')^2$$

$$77.28 = (0.1789)(240 + k_C)$$

$$k_C = 191.97 \text{ lb/ft}$$

$k_C = 192.0$ lb/ft ◀



PROBLEM 19.26

The period of vibration for a barrel floating in salt water is found to be 0.58 s when the barrel is empty and 1.8 s when it is filled with 55 gallons of crude oil. Knowing that the density of the oil is 900 kg/m^3, determine (*a*) the mass of the empty barrel, (*b*) the density of the salt water, ρ_{sw}. [*Hint*: the force of the water on the bottom of the barrel can be modeled as a spring with constant $k = \rho_{sw} gA$.]

SOLUTION

Area of bottom of barrel:

$$A = \frac{\pi D^2}{4} = \frac{\pi(0.572 \text{ m})^2}{4} = 0.2570 \text{ m}^2$$

Mass of oil:

$$m_{\text{oil}} = (55 \text{ gal})\left(\frac{1 \text{ m}^3}{264.172 \text{ gal}}\right)(900 \text{ kg/m}^3) = 187.378 \text{ kg}$$

Barrel empty:

$$\tau_1 = 0.58 \text{ s}$$

$$\omega_{n1} = \frac{2\pi}{\tau_1} = \frac{2\pi}{0.58 \text{ s}} = 10.833 \text{ rad/s}$$

$$\omega_{n1} = \sqrt{\frac{k}{m_b}} \qquad (1)$$

Barrel full:

$$\tau_2 = 1.8 \text{ s}$$

$$\omega_{n2} = \frac{2\pi}{\tau_2} = \frac{2\pi}{1.8 \text{ s}} = 3.4907 \text{ rad/s}$$

$$\omega_{n2} = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{m_{\text{oil}} + m_b}} \qquad (2)$$

(*a*) Mass m_b of empty barrel.

Divide Eq. (1) by Eq. (2) and square both sides.

$$\frac{\omega_{n1}^2}{\omega_{n2}^2} = \frac{(10.833)^2}{(3.4907)^2} = 9.6310 = \frac{m_{\text{oil}} + m_b}{m_b}$$

$$9.6310\, m_b = m_{\text{oil}} + m_b$$

$$m_b = \frac{m_{\text{oil}}}{9.6310 - 1} = \frac{187.378 \text{ kg}}{8.6310} = 21.710 \text{ kg}$$

$$m_b = 21.7 \text{ kg} \blacktriangleleft$$

Spring constant:

$$k = m_b \omega_{n1}^2 = (21.710)(10.833)^2 = 2.5477 \times 10^3 \text{ N/m}$$

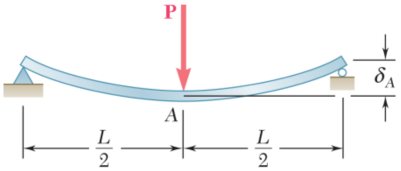
PROBLEM 19.26 (Continued)

(*b*) Density of the salt water.

$$k = \rho_{\text{sw}} gA$$

$$\rho_{\text{sw}} = \frac{k}{gA} = \frac{2.5477 \times 10^3 \text{ N/m}}{(9.81 \text{ m/s}^2)(0.2570 \text{ m}^2)}$$

$$\rho_{\text{sw}} = 1011 \text{ kg/m}^3 \blacktriangleleft$$



PROBLEM 19.27

From mechanics of materials it is known that for a simply supported beam of uniform cross section a static load **P** applied at the center will cause a deflection $\delta_A = PL^3/48EI$, where L is the length of the beam, E is the modulus of elasticity, and I is the moment of inertia of the cross-sectional area of the beam. Knowing that $L = 15$ ft, $E = 30 \times 10^6$ psi, and $I = 2 \times 10^{-3}$ ft^4, determine (*a*) the equivalent spring constant of the beam, (*b*) the frequency of vibration of a 1500-lb block attached to the center of the beam. Neglect the mass of the beam and assume that the load remains in contact with the beam.

SOLUTION

(*a*)

$$k = \frac{P}{\delta} = \frac{48EI}{L^3} = \frac{48\left(30 \times 10^6 \text{ lb/in.}^2\right)\left(144 \text{ in.}^2/\text{ft}^2\right)\left(2 \times 10^{-3}\text{ft}^4\right)}{\left(15 \text{ ft}\right)^3}$$

$$= 122880 \text{ lb/ft}$$

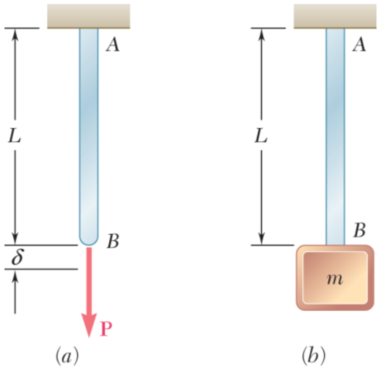
$$k = 1.229 \times 10^5 \text{ lb/ft} \blacktriangleleft$$

(*b*)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{122880 \text{ lb/ft}}{\left(1500 \text{ lb}/32.2 \text{ ft/s}^2\right)}} = 51.360 \text{ rad/s}$$

or

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{\omega_n}{2\pi} = \frac{51.36}{2\pi} = 8.17 \text{ Hz} \blacktriangleleft$$



PROBLEM 19.28

From mechanics of materials it is known that when a static load **P** is applied at the end *B* of a uniform metal rod fixed at end *A*, the length of the rod will increase by an amount $\delta = PL/AE$, where *L* is the length of the undeformed rod. *A* is its cross-sectional area, and *E* is the modulus of elasticity of the metal. Knowing that $L = 450$ mm and $E = 200$ GPa and that the diameter of the rod is 8 mm, and neglecting the mass of the rod, determine (*a*) the equivalent spring constant of the rod, (*b*) the frequency of the vertical vibrations of a block of mass $m = 8$ kg attached to end *B* of the same rod.

SOLUTION

(*a*)

$$P = k_e \delta$$

$$\delta = \frac{PL}{AE}$$

$$P = \left(\frac{AE}{L}\right)\delta$$

$$k_e = \frac{AE}{L}$$

$$A = \frac{\pi d^2}{4} = \frac{\pi (8 \times 10^{-3}\ \text{m})^2}{4}$$

$$A = 5.027 \times 10^{-5}\ \text{m}^2$$

$$L = 0.450\ \text{m}$$

$$E = 200 \times 10^9\ \text{N/m}^2$$

$$k_e = \frac{(5.027 \times 10^{-5}\ \text{m}^2)(200 \times 10^9\ \text{N/m}^2)}{(0.450\ \text{m})}$$

$$k_e = 22.34 \times 10^6\ \text{N/m} \qquad k_e = 22.3\ \text{MN/m} \blacktriangleleft$$

(*b*)

$$f_n = \frac{\sqrt{\frac{k_e}{m}}}{2\pi}$$

$$= \frac{\sqrt{\frac{22.3 \times 10^6}{8}}}{2\pi}$$

$$= 265.96\ \text{Hz} \qquad f_n = 266\ \text{Hz} \blacktriangleleft$$

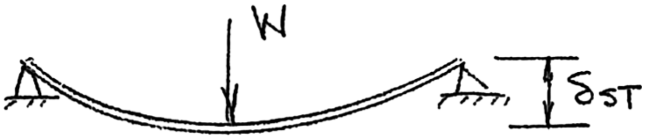
PROBLEM 19.29

Denoting by δ_{st} the static deflection of a beam under a given load, show that the frequency of vibration of the load is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

Neglect the mass of the beam, and assume that the load remains in contact with the beam.

SOLUTION



$$k = \frac{W}{\delta_{st}}$$

$$m = \frac{W}{g}$$

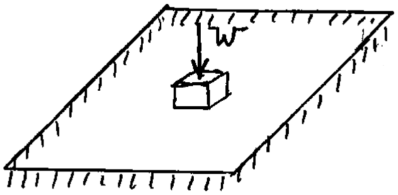
$$\omega_n^2 = \frac{k}{m} = \frac{\frac{W}{\delta_{st}}}{\frac{W}{g}} = \frac{g}{\delta_{st}}$$

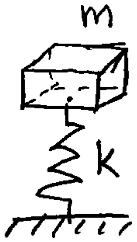
$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} \blacktriangleleft$$

PROBLEM 19.30

A 40-mm deflection of the second floor of a building is measured directly under a newly installed 3500-kg piece of rotating machinery, which has a slightly unbalanced rotor. Assuming that the deflection of the floor is proportional to the load it supports, determine (*a*) the equivalent spring constant of the floor system, (*b*) the speed in rpm of the rotating machinery that should be avoided if it is not to coincide with the natural frequency of the floor-machinery system.

SOLUTION





(*a*) Equivalent spring constant.

$$W = k_e \delta_s$$

$$k_e = \frac{mg}{\delta}$$

$$= \frac{3500(9.81)\text{ N}}{40\text{ mm}}$$

$k_e = 858\text{ N/mm}$ ◀

(*b*) Natural frequency.

$$f_n = \frac{\sqrt{\frac{k_e}{m}}}{2\pi}$$

$$= \frac{\sqrt{\frac{(858.38 \times 1000\text{ N/m})}{(3500\text{ kg})}}}{2\pi}$$

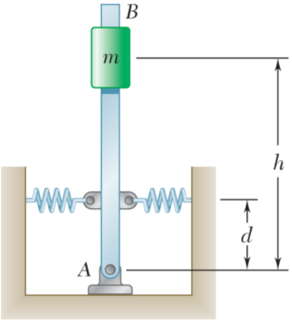
$$f_n = 2.4924\text{ Hz}$$

$$1\text{ Hz} = 1\text{ cycle/s}$$

$$= 60\text{ rpm}$$

$$\text{Speed} = (2.424\text{ Hz})\frac{(60\text{ rpm})}{\text{Hz}}$$

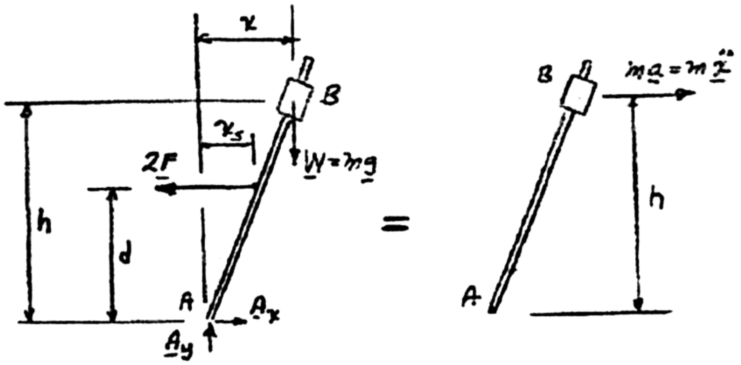
$\text{Speed} = 149.5\text{ rpm}$ ◀



PROBLEM 19.31

If $h = 700$ mm and $d = 500$ mm and each spring has a constant $k = 600$ N/m, determine the mass m for which the period of small oscillations is (*a*) 0.50 s, (*b*) infinite. Neglect the mass of the rod and assume that each spring can act in either tension or compression.

SOLUTION



$$x_s = x\frac{d}{h}$$

$$2F = 2kx_s = 2k\frac{d}{h}x$$

$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \quad 2Fd - mgx = -(m\ddot{x})h$$

$$2k\left(\frac{d}{h}x\right)d - mgx = -m\ddot{x}h$$

$$\ddot{x} + \left[\frac{2kd^2}{mh^2} - \frac{g}{h}\right]x = 0$$

$$\omega_n^2 = \left[\frac{2kd^2}{mh^2} - \frac{g}{h}\right]$$

$$\omega_n^2 = \frac{2k}{m}\left(\frac{d}{h}\right)^2 - \frac{g}{h} \qquad (1)$$

Data:

$d = 0.5$ m

$h = 0.7$ m

$k = 600$ N/m

PROBLEM 19.31 (Continued)

(*a*) For $\tau = 0.5$ s: $\tau = \dfrac{2\pi}{\omega_n}$; $0.5 = \dfrac{2\pi}{\omega_n}$ $\omega_n = 4\pi$

Eq. (1): $$(4\pi)^2 = \frac{2(600)}{m}\left(\frac{0.5}{0.7}\right)^2 - \frac{9.81}{0.7}$$

$m = 3.561 \text{ kg}$ $m = 3.56 \text{ kg}$ ◀

(*b*) For $\tau =$ infinite: $\tau = \dfrac{2\pi}{\omega_n}$ $\omega_n = 0$

Eq. (1): $$0 = \frac{2(600)}{m}\left(\frac{0.5}{0.7}\right)^2 - \frac{9.81}{0.7}$$

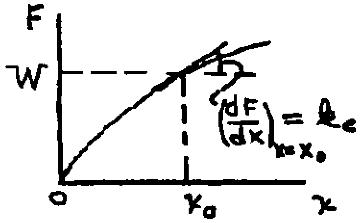
$m = 43.69 \text{ kg}$ $m = 43.7 \text{ kg}$ ◀

PROBLEM 19.32

The force-deflection equation for a nonlinear spring fixed at one end is $F = 1.5x^{1/2}$ where F is the force, expressed in newtons, applied at the other end, and x is the deflection expressed in meters. (*a*) Determine the deflection x_0 if a 4-oz block is suspended from the spring and is at rest. (*b*) Assuming that the slope of the force-deflection curve at the point corresponding to this loading can be used as an equivalent spring constant, determine the frequency of vibration of the block if it is given a very small downward displacement from its equilibrium position and released.

SOLUTION

(*a*) Deflection x_0.



$$W = 4\text{ oz} = 0.25\text{ lb}$$

$$F = W$$

$$= 1.5x_0^{1/2}$$

$$x_0 = \left(\frac{0.25}{1.5}\right)^2$$

$$= 0.027778\text{ ft}$$

$x_0 = 0.333$ in. ◀

Equivalent spring constant.

At x_0,

$$\left(\frac{dF}{dx}\right)_{x_0} = \frac{1.5}{2}(x_0)^{-1/2} = \frac{1.5}{2}(0.027778)^{-1/2}$$

$$\left(\frac{dF}{dx}\right)_{x_0} = 4.5\text{ lb/ft}$$

$$k_e = 4.5\text{ lb/ft}$$

(*b*) Natural frequency.

$$f_n = \frac{\sqrt{\frac{k_e}{m}}}{2\pi}$$

$$= \frac{\sqrt{\frac{(4.5\text{ lb/ft})}{(0.25/32.2)}}}{2\pi}$$

$$f_n = 3.8316\text{ Hz}$$

$f_n = 3.83$ Hz ◀

PROBLEM 19.33*

Expanding the integrand in Equation (19.19) of Section 19.4 into a series of even powers of $\sin\phi$ and integrating, show that the period of a simple pendulum of length l may be approximated by the formula

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\theta_m}{2}\right)$$

where θ_m is the amplitude of the oscillations.

SOLUTION

Using the Binomial Theorem, we write

$$\frac{1}{\sqrt{1-\sin^2\left(\frac{\theta_m}{2}\right)\sin^2\phi}} = \left[1-\sin^2\left(\frac{\theta_m}{2}\right)\sin\phi\right]^{-1/2}$$

$$= 1+\frac{1}{2}\sin^2\frac{\theta_m}{2}\sin^2\phi+\cdots$$

Neglecting terms of order higher than 2 and setting $\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$, we have

$$\tau_n = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\left\{1+\frac{1}{2}\sin^2\frac{\theta_m}{2}\left[\frac{1}{2}(1-\cos 2\phi)\right]\right\}d\phi$$

$$= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\left\{1+\frac{1}{4}\sin^2\frac{\theta_m}{2}-\frac{1}{4}\sin^2\frac{\theta_m}{2}\cos 2\phi\right\}d\phi$$

$$= 4\sqrt{\frac{l}{g}}\left[\phi+\frac{1}{4}\left(\sin^2\frac{\theta_m}{2}\right)\phi-\frac{1}{8}\sin^2\frac{\theta_m}{2}\sin 2\phi\right]_0^{\pi/2}$$

$$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2}+\frac{1}{4}\left(\sin^2\frac{\theta_m}{2}\right)\frac{\pi}{2}+0\right]$$

$$\tau_n = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\theta_m}{2}\right) ◀$$

PROBLEM 19.34*

Using the formula given in Problem 19.33, determine the amplitude θ_m for which the period of a simple pendulum is $\frac{1}{2}$ percent longer than the period of the same pendulum for small oscillations.

SOLUTION

For small oscillations,

$$(\tau_n)_0 = 2\pi\sqrt{\frac{l}{g}}$$

We want

$$\tau_n = 1.005(\tau_n)_0$$

$$= 1.005\, 2\pi\sqrt{\frac{l}{g}}$$

Using the formula of Problem 19.33, we write

$$\tau_n = (\tau_n)_0\left(1 + \frac{1}{4}\sin^2\frac{\theta_m}{2}\right)$$

$$= 1.005(\tau_n)_0$$

$$\sin^2\frac{\theta_m}{2} = 4[1.005 - 1] = 0.02$$

$$\sin\frac{\theta_m}{2} = \sqrt{0.02}$$

$$\frac{\theta_m}{2} = 8.130°$$

$\theta_m = 16.26°$ ◀

PROBLEM 19.35*

Using the data of Table 19.1, determine the period of a simple pendulum of length $l = 750\text{ mm}$ (*a*) for small oscillations, (*b*) for oscillations of amplitude $\theta_m = 60°$, (*c*) for oscillations of amplitude $\theta_m = 90°$.

SOLUTION

(*a*) $\tau_n = 2\pi\sqrt{\dfrac{l}{g}}$ (Equation 19.18 for small oscillations):

$$\tau_n = 2\pi\sqrt{\frac{0.750\text{ m}}{9.81\text{ m/s}^2}}$$

$$= 1.737\text{ s}$$

$\tau_n = 1.737\text{ s}$ ◀

(*b*) For large oscillations (Eq. 19.20),

$$\tau_n = \left(\frac{2K}{\pi}\right)\left(2\pi\sqrt{\frac{l}{g}}\right)$$

$$= \frac{2K}{\pi}(1.737\text{ s})$$

For $\theta_m = 60°$, $K = 1.686$ (Table 19.1)

$$\tau_n(60°) = \frac{2(1.686)(1.737\text{ s})}{\pi}$$

$$= 1.864\text{ s}$$

$\tau_n = 1.864\text{ s}$ ◀

(*c*) For $\theta_m = 90°$, $K = 1.854$ $\tau_n = \dfrac{2(1.854)(1.737\text{ s})}{\pi} = 2.05\text{ s}$ ◀

PROBLEM 19.36*

Using the data of Table 19.1, determine the length in inches of a simple pendulum which oscillates with a period of 2 s and an amplitude of 90°.

SOLUTION

For large oscillations (Eq. 19.20),

$$\tau_n = \left(\frac{2K}{\pi}\right)\left(2\pi\sqrt{\frac{l}{g}}\right)$$

for $\theta_m = 90°$

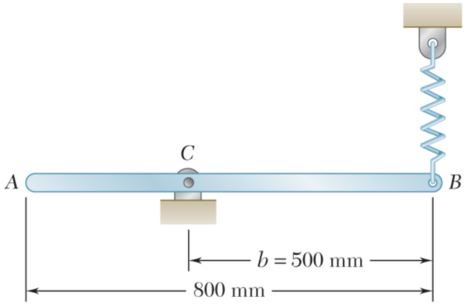
$$K = 1.854 \ \text{(Table 19.1)}$$

$$(2\text{ s}) = (2)(1.854)(2)\sqrt{\frac{l}{32.2\text{ ft/s}^2}}$$

$$l = \frac{(2\text{ s})^2(32.2\text{ ft/s}^2)}{[(4)(1.854)]^2}$$

$$= 2.342\text{ ft}$$

$l = 28.1$ in. ◀

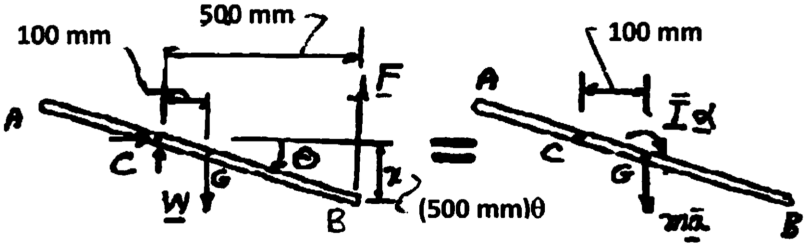


PROBLEM 19.37

The uniform rod shown has mass 6 kg and is attached to a spring of constant $k = 700$ N/m. If end B of the rod is depressed 10 mm and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of end B.

SOLUTION

$$k = 700 \text{ N/m}$$
$$W = mg$$



where
$$F = k(x + \delta_{\text{st}})$$
$$= k(0.5\theta + \delta_{\text{st}})$$
$$m\bar{a} = m\bar{r}\alpha = 6(0.1 \text{ m})\ddot{\theta} = 0.6\ddot{\theta}$$
$$\bar{I}\alpha = \frac{1}{12}(6)(0.8 \text{ m})^2\ddot{\theta}$$
$$= 0.32\ddot{\theta}$$

(*a*) Equation of motion.

$$+\circlearrowright \Sigma M_C = \bar{I}\alpha + m\bar{a}d: \quad W(0.1 \text{ m}) - F(0.5 \text{ m}) = \bar{I}\alpha + m\bar{a}(0.1 \text{ m})$$
$$W(0.1) - k(0.5\theta + \delta_{\text{st}})(0.5 \text{ m}) = 0.32\ddot{\theta} + 0.6\ddot{\theta}(0.1)$$

But in equilibrium, we have $+\circlearrowright W(0.1 \text{ m}) - k\delta_{\text{st}}(0.5 \text{ m}) = 0$

Thus,
$$-k(0.5)^2\theta = [0.32 + 0.06]\ddot{\theta}$$
$$-(700 \text{ N/m})(0.5)^2\theta = 0.38\ddot{\theta}$$
$$\ddot{\theta} + (460.53)\theta = 0$$

Natural frequency and period.

$$\omega_n^2 = 460.53$$

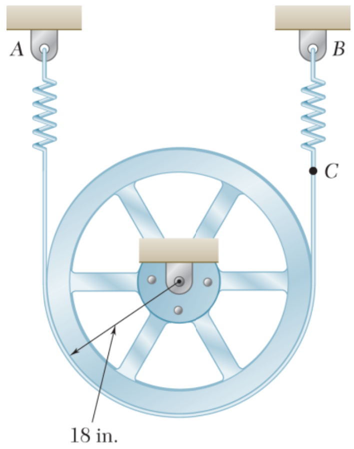
$$\omega_n = 21.46 \text{ rad/s}$$

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{21.46 \text{ rad/s}} \qquad \tau = 0.293 \text{ s} \blacktriangleleft$$

(*b*) At end *B*.

$$x_m = 0.010 \text{ m}$$

$$\begin{aligned} v_m &= x_m \omega_n \\ &= (10 \text{ mm})(21.46 \text{ rad/s}) \\ &= 214.6 \text{ mm/s} \end{aligned} \qquad v_m = 0.215 \text{ m/s} \blacktriangleleft$$

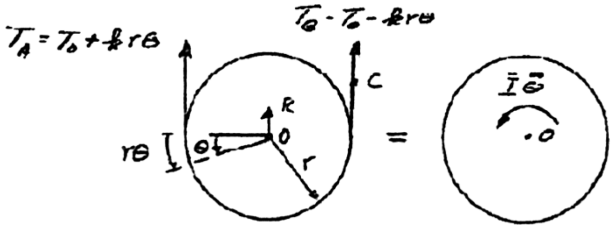


PROBLEM 19.38

A belt is placed around the rim of a 500-lb flywheel and attached as shown to two springs, each of constant $k = 85$ lb/in. If end C of the belt is pulled 1.5 in. down and released, the period of vibration of the flywheel is observed to be 0.5 s. Knowing that the initial tension in the belt is sufficient to prevent slipping, determine (*a*) the maximum angular velocity of the flywheel, (*b*) the centroidal radius of gyration of the flywheel.

SOLUTION

Denote the initial tension by T_0.



Equation of motion. $\quad +\circlearrowleft \Sigma M_0 = \bar{I}\ddot{\theta}: \quad -T_A r + T_B r = \bar{I}\ddot{\theta}$

$$-(T_0 + kr\theta)r + (T_0 - kr\theta)\theta = \bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{2kr^2}{\bar{I}}\theta = 0$$

$$\omega_n^2 = \frac{2kr^2}{\bar{I}} \qquad (1)$$

Data:

$$m = \frac{W}{g} = \frac{500 \text{ lb}}{32.2} \qquad k = 85 \text{ lb/in.} = 1020 \text{ lb/ft}$$

$$\tau = 0.5 \text{ s} \qquad r = 18 \text{ in.} = 1.5 \text{ ft}$$

$$\tau = \frac{2\pi}{\omega_n}; \qquad \omega_n = \frac{2\pi}{\tau} = \frac{2\pi}{0.5} = 4\pi \text{ rad/s}$$

(*a*) Maximum angular velocity. If Point C is pulled down 1.5 in. and released,

$$\theta_m = \theta_{\max} = \left(\frac{1.5 \text{ in.}}{18 \text{ in.}}\right) = 83.333 \times 10^{-3} \text{ rad}$$

$$\dot{\theta}_m = \theta_m \omega_n = (83.333 \times 10^{-3} \text{ rad})(4\pi \text{ rad/s}) \qquad \dot{\theta}_m = 1.047 \text{ rad/s} \blacktriangleleft$$

(*b*) Centroidal radius of gyration.

$$\omega_n^2 = \frac{2kr^2}{I}$$

$$(4\pi \text{ rad/s})^2 = \frac{2(1020 \text{ lb/ft})(1.5 \text{ ft})^2}{\bar{I}}$$

$$\bar{I} = 29.067 \text{ slug} \cdot \text{ft}^2$$

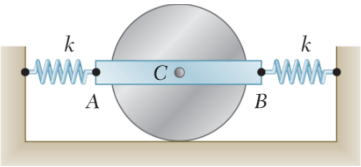
or since

$$\bar{I} = m\bar{k}^2$$

$$\left(\frac{500 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\bar{k}^2 = 29.067 \text{ slug} \cdot \text{ft}^2$$

$$\bar{k} = 1.3682 \text{ ft}$$

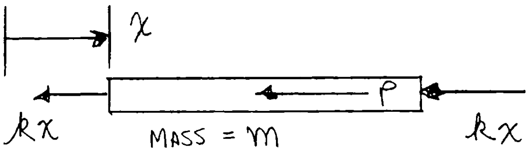
$\bar{k} = 16.42$ in. ◀

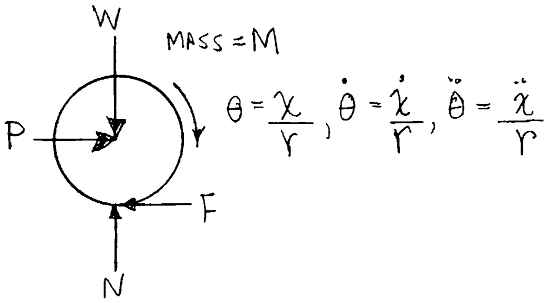


PROBLEM 19.39

A 6-kg uniform cylinder can roll without sliding on a horizontal surface and is attached by a pin at point C to the 4-kg horizontal bar AB. The bar is attached to two springs, each of constant $k = 5$ kN/m as shown. Knowing that the bar is moved 12 mm to the right of the equilibrium position and released, determine (*a*) the period of vibration of the system, (*b*) the magnitude of the maximum velocity of bar AB.

SOLUTION





$$m\ddot{x} = 2kx - P$$

$$M\ddot{x} = P - F$$

$$\frac{1}{2}Mr^2\ddot{\theta} = Fr$$

Eliminate P, F and θ

$$\left(m + \frac{3M}{2}\right)\ddot{x} + 2kx = 0$$

$$\omega_n^2 = \frac{2k}{m + \dfrac{3M}{2}} = \frac{2(5000\text{ N/m})}{4\text{ kg} + 9\text{ kg}} = 769.23$$

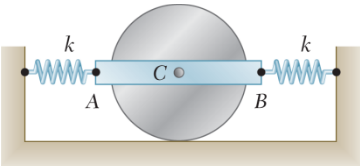
$$\omega_n = 27.735\text{ rad/s}$$

(*a*)

$$\tau_n = \frac{2\pi}{\omega_n} = 0.22654\text{ s}$$

$\tau_n = 0.227$ s ◀

(*b*) $|v_{\max}| = (12\text{ mm})\omega_n = 333$ mm/s ◀



PROBLEM 19.40

A 6-kg uniform cylinder is assumed to roll without sliding on a horizontal surface and is attached by a pin at point *C* to the 4-kg horizontal bar *AB*. The bar is attached to two springs, each of constant k = 3.5 kN/m as shown. Knowing that the coefficient of static friction between the cylinder and the surface is 0.5, determine the maximum amplitude of the motion of point *C* which is compatible with the assumption of rolling.

SOLUTION

From Problem 19.39

$$\omega_n^2 = \frac{2k}{m + \frac{3M}{2}} = \frac{2(3500\text{ N/m})}{4\text{ kg} + 9\text{ kg}} = 538.46; \qquad \omega_n = 23.2048\text{ rad/s}$$

$$N = W = Mg + mg = (6\text{ kg})(9.81\text{ m/s}^2) + (4\text{ kg})(9.81\text{ m/s}^2) = 98.1\text{ N}$$

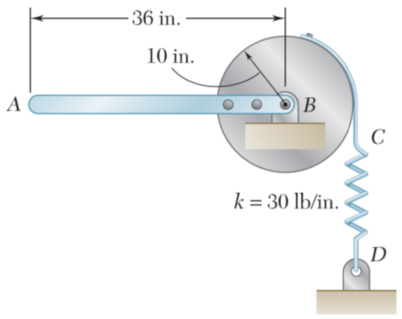
$$Fr = \frac{1}{2}Mr^2\ddot{\theta} = \frac{1}{2}Mr^2\left(\frac{\ddot{x}}{r}\right)$$

$$\max\frac{F}{N} = \frac{\frac{1}{2}(6\text{ kg})}{98.1\text{ N}}\ddot{x}_m, \qquad \ddot{x}_m = \omega_n^2 A; \qquad A = \text{Amplitude}$$

$$\mu_s = 0.5 = \frac{\frac{1}{2}(6\text{ kg})}{98.1\text{ N}}(23.2048\text{ rad/s})^2 A$$

or $$A = 0.03036\text{ m}$$

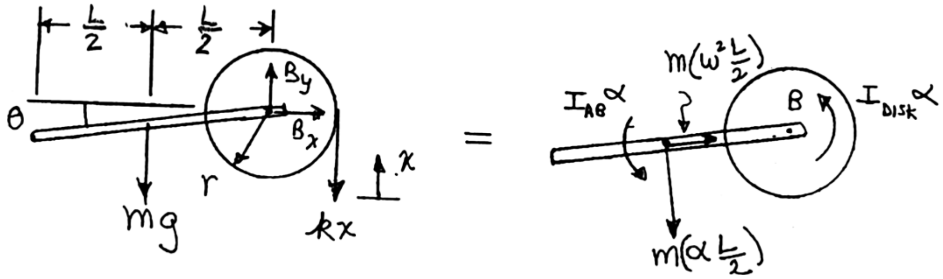
$A = 30.4$ mm ◀



PROBLEM 19.41

A 15-lb slender rod *AB* is riveted to a 12-lb uniform disk as shown. A belt is attached to the rim of the disk and to a spring which holds the rod at rest in the position shown. If end *A* of the rod is moved 0.75 in. down and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of end *A*.

SOLUTION



Equation of motion. $\circlearrowleft\!+\ \Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad mg\frac{L}{2}\cos\theta - kxr = I_{AB}\alpha + m\left(\alpha\frac{L}{2}\right)\left(\frac{L}{2}\right) + I_{\text{disk}}\alpha$ (1)

where $x = r\theta + \delta_{\text{st}}$ and from statics, $mg\frac{L}{2} = k\delta_{\text{st}}r$

Assuming small angles $(\cos\theta \approx 1)$, Equation (1) becomes

$$\cancel{mg\frac{L}{2}}\theta - kr^2\theta - \cancel{kr\delta_{\text{st}}} = \left(I_{AB} + m\left(\frac{L}{2}\right)^2 + I_{\text{disk}}\right)\alpha$$

$$\left(I_{AB} + \frac{mL^2}{4} + I_{\text{disk}}\right)\ddot{\theta} + kr^2\theta = 0$$

Data:

$$m = \frac{15}{32.2}$$
$$= 0.46584\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$m_{\text{disk}} = \frac{12}{32.2}$$
$$= 0.37267\ \text{lb}\cdot\text{s}^2/\text{ft}$$
$$L = 36\ \text{in.} = 3.0\ \text{ft}$$
$$r = 10\ \text{in.} = 0.83333\ \text{ft}$$
$$k = 30\ \text{lb/in.} = 360\ \text{lb/ft}$$

PROBLEM 19.41 (Continued)

$$I_{AB} = \frac{1}{12}mL^2$$

$$= \frac{1}{12}(0.46584)(3.0)^2$$

$$= 0.34938\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

$$I_{\text{disk}} = \frac{1}{2}m_{\text{disk}}r^2$$

$$= \frac{1}{2}(0.37267)(0.83333)^2$$

$$= 0.1294\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\left[0.34935 + \frac{1}{4}(0.46584)(3.0)^2 + 0.1294\right]\ddot{\theta} + (360)(0.83333)^2\theta = 0$$

$$1.5269\ddot{\theta} + 250\theta = 0 \quad \text{or} \quad \ddot{\theta} + 163.73\theta = 0$$

(*a*) Natural frequency and period. $\omega_n^2 = 163.73\ (\text{rad/s})^2$

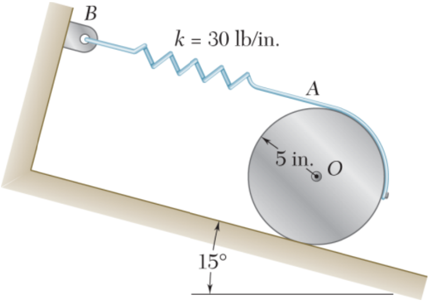
$$\omega_n = 12.796\ \text{rad/s}$$

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{12.796}$$

$\tau = 0.491\ \text{s}$ ◀

(*b*) Maximum velocity. $v_m = \omega_n x_m = (12.796)(0.75)$

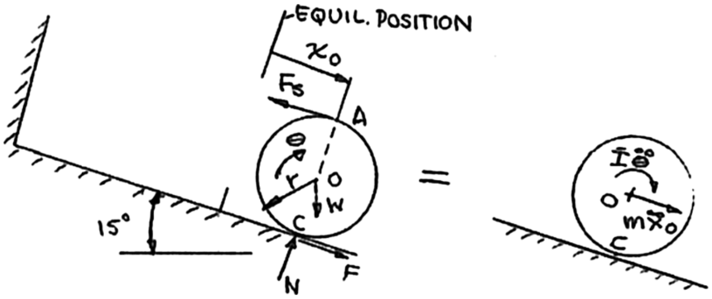
$v_m = 9.60\ \text{in./s}$ ◀



PROBLEM 19.42

A 30-lb uniform cylinder can roll without sliding on a 15°-incline. A belt is attached to the rim of the cylinder, and a spring holds the cylinder at rest in the position shown. If the center of the cylinder is moved 2 in. down the incline and released, determine (*a*) the period of vibration, (*b*) the maximum acceleration of the center of the cylinder.

SOLUTION



Spring deflection.

$$x_A = x_0 + x_{A/0}$$

$$x_{A/0} = r\theta$$

$$\theta = \frac{x_0}{r}$$

$$x_A = 2x_0$$

$$F_S = k(x_A + \delta_{st}) = k(2x_0 + \delta_{st})$$

$$\overset{\curvearrowright}{+}\ \Sigma M_C = (\Sigma M)_{\text{eff}}:\quad -2rk(2x_0 + \delta_{st}) + rW\sin 15^\circ = rm\ddot{x}_0 + \bar{I}\ddot{\theta} \qquad (1)$$

But in equilibrium, $x_0 = 0$

$$\Sigma M_C = 0 = -2rk\delta_{st} + rW\sin 15^\circ \qquad (2)$$

Substituting Eq. (2) into Eq. (1) and noting that $\theta = \dfrac{x_0}{r}, \quad \ddot{\theta} = \dfrac{\ddot{x}_0}{r}$

$$rm\ddot{x}_0 + \bar{I}\frac{\ddot{x}_0}{r} + 4rkx_0 = 0$$

$$\bar{I} = \frac{1}{2}mr^2$$

$$\frac{3}{2}mr\ddot{x}_0 + 4rkx_0 = 0$$

$$\ddot{x}_0 + \left(\frac{8}{3}\frac{k}{m}\right)x_0 = 0$$

PROBLEM 19.42 (Continued)

Natural frequency. $\omega_n = \sqrt{\dfrac{8}{3}\dfrac{k}{m}} = \sqrt{\dfrac{(8)(30 \times 12 \text{ lb/ft})}{(3)\frac{(30 \text{ lb})}{(32.2 \text{ ft/s}^2)}}} = 32.1 \text{ s}^{-1}$

(*a*) Period. $\tau_n = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{32.1} = 0.1957 \text{ s}$ $\tau_n = 0.1957 \text{ s}$ ◀

(*b*) $x_0 = (x_0)_m \sin(\omega_n t + \phi)$

At $t = 0$, $x_0 = \dfrac{2}{12} \text{ ft} \qquad \dot{x}_0 = 0$

$$\dot{x}_0 = (x_0)_m \omega_n \cos(\omega_n t + \phi)$$

$$t = 0$$

$$0 = (x_0)_m \omega_n \cos\phi$$

Thus, $\phi = \dfrac{\pi}{2}$

$$t = 0$$

$$x_0(0) = \frac{1}{6} \text{ ft} = (x_0)_m \sin\phi = (x_0)_m (1)$$

$$(x_0)_m = \frac{1}{6} \text{ ft}$$

$$\ddot{x}_0 = -(x_0)_m \omega_n^2 \sin(\omega_n t + \phi)$$

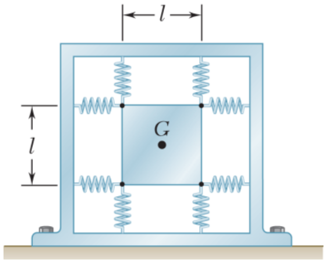
$$(a_0)_{\max} = (\ddot{x}_0)_{\max}$$

$$= -(x_0)_m \omega_n^2$$

$$= -\left(\frac{1}{6} \text{ ft}\right)(32.1 \text{ s}^{-1})^2$$

$$= 171.7 \text{ ft/s}^2$$

$(a_0)_{\max} = 171.7 \text{ ft/s}^2$ ◀



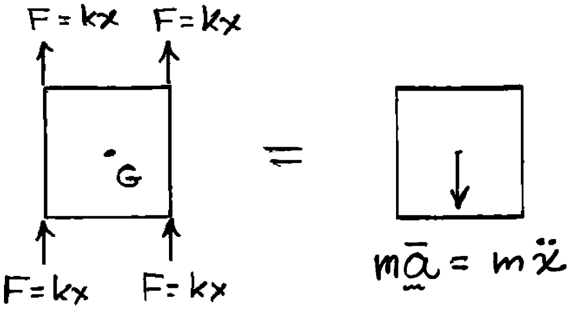
PROBLEM 19.43

A square plate of mass m is held by eight springs, each of constant k. Knowing that each spring can act in either tension or compression, determine the frequency of the resulting vibration (*a*) if the plate is given a small vertical displacement and released, (*b*) if the plate is rotated through a small angle about G and released.

SOLUTION

(*a*) Small vertical displacement.

Let the plate be displaced downward a distance x from the equilibrium position. Each corner moves downward a distance x and the four vertical springs exert additional forces kx for each spring. The horizontal springs exert negligible change.



$$+\downarrow \Sigma F = ma: \quad -4kx = m\ddot{x}$$

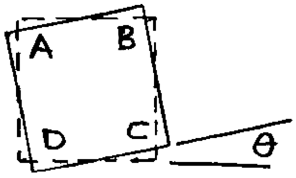
$$\ddot{x} + \frac{4k}{m}x = 0$$

$$\omega_n^2 = \frac{4k}{m}$$

Frequency: $$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{4k}{m}}$$ $$f = 0.318\sqrt{\frac{k}{m}} \blacktriangleleft$$

(*b*) Small rotation about G.

Let the plate be rotated through a small counterclockwise angle θ from the equilibrium position. The corners A, B, C, and D move as indicated below:



A: $(l/2)\theta\downarrow + (l/2)\theta\leftarrow$

B: $(l/2)\theta\leftarrow + (l/2)\theta\uparrow$

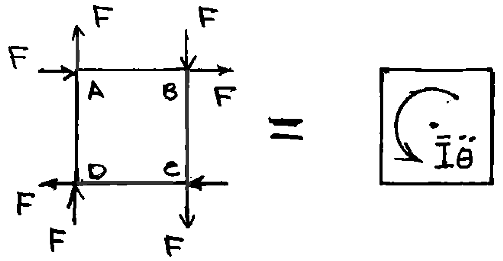
C: $(l/2)\theta\uparrow + (l/2)\theta\leftarrow$

D: $(l/2)\theta\rightarrow + (l/2)\theta\downarrow$

The additional force exerted by each of the eight springs is $F = (kl/2)\theta$ and directed as shown on the free body diagram. The eight forces reduce to four clockwise couples, each of magnitude Fl. For a square plate

$$\bar{I} = \frac{1}{6}ml^2$$

PROBLEM 19.43 (Continued)



$$\overset{\curvearrowleft}{+} M_G = \bar{I}\alpha: \quad 4Fl = \bar{I}\ddot{\theta} - 4(kl^2/2)\theta$$

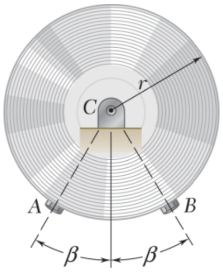
$$= \frac{1}{6}ml^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{12k}{\text{m}}\theta = 0$$

$$\omega_n^2 = \frac{12k}{\text{m}}$$

Frequency: $f = \dfrac{\omega_n}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{12k}{m}}$

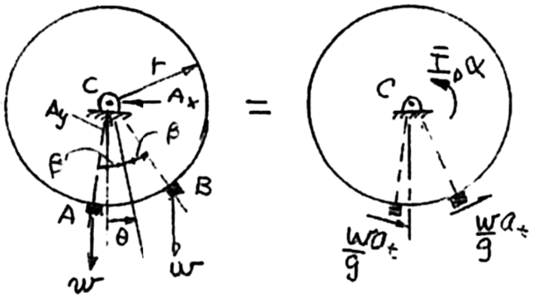
$$f = 0.551\sqrt{\frac{k}{m}} \blacktriangleleft$$



PROBLEM 19.44

Two small weights w are attached at A and B to the rim of a uniform disk of radius r and weight W. Denoting by τ_0 the period of small oscillations when $\beta = 0$, determine the angle β for which the period of small oscillations is $2\tau_0$.

SOLUTION



$$\alpha = \ddot{\theta}$$

$$a_t = r\alpha = r\ddot{\theta}$$

$$\bar{I}_D = \frac{1}{2}\frac{W}{g}r^2$$

Equation of motion.

$$\Sigma M_C = (\Sigma M_C)_{\text{eff}}: \quad wr\sin(\beta - \theta) - wr\sin(\beta + \theta) = \frac{2w}{g}ra_t + \bar{I}\alpha$$

$$wr[\sin(\beta - \theta) - \sin(\beta + \theta)] = -2wr\sin\theta\cos\beta$$

$$\sin\theta \approx \theta$$

$$\left(\frac{2w}{g}r^2 + \frac{W}{2g}r^2\right)\ddot{\theta} + (2\,wr\cos\beta)\theta = 0$$

Natural frequency.

$$\omega_n = \sqrt{\frac{2wg\cos\beta}{\left(2w + \frac{W}{2}\right)r}} = \sqrt{\frac{4g\cos\beta}{(4 + \frac{W}{w})r}} \qquad (1)$$

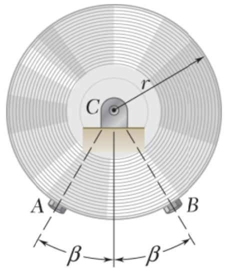
$$\beta = 0 \qquad \tau_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{4g}{(4+\frac{W}{w})r}}}$$

$$\tau_n = \frac{2\pi}{\sqrt{\frac{\cos\beta}{(4+\frac{W}{w})r}}} = 2\tau_0 = \frac{4\pi}{\sqrt{\frac{4g}{(4+\frac{W}{w})r}}}$$

$$\cos\beta = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

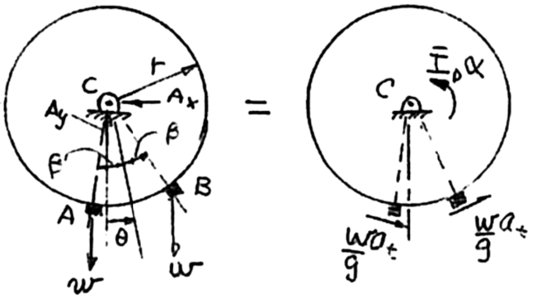
$\beta = 75.5°$ ◀

PROBLEM 19.45



Two 40-g weights are attached at *A* and *B* to the rim of a 1.5-kg uniform disk of radius $r = 100$ mm. Determine the frequency of small oscillations when $\beta = 60°$.

SOLUTION



$$\alpha = \ddot{\theta}$$

$$a_t = r\alpha = r\ddot{\theta}$$

$$\bar{I}_D = \frac{1}{2}m_D r^2$$

Equation of motion.

$$\Sigma M_C = \bar{I}\alpha + m\bar{a}d: \quad wr\sin(\beta-\theta) - wr\sin(\beta+\theta) = \frac{2w}{g}ra_t + \bar{I}\alpha$$

$$wr[\sin(\beta-\theta) - \sin(\beta+\theta)] = -2wr\sin\theta\cos\beta$$

$$\sin\theta \approx \theta$$

$$\left(\frac{2w}{g}r^2 + \frac{W}{2g}r^2\right)\ddot{\theta} + (2\,wr\cos\beta)\theta = 0$$

Natural frequency.

$$\omega_n = \sqrt{\frac{2wg\cos\beta}{\left(2w+\frac{W}{2}\right)r}} = \sqrt{\frac{4g\cos\beta}{(4+\frac{W}{w})r}} \qquad (1)$$

Data:

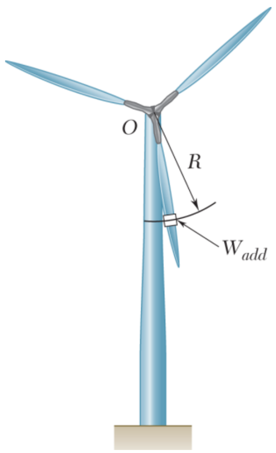
$$w = mg = 0.04g \qquad W = m_D g = 1.5g \qquad \frac{W}{w} = \frac{1.5g}{0.04g} = 37.5$$

$$r = 0.100 \text{ m} \qquad \beta = 60°$$

$$\omega_n = \sqrt{\frac{(4)(9.81)\cos 60°}{(4+37.5)(0.10)}} = 2.1743 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{2.1743}{2\pi}$$

$f_n = 0.346$ Hz ◀

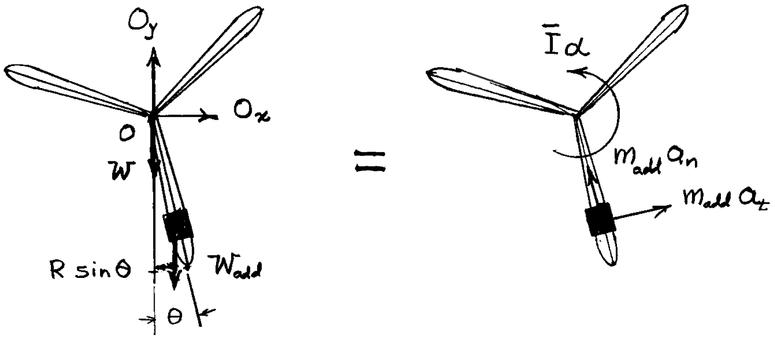


PROBLEM 19.46

A three-bladed wind turbine used for research is supported on a shaft so that it is free to rotate about O. One technique to determine the centroidal mass moment of inertia of an object is to place a known weight at a known distance from the axis of rotation and to measure the frequency of oscillations after releasing it from rest with a small initial angle. In this case, a weight of $W_{\text{add}} = 50$ lb is attached to one of the blades at a distance $R = 20$ ft from the axis of rotation. Knowing that when the blade with the added weight is displaced slightly from the vertical axis, the system is found to have a period of 7.6 s, determine the centroidal mass moment of inertia of the 3-bladed rotor.

SOLUTION

Let the turbine rotor be turned counterclockwise through a small angle θ. The moment of the added weight about Point O is



$$M = -W_{\text{add}} R \sin\theta$$

$$+\circlearrowleft \Sigma M_0 = \Sigma (M_0)_{\text{eff}}: \quad -W_{\text{add}} R \sin\theta = \bar{I}\alpha + m_{\text{add}} R a_a$$

$$= (\bar{I} + m_{\text{add}} R^2)\alpha$$

$$= (\bar{I} + m_{\text{add}} R^2)\ddot{\theta}$$

$$\ddot{\theta} + \frac{W_{\text{add}} R}{\bar{I} + m_{\text{add}} R^2} \sin\theta = 0$$

Using $\sin\theta \approx \theta$ gives

$$\ddot{\theta} + \frac{W_{\text{add}} R}{\bar{I} + m_{\text{add}} R^2} \theta = 0$$

$$\ddot{\theta} + \omega_n^2 \theta = 0 \qquad \omega_n^2 = \frac{W_{\text{add}} R}{\bar{I} + m_{\text{add}} R^2}$$

Solving for $\bar{I}$,

$$\bar{I} = \frac{W_{\text{add}} R}{\omega_n^2} - m_{\text{add}} R^2 \tag{1}$$

Data: $R = 20 \text{ ft}, \qquad W_{\text{add}} = 50 \text{ lb}$

$$m_{\text{add}} = \frac{W_{\text{add}}}{g} = \frac{50 \text{ lb}}{32.2 \text{ ft/s}^2} = 1.5528 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Period and frequency: $\tau = 7.6 \text{ s}$

$$f = \frac{1}{\tau} = \frac{1}{7.6} \text{Hz}$$

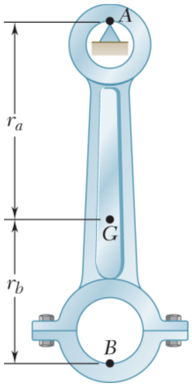
$$\omega_n = 2\pi f = \frac{2\pi}{7.6} = 0.82673 \text{ rad/s}$$

From Eq. (1),

$$\bar{I} = \frac{(50 \text{ lb})(20 \text{ ft})}{(0.82673 \text{ rad/s})^2} - (1.5528 \text{ lb} \cdot \text{s}^2/\text{ft})(20 \text{ ft})^2$$

$$= 1463.10 - 621.12$$

$$\bar{I} = 842 \text{ lb} \cdot \text{s}^2 \cdot \text{ft} \blacktriangleleft$$

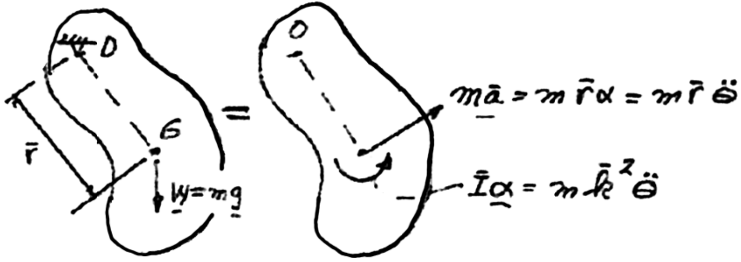


PROBLEM 19.47

A connecting rod is supported by a knife-edge at Point A; the period of its small oscillations is observed to be 0.87 s. The rod is then inverted and supported by a knife-edge at Point B and the period of small oscillations is observed to be 0.78 s. Knowing that $r_a + r_b = 10$ in. determine (a) the location of the mass center G, (b) the centroidal radius of gyration $\bar{k}$.

SOLUTION

Consider general pendulum of centroidal radius of gyration $\bar{k}$.


D
$\bar{r}$
G
W=mg
=
O
$m\bar{a} = m\bar{r}\alpha = m\bar{r}\ddot{\theta}$
$\bar{I}\alpha = m\bar{k}^2\ddot{\theta}$

Equation of motion. $+\circlearrowright \Sigma M_0 = \Sigma(M_0)_{\text{eff}}: \quad -mg\bar{r}\sin\theta = (m\bar{r}\ddot{\theta})\bar{r} + m\bar{k}^2\ddot{\theta}$

$$\ddot{\theta} + \left[\frac{g\bar{r}}{\bar{r}^2 + \bar{k}^2}\right]\sin\theta = 0$$

For small oscillations, $\sin\theta \approx \theta$, we have

$$\ddot{\theta} + \left[\frac{g\bar{r}}{\bar{r}^2 + \bar{k}^2}\right]\theta = 0$$

$$\omega_n^2 = \frac{g\bar{r}}{\bar{r}^2 + \bar{k}^2}$$

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\bar{r}^2 + \bar{k}^2}{g\bar{r}}}$$

For rod suspended at A,

$$\tau_A = 2\pi\sqrt{\frac{r_a^2 + \bar{k}^2}{gr_a}}$$

$$g\tau_A^2 r_a = 4\pi^2\left(r_a^2 + \bar{k}^2\right) \tag{1}$$

For rod suspended at B,

$$\tau_B = 2\pi\sqrt{\frac{r_b^2 + \bar{k}^2}{gr_b}}$$

$$g\tau_B^2 r_b = 4\pi^2\left(r_b^2 + \bar{k}^2\right) \tag{2}$$

PROBLEM 19.47 (Continued)

(*a*) Value of r_a.

Subtracting Eq. (2) from Eq. (1), $g\tau_A^2 r_a - g\tau_B^2 r_b = 4\pi^2\left(r_a^2 - r_b^2\right)$

$$g\tau_A^2 r_a - g\tau_B^2 r_b = 4\pi^2(r_a + r_b)(r_a - r_b)$$

Applying the numerical data with $r_a + r_b = 10 \text{ in.} = 0.83333 \text{ ft}$

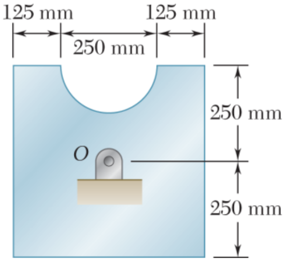
$$(32.2)(0.87)^2 r_a - (32.2)(0.78)^2 r_b = 4\pi^2(0.83333)(r_a - r_b)$$
$$24.372 r_a - 19.590 r_b = 32.899(r_a - r_b)$$

$$13.309 r_b = 8.527 r_a \qquad r_b = 0.6407 r_a$$
$$0.83333 = r_a + 0.6407 r_a \qquad r_a = 0.5079 \text{ ft} \qquad r_a = 6.09 \text{ in.} \blacktriangleleft$$
$$r_b = 0.83333 - 0.5079 \qquad r_b = 0.32543 \text{ ft} \qquad r_b = 3.91 \text{ in.} \blacktriangleleft$$

(*b*) Centroidal radius of gyration.

From Eq. (1), $4\pi^2 \bar{k}^2 = g\,\tau_A^2\, r_a - 4\pi^2 r_a^2$

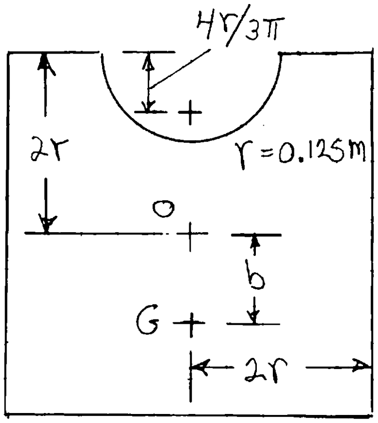
$$= (32.2)(0.87)^2(0.5079) - 4\pi^2(0.5079)^2 = 2.1947 \text{ ft}^2$$
$$\bar{k} = 0.2398 \text{ ft} \qquad \bar{k} = 2.83 \text{ in.} \blacktriangleleft$$



PROBLEM 19.48

A semicircular hole is cut in a uniform square plate which is attached to a frictionless pin at its geometric center *O*. Determine (*a*) the period of small oscillations of the plate, (*b*) the length of a simple pendulum which has the same period.

SOLUTION



$$b = \frac{\left(\frac{\pi r^2}{2}\right)\left(2r - \frac{4r}{3\pi}\right)}{16r^2 - \frac{\pi r^2}{2}} = 0.021440 \text{ m}$$

$$m = \rho t r^2\left(16 - \frac{\pi}{2}\right) = \rho t(0.22546) \text{ kg}$$

$$I_o = \frac{\rho t}{6}(4r)^4 - \rho t\left[\frac{\pi r^4}{4} - \frac{8r^4}{9\pi} + \frac{\pi r^2}{2}\left(2r - \frac{4r}{3\pi}\right)^2\right]$$

$$= \rho t r^4\left[\frac{136}{3} - \frac{9\pi}{4}\right] = \rho t(0.009342)$$

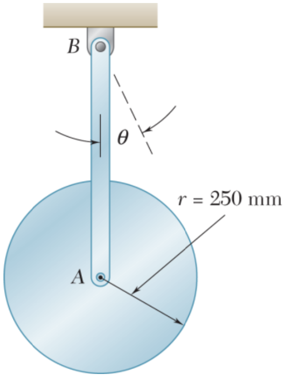
(*a*) $I_o\ddot{\theta} + mgb\theta = 0, \quad \omega_n^2 = \frac{mgb}{I_o} = \frac{\cancel{\rho t}(0.22546)(9.81)(0.021440)}{\cancel{\rho t}(0.009342)}$

$$\omega_n^2 = 5.07602, \quad \omega_n = 2.2530 \text{ rad/s}$$

$$\tau = \frac{2\pi}{\omega_n} = 2.79 \text{ s} \blacktriangleleft$$

(*b*) $\omega_n^2 = \frac{g}{l}, \quad l = \frac{g}{\omega_n^2} = \frac{9.81 \text{ m/s}^2}{5.07602 \text{ rad}^2/\text{s}^2} = 1.9326 \text{ m}$

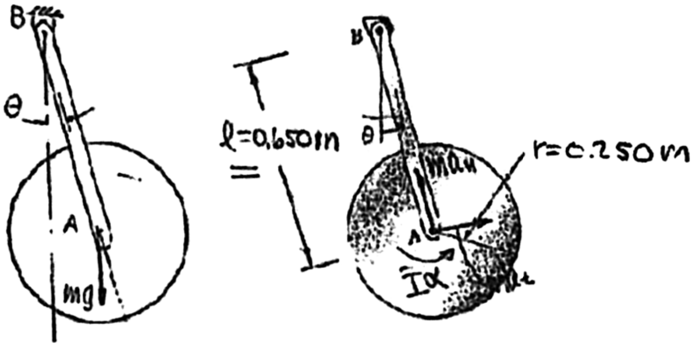
$$l = 1.933 \text{ m} \blacktriangleleft$$



PROBLEM 19.49

A uniform disk of radius $r = 250$ mm is attached at A to a 650-mm rod AB of negligible mass, which can rotate freely in a vertical plane about B. Determine the period of small oscillations (*a*) if the disk is free to rotate in a bearing at A, (*b*) if the rod is riveted to the disk at A.

SOLUTION



$$\bar{I} = \frac{1}{2}mr^2$$

$$= \frac{1}{2}(0.250)^2 m = \frac{m}{32}$$

$$a_t = l\alpha = 0.650\alpha$$

$$\alpha = \ddot{\theta}$$

(*a*) The disk is free to rotate and is in curvilinear translation.

Thus, $\bar{I}\alpha = 0$

$$\Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad \circlearrowleft + -mgl\sin\theta = lma_t \qquad \sin\theta \approx \theta$$

$$ml^2\ddot{\theta} - mgl\theta = 0$$

$$\omega_n^2 = \frac{g}{I} = \frac{9.81\text{ m/s}^2}{0.650\text{ m}}$$

$$= 15.092$$

$$\omega_n = 3.885\text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.885}$$

$\tau_n = 1.617$ s ◀

PROBLEM 19.49 (Continued)

(*b*) When the disk is riveted at A, it rotates at an angular acceleration α.

$$\Sigma M_B = \Sigma(M_B)_{\text{eff}}: \quad \circlearrowleft + -mgl\sin\theta = \bar{I}\alpha + lma_t \qquad I = \frac{1}{2}mr^2$$

$$\left(\frac{1}{2}mr^2 + ml^2\right)\ddot{\theta} + mgl\theta = 0$$

$$\omega_n^2 = \frac{gl}{\left(\frac{r^2}{2} + l^2\right)}$$

$$= \frac{(9.81\text{ m/s}^2)(0.650\text{ m})}{\left[\left(\frac{0.250^2}{2}\right) + (0.650)^2\right]}$$

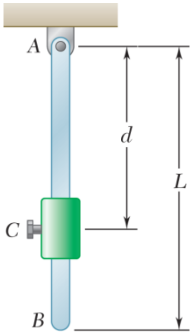
$$= 14.053$$

$$\omega_n = 3.749\text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$= \frac{2\pi}{3.749}$$

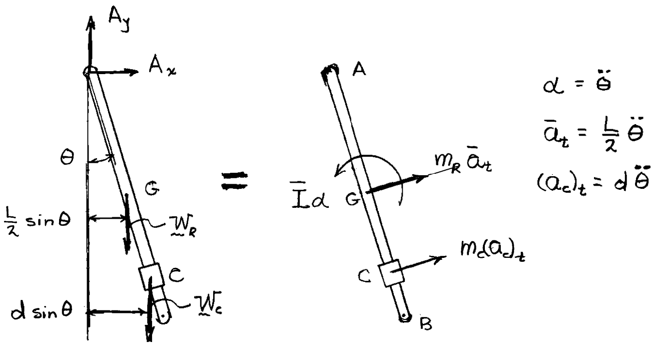
$\tau_n = 1.676$ s ◀



PROBLEM 19.50

A small collar of mass 1 kg is rigidly attached to a 3-kg uniform rod of length $L = 750$ mm. Determine (*a*) the distance d to maximize the frequency of oscillation when the rod is given a small initial displacement, (*b*) the corresponding period of oscillation.

SOLUTION



Equation of motion.

$$\circlearrowleft + \Sigma M_A = (\Sigma M_A)_{\text{eff}}: \quad -W_R \frac{L}{2}\sin\theta - W_C d\sin\theta = \bar{I}_R\alpha + m_R\frac{L}{2}(\bar{a}_t)_R + m_C d(a_t)_C$$

$$\sin\theta \approx \theta \qquad \alpha = \ddot{\theta}, \quad (\bar{a}_t)_R = \frac{L}{2}\alpha = \frac{L}{2}\ddot{\theta}, \quad (a_t)_C = d\alpha = d\ddot{\theta}$$

$$\left(\bar{I}_R + m_R\left(\frac{L}{2}\right)^2 + m_C d^2\right)\ddot{\theta} + \left(m_R g\frac{L}{2} + m_C g d\right)\theta = 0$$

$$\bar{I}_R = \frac{1}{12}m_R L^2$$

$$\bar{I}_R + m_R\left(\frac{L}{2}\right)^2 = \frac{m_R L^2}{3}$$

$$\left(\frac{m_R L^2}{3} + m_C d^2\right)\ddot{\theta} + \left(m_R g\frac{L}{2} + m_C g d\right)\theta = 0$$

$$\ddot{\theta} + \frac{\left(\frac{L}{2} + \frac{m_C}{m_R}d\right)g}{\left(\frac{L^2}{3} + \frac{m_C}{m_R}d^2\right)}\theta = 0$$

$$\frac{m_C}{m_R} = \frac{1}{3}$$

Natural frequency.
$$\omega_n^2 = \frac{\left(\frac{L}{2} + \frac{m_C}{m_R} d\right) g}{\frac{L^2}{3} + \frac{m_C}{m_R} d^2} = \frac{\left(\frac{3L}{2} + d\right) g}{(L^2 + d^2)}$$

(*a*) To maximize the frequency, we need to take the derivative with respect to *d* and set it equal to zero.

$$\frac{1}{g}\frac{d\left(\omega_n^2\right)}{d(d)} = \frac{(L^2 + d^2)(1) - \left(\frac{3L}{2} + d\right)(2d)}{(L^2 + d^2)^2} = 0$$

$$d^2 + L^2 - 3Ld - 2d^2 = 0$$

$$d^2 + 3Ld - L^2 = 0$$

Solve for *d* knowing that $L = 0.75$ m $\quad d = 0.22708 \quad \text{or} \quad -2.4771$

$$d = 0.22708 \text{ m}$$

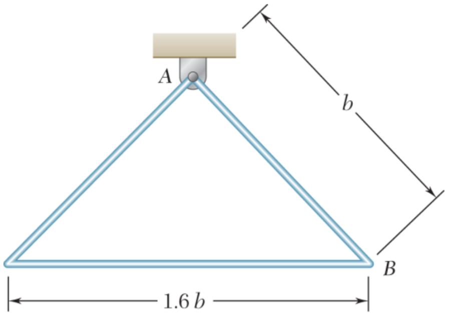
$d = 227$ mm ◀

$$\omega_n^2 = \frac{\left(\frac{3(0.75)}{2} + 0.22708\right) 9.81}{(0.75^2 + 0.22708^2)}$$
$$= 21.6$$

$\omega_n = 4.6476$ rad/s

(*b*) Period of oscillation.
$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{4.6476}$$

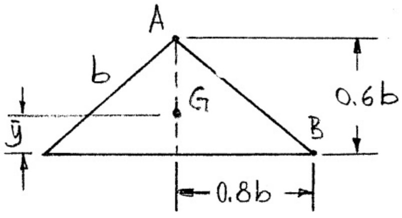
$\tau_n = 1.352$ s ◀



PROBLEM 19.51

A thin homogeneous wire is bent into the shape of an isosceles triangle of sides b, b, and $1.6b$. Determine the period of small oscillations if the wire (a) is suspended from point A as shown, (b) is suspended from point B.

SOLUTION



$$I_A = 2\left[\frac{1}{3}\left(\frac{m}{3.6}\right)b^2\right] + \left(\frac{1.6m}{3.6}\right)\left[\frac{(1.6b)^2}{12} + (0.6b)^2\right]$$

$$= \frac{11}{25}mb^2$$

$$\bar{y} = \frac{2b(0.3b)}{3.6b} = \frac{b}{6}$$

$$AG = 0.6b - \frac{b}{6} = \frac{13}{30}b$$

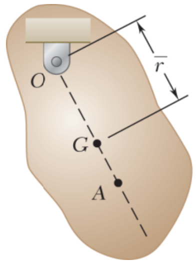
$$I_B = \frac{11}{25}mb^2 - m\left(\frac{13}{30}b\right) + m\left[0.64b^2 + \frac{b^2}{36}\right]$$

$$= \frac{23}{25}mb^2$$

$$GB = \sqrt{24.04}\,\frac{b}{6}$$

(a) $$\frac{11mb^2}{25}\ddot{\theta} + mg\frac{13b}{30}\theta = 0, \quad \tau_n = \frac{2\pi}{\omega_n} = 6.33\sqrt{\frac{b}{g}} \blacktriangleleft$$

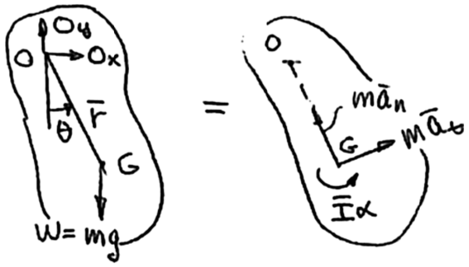
(b) $$\frac{23mb^2}{25}\ddot{\theta} + mg\left(\frac{\sqrt{24.04}}{6}b\right)\theta = 0, \quad \tau_n = \frac{2\pi}{\omega_n} = 6.67\sqrt{\frac{b}{g}} \blacktriangleleft$$



PROBLEM 19.52

A *compound pendulum* is defined as a rigid slab which oscillates about a fixed Point O, called the center of suspension. Show that the period of oscillation of a compound pendulum is equal to the period of a simple pendulum of length OA, where the distance from A to the mass center G is $GA = \bar{k}^2/\bar{r}$. Point A is defined as the center of oscillation and coincides with the center of percussion defined in Problem 17.66.

SOLUTION

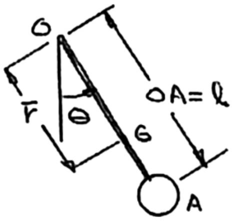


$$+\circlearrowleft \Sigma M_0 = \Sigma (M_0)_{\text{eff}}: \quad -W\bar{r}\sin\theta = \bar{I}\alpha + m\bar{a}_t\bar{r}$$

$$-mg\bar{r}\sin\theta = m\bar{k}^2\ddot{\theta} + m\bar{r}^2\ddot{\theta}$$

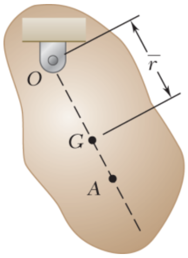
$$\ddot{\theta} + \frac{g\bar{r}}{\bar{r}^2 + \bar{k}^2}\sin\theta = 0 \qquad (1)$$

For a simple pendulum of length $OA = l$,



$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad (2)$$

Comparing Equations (1) and (2), $\quad l = \dfrac{\bar{r}^2 + \bar{k}^2}{\bar{r}}$ $\qquad GA = l - \bar{r} = \dfrac{\bar{k}^2}{\bar{r}}$ Q.E.D. ◀



PROBLEM 19.53

A rigid slab oscillates about a fixed Point O. Show that the smallest period of oscillation occurs when the distance $\bar{r}$ from Point O to the mass center G is equal to $\bar{k}$.

SOLUTION

See Solution to Problem 19.52 for derivation of

$$\ddot{\theta}+\frac{g\bar{r}}{\bar{r}^2+\bar{k}^2}\sin\theta=0$$

For small oscillations, $\sin\theta\approx\theta$ and

$$\tau_n=\frac{2\pi}{\omega_n}=2\pi\sqrt{\frac{\bar{r}^2+\bar{k}^2}{g\bar{r}}}=\frac{2\pi}{\sqrt{g}}\sqrt{\bar{r}+\frac{\bar{k}^2}{\bar{r}}}$$

For smallest τ_n, we must have $\bar{r}+\frac{\bar{k}^2}{\bar{r}}$ as a minimum:

$$\frac{d\left(\bar{r}+\frac{\bar{k}^2}{\bar{r}}\right)}{d\bar{r}}=1-\frac{\bar{k}^2}{\bar{r}^2}=0$$

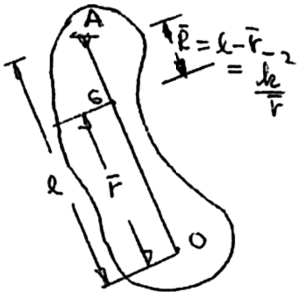
$$\bar{r}^2=\bar{k}^2$$

$\bar{r}=\bar{k}$ Q.E.D. ◀

PROBLEM 19.54

Show that if the compound pendulum of Problem 19.52 is suspended from *A* instead of *O*, the period of oscillation is the same as before and the new center of oscillation is located at *O*.

SOLUTION



Same derivation as in Problem 19.52 with $\bar{r}$ replaced by $\bar{R}$.

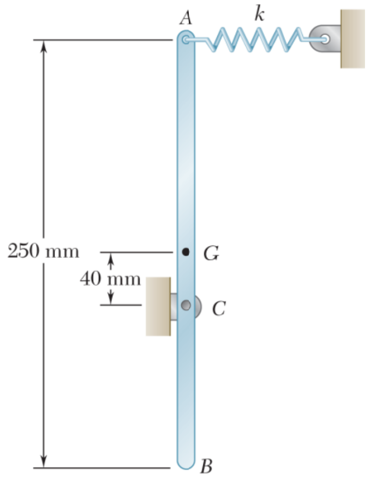
Thus,
$$\ddot{\theta} + \frac{g\bar{R}}{\bar{R}^2 + \bar{k}}\theta = 0$$

Length of the equivalent simple pendulum is

$$L = \frac{\bar{R}^2 + \bar{k}^2}{\bar{R}} = \bar{R} + \frac{\bar{k}^2}{\bar{R}}$$

$$L = (l - \bar{r}) + \frac{\bar{k}^2}{\frac{\bar{k}^2}{\bar{r}}} = l$$

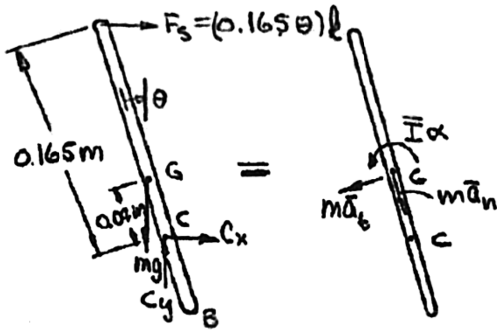
Thus, the length of the equivalent simple pendulum is the same as in Problem 19.52. It follows that the period is the same and that the new center of oscillation is at *O*. Q.E.D.



PROBLEM 19.55

The 8-kg uniform bar AB is hinged at C and is attached at A to a spring of constant $k = 500$ N/m. If end A is given a small displacement and released, determine (a) the frequency of small oscillations, (b) the smallest value of the spring constant k for which oscillations will occur.

SOLUTION



$$\bar{I} = \frac{1}{12}ml^2 = \left(\frac{1}{12}\right)(8)(0.250)^2$$

$$\bar{I} = 0.04167 \text{ kg}\cdot\text{m}^2$$

$$\alpha = \ddot{\theta}$$

$$a_t = 0.04\alpha = 0.04\ddot{\theta}$$

$$\sin\theta \approx \theta$$

Equation of motion.

$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad -(0.165)^2 k\theta + 0.04mg\theta = \bar{I}\ddot{\theta} + (0.04)^2 m\ddot{\theta}$$

$$(0.04167 + 0.01280)\ddot{\theta} + (0.02722k - 0.32g)\theta = 0 \qquad (1)$$

(a) Frequency if $k = 500 \text{ N}\cdot\text{m}$.

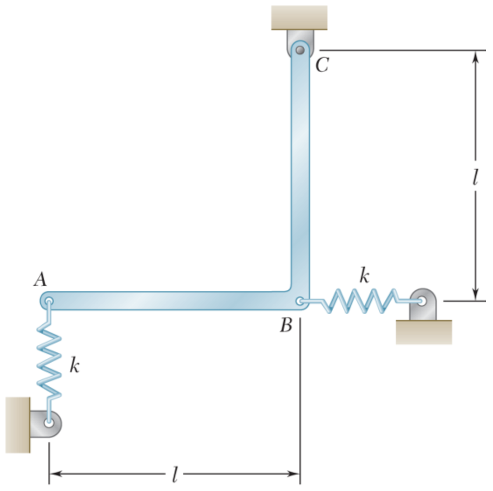
$$0.05447\ddot{\theta} + (10.47)\theta = 0$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{\left(\sqrt{\frac{10.47}{0.05447}}\right)}{2\pi} \qquad f_n = 2.21 \text{ Hz} \blacktriangleleft$$

(b) For $\tau_n \longrightarrow \infty$ or $\omega_n \longrightarrow 0$, oscillations will not occur.

From Equation (1),
$$\omega_n^2 = \frac{0.02722k - 0.32g}{(0.05447)} = 0$$

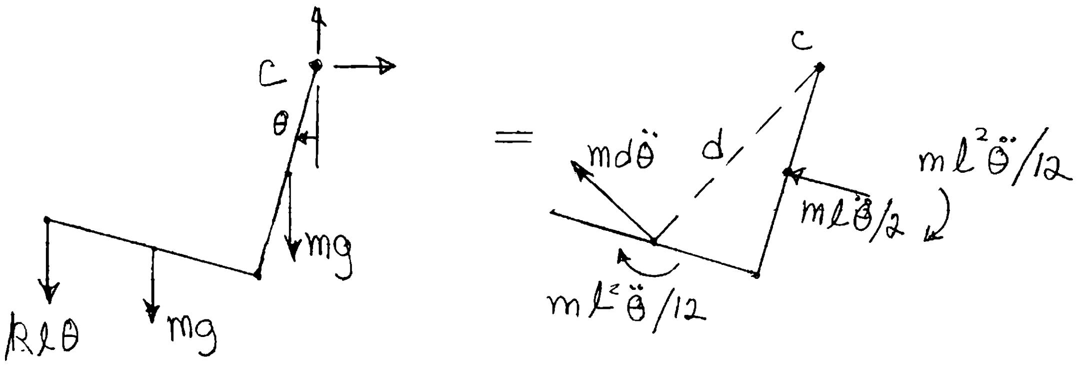
$$k = \frac{0.32g}{0.02722} = \frac{(0.32)(9.81)}{(0.02722)} \qquad k = 115.3 \text{ N/m} \blacktriangleleft$$



PROBLEM 19.56

Two uniform rods, each of mass m and length l, are welded together to form an L-shaped assembly. The assembly is constrained by two springs, each of constant k, and is in equilibrium in a vertical plane in the position shown. Determine the frequency of small oscillations of the system.

SOLUTION


C
θ
mg
klθ
mg
=
C
$md\ddot{\theta}$
d
$ml^2\ddot{\theta}/12$
$ml\ddot{\theta}/2$
$ml^2\ddot{\theta}/12$

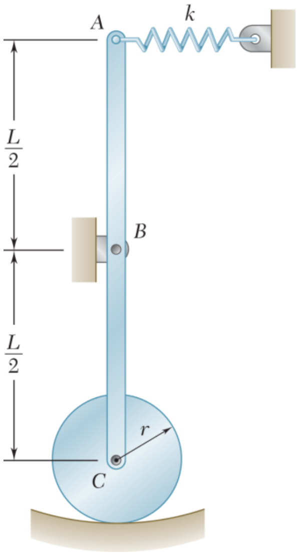
$$d^2 = l^2 + \frac{l^2}{4} = 5\frac{l^2}{4}$$

Ignore static terms: $$+\!\circlearrowright\ \Sigma M_C = -mg\frac{l}{2}\theta - kl^2\theta$$

$$-mgl\theta - kl^2\theta = 2ml^2\frac{\ddot{\theta}}{12} + ml^2\frac{\ddot{\theta}}{4} + \left(\frac{5ml^2}{4}\right)\ddot{\theta}$$

$$\omega^2 = \frac{2k + \frac{3mg}{2l}}{\frac{5m}{3}}$$

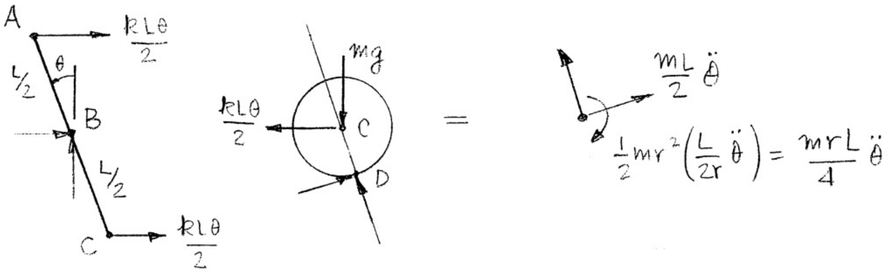
$$f_n = \frac{1}{2\pi}\sqrt{\frac{6k}{5m} + \frac{9g}{10l}}\ \text{Hz}\blacktriangleleft$$



PROBLEM 19.57

A uniform disk of radius *r* and mass *m* can roll without slipping on a cylindrical surface and is attached to bar *ABC* of length *L* and negligible mass. The bar is attached at point *A* to a spring of constant *k* and can rotate freely about point *B* in the vertical plane. Knowing that end *A* is given a small displacement and released, determine the frequency of the resulting vibration in terms of *m*, *L*, *k*, and *g*.

SOLUTION

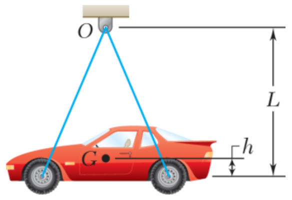


$$\circlearrowright\ \Sigma M_D = \Sigma\left(M_D\right)_{\text{eff}}:\quad -mgr\theta - \frac{kL\theta r}{4} = \frac{mrL}{4}\ddot{\theta} + \frac{mLr}{2}\ddot{\theta}$$

$$\frac{3mLr}{4}\ddot{\theta} + \left(mgr + \frac{kLr}{2}\right)\theta = 0$$

$$\omega_n^2 = \frac{4g}{3L} + \frac{2k}{3m}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3m} + \frac{4g}{3L}}\ \text{Hz} \blacktriangleleft$$



PROBLEM 19.58

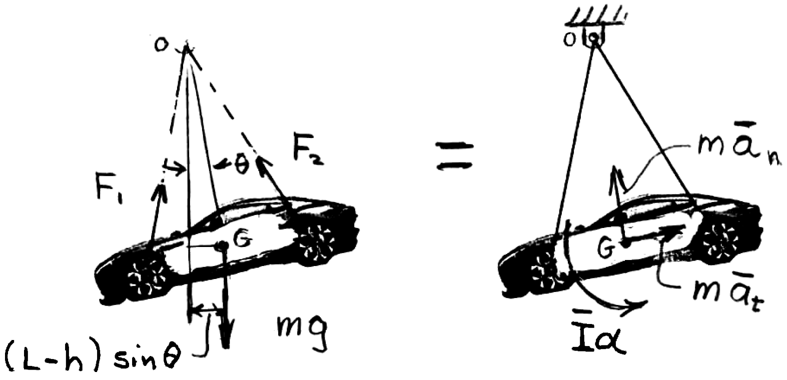
A 1300-kg sports car has a center of gravity *G* located a distance *h* above a line connecting the front and rear axles. The car is suspended from cables that are attached to the front and rear axles as shown. Knowing that the periods of oscillation are 4.04 s when $L = 4$ m and 3.54 s when $L = 3$ m, determine *h* and the centroidal radius of gyration.

SOLUTION

Let the mass center of the car be displaced a small distance *x* to the right. The mass center is moves on a circular arc of radius $L - h$, so that $x = (L - h)\sin\theta$, where θ is the counterclockwise rotation of the car. From kinematics

$$\alpha = \ddot{\theta} \qquad \bar{a}_t = (L-h)\ddot{\theta}$$

The moment of the weight force about *O* is



$$M_0 = -mg(L-h)\sin\theta$$

$$+\circlearrowleft M_0 = \bar{I}\alpha + (L-h)m\bar{a}_t$$

$$-mg(L-h)\sin\theta = \bar{I}\ddot{\theta} + m(L-h)^2\ddot{\theta}$$

Dividing by *m* and transposing terms yields

$$[\bar{k}^2 + (L-h)^2]\ddot{\theta} + g(L-h)\sin\theta = 0$$

For small angle θ, $\sin\theta \approx \theta$

$$\ddot{\theta} + \frac{g(L-h)}{\bar{k}^2 + (L-h)^2}\theta = 0$$

$$\ddot{\theta} + \omega_n^2\theta = 0 \qquad \omega_n^2 = \frac{g(L-h)}{\bar{k}^2 + (L-h)^2}$$

$$\bar{k}^2 + (L-h)^2 = \frac{g}{\omega_n^2}(L-h)$$

Using the two different values (L_1 and L_2) for L,

$$\bar{k}^2 + (L_1 - h)^2 = \frac{g}{\omega_{n1}^2}(L_1 - h) \qquad (1)$$

$$\bar{k}^2 + (L_2 - h)^2 = \frac{g}{\omega_{n2}^2}(L_2 - h) \qquad (2)$$

Subtracting Eq. (2) from Eq. (1) to eliminate $\bar{k}^2$,

$$(L_1 - h)^2 - (L_2 - h)^2 = \frac{gL_1}{\omega_{n1}^2} - \frac{gL_2}{\omega_{n2}^2} - \left(\frac{g}{\omega_{n1}^2} - \frac{g}{\omega_{n2}^2}\right)h$$

$$\left(L_1^2 - L_2^2\right) - 2(L_1 - L_2)h = A - Bh$$

where $$A = \frac{gL_1}{\omega_{n1}^2} - \frac{gL_2}{\omega_{n2}^2}$$

and $$B = \frac{g}{\omega_{n1}^2} - \frac{g}{\omega_{n2}^2}$$

$$h = \frac{L_1^2 - L_2^2 - A}{2(L_1 - L_2) - B}$$

Data: $L_1 = 4\text{ m}, \quad L_2 = 3\text{ m}, \quad g = 9.81\text{ m/s}^2$

$$\omega_{n1} = \frac{2\pi}{\tau_1} = \frac{2\pi}{4.04\text{ s}} = 1.55524\text{ rad/s}$$

$$\omega_{n2} = \frac{2\pi}{\tau_2} = \frac{2\pi}{3.54\text{ s}} = 1.77491\text{ rad/s}$$

$$A = \frac{(9.81)(4)}{(1.55524)^2} - \frac{(9.81)(3)}{(1.77491)^2} = 6.8812\text{ m}^2$$

$$B = \frac{9.81}{(1.55524)^2} - \frac{9.81}{(1.77491)^2} = 0.94190\text{ m}$$

$$h = \frac{(4)^2 - (3)^2 - 6.8812}{2(4-3) - 0.94190} = 0.11228\text{ m}$$

$h = 0.1123\text{ m}$ ◀

$$L_1 - h = 3.88772\text{ m} \qquad L_2 - h = 2.88772\text{ m}$$

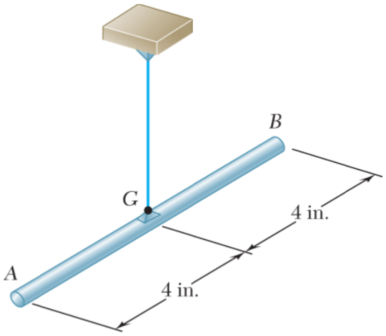
From Eq. (1), $$\bar{k}^2 + (3.88772)^2 = \frac{(9.81)(3.88772)}{(1.55524)^2}$$

$$\bar{k}^2 = 0.65336\text{ m}^2$$

$\bar{k} = 0.808\text{ m}$ ◀

Checking, using Eq. (2), $$\bar{k}^2 + (2.88772)^2 = \frac{(9.81)(2.88772)}{(1.77491)^2}$$

$$\bar{k}^2 = 0.65339\text{ m}^2$$



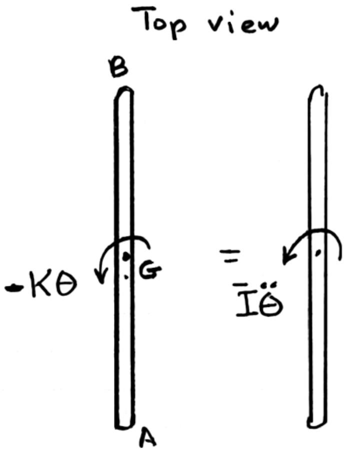
PROBLEM 19.59

A 6-lb slender rod is suspended from a steel wire which is known to have a torsional spring constant $K = 1.5$ ft·lb/rad. If the rod is rotated through 180° about the vertical and released, determine (*a*) the period of oscillation, (*b*) the maximum velocity of end *A* of the rod.

SOLUTION

Equation of motion. $\Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad -K\theta = \bar{I}\ddot{\theta} \qquad \ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$

$$\ddot{\theta} + \omega_n^2\theta = 0 \qquad \omega_n^2 = \frac{K}{\bar{I}}$$



Data:

$$W = 6 \text{ lb.}$$

$$m = \frac{W}{g} = \frac{6}{32.2} = 0.186335 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$l = 8 \text{ in.} = \frac{2}{3} \text{ ft}$$

$$\bar{I} = \frac{1}{12}ml^2 = \frac{1}{12}(0.186335)\left(\frac{2}{3}\right)^2$$

$$= 0.006901 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$K = 1.5 \text{ lb}\cdot\text{ft/rad}$$

$$\omega_n^2 = \frac{1.5}{0.006901} = 217.35$$

$$\omega_n = 14.743 \text{ rad/s}$$

(*a*) Period of oscillation. $\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{14.743}$ $\qquad \tau_n = 0.426 \text{ s}$ ◀

Simple harmonic motion:

$$\theta = \theta_m \sin(\omega_n t + \varphi)$$

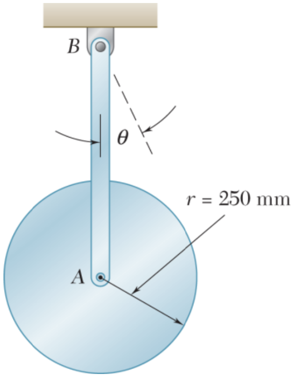
$$\dot{\theta} = \omega_n\theta_m \cos(\omega_n t + \varphi)$$

$$\dot{\theta}_m = \omega_n\theta_m$$

$$(v_A)_m = \frac{l}{2}\dot{\theta}_m = \frac{1}{2}l\omega_A\theta_m$$

$$\theta_m = 180° = \pi \text{ radians}$$

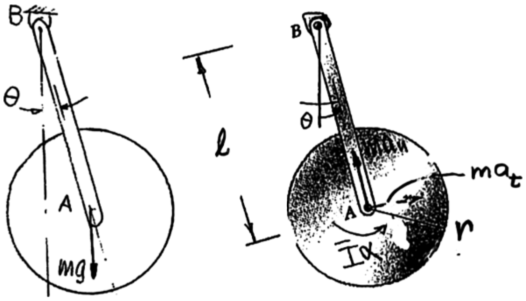
(*b*) Maximum velocity at end *A*. $(v_A)_m = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(14.743)(\pi)$ $\qquad (v_A)_m = 15.44 \text{ ft/s}$ ◀



PROBLEM 19.60

A uniform disk of radius $r = 250$ mm is attached at A to a 650-mm rod AB of negligible mass which can rotate freely in a vertical plane about B. If the rod is displaced 2° from the position shown and released, determine the magnitude of the maximum velocity of Point A, assuming that the disk (a) is free to rotate in a bearing at A, (b) is riveted to the rod at A.

SOLUTION



$$\bar{I} = \frac{1}{2}mr^2$$

Kinematics: $\alpha = \ddot{\theta}$

$$a_t = l\alpha = l\ddot{\theta}$$

(a) The disk is free to rotate and is in curvilinear translation.

Thus, $\bar{I}\alpha = 0$

Equation of motion. $\Sigma M_B = (\Sigma M_B)_{\text{eff}}: \quad -mgl\sin\theta = lma_t, \quad \sin\theta \approx \theta$

$$ml^2\ddot{\theta} + mgl\theta = 0$$

Frequency.

$$\omega_n^2 = \frac{g}{l} = \frac{9.81}{0.650} = 15.092$$

$$\omega_n = 3.8849 \text{ rad/s}$$

$$\theta_m = 2° = 0.034906 \text{ rad}$$

$$\dot{\theta}_m = \omega_n\theta_m = (3.8849)(0.034906) = 0.13561 \text{ rad/s}$$

$$(v_A)_m = l\dot{\theta}_m = (0.650)(0.13561)$$

$(v_A)_m = 0.0881$ m/s ◀

(*b*) When the disk is riveted at *A*, it rotates at an angular acceleration α.

Equation of motion. $\Sigma M_B = (\Sigma M_B)_{\text{eff}}: \quad -mgl\sin\theta = \bar{I}\alpha + lma_t, \quad \bar{I} = \frac{1}{2}mr^2, \quad \sin\theta \approx \theta$

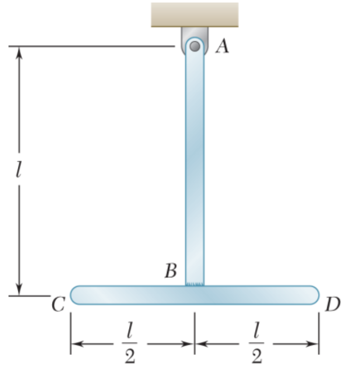
$$\left(\frac{1}{2}mr^2 + ml^2\right)\ddot{\theta} + mgl\theta = 0$$

Frequency.

$$\begin{aligned}\omega_n^2 &= \frac{gl}{\left(\frac{r^2}{2} + l^2\right)} \\ &= \frac{(9.81)(0.650)}{\frac{1}{2}(0.250)^2 + (0.650)^2} \\ &= 14.053 \\ \omega_n &= 3.7487 \text{ rad/s}\end{aligned}$$

$$\begin{aligned}\theta_m &= 2° = 0.034906 \text{ rad} \\ \dot{\theta}_m &= \omega_n\theta_m = (3.7487)(0.034906) = 0.13085 \text{ rad/s} \\ (v_A)_m &= l\dot{\theta}_m = (0.650)(0.13085)\end{aligned}$$

$(v_A)_m = 0.0851$ m/s ◀

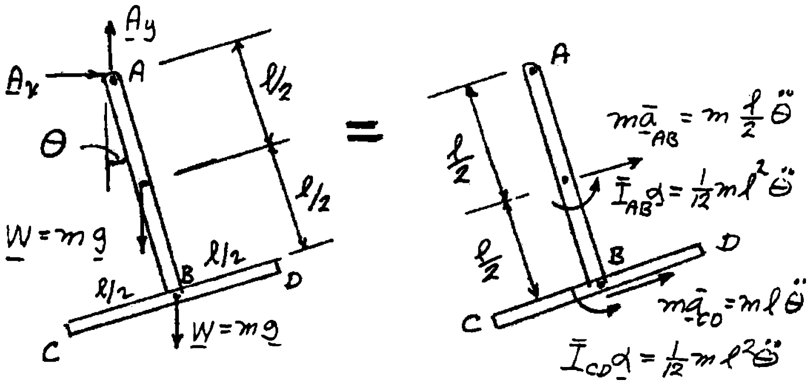

A
l
B
C
D
$\frac{l}{2}$
$\frac{l}{2}$

PROBLEM 19.61

Two uniform rods, each of mass m and length l, are welded together to form the T-shaped assembly shown. Determine the frequency of small oscillations of the assembly.

SOLUTION

Let the assembly be rotated counterclockwise through the small angle θ about the fixed Point A.


A_y
A_x
A
θ
$W = mg$
$l/2$
$l/2$
$l/2$
$l/2$
B
D
C
$W = mg$
=
A
$\frac{l}{2}$
$\frac{l}{2}$
$m\bar{a}_{AB} = m\frac{l}{2}\ddot{\theta}$
$\bar{I}_{AB}\alpha = \frac{1}{12}ml^2\ddot{\theta}$
B
D
C
$m\bar{a}_{CD} = ml\ddot{\theta}$
$\bar{I}_{CD}\alpha = \frac{1}{12}ml^2\ddot{\theta}$

Equation of motion. $\circlearrowleft + \Sigma M_A = \Sigma (M_A)_{\text{eff}}$: $-mg\frac{l}{2}\sin\theta - mgl\sin\theta = \bar{I}_{AB}\alpha + m\bar{a}_{AB}\frac{l}{2} + \bar{I}_{CD}\alpha + m\bar{a}_{CD}l$

$$-\frac{3}{2}mgl\sin\theta = \frac{1}{12}ml^2\ddot{\theta} + m\left(\frac{l}{2}\right)^2\ddot{\theta} + \frac{1}{12}ml^2\ddot{\theta} + ml^2\ddot{\theta}$$

$$-\frac{3}{2}mgl\sin\theta = \frac{17}{12}ml^2\ddot{\theta}$$

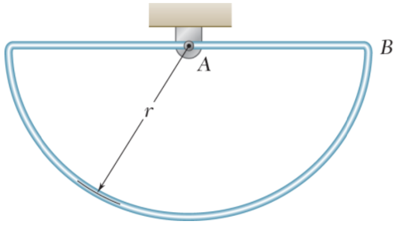
$$\ddot{\theta} + \frac{18}{17}\frac{g}{l}\sin\theta = 0$$

For small oscillations, $\sin\theta \approx \theta$

$$\ddot{\theta} + \frac{18}{17}\frac{g}{l}\theta = 0$$

$$\omega_n^2 = \frac{18}{17}\frac{g}{l} \qquad \omega_n = \sqrt{\frac{18g}{17l}}$$

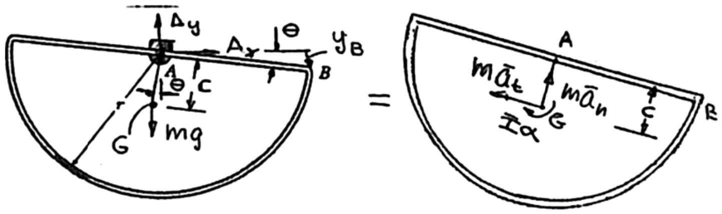
Frequency. $f = \frac{\omega_n}{2\pi} \qquad f = \frac{1}{2\pi}\sqrt{\frac{18g}{17l}}$ $\qquad f = 0.1638\sqrt{\frac{g}{l}}$ ◀



PROBLEM 19.62

A homogeneous wire bent to form the figure shown is attached to a pin support at A. Knowing that $r = 220\text{ mm}$ and that Point B is pushed down 20 mm and released, determine the magnitude of the velocity of B, 8 s later.

SOLUTION

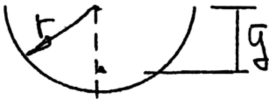


Determine location of the centroid G.

Let $\rho = \text{mass per unit length}$

Then total mass $m = \rho(2r + \pi r) = \rho r(2 + \pi)$

About C:
$$mgc = 0 + \pi r\rho\left(\frac{2r}{\pi}\right)g = 2r^2\rho g$$



$$\bar{y} = \frac{2r}{\pi} \quad \text{for a semicircle}$$

$$\rho r(2+\pi)c = 2r^2\rho, \quad c = \frac{2r}{(2+\pi)}$$

Equation of motion.
$$\overset{\curvearrowright}{+}\ \Sigma M_0 = \Sigma(M_0)_{\text{eff}}: \quad \alpha = \ddot{\theta} \quad a_t = c\alpha = c\ddot{\theta}$$

$$-mgc\sin\theta = \bar{I}\alpha + mca_n \qquad \sin\theta \approx \theta$$

$$(\bar{I} + mc^2)\ddot{\theta} + mgc\theta = 0 \qquad I_0\ddot{\theta} + mgc\theta = 0$$

Moment of inertia.
$$\bar{I} + mc^2 = I_0$$

$$I_0 = (I_0)_{\text{semicirc.}} + (I_0)_{\text{line}} = m_{\text{semicirc.}}r^2 + m_{\text{line}}\frac{(2r)^2}{12}$$

$$m_{\text{semicirc.}} = \rho\pi r \qquad m_{\text{line}} = 2\rho r \qquad \rho = \frac{m}{(2+\pi)r}$$

$$I_0 = \rho\left[\pi r^2 \cdot r + 2r\cdot\frac{r^2}{3}\right] = \frac{mr^2}{(2+\pi)}\left[\pi + \frac{2}{3}\right]$$

$$\frac{mr^2}{(2+\pi)}\left[\pi + \frac{2}{3}\right]\ddot{\theta} + mg\frac{2r}{(2+\pi)}\theta = 0$$

PROBLEM 19.62 (Continued)

Frequency. $$\omega_n^2 = \frac{2g}{\left(\pi + \frac{2}{3}\right)r} = \frac{(2)(9.81)}{\left(\pi + \frac{2}{3}\right)(0.220)}$$

$$\omega_n^2 = 23.418 \text{ s}^{-2} \quad \omega_n = 4.8392 \text{ rad/s}$$

$$\theta = \theta_m \sin(\omega_n t + \phi) \quad y_B = r\theta$$

At $t = 0$, $$y_B = 20 \text{ mm}, \quad \dot{y}_B = 0$$

$$\dot{y}_B = 0 = (y_B)_m \omega_n \cos(0 + \phi), \quad \phi = \frac{\pi}{2}$$

$$y_B = 20 \text{ mm } = (y_B)_m \sin\left(0 + \frac{\pi}{2}\right), \quad (y_B)_m = 20 \text{ mm}$$

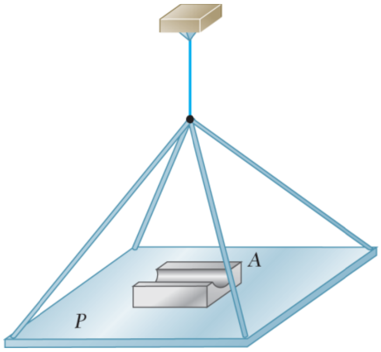
$$y_B = (20 \text{ mm})\sin\left(\omega_n t + \frac{\pi}{2}\right) \quad \omega_n = 4.8392 \text{ rad/s}$$

$$\dot{y}_B = 20\omega\cos\left(\omega_n t + \frac{\pi}{2}\right) = -(20 \text{ mm})\omega_n \sin \omega_n t$$

At $t = 8$ s, $$\dot{y}_B = -(20)(4.8392)\sin[(4.8392)(8)] = -(96.78)(0.8492)$$

$$= -82.2 \text{ mm/s}$$

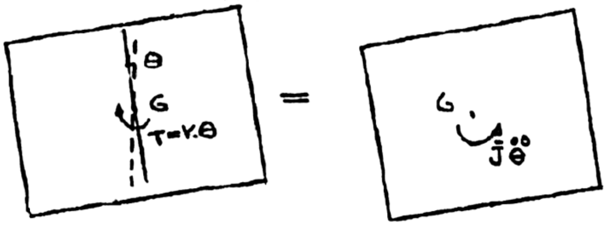
$v_B = 82.2$ mm/s ◀



PROBLEM 19.63

A horizontal platform P is held by several rigid bars which are connected to a vertical wire. The period of oscillation of the platform is found to be 2.2 s when the platform is empty and 3.8 s when an object A of unknown moment of inertia is placed on the platform with its mass center directly above the center of the plate. Knowing that the wire has a torsional constant $K = 27\ \text{N}\cdot\text{m/rad}$, determine the centroidal moment of inertia of object A.

SOLUTION



Equation of motion. $\Sigma M_G = \bar{I}\alpha: \quad -K\theta = \bar{I}\ddot{\theta}$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0 \qquad \omega_n^2 = \frac{K}{\bar{I}}$$

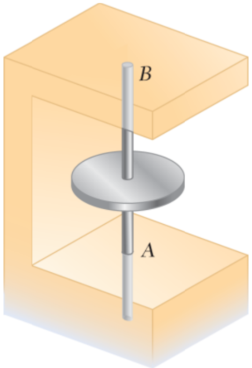
Case 1. The platform is empty. $\omega_{n1} = \frac{2\pi}{\tau_1} = \frac{2\pi}{2.2} = 2.856 \text{ rad/s}$

$$\bar{I}_1 = \frac{K}{\omega_{n1}^2} = \frac{27}{(2.856)^2} = 3.31\ \text{kg}\cdot\text{m}^2$$

Case 2. Object A is on the platform. $\omega_{n2} = \frac{2\pi}{\tau_2} = \frac{2\pi}{3.8} = 1.653 \text{ rad/s}$

$$\bar{I}_2 = \frac{K}{\omega_{n2}^2} = \frac{27}{(1.653)^2} = 9.8814\ \text{kg}\cdot\text{m}^2$$

Moment of inertia of object A. $\bar{I}_A = \bar{I}_2 - \bar{I}_1$ $\qquad \bar{I}_A = 6.57\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM 19.64

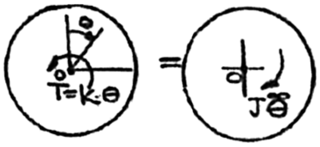
A uniform disk of radius $r = 120$ mm is welded at its center to two elastic rods of equal length with fixed ends at A and B. Knowing that the disk rotates through an 8° angle when a 500-mN·m couple is applied to the disk and that it oscillates with a period of 1.3 s when the couple is removed, determine (*a*) the mass of the disk, (*b*) the period of vibration if one of the rods is removed.

SOLUTION

Torsional spring constant.

$$k = \frac{T}{\theta} = \frac{0.5\ \text{N}\cdot\text{m}}{(8)\left(\frac{\pi}{180}\right)}$$

$$k = 3.581\ \text{N}\cdot\text{m/rad}$$



Equation of motion.

$$\Sigma M_0 = \Sigma(M_0)_{\text{eff}}: \quad -K\theta = I\ddot{\theta} \quad \ddot{\theta} + \frac{K}{I}\theta = 0$$

Natural frequency and period. $\omega_n^2 = \frac{K}{I}$

Period. $\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I}{K}}$

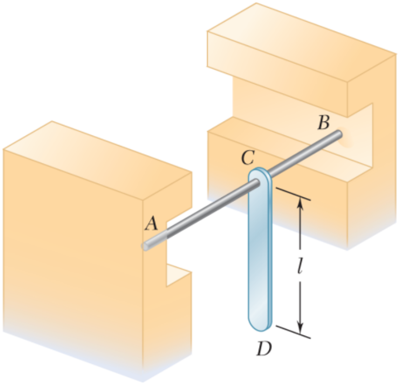
Mass moment of inertia. $I = \frac{\tau_n^2 K}{(2\pi)^2} = \frac{(1.35)^2(3581\ \text{N}\cdot\text{m/r})}{(2\pi)^2}$

$$I = 0.1533\ \text{N}\cdot\text{m}\cdot\text{s}^2 = \frac{1}{2}mr^2 = \frac{1}{2}m(0.120\ \text{m})^2$$

(*a*) Mass of the disk. $m = \frac{(0.1533\,\text{N}\cdot\text{m}\cdot\text{s}^2)(2)}{(0.120\ \text{m})^2}$ $m = 21.3$ kg ◀

(*b*) With one rod removed: $K' = \frac{K}{2} = \frac{3.581}{2} = 1.791\ \text{N}\cdot\text{m/rad}$

Period. $\tau = 2\pi\sqrt{\frac{I}{K'}} = 2\pi\sqrt{\frac{(0.1533\ \text{N}\cdot\text{m}\cdot\text{s}^2)}{1.791\ \text{N}\cdot\text{m/rad}}}$ $\tau = 1.838\ \text{s}$ ◀

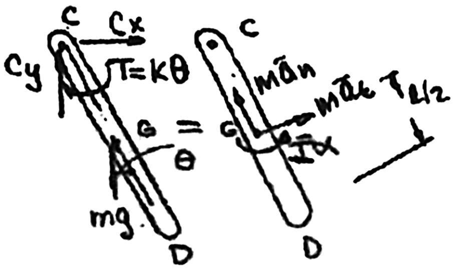


PROBLEM 19.65

A 5-kg uniform rod *CD* of length $l = 0.7$ m is welded at *C* to two elastic rods, which have fixed ends at *A* and *B* and are known to have a combined torsional spring constant $K = 24\ \text{N}\cdot\text{m/rad}$. Determine the period of small oscillation, if the equilibrium position of *CD* is (*a*) vertical as shown, (*b*) horizontal.

SOLUTION

(*a*) Equation of motion.



$$\alpha = \ddot{\theta} \qquad a_t = \frac{l}{2}\alpha = \frac{l}{2}\ddot{\theta}$$

$$+\circlearrowleft \Sigma M_C = \bar{I}\alpha + m\bar{a}d: \quad -K\theta - (mg)\frac{l}{2}\sin\theta = \bar{I}\alpha + \frac{l}{2}(ma_t)$$

$$-K\theta - \frac{1}{2}mgl\sin\theta = \bar{I}\ddot{\theta} + \frac{1}{4}ml^2\ddot{\theta}$$

$$\left(\bar{I} + \frac{1}{4}ml^2\right)\ddot{\theta} + \frac{1}{2}mgl\sin\theta + K\theta = 0$$

$$\left(\frac{1}{12}ml^2 + \frac{1}{4}ml^2\right)\ddot{\theta} + K\theta + \frac{1}{2}mgl\theta = 0$$

$$\frac{1}{3}ml^2\ddot{\theta} + \left(K + \frac{1}{2}mgl\right)\theta = 0$$

$$\ddot{\theta} + \left(\frac{3K}{ml^2} + \frac{3g}{2l}\right)\theta = 0$$

PROBLEM 19.65 (Continued)

Data: $K = 24\ \text{N}\cdot\text{m/rad}, \quad m = 5\ \text{kg}, \quad l = 0.7\ \text{m}$

$$\ddot{\theta} + \left[\frac{(3)(24)}{(5)(0.7)^2} + \frac{(3)(9.81)}{(2)(0.7)}\right]\theta = 0$$

$$\ddot{\theta} + 50.409\theta = 0$$

Frequency. $\omega_n^2 = 50.409 \qquad \omega_n = 7.1\ \text{rad/s}$

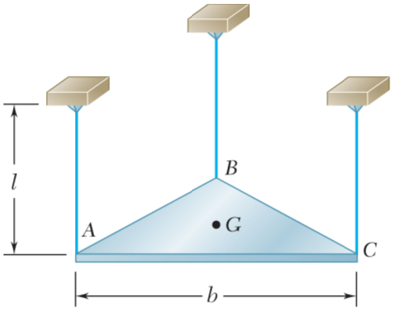
Period. $\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{7.1}$ $\qquad \tau_n = 0.885\ \text{s}$ ◀

(*b*) If the rod is horizontal, the gravity term is not present and the equation of motion is

$$\ddot{\theta} + \frac{3K}{ml^2}\theta = 0$$

$$\omega_n^2 = \frac{3K}{ml^2} = \frac{(3)(24)}{(5)(0.7)^2} = 29.388$$

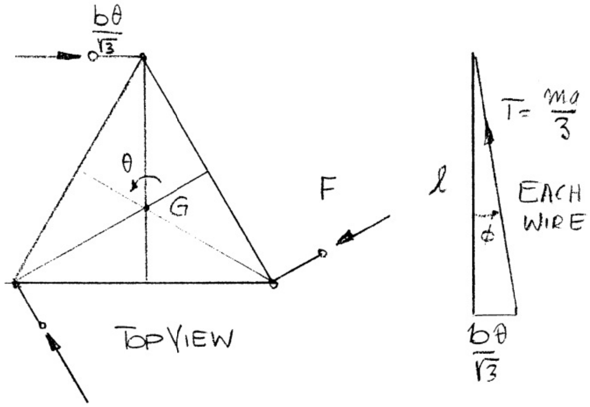
$\omega_n = 5.4210\ \text{rad/s} \qquad \tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{5.4210}$ $\qquad \tau_n = 1.159\ \text{s}$ ◀



PROBLEM 19.66

A uniform equilateral triangular plate of side b is suspended from three vertical wires of the same length l. Determine the period of small oscillations of the plate when (*a*) it is rotated through a small angle about a vertical axis through its mass center G, (*b*) it is given a small horizontal displacement in a direction perpendicular to AB.

SOLUTION



Geometry

$$\phi = \frac{b\theta}{l\sqrt{3}}$$

$$F = \frac{mg}{3}\frac{b\theta}{l\sqrt{3}}$$

Then

$$+\circlearrowleft\ \Sigma M_G = I_G\ddot{\theta} = -F\left(\frac{b}{\sqrt{3}}\right)\theta$$

or

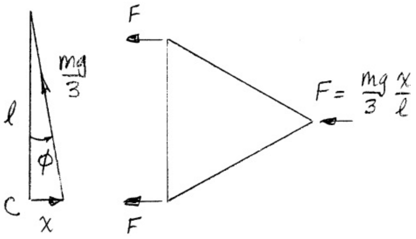
$$\frac{mb^2}{12}\ddot{\theta} + \frac{mgb^2}{3L}\theta = 0$$

(*a*)

$$\omega_n^2 = \frac{4g}{l}, \quad \tau = \frac{2\pi}{\omega_n} = \pi\sqrt{\frac{l}{g}} \blacktriangleleft$$

PROBLEM 19.66 (Continued)

(*b*)

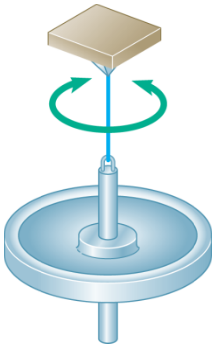

F
$\frac{mg}{3}$
ℓ
ϕ
C
x
$F = \frac{mg}{3}\frac{x}{\ell}$
F

$$m\ddot{x} = -3F$$

$$= -\frac{mgx}{l}$$

$$\omega_n^2 = \frac{g}{l}$$

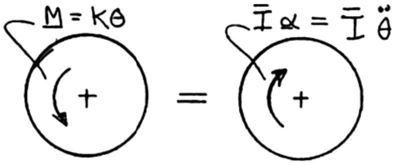
$$\tau = 2\pi\sqrt{\frac{l}{g}} \blacktriangleleft$$



PROBLEM 19.67

A period of 6.00 s is observed for the angular oscillations of a 4-oz gyroscope rotor suspended from a wire as shown. Knowing that a period of 3.80 s is obtained when a 1.25-in.-diameter steel sphere is suspended in the same fashion, determine the centroidal radius of gyration of the rotor. (Specific weight of steel = 490 lb/ft^3.)

SOLUTION



$$\circlearrowleft^{+} \Sigma M = \Sigma(M)_{\text{eff}}: \quad -K\theta = \bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$

$$\omega_n^2 = \frac{K}{\bar{I}}$$

$$\tau = 2\pi\sqrt{\frac{\bar{I}}{K}} \qquad (1)$$

$$K = \frac{4\pi^2\bar{I}}{\tau^2} \qquad (2)$$

$$\bar{I} = \frac{K\tau^2}{4\pi^2} \qquad (3)$$

For the sphere, $r = \frac{d}{2} = 0.625 \text{ in.} = 52.083 \times 10^{-3} \text{ ft}$

Volume: $V_s = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(52.083 \times 10^{-3})^3$

$= 591.81 \times 10^{-6} \text{ ft}^3$

Weight: $W_s = \gamma V_s$

$= (490 \text{ lb/ft}^3)(591.81 \times 10^{-6} \text{ ft}^3)$

$= 0.28999 \text{ lb}$

Mass: $m_s = \frac{W_s}{g} = \frac{0.28999}{32.2}$

$= 9.0059 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$

Moment of inertia: $\bar{I} = \frac{2}{5}m_s r^2 = \frac{2}{5}(9.0059 \times 10^{-3})(52.083 \times 10^{-3})^2$

$= 9.7719 \times 10^{-6} \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$

Period: $\tau_s = 3.80 \text{ s}$

PROBLEM 19.67 (Continued)

From Eq. (2): $$K = \frac{4\pi^2(9.7719\times10^{-6})}{(3.80)^2}$$

$$= 26.716\times10^{-6}\ \text{lb}\cdot\text{ft/rad}$$

For the rotor, $$m = \frac{W}{g} = \left(\frac{4}{16}\right)\left(\frac{1}{32.2}\right)$$

$$= 7.764\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$\tau = 6.00\ \text{s}$$

From Eq. (3): $$\bar{I} = (26.716\times10^{-6})\frac{(6.00)^2}{4\pi^2}$$

$$= 24.362\times10^{-6}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

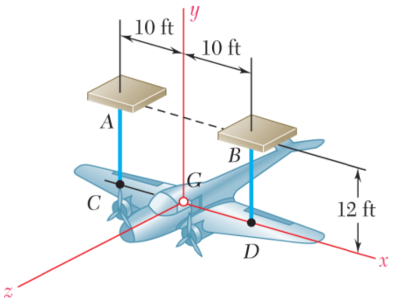
Radius of gyration. $$\bar{I} = m\bar{k}^2$$

$$\bar{k} = \sqrt{\frac{\bar{I}}{m}}$$

$$= \sqrt{\frac{24.362\times10^{-6}}{7.764\times10^{-3}}}$$

$$= 0.056016\ \text{ft}$$

$\bar{k} = 0.672$ in. ◀



PROBLEM 19.68

The centroidal radius of gyration $\bar{k}_y$ of an airplane is determined by suspending the airplane by two 12-ft-long cables as shown. The airplane is rotated through a small angle about the vertical through G and then released. Knowing that the observed period of oscillation is 3.3 s, determine the centroidal radius of gyration $\bar{k}_y$.

SOLUTION

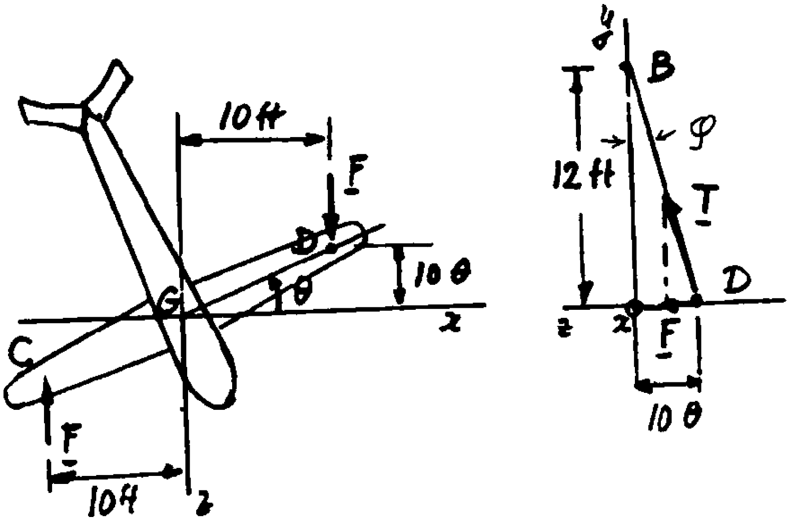
Let the airplane rotate through the small angle θ about a vertical axis. Suspension Points C and D on the airplane each move horizontally a distance (10 ft) sin $\theta \approx$ (10 ft) θ. Let φ be the angle between a cable and the vertical direction. Then, $\sin\varphi = (10\text{ ft})\theta/(12\text{ ft}) = \frac{5}{6}\theta$.

$$+\!\uparrow \Sigma F = 0: \quad 2T\cos\varphi - W = 0$$

$$T = \frac{W}{2\cos\varphi} \approx \frac{W}{2}$$

Let F be the horizontal component of T.

$$F = T\sin\varphi \approx \frac{W}{2}\cdot\frac{5}{6}\theta = \frac{5}{12}W\theta$$



The two forces F form a couple of moment

$$M = -(20\text{ ft})F = -(20)\left(\frac{5}{12}\right)W\theta$$

Equation of motion:

$$+\circlearrowleft \Sigma M_y = \bar{I}_y\alpha: \quad -20\left(\frac{5W}{12}\theta\right) = \frac{W}{g}\bar{k}_y^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{(20)(5)g}{12\bar{k}_y^2}\theta = 0$$

PROBLEM 19.68 (Continued)

Natural frequency:

$$\omega_n^2 = \frac{(20)(5)g}{12\,\bar{k}_y^2} = \frac{(20)(5)(32.2)}{12\,\bar{k}_y^2} = \frac{268.33}{\bar{k}_y^2}$$

$$\bar{k}_y^2 = \frac{268.33}{\omega_n^2}$$

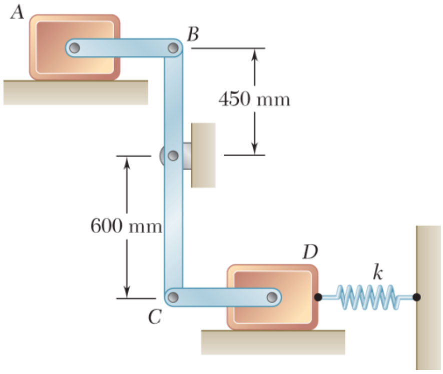
$$\bar{k}_y = \frac{16.381}{\omega_n} = \frac{16.381}{2\pi f}$$

$$= \frac{2.607}{f} = 2.607\tau$$

With $\tau = 3.3$ s, $\bar{k}_y = (2.607)(3.3)$

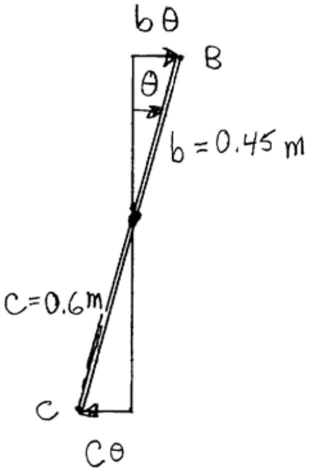
$\bar{k}_y = 8.60$ ft ◀

PROBLEM 19.69



Two blocks, each of mass 1.5 kg, are attached to links which are pin-connected to bar *BC* as shown. The masses of the links and bar are negligible, and the blocks can slide without friction. Block *D* is attached to a spring of constant $k = 720$ N/m. Knowing that block *A* is at rest when it is struck horizontally with a mallet and given an initial velocity of 250 mm/s determine the magnitude of the maximum displacement of block *D* during the resulting motion.

SOLUTION



$$T = \frac{1}{2}m\left(b^2 + c^2\right)\dot{\theta}^2$$

$$V = \frac{1}{2}kc^2\theta^2$$

$$\omega_n^2 = \frac{kc^2}{m\left(b^2 + c^2\right)}$$

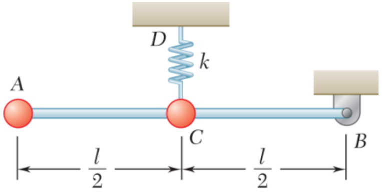
$$k = 720 \text{ N/m}, \qquad m = 1.5 \text{ kg}$$

$$\omega_n = 17.5271 \text{ rad/s}$$

$$\dot{\theta}_0 = \frac{(0.25 \text{ m/s})}{0.45 \text{ m}} = 0.5555 \text{ rad/s}$$

$$\max\left|x_D\right| = \frac{c\dot{\theta}_0}{\omega_n} = 0.01902 \text{ m}$$

$$\max\left|x_D\right| = 19.02 \text{ mm} \blacktriangleleft$$



PROBLEM 19.70

Two small spheres, *A* and *C*, each of mass *m*, are attached to rod *AB*, which is supported by a pin and bracket at *B* and by a spring *CD* of constant *k*. Knowing that the mass of the rod is negligible and that the system is in equilibrium when the rod is horizontal, determine the frequency of the small oscillations of the system.

SOLUTION

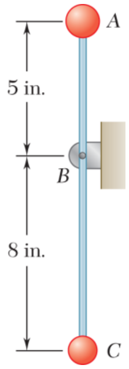
$$T = \frac{1}{2}I_B\dot{\theta}^2 = \frac{1}{2}\left[ml^2 + m\left(\frac{l}{2}\right)^2\right]\dot{\theta}^2$$

$$V = \frac{1}{2}k\left(\frac{l\theta}{2}\right)^2$$

$$\omega_n^2 = \frac{\frac{kl^2}{8}}{\frac{5ml^2}{8}}$$

$$= \frac{k}{5m}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{5m}}\ \text{Hz} \blacktriangleleft$$

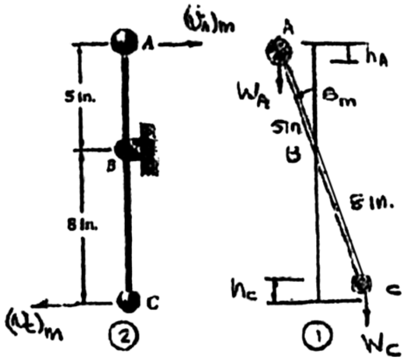


PROBLEM 19.71

A 14-oz sphere A and a 10-oz sphere C are attached to the ends of a rod AC of negligible weight which can rotate in a vertical plane about an axis at B. Determine the period of small oscillations of the rod.

SOLUTION

Datum at ①:



Position ①

$$T_1 = 0$$

$$V_1 = W_C h_C - W_A h_A$$

$$h_C = BC(1-\cos\theta_m)$$

$$h_A = BA(1-\cos\theta_m)$$

Small angles.

$$1-\cos\theta_m \approx \frac{\theta_m^2}{2}$$

$$V_1 = [(W_C)(BC)-(W_A)(BA)]\frac{\theta_m^2}{2}$$

$$V_1 = \left[\left(\frac{10}{16}\text{lb}\right)\left(\frac{8}{12}\text{ft}\right)-\left(\frac{14}{16}\text{lb}\right)\left(\frac{5}{12}\text{ft}\right)\right]\frac{\theta_m^2}{2}$$

$$V_1 = (0.4167-0.3646)\frac{\theta_m^2}{2} = 0.05208\frac{\theta_m^2}{2}$$

Position ②

$$V_2 = 0$$

$$(v_C)_m = \frac{8}{12}\dot{\theta}_m \qquad (v_A)_m = \frac{5}{12}\dot{\theta}_m$$

$$m_C = \frac{W_C}{g} = \frac{10}{(16)(32.2)} = 0.019410\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$m_A = \frac{W_A}{g} = \frac{14}{(16)(32.2)} = 0.027174\ \text{lb}\cdot\text{s}^2/\text{ft}$$

PROBLEM 19.71 (Continued)

$$T_2 = \frac{1}{2} m_C (v_C)_m^2 + \frac{1}{2} m_A (v_A)_m^2$$

$$= \frac{1}{2}\left[(0.019410)\left(\frac{8}{12}\right)^2 + \frac{1}{2}(0.027174)\left(\frac{5}{12}\right)^2\right]\dot{\theta}_m^2$$

$$= 0.013344 \frac{\dot{\theta}_m^2}{2}$$

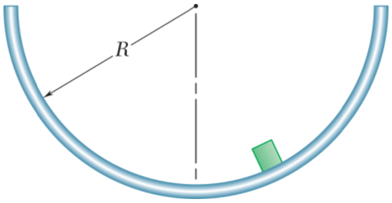
$$= 0.013344 \frac{\omega_n^2 \theta_m^2}{2}$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2: \quad 0 + 0.05208 \frac{\theta_m^2}{2} = 0.013344 \frac{\omega_n^2 \theta_m^2}{2} + 0$

$$\omega_n^2 = \frac{0.05208}{0.013344} = 3.902$$

$$\omega_n = 1.9755 \text{ rad/s}$$

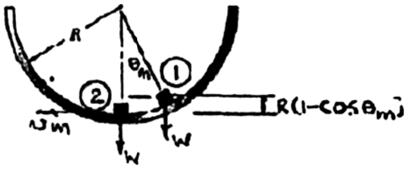
Period of oscillations. $\tau_n = \frac{2\pi}{\omega_n}$ $\tau_n = 3.18 \text{ s}$ ◀



PROBLEM 19.72

Determine the period of small oscillations of a small particle which moves without friction inside a cylindrical surface of radius *R*.

SOLUTION



Datum at ②:

Position ①

$$T_1 = 0$$

$$V_1 = WR(1-\cos\theta_m)$$

Small oscillations:

$$(1-\cos\theta_m) = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$V_1 = \frac{WR\theta_m^2}{2}$$

Position ②

$$v_m = R\dot{\theta}_m \quad T_2 = \frac{1}{2}mv_m^2 = \frac{1}{2}mR^2\dot{\theta}_m^2$$

$$V_2 = 0$$

Conservation of energy. $T_1 + Y_1 = T_2 + V_2$

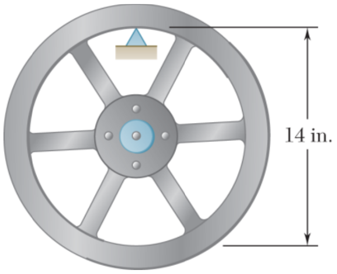
$$0 + WR\frac{\theta_m^2}{2} = \frac{1}{2}mR^2\dot{\theta}_m^2 + 0 \qquad \dot{\theta}_m = \omega_n\theta_m$$

$$W = mg$$

$$mgR\frac{\theta_m^2}{2} = \frac{1}{2}mR^2\omega_n^2\theta_m^2$$

$$\omega_n = \sqrt{\frac{g}{R}}$$

Period of oscillations. $\tau_n = \dfrac{2\pi}{\omega_n} = 2\pi\sqrt{\dfrac{R}{g}}$ ◀

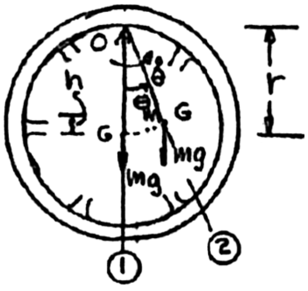


PROBLEM 19.73

The inner rim of an 85-lb flywheel is placed on a knife edge, and the period of its small oscillations is found to be 1.26 s. Determine the centroidal moment of inertia of the flywheel.

SOLUTION

Datum at ①:



Position ①

$$T_1 = \frac{1}{2} I_0 \dot{\theta}_m^2 \quad V_1 = 0$$

Position ②

$$T_2 = 0 \quad V_2 = mgh$$

$$h = r(1 - \cos\theta_m) = r\left(2\sin^2\frac{\theta_m}{2}\right)$$

$$\approx r\frac{\theta_m^2}{2}$$

$$V_2 = mgr\frac{\theta_m^2}{2}$$

Conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2} I_0 \dot{\theta}_m^2 + 0 = 0 + mgr\frac{\theta_m^2}{2}$$

For simple harmonic motion, $\dot{\theta}_m = \omega_n \theta_m$

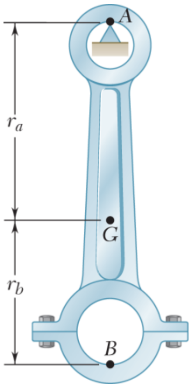
$$I_0 \omega_n^2 \theta_m^2 = mgr\,\theta_m^2 \qquad \omega_n^2 = \frac{mgr}{I_0} \qquad \tau_n^2 = \frac{4\pi^2}{\omega_n^2} = \frac{(4\pi^2)I_0}{mgr}$$

Moment of inertia. $I_0 = \overline{I} + mr^2 \quad \overline{I} + mr^2 = \dfrac{\left(\tau_n^2\right)(mgr)}{4\pi^2}$

$$\overline{I} = \frac{\left(\tau_n^2\right)(mgr)}{4\pi^2} - mr^2 = \frac{(1.26\text{ s})^2(85\text{ lb})}{4\pi^2}\left(\frac{7}{12}\text{ ft}\right) - \frac{(85\text{ lb})}{(32.2\text{ ft/s}^2)}\left(\frac{7}{12}\text{ ft}\right)^2$$

$$\overline{I} = 1.994 - 0.8983$$

$\overline{I} = 1.096\text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 19.74

A connecting rod is supported by a knife edge at Point A; the period of its small oscillations is observed to be 1.03 s. Knowing that the distance r_a is 6 in. determine the centroidal radius of gyration of the connecting rod.

SOLUTION

Position ① Displacement is maximum.

$$T_1 = 0, \quad V_1 = mgr_a(1-\cos\theta_m) \approx \frac{1}{2}mgr_a\theta_m^2$$

Position ② Velocity is maximum.

$$(v_G)_m = r_a\dot{\theta}_m$$

$$T_2 = \frac{1}{2}mv_G^2 + \frac{1}{2}\bar{I}\dot{\theta}_m^2 = \frac{1}{2}mr_a^2\dot{\theta}_m^2 + \frac{1}{2}m\bar{k}^2\dot{\theta}_m^2$$

$$= \frac{1}{2}m\left(r_a^2 + \bar{k}^2\right)\dot{\theta}_m^2$$

$$V_2 = 0$$

For simple harmonic motion, $\dot{\theta}_m = \omega_n\theta_m$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$

$$0 + \frac{1}{2}mgr_a\theta_m^2 = \frac{1}{2}m\left(r_a^2 + \bar{k}^2\right)\omega_n^2\,\theta_m^2 + 0$$

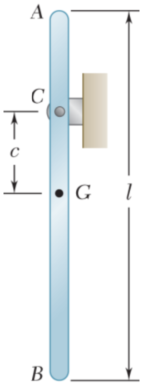
$$\omega_n^2 = \frac{gr_a}{r_a^2 + \bar{k}^2} \quad \text{or} \quad \bar{k}^2 = \frac{gr_a}{\omega_n^2} - r_a^2$$

Data: $\tau_n = 1.03\text{ s} \qquad \omega_n = \dfrac{2\pi}{\tau_n} = \dfrac{2\pi}{1.03} = 6.1002\text{ rad/s}$

$$r_a = 6\text{ in.} = 0.5\text{ ft} \qquad g = 32.2\text{ ft/s}^2$$

$$\bar{k}^2 = \frac{(32.2)(0.5)}{(6.1002)^2} - (0.5)^2 = 0.43265 - 0.25 = 0.18265\text{ ft}^2$$

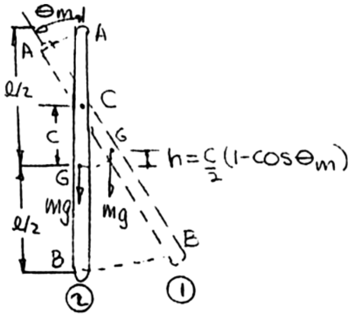
$\bar{k} = 0.42738\text{ ft}$ $\qquad \bar{k} = 5.13\text{ in.}$ ◀



PROBLEM 19.75

A uniform rod AB can rotate in a vertical plane about a horizontal axis at C located at a distance c above the mass center G of the rod. For small oscillations, determine the value of c for which the frequency of the motion will be maximum.

SOLUTION



Find ω_n as a function of c.

Datum at ②:

Position ①

$$T_1 = 0 \qquad V_1 = mgh$$

$$V_1 = mgc(1-\cos\theta_m)$$

$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$V_1 = mgc\frac{\theta_m^2}{2}$$

Position ②

$$T_2 = \frac{1}{2}I_C\dot{\theta}_m^2$$

$$I_C = \overline{I} + mc^2 = \frac{1}{12}ml^2 + mc^2$$

$$T_2 = \frac{1}{2}m\left(\frac{l^2}{12}+c^2\right)\dot{\theta}_m^2 \qquad V_2 = 0$$

$$T_1 + V_1 = T_2 + V_2 \quad 0 + mgc\frac{\theta_m^2}{2} = m\left(\frac{l^2}{12}+c^2\right)\frac{\dot{\theta}_m^2}{2} + 0$$

$$\dot{\theta}_m = \omega_n\theta_m$$

$$gc = m\left(\frac{l^2}{12}+c^2\right)\omega_n^2$$

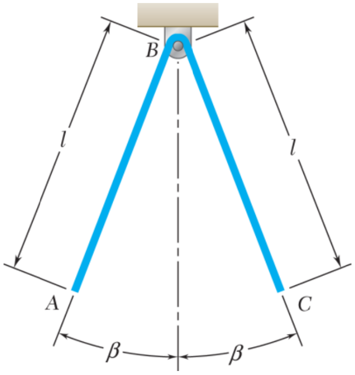
$$\omega_n^2 = \frac{gc}{\left(\frac{l^2}{12}+c^2\right)}$$

Maximum c, when

$$\frac{d\omega_n^2}{dc} = 0 = \frac{g\left(\frac{l^2}{12}+c^2\right) - 2c^2g}{\left(\frac{l^2}{12}+c^2\right)} = 0$$

$$\frac{l^2}{12} - c^2 = 0 \qquad c = \frac{l}{\sqrt{12}} \blacktriangleleft$$

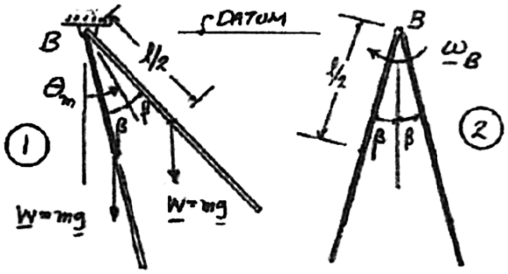
PROBLEM 19.76



A homogeneous wire of length $2l$ is bent as shown and allowed to oscillate about a frictionless pin at B. Denoting by τ_0 the period of small oscillations when $\beta = 0$, determine the angle β for which the period of small oscillations is $2\tau_0$.

SOLUTION

We denote by m the mass of half the wire.



Position ① Maximum deflections:

$$T_1 = 0, \quad V_1 = -mg\frac{l}{2}\cos(\theta_m - \beta) - mg\frac{l}{2}\cos(\theta_m + \beta)$$

$$= -mg\frac{l}{2}(\cos\theta_m \cos\beta + \sin\theta_m \sin\beta + \cos\theta_m \cos\beta - \sin\theta_m \sin\beta)$$

$$V_1 = -mgl\cos\beta\cos\theta_m$$

For small oscillations,

$$\cos\theta_m \approx 1 - \frac{1}{2}\theta_m^2$$

$$V_1 = -mgl\cos\beta + \frac{1}{2}mgl\cos\beta\,\theta_m^2$$

Position ② Maximum velocity:

$$T_2 = \frac{1}{2}I_B\dot{\theta}_m^2 \quad \text{but} \quad I_B = 2\left(\frac{1}{3}ml^2\right)$$

Thus,

$$T_2 = \frac{1}{2}\left(\frac{2}{3}ml^2\right)\dot{\theta}_m^2$$

$$V_2 = -2mg\left(\frac{l}{2}\cos\beta\right) = -mgl\cos\beta$$

NPROBLEM 19.76 (Continued)

Conservation of energy. $T_1 + V_1 = T_2 + V_2$

$$-mgl\cos\beta + \frac{1}{2}mgl\cos\beta\,\theta_m^2 = \frac{1}{3}ml^2\dot{\theta}_m^2 - mgl\cos\beta$$

Setting $\dot{\theta}_m = \theta_m\omega_n$, $\frac{1}{2}mgl\cos\beta\,\theta_m^2 = \frac{1}{3}ml^2\,\theta_m^2\omega_n^2$

$$\omega_n^2 = \frac{3}{2}\frac{g}{l}\cos\beta \qquad \tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{2l}{3g\cos\beta}} \tag{1}$$

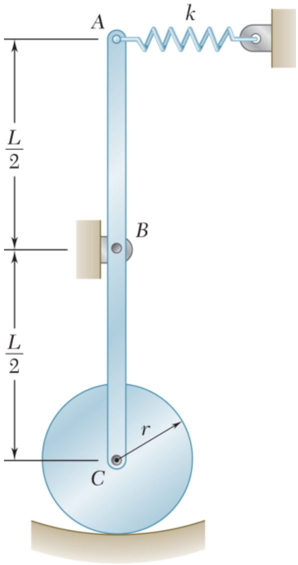
But for $\beta = 0$, $\tau_0 = 2\pi\sqrt{\frac{2l}{3g}}$

For $\tau = 2\tau_0$, $2\pi\sqrt{\frac{2l}{3g\cos\beta}} = 2\left(2\pi\sqrt{\frac{2l}{3g}}\right)$

Squaring and reducing,

$$\frac{1}{\cos\beta} = 4 \qquad \cos\beta = \frac{1}{4}$$

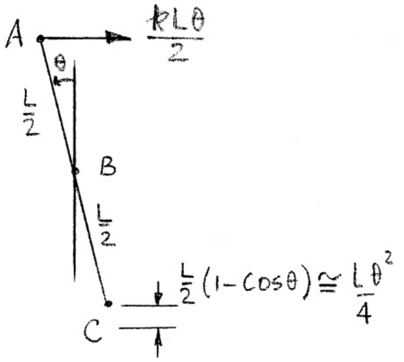
$\beta = 75.5°$ ◀


A
k
B
C
r
$\frac{L}{2}$
$\frac{L}{2}$

PROBLEM 19.77

A uniform disk of radius *r* and mass *m* can roll without slipping on a cylindrical surface and is attached to bar *ABC* of length *L* and negligible mass. The bar is attached to a spring of constant *k* and can rotate freely in the vertical plane about Point *B*. Knowing that end *A* is given a small displacement and released, determine the frequency of the resulting oscillations in terms of *m*, *L*, *k*, and *g*.

SOLUTION


A
$\frac{kL\theta}{2}$
θ
$\frac{L}{2}$
B
$\frac{L}{2}$
$\frac{L}{2}(1-\cos\theta) \cong \frac{L\theta^2}{4}$
C

$$V = \frac{1}{2}k\left(\frac{L\theta}{2}\right)^2 + mg\frac{L\theta^2}{4}$$

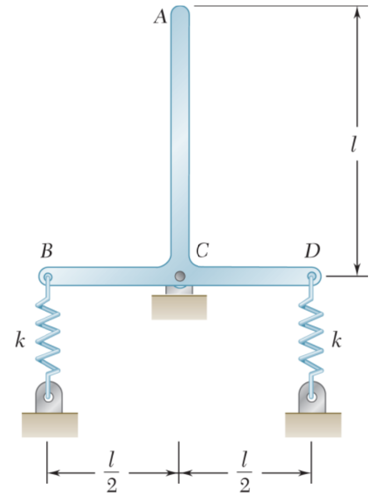
$$T = \frac{1}{2}m\left(\frac{L^2\dot{\theta}^2}{4}\right) + \frac{1}{2}\frac{mr^2}{2}\left(\frac{L^2\dot{\theta}^2}{4r^2}\right)$$

$$= \frac{3mL^2\dot{\theta}^2}{16}$$

$$\omega_n^2 = \frac{\frac{kL^2}{8} + \frac{mgL}{4}}{\frac{3mL^2}{16}}$$

$$= \frac{2}{3}\left(\frac{k}{m} + \frac{2g}{L}\right)$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3m} + \frac{4g}{3L}} \blacktriangleleft$$


A
l
B
C
D
k
k
$\frac{l}{2}$
$\frac{l}{2}$

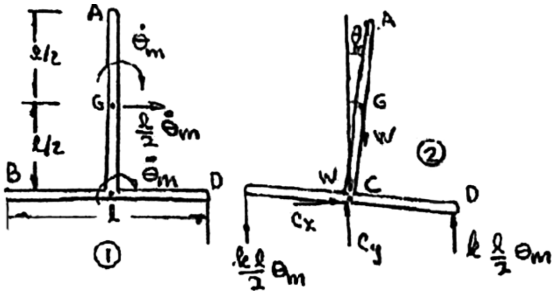
PROBLEM 19.78

Two uniform rods, each of weight $W = 1.2$ lb and length $l = 8$ in., are welded together to form the assembly shown. Knowing that the constant of each spring is $k = 0.6$ lb/in. and that end A is given a small displacement and released, determine the frequency of the resulting motion.

SOLUTION

Mass and moment of inertia of one rod.
$$m = \frac{W}{g} = \frac{1.2}{32.2} = 0.037267 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\overline{I} = \frac{1}{12}ml^2 = \frac{1}{12}(0.037267)\left(\frac{8}{12}\right)^2 = 1.38026\times10^{-3} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$


A
$\dot{\theta}_m$
$l/2$
G
$\frac{l}{2}\dot{\theta}_m$
$l/2$
$\dot{\theta}_m$
B
D
l
①
A
θ
G
W
②
W
C
D
C_x
C_y
$k\frac{l}{2}\theta_m$
$k\frac{l}{2}\theta_m$

Approximation.

$$\sin\theta_m \approx \tan\theta_m \approx \theta_m$$
$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{1}{2}\theta_m^2$$

Spring constant: $k = 0.6 \text{ lb/in.} = 7.2 \text{ lb/ft}$

Position ①

$$T_1 = 2\left(\frac{1}{2}\overline{I}\dot{\theta}_m^2\right) + \frac{1}{2}m\left(\frac{l}{2}\dot{\theta}_m\right)^2$$
$$= (2)\left(\frac{1}{2}\right)(1.38026\times10^{-3})\dot{\theta}_m^2 + \left(\frac{1}{2}\right)(0.037267)\left(\frac{4}{12}\dot{\theta}_m\right)^2$$
$$= 3.4506\times10^{-3}\dot{\theta}_m^2$$
$$V_1 = 0$$

PROBLEM 19.78 (Continued)

Position ②

$$T_2 = 0$$

$$V_2 = -\frac{Wl}{2}(1-\cos\theta_m) + 2\left(\frac{1}{2}\right)k\left(\frac{l}{2}\theta_m\right)^2$$

$$\approx -\frac{(1.2)(0.66667)}{2}\left(\frac{1}{2}\theta_m^2\right) + (2)\left(\frac{1}{2}\right)(7.2)\left(\frac{0.66667}{2}\theta_m\right)^2$$

$$\approx 0.6\theta_m^2$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$

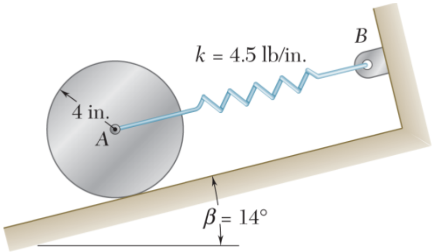
$$3.4506\times 10^{-3}\dot{\theta}_m^2 + 0 = 0 + 0.6\theta_m^2$$

$$\dot{\theta}_m = 13.186\theta_m$$

Simple harmonic motion. $\dot{\theta}_m = \omega_n \theta_m$

$$\omega_n = 13.186 \text{ rad/s}$$

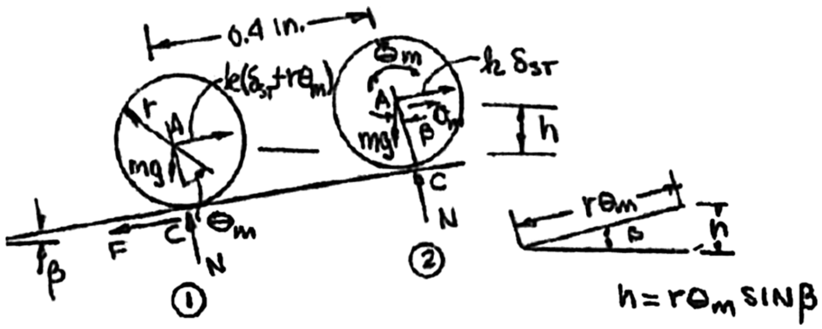
Frequency. $f_n = \dfrac{\omega_n}{2\pi}$ $f_n = 2.10$ Hz ◀



PROBLEM 19.79

A 15-lb uniform cylinder can roll without sliding on an incline and is attached to a spring *AB* as shown. If the center of the cylinder is moved 0.4 in. down the incline and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of the center of the cylinder.

SOLUTION



(*a*) Position ①

$$T_1 = 0 \quad V_1 = \frac{1}{2}k(\delta_{\text{st}} + r\theta_m)^2$$

Position ②

$$T_2 = \frac{1}{2}\bar{I}\dot{\theta}_m^2 + \frac{1}{2}m\bar{v}_m^2$$

$$V_2 = mgh + \frac{1}{2}k(\delta_{\text{st}})^2$$

Conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad 0 + \frac{1}{2}k(\delta_{\text{st}} + r\theta_m)^2 = \frac{1}{2}\bar{I}\dot{\theta}_m^2 + \frac{1}{2}m\bar{v}_m^2 + mgh + \frac{1}{2}k(\delta_{\text{st}})^2$$

$$k\delta_{\text{st}}^2 + 2k\delta_{\text{st}}r\theta_m + kr^2\theta_m^2 = \bar{I}\dot{\theta}_m^2 + mv_m^2 + 2mgh + k\delta_{\text{st}}^2 \tag{1}$$

When the disk is in equilibrium,

$$\circlearrowleft + \Sigma M_c = 0 = mg\sin\beta r - k\delta_{\text{st}}r$$

Also,

$$h = r\sin\beta\,\theta_m$$

Thus,

$$mgh - k\delta_{\text{st}}r = 0 \tag{2}$$

Substituting Eq. (2) into Eq. (1)

$$kr^2\theta_m^2 = \bar{I}\dot{\theta}_m^2 + m\bar{v}_m^2$$

$$\dot{\theta}_m = \omega_n\theta_m \quad v_m = r\dot{\theta}_m = r\omega_n\theta_m$$

$$kr^2\theta_m^2 = (\bar{I} + mr^2)\theta_m^2\omega_n^2$$

$$\omega_n^2 = \frac{kr^2}{\bar{J} + mr^2} \qquad \bar{I} = \frac{1}{2}mr^2$$

$$\omega_n^2 = \frac{kr^2}{\frac{1}{2}mr^2 + mr^2} = \frac{2}{3}\frac{k}{m}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{2}{3}\frac{(4.5\times 12\ \text{lb/ft})}{\left(\frac{15\ \text{lb}}{32.2}\ \text{ft/s}^2\right)}}} \qquad \tau_n = 0.715\ \text{s} \blacktriangleleft$$

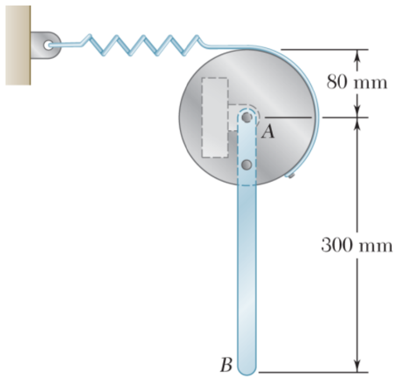
(*b*) Maximum velocity.

$$v_m = r\dot{\theta}_m$$

$$\dot{\theta}_m = \theta_m\omega_n$$

$$v_m = r\theta_m\omega_n \qquad r\theta_m = \frac{0.4}{12}\ \text{ft}$$

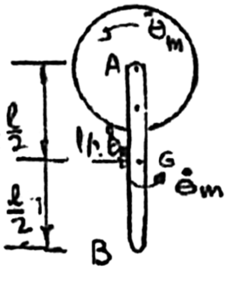
$$v_m = \left(\frac{0.4}{12}\ \text{ft}\right)\left(\frac{2\pi}{0.715\ \text{s}}\right) \qquad v_m = 0.293\ \text{ft/s} \blacktriangleleft$$



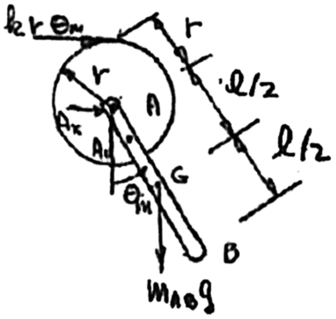
PROBLEM 19.80

A 3-kg slender rod AB is bolted to a 5-kg uniform disk. A spring of constant 280 N/m is attached to the disk and is unstretched in the position shown. If end B of the rod is given a small displacement and released, determine the period of vibration of the system.

SOLUTION



①



②

$$r = 0.08 \text{ m}$$

$$l = 0.3 \text{ m}$$

Position ①

$$T_1 = \frac{1}{2}\bar{I}_{\text{disk}}\dot{\theta}_m^2 + \frac{1}{2}(I_A)_{\text{rod}}\dot{\theta}_m^2$$

$$V_1 = 0$$

$$I_{\text{disk}} = \frac{1}{2}m_0 r^2$$

$$I_{A \text{ rod}} = \frac{1}{3}m_{AB}l^2$$

Position ②

$$T_2 = 0$$

$$V_2 = \frac{1}{2}k(r\theta_m)^2 + m_{AB}g\frac{l}{2}(1-\cos\theta_m)$$

$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$V_2 = \frac{1}{2}kr^2\theta_m^2 + \frac{m_{AB}g\left(\frac{l}{2}\right)}{2}\theta_m^2$$

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PROBLEM 19.80 (Continued)

Conservation of energy.

$$T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2}\left(\frac{1}{2}m_0 r^2 + \frac{1}{3}m_{AB}l^2\right)\dot{\theta}_m^2 + 0 = 0 + \frac{1}{2}kr^2\theta_m^2 + \frac{1}{2}m_{AB}g\frac{l}{2}\theta_m^2$$

For simple harmonic motion,

$$\dot{\theta}_m = \omega_n \theta_m$$

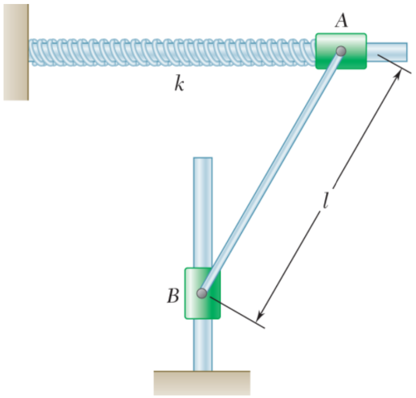
$$\left(\frac{1}{2}m_0 t^2 + \frac{1}{3}m_{AB}l^2\right)\omega_n^2\theta_m^2 = \left(kr^2 + m_{AB}g\frac{l}{2}\right)\theta_m^2$$

$$\omega_n^2 = \frac{kr^2 + m_{AB}gl}{\frac{1}{2}m_0 r^2 + \frac{1}{3}m_{AB}l^2}$$

$$\omega_n^2 = \frac{(280\text{ N/m})(0.08\text{ m})^2 + (3\text{ kg})(9.81\text{ m/s}^2)\left(\frac{0.3}{2}\text{ m}\right)}{\frac{1}{2}(5\text{ kg})(0.08\text{ m})^2 + \frac{1}{3}(3\text{ kg})(0.300\text{ m})^2}$$

$$\omega_n^2 = \frac{6.207}{0.106} = 58.55$$

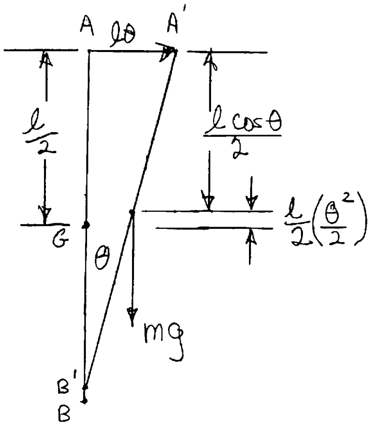
Period of vibration. $$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{58.55}}$$ $\tau_n = 0.821\text{ s}$ ◀



PROBLEM 19.81

A slender 10-kg bar *AB* of length $l = 0.6$ m is connected to two collars of negligible weight. Collar *A* is attached to a spring of constant $k = 1.5$ kN/m and can slide on a horizontal rod, while collar *B* can slide freely on a vertical rod. Knowing that the system is in equilibrium when bar *AB* is vertical and that collar *A* is given a small displacement and released, determine the period of the resulting vibrations.

SOLUTION


A
$l\theta$
A'
$\frac{l}{2}$
$\frac{l\cos\theta}{2}$
$\frac{l}{2}\left(\frac{\theta^2}{2}\right)$
G
θ
mg
B'
B

$$V = \frac{1}{2}k(l\theta)^2 + mg\left(\frac{l\theta^2}{4}\right)$$

$$T = \frac{1}{2}\left[\frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2\right]\dot{\theta}^2$$

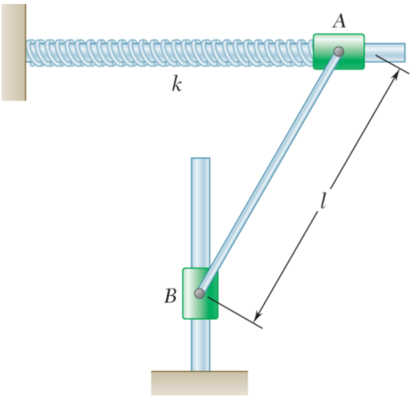
$$\omega_n^2 = \frac{\frac{kl^2}{2} + \frac{mgl}{4}}{\frac{ml^2}{6}}$$

$$\omega_n^2 = \frac{3k}{m} + \frac{3g}{2l}$$

With $k = 1500$ N/m, $l = 0.6$ m, $m = 10$ kg

$$\omega_n^2 = 474.525 \text{ rad}^2/\text{s}^2$$

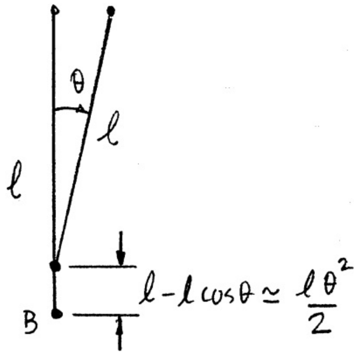
$$\tau = \frac{2\pi}{\omega_n} = 0.288 \text{ s} ◀$$

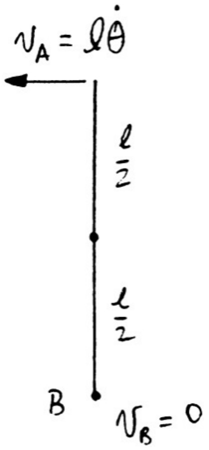


PROBLEM 19.82

A slender 5-kg bar *AB* of length $l = 0.6$ m is connected to two collars, each of mass 2.5 kg. Collar *A* is attached to a spring of constant $k = 1.5$ kN/m and can slide on a horizontal rod, while collar *B* can slide freely on a vertical rod. Knowing that the system is in equilibrium when bar *AB* is vertical and that collar *A* is given a small displacement and released, determine the period of the resulting vibrations.

SOLUTION





Let m be the mass of rod and m_C be mass of each collar

Then
$$V = \left(\frac{kl^2\theta^2}{2} + \frac{mgl\theta^2}{4}\right) + m_C g\frac{l\theta^2}{2}$$

$$T = \frac{1}{2}\left(m\frac{l^2}{3}\right)\dot{\theta}^2 + \frac{1}{2}m_C\left(l\dot{\theta}\right)^2$$

$$\omega_n^2 = \frac{\frac{kl^2}{2} + \frac{mgl}{4} + \frac{m_C gl}{2}}{\frac{ml^2}{6} + \frac{m_C l^2}{2}}$$

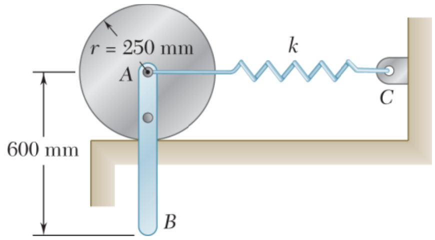
$$= \frac{\frac{k}{2} + \frac{mg}{4l} + \frac{m_C g}{2l}}{\frac{m}{6} + \frac{m_C}{2}}$$

$$\omega_n^2 = \frac{\left(\frac{1500 \text{ N/m}}{2} + \frac{(5 \text{ kg})(9.81 \text{ m/s}^2)}{4(0.6 \text{ m})} + \frac{(2.5 \text{ kg})(9.81 \text{ m/s}^2)}{2(0.6 \text{ m})}\right)}{\left(\frac{5 \text{ kg}}{6} + \frac{2.5 \text{ k}}{2}\right)}$$

$$= 397.62 \text{ s}^{-2}$$

$$\omega_n = 19.4838 \text{ rad/s}$$

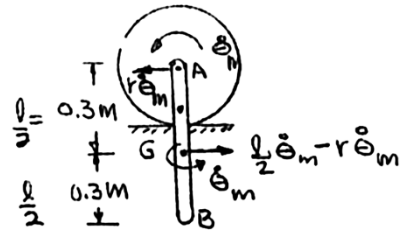
$$\tau = \frac{2\pi}{\omega_n} = 0.322 \text{ s} \blacktriangleleft$$



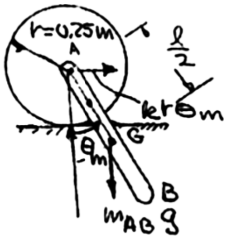
PROBLEM 19.83

An 800-g rod AB is bolted to a 1.2-kg disk. A spring of constant $k = 12$ N/m is attached to the center of the disk at A and to the wall at C. Knowing that the disk rolls without sliding, determine the period of small oscillations of the system.

SOLUTION



①



②

Position ①

$$T_1 = \frac{1}{2}(\bar{I}_G)_{AB}\dot{\theta}_m^2 + \frac{1}{2}m_{AB}\left(\frac{l}{2} - r\right)^2 \dot{\theta}_m + \frac{1}{2}(\bar{I}_G)_{\text{disk}}\dot{\theta}_m^2 + \frac{1}{2}m_{\text{disk}}r^2\dot{\theta}_m^2$$

$$(\bar{I}_G)_{AB} = \frac{1}{12}ml^2 = \frac{1}{12}(0.8)(0.6)^2 = 0.024 \text{ kg}\cdot\text{m}^2$$

$$m_{AB}\left(\frac{l}{2} - r\right)^2 = (0.8)(0.3 - 0.25)^2 = 0.002 \text{ kg}\cdot\text{m}^2$$

$$(I_G)_{\text{disk}} = \frac{1}{2}m_{\text{disk}}r^2 = \frac{1}{2}(1.2)(0.25)^2 = 0.0375 \text{ kg}\cdot\text{m}^2$$

$$m_{\text{disk}}r^2 = 1.2(0.25)^2 = 0.0750 \text{ kg}\cdot\text{m}^2$$

$$T_1 = \frac{1}{2}[0.024 + 0.002 + 0.0375 + 0.0750]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2}[0.1385]\dot{\theta}_m^2$$

$$V_1 = 0$$

Position ②

$$T_2 = 0$$

$$V_2 = \frac{1}{2}k(r\theta_m)^2 + m_{AB}g\frac{l}{2}(1 - \cos\theta_m)$$

$$1 - \cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2} \text{ (small angles)}$$

$$V_2 = \frac{1}{2}(12 \text{ N/m})(0.25 \text{ m})^2\theta_m^2 + (0.8 \text{ kg})(9.81 \text{ m/s}^2)\left(\frac{0.6 \text{ m}}{2}\right)\frac{\theta_m^2}{2}$$

PROBLEM 19.83 (Continued)

$$V_2 = \frac{1}{2}[0.750 + 2.354]\theta_m^2$$

$$= \frac{1}{2}(3.104)\,\theta_m^2 \text{ N}\cdot\text{m}$$

$$T_1 + V_1 = T_2 + V_2$$

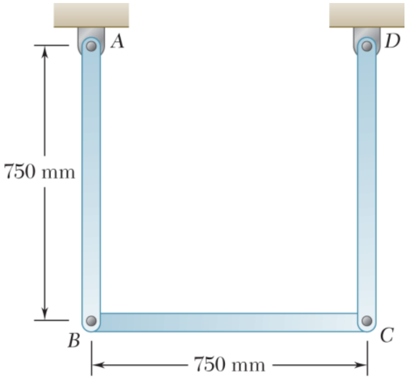
$$\dot{\theta}_m^2 = \omega_n^2 \theta_m^2$$

$$\frac{1}{2}(0.1385)\theta_m^2\omega_n^2 + 0 = 0 + \frac{1}{2}(3.104)\theta_m^2$$

$$\omega_n^2 = \frac{(3.104 \text{ N}\cdot\text{m})}{(0.1385 \text{ kg}\cdot\text{m}^2)}$$

$$= 22.41 \text{ s}^{-2}$$

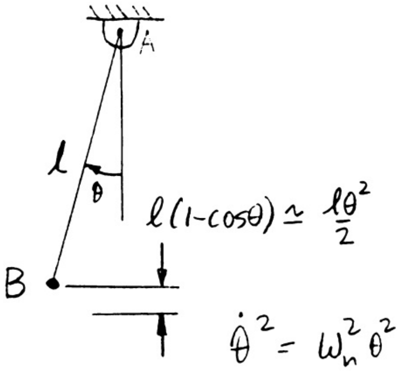
$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{22.41}} \qquad \tau_n = 1.327 \text{ s} \blacktriangleleft$$


A
D
750 mm
B
C
750 mm

PROBLEM 19.84

Three identical 3.6-kg uniform slender bars are connected by pins as shown and can move in a vertical plane. Knowing that bar *BC* is given a small displacement and released, determine the period of vibration of the system.

SOLUTION


A
l
θ
$l(1-\cos\theta) \simeq \frac{l\theta^2}{2}$
B
$\dot{\theta}^2 = \omega_n^2\theta^2$

$$V_1 = 0, \quad V_2 = mgl\frac{\theta^2}{2} + 2mg\frac{l}{2}\left(\frac{\theta^2}{2}\right)$$

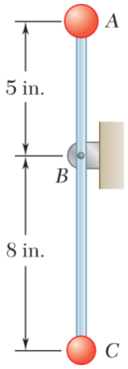
$$= mgl\theta^2$$

$$T_2 = 0, \quad T_1 = \frac{1}{2}ml^2\dot{\theta}^2 + 2\left[\frac{1}{2}\frac{ml^2}{3}\dot{\theta}^2\right]$$

$$= \frac{5ml^2\dot{\theta}^2}{6}$$

$$\dot{\theta}^2 = \omega_n^2\theta^2: \quad mgl = \frac{5ml^2}{6}\omega_n^2, \quad \omega_n^2 = \frac{6g}{5l}$$

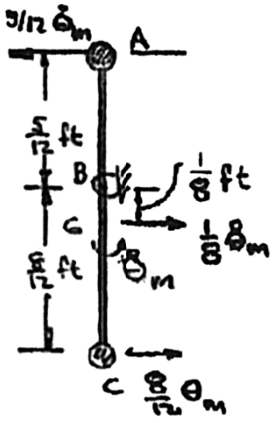
$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{6}{5}\frac{(9.81 \text{ m/s})}{(0.75 \text{ m})}}} = 1.586 \text{ s} \blacktriangleleft$$



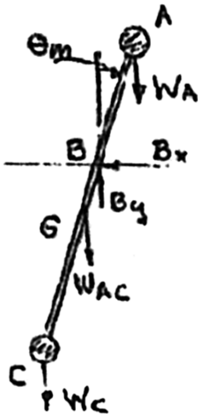
PROBLEM 19.85

A 14-oz sphere A and a 10-oz sphere C are attached to the ends of a 20-oz rod AC which can rotate in a vertical plane about an axis at B. Determine the period of small oscillations of the rod.

SOLUTION



①



②

Position ①

$$T_1 = \frac{1}{2}\frac{W_A}{g}\left(\frac{5}{12}\dot{\theta}_m\right)^2 + \frac{1}{2}\frac{W_C}{g}\left(\frac{8}{12}\dot{\theta}_m\right)^2 + \frac{1}{2}\frac{W_{AC}}{g}\left(\frac{1}{8}\dot{\theta}_m\right)^2 + \frac{1}{2}\bar{I}_{AC}\dot{\theta}_m^2$$

$$\bar{I}_{AC} = \frac{1}{12}\frac{W_{AC}}{g}\left(\frac{13}{12}\right)^2$$

$$T_1 = \frac{1}{2g}\left[\frac{14}{16}\left(\frac{5}{12}\right)^2 + \frac{10}{16}\left(\frac{8}{12}\right)^2 + \frac{20}{16}\left(\frac{1}{8}\right)^2 + \frac{1}{12}\left(\frac{20}{16}\right)\left(\frac{13}{12}\right)^2\right]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2(32.2\text{ ft/s}^2)}[0.1519 + 0.2778 + 0.01953 + 0.1223]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2}\left(\frac{0.5715\text{ lb}\cdot\text{ft}^2}{32.2\text{ ft/s}^2}\right)\dot{\theta}_m^2 = \frac{1}{2}(0.01775)\dot{\theta}_m^2(\text{lb}\cdot\text{ft})$$

$$V_1 = 0$$

Position ②

$$T_2 = 0$$

$$V_2 = -W_A \frac{5}{12}(1-\cos\theta_m) + W_C \frac{8}{12}(1-\cos\theta_m) + W_{AC}\frac{1}{8}(1-\cos\theta_m)$$

$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$V_2 = \left[-\left(\frac{14}{16}\right)\left(\frac{5}{12}\right)+\left(\frac{10}{16}\right)\left(\frac{8}{12}\right)+\left(\frac{20}{16}\right)\left(\frac{1}{8}\right)\right]\frac{\theta_m^2}{2}\ (\text{lb}\cdot\text{ft})$$

$$V_2 = [-0.3646 + 0.4167 + 0.1563]\frac{\theta_m^2}{2}$$

$$V_2 = \frac{0.2084\theta_m^2}{2}$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$: $\frac{1}{2}(0.01775)\dot{\theta}_m^2 + 0 = 0 + \frac{0.2084}{2}\theta_m^2$

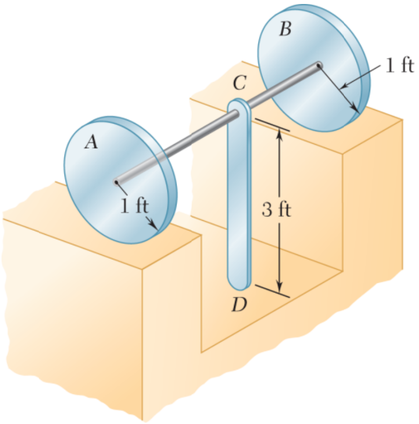
Simple harmonic motion.

$$\dot{\theta}_m = \omega_n\theta_m$$

$$\omega_n^2 = \frac{0.2084}{0.01775} = 11.738$$

$$\tau_n = \frac{2\pi}{\sqrt{\omega_n}} = \frac{2\pi}{\sqrt{11.738}}$$

$\tau_n = 1.834$ s ◀

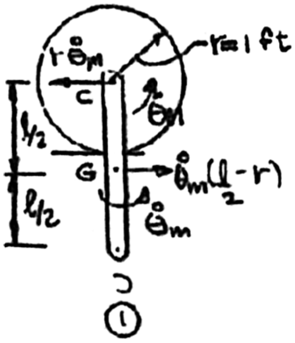

B
1 ft
C
A
1 ft
3 ft
D

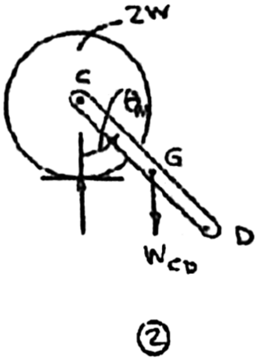
PROBLEM 19.86

A 10-lb uniform rod *CD* is welded at *C* to a shaft of negligible mass which is welded to the centers of two 20-lb uniform disks *A* and *B*. Knowing that the disks roll without sliding, determine the period of small oscillations of the system.

SOLUTION

$$W_A = W_B = W_{\text{disk}}$$


$r\dot{\theta}_m$
r=1 ft
C
$\dot{\theta}_m$
l/2
G
$\dot{\theta}_m(\frac{l}{2}-r)$
l/2
$\dot{\theta}_m$
D
①


2W
C
θ_m
G
D
W_{CD}
②

Position ①

$$T_1 = \frac{1}{2}2(\bar{I}_A)_{\text{disk}}\dot{\theta}_m^2 + \frac{1}{2}\left(\frac{2W_{\text{disk}}}{g}\right)(r\dot{\theta}_m)^2$$

$$+\frac{1}{2}\bar{I}_{CD}\dot{\theta}_m^2 + \frac{1}{2}\frac{W_{CD}}{g}\left(\frac{l}{2}-r\right)^2\dot{\theta}_m^2$$

$$(\bar{I}_A)_{\text{disk}} = \frac{1}{2}\frac{W_{\text{disk}}}{g}r^2 = \frac{1}{2}\frac{(20)}{g}(1)^2 = \frac{10}{g}$$

$$\bar{I}_{CD} = \frac{1}{12}\frac{W_{CD}}{g}l^2 = \frac{1}{12}\frac{(10)}{g}(3)^2 = \frac{15}{2g}$$

$$T_1 = \frac{1}{2g}\left[20 + 40 + \frac{15}{2} + \frac{5}{2}\right]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2}g(70)\dot{\theta}_m^2 \qquad V_1 = 0$$

Position ②

$$T_2 = 0$$

$$V_2 = W_{CD}\frac{l}{2}(1-\cos\theta_m)$$

PROBLEM 19.86 (Continued)

Small angles: $$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$V_2 = \frac{1}{2}W_{CD}l\frac{\theta_m^2}{2}$$

$$= \frac{1}{2}(10)(1.5)\theta_m^2$$

$$= \frac{1}{2}(15)\theta_m^2$$

Conservation of energy and simple harmonic motion.

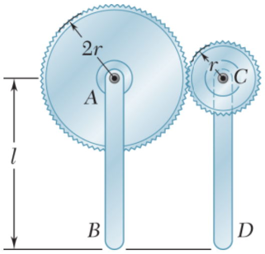
$$T_1 + V_1 = T_2 + V_2$$

$$\dot{\theta}_m = \omega_n\theta_m$$

$$\frac{1}{2g}(70)\omega_n^2\theta_m^2 + 0 = 0 + \frac{1}{2}(15)\theta_m^2$$

$$\omega_n^2 = \frac{15g}{70}$$

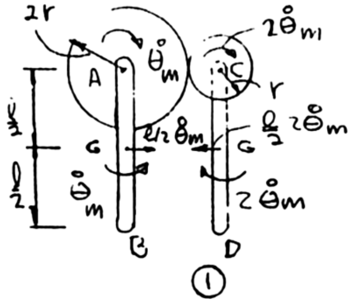
Period of oscillations. $$\tau_n = \frac{2\pi}{\sqrt{\omega_n}} = 2\pi\sqrt{\frac{70}{(15)(32.2)}}$$ $\tau_n = 2.39$ s ◀



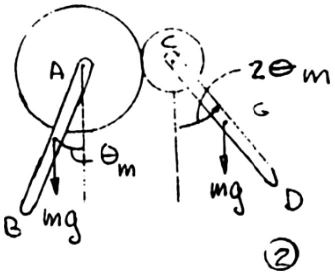
PROBLEM 19.87

Two uniform rods *AB* and *CD*, each of length *l* and mass *m*, are attached to gears as shown. Knowing that the mass of gear *C* is *m* and that the mass of gear *A* is 4*m*, determine the period of small oscillations of the system.

SOLUTION


2r
A
$\dot\theta_m$
$2\dot\theta_m$
C
r
$\frac{l}{2}$
$\frac{l}{2}$
G
$\frac{l}{2}\dot\theta_m$
G
$\frac{l}{2}2\dot\theta_m$
$\dot\theta_m$
$2\dot\theta_m$
B
D

①


A
C
$2\theta_m$
G
θ_m
mg
mg
B
D

②

Kinematics:

$$2r\theta_A = r\theta_C$$
$$2\theta_A = \theta_C$$
$$2\dot\theta_A = \dot\theta_C$$

Let

$$\theta_A = \theta_m$$
$$2\theta_m = (\theta_C)_m$$
$$2\dot\theta_m = (\dot\theta_C)_m$$

Position ①

$$T_1 = \frac{1}{2}\bar{I}_A\dot\theta_m^2 + \frac{1}{2}\bar{I}_C(2\dot\theta_m)^2 + \frac{1}{2}\bar{I}_{AB}\dot\theta_m^2 + \frac{1}{2}\bar{I}_{CD}(2\dot\theta_m)^2 + \frac{1}{2}m_{AB}\left(\frac{l}{2}\dot\theta_m\right)^2 + \frac{1}{2}m_{CD}\left(\frac{l}{2}2\dot\theta_m\right)^2$$

$$\bar{I}_A = \frac{1}{2}(4m)(2r)^2 = 8mr^2$$

$$\bar{I}_C = \frac{1}{2}(m)(r)^2 = \frac{1}{2}mr^2$$

$$\bar{I}_{AB} = \frac{1}{12}ml^2 \qquad \bar{I}_{CD} = \frac{1}{12}ml^2$$

$$T_1 = \frac{1}{2}m\left[8r^2 + \left(\frac{r^2}{2}\right)4 + \frac{l^2}{12} + \frac{l^2}{3} + \frac{l^2}{4} + l^2\right]$$

$$T_1 = \frac{1}{2}m\left[10r^2 + \frac{5}{3}l^2\right]\dot\theta_m^2$$

$$V_1 = 0$$

Position ②

$$T_1 = 0$$

$$V_1 = mg\frac{l}{2}(1-\cos\theta_m) + \frac{mgl}{2}(1-\cos 2\theta_m)$$

For small angles,

$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$1-\cos 2\theta_m = 2\sin^2\theta_m \approx 2\theta_m^2$$

$$V_1 = \frac{1}{2}mgl\left(\frac{\theta_m^2}{2} + 2\theta_m^2\right) = \frac{1}{2}mgl\frac{5\theta_m^2}{2}$$

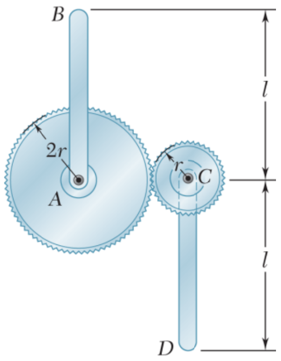
$$T_1 + V_1 = T_2 + V_2 \qquad \dot{\theta}_m^2 = \omega_n^2\theta_m^2$$

$$\frac{1}{2}m\left[10r^2 + \frac{5}{3}l^2\right]\omega_n^2\theta_m^2 + 0 = 0 + \frac{1}{2}mgl\frac{5\theta_m^2}{2}$$

$$\omega_n^2 = \frac{\frac{5}{2}gl}{10r^2 + \frac{5}{3}l^2}$$

$$= \frac{3gl}{12r^2 + 2l^2}$$

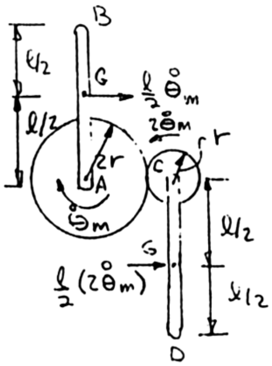
$$\tau_n = \frac{2\pi}{\sqrt{\omega_n}} = 2\pi\sqrt{\frac{12r^2 + 2l^2}{3gl}} \blacktriangleleft$$



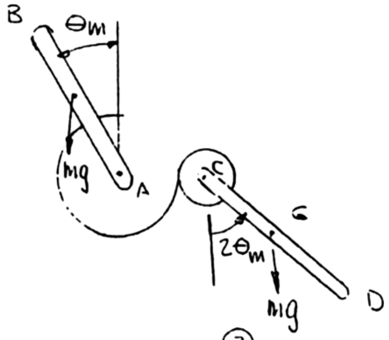
PROBLEM 19.88

Two uniform rods *AB* and *CD*, each of length *l* and mass *m*, are attached to gears as shown. Knowing that the mass of gear *C* is *m* and that the mass of gear *A* is 4*m*, determine the period of small oscillations of the system.

SOLUTION


B
l/2
l/2
G
$\frac{l}{2}\dot{\theta}_m$
$2\dot{\theta}_m$
r
2r
A
C
$\dot{\theta}_m$
l/2
G
$\frac{l}{2}(2\dot{\theta}_m)$
l/2
D

①


B
θm
mg
A
C
G
2θm
mg
D

②

Kinematics: $2r\theta_A = r\theta_C \qquad 2\theta_A = \theta_C$

$2\dot{\theta}_A = \dot{\theta}_C$

Let $\theta_A = \theta_m \qquad 2\theta_m = (\theta_C)_m$

$2\dot{\theta}_m = (\dot{\theta}_C)_m$

Position ①

$$T_1 = \frac{1}{2}\overline{I}_A\dot{\theta}_m^2 + \frac{1}{2}\overline{I}_C(2\dot{\theta}_m)^2 + \frac{1}{2}\overline{I}_{AB}\dot{\theta}_m^2 + \frac{1}{2}\overline{I}_{CD}(2\dot{\theta}_m)^2 + \frac{1}{2}m_{AB}\left(\frac{l}{2}\dot{\theta}_m\right)^2 + \frac{1}{2}m_{CD}\left(\frac{l}{2}2\dot{\theta}_m\right)^2$$

$$\overline{I}_A = \frac{1}{2}(4m)(2r)^2 = 8mr^2$$

$$\overline{I}_C = \frac{1}{2}(m)(r^2) = \frac{1}{2}mr^2$$

$$\overline{I}_{AB} = \frac{1}{12}ml^2 \qquad \overline{I}_{CD} = \frac{1}{12}ml^2$$

$$T_1 = \frac{1}{2}m\left[8r^2 + \left(\frac{r^2}{2}\right)4 + \frac{l^2}{12} + \frac{l^2}{3} + \frac{l^2}{4} + l^2\right]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2}m\left[10r^2 + \frac{5}{3}l^2\right]\dot{\theta}_m^2 \qquad V_1 = 0$$

PROBLEM 19.88 (Continued)

Position ②

$$T_2 = 0$$

$$V_2 = -mg\frac{l}{2}(1-\cos\theta_m) + \frac{mgl}{2}(1-\cos 2\theta_m)$$

For small angles,

$$1-\cos\theta_m = 2\sin^2\frac{\theta_m}{2} \approx \frac{\theta_m^2}{2}$$

$$1-\cos 2\theta_m = 2\sin^2\theta_m \approx 2\theta_m^2$$

$$V_2 = -mg\frac{l}{2}\frac{\theta_m^2}{2} + \frac{mgl}{2}2\theta_m^2$$

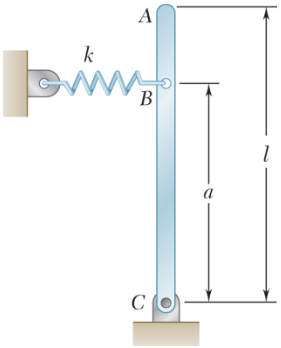
$$= \frac{1}{2}mgl\frac{3}{2}\theta_m^2$$

$$T_1 + V_1 = T_2 + V_2 \qquad \dot{\theta}_m = \omega_n\theta_m$$

$$\frac{1}{2}m\left[10r^2 + \frac{5}{3}l^2\right]\theta_m^2\omega_n^2 + 0 = 0 + \frac{1}{2}mgl\frac{3}{2}\theta_m^2$$

$$\omega_n^2 = \frac{\frac{3}{2}gl}{10r^2 + \frac{5}{3}l^2}$$

$$= \frac{9gl}{60r^2 + 10l^2} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{60r^2 + 10l^2}{9gl}} \quad ◀$$



PROBLEM 19.89

An inverted pendulum consisting of a rigid bar *ABC* of length l and mass m is supported by a pin and bracket at C. A spring of constant k is attached to the bar at B and is undeformed when the bar is in the vertical position shown. Determine (*a*) the frequency of small oscillations, (*b*) the smallest value of a for which these oscillations will occur.

SOLUTION

Moment of inertia: $\bar{I} = \frac{1}{12}ml^2$

Position ① Maximum deflection. Let rod *AC* rotate through angle θ_m. The spring stretches an amount

$$x_m = a\sin\theta_m$$

and the center of gravity moves down an amount

$$-y_m = \frac{l}{2}(1-\cos\theta_m)$$

$$\begin{aligned}
V_1 &= \frac{1}{2}kx_m^2 + mgy_m \\
&= \frac{1}{2}k(a\sin\theta_m)^2 - mg\frac{l}{2}(1-\cos\theta_m) \\
&\approx \frac{1}{2}ka^2\theta_m^2 - mg\left(\frac{l}{2}\right)\left(\frac{1}{2}\theta_m^2\right) \\
&= \frac{1}{2}\left(ka^2 - \frac{1}{2}mgl\right)\theta_m^2
\end{aligned}$$

$$T_1 = 0$$

Position ② Maximum velocity:

For simple harmonic motion, $\dot{\theta} = -\omega_n\theta_m$

Velocity of the mass center of the rod: $\bar{v} = \frac{l}{2}\dot{\theta}$

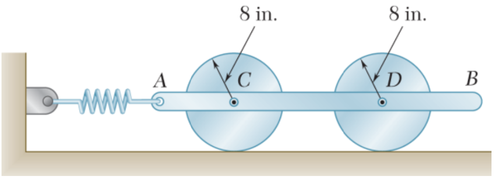
PROBLEM 19.89 (Continued)

Kinetic energy:
$$T_2 = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\dot{\theta}^2$$
$$= \frac{1}{2}\left[m\left(\frac{l\dot{\theta}}{2}\right)^2 + \frac{1}{12}ml^2\dot{\theta}^2\right]$$
$$= \frac{1}{2}\left[\frac{1}{3}ml^2\dot{\theta}^2\right]$$
$$= \frac{1}{2}\left(\frac{1}{3}ml^2\omega_n^2\theta_m^2\right)$$
$$V_2 = 0$$

Conservation of energy:
$$T_1 + V_1 = T_2 + V_2$$
$$0 + \frac{1}{2}\left(ka^2 - \frac{1}{2}mgl\right)\theta_m^2 = \frac{1}{2}\left(\frac{1}{3}ml^2\omega_n^2\theta_m^2\right)$$
$$\omega_n^2 = \frac{6\,ka^2 - 3mgl}{2ml^2}$$

(*a*) Frequency:
$$f = \frac{\omega_n}{2\pi}$$
$$f = \frac{1}{2\pi}\sqrt{(6ka^2 - 3mgl)/2ml^2} \blacktriangleleft$$

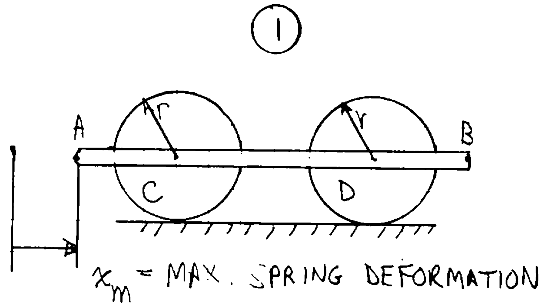
(*b*) Smallest value of *a* for oscillations. *f* is real for $6\,ka^2 > 3mgl$
$$a > \sqrt{\frac{mgl}{2k}} \qquad a_{\min} = \sqrt{\frac{mgl}{2k}} \blacktriangleleft$$

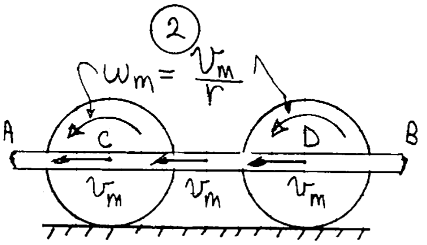


PROBLEM 19.90

Two 12-lb uniform disks are attached to the 20-lb rod *AB* as shown. Knowing that the constant of the spring is 30 lb/in. and that the disks roll without sliding, determine the frequency of vibration of the system.

SOLUTION





Position 1: $T_1 = 0,\qquad V_1 = \frac{1}{2}kx_m^2$

Position 2: $V_2 = 0,\qquad T_2 = \frac{1}{2}m_{AB}v_m^2 + 2\left(\frac{1}{2}\bar{I}\omega_m^2 + \frac{1}{2}m_{\text{disk}}\bar{v}_m^2\right)$

$$T_2 = \frac{1}{2}m_{AB}v_m^2 + \left(\frac{1}{2}m_{\text{disk}}r^2\right)\left(\frac{v_m}{r}\right)^2 + m_{\text{disk}}v_m^2$$

$$T_2 = \frac{1}{2}(m_{AB} + 3m_{\text{disk}})v_m^2$$

Conservation of energy

$$T_1 + V_1 = T_2 + V_2:\qquad 0 + \frac{1}{2}kx_m^2 = \frac{1}{2}(m_{AB} + 3m_{\text{disk}})v_m^2$$

But for simple harmonic motion, $v_m = \omega_n x_m$:

$$\frac{1}{2}kx_m^2 = \frac{1}{2}(m_{AB} + 3m_{\text{disk}})(\omega_n x_m)^2$$

$$\omega_n^2 = \frac{k}{m_{AB} + 3m_{\text{disk}}}$$

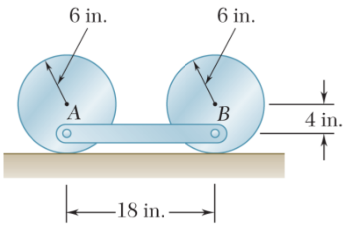
Note: Result is independent of *r*

Data: $k = 30$ lb/in. $\quad W_{AB} = 20$ lb $\quad W_{\text{disk}} = 12$ lb $\quad m_{AB} = \frac{W_{AB}}{g} \quad m_{\text{disk}} = \frac{W_{\text{disk}}}{g}$

$$\omega_n^2 = \frac{30(12)\text{ lb/ft}}{20\text{ lb}/32.2 + 3(12/32.2)} = 207 \qquad \omega_n = 14.387\text{ rad/s}$$

$$f = \frac{\omega_n}{2\pi} = \frac{14.387\text{ rad/s}}{2\pi}$$

$f = 2.29$ Hz ◀


6 in.
6 in.
A
B
4 in.
18 in.

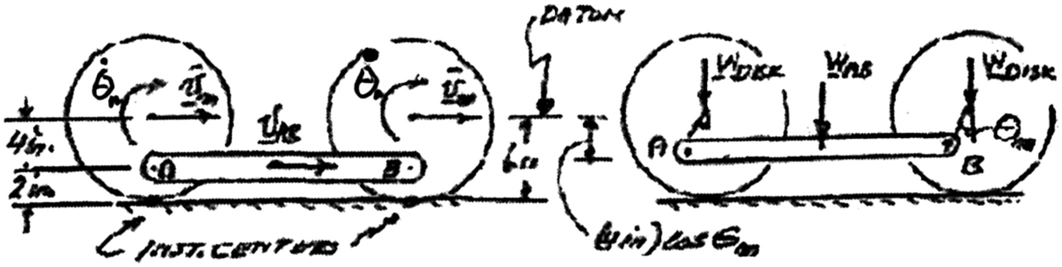
PROBLEM 19.91

The 20-lb rod AB is attached to two 8-lb disks as shown. Knowing that the disks roll without sliding, determine the frequency of small oscillations of the system.

SOLUTION

Position ② *Position ①*

$r = 6\text{ in.}$


DATUM
$\dot\theta_m$
$\bar v_m$
$\dot\theta_m$
$\bar v_m$
v_{AB}
4 in.
2 in
6 in
A
B
INST. CENTERS
(6 in) cos θ_m
W_{DISK}
W_{AB}
W_{DISK}
A
B
θ_m

Masses and moments of inertia.

$$m_A = m_B = \frac{8}{32.2} = 0.24845\text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\bar I_A = \bar I_B = \frac{1}{2}m_A r_A^2 = \frac{1}{2}(0.24845)\left(\frac{6}{12}\right)^2$$

$$= 0.031056\text{ lb}\cdot\text{s}\cdot\text{ft}$$

$$m_{AB} = \frac{20}{32.2} = 0.62112\text{ lb}\cdot\text{s}^2/\text{ft}$$

Kinematics:

$$\bar v_m = r_A\dot\theta_m = \frac{6}{12}\dot\theta_m = 0.5\dot\theta_m$$

$$v_{AB} = \left(\frac{2}{12}\right)\dot\theta_m = \frac{1}{6}\dot\theta_m$$

Position ① (Maximum displacement)

$$T_1 = 0$$

$$V_1 = -W_{AB}\left(\frac{4}{12}\cos\theta_m\right) = -\frac{80}{12}\cos\theta_m$$

Position ② (Maximum speed)

$$T_2 = \frac{1}{2}m_A v_m^2 + \frac{1}{2}\bar I_A\dot\theta_m^2 + \frac{1}{2}m_B v_m^2 + \frac{1}{2}\bar I_B\dot\theta_m^2 + \frac{1}{2}m_{AB}v_{AB}^2$$

$$= 2\left[\frac{1}{2}(0.24845)(0.5\dot\theta_m)^2 + \frac{1}{2}(0.031056)\dot\theta_m^2\right] + \frac{1}{2}(0.62112)\left(\frac{1}{6}\dot\theta_m\right)^2$$

$$= 0.101795\dot\theta_m^2$$

$$V_2 = -W_{AB}\left(\frac{4}{12}\right) = -\frac{80}{12}$$

PROBLEM 19.91 (Continued)

Conservation of energy. $T_1 + V_1 = T_2 + V_2$

$$0 - \frac{80}{12}\cos\theta_m = 0.101795\dot{\theta}_m^2 - \frac{80}{12}$$

$$\dot{\theta}_m^2 = 65.491(1 - \cos\theta_m)$$

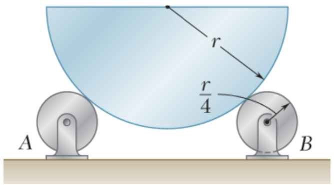
$$\approx 65.491\left(\frac{1}{2}\theta_m^2\right)$$

$$= 32.745\theta_m^2$$

$$\theta_m = 5.7224\,\theta_m$$

Simple harmonic motion. $\dot{\theta}_m = \omega_n\theta_m$ $\omega_n = 5.7224$ rad/s

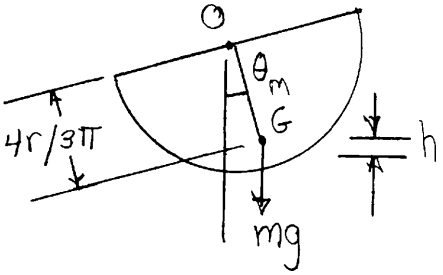
Frequency. $f_n = \frac{\omega_n}{2\pi} = \frac{5.7224}{2\pi}$ $f_n = 0.911$ Hz ◀



PROBLEM 19.92

A half section of a uniform cylinder of radius r and mass m rests on two casters A and B, each of which is a uniform cylinder of radius $r/4$ and mass $m/8$. Knowing that the half cylinder is rotated through a small angle and released and that no slipping occurs, determine the frequency of small oscillations.

SOLUTION



$$V_1 = mgh = mg\left(\frac{4r}{3\pi}\right)(1-\cos\theta)$$

$$1-\cos\theta \approx \frac{\theta_m^2}{2}$$

$$V_1 = 2mgr\,\frac{\theta_m^2}{3\pi}$$

$$T_2 = \frac{1}{2}I_A\omega_A^2 + \frac{1}{2}I_B\omega_B^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2$$

Where $$I_A = I_B = \frac{1}{2}\left(\frac{m}{8}\right)\left(\frac{r}{4}\right)^2 = \frac{mr^2}{256}$$

and $$\omega_A = \omega_B = 4\omega$$

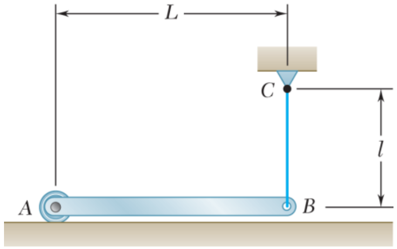
$$\therefore\ T_2 = \left(\frac{mr^2}{16} + \frac{mr^2}{4}\right)\omega^2 = \frac{5mr^2\omega^2}{16}$$

$$V_1 = T_2,\quad \frac{2\cancel{m}gr\cancel{\theta_m^2}}{3\pi} = \frac{5\cancel{m}r^2\omega_n^2\cancel{\theta_m^2}}{16}$$

$$\omega_n^2 = \frac{32g}{15\pi r},$$

$$f_n = \left(\frac{1}{2\pi}\right)\sqrt{\frac{32g}{15\pi r}}$$

$$f_n = 0.1312\sqrt{\frac{g}{r}}$$ ◀



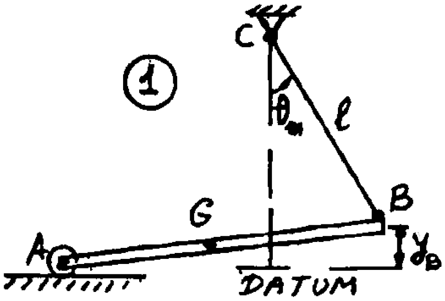
PROBLEM 19.93

The motion of the uniform rod AB is guided by the cord BC and by the small roller at A. Determine the frequency of oscillation when the end B of the rod is given a small horizontal displacement and released.

SOLUTION

Position ①. (Maximum deflection):

Let θ_m be the small angle between the cord CB and the vertical. As the rod is moved from the equilibrium position the center of gravity G moves up an amount $\overline{y}_m$.



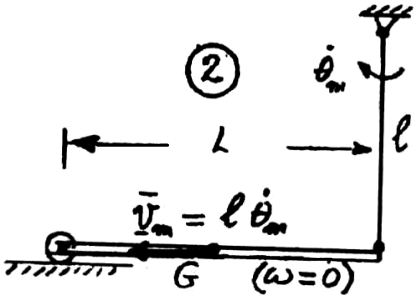
$$y_B = l(1-\cos\theta_m) \approx l\left(1-1+\frac{1}{2}\theta_m^2\right) = \frac{1}{2}l\theta_m^2$$

$$\overline{y}_m = y_G = \frac{1}{2}y_B = \frac{1}{4}l\theta_m^2$$

$$V_1 = mg\overline{y}_m = \frac{1}{4}mgl\theta_m^2$$

$$T_1 = 0,$$

Position ②. (Maximum velocity): At the equilibrium position the motion of the rod is a translation.



$$\overline{v}_m = l\omega = l\dot{\theta}_m$$

$$T_2 = \frac{1}{2}m\overline{v}_m^2 = \frac{1}{2}ml^2\dot{\theta}_m^2$$

$$V_2 = 0$$

Conservation of energy: $T_1 + V_1 = T_2 + V_2$: $\frac{1}{4}mgl\theta_m^2 = \frac{1}{2}ml^2\dot{\theta}_m^2$

For simple harmonic motion; $\dot{\theta}_m = \omega_n\theta_m$ so that

$$\frac{1}{4}mgl\theta_m^2 = \frac{1}{2}ml^2\omega_n^2\theta_m^2$$

Natural frequency: $\omega_n^2 = \frac{g}{2l}$

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{2l}}$$

$f = 0.1125\sqrt{\frac{g}{l}}$ ◀

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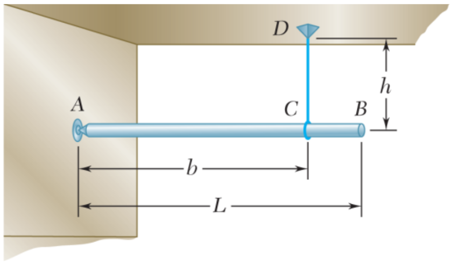
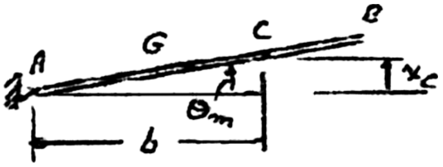


PROBLEM 19.94

A uniform rod of length L is supported by a ball-and-socket joint at A and by a vertical wire CD. Derive an expression for the period of oscillation of the rod if end B is given a small horizontal displacement and then released.

SOLUTION

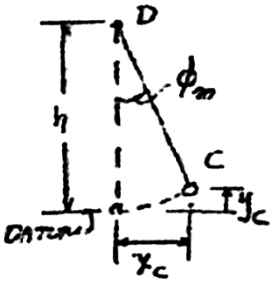


Position ① (Maximum deflection)

Looking from above:

Horizontal displacement of C: $x_C = b\theta_m$

Looking from right: $\phi_m = \dfrac{x_C}{h} = \dfrac{b}{h}\theta_m$



$$y_C = h(1-\cos\phi_m) \approx \frac{1}{2}h\phi_m^2$$

$$y_C = \frac{1}{2}h\left(\frac{b}{h}\theta_m\right)^2 = \frac{1}{2}\frac{b^2}{h}\theta_m^2$$

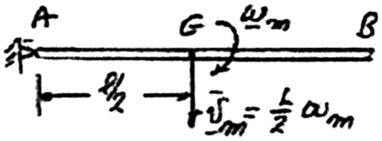
$$\bar{y}_m = y_G = \frac{AG}{AC} - y_C = \frac{\frac{1}{2}L}{b}\left(\frac{1}{2}\frac{b^2}{h}\theta_m^2\right)$$

$$\bar{y}_m = \frac{1}{4}\cdot\frac{bL}{h}\theta_m^2$$

We have $T_1 = 0$

$$V_1 = mgy_m = \frac{1}{4}\frac{mgbL}{h}\theta_m^2$$

Position ② (Maximum velocity)



Looking from above: $T_2 = \dfrac{1}{2}\bar{I}\dot{\theta}_m^2 + \dfrac{1}{2}m\bar{v}_m^2$

$$= \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}_m^2 + \frac{1}{2}m\left(\frac{L}{2}\dot{\theta}\right)^2$$

$$T_2 = \frac{1}{6}mL^2\dot{\theta}_m^2$$

$$V_2 = 0$$

Conservation of energy. $T_1 + V_1 = T_2 + V_2$: $\quad 0 + \dfrac{1}{4}\dfrac{mgbL}{h}\theta_m^2 = \dfrac{1}{6}mL^2\dot{\theta}_m^2$

PROBLEM 19.94 (Continued)

But for simple harmonic motion,

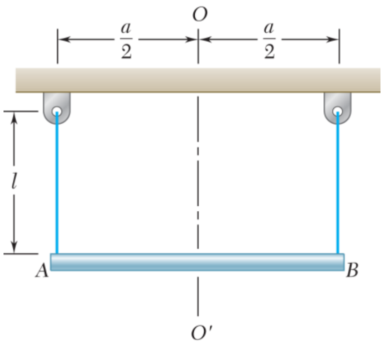
$$\dot{\theta}_m = \omega_n \theta_m$$

$$\frac{1}{4}\frac{mgbL}{h}\theta_m^2 = \frac{1}{6}mL^2(\omega_n\theta_m)^2$$

$$\omega_n^2 = \frac{3}{2}\frac{bg}{hL}$$

Period of vibration.

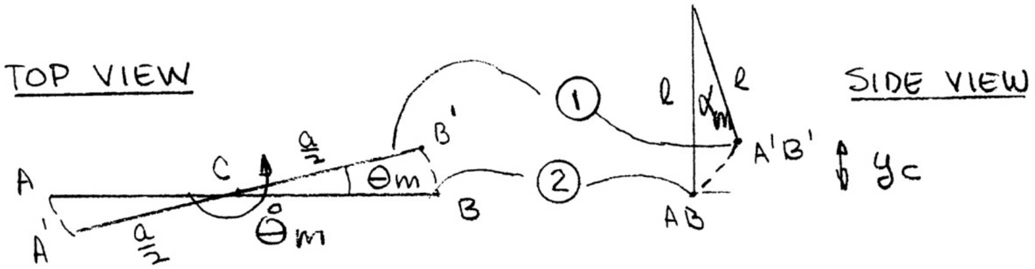
$$\tau_n = \frac{2\pi}{\omega_n} \qquad \tau_n = 2\pi\sqrt{\frac{2hL}{3bg}} \blacktriangleleft$$



PROBLEM 19.95

A section of uniform pipe is suspended from two vertical cables attached at A and B. Determine the frequency of oscillation when the pipe is given a small rotation about the centroidal axis OO' and released.

SOLUTION



$$AA' = BB' = \frac{a}{2}\theta_m = l\alpha_m \qquad \alpha_m = \frac{a}{2l}\theta_m$$

Position ①

$$T_1 = 0 \qquad V_1 = mgy_c = mgl(1 - \cos\alpha)$$

For small angles

$$1 - \cos\alpha_m = 2\sin\frac{\alpha_m}{2} \approx \frac{\alpha_m^2}{2} = \frac{a^2}{8l^2}\theta_m^2$$

$$V_1 = mgl\left(\frac{a^2}{8l^2}\right)\theta_m^2$$

Position ②

$$T_2 = \frac{1}{2}\bar{I}\dot{\theta}_m^2 = \frac{1}{2}\left(\frac{1}{12}ma^2\right)\dot{\theta}_m^2 \qquad V_2 = 0$$

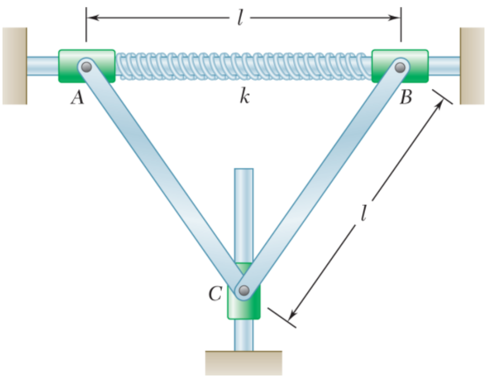
$$\dot{\theta}_m = \omega_n\theta_m$$

$$T_1 + V_1 = T_2 + V_2$$

$$mgl\left(\frac{a^2}{8l^2}\right) + 0 + \frac{1}{24}ma^2\omega_n^2\theta_m^2$$

$$\omega_n^2 = \frac{3g}{l}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{3g}{l}} \blacktriangleleft$$

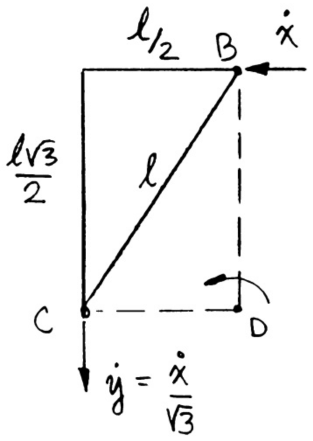


PROBLEM 19.96

Three collars, each of mass *m*, are connected by pins to bars *AC* and *BC*, each of length *l* and negligible mass. Collars *A* and *B* can slide without friction on a horizontal rod and are connected by a spring of constant *k*. Collar *C* can slide without friction on a vertical rod and the system is in equilibrium in the position shown. Knowing that collar *C* is given a small displacement and released, determine the frequency of the resulting motion of the system.

SOLUTION

$$T = 2\left[\frac{1}{2}m\dot{x}^2\right] + \frac{1}{2}m\dot{y}^2 = 7m\frac{\dot{x}^2}{6}$$



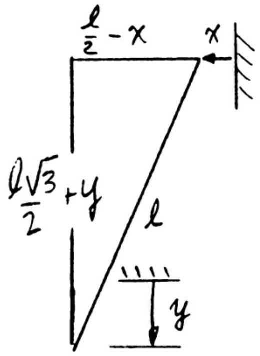
Equilibrium of *BC*:

$$\Sigma M_D = 0 = \frac{mg}{2}\cdot\frac{l}{2} - ku\left(\frac{\sqrt{3}l}{2}\right)$$

Where $u = \dfrac{mg}{2\sqrt{3}k}$, $l_0 =$ natural length $= l + u$

$$V_k = \frac{1}{2}k\left[(l-2x) - l_0\right]^2 = \frac{k}{2}(2x+u)^2$$

$$V_g = -mgy, \quad \text{where} \quad l^2 = \left(\frac{l}{2} - x\right)^2 + \left(\frac{l\sqrt{3}}{2} + y\right)^2$$



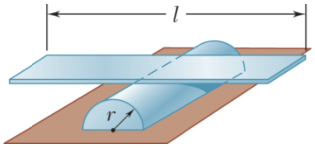
And $$y^2 + \sqrt{3}ly + \left(x^2 - lx\right) = 0, \quad y = \frac{-\sqrt{3}l + \sqrt{3l^2 - 4\left(x^2 - lx\right)}}{2}$$

So $$y = \left(\frac{1}{\sqrt{3}}\right)x + \frac{1}{2}\left(-\frac{8}{3\sqrt{3}l}\right)x^2 + \text{ H.O.T.}$$

Then $$V = 2kx^2 + 2kux + \text{ constant} - mg\frac{x}{\sqrt{3}} + \frac{4mgx^2}{3\sqrt{3}l}$$

$$\omega_n^2 = \frac{2k + \frac{4mg}{3\sqrt{3}l}}{\frac{7m}{6}}$$

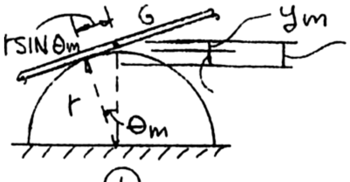
$$f_n = \frac{1}{2\pi}\sqrt{\frac{12k}{7m} + \frac{8g}{7\sqrt{3}l}} \text{ Hz} ◀$$



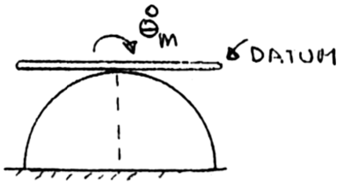
PROBLEM 19.97*

A thin plate of length l rests on a half cylinder of radius r. Derive an expression for the period of small oscillations of the plate.

SOLUTION



①



②

$$(r\sin\theta_m)\sin\theta_m \approx r\theta_m^2$$

$$r(1-\cos\theta_m) \approx r\frac{\theta_m^2}{2}$$

Position ① (Maximum deflection)

$$T_1 = 0$$

$$V_1 = Wy_m$$

$$= mgr\frac{\theta_m^2}{2}$$

Position ② ($\theta = 0$):

$$T_2 = \frac{1}{2}\bar{I}\dot{\theta}_m^2$$

$$= \frac{1}{2}\left(\frac{1}{12}\right)ml^2\dot{\theta}_m^2$$

$$\dot{\theta}_m = \omega_n\theta_m$$

$$T_2 = \frac{1}{2}\left(\frac{1}{12}\right)ml^2\omega_n^2\theta_m^2$$

$$T_1 + V_1 = T_2 + V_2$$

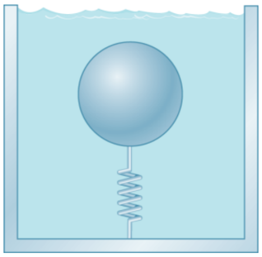
$$0 + \frac{1}{2}mgr\theta_m^2 = \frac{1}{2}\left(\frac{1}{12}\right)ml^2\omega_n^2\theta_m^2$$

$$\omega_n^2 = \frac{12gr}{l^2}$$

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l^2}{12gr}}$$

$$\tau_n = \frac{\pi l}{\sqrt{3gr}} \blacktriangleleft$$

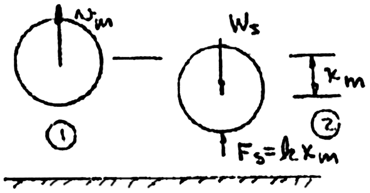
PROBLEM 19.98*



As a submerged body moves through a fluid, the particles of the fluid flow around the body and thus acquire kinetic energy. In the case of a sphere moving in an ideal fluid, the total kinetic energy acquired by the fluid is $\frac{1}{4}\rho V v^2$, where ρ is the mass density of the fluid, V is the volume of the sphere, and v is the velocity of the sphere. Consider a 500-g hollow spherical shell of radius 80 mm, which is held submerged in a tank of water by a spring of constant 500 N/m. (*a*) Neglecting fluid friction, determine the period of vibration of the shell when it is displaced vertically and then released. (*b*) Solve Part *a,* assuming that the tank is accelerated upward at the constant rate of 8 m/s^2.

SOLUTION

This is not a damped vibration. However, the kinetic energy of the fluid must be included.



(*a*) *Position* ②

$$T_2 = 0$$

$$V_2 = \frac{1}{2}kx_m^2$$

Position ①

$$T_1 = T_{\text{spere}} + T_{\text{fluid}} = \frac{1}{2}m_s v_m^2 + \frac{1}{4}\rho V v_m^2$$

$$V_1 = 0$$

Conservation of energy and simple harmonic motion.

$$T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2}m_s v_m^2 + \frac{1}{4}\rho V v_m^2 + 0 = 0 + \frac{1}{2}kx_m^2$$

$$v_m = \dot{x}_m = x_m\omega_n$$

$$\frac{1}{2}\left(m_s + \frac{1}{2}\rho V\right)x_m^2\omega_n^2 = \frac{1}{2}kx_m^2$$

$$\omega_n^2 = \frac{k}{m_s + \frac{1}{2}\rho V}$$

$$\omega_n^2 = \frac{500\ \text{N/m}}{(0.5\ \text{kg}) + \left(\frac{1}{2}\rho V\right)}$$

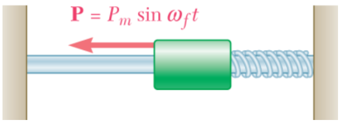
$$\frac{1}{2}\rho V = \frac{1}{2}(1000\ \text{kg/m}^3)\left(\frac{4}{3}\pi(0.08\ \text{m})^3\right)$$

$$\frac{1}{2}\rho V = 1.0723\ \text{kg}$$

$$\omega_n^2 = \frac{500\ \text{N/m}}{(0.5\ \text{kg}) + (1.0723\ \text{kg})} = 318\ \text{s}^{-2}$$

Period of vibration. $\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{318}}$ $\tau_n = 0.352$ s ◀

(*b*) Acceleration does not change mass. $\tau_n = 0.352$ s ◀



PROBLEM 19.99

A 4-kg collar can slide on a frictionless horizontal rod and is attached to a spring of constant 450 N/m. It is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $P_m = 13$ N. Determine the amplitude of the motion of the collar if (*a*) $\omega_f = 5$ rad/s, (*b*) $\omega_f = 10$ rad/s.

SOLUTION

Eq. (19.33)

$$x_m = \frac{\frac{P_m}{k}}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{450 \text{ N/m}}{4 \text{ kg}}} = 10.607 \text{ s}^{-1}$$

$$\frac{P_m}{k} = \frac{13 \text{ N}}{450 \text{ N/m}} = 0.28889 \text{ m}$$

$$x_m = \frac{0.28889 \text{ m}}{1 - \left(\frac{\omega_f}{10.607}\right)^2}$$

(*a*)

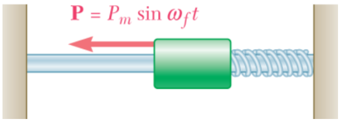
$$\omega_f = 5: \quad x_m = \frac{0.28889 \text{ m}}{1 - \left(\frac{5}{10.607}\right)^2} = 0.03714 \text{ m}$$

(In Phase) $x_m = 37.1$ mm ◀

(*b*)

$$\omega_f = 10: \quad x_m = \frac{0.28889 \text{ m}}{1 - \left(\frac{10}{10.607}\right)^2} = 0.25984 \text{ m}$$

(In Phase) $x_m = 260$ mm ◀



PROBLEM 19.100

A 4-kg collar can slide on a frictionless horizontal rod and is attached to a spring of constant k. It is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $P_m = 9$ N and $\omega_f = 5$ rad/s. Determine the value of the spring constant k knowing that the motion of the collar has an amplitude of 150 mm and is (*a*) in phase with the applied force, (*b*) out of phase with the applied force.

SOLUTION

Eq. (19.33)

$$x_m = \frac{\dfrac{P_m}{k}}{1 - \left(\dfrac{\omega_f}{\omega_n}\right)^2}, \quad \omega_n^2 = \frac{k}{m}$$

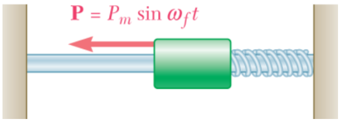
$$x_m = \frac{P_m}{\left(k - m\omega_f^2\right)}, \qquad \text{or} \qquad k = \frac{P_m}{x_m + m\omega_f^2}$$

(*a*) In phase

$$k = \frac{9 \text{ N}}{0.15 \text{ m} + (4 \text{ kg})\left(5 \text{ s}^{-1}\right)} = 160.0 \text{ N/m} \blacktriangleleft$$

(*b*) Out of phase $\qquad x_m = -0.15 \text{ m}$

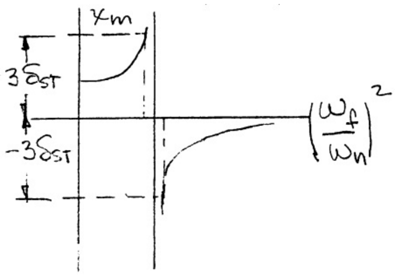
$$k = \frac{9 \text{ N}}{-0.15 \text{ m} + (4 \text{ kg})\left(5 \text{ s}^{-1}\right)} = 40.0 \text{ N/m} \blacktriangleleft$$



PROBLEM 19.101

A collar of mass m which slides on a frictionless horizontal rod is attached to a spring of constant k and is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$. Determine the range of values of ω_f for which the amplitude of the vibration exceeds three times the static deflection caused by a constant force of magnitude P_m.

SOLUTION



Eq. (19.33)

$$x_m = \frac{\frac{P_m}{k}}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

$$\frac{P_m}{k} = \delta_{st} \qquad \omega_n^2 = \frac{k}{m}$$

$$\frac{\delta_{st}}{1 - \frac{\omega_f^2}{\omega_n^2}} \geq 3\delta_{st}$$

$$1 - \frac{\omega_f^2}{\omega_n^2} \leq \frac{1}{3}$$

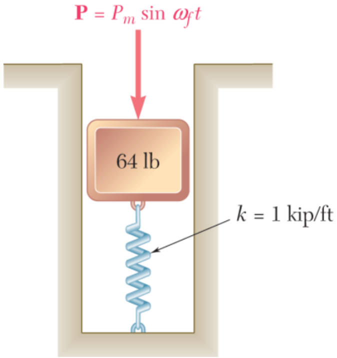
$$\frac{\omega_f^2}{\omega_n^2} > \frac{2}{3}$$

Also

$$\frac{\delta_{st}}{1 - \frac{\omega_f^2}{\omega_n^2}} < -3\delta_{st}$$

$$1 - \frac{\omega_f^2}{\omega_n^2} < -\frac{1}{3} \qquad \frac{\omega_f^2}{\omega_n^2} < \frac{4}{3}$$

$$\frac{2}{3} < \frac{\omega_f^2}{\omega_n^2} < \frac{4}{3} \quad \sqrt{\frac{2k}{3m}} < \omega_f < \sqrt{\frac{4k}{3m}} \blacktriangleleft$$



PROBLEM 19.102

A 64-lb block is attached to a spring of constant $k = 1$ kip/ft and can move without friction in a vertical slot as shown. It is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $\omega_f = 10$ rad/s. Knowing that the amplitude of the motion is 0.75 in., determine P_m.

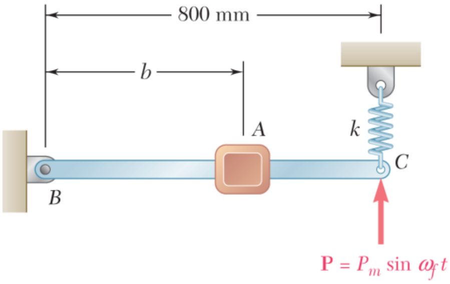
SOLUTION

$$x_m = \frac{\frac{P_m}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}, \quad P_m = x_m\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]k$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000 \text{ lb/ft}}{\frac{64 \text{ lb}}{32.2 \text{ ft/s}^2}}} = 22.43 \text{ s}^{-1}$$

$$P_m = \left(\frac{0.75 \text{ in.}}{12 \text{ in./ft}}\right)\left[1-\left(\frac{10 \text{ s}^{-1}}{22.43 \text{ s}^{-1}}\right)^2\right](1000 \text{ lb/ft}) = 50.077 \text{ lb}$$

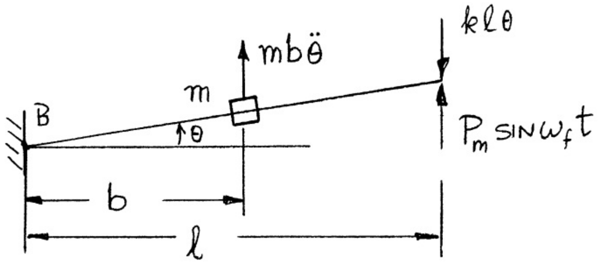
$P_m = 50.1$ lb ◀



PROBLEM 19.103

A small 20-kg block A is attached to the rod BC of negligible mass which is supported at B by a pin and bracket and at C by a spring of constant $k = 2$ kN/m. The system can move in a vertical plane and is in equilibrium when the rod is horizontal. The rod is acted upon at C by a periodic force **P** of magnitude $P = P_m \sin \omega_f t$, where $P_m = 6$ N. Knowing that $b = 200$ mm, determine the range of values of ω_f for which the amplitude of vibration of block A exceeds 3.5 mm.

SOLUTION



$$+\circlearrowleft \Sigma M_B = mb^2\ddot{\theta} = -kl^2\theta + P_m l \sin\omega_f t$$

$$mb^2\ddot{\theta} + kl^2\theta = P_m l \sin\omega_f t$$

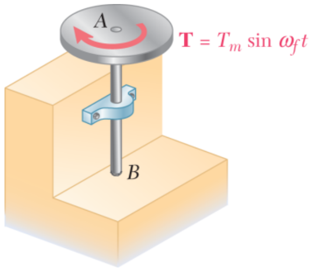
$$\omega_n = \sqrt{\frac{kl^2}{mb^2}} = 40 \text{ rad/s}, \ \theta = \theta_m \sin\omega_f t$$

$$\theta_m = \frac{\pm 3.5 \text{ mm}}{b} = \pm 0.0175 \text{ rad} = \frac{\frac{P_m l}{mb^2}}{\omega_n^2 - \omega_f^2} = \frac{6}{1600 - \omega_f^2}$$

Lower frequency: $6 = 0.0175\left(1600 - \omega_f^2\right), \quad \omega_f = 35.5 \text{ rad/s}$

Upper frequency: $6 = -0.0175\left(1600 - \omega_f^2\right), \quad \omega_f = 44.1 \text{ rad/s}$

$35.5 \text{ rad/s} < \omega_f < 44.1 \text{ rad/s}$ ◀



PROBLEM 19.104

An 8-kg uniform disk of radius 200 mm is welded to a vertical shaft with a fixed end at *B*. The disk rotates through an angle of 3° when a static couple of magnitude $50\text{ N}\cdot\text{m}$ is applied to it. If the disk is acted upon by a periodic torsional couple of magnitude $T = T_m \sin \omega_f t$, where $T_m = 60\text{ N}\cdot\text{m}$, determine the range of values of ω_f for which the amplitude of the vibration is less than the angle of rotation caused by a static couple of magnitude T_m.

SOLUTION

Mass moment of inertia:

$$\bar{I} = \frac{1}{2}mr^2 = \frac{1}{2}(8)(0.200)^2 = 0.16\text{ kg}\cdot\text{m}^2$$

Torsional spring constant:

$$K = \frac{T}{\theta}$$

$$T = 50\text{ N}\cdot\text{m}$$

$$\theta = 3° = 0.05236\text{ rad}$$

$$K = \frac{50}{0.05236}$$

$$= 954.93\text{ N}\cdot\text{m/rad}$$

Natural circular frequency:

$$\omega_n = \sqrt{\frac{K}{I}} = \sqrt{\frac{954.93}{0.16}} = 77.254\text{ rad/s}$$

For forced vibration,

$$\theta_m = \frac{\frac{T_m}{K}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{\theta_{\text{st}}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

For the amplitude $|\theta_m|$ to be less than θ_{st}, we must have $\omega_f > \omega_n$.

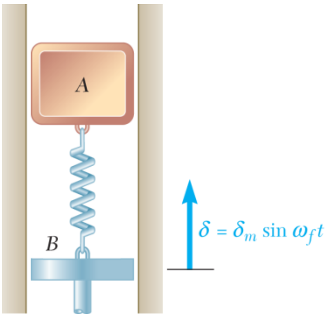
Then

$$|\theta_m| = \frac{\theta_{\text{st}}}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} < \theta_{\text{st}}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 - 1 > 1$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 > 2 \qquad \omega_f > \sqrt{2}\omega_n = (\sqrt{2})(77.254)$$

$$\omega_f > 109.3\text{ rad/s} \blacktriangleleft$$



PROBLEM 19.105

An 18-lb block A slides in a vertical frictionless slot and is connected to a moving support B by means of a spring AB of constant $k = 10$ lb/in. Knowing that the displacement of the support is $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 6$ in., determine the range of values of ω_f for which the amplitude of the fluctuating force exerted by the spring on the block is less than 30 lb.

SOLUTION

Natural circular frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10 \times 12}{\frac{18 \text{ lb}}{32.2}}} = 14.652 \text{ rad/s}$$

Eq. (19.33′):

$$x_m = \frac{\delta_m}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

Spring force:

$$F_m = -k(x_m - \delta_m) = -k\delta_m \left[1 - \frac{1}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}\right]$$

$$= k\delta_m \frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

$$= (120)(0.50) \frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} = 60 \frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

Limit on spring force: $|F_m| < 30$ lb

$$60 \left| \frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} \right| < 30 \quad \text{or} \quad \left| \frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} \right| < \frac{1}{2}$$

PROBLEM 19.105 (Continued)

In phase motion.

$$\frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} < \frac{1}{2}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 < \frac{1}{2} - \frac{1}{2}\left(\frac{\omega_f}{\omega_n}\right)^2$$

$$\frac{3}{2}\left(\frac{\omega_f}{\omega_n}\right)^2 < \frac{1}{2} \qquad \frac{\omega_f}{\omega_n} > \frac{1}{3}$$

$$\omega_f < \frac{1}{\sqrt{3}}\omega_n$$

$\omega_f < 8.46$ rad/s ◀

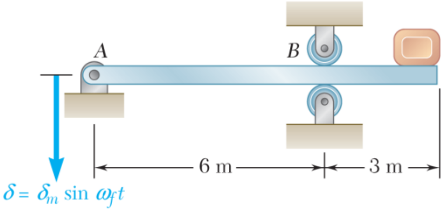
Out of phase motion.

$$\frac{\left(\frac{\omega_f}{\omega_n}\right)^2}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} < \frac{1}{2}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 < \frac{1}{2}\left(\frac{\omega_f}{\omega_n}\right)^2 - \frac{1}{2}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 < -\frac{1}{2}$$

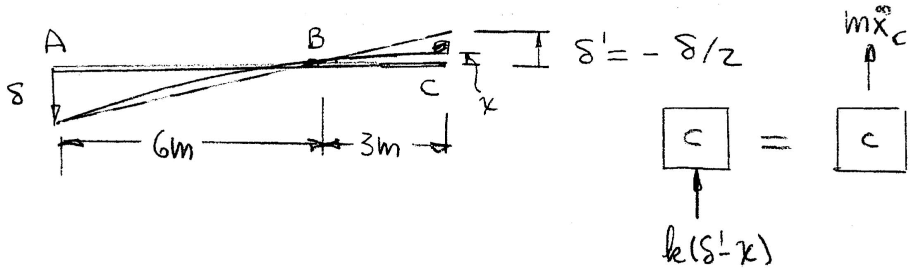
No solution for ω_f.



PROBLEM 19.106

A beam *ABC* is supported by a pin connection at *A* and by rollers at *B*. A 120-kg block placed on the end of the beam causes a static deflection of 15 mm at *C*. Assuming that the support at *A* undergoes a vertical periodic displacement $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 10$ mm and $\omega_f = 18$ rad/s, and the support at *B* does not move, determine the maximum acceleration of the block at *C*. Neglect the weight of the beam and assume that the block does not leave the beam.

SOLUTION



$$\Sigma F = ma$$

$$m\ddot{x}_c = k\left(\delta' - x_c\right) \qquad \delta' = -\frac{\delta}{2}$$

$$\ddot{x}_c + \frac{k}{m}x_c = -k\frac{\delta}{2} = -k\frac{\delta_m}{2}\sin\omega_f t$$

From Equation (19.31 and 19.33′)

$$\left(x_c\right)_m = \frac{-\dfrac{\delta_m}{2}}{1 - \dfrac{\omega_f^2}{\omega_n^2}}$$

Thus

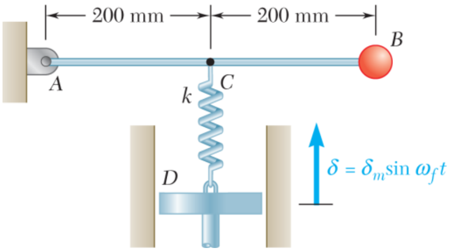
$$\left(x_c\right)_m = \frac{-\dfrac{0.010}{2}\text{ m}}{1 - \dfrac{(18)^2}{654}} \qquad \omega_n^2 = \frac{g}{\left(\delta_{ST}\right)_c} = \frac{9.81 \text{ m/s}^2}{0.015 \text{ m}}$$

$$\left(x_c\right)_m = -0.009909 \text{ m} \qquad \omega_n^2 = 654 \text{ s}^{-2}$$

$$X_c = \left(x_c\right)_m \sin\omega_f t$$

$$\ddot{X}_c = \left(a_c\right)_m = -\left(x_c\right)_m \omega_f^2 \qquad \left(a_c\right)_m = \left(0.009909 \text{ m}\right)\left(18 \text{ s}^{-1}\right)^2$$

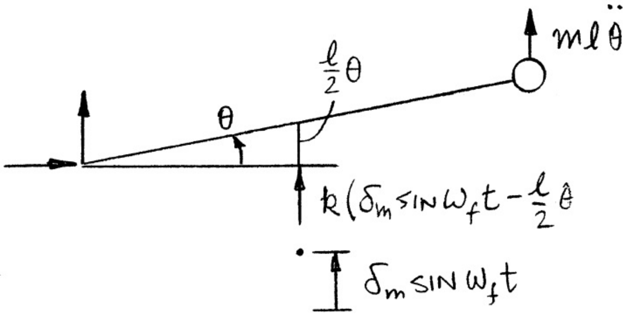
$$\left(a_c\right)_m = 3.21 \text{ m/s}^2 \blacktriangleleft$$


200 mm
200 mm
B
A
C
k
D
$\delta = \delta_m \sin \omega_f t$

PROBLEM 19.107

A small 2-kg sphere *B* is attached to the bar *AB* of negligible mass which is supported at *A* by a pin and bracket and connected at *C* to a moving support *D* by means of a spring of constant $k = 3.6$ kN/m. Knowing that support *D* undergoes a vertical displacement $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 3$ mm and $\omega_f = 15$ rad/s, determine (*a*) the magnitude of the maximum angular velocity of bar *AB*, (*b*) the magnitude of the maximum acceleration of sphere *B*.

SOLUTION


$ml\ddot{\theta}$
$\frac{l}{2}\theta$
θ
$k(\delta_m \sin \omega_f t - \frac{l}{2}\theta$
$\delta_m \sin \omega_f t$

$$+\circlearrowleft \Sigma M_A;\; ml^2\ddot{\theta} = \frac{l}{2}\left(k\delta_m \sin \omega_f t - \frac{kl}{2}\theta\right)$$

$$ml^2\ddot{\theta} + \frac{kl^2}{4}\theta = \frac{kl\delta_m}{2}\sin \omega_f t, \qquad \theta = \theta_m \sin \omega_f t$$

So

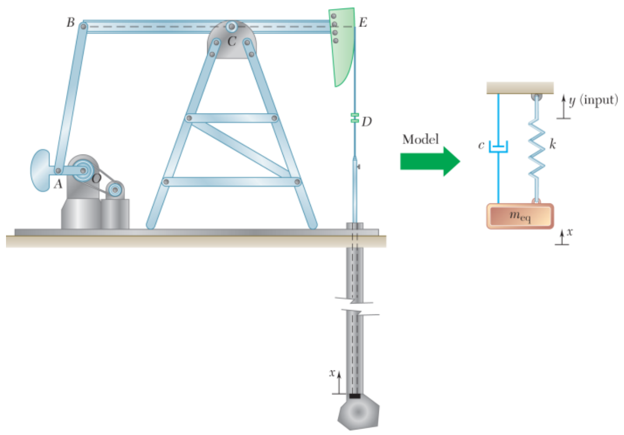
$$\omega_n^2 = \frac{k}{4m} = \frac{(3600\text{ N/m})}{4(2\text{ kg})} = 450\text{ s}^{-2}$$

$$\theta_m = \frac{\frac{k\delta_m}{2ml}}{\omega_n^2 - \omega_f^2} = \frac{\frac{(3600\text{ N/m})(0.003\text{ m})}{2(2\text{ kg})(0.4\text{ m})}}{(450 - 225)\text{s}^{-2}}$$

$$= 0.03\text{ rad} = 1.719°$$

(*a*) $\omega_f \theta_m = 0.450$ rad/s ◀

(*b*) $l\omega_f^2 \theta_m = 2.70\text{ m/s}^2$ ◀



PROBLEM 19.108

The crude-oil-pumping rig shown in the accompanying figure is driven at 20 rpm. The inside diameter of the well pipe is 2 in., and the diameter of the pump rod is 0.75 in. The length of the pump rod and the length of the column of oil lifted during the stroke are essentially the same, and equal to 6000 ft. During the downward stroke, a valve at the lower end of the pump rod opens to let a quantity of oil into the well pipe, and the column of oil is then lifted to obtain a discharge into the connecting pipeline. Thus, the amount of oil pumped in a given time depends upon the stroke of the lower end of the pump rod. Knowing that the upper end of the rod at *D* is essentially sinusoidal with a stroke of 45 in. and the specific weight of crude oil is 56.2 lb/ft^3, determine (*a*) the output of the well in ft^3/min if the shaft is rigid, (*b*) the output of the well in ft^3/min if the stiffness of the rod is 2210 N/m, the equivalent mass of the oil and shaft is 290 kg and damping is negligible.

SOLUTION

Forcing frequency: $\omega_f = 20 \text{ rpm} = 2.0944 \text{ rad/s}$

Cross sectional area of the flow chamber

$$A_{\text{oil}} = \frac{\pi}{4}\left[(2 \text{ in.})^2 - (0.75 \text{ in.})^2\right] = 2.6998 \text{ in}^2 = 0.018749 \text{ ft}^2$$

Let *s* be the stroke at the lower end of the pump in feet. Stroke is twice the amplitude. $s = 2x_m$

Volume of oil pumped per revolution:

$$V_{\text{oil}} = A_{\text{oil}}s = 0.018749s$$

Amplitude of motion at top of shaft:

$$\delta_m = \frac{1}{2}(45 \text{ in.}) = 22.5 \text{ in.} = 1.875 \text{ ft}$$

Amplitude of motion at bottom of shaft:

$$x_m = \frac{\delta_m}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

(*a*) Rigid shaft:

$$\omega_n = \infty$$

$$x_m = \delta_m = 1.875 \text{ ft}$$

$$s = (2)(1.875) = 3.75 \text{ ft}$$

$$V_{\text{oil}} = (0.018749 \text{ ft}^2)(3.75 \text{ ft}) = 0.070309 \text{ ft}^3/\text{rev}$$

PROBLEM 19.108 (Continued)

output rate: $(0.070309\ \text{ft}^3/\text{rev})(20\ \text{rev/min})$ $1.406\ \text{ft}^3/\text{min}$ ◀

(*b*) Flexible shaft.

$$k = 2210\ \text{N/m} \qquad m_{\text{eq}} = 290\ \text{kg}$$

$$\omega_n = \sqrt{\frac{k}{m_{\text{eq}}}} = \sqrt{\frac{2210\ \text{N/m}}{290\ \text{kg}}} = 2.7606\ \text{rad/s}$$

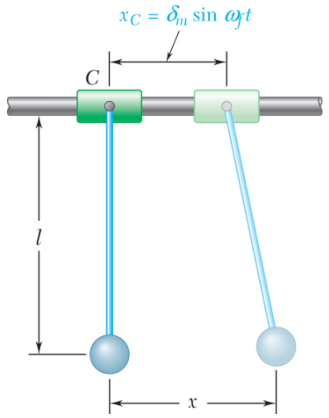
$$\frac{\omega_f}{\omega_n} = \frac{2.0944}{2.7606} = 0.75869$$

$$x_m = \frac{1.875}{1-(0.75869)^2} = 4.4178\ \text{ft}$$

$$s = (2)(4.4178) = 8.8358\ \text{ft}$$

$$V_{\text{oil}} = (0.018749\ \text{ft}^2)(8.8358\ \text{ft}) = 0.16566\ \text{ft}^3/\text{rev}$$

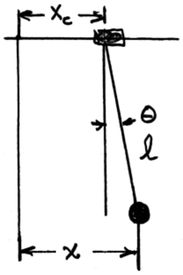
output rate: $(0.16566\ \text{ft}^3/\text{rev})(20\ \text{rev/min})$ $3.31\ \text{ft}^3/\text{min}$ ◀



PROBLEM 19.109

A simple pendulum of length l is suspended from a collar C which is forced to move horizontally according to the relation $x_C = \delta_m \sin \omega_f t$. Determine the range of values of ω_f for which the amplitude of the motion of the bob is less than δ_m. (assume that δ_m is small compared with the length l of the pendulum).

SOLUTION



Geometry.

$$x = x_C + l \sin\theta$$

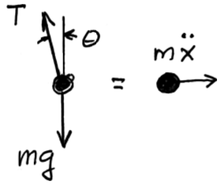
$$\sin\theta = \frac{x - x_C}{l}$$

$$+\!\uparrow \Sigma F_y = ma_y \approx 0: \quad T\cos\theta - mg = 0 \qquad T \approx mg$$

$$\xrightarrow{+} \Sigma F_x = ma_x: \qquad -T\sin\theta = m\ddot{x}$$

$$m\ddot{x} + \frac{mg(x - x_C)}{l} = 0$$

$$\ddot{x} + \frac{g}{l}x = \frac{g}{l}x_C$$



Using the given motion of x_C,

$$\ddot{x} + \frac{g}{l}x = \frac{g}{l}\delta_m \sin\omega_f t$$

Circular natural frequency.

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$\ddot{x} + \omega_n^2 x = \omega_n^2 \delta_m \sin\omega_f t$$

The steady state response is

$$x_m = \frac{\delta_m}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

$$x_m^2 = \frac{\delta_m^2}{\left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right]^2} \leq \delta_m^2$$

Consider

$$x_m^2 = \delta_m^2.$$

PROBLEM 19.109 (Continued)

Then $$\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 = 1-2\left(\frac{\omega_f}{\omega_n}\right)^2+\left(\frac{\omega_f}{\omega_n}\right)^4 = 1$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 = 0 \quad \text{and} \quad \left(\frac{\omega_f}{\omega_n}\right)^2 = 2$$

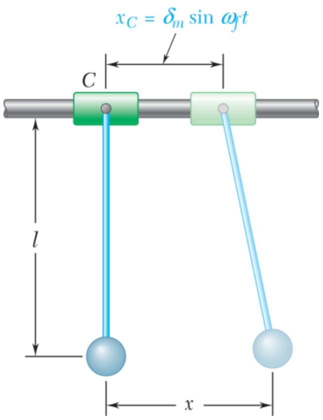
$$\left(\frac{\omega_f}{\omega_n}\right) = 0 \quad \text{and} \quad \frac{\omega_f}{\omega_n} = \sqrt{2}$$

For $$0 < \frac{\omega_f}{\omega_n} < \sqrt{2}, \quad |x_m| > \delta_m$$

For $$\frac{\omega_f}{\omega_n} > \sqrt{2}, \quad |x_m| < \delta_n$$

Then $$\omega_f > \sqrt{2}\,\omega_n = \sqrt{\frac{2g}{l}}$$

$$\omega_f > \sqrt{\frac{2g}{l}} \blacktriangleleft$$

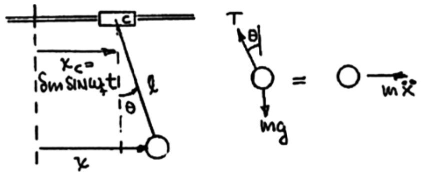


PROBLEM 19.110

The 2.75-lb bob of a simple pendulum of length $l = 24$ in. is suspended from a 3-lb collar C. The collar is forced to move according to the relation $x_C = \delta_m \sin \omega_f t$, with an amplitude $\delta_m = 0.4$ in. and a frequency $f_f = 0.5$ Hz. Determine (*a*) the amplitude of the motion of the bob, (*b*) the force that must be applied to collar C to maintain the motion.

SOLUTION

(*a*)



$$\Sigma F_x = ma_x$$
$$-T\sin\theta = m\ddot{x}$$
$$\Sigma F_y = T\cos\theta - mg = 0$$

For small angles $\cos\theta \approx 1$. Acceleration in the y direction is second order and is neglected.

$$T = mg$$
$$m\ddot{x} = -mg\sin\theta$$
$$\sin\theta = \frac{x - x_c}{l}$$
$$m\ddot{x} + \frac{mg}{l}x = \frac{g}{l}x_c = \frac{mg}{l}\delta_m \sin\omega_f t$$
$$\omega_n^2 = \frac{g}{l}$$
$$\ddot{x} + \omega_n^2 x = \omega_n^2 \delta_m \sin\omega_f t$$

From Equation (19.33′):

$$x_m = \frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

So $\quad \omega_f^2 = (2\pi f_f)^2 = 4\pi^2(0.5)^2 = \pi^2 \ s^{-2}$

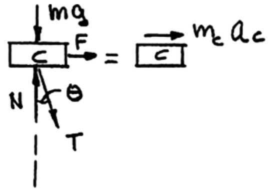
$$\omega_n^2 = \frac{g}{l} = \frac{32.2 \text{ ft/s}^2}{2 \text{ ft}} = 16.1 \text{ s}^{-2}$$

$$x_m = \frac{\frac{0.4}{12}\text{ ft}}{1 - \frac{\pi^2}{16.1}} = 0.086137 \text{ ft}$$

$x_m = 1.034$ in. ◀

PROBLEM 19.110 (Continued)

(*b*)



$$a_c = \ddot{x}_c = -\delta_m \omega_f^2 \sin \omega_f t$$

$$\xrightarrow{+} \Sigma F_x = m_c a_c$$

$$F - T\sin\theta = m_c a_c$$

From Part (*a*): $$T = mg, \quad \sin\theta = \frac{x - x_c}{l}$$

Thus, $$F = -mg\left[\frac{x - x_c}{l}\right] + m_c \ddot{x}_c$$

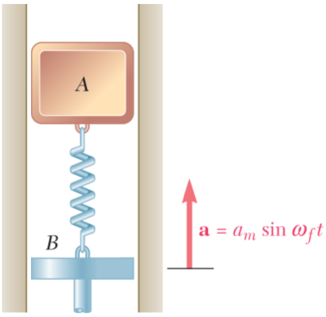
$$= -m\omega_n^2 x + m\omega_n^2 x_c + m_c \ddot{x}_c$$

$$= -m\omega_n^2 x_m \sin\omega_f t + m\omega_n^2 \delta_m \sin\omega_f t - m_c \omega_f^2 \delta_m \sin\omega_f t$$

$$= \left[-\left(\frac{2.75 \text{ lb}}{32.2}\right)(16.1)(0.086137) + \left(\frac{2.75 \text{ lb}}{32.2}\right)(16.1)\left(\tfrac{0.4}{12}\right) - \left(\frac{3 \text{ lb}}{32.2}\right)\pi^2\left(\tfrac{0.4}{12}\right)\right]\sin\pi t$$

$$= -0.10326 \sin\pi t$$

$F = -0.1033 \sin\pi t$ (lb) ◀

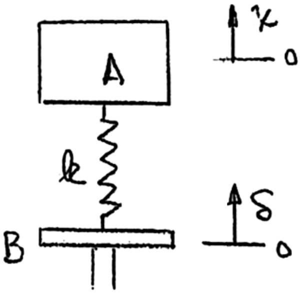


PROBLEM 19.111

An 18-lb block A slides in a vertical frictionless slot and is connected to a moving support B by means of a spring AB of constant $k = 8$ lb/ft. Knowing that the acceleration of the support is $a = a_m \sin \omega_f t$, where $a_m = 5 \text{ ft/s}^2$ and $\omega_f = 6$ rad/s, determine (*a*) the maximum displacement of block A, (*b*) the amplitude of the fluctuating force exerted by the spring on the block.

SOLUTION

(*a*) Support motion.



$$a = \ddot{\delta} = a_m \sin \omega_f t$$

$$\delta = -\left(\frac{a_m}{\omega_f^2}\right)\sin \omega_f t$$

$$\delta_m = \frac{-a_m}{\omega_f^2} = -\frac{5 \text{ ft/s}^2}{(6 \text{ rad/s})^2} = -0.13889 \text{ ft}$$

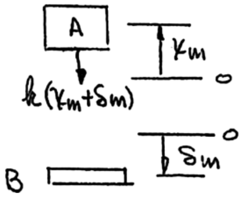
From Equations (19.31 and 19.33′):

$$x_m = \frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}} \qquad \omega_n^2 = \frac{k}{m} = \frac{8 \text{ lb/ft}}{\frac{18}{32.2}} = 14.311 \text{ (rad/s)}^2$$

$$x_m = \frac{-0.13889}{1 - \left(\frac{36}{14.311}\right)} = 0.091643 \text{ ft}$$

$x_m = 1.100$ in. ◀

(*b*) x is out of phase with δ for $\omega_f = 6$ rad/s.

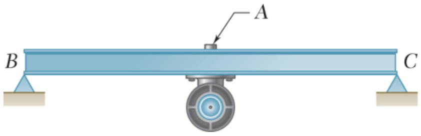


Thus,

$$F_m = k(x_m + \delta_m) = 8 \text{ lb/ft}(0.091643 \text{ ft} + 0.13889 \text{ ft})$$

$$= 1.8443 \text{ lb}$$

$F_m = 1.844$ lb ◀



PROBLEM 19.112

A variable-speed motor is rigidly attached to a beam *BC*. When the speed of the motor is less than 600 rpm or more than 1200 rpm, a small object placed at *A* is observed to remain in contact with the beam. For speeds between 600 and 1200 rpm the object is observed to "dance" and actually to lose contact with the beam. Determine the speed at which resonance will occur.

SOLUTION

Let m be the unbalanced mass and $\bar{r}$ the eccentricity of the unbalanced mass. The vertical force exerted on the beam due to the rotating unbalanced mass is

$$P = m\bar{r}\omega_f^2 \sin\omega_f t = P_m \sin\omega_f t$$

Then from Eq. 19.33,

$$x_m = \frac{\frac{P_m}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{\frac{m\bar{r}\omega_f^2}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

For simple harmonic motion, the acceleration is

$$a_m = -\omega_f^2 x_m = \frac{\frac{m\bar{r}\omega_f^4}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

When the object loses contact with the beam, the acceleration $|a_m|$ is greater than g.
Let $\omega_1 = 600 \text{ rpm} = 62.832 \text{ rad/s}$.

$$|a_m|_1 = \frac{\frac{m\bar{r}\omega_1^4}{k}}{1-\left(\frac{\omega_1}{\omega_n}\right)^2} = \frac{\frac{mr\omega_n^4 U^4}{k}}{1-U^2} \qquad (1)$$

where $U = \dfrac{\omega_1}{\omega_n}$.

Let $\omega_2 = 1200 \text{ rpm} = 125.664 \text{ rad/s} = 2\omega_1$

$$|a_m|_2 = \frac{\frac{m\bar{r}\omega_2^4}{k}}{\left(\frac{\omega_2}{\omega_n}\right)^2 - 1} = \frac{\frac{mr\omega_n^4 (2U)^4}{k}}{4U^2 - 1} \qquad (2)$$

PROBLEM 19.112 (Continued)

Dividing Eq. (1) by Eq. (2),

$$1 = \frac{4U^2 - 1}{16(1-U^2)} \quad \text{or} \quad 16 - 16U^2 = 4U^2 - 1$$

$$20U^2 = 17 \qquad U = \sqrt{\frac{17}{20}}$$

$$\frac{\omega_1}{\omega_n} = \sqrt{\frac{17}{20}} \qquad \omega_n = \sqrt{\frac{20}{17}}\omega_1 = 1.08465\,\omega_1$$

$$\omega_n = (1.08465)(600\text{ rpm}) \qquad \omega_n = 651\text{ rpm} \blacktriangleleft$$

PROBLEM 19.113

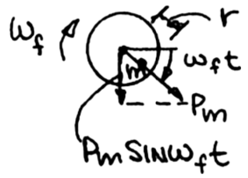
A motor of mass M is supported by springs with an equivalent spring constant k. The unbalance of its rotor is equivalent to a mass m located at a distance r from the axis of rotation. Show that when the angular velocity of the motor is ω_f, the amplitude x_m of the motion of the motor is

$$x_m = \frac{r\left(\frac{m}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

where $\omega_n = \sqrt{\frac{k}{M}}$.

SOLUTION

Rotor



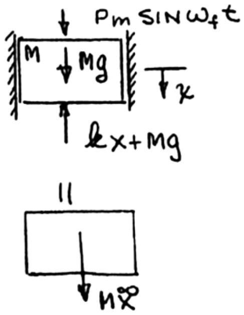
$$+\!\downarrow \Sigma F = ma \quad P_m \sin\omega_f t - kx = M\ddot{x}$$

$$M\ddot{x} + kx = P_m \sin\omega_f t$$

$$\ddot{x} + \frac{k}{M}x = \frac{P_m}{M}\sin\omega_f t$$

$$\omega_n^2 = \frac{k}{M}$$

Motor



From Equation (19.33):

$$x_m = \frac{\frac{P_m}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

But

$$\frac{P_m}{k} = \frac{mr\omega_f^2}{k} \qquad k = M\omega_n^2$$

$$\frac{P_m}{k} = r\left(\frac{m}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2$$

Thus,

$$x_m = \frac{r\left(\frac{m}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} \quad \text{Q.E.D.} \blacktriangleleft$$

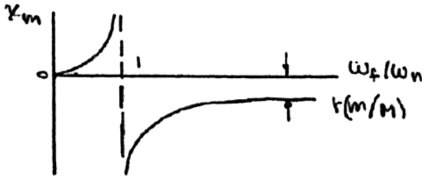
PROBLEM 19.114

As the rotational speed of a spring-supported 100-kg motor is increased, the amplitude of the vibration due to the unbalance of its 15-kg rotor first increases and then decreases. It is observed that as very high speeds are reached, the amplitude of the vibration approaches 3.3 mm. Determine the distance between the mass center of the rotor and its axis of rotation. (*Hint:* Use the formula derived in Problem 19.113.)

SOLUTION

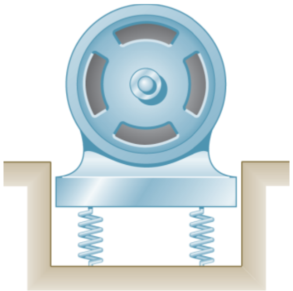
Use the equation derived in Problem 19.113.

$$x_m = \frac{r\left(\frac{m}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{r\left(\frac{m}{M}\right)}{\frac{1}{\left(\frac{\omega_f}{\omega_n}\right)^2}-1}$$



For very high speeds, $\frac{1}{\left(\frac{\omega_f}{\omega_n}\right)^2} \longrightarrow 0$ and $x_m \longrightarrow \frac{rm}{M}$,

thus, $3.3 \text{ mm} = r\left(\frac{15}{100}\right)$ $r = 22 \text{ mm}$ ◀



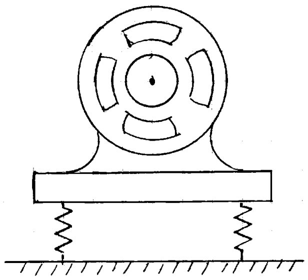
PROBLEM 19.115

A motor of weight 40 lb is supported by four springs, each of constant 225 lb/in. The motor is constrained to move vertically, and the amplitude of its motion is observed to be 0.05 in. at a speed of 1200 rpm. Knowing that the weight of the rotor is 9 lb, determine the distance between the mass center of the rotor and the axis of the shaft.

SOLUTION

W = 40 lb

Four springs each of constant 225 lb/in.



We note that the motor is constrained to move vertically.

$$4(225 \text{ lb/in.}) = 900 \text{ lb/in.} = 10800 \text{ lb/ft}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10800 \text{ lb/ft}}{(40 \text{ lb}/32.2)}} = 93.242 \text{ rad/s}$$

For $\omega = 1200 \text{ rpm} = 125.664 \text{ rad/s}$ we have

$$x_m = 0.05 \text{ in.} = 4.1667 \times 10^{-3} \text{ ft}$$

Eq. (19.33): $$x_m = \frac{\frac{P_m}{k}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

Thus: $$\frac{P_m}{k} = x_m\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]$$

$$= (4.1667 \times 10^{-3} \text{ ft})\left[1 - \left(\frac{125.664}{93.242}\right)^2\right] = -3.4015 \times 10^{-3} \text{ ft (out of phase)}$$

$$P_m = (10800 \text{ lb/ft})(3.4015 \times 10^{-3} \text{ ft}) = 36.736 \text{ lb}$$

We have found: $P_m = 36.736$ lb

For an unbalanced rotor of weight $W_R = 9$ lb, rotating at $\omega = 1200 \text{ rpm} = 125.664 \text{ rad/s}$, with the mass center at a distance $\overline{r}$ from the axis of rotation, we have,

$$P_m = m_R \overline{r} \omega^2$$

$$\overline{r} = \frac{P_m}{m_R \omega^2} = \frac{36.736 \text{ lb}}{(9 \text{ lb}/32.2)(125.664 \text{ rad/s})^2} = 8.3231 \times 10^{-3} \text{ ft}$$

$\overline{r} = 0.0999$ in. ◀

PROBLEM 19.116

A motor weighing 400 lb is supported by springs having a total constant of 1200 lb/in. The unbalance of the rotor is equivalent to a 1-oz weight located 8 in. from the axis of rotation. Determine the range of allowable values of the motor speed if the amplitude of the vibration is not to exceed 0.06 in.

SOLUTION

Let M = mass of motor, m = unbalance mass, r = eccentricity

$$M = \frac{400}{32.2} = 12.4224 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m = \left(\frac{1}{16}\right)\left(\frac{1}{32.2}\right) = 0.001941 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$r = 8 \text{ in.} = 0.66667 \text{ ft} \quad k = 1200 \text{ lb/in.} = 14{,}400 \text{ lb/ft}$$

Natural circular frequency:

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{14{,}400}{12.4224}} = 34.047 \text{ rad/s}$$

$$\frac{rm}{M} = \frac{(0.66667)(0.001941)}{12.4224}$$

$$= 0.00014017 \text{ ft} = 0.00125 \text{ in.}$$

From the derivation given in Problem 19.113,

$$x_m = \frac{\left(\frac{rm}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{0.00125\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} \text{ in.}$$

In phase motion with $|x_m| < 0.06$ in.

$$\frac{0.00125\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} < 0.06$$

$$0.00125\left(\frac{\omega_f}{\omega_n}\right)^2\left(\frac{\omega_f}{\omega_n}\right)^2 < 0.06 - 0.06\left(\frac{\omega_f}{\omega_n}\right)^2$$

$$0.06125\left(\frac{\omega_f}{\omega_n}\right)^2 < 0.06$$

$$\frac{\omega_f}{\omega_n} < \sqrt{\frac{0.06}{0.06125}} = 0.98974$$

$$\omega_f < (0.98974)(34.047) = 33.698 \text{ rad/s} \qquad \omega_f < 322 \text{ rpm} \blacktriangleleft$$

PROBLEM 19.116 (Continued)

Out of phase motion with $|x_m| = 0.06$ in.

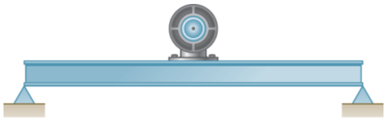
$$\frac{0.00125\left(\frac{\omega_f}{\omega_n}\right)^2}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} < 0.06$$

$$0.00125\left(\frac{\omega_f}{\omega_n}\right)^2 < 0.06\left(\frac{\omega_f}{\omega_n}\right)^2 - 0.06$$

$$0.06 < 0.05875\left(\frac{\omega_f}{\omega_n}\right)^2$$

$$\frac{\omega_f}{\omega_n} > \sqrt{\frac{0.06}{0.05875}} = 1.01058$$

$$\omega_f > (1.01058)(34.047) = 34.407 \text{ rad/s} \qquad \omega_f > 329 \text{ rpm} \blacktriangleleft$$



PROBLEM 19.117

A 180-kg motor is bolted to a light horizontal beam. The unbalance of its rotor is equivalent to a 28-g mass located 150 mm from the axis of rotation, and the static deflection of the beam due to the weight of the motor is 12 mm. The amplitude of the vibration due to the unbalance can be decreased by adding a plate to the base of the motor. If the amplitude of vibration is to be less than 60 μm for motor speeds above 300 rpm, determine the required mass of the plate.

SOLUTION

Before the plate is added, $M_1 = 180\text{ kg}, \quad m = 28\times10^{-3}\text{ kg}$

$$r = 150\text{ mm} = 0.150\text{ m}$$

Equivalent spring constant: $k = \dfrac{W_1}{\delta_{\text{st}}} = \dfrac{M_1 g}{\delta_{\text{st}}}$

$$k = \frac{(180)(9.81)}{12\times10^{-3}} = 147.15\times10^3\text{ N/m}$$

Let M_2 be the mass of motor plus the plate.

Natural circular frequency. $\omega_n = \sqrt{\dfrac{k}{M_2}}$

Forcing frequency: $\omega_f = 300\text{ rpm} = 31.416\text{ rad/s}$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 = \frac{\omega_f^2 M_2}{k} = \frac{(31.416)^2 M_2}{147.15\times10^3} = 0.006707M_2$$

From the derivation in Problem 19.113,

$$x_m = \frac{\left(\frac{rm}{M_2}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

For out of phase motion with $x_m = -60\times10^{-6}\text{ m},$

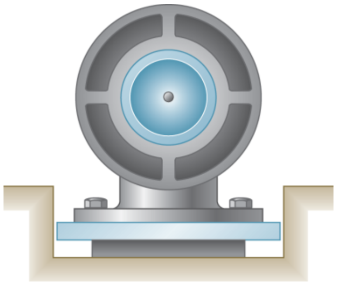
$$-60\times10^{-6} = \frac{\left[\frac{(0.150)(28\times10^{-3})}{M_2}\right](0.006707M_2)}{1-0.006707M_2}$$

$$-60\times10^{-6} + (60\times10^{-6})(0.006707)M_2 = 28.170\times10^{-6}$$

$$402.49\times10^{-9}M_2 = 88.170\times10^{-6}$$

$$M_2 = 219.10\text{ kg}$$

Added mass: $\Delta M = M_2 - M_1 = 219.10 - 180$ $\quad \Delta M = 39.1\text{ kg}$ ◀



PROBLEM 19.118

The unbalance of the rotor of a 400-lb motor is equivalent to a 3-oz weight located 6 in. from the axis of rotation. In order to limit to 0.2 lb the amplitude of the fluctuating force exerted on the foundation when the motor is run at speeds of 100 rpm and above, a pad is to be placed between the motor and the foundation. Determine (*a*) the maximum allowable spring constant *k* of the pad, (*b*) the corresponding amplitude of the fluctuating force exerted on the foundation when the motor is run at 200 rpm.

SOLUTION

Mass of motor.
$$M = \frac{400}{32.2} = 12.422 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Unbalance mass.
$$m = \left(\frac{3}{16}\right)\left(\frac{1}{32.2}\right) = 0.005823 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Eccentricity.
$$r = 6 \text{ in.} = 0.5 \text{ ft}$$

Equation of motion:
$$M\ddot{x} + kx = P_m \sin \omega_f t = mr\omega_f^2 \sin \omega_f t$$

$$(-M\omega_f^2 + k)x_m = mr\omega_f^2$$

$$x_m = \frac{mr\omega_f^2}{k - M\omega_f^2}$$

Transmitted force.
$$F_m = kx_m = \frac{kmr\omega_f^2}{k - M\omega_f^2}$$

For out of phase motion,
$$|F_m| = \frac{kmr\omega_f^2}{M\omega_f^2 - k} \qquad (1)$$

(*a*) Required value of *k*.

Solve Eq. (1) for *k*.
$$|F_m|(M\omega_f^2 - k) = kmr\omega_f^2$$

$$k(mr\omega_f^2 + |F_m|) = |F_m| M\omega_f^2$$

$$k = \frac{|F_m| M\omega_f^2}{mr\omega_f^2 + |F_m|}$$

Data: $|F_m| = 0.2$ lb $\quad \omega_f = 100 \text{ rpm} = 10.472 \text{ rad/s}$

$$k = \frac{(0.2)(12.422)(10.472)^2}{(0.005823)(0.5)(10.472)^2 + 0.2} = 524.65$$

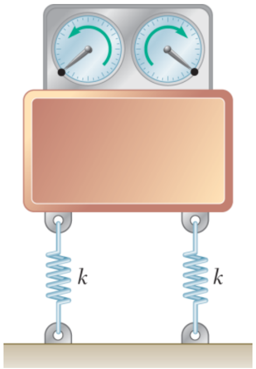
$k = 525$ lb/ft ◀

PROBLEM 19.118 (Continued)

(*b*) Force amplitude at 200 rpm. $\omega_f = 20.944\text{ rad/s}$

From Eq. (1), $$|F_m| = \frac{(524.65)(0.005823)(0.5)(20.944)^2}{(12.422)(20.944)^2 - 524.65}$$

$$|F_m| = 0.1361\text{ lb} \blacktriangleleft$$



PROBLEM 19.119

A counter-rotating eccentric mass exciter consisting of two rotating 100-g masses describing circles of radius *r* at the same speed but in opposite senses is placed on a machine element to induce a steady-state vibration of the element. The total mass of the system is 300 kg, the constant of each spring is $k = 600$ kN/m, and the rotational speed of the exciter is 1200 rpm. Knowing that the amplitude of the total fluctuating force exerted on the foundation is 160 N, determine the radius *r*.

SOLUTION

$$P_m = 2mr\omega_f^2, \qquad x_m = \frac{\frac{2mr\omega_f^2}{2k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}, \qquad \omega_n^2 = \frac{2k}{M}$$

With
$$2kx_m = 160\text{ N} = \pm\frac{2mr\omega_f^2}{1-\frac{M\omega_f^2}{2k}}, \qquad \omega_f = 40\pi\text{ rad/s}$$

Solving for *r*,

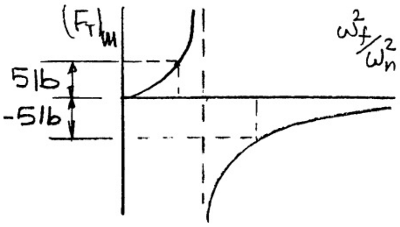
$$r = \pm\frac{160\text{ N}\left[1-\frac{(300\text{ kg})\left(40\pi\text{ s}^{-1}\right)^2}{1200000\text{ N/m}}\right]}{2(0.1\text{ kg})(40\pi\text{ s}^{-1})^2} = 0.1493\text{ m}$$

$r = 149.3$ mm ◀

PROBLEM 19.120

A 360-lb motor is supported by springs of total constant 12.5 kips/ft. The unbalance of the rotor is equivalent to a 0.9-oz weight located 7.5 in. from the axis of rotation. Determine the range of speeds of the motor for which the amplitude of the fluctuating force exerted on the foundation is less than 5 lb.

SOLUTION



From Problem 19.113 $$x_m = \frac{r\left(\frac{m}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

And $$(F_T)_m = kx_m, \quad \frac{k}{M} = \omega_n^2, \quad (F_T)_m = \frac{rm\omega_f^2}{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]}$$

Then $$rm = \left(\frac{7.5}{12}\text{ ft}\right)\left(\frac{\left(\frac{0.9}{16}\text{ lb}\right)}{32.2\text{ ft/s}^2}\right) = 0.0010918\text{ lb}\cdot\text{s}^2$$

$$\omega_n^2 = \frac{k}{M} = \frac{12500\text{ lb/ft}}{\frac{360\text{ lb}}{32.2\text{ ft/s}^2}} = 1118.1\text{ s}^{-2}$$

$$(F_T)_m = (0.0010918\text{ lb}\cdot\text{s}^2)\frac{\omega_f^2}{1-\frac{\omega_f^2}{1118.1}}$$

or $$(F_T)_m\left[1-\frac{\omega_f^2}{1118.1}\right] = (0.0010918\text{ lb}\cdot\text{s}^2)\omega_f^2$$

$$(F_T)_m = \left[\frac{(F_T)_m}{1118.1} + (0.0010918)\right]\omega_f^2$$

Then $$\omega_f^2 = \frac{1118.1(F_T)_m}{(F_T)_m + 1.2207}$$

(*a*) $$(F_T)_m = +5: \quad \omega_f^2 = \frac{1118.1(5)}{5+1.2207} = 898.69\text{ s}^{-2},$$

$$\omega_f \le 29.978\text{ rad/s}$$

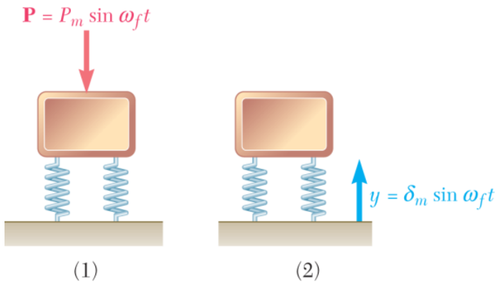
$$\le 286.26\text{ rpm} \qquad \omega_f \le 286\text{ rpm} \blacktriangleleft$$

(*b*) $$(F_T)_m = -5: \quad \omega_f^2 = \frac{1118.1\,(-5)}{-5+1.2207} = 1479.2\text{ s}^{-2},$$

$$\omega_f > 38.461\text{ rad/s}$$

$$> 367.27\text{ rpm} \qquad \omega_f > 367\text{ rpm} \blacktriangleleft$$

PROBLEM 19.121



$\mathbf{P} = P_m \sin \omega_f t$
$y = \delta_m \sin \omega_f t$
(1)
(2)

Figures (1) and (2) show how springs can be used to support a block in two different situations. In Figure (1), they help decrease the amplitude of the fluctuating force transmitted by the block to the foundation. In Figure (2), they help decrease the amplitude of the fluctuating displacement transmitted by the foundation to the block. The ratio of the transmitted force to the impressed force or the ratio of the transmitted displacement to the impressed displacement is called the *transmissibility*. Derive an equation for the transmissibility for each situation. Give your answer in terms of the ratio ω_f/ω_n of the frequency ω_f of the impressed force or impressed displacement to the natural frequency ω_n of the spring-mass system. Show that in order to cause any reduction in transmissibility, the ratio ω_f/ω_n must be greater than $\sqrt{2}$.

SOLUTION

(1) From Equation (19.33):

$$x_m = \frac{\frac{P_m}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

Force transmitted:

$$(P_T)_m = kx_m = k\left[\frac{\frac{P_m}{k}}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}\right]$$

Thus,

$$\text{Transmissibility} = \frac{(P_T)_m}{P_m} = \frac{1}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} \blacktriangleleft$$

(2) From Equation (19.33′):

Displacement transmitted:

$$x_m = \frac{\delta_m}{1-\left(\frac{\omega_f}{\omega_n}\right)^2}$$

$$\text{Transmissibility} = \frac{x_m}{\delta_m} = \frac{1}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} \blacktriangleleft$$

For $\frac{(P_T)_m}{P_m}$ or $\frac{x_m}{\delta_m}$ to be less than 1,

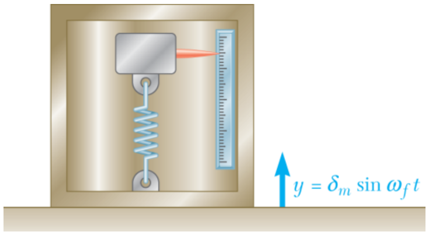
$$\frac{1}{\left|1-\left(\frac{\omega_f}{\omega_n}\right)^2\right|} < 1$$

$$1 < \left|1-\left(\frac{\omega_f}{\omega_n}\right)^2\right|$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 > 2$$

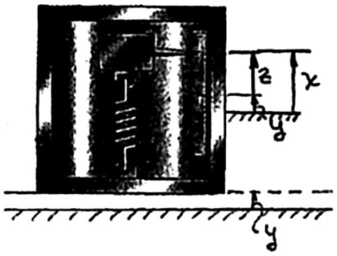
$$\frac{\omega_f}{\omega_n} > \sqrt{2} \quad \text{Q.E.D.} \blacktriangleleft$$

PROBLEM 19.122


$y = \delta_m \sin \omega_f t$

A vibrometer used to measure the amplitude of vibrations consists essentially of a box containing a mass-spring system with a known natural frequency of 120 Hz. The box is rigidly attached to a surface, which is moving according to the equation $y = \delta_m \sin \omega_f t$. If the amplitude z_m of the motion of the mass relative to the box is used as a measure of the amplitude δ_m of the vibration of the surface, determine (*a*) the percent error when the frequency of the vibration is 600 Hz, (*b*) the frequency at which the error is zero.

SOLUTION



$$x = \left(\frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}} \right) \sin \omega_f t$$

$$y = \delta_m \sin \omega_f t$$

z = relative motion

$$z = x - y = \left[\frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}} - \delta_m \right] \sin \omega_f t$$

$$z_m = \delta_m \left[\frac{1}{1 - \frac{\omega_f^2}{\omega_n^2}} - 1 \right] = \frac{\delta_m \frac{\omega_f^2}{\omega_n^2}}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

(*a*)

$$\frac{z_m}{\delta_m} = \frac{\frac{\omega_f^2}{\omega_n^2}}{1 - \frac{\omega_f^2}{\omega_n^2}} = \frac{\left(\frac{600}{120}\right)^2}{1 - \left(\frac{600}{120}\right)^2} = \frac{25}{24} = 1.0417 \quad ◀$$

Error = 4.17%

(*b*)

$$\frac{z_m}{\delta_m} = 1 = \frac{\frac{\omega_f^2}{\omega_n^2}}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

$$1 = 2 \frac{\omega_f^2}{\omega_n^2}$$

$$f_f = \frac{\sqrt{2}}{2} f_n = \frac{\sqrt{2}}{2}(120) = 84.853 \text{ Hz}$$

$f_n = 84.9$ Hz ◀

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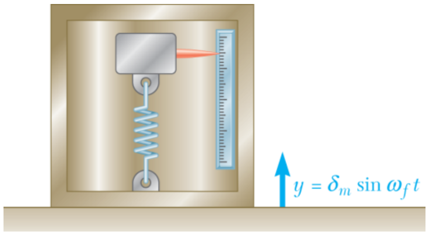
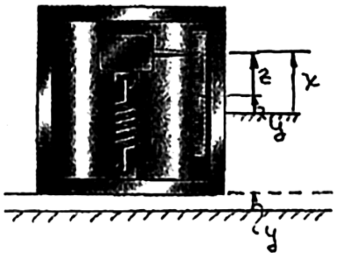

$y = \delta_m \sin \omega_f t$

PROBLEM 19.123

A certain accelerometer consists essentially of a box containing a mass-spring system with a known natural frequency of 2200 Hz. The box is rigidly attached to a surface, which is moving according to the equation $y = \delta_m \sin \omega_f t$. If the amplitude z_m of the motion of the mass relative to the box times a scale factor ω_n^2 is used as a measure of the maximum acceleration $\alpha_m = \delta_m \omega_f^2$ of the vibrating surface, determine the percent error when the frequency of the vibration is 600 Hz.

SOLUTION



$$x = \left(\frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}} \right) \sin \omega_f t$$

$$y = \delta_m \sin \omega_f t$$

$z =$ relative motion

$$z = x - y = \left[\frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_n^2}} - \delta_m \right] \sin \omega_f t$$

$$z_m = \delta_m \left[\frac{1}{1 - \frac{\omega_f^2}{\omega_n^2}} - 1 \right] = \frac{\delta_m \frac{\omega_f^2}{\omega_n^2}}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

The actual acceleration is $a_m = -\omega_f^2 \delta_m$

The measurement is proportional to $z_m \omega_n^2$.

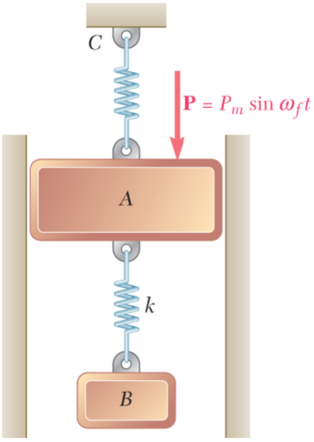
Then
$$\frac{z_m \omega_n^2}{a_m} = \frac{z_m}{\delta_m} \left(\frac{\omega_n}{\omega_f} \right)^2$$

$$= \frac{1}{1 - \left(\frac{\omega_f}{\omega_n} \right)^2}$$

$$= \frac{1}{1 - \left(\frac{600}{2200} \right)^2}$$

$= 1.0804$

Error $= 8.04\%$ ◀



PROBLEM 19.124

Block A can move without friction in the slot as shown and is acted upon by a vertical periodic force of magnitude $P = P_m \sin \omega_f t$, where $\omega_f = 2$ rad/s and $P_m = 20$ N. A spring of constant k is attached to the bottom of block A and to a 22-kg block B. Determine (a) the value of the constant k which will prevent a steady-state vibration of block A, (b) the corresponding amplitude of the vibration of block B.

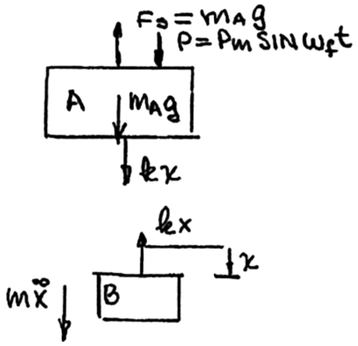
SOLUTION

In steady state vibration, block A does not move and therefore, remains in its original equilibrium position.

Block A: $+\!\downarrow \Sigma F = 0$

$$kx = -P_m \sin \omega_f t \qquad (1)$$

Block B: $+\!\downarrow \Sigma F = m_B \ddot{x}$



$$m_B \ddot{x} + kx = 0$$

$$x = x_m \sin \omega_n t$$

$$\omega_n^2 = k/m_B$$

From Eq. (1): $kx_m \sin \omega_n t = -P_m \sin \omega_f t$

$$\omega_n = \omega_f = 2 \text{ rad/s}$$

$$kx_m = -P_m$$

$$\omega_n = \sqrt{\frac{k}{m_B}}$$

$$k = m_B \omega_n^2$$

(a) Required spring constant. $k = (22)(2)^2$ $\qquad k = 88.0$ N/m ◀

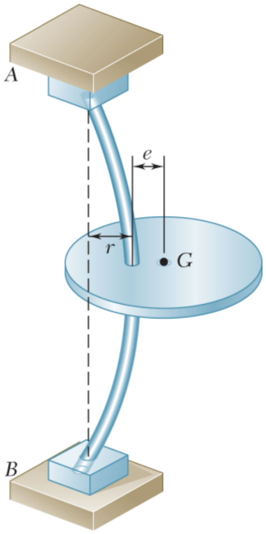
(b) Corresponding amplitude of vibration of B.

$$kx_m = -P_m$$

$$x_m = -\frac{P_m}{k}$$

$$x_m = -\frac{20 \text{ N}}{88 \text{ N/m}}$$

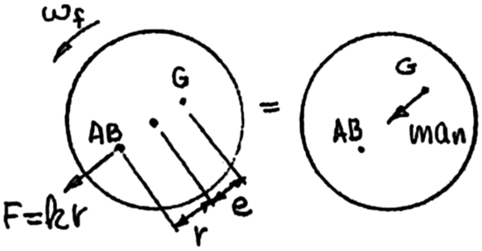
$x_m = -0.227$ m ◀



PROBLEM 19.125

A 60-lb disk is attached with an eccentricity $e = 0.006$ in. to the midpoint of a vertical shaft *AB*, which revolves at a constant angular velocity ω_f. Knowing that the spring constant k for horizontal movement of the disk is 40,000 lb/ft, determine (*a*) the angular velocity ω_f at which resonance will occur, (*b*) the deflection r of the shaft when $\omega_f = 1200$ rpm.

SOLUTION



G describes a circle about the axis *AB* of radius $r + e$.

Thus, $$a_n = (r + e)\omega_f^2$$

Deflection of the shaft is $$F = kr$$

Thus, $$F = ma_n$$

$$kr = m(r + e)\omega_f^2$$

$$\omega_n^2 = \frac{k}{m} \qquad m = \frac{k}{\omega_n^2}$$

$$\cancel{k}r = \frac{\cancel{k}}{\omega_n^2}(r + e)\omega_f^2$$

$$r = \frac{e\frac{\omega_f^2}{\omega_n^2}}{1 - \frac{\omega_f^2}{\omega_n^2}}$$

PROBLEM 19.125 (Continued)

(*a*) Resonance occurs when $\omega_f = \omega_n$, i.e., $r \longrightarrow \infty$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}}$$

$$= \sqrt{\frac{(40,000)(32.2)}{60}}$$

$$= 146.52 \text{ rad/s}$$

$$= 1399.1 \text{ rpm}$$

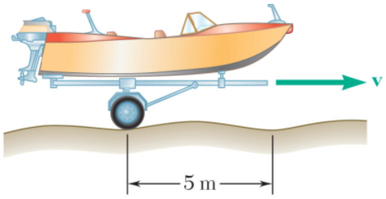
$\omega_n = \omega_f = 1399$ rpm ◀

(*b*)

$$r = \frac{(0.006 \text{ in.})\left(\frac{1200}{1399.1}\right)^2}{1-\left(\frac{1200}{1399.1}\right)^2}$$

$$= 0.01670 \text{ in.}$$

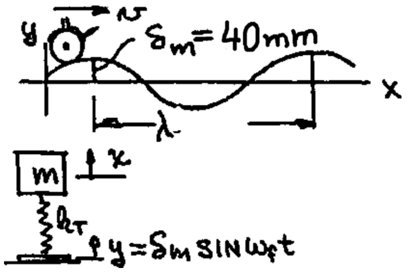
$r = 0.01670$ in. ◀



PROBLEM 19.126

A small trailer and its load have a total mass of 250-kg. The trailer is supported by two springs, each of constant 10 kN/m, and is pulled over a road, the surface of which can be approximated by a sine curve with an amplitude of 40 mm and a wavelength of 5 m (i.e., the distance between successive crests is 5 m and the vertical distance from crest to trough is 80 mm). Determine (*a*) the speed at which resonance will occur, (*b*) the amplitude of the vibration of the trailer at a speed of 50 km/h.

SOLUTION



Total spring constant $\quad k = 2(10\times10^3 \text{ N/m})$

$$= 20\times10^3 \text{ N/m}$$

(*a*) $\quad \omega_n^2 = \dfrac{k}{m} = \dfrac{20\times10^3 \text{ N/m}}{250 \text{ kg}} = 80 \text{ s}^{-2}$

$$\lambda = 5 \text{ m}$$

$$\delta_m = 40 \text{ mm} = 40\times10^{-3} \text{ m}$$

$$y = \delta_m \sin\frac{2\pi x}{\lambda} \quad \text{where} \quad x = vt$$

$$y = \delta_m \sin\omega_f t$$

$$\omega_f = \frac{2\pi v}{\lambda}$$

$$\delta_m = 40 \text{ mm} = 40\times10^{-3} \text{ m}$$

From Equation (19.33′): $\quad x_m = \dfrac{\delta_m}{\left(1-\frac{\omega_f^2}{\omega_n^2}\right)}$

Resonance: $\quad \omega_f = \dfrac{2\pi v}{5} = \omega_n = \sqrt{80} \text{ s}^{-1},$

$$v = 7.1176 \text{ m/s}$$

$v = 25.6$ km/h ◀

(*b*) Amplitude at $\quad v = 50 \text{ km/h} = 13.8889 \text{ m/s}$

$$\omega_f = \frac{2\pi(13.8889)}{5} = 17.4533 \text{ rad/s}$$

$$\omega_f^2 = 304.60 \text{ s}^{-2}$$

$$x_m = \frac{40\times10^{-3}}{1-\frac{304.62}{80}} = -14.246\times10^{-3} \text{ m}$$

$x_m = -14.25$ mm ◀

PROBLEM 19.127

Show that in the case of heavy damping $(c > c_c)$, a body never passes through its position of equilibrium O (*a*) if it is released with no initial velocity from an arbitrary position or (*b*) if it is started from O with an arbitrary initial velocity.

SOLUTION

Since $c > c_c$, we use Equation (19.42), where

$$\lambda_1 < 0,\ \lambda_2 < 0$$

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{1}$$

$$v = \frac{dx}{dt} = C_1 \lambda_1 e^{\lambda_1 t} + C_2 \lambda_2 e^{\lambda_2 t} \tag{2}$$

(*a*) $t = 0,\ x = x_0, \quad v = 0:$

From Eqs. (1) and (2):

$$x_0 = C_1 + C_2$$

$$0 = C_1 \lambda_1 + C_2 \lambda_2$$

Solving for c_1 and c_2,

$$C_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} x_0$$

$$C_2 = \frac{-\lambda_1}{\lambda_2 - x_1} x_0$$

Substituting for C_1 and C_2 in Eq. (1), $x = \frac{x_2}{\lambda_2 - \lambda_1}\left[\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}\right]$

For $x = 0$: when $t \neq \infty$, we must have

$$\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t} = 0 \qquad \frac{\lambda_2}{\lambda_1} = e^{(\lambda_2 - \lambda_1)t} \tag{3}$$

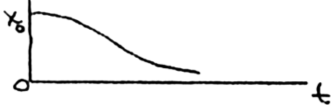
Recall that

$\lambda_1 < 0,\ \lambda_2 < 0.$ Choosing λ_1 and λ_2 so that $\lambda_1 < \lambda_2 < 0,$ we have

$$0 < \frac{\lambda_2}{\lambda_1} < 1 \quad \text{and} \quad \lambda_2 - \lambda_1 > 0$$

Thus a positive solution for $t > 0$ for Equation (3) cannot exist, since it would require that e raised to a positive power be less than 1, which is impossible. Thus, x is never 0.

The $x-t$ curve for this case is as shown.



◀

PROBLEM 19.127 (Continued)

(*b*) $t = 0,\ x = 0,\ v = v_0$: Equations (1) and (2) yield

$$0 = C_1 + C_2$$

$$v_0 = C_1\lambda_1 + C_2\lambda_2$$

Solving for C_1 and C_2,

$$C_1 = -\frac{v_0}{\lambda_2 - \lambda_1}$$

$$C_2 = \frac{v_2}{\lambda_2 - \lambda_1}$$

Substituting into Eq. (1),

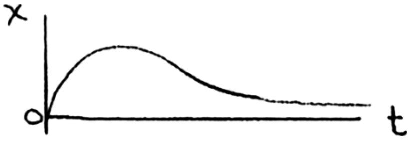
$$x = \frac{v_0}{\lambda_2 - \lambda_1}[e^{\lambda_2 t} - e^{\lambda_1 t}]$$

For $x = 0$, and $t > 0$

$$e^{\lambda_2 t} = e^{\lambda_1 t}$$

For $c > c_c$, $\lambda_1 \neq \lambda_2$; thus, no solution can exist for *t*, and *x* is never 0 when $t > 0$.

The $x - t$ curve for this motion is as shown.



◀

PROBLEM 19.128

Show that in the case of heavy damping $(c > c_c)$, a body released from an arbitrary position with an arbitrary initial velocity cannot pass more than once through its equilibrium position.

SOLUTION

Substitute the initial conditions, $t = 0, x = x_0, v = v_0$ in Equations (1) and (2) of Problem 19.127.

$$x_0 = C_1 + C_2 \qquad v_0 = C_1\lambda_1 + C_2\lambda_2$$

Solving for C_1 and C_2,

$$C_1 = -\frac{(v_0 - \lambda_2 x_0)}{\lambda_2 - \lambda_1}$$

$$C_2 = \frac{(v_0 - \lambda_1 x_0)}{\lambda_2 - \lambda_1}$$

And substituting in Eq. (1)

$$x = \frac{1}{\lambda_2 - \lambda_1}\left[(v_0 - \lambda_1 x_0)e^{\lambda_2 t} - (v_0 - \lambda_2 x_0)e^{\lambda_1 t}\right]$$

For $x = 0,\ t \neq \infty$:

$$(v_0 - \lambda_1 x_0)e^{\lambda_2 t} = (v_0 - \lambda_2 x_0)e^{\lambda_1 t}$$

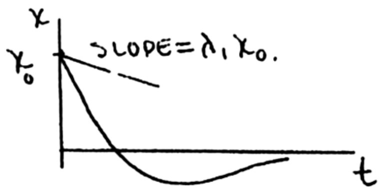
$$e^{(\lambda_2 - \lambda_1)_t} = \frac{(v_0 - \lambda_2 x_0)}{(v_0 - \lambda_1 x_0)}$$

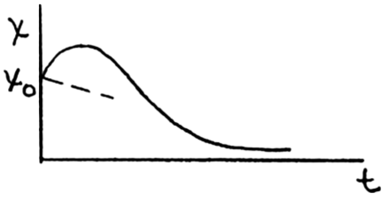
$$t = \frac{1}{(\lambda_2 - \lambda_1)}\ln\frac{v_0 - \lambda_2 x_0}{v_0 - \lambda_1 x_0}$$

This defines one value of t only for $x = 0$, which will exist if the argument of the natural logarithm is positive,

i.e., if $\frac{v_0 - \lambda_2 x_0}{v_0 - \lambda_1 x_0} > 1$. Assuming $\lambda_1 < \lambda_2 < 0$,

this occurs if $v_0 < \lambda_1 x_0$.





◀

PROBLEM 19.129

In the case of light damping, the displacements $x_1,\ x_2,\ x_3,$ shown in Figure 19.11 may be assumed equal to the maximum displacements. Show that the ratio of any two successive maximum displacements x_n and x_{n+1} is a constant and that the natural logarithm of this ratio, called the *logarithmic decrement*, is

$$\ln\frac{x_n}{x_{n+1}} = \frac{2\pi(c/c_c)}{\sqrt{1-(c/c_c)^2}}$$

SOLUTION

For light damping,

Equation (19.46): $$x = x_0 e^{-\left(\frac{c}{2m}\right)t}\sin(\omega_0 t + \phi)$$

At given maximum displacement, $$t = t_n,\ x = x_n$$

$$\sin(\omega_0 t_n + \phi) = 1$$

$$x_n = x_0 e^{-\left(\frac{c}{2m}\right)t_n}$$

At next maximum displacement, $$t = t_{n+1},\ x = x_{n+1}$$

$$\sin(\omega_0 t_{n+1} + \phi) = 1$$

$$x_{n+1} = x_0 e^{-\left(\frac{c}{2m}\right)t_{n+1}}$$

But $$\omega_D t_{n+1} - \omega_D t_n = 2\pi$$

$$t_{n+1} - t_n = \frac{2\pi}{\omega_D}$$

Ratio of successive displacements: $$\frac{x_n}{x_{n+1}} = \frac{x_0 e^{-\frac{c}{2m}t_n}}{x_0 e^{-\frac{c}{2m}t_{n+1}}}$$

$$= e^{-\frac{c}{2m}(t_n - t_{n+1})} = e^{+\frac{c}{2m}\frac{2\pi}{\omega_D}}$$

Thus, $$\ln\frac{x_n}{x_{n+1}} = \frac{c\pi}{m\omega_D} \qquad (1)$$

From Equations (19.45) and (19.41): $$\omega_D = \omega_n\sqrt{1-\left(\frac{c}{c_c}\right)^2}$$

$$\omega_D = \frac{c_c}{2m}\sqrt{1-\left(\frac{c}{c}\right)^2}$$

Thus, $$\ln\frac{x_n}{x_{n+1}} = \frac{c\pi}{m}\frac{2m}{c_c}\frac{1}{\sqrt{1-\left(\frac{c}{c_c}\right)^2}} \qquad \ln\frac{x_n}{x_{n+1}} = \frac{2\pi\left(\frac{c}{c_c}\right)}{\sqrt{1-\left(\frac{c}{c_c}\right)^2}}$$ Q.E.D. ◀

PROBLEM 19.130

In practice, it is often difficult to determine the logarithmic decrement of a system with light damping defined in Problem 19.129 by measuring two successive maximum displacements. Show that the logarithmic decrement can also be expressed as $(1/k)\ln(x_n/x_{n+k})$, where k is the number of cycles between readings of the maximum displacement.

SOLUTION

As in Problem 19.129, for maximum displacements x_n and x_{n+k} at t_n and t_{n+k}, $\sin(\omega_0 t_n + \phi) = 1$

and $\sin(\omega_n t_{n+k} + \phi) = 1$.

$$x_n = x_0 e^{-\left(\frac{c}{2m}\right)t_n}$$

$$x_{n+k} = x_0 e^{-\left(\frac{c}{2m}\right)(t_{n+k})}$$

Ratio of maximum displacements:

$$\frac{x_n}{x_{n+k}} = \frac{x_0 e^{\left(\frac{-c}{2m}\right)t_n}}{x_0 e^{\left(\frac{-c}{2m}\right)t_{n+k}}} = e^{\frac{-c}{2m}(t_n - t_{n+k})}$$

But

$$\omega_D t_{n+k} - \omega_D t_n = k(2\pi)$$

$$t_n - t_{n+k} = k\frac{2\pi}{\omega_D}$$

Thus,

$$\frac{x_n}{x_{n+k}} = +\frac{c}{2m}\left(\frac{2k\pi}{\omega_D}\right)$$

$$\ln\frac{x_n}{x_{n+k}} = k\frac{c\pi}{m\omega_D} \qquad (2)$$

But from Problem 19.129, Equation (1):

$$\text{log decrement} = \ln\frac{x_n}{x_{n+1}} = \frac{c\pi}{m\omega_D}$$

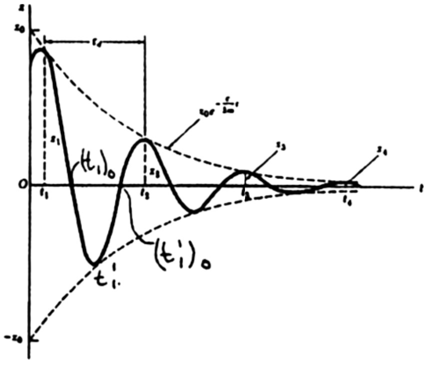
Comparing with Equation (2),

$$\text{log decrement} = \frac{1}{k}\ln\frac{x_n}{x_{n+k}} \quad \text{Q.E.D.} \blacktriangleleft$$

PROBLEM 19.131

In an underdamped system $(c < c_c)$, the period of vibration is commonly defined as the time interval $\tau_d = 2\pi/\omega_d$ corresponding to two successive points where the displacement-time curve touches one of the limiting curves shown in Figure 19.11. Show that the interval of time (*a*) between a maximum positive displacement and the following maximum negative displacement is $\frac{1}{2}\tau_d$, (*b*) between two successive zero displacements is $\frac{1}{2}\tau_d$, (*c*) between a maximum positive displacement and the following zero displacement is greater than $\frac{1}{4}\tau_d$.

SOLUTION



Equation (19.46): $x = x_0 e^{-\left(\frac{c}{2m}\right)t}\sin(\omega_d t + \phi)$

(*a*) Maxima (positive or negative) when $\dot{x} = 0$:

$$\dot{x} = x_0\left(\frac{-c}{2m}\right)e^{-\left(\frac{c}{2m}\right)t}\sin(\omega_d t + \phi) + x_0\omega_d e^{-\left(\frac{c}{2m}\right)t}\cos(\omega_d t + \phi)$$

Thus, zero velocities occur at times when

$$\dot{x} = 0, \quad \text{or} \quad \tan(\omega_d t + \phi) = \frac{2m\omega_d}{c} \tag{1}$$

The time to the first zero velocity, t_1, is

$$t_1 = \frac{\left[\tan^{-1}\left(\frac{2m\omega_d}{c}\right) - \phi\right]}{\omega_d} \tag{2}$$

The time to the next zero velocity where the displacement is negative is

$$t_1' = \frac{\left[\tan^{-1}\left(\frac{2m\omega_d}{c}\right) - \phi + \pi\right]}{\omega_d} \tag{3}$$

Subtracting Eq. (2) from Eq. (3),

$$t_1' - t_1 = \frac{\pi}{\omega_d} = \frac{\pi\cdot\tau_d}{2\pi} = \frac{\tau_d}{2} \quad \text{Q.E.D.}$$

PROBLEM 19.131 (Continued)

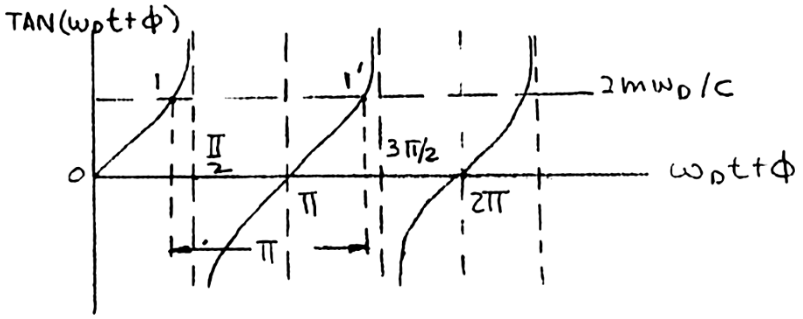
(*b*) Zero displacements occur when

$$\sin(\omega_\theta t + \phi) = 0 \quad \text{or at intervals of}$$

$$\omega_d t + \phi = \pi,\ 2\pi\ n\pi$$

Thus, $$\frac{(t_1)_0 = (\pi - \phi)}{\omega_d} \quad \text{and} \quad (t_1')_0 = \frac{(2\pi - \phi)}{\omega_d}$$

Time between $$0_s' = (t_1')_0 - (t_1)_0 = \frac{2\pi - \pi}{\omega_d} = \frac{\pi\tau_d}{2\pi} = \frac{\tau_d}{2} \quad \text{Q.E.D.}$$



Plot of Equation (1)

(*c*) The first maximum occurs at 1: $(\omega_d t_1 + \phi)$

The first zero occurs at $(\omega_d (t_1)_0 + \phi) = \pi$

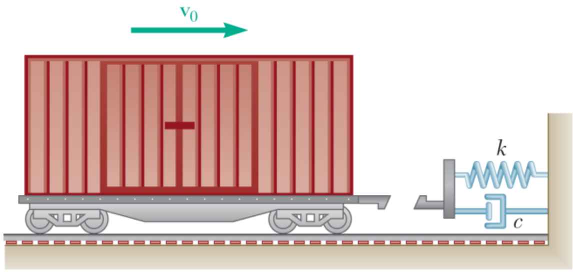
From the above plot, $$(\omega_d (t_1)_0 + \phi) - (\omega_D t_1 + \phi) > \frac{\pi}{2}$$

or $$(t_1)_0 - t_1 > \frac{\pi}{2\omega_d} \quad (t_1)_0 - t_1 > \frac{\tau_d}{4} \quad \text{Q.E.D.}$$

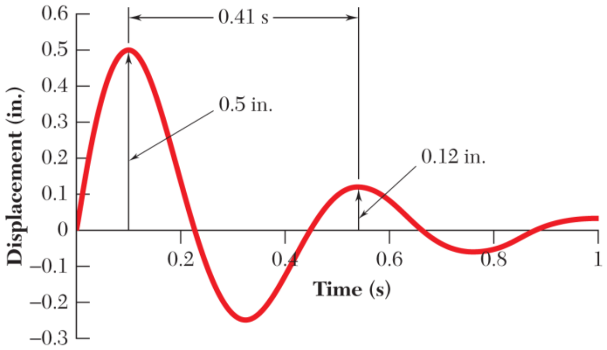
Similar proofs can be made for subsequent maximum and minimum.

PROBLEM 19.132

A loaded railroad car weighing 30,000 lb is rolling at a constant velocity $\mathbf{v}_0$ when it couples with a spring and dashpot bumper system (Figure 1). The recorded displacement-time curve of the loaded railroad car after coupling is as shown (Figure 2). Determine (*a*) the damping constant, (*b*) the spring constant. (*Hint:* Use the definition of logarithmic decrement given in Problem 19.129.)



(1)



(2)

SOLUTION

Mass of railroad car:

$$m = \frac{W}{g} = \frac{30,000}{32.2}$$

$$= 931.67 \text{ lb}\cdot\text{s}^2/\text{ft}$$

The differential equation of motion for the system is

$$m\ddot{x} + c\dot{x} + kx = 0$$

For light damping, the solution is given by Eq. (19.44):

$$x = e^{-\left(\frac{c}{2m}\right)t}(C_1 \sin \omega_d t + C_2 \cos \omega_d t)$$

From the displacement versus time curve,

$$\tau_d = 0.41 \text{ s}$$

$$\omega_d = \frac{2\pi}{\tau_d} = \frac{2\pi}{0.41} = 15.325 \text{ rad/s}$$

At the first peak, $x_1 = 0.5$ in. and $t = t_1$.

At the second peak, $x_2 = 0.12$ in. and $t = t_1 + \tau_d$.

Forming the ratio $\frac{x_2}{x_1}$,

$$\frac{x_2}{x_1} = \frac{e^{-\left(\frac{c}{2m}\right)(t_1+\tau_d)}}{e^{-\left(\frac{c}{2m}\right)t_1}} = e^{-\left(\frac{c}{2m}\right)\tau_d} \qquad (1)$$

$$\frac{x_1}{x_2} = e^{\frac{c\tau_d}{2m}}$$

PROBLEM 19.132 (Continued)

(*a*) Damping constant.

From Eq. (1):
$$\frac{c\tau_d}{2m} = \ln\left(\frac{x_1}{x_2}\right)$$

$$c = \frac{2m}{\tau_d}\ln\frac{x_1}{x_2}$$
$$= \frac{(2)(931.67)}{0.41}\ln\frac{0.5}{0.12}$$
$$= 6485.9 \text{ lb}\cdot\text{s/ft} \qquad c = 6.49 \text{ kip}\cdot\text{s/ft} \blacktriangleleft$$

(*b*) Spring constant.

Equation for ω_d:
$$\omega_d^2 = \frac{k}{m} - \left(\frac{c}{2m}\right)^2$$

$$k = m\omega_d^2 + \frac{c^2}{4m}$$
$$= (931.67)(15.325)^2 + \frac{(6485.9)^2}{(4)(931.67)}$$
$$= 230\times 10^3 \text{ lb/ft} \qquad k = 230 \text{ kips/ft} \blacktriangleleft$$

PROBLEM 19.133

A torsional pendulum has a centroidal mass moment of inertia of 0.3 kg-m^2 and when given an initial twist and released is found to have a frequency of oscillation of 200 rpm. Knowing that when this pendulum is immersed in oil and given the same initial condition it is found to have a frequency of oscillation of 180 rpm, determine the damping constant for the oil.

SOLUTION

Let the mass be rotated through the small angle θ from the equilibrium position.

Couples acting on the mass: Shaft: $-K\theta$

Oil: $-C\dot{\theta}$

Equation of motion: $\Sigma M = \bar{I}\ddot{\theta} : -K\theta - C\dot{\theta} = \bar{I}\ddot{\theta}$

$$\bar{I}\ddot{\theta} + C\dot{\theta} + K\theta = 0$$

Solution for light damping: $\theta = e^{-\lambda t}(C_1 \sin \omega_d t + C_2 \sin \omega_d t)$

where $\lambda = \dfrac{C}{2\bar{I}}$

$$\omega_n = \sqrt{\frac{K}{\bar{I}}}$$

$$\omega_d = \sqrt{\omega_n^2 - \lambda^2}$$

When there is no oil, assume $C \approx 0.$

$$\omega_n = 200 \text{ rpm} = 20.944 \text{ rad/s}$$

When oil is present,

$$\omega_d = 180 \text{ rpm} = 18.8496 \text{ rad/s}$$

$$\lambda = \sqrt{\omega_n^2 - \omega_d^2} = 9.1293 \text{ s}^{-1}$$

Damping constant for oil.

$$C = 2I\lambda = (2)(0.3 \text{ kg}\cdot\text{m}^2)(9.1293 \text{ s}^{-1})$$

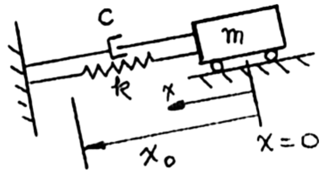
$C = 5.48 \text{ N}\cdot\text{m}\cdot\text{s}$ ◀

PROBLEM 19.134

The barrel of a field gun weighs 1500 lb and is returned into firing position after recoil by a recuperator of constant $c = 1100$ lb·s/ft. Determine (*a*) the constant k which should be used for the recuperator to return the barrel into firing position in the shortest possible time without any oscillation, (*b*) the time needed for the barrel to move back two-thirds of the way from its maximum-recoil position to its firing position.

SOLUTION

(*a*) A critically damped system regains its equilibrium position in the shortest time.



$$c = c_c$$
$$= 1100$$
$$= 2m\sqrt{\frac{k}{m}}$$

Then $k = \dfrac{\left(\frac{c_c}{2}\right)^2}{m} = \dfrac{\left(\frac{1100}{2}\right)^2}{\frac{1500\text{ lb}}{32.2\text{ ft/s}^2}} = 6494\text{ lb/ft}$ $\qquad k = 6490\text{ lb/ft}$ ◀

(*b*) For a critically damped system, Equation (19.43):

$$x = (C_1 + C_2 t)e^{-\omega_n t}$$

We take $t = 0$ at maximum deflection x_0.

Thus, $\dot{x}(0) = 0$

$x(0) = x_0$

Using the initial conditions, $x(0) = x_0 = (C_1 + 0)e^0$, so $C_1 = x_0$

$$x = (x_0 + C_2 t)e^{-\omega_n t}$$

and $\dot{x} = -\omega_n(x_0 + C_2 t)e^{-\omega_n t} + C_2 e^{-\omega_n t}$

$\dot{x}(0) = 0 = -\omega_n x_0 + C_2$, so $C_2 = \omega_n x_0$

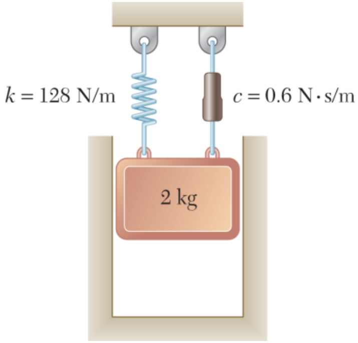
Thus, $x = x_0(1 + \omega_n t)e^{-\omega_n t}$

For $x = \dfrac{x_0}{3}$, $\dfrac{1}{3} = (1 + \omega_n t)e^{-\omega_n t}$

Solving by trial for $\omega_n t$ gives: $\omega_n t = 2.289$

But $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{6494\text{ lb/ft}}{\left(\frac{1500\text{ lb}}{32.2\text{ ft/s}^2}\right)}} = 11.807\text{ s}^{-1}$

Then $t = \dfrac{\omega_n t}{\omega_n} = \dfrac{2.289}{11.807} = 0.19387$ $\qquad t = 0.1939\text{ s}$ ◀



PROBLEM 19.135

A 2-kg block is supported by a spring of constant $k = 128$ N/m and a dashpot with a coefficient of viscous damping $c = 0.6$ N·s/m. The block is in equilibrium when it is struck from below by a hammer which imparts to the block an upward velocity of 0.4 m/s. Determine (*a*) the logarithmic decrement, (*b*) the maximum upward displacement of the block from equilibrium after two cycles. (*Hint*: See Probs. 19.128 and 19.129.)

SOLUTION

$$\omega_n = \sqrt{\frac{k}{m}} = 8 \text{ rad/s}, \qquad c_c = 2m\omega_n = 32 \text{ N}\cdot\text{s/m}, \qquad \frac{c}{c_c} = 0.01875$$

$$\omega_d = \omega_n\sqrt{1-\left(\frac{c}{c_c}\right)^2} = 7.99859363 \text{ rad/s}, \qquad x = c_1 e^{-0.15t}\sin\omega_d t\,(+\uparrow)$$

At $t = 0$, $\qquad \dot{x} = c_1\,\omega_d = 0.4$ m/s, $\qquad c_1 = 0.050008791$ m

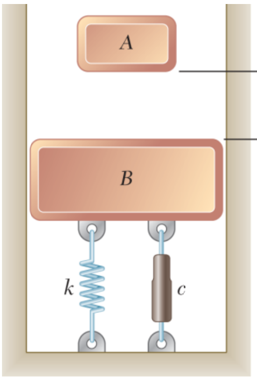
First peak at $\qquad t = \frac{1}{4}\left(\frac{2\pi}{\omega_d}\right) = 0.196384065$

$$\therefore x_1 = c_1 e^{-0.15(0.19638)} = 0.04855714 \text{ m} \uparrow$$

(*a*) $\qquad$ Log decrement $= \dfrac{2\pi(0.01875)}{\sqrt{1-(0.01875)^2}}$, $\qquad$ Log decrement $= 0.1178$ ◀

(*b*) $\qquad \therefore \dfrac{x_1}{x_3} = 1.2657$

$x_3 = 0.0384 \text{ m} = 38.4 \text{ mm}$ ◀



PROBLEM 19.136

A 4-kg block *A* is dropped from a height of 800 mm onto a 9-kg block *B* which is at rest. Block *B* is supported by a spring of constant $k = 1500$ N/m and is attached to a dashpot of damping coefficient $c = 230$ N·s/m. Knowing that there is no rebound, determine the maximum distance the blocks will move after the impact.

SOLUTION

Velocity of Block *A* just before impact.

$$\begin{aligned} v_A &= \sqrt{2gh} \\ &= \sqrt{2(9.81)(0.8)} \\ &= 3.962 \text{ m/s} \end{aligned}$$

Velocity of Blocks *A* and *B* immediately after impact.

Conservation of momentum.

$$m_A v_A + m_B v_B = (m_A + m_B)v'$$

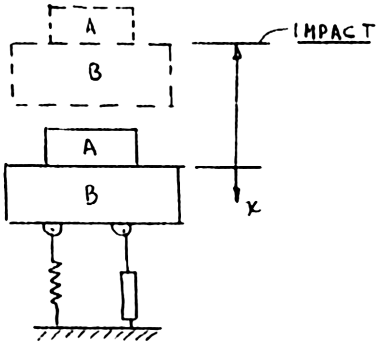
$$(4)(3.962) + 0 = (4 + 9)v'$$

$$v' = 1.219 \text{ m/s}$$

$$\dot{x}_0 = +1.219 \text{ m/s} \downarrow = \dot{x}_0$$

Static deflection (Block *A*):

$$\begin{aligned} x_0 &= -\frac{m_A g}{k} \\ &= -\frac{(4)(9.82)}{1500} \\ &= -0.02619 \text{ m} \end{aligned}$$



$x = 0$, Equilibrium position for both blocks:

$$\begin{aligned} c_c &= 2\sqrt{km} \\ &= 2\sqrt{(1500)(13)} \\ &= 279.3 \text{ N}\cdot\text{s/m} \end{aligned}$$

Since $c < c_c$, Equation (19.44):

$$x = e^{-\left(\frac{c}{2m}\right)t}[C_1 \sin \omega_d t + C_2 \cos \omega_d t]$$

$$\begin{aligned} \frac{c}{2m} &= \frac{230}{(2)(13)} \\ &= 8.846 \text{ s}^{-1} \end{aligned}$$

Expression for ω_d:

$$\omega_d^2 = \frac{k}{m} - \left(\frac{c}{2m}\right)^2$$

$$\omega_d = \sqrt{\frac{1500}{13} - \left(\frac{230}{(2)(13)}\right)^2}$$

$$= 6.094 \text{ rad/s}$$

$$x = e^{-8.846t}(C_1 \sin 6.094t + C_2 \cos 6.094t)$$

Initial conditions:

$$x_0 = -0.02619 \text{ m}$$

$$(t = 0) \quad \dot{x}_0 = +1.219 \text{ m/s}$$

$$x_0 = -0.02619 = e^0[C_1(0) + C_2(1)]$$

$$C_2 = -0.02619$$

$$\dot{x}(0) = -8.846e^{(-8.846)0}[C_1(0) + (-0.02619)(1)]$$

$$+e^{(-8.846)(0)}[6.094C_1(1) + C_2(0)] = 1.219$$

$$1.219 = (-8.846)(-0.02619) + 6.094C_1$$

$$C_1 = 0.16202$$

$$x = e^{-8.846t}(0.16202 \sin 6.094t - 0.02619 \cos 6.094t)$$

Maximum deflection occurs when $\dot{x} = 0$

$$\dot{x} = 0 = -8.846e^{-8.846t_m}(0.16202 \sin 6.094t_m - 0.02619 \cos 6.094t_m)$$

$$+\ e^{-8.846t_m}[6.094][0.1620 \cos 6.094t_m + 0.02619 \sin 6.094t_m]$$

$$0 = [(-8.846)(0.16202) + (6.094)(0.02619)] \sin 6.094t_m$$

$$+[(-8.846)(-0.02619) + (6.094)(0.1620)] \cos 6.094t_m$$

$$0 = -1.274 \sin 6.094t_m + 1.219 \cos 6.094t_m$$

$$\tan 6.094t_m = \frac{1.219}{1.274} = 0.957$$

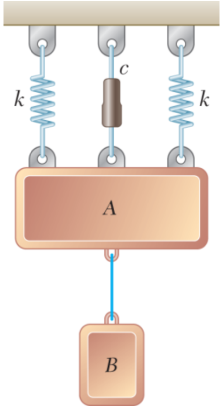
$$\text{Time at maximum deflection} = t_m = \frac{\tan^{-1} 0.957}{6.094} = 0.1253 \text{ s}$$

$$x_m = e^{-(8.846)(0.1253)}[0.1620 \sin(6.094)(0.1253)$$

$$-0.02619 \cos(6.094)(0.1253)]$$

$$x_m = (0.3301)(0.1120 - 0.0189) = 0.0307 \text{ m}$$

Blocks move, static deflection $+\ x_m$ Total distance $= 0.02619 + 0.0307 = 0.0569$ m $= 56.9$ mm ◀



PROBLEM 19.137

A 0.9-kg block B is connected by a cord to a 2.4-kg block A which is suspended as shown from two springs, each of constant $k = 180$ N/m, and a dashpot of damping coefficient $c = 7.5$ N·s/m. Knowing that the system is at rest when the cord connecting A and B is cut, determine the minimum tension that will occur in each spring during the resulting motion.

SOLUTION

From the given data

$$\omega_n = \sqrt{\frac{2k}{m}} = \sqrt{150}\ \text{rad/s}, \qquad c_c = 2m\omega_n = 58.7878\ \text{N}\cdot\text{s/m}$$

$$\text{Stretch at equilibrium} = \frac{(2.4\ \text{kg})(9.81\ \text{m/s}^2)}{2(180\ \text{N/m})} = 0.0654\ \text{m}$$

$$\text{Initial stretch} = \frac{(3.3\ \text{kg})(9.81\ \text{m/s}^2)}{2(180\ \text{N/m})} = 0.089925\ \text{m}$$

With $$c < c_c, \qquad \omega_d = \sqrt{150}\sqrt{1-\left(\frac{7.5}{58.7878}\right)^2} = 12.1474\ \text{rad/s}$$

From Figure 19.11 $$\frac{1}{2}\tau_D = \frac{1}{2}\left(\frac{2\pi}{12.1474}\right) = 0.25862\ \text{s}$$

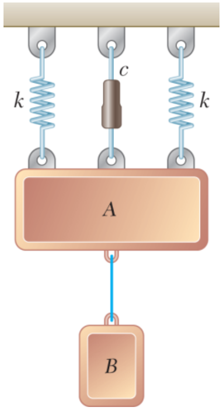
$$\downarrow + \ \text{Minimum } x = -0.024525e^{-1.5625(0.25862)}$$

$$= -0.001637\ \text{m}$$

$$\text{Minimum force in one spring} = (180\ \text{N/m})(0.0654 - 0.01637)$$

$$= 8.825\ \text{N}$$

8.82 N ◀



PROBLEM 19.138

A 0.9-kg block B is connected by a cord to a 2.4-kg block A which is suspended as shown from two springs, each of constant $k = 180$ N/m, and a dashpot of damping coefficient $c = 60$ N·s/m. Knowing that the system is at rest when the cord connecting A and B is cut, determine the velocity of block A after 0.1 s.

SOLUTION

From the given data

$$\omega_n = \sqrt{\frac{2k}{m}} = \sqrt{150}\ \text{rad/s}$$

$$c_c = 2m\omega_n = 58.7878\ \text{N}\cdot\text{s/m}$$

$$\text{Stretch at equilibrium} = \frac{(2.4\ \text{kg})(9.81\ \text{m/s}^2)}{2(180\ \text{N/m})} = 0.0654\ \text{m}$$

$$\text{Initial stretch} = \frac{(3.3\ \text{kg})(9.81\ \text{m/s}^2)}{2(180\ \text{N/m})} = 0.089925\ \text{m}$$

$$c > c_c: \quad x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

So

$$(2.4\ \text{kg})\lambda^2 + (60\ \text{N}\cdot\text{s/m})\lambda + 360 = 0$$

$$\lambda = \frac{-60 \pm \sqrt{(60)^2 - (4)(360)(2.4)}}{2(2.4)}$$

$$\lambda_1 = -10, \qquad \lambda_2 = -15$$

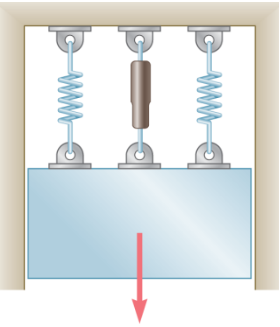
$$+\!\downarrow\ t = , \qquad \left.\begin{aligned} x &= 0.024524\ \text{m} = c_1 + c_2 \\ \dot{x} &= 0 = c_1\lambda_1 + c_2\lambda_2 \end{aligned}\right\}\ \text{solve for } c_1,\ c_2$$

$$c_1 = 0.073575; \qquad c_2 = -0.04905$$

$$\dot{x} = c_1\lambda_1 e^{0.1\lambda_1} + c_2\lambda_2 e^{0.1\lambda_2}$$

$$\dot{x} = -0.1065\ \text{m/s}$$

$v = 106.5$ mm/s ↑ ◀



PROBLEM 19.139

A machine element weighing 800 lb is supported by two springs, each having a constant of 200 lb/in. A periodic force of maximum value 30 lb is applied to the element with a frequency of 2.5 cycles per second. Knowing that the coefficient of damping is $8 \text{ lb} \cdot \text{s/in.}$, determine the amplitude of the steady-state vibration of the element.

SOLUTION

Equivalent spring: $k = 2(200) = 400 \text{ lb/in.} = 4800 \text{ lb/ft}$

Undamped natural frequency: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4800}{800/32.2}} = 13.90 \text{ rad/s}$

$$f_n = \frac{\omega_n}{2\pi} = \frac{13.90}{2\pi} = 2.212 \text{ Hz}$$

Critical damping coefficient: $c_c = 2\, m\omega_n = 2\left(\frac{800}{32.2}\right)(13.90) = 691 \text{ lb} \cdot \text{s/ft}$

Damping coefficient: $c = 8 \text{ lb} \cdot \text{s/in.} = 96 \text{ lb} \cdot \text{s/ft}$

Damping ratio: $\frac{c}{c_c} = \frac{96}{691} = 0.1390$

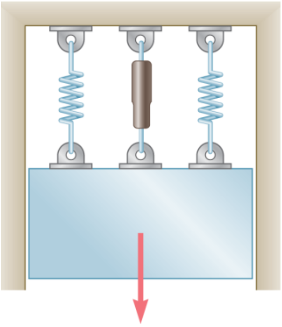
Amplitude:

$$x_m = \frac{P_m/k}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right]^2}} \qquad (1)$$

where $\frac{\omega_f}{\omega_n} = \frac{f_f}{f_n} = \frac{2.5 \text{ Hz}}{2.212 \text{ Hz}} = 1.130$

Substituting into Eq. (1) with $P_m = 30$ lb, we have

$$x_m = \frac{30 \text{ lb}/400 \text{ lb/in.}}{\sqrt{[1-(1.130)^2]^2 + [2(0.1390)(1.130)]^2}} \qquad x_m = 0.1791 \text{ in.} ◀$$



PROBLEM 19.140

In Problem 19.139, determine the required value of the coefficient of damping if the amplitude of the steady-state vibration of the element is to be 0.15 in.

PROBLEM 19.139 A machine element weighing 800 lb is supported by two springs, each having a constant of 200 lb/in. A periodic force of maximum value 30 lb is applied to the element with a frequency of 2.5 cycles per second. Knowing that the coefficient of damping is $8 \text{ lb}\cdot\text{s/in.}$, determine the amplitude of the steady-state vibration of the element.

SOLUTION

Equivalent spring: $$k = 2(200) = 400 \text{ lb/in.} = 4800 \text{ lb/ft}$$

Undamped natural frequency: $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4800}{800/32.2}} = 13.90 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{13.90}{2\pi} = 2.212 \text{ Hz}$$

Critical damping coefficient: $$c_c = 2\,m\omega_n = 2\left(\frac{800}{32.2}\right)(13.90) = 691 \text{ lb}\cdot\text{s/ft}$$

Amplitude: $$x_m = \frac{P_m/k}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right]^2}} \qquad (1)$$

where $$\frac{\omega_f}{\omega_n} = \frac{f_f}{f_n} = \frac{2.5 \text{ Hz}}{2.212 \text{ Hz}} = 1.130$$

Using $x_m = 0.15$ in., $P_m = 30$ lb, and $k = 400$ lb/in.

$$0.15 \text{ in.} = \frac{30 \text{ lb}/400 \text{ lb/in.}}{\sqrt{[1-(1.130)^2]^2 + \left[2\,\frac{c}{c_c}(1.130)\right]^2}}$$

Solving for $\frac{c}{c_c}$, we find $\frac{c}{c_c} = 0.1842.$

Since $c_c = 691 \text{ lb}\cdot\text{s/ft},$ we have

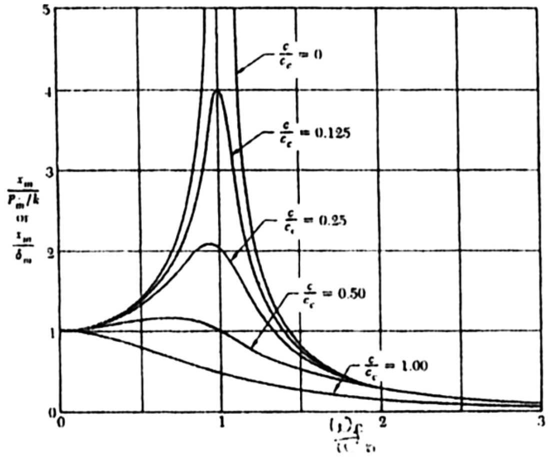
$$c = (0.1842)(691) \qquad c = 1273 \text{ lb}\cdot\text{s/ft}$$

or $$c = 10.61 \text{ lb}\cdot\text{s/in.} \blacktriangleleft$$

PROBLEM 19.141

In the case of the forced vibration of a system, determine the range of values of the damping factor c/c_c for which the magnification factor will always decrease as the frequency ratio ω_f/ω_n increases.

SOLUTION



From Eq. (19.53)′:

Magnification factor:

$$\frac{x_m}{\frac{P_m}{k}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$

Find value of $\frac{c}{c_c}$ for which there is no maximum for $\frac{x_m}{\frac{P_m}{k}}$ as $\frac{\omega_f}{\omega_n}$ increases.

$$\frac{d\left(\frac{x_m}{\frac{P_m}{k}}\right)^2}{d\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{-\left[2\left(1-\left(\frac{\omega_f}{\omega_n}\right)^2\right)(-1) + 4\frac{c^2}{c_c^2}\right]}{\left\{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)^2\right]\right\}^2} = 0$$

$$-2 + 2\left(\frac{\omega_f}{\omega_n}\right)^2 + 4\frac{c^2}{c_c^2} = 0$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 = 1 - 2\frac{c^2}{c_c^2}$$

For $\frac{c^2}{c_c^2} \geq \frac{1}{2}$, there is no maximum for $\frac{x_m}{\left(\frac{P_m}{k}\right)}$ and the magnification factor will decrease as $\frac{\omega_f}{\omega_n}$ increases.

$$\frac{c}{c_c} \geq \frac{1}{\sqrt{2}} \qquad\qquad \frac{c}{c_c} \geq 0.707 \blacktriangleleft$$

PROBLEM 19.142

Show that for a small value of the damping factor c/c_c, the maximum amplitude of a forced vibration occurs when $\omega_f \approx \omega_n$ and that the corresponding value of the magnification factor is $\frac{1}{2}(c/c_c)$.

SOLUTION

From Eq. (19.53′):

$$\text{Magnification factor} = \frac{x_m}{\frac{P_m}{k}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$

Find value of $\dfrac{\omega_f}{\omega_n}$ for which $\dfrac{x_m}{\frac{P_m}{k}}$ is a maximum.

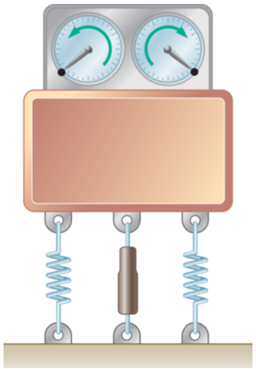
$$0 = \frac{d\left(\frac{x_m}{\frac{P_m}{k}}\right)^2}{d\left(\frac{\omega_f}{\omega_n}\right)^2} = -\frac{\left[2\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right](-1) + 4\left(\frac{c^2}{c_c^2}\right)\right]}{\left\{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2\right\}^2}$$

$$-2 + 2\left(\frac{\omega_f}{\omega_n}\right)^2 + 4\left(\frac{c}{c_c}\right)^2 = 0$$

For small $\dfrac{c}{c_c}$, $\qquad \dfrac{\omega_f}{\omega_n} \approx 1 \qquad \omega_f \approx \omega_n$

For $\qquad \dfrac{\omega_f}{\omega_n} = 1$

$$\frac{x_m}{\frac{P_m}{k}} = \frac{1}{\sqrt{[1-1]^2 + \left[2\left(\frac{c}{c_c}\right)1\right]^2}} \qquad \frac{x_m}{\left(\frac{P_m}{k}\right)} = \frac{1}{2}\frac{c_c}{c} \blacktriangleleft$$



PROBLEM 19.143

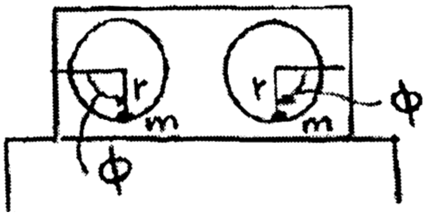
A counter-rotating eccentric mass exciter consisting of two rotating 14-oz masses describing circles of 6-in. radius at the same speed but in opposite senses is placed on a machine element to induce a steady-state vibration of the element and to determine some of the dynamic characteristics of the element. At a speed of 1200 rpm a stroboscope shows the eccentric masses to be exactly under their respective axes of rotation and the element to be passing through its position of static equilibrium. Knowing that the amplitude of the motion of the element at that speed is 0.6 in. and that the total mass of the system is 300 lb, determine (*a*) the combined spring constant *k*, (*b*) the damping factor c/c_c.

SOLUTION

Forcing frequency: $\omega_f = 1200 \text{ rpm} = 125.664 \text{ rad/s}$

Unbalance of one mass: $w = 14 \text{ oz} = 0.875 \text{ lb}$

$r = 6 \text{ in.} = 0.5 \text{ ft}$



Shaking force:

$$P = 2\,mr\omega_f^2 \sin \omega_f t$$

$$= (2)\left(\tfrac{0.875}{32.2}\right)(0.5)(125.664)^2 \sin \omega_f t$$

$$= 429.11 \sin \omega_f t$$

$$P_m = 429 \text{ lb}$$

Total weight: $W = 300 \text{ lb}$

By Eqs. (19.48) and (19.52), the vibratory response of the system is

$$x = x_m \sin(\omega_f t - \varphi)$$

where

$$x_m = \frac{P_m}{\sqrt{\left(k - M\omega_f^2\right)^2 + (c\omega_f)^2}} \quad (1)$$

and

$$\tan \varphi = \frac{c\omega_f}{k - M\omega_f^2} \quad (2)$$

Since $\varphi = 90° = \dfrac{\pi}{2}$, $\tan \varphi = \infty$ and $k - M\omega_f^2 = 0.$

(*a*) Combined spring constant.

$$k = M\omega_f^2$$

$$= \left(\tfrac{300}{32.2}\right)(125.664)^2$$

$$= 147.12 \times 10^3 \text{ lb/ft}$$

$k = 147 \text{ kip/ft}$ ◀

PROBLEM 19.143 (Continued)

The observed amplitude is $x_m = 0.6 \text{ in.} = 0.05 \text{ ft}$

From Eq. (1):

$$c = \frac{1}{\omega_f}\sqrt{\left(\frac{P_m}{x_m}\right)^2 - (k - M\omega_f)^2} = \frac{P_m}{\omega_f x_m}$$

$$= \frac{429.11}{(125.664)(0.05)}$$

$$= 68.296 \text{ lb}\cdot\text{s/ft}$$

Critical damping coefficient:

$$c_c = 2\sqrt{kM}$$

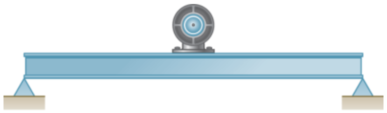
$$= 2\sqrt{(147.12\times 10^3)\left(\tfrac{300}{32.2}\right)}$$

$$= 2.3416\times 10^3 \text{ lb}\cdot\text{s/ft}$$

(*b*) Damping factor.

$$\frac{c}{c_c} = \frac{68.296}{2.3416\times 10^3}$$

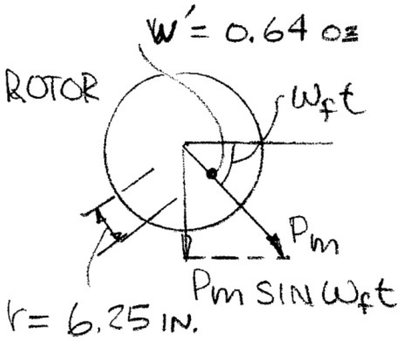
$$\frac{c}{c_c} = 0.0292 \blacktriangleleft$$



PROBLEM 19.144

A 36-lb motor is bolted to a light horizontal beam which has a static deflection of 0.075 in. due to the weight of the motor. Knowing that the unbalance of the rotor is equivalent to a weight of 0.64 oz located 6.25 in. from the axis of rotation, determine the amplitude of the vibration of the motor at a speed of 900 rpm, assuming (*a*) that no damping is present, (*b*) that the damping factor c/c_c is equal to 0.055.

SOLUTION



From Eq. (19.52)

$$x_m = \frac{P_m}{\sqrt{\left(k - m\omega_f^2\right)^2 + \left(c\omega_f\right)^2}}$$

$$\omega_f^2 = \left[900\left(\frac{2\pi}{60}\right)\text{rad/s}\right]^2 = 8882.6\ \text{s}^{-2}$$

$$k = \frac{W}{\delta_{st}} = \frac{36\ \text{lb}}{0.075\ \text{in.}} = 480\ \text{lb/in.} = 5760\ \text{lb/ft}$$

$$P_m = m'r\omega_f^2 = \frac{\left(\frac{0.64}{16}\ \text{lb}\right)}{32.2\ \text{ft/s}^2}\left(\frac{6.25}{12}\ \text{ft}\right)\left(8882.6\ \text{s}^{-2}\right)$$

$$= 5.7470\ \text{lb}$$

(*a*) $c = 0$

$$x_m = \frac{5.747\ \text{lb}}{\left[5760\ \text{lb/ft} - \left(\frac{36\ \text{lb}}{32.2\ \text{ft/s}^2}\right)\left(8882.6\ \text{s}^{-2}\right)\right]}$$

$$= \frac{5.747\ \text{lb}}{-4170.9\ \text{lb/ft}}$$

$$= -0.0013779\ \text{ft}$$

$x_m = 0.01653$ in. ◀

(out of phase)

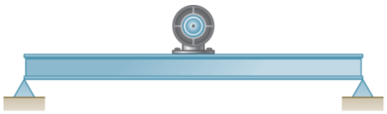
(*b*)

$$c_c = 2\sqrt{km} = 2\sqrt{\left(5760\ \text{lb/ft}\right)\left(\frac{36\ \text{lb}}{32.2\ \text{ft/s}^2}\right)} = 160.496\ \text{lb}\cdot\text{s/ft}$$

$$c = 0.055\left(160.496\ \text{lb}\cdot\text{s/ft}\right) = 8.8273\ \text{lb}\cdot\text{s/ft}$$

$$x_m = 0.0013513\ \text{ft}$$

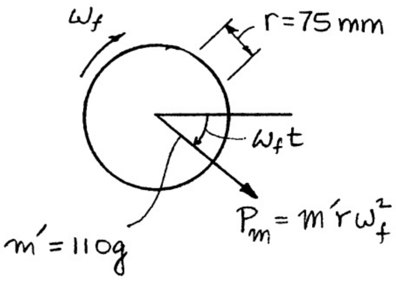
$x_m = 0.01622$ in. ◀



PROBLEM 19.145

A 45-kg motor is bolted to a light horizontal beam which has a static deflection of 6 mm due to the weight of the motor. The unbalance of the motor is equivalent to a mass of 110 g located 75 mm from the axis of rotation. Knowing that the amplitude of the vibration of the motor is 0.25 mm at a speed of 300 rpm, determine (*a*) the damping factor c/c_c, (*b*) the coefficient of damping c.

SOLUTION



From Eq. (19.53′)

$$x_m = \frac{\frac{P_m}{k}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}} \qquad (1)$$

$$\omega_n^2 = \frac{g}{\delta_{st}} = \frac{9.81 \text{ m/s}^2}{0.06 \text{ m}} = 1635 \text{ s}^{-2}$$

$$\omega_f^2 = \left[(300)\left(\frac{\pi}{30}\right)\right]^2 = 987 \text{ s}^{-2}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 = \frac{987}{1635} = 0.60365$$

$$P_m = m'r\omega_f^2 = \left[(0.11 \text{ kg})(0.075 \text{ m})(987 \text{ s}^{-2})\right] = 8.1424 \text{ N}$$

$$k = m\omega_n^2 = (45 \text{ N/m})(1635 \text{ s}^{-2}) = 73.575 \text{ kN/m},$$

$$\frac{P_m}{k} = 0.11067 \text{ mm}$$

(*a*) Then, from Equation (1)

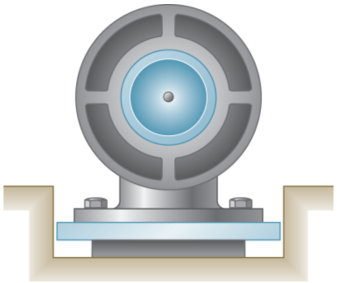
$$0.25 \times 10^{-3} \text{ m} = \frac{0.11067 \times 10^{-3} \text{ m}}{\sqrt{(1-0.60365)^2 + 4(0.60365)\left(\frac{c}{c_c}\right)^2}}$$

or $$0.19597 = 0.15707 + 2.4147\left(\frac{c}{c_c}\right)^2$$

Then $$\frac{c}{c_c} = 0.1269 \blacktriangleleft$$

(*b*) $$c_c = 2m\omega_n = 2(45 \text{ kg})(1635 \text{ s}^{-2})^{\frac{1}{2}} = 3639.2 \text{ N}\cdot\text{s/m}$$

$$c = (0.1269)(3639.2 \text{ N}\cdot\text{s/m}) = 462 \text{ N}\cdot\text{s/m} \blacktriangleleft$$



PROBLEM 19.146

The unbalance of the rotor of a 180-kg motor is equivalent to a mass of 85 g located 150 mm from the axis of rotation. The pad which is placed between the motor and the foundation is equivalent to a spring of constant k = 7.5 kN/m in parallel with a dashpot of constant c. Knowing that the magnitude of the maximum acceleration of the motor is 9 mm/s^2 at a speed of 100 rpm, determine the damping factor c/c_c.

SOLUTION

$$P_m = mr\omega_f^2 = \left(85 \times 10^{-3}\text{ kg}\right)(0.150\text{ m})\left(\frac{10\pi}{3}\text{s}^{-1}\right)^2$$

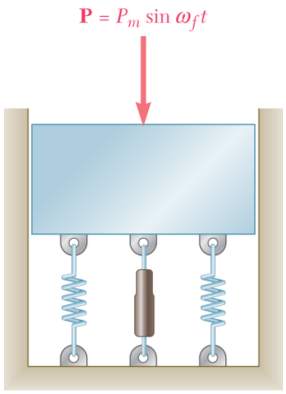
$$\omega_n^2 = \frac{7.5 \times 10^3\text{ N}}{180\text{ kg}} = 41.666$$

Then $$\ddot{x} = -\omega_f^2 x_m \sin\left(\omega_f t - \phi\right), \qquad \left|-\omega_f^2 x_m\right| = 9 \times 10^{-3}\text{ m/s}^2$$

So $$0.009\text{ m/s}^2 = \frac{\omega_f^2 \dfrac{P_m}{k}}{\sqrt{\left[1-\left(\dfrac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\dfrac{c}{c_c}\left(\dfrac{\omega_f}{\omega_n}\right)\right]^2}}$$

$$\left(\frac{c}{c_c}\right)^2 = \frac{\left(\dfrac{\omega_f^4 P_m^2}{0.009k^2}\right) - \left[1-\left(\dfrac{\omega_f}{\omega_n}\right)^2\right]^2}{4\left(\dfrac{\omega_f}{\omega_n}\right)^2}$$

$\dfrac{c}{c_c} = 0.487$ ◀



PROBLEM 19.147

A machine element is supported by springs and is connected to a dashpot as shown. Show that if a periodic force of magnitude $P = P_m \sin \omega_f t$ is applied to the element, the amplitude of the fluctuating force transmitted to the foundation is

$$F_m = P_m \sqrt{\frac{1+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2}}$$

SOLUTION

From Equation (19.48), the motion of the machine is $x = x_m \sin(\omega_f t - \phi)$

The force transmitted to the foundation is

Springs: $F_s = kx = kx_m \sin(\omega_f t - \phi)$

Dashpot: $F_d = c\dot{x} = cx_m\omega_f \cos(\omega_f t - \phi)$

$$F_t = x_m[k \sin(\omega_f t - \phi) + c\omega_f \cos(\omega_f t - \phi)]$$

or recalling the identity,

$$A \sin y + B \cos y = \sqrt{A^2 + B^2} \sin(y + \psi)$$

$$\sin \psi = \frac{B}{\sqrt{A^2 + B^2}}$$

$$\cos \psi = \frac{A}{\sqrt{A^2 + B^2}}$$

$$F_t = \left[x_m\sqrt{k^2 + (c\omega_f)^2}\right]\sin(\omega_f t - \phi + \psi)$$

Thus, the amplitude of F_t is $F_m = x_m\sqrt{k^2 + (c\omega_f)^2}$ (1)

From Equation (19.53):

$$x_m = \frac{\frac{P_m}{k}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2+\left[2\left(\frac{c}{c_c}\right)\frac{\omega_f}{\omega_n}\right]^2}}$$

Substituting for x_m in Equation (1),

$$F_m = \frac{P_m\sqrt{1+\left(\frac{c\omega_f}{k}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n^2}\right)\right]}} \quad (2)$$

$$\omega_n^2 = \frac{k}{m}$$

and Equation (19.41),

$$c_c = 2m\omega_n$$

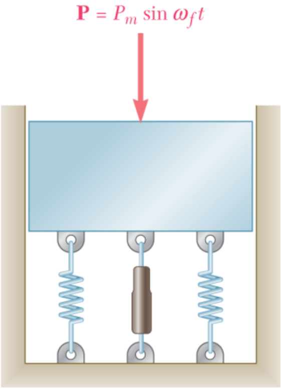
$$m = \frac{c\omega_n}{2}$$

$$\frac{c\omega_f}{k} = \frac{c\omega_f}{m\omega_n^2} = 2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)$$

Substituting in Eq. (2),

$$F_m = \frac{P_m\sqrt{1+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$

Q.E.D. ◀



PROBLEM 19.148

A 91-kg machine element supported by four springs, each of constant $k = 175$ N/m, is subjected to a periodic force of frequency 0.8 Hz and amplitude 89 N. Determine the amplitude of the fluctuating force transmitted to the foundation if (*a*) a dashpot with a coefficient of damping $c = 365$ N · s/m is connected to the machine element and to the ground, (*b*) the dashpot is removed.

SOLUTION

Forcing frequency: $\omega_f = 2\pi f_f = (2\pi)(0.8) = 1.6\pi \text{ rad/s}$

Exciting force amplitude: $P_m = 89 \text{ N}$

Equivalent spring constant: $k = (4)(175 \text{ N/m}) = 700 \text{ N/m}$

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{700}{91}} = 2.7735 \text{ rad/s}$$

Frequency ratio:

$$\frac{\omega_f}{\omega_n} = \frac{1.6\pi}{2.7735} = 1.8123$$

Critical damping coefficient:

$$c_c = 2\sqrt{km} = 2\sqrt{(700)(91)} = 504.78 \text{ N}\cdot\text{s/m}$$

From the derivation given in Problem 19.147, the amplitude of the force transmitted to the foundation is

$$F_m = \frac{P_m\sqrt{1+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}{\sqrt{\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2+\left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}} \quad (1)$$

$$1-\left(\frac{\omega_f}{\omega_n}\right)^2 = 1-(1.8123)^2 = -2.2844$$

PROBLEM 19.148 (Continued)

(*a*) F_m when $c = 365\text{ N}\cdot\text{s/m}$:

$$\frac{c}{c_c} = \frac{365}{504.78}$$

$$= 0.72309$$

$$2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right) = (2)(0.72309)(1.8123)$$

$$= 2.6209$$

From Eq. (1):

$$F_m = \frac{89\sqrt{1+(2.6209)^2}}{\sqrt{(-2.2844)^2+(2.6209)^2}}$$

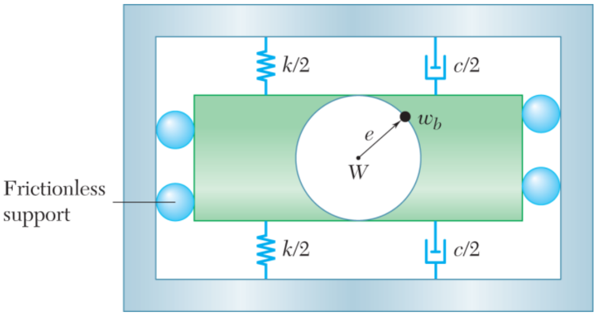
$$= \frac{89\sqrt{7.8692}}{\sqrt{12.088}}$$

$F_m = 71.8\text{ N}$ ◀

(*b*) F_m when $c = 0$:

$$F_m = \frac{P_m}{\left|1-\left(\frac{\omega_f}{\omega_n}\right)^2\right|} = \frac{89}{2.2844}$$

$F_m = 39.0\text{ N}$ ◀



PROBLEM 19.149

A simplified model of a washing machine is shown. A bundle of wet clothes forms a weight w_b of 20 lb in the machine and causes a rotating unbalance. The rotating mass is 40 lb (including m_b) and the radius of the washer basket e is 9 in. Knowing the washer has an equivalent spring constant k = 70 lb/ft and damping ratio $\zeta = c/c_c = 0.05$ and during the spin cycle the drum rotates at 250 rpm, determine the amplitude of the motion and the magnitude of the force transmitted to the sides of the washing machine.

SOLUTION

Forced circular frequency: $\omega_f = \dfrac{(2\pi)(250)}{60} = 26.18 \text{ rad/s}$

System mass: $m = \dfrac{W}{g} = \dfrac{40 \text{ lb}}{32.2}$

Spring constant: $k = 70 \text{ lb/ft}$

Natural circular frequency: $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{70}{\frac{40}{32.2}}} = 7.5067 \text{ rad/s}$

Critical damping constant: $c_c = 2\sqrt{km} = 2\sqrt{(70)\left(\frac{40}{32.2}\right)} = 18.650 \text{ lb}\cdot\text{s/ft}$

Damping constant: $c = \left(\dfrac{c}{c_c}\right)c_c = (0.05)(18.650) = 0.9325 \text{ lb}\cdot\text{s/ft}$

Unbalance force: $m_b = \dfrac{w_b}{g}$

$$P_m = m_b e \omega_f^2$$

$$P_m = \left(\frac{20 \text{ lb}}{32.2}\right)\left(\frac{9}{12}\text{ ft}\right)(26.18 \text{ rad/s})^2 = 319.28 \text{ lb}$$

The differential equation of motion is

$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t$$

The steady state response is

$$x = x_m \sin(\omega_f t - \varphi) \qquad \dot{x} = \omega_f x_m \cos(\omega_f t - \varphi)$$

where
$$x_m = \frac{P_m}{\sqrt{\left(k - m\omega_f^2\right)^2 + (c\omega_f)^2}}$$
$$= \frac{319.28}{\sqrt{[70 - \left(\frac{40}{32.2}\right)(26.18)^2]^2 + [(0.9325)(26.18)]^2}}$$
$$= \frac{319.28}{\sqrt{(-781.42)^2 + (24.413)^2}} = \frac{319.28}{781.796} = 0.40839 \text{ ft}$$

(*a*) Amplitude of vibration. $x_m = 4.90$ in. ◀

$$x = 0.40839\sin(\omega_f t - \varphi)$$
$$\dot{x} = (26.18)(0.40839)\cos(\omega_f t - \varphi)$$
$$= 10.6917\cos(\omega_f t - \varphi)$$

Spring force:
$$kx = (70)(0.40839)\sin(\omega_f t - \varphi)$$
$$= 28.588\sin(\omega_f t - \varphi)$$

Damping force:
$$c\dot{x} = (0.9325)(10.6917)\cos(\omega_f t - \varphi)$$
$$= 9.9701\cos(\omega_f t - \varphi)$$

(*b*) Total force:
$$F = 28.588\sin(\omega_f t - \varphi) + 9.9701\cos(\omega_f t - \varphi)$$

Let
$$F = F_m\cos\psi\sin(\omega_f t - \varphi) + F_m\sin\psi\sin(\omega_f t - \varphi)$$
$$= F_m\sin(\omega_f t - \varphi + \psi)$$

Maximum force.
$$F_m^2 = F_m^2\cos^2\psi + F_m^2\sin^2\psi$$
$$= (28.588)^2 + (9.9701)^2$$
$$= 916.65$$

$F_m = 30.3$ lb ◀

PROBLEM 19.150*

For a steady-state vibration with damping under a harmonic force, show that the mechanical energy dissipated per cycle by the dashpot is $E = \pi c x_m^2 \omega_f$, where c is the coefficient of damping, x_m is the amplitude of the motion, and ω_f is the circular frequency of the harmonic force.

SOLUTION

Energy is dissipated by the dashpot.

From Equation (19.48), the deflection of the system is

$$x = x_m \sin(\omega_f t - \varphi)$$

The force on the dashpot.

$$F_d = c\dot{x}$$

$$F_d = c x_m \omega_f \cos(\omega_f t - \varphi)$$

The work done in a complete cycle with

$$\tau_f = \frac{2\pi}{\omega_f}$$

$$E = \int F_d dx \text{ (i.e., force} \times \text{distance)}$$

$$dx = x_m \omega_f \cos(\omega_f t - \varphi) dt$$

$$E = \int_0^{2\pi/\omega_f} c x_m^2 \omega_f^2 \cos^2(\omega_f t - \varphi) dt$$

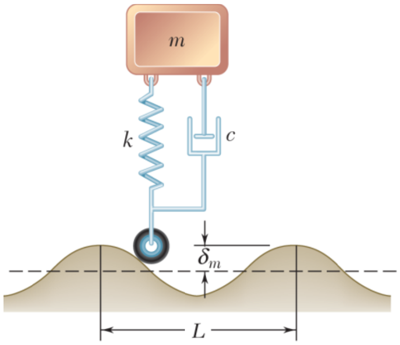
$$\cos^2(\omega_D t - \varphi) = \frac{[1 - 2\cos(\omega_f t - \phi)]}{2}$$

$$E = c x_m^2 \omega_f^2 \int_0^{2\pi/\omega_f} \frac{1 - 2\cos(\omega_f t - \phi)}{2} dt$$

$$E = \frac{c x_m^2 \omega_f^2}{2} \left[t - \frac{2\sin(\omega_f t - \phi)}{\omega_f} \right]_0^{2\pi/\omega_f}$$

$$E = \frac{c x_m^2 \omega_f^2}{2} \left[\frac{2\pi}{\omega_f} - \frac{2}{\omega_f} (\sin(2\pi - \varphi) + \sin\varphi) \right]$$

$E = \pi c x_m^2 \omega_f$ Q.E.D. ◀

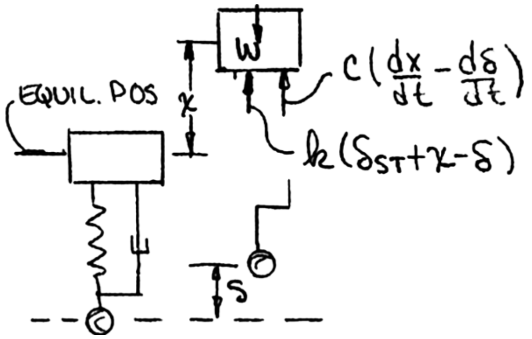


PROBLEM 19.151*

The suspension of an automobile can be approximated by the simplified spring-and-dashpot system shown. (*a*) Write the differential equation defining the vertical displacement of the mass *m* when the system moves at a speed *v* over a road with a sinusoidal cross section of amplitude δ_m and wave length *L*. (*b*) Derive an expression for the amplitude of the vertical displacement of the mass *m*.

SOLUTION

(*a*)


EQUIL. POS
x
W
$c\left(\frac{dx}{dt}-\frac{d\delta}{dt}\right)$
$k(\delta_{ST}+x-\delta)$
δ

$$+\downarrow \Sigma F = ma: \quad W - k(\delta_{\text{st}} + x - \delta) - c\left(\frac{dx}{dt} - \frac{d\delta}{dt}\right) = m\frac{d^2x}{dt^2}$$

Recalling that $W = k\delta_{\text{st}}$, we write

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = k\delta + c\frac{d\delta}{dt} \tag{1}$$

Motion of wheel is a sine curve, $\delta = \delta_m \sin \omega_f t$. The interval of time needed to travel a distance *L* at a speed *v* is $t = \frac{L}{v}$.

Thus,

$$\omega_f = \frac{2\pi}{\tau_f} = \frac{2\pi}{\frac{L}{v}} = \frac{2\pi v}{L}$$

and

$$\delta = \delta_m \sin \omega_f t \quad \frac{d\delta}{dt} = \frac{\delta_m 2\pi}{\frac{L}{v}} \cos \omega_f t$$

Thus, Equation (1) is

$$m\frac{d^2x}{dt} + c\frac{dx}{dt} + kx = (k \sin \omega_f t + c\omega_f \cos \omega_f t)\delta_m \blacktriangleleft$$

PROBLEM 19.151* (Continued)

(*b*) From the identity

$$A\sin y + B\cos y = \sqrt{A^2+B^2}\,\sin(y+\psi)$$

$$\sin\psi = \frac{B}{\sqrt{A^2+B^2}}$$

$$\cos\psi = \frac{A}{\sqrt{A^2+B^2}}$$

We can write the differential equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = \delta_m\sqrt{k^2+(c\omega_f)^2}\,\sin(\omega_f t+\psi)$$

$$\psi = \tan^{-1}\frac{c\omega_f}{k}$$

The solution to this equation is analogous to Equations 19.47 and 19.48, with

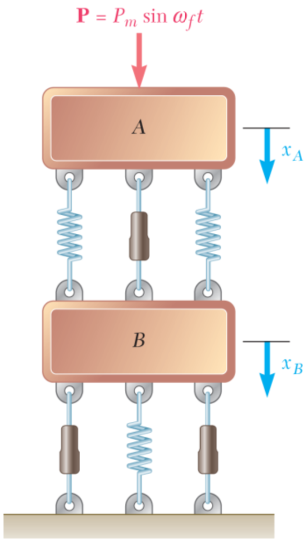
$$P_m = \delta_m\sqrt{k^2+(c\omega_f)^2}$$

$x = x_m \sin(\omega_f t - \varphi + \psi)$ (where analogous to Equations (19.52)) ◀

$$x_m = \frac{\delta_m\sqrt{k^2+(c\omega_f)^2}}{\sqrt{\left(k-m\omega_f^2\right)^2+(c\omega_f)^2}} \quad ◀$$

$$\tan\varphi = \frac{c\omega_f}{k-m\omega_f^2} \quad ◀$$

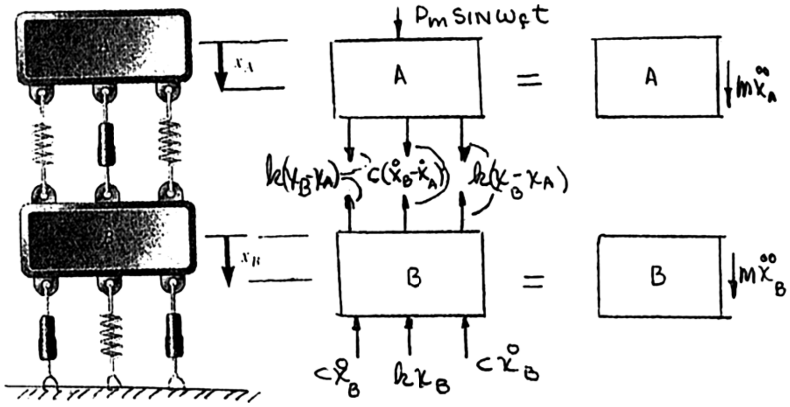
$$\tan\psi = \frac{c\omega_f}{k} \quad ◀$$


$P = P_m \sin \omega_f t$
A
x_A
B
x_B

PROBLEM 19.152*

Two blocks A and B, each of mass m, are supported as shown by three springs of the same constant k. Blocks A and B are connected by a dashpot, and block B is connected to the ground by two dashpots, each dashpot having the same coefficient of damping c. Block A is subjected to a force of magnitude $P = P_m \sin \omega_f t$. Write the differential equations defining the displacements x_A and x_B of the two blocks from their equilibrium positions.

SOLUTION


x_A
x_B
$P_m \sin \omega_f t$
A
=
A
$m\ddot{x}_A$
$k(x_B - x_A)$
$c(\dot{x}_B - \dot{x}_A)$
$k(x_B - x_A)$
B
=
B
$m\ddot{x}_B$
$c\dot{x}_B$
kx_B
$c\dot{x}_B$

Since the origins of coordinates are chosen from the equilibrium position, we may omit the initial spring compressions and the effect of gravity

For load A,

$$+\downarrow \Sigma F = ma_A: \quad P_m \sin \omega_f t + 2k(x_B - x_A) + c(\dot{x}_B - \dot{x}_A) = m\ddot{x}_A \qquad (1)$$

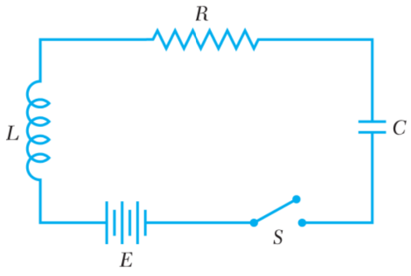
For load B,

$$+\downarrow \Sigma F = ma_B: \quad -2k(x_B - x_A) - c(\dot{x}_B - \dot{x}_A) - kx_B - 2c\dot{x}_B = m\ddot{x}_B \qquad (2)$$

Rearranging Equations (1) and (2), we find:

$$m\ddot{x}_A + c(\dot{x}_A - \dot{x}_B) + 2k(x_A - x_B) = P_m \sin \omega_f t \blacktriangleleft$$

$$m\ddot{x}_B + 3c\dot{x}_B - c\dot{x}_A + 3kx_B - 2kx_A = 0 \blacktriangleleft$$



PROBLEM 19.153

Express in terms of L, C, and E the range of values of the resistance R for which oscillations will take place in the circuit shown when switch S is closed.

SOLUTION

For a mechanical system, oscillations take place if $c < c_c$ (lightly damped).

But from Equation (19.41), $$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km}$$

Therefore, $$c < 2\sqrt{km} \qquad (1)$$

From Table 19.2:

$$c \longrightarrow R$$

$$m \longrightarrow L$$

$$k \longrightarrow \frac{1}{C} \qquad (2)$$

Substituting in Eq. (1) the analogous electrical values in Eq. (2), we find that oscillations will take place if

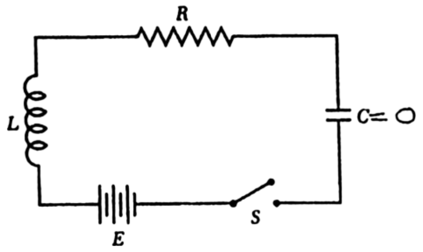
$$R < 2\sqrt{\left(\frac{1}{C}\right)(L)} \qquad R < 2\sqrt{\frac{L}{C}} \blacktriangleleft$$

PROBLEM 19.154

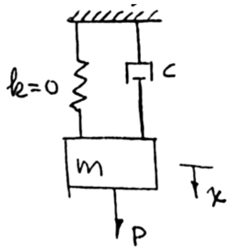
Consider the circuit of Problem 19.153 when the capacitor *C* is removed. If switch *S* is closed at time $t = 0$, determine (*a*) the final value of the current in the circuit, (*b*) the time *t* at which the current will have reached $(1 - 1/e)$ times its final value. (The desired value of *t* is known as the *time constant* of the circuit.)

SOLUTION

Electrical system

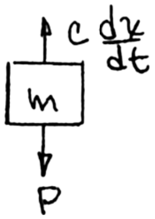


Mechanical system



The mechanical analogue of closing a switch *S* is the sudden application of a constant force of magnitude *P* to the mass.

(*a*) Final value of the current corresponds to the final velocity of the mass, and since the capacitance is zero, the spring constant is also zero



$$+\downarrow \Sigma F = ma: \quad P - c\frac{dx}{dt} = m\frac{d^2x}{dt^2} \tag{1}$$

Final velocity occurs when $\dfrac{d^2x}{dt^2} = 0$

$$P - c\left.\frac{dx}{dt}\right|_{\text{final}} = 0 \qquad \left.\frac{dx}{dt}\right|_{\text{final}} = v_{\text{final}}$$

$$v_{\text{final}} = \frac{P}{c}$$

From Table 19.2: $v \longrightarrow i, \quad P \longrightarrow E, \quad c \longrightarrow R$

Thus, $i_{\text{final}} = \dfrac{E}{R}$ ◀

PROBLEM 19.154 (Continued)

(*b*) Rearranging Equation (1), we have

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}=P$$

Substitute $$\frac{dx}{dt}=Ae^{-\lambda t}+\frac{P}{c}; \qquad \frac{d^2x}{dt}=-A\lambda e^{-\lambda t}$$

$$m\left[-A\lambda e^{-\lambda t}\right]+c\left[Ae^{-\lambda t}+\frac{P}{c}\right]=P$$

$$-m\lambda+c=0 \qquad \lambda=\frac{c}{m}$$

Thus, $$\frac{dx}{dt}=Ae^{-(c/m)t}+\frac{P}{c}$$

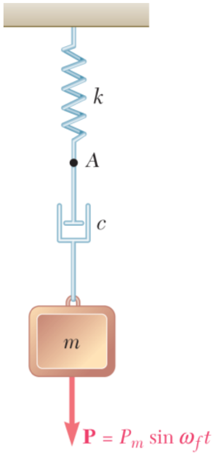
At $t=0$, $$\frac{dx}{dt}=0 \quad 0=A+\frac{P}{c} \qquad A=-\frac{P}{c}$$

$$v=\frac{dx}{dt}=\frac{p}{c}\left[1-e^{-(c/m)t}\right]$$

From Table 19.2: $$v\longrightarrow i,\ \ P\longrightarrow E,\ \ c\longrightarrow R,\ \ m\longrightarrow L$$

$$L=\frac{E}{R}\left[1-e^{-(R/L)t}\right]$$

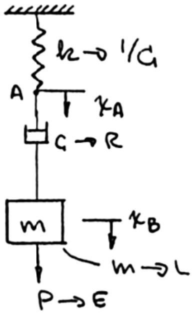
For $i=\left(\frac{E}{R}\right)\left(1-\frac{1}{e}\right)$, $$\left(\frac{R}{L}\right)t=1 \qquad t=\frac{L}{R}$$ ◀


k
A
c
m
$\mathbf{P} = P_m \sin \omega_f t$

PROBLEM 19.155

Draw the electrical analogue of the mechanical system shown. (*Hint:* Draw the loops corresponding to the free bodies m and A.)

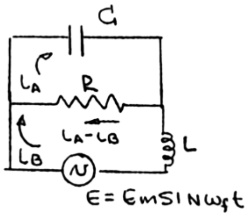
SOLUTION


k → 1/C
A
x_A
c → R
m
x_B
m → L
P → E

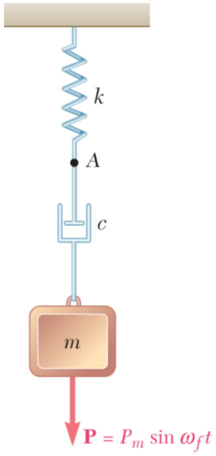
We note that both the spring and the dashpot affect the motion of Point A. Thus, one loop in the electrical circuit should consist of a capacitor $\left(k \Rightarrow \frac{1}{C}\right)$ and a resistance $(c \Rightarrow R)$.

The other loop consists of $(P_m \sin \omega_f t \longrightarrow E_m \sin \omega_f t)$, an inductor $(m \longrightarrow L)$ and the resistor $(c \longrightarrow R)$.

Since the resistor is common to both loops, the circuit is


C
i_A
R
$i_A - i_B$
i_B
L
$E = E_m \sin \omega_f t$

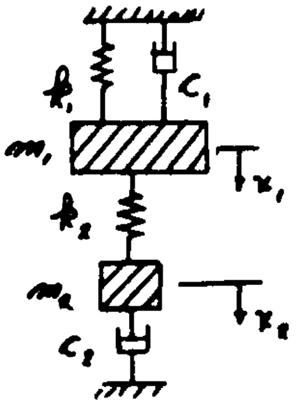
◀

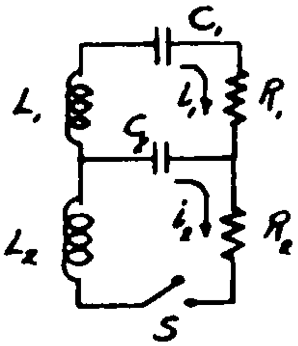

k
A
c
m
P = $P_m \sin \omega_f t$

PROBLEM 19.156

Draw the electrical analogue of the mechanical system shown. (*Hint:* Draw the loops corresponding to the free bodies *m* and *A*.)

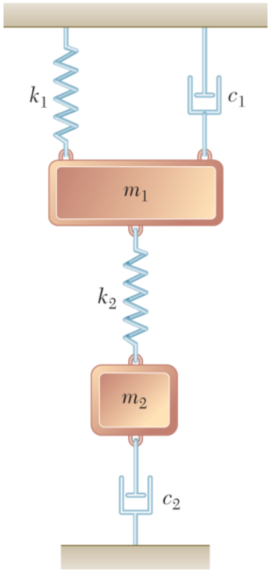
SOLUTION


k_1
c_1
m_1
x_1
k_2
m_2
x_2
c_2


C_1
L_1
i_1
R_1
C_2
L_2
i_2
R_2
S

Loop 1 (Mass 1)	Loop 2 (Mass 2)
$k_1 \longrightarrow 1/C_1$	$k_2 \longrightarrow 1/C_2$
$c_1 \longrightarrow R_1$	$c_2 \longrightarrow R_2$
$m_1 \longrightarrow L_1$	$m_2 \longrightarrow L_2$
$x_1 \longrightarrow q_1$	$x_2 \longrightarrow q_2$
$\dot{x}_1 \longrightarrow \dot{x}_1$	$\dot{x}_2 \longrightarrow \dot{x}_2$

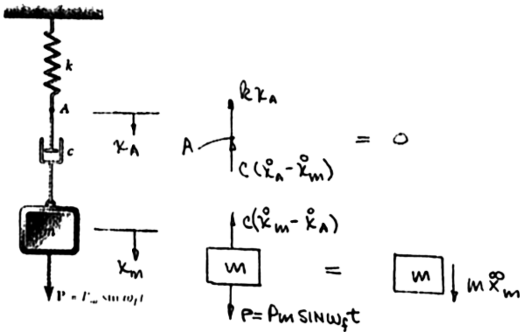
k_2 is connected to both masses, so C_2 is common to both loops.


k_1
c_1
m_1
k_2
m_2
c_2

PROBLEM 19.157

Write the differential equations defining (*a*) the displacements of the mass *m* and of the Point *A*, (*b*) the charges on the capacitors of the electrical analogue.

SOLUTION


k
A
c
x_A
A
kx_A
$c(\dot{x}_A - \dot{x}_m)$
= 0
x_m
$P = P_m \sin \omega_f t$
$c(\dot{x}_m - \dot{x}_A)$
m
$P = P_m \sin \omega_f t$
=
m
$m\ddot{x}_m$

(*a*) Mechanical system.

Point *A*: $+\uparrow \Sigma F = 0$:

$$c\frac{d}{dt}(x_A - x_m) + kx_A = 0 \quad ◀$$

Mass *m*:

$$+\uparrow \Sigma F = ma: \quad c\frac{d}{dt}(x_m - x_A) - P_m \sin \omega_f t = -m\frac{d^2 x_m}{dt^2}$$

$$m\frac{d^2 x_m}{dt^2} + c\frac{d}{dt}(x_m - x_A) = P_m \sin \omega_f t \quad ◀$$

(*b*) Electrical analogue.

From Table 19.2:

$m \longrightarrow L$

$c \longrightarrow R$

$k \longrightarrow \dfrac{1}{C}$

$x \longrightarrow q$

$P \longrightarrow E$

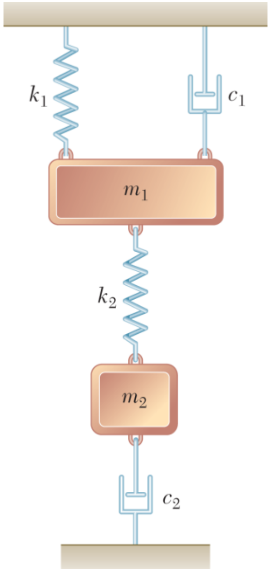
PROBLEM 19.157 (Continued)

Substituting into the results from Part (*a*), the analogous electrical characteristics,

$$R\frac{d}{dt}(q_A - q_m) + \left(\frac{1}{C}\right)q_n = 0 \blacktriangleleft$$

$$L\frac{d^2q_m}{dt^2} + R\frac{d}{dt}(q_m - q_A) = E_m \sin \omega_f t \blacktriangleleft$$

Note: These equations can also be obtained by summing the voltage drops around the loops in the circuit of Problem 19.155.



PROBLEM 19.158

Write the differential equations defining (*a*) the displacements of the masses m_1 and m_2, (*b*) the charges on the capacitors of the electrical analogue.

SOLUTION

(*a*) Displacements at masses m_1 and m_2

$$m_1 \frac{d^2x_1}{dt^2} + c_1 \frac{dx_1}{dt} + k_1 x_1 + k_2(x_1 - x_2) = 0 \quad ◀$$

$$m_2 \frac{d^2x_2}{dt^2} + c_2 \frac{dx_2}{dt} + k_2(x_2 - x_1) = 0 \quad ◀$$

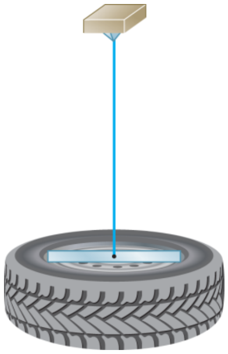
(*b*) Electrical analogues.

We let: $q_1 = \int i_1 dt \qquad q_2 = \int i_2 dt$

Thus, $i_1 = \frac{dq_1}{dt} \qquad i_2 = \frac{dq_2}{dt}$

$$L_1 \frac{d^2q_1}{dt^2} + R_1 \frac{dq_1}{dt} + \frac{q_1}{C_1} + \frac{(q_1 - q_2)}{C_2} = 0 \quad ◀$$

$$L_2 \frac{d^2q_2}{dt^2} + R_2 \frac{dq_2}{dt} + \frac{q_2 - q_1}{C_2} = 0 \quad ◀$$



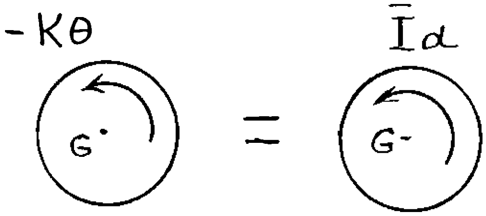
PROBLEM 19.159

An automobile wheel-and-tire assembly of total weight 47 lb is attached to a mounting plate of negligible weight which is suspended from a steel wire. The torsional spring constant of the wire is known to be $K = 0.40\ \text{lb}\cdot\text{in./rad}$. The wheel is rotated through 90° about the vertical and then released. Knowing that the period of oscillation is observed to be 30 s, determine the centroidal mass moment of inertia and the centroidal radius of gyration of the wheel-and-tire assembly.

SOLUTION

Torsional spring constant: $K = 0.40\ \text{lb}\cdot\text{in/rad} = 33.333\times10^{-3}\ \text{lb}\cdot\text{ft/rad}$

Let the wheel-and-tire assembly be rotated through the small angle θ. The moment that the wire exerts on the assembly is



$$M = -K\theta$$

$$\Sigma M = \Sigma M_{\text{eff}} = \bar{I}\alpha: \quad -K\theta = \bar{I}\alpha = \bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$

$$\omega_n^2 = \frac{K}{\bar{I}} \qquad (1)$$

Frequency: $f = \dfrac{1}{\tau} = \dfrac{1}{30\ \text{s}} = 0.033333\ \text{H}_z$

$$\omega_n = 2\pi f = (2\pi)(0.033333) = 0.20944\ \text{rad/s}$$

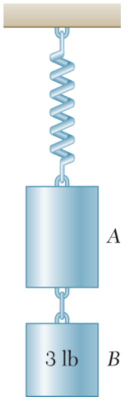
From Eq. (1), $\bar{I} = \dfrac{K}{\omega_n^2} = \dfrac{33.333\times10^{-3}\ \text{lb}\cdot\text{ft/rad}}{(0.20944\ \text{rad/s})^2}$

Centroidal mass moment of inertia: $\bar{I} = 0.75990\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$ $\qquad \bar{I} = 0.760\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$ ◀

Mass: $m = \dfrac{W}{g} = \dfrac{47\ \text{lb}}{32.2\ \text{ft/s}^2} = 1.4596\ \text{lb}\cdot\text{s}^2/\text{ft}$

Centroidal radius of gyration: $\bar{k}^2 = \dfrac{\bar{I}}{m} = \dfrac{0.75990\ \text{lb}\cdot\text{s}^2\cdot\text{ft}}{1.4596\ \text{lb}\cdot\text{s}^2/\text{ft}} = 0.52061\ \text{ft}^2$

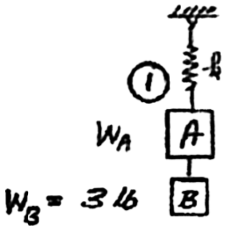
$\bar{k} = 0.7215\ \text{ft}$ $\qquad \bar{k} = 8.66\ \text{in.}$ ◀



PROBLEM 19.160

The period of vibration of the system shown is observed to be 0.6 s. After cylinder B has been removed, the period is observed to be 0.5 s. Determine (*a*) the weight of cylinder A, (*b*) the constant of the spring.

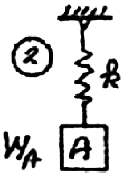
SOLUTION



$$m_1 = \frac{W_A + 3}{g} \qquad \tau_1 = 0.6 \text{ s}$$

$$\tau_1 = \frac{2\pi}{\omega_1} \qquad \omega_1 = \frac{2\pi}{\tau_1} = \frac{2\pi}{0.6} = 3.333\pi \text{ rad/s}$$

$$\omega_1^2 = \frac{k}{m_1} \qquad k = m_1\omega_1^2 = \left(\frac{W_A + 3}{g}\right)(3.333\pi)^2 \tag{1}$$



$$m_2 = \frac{W_A}{g} \qquad \tau_2 = 0.5 \text{ s}$$

$$\tau_2 = \frac{2\pi}{\omega_2} \qquad \omega_2 = \frac{2\pi}{\tau_2} = \frac{2\pi}{0.5} = 4\pi \text{ rad/s}$$

$$\omega_2^2 = \frac{k}{m_2} \qquad k = m_2\omega_2^2 = \frac{W_A}{g}(4\pi)^2 \tag{2}$$

(*a*) Equating the expressions found for k in Eqs. (1) and (2):

$$\frac{W_A + 3}{g}(3.333\pi)^2 = \frac{W_A}{g}(4\pi)^2$$

$$(11.111)(W_A + 3) = 16W_A$$

$$4.889W_A = 33.333$$

$$W_A = 6.818 \text{ lb}$$

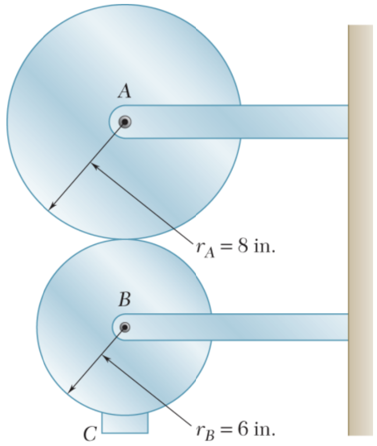
$W_A = 6.82$ lb ◀

(*b*) Eq. (1):

$$k = \frac{6.818 \text{ lb} + 3 \text{ lb}}{32.2 \text{ ft/s}^2}(3.333\pi \text{ rad/s})^2$$

$$k = 33.44 \text{ lb/ft}$$

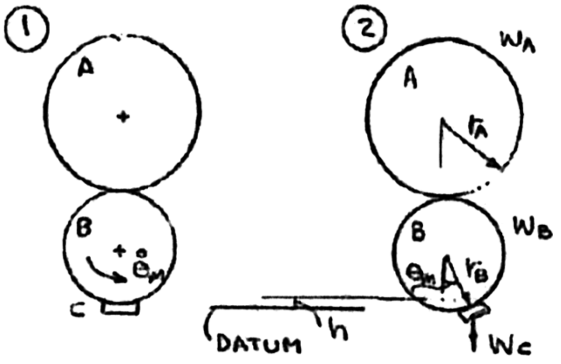
$k = 33.4$ lb/ft ◀



PROBLEM 19.161

Disks *A* and *B* weigh 30 lb and 12 lb, respectively, and a small 5-lb block *C* is attached to the rim of disk *B*. Assuming that no slipping occurs between the disks, determine the period of small oscillations of the system.

SOLUTION



Small oscillations:

$$h = r_B(1 - \cos\theta_m) \approx \frac{r_B\theta_B^2}{2}$$

Position ①

$$r_B\dot{\theta}_B = r_A\dot{\theta}_A$$

$$T_1 = \frac{1}{2}m_C(r_B\dot{\theta}_m)^2 + \frac{1}{2}\bar{I}_B\dot{\theta}_m^2 + \frac{1}{2}\bar{I}_A\left(\frac{r_B}{r_A}\dot{\theta}_m\right)^2$$

$$\bar{I}_B = \frac{m_B r_B^2}{2}$$

$$\bar{I}_A = \frac{m_A r_A^2}{2}$$

$$T_1 = \frac{1}{2}\left[m_C r_B^2 + \frac{m_B r_B^2}{2} + \left(\frac{m_A r_A^2}{2}\right)\left(\frac{r_B}{r_A}\right)^2\right]\dot{\theta}_m^2$$

$$T_1 = \frac{1}{2}\left[\left(m_C + \frac{m_B}{2} + \frac{m_A}{2}\right)\right]r_B^2\dot{\theta}_m^2$$

$$V_1 = 0$$

Position ②

$$T_2 = 0$$
$$V_2 = m_C g h$$
$$= \frac{m_C g \dot{\theta}_m^2}{2}$$

Conservation of energy and simple harmonic motion.

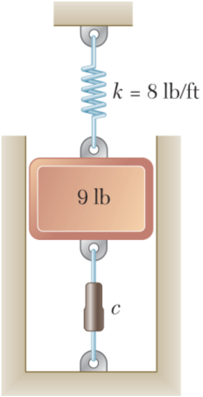
$$T_1 + V_1 = T_2 + V_2$$
$$\dot{\theta}_m = \omega_n \theta_m$$
$$\frac{1}{2}\left[\left(m_C + \frac{m_B}{2} + \frac{m_A}{2}\right)\right] r_B^2 \omega_n^2 \theta_m^2 + 0 = 0 + \frac{m_C g r_B \theta_m^2}{2}$$
$$\omega_n^2 = \frac{m_C}{m_C + \frac{(m_B + m_A)}{2}} \frac{g}{r_B}$$
$$\omega_n^2 = \frac{5}{5 + \frac{(12+30)}{2}} \frac{(32.2 \text{ ft/s}^2)}{\left(\frac{6}{12}\right)\text{ft}}$$
$$\omega_n^2 = 12.39 \text{ s}^{-2}$$

Period of small oscillations.

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{12.39}}$$

$\tau_n = 1.785$ s ◀

PROBLEM 19.162



The block shown is depressed 1.2 in. from its equilibrium position and released. Knowing that after 10 cycles the maximum displacement of the block is 0.5 in., determine (*a*) the damping factor *c/c*, (*b*) the value of the coefficient of viscous damping. (*Hint:* See Problems 19.129 and 19.130.)

SOLUTION

From Problems 19.130 and 19.129:

$$\left(\frac{1}{k}\right)\ln\left(\frac{x_n}{x_{n+k}}\right)=\frac{2\pi\frac{c}{c_c}}{\sqrt{1-\left(\frac{c}{c_c}\right)^2}}$$

where $k=$ number of cycles $=10$

(*a*) First maximum is $x_1=1.2$ in.

Thus, $n=1$

$$\frac{x_1}{x_{1+10}}=\frac{1.2}{0.5}=2.4$$

$$\frac{1}{10}\ln 2.4=0.08755$$

$$=\frac{2\pi\frac{c}{c_c}}{\sqrt{1-\left(\frac{c}{c_c}\right)^2}}$$

Damping factor.

$$1-\left(\frac{c}{c_c}\right)^2=\left(\frac{2\pi}{0.08755}\right)^2\left(\frac{c}{c_c}\right)^2$$

$$\left(\frac{c}{c_c}\right)^2\left[\left(\frac{2\pi}{0.08755}\right)^2+1\right]=1$$

$$\left(\frac{c}{c_c}\right)^2=\frac{1}{(5150+1)}$$

$$=0.0001941$$

$$\frac{c}{c_c}=0.01393 \blacktriangleleft$$

(*b*) Critical damping coefficient. $c_c = 2\,m\sqrt{\dfrac{k}{m}}$ (Eq. 19.41)

or $c_c = 2\sqrt{km}$

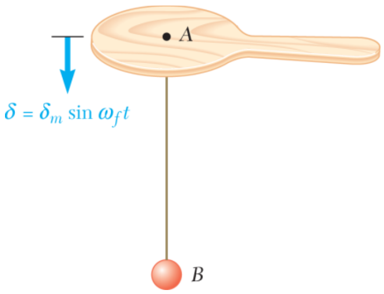
$$c_c = 2\sqrt{(8\text{ lb/ft})\left(\tfrac{9\text{ lb}}{32.2\text{ ft/s}^2}\right)}$$

$$c_c = 2.991\text{ lb}\cdot\text{s/ft}$$

From Part (*a*), $\dfrac{c}{c_c} = 0.01393$

$$c = (0.01393)(2.991)$$

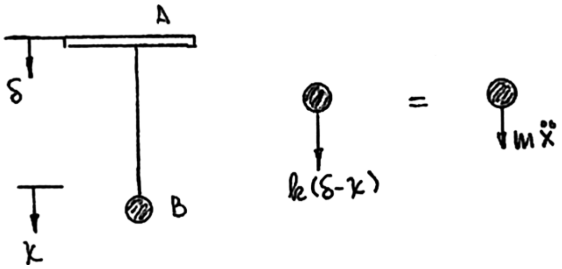
Coefficient of viscous damping. $c = 0.0417\text{ lb}\cdot\text{s/ft}$ ◀



PROBLEM 19.163

An 0.8-lb ball is connected to a paddle by means of an elastic cord *AB* of constant $k = 5$ lb/ft. Knowing that the paddle is moved vertically according to the relation $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 8$ in., determine the maximum allowable circular frequency ω_f if the cord is not to become slack.

SOLUTION



$$\Sigma F = ma \qquad k(\delta - x) = m\ddot{x} \qquad \ddot{x} + \left(\frac{k}{m}\right)x = \delta$$

From Equation (19.31 and 19.33′):
$$x_m = \frac{\delta_m}{\left(1 - \frac{\omega_f^2}{\omega_n^2}\right)}$$

Data:
$$m = \frac{W}{g}$$
$$= \frac{0.8}{32.2}$$
$$= 0.024845 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$k = 5 \text{ lb/ft} \qquad \delta_m = 8 \text{ in.} = 0.66667 \text{ ft}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$
$$= \sqrt{\frac{5}{0.024845}}$$
$$= 14.186 \text{ rad/s}$$

The cord becomes slack if $x_m - \delta_m$ exceeds δ_{st}, where

$$\delta_{\text{st}} = \frac{W}{k} = \frac{0.8 \text{ lb}}{5 \text{ lb/ft}} = 0.16 \text{ ft}$$

PROBLEM 19.163 (Continued)

Then
$$\frac{0.66667}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} - 0.66667 < 0.16$$

$$0.66667 - 0.66667 + 0.66667\left(\frac{\omega_f}{\omega_n}\right)^2 < 0.16 - 0.16\left(\frac{\omega_f}{\omega_n}\right)^2$$

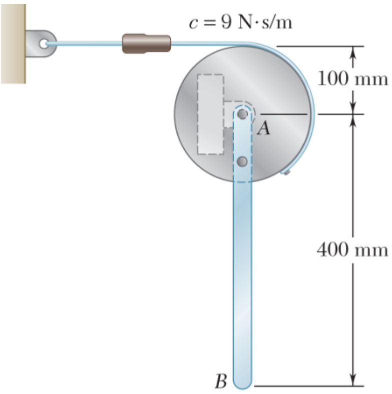
$$0.82667\left(\frac{\omega_f}{\omega_n}\right)^2 < 0.16$$

$$\frac{\omega_f}{\omega_n} < \sqrt{\frac{0.16}{0.82667}} = 0.43994$$

Maximum allowable circular frequency.

$$\omega_f < (0.43994)(14.186\text{ rad/s})$$

$\omega_f < 6.24$ rad/s ◀


$c = 9$ N·s/m
100 mm
A
400 mm
B

PROBLEM 19.164

A 3-kg slender rod AB is bolted to a 5-kg uniform disk. A dashpot of damping coefficient $c = 9\ \text{N}\cdot\text{s/m}$ is attached to the disk as shown. Determine (*a*) the differential equation of motion for small oscillations, (*b*) the damping factor c/c_c.

SOLUTION

Data: $r = 100\ \text{mm} = 0.100\ \text{m},\ \ l = 400\ \text{mm} = 0.400\ \text{m}$

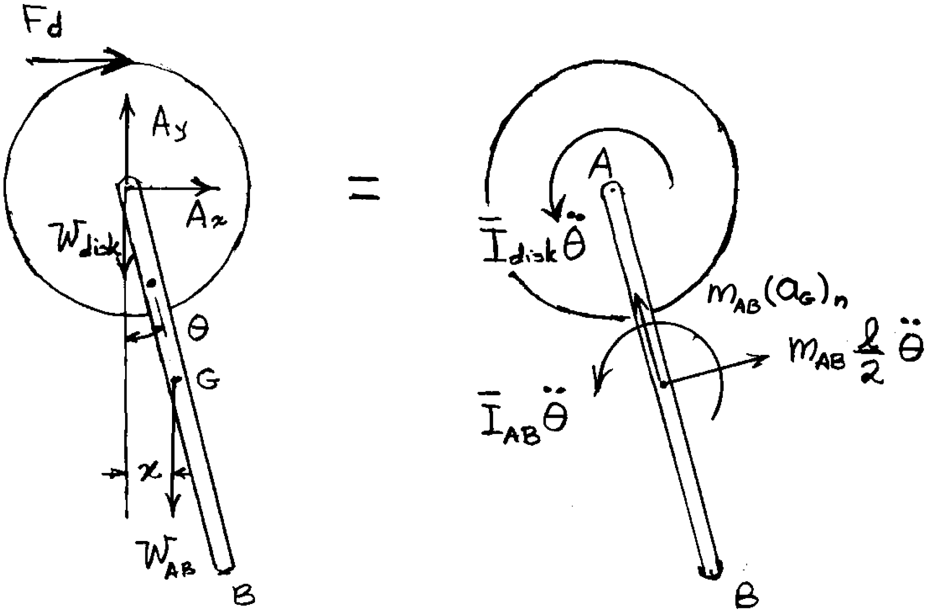
$$\overline{I}_{\text{disk}} = \frac{1}{2}m_{\text{disk}}r^2 = \frac{1}{2}(5\ \text{kg})(0.100\ \text{m})^2 = 0.025\ \text{kg}\cdot\text{m}^2$$

$$\overline{I}_{AB} = \frac{1}{2}m_{AB}l^2 = \frac{1}{12}(3\ \text{kg})(0.400\ \text{m})^2 = 0.040\ \text{kg}\cdot\text{m}^2$$

$$W_{AB} = m_{AB}g = (3\ \text{kg})(9.81\ \text{m/s}^2) = 29.43\ \text{N}$$

$$c = 9\ \text{N}\cdot\text{s/m}$$

Equation of motion: Let the disk and rod assembly be rotated through a small counterclockwise angle θ


F_d
A_y
A_x
W_{disk}
θ
G
x
W_{AB}
B
=
A
$\overline{I}_{disk}\ddot{\theta}$
$m_{AB}(a_G)_n$
$m_{AB}\frac{l}{2}\ddot{\theta}$
$\overline{I}_{AB}\ddot{\theta}$
B

$$\Sigma M_A = \Sigma(M_A)_{\text{eff}}:$$

$$-W_{AB}x - F_d r = \overline{I}_{\text{disk}}\ddot{\theta} + \overline{I}_{AB}\ddot{\theta} + m_{AB}\left(\frac{l}{2}\ddot{\theta}\right)\frac{l}{2}$$

PROBLEM 19.164 (Continued)

where $W_{AB}x = -W_{AB}\dfrac{l}{2}\sin\theta$

$= (29.43\text{ N})(0.2\text{ m})\sin\theta$

$= 5.886\sin\theta\ \text{N}\cdot\text{m}$

$\approx -5.886\,\theta$

Damping force: $F_d = cr\dot{\theta}$

$$F_d r = cr^2\dot{\theta} = (9\text{ N}\cdot\text{s}\cdot\text{m})(0.100\text{ m})^2\dot{\theta} = 0.09\dot{\theta} = C\dot{\theta}$$

Inertia: $\bar{I}_{\text{disk}} + \bar{I}_{AB} + m_{AB}\left(\dfrac{l}{2}\right)^2 = 0.025 + 0.040 + (3)(0.2)^2 = 0.185\text{ kg}\cdot\text{m}^2$

$$-5.886\theta - 0.09\dot{\theta} = 0.185\ddot{\theta}$$

$$0.185\ddot{\theta} + 0.09\dot{\theta} + 5.886\theta = 0$$

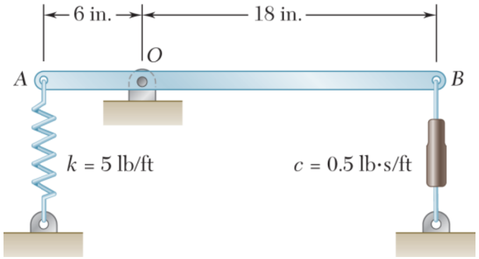
$$M\ddot{\theta} + C\dot{\theta} + K\theta = 0$$

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{5.886}{0.185}} = 5.6406\text{ rad/s}$$

$$C_c = 2\,M\omega_2 = (2)(0.185)(5.6406) = 2.087$$

$$\frac{c}{c_c} = \frac{C}{C_c} = \frac{0.09}{2.087}$$

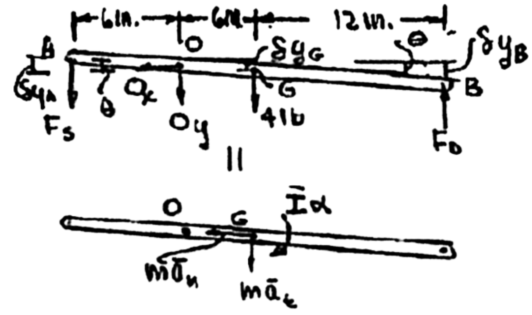
$\dfrac{c}{c_c} = 0.0431$ ◀



PROBLEM 19.165

A 4-lb uniform rod is supported by a pin at *O* and a spring at *A*, and is connected to a dashpot at *B*. Determine (*a*) the differential equation of motion for small oscillations, (*b*) the angle that the rod will form with the horizontal 5 s after end *B* has been pushed 0.9 in. down and released.

SOLUTION



Small angles: $\sin\theta \approx \theta, \quad \cos\theta \approx 1$

$$\delta y_A = \left(\frac{6}{12}\text{ ft}\right)\theta = \frac{\theta}{2}$$

$$\delta y_C = \left(\frac{6}{12}\text{ ft}\right)\theta = \frac{\theta}{2}$$

$$\delta y_B = \left(\frac{18}{12}\text{ ft}\right)\theta = \frac{3\theta}{2}$$

(*a*) Newton's Law: $\Sigma M_0 = (\Sigma M_0)_{\text{eff}}$

$$\overset{\curvearrowright}{+}\ -\left(\frac{6}{12}\text{ ft}\right)F_s + \left(\frac{6}{12}\text{ft}\right)(4) - \left(\frac{18}{12}\text{ ft}\right)F_D$$

$$= \bar{I}\alpha + \left(\frac{6}{12}\text{ ft}\right)m\bar{a}_t \qquad (1)$$

$$F_s = k(\delta y_A + (\delta_{\text{st}})_A) = k\left(\frac{\theta}{2} + (\delta_{\text{st}})_A\right)$$

$$F_D = c\delta\dot{y}_B = c\frac{3}{2}\dot{\theta}$$

$$\bar{I} = \frac{1}{12}ml^2 = \frac{1}{12}m\left(\frac{24}{12}\text{ft}\right)^2 = \frac{1}{3}m$$

Kinematics: $\alpha = \ddot{\theta}, \quad \bar{a}_t = \left(\frac{6}{12}\text{ft}\right)\alpha = \frac{\ddot{\theta}}{2}$

Thus, from Eq. (1),

$$\left[\frac{m}{3} + \frac{m}{4}\right]\ddot{\theta} + \left(\frac{3}{2}\right)^2 c\ddot{\theta} + \left(\frac{k}{2}\right)\left(\frac{\theta}{2} + (\delta_{\text{st}})_A\right) - 2 = 0 \qquad (2)$$

But in equilibrium, $\Sigma M_0 = 0$

$$\overset{\curvearrowright}{+}\ k(\delta_{\text{st}})_A\left(\frac{6}{12}\right) - (4)\left(\frac{6}{12}\right) = 0, \quad \frac{k}{2}(\delta_{\text{st}})_A = 2$$

Equation (2) becomes

$$\left(\frac{7}{12}\right)m\ddot{\theta}+\left(\frac{9}{4}\right)\left(\ddot{\theta}+\frac{k}{4}\right)\theta=0$$

$$\frac{7}{12}m=\left(\frac{7}{12}\right)\left(\frac{4}{32.2}\right)=0.07246$$

$$\frac{9}{4}c=\left(\frac{9}{4}\right)(0.15)=0.3375$$

$$\frac{k}{4}=\frac{5}{4}=1.25 \qquad 0.07246\ddot{\theta}+0.3375\dot{\theta}+1.25\theta=0 \blacktriangleleft$$

(*b*) Substituting $e^{\lambda t}$ into the above differential equation,

$$0.07246\lambda^2+0.3375\lambda+1.25=0$$

$$\lambda=\frac{(-0.33757\ \sqrt{(0.3375)^2-4(.07246)(1.25)})}{(2)(0.07246)}$$

$$\lambda=\frac{(-0.33757\ \sqrt{(-0.2484)})}{(2)(0.07246)}$$

$$\lambda=-2.3296\ 3.439i$$

Since the roots are complex and conjugate (light damping), the solution to the differential equation is (Eq. 19.46):

$$\theta=\theta_0 e^{-2.329t}\sin(3.939t+\phi) \qquad (3)$$

Initial conditions. $(\delta y_B)(0)=0.9$ in.

$$\theta(0)=\frac{(\delta y_B)}{18\text{ in.}}=\frac{0.9}{18}$$

$$\theta(0)=0.05\text{ rad}$$

$$\dot{\theta}(0)=0$$

From Eq. (3):

$$\theta(0)=0.05=\theta_0\sin\phi$$

$$\dot{\theta}(0)=0=-2.329\theta_0\sin\phi+3.439\theta_0\cos\phi$$

$$\tan\phi=\frac{3.439}{2.329}$$

$$\phi=0.9755\text{ rad}$$

$$\theta_0=\frac{0.05}{\sin(0.9755)}=0.06039\text{ rad}$$

PROBLEM 19.165 (Continued)

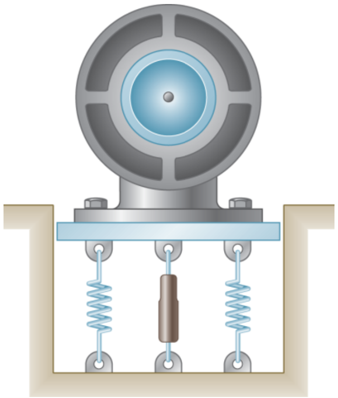
Substituting into Eq. (3), $\theta = 0.06039e^{-2.329t}\sin(3.439t + 0.9752)$

At $t = 5$ s, $\theta = 0.06039e^{-(2.329)(5)}\sin[(3.439)(5) + 0.9752]$

$$= 0.06039e^{-11.645}\sin(18.1702)$$

$$= (0.06039)(8.7627\times 10^{-6})(-0.6283)$$

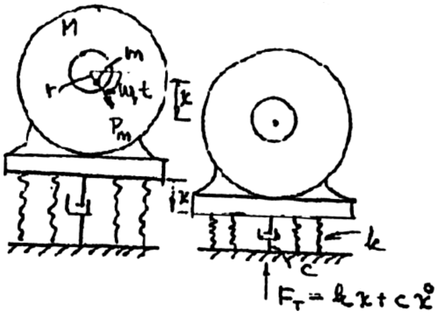
$$= -0.332\times 10^{-6}\text{ rad} \qquad \theta = -19.05\times 10^{-6}\text{ degrees} \blacktriangleleft$$



PROBLEM 19.166

A 400-kg motor supported by four springs, each of constant 150 kN/m, and a dashpot of constant $c = 6500\ \text{N}\cdot\text{s/m}$ is constrained to move vertically. Knowing that the unbalance of the rotor is equivalent to a 23-g mass located at a distance of 100 mm from the axis of rotation, determine for a speed of 800 rpm (*a*) the amplitude of the fluctuating force transmitted to the foundation, (*b*) the amplitude of the vertical motion of the motor.

SOLUTION



Total mass: $M = 400$ kg

Unbalance: $m = 23\text{ g} = 0.023$ kg

$r = 100\text{ mm} = 0.100$ m

Forcing frequency: $\omega_f = 800$ rpm

$= 83.776$ rad/s

Spring constant: $(4)(150\times10^3\ \text{N/m}) = 600\times10^3\ \text{N/m}$

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{600\times10^3}{400}}$$

$= 38.730$ rad/s

Frequency ratio:

$$\frac{\omega_f}{\omega_n} = 2.1631$$

Viscous damping coefficient: $c = 6500\ \text{N}\cdot\text{s/m}$

Critical damping coefficient:

$$c_c = 2\sqrt{kM} = 2\sqrt{(600\times10^3)(400)}$$

$= 30{,}984\ \text{N}\cdot\text{s/m}$

Damping factor:

$$\frac{c}{c_c} = 0.20978$$

Unbalance force:

$$P_m = mr\omega_f^2 = (0.023)(0.100)(83.776)^2$$

$= 16.1424$ N

Static deflection:

$$\delta_{\text{st}} = \frac{P_m}{k} = \frac{16.1424}{600\times10^3}$$

$= 26.904\times10^{-6}$ m

PROBLEM 19.166 (Continued)

Amplitude of vibration. Use Eq. (19.53).

$$x_m = \frac{\frac{P_m}{k}}{\sqrt{1\left[1-\left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$

Where $$1-\left(\frac{\omega_f}{\omega_n}\right)^2 = 1-(2.1631)^2 = -3.679$$

and $$2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right) = (2)(0.20978)(2.1631) = 0.90755$$

$$x_m = \frac{26.904\times 10^{-6}}{\sqrt{(-3.679)^2+(0.90755)^2}}$$
$$= 7.1000\times 10^{-6}\text{ m}$$

Resulting motion: $$x = x_m \sin(\omega_f t - \varphi)$$
$$\dot{x} = \omega_f x_m \cos(\omega_f t - \varphi)$$

Spring force: $$F_s = kx = kx_m \sin(\omega_f t - \varphi) = 4.26\sin(\omega_f t - \varphi)$$

Damping force: $$F_d = c\dot{x} = c\omega_f x_m \cos(\omega_f t - \varphi) = 3.8663\cos(\omega_f t - \varphi)$$

Let $F_s = F_m \cos\psi \sin(\omega_f t - \varphi)$ and $F_d = F_m \sin\psi \cos(\omega_f t - \varphi)$

Total force: $$F = F_m \cos\psi \sin(\omega_f t - \varphi) + F_m \sin\psi \cos(\omega_f t - \varphi)$$
$$= F_m \sin(\omega_f t - \varphi + \psi)$$

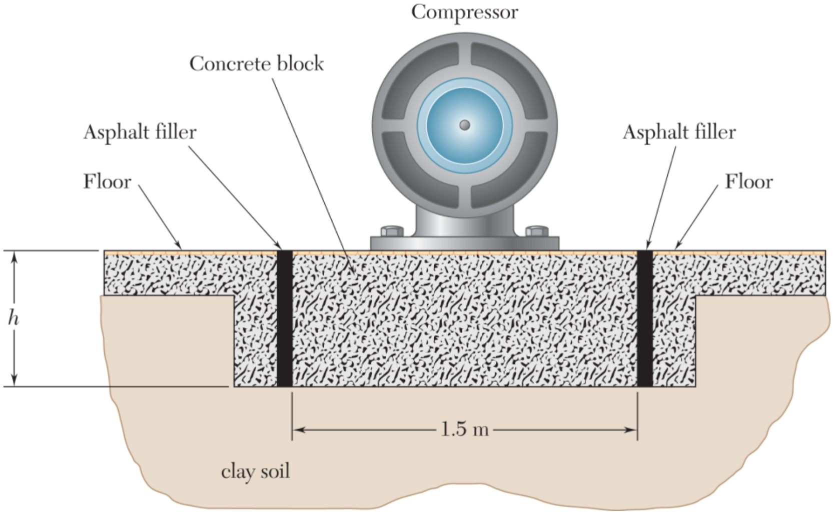
(*a*) Force amplitude. $$F_m = \sqrt{(F_m\cos\psi)^2 + (F_m\sin\psi)^2}$$
$$F_m = \sqrt{(kx_m)^2 + (c\omega_f x_m)^2}$$
$$= \sqrt{(4.26)^2 + (3.8663)^2}$$

$F_m = 5.75$ N ◀

(*b*) Amplitude of vibration. $x_m = 0.00710$ mm ◀

PROBLEM 19.167

The compressor shown has a mass of 250 kg and operates at 2000 rpm. At this operating condition the force transmitted to the ground is excessively high and is found to be $mr\omega_f^2$ where mr is the unbalance and ω_f is the forcing frequency. To fix this problem, it is proposed to isolate the compressor by mounting it on a square concrete block separated from the rest of the floor as shown. The density of concrete is 2400 kg/m^3 and the spring constant for the soil is found to be 80×10^6 N/m. The geometry of the compressor leads to choosing a block that is 1.5 m by 1.5 m. Determine the depth h that will reduce the force transmitted to the ground by 75%.



SOLUTION

Forced circular frequency corresponding to 2000 rpm.

$$\omega_f = \frac{(2\pi)(2000)}{60} = 209.44 \text{ rad/s}$$

In the first case the natural frequency is very large so that the transmitted force is $mr\omega_f^2$.

After the problem is fixed, the transmitted force is

$$P = kx_m = \frac{P_m}{\left|1-\left(\frac{\omega_f}{\omega_n}\right)^2\right|}$$

Since the motion is out-of-phase,

$$P = \frac{P_m}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} = \frac{mr\omega_f^2}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} \tag{1}$$

But $$P = (1-0.75)mr\omega_f^2 = 0.25\, mr\omega_f^2 \tag{2}$$

PROBLEM 19.167 (Continued)

Equating expressions (1) and (2) dividing by $mr\omega_f^2$,

$$\frac{1}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} = 0.25$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 - 1 = 4$$

$$\frac{\omega_f}{\omega_n} = \sqrt{5}$$

$$\omega_n = \frac{1}{\sqrt{5}}\omega_f = \frac{1}{\sqrt{5}}(209.44) = 93.664 \text{ rad/s}$$

$$\sqrt{\frac{k}{m}} = \omega_n$$

$$m = \frac{k}{\omega_n^2} = \frac{80 \times 10^6 \text{ N/m}}{(93.664 \text{ rad/s})^2} = 9119 \text{ kg}$$

Required properties of the attached concrete block.

Mass: $m - 250 \text{ kg} = 8869 \text{ kg}$

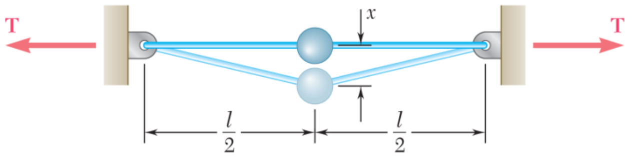
$$\text{volume} = \frac{\text{mass}}{\text{density}} = \frac{8869 \text{ kg}}{2400 \text{ kg/m}^3} = 3.6954 \text{ m}^3$$

$$\text{area} = 1.5 \text{ m} \times 1.5 \text{ m} = 2.25 \text{ m}^2$$

$$\text{depth} = \frac{\text{volume}}{\text{area}} = \frac{3.6954 \text{ m}^3}{2.25 \text{ m}^2}$$

$h = 1.642$ m ◀

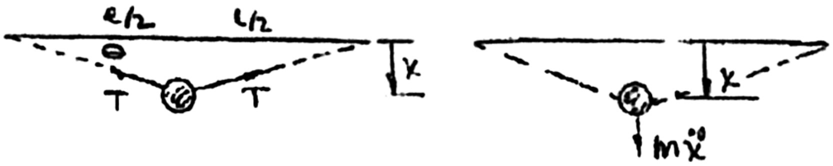
PROBLEM 19.168



A small ball of mass m attached at the midpoint of a tightly stretched elastic cord of length l can slide on a horizontal plane. The ball is given a small displacement in a direction perpendicular to the cord and released. Assuming the tension T in the cord to remain constant, (*a*) write the differential equation of motion of the ball, (*b*) determine the period of vibration.

SOLUTION

(*a*) Differential equation of motion.



$$+\uparrow \Sigma F = ma: \quad 2T\sin\theta = -m\ddot{x}$$

For small x,

$$\sin\theta \approx \tan\theta = \frac{x}{\left(\frac{l}{2}\right)} = \frac{2x}{l}$$

$$m\ddot{x} + (2T)\left(\frac{2x}{l}\right) = 0 \quad ◀$$

$$m\ddot{x} + \left(\frac{4T}{l}\right)x = 0$$

Natural circular frequency.

$$\omega_n^2 = \frac{4T}{ml}$$

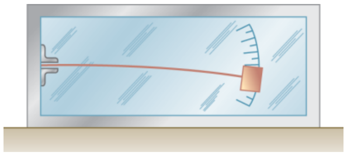
$$\omega_n = 2\sqrt{\frac{T}{ml}}$$

(*b*) Period of vibration.

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\tau_n = \pi\sqrt{\frac{ml}{T}} \quad ◀$$

PROBLEM 19.169



A certain vibrometer used to measure vibration amplitudes consists essentially of a box containing a slender rod to which a mass *m* is attached; the natural frequency of the mass-rod system is known to be 5 Hz. When the box is rigidly attached to the casing of a motor rotating at 600 rpm, the mass is observed to vibrate with an amplitude of 0.06 m. relative to the box. Determine the amplitude of the vertical motion of the motor.

SOLUTION

Natural frequency:

$$f_n = 5\text{ Hz}$$

$$\omega_n = 2\pi f_n = 31.416\text{ rad/s}$$

Forcing frequency:

$$f_f = 600\text{ rpm} = 10\text{ Hz}$$

$$\omega_f = 2\pi f_f = 62.832\text{ rad/s}$$

Ratio:

$$\frac{\omega_f}{\omega_n} = 2.000$$

$$x_m = \frac{\delta_m}{1-\left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{\delta_m}{1-(2)^2} = \frac{\delta_m}{-3}$$

$$x_m = -\frac{1}{3}\delta_m$$

Relative motion:

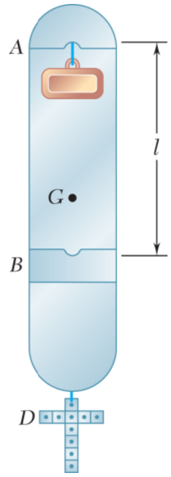
$$y_m = x_m - \delta_m = \frac{4}{3}x_m$$

The observed relative motion is

$$y_m = 0.06\text{ in.}$$

$$x_m = \frac{3}{4}y_m = \frac{3}{4}(0.06\text{ in.})$$

$x_m = 0.045$ in. ◀

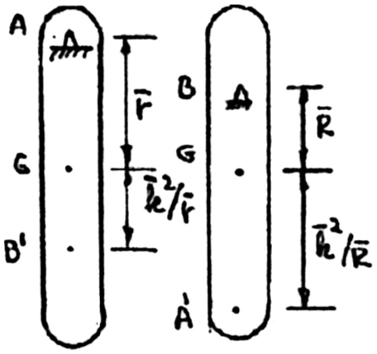


PROBLEM 19.170

If either a simple or a compound pendulum is used to determine experimentally the acceleration of gravity *g*, difficulties are encountered. In the case of the simple pendulum, the string is not truly weightless, while in the case of the compound pendulum, the exact location of the mass center is difficult to establish. In the case of a compound pendulum, the difficulty can be eliminated by using a reversible, or Kater, pendulum. Two knife edges *A* and *B* are placed so that they are obviously not at the same distance from the mass center *G*, and the distance *l* is measured with great precision. The position of a counterweight *D* is then adjusted so that the period of oscillation τ is the same when either knife edge is used. Show that the period τ obtained is equal to that of a true simple pendulum of length *l* and that $g = 4\pi^2 l/\tau^2$.

SOLUTION

From Problem 19.52, the length of an equivalent simple pendulum is:



$$l_A = \bar{r} + \frac{\bar{k}^2}{\bar{r}}$$

and
$$l_B = \bar{R} + \frac{\bar{k}^2}{\bar{R}}$$

But
$$\tau_A = \tau_B$$

$$2\pi\sqrt{\frac{l_A}{g}} = 2\pi\sqrt{\frac{l_B}{g}}$$

Thus,
$$l_A = l_B$$

For
$$l_A = l_B$$

$$\bar{r} + \frac{\bar{k}^2}{\bar{r}} = \bar{R} + \frac{\bar{k}^2}{\bar{R}}$$

$$\bar{r}^2\bar{R} + \bar{k}^2\bar{R} = \bar{r}\,\bar{R}^2 + \bar{k}^2\bar{r}$$

$$\bar{r}\,\bar{R}[\bar{r} - \bar{R}] = \bar{k}^2[\bar{r} - \bar{R}]$$

$$(\bar{r} - \bar{R}) \neq 0$$

Thus,
$$\bar{r}\,\bar{R} = \bar{k}^2$$

or
$$\bar{r} = \frac{\bar{k}^2}{\bar{R}}$$

$$\bar{R} = \frac{\bar{k}^2}{\bar{r}}$$

Thus, $AG = GA'$ and $BG = GB'$

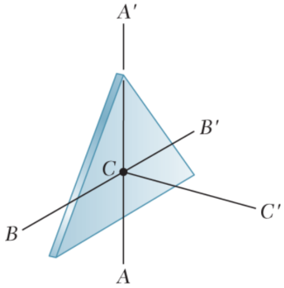
That is, $A = A'$ and $B = B'$

Noting that $l_A = l_B = l$

$$\tau = 2\pi\sqrt{\frac{l}{g}}$$

or $$g = \frac{4\pi^2 l}{\tau^2}$$

APPENDIX B

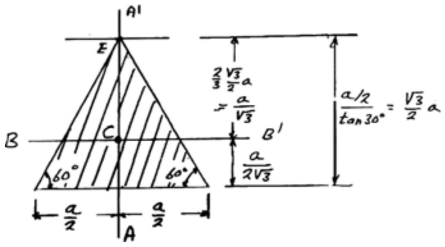


PROBLEM B.1

A thin plate of mass *m* is cut in the shape of an equilateral triangle of side *a*. Determine the mass moment of inertia of the plate with respect to (*a*) the centroidal axes *AA*′ and *BB*′, (*b*) the centroidal axis *CC*′ that is perpendicular to the plate.

SOLUTION

$$\text{Area} = \frac{1}{2}a\left(\frac{\sqrt{3}}{2}a\right) = \frac{\sqrt{3}}{4}a^2$$


A′
E
$\frac{2}{3}\frac{\sqrt{3}}{2}a$
$= \frac{a}{\sqrt{3}}$
$\frac{a/2}{\tan 30°} = \frac{\sqrt{3}}{2}a$
B
C
B′
60°
60°
$\frac{a}{2\sqrt{3}}$
$\frac{a}{2}$
$\frac{a}{2}$
A

$$\text{Mass} = m = \rho V = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

(*a*) Axis *AA*′:

$$I_{AA',\text{area}} = 2\left[\frac{1}{12}\left(\frac{\sqrt{3}}{2}a\right)\left(\frac{a}{2}\right)^3\right] = \frac{\sqrt{3}}{96}a^4$$

$$I_{AA',\text{ mass}} = \frac{m}{A} I_{AA',\text{ area}} = \frac{\frac{m}{\sqrt{3}}}{4}a^2\left(\frac{\sqrt{3}}{96}a^4\right) = \frac{1}{24}ma^2 \quad ◀$$

Axis *BB*″:

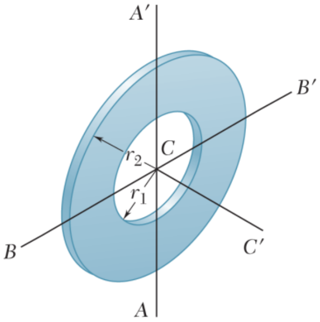
$$I_{BB',\text{ area}} = \frac{1}{36}a\left(\frac{\sqrt{3}}{2}a\right)^3 = \frac{\sqrt{3}}{96}a^4 \quad \text{(we check that } I_{AA} = I_{BB}\text{)}$$

$$I_{BB',\text{mass}} = \frac{m}{A} I_{BB',\text{ area}} = \frac{\frac{m}{\sqrt{3}}}{4}a^2\left(\frac{\sqrt{3}}{96}a^4\right) = \frac{1}{24}ma^2 \quad ◀$$

(b) Eq. (9.38):

$$I_{CC'} = I_{AA'} + I_{BB'} = 2\left(\frac{1}{24}ma^2\right)$$

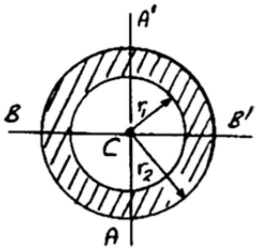
$$I_{CC'} = \frac{1}{12}ma^2 \quad ◀$$



PROBLEM B.2

A ring with a mass m is cut from a thin uniform plate. Determine the mass moment of inertia of the ring with respect to (*a*) the axis AA', (*b*) the centroidal axis CC' that is perpendicular to the plane of the ring.

SOLUTION



$$\text{Mass} = m = \rho V = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{A} I_{\text{area}}$$

We first determine

$$A = \pi r_2^2 - \pi r_1^2 = \pi\left(r_2^2 - r_1^2\right)$$

$$I_{AA',\text{area}} = \frac{\pi}{4} r_2^4 - \frac{\pi}{4} r_1^4 = \frac{\pi}{4}\left(r_2^4 - r_1^4\right)$$

(*a*)

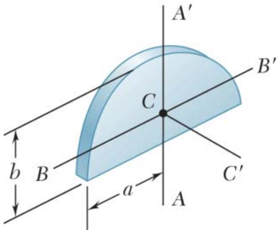
$$I_{AA',\text{ mass}} = \frac{m}{A} I_{AA',\text{ area}} = \frac{m}{\pi\left(r_2^2 - r_1^2\right)} \frac{\pi}{4}\left(r_2^4 - r_1^4\right) = \frac{1}{4} m\left(r_2^2 + r_1^2\right)$$

$$I_{AA'} = \frac{1}{4} m\left(r_1^2 + r_2^2\right) \blacktriangleleft$$

(*b*) By symmetry: $I_{BB'} = I_{AA'}$

Eq. (9.38): $I_{CC'} = I_{AA'} + I_{BB'} = 2I_{AA'}$

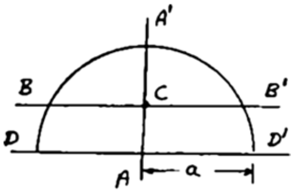
$$I_{CC'} = \frac{1}{2} m\left(r_1^2 + r_2^2\right) \blacktriangleleft$$



PROBLEM B.3

A thin, semielliptical plate has a mass m. Determine the mass moment of inertia of the plate with respect to (a) the centroidal axis BB', (b) the centroidal axis CC' that is perpendicular to the plate.

SOLUTION



$$\text{mass} = m = \rho t A$$

$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{2m}{\pi ab} I_{\text{area}}$$

Area: $$A = \frac{1}{2}\pi a^2$$

$$I_{x,\text{area}} = I_{BB',\text{area}} + A\bar{y}^2$$

$$I_{BB',\text{area}} = \frac{\pi}{8}ab^3 - \left(\frac{\pi}{2}ab\right)\left(\frac{4b}{3\pi}\right)^2$$

$$I_{BB',\text{area}} = \frac{\pi}{8}ab^3\left(1 - \frac{64}{9\pi^2}\right)$$

(a) $$I_{BB',\text{mass}} = \frac{2m}{\pi ab}\left(\frac{\pi}{8}ab^3\right)\left(1 - \frac{64}{9\pi^2}\right)$$

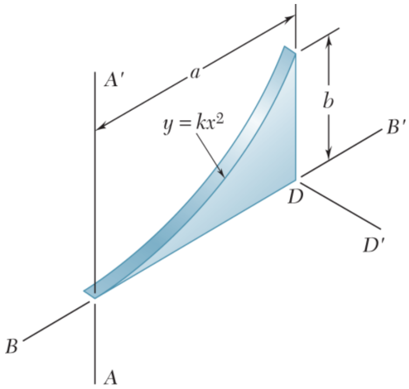
$$= \frac{1}{4}mb^2\left(1 - \frac{64}{9\pi^2}\right)$$

$$I_{BB'} = 0.0699mb^2 \blacktriangleleft$$

(b) Fig. (9.12): $I_{AA',area} = \frac{\pi}{8}a^3 b$, then $I_{AA',mass} = \left(\frac{2m}{\pi ab}\right)\left(\frac{\pi}{8}a^3 b\right) = \frac{1}{4}ma^2$

$$I_{CC',mass} = I_{AA',mass} + I_{BB',mass} = \frac{1}{4}ma^2 + \frac{1}{4}mb^2\left(1 - \frac{64}{9\pi^2}\right)$$

$$I_{CC'} = \frac{1}{4}\text{m}(a^2 + 0.279b^2) \blacktriangleleft$$



PROBLEM B.4

The parabolic spandrel shown was cut from a thin, uniform plate. Denoting the mass of the spandrel by *m*, determine its mass moment of inertia with respect to (*a*) the axis *BB*′, (*b*) the axis *DD*′ that is perpendicular to the spandrel. (*Hint:* See Sample Problem 9.3.)

SOLUTION

First note

$$\text{mass} = m = \rho V = \rho t A$$

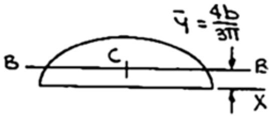
$$= \rho t\left(\frac{\pi}{2}ab\right)$$

Also

$$I_{\text{mass}} = \rho t I_{\text{area}}$$

$$= \frac{2m}{\pi ab} I_{\text{area}}$$

(*a*) We have



$$I_{x,\text{area}} = I_{BB',\text{area}} + A\bar{y}^2$$

or

$$I_{BB',\text{area}} = \frac{\pi}{8}ab^3 - \left(\frac{\pi}{2}ab\right)\left(\frac{4b}{3\pi}\right)^2$$

(*a*) From Sample Problem 9.3:

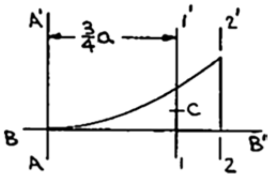
$$I_{BB',\text{area}} = \frac{1}{21}ab^3$$

Then

$$I_{BB',\text{mass}} = \frac{3m}{ab} \times \frac{1}{21}ab^3$$

or $I_{BB'} = \frac{1}{7}mb^2$ ◀

(*b*) From Sample Problem 9.3:



$$I_{AA',\text{area}} = \frac{1}{5}a^3b$$

Now

$$I_{AA',\text{area}} = I_{11',\text{area}} + A\left(\frac{3}{4}a\right)^2$$

and

$$I_{22',\text{area}} = I_{11',\text{area}} + A\left(\frac{1}{4}a\right)^2$$

Then

$$\begin{aligned}
I_{22',\text{area}} &= I_{AA',\text{area}} + A\left[\left(\frac{1}{4}a\right)^2 - \left(\frac{3}{4}a\right)^2\right] \\
&= \frac{1}{5}a^3b + \frac{1}{3}ab\left(\frac{1}{16}a^2 - \frac{9}{16}a^2\right) \\
&= \frac{1}{30}a^3b
\end{aligned}$$

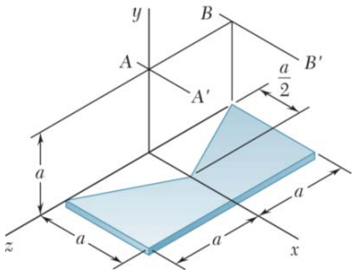
and

$$\begin{aligned}
I_{22',\text{mass}} &= \frac{3m}{ab} \times \frac{1}{30}a^3b \\
&= \frac{1}{10}ma^2
\end{aligned}$$

Finally,

$$\begin{aligned}
I_{DD',\text{mass}} &= I_{BB',\text{mass}} + I_{22',\text{mass}} \\
&= \frac{1}{7}mb^2 + \frac{1}{10}ma^2
\end{aligned}$$

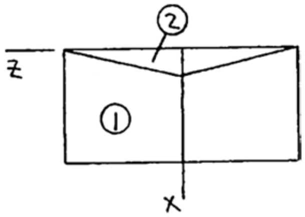
or $I_{DD'} = \frac{1}{70}m(7a^2 + 10b^2)$ ◀



PROBLEM B.5

A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by *m*, determine its mass moment of inertia with respect to (*a*) the *x* axis, (*b*) the *y* axis.

SOLUTION



First note
$$\text{mass} = m = \rho V = \rho t A$$
$$= \rho t\left[(2a)(a) - \frac{1}{2}(2a)\left(\frac{a}{2}\right)\right]$$
$$= \frac{3}{2}\rho t a^2$$

Also
$$I_{\text{mass}} = \rho t I_{\text{area}}$$
$$= \frac{2m}{3a^2} I_{\text{area}}$$

(*a*) Now
$$\bar{I}_{x,\text{area}} = (I_x)_{1,\text{area}} - 2(I_x)_{2,\text{area}}$$
$$= \frac{1}{12}(a)(2a)^3 - 2\left[\frac{1}{12}\left(\frac{a}{2}\right)(a)^3\right]$$
$$= \frac{7}{12}a^4$$

Then
$$\bar{I}_{x,\text{mass}} = \frac{2m}{3a^2} \times \frac{7}{12}a^4$$

or $\bar{I}_x = \dfrac{7}{18}ma^2$ ◀

(*b*) We have
$$\bar{I}_{z,\text{area}} = (I_z)_{1,\text{area}} - 2(I_z)_{2,\text{area}}$$
$$= \frac{1}{3}(2a)(a)^3 - 2\left[\frac{1}{12}(a)\left(\frac{a}{2}\right)^3\right]$$
$$= \frac{31}{48}a^4$$

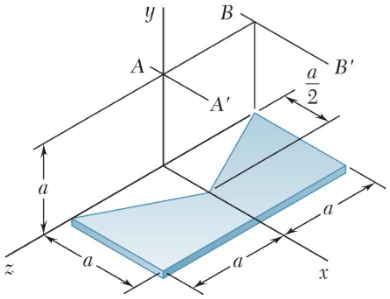
Then
$$I_{z,\text{mass}} = \frac{2m}{3a^2} \times \frac{31}{48}a^4$$
$$= \frac{31}{72}ma^2$$

PROBLEM B.5 (Continued)

Finally,

$$\begin{aligned} I_{y,\text{mass}} &= \bar{I}_{x,\text{mass}} + I_{z,\text{mass}} \\ &= \frac{7}{18}ma^2 + \frac{31}{72}ma^2 \\ &= \frac{59}{72}ma^2 \end{aligned}$$

or $I_y = 0.819ma^2$ ◀



PROBLEM B.6

A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by m, determine its mass moment of inertia with respect to (*a*) the axis AA', (*b*) the axis BB', where the AA' and BB' axes are parallel to the x axis and lie in a plane parallel to and at a distance a above the xz plane.

SOLUTION

First note that the x axis is a centroidal axis so that

$$I = \bar{I}_{x,\text{mass}} + md^2$$

and that from the solution to Problem 9.115,

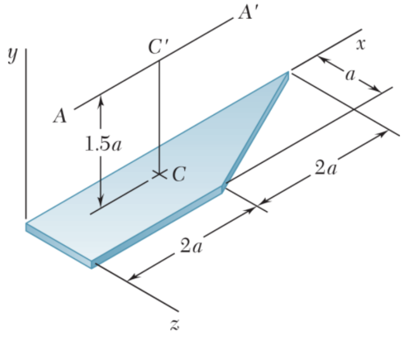
$$\bar{I}_{x,\text{mass}} = \frac{7}{18}ma^2$$

(*a*) We have $$I_{AA',\text{mass}} = \frac{7}{18}ma^2 + m(a)^2$$

or $I_{AA'} = 1.389ma^2$ ◀

(*b*) We have $$I_{BB',\text{mass}} = \frac{7}{18}ma^2 + m(a\sqrt{2})^2$$

or $I_{BB'} = 2.39ma^2$ ◀



PROBLEM B.7

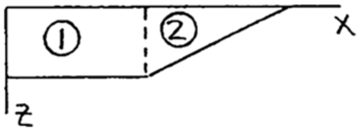
A thin plate of mass *m* has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (*a*) the *x* axis, (*b*) the *y* axis.

SOLUTION

First note
$$\text{mass} = m = \rho V = \rho t A$$
$$= \rho t\left[(2a)(a) + \frac{1}{2}(2a)(a)\right] = 3\rho t a^2$$

Also
$$I_{\text{mass}} = \rho t I_{\text{area}} = \frac{m}{3a^2} I_{\text{area}}$$

(*a*) Now
$$I_{x,\text{area}} = (I_x)_{1,\text{area}} + (I_x)_{2,\text{area}}$$
$$= \frac{1}{3}(2a)(a)^3 + \frac{1}{12}(2a)(a)^3$$
$$= \frac{5}{6}a^4$$

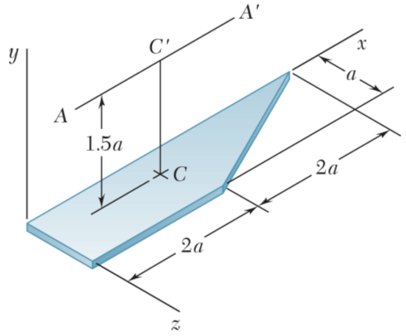


Then
$$I_{x,\text{mass}} = \frac{m}{3a^2} \times \frac{5}{6}a^4$$
or $I_{x,\text{ mass}} = \frac{5}{18}ma^2$ ◀

(*b*) We have
$$I_{z,\text{area}} = (I_z)_{1,\text{area}} + (I_z)_{2,\text{ area}}$$
$$= \left[\frac{1}{3}(a)(2a)^3\right] + \left[\frac{1}{36}(a)(2a)^3 + \frac{1}{2}(2a)(a)\left(2a + \frac{1}{3} \times 2a\right)^2\right] = 10a^4$$

Then
$$I_{x,\text{mass}} = \frac{m}{3a^2} \times 10a^4 = \frac{10}{3}ma^2$$

Finally,
$$I_{y,\text{mass}} = I_{x,\text{mass}} + I_{z,\text{mass}}$$
$$= \frac{5}{18}ma^2 + \frac{10}{3}ma^2$$
$$= \frac{65}{18}ma^2$$
or $I_{y,\text{mass}} = 3.61ma^2$ ◀



PROBLEM B.8

A thin plate of mass *m* has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (*a*) the centroidal axis *CC*′ that is perpendicular to the plate, (*b*) the axis *AA*′ that is parallel to the *x* axis and is located at a distance 1.5*a* from the plate.

SOLUTION

First locate the centroid *C*.

$$\bar{X}\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(2a^2 + a^2) = a(2a^2) + \left(2a + \frac{1}{3}\times 2a\right)(a^2)$$

or

$$\bar{X} = \frac{14}{9}a$$

$$\bar{Z}\Sigma A = \Sigma \bar{z}A: \quad \bar{Z}(2a^2 + a^2) = \left(\frac{1}{2}a\right)(2a^2) + \left(\frac{1}{3}a\right)(a^2)$$

or

$$\bar{Z} = \frac{4}{9}a$$

(*a*) We have $I_{y,\,\text{mass}} = \bar{I}_{CC',\,\text{mass}} + m(\bar{X}^2 + \bar{Z}^2)$

From the solution to Problem 9.117:

$$I_{y,\text{mass}} = \frac{65}{18}ma^2$$

Then $\bar{I}_{cc',\text{mass}} = \frac{65}{18}ma^2 - m\left[\left(\frac{14}{9}a\right)^2 + \left(\frac{4}{9}a\right)^2\right]$

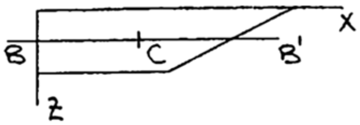
or $\bar{I}_{cc'} = 0.994ma^2$ ◀

(*b*) We have $I_{x,\text{mass}} = \bar{I}_{BB',\text{mass}} + m(\bar{Z})^2$

and $I_{AA',\,\text{mass}} = \bar{I}_{BB',\,\text{mass}} + m(1.5a)^2$

Then $I_{AA',\,\text{mass}} = I_{x,\,\text{mass}} + m\left[(1.5a)^2 - \left(\frac{4}{9}a\right)^2\right]$

PROBLEM B.8 (Continued)

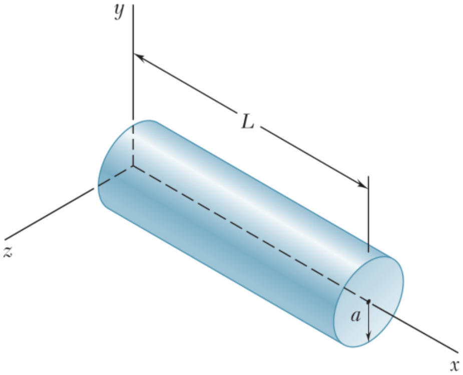


From the solution to Problem 9.193:

$$I_{x,\text{mass}} = \frac{5}{18}ma^2$$

Then $$I_{AA',\text{mass}} = \frac{5}{18}ma^2 + m\left[(1.5a)^2 - \left(\frac{4}{9}a\right)^2\right]$$

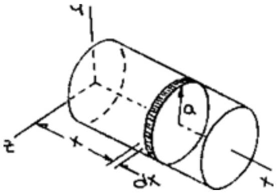
or $I_{AA'} = 2.33ma^2$ ◀



PROBLEM B.9

Determine by direct integration the mass moment of inertia with respect to the *z* axis of the right circular cylinder shown, assuming that it has a uniform density and a mass *m*.

SOLUTION



For the cylinder: $m = \rho V = \rho \pi a^2 L$

For the element shown: $dm = \rho \pi a^2 dx$

$$= \frac{m}{L} dx$$

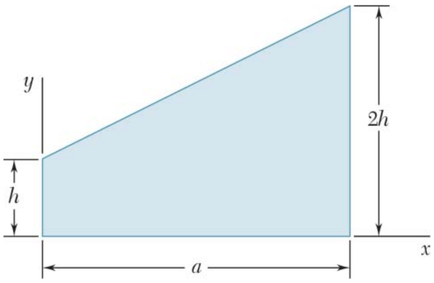
and $dI_z = dI_z + x^2 dm$

$$= \frac{1}{4} a^2 dm + x^2 dm$$

Then $I_z = \int dI_z = \int_0^L \left(\frac{1}{4} a^2 + x^2 \right) \left(\frac{m}{L} dx \right)$

$$= \frac{m}{L} \left[\frac{1}{4} a^2 x + \frac{1}{3} x^3 \right]_0^L$$

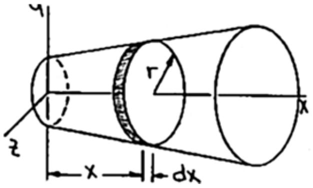
$$I_z = \frac{1}{12} m(3a^2 + 4L^2) \blacktriangleleft$$



PROBLEM B.10

The area shown is revolved about the *x* axis to form a homogeneous solid of revolution of mass *m*. Using direct integration, express the mass moment of inertia of the solid with respect to the *x* axis in terms of *m* and *h*.

SOLUTION



We have $$y = \frac{2h-h}{a}x + h$$

so that $$r = \frac{h}{a}(x+a)$$

For the element shown:

$$dm = \rho\pi r^2 dx \quad dI_x = \frac{1}{2}r^2 dm$$

$$= \rho\pi\left[\frac{h}{a}(x+a)\right]^2 dx$$

Then $$m = \int dm = \int_0^a \rho\pi\frac{h^2}{a^2}(x+a)^2 dx = \frac{1}{3}\rho\pi\frac{h^2}{a^2}[(x+a)^3]_0^a$$

$$= \frac{1}{3}\rho\pi\frac{h^2}{a^2}(8a^3 - a^3) = \frac{7}{3}\rho\pi a h^2$$

Now $$I_x = \int dI_x = \int\frac{1}{2}r^2(\rho\pi r^2 dx) = \frac{1}{2}\rho\pi\int_0^a\left[\frac{h}{a}(x+a)\right]^4 dx$$

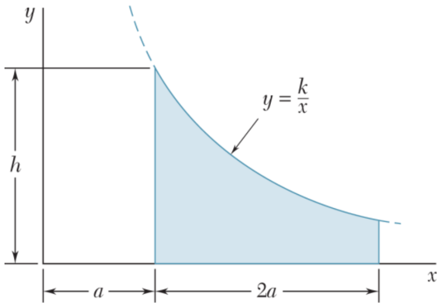
$$= \frac{1}{2}\rho\pi \times \frac{1}{5}\frac{h^4}{a^4}[(x+a)^5]_0^a = \frac{1}{10}\rho\pi\frac{h^4}{a^4}(32a^5 - a^5)$$

$$= \frac{31}{10}\rho\pi a h^4$$

From above: $$\rho\pi a h^2 = \frac{3}{7}m$$

Then $$I_x = \frac{31}{10}\left(\frac{3}{7}m\right)h^2 = \frac{93}{70}mh^2$$

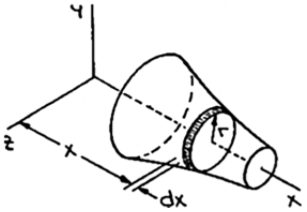
or $I_x = 1.329\ mh^2$ ◀



PROBLEM B.11

The area shown is revolved about the *x* axis to form a homogeneous solid of revolution of mass *m*. Determine by direct integration the mass moment of inertia of the solid with respect to (*a*) the *x* axis, (*b*) the *y* axis. Express your answers in terms of *m* and the dimensions of the solid.

SOLUTION



We have at (a, h): $h = \frac{k}{a}$

or $k = ah$

For the element shown:

$$r = 4$$

$$dm = \rho\pi r^2 dx$$

$$= \rho\pi\left(\frac{ah}{x}\right)^2 dx$$

Then

$$m = \int dm = \int_a^{3a} \rho\pi\left(\frac{ah}{x}\right)^2 dx$$

$$= \rho\pi a^2h^2\left[-\frac{1}{x}\right]_a^{3a}$$

$$= \rho\pi a^2h^2\left[-\frac{1}{3a} - \left(-\frac{1}{a}\right)\right] = \frac{2}{3}\rho\pi ah^2$$

(*a*) For the element:

$$dI_x = \frac{1}{2}r^2dm = \frac{1}{2}\rho\pi r^4dx$$

Then

$$I_x = \int dI_x = \int_a^{3a}\frac{1}{2}\rho\pi\left(\frac{ah}{x}\right)^4 dx = \frac{1}{2}\rho\pi a^4h^4\left[-\frac{1}{3}\frac{1}{x^3}\right]_a^{3a}$$

$$= -\frac{1}{6}\rho\pi a^4h^4\left[\left(\frac{1}{3a}\right)^3 - \left(\frac{1}{a}\right)^3\right] = \frac{1}{6}\times\frac{26}{27}\rho\pi ah^4$$

$$= \frac{1}{6}\times\frac{2}{3}\rho\pi ah^2\times\frac{13}{9}h^2 = \frac{13}{54}mh^2$$

or $I_x = 0.241\,mh^2$ ◀

PROBLEM B.11 (Continued)

(*b*) For the element:

$$dI_y = dI_y + x^2 dm$$

$$= \frac{1}{4} r^2 dm + x^2 dm$$

Then

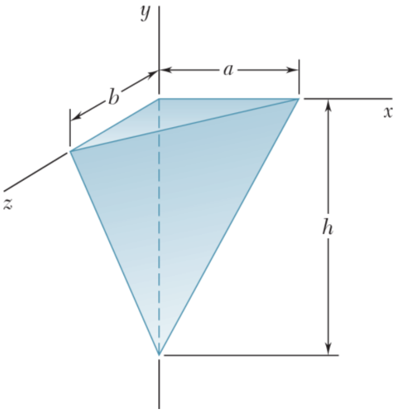
$$I_y = \int dI_y = \int_a^{3a} \left[\frac{1}{4} \left(\frac{ah}{x} \right)^2 + x^2 \right] \rho\pi \left(\frac{ah}{x} \right)^2 dx$$

$$= \rho\pi a^2 h^2 \int_a^{3a} \left(\frac{1}{4} \frac{a^2 h^2}{x^4} + 1 \right) dx = \rho\pi a^2 h^2 \left[-\frac{1}{12} \frac{a^2 h^2}{x^3} + x \right]_a^{3a}$$

$$= \left(\frac{3}{2} m \right) a \left\{ \left[-\frac{1}{12} \frac{a^2 h^2}{(3a)^3} + 3a \right] - \left[-\frac{1}{12} \frac{a^2 h^2}{(a)^3} + a \right] \right\}$$

$$= \frac{3}{2} ma \left(\frac{1}{12} \times \frac{26}{27} \frac{h^2}{a} + 2a \right) = m \left(\frac{13}{108} h^2 + 3a^2 \right)$$

or $I_y = m(3a^2 + 0.1204h^2)$ ◀



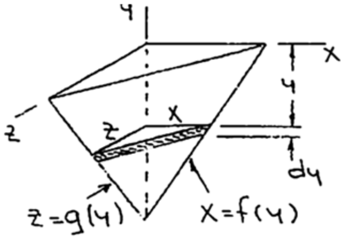
PROBLEM B.12

Determine by direct integration the mass moment of inertia with respect to the *x* axis of the tetrahedron shown, assuming that it has a uniform density and a mass *m*.

SOLUTION

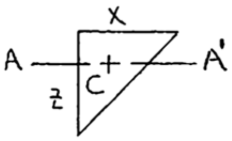
We have
$$x = \frac{a}{h}y + a = a\left(1 + \frac{y}{h}\right)$$

and
$$z = \frac{b}{h}y + b = b\left(1 + \frac{y}{h}\right)$$



For the element shown:
$$dm = \rho\left(\frac{1}{2}xz\,dy\right) = \frac{1}{2}\rho ab\left(1 + \frac{y}{h}\right)^2 dy$$

Then
$$m = \int dm = \int_{-h}^{0} \frac{1}{2}\rho ab\left(1 + \frac{y}{h}\right)^2 dy$$
$$= \frac{1}{2}\rho ab \times \frac{h}{3}\left[\left(1 + \frac{y}{h}\right)^3\right]_{-h}^{0}$$
$$= \frac{1}{6}\rho abh\left[(1)^3 - (1-1)^3\right]$$
$$= \frac{1}{6}\rho abh$$



Now, for the element:
$$I_{AA',\text{area}} = \frac{1}{36}xz^3 = \frac{1}{36}ab^3\left(1 + \frac{y}{h}\right)^4$$

Then
$$dI_{AA',\text{mass}} = \rho t I_{AA',\text{area}} = \rho(dy)\left[\frac{1}{3}ab^3\left(1 + \frac{y}{h}\right)^4\right]$$

PROBLEM B.12 (Continued)

Now

$$dI_x = dI_{AA',\text{mass}} + \left[y^2 + \left(\frac{1}{3}z\right)^2\right]dm$$

$$= \frac{1}{36}\rho ab^3\left(1+\frac{y}{h}\right)^4 dy$$

$$+\left\{y^2 + \left[\frac{1}{3}b\left(1+\frac{y}{h}\right)\right]^2\right\}\left[\frac{1}{2}\rho ab\left(1+\frac{y}{h}\right)^2 dy\right]$$

$$= \frac{1}{12}\rho ab^3\left(1+\frac{y}{h}\right)^4 dy + \frac{1}{2}\rho ab\left(y^2 + 2\frac{y^3}{h} + \frac{y^4}{h^2}\right)dy$$

Now $$m = \frac{1}{6}\rho abh$$

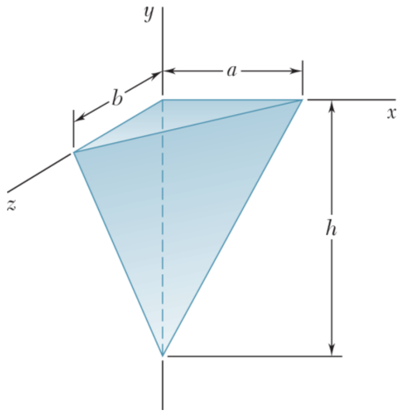
Then $$dI_x = \left[\frac{1}{2}m\frac{b^2}{h}\left(1+\frac{y}{h}\right)^4 + \frac{3m}{h}\left(y^2 + 2\frac{y^3}{h} + \frac{y^4}{h^2}\right)\right]dy$$

and $$I_x = \int dI_x = \int_{-h}^{0}\frac{m}{2h}\left[b^2\left(1+\frac{y}{h}\right)^4 + 6\left(y^2 + 2\frac{y^3}{h} + \frac{y^4}{h^2}\right)\right]dy$$

$$= \frac{m}{2h}\left[b^2 \times \frac{h}{5}\left(1+\frac{y}{h}\right)^5 + 6\left(\frac{1}{3}y^3 + \frac{1}{2}\frac{y^4}{h} + \frac{y^5}{5h^2}\right)\right]_{-h}^{0}$$

$$= \frac{m}{2h}\left\{\frac{1}{5}b^2h(1)^5 - 6\left[\frac{1}{3}(-h)^3 + \frac{1}{2h}(-h)^4 + \frac{1}{5h^2}(-h)^5\right]\right\}$$

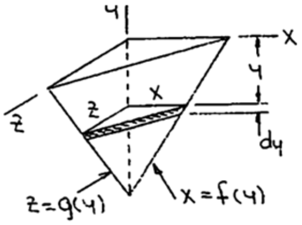
or $$I_x = \frac{1}{10}m(b^2 + h^2)$$ ◀

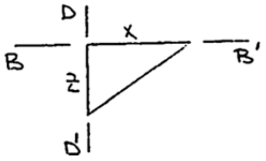


PROBLEM B.13

Determine by direct integration the mass moment of inertia with respect to the *y* axis of the tetrahedron shown, assuming that it has a uniform density and a mass *m*.

SOLUTION





We have

$$x = \frac{a}{h}y + a = a\left(1 + \frac{y}{h}\right)$$

and

$$z = \frac{b}{h}y + b = b\left(1 + \frac{y}{h}\right)$$

For the element shown:

$$dm = \rho\left(\frac{1}{2}xz\,dy\right) = \frac{1}{2}\rho ab\left(1 + \frac{y}{h}\right)^2 dy$$

Then

$$m = \int dm = \int_{-h}^{0} \frac{1}{2}\rho ab\left(1 + \frac{y}{h}\right)^2 dy$$

$$= \frac{1}{2}\rho ab \times \frac{h}{3}\left[\left(1 + \frac{y}{h}\right)^3\right]_{-h}^{0}$$

$$= \frac{1}{6}\rho abh\left[(1)^3 - (1-1)^3\right]$$

$$= \frac{1}{6}\rho abh$$

Also

$$I_{BB',\text{area}} = \frac{1}{12}xz^3 \qquad I_{DD',\text{area}} = \frac{1}{12}zx^3$$

Then, using

$$I_{\text{mass}} = \rho t I_{\text{area}}$$

we have

$$dI_{BB',\text{mass}} = \rho(dy)\left(\frac{1}{12}xz^3\right) \qquad dI_{DD',\text{mass}} = \rho(dy)\left(\frac{1}{12}zx^3\right)$$

PROBLEM B.13 (Continued)

Now

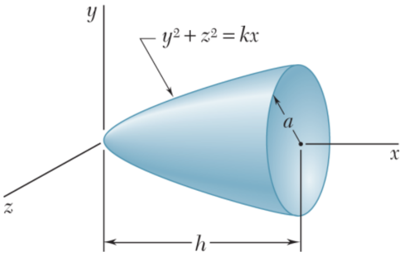
$$\begin{aligned} dI_y &= dI_{BB',\,\text{mass}} + dI_{DD',\,\text{mass}} \\ &= \frac{1}{12}\rho xz(x^2+z^2)dy \\ &= \frac{1}{12}\rho ab\left(1+\frac{y}{h}\right)^2\left[(a^2+b^2)\left(1+\frac{y}{h}\right)^2\right]dy \end{aligned}$$

We have $$m = \frac{1}{6}\rho abh \Rightarrow dI_y = \frac{m}{2h}(a^2+b^2)\left(1+\frac{y}{h}\right)^4 dy$$

Then

$$\begin{aligned} I_y = \int dI_y &= \int_{-h}^{0}\frac{m}{2h}(a^2+b^2)\left(1+\frac{y}{h}\right)^4 dy \\ &= \frac{m}{2h}\left(a^2+b^2\right)\times\frac{h}{5}\left[\left(1+\frac{y}{h}\right)^5\right]_{-h}^{0} \\ &= \frac{m}{10}\left(a^2+b^2\right)\left[(1)^5-(1-1)^5\right] \end{aligned}$$

or $$I_y = \frac{1}{10}m(a^2+b^2)\ ◀$$



PROBLEM B.14

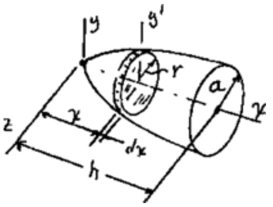
Determine by direct integration the mass moment of inertia and the radius of gyration with respect to the *x* axis of the paraboloid shown, assuming that it has a uniform density and a mass *m*.

SOLUTION

$r^2 = y^2 + z^2 = kx$:

at $x = h,\ r = a;\quad a^2 = kh;\quad k = \dfrac{a^2}{h}$

Thus: $r^2 = \dfrac{a^2}{h}x$



$$dm = \rho\pi r^2 dx = \rho\pi\frac{a^2}{h}xdy$$

$$m = \int_0^h dm = \rho\pi\frac{a^2}{h}\int_0^h xdx;\quad m = \frac{1}{2}\rho\pi a^2 h$$

$$dI_x = \left(\frac{1}{2}r^2\right)dm$$

$$I_x = \int_0^h dI_x = \frac{1}{2}\int_0^h \left(r^2\right)\left(\rho\pi r^2 dx\right) = \frac{1}{2}\rho\pi\int_0^h r^4 dx$$

$$= \frac{1}{2}\rho\pi\int_0^h\left(\frac{a^2}{h}x\right)^2 dx = \frac{\rho\pi a^4}{2h^2}\int_0^h x^2 dx$$

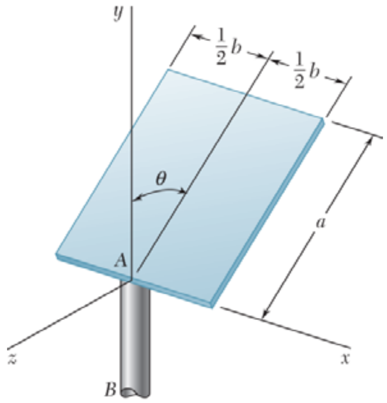
$$I_x = \frac{1}{6}\rho\pi a^4 h \ \triangleleft$$

Recall: $m = \dfrac{1}{2}\rho\pi a^2 h$; $\quad I_x = \left(\dfrac{1}{2}\rho\pi a^2 h\right)\left(\dfrac{1}{3}\right)a^2$

$$I_x = \frac{1}{3}ma^2 \ \blacktriangleleft$$

$$\left(k_x\right)^2 = \frac{I}{m} = \left(\frac{\frac{1}{3}ma^2}{m}\right) = \frac{1}{3}a^2$$

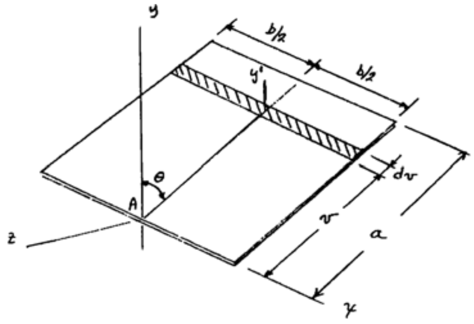
$$k_x = \frac{a}{\sqrt{3}} \ \blacktriangleleft$$



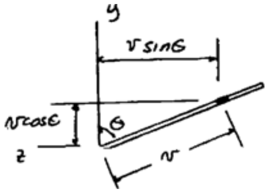
PROBLEM B.15

A thin rectangular plate of mass *m* is welded to a vertical shaft *AB* as shown. Knowing that the plate forms an angle θ with the *y* axis, determine by direct integration the mass moment of inertia of the plate with respect to (*a*) the *y* axis, (*b*) the *z* axis.

SOLUTION



Projection on *yz* plane



Mass of plate: $m = \rho tab$

(*a*) For element shown: $dm = \rho btdv$

$$d\bar{I}_y = d\bar{I}_{y'} + (v\sin\theta)^2\,dm = \frac{1}{12}b^2dm + v^2\sin^2\theta dm$$

$$= \left(\frac{1}{12}b^2 + v^2\sin^2\theta\right)\rho btdv$$

$$I_y = \int dI_y = \rho bt\int_0^a \left(\frac{1}{12}b^2 + v^2\sin^2\theta\right)dv$$

$$= \rho bt\left|\frac{1}{12}b^2v + \frac{v^3}{3}\sin\theta\right|_0^a = \rho bt\left(\frac{1}{12}ab^2 + \frac{a^3}{3}\sin^2\theta\right)$$

$$= (\rho tab)\frac{1}{12}(b^2 + 4a^2\sin^2\theta)$$

PROBLEM B.15 (Continued)

$$I_y = \frac{1}{12}m(b^2 + 4a^2 \sin^2 \theta) \blacktriangleleft$$

(b)

$$dI_z = d\bar{I}_{z'} + (v\cos\theta)^2\, dm = \frac{1}{12}b^2 dm + v^2 \cos^2 \theta dm$$

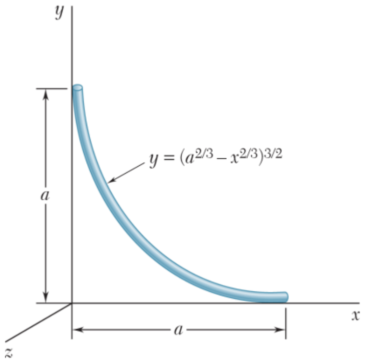
$$= \left(\frac{1}{12}b^2 + v^2 \cos^2 \theta\right)\rho btdv$$

$$I_z = \int dI_z = \rho bt \int_0^a \left(\frac{1}{12}b^2 + v^2 \cos^2 \theta\right)dv$$

$$= \rho bt \left|\frac{1}{12}b^2 v + \frac{v^3}{3}\cos^2 \theta\right|_0^a = \rho bt\left(\frac{1}{12}ab^2 + \frac{a^3}{3}\cos^2 \theta\right)$$

$$= (\rho tab)\frac{1}{12}(b^2 + 4a^2 \cos^2 \theta)$$

$$I_z = \frac{1}{12}m(b^2 + 4a^2 \cos^2 \theta) \blacktriangleleft$$



PROBLEM B.16*

A thin steel wire is bent into the shape shown. Denoting the mass per unit length of the wire by m', determine by direct integration the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION

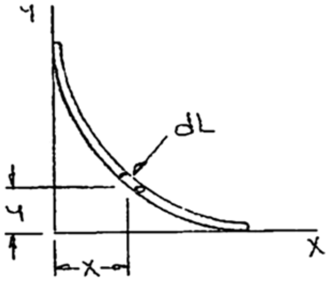
First note

$$\frac{dy}{dx} = -x^{-1/3}(a^{2/3} - x^{2/3})^{1/2}$$

Then

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^{-2/3}(a^{2/3} - x^{2/3})$$

$$= \left(\frac{a}{x}\right)^{2/3}$$



For the element shown:

$$dm = m'dL = m'\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= m'\left(\frac{a}{x}\right)^{1/3} dx$$

Then

$$m = \int dm = \int_0^a m'\frac{a^{1/3}}{x^{1/3}}dx = \frac{3}{2}m'a^{1/3}\left[x^{2/3}\right]_0^a = \frac{3}{2}m'a$$

Now

$$I_x = \int y^2 dm = \int_0^a (a^{2/3} - x^{2/3})^3 \left(m'\frac{a^{1/3}}{x^{1/3}}dx\right)$$

$$= m'a^{1/3}\int_0^a \left(\frac{a^2}{x^{1/3}} - 3a^{4/3}x^{1/3} + 3a^{2/3}x - x^{5/3}\right)dx$$

$$= m'a^{1/3}\left[\frac{3}{2}a^2x^{2/3} - \frac{9}{4}a^{4/3}x^{4/3} + \frac{3}{2}a^{2/3}x^2 - \frac{3}{8}x^{8/3}\right]_0^a$$

$$= m'a^3\left(\frac{3}{2} - \frac{9}{4} + \frac{3}{2} - \frac{3}{8}\right) = \frac{3}{8}m'a^3$$

or $I_x = \frac{1}{4}ma^2$ ◀

Symmetry implies $I_y = \frac{1}{4}ma^2$ ◀

PROBLEM B.16* (Continued)

Alternative solution:

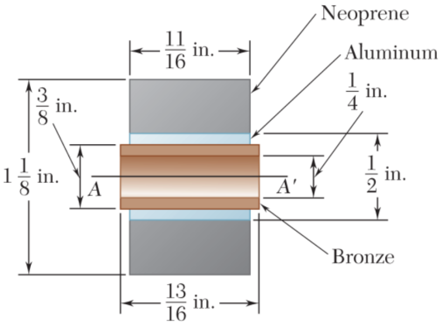
$$I_y = \int x^2 dm = \int_0^a x^2 \left(m' \frac{a^{1/3}}{x^{1/3}} dx \right) = m'a^{1/3} \int_0^a x^{5/3} dx$$

$$= m'a^{1/3} \times \frac{3}{8} \left[x^{8/3} \right]_0^a = \frac{3}{8} m'a^3$$

$$= \frac{1}{4} ma^2$$

Also $$I_z = \int (x^2 + y^2)\, dm = I_y + I_x$$

or $$I_z = \frac{1}{2} ma^2 \blacktriangleleft$$



PROBLEM B.17

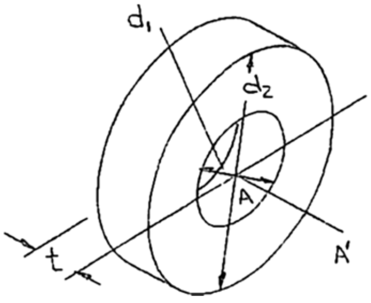
Shown is the cross section of an idler roller. Determine its mass moment of inertia and its radius of gyration with respect to the axis *AA′*. (The specific weight of bronze is 0.310 lb/in^3; of aluminum, 0.100 lb/in^3; and of neoprene, 0.0452 lb/in^3.)

SOLUTION

First note for the cylindrical ring shown that

$$m = \rho V = \rho t \times \frac{\pi}{4}\left(d_2^2 - d_1^2\right) = \frac{\pi}{4}\rho t\left(d_2^2 - d_1^2\right)$$

and, using Figure 9.28, that



$$\begin{aligned}
I_{AA'} &= \frac{1}{2}m_2\left(\frac{d_2}{2}\right)^2 - \frac{1}{2}m_1\left(\frac{d_1}{2}\right)^2 \\
&= \frac{1}{8}\left[\left(\rho t \times \frac{\pi}{4}d_2^2\right)d_2^2 - \left(\rho t \times \frac{\pi}{4}d_1^2\right)d_1^2\right] \\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^4 - d_1^4\right) \\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^2 - d_1^2\right)\left(d_2^2 + d_1^2\right) \\
&= \frac{1}{8}m\left(d_1^2 + d_2^2\right)
\end{aligned}$$

Now treat the roller as three concentric rings and, working from the bronze outward, we have

$$\begin{aligned}
m = \frac{\pi}{4} \times \frac{1}{32.2}\,\text{ft/s}^2 \Bigg\{&(0.310\text{ lb/in}^3)\left(\frac{13}{16}\text{in.}\right)\left[\left(\frac{3}{8}\right)^2 - \left(\frac{1}{4}\right)^2\right]\text{in}^2 \\
&+ (0.100\text{ lb/in}^3)\left(\frac{11}{16}\text{ in.}\right)\left[\left(\frac{1}{2}\right)^2 - \left(\frac{3}{8}\right)^2\right]\text{in}^2 \\
&+ (0.0452\text{ lb/in}^3)\left(\frac{11}{16}\text{in.}\right)\left[\left(1\frac{1}{8}\right)^2 - \left(\frac{1}{2}\right)^2\right]\text{in}^2\Bigg\} \\
= {}&(479.96 + 183.41 + 769.80) \times 10^{-6}\text{ lb}\cdot\text{s}^2/\text{ft} \\
= {}&1.4332 \times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft}
\end{aligned}$$

PROBLEM B.17 (Continued)

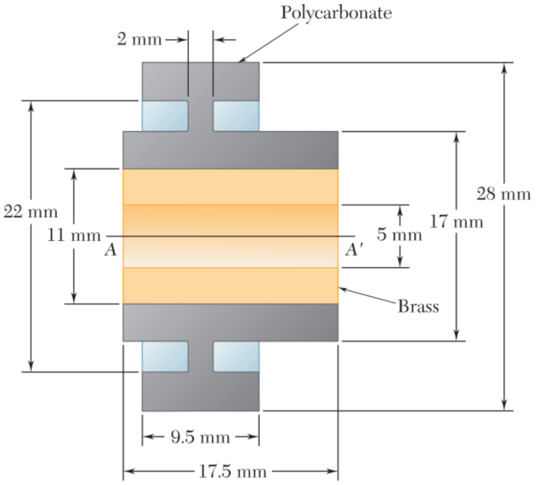
and
$$I_{AA'} = \frac{1}{8}\left\{(479.96)\left[\left(\frac{1}{4}\right)^2 + \left(\frac{3}{8}\right)^2\right] + (183.41)\left[\left(\frac{3}{8}\right)^2 + \left(\frac{1}{2}\right)^2\right]\right.$$
$$\left. + (769.80)\left[\left(1\frac{1}{8}\right)^2 + \left(\frac{1}{2}\right)^2\right]\right\} \times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft} \times \text{in}^2 \times \frac{1\ \text{ft}^2}{144\ \text{in}^2}$$
$$= (84.628 + 62.191 + 1012.78) \times 10^{-9}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$
$$= 1.15960 \times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_{AA'} = 1.160 \times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

Now
$$k_{AA'}^2 = \frac{I_{AA'}}{m} = \frac{1.15960 \times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2}{1.4332 \times 10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft}} = 809.09 \times 10^{-6}\ \text{ft}^2$$

Then
$$k_{AA'} = 28.445 \times 10^{-3}\ \text{ft}$$

or $k_{AA'} = 0.341\ \text{in.}$ ◀



PROBLEM B.18

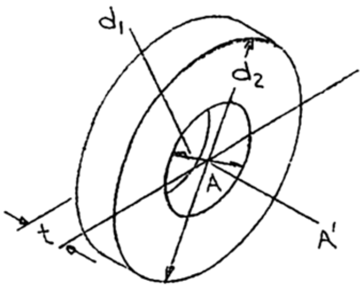
Shown is the cross section of a molded flat-belt pulley. Determine its mass moment of inertia and its radius of gyration with respect to the axis *AA*′. (The density of brass is 8650 kg/m^3 and the density of the fiber-reinforced polycarbonate used is 1250 kg/m^3.)

SOLUTION

First note for the cylindrical ring shown that

$$m = \rho V = \rho t \times \frac{\pi}{4}\left(d_2^2 - d_1^2\right)$$

and, using Figure 9.28, that



$$\begin{aligned}
I_{AA'} &= \frac{1}{2}m_2\left(\frac{d_2}{2}\right)^2 - \frac{1}{2}m_1\left(\frac{d_1}{2}\right)^2\\
&= \frac{1}{8}\left[\left(\rho t \times \frac{\pi}{4}d_2^2\right)d_2^2 - \left(\rho t \times \frac{\pi}{4}d_1^2\right)d_1^2\right]\\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^4 - d_1^4\right)\\
&= \frac{1}{8}\left(\frac{\pi}{4}\rho t\right)\left(d_2^2 - d_1^2\right)\left(d_2^2 + d_1^2\right)\\
&= \frac{1}{8}m\left(d_1^2 + d_2^2\right)
\end{aligned}$$

Now treat the pulley as four concentric rings and, working from the brass outward, we have

$$\begin{aligned}
m = \frac{\pi}{4}\{&8650\text{ kg/m}^3 \times (0.0175\text{ m}) \times (0.011^2 - 0.005^2)\text{ m}^2\\
&+1250\text{ kg/m}^3[(0.0175\text{ m}) \times (0.017^2 - 0.011^2)\text{ m}^2\\
&+(0.002\text{ m}) \times (0.022^2 - 0.017^2)\text{ m}^2\\
&+(0.0095\text{ m}) \times (0.028^2 - 0.022^2)\text{ m}^2]\}\\
=\ &(11.4134 + 2.8863 + 0.38288 + 2.7980) \times 10^{-3}\text{ kg}\\
=\ &17.4806 \times 10^{-3}\text{ kg}
\end{aligned}$$

PROBLEM B.18 (Continued)

and

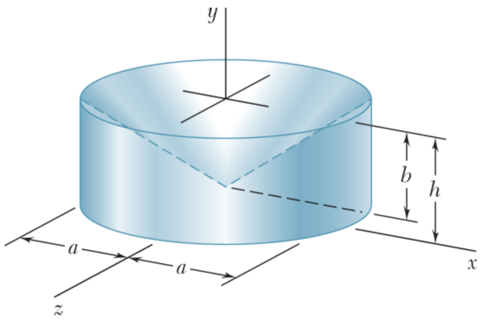
$$I_{AA'} = \frac{1}{8}[(11.4134)(0.005^2 + 0.011^2) + (2.8863)(0.011^2 + 0.017^2)$$
$$+ (0.38288)(0.017^2 + 0.022^2)$$
$$+ (2.7980)(0.022^2 + 0.028^2)] \times 10^{-3} \text{ kg} \times \text{m}^2$$
$$= (208.29 + 147.92 + 37.00 + 443.48) \times 10^{-9} \text{ kg} \cdot \text{m}^2$$
$$= 836.69 \times 10^{-9} \text{ kg} \cdot \text{m}^2$$

or $I_{AA'} = 837 \times 10^{-9} \text{ kg} \cdot \text{m}^2$ ◀

Now

$$k_{AA'}^2 = \frac{I_{AA'}}{m} = \frac{836.69 \times 10^{-9} \text{ kg} \cdot \text{m}^2}{17.4806 \times 10^{-3} \text{ kg}} = 47.864 \times 10^{-6} \text{m}^2$$

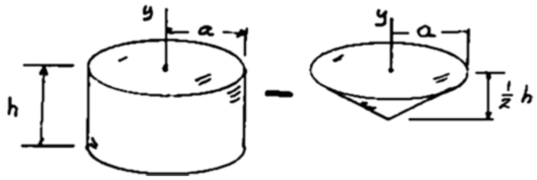
or $k_{AA'} = 6.92 \text{ mm}$ ◀



PROBLEM B.19

The machine part shown is formed by machining a conical surface into a circular cylinder. For $b = \frac{1}{2}h$, determine the mass moment of inertia and the radius of gyration of the machine part with respect to the y axis.

SOLUTION



Mass:

$$m_{\text{cyl}} = \rho\pi a^2 h \qquad m_{\text{cone}} = \frac{1}{3}\rho\pi a^2 \frac{h}{2} = \frac{1}{6}\rho\pi a^2 h$$

$$I_y: \quad I_{\text{cyl}} = \frac{1}{2} m_{\text{cyl}} a^2 \qquad I_{\text{cone}} = \frac{3}{10} m_{\text{cone}} a^2$$

$$= \frac{1}{2}\rho\pi a^4 h \qquad = \frac{1}{20}\rho\pi a^4 h$$

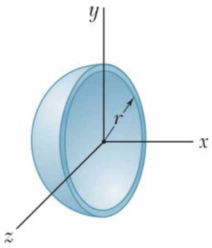
For entire machine part:

$$m = m_{\text{cyl}} - m_{\text{cone}} = \rho\pi a^2 h - \frac{1}{6}\rho\pi a^2 h = \frac{5}{6}\rho\pi a^2 h$$

$$I_y = I_{\text{cyl}} - I_{\text{cone}} = \frac{1}{2}\rho\pi a^4 h - \frac{1}{20}\rho\pi a^4 h = \frac{9}{20}\rho\pi a^4 h$$

or $$I_y = \left(\frac{5}{6}\rho\pi a^2 h\right)\left(\frac{6}{5}\right)\left(\frac{9}{20}\right)a^2 \qquad I_y = \frac{27}{50} m a^2 \blacktriangleleft$$

Then $$k_y^2 = \frac{I}{m} = \frac{27}{50}a^2 \qquad k_y = 0.735a \blacktriangleleft$$



PROBLEM B.20

Knowing that the thin hemispherical shell shown has a mass *m* and thickness *t*, determine the mass moment of inertia and the radius of gyration of the shell with respect to the *x* axis. (*Hint:* Consider the shell as formed by removing a hemisphere of radius *r* from a hemisphere of radius *r* + *t*; then neglect the terms containing t^2 and t^3 and keep those terms containing *t*.)

SOLUTION

Consider shell to be formed by removing hemisphere of radius *r* from hemisphere of radius $r+t$.

For a hemisphere,

$$I=\frac{2}{5}mr^2;\quad A=\frac{1}{2}\left(4\pi r^2\right)=2\pi r^2$$

$$m=\rho V=\frac{1}{2}\rho\left(\frac{4}{3}\pi r^3\right)=\frac{2}{3}\rho\pi r^3$$

Then

$$I=\frac{2}{5}\left(\frac{2}{3}\rho\pi r^3\right)r^2$$

$$=\frac{4}{15}\rho\pi r^5$$

Now following the hint, for a hemispherical shell,

$$I=\frac{4}{15}\rho\pi\left[\left(r+t\right)^5-r^5\right]$$

$$I=\frac{4}{15}\rho\pi\left[r^5+5r^4t+10r^3t^2+\ldots-r^5\right]$$

Neglect terms with powers of $t>1$. $I=\frac{4}{3}\rho\pi r^4t=\left(2\rho\pi r^2\right)\frac{2}{3}r^2t$

Mass of shell,

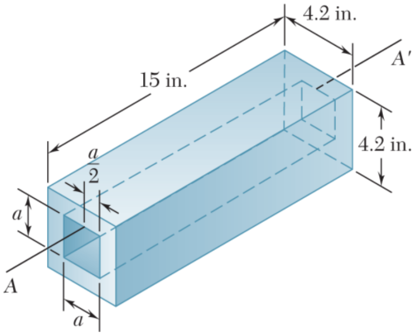
$$m=\rho V=\rho tA=\rho t\left(2\pi r^2\right)=2\rho\pi r^2t$$

Substituting with expression for *m* to simplify,

$$I=\frac{2}{3}mr^2 \blacktriangleleft$$

Radius of gyration, $k^2=\frac{I}{m}=\frac{2}{3}r^2$

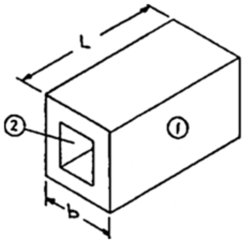
$$k=0.816r \blacktriangleleft$$



PROBLEM B.21

A square hole is centered in and extends through the aluminum machine component shown. Determine (*a*) the value of *a* for which the mass moment of inertia of the component with respect to the axis *AA*′, which bisects the top surface of the hole, is maximum, (*b*) the corresponding values of the mass moment of inertia and the radius of gyration with respect to the axis *AA*′. (The specific weight of aluminum is 0.100 lb/in^3.)

SOLUTION



First note $m_1 = \rho V_1 = \rho b^2 L$

and $m_2 = \rho V_2 = \rho a^2 L$

(*a*) Using Figure 9.28 and the parallel-axis theorem, we have

$$
\begin{aligned}
I_{AA'} &= (I_{AA'})_1 - (I_{AA'})_2 \\
&= \left[\frac{1}{12}m_1(b^2+b^2) + m_1\left(\frac{a}{2}\right)^2\right] \\
&\quad - \left[\frac{1}{12}m_2(a^2+a^2) + m_2\left(\frac{a}{2}\right)^2\right] \\
&= (\rho b^2 L)\left(\frac{1}{6}b^2 + \frac{1}{4}a^2\right) - (\rho a^2 L)\left(\frac{5}{12}a^2\right) \\
&= \frac{\rho L}{12}(2b^4 + 3b^2a^2 - 5a^4)
\end{aligned}
$$

Then $$\frac{dI_{AA'}}{da} = \frac{\rho L}{12}(6b^2 a - 20a^3) = 0$$

or $$a = 0 \qquad \text{and} \qquad a = b\sqrt{\frac{3}{10}}$$

Also $$\frac{d^2 I_{AA'}}{da^2} = \frac{\rho L}{12}(6b^2 - 60a^2) = \frac{1}{2}\rho L(b^2 - 10a^2)$$

Now for $a = 0$, $$\frac{d^2 I_{AA'}}{da^2} > 0$$

and for $a = b\sqrt{\frac{3}{10}}$, $$\frac{d^2 I_{AA'}}{da^2} < 0$$

PROBLEM B.21 (Continued)

$(I_{AA'})_{\max}$ occurs when $\quad a = b\sqrt{\frac{3}{10}}$

Then $\quad a = (4.2 \text{ in.})\sqrt{\frac{3}{10}}$

or $\quad a = 2.30$ in. ◀

(*b*) From part *a*:

$$(I_{AA'})_{\max} = \frac{\rho L}{12}\left[2b^4 + 3b^2\left(b\sqrt{\frac{3}{10}}\right)^2 - 5\left(b\sqrt{\frac{3}{10}}\right)^4\right] = \frac{49}{240}\rho L b^4$$

$$= \frac{49}{240}\frac{\gamma_{AL}}{g}Lb^4 = \frac{49}{240}\times\frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2}\times(15 \text{ in.})(4.2 \text{ in.})^4\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

or $\quad (I_{AA'})_{\max} = 20.6\times10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

Now $\quad k_{AA'}^2 = \frac{(I_{AA'})_{\max}}{m}$

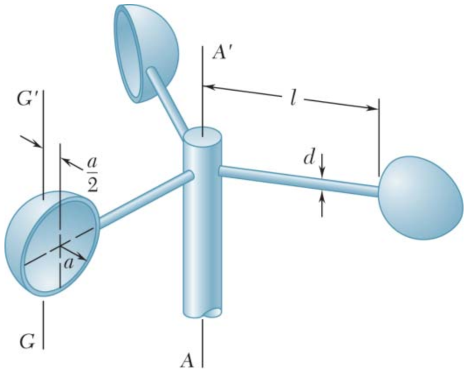
where $\quad m = m_1 - m_2 = \rho L(b^2 - a^2)$

$$= \rho L\left[b^2 - \left(b\sqrt{\frac{3}{10}}\right)^2\right]$$

$$= \frac{7}{10}\rho L b^2$$

Then $\quad k_{AA'}^2 = \frac{\frac{49}{240}\rho L b^4}{\frac{7}{10}\rho L b^2} = \frac{7}{24}b^2 = \frac{7}{24}(4.2 \text{ in.})^2$

or $\quad k_{AA'} = 2.27$ in. ◀



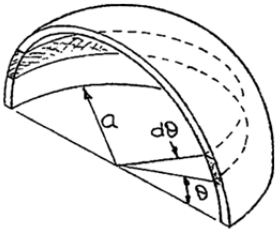
PROBLEM B.22

The cups and the arms of an anemometer are fabricated from a material of density ρ. Knowing that the mass moment of inertia of a thin, hemispherical shell of mass m and thickness t with respect to its centroidal axis GG' is $5ma^2/12$, determine (*a*) the mass moment of inertia of the anemometer with respect to the axis AA', (*b*) the ratio of a to l for which the centroidal moment of inertia of the cups is equal to 1 percent of the moment of inertia of the cups with respect to the axis AA'.

SOLUTION

(*a*) First note

$$m_{\text{arm}} = \rho V_{\text{arm}} = \rho \times \frac{\pi}{4} d^2 l$$



and

$$dm_{\text{cup}} = \rho dV_{\text{cup}}$$
$$= \rho[(2\pi a\cos\theta)(t)(a d\theta)]$$

Then

$$m_{\text{cup}} = \int dm_{\text{cup}} = \int_0^{\pi/2} 2\pi\rho a^2 t\cos\theta d\theta$$
$$= 2\pi\rho a^2 t[\sin\theta]_0^{\pi/2}$$
$$= 2\pi\rho a^2 t$$

Now

$$(I_{AA'})_{\text{anem}} = (I_{AA'})_{\text{cups}} + (I_{AA'})_{\text{arms}}$$

Using the parallel-axis theorem and assuming the arms are slender rods, we have

$$(I_{AA'})_{\text{anem}} = 3\left[(I_{GG'})_{\text{cup}} + m_{\text{cup}} d_{AG}^2\right] + 3\left[\bar{I}_{\text{arm}} + m_{\text{arm}} d_{AG_{\text{arm}}}\right]$$
$$= 3\left\{\frac{5}{12} m_{\text{cup}} a^2 + m_{\text{cup}}\left[(l+a)^2 + \left(\frac{a}{2}\right)^2\right]\right\} + 3\left[\frac{1}{2} m_{\text{arm}} l^2 + m_{\text{arm}}\left(\frac{l}{2}\right)^2\right]$$
$$= 3m_{\text{cup}}\left(\frac{5}{3}a^2 + 2la + l^2\right) + m_{\text{arm}} l^2$$
$$= 3(2\pi\rho a^2 t)\left(\frac{5}{3}a^2 + 2la + l^2\right) + \left(\frac{\pi}{4}\rho d^2 l\right)(l^2)$$

or $$(I_{AA'})_{\text{anem}} = \pi\rho l^2\left[6a^2 t\left(\frac{5}{3}\frac{a^2}{l^2} + 2\frac{a}{l} + 1\right) + \frac{d^2 l}{4}\right] \blacktriangleleft$$

PROBLEM B.22 (Continued)

(*b*) We have
$$\frac{(I_{GG'})_{\text{cup}}}{(I_{AA'})_{\text{cup}}} = 0.01$$

or
$$\frac{5}{12} m_{\text{cup}} a^2 = 0.01 m_{\text{cup}} \left(\frac{5}{3} a^2 + 2la + l^2 \right) \quad \text{(from part } a\text{)}$$

Now let $\zeta = \dfrac{a}{l}$.

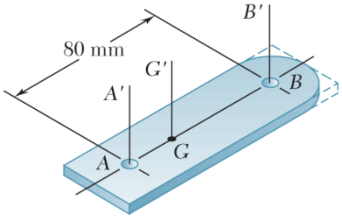
Then
$$5\zeta^2 = 0.12\left(\frac{5}{3}\zeta^2 + 2\zeta + 1 \right)$$

or
$$40\zeta^2 - 2\zeta - 1 = 0$$

Then
$$\zeta = \frac{2 \pm \sqrt{(-2)^2 - 4(40)(-1)}}{2(40)}$$

or
$$\zeta = 0.1851 \quad \text{and} \quad \zeta = -0.1351$$

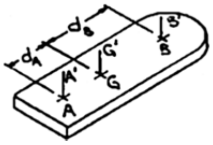
$$\frac{a}{l} = 0.1851 \blacktriangleleft$$



PROBLEM B.23

After a period of use, one of the blades of a shredder has been worn to the shape shown and is of mass 0.18 kg. Knowing that the mass moments of inertia of the blade with respect to the *AA*′ and *BB*′ axes are $0.320\ \text{g}\cdot\text{m}^2$ and $0.680\ \text{g}\cdot\text{m}^2$, respectively, determine (*a*) the location of the centroidal axis *GG*′, (*b*) the radius of gyration with respect to axis *GG*′.

SOLUTION



(*a*) We have $d_B = (0.08 - d_A)\ \text{m}$

and, using the parallel axis

$$I_{AA'} = \bar{I}_{GG'} + md_A^2$$
$$I_{BB'} = \bar{I}_{GG'} + md_B^2$$

Then
$$I_{BB'} - I_{AA'} = m\left(d_B^2 - d_A^2\right)$$
$$= m\left[(0.08 - d_A)^2 - d_A^2\right]$$
$$= m(0.0064 - 0.16d_A)$$

Substituting:
$$(0.68 - 0.32)\times 10^{-3}\ \text{kg}\cdot\text{m}^2 = 0.18\ \text{kg}(0.0064 - 0.16d_A)\ \text{m}^2$$

or $d_A = 27.5\ \text{mm}$ ◀

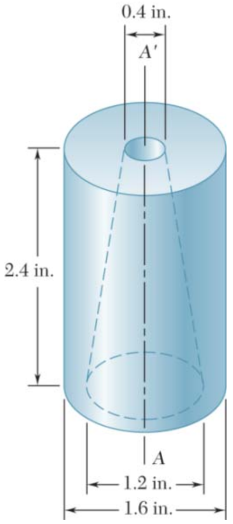
to the right of *A*

(*b*) We have
$$I_{AA'} = I_{GG'} + md_A^2$$

or
$$I_{GG'} = 0.32\times 10^{-3}\ \text{kg}\cdot\text{m}^2 - 0.18\ \text{kg}\cdot(0.0275\ \text{m})^2$$
$$= 0.183875\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

Then
$$k_{GG'}^2 = \frac{I_{GG'}}{m} = \frac{0.183875\times 10^{-3}\ \text{kg}\cdot\text{m}^2}{0.18\ \text{kg}} = 1.02153\times 10^{-3}\ \text{m}^2$$

or $k_{GG'} = 32.0\ \text{mm}$ ◀

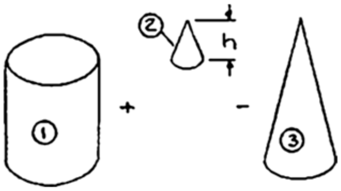


PROBLEM B.24

Determine the mass moment of inertia of the 0.9-lb machine component shown with respect to the axis *AA*′.

SOLUTION

First note that the given shape can be formed adding a small cone to a cylinder and then removing a larger cone as indicated.



Now $$\frac{h}{0.4} = \frac{h+2.4}{1.2} \quad \text{or} \quad h = 1.2 \text{ in.}$$

The weight of the body is given by

$$W = mg = g(m_1 + m_2 - m_3) = \rho g(V_1 + V_2 - V_3)$$

or $$0.9 \text{ lb} = \rho \times 32.2 \text{ ft/s}^2$$

$$\left[\pi(0.8)^2(2.4) + \frac{\pi}{3}(0.2)^2(1.2) - \frac{\pi}{3}(0.6)^2(3.6)\right] \text{in}^3 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$

$$= \rho \times 32.2 \text{ ft/s}^2 (2.79253 + 0.02909 - 0.78540) \times 10^{-3} \text{ ft}^3$$

or $$\rho = 13.7266 \text{ lb}\cdot\text{s}^2/\text{ft}^4$$

Then $$m_1 = (13.7266)(2.79253 \times 10^{-3}) = 0.038332 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m_2 = (13.7266)(0.02909 \times 10^{-3}) = 0.000399 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m_3 = (13.7266)(0.78540 \times 10^{-3}) = 0.010781 \text{ lb}\cdot\text{s}^2/\text{ft}$$

PROBLEM B.24 (Continued)

Finally, using Figure 9.28, we have

$$I_{AA'} = (I_{AA'})_1 + (I_{AA'})_2 - (I_{AA'})_3$$

$$= \frac{1}{2} m_1 a_1^2 + \frac{3}{10} m_2 a_2^2 - \frac{3}{10} m_3 a_3^2$$

$$= \left[\frac{1}{2}(0.038332)(0.8)^2 + \frac{3}{10}(0.000399)(0.2)^2 - \frac{3}{10}(0.010781)(0.6)^2\right](\text{lb}\cdot\text{s}^2/\text{ft}) \times \text{in}^2 \times \left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$

$$= (85.1822 + 0.0333 - 8.0858) \times 10^{-6}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2$$

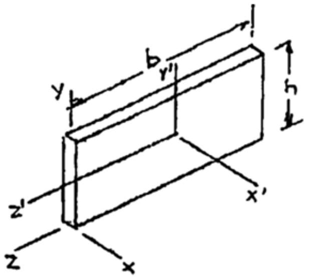
or $I_{AA'} = 77.1 \times 10^{-6}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

To the instructor:

The following formulas for the mass moment of inertia of thin plates and a half cylindrical shell are derived at this time for use in the solutions of Problems 9.135 through 9.140.

Thin rectangular plate

$$
\begin{aligned}
(I_x)_m &= (\bar{I}_{x'})_m + md^2 \\
&= \frac{1}{12}m(b^2+h^2) + m\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right] \\
&= \frac{1}{3}m(b^2+h^2)
\end{aligned}
$$

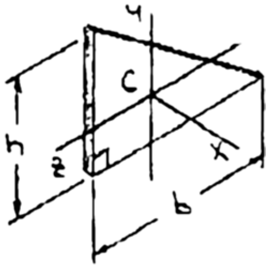


$$
\begin{aligned}
(I_y)_m &= (\bar{I}_{y'})_m + md^2 \\
&= \frac{1}{12}mb^2 + m\left(\frac{b}{2}\right)^2 = \frac{1}{3}mb^2
\end{aligned}
$$

$$
\begin{aligned}
I_z &= (\bar{I}_{z'})_m + md^2 \\
&= \frac{1}{12}mh^2 + m\left(\frac{h}{2}\right)^2 = \frac{1}{3}mh^2
\end{aligned}
$$

Thin triangular plate

We have $$m = \rho V = \rho\left(\frac{1}{2}bht\right)$$



and $$\bar{I}_{z,\text{area}} = \frac{1}{36}bh^3$$

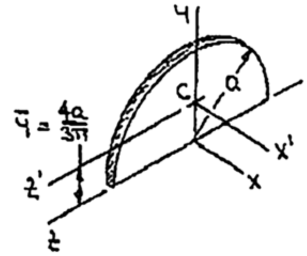
Then
$$
\begin{aligned}
\bar{I}_{z,\text{mass}} &= \rho t I_{z,\text{area}} \\
&= \rho t \times \frac{1}{36}bh^3 \\
&= \frac{1}{18}mh^2
\end{aligned}
$$

Similarly, $$\bar{I}_{y,\text{mass}} = \frac{1}{18}mb^2$$

Now $$\bar{I}_{x,\text{mass}} = \bar{I}_{y,\text{mass}} + \bar{I}_{z,\text{mass}} = \frac{1}{18}m(b^2+h^2)$$

Thin semicircular plate

We have $$m = \rho V = \rho\left(\frac{\pi}{2}a^2 t\right)$$



and $$\bar{I}_{y,\text{area}} = I_{z,\text{area}} = \frac{\pi}{8}a^4$$

(Continued)

Then $$\bar{I}_{y,\text{mass}} = I_{z,\text{mass}} = \rho t \bar{I}_{y,\text{area}}$$

$$= \rho t \times \frac{\pi}{8} a^4$$

$$= \frac{1}{4} m a^2$$

Now $$I_{x,\text{mass}} = \bar{I}_{y,\text{mass}} + I_{z,\text{mass}} = \frac{1}{2} m a^2$$

Also $$I_{x,\text{mass}} = \bar{I}_{x',\text{mass}} + m\bar{y}^2 \quad \text{or} \quad \bar{I}_{x',\text{mass}} = m\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)a^2$$

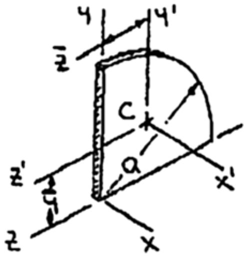
and $$I_{z,\text{mass}} = \bar{I}_{z',\text{mass}} + m\bar{y}^2 \quad \text{or} \quad \bar{I}_{z',\text{mass}} = m\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)a^2$$

$$\bar{y} = \bar{z} = \frac{4a}{3\pi}$$

Thin Quarter-Circular Plate

We have $$m = \rho V = \rho\left(\frac{\pi}{4} a^2 t\right)$$

and $$I_{y,\text{area}} = I_{z,\text{area}} = \frac{\pi}{16} a^4$$



Then $$I_{y,\text{mass}} = I_{z,\text{mass}} = \rho t I_{y,\text{area}}$$

$$= \rho t \times \frac{\pi}{16} a^4$$

$$= \frac{1}{4} m a^2$$

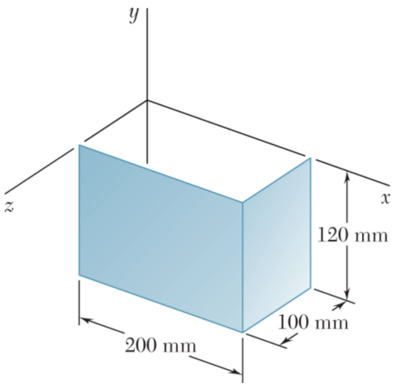
Now $$I_{x,\text{mass}} = I_{y,\text{mass}} + I_{z,\text{mass}} = \frac{1}{2} m a^2$$

Also $$I_{x,\text{mass}} = I_{x',\text{mass}} + m(\bar{y}^2 + \bar{z}^2)$$

or $$\bar{I}_{x',\text{mass}} = m\left(\frac{1}{2} - \frac{32}{9\pi^2}\right)a^2$$

and $$I_{y,\text{mass}} = I_{y',\text{mass}} + m\bar{z}^2$$

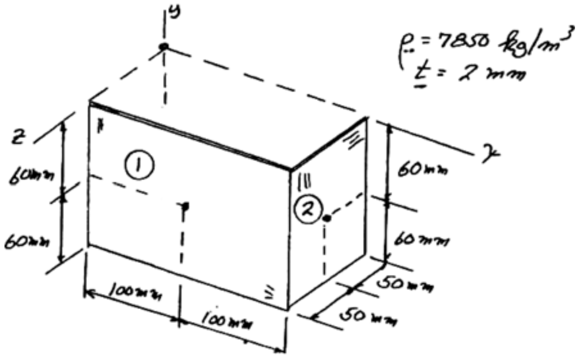
or $$\bar{I}_{y',\text{mass}} = m\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)a^2$$



PROBLEM B.25

A 2-mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION



$$m_1 = \rho V = 7850\ \text{kg/m}^3\ (0.120\ \text{m})(0.200\ \text{m})(0.002\ \text{m}) = 0.3768\ \text{kg}$$

$$m_2 = \rho V = 7850\ \text{kg/m}^3\ (0.120\ \text{m})(0.100\ \text{m})(0.002\ \text{m}) = 0.1884\ \text{kg}$$

Panel 1:

$$I_x = \bar{I}_x + md^2 = \frac{1}{12}(0.3768\ \text{kg})(0.12\ \text{m})^2 + (0.3768\ \text{kg})\left[(0.1\ \text{m})^2 + (0.06\ \text{m})^2\right]$$

$$I_x = 5.577\times10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_y = \bar{I}_y + md^2 = \frac{1}{12}(0.3768\ \text{kg})(0.2\ \text{m})^2 + (0.3768\ \text{kg})\left[(0.1\ \text{m})^2 + (0.1\ \text{m})^2\right]$$

$$I_y = 8.792\times10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_z = \bar{I}_z + md^2 = \frac{1}{12}(0.3768\ \text{kg})\left[(0.2\ \text{m})^2 + (0.12\ \text{m})^2\right] + (0.3768\ \text{kg})\left[(0.1\ \text{m})^2 + (0.06\ \text{m})^2\right]$$

$$I_z = 6.833\times10^{-3}\ \text{kg}\cdot\text{m}^2$$

PROBLEM B.25 (Continued)

Panel 2:

$$I_x = \bar{I}_x + md^2 = \frac{1}{12}(0.1884\text{ kg})\left[(0.12\text{ m})^2 + (0.1\text{ m})^2\right] + (0.1884\text{ kg})\left[(0.05\text{ m})^2 + (0.06\text{ m})^2\right]$$

$I_x = 1.532\times10^{-3}\text{ kg}\cdot\text{m}^2$

$$I_y = \bar{I}_y + md^2 = \frac{1}{12}(0.1884\text{ kg})(0.1\text{ m})^2 + (0.1884\text{ kg})\left[(0.05\text{ m})^2 + (0.2\text{ m})^2\right]$$

$I_y = 8.164\times10^{-3}\text{ kg}\cdot\text{m}^2$

$$I_z = \bar{I}_z + md^2 = \frac{1}{12}(0.1884\text{ kg})(0.12\text{ m})^2 + (0.1884\text{ kg})\left[(0.06\text{ m})^2 + (0.2\text{ m})^2\right]$$

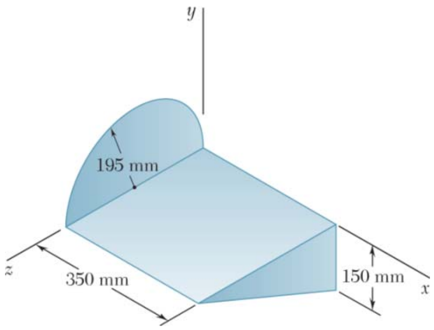
$I_z = 8.440\times10^{-3}\text{ kg}\cdot\text{m}^2$

Total Mass Moments of Inertia: $I_T = I_1 + I_2$

$I_x = 7.11\times10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

$I_y = 16.96\times10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

$I_z = 15.27\times10^{-3}\text{ kg}\cdot\text{m}^2$ ◀



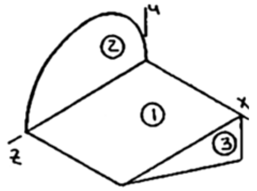
PROBLEM B.26

A 2-mm thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to each of the coordinate axes.

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \rho_{\text{ST}} t A$$



Then

$$\begin{aligned} m_1 &= (7850\ \text{kg/m}^3)(0.002\ \text{m})(0.35 \times 0.39)\ \text{m}^2 \\ &= 2.14305\ \text{kg} \end{aligned}$$

$$m_2 = (7850\ \text{kg/m}^3)(0.002\ \text{m})\left(\frac{\pi}{2} \times 0.195^2\right)\text{m}^2 = 0.93775\ \text{kg}$$

$$m_3 = (7850\ \text{kg/m}^3)(0.002\ \text{m})\left(\frac{1}{2} \times 0.39 \times 0.15\right)\text{m}^2 = 0.45923\ \text{kg}$$

Using Figure 9.28 for component 1 and the equations derived above for components 2 and 3, we have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 + (I_x)_3 \\ &= \left[\frac{1}{12}(2.14305\ \text{kg})(0.39\ \text{m})^2 + (2.14305\ \text{kg})\left(\frac{0.39}{2}\ \text{m}\right)^2\right] \\ &\quad + \left\{\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(0.93775\ \text{kg})(0.195\ \text{m})^2 + (0.93775\ \text{kg})\left[\left(\frac{4 \times 0.195}{3\pi}\right)^2 + (0.195)^2\right]\text{m}^2\right\} \\ &\quad + \left\{\frac{1}{18}(0.45923\ \text{kg})[(0.39)^2 + (0.15)^2]\ \text{m}^2 + (0.45923\ \text{kg})\left[\left(\frac{0.39}{3}\right)^2 + \left(\frac{0.15}{3}\right)^2\right]\text{m}^2\right\} \\ &= [(0.027163 + 0.081489) + (0.011406 + 0.042081) + (0.004455 + 0.008909)]\ \text{kg} \cdot \text{m}^2 \\ &= (0.108652 + 0.053487 + 0.013364)\ \text{kg} \cdot \text{m}^2 \\ &= 0.175503\ \text{kg} \cdot \text{m}^2 \end{aligned}$$

or $I_x = 175.5 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ ◀

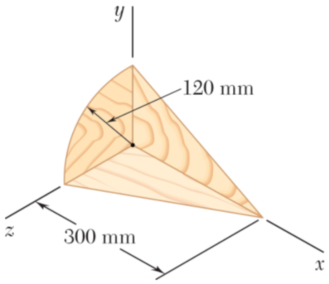
PROBLEM B.26 (Continued)

$$
\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 \\
&= \left\{\frac{1}{12}(2.14305\text{ kg})[(0.35)^2 + (0.39)^2]\text{ m}^2 + (2.14305\text{ kg})\left[\left(\frac{0.35}{2}\right)^2 + \left(\frac{0.39}{2}\right)^2\right]\text{ m}^2\right\} \\
&\quad + \left[\frac{1}{4}(0.93775\text{ kg})(0.195\text{ m})^2 + (0.93775\text{ kg})(0.195\text{ m})^2\right] \\
&\quad + \left\{\frac{1}{18}(0.45923\text{ kg})(0.39\text{ m})^2 + (0.45923\text{ kg})\left[(0.35)^2 + \left(\frac{0.39}{3}\right)^2\right]\text{ m}^2\right\} \\
&= [(0.049040 + 0.147120) + (0.008914 + 0.035658) \\
&\quad + (0.003880 + 0.064017)]\text{ kg}\cdot\text{m}^2 \\
&= (0.196160 + 0.044572 + 0.067897)\text{ kg}\cdot\text{m}^2 \\
&= 0.308629\text{ kg}\cdot\text{m}^2
\end{aligned}
$$

or $I_y = 309\times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

$$
\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 \\
&= \left[\frac{1}{12}(2.14305\text{ kg})(0.35\text{ m})^2 + (2.14305\text{ kg})\left(\frac{0.35}{2}\text{ m}\right)^2\right] \\
&\quad + \left[\frac{1}{4}(0.93775\text{ kg})(0.195\text{ m})^2\right] \\
&\quad + \left\{\frac{1}{18}(0.45923\text{ kg})(0.15\text{ m})^2 + (0.45923\text{ kg})\left[(0.35)^2 + \left(\frac{0.15}{3}\right)^2\right]\text{ m}^2\right\} \\
&= [(0.021877 + 0.065631) + 0.008914) + (0.000574 + 0.057404)]\text{ kg}\cdot\text{m}^2 \\
&= (0.087508 + 0.008914 + 0.057978)\text{ kg}\cdot\text{m}^2 \\
&= 0.154400\text{ kg}\cdot\text{m}^2
\end{aligned}
$$

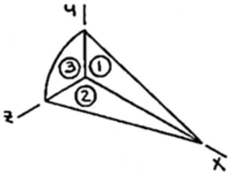
or $I_z = 154.4\times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀



PROBLEM B.27

A subassembly for a model airplane is fabricated from three pieces of 1.5-mm plywood. Neglecting the mass of the adhesive used to assemble the three pieces, determine the mass moment of inertia of the subassembly with respect to each of the coordinate axes. (The density of the plywood is 780 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho V = \rho t A$$

Then

$$m_1 = m_2 = (780 \text{ kg/m}^3)(0.0015 \text{ m})\left(\frac{1}{2}\times 0.3\times 0.12\right)\text{m}^2$$

$$= 21.0600\times 10^{-3} \text{ kg}$$

$$m_3 = (780 \text{ kg/m}^3)(0.0015 \text{ m})\left(\frac{\pi}{4}\times 0.12^2\right)\text{m}^2 = 13.2324\times 10^{-3} \text{ kg}$$

Using the equations derived above and the parallel-axis theorem, we have

$$(I_x)_1 = (I_x)_2$$

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$

$$= 2\left[\frac{1}{18}(21.0600\times 10^{-3} \text{ kg})(0.12 \text{ m})^2 + (21.0600\times 10^{-3} \text{ kg})\left(\frac{0.12}{3}\text{ m}\right)^2\right]$$

$$+\left[\frac{1}{2}(13.2324\times 10^{-3} \text{ kg})(0.12 \text{ m})^2\right]$$

$$= [2(16.8480 + 33.6960) + (95.2733)]\times 10^{-6} \text{ kg}\cdot\text{m}^2$$

$$= [2(50.5440) + (95.2733)]\times 10^{-6} \text{ kg}\cdot\text{m}^2$$

or $I_x = 196.4\times 10^{-6} \text{ kg}\cdot\text{m}^2$ ◀

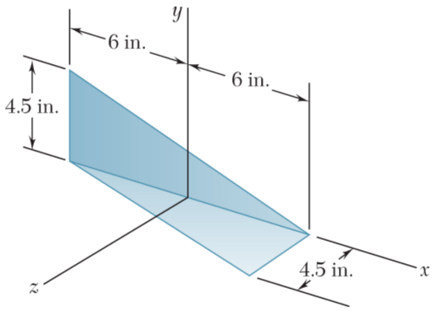
PROBLEM B.27 (Continued)

$$\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 \\
&= \left[\frac{1}{18}(21.0600\times10^{-3}\text{ kg})(0.3\text{ m})^2 + (21.0600\text{ kg})\left(\frac{0.3}{3}\text{m}\right)^2\right] \\
&\quad + \left\{\frac{1}{18}(21.0600\times10^{-3}\text{ kg})[(0.3)^2+(0.12)^2]\text{ m}^2\right. \\
&\quad \left. + (21.0600\times10^{-3}\text{ kg})\left[\left(\frac{0.3}{3}\right)^2 + \left(\frac{0.12}{3}\right)^2\right]\text{m}^2\right\} \\
&\quad + \left[\frac{1}{4}(13.2324\times10^{-3}\text{ kg})(0.12\text{ m})^2\right] \\
&= [(105.300 + 210.600) + (122.148 + 244.296) \\
&\quad + (47.637)]\times10^{-6}\text{ kg}\cdot\text{m}^2 \\
&= (315.900 + 366.444 + 47.637)\times10^{-6}\text{ kg}\cdot\text{m}^2
\end{aligned}$$

or $I_y = 730\times10^{-6}\text{ kg}\cdot\text{m}^2$ ◀

Symmetry implies $I_y = I_z$

$I_z = 730\times10^{-6}\text{ kg}\cdot\text{m}^2$ ◀



PROBLEM B.28

A section of sheet steel 0.03 in. thick is cut and bent into the sheet metal machine component shown. Determine the mass moment of inertia of the component with respect to each of the coordinate axes. (The specific weight of steel is 490 lb/ft^3.)

SOLUTION

First consider one triangular plate with local axes *u*-*v*. We have

$$m = \rho V = \frac{\gamma}{g}tA = \left(\frac{490 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2}\right)\frac{1}{2}(1 \text{ ft})\left(\frac{4.5 \text{ in.}}{12 \text{ in./ft}}\right)\left(\frac{0.03 \text{ in.}}{12 \text{ in./ft}}\right) = 7.1332\times10^{-3} \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$I_{mass} = \rho t I_{area} = \frac{m}{A}I_{area}$$

Then $\quad \bar{I}_u = \left(\dfrac{m}{\frac{1}{2}bh}\right)\left(\dfrac{1}{36}bh^3\right) = \dfrac{1}{18}mh^2, \qquad \bar{I}_v = \dfrac{1}{18}mb^2$

$$I_u = \frac{1}{18}(7.1332\times10^{-3})\left(\frac{4.5 \text{ in.}}{12 \text{ in./ft}}\right)^2 = 55.728\times10^{-6} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_v = \frac{1}{18}(7.1332\times10^{-3})(1 \text{ ft})^2 = 396.29\times10^{-6} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

For composite area:

$$\bar{I}_x = 2\left[I_u + md^2\right] = 2\left[55.728\times10^{-6} + 7.1332\times10^{-3}\left(\frac{1.5 \text{ in.}}{12 \text{ in./ft}}\right)^2\right]$$

$$\bar{I}_x = 334\times10^{-6} \text{ lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$

For $\bar{I}_y$, consider plates separately.

Plate 1 in *x*-*y* plane:

$$\bar{I}_{y_1} = I_v + md^2 = 396.29\times10^{-6} + 7.1332\times10^{-3}\left(\frac{2 \text{ in.}}{12 \text{ in./ft}}\right)^2 = 594.43\times10^{-6} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

PROBLEM B.28 (Continued)

Plate 2 in *x*-*z* plane:
Note: By Eqn. 9.38,

$$\bar{I}_{y'} = I_u + I_v = 55.728\times10^{-6} + 396.29\times10^{-6}$$

$$\bar{I}_{y'} = 452.02\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

Now applying the Parallel Axis Theorem,

$$\bar{I}_{y_2} = I_{y'} + md^2$$

$$\bar{I}_{y_2} = 452.02\times10^{-6} + \left(7.1332\times10^{-3}\right)\left[\left(\frac{2}{12}\right)^2 + \left(\frac{1.5}{12}\right)^2\right]$$

$$\bar{I}_{y_2} = 761.62\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

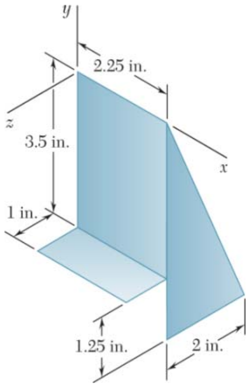
Finally, total for both plates 1 and 2 gives,

$$\bar{I}_y = \bar{I}_{y_1} + \bar{I}_{y_2} = 594.43\times10^{-6} + 761.62\times10^{-6}$$

$$I_y = 1.356\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$

By symmetry, $I_y = I_z$

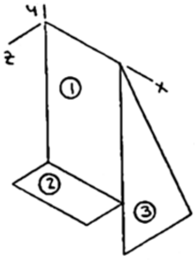
$$I_z = 1.356\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$



PROBLEM B.29

A framing anchor is formed of 0.05-in.-thick galvanized steel. Determine the mass moment of inertia of the anchor with respect to each of the coordinate axes. (The specific weight of galvanized steel is 470 lb/ft^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho V = \frac{\gamma_{\text{G.S.}}}{g} tA$$

Then

$$m_1 = \frac{470 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times (2.25 \times 3.5) \text{ in}^2 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 3325.97 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{470 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times (2.25 \times 1) \text{ in}^2 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 950.28 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{470 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2} \times 0.05 \text{ in.} \times \left(\frac{1}{2} \times 2 \times 4.75\right) \text{ in}^2 \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^3$$
$$= 2006.14 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft}$$

Using Figure 9.28 for components 1 and 2 and the equations derived above for component 3, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3$$
$$= \left[\frac{1}{12}(3325.97 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})(3.5 \text{ in.})^2\right.$$
$$\left. + (3325.97 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})\left(\frac{3.5}{2} \text{ in}^2\right)\right]\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$
$$+ \left\{\frac{1}{12}(950.28 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})(1 \text{ in.})^2\right.$$
$$\left. + (950.28 \times 10^{-6} \text{ lb} \cdot \text{s}^2/\text{ft})\left[(3.5)^2 + \left(\frac{1}{2}\right)^2\right] \text{ in}^2\right\}\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

PROBLEM B.29 (Continued)

$$+\left\{\frac{1}{18}(2006.14\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})[(4.75)^2+(2)^2]\ \text{in}^2\right.$$

$$\left.+(2006.14\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2}{3}\times4.75\right)^2+\left(\frac{1}{3}\times2\right)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$=[(23.578+70.735)+(0.550+82.490)$$

$$+(20.559+145.894)]\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$=(94.313+83.040+166.453)\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_x=344\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$I_y=(I_y)_1+(I_y)_2+(I_y)_3$$

$$=\left[\frac{1}{12}(3325.97\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})(2.25\ \text{in.})^2\right.$$

$$\left.+(3325.97\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left(\frac{2.25}{2}\ \text{in.}\right)^2\right]\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+\left\{\frac{1}{12}(950.28\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})[(2.25)^2+(1)^2]\ \text{in}^2\right.$$

$$\left.+(950.28\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2.25}{2}\right)^2+\left(\frac{1}{2}\right)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+\left\{\frac{1}{18}(2006.14\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})(2\ \text{in.})^2\right.$$

$$\left.+(2006.14\times10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[(2.25)^2+\left(\frac{1}{3}\times2\right)^2\right]\text{in}^2\right\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$=[(9.744+29.232)+(3.334+10.002)$$

$$+(3.096+76.720)]\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

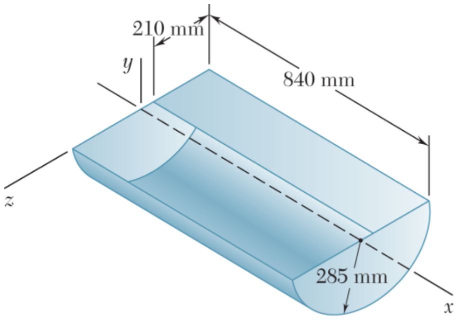
$$=(38.976+13.336+79.816)\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_y=132.1\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM B.29 (Continued)

$$
\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 \\
&= \biggl\{ \frac{1}{12}(3325.97\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})[(2.25)^2 + (3.5)^2]\ \text{in}^2 \\
&\quad + (3325.97\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2.25}{2}\right)^2 + \left(\frac{3.5}{2}\right)^2\right]\ \text{in}^2 \biggr\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&\quad + \biggl\{ \frac{1}{12}(950.28\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})(2.25\ \text{in.})^2 \\
&\quad + (950.28\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2.25}{2}\right)^2 + (3.5)^2\right]\ \text{in}^2 \biggr\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&\quad + \biggl\{ \frac{1}{18}(2006.14\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})(4.75\ \text{in.})^2 \\
&\quad + (2006.14\times 10^{-6}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[(2.25)^2 + \left(\frac{2}{3}\times 4.75\right)^2\right]\ \text{in}^2 \biggr\}\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\
&= [(33.322 + 99.967) + (2.784 + 89.192) \\
&\quad + (17.463 + 210.231)]\times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \\
&= (133.289 + 91.976 + 227.694)\times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2
\end{aligned}
$$

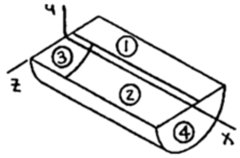
or $I_z = 453\times 10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM B.30*

A farmer constructs a trough by welding a rectangular piece of 2-mm-thick sheet steel to half of a steel drum. Knowing that the density of steel is $7850\ \text{kg/m}^3$ and that the thickness of the walls of the drum is 1.8 mm, determine the mass moment of inertia of the trough with respect to each of the coordinate axes. Neglect the mass of the welds.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \rho_{\text{ST}} t A$$

Then

$$m_1 = (7850\ \text{kg/m}^3)(0.002\ \text{m})(0.84 \times 0.21)\ \text{m}^2$$
$$= 2.76948\ \text{kg}$$

$$m_2 = (7850\ \text{kg/m}^3)(0.0018\ \text{m})(\pi \times 0.285 \times 0.84)\ \text{m}^2$$
$$= 10.62713\ \text{kg}$$

$$m_3 = m_4 = (7850\ \text{kg/m}^3)(0.0018\ \text{m})\left(\frac{\pi}{2} \times 0.285^2\right)\ \text{m}^2$$
$$= 1.80282\ \text{kg}$$

Using Figure 9.28 for component 1 and the equations derived above for components 2 through 4, we have

$$(I_x)_3 = (I_x)_4$$
$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4$$
$$= \left[\frac{1}{12}(2.76948\ \text{kg})(0.21\ \text{m})^2 + (2.76948\ \text{kg})\left(0.285 - \frac{0.21}{2}\right)^2\ \text{m}^2\right]$$
$$+ [(10.62713\ \text{kg})(0.285\ \text{m})^2] + 2\left[\frac{1}{2}(1.80282\ \text{kg})(0.285\ \text{m})^2\right]$$
$$= [(0.01018 + 0.08973) + (0.86319) + 2(0.07322)]\ \text{kg}\cdot\text{m}^2$$
$$= [(0.09991 + 0.86319 + 2(0.07322)]\ \text{kg}\cdot\text{m}^2$$

or $I_x = 1.110\ \text{kg}\cdot\text{m}^2$ ◀

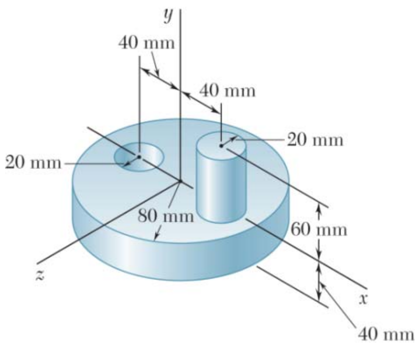
PROBLEM B.30* (Continued)

$$
\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 \\
&= \left\{ \frac{1}{12}(2.76948\ \text{kg})[(0.84)^2 + (0.21)^2]\ \text{m}^2 \right. \\
&\qquad \left. + (2.76948\ \text{kg})\left[\left(\frac{0.84}{2}\right)^2 + \left(0.285 - \frac{0.21}{2}\right)^2\right]\text{m}^2 \right\} \\
&\qquad + \left\{ \frac{1}{12}(10.62713\ \text{kg})[(0.84)^2 + 6(0.285)^2]\ \text{m}^2 + (10.62713\ \text{kg})\left(\frac{0.84}{2}\ \text{m}\right)^2 \right\} \\
&\qquad + \left[\frac{1}{4}(1.80282\ \text{kg})(0.285\ \text{m})^2 \right] \\
&\qquad + \left[\frac{1}{4}(1.80282\ \text{kg})(0.285\ \text{m})^2 + (1.80282\ \text{kg})(0.84\ \text{m})^2 \right] \\
&= [(0.17302 + 0.57827) + (1.05647 + 1.87463) \\
&\qquad + (0.03661) + (0.03661 + 1.27207)]\ \text{kg}\cdot\text{m}^2 \\
&= (0.75129 + 2.93110 + 0.03661 + 1.30868)\ \text{kg}\cdot\text{m}^2
\end{aligned}
$$

or $I_y = 5.03\ \text{kg}\cdot\text{m}^2$ ◀

$$
\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 + (I_z)_4 \\
&= \left[\frac{1}{12}(2.76948\ \text{kg})(0.84\ \text{m})^2 + (2.76948\ \text{kg})\left(\frac{0.84}{2}\ \text{m}\right)^2 \right] \\
&\qquad + \left\{ \frac{1}{12}(10.62713\ \text{kg})[(0.84)^2 + 6(0.285)^2]\ \text{m}^2 + (10.62713\ \text{kg})\left(\frac{0.84}{2}\ \text{m}\right)^2 \right\} \\
&\qquad + \left[\frac{1}{4}(1.80282\ \text{kg})(0.285\ \text{m})^2 \right] \\
&\qquad + \left[\frac{1}{4}(1.80282\ \text{kg})(0.285\ \text{m})^2 + (1.80282\ \text{kg})(0.84\ \text{m})^2 \right] \\
&= [(0.16285 + 0.48854) + (1.05647 + 1.87463) \\
&\qquad + (0.03661) + (0.03661 + 1.27207)]\ \text{kg}\cdot\text{m}^2 \\
&= (0.65139 + 2.93110 + 0.03661 + 1.30868)\ \text{kg}\cdot\text{m}^2
\end{aligned}
$$

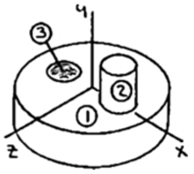
or $I_z = 4.93\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM B.31

The machine element shown is fabricated from steel. Determine the mass moment of inertia of the assembly with respect to (*a*) the x axis, (*b*) the y axis, (*c*) the z axis. (The density of steel is $7850\ \text{kg/m}^3$.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V$$

Then

$$m_1 = (7850\ \text{kg/m}^3)(\pi(0.08\ \text{m})^2(0.04\ \text{m})]$$
$$= 6.31334\ \text{kg}$$

$$m_2 = (7850\ \text{kg/m}^3)[\pi(0.02\ \text{m})^2(0.06\ \text{m})] = 0.59188\ \text{kg}$$

$$m_3 = (7850\ \text{kg/m}^3)[\pi(0.02\ \text{m})^2(0.04\ \text{m})] = 0.39458\ \text{kg}$$

Using Figure 9.28 and the parallel-axis theorem, we have

(*a*)

$$I_x = (I_x)_1 + (I_x)_2 - (I_x)_3$$

$$= \left\{\frac{1}{12}(6.31334\ \text{kg})[3(0.08)^2 + (0.04)^2]\ \text{m}^2 + (6.31334\ \text{kg})(0.02\ \text{m})^2\right\}$$

$$+ \left\{\frac{1}{12}(0.59188\ \text{kg})[3(0.02)^2 + (0.06)^2]\ \text{m}^2 + (0.59188\ \text{kg})(0.03\ \text{m})^2\right\}$$

$$- \left\{\frac{1}{12}(0.39458\ \text{kg})[3(0.02)^2 + (0.04)^2]\ \text{m}^2 + (0.39458\ \text{kg})(0.02\ \text{m})^2\right\}$$

$$= [(10.94312 + 2.52534) + (0.23675 + 0.53269)$$
$$- (0.09207 + 0.15783)] \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$= (13.46846 + 0.76944 - 0.24990) \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$= 13.98800 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

or $I_x = 13.99 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

PROBLEM B.31 (Continued)

(*b*)

$$
\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 - (I_y)_3 \\
&= \left[\frac{1}{2}(6.31334\ \text{kg})(0.08\ \text{m})^2\right] \\
&\quad + \left[\frac{1}{2}(0.59188\ \text{kg})(0.02\ \text{m})^2 + (0.59188\ \text{kg})(0.04\ \text{m})^2\right] \\
&\quad - \left[\frac{1}{2}(0.39458\ \text{kg})(0.02\ \text{m}^2) + (0.39458\ \text{kg})(0.04\ \text{m})^2\right] \\
&= [(20.20269) + (0.11838 + 0.94701) \\
&\quad - (0.07892 + 0.63133)] \times 10^{-3}\ \text{kg}\cdot\text{m}^2 \\
&= (20.20269 + 1.06539 - 0.71025) \times 10^{-3}\ \text{kg}\cdot\text{m}^2 \\
&= 20.55783 \times 10^{-3}\ \text{kg}\cdot\text{m}^2
\end{aligned}
$$

or $I_y = 20.6 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

(*c*)

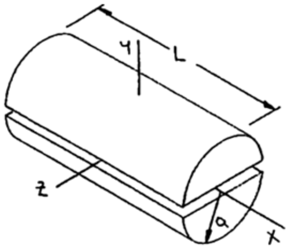
$$
\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 - (I_z)_3 \\
&= \left\{\frac{1}{12}(6.31334\ \text{kg})[3(0.08)^2 + (0.04)^2]\ \text{m}^2 + (6.31334\ \text{kg})(0.02\ \text{m})^2\right\} \\
&\quad + \left\{\frac{1}{12}(0.59188\ \text{kg})[3(0.02)^2 + (0.06)^2]\ \text{m}^2 + (0.59188\ \text{kg})[(0.04)^2 + (0.03)^2]\ \text{m}^2\right\} \\
&\quad - \left\{\frac{1}{12}(0.39458\ \text{kg})[3(0.02)^2 + (0.04)^2]\ \text{m}^2 + (0.03958\ \text{kg})[(0.04)^2 + (0.02)^2]\ \text{m}^2\right\} \\
&= [(10.94312 + 2.52534) + (0.23675 + 1.47970) \\
&\quad - (0.09207 + 0.78916)] \times 10^{-3}\ \text{kg}\cdot\text{m}^2 \\
&= (13.46846 + 1.71645 - 0.88123) \times 10^{-3}\ \text{kg}\cdot\text{m}^2 \\
&= 14.30368 \times 10^{-3}\ \text{kg}\cdot\text{m}^2
\end{aligned}
$$

or $I_z = 14.30 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

To the Instructor:

The following formulas for the mass of inertia of a semicylinder are derived at this time for use in the solutions of Problems 9.142 through 9.145.

From Figure 9.28:



Cylinder $$(I_x)_{\text{cyl}} = \frac{1}{2}m_{\text{cyl}}a^2$$

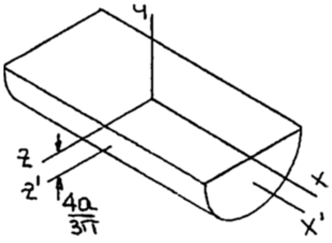
$$(I_y)_{\text{cyl}} = (I_z)_{\text{cyl}} = \frac{1}{12}m_{\text{cyl}}(3a^2 + L^2)$$

Symmetry and the definition of the mass moment of inertia $(I = \int r^2 dm)$ imply

$$(I)_{\text{semicylinder}} = \frac{1}{2}(I)_{\text{cylinder}}$$

$$(I_x)_{\text{sc}} = \frac{1}{2}\left(\frac{1}{2}m_{\text{cyl}}a^2\right)$$

and $$(I_y)_{\text{sc}} = (I_z)_{\text{sc}} = \frac{1}{2}\left[\frac{1}{12}m_{\text{cyl}}(3a^2 + L^2)\right]$$



However, $$m_{\text{sc}} = \frac{1}{2}m_{\text{cyl}}$$

Thus, $$(I_x)_{\text{sc}} = \frac{1}{2}m_{\text{sc}}a^2$$

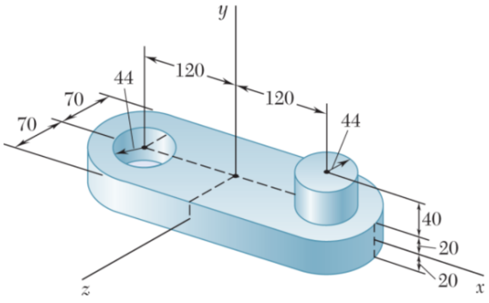
and $$(I_y)_{\text{sc}} = (I_z)_{\text{sc}} = \frac{1}{12}m_{\text{sc}}(3a^2 + L^2)$$

Also, using the parallel axis theorem find

$$\bar{I}_{x'} = m_{\text{sc}}\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)a^2$$

$$\bar{I}_{z'} = m_{\text{sc}}\left[\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)a^2 + \frac{1}{12}L^2\right]$$

where x' and z' are centroidal axes.

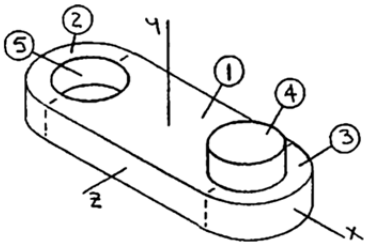


Dimensions in mm

PROBLEM B.32

Determine the mass moments of inertia and the radii of gyration of the steel machine element shown with respect to the x and y axes. (The density of steel is 7850 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V$$

Then

$$m_1 = (7850 \text{ kg/m}^3)(0.24 \times 0.04 \times 0.14) \text{ m}^3$$
$$= 10.5504 \text{ kg}$$

$$m_2 = m_3 = (7850 \text{ kg/m}^3)\left[\frac{\pi}{2}(0.07)^2 \times 0.04\right] \text{m}^3 = 2.41683 \text{ kg}$$

$$m_4 = m_5 = (7850 \text{ kg/m}^3)[\pi(0.044)^2 \times (0.04)] \text{ m}^3 = 1.90979 \text{ kg}$$

Using Figure 9.28 for components 1, 4, and 5 and the equations derived above (before the solution to Problem 9.144) for a semicylinder, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 - (I_x)_5 \quad \text{where} \quad (I_x)_2 = (I_x)_3$$
$$= \left[\frac{1}{12}(10.5504 \text{ kg})(0.04^2 + 0.14^2) \text{ m}^2\right] + 2\left\{\frac{1}{12}(2.41683 \text{ kg})[3(0.07 \text{ m})^2 + (0.04 \text{ m})^2]\right\}$$
$$+ \left\{\frac{1}{12}(1.90979 \text{ kg})[3(0.044 \text{ m})^2 + (0.04 \text{ m})^2] + (1.90979 \text{ kg})(0.04 \text{ m})^2\right\}$$
$$- \left\{\frac{1}{12}(1.90979 \text{ kg})[3(0.044 \text{ m})^2 + (0.04 \text{ m})^2]\right\}$$
$$= [(0.0186390) + 2(0.0032829) + (0.0011790 + 0.0030557) - (0.0011790)] \text{ kg} \cdot \text{m}^2$$
$$= 0.0282605 \text{ kg} \cdot \text{m}^2$$

or $\quad I_x = 28.3 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

PROBLEM B.32 (Continued)

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 - (I_y)_5$$

where $(I_y)_2 = (I_y)_3 \quad (I_y)_4 = |(I_y)_5|$

Then
$$I_y = \left[\frac{1}{12}(10.5504\text{ kg})(0.24^2 + 0.14^2)\text{ m}^2\right]$$
$$+2\left[(2.41683\text{ kg})\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(0.07\text{ m}^2) + (2.41683\text{ kg})\left(0.12 + \frac{4\times 0.07}{3\pi}\right)^2\text{ m}^2\right]$$
$$= [(0.0678742) + 2(0.0037881 + 0.0541678)]\text{ kg}\cdot\text{m}^2$$
$$= 0.1837860\text{ kg}\cdot\text{m}^2$$

or $I_y = 183.8\times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

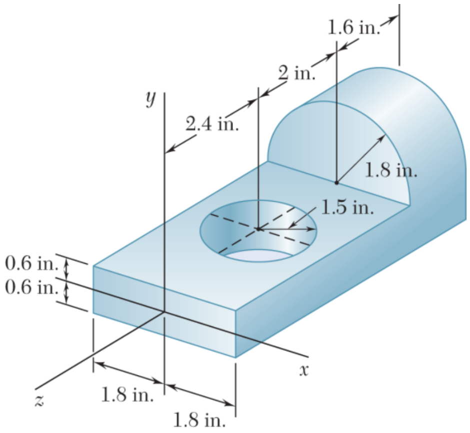
Also
$$m = m_1 + m_2 + m_3 + m_4 - m_5 \text{ where } m_2 = m_3,\ \ m_4 = |m_5|$$
$$= (10.5504 + 2\times 2.41683)\text{ kg} = 15.38406\text{ kg}$$

Then
$$k_x^2 = \frac{I_x}{m} = \frac{0.0282605\text{ kg}\cdot\text{m}^2}{15.38406\text{ kg}}$$

or $k_x = 42.9\text{ mm}$ ◀

and
$$k_y^2 = \frac{I_y}{m} = \frac{0.1837860\text{ kg}\cdot\text{m}^2}{15.38406\text{ kg}}$$

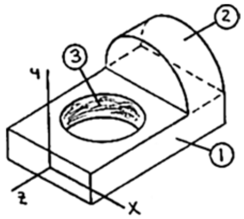
or $k_y = 109.3\text{ mm}$ ◀



PROBLEM B.33

Determine the mass moment of inertia of the steel machine element shown with respect to the x axis. (The density of steel is $490\ \text{lb/ft}^3$.)

SOLUTION



First compute the mass of each component. We have

$$\rho = \frac{\gamma}{g} = \frac{490\ \text{lb}\cdot\text{s}^2}{32.2\ \text{ft}}; \qquad m = \rho V$$

Then

$$m_1 = \rho\left(\frac{3.6\times 1.2\times 6}{12^3}\right)\ \text{ft}^3$$

$$= 15\times 10^{-3}\rho$$

$$m_2 = \rho\left[\frac{\pi}{2}\left(\frac{1.8}{12}\right)^2\left(\frac{1.6}{12}\right)\right]\ \text{ft}^3$$

$$= 4.7124\times 10^{-3}\rho$$

$$m_3 = \rho\left[\pi\left(\frac{1.5}{12}\right)^2\left(\frac{1.2}{12}\right)\right]\ \text{ft}^3 = 4.9087\times 10^{-3}\rho$$

Using Figure 9.28 for components 1 and 3 and the equation derived above (before the solution to Problem 9.142) for a semicylinder, we have

$$I_x = (I_x)_1 + (I_x)_2 - (I_x)_3$$

$$I_{x_1} = \left[\frac{1}{12}\left(15\times 10^{-3}\rho\right)\left(\left(\frac{1.2}{12}\right)^2 + \left(\frac{6}{12}\right)^2\right) + \left(15\times 10^{-3}\rho\right)\left(\frac{3}{12}\right)^2\right]$$

$$I_{x_1} = 1.26250\times 10^{-3}\rho$$

Half-cylinder, centroidal, $$I_{x_{2''}} = \frac{m}{12}\left(3r^2 + L^2\right) = \frac{4.7124\times 10^{-3}\rho}{12}\left(3\left(\frac{1.8}{12}\right)^2 + \left(\frac{1.6}{12}\right)^2\right) = 3.3489\times 10^{-5}\rho$$

PROBLEM B.33 (Continued)

Half-cylinder about base,

$$I_{x_{2''}} = \bar{I}_{x_{2'}} + md^2$$

$$3.3489\times10^{-5}\rho = \bar{I}_{x_{2'}} + 4.7124\times10^{-3}\rho\left(\frac{0.7639}{12}\right)^2$$

$$\bar{I}_{x_{2'}} = 1.43925\times10^{-5}\rho$$

$$I_{x_2} = \bar{I}_{x_{2'}} + md^2$$

$$I_{x_2} = 1.43925\times10^{-5}\rho + 4.7124\times10^{-3}\rho\left[\left(\frac{5.2}{12}\right)^2 + \left(\frac{0.6+0.7639}{12}\right)^2\right]$$

$$I_{x_2} = 9.6015\times10^{-4}\rho$$

$$I_{x_3} = \frac{m}{12}\left(3r^2 + L^2\right) + md^2$$

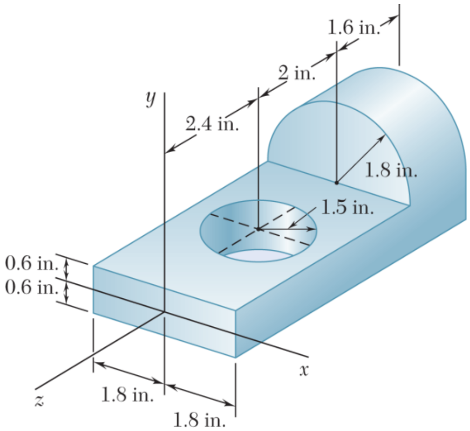
$$I_{x_3} = \frac{4.9087\times10^{-3}\rho}{12}\left[3\left(\frac{1.5}{12}\right)^2 + \left(\frac{1.2}{12}\right)^2\right] + 4.9087\times10^{-3}\rho\left(\frac{2.4}{12}\right)^2$$

$$I_{x_3} = 2.1961\times10^{-4}\rho$$

Now combining all components and substituting for ρ,

$$I_x = \frac{490}{32.2}\left(1.26250\times10^{-3} + 9.6015\times10^{-4} - 2.1961\times10^{-4}\right)$$

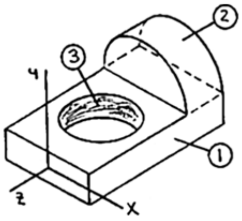
or $I_x = 30.5\times10^{-3}$ lb · ft · s^2 ◀



PROBLEM B.34

Determine the mass moment of inertia of the steel machine element shown with respect to the *y*axis. (The density of steel is 490 lb/ft^3.)

SOLUTION



First compute the mass of each component. We have

$$\rho = \frac{\gamma}{g} = \frac{490 \text{ lb} \cdot \text{s}^2}{32.2 \text{ ft}}; \quad m = \rho V$$

Then

$$m_1 = \rho\left(\frac{3.6 \times 1.2 \times 6}{12^3}\right) \text{ ft}^3$$

$$= 15 \times 10^{-3} \rho$$

$$m_2 = \rho\left[\frac{\pi}{2}\left(\frac{1.8}{12}\right)^2\left(\frac{1.6}{12}\right)\right] \text{ ft}^3$$

$$= 4.7124 \times 10^{-3} \rho$$

$$m_3 = \rho\left[\pi\left(\frac{1.5}{12}\right)^2\left(\frac{1.2}{12}\right)\right] \text{ ft}^3 = 4.9087 \times 10^{-3} \rho$$

Using Figure 9.28 for components 1 and 3 and the equation derived above (before the solution to Problem 9.142) for a semicylinder, we have

$$I_y = (I_y)_1 + (I_y)_2 - (I_y)_3$$

$$I_{y_1} = \left[\frac{1}{12}\left(15 \times 10^{-3} \rho\right)\left(\left(\frac{3.6}{12}\right)^2 + \left(\frac{6}{12}\right)^2\right) + \left(15 \times 10^{-3} \rho\right)\left(\frac{3}{12}\right)^2\right]$$

$$I_{y_1} = 1.36250 \times 10^{-3} \rho$$

PROBLEM B.34 (Continued)

$$I_{y_2} = \overline{I}_{y_2'} + md^2$$

$$I_{y_2} = \frac{4.7124\times10^{-3}\rho}{12}\left(3\left(\frac{1.8}{12}\right)^2 + \left(\frac{1.6}{12}\right)^2\right) + 4.7124\times10^{-3}\rho\left(\frac{5.2}{12}\right)^2$$

$$I_{y_2} = 9.1837\times10^{-4}\rho$$

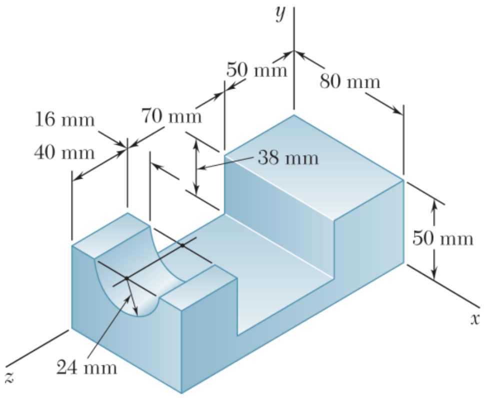
$$I_{y_3} = \frac{4.9087\times10^{-3}\rho}{2}\left(\frac{1.5}{12}\right)^2 + 4.9087\times10^{-3}\rho\left(\frac{2.4}{12}\right)^2$$

$$I_{y_3} = 2.3470\times10^{-4}\rho$$

Now combining all components and substituting for ρ,

$$I_y = \frac{490}{32.2}\left(1.36250\times10^{-3} + 9.1837\times10^{-4} - 2.3470\times10^{-4}\right)$$

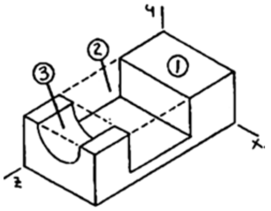
or $I_y = 31.1\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM B.35

Determine the mass moment of inertia of the steel fixture shown with respect to (*a*) the *x* axis, (*b*) the *y* axis, (*c*) the *z* axis. (The density of steel is 7850 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V$$

Then
$$m_1 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.05 \times 0.160) \text{ m}^3$$
$$= 5.02400 \text{ kg}$$

$$m_2 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.038 \times 0.07) \text{ m}^3 = 1.67048 \text{ kg}$$

$$m_3 = 7850 \text{ kg/m}^3 \times \left(\frac{\pi}{2} \times 0.024^2 \times 0.04\right) \text{m}^3 = 0.28410 \text{ kg}$$

Using Figure 9.28 for components 1 and 2 and the equations derived above for component 3, we have

(*a*) $I_x = (I_x)_1 - (I_x)_2 - (I_x)_3$

$$= \left\{\frac{1}{12}(5.02400 \text{ kg})[(0.05)^2 + (0.16)^2] \text{ m}^2 + (5.02400 \text{ kg})\left[\left(\frac{0.05}{2}\right)^2 + \left(\frac{0.16}{2}\right)^2\right] \text{m}^2\right\}$$

$$- \left\{\frac{1}{12}(1.67048 \text{ kg})[(0.038)^2 + (0.07)^2] \text{ m}^2 + (1.67048 \text{ kg})\left[\left(0.05 - \frac{0.038}{2}\right)^2 + \left(0.05 + \frac{0.07}{2}\right)^2\right] \text{m}^2\right\}$$

$$- \left\{(0.28410 \text{ kg})\left[\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)(0.024)^2 + \frac{1}{12}(0.04)^2\right] \text{m}^2\right.$$

$$\left. + (0.28410 \text{ kg})\left[\left(0.05 - \frac{4 \times 0.024}{3\pi}\right)^2 + \left(0.16 - \frac{0.04}{2}\right)^2\right] \text{m}^2\right\}$$

$$= [(11.7645 + 35.2936) - (0.8831 + 13.6745) - (0.0493 + 6.0187)] \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= (47.0581 - 14.5576 - 6.0680) \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$= 26.4325 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

or $I_x = 26.4 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

PROBLEM B.35 (Continued)

(*b*) $I_y = (I_y)_1 - (I_y)_2 - (I_y)_3$

$$= \left\{\frac{1}{12}(5.02400\text{ kg})[(0.08)^2 + (0.16)^2]\text{ m}^2 + (5.02400\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(\frac{0.16}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(1.67048\text{ kg})[(0.08)^2 + (0.07)^2]\text{ m}^2 + (1.67048\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 + \frac{0.07}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(0.28410\text{ kg})[3(0.024)^2 + (0.04)^2]\text{ m}^2 + (0.28410\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.16 - \frac{0.04}{2}\right)^2\right]\text{m}^2\right\}$$

$$= [(13.3973 + 40.1920) - (1.5730 + 14.7420) - (0.0788 + 6.0229)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (53.5893 - 16.3150 - 6.1017) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= 31.1726 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_y = 31.2 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

(*c*) $I_z = (I_z)_1 - (I_z)_2 - (I_z)_3$

$$= \left\{\frac{1}{12}(5.02400\text{ kg})[(0.08)^2 + (0.05)^2]\text{ m}^2 + (5.02400\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(\frac{0.05}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{\frac{1}{12}(1.67048\text{ kg})[(0.08)^2 + (0.038)^2]\text{ m}^2 + (1.67048\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 - \frac{0.038}{2}\right)^2\right]\text{m}^2\right\}$$

$$-\left\{(0.28410\text{ kg})\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)(0.024\text{ m})^2 + (0.28410\text{ kg})\left[\left(\frac{0.08}{2}\right)^2 + \left(0.05 - \frac{4 \times 0.024}{3\pi}\right)^2\right]\text{m}^2\right\}$$

$$= [(3.7261 + 11.1784) - (1.0919 + 4.2781) - (0.0523 + 0.9049)] \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (14.9045 - 5.3700 - 0.9572) \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

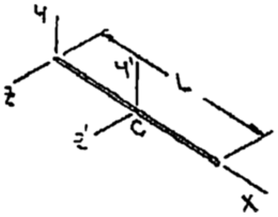
$$= 8.5773 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

or $I_z = 8.58 \times 10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

To the instructor:

The following formulas for the mass moment of inertia of wires are derived or summarized at this time for use in the solutions of Problems 9.146 through 9.148.

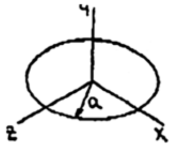
Slender Rod

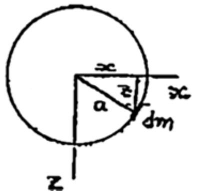


$$I_x = 0 \qquad \bar{I}_{y'} = \bar{I}_{z'} = \frac{1}{12}mL^2 \text{ (Figure 9.28)}$$

$$I_y = I_z = \frac{1}{3}mL^2 \text{ (Sample Problem 9.9)}$$

Circle





We have $\bar{I}_y = \int r^2 dm = ma^2$

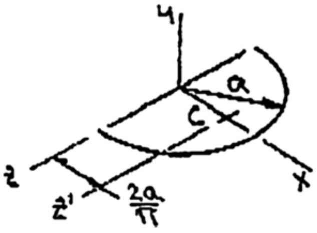
Now $\bar{I}_y = \bar{I}_x + \bar{I}_z$

And symmetry implies $\bar{I}_x = \bar{I}_z$

$$\bar{I}_x = \bar{I}_z = \frac{1}{2}ma^2$$

Semicircle

Following the above arguments for a circle, We have

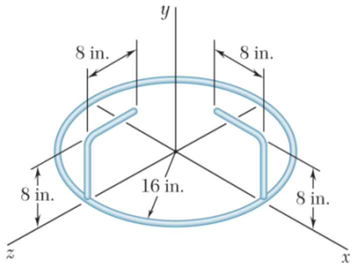


$$\bar{I}_x = I_z = \frac{1}{2}ma^2 \qquad I_y = ma^2$$

Using the parallel-axis theorem

$$I_z = \bar{I}_{z'} + m\bar{x}^2 \qquad x = \frac{2a}{\pi}$$

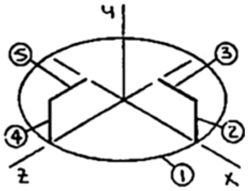
or $$I_{z'} = m\left(\frac{1}{2} - \frac{4}{\pi^2}\right)a^2$$



PROBLEM B.36

Aluminum wire with a weight per unit length of 0.033 lb/ft is used to form the circle and the straight members of the figure shown. Determine the mass moment of inertia of the assembly with respect to each of the coordinate axes.

SOLUTION



First compute the mass of each component. We have

$$m = \frac{W}{g} = \frac{1}{g}\left(\frac{W}{L}\right)_{\text{AL}} L$$

Then

$$m_1 = \frac{1}{32.2 \text{ ft/s}^2} \times 0.033 \text{ lb/ft} \times (2\pi \times 16 \text{ in.}) \times \frac{1 \text{ ft}}{12 \text{ in.}}$$

$$= 8.5857 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = m_3 = m_4 = m_5 = \frac{1}{32.2 \text{ ft/s}^2} \times 0.033 \text{ lb/ft} \times 8 \text{ in.} \times \frac{1 \text{ ft}}{12 \text{ in.}}$$

$$= 0.6832 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Using the equations given above and the parallel-axis theorem, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 + (I_x)_5$$

$$= \left[\frac{1}{2}(8.5857 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft})(16 \text{ in.})^2\right] \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$+ \left[\frac{1}{3}(0.6832 \text{ lb} \cdot \text{s}^2/\text{ft})(8 \text{ in.})^2\right] \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$+ [0 + (0.6832 \text{ lb} \cdot \text{s}^2/\text{ft})(8 \text{ in.})^2] \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$+ \left[\frac{1}{12}(0.6832 \text{ lb} \cdot \text{s}^2/\text{ft})(8 \text{ in.})^2 + (0.6832 \text{ lb} \cdot \text{s}^2/\text{ft})(4^2 + 16^2) \text{ in}^2\right] \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$+ \left[\frac{1}{12}(0.6832 \text{ lb} \cdot \text{s}^2/\text{ft})(8 \text{ in.})^2 + (0.6832 \text{ lb} \cdot \text{s}^2/\text{ft})(8^2 + 12^2) \text{ in}^2\right] \times \left(\frac{1 \text{ ft}}{12 \text{ in.}}\right)^2$$

$$= [(7.6315) + (0.1012) + (0.3036) + (0.0253 + 1.2905) + (0.0253 + 0.9868)] \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$= (7.6315 + 0.1012 + 0.3036) + 1.3158 + 1.0121) \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$= 10.3642 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

or $I_x = 10.36 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ◀

$$(I_y)_2 = (I_y)_4, \qquad (I_y)_3 = (I_y)_5$$

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 + (I_y)_5$$

$$= [(8.5857\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(16\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+2[0+(0.6832\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(16\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

$$+2\left[\frac{1}{12}(0.6832\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(8\ \text{in.})^2+(0.6832\ \text{lb}\cdot\text{s}^2/\text{ft})(12\ \text{in.})^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2$$

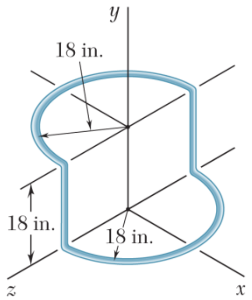
$$= [(15.2635)+2(1.2146)+2(0.0253+0.6832)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= [15.2635+2(1.2146)+2(0.7085)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 19.1097\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_y = 19.11\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

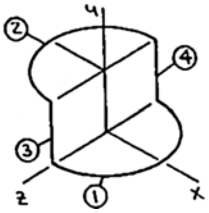
Symmetry implies $I_x = I_z$ $I_z = 10.36\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM B.37

The figure shown is formed of $\frac{1}{8}$-in.-diameter steel wire. Knowing that the specific weight of the steel is 490 lb/ft^3, determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\mathrm{ST}} V = \frac{\gamma_{\mathrm{ST}}}{g} AL$$

Then

$$m_1 = m_2 = \frac{490\ \mathrm{lb/ft}^3}{32.2\ \mathrm{ft/s}^2} \times \left[\frac{\pi}{4}\left(\frac{1}{8}\ \mathrm{in.}\right)^2\right] \times (\pi \times 18\ \mathrm{in.}) \times \left(\frac{1\ \mathrm{ft}}{12\ \mathrm{in.}}\right)^3$$

$$= 6.1112 \times 10^{-3}\ \mathrm{lb \cdot s^2/ft}$$

$$m_3 = m_4 = \frac{490\ \mathrm{lb/ft}^3}{32.2\ \mathrm{ft/s}^2} = \left[\frac{\pi}{4}\left(\frac{1}{8}\mathrm{in.}\right)^2\right] \times 18\ \mathrm{in.} \times \left(\frac{1\ \mathrm{ft}}{12\ \mathrm{in.}}\right)^3$$

$$= 1.9453 \times 10^{-3}\ \mathrm{lb \cdot s^2/ft}$$

Using the equations given above and the parallel-axis theorem, we have

$$(I_x)_3 = (I_x)_4$$

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4$$

$$= \left[\frac{1}{2}(6.1112 \times 10^{-3}\ \mathrm{lb \cdot s^2/ft})(18\ \mathrm{in.})^2\right] \times \left(\frac{1\ \mathrm{ft}}{12\ \mathrm{in.}}\right)^2$$

$$+ \left[\frac{1}{2}(6.1112 \times 10^{-3}\ \mathrm{lb \cdot s^2/ft})(18\ \mathrm{in.})^2 + (6.1112 \times 10^{-3}\ \mathrm{lb \cdot s^2/ft})(18\ \mathrm{in.})^2\right] \times \left(\frac{1\ \mathrm{ft}}{12\ \mathrm{in.}}\right)^2$$

$$+ 2\left[\frac{1}{12}(1.9453 \times 10^{-3}\ \mathrm{lb \cdot s^2/ft})(18\ \mathrm{in.})^2 + (6.1112 \times 10^{-3}\ \mathrm{lb \cdot s^2/ft})(9^2 + 18^2)\ \mathrm{in}^2\right] \times \left(\frac{1\ \mathrm{ft}}{12\ \mathrm{in.}}\right)^2$$

$$= [(6.8751) + (6.8751 + 13.1502) + 2(0.3647 + 5.4712)] \times 10^{-3}\ \mathrm{lb \cdot ft \cdot s^2}$$

$$= [6.8751 + 20.6252 + 2(5.8359)] \times 10^{-3}\ \mathrm{lb \cdot ft \cdot s^2}$$

$$= 39.1721 \times 10^{-3}\ \mathrm{lb \cdot ft \cdot s^2}$$

or $I_x = 0.0392\ \mathrm{lb \cdot ft \cdot s^2}$ ◀

PROBLEM B.37 (Continued)

$$(I_y)_1 = (I_y)_2, \quad (I_y)_3 = (I_y)_4$$

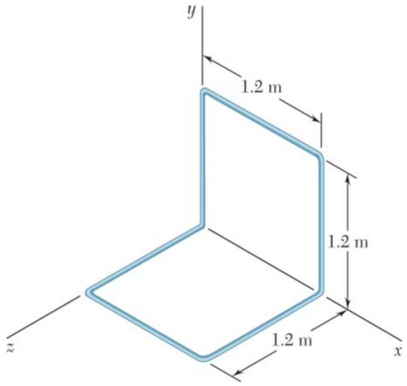
$$\begin{aligned} I_y &= (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 \\ &= 2[(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\ &\quad + 2[(0 + 1.9453\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\ &= [2(13.7502) + 2(4.3769)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \\ &= 36.2542\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \end{aligned}$$

or $I_y = 0.0363\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$(I_z)_3 = (I_z)_4$$

$$\begin{aligned} I_z &= (I_z)_1 + (I_z)_2 + (I_z)_3 + (I_z)_4 \\ &= \left[\frac{1}{2}(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\ &\quad + \left\{(6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left(\frac{1}{2} - \frac{4}{\pi^2}\right)(18\ \text{in.})^2\right. \\ &\quad \left. + (6.1112\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})\left[\left(\frac{2\times18}{\pi}\right)^2 + (18)^2\right]\ \text{in}^2\right\}\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\ &\quad + 2\left[\frac{1}{3}(1.9453\times10^{-3}\ \text{lb}\cdot\text{s}^2/\text{ft})(18\ \text{in.})^2\right]\times\left(\frac{1\ \text{ft}}{12\ \text{in.}}\right)^2 \\ &= [(6.8751) + (1.3024 + 19.3229) + 2(1.4590)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \\ &= [6.8751 + 20.6253 + 2(1.4590)]\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \\ &= 30.4184\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \end{aligned}$$

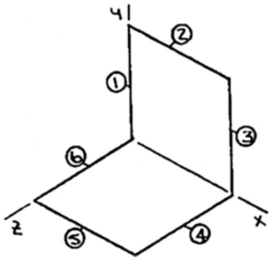
or $I_z = 0.0304\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM B.38

A homogeneous wire with a mass per unit length of 0.056 kg/m is used to form the figure shown. Determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

SOLUTION



First compute the mass *m* of each component. We have

$$m = (m/L)L$$
$$= 0.056 \text{ kg/m} \times 1.2 \text{ m}$$
$$= 0.0672 \text{ kg}$$

Using the equations given above and the parallel-axis theorem, we have

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 + (I_x)_4 + (I_x)_5 + (I_x)_6$$

Now $(I_x)_1 = (I_x)_3 = (I_x)_4 = (I_x)_6 \quad \text{and} \quad (I_x)_2 = (I_x)_5$

Then $$I_x = 4\left[\frac{1}{3}(0.0672 \text{ kg})(1.2 \text{ m})^2\right] + 2[0 + (0.0672 \text{ kg})(1.2 \text{ m})^2]$$
$$= [4(0.03226) + 2(0.09677)] \text{ kg}\cdot\text{m}^2$$
$$= 0.32258 \text{ kg}\cdot\text{m}^2$$

or $I_x = 0.323 \text{ kg}\cdot\text{m}^2$ ◀

$$I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 + (I_y)_4 + (I_y)_5 + (I_y)_6$$

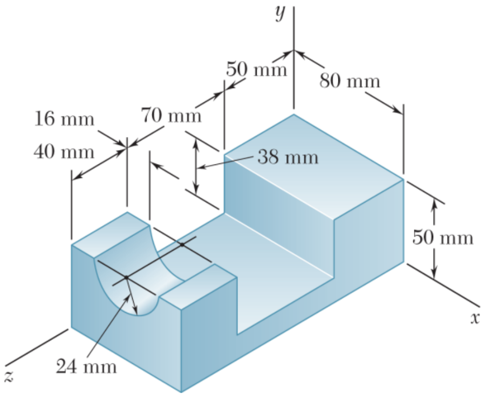
Now $(I_y)_1 = 0, \quad (I_y)_2 = (I_y)_6, \quad \text{and} \quad (I_y)_4 = (I_y)_5$

Then $$I_y = 2\left[\frac{1}{3}(0.0672 \text{ kg})(1.2 \text{ m})^2\right] + [0 + (0.0672 \text{ kg})(1.2 \text{ m})^2]$$
$$+ 2\left[\frac{1}{12}(0.0672 \text{ kg})(1.2 \text{ m})^2 + (0.0672 \text{ kg})(1.2^2 + 0.6^2)\text{ m}^2\right]$$
$$= [2(0.03226) + (0.09677) + 2(0.00806 + 0.12096)] \text{ kg}\cdot\text{m}^2$$
$$= [2(0.03226) + (0.09677) + 2(0.12902)] \text{ kg}\cdot\text{m}^2$$
$$= 0.41933 \text{ kg}\cdot\text{m}^2$$

or $I_y = 0.419 \text{ kg}\cdot\text{m}^2$ ◀

Symmetry implies $I_y = I_z$

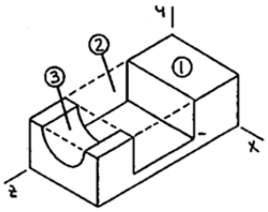
$I_z = 0.419 \text{ kg}\cdot\text{m}^2$ ◀



PROBLEM B.39

Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the steel fixture shown. (The density of steel is 7850 kg/m^3.)

SOLUTION



First compute the mass of each component. We have

$$m = \rho V$$

Then
$$m_1 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.05 \times 0.16) \text{ m}^3 = 5.02400 \text{ kg}$$
$$m_2 = 7850 \text{ kg/m}^3 \times (0.08 \times 0.038 \times 0.07) \text{ m}^3 = 1.67048 \text{ kg}$$
$$m_3 = 7850 \text{ kg/m}^3 \times \left(\frac{\pi}{2} \times 0.024^2 \times 0.04\right) \text{m}^3 = 0.28410 \text{ kg}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Now

$$\bar{y}_2 = \left(0.05 - \frac{0.038}{2}\right) \text{m} = 0.031 \text{ m}$$
$$\bar{y}_3 = \left(0.05 - \frac{4 \times 0.024}{3\pi}\right) \text{m} = 0.039814 \text{ m}$$

and then

	m, kg	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$m\bar{x}\bar{y}$, kg·m^2	$m\bar{y}\bar{z}$, kg·m^2	$m\bar{z}\bar{x}$, kg·m^2
1	5.02400	0.04	0.025	0.08	5.0240×10^{-3}	10.0480×10^{-3}	16.0768×10^{-3}
2	1.67048	0.04	0.031	0.085	2.0714×10^{-3}	4.4017×10^{-3}	5.6796×10^{-3}
3	0.28410	0.04	0.039814	0.14	0.4524×10^{-3}	1.5836×10^{-3}	1.5910×10^{-3}

Finally,

$$I_{xy} = (I_{xy})_1 - (I_{xy})_2 - (I_{xy})_3 = [(0 + 5.0240) - (0 + 2.0714) - (0 + 0.4524)] \times 10^{-3}$$
$$= 2.5002 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

or $I_{xy} = 2.50 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ ◀

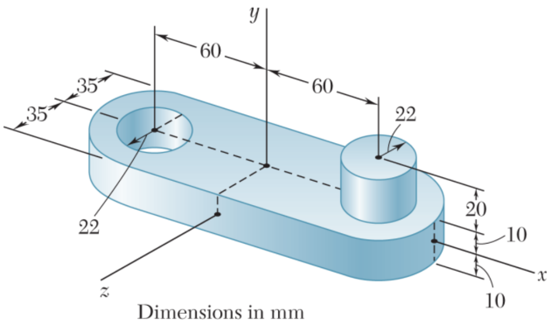
PROBLEM B.39 (Continued)

$$I_{yz} = (I_{yz})_1 - (I_{yz})_2 - (I_{yz})_3 = [(0+10.0480) - (0+4.4017) - (0+1.5836)] \times 10^{-3}$$
$$= 4.0627 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

or $I_{yz} = 4.06 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ ◀

$$I_{zx} = (I_{zx})_1 - (I_{zx})_2 - (I_{zx})_3 = [(0+16.0768) - (0+5.6796) - (0+1.5910)] \times 10^{-3}$$
$$= 8.8062 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

or $I_{zx} = 8.81 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ ◀


y
60
60
35
35
22
22
20
10
10
x
z
Dimensions in mm

PROBLEM B.40

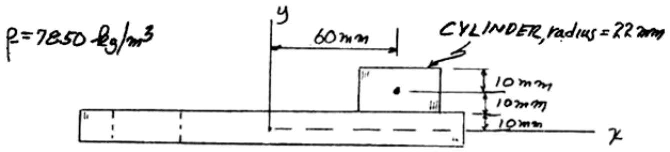
Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the steel machine element shown. (The density of steel is 7850 kg/m^3.)

SOLUTION

Since the machine element is symmetrical with respect to the *xy* plane, $I_{yz} = I_{zx} = 0$ ◀

Also, $I_{xy} = 0$

for all components except the cylinder, since these are symmetrical with respect to the *xz* plane.

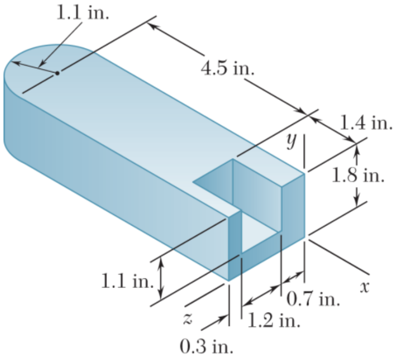

ρ=7850 kg/m³
y
60mm
CYLINDER, radius = 22 mm
10mm
10mm
10mm
x

For the cylinder

$$m = \rho V = (7850\ \text{kg/m}^3)\pi(0.022\ \text{m})^2(0.02\ \text{m}) = 0.2387\ \text{kg}$$

$$I'_{xy} = \bar{I}_{x'y'} + m\bar{x}\,\bar{y} = 0 + (0.2387\ \text{kg})(0.06\ \text{m})(0.02\ \text{m}) = +286.4\times10^{-6}\ \text{kg}\cdot\text{m}^2$$

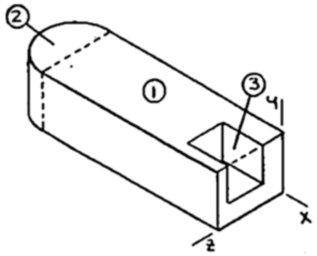
$$I_{xy} = 286\times10^{-6}\ \text{kg}\cdot\text{m}^2$$ ◀



PROBLEM B.41

Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the cast aluminum machine component shown. (The specific weight of aluminum is 0.100 lb/in^3)

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{AL}} V = \frac{\gamma_{\text{AL}}}{g} V$$

Then

$$m_1 = \frac{0.100\ \text{lb/in}^3}{32.2\ \text{ft/s}^2} \times (5.9 \times 1.8 \times 2.2)\ \text{in}^3$$

$$= 72.5590 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{0.100\ \text{lb/in}^3}{32.2\ \text{ft/s}^2} \times \left[\frac{\pi}{2}(1.1)^2 \times 1.8\right]\ \text{in}^3$$

$$= 10.6248 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{0.100\ \text{lb/in}^3}{32.2\ \text{ft/s}^2} \times (1.4 \times 1.1 \times 1.2)\ \text{in}^3$$

$$= 5.7391 \times 10^{-3}\ \text{lb} \cdot \text{s}^2/\text{ft}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Now

$$\bar{x}_2 = -\left(5.9 + \frac{4 \times 1.1}{3\pi}\right)\ \text{in.}$$

$$= -6.36685\ \text{in.}$$

$$\bar{y}_3 = \left(1.8 - \frac{1.1}{2}\right)\ \text{in.}$$

$$= 1.25\ \text{in.}$$

PROBLEM B.41 (Continued)

and then

	m, $\text{lb}\cdot\text{s}^2/\text{ft}$	$\bar{x}$, ft	$\bar{y}$, ft	$\bar{z}$, ft	$m\bar{x}\,\bar{y}$, $\text{lb}\cdot\text{ft}\cdot\text{s}^2$	$m\bar{y}\,\bar{z}$, $\text{lb}\cdot\text{ft}\cdot\text{s}^2$	$m\bar{z}\,\bar{x}$, $\text{lb}\cdot\text{ft}\cdot\text{s}^2$
1	72.5590×10^{-3}	$-\frac{2.95}{12}$	$\frac{0.9}{12}$	$\frac{1.1}{12}$	-1.33781×10^{-3}	0.49884×10^{-3}	-1.63510×10^{-3}
2	10.6248×10^{-3}	$-\frac{6.36685}{12}$	$\frac{0.9}{12}$	$\frac{1.1}{12}$	-0.42279×10^{-3}	0.07305×10^{-3}	-0.51674×10^{-3}
3	5.7391×10^{-3}	$-\frac{0.7}{12}$	$\frac{1.25}{12}$	$\frac{1.3}{12}$	-0.03487×10^{-3}	0.06476×10^{-3}	-0.03627×10^{-3}

Finally,

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 - (I_{xy})_3$$
$$= [(0-1.33781)+(0-0.42279)-(0-0.03487)]\times10^{-3}$$

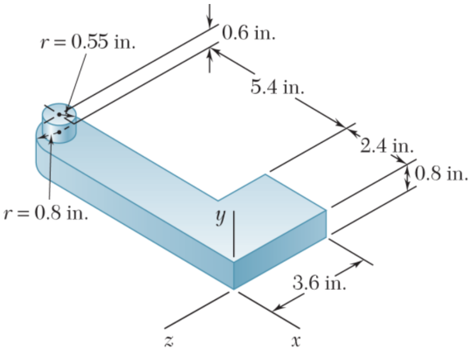
or $I_{xy} = -1.726\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$I_{yz} = (I_{yz})_1 + (I_{yz})_2 - (I_{yz})_3$$
$$= [(0+0.49884)+(0+0.07305)-(0+0.06476)]\times10^{-3}$$

or $I_{yz} = 0.507\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$I_{zx} = (I_{zx})_1 + (I_{zx})_2 - (I_{zx})_3$$
$$= [(0-1.63510)+(0-0.51674)-(0-0.3627)]\times10^{-3}$$

or $I_{zx} = -2.12\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



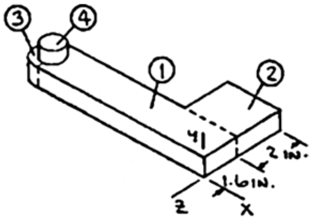
PROBLEM B.42

Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the cast aluminum machine component shown. (The specific weight of aluminum is 0.100 lb/in^3)

SOLUTION

First compute the mass of each component. We have

$$m = \rho_{\text{AL}} V = \frac{\gamma_{\text{AL}}}{g} V$$



Then

$$m_1 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times (7.8 \times 0.8 \times 1.6) \text{ in}^3$$

$$= 31.0062 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_2 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times (2.4 \times 0.8 \times 2) \text{ in}^3$$

$$= 11.9255 \times 10^{-3} \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_3 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times \left[\frac{\pi}{2} (0.8)^2 \times 0.8 \right] \text{ in}^3 = 2.4977 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$m_4 = \frac{0.100 \text{ lb/in}^3}{32.2 \text{ ft/s}^2} \times [\pi (0.55)^2 \times 0.6] \text{ in}^3 = 1.7708 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}, \text{and } \bar{I}_{z'x'},$ of each component are zero because of symmetry. Now

$$\bar{x}_3 = -\left(7.8 + \frac{4 \times 0.8}{3\pi} \right) \text{ in.} = -8.13953 \text{ in.}$$

PROBLEM B.42 (Continued)

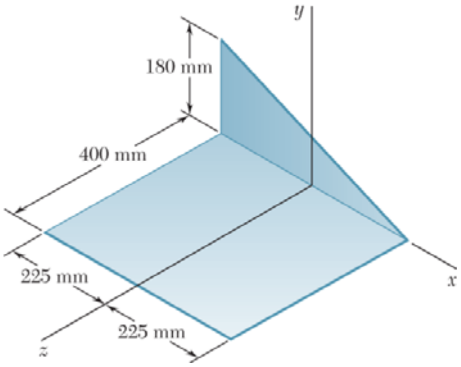
and then

	m, lb·s²/ft	$\bar{x}$, ft	$\bar{y}$, ft	$\bar{z}$, ft	$m\bar{x}\,\bar{y}$, lb·ft·s²	$m\bar{y}\,\bar{z}$, lb·ft·s²	$m\bar{z}\,\bar{x}$, lb·ft·s²
1	31.0062×10^{-3}	$-\frac{3.9}{12}$	$\frac{0.4}{12}$	$-\frac{0.8}{12}$	-335.901×10^{-6}	-68.903×10^{-6}	671.801×10^{-6}
2	11.9255×10^{-3}	$-\frac{1.2}{12}$	$\frac{0.4}{12}$	$-\frac{2.6}{12}$	-39.752×10^{-6}	-86.129×10^{-6}	258.386×10^{-6}
3	2.4977×10^{-3}	$-\frac{8.13953}{12}$	$\frac{0.4}{12}$	$-\frac{0.8}{12}$	-56.473×10^{-6}	-5.550×10^{-6}	112.945×10^{-6}
4	1.7708×10^{-3}	$-\frac{7.8}{12}$	$\frac{1.1}{12}$	$-\frac{0.8}{12}$	-105.511×10^{-6}	-10.822×10^{-6}	76.735×10^{-6}
Σ					-537.637×10^{-6}	-171.404×10^{-6}	1119.867×10^{-6}

Then $I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y})$ or $I_{xy} = -538\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z})$ or $I_{yz} = -171.4\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

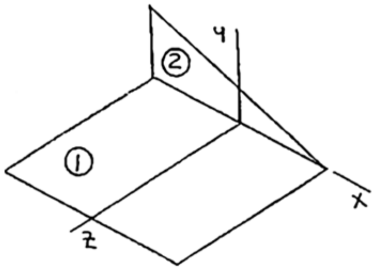
$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x})$ or $I_{zx} = 1120\times10^{-6}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM B.43

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



First compute the mass of each component.

We have
$$m = \rho_{\mathrm{ST}} V = \rho_{\mathrm{ST}} tA$$

Then
$$m_1 = (7850\ \mathrm{kg/m^3})[(0.002)(0.4)(0.45)]\mathrm{m^3}$$
$$= 2.8260\ \mathrm{kg}$$
$$m_2 = 7850\ \mathrm{kg/m^3}\left[(0.002)\left(\frac{1}{2}\times 0.45\times 0.18\right)\right]\mathrm{m^3}$$
$$= 0.63585\ \mathrm{kg}$$

Now observe that
$$(\bar{I}_{x'y'})_1 = (\bar{I}_{y'z'})_1 = (\bar{I}_{z'x'})_1 = 0$$
$$(\bar{I}_{y'z'})_2 = (\bar{I}_{z'x'})_2 = 0$$

From Sample Problem 9.6: $(\bar{I}_{x'y'})_{2,\mathrm{area}} = -\dfrac{1}{72} b_2^2 h_2^2$

Then
$$(\bar{I}_{x'y'})_2 = \rho_{\mathrm{ST}} t (I_{x'y'})_{2,\mathrm{area}} = \rho_{\mathrm{ST}} t\left(-\frac{1}{72} b_2^2 h_2^2\right) = -\frac{1}{36} m_2 b_2 h_2$$

Also
$$\bar{x}_1 = \bar{y}_1 = \bar{z}_2 = 0 \qquad \bar{x}_2 = \left(-0.225 + \frac{0.45}{3}\right)\mathrm{m} = -0.075\ \mathrm{m}$$

PROBLEM B.43 (Continued)

Finally,
$$I_{xy}=\Sigma(\bar{I}_{xy}+m\bar{x}\,\bar{y})=(0+0)+\left[-\frac{1}{36}(0.63585\text{ kg})(0.45\text{ m})(0.18\text{ m})\right.$$
$$\left.+(0.63585\text{ kg})(-0.075\text{ m})\left(\frac{0.18\text{ m}}{3}\right)\right]$$
$$=(-1.43066\times10^{-3}-2.8613\times10^{-3})\text{ kg}\cdot\text{m}^2$$

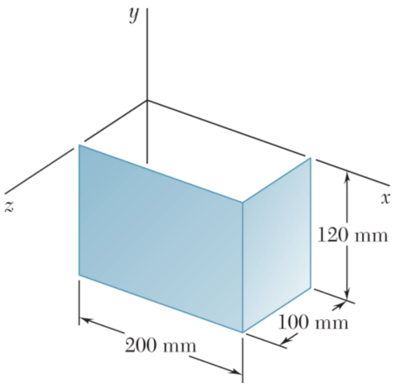
or $I_{xy}=-4.29\times10^{-3}\text{ kg}\cdot\text{m}^2$ ◀

and
$$I_{yz}=\Sigma(\bar{I}_{y'z'}+m\,\bar{y}\,\bar{z})=(0+0)+(0+0)=0$$

or $I_{yz}=0$ ◀

$$I_{zx}=\Sigma(\bar{I}_{z'x'}+m\,\bar{z}\,\bar{x})=(0+0)+(0+0)=0$$

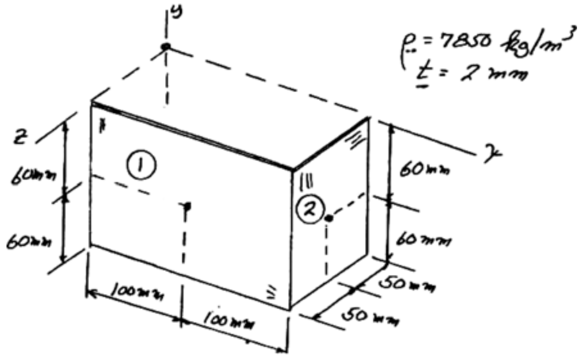
or $I_{zx}=0$ ◀



PROBLEM B.44

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



$$m_1 = \rho V = 7850 \text{ kg/m}^3 (0.120 \text{ m})(0.200 \text{ m})(0.002 \text{ m}) = 0.3768 \text{ kg}$$

$$m_2 = \rho V = 7850 \text{ kg/m}^3 (0.120 \text{ m})(0.100 \text{ m})(0.002 \text{ m}) = 0.1884 \text{ kg}$$

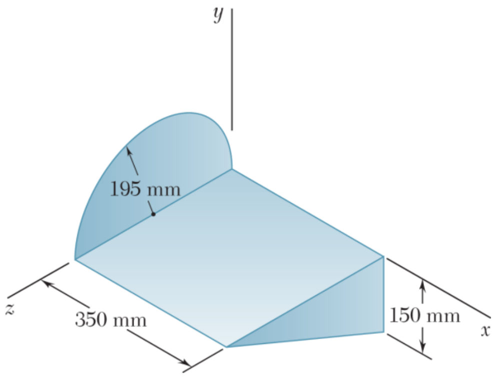
For each panel the centroidal product of inertia is zero with respect to each pair of coordinate axes.

	m, kg	$\bar{x}, m$	$\bar{y}, m$	$\bar{z}, m$	$m\bar{x}\bar{y}$	$m\bar{y}\bar{z}$	$m\bar{z}\bar{x}$
①	0.3768	0.1	–0.06	0.1	-2.261×10^{-3}	-2.261×10^{-3}	$+3.768\times10^{-3}$
②	0.1884	0.2	–0.06	0.05	-2.261×10^{-3}	-0.565×10^{-3}	$+1.884\times10^{-3}$
Σ					-4.522×10^{-3}	-2.826×10^{-3}	$+5.652\times10^{-3}$

$$I_{xy} = -4.52\times10^{-3} \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

$$I_{yz} = -2.83\times10^{-3} \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

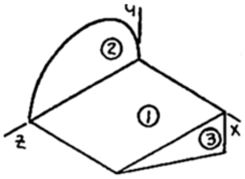
$$I_{zx} = +5.65\times10^{-3} \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$



PROBLEM B.45

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \rho_{\text{ST}} t A$$

Then

$$m_1 = (7850\ \text{kg/m}^3)(0.002\ \text{m})(0.35 \times 0.39)\ \text{m}^2$$
$$= 2.14305\ \text{kg}$$
$$m_2 = (7850\ \text{kg/m}^3)(0.002\ \text{m})\left(\frac{\pi}{2} \times 0.195^2\right)\ \text{m}^2$$
$$= 0.93775\ \text{kg}$$
$$m_3 = (7850\ \text{kg/m}^3)(0.002\ \text{m})\left(\frac{1}{2} \times 0.39 \times 0.15\right)\ \text{m}^2$$
$$= 0.45923\ \text{kg}$$

Now observe that because of symmetry the centroidal products of inertia of components 1 and 2 are zero and

$$(\bar{I}_{x'y'})_3 = (\bar{I}_{z'x'})_3 = 0$$

Also

$$(\bar{I}_{y'z'})_{3,\text{mass}} = \rho_{\text{ST}} t (\bar{I}_{y'z'})_{3,\text{area}}$$

Using the results of Sample Problem 9.6 and noting that the orientation of the axes corresponds to a 90° rotation, we have

$$(\bar{I}_{y'z'})_{3,\text{area}} = \frac{1}{72} b_3^2 h_3^2$$

Then

$$(I_{y'z'})_3 = \rho_{\text{ST}} t \left(\frac{1}{72} b_3^2 h_3^2\right) = \frac{1}{36} m_3 b_3 h_3$$

Also

$$\bar{y}_1 = \bar{x}_2 = 0$$
$$\bar{y}_2 = \frac{4 \times 0.195}{3\pi}\ \text{m} = 0.082761\ \text{m}$$

PROBLEM B.45 (Continued)

Finally,

$$I_{xy} = \Sigma(\bar{I}_{x'y'} + m\bar{x}\,\bar{y})$$

$$= (0+0) + (0+0) + \left[0 + (0.45923\text{ kg})(0.35\text{ m})\left(-\frac{0.15}{3}\text{ m}\right)\right]$$

$$= -8.0365\times 10^{-3}\text{ kg}\cdot\text{m}^2 \qquad \text{or } I_{xy} = -8.04\times 10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

$$I_{yz} = \Sigma(\bar{I}_{y'z'} + m\bar{y}\,\bar{z})$$

$$= (0+0) + [0 + (0.93775\text{ kg})(0.082761\text{ m})(0.195\text{ m})]$$

$$+\left[\frac{1}{36}(0.45923\text{ kg})(0.39\text{ m})(0.15\text{ m}) + (0.45923\text{ kg})\left(-\frac{0.15}{3}\text{ m}\right)\left(\frac{0.39}{3}\text{ m}\right)\right]$$

$$= [(15.1338) + (0.7462 - 2.9850)]\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= (15.1338 - 2.2388)\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

$$= 12.8950\times 10^{-3}\text{ kg}\cdot\text{m}^2 \qquad \text{or } I_{yz} = 12.90\times 10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

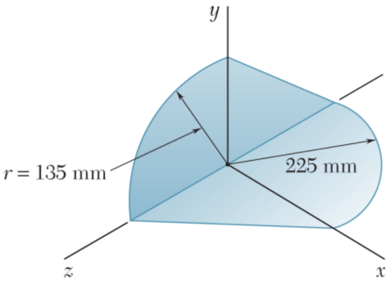
$$I_{zx} = \Sigma(\bar{I}_{z'x'} + m\bar{z}\,\bar{x})$$

$$= [0 + (2.14305\text{ kg})(0.175\text{ m})(0.195\text{ m})] + (0+0)$$

$$+\left[0 + (0.45923\text{ kg})\left(\frac{0.39}{3}\text{ m}\right)(0.35\text{ m})\right]$$

$$= (73.1316 + 20.8950)\times 10^{-3}\text{ kg}\cdot\text{m}^2$$

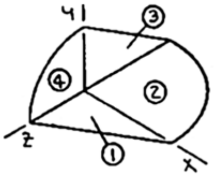
$$= 94.0266\times 10^{-3}\text{ kg}\cdot\text{m}^2 \qquad \text{or } I_{zx} = 94.0\times 10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$



PROBLEM B.46

A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}}V = \rho_{\text{ST}}tA$$

Then

$$m_1 = m_3 = (7850\text{ kg/m}^3)(0.002\text{ m})\left(\frac{1}{2}\times 0.225\times 0.135\right)\text{m}^2 = 0.23844\text{ kg}$$

$$m_2 = (7850\text{ kg/m}^3)(0.002\text{ m})\left(\frac{\pi}{4}\times 0.225^2\right)\text{m}^2 = 0.62424\text{ kg}$$

$$m_4 = (7850\text{ kg/m}^3)(0.002\text{ m})\left(\frac{\pi}{4}\times 0.135^2\right)\text{m}^2 = 0.22473\text{ kg}$$

Now observe that the following centroidal products of inertia are zero because of symmetry.

$$(\bar{I}_{x'y'})_1 = (\bar{I}_{y'z'})_1 = 0 \qquad (\bar{I}_{x'y'})_2 = (\bar{I}_{y'z'})_2 = 0$$

$$(\bar{I}_{x'y'})_3 = (\bar{I}_{z'x'})_3 = 0 \qquad (\bar{I}_{x'y'})_4 = (\bar{I}_{z'x'}) = 0$$

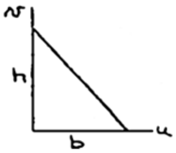
Also $\quad \bar{y}_1 = \bar{y}_2 = 0 \qquad \bar{x}_3 = \bar{x}_4 = 0$

Now $\quad I_{xy} = \Sigma(\bar{I}_{x'y'} + m\bar{x}\,\bar{y})$

so that $\quad I_{xy} = 0$ ◀

Using the results of Sample Problem 9.6, we have

$$I_{uv,\text{area}} = \frac{1}{24}b^2h^2$$



Now

$$I_{uv,\text{mass}} = \rho_{\text{ST}}t\, I_{uv,\text{area}} = \rho_{\text{ST}}t\left(\frac{1}{24}b^2h^2\right) = \frac{1}{12}mbh$$

Thus,

$$(I_{zx})_1 = \frac{1}{12}m_1b_1h_1$$

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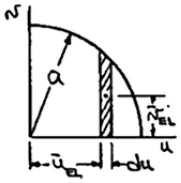
PROBLEM B.46 (Continued)

While $$(I_{yz})_3 = -\frac{1}{12} m_3 b_3 h_3$$

because of a 90° rotation of the coordinate axes.

To determine I_{uv} for a quarter circle, we have



$$dI_{uv} = d\cancelto{0}{\bar{I}_{u'v'}} + \bar{u}_{EL}\bar{v}_{EL}\, dm$$

Where $$\bar{u}_{EL} = u \quad \bar{v}_{EL} = \frac{1}{2}v = \frac{1}{2}\sqrt{a^2 - u^2}$$

$$dm = \rho_{ST} t\, dA = \rho_{ST} tv\, du = \rho_{ST} t\sqrt{a^2 - u^2}\, du$$

Then $$I_{uv} = \int dI_{uv} = \int_0^a (u)\left(\frac{1}{2}\sqrt{a^2 - u^2}\right)\left(\rho_{ST} t\sqrt{a^2 - u^2}\, du\right)$$

$$= \frac{1}{2}\rho_{ST} t\int_0^a u(a^2 - u^2)\, du$$

$$= \frac{1}{2}\rho_{ST} t\left[\frac{1}{2}a^2u^2 - \frac{1}{4}u^4\right]_0^a = \frac{1}{8}\rho_{ST} t\, a^4 = \frac{1}{2\pi} ma^4$$

Thus $$(I_{zx})_2 = -\frac{1}{2\pi} m_2 a_2^2$$

because of a 90° rotation of the coordinate axes. Also

$$(I_{yz})_4 = \frac{1}{2\pi} m_4 a_4^2$$

Finally,

$$I_{yz} = \Sigma(I_{yz}) = [(\cancelto{0}{\bar{I}_{y'z'}}) + m_1\bar{y}_1\bar{z}_1] + [(\cancelto{0}{\bar{I}_{y'z'}}) + m_2\bar{y}_2\bar{z}_2] + (I_{yz})_3 + (I_{yz})_4$$

$$= \left[-\frac{1}{12}(0.23844 \text{ kg})(0.225 \text{ m})(0.135 \text{ m})\right] + \left[\frac{1}{2\pi}(0.22473 \text{ kg})(0.135 \text{ m})^2\right]$$

$$= (-0.60355 + 0.65185)\times 10^{-3} \text{ kg}\cdot\text{m}^2$$

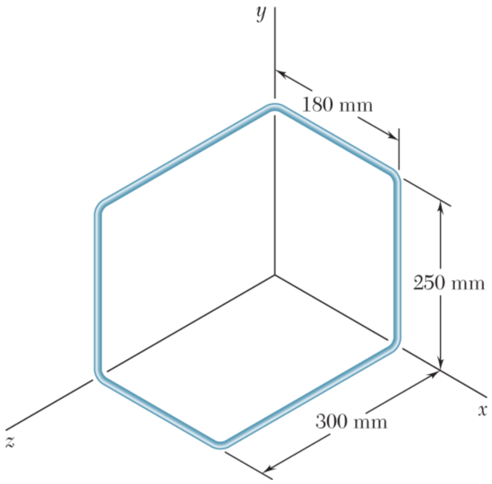
or $I_{yz} = 48.3\times 10^{-6} \text{ kg}\cdot\text{m}^2$ ◀

$$I_{zx} = \Sigma(I_{zx}) = (I_{zx})_1 + (I_{zx})_2 + [(\cancelto{0}{\bar{I}_{z'x'}})_3 + m_3\bar{z}_3\bar{x}_3] + [(\cancelto{0}{\bar{I}_{z'x'}})_4 + m_4\bar{z}_4\bar{x}_4]$$

$$= \left[\frac{1}{12}(0.23844 \text{ kg})(0.225 \text{ m})(0.135 \text{ m})\right] + \left[-\frac{1}{2\pi}(0.62424 \text{ kg})(0.225 \text{ m})^2\right]$$

$$= (0.60355 - 5.02964)\times 10^{-3} \text{ kg}\cdot\text{m}^2$$

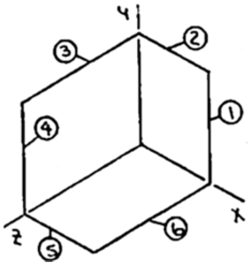
or $I_{zx} = -4.43\times 10^{-3} \text{ kg}\cdot\text{m}^2$ ◀



PROBLEM B.47

The figure shown is formed of 1.5-mm-diameter aluminum wire. Knowing that the density of aluminum is 2800 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{AL}} V = \rho_{\text{AL}} AL$$

Then

$$m_1 = m_4 = (2800\ \text{kg/m}^3)\left[\frac{\pi}{4}(0.0015\ \text{m})^2\right](0.25\ \text{m})$$

$$= 1.23700 \times 10^{-3}\ \text{kg}$$

$$m_2 = m_5 = (2800\ \text{kg/m}^3)\left[\frac{\pi}{4}(0.0015\ \text{m})^2\right](0.18\ \text{m})$$

$$= 0.89064 \times 10^{-3}\ \text{kg}$$

$$m_3 = m_6 = (2800\ \text{kg/m}^3)\left[\frac{\pi}{4}(0.0015\ \text{m})^2\right](0.3\ \text{m})$$

$$= 1.48440 \times 10^{-3}\ \text{kg}$$

PROBLEM B.47 (Continued)

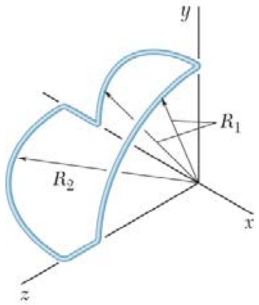
Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

	m, kg	$\bar{x}$, m	$\bar{y}$, m	$\bar{z}$, m	$m\bar{x}\,\bar{y}$, $\text{kg}\cdot\text{m}^2$	$m\bar{y}\,\bar{z}$, $\text{kg}\cdot\text{m}^2$	$m\bar{z}\,\bar{x}$, $\text{kg}\cdot\text{m}^2$
1	1.23700×10^{-3}	0.18	0.125	0	27.8325×10^{-6}	0	0
2	0.89064×10^{-3}	0.09	0.25	0	20.0394×10^{-6}	0	0
3	1.48440×10^{-3}	0	0.25	0.15	0	55.6650×10^{-6}	0
4	1.23700×10^{-3}	0	0.125	0.3	0	46.3875×10^{-6}	0
5	0.89064×10^{-3}	0.09	0	0.3	0	0	24.0473×10^{-6}
6	1.48440×10^{-3}	0.18	0	0.15	0	0	40.0788×10^{-6}
Σ					47.8719×10^{-6}	102.0525×10^{-6}	64.1261×10^{-6}

Then $$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) \quad \text{or} \quad I_{xy} = 47.9\times10^{-6}\ \text{kg}\cdot\text{m}^2 \blacktriangleleft$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) \quad \text{or} \quad I_{yz} = 102.1\times10^{-6}\ \text{kg}\cdot\text{m}^2 \blacktriangleleft$$

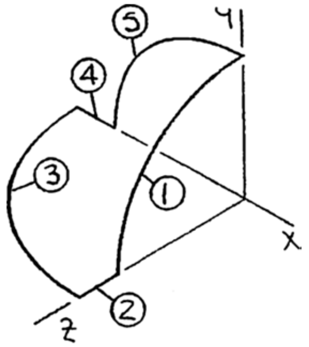
$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) \quad \text{or} \quad I_{zx} = 64.1\times10^{-6}\ \text{kg}\cdot\text{m}^2 \blacktriangleleft$$



PROBLEM B.48

Thin aluminum wire of uniform diameter is used to form the figure shown. Denoting by m' the mass per unit length of the wire, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION



First compute the mass of each component. We have

$$m = \left(\frac{m}{L}\right)L = m'L$$

Then $$m_1 = m_5 = m'\left(\frac{\pi}{2}R_1\right) = \frac{\pi}{2}m'R_1$$

$$m_2 = m_4 = m'(R_2 - R_1)$$

$$m_3 = m'\left(\frac{\pi}{2}R_2\right) = \frac{\pi}{2}m'R_2$$

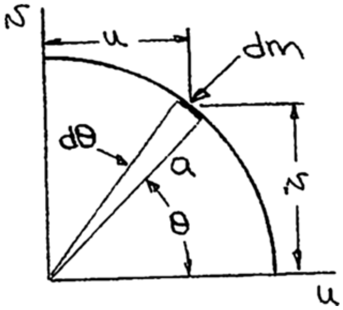
Now observe that because of symmetry the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of components 2 and 4 are zero and

$$(\bar{I}_{x'y'})_1 = (\bar{I}_{z'x'})_1 = 0 \qquad (\bar{I}_{x'y'})_3 = (\bar{I}_{y'z'})_3 = 0$$

$$(\bar{I}_{y'z'})_5 = (\bar{I}_{z'x'})_5 = 0$$

Also $$\bar{x}_1 = \bar{x}_2 = 0 \qquad \bar{y}_2 = \bar{y}_3 = \bar{y}_4 = 0 \qquad \bar{z}_4 = \bar{z}_5 = 0$$

Using the parallel-axis theorem [Equations (9.47)], it follows that $I_{xy} = I_{yz} = I_{zx}$ for components 2 and 4.



To determine I_{uv} for one quarter of a circular arc, we have $dI_{uv} = uvdm$

where $$u = a\cos\theta \qquad v = a\sin\theta$$

and $$dm = \rho dV = \rho[A(ad\theta)]$$

where A is the cross-sectional area of the wire. Now

$$m = m'\left(\frac{\pi}{2}a\right) = \rho A\left(\frac{\pi}{2}a\right)$$

so that $$dm = m'ad\theta$$

and $$dI_{uv} = (a\cos\theta)(a\sin\theta)(m'ad\theta)$$
$$= m'a^3\sin\theta\cos\theta d\theta$$

PROBLEM B.48 (Continued)

Then
$$I_{uv} = \int dI_{uv} = \int_0^{\pi/2} m'a^3 \sin\theta \cos\theta d\theta$$
$$= m'a^3 \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} = \frac{1}{2}m'a^3$$

Thus,
$$(I_{yz})_1 = \frac{1}{2}m'R_1^3$$

and
$$(I_{zx})_3 = -\frac{1}{2}m'R_2^3 \quad (I_{xy})_5 = -\frac{1}{2}m'R_1^3$$

because of 90° rotations of the coordinate axes. Finally,

$$I_{xy} = \Sigma(I_{xy}) = [\cancelto{0}{(\bar{I}_{x'y'})_1} + m_1\bar{x}_1\bar{y}_1] + [\cancelto{0}{(\bar{I}_{x'y'})_3} + m_3\bar{x}_3\bar{y}_3] + (I_{xy})_5$$

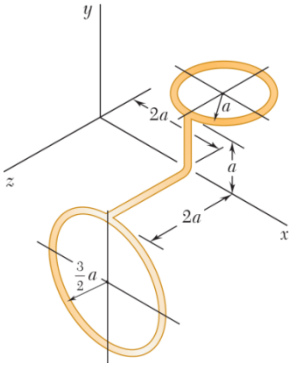
or $I_{xy} = -\frac{1}{2}m'R_1^3$ ◀

$$I_{yz} = \Sigma(I_{yz}) = (I_{yz})_1 + [\cancelto{0}{(\bar{I}_{y'z'})_3} + m_3\bar{y}_3\bar{z}_3] + [\cancelto{0}{(\bar{I}_{y'z'})_5} + m_5\bar{y}_5\bar{z}_5]$$

or $I_{yz} = \frac{1}{2}m'R_1^3$ ◀

$$I_{zx} = \Sigma(I_{zx}) = [\cancelto{0}{(\bar{I}_{z'x'})_1} + m_1\bar{z}_1\bar{x}_1] + (I_{zx})_3 + [\cancelto{0}{(\bar{I}_{z'x'})_5} + m_5\bar{z}_5\bar{x}_5]$$

or $I_{zx} = -\frac{1}{2}m'R_2^3$ ◀

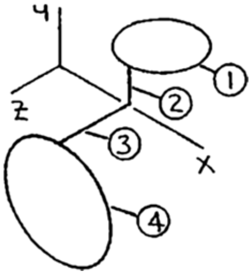


PROBLEM B.49

Brass wire with a weight per unit length w is used to form the figure shown. Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION

First compute the mass of each component. We have



$$m = \frac{W}{g} = \frac{1}{g}wL$$

Then

$$m_1 = \frac{W}{g}(2\pi \times a) = 2\pi\frac{w}{g}a$$

$$m_2 = \frac{w}{g}(a) = \frac{w}{g}a$$

$$m_3 = \frac{w}{g}(2a) = 2\frac{w}{g}a$$

$$m_4 = \frac{w}{g}\left(2\pi \times \frac{3}{2}a\right) = 3\pi\frac{w}{g}a$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

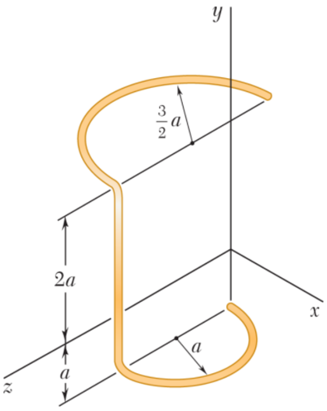
	m	$\bar{x}$	$\bar{y}$	$\bar{z}$	$m\bar{x}\,\bar{y}$	$m\bar{y}\,\bar{z}$	$m\bar{z}\,\bar{x}$
1	$2\pi\frac{w}{g}a$	$2a$	a	$-a$	$4\pi\frac{w}{g}a^3$	$-2\pi\frac{w}{g}a^3$	$-4\pi\frac{w}{g}a^3$
2	$\frac{w}{g}a$	$2a$	$\frac{1}{2}a$	0	$\frac{w}{g}a^3$	0	0
3	$2\frac{w}{g}a$	$2a$	0	a	0	0	$4\frac{w}{g}a^3$
4	$3\pi\frac{w}{g}a$	$2a$	$-\frac{3}{2}a$	$2a$	$-9\pi\frac{w}{g}a^3$	$-9\pi\frac{w}{g}a^3$	$12\pi\frac{w}{g}a^3$
Σ					$\frac{w}{g}(1-5\pi)a^3$	$-11\pi\frac{w}{g}a^3$	$4\frac{w}{g}(1+2\pi)a^3$

PROBLEM B.49 (Continued)

Then $I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y})$ or $I_{xy} = \dfrac{w}{g}a^3(1-5\pi)$ ◀

$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z})$ or $I_{yz} = -11\pi\dfrac{w}{g}a^3$ ◀

$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x})$ or $I_{zx} = 4\dfrac{w}{g}a^3(1+2\pi)$ ◀



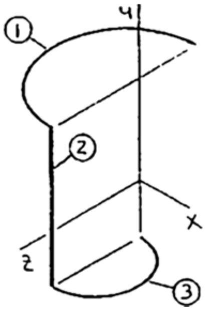
PROBLEM B.50

Brass wire with a weight per unit length w is used to form the figure shown. Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION

First compute the mass of each component. We have

$$m = \frac{W}{g} = \frac{1}{g} wL$$



Then

$$m_1 = \frac{W}{g}\left(\pi \times \frac{3}{2}a\right) = \frac{3}{2}\pi \frac{w}{g} a$$

$$m_2 = \frac{w}{g}(3a) = 3\frac{w}{g} a$$

$$m_3 = \frac{w}{g}(\pi \times a) = \pi \frac{w}{g} a$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry.

	m	$\bar{x}$	$\bar{y}$	$\bar{z}$	$m\bar{x}\,\bar{y}$	$m\bar{y}\,\bar{z}$	$m\bar{z}\,\bar{x}$
1	$\frac{3}{2}\pi\frac{w}{g}a$	$-\frac{2}{\pi}\left(\frac{3}{2}a\right)$	$2a$	$\frac{1}{2}a$	$-9\frac{w}{g}a^3$	$\frac{3}{2}\pi\frac{w}{g}a^3$	$-\frac{9}{4}\frac{w}{g}a^3$
2	$3\frac{w}{g}a$	0	$\frac{1}{2}a$	$2a$	0	$3\frac{w}{g}a^3$	0
3	$\pi\frac{w}{g}a$	$\frac{2}{\pi}(a)$	$-a$	a	$-2\frac{w}{g}a^3$	$-\pi\frac{w}{g}a^3$	$2\frac{w}{g}a^3$
Σ					$-11\frac{w}{g}a^3$	$\frac{w}{g}\left(\frac{\pi}{2}+3\right)a^3$	$-\frac{1}{4}\frac{w}{g}a^3$

PROBLEM B.50 (Continued)

Then $$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) \qquad \text{or} \qquad I_{xy} = -11\frac{w}{g}a^3 \blacktriangleleft$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) \qquad \text{or} \qquad I_{yz} = \frac{1}{2}\frac{w}{g}a^3(\pi + 6) \blacktriangleleft$$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) \qquad \text{or} \qquad I_{zx} = -\frac{1}{4}\frac{w}{g}a^3 \blacktriangleleft$$

PROBLEM B.51

Complete the derivation of Eqs. (9.47), which express the parallel-axis theorem for mass products of inertia.

SOLUTION

We have $$I_{xy} = \int xy\,dm \quad I_{yz} = \int yz\,dm \quad I_{zx} = \int zx\,dm \qquad (9.45)$$

and $$x = x' + \overline{x} \quad y = y' + \overline{y} \quad z = z' + \overline{z} \qquad (9.31)$$

Consider $$I_{xy} = \int xy\,dm$$

Substituting for *x* and for *y*

$$\begin{aligned} I_{xy} &= \int (x' + \overline{x})(y' + \overline{y})\,dm \\ &= \int x'y'dm + \overline{y}\int x'dm + \overline{x}\int y'dm + \overline{x}\,\overline{y}\int dm \end{aligned}$$

By definition $$\overline{I}_{x'y'} = \int x'y'dm$$

and $$\int x'dm = m\overline{x}'$$

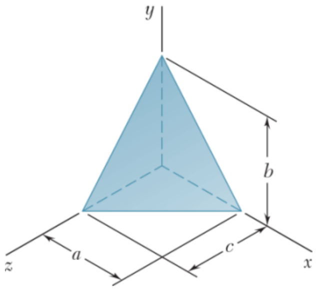
$$\int y'dm = m\overline{y}'$$

However, the origin of the primed coordinate system coincides with the mass center *G*, so that

$$\overline{x}' = \overline{y}' = 0$$

$$I_{xy} = \overline{I}_{x'y'} + m\overline{x}\,\overline{y} \quad \text{Q.E.D.} \blacktriangleleft$$

The expressions for I_{yz} and I_{zx} are obtained in a similar manner.

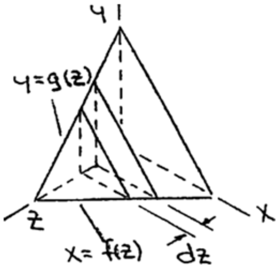


PROBLEM B.52

For the homogeneous tetrahedron of mass *m* shown, (*a*) determine by direct integration the mass product of inertia I_{zx}, (*b*) deduce I_{yz} and I_{xy} from the result obtained in part *a*.

SOLUTION

(*a*) First divide the tetrahedron into a series of thin vertical slices of thickness *dz* as shown.



Now $$x = -\frac{a}{c}z + a = a\left(1-\frac{z}{c}\right)$$

and $$y = -\frac{b}{c}z + b = b\left(1-\frac{z}{c}\right)$$

The mass *dm* of the slab is

$$dm = \rho dV = \rho\left(\frac{1}{2}xydz\right) = \frac{1}{2}\rho ab\left(1-\frac{z}{c}\right)^2 dz$$

Then $$m = \int dm = \int_0^c \frac{1}{2}\rho ab\left(1-\frac{z}{c}\right)^2 dz$$

$$= \frac{1}{2}\rho ab\left[\left(-\frac{c}{3}\right)\left(1-\frac{z}{c}\right)^3\right]_0^c = \frac{1}{6}\rho abc$$

Now $$dI_{zx} = d\bar{I}_{z'x'} + \bar{z}_{\text{EL}}\bar{x}_{\text{EL}}dm$$

where $$d\bar{I}_{z'x'} = 0 \quad \text{(symmetry)}$$

and $$\bar{z}_{\text{EL}} = z \quad \bar{x}_{\text{EL}} = \frac{1}{3}x = \frac{1}{3}a\left(1-\frac{z}{c}\right)$$

Then $$I_{zx} = \int dI_{zx} = \int_0^c z\left[\frac{1}{3}a\left(1-\frac{z}{c}\right)\right]\left[\frac{1}{2}\rho ab\left(1-\frac{z}{c}\right)^2 dz\right]$$

$$= \frac{1}{6}\rho a^2 b\int_0^c \left(z - 3\frac{z^2}{c} + 3\frac{z^3}{c^2} - \frac{z^4}{c^3}\right)dz$$

$$= \frac{m}{c}a\left[\frac{1}{2}z^2 - \frac{z^3}{c} + \frac{3}{4}\frac{z^4}{c^2} - \frac{1}{5}\frac{z^5}{c^3}\right]_0^c$$

or $$I_{zx} = \frac{1}{20}mac \blacktriangleleft$$

PROBLEM B.52 (Continued)

(*b*) Because of the symmetry of the body, I_{xy} and I_{yz} can be deduced by considering the circular permutation of (x, y, z) and (a, b, c).

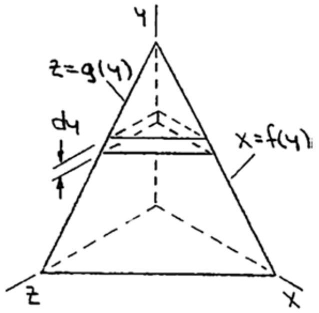
Thus,

$$I_{xy} = \frac{1}{20} mab \blacktriangleleft$$

$$I_{yz} = \frac{1}{20} mbc \blacktriangleleft$$

Alternative solution for part *a*:

First divide the tetrahedron into a series of thin horizontal slices of thickness dy as shown.



Now $$x = -\frac{a}{b}y + a = a\left(1 - \frac{y}{b}\right)$$

and $$z = -\frac{c}{b}y + c = c\left(1 - \frac{y}{b}\right)$$

The mass dm of the slab is

$$dm = \rho dV = \rho\left(\frac{1}{2}xzdy\right) = \frac{1}{2}\rho ac\left(1 - \frac{y}{b}\right)^2 dy$$

Now $$dI_{zx} = \rho t dI_{zx,\text{area}}$$

where $$t = dy$$

and $dI_{zx,\text{area}} = \frac{1}{24}x^2z^2$ from the results of Sample Problem 9.6.

Then $$dIzx = \rho(dy)\left\{\frac{1}{24}\left[a\left(1 - \frac{y}{b}\right)^2\right]\left[c\left(1 - \frac{y}{b}\right)\right]^2\right\}$$

$$= \frac{1}{24}\rho a^2c^2\left(1 - \frac{y}{b}\right)^4 dy = \frac{1}{4}\frac{m}{b}ac\left(1 - \frac{y}{b}\right)^4 dy$$

Finally, $$I_{zx} = \int dI_{zx} = \int_0^b \frac{1}{4}\frac{m}{b}ac\left(1 - \frac{y}{b}\right)^4 dy$$

$$= \frac{1}{4}\frac{m}{b}ac\left[\left(-\frac{b}{5}\right)\left(1 - \frac{y}{b}\right)^5\right]_0^b$$

or $$I_{zx} = \frac{1}{20} mac \blacktriangleleft$$

PROBLEM B.52 (Continued)

Alternative solution for part *a*:

The equation of the included face of the tetrahedron is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

so that $$y=b\left(1-\frac{x}{a}-\frac{z}{c}\right)$$

For an infinitesimal element of sides $dx, dy,$ and dz:

$$dm=\rho dV=\rho dydxdz$$

From part *a* $$x=a\left(1-\frac{z}{c}\right)$$

Now

$$I_{zx}=\int zxdm=\int_0^c\int_0^{a(1-z/c)}\int_0^{b(1-x/a-z/c)} zx(\rho dydxdz)$$

$$=\rho\int_0^c\int_0^{a(1-z/c)} zx\left[b\left(1-\tfrac{x}{a}-\tfrac{z}{c}\right)\right]dxdz$$

$$=\rho b\int_0^c z\left[\frac{1}{2}x^2-\frac{1}{3}\frac{x^3}{a}-\frac{1}{2}\frac{z}{c}x^2\right]_0^{a(1-z/c)}dz$$

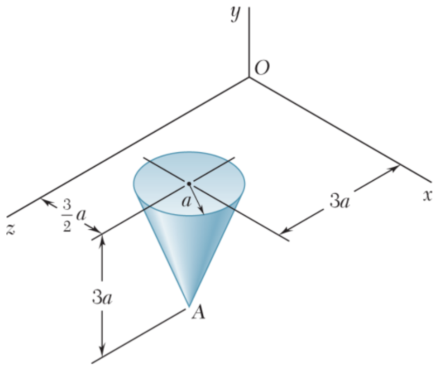
$$=\rho b\int_0^c z\left[\frac{1}{2}a^2\left(1-\frac{z}{c}\right)^2-\frac{1}{3a}a^3\left(1-\frac{z}{c}\right)^3-\frac{1}{2}\frac{z}{c}a^2\left(1-\frac{z}{c}\right)^2\right]dz$$

$$=\rho b\int_0^c \frac{1}{6}a^2z\left(1-\frac{z}{c}\right)^3 dz$$

$$=\frac{1}{6}\rho a^2b\int_0^c\left(z-3\frac{z^2}{c}+3\frac{z^3}{c^2}-\frac{z^4}{c^3}\right)dz$$

$$=\frac{m}{c}a\left[\frac{1}{2}z^2-\frac{z^3}{c}+\frac{3}{4}\frac{z^4}{c^2}-\frac{1}{5}\frac{z^5}{c^3}\right]_0^c$$

or $I_{zx}=\dfrac{1}{20}mac$ ◀



PROBLEM B.53

The homogeneous circular cone shown has a mass *m*. Determine the mass moment of inertia of the cone with respect to the line joining the origin *O* and Point *A*.

SOLUTION

First note that

$$d_{OA} = \sqrt{\left(\frac{3}{2}a\right)^2 + (-3a)^2 + (3a)^2} = \frac{9}{2}a$$

Then

$$\lambda_{OA} = \frac{1}{\frac{9}{2}a}\left(\frac{3}{2}a\mathbf{i} - 3a\mathbf{j} + 3a\mathbf{k}\right) = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

For a rectangular coordinate system with origin at Point *A* and axes aligned with the given x, y, z axes, we have (using Figure 9.28)

$$I_x = I_z = \frac{3}{5}m\left[\frac{1}{4}a^2 + (3a)^2\right] \qquad I_y = \frac{3}{10}ma^2$$

$$= \frac{111}{20}ma^2$$

Also, symmetry implies

$$I_{xy} = I_{yz} = I_{zx} = 0$$

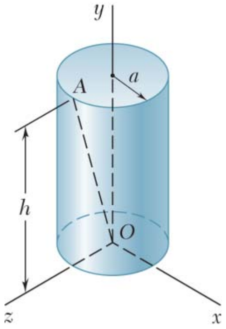
With the mass products of inertia equal to zero, Equation (9.46) reduces to

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2$$

$$= \frac{111}{20}ma^2\left(\frac{1}{3}\right)^2 + \frac{3}{10}ma^2\left(-\frac{2}{3}\right)^2 + \frac{111}{20}ma^2\left(\frac{2}{3}\right)^2$$

$$= \frac{193}{60}ma^2$$

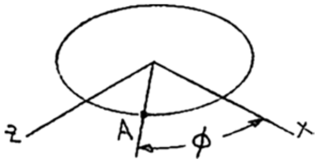
or $I_{OA} = 3.22ma^2$ ◀



PROBLEM B.54

The homogeneous circular cylinder shown has a mass m. Determine the mass moment of inertia of the cylinder with respect to the line joining the origin O and Point A that is located on the perimeter of the top surface of the cylinder.

SOLUTION



From Figure 9.28: $I_y = \frac{1}{2}ma^2$

and using the parallel-axis theorem

$$I_x = I_z = \frac{1}{12}m(3a^2 + h^2) + m\left(\frac{h}{2}\right)^2 = \frac{1}{12}m(3a^2 + 4h^2)$$

Symmetry implies $I_{xy} = I_{yz} = I_{zx} = 0$

For convenience, let Point A lie in the yz plane. Then

$$\lambda_{OA} = \frac{1}{\sqrt{h^2 + a^2}}(h\mathbf{j} + a\mathbf{k})$$

With the mass products of inertia equal to zero, Equation (9.46) reduces to

$$I_{OA} = I_x\cancelto{0}{\lambda_x^2} + I_y\lambda_y^2 + I_z\lambda_z^2$$

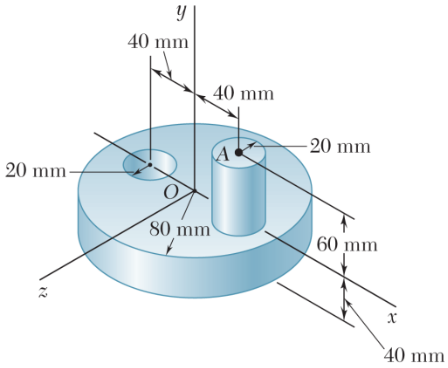
$$= \frac{1}{2}ma^2\left(\frac{h}{\sqrt{h^2 + a^2}}\right)^2 + \frac{1}{12}m(3a^2 + 4h^2)\left(\frac{a}{\sqrt{h^2 + a^2}}\right)^2$$

or $$I_{OA} = \frac{1}{12}ma^2\frac{10h^2 + 3a^2}{h^2 + a^2} \blacktriangleleft$$

Note: For Point A located at an arbitrary point on the perimeter of the top surface, λ_{OA} is given by

$$\lambda_{OA} = \frac{1}{\sqrt{h^2 + a^2}}(a\cos\phi\mathbf{i} + h\mathbf{j} + a\sin\phi\mathbf{k})$$

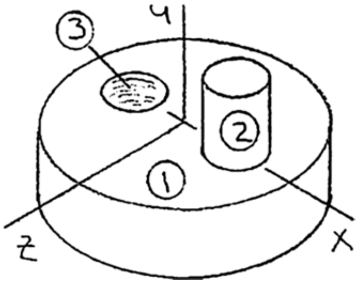
which results in the same expression for I_{OA}.



PROBLEM B.55

Shown is the machine element of Problem 9.141. Determine its mass moment of inertia with respect to the line joining the origin *O* and Point *A*.

SOLUTION



First compute the mass of each component.

We have $$m = \rho_{\text{ST}} V = \frac{0.284\ \text{lb/in}^3}{32.2\ \text{ft/s}^2} V = (0.008819\ \text{lb} \cdot \text{s}^2/\text{ft} \cdot \text{in}^3)V$$

Then

$$m_1 = (7850\ \text{kg/m}^3)[\pi(0.08\ \text{m})^2(0.04\ \text{m})] = 6.31334\ \text{kg}$$
$$m_2 = (7850\ \text{kg/m}^3)[\pi(0.02\ \text{m})^2(0.06\ \text{m})] = 0.59188\ \text{kg}$$
$$m_3 = (7850\ \text{kg/m}^3)[\pi(0.02\ \text{m})^2(0.04\ \text{m})] = 0.39458\ \text{kg}$$

Symmetry implies $$I_{yz} = I_{zx} = 0 \qquad (I_{xy})_1 = 0$$

and $$(\bar{I}_{x'y'})_2 = (\bar{I}_{x'y'})_3 = 0$$

Now
$$\begin{aligned} I_{xy} &= \Sigma(\bar{I}_{x'y'} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 - m_3\bar{x}_3\bar{y}_3 \\ &= [0.59188\ \text{kg}\ (0.04\ \text{m})(0.03\ \text{m})] - [0.39458\ \text{kg}\ (-0.04\ \text{m})(-0.02\ \text{m})] \\ &= (0.71026 - 0.31566) \times 10^{-3}\ \text{kg} \cdot \text{m}^2 \\ &= 0.39460 \times 10^{-3}\ \text{kg} \cdot \text{m}^2 \end{aligned}$$

PROBLEM B.55 (Continued)

From the solution to Problem 9.141, we have

$$I_x = 13.98800 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$
$$I_y = 20.55783 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$
$$I_z = 14.30368 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

By observation

$$\boldsymbol{\lambda}_{OA} = \frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{j})$$

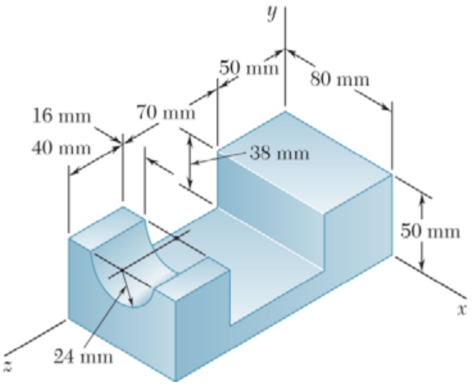
Substituting into Eq. (9.46)

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\cancelto{0}{\lambda_z^2} - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\cancelto{0}{\lambda_y\lambda_z} - 2I_{zx}\cancelto{0}{\lambda_z\lambda_x}$$

$$= \left[(13.98800)\left(\frac{2}{\sqrt{13}}\right)^2 + (20.55783)\left(\frac{3}{\sqrt{13}}\right)^2 - 2(0.39460)\left(\frac{2}{\sqrt{13}}\right)\left(\frac{2}{\sqrt{13}}\right)\right] \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

$$= (4.30400 + 14.23234 - 0.36425) \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

or $I_{OA} = 18.17 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ ◀



PROBLEM B.56

Determine the mass moment of inertia of the steel fixture of Problems 9.145 and 9.149 with respect to the axis through the origin that forms equal angles with the *x*, *y*, and *z* axes.

SOLUTION

From the solutions to Problems 9.145 and 9.149, we have

Problem 9.145:

$$I_x = 26.4325\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_y = 31.1726\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_z = 8.5773\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

Problem 9.149:

$$I_{xy} = 2.5002\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_{yz} = 4.0627\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_{zx} = 8.8062\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

From the problem statement it follows that

$$\lambda_x = \lambda_y = \lambda_z$$

Now $$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \Rightarrow 3\lambda_x^2 = 1$$

or $$\lambda_x = \lambda_y = \lambda_z = \frac{1}{\sqrt{3}}$$

Substituting into Eq. (9.46)

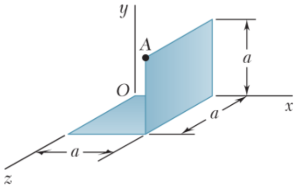
$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

Noting that $$\lambda_x^2 = \lambda_y^2 = \lambda_z^2 = \lambda_x\lambda_y = \lambda_y\lambda_z = \lambda_z\lambda_x = \frac{1}{3}$$

We have $$I_{OL} = \frac{1}{3}[26.4325 + 31.1726 + 8.5773$$

$$-2(2.5002 + 4.0627 + 8.8062)]\times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

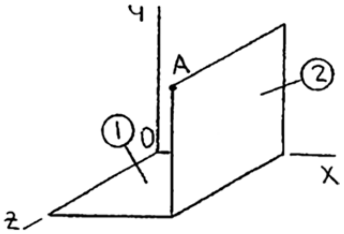
or $I_{OL} = 11.81\times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM B.57

The thin bent plate shown is of uniform density and weight *W*. Determine its mass moment of inertia with respect to the line joining the origin *O* and Point *A*.

SOLUTION



First note that $$m_1 = m_2 = \frac{1}{2}\frac{W}{g}$$

and that $$\lambda_{OA} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Using Figure 9.28 and the parallel-axis theorem, we have

$$\begin{aligned}
I_x &= (I_x)_1 + (I_x)_2 \\
&= \left[\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)^2\right] \\
&\quad + \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)(a^2 + a^2) + \frac{1}{2}\frac{W}{g}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right]\right\} \\
&= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{12} + \frac{1}{4}\right)a^2 + \left(\frac{1}{6} + \frac{1}{2}\right)a^2\right] = \frac{1}{2}\frac{W}{g}a^2
\end{aligned}$$

$$\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 \\
&= \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)(a^2 + a^2) + \frac{1}{2}\frac{W}{g}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right]\right\} \\
&\quad + \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left[(a)^2 + \left(\frac{a}{2}\right)^2\right]\right\} \\
&= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{6} + \frac{1}{2}\right)a^2 + \left(\frac{1}{12} + \frac{5}{4}\right)a^2\right] = \frac{W}{g}a^2
\end{aligned}$$

$$\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 \\
&= \left[\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)^2\right] \\
&\quad + \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left[(a)^2 + \left(\frac{a}{2}\right)^2\right]\right\} \\
&= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{12} + \frac{1}{4}\right)a^2 + \left(\frac{1}{12} + \frac{5}{4}\right)a^2\right] = \frac{5}{6}\frac{W}{g}a^2
\end{aligned}$$

PROBLEM B.57 (Continued)

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of both components are zero because of symmetry. Also, $\bar{y}_1 = 0$

Then

$$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 = \frac{1}{2}\frac{W}{g}(a)\left(\frac{a}{2}\right) = \frac{1}{4}\frac{W}{g}a^2$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) = m_2\bar{y}_2\bar{z}_2 = \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) = \frac{1}{8}\frac{W}{g}a^2$$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) = m_1\bar{z}_1\bar{x}_1 + m_2\bar{z}_2\bar{x}_2$$

$$= \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)(a) = \frac{3}{8}\frac{W}{g}a^2$$

Substituting into Equation (9.46)

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

Noting that

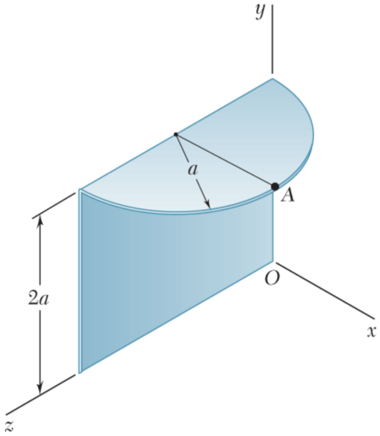
$$\lambda_x^2 = \lambda_y^2 = \lambda_z^2 = \lambda_x\lambda_y = \lambda_y\lambda_z = \lambda_z\lambda_x = \frac{1}{3}$$

We have

$$I_{OA} = \frac{1}{3}\left[\frac{1}{2}\frac{W}{g}a^2 + \frac{W}{g}a^2 + \frac{5}{6}\frac{W}{g}a^2 - 2\left(\frac{1}{4}\frac{W}{g}a^2 + \frac{1}{8}\frac{W}{g}a^2 + \frac{3}{8}\frac{W}{g}a^2\right)\right]$$

$$= \frac{1}{3}\left[\frac{14}{6} - 2\left(\frac{3}{4}\right)\right]\frac{W}{g}a^2$$

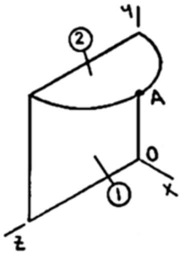
or $I_{OA} = \dfrac{5}{18}\dfrac{W}{g}a^2$ ◀



PROBLEM B.58

A piece of sheet steel of thickness t and specific weight γ is cut and bent into the machine component shown. Determine the mass moment of inertia of the component with respect to the joining the origin O and Point A.

SOLUTION



First note that

$$\boldsymbol{\lambda}_{OA} = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

Next compute the mass of each component. We have

$$m = \rho V = \frac{\gamma}{g}(tA)$$

Then

$$m_1 = \frac{\gamma}{g}t(2a \times 2a) = 4\frac{\gamma t}{g}a^2$$

$$m_2 = \frac{\gamma}{g}t\left(\frac{\pi}{2} \times a^2\right) = \frac{\pi}{2}\frac{\gamma t}{g}a^2$$

Using Figure 9.28 for component 1 and the equations derived above (following the solution to Problem 9.134) for a semicircular plate for component 2, we have

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 \\ &= \left\{\frac{1}{12}\left(4\frac{\gamma t}{g}a^2\right)[(2a)^2 + (2a)^2] + 4\frac{\gamma t}{g}a^2(a^2 + a^2)\right\} \\ &\quad + \left\{\frac{1}{4}\left(\frac{\pi}{2}\frac{\gamma t}{g}a^2\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2[(2a)^2 + (a)^2]\right\} \\ &= 4\frac{\gamma t}{g}a^2\left(\frac{2}{3} + 2\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{4} + 5\right)a^2 \\ &= 18.91335\frac{\gamma t}{g}a^4 \end{aligned}$$

PROBLEM B.58 (Continued)

$$\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 \\
&= \left[\frac{1}{12}\left(4\frac{\gamma t}{g}a^2\right)(2a)^2 + 4\frac{\gamma t}{g}a^2(a^2)\right] \\
&\quad + \left\{\frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{2}-\frac{16}{9\pi^2}\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left[\left(\frac{4a}{3\pi}\right)^2 + (a)^2\right]\right\} \\
&= 4\frac{\gamma t}{g}a^2\left(\frac{1}{3}+1\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{2}-\frac{16}{9\pi^2}+\frac{16}{9\pi^2}+1\right)a^2 \\
&= 7.68953\frac{\gamma t}{g}a^4
\end{aligned}$$

$$\begin{aligned}
I_z &= (I_z)_1 + (I_z)_2 \\
&= \left[\frac{1}{12}\left(4\frac{\gamma t}{g}a^2\right)(2a)^2 + 4\frac{\gamma t}{g}a^2(a^2)\right] \\
&\quad + \left\{\frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{4}-\frac{16}{9\pi^2}\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left[\left(\frac{4a}{3\pi}\right)^2 + (2a)^2\right]\right\} \\
&= 4\frac{\gamma t}{g}a^2\left(\frac{1}{3}+1\right)a^2 + \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{1}{4}-\frac{16}{9\pi^2}+\frac{16}{9\pi^2}+4\right)a^2 \\
&= 12.00922\frac{\gamma t}{g}a^4
\end{aligned}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of both components are zero because of symmetry. Also $\bar{x}_1 = 0$.

Then

$$\begin{aligned}
I_{xy} &= \Sigma(\overset{0}{\cancel{\bar{I}_{x'y'}}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 = \frac{\pi}{2}\frac{\gamma t}{g}a^2\left(\frac{4a}{3\pi}\right)(2a) \\
&= 1.33333\frac{\gamma t}{g}a^4
\end{aligned}$$

$$\begin{aligned}
I_{yz} &= \Sigma(\overset{0}{\cancel{\bar{I}_{y'z'}}} + m\bar{y}\,\bar{z}) = m_1\bar{y}_1\bar{z}_1 + m_2\bar{y}_2\bar{z}_2 \\
&= 4\frac{\gamma t}{g}a^2(a)(a) + \frac{\pi}{2}\frac{\gamma t}{g}a^2(2a)(a) \\
&= 7.14159\frac{\gamma t}{g}a^4
\end{aligned}$$

$$\begin{aligned}
I_{zx} &= \Sigma(\overset{0}{\cancel{\bar{I}_{z'x'}}} + m\bar{z}\,\bar{x}) = m_2\bar{z}_2\bar{x}_2 = \frac{\pi}{2}\frac{\gamma t}{g}a^2(a)\left(\frac{4a}{3\pi}\right) \\
&= 0.66667\frac{\gamma t}{g}a^4
\end{aligned}$$

PROBLEM B.58 (Continued)

Substituting into Eq. (9.46)

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

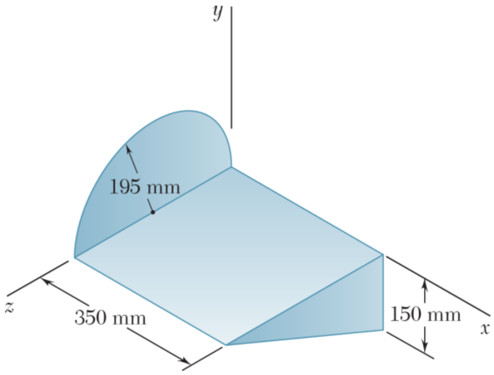
$$= 18.91335\frac{\gamma t}{g}a^4\left(\frac{1}{\sqrt{6}}\right)^2 + 7.68953\frac{\gamma t}{g}a^4\left(\frac{2}{\sqrt{6}}\right)^2$$

$$+12.00922\frac{\gamma t}{g}a^4\left(\frac{1}{\sqrt{6}}\right)^2 - 2\left(1.33333\frac{\gamma t}{g}a^4\right)\left(\frac{1}{\sqrt{6}}\right)\left(\frac{2}{\sqrt{6}}\right)$$

$$-2\left(7.14159\frac{\gamma t}{g}a^4\right)\left(\frac{2}{\sqrt{6}}\right)\left(\frac{2}{\sqrt{6}}\right) - 2\left(0.66667\frac{\gamma t}{g}a^4\right)\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right)$$

$$= (3.15223 + 5.12635 + 2.00154 - 0.88889 - 4.76106 - 0.22222)\frac{\gamma t}{g}a^4$$

or $I_{OA} = 4.41\frac{\gamma t}{g}a^4$ ◀



PROBLEM B.59

Determine the mass moment of inertia of the machine component of Problems 9.136 and 9.155 with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-4\mathbf{i} + 8\mathbf{j} + \mathbf{k})/9$.

SOLUTION

From the solutions to Problems 9.136 and 9.155. We have

Problem 9.136:

$$I_x = 175.503 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_y = 308.629 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_z = 154.400 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

Problem 9.155:

$$I_{xy} = -8.0365 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

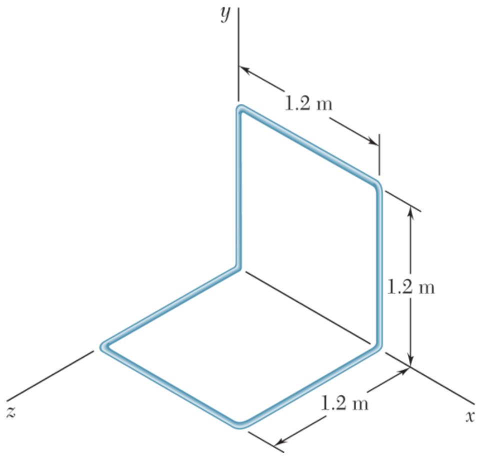
$$I_{yz} = 12.8950 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_{zx} = 94.0266 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

Substituting into Eq. (9.46)

$$\begin{aligned} I_{OL} &= I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x \\ &= \left[175.503\left(-\frac{4}{9}\right)^2 + 308.629\left(\frac{8}{9}\right)^2 + 154.400\left(\frac{1}{9}\right)^2\right. \\ &\quad -2(-8.0365)\left(-\frac{4}{9}\right)\left(\frac{8}{9}\right) - 2(12.8950)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right) \\ &\quad \left.-2(94.0266)\left(\frac{1}{9}\right)\left(-\frac{4}{9}\right)\right]\times 10^{-3}\ \text{kg}\cdot\text{m}^2 \\ &= (34.6673 + 243.855 + 1.906 - 6.350 \\ &\quad -2.547 + 9.287)\times 10^{-3}\ \text{kg}\cdot\text{m}^2 \end{aligned}$$

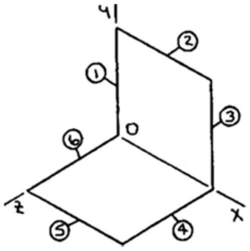
or $I_{OL} = 281 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM B.60

For the wire figure of Problem 9.148, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION



First compute the mass of each component. We have

$$m = \left(\frac{m}{L}\right)L = 0.056\ \text{kg/m} \times 1.2\ \text{m}$$
$$= 0.0672\ \text{kg}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, for each component are zero because of symmetry.

Also

$$\bar{x}_1 = \bar{x}_6 = 0 \qquad \bar{y}_4 = \bar{y}_5 = \bar{y}_6 = 0 \qquad \bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$$

Then
$$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 + m_3\bar{x}_3\bar{y}_3$$
$$= (0.0672\ \text{kg})(0.6\ \text{m})(1.2\ \text{m}) + (0.0672\ \text{kg})(1.2\ \text{m})(0.6\ \text{m})$$
$$= 0.096768\ \text{kg}\cdot\text{m}^2$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) = 0$$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) = m_4\bar{z}_4\bar{x}_4 + m_5\bar{z}_5\bar{x}_5$$
$$= (0.0672\ \text{kg})(0.6\ \text{m})(1.2\ \text{m}) + (0.0672\ \text{kg})(1.2\ \text{m})(0.6\ \text{m})$$
$$= 0.096768\ \text{kg}\cdot\text{m}^2$$

From the solution to Problem 9.148, we have

$$I_x = 0.32258\ \text{kg}\cdot\text{m}^2$$
$$I_y = I_z = 0.41933\ \text{kg}\cdot\text{m}^2$$

PROBLEM B.60 (Continued)

Substituting into Eq. (9.46)

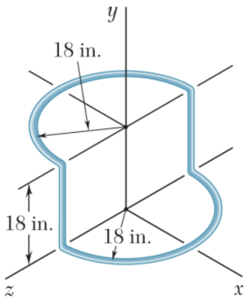
$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2\cancelto{0}{I_{yz}}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

$$= \left[0.32258\left(-\frac{3}{7}\right)^2 + 0.41933\left(-\frac{6}{7}\right)^2 + 0.41933\left(\frac{2}{7}\right)^2 \right.$$

$$\left. - 2(0.096768)\left(-\frac{3}{7}\right)\left(-\frac{6}{7}\right) - 2(0.096768)\left(\frac{2}{7}\right)\left(-\frac{3}{7}\right)\right] \text{kg}\cdot\text{m}^2$$

$$I_{OL} = (0.059249 + 0.30808 + 0.034231 - 0.071095 + 0.023698)\ \text{kg}\cdot\text{m}^2$$

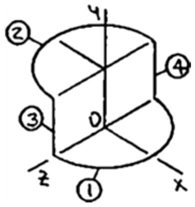
or $I_{OL} = 0.354\ \text{kg}\cdot\text{m}^2$ ◀



PROBLEM B.61

For the wire figure of Problem 9.147, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION



First compute the mass of each component. We have

$$m = \rho_{\text{ST}} V = \frac{\gamma_{\text{ST}}}{g} AL$$

Then

$$m_1 = m_2 = \frac{490\text{ lb/ft}^3}{32.2\text{ ft/s}^2} \times \left[\frac{\pi}{4}\left(\frac{1}{8}\text{ in.}\right)^2\right] \times (\pi \times 18\text{ in.}) \times \left(\frac{1\text{ ft}}{12\text{ in.}}\right)^3$$
$$= 6.1112 \times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft}$$

$$m_3 = m_4 = \frac{490\text{ lb/ft}^3}{32.2\text{ ft/s}^2} \times \left[\frac{\pi}{4}\left(\frac{1}{8}\text{ in.}\right)^2\right] \times 18\text{ in.} \times \left(\frac{1\text{ ft}}{12\text{ in.}}\right)^3$$
$$= 1.9453\text{ lb}\cdot\text{s}^2/\text{ft}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, for each component are zero because of symmetry.

Also $\qquad \bar{x}_3 = \bar{x}_4 = 0 \qquad \bar{y}_1 = 0 \qquad \bar{z}_1 = \bar{z}_2 = 0$

Then $\qquad I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2$

$$= (6.1112 \times 10^{-3}\text{ lb}\cdot\text{s}^2/\text{ft})\left(-\frac{2\times 18}{\pi}\text{ in.}\right)(18\text{ in.})\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2$$
$$= -8.75480 \times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_{yz} = \Sigma(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}) = m_3\bar{y}_3\bar{z}_3 + m_4\bar{y}_4\bar{z}_4$$

Now $\qquad m_3 = m_4, \qquad \bar{y}_3 = \bar{y}_4, \qquad \bar{z}_4 = -\bar{z}_3 \qquad I_{yz} = 0$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) \quad \text{or} \quad I_{zx} = 0$$

PROBLEM B.61 (Continued)

From the solution to Problem 9.147, we have

$$I_x = 39.1721 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_y = 36.2542 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_z = 30.4184 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

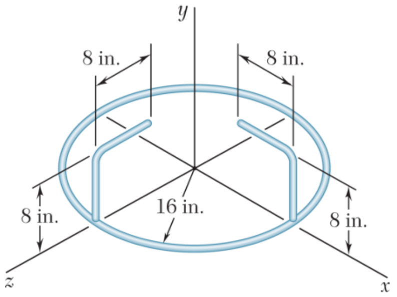
Substituting into Eq. (9.46)

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2\cancelto{0}{I_{yz}}\lambda_y\lambda_z - 2\cancelto{0}{I_{zx}}\lambda_z\lambda_x$$

$$= \left[39.1721\left(-\frac{3}{7}\right)^2 + 36.2542\left(-\frac{6}{7}\right)^2 + 30.4184\left(\frac{2}{7}\right)^2 - 2(-8.75480)\left(-\frac{3}{7}\right)\left(-\frac{6}{7}\right)\right] \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

$$= (7.19488 + 26.6357 + 2.48313 + 6.43210) \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$$

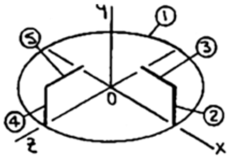
or $I_{OL} = 0.0427\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$ ◀



PROBLEM B.62

For the wire figure of Problem 9.146, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

SOLUTION



First compute the mass of each component. We have

$$m = \frac{W}{g} = \frac{1}{g}(W/L)_{\mathrm{AL}}\,L$$

Then

$$m_1 = \frac{1}{32.2\ \mathrm{ft/s^2}}(0.033\ \mathrm{lb/ft})(2\pi \times 16\ \mathrm{in.}) \times \frac{1\ \mathrm{ft}}{12\ \mathrm{in.}}$$

$$= 8.5857 \times 10^{-3}\ \mathrm{lb\cdot s^2/ft}$$

$$m_2 = m_3 = m_4 = m_5 = \frac{1}{32.2\ \mathrm{ft/s^2}}(0.033\ \mathrm{lb/ft})(8\ \mathrm{in.}) \times \frac{1\ \mathrm{ft}}{12\ \mathrm{in.}}$$

$$= 0.6832 \times 10^{-3}\ \mathrm{lb\cdot ft/s^2}$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'}$, and $\bar{I}_{z'x'}$, of each component are zero because of symmetry. Also

$$\bar{x}_1 = \bar{x}_4 = \bar{x}_5 = 0 \qquad \bar{y}_1 = 0 \qquad \bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$$

Then

$$I_{xy} = \Sigma(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}) = m_2\bar{x}_2\bar{y}_2 + m_3\bar{x}_3\bar{y}_3$$

$$= 0.6832 \times 10^{-3}\ \mathrm{lb\cdot s^2/ft}\left(\frac{16}{12}\ \mathrm{ft}\right)\left(\frac{4}{12}\ \mathrm{ft}\right)$$

$$+ 0.6832 \times 10^{-3}\ \mathrm{lb\cdot s^2/ft}\left(\frac{12}{12}\ \mathrm{ft}\right)\left(\frac{8}{12}\ \mathrm{ft}\right)$$

$$= (0.30364 + 0.45547) \times 10^{-3}\ \mathrm{lb\cdot ft\cdot s^2}$$

$$= 0.75911 \times 10^{-3}\ \mathrm{lb\cdot ft\cdot s^2}$$

Symmetry implies

$$I_{yz} = I_{xy} \qquad I_{vz} = 0.75911 \times 10^{-3}\ \mathrm{lb\cdot ft\cdot s^2}$$

$$I_{zx} = \Sigma(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}) = 0$$

PROBLEM B.62 (Continued)

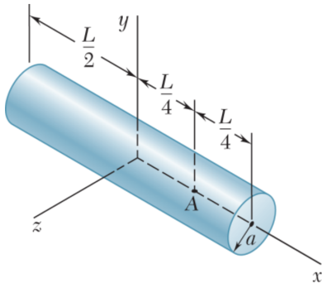
From the solution to Problem 9.146, we have

$$I_x = I_z = 10.3642 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$
$$I_y = 19.1097 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Substituting into Eq. (9.46)

$$\begin{aligned}
I_{OL} &= I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2\cancelto{0}{I_{zx}}\lambda_z\lambda_x \\
&= \left[10.3642\left(-\frac{3}{7}\right)^2 + 19.1097\left(-\frac{6}{7}\right)^2 + 10.3642\left(\frac{2}{7}\right)^2\right. \\
&\quad \left. - 2(0.75911)\left(-\frac{3}{7}\right)\left(-\frac{6}{7}\right) - 2(0.75911)\left(-\frac{6}{7}\right)\left(\frac{2}{7}\right)\right] \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \\
&= (1.90663 + 14.03978 + 0.84606 - 0.55771 + 0.37181) \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2
\end{aligned}$$

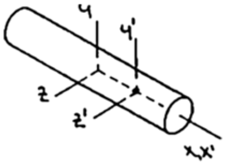
or $I_{OL} = 16.61 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$ ◀



PROBLEM B.63

For the homogeneous circular cylinder shown, of radius a and length L, determine the value of the ratio a/L for which the ellipsoid of inertia of the cylinder is a sphere when computed (*a*) at the centroid of the cylinder, (*b*) at Point A.

SOLUTION



(*a*) From Figure 9.28:

$$\bar{I}_x = \frac{1}{2}ma^2$$

$$\bar{I}_y = \bar{I}_z = \frac{1}{12}m(3a^2 + L^2)$$

Now observe that symmetry implies $I_{xy} = I_{yz} = I_{zx} = 0$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_x x^2 + I_y y^2 + I_z z^2 = 1: \quad \frac{1}{2}ma^2x^2 + \frac{1}{12}m(3a^2 + L^2)y^2 + \frac{1}{12}m(3a^2 + L^2) = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore

$$\frac{1}{2}ma^2 = \frac{1}{12}m(3a^2 + L^2) \qquad \text{or} \qquad \frac{a}{L} = \frac{1}{\sqrt{3}} \blacktriangleleft$$

(*b*) Using Figure 9.28 and the parallel-axis theorem, we have

$$I_{x'} = \frac{1}{2}ma^2$$

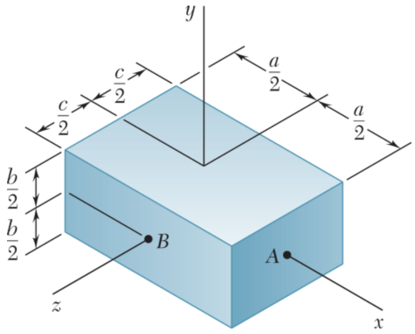
$$I_{y'} = I_{z'} = \frac{1}{12}m(3a^2 + L^2) + m\left(\frac{L}{4}\right)^2 = m\left(\frac{1}{4}a^2 + \frac{7}{48}L^2\right)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

From Part *a* it then immediately follows that

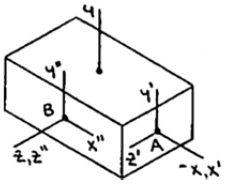
$$\frac{1}{2}ma^2 = m\left(\frac{1}{4}a^2 + \frac{7}{48}L^2\right) \qquad \text{or} \qquad \frac{a}{L} = \sqrt{\frac{7}{12}} \blacktriangleleft$$



PROBLEM B.64

For the rectangular prism shown, determine the values of the ratios *b*/*a* and *c*/*a* so that the ellipsoid of inertia of the prism is a sphere when computed (*a*) at Point *A*, (*b*) at Point *B*.

SOLUTION



(*a*) Using Figure 9.28 and the parallel-axis theorem, we have at Point *A*

$$I_{x'} = \frac{1}{12}m(b^2 + c^2)$$

$$I_{y'} = \frac{1}{12}m(a^2 + c^2) + m\left(\frac{a}{2}\right)^2$$

$$= \frac{1}{12}m(4a^2 + c^2)$$

$$I_{z'} = \frac{1}{12}m(a^2 + b^2) + m\left(\frac{a}{2}\right)^2 = \frac{1}{12}m(4a^2 + b^2)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_{x'}x^2 + I_{y'}y^2 + I_{z'}z^2 = 1: \quad \frac{1}{12}m(b^2 + c^2)x^2 + \frac{1}{12}m(4a^2 + c^2)y^2 + \frac{1}{12}m(4a^2 + b^2)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore

$$\frac{1}{12}m(b^2 + c^2) = \frac{1}{12}m(4a^2 + c^2)$$

$$= \frac{1}{12}m(4a^2 + b^2)$$

Then $\quad b^2 + c^2 = 4a^2 + c^2 \quad$ or $\quad \frac{b}{a} = 2$ ◀

and $\quad b^2 + c^2 = 4a^2 + b^2 \quad$ or $\quad \frac{c}{a} = 2$ ◀

PROBLEM B.64 (Continued)

(*b*) Using Figure 9.28 and the parallel-axis theorem, we have at Point *B*

$$I_{x''} = \frac{1}{12}m(b^2+c^2) + m\left(\frac{c}{2}\right)^2 = \frac{1}{12}m(b^2+4c^2)$$

$$I_{y''} = \frac{1}{12}m(a^2+c^2) + m\left(\frac{c}{2}\right)^2 = \frac{1}{12}m(a^2+4c^2)$$

$$I_{z''} = \frac{1}{12}m(a^2+b^2)$$

Now observe that symmetry implies

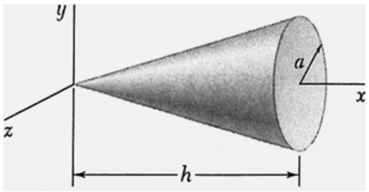
$$I_{x''y''} = I_{y''z''} = I_{z''x''} = 0$$

From part *a* it then immediately follows that

$$\frac{1}{12}m(b^2+4c^2) = \frac{1}{12}m(a^2+4c^2) = \frac{1}{12}m(a^2+b^2)$$

Then $b^2+4c^2 = a^2+4c^2$ or $\frac{b}{a} = 1$ ◀

and $b^2+4c^2 = a^2+b^2$ or $\frac{c}{a} = \frac{1}{2}$ ◀

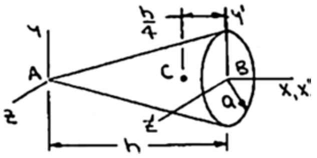


PROBLEM B.65

For the right circular cone of Sample Problem 9.11, determine the value of the ratio *a/h* for which the ellipsoid of inertia of the cone is a sphere when computed (*a*) at the apex of the cone, (*b*) at the center of the base of the cone.

SOLUTION

(*a*) From Sample Problem 9.11, we have at the apex *A*



$$I_x = \frac{3}{10}ma^2$$

$$I_y = I_z = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)$$

Now observe that symmetry implies $I_{xy} = I_{yz} = I_{zx} = 0$

Using Eq. (9.48), the equation of the ellipsoid of inertia is then

$$I_x x^2 + I_y y^2 + I_z z^2 = 1: \quad \frac{3}{10}ma^2x^2 + \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)y^2 + \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore,

$$\frac{3}{10}ma^2 = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right) \qquad \text{or} \qquad \frac{a}{h} = 2 ◀$$

(*b*) From Sample Problem 9.11, we have

$$I_{x'} = \frac{3}{10}ma^2$$

and at the centroid *C*

$$I_{y''} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right)$$

Then

$$I_{y'} = I_{z'} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right) + m\left(\frac{h}{4}\right)^2 = \frac{1}{20}m(3a^2 + 2h^2)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

From Part *a* it then immediately follows that

$$\frac{3}{10}ma^2 = \frac{1}{20}m(3a^2 + 2h^2) \qquad \text{or} \qquad \frac{a}{h} = \sqrt{\frac{2}{3}} ◀$$

PROBLEM B.66

Given an arbitrary body and three rectangular axes *x*, *y*, and *z*, prove that the mass moment of inertia of the body with respect to any one of the three axes cannot be larger than the sum of the mass moments of inertia of the body with respect to the other two axes. That is, prove that the inequality $I_x \le I_y + I_z$ and the two similar inequalities are satisfied. Further, prove that $I_y \ge \frac{1}{2} I_x$ if the body is a homogeneous solid of revolution, where *x* is the axis of revolution and *y* is a transverse axis.

SOLUTION

(i) To prove $I_y + I_z \ge I_x$

By definition

$$I_y = \int (z^2 + x^2)dm$$

$$I_z = \int (x^2 + y^2)dm$$

Then

$$I_y + I_z = \int (z^2 + x^2)dm + \int (x^2 + y^2)dm$$

$$= \int (y^2 + z^2)dm + 2\int x^2 dm$$

Now $\int (y^2 + z^2)dm = I_x$ and $\int x^2 dm \ge 0$

$$I_y + I_z \ge I_x \qquad \text{Q.E.D.}$$

The proofs of the other two inequalities follow similar steps.

(ii) If the *x* axis is the axis of revolution, then

$$I_y = I_z$$

and from part (i) $I_y + I_z \ge I_x$

or $2I_y \ge I_x$

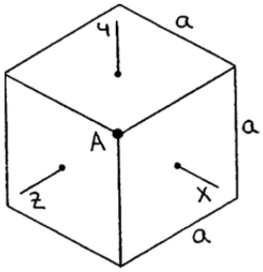
or

$$I_y \ge \frac{1}{2} I_x \qquad \text{Q.E.D.}$$

PROBLEM B.67

Consider a cube of mass m and side a. (*a*) Show that the ellipsoid of inertia at the center of the cube is a sphere, and use this property to determine the moment of inertia of the cube with respect to one of its diagonals. (*b*) Show that the ellipsoid of inertia at one of the corners of the cube is an ellipsoid of revolution, and determine the principal moments of inertia of the cube at that point.

SOLUTION



(*a*) At the center of the cube have (using Figure 9.28)

$$I_x = I_y = I_z = \frac{1}{12}m(a^2 + a^2) = \frac{1}{6}ma^2$$

Now observe that symmetry implies

$$I_{xy} = I_{yz} = I_{zx} = 0$$

Using Equation (9.48), the equation of the ellipsoid of inertia is

$$\left(\frac{1}{6}ma^2\right)x^2 + \left(\frac{1}{6}ma^2\right)y^2 + \left(\frac{1}{6}ma^2\right)z^2 = 1$$

$$\text{or} \quad x^2 + y^2 + z^2 = \frac{6}{ma^2} \;(= R^2) \blacktriangleleft$$

which is the equation of a sphere.

Since the ellipsoid of inertia is a sphere, the moment of inertia with respect to any axis OL through the center O of the cube must always be the same $\left(R = \frac{1}{\sqrt{I_{OL}}}\right)$.

$$I_{OL} = \frac{1}{6}ma^2 \blacktriangleleft$$

(*b*) The above sketch of the cube is the view seen if the line of sight is along the diagonal that passes through corner A. For a rectangular coordinate system at A and with one of the coordinate axes aligned with the diagonal, an ellipsoid of inertia at A could be constructed. If the cube is then rotated 120° about the diagonal, the mass distribution will remain unchanged. Thus, the ellipsoid will also remain unchanged after it is rotated. As noted at the end of Section 9.17, this is possible only if the ellipsoid is an ellipsoid of revolution, where the diagonal is both the axis of revolution and a principal axis.

It then follows that

$$I_{x'} = I_{OL} = \frac{1}{6}ma^2 \blacktriangleleft$$

In addition, for an ellipsoid of revolution, the two transverse principal moments of inertia are equal and any axis perpendicular to the axis of revolution is a principal axis. Then, applying the parallel-axis theorem between the center of the cube and corner A for any perpendicular axis

$$I_{y'} = I_{z'} = \frac{1}{6}ma^2 + m\left(\frac{\sqrt{3}}{2}a\right)^2$$

$$\text{or} \quad I_{y'} = I_{z'} = \frac{11}{12}ma^2 \blacktriangleleft$$

(*Note:* Part *b* can also be solved using the method of Section 9.18.)

PROBLEM B.67 (Continued)

First note that at corner *A*

$$I_x = I_y = I_z = \frac{2}{3}ma^2$$

$$I_{xy} = I_{yz} = I_{zx} = \frac{1}{4}ma^2$$

Substituting into Equation (9.56) yields

$$k^3 - 2ma^2k^2 + \frac{55}{48}m^2a^6k - \frac{121}{864}m^3a^9 = 0$$

For which the roots are

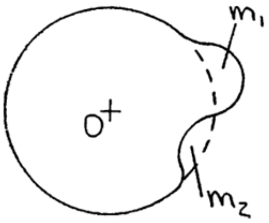
$$k_1 = \frac{1}{6}ma^2$$

$$k_2 = k_3 = \frac{11}{12}ma^2$$

PROBLEM B.68

Given a homogeneous body of mass m and of arbitrary shape and three rectangular axes x, y, and z with origin at O, prove that the sum $I_x + I_y + I_z$ of the mass moments of inertia of the body cannot be smaller than the similar sum computed for a sphere of the same mass and the same material centered at O. Further, using the result of Problem 9.176, prove that if the body is a solid of revolution, where x is the axis of revolution, its mass moment of inertia I_y about a transverse axis y cannot be smaller than $3ma^2/10$, where a is the radius of the sphere of the same mass and the same material.

SOLUTION



(i) Using Equation (9.30), we have

$$I_x + I_y + I_z = \int (y^2 + z^2)dm + \int (z^2 + x^2)\,dm + \int (x^2 + y^2)dm$$

$$= 2\int (x^2 + y^2 + z^2)dm$$

$$= 2\int r^2 dm$$

where r is the distance from the origin O to the element of mass dm. Now assume that the given body can be formed by adding and subtracting appropriate volumes V_1 and V_2 from a sphere of mass m and radius a which is centered at O; it then follows that

$$m_1 = m_2 \ (m_{\text{body}} = m_{\text{sphere}} = m)$$

Then

$$(I_x + I_y + I_z)_{\text{body}} = (I_x + I_y + I_z)_{\text{sphere}} + (I_x + I_y + I_z)_{V_1}$$

$$-(I_x + I_y + I_z)_{V_2}$$

or

$$(I_x + I_y + I_z)_{\text{body}} = (I_x + I_y + I_z)_{\text{sphere}} + 2\int_{m_1} r^2 dm - 2\int_{m_2} r^2 dm$$

Now, $m_1 = m_2$ and $r_1 \geq r_2$ for all elements of mass dm in volumes 1 and 2.

$$\int_{m_1} r^2 dm - \int_{m_2} r^2 dm \geq 0$$

so that

$$(I_x + I_y + I_z)_{\text{body}} \geq (I_x + I_y + I_z)_{\text{sphere}} \quad \text{Q.E.D.}$$

PROBLEM B.68 (Continued)

(ii) First note from Figure 9.28 that for a sphere

$$I_x = I_y = I_z = \frac{2}{5}ma^2$$

Thus $$(I_x + I_y + I_z)_{\text{sphere}} = \frac{6}{5}ma^2$$

For a solid of revolution, where the *x* axis is the axis of revolution, we have

$$I_y = I_z$$

Then, using the results of part (i)

$$(I_x + 2I_y)_{\text{body}} \geq \frac{6}{5}ma^2$$

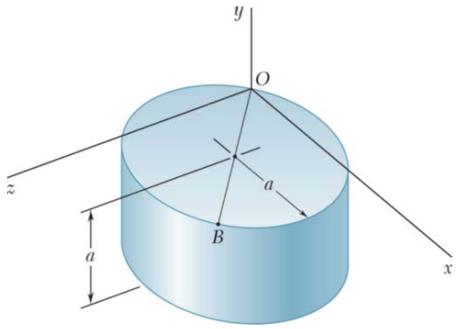
From Problem 9.178 we have $$I_y \geq \frac{1}{2}I_x$$

or $$(2I_y - I_x)_{\text{body}} \geq 0$$

Adding the last two inequalities yields

$$(4I_y)_{\text{body}} \geq \frac{6}{5}ma^2$$

or $$(I_y)_{\text{body}} \geq \frac{3}{10}ma^2 \quad \text{Q.E.D.}$$

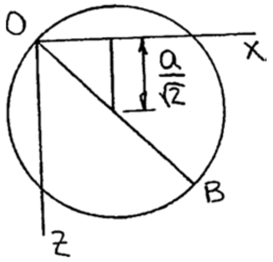


PROBLEM B.69*

The homogeneous circular cylinder shown has a mass *m*, and the diameter *OB* of its top surface forms 45° angles with the *x* and *z* axes. (*a*) Determine the principal mass moments of inertia of the cylinder at the origin *O*. (*b*) Compute the angles that the principal axes of inertia at *O* form with the coordinate axes. (*c*) Sketch the cylinder, and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) First compute the moments of inertia using Figure 9.28 and the parallel-axis theorem.



$$I_x = I_z = \frac{1}{12}m(3a^2 + a^2) + m\left[\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{2}\right)^2\right] = \frac{13}{12}ma^2$$

$$I_y = \frac{1}{2}ma^2 + m(a)^2 = \frac{3}{2}ma^2$$

Next observe that the centroidal products of inertia are zero because of symmetry. Then

$$I_{xy} = \cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y} = m\left(\frac{a}{\sqrt{2}}\right)\left(-\frac{a}{2}\right) = -\frac{1}{2\sqrt{2}}ma^2$$

$$I_{yz} = \cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z} = m\left(-\frac{a}{2}\right)\left(\frac{a}{\sqrt{2}}\right) = -\frac{1}{2\sqrt{2}}ma^2$$

$$I_{zx} = \cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x} = m\left(\frac{a}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}}\right) = \frac{1}{2}ma^2$$

Substituting into Equation (9.56)

$$\begin{aligned}
&K^3 - \left(\frac{13}{12} + \frac{3}{2} + \frac{13}{12}\right)ma^2K^2 \\
&+\left[\left(\frac{13}{12}\times\frac{3}{2}\right) + \left(\frac{3}{2}\times\frac{13}{12}\right) + \left(\frac{13}{12}\times\frac{13}{12}\right) - \left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{1}{2}\right)^2\right](ma^2)^2K \\
&-\left[\left(\frac{13}{12}\times\frac{3}{2}\times\frac{13}{12}\right) - \left(\frac{13}{12}\right)\left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2\right. \\
&\left.-\left(\frac{13}{12}\right)\left(-\frac{1}{2\sqrt{2}}\right)^2 - 2\left(-\frac{1}{2\sqrt{2}}\right)\left(-\frac{1}{2\sqrt{2}}\right)\left(\frac{1}{2}\right)\right](ma^2)^3 = 0
\end{aligned}$$

PROBLEM B.69* (Continued)

Simplifying and letting $K = ma^2\zeta$ yields

$$\zeta^3 - \frac{11}{3}\zeta^2 + \frac{565}{144}\zeta - \frac{95}{96} = 0$$

Solving yields

$$\zeta_1 = 0.363383 \qquad \zeta_2 = \frac{19}{12} \qquad \zeta_3 = 1.71995$$

The principal moments of inertia are then

$$K_1 = 0.363ma^2 \blacktriangleleft$$

$$K_2 = 1.583ma^2 \blacktriangleleft$$

$$K_3 = 1.720ma^2 \blacktriangleleft$$

(*b*) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, we use two of the equations of Equations (9.54) and (9.57).

Thus,

$$(I_x - K)\lambda_x - I_{xy}\lambda_y - I_{zx}\lambda_z = 0 \tag{9.54a}$$

$$-I_{zx}\lambda_x - I_{yz}\lambda_y + (I_z - K)\lambda_z = 0 \tag{9.54c}$$

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \tag{9.57}$$

(*Note:* Since $I_{xy} = I_{yz}$, Equations (9.54a) and (9.54c) were chosen to simplify the "elimination" of λ_y during the solution process.)

Substituting for the moments and products of inertia in Equations (9.54a) and (9.54c)

$$\left(\frac{13}{12}ma^2 - K\right)\lambda_x - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_y - \left(\frac{1}{2}ma^2\right)\lambda_z = 0$$

$$-\left(\frac{1}{2}ma^2\right)\lambda_x - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_y + \left(\frac{13}{12}ma^2 - K\right)\lambda_z = 0$$

or

$$\left(\frac{13}{12} - \zeta\right)\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y - \frac{1}{2}\lambda_z = 0 \tag{i}$$

and

$$-\frac{1}{2}\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y + \left(\frac{13}{12} - \zeta\right)\lambda_z = 0 \tag{ii}$$

Observe that these equations will be identical, so that one will need to be replaced, if

$$\frac{13}{12} - \zeta = -\frac{1}{2} \qquad \text{or} \qquad \zeta = \frac{19}{12}$$

Thus, a third independent equation will be needed when the direction cosines associated with K_2 are determined. Then for K_1 and K_3

Eq. (i) through Eq. (ii): $$\left[\frac{13}{12}-\zeta-\left(-\frac{1}{2}\right)\right]\lambda_x+\left[-\frac{1}{2}-\left(\frac{13}{12}-\zeta\right)\right]\lambda_z=0$$

or $$\lambda_z=\lambda_x$$

Substituting into Eq. (i): $$\left(\frac{13}{12}-\zeta\right)\lambda_x+\frac{1}{2\sqrt{2}}\lambda_y-\frac{1}{2}\lambda_x=0$$

or $$\lambda_y=2\sqrt{2}\left(\zeta-\frac{7}{12}\right)\lambda_x$$

Substituting into Equation (9.57):

$$\lambda_x^2+\left[2\sqrt{2}\left(\zeta-\frac{7}{12}\right)\lambda_x\right]^2+(\lambda_x)^2=1$$

or $$\left[2+8\left(\zeta-\frac{7}{12}\right)^2\right]\lambda_x^2=1 \qquad \text{(iii)}$$

$\mathbf{K}_1$: Substituting the value of ζ_1 into Eq. (iii):

$$\left[2+8\left(0.363383-\frac{7}{12}\right)^2\right](\lambda_x)_1^2=1$$

or $$(\lambda_x)_1=(\lambda_z)_1=0.647249$$

and then $$(\lambda_y)_1=2\sqrt{2}\left(0.363383-\frac{7}{12}\right)(0.647249)$$

$$=-0.402662$$

$$(\theta_x)_1=(\theta_z)_1=49.7° \quad (\theta_y)_1=113.7° \blacktriangleleft$$

$\mathbf{K}_3$: Substituting the value of ζ_3 into Eq. (iii):

$$\left[2+8\left(1.71995-\frac{7}{12}\right)^2\right](\lambda_x)_3^2=1$$

or $$(\lambda_x)_3=(\lambda_z)_3=0.284726$$

and then $$(\lambda_y)_3=2\sqrt{2}\left(1.71995-\frac{7}{12}\right)(0.284726)$$

$$=0.915348$$

$$(\theta_x)_3=(\theta_z)_3=73.5° \quad (\theta_y)_3=23.7° \blacktriangleleft$$

PROBLEM B.69* (Continued)

K$_2$: For this case, the set of equations to be solved consists of Equations (9.54a), (9.54b), and (9.57).

Now

$$-I_{xy}\lambda_x + (I_y - K)\lambda_y - I_{yz}\lambda_z = 0 \qquad (9.54b)$$

Substituting for the moments and products of inertia.

$$-\left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_x + \left(\frac{3}{2}ma^2 - K\right)\lambda_y - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_z = 0$$

or

$$\frac{1}{2\sqrt{2}}\lambda_x + \left(\frac{3}{2} - \xi\right)\lambda_y + \frac{1}{2\sqrt{2}}\lambda_z = 0 \qquad (iv)$$

Substituting the value of ξ_2 into Eqs. (i) and (iv):

$$\left(\frac{13}{12} - \frac{19}{12}\right)(\lambda_x)_2 + \frac{1}{2\sqrt{2}}(\lambda_y)_2 - \frac{1}{2}(\lambda_z)_2 = 0$$

$$\frac{1}{2\sqrt{2}}(\lambda_x)_2 + \left(\frac{3}{2} - \frac{19}{12}\right)(\lambda_y)_2 + \frac{1}{2\sqrt{2}}(\lambda_z)_2 = 0$$

or

$$-(\lambda_x)_2 + \frac{1}{\sqrt{2}}(\lambda_y)_2 - (\lambda_z)_2 = 0$$

and

$$(\lambda_x)_2 - \frac{\sqrt{2}}{6}(\lambda_y)_2 + (\lambda_z)_2 = 0$$

Adding yields

$$(\lambda_y)_2 = 0$$

and then

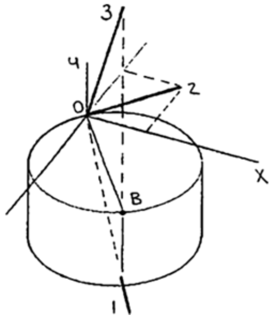
$$(\lambda_y)_2 = -(\lambda_x)_2$$

Substituting into Equation (9.57)

$$(\lambda_x)_2^2 + \overset{0}{\cancel{(\lambda_y)_2^2}} + (-\lambda_x)_2^2 = 1$$

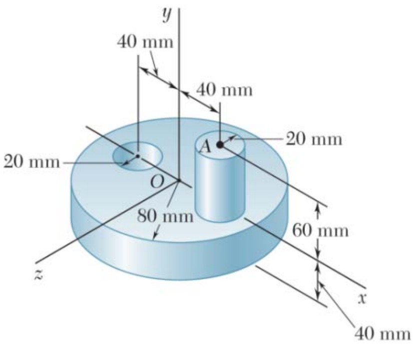
or

$$(\lambda_x)_2 = \frac{1}{\sqrt{2}} \quad \text{and} \quad (\lambda_z)_2 = -\frac{1}{\sqrt{2}}$$



$$(\theta_x)_2 = 45.0° \quad (\theta_y)_2 = 90.0° \quad (\theta_z)_2 = 135.0° \blacktriangleleft$$

(*c*) Principal axes 1 and 3 lie in the vertical plane of symmetry passing through Points *O* and *B*. Principal axis 2 lies in the *xz* plane.



PROBLEM B.70

For the component described in Problem 9.165, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solutions to Problems 9.141 and 9.165 we have

Problem 9.141:

$$I_x = 13.98800 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

$$I_y = 20.55783 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

$$I_z = 14.30368 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

Problem 9.165:

$$I_{yz} = I_{zx} = 0$$

$$I_{xy} = 0.39460 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

Eq. (9.55) then becomes

$$\begin{vmatrix} I_x - K & -I_{xy} & 0 \\ -I_{xy} & I_y - K & 0 \\ 0 & 0 & I_z - K \end{vmatrix} = 0 \quad \text{or} \quad (I_x - K)(I_y - K)(I_z - K) - (I_z - K)I_{xy}^2 = 0$$

Thus

$$I_z - K = 0 \qquad I_x I_y - (I_x + I_y)K + K^2 - I_{xy}^2 = 0$$

Substituting:

$$K_1 = 14.30368 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

or $K_1 = 14.30 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ ◀

and

$$(13.98800 \times 10^{-3})(20.55783 \times 10^{-3}) - (13.98800 + 20.55783)(10^{-3})K + K^2 - (0.39460 \times 10^{-3})^2 = 0$$

or

$$K^2 - (34.54583 \times 10^{-3})K + 287.4072 \times 10^{-6} = 0$$

Solving yields

$$K_2 = 13.96438 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$$

or $K_2 = 13.96 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ ◀

and $K_3 = 20.58145 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ or $K_3 = 20.6 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$ ◀

PROBLEM B.70 (Continued)

(*b*) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, use two of the equations of Eq. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*) with $I_{yz} = I_{zx} = 0.$

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 = 0$$
$$-I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 = 0$$

Substituting:

$$[(13.98800 - 14.30368) \times 10^{-3}](\lambda_x)_1 - (0.39460 \times 10^{-3})(\lambda_y)_1 = 0$$
$$-(0.39460 \times 10^{-3})(\lambda_x)_1 + [(20.55783 - 14.30368) \times 10^{-3}](\lambda_y)_1 = 0$$

Adding yields $(\lambda_x)_1 = (\lambda_y)_1 = 0$

Then using Eq. (9.57) $\cancelto{0}{(\lambda_x)}{}_1^2 + \cancelto{0}{(\lambda_y)}{}_1^2 + (\lambda_z)_1^2 = 1$

or $(\lambda_z)_1 = 1$

$$(\theta_x)_1 = 90.0° \quad (\theta_y)_1 = 90.0° \quad (\theta_z)_1 = 0°$$ ◀

K_2: Begin with Eqs. (9.54*b*) and (9.54*c*) with $I_{yz} = I_{zx} = 0.$

$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 = 0 \tag{i}$$
$$(I_z - K_2)(\lambda_z)_2 = 0$$

Now $I_z \neq K_2 \Rightarrow (\lambda_z)_2 = 0$

Substituting into Eq. (i):

$$-(0.39460 \times 10^{-3})(\lambda_x)_z + [(20.55783 - 13.96438) \times 10^{-3}](\lambda_y)_2 = 0$$

or $(\lambda_y)_2 = 0.059847(\lambda_x)_2$

Using Eq. (9.57): $(\lambda_x)_2^2 + [0.05984](\lambda_x)_2]^2 + \cancelto{0}{(\lambda_z)}{}_2^2 = 1$

or $(\lambda_x)_2 = 0.998214$

and $(\lambda_y)_2 = 0.059740$

$$(\theta_x)_2 = 3.4° \qquad (\theta_y)_2 = 86.6° \qquad (\theta_z)_2 = 90.0°$$ ◀

K_3: Begin with Eqs. (9.54*b*) and (9.54*c*) with $I_{yz} = I_{zx} = 0.$

$$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 = 0 \tag{ii}$$
$$(I_z - K_3)(\lambda_z)_3 = 0$$

PROBLEM B.70 (Continued)

Now $$I_z \neq K_3 \Rightarrow (\lambda_z)_3 = 0$$

Substituting into Eq. (ii):

$$-(0.39460 \times 10^{-3})(\lambda_x)_3 + [(20.55783 - 20.58145) \times 10^{-3}](\lambda_y)_3 = 0$$

or $$(\lambda_y)_3 = -16.70618(\lambda_x)_3$$

Using Eq. (9.57): $$(\lambda_x)_3^2 + [-16.70618(\lambda_x)_3]^2 + \cancelto{0}{(\lambda_z)_3^2} = 1$$

or $$(\lambda_x)_3 = -0.059751 \quad \text{(axes right-handed set } \Rightarrow \text{"\_")}$$

and $$(\lambda_y)_3 = 0.998211$$

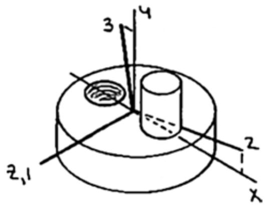
$$(\theta_x)_3 = 93.4° \qquad (\theta_y)_3 = 3.43° \qquad (\theta_z)_3 = 90.0° \blacktriangleleft$$

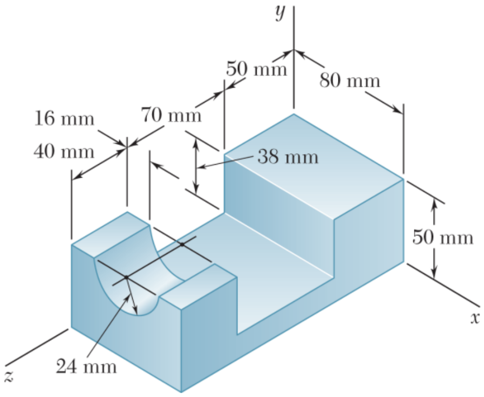
Note: Since the principal axes are orthogonal and $(\theta_z)_2 = (\theta_z)_3 = 90°$, it follows that

$$|(\lambda_x)_2| = |(\lambda_y)_3| \qquad |(\lambda_y)_2| = |(\lambda_z)_3|$$

The differences in the above values are due to round-off errors.

(*c*) Principal axis 1 coincides with the *z* axis, while principal axes 2 and 3 lie in the *xy* plane.





PROBLEM B.71*

For the component described in Problems 9.145 and 9.149, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solutions to Problems 9.145 and 9.149, we have

Problem 9.145: $I_x = 26.4325\times10^{-3}\ \text{kg}\cdot\text{m}^2$ Problem 9.149: $I_{xy} = 2.5002\times10^{-3}\ \text{kg}\cdot\text{m}^2$

$I_y = 31.1726\times10^{-3}\ \text{kg}\cdot\text{m}^2$ $I_{yz} = 4.0627\times10^{-3}\ \text{kg}\cdot\text{m}^2$

$I_z = 8.5773\times10^{-3}\ \text{kg}\cdot\text{m}^2$ $I_{zx} = 8.8062\times10^{-3}\ \text{kg}\cdot\text{m}^2$

Substituting into Eq. (9.56):

$$\begin{aligned}K^3 &- [(26.4325+31.1726+8.5773)(10^{-3})]K^2\\ &+ [(26.4325)(31.1726)+(31.1726)(8.5773)+(8.5773)(26.4325)\\ &- (2.5002)^2-(4.0627)^2-(8.8062)^2](10^{-6})K\\ &- [(26.4325)(31.1726)(8.5773)-(26.4325)(4.0627)^2\\ &- (31.1726)(8.8062)^2-(8.5773)(2.5002)^2\\ &- 2(2.5002)(4.0627)(8.8062)](10^{-9})=0\end{aligned}$$

or $$K^3-(66.1824\times10^{-3})K^2+(1217.76\times10^{-6})K-(3981.23\times10^{-9})=0$$

Solving yields

$K_1 = 4.1443\times10^{-3}\ \text{kg}\cdot\text{m}^2$ or $K_1 = 4.14\times10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

$K_2 = 29.7840\times10^{-3}\ \text{kg}\cdot\text{m}^2$ or $K_2 = 29.8\times10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

$K_3 = 32.2541\times10^{-3}\ \text{kg}\cdot\text{m}^2$ or $K_3 = 32.3\times10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

PROBLEM B.71* (Continued)

(*b*) To determine the direction cosines $\lambda_x,\ \lambda_y,\ \lambda_z$ of each principal axis, use two of the equations of Eqs. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$
$$I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting:

$$[(26.4325 - 4.1443)(10^{-3})](\lambda_x)_1 - (2.5002 \times 10^{-3})(\lambda_y)_1 - (8.8062 \times 10^{-3})(\lambda_z)_1 = 0$$

$$-(2.5002 \times 10^{-3})(\lambda_x)_1 + [(31.1726 - 4.1443)(10^{-3})](\lambda_y)_1 - (4.0627 \times 10^{-3})(\lambda_z)_1 = 0$$

Simplifying

$$8.9146(\lambda_x)_1 - (\lambda_y)_1 - 3.5222(\lambda_z)_1 = 0$$
$$-0.0925(\lambda_x)_1 + (\lambda_y)_1 - 0.1503(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$:

$$(\lambda_z)_1 = 2.4022(\lambda_x)_1$$

and then
$$(\lambda_y)_1 = [8.9146 - 3.5222(2.4022)](\lambda_x)_1$$
$$= 0.45357(\lambda_x)_1$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_1^2 + [0.45357(\lambda_x)_1]^2 + [2.4022(\lambda_x)_1]^2 = 1$$

or $(\lambda_x)_1 = 0.37861$

and $(\lambda_y)_1 = 0.17173$ $(\lambda_z)_1 = 0.90950$

$(\theta_x)_1 = 67.8°$ $(\theta_y)_1 = 80.1°$ $(\theta_z)_1 = 24.6°$ ◀

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

Substituting:

$$[(26.4325 - 29.7840)(10^{-3})](\lambda_x)_2 - (2.5002 \times 10^{-3})(\lambda_y)_2 - (8.8062 \times 10^{-3})(\lambda_z)_2 = 0$$

$$-(2.5002 \times 10^{-3})(\lambda_x)_2 + [(31.1726 - 29.7840)(10^{-3})](\lambda_y)_2 - (4.0627 \times 10^{-3})(\lambda_z)_2 = 0$$

PROBLEM B.71* (Continued)

Simplifying

$$-1.3405(\lambda_x)_2 - (\lambda_y)_2 - 3.5222(\lambda_z)_2 = 0$$
$$-1.8005(\lambda_x)_2 + (\lambda_y)_2 - 2.9258(\lambda_z)_2 = 0$$

Adding and solving for $(\lambda_z)_2$:

$$(\lambda_z)_2 = -0.48713(\lambda_x)_2$$

and then

$$(\lambda_y)_2 = [-1.3405 - 3.5222(-0.48713)](\lambda_x)_2$$
$$= 0.37527(\lambda_x)_2$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2 + [0.37527(\lambda_x)_2]^2 + [-0.48713(\lambda_x)_2]^2 = 1$$

or

$$(\lambda_x)_2 = 0.85184$$

and

$$(\lambda_y)_2 = 0.31967 \qquad (\lambda_z)_2 = -0.41496$$

$$(\theta_x)_1 = 31.6° \quad (\theta_y)_2 = 71.4° \quad (\theta_z)_2 = 114.5° \blacktriangleleft$$

K_3: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_3)(\lambda_x)_3 - I_{xy}(\lambda_y)_3 - I_{zx}(\lambda_z)_3 = 0$$
$$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 - I_{yz}(\lambda_z)_3 = 0$$

Substituting:

$$[(26.4325 - 32.2541)(10^{-3})](\lambda_x)_3 - (2.5002 \times 10^{-3})(\lambda_y)_3 - (8.8062 \times 10^{-3})(\lambda_z)_3 = 0$$

$$-(2.5002 \times 10^{-3})(\lambda_x)_3 + [(31.1726 - 32.2541)(10^{-3})](\lambda_y)_3 - (4.0627 \times 10^{-3})(\lambda_z)_3 = 0$$

Simplifying

$$-2.3285(\lambda_x)_3 - (\lambda_y)_3 - 3.5222(\lambda_z)_3 = 0$$
$$2.3118(\lambda_x)_3 + (\lambda_y)_3 + 3.7565(\lambda_z)_3 = 0$$

Adding and solving for $(\lambda_z)_3$:

$$(\lambda_z)_3 = 0.071276(\lambda_x)_3$$

and then

$$(\lambda_y)_3 = [-2.3285 - 3.5222(0.071276)](\lambda_x)_3$$
$$= -2.5795(\lambda_x)_3$$

PROBLEM B.71* (Continued)

Now substitute into Eq. (9.57):

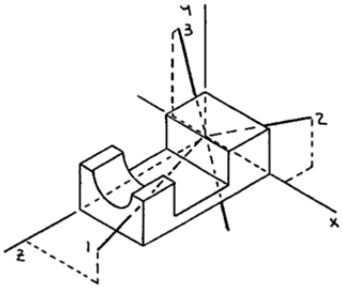
$$(\lambda_x)_3^2 + [-2.5795(\lambda_x)_3]^2 + [0.071276(\lambda_x)_3]^2 = 1 \qquad \text{(i)}$$

or $$(\lambda_x)_3 = 0.36134$$

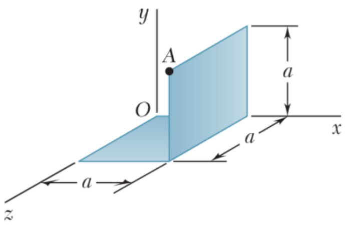
and $$(\lambda_y)_3 = 0.93208 \qquad (\lambda_z)_3 = 0.025755$$

$$(\theta_x)_3 = 68.8° \quad (\theta_y)_3 = 158.8° \quad (\theta_z)_3 = 88.5° \blacktriangleleft$$

(*c*) *Note:* Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.36134$.



Then $$(\theta_x)_3 = 111.2° \quad (\theta_y)_3 = 21.2° \quad (\theta_z)_3 = 91.5° \blacktriangleleft$$



PROBLEM B.72*

For the component described in Problem 9.167, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solution of Problem 9.167, we have

$$I_x = \frac{1}{2}\frac{W}{g}a^2 \qquad I_{xy} = \frac{1}{4}\frac{W}{g}a^2$$

$$I_y = \frac{W}{g}a^2 \qquad I_{yz} = \frac{1}{8}\frac{W}{g}a^2$$

$$I_z = \frac{5}{6}\frac{W}{g}a^2 \qquad I_{zx} = \frac{3}{8}\frac{W}{g}a^2$$

Substituting into Eq. (9.56):

$$K^3 - \left[\left(\frac{1}{2}+1+\frac{5}{6}\right)\left(\frac{W}{g}a^2\right)\right]K^2$$

$$+\left[\left(\frac{1}{2}\right)(1)+(1)\left(\frac{5}{6}\right)+\left(\frac{5}{6}\right)\left(\frac{1}{2}\right)-\left(\frac{1}{4}\right)^2-\left(\frac{1}{8}\right)^2-\left(\frac{3}{8}\right)^2\right]\left(\frac{W}{g}a^2\right)^2 K$$

$$-\left[\left(\frac{1}{2}\right)(1)\left(\frac{5}{6}\right)-\left(\frac{1}{2}\right)\left(\frac{1}{8}\right)^2-(1)\left(\frac{3}{8}\right)^2-\left(\frac{5}{6}\right)\left(\frac{1}{4}\right)^2-2\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)\left(\frac{3}{8}\right)\right]\left(\frac{W}{g}a^2\right)^3=0$$

Simplifying and letting $K = \frac{W}{g}a^2 K$ yields

$$K^3 - 2.33333K^2 + 1.53125K - 0.192708 = 0$$

Solving yields

$$K_1 = 0.163917 \quad K_2 = 1.05402 \quad K_3 = 1.11539$$

The principal moments of inertia are then $K_1 = 0.1639\frac{W}{g}a^2$ ◀

$K_2 = 1.054\frac{W}{g}a^2$ ◀

$K_3 = 1.115\frac{W}{g}a^2$ ◀

PROBLEM B.72* (Continued)

(*b*) To determine the direction cosines $\lambda_x,\ \lambda_y,\ \lambda_z$ of each principal axis, use two of the equations of Eq. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*).

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$
$$-I_{xy}(\lambda_x)_1 + (I_y - K_2)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting

$$\left[\left(\frac{1}{2} - 0.163917\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_1 - \left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_1 - \left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_1 = 0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_1 + \left[(1 - 0.163917)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_1 - \left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_1 = 0$$

Simplifying

$$1.34433(\lambda_x)_1 - (\lambda_y)_1 - 1.5(\lambda_z)_1 = 0$$

$$-0.299013(\lambda_x)_1 + (\lambda_y)_1 - 0.149507(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$:

$$(\lambda_z)_1 = 0.633715(\lambda_x)_1$$

and then

$$(\lambda_y)_1 = [1.34433 - 1.5(0.633715)](\lambda_x)_1$$
$$= 0.393758(\lambda_x)_1$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_1^2 + [0.393758(\lambda_x)_1]^2 + (0.633715(\lambda_x)_1]^2 = 1$$

or $(\lambda_x)_1 = 0.801504$

and $(\lambda_y)_1 = 0.315599$ $\quad (\lambda_z)_1 = 0.507925$

$(\theta_x)_1 = 36.7°$ $\quad (\theta_y)_1 = 71.6°$ $\quad (\theta_z)_1 = 59.5°$ ◀

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - k_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - k_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

PROBLEM B.72* (Continued)

Substituting

$$\left[\left(\frac{1}{2}-1.05402\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_2-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_2-\left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_2=0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_2+\left[(1-1.05402)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_2-\left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_2=0$$

Simplifying

$$-2.21608(\lambda_x)_2-(\lambda_y)_2-1.5(\lambda_z)_2=0$$

$$4.62792(\lambda_x)_2+(\lambda_y)_2+2.31396(\lambda_z)_2=0$$

Adding and solving for $(\lambda_z)_2$

$$(\lambda_z)_2=-2.96309(\lambda_x)_2$$

and then

$$\begin{aligned}(\lambda_y)_2&=[-2.21608-1.5(-2.96309)](\lambda_x)_2\\&=2.22856(\lambda_x)_2\end{aligned}$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2+[2.22856(\lambda_x)_2]^2+[-2.96309(\lambda_x)_2]^2=1$$

or $(\lambda_x)_2=0.260410$

and $(\lambda_y)_2=0.580339$ $(\lambda_z)_2=-0.771618$

$(\theta_x)_2=74.9°$ $(\theta_y)_2=54.5°$ $(\theta_z)_2=140.5°$ ◀

K_3: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x-K_3)(\lambda_x)_3-I_{xy}(\lambda_y)_3-I_{zx}(\lambda_z)_3=0$$

$$-I_{xy}(\lambda_x)_3+(I_y-K_3)(\lambda_y)_3-I_{yz}(\lambda_z)_3=0$$

Substituting

$$\left[\left(\frac{1}{2}-1.11539\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_3-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_3-\left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_3=0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_3+\left[(1-1.11539)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_3-\left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_3=0$$

Simplifying

$$-2.46156(\lambda_x)_3-(\lambda_y)_3-1.5(\lambda_z)_3=0$$

$$2.16657(\lambda_x)_3+(\lambda_y)_3+1.08328(\lambda_z)_3=0$$

PROBLEM B.72* (Continued)

Adding and solving for $(\lambda_z)_3$

$$(\lambda_z)_3 = -0.707885(\lambda_x)_3$$

and then

$$(\lambda_y)_3 = [-2.46156 - 1.5(-0.707885)](\lambda_x)_3$$
$$= -1.39973(\lambda_x)_3$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_3^2 + [-1.39973(\lambda_x)_3]^2 + [-0.707885(\lambda_x)_3]^2 = 1 \qquad \text{(i)}$$

or

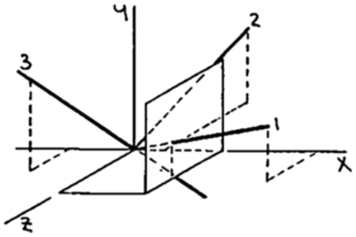
$$(\lambda_x)_3 = 0.537577$$

and

$$(\lambda_y)_3 = -0.752463 \qquad (\lambda_z)_3 = -0.380543$$

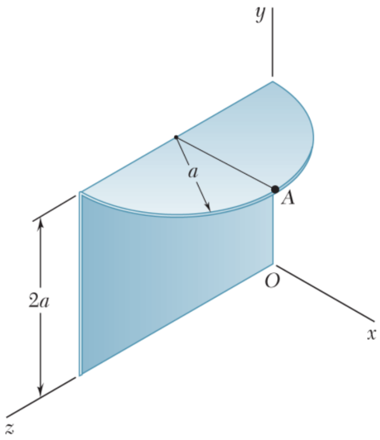
$$(\theta_x)_3 = 57.5° \quad (\theta_y)_3 = 138.8° \quad (\theta_z)_3 = 112.4° \blacktriangleleft$$

(*c*) *Note:* Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.537577$.



Then

$$(\theta_x)_3 = 122.5° \quad (\theta_y)_3 = 41.2° \quad (\theta_z)_3 = 67.6° \blacktriangleleft$$



PROBLEM B.73*

For the component described in Problem 9.168, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solution to Problem 9.168, we have

$$I_x = 18.91335\frac{\gamma t}{g}a^4 \qquad I_{xy} = 1.33333\frac{\gamma t}{g}a^4$$

$$I_y = 7.68953\frac{\gamma t}{g}a^4 \qquad I_{yz} = 7.14159\frac{\gamma t}{g}a^4$$

$$I_z = 12.00922\frac{\gamma t}{g}a^4 \qquad I_{zx} = 0.66667\frac{\gamma t}{g}a^4$$

Substituting into Eq. (9.56):

$$\begin{aligned}
K^3 &- \left[(18.91335 + 7.68953 + 12.00922)\left(\frac{\gamma t}{g}a^4\right)\right]K^2 \\
&+ [(18.91335)(7.68953) + (7.68953)(12.00922) + (12.00922)(18.91335) \\
&- (1.33333)^2 - (7.14159)^2 + (0.66667)^2]\left(\frac{\gamma t}{g}a^4\right)^2 K \\
&- [(18.91335)(7.68953)(12.00922) - (18.91335)(7.14159)^2 \\
&- (7.68953)(0.66667)^2 - (12.00922)(1.33333)^2 \\
&- 2(1.33333)(7.14159)(0.66667)]\left(\frac{\gamma t}{g}a^4\right)^3 = 0
\end{aligned}$$

Simplifying and letting

$$K = \frac{\gamma t}{g}a^4 K \quad \text{yields}$$

$$K^3 - 38.61210K^2 + 411.69009K - 744.47027 = 0$$

PROBLEM B.73* (Continued)

Solving yields

$$K_1 = 2.25890 \quad K_2 = 17.27274 \quad K_3 = 19.08046$$

The principal moments of inertia are then $K_1 = 2.26\dfrac{\gamma t}{g}a^4$ ◀

$$K_2 = 17.27\frac{\gamma t}{g}a^4 \text{ ◀}$$

$$K_3 = 19.08\frac{\gamma t}{g}a^4 \text{ ◀}$$

(*b*) To determine the direction cosines $\lambda_x,\ \lambda_y,\ \lambda_z$ of each principal axis, use two of the equations of Eq. (9.54) and Eq. (9.57). Then

K_1: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$
$$-I_{xy}(\lambda_x)_1 + (I_y - K_2)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting

$$\left[(18.91335 - 2.25890)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_x)_1 - \left(1.33333\frac{\gamma t}{a}a^4\right)(\lambda_y)_1 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_1 = 0$$

$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_1 + \left[(7.68953 - 2.25890)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_1 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_1 = 0$$

Simplifying

$$12.49087(\lambda_x)_1 - (\lambda_y)_1 - 0.5(\lambda_z)_1 = 0$$
$$-0.24552(\lambda_x)_1 + (\lambda_y)_1 - 1.31506(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$

$$(\lambda_z)_1 = 6.74653(\lambda_x)_1$$

and then

$$(\lambda_y)_1 = [12.49087 - (0.5)(6.74653)](\lambda_x)_1$$
$$= 9.11761(\lambda_x)_1$$

Now substitute into Eq. (9.57): $(\lambda_x)_1^2 + [9.11761(\lambda_x)_1]^2 + [6.74653(\lambda_x)_1]^2 = 1$

or $(\lambda_x)_1 = 0.087825$

and $(\lambda_y)_1 = 0.80075 \qquad (\lambda_z)_1 = 0.59251$

$$(\theta_x)_1 = 85.0° \quad (\theta_y)_1 = 36.8° \quad (\theta_z)_1 = 53.7° \text{ ◀}$$

PROBLEM B.73* (Continued)

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

Substituting

$$[(18.91335 - 17.27274)\left(\frac{\gamma t}{g}a^4\right)(\lambda_x)_2 - \left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_y)_2 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_2 = 0$$
$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_2 + \left[(7.68953 - 17.27274)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_2 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_2 = 0$$

Simplifying

$$1.23046(\lambda_x)_2 - (\lambda_y)_2 - 0.5(\lambda_z)_2 = 0$$
$$0.13913(\lambda_x)_2 + (\lambda_y)_2 + 0.74522(\lambda_z)_2 = 0$$

Adding and solving for $(\lambda_z)_2$

$$(\lambda_z)_2 = -5.58515(\lambda_x)_2$$

and then $(\lambda_y)_2 = [1.23046 - (0.5)(-5.58515)](\lambda_x)_2$

$$= 4.02304\,(\lambda_x)_2$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_2^2 + [4.02304(\lambda_x)_2]^2 + [-5.58515(\lambda_x)_2]^2 = 1$$

or $(\lambda_x)_2 = 0.14377$

and $(\lambda_y)_2 = 0.57839 \qquad (\lambda_z)_2 = -0.80298$

$$(\theta_x)_2 = 81.7^\circ \quad (\theta_y)_2 = 54.7^\circ \quad (\theta_z)_2 = 143.4^\circ \blacktriangleleft$$

K_3: Begin with Eqs. (9.54*a*) and (9.54*b*):

$$(I_x - K_3)(\lambda_x)_3 - I_{xy}(\lambda_y)_3 - I_{zx}(\lambda_z)_3 = 0$$
$$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 - I_{yz}(\lambda_y)_3 = 0$$

Substituting

$$[(18.91335 - 19.08046)\left(\frac{\gamma t}{g}a^4\right)(\lambda_x)_3 - \left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_y)_3 - \left(0.66667\frac{\gamma t}{g}a^4\right)(\lambda_z)_3 = 0$$
$$-\left(1.33333\frac{\gamma t}{g}a^4\right)(\lambda_x)_3 + \left[(7.68953 - 19.08046)\left(\frac{\gamma t}{g}a^4\right)\right](\lambda_y)_3 - \left(7.14159\frac{\gamma t}{g}a^4\right)(\lambda_z)_3 = 0$$

PROBLEM B.73* (Continued)

Simplifying

$$-0.12533(\lambda_x)_3 - (\lambda_y)_3 - 0.5(\lambda_z)_3 = 0$$
$$0.11705(\lambda_x)_3 + (\lambda_y)_3 + 0.62695(\lambda_z)_3 = 0$$

Adding and solving for $(\lambda_z)_3$

$$(\lambda_z)_3 = 0.06522(\lambda_x)_3$$

and then

$$(\lambda_y)_3 = [-0.12533 - (0.5)(0.06522)](\lambda_x)_3$$
$$= -0.15794(\lambda_x)_3$$

Now substitute into Eq. (9.57):

$$(\lambda_x)_3^2 + [-0.15794(\lambda_x)_3]^2 + [0.06522(\lambda_x]_3]^2 = 1 \qquad \text{(i)}$$

or

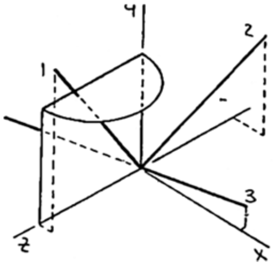
$$(\lambda_x)_3 = 0.98571$$

and

$$(\lambda_y)_3 = -0.15568 \qquad (\lambda_z)_3 = 0.06429$$

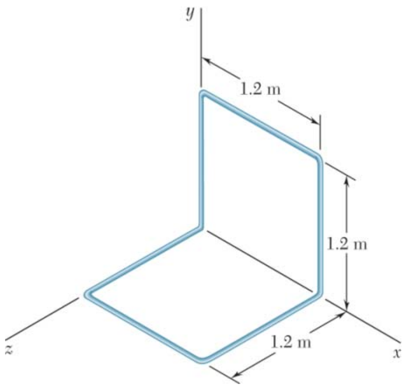
$$(\theta_x)_3 = 9.7° \quad (\theta_y)_3 = 99.0° \quad (\theta_z)_3 = 86.3° ◀$$

(*c*) *Note:* Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen; that is, $(\lambda_x)_3 = -0.98571$.



Then

$$(\theta_x)_3 = 170.3° \quad (\theta_y)_3 = 81.0° \quad (\theta_z)_3 = 93.7° ◀$$



PROBLEM B.74*

For the component described in Problems 9.148 and 9.170, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solutions to Problems 9.148 and 9.170. We have

$$I_x = 0.32258 \text{ kg}\cdot\text{m}^2 \qquad I_y = I_z = 0.41933 \text{ kg}\cdot\text{m}^2$$

$$I_{xy} = I_{zx} = 0.096768 \text{ kg}\cdot\text{m}^2 \qquad I_{yz} = 0$$

Substituting into Eq. (9.56) and using

$$I_y = I_z \qquad I_{xy} = I_{zx} \qquad I_{yz} = 0$$

$$\begin{aligned} &K^3 - [0.32258 + 2(0.41933)]K^2 \\ &\quad + [2(0.32258)(0.41933) + (0.41933)^2 - 2(0.096768)^2]K \\ &\quad - [(0.32258)(0.41933)^2 - 2(0.41933)(0.096768)^2] = 0 \end{aligned}$$

Simplifying

$$K^3 - 1.16124K^2 + 0.42764K - 0.048869 = 0$$

Solving yields

$$K_1 = 0.22583 \text{ kg}\cdot\text{m}^2 \qquad \text{or} \qquad K_1 = 0.226 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

$$K_2 = 0.41920 \text{ kg}\cdot\text{m}^2 \qquad \text{or} \qquad K_2 = 0.419 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

$$K_3 = 0.51621 \text{ kg}\cdot\text{m}^2 \qquad \text{or} \qquad K_3 = 0.516 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

(*b*) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, use two of the equations of Eqs. (9.54) and (9.57). Then

K_1: Begin with Eqs. (9.54*b*) and (9.54*c*):

$$-I_{xy}(\lambda_x)_1 + (I_y - K_1)(\lambda_y)_1 - \cancelto{0}{I_{yz}}(\lambda_z)_3 = 0$$

$$-I_{zx}(\lambda_x)_1 - \cancelto{0}{I_{yz}}(\lambda_y)_1 + (I_z - K_1)(\lambda_z)_3 = 0$$

PROBLEM B.74* (Continued)

Substituting

$$-(0.096768)(\lambda_x)_1 + (0.41933 - 0.22583)(\lambda_y)_1 = 0$$
$$-(0.096768)(\lambda_x)_1 + (0.41933 - 0.22583)(\lambda_z)_1 = 0$$

Simplifying yields

$$(\lambda_y)_1 = (\lambda_z)_1 = 0.50009(\lambda_x)_1$$

Now substitute into Eq. (9.54):

$$(\lambda_x)^2 + 2[0.50009(\lambda_x)_1]^2 = 1$$

or $$(\lambda_x)_1 = 0.81645$$

and $$(\lambda_y)_1 = (\lambda_z)_1 = 0.40830$$

$$(\theta_x)_1 = 35.3° \quad (\theta_y)_1 = (\theta_z)_1 = 65.9° \blacktriangleleft$$

K_2: Begin with Eqs. (9.54*a*) and (9.54*b*)

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$
$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - \cancelto{0}{I_{yz}}(\lambda_z)_2 = 0$$

Substituting

$$(0.32258 - 0.41920)(\lambda_x)_2 - (0.096768)(\lambda_y)_2 - (0.096768)(\lambda_z)_2 = 0 \qquad \text{(i)}$$

$$-(0.96768)(\lambda_x)_2 + (0.41933 - 0.41920)(\lambda_y)_2 = 0 \qquad \text{(ii)}$$

Eq. (ii) $\Rightarrow (\lambda_x)_2 = 0$

and then Eq. (i) $\Rightarrow (\lambda_z)_2 = -(\lambda_y)_2$

Now substitute into Eq. (9.57):

$$\cancelto{0}{(\lambda_x)_2^2} + (\lambda_y)_2^2 + [-(\lambda_y)_2]^2 = 1$$

or $$(\lambda_y)_2 = \frac{1}{\sqrt{2}}$$

and $$(\lambda_z)_2 = -\frac{1}{\sqrt{2}}$$

$$(\theta_x)_2 = 90.0° \quad (\theta_y)_2 = 45.0° \quad (\theta_z)_2 = 135.0° \blacktriangleleft$$

PROBLEM B.74* (Continued)

K_3: Begin with Eqs. (9.54*b*) and (9.54*c*):

$$I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 + \cancelto{0}{I_{yz}}(\lambda_z)_3 = 0$$

$$-I_{zx}(\lambda_x)_3 - \cancelto{0}{I_{yz}}(\lambda_y)_3 + (I_z - K_3)(\lambda_z)_3 = 0$$

Substituting

$$-(0.096768)(\lambda_x)_3 + (0.41933 - 0.51621)(\lambda_y)_3 = 0$$

$$-(0.096768)(\lambda_x)_3 + (0.41933 - 0.51621)(\lambda_z)_3 = 0$$

Simplifying yields

$$(\lambda_y)_3 = (\lambda_z)_3 = -(\lambda_x)_3$$

Now substitute into Eq. (9.57):

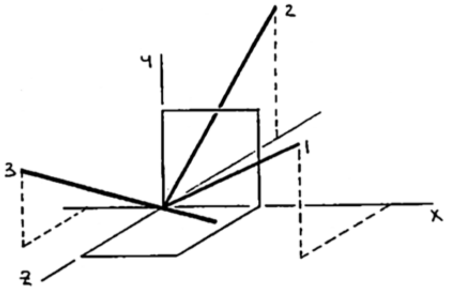
$$(\lambda_x)_3^2 + 2[-(\lambda_x)_3]^2 = 1 \qquad \text{(i)}$$

or $$(\lambda_x)_3 = \frac{1}{\sqrt{3}}$$

and $$(\lambda_y)_3 = (\lambda_z) = -\frac{1}{\sqrt{3}}$$

$$(\theta_x)_3 = 54.7^\circ \quad (\theta_y)_3 = (\theta_z)_3 = 125.3^\circ \blacktriangleleft$$

(*c*)



Note: Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Eq. (i) must be chosen;

That is, $$(\lambda_x)_3 = -\frac{1}{\sqrt{3}}$$

Then $$(\theta_x)_3 = 125.3^\circ \qquad (\theta_y)_3 = (\theta_z)_3 = 54.7^\circ \blacktriangleleft$$